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Function and Operator Theory on Large Bergman spaces

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Declaration

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Contents

Introduction

The theory of Bergman spaces has been a central subject of study in complex analysis during the past decades. The book [7] by S. Bergman contains the first systematic treatment of the Hilbert space of square integrable analytic functions with respect to Lebesgue area measure on a domain. His approach was based on a reproducing kernel that became known as the Bergman kernel function. When attention was later directed to the spaces A^p over the unit disk, it was natural to call them Bergman spaces. As counterparts of Hardy spaces, they presented analogous problems. However, although many problems in Hardy spaces were well understood by the 1970s, their counterparts for Bergman spaces were generally viewed as intractable, and only some isolated progress was done. The 1980s saw the emerging of operator theoretic studies related to Bergman spaces with important contributions by several authors. Their achievements on Bergman spaces with standard weights are presented in Zhu's book [77]. The main breakthroughs came in the 1990s, where in a flurry of important advances, problems previously considered intractable began to be solved. First came Hedenmalm‘s construction of canonical divisors [26], then Seip's description [59] of sampling and interpolating sequences on Bergman spaces, and later on, the study of Aleman, Richter and Sundberg [1] on the invariant subspaces of A^2, among others. This attracted other workers to the field and inspired a period of intense research on Bergman spaces and related topics. Nowadays there are rich theories on Bergman spaces that can be found on the textbooks [27] and [22].

Meanwhile, also in the nineties, some isolated problems on Bergman spaces with exponential type weights began to be studied. These spaces are large in the sense that they contain all the Bergman spaces with standard weights, and their study presented new difficulties, as the techniques and ideas that led to success when working on the analogous problems for standard Bergman spaces, failed to work on that context. It is the main goal of this work to do a deep study of the function theoretic properties of such spaces, as well as of some operators acting on them. It turns out that large Bergman spaces are close in spirit to Fock spaces [79], and many times mixing classical techniques from both Bergman and Fock spaces in an appropriate way, can led to some success when studying large Bergman spaces.

This dissertation is structured into five chapters.

In Chapter 1, we give some basic properties of the large Bergman spaces $A^p(\omega)$, for weights ω in a certain class $\mathcal{W}$ of rapidly decreasing weights, considered previously in [23] and [51]. The concrete definition of the class is given, and it is seen that contains the family of exponential type weights

$$\omega_\sigma(z) = \exp\left(\frac{-A}{(1-|z|^2)^\sigma}\right), \qquad \sigma > 0,\ A > 0, \tag{0.1}$$

and also weights decreasing even faster such as some double exponential weights. We collect some of the well known results proved in earlier works (pointwise estimates, completeness, construction of suitable test functions, description of Carleson-type measures, etc.), but we also include some new ones as the atomic decomposition for the Hilbert space $A^2(\omega)$, or the estimates for the solutions of the d-bar equation obtained in Theorem 1.4, a result used later in Chapter 4 in order to characterize bounded and compact Hankel operators with conjugate analytic symbols. The test functions given in Lemma C are useful in order to study several problems on $A^p(\omega)$ even for $p \neq 2$ (see [9] for the study of sampling and interpolating sequences, or [51] for the study of the boundedness of certain Cesaro type integration operators acting between large weighted Bergman spaces), but the fact that estimates for the norm of the reproducing kernels K_z are only available when $p = 2$ allows us to consider other problems only in the Hilbert space case, as for example, the study of Toeplitz operators in the next chapter. The topics of this chapter will serve as a basis for our later chapters.

Chapter 2 is devoted to characterize the boundedness, compactness and membership in Schatten classes of the Toeplitz operator T_μ acting on $A^2(\omega)$ for weights in our class $\mathcal{W}$. If μ is a finite positive Borel measue on $\mathbb{D}$, the Toeplitz operator T_μ is the integral type operator given by

$$T_\mu f(z) = \int_{\mathbb{D}} f(\xi)\overline{K_z(\xi)}\,\omega(\xi)\,d\mu(\xi), \qquad f \in H(\mathbb{D}),$$

where K_z denotes the reproducing kernel at the point $z \in \mathbb{D}$. The operator T_μ acting on standard weighted Bergman spaces has been extensively studied [75]. Luecking was probably one of the first who considered T_μ with measures as symbols, and the study of Toeplitz operators acting on large weighted Bergman spaces was initiated by Lin and Rochberg [35], where they proved descriptions of the boundedness and compactness of the Toeplitz operator in terms of the behavior of a certain averaging function of μ. Since their class of weights is slightly different of our class, we will offer a proof for the both descriptions in Theorem 2.1.

The main result of this chapter is Theorem 2.8 where a complete description, valid for all $0 < p < \infty$, of when the Toeplitz operator T_μ acting on $A^2(\omega)$ belongs to the Schatten ideal S_p. This extends the description obtained for standard Bergman spaces [78], and solves a problem posed by Lin and Rochberg in [32], where only one implication was proved. Just to mention here that the results of this chapter has been recently published in our paper [4].

In Chapter 3 we study the area operator A_μ acting on large weighted Bergman spaces. Given a positive Borel measure μ on the unit disk $\mathbb{D}$, the area operator A_μ is the sublinear operator defined by

$$A_\mu(f)(\zeta) = \int_{\Gamma(\zeta)} |f(z)| \, \frac{d\mu(z)}{1-|z|^2}, \qquad \zeta \in \mathbb{T} := \partial\mathbb{D},$$

where $\Gamma(\zeta)$ is a typical non-tangential approach region (Stolz region) with vertex $\zeta \in \mathbb{T}$.

The area operator is useful in harmonic analysis, and is closely related to, for example, non-tangential maximal functions, Poisson integrals, Littlewood-Paley operators, tent spaces, etc. The study of the area operator acting on the classical Hardy spaces H^p was initiated by W. Cohn [14]. He proved that, for $0 < p < \infty$, the area operator $A_\mu : H^p \to L^p(\mathbb{T})$ is bounded if and only if μ is a classical Carleson measure. This was pursued later in [24], where a full description of the boundedness of $A_\mu : H^p \to L^q(\mathbb{T})$ for the case $0 < p < q < \infty$ and $1 \le q < p < \infty$ was obtained. In the setting of standard Bergman spaces A^p_α, the study of the area operator was initiated in [69] by Z. Wu, who obtained a characterization of the boundedness of $A_\mu : A^p_\alpha \to L^q(\mathbb{T})$ for $1 \le p, q < \infty$. In this chapter we are going to extend these results to our large Bergman spaces $A^p(\omega)$ for weights ω in the class $\mathcal{W}$ and characterize those positive Borel measures μ for which the area operator $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$, $1 \le p, q < \infty$ is bounded. Some of the key tools used in the proofs are: the test functions given in Lemma C, the classical description of Carleson measures for Hardy spaces, and the recent description [51] of Carleson type measures for large weighted Bergman spaces.

In Chapter 4 we add an extra condition to our class of weights $\mathcal{W}$, and consider the class $\mathcal{E}$ that consists of those weights $\omega \in \mathcal{W}$ satisfying

$$\int_{\mathbb{D}} |K_z(\xi)| \, \omega(\xi)^{1/2} \, dA(\xi) \le C \, \omega(z)^{-1/2}, \quad z \in \mathbb{D}. \tag{0.2}$$

This condition will allow us to extend some of the previous results to the non-Hilbert space setting. It has been recently proved in [16] that the exponential type weights ω_σ given in (0.1) with $\sigma = 1$ satisfy the previous condition and, therefore, they are in the class $\mathcal{E}$. We are able to show, following the proof given in [16] with non-trivial modifications, that all the family of exponential type weights given in (0.1) are in the class $\mathcal{E}$ for all $0 < \sigma < \infty$. The integral estimate (0.2) allows to study other properties and operators, such as the Bergman projection which is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\xi) \, \overline{K_z(\xi)} \, \omega(\xi) \, dA(\xi), \quad z \in \mathbb{D}.$$

The boundedness of the Bergman projection P_ω on $L^2(\omega)$ is trivial from the general theory of Hilbert spaces. In contrast with the case of standard Bergman spaces (where the Bergman projection is bounded for $1 < p < \infty$), in the case of exponential type weights it turns out that the the natural Bergman projection is not bounded on $L^p(\omega)$ unless $p = 2$

(see [18] and [71]). At first glance, this may look as a surprise, but when one takes into account the similarities with Fock spaces, this seems to be more natural. It turns out that, similarly as in the setting of Fock spaces, when studying problems where the reproducing kernels are involved, the most convenient setting are the spaces $A^p(\omega^{p/2})$. As a consequence of condition (0.2), we get the right estimates for the norm of the reproducing kernels in $A^p(\omega^{p/2})$ for $1 \leq p \leq \infty$. Also, we prove in Theorem 4.4 that, for weights in the class $\mathcal{E}$, the Bergman projection $P_\omega : L^p(\omega^{p/2}) \to A^p(\omega^{p/2})$ is bounded for $1 \leq p \leq \infty$. A consequence of that result will be the identification of the dual space of $A^p(\omega^{p/2})$ with the space $A^{p'}(\omega^{p'/2})$ and $A^1(\omega^{1/2})$ with $A^\infty(\omega^{1/2})$ under the natural integral pairing $\langle \cdot, \cdot \rangle_\omega$, where p' denotes the conjugate exponent of p. Afterwards, by using the duality and the estimates for the p-norms of reproducing kernels, we extend in Theorem 4.12 the atomic decomposition obtained in Chapter 1 to the non-Hilbert space setting: for weights $\omega \in \mathcal{E}$, every function in the weighted Bergman space $A^p(\omega^{p/2})$, $1 \leq p < \infty$ can be decomposed into a series of very nice functions (called atoms). These atoms are defined in terms of kernels functions and in some sense act as a basis for the space $A^p(\omega^{p/2})$. The atomic decomposition for Bergman space with standard weights was obtained by Coifman and Rochberg [15], and has become a powerful tool in the study of the properties of weighted Bergman spaces having found many applications.

The norm estimates for the reproducing kernels in $A^p(\omega^{p/2})$ for $1 \leq p < \infty$ permits to extend the results on the boundedness and compactness of Toeplitz operators T_μ and to consider the action of T_μ between different large weighted Bergman spaces, and to find a general description of when $T_\mu : A^p(\omega^{p/2}) \to A^q(\omega^{q/2})$ is bounded or compact for all values of $1 \leq p, q < \infty$. Furthermore, we also generalize the results obtained in [23] on the boundedness and compactness of big Hankel operators $H_{\overline{g}}$ with conjugate analytic symbols to the non-Hilbert space setting, characterizing for all $1 < p, q < \infty$, when $H_{\overline{g}} : A^p(\omega^{p/2}) \to L^q(\omega^{q/2})$, is bounded or compact. As mentioned earlier, one of the key tools used is the estimates for the d-bar equation obtained in the first chapter.

Finally, in Chapter 5, we discuss some problems we have not been able to solve, as well other interesting problems to look on the future.

Chapter 1

Large Bergman spaces

1.1 Basic properties

Let $\mathbb{D}$ be the unit disk in the complex plane, $dA(z) = \frac{dxdy}{\pi}$ be the normalized area measure on $\mathbb{D}$, and let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. A weight is a positive function $\omega \in L^1(\mathbb{D}, dA)$. When $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, we say that ω is radial.

For $0 < p < \infty$, the weighted Bergman space $A^p(\omega)$ is the space of all functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p(\omega)} = \left(\int_{\mathbb{D}} |f(z)|^p \, \omega(z) \, dA(z) \right)^{\frac{1}{p}} < \infty.$$

Our main goal is to study the Bergman spaces $A^p(\omega)$ for a large class of weights, which includes certain rapidly radial decreasing weights, that is, weights that are going to decrease faster than any standard weight $(1-|z|^2)^\alpha$, $\alpha > 0$, such as the exponential type weights

$$\omega_\sigma(z) = \exp\left(\frac{-A}{(1-|z|^2)^\sigma} \right), \quad \sigma > 0, \; A > 0. \tag{1.1}$$

Definition 1.1. *We say that a positive function τ belongs to the class $\mathcal{L}$ if it satisfies the following two properties:*
(A) there exists $c_1 > 0$ such that $\tau(z) \le c_1(1-|z|)$, for all $z \in \mathbb{D}$.
(B) there exists $c_2 > 0$ such that $|\tau(z) - \tau(w)| \le c_2|z-w|$, for all $z, w \in \mathbb{D}$.

For $a \in \mathbb{D}$ and $\delta > 0$ we use too the notation $D(\delta\tau(a))$ for the euclidian disc centered at a and radius $\delta\tau(a)$ and

$$m_\tau := \frac{\min(1, c_1,^{-1} c_2^{-1})}{4},$$

where c_1 and c_2 are the constants appearing in the previous definition. From the above definition it is easy to show (see [51, Lemma 2.1]) that if $\tau \in \mathcal{L}$ and $z \in D(\delta\tau(a))$, then

$$\frac{1}{2}\,\tau(a) \le \tau(z) \le 2\,\tau(a), \tag{1.2}$$

for sufficiently small $\delta > 0$, that is, for $\delta \in (0, m_\tau)$. This fact will be used repeatedly throughout this work.

Also, throughout this manuscript, the letter C will denote an absolute constant whose value may change at different occurrences. We also use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \leq Cb$, and the notation $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$.

Definition 1.2. *We say that a weight ω is in the class $\mathcal{L}^*$ if it is of the form $\omega = e^{-2\varphi}$, where $\varphi \in \mathcal{C}^2(\mathbb{D})$ with $\Delta\varphi > 0$, and $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$, with $\tau(z)$ being a function in the class $\mathcal{L}$. Here Δ denotes the classical Laplace operator.*

The following result from [51, Lemma 2.2] will play an essential role in the proof of the main theorems of this work and can be thought as some type of generalized sub-mean value property for $|f|^p\,\omega$ that gives the boundedness of the point evaluation functionals on $A^p(\omega)$.

Lemma A. *Let $\omega \in \mathcal{L}^*$, $0 < p < \infty$ and $z \in \mathbb{D}$. If $\beta \in \mathbb{R}$ there exists $M \geq 1$ such that*

$$|f(z)|^p\,\omega(z)^\beta \leq \frac{M}{\delta^2\tau(z)^2}\int_{D(\delta\tau(z))} |f(\xi)|^p\,\omega(\xi)^\beta\,dA(\xi),$$

for all $f \in H(\mathbb{D})$ and all $\delta > 0$ sufficiently small.

It can be seen from the proof given in [51] that one only needs f to be holomorphic in a neighbourhood of $D(\delta\tau(z))$.

Another consequence of the above result is that the Bergman space $A^p(\omega^\beta)$ is a Banach space when $1 \leq p < \infty$ and a complete metric space when $0 < p < 1$. In particular $A^2(\omega)$ is a Hilbert space with the inner product inherited from $L^2(\mathbb{D}, \omega\,dA)$. Another immediate consequence is that, for any $z \in \mathbb{D}$, the point evaluations $\ell_z : f \longmapsto f(z)$ are bounded linear functionals on $A^p(\omega^\beta)$. In particular, when $p = 2$, it follows from the Riesz representation theorem that there are functions $K_z \in A^2(\omega)$ with $\|\ell_z\| = \|K_z\|_{A^2(\omega)}$ such that

$$f(z) = \langle f, K_z\rangle_\omega := \int_{\mathbb{D}} f(\xi)\overline{K_z(\xi)}\;\omega(\xi)dA(\xi)$$

for all $f \in A^2(\omega)$. The function K_z has the property that $K_z(\xi) = \overline{K_\xi(z)}$, and is called the reproducing kernel for the Bergman space $A^2(\omega)$. It is straightforward to see from the previous formula that the orthogonal (Bergman) projection from $L^2(\mathbb{D}, \omega dA)$ to $A^2(\omega)$ is given by

$$P_\omega f(z) = \int_{\mathbb{D}} f(\xi)\overline{K_z(\xi)}\,\omega(\xi)\;dA(\xi).$$

We also need a similar estimate as in Lemma A, but for the gradient of $|f|\,\omega^{1/2}$.

Lemma 1.1. *Let $\omega \in \mathcal{L}^*$ and $0 < p < \infty$. For any $\delta_0 > 0$ sufficiently small there exists a constant $C(\delta_0) > 0$ such that*

$$\left|\nabla(|f|\,\omega^{1/2})(z)\right| \leq \frac{C(\delta_0)}{\tau(z)^{1+\frac{2}{p}}}\left(\int_{D(\delta_0\tau(z)/2)} |f(\xi)|^p\,\omega(\xi)^{p/2} dA(\xi)\right)^{1/p},$$

for all $f \in H(\mathbb{D})$.

Proof. We follow the method used in [49]. Without loss of generality we can assume $z = 0$. Then, applying the Riesz's decomposition (see for example [56]) of the subharmonic function φ in $D(0, \frac{\delta_0}{2}\tau(0))$, we obtain

$$\varphi(\xi) = u(\xi) + \int_{D(\frac{r}{2})} G(\xi,\eta)\Delta\varphi(\eta)dA(\eta), \tag{1.3}$$

where $r = \delta_0\tau(0)$, u is the least harmonic majorant of φ in $D(0, \frac{r}{2})$ and G is the Green function defined for every $\xi, \eta \in D(0, r)$, $\xi \neq \eta$ by

$$G(\xi,\eta) := \log\left|\frac{r(\xi-\eta)}{r^2 - \bar{\eta}\xi}\right|^2.$$

For $\xi, \eta \in D(0, \frac{r}{2})$ we have

$$\begin{aligned}\left|\frac{\partial G}{\partial \xi}(\xi,\eta)\right| &= \frac{r^2 - |\eta|^2}{|\xi-\eta||r^2 - \eta\bar{\xi}|}\\ &\leq \frac{r^2}{|\xi-\eta|\cdot\big|r^2 - |\eta||\xi|\big|} \leq \frac{4}{3|\xi-\eta|}.\end{aligned}$$

Then,

$$\begin{aligned}\left|\frac{\partial\varphi(0)}{\partial\xi} - \frac{\partial u(0)}{\partial\xi}\right| &\leq \int_{D(\frac{r}{2})}\left|\frac{\partial G}{\partial\xi}(0,\eta)\right|\Delta\varphi(\eta)dA(\eta)\\ &\lesssim \frac{1}{\tau(0)^2}\int_{D(\frac{r}{2})}\frac{dA(\eta)}{|\eta|} = \frac{\delta_0}{\tau(0)}.\end{aligned} \tag{1.4}$$

We pick a function $h \in H(\mathbb{D})$ such that $Re(h) = u$. Also,

$$\begin{aligned}\left|\nabla(|f|e^{-\varphi})(\xi)\right| &= \left|\frac{1}{2}\frac{f'(\xi)\overline{f(\xi)}}{|f(\xi)|} - \frac{\partial\varphi}{\partial\xi}(\xi)|f(\xi)|\right|e^{-\varphi(\xi)}\\ &= \frac{1}{2}\left|f'(\xi) - 2\frac{|f(\xi)|^2}{\overline{f(\xi)}}\frac{\partial\varphi}{\partial\xi}(\xi)\right|e^{-\varphi(\xi)}.\end{aligned}$$

Therefore, since $h'(0) = 2\frac{\partial u}{\partial \xi}(0)$, we get

$$\left|\nabla(|f|e^{-\varphi})(0)\right| = \frac{1}{2}\left|f'(0) - 2f(0)\frac{\partial \varphi}{\partial \xi}(0)\right| e^{-\varphi(0)}$$

$$\leq \frac{1}{2}\left|f'(0) - 2f(0)\frac{\partial u}{\partial \xi}(0)\right| e^{-\varphi(0)} + \left|\frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0)\right| |f(0)| e^{-\varphi(0)}$$

$$\lesssim \left|\frac{\partial (f e^{-h})(0)}{\partial \xi}\right| e^{u(0)-\varphi(0)} + \left|\frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0)\right| |f(0)| e^{-\varphi(0)}.$$

By (1.4) we have

$$\left|\frac{\partial u}{\partial \xi}(0) - \frac{\partial \varphi}{\partial \xi}(0)\right| |f(0)| e^{-\varphi(0)} \lesssim \frac{\delta_0}{\tau(0)}\, |f(0)|\, e^{-\varphi(0)}.$$

This gives

$$\left|\nabla(|f|e^{-\varphi})(0)\right| \lesssim \left|\frac{\partial (f e^{-h})(0)}{\partial \xi}\right| e^{u(0)-\varphi(0)} + \frac{|f(0)|}{\tau(0)}\, e^{-\varphi(0)}. \tag{1.5}$$

It follows from Lemma A that

$$\frac{|f(0)|}{\tau(0)}\, e^{-\varphi(0)} \lesssim \frac{1}{\tau(0)^{1+\frac{2}{p}}} \left(\int_{D(\delta_0 \tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} dA(z) \right)^{1/p}. \tag{1.6}$$

To manage the other term appearing in (1.5), notice that if we use the identity (1.3) with the function $\phi(\xi) = |\xi|^2 - (r/2)^2$ (since $\Delta\phi(\xi) = 4$ and its least harmonic majorant is $u_\phi = 0$), we obtain

$$\int_{D(\frac{r}{2})} G(\xi, \eta)\, dA(\eta) = \frac{1}{4}\big(|\xi|^2 - (r/2)^2\big).$$

Therefore, since $\Delta\varphi(\eta) = \frac{1}{\tau(\eta)^2} \lesssim \frac{1}{\tau(0)^2} = \Delta\varphi(0)$ and the Green's function $G \leq 0$, we obtain for every $\xi \in D(0, \frac{r}{2})$

$$u(\xi) - \varphi(\xi) = -\int_{D(\frac{r}{2})} G(\xi, \eta) \Delta\varphi(\eta) dA(\eta)$$

$$\lesssim \frac{\Delta\varphi(0)}{4}\big((r/2)^2 - |\xi|^2\big) = \frac{1}{4\tau(0)^2}\big((r/2)^2 - |\xi|^2\big).$$

This gives

$$e^{u(0)-\varphi(0)} \leq e^{C\delta_0^2}.$$

Therefore

$$\left|\frac{\partial(fe^{-h})(0)}{\partial\xi}\right| e^{u(0)-\varphi(0)} \lesssim \left|\frac{\partial(fe^{-h})(0)}{\partial\xi}\right|. \tag{1.7}$$

On the other hand, using Cauchy's inequality, the fact that $\varphi - u \leq 0$ and Lemma A, we get

$$\left|\frac{\partial(fe^{-h})}{\partial\xi}(0)\right| \lesssim \left|\int_{|\eta|=\frac{\delta_0\tau(0)}{4}} \frac{f(\eta)e^{-h(\eta)}}{\eta^2} d\eta\right|$$

$$\lesssim \frac{1}{\delta_0^2\tau(0)^2} \int_{|\eta|=\frac{\delta_0\tau(0)}{4}} |f(\eta)|e^{-\varphi(\eta)}e^{\varphi(\eta)-u(\eta)}|d\eta|$$

$$\lesssim \frac{1}{\tau(0)^2} \int_{|\eta|=\frac{\delta_0\tau(0)}{4}} \left(\frac{1}{\tau(\eta)^2}\int_{D(\delta_0\tau(\eta)/4)} |f(z)|^p e^{-p\varphi(z)} dA(z)\right)^{1/p} |d\eta|.$$

Finally, using $\tau(\eta) \asymp \tau(0)$, we obtain

$$\left|\frac{\partial fe^{-h}}{\partial\xi}(0)\right| \lesssim \frac{1}{\tau(0)^2} \int_{|\eta|=\frac{\delta_0\tau(0)}{4}} \left(\frac{1}{\tau(0)^2}\int_{D(\delta_0\tau(0)/2)} |f(z)|^p e^{-pu(z)} dA(z)\right)^{1/p} |d\eta|$$

$$\lesssim \frac{1}{\tau(0)^{1+\frac{2}{p}}} \left(\int_{D(\delta_0\tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} dA(z)\right)^{1/p}.$$

Bearing in mind (1.7) this gives

$$\left|\frac{\partial(fe^{-h})(0)}{\partial\xi}\right| e^{u(0)-\varphi(0)} \lesssim \frac{1}{\tau(0)^{1+\frac{2}{p}}} \left(\int_{D(\delta_0\tau(0)/2)} |f(z)|^p e^{-p\varphi(z)} dA(z)\right)^{1/p}.$$

Putting this and (1.6) into (1.5) we get the result. □

We shall also need the following lemma on coverings due to Oleinik [47]

Lemma B. *Let $\tau \in \mathcal{L}$ and $\delta \in (0, m_\tau)$. Then there exists a sequence of points $\{z_j\} \subset \mathbb{D}$, such that the following conditions are satisfied:*

(1) $z_j \notin D(\delta\tau(z_k)), j \neq k$.

(2) $\bigcup_j D(\delta\tau(z_j)) = \mathbb{D}$

(3) $\widetilde{D}(\delta\tau(z_j)) \subset D(3\delta\tau(z_j))$, *where* $\widetilde{D}(\delta\tau(z_j)) = \bigcup_{z\in D(\delta\tau(z_j))} D(\delta\tau(z))$

$j = 1, 2, 3, ...$

(4) $\{D(3\delta\tau(z_j))\}$ *is a covering of* $\mathbb{D}$ *of finite multiplicity* N*.*

Definition 1.3. *A sequence of points* $\{z_n\}$ *in* $\mathbb{D}$ *satisfying the conditions of the above Lemma will be called a* (δ,τ)*-lattice on* $\mathbb{D}$*.*

Remark 1.1. *The multiplicity* N *in the previous Lemma is independent of* δ*, and one can take* $N = 256$*. Moreover, if* $\{z_j\}$ *is a* (δ,τ)*-lattice on* $\mathbb{D}$*, then there exists a positive constant (independent of* δ*) such that every point* z *in* $\mathbb{D}$ *belongs to at most* $N \le C(\frac{\delta_0}{\delta})^2$ *of the sets* $D(3\delta_0\tau(z_j))$*, where* $\delta_0 \in (\delta, m_\tau)$*.*

Proof. Let $z \in \bigcap_{j=1}^{N} D(3\delta_0\tau(z_j))$. Applying (1.2), that $Area(D(\delta\tau(z_j)) = \delta^2\tau(z_j)$ and

$$D(\frac{\delta}{2}\tau(z_j)) \cap D(\frac{\delta}{2}\tau(z_k)) = \emptyset \qquad \text{if} \quad j \neq k,$$

we obtain

$$\begin{aligned} N\tau(z)^2 = \sum_{j=1}^{N} \tau(z)^2 \le 4\sum_{j=1}^{N} \tau(z_j)^2 = \frac{16}{\delta^2}\sum_{j=1}^{N} Area(D(\frac{\delta}{2}\tau(z_j)) \\ \le \frac{16}{\delta^2} Area\left(\bigcup_{j=1}^{N} D(\frac{\delta}{2}\tau(z_j))\right) \end{aligned}$$

Because of the inclusion $\bigcup_{j=1}^{N} D(\frac{\delta}{2}\tau(z_j)) \subset D(z, 4\delta_0\tau(z))$, we get

$$N \le C\left(\frac{\delta_0}{\delta}\right)^2,$$

where $C = 256$. □

The next result on the density of holomorphic polynomials it is certainly well known, but for completeness we offer a proof here.

Proposition 1.2. *Let* ω *be a radial weight and* $0 < p < \infty$*. Then the holomorphic polynomials are dense in* $A^p(\omega)$*.*

Proof. Given $f \in A^p(\omega)$ and $0 < \rho < 1$, let f_ρ be the dilated function defined by $f_\rho(z) = f(\rho z)$, $z \in \mathbb{D}$. First we want to prove that $\|f - f_\rho\|_{A^p(\omega)} \to 0$ as $\rho \to 1^-$. For $0 \le r < 1$, the classical integral means of f are defined by

$$M_p(f, r) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty.$$

Since $0 < \rho < 1$ and $M_p(f, r)$ is increasing function of $r \in [0, 1)$ (see [76, Corollary 4.21]) we get

$$M_p(f_\rho, r) = M_p(f, \rho r) \le M_p(f, r).$$

Therefore,

$$M_p^p(f - f_\rho, r) \le 2^p \left(M_p^p(f, r) + M_p^p(f_\rho, r) \right) \le 2^{p+1} M_p^p(f, r).$$

But the hypothesis that $f \in A^p(\omega)$ is equivalent to saying that $M_p^p(f, r)$ is integrable over the interval $[0, 1)$ with respect to the measure $\omega(r) r dr$, and it is clear that $f_\rho(z) \to f(z)$ uniformly on compact subsets on $\mathbb{D}$ as $\rho \to 1^-$, which implies that $M_p^p(f - f_\rho, r) \to 0$ for each $r \in [0, 1)$. Thus by the Lebesgue dominated convergence theorem, we conclude that

$$\|f - f_\rho\|_{A^p(\omega)}^p = 2 \int_0^1 M_p^p(f - f_\rho, r) \omega(r) \, r dr \to 0$$

as $\rho \to 1^-$. Since f_ρ is analytic on $\overline{\mathbb{D}}$, using Runge's theorem there is a polynomial P such that

$$|f_\rho(z) - P(z)| < \varepsilon/2,$$

for all $z \in \mathbb{D}$ and $\varepsilon > 0$. Finally,

$$\|f - P\|_{A^p(\omega)} \le \|f - f_\rho\| + \|f_\rho - P\| < \varepsilon,$$

which completes the proof. □

Proposition 1.3. *Let $\omega \in \mathcal{L}^*$ radial and $0 < p \le 2$. Then the set E of finite linear combinations of reproducing kernels is dense in $A^p(\omega^{p/2})$.*

Proof. Suppose first that $p = 2$. Since E is a linear subspace of $A^2(\omega)$, it is enough to prove that $g \equiv 0$ if $g \in A^2(\omega)$ satisfies $\langle f, g \rangle_\omega = 0$ for each f in E. But, taking $f = K_z$, for each $z \in \mathbb{D}$ we get $g(z) = \langle g, K_z \rangle_\omega = 0$. This finishes the proof for $p = 2$.

If $0 < p < 2$, then $A^2(\omega) \subset A^p(\omega^{p/2})$ continuously, that is, $\|f\|_{A^p(\omega^{p/2})} \lesssim \|f\|_{A^2(\omega)}$. Thus, for a polynomial f and points $a_1, \dots, a_n$, we have

$$\Big\| f - \sum_{k=1}^n \alpha_k K_{a_k} \Big\|_{A^p(\omega^{p/2})} \lesssim \Big\| f - \sum_{k=1}^n \alpha_k K_{a_k} \Big\|_{A^2(\omega)}.$$

Now, by the case $p = 2$ and the density of the polynomials in $A^p(\omega^{p/2})$, the result follows. □

For the case $p > 2$, one can obtain the density of the span of reproducing kernels K_z^* corresponding to some associated Bergman space $A^2(\omega_*)$ with a weight of the form $\omega_*(z) = \omega(z)\tau(z)^\alpha$ via the embedding $A^2(\omega_*) \subset A^p(\omega^{p/2})$. For the case of the exponential weight, we will see in Chapter 4 that we can use reproducing kernels of $A^2(\omega)$.

1.2 The class of weights $\mathcal{W}$ and examples

In this section, we introduce the class of weights $\mathcal{W}$ for which we are going to study the corresponding weighted Bergman spaces

Definition 1.4. *The class $\mathcal{W}$ consists of those radial decreasing weights of the form $\omega(z) = e^{-2\varphi(z)}$, where $\varphi \in C^2(\mathbb{D})$ is a radial function such that $(\Delta\varphi(z))^{-1/2} \asymp \tau(z)$, for a radial positive function $\tau(z)$ that decreases to 0 as $|z| \to 1^-$, and $\lim_{r\to 1^-} \tau'(r) = 0$. Furthermore, we shall also suppose that either there exist a constant $C > 0$ such that $\tau(r)(1-r)^{-C}$ increases for r close to 1 or*

$$\lim_{r\to 1^-} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

The class $\mathcal{W}$ is the same class of weights considered in [9], [51] and [23], and it is straightforward to see that $\mathcal{W} \subset \mathcal{L}^*$. Next we give several examples of weights in the class $\mathcal{W}$ (these examples already appears in [51], but we give some details for completeness).

Example 1: The exponential type weights

$$\omega_{\gamma,\sigma}(z) = (1-|z|^2)^{\gamma} \exp\left(\frac{-A}{(1-|z|^2)^{\sigma}}\right), \quad \gamma \geq 0,\ \sigma > 0,\ A > 0,$$

are in the class $\mathcal{W}$ with associated subharmonic function

$$\varphi_{\gamma,\sigma}(z) = -\gamma \log(1-|z|^2) + c\,(1-|z|^2)^{-\sigma}.$$

We have that

$$\left(\Delta\varphi_{\gamma,\sigma}(z)\right)^{-1} \asymp \tau(z)^2 = (1-|z|^2)^{2+\sigma},$$

and it is easy to see that $\tau(z)$ satisfies the conditions in the definition of the class $\mathcal{W}$.

Example 2: For $\alpha > 1$ and $A > 0$ the weights

$$\omega(r) = \exp\left(-A\left(\log\frac{e}{1-r}\right)^{\alpha}\right),$$

with associated subharmonic function $\varphi(z) = A\left(\log \frac{e}{1-|z|}\right)^{\alpha}$, belong to the class $\mathcal{W}$. Indeed, it is easy to see that

$$\Delta\varphi(z) \asymp (1-|z|)^{-2}\left(\log\frac{e}{1-|z|}\right)^{\alpha-1},$$

so $\tau(z) = (1-|z|)\left(\log\frac{e}{1-|z|}\right)^{\frac{1-\alpha}{2}}$, and since $\alpha > 1$

$$\tau'(r) \asymp \left(\log\frac{e}{1-r}\right)^{\frac{-\alpha+1}{2}}, \quad r \to 1^-,$$

which implies that $\lim_{r\to 1^-} \tau(r) = \lim_{r\to 1^-} \tau'(r) = 0$. Moreover, the function $\tau(r)(1-r)^{-2}$ increases for r close to 1. This proves that $\omega \in \mathcal{W}$.

Example 3: For $\alpha, \beta, \gamma > 0$, the double exponential weight

$$\omega(r) = \exp\left(-\gamma \exp\left(\frac{\beta}{(1-r)^\alpha}\right)\right)$$

belongs to $\mathcal{W}$. Indeed, the associated subharmonic function is $\varphi(z) = \gamma \exp\left(\frac{\beta}{(1-|z|)^\alpha}\right)$, and a straightforward computation gives

$$\Delta\varphi(z) \asymp (1-|z|)^{-2\alpha-2} \exp\left(\frac{\beta}{(1-|z|)^\alpha}\right).$$

Then we can take $\tau(z) = (1-|z|)^{\alpha+1} \exp\left(\frac{-\beta/2}{(1-|z|)^\alpha}\right)$. Since

$$\tau'(r) \asymp \exp\left(\frac{-\beta/2}{(1-r)^\alpha}\right), \quad r \to 1^-,$$

we obtain $\lim_{r\to 1^-} \tau(r) = \lim_{r\to 1^-} \tau'(r) = 0$. Also, it is easy to see that $\lim_{r\to 1^-} \tau'(r) \log \frac{1}{\tau(r)} = 0$. This proves that $\omega \in \mathcal{W}$.

1.3 Test functions and some estimates

Since the norm of the point evaluation functional equals the norm of the reproducing kernel in $A^2(\omega)$, the result of Lemma A also gives an upper bound for $\|K_z\|_{A^2(\omega)}$. In order to see that this upper bound is the corresponding growth of the reproducing kernel, one needs an appropriate family of test functions, a family that can also be used in other problems such as to characterize the Carleson type measures for large weighted Bergman spaces. The following result on test functions was obtained in [51].

Lemma C. *Assume that $0 < p < \infty$, $N \in \mathbb{N} \setminus \{0\}$ with $Np \geq 1$ and $\omega \in \mathcal{W}$. Then, for each $a \in \mathbb{D}$, there is a function $F_{a,N,p}$ analytic in $\mathbb{D}$ with*

$$|F_{a,N,p}(z)|^p \, \omega(z) \asymp 1 \quad \text{if} \quad |z-a| < \tau(a), \tag{1.8}$$

and

$$|F_{a,N,p}(z)| \, \omega(z)^{1/p} \lesssim \min\left(1, \frac{\min\left(\tau(a), \tau(z)\right)}{|z-a|}\right)^{3N}, \quad z \in \mathbb{D}. \tag{1.9}$$

Moreover, the function $F_{a,N,p}$ belongs to $A^p(\omega)$ with

$$\|F_{a,N,p}\|_{A^p(\omega)} \asymp \tau(a)^{2/p}.$$

As a consequence we have the following estimate for the norm of the reproducing kernel, result that can be found in [9], [51].

Lemma D. *Let $\omega \in \mathcal{W}$. Then*

$$\|K_z\|_{A^2(\omega)}^2\, \omega(z) \asymp \frac{1}{\tau(z)^2}, \quad z \in \mathbb{D}. \tag{1.10}$$

Next result is an estimate of the reproducing kernel function for points close to the diagonal. Despite that this result is stated in [35, Lemma 3.6] we offer here a proof based on Lemma 1.1, since the conditions on the weights are slightly different.

Lemma E. *Let $\omega \in \mathcal{W}$ and $z \in \mathbb{D}$. For $\varepsilon > 0$ sufficiently small we have*

$$|K_z(\xi)| \asymp \|K_z\|_{A^2(\omega)} \cdot \|K_\xi\|_{A^2(\omega)}, \tag{1.11}$$

if $\xi \in D(\varepsilon\tau(z))$.

Proof. The upper estimate is trivial from Cauchy-Schwarz. To prove the other inequality, let $\varepsilon \in (0, m_\tau)$ be sufficiently small and $z \in \mathbb{D}$ be fixed such that $\xi \in D(\varepsilon\tau(z))$. We have

$$|K_z(z)|\omega(z) = \left(|K_z(z)|\omega(z)^{1/2} - |K_z(\xi)|\omega(\xi)^{1/2}\right)\omega(z)^{1/2} + |K_z(\xi)|\omega(\xi)^{1/2}\omega(z)^{1/2}. \tag{1.12}$$

Consider

$$I(z,\xi) := \left(|K_z(z)|\,\omega(z)^{1/2} - |K_z(\xi)|\,\omega(\xi)^{1/2}\right)\omega(z)^{1/2}.$$

By Lemma 1.1 with $p = 2$, we can estimate the first term on the righthand side as follows: by Cauchy's estimates there exists $s \in [z, \xi]$ such that

$$\begin{aligned} I(z,\xi) &\le \nabla\left(|K_z|\omega^{1/2}\right)(s)\,|z-\xi|\,\omega(z)^{1/2} \\ &\le \frac{\varepsilon C(\delta_0)\,\omega(z)^{1/2}\,\tau(z)}{\tau(s)^2}\left(\int_{D(\delta_0\tau(s))} |K_z(t)|^2\,\omega(t)\,dA(t)\right)^{1/2}, \end{aligned}$$

for $\delta_0 \in (0, m_\tau)$ fixed. Using $\tau(s) \asymp \tau(z)$ and Lemma D we get

$$\begin{aligned} I(z,\xi) &\le \frac{\varepsilon\, C(\delta_0)\,\omega(z)^{1/2}}{\tau(z)}\left(\int_{D(\delta_0\tau(z))} |K_z(t)|^2\,\omega(t)\,dA(t)\right)^{1/2} \\ &\le \frac{\varepsilon\, C(\delta_0)\,\omega(z)^{1/2}}{\tau(z)}\,\|K_z\|_{A^2(\omega)} \\ &\le \varepsilon\, C(\delta_0)\,\omega(z)\,\|K_z\|_{A^2(\omega)}^2 = \varepsilon\, C(\delta_0)\,\omega(z)\,|K_z(z)|. \end{aligned}$$

Putting this into (1.12) and taking ε small enough such that $\varepsilon\, C(\delta_0) < 1/2$, we obtain

$$|K_z(z)|\,\omega(z) \le 2|K_z(\xi)|\,\omega(\xi)^{1/2}\,\omega(z)^{1/2}.$$

Therefore, again using Lemma D and the fact that $\tau(z) \asymp \tau(\xi)$ for $\xi \in D(\varepsilon\tau(z))$, we have

$$\|K_z\|_{A^2(\omega)} \cdot \|K_\xi\|_{A^2(\omega)} \leq C\,|K_z(\xi)|$$

obtaining the desired result. $\square$

Estimates for the $\overline{\partial}$-equation. We recall first the classical Hörmander's theorem [28] on L^2-estimates for solutions of the $\overline{\partial}$-equation.

Theorem A (Hörmander)**.** *Let $\varphi \in \mathcal{C}^2(\mathbb{D})$ with $\Delta\varphi > 0$ on $\mathbb{D}$. Then there exists a solution u of the equation $\overline{\partial}u = f$ such that*

$$\int_{\mathbb{D}} |u(z)|^2 e^{-2\varphi(z)} dA(z) \leq \int_{\mathbb{D}} \frac{|f(z)|^2}{\Delta\varphi(z)}\, e^{-2\varphi(z)} dA(z),$$

provided the right hand side integral is finite.

In some situations, one can use the associated weights ω_* given by

$$\omega_*(z) = \omega(z)\,\tau(z)^\alpha, \qquad \alpha \in \mathbb{R}. \tag{1.13}$$

The following result, that provides more estimates on the solutions of the $\overline{\partial}$-equation, can be of independent interest.

Theorem 1.4. *Let $\omega \in \mathcal{W}$, and consider the associated weight $\omega_*(z) := \omega(z)\,\tau(z)^\alpha, z \in \mathbb{D}$ and $\alpha \in \mathbb{R}$. Then there exists a solution u of the equation $\overline{\partial}u = f$ such that*

$$\int_{\mathbb{D}} |u(z)|^p\,\omega_*(z)^{p/2}\,dA(z) \leq C \int_{\mathbb{D}} |f(z)|^p\,\omega_*(z)^{p/2}\,\tau(z)^p\,dA(z)$$

for all $1 \leq p < \infty$, provided the right hand side integral is finite. Moreover, one also has the L^∞-estimate

$$\sup_{z\in\mathbb{D}} |u(z)|\,\omega_*(z)^{1/2} \leq C \sup_{z\in\mathbb{D}} |f(z)|\,\omega_*(z)^{1/2}\,\tau(z).$$

Proof. We follow the method used in [12] where the case $\alpha = 0$ was proved. By Lemma C, there are holomorphic functions F_a and some $\delta_0 \in (0, m_\tau)$ such that

$$\begin{aligned} &(i)\ |F_a(\xi)| \asymp \omega(\xi)^{-1/2}, \qquad \xi \in D(\delta_0\tau(a)). \\ &(ii)\ |F_a(\xi)| \leq C\omega(\xi)^{-1/2} \left(\frac{\min\big(\tau(\xi),\tau(a)\big)}{|a-\xi|}\right)^M, \quad (a,\xi) \in \mathbb{D}\times\mathbb{D}, \end{aligned} \tag{1.14}$$

Let $\delta_1 < \delta_0$. Then, there is a sequence $\{z_n\}_{n\geq 1}$ such that $\{D(\delta_1\tau(z_n))\}$ is a covering of $\mathbb{D}$ of finite multiplicity N and satisfy the other statements of Lemma B. Let χ_n be a partition of unity subordinate to the covering $D(\delta_1\tau(z_n))$. Consider

$$S_nf(z) = F_{z_n}(z) \int_{\mathbb{D}} \frac{f(\xi)\,\chi_n(\xi)}{(\xi - z)\,F_{z_n}(\xi)}\,dA(\xi).$$

Since F_{z_n} are holomorphic functions on $\mathbb{D}$, by the Cauchy-Pompieu formula, we have

$$\overline{\partial} S_n f(z) = f(z)\,\chi_n(z), \quad n = 1, 2....$$

Then,

$$Sf(z) = \sum_{n=1}^{\infty} S_n f(z) = \omega_*(z)^{-1/2} \int_{\mathbb{D}} G(z,\xi)\, f(\xi)\, \omega_*(\xi)^{1/2}\, dA(\xi),$$

where

$$G(z,\xi) = \sum_{n=1}^{\infty} \frac{F_{z_n}(z)}{\xi - z}\, \frac{\chi_n(\xi)}{F_{z_n}(\xi)}\, \omega_*(\xi)^{-1/2}\, \omega_*(z)^{1/2}.$$

Since χ_n is a partition of the unity, we have

$$\overline{\partial}(Sf) = \sum_{n=1}^{\infty} \overline{\partial}(S_n f) = f \sum_{n=1}^{\infty} \chi_n = f$$

on $\mathbb{D}$, so that Sf solves the equation $\overline{\partial} Sf(z) = f(z)$.

Now we are going to prove that

$$\int_{\mathbb{D}} |G(z,\xi)| \frac{dA(\xi)}{\tau(\xi)} \lesssim 1. \tag{1.15}$$

First we consider the covering of $\{\xi \in \mathbb{D} : |z - \xi| > \delta_2 \tau(z)\}$ given by

$$R_k(z) = \{\xi \in \mathbb{D} : 2^{k-1}\delta_2\tau(z) < |z-\xi| \le 2^k \delta_2 \tau(z)\}, \qquad k = 0, 1, 2, \dots$$

Let $4\,\delta_1 < \delta_2 < \frac{\delta_0}{5}$ and $z \in \mathbb{D}$ be fixed. If $\xi \in D(\delta_2\tau(z)) \cap D(\delta_1 \tau(z_n))$, using (1.2)

$$|z - z_n| \le |z - \xi| + |\xi - z_n| \le \delta_2 \tau(z) + \delta_1 \tau(z_n)$$

$$\le 4\delta_2 \tau(z_n) + \delta_1 \tau(z_n) < \delta_0 \tau(z_n),$$

that implies $z \in D(\delta_0 \tau(z_n))$. Using (1.2) and property (i) of (1.14), it follows

$$|G(z,\xi)| \lesssim \frac{\omega(z)^{-1/2}}{|\xi - z|}\, \frac{\omega_*(\xi)^{-1/2}\omega_*(z)^{1/2}}{\omega(\xi)^{-1/2}} \sum_{n=1}^{\infty} \chi_n(\xi) \lesssim \frac{1}{|\xi - z|}.$$

Therefore, using (1.2) and polar coordinates, we get

$$\int_{D(\delta_2\tau(z))} |G(z,\xi)| \frac{dA(\xi)}{\tau(\xi)} \lesssim \frac{1}{\tau(z)} \int_{D(\delta_2\tau(z))} \frac{1}{|\xi - z|}\, dA(\xi) \lesssim 1. \tag{1.16}$$

If $\xi \in \big(\mathbb{D} \setminus D(\delta_2\tau(z))\big) \cap D(\delta_1\tau(z_n))$, we show that $z \notin D(\delta_1\tau(z_n))$. In fact, if not, $z \in D(\delta_1\tau(z_n))$, then using (1.2) we have

$$|z - z_n| > |z - \xi| - |\xi - z_n| > \delta_2\tau(z) - \delta_1\tau(z_n) \geq (\delta_2/2 - \delta_1)\tau(z_n) \geq \delta_1\tau(z_n),$$

this implies a contradiction with our assumption. Thus,

$$|z - \xi| \leq |z - z_n| + |z_n - \xi| \leq |z - z_n| + \delta_1\tau(z_n)$$

$$\leq 2|z - z_n|.$$

Also, using $\tau(\xi) \asymp \tau(z_n)$ we get

$$\frac{|z - z_n|}{\min\big(\tau(z), \tau(z_n)\big)} \geq C\, \frac{|z - \xi|}{\min\big(\tau(z), \tau(\xi)\big)}.$$

Then, again using $\tau(\xi) \asymp \tau(z_n)$ and property (ii) of (1.14) with

$$M > \max(1; -\alpha/2; 1 + \alpha/2),$$

we have

$$|G(z,\xi)| \leq C \sum_{n=1}^{\infty} \frac{|F_{z_n}(z)|}{|\xi - z|}\, \frac{\chi_n(\xi)}{|F_{z_n}(\xi)|}\, \omega_*(\xi)^{-1/2}\omega_*(z)^{1/2}$$

$$\leq C\, \frac{\omega(z)^{-1/2}}{|\xi - z|}\, \frac{\omega_*(\xi)^{-1/2}\omega_*(z)^{1/2}}{\omega(\xi)^{-1/2}} \left(\frac{\min\big(\tau(z), \tau(\xi)\big)}{|\xi - z|}\right)^M \sum_{n=1}^{\infty} \chi_n(\xi) \tag{1.17}$$

$$\leq C \frac{\tau(z)^{\frac{\alpha}{2}}}{\tau(\xi)^{\frac{\alpha}{2}}|\xi - z|} \left(\frac{\min\big(\tau(z), \tau(\xi)\big)}{|\xi - z|}\right)^M.$$

Then,

$$\int_{\mathbb{D}\setminus D(\delta_2\tau(z))} |G(z,\xi)| \frac{dA(\xi)}{\tau(\xi)}$$

$$\leq C\tau(z)^{\alpha/2} \int_{\mathbb{D}\setminus D(\delta_2\tau(z))} \frac{\tau(\xi)^{\frac{-(2+\alpha)}{2}}}{|\xi - z|} \left(\frac{\min\big(\tau(z), \tau(\xi))\big)}{|\xi - z|}\right)^M dA(\xi).$$

• If $2+\alpha>0$.

$$\int_{\mathbb{D}\setminus D(\delta_2\tau(z))}|G(z,\xi)|\frac{dA(\xi)}{\tau(\xi)}\le C\tau(z)^{M-1}\sum_{k=1}^{\infty}\int_{R_k(z)}\frac{1}{|\xi-z|^{M+1}}dA(\xi)$$

$$\le C\tau(z)^{M-1}\sum_{k=1}^{\infty}\int_{R_k(z)}\frac{1}{\big(2^k\tau(z)\big)^{M+1}}dA(\xi)$$

$$\le C\sum_{k=1}^{\infty}\frac{1}{2^{k(M-1)}}\lesssim 1.$$

• If $2+\alpha\le 0$. Using the condition (B) in the definition of the class $\mathcal{L}$, it follows that

$$\tau(\xi)\le C2^k\delta_2\tau(z),\quad \xi\in R_k(z)\qquad k=0,1,2,\dots$$

So,

$$\int_{\mathbb{D}\setminus D(\delta_2\tau(z))}|G(z,\xi)|\frac{dA(\xi)}{\tau(\xi)}\le C\tau(z)^{\alpha/2}\int_{\mathbb{D}\setminus D(\delta_2\tau(z))}\frac{\tau(\xi)^{-(2+\alpha)/2}\tau(z)^M}{|\xi-z|^{M+1}}dA(\xi)$$

$$\le C\tau(z)^{M+\alpha/2}\sum_{k=1}^{\infty}\int_{R_k(z)}\frac{\big(2^k\tau(z)\big)^{-(2+\alpha)/2}}{\big(2^k\tau(z)\big)^{M+1}}dA(\xi)$$

$$\le C\sum_{k=1}^{\infty}\frac{1}{2^{k(M+\alpha/2)}}\lesssim 1.$$

This together with (1.16) establish (1.15).

Using (1.15), it is straightforward that the L^∞-estimate holds. Our next goal is to prove the inequality

$$\int_{\mathbb{D}}|Sf(z)|^p\omega_*(z)^{p/2}dA(z)\lesssim\int_{\mathbb{D}}|f(z)|^p\omega_*(z)^{p/2}\tau(z)^p dA(z).$$

Consider $g(\xi):=f(\xi)\,\omega_*(\xi)^{1/2}$ and $Tg(z):=\int_{\mathbb{D}}G(z,\xi)\,g(\xi)dA(\xi)$. Then, the last inequality translates

$$\int_{\mathbb{D}}|Tg(z)|^p dA(z)\lesssim\int_{\mathbb{D}}|g(z)|^p\,\tau(z)^p dA(z).$$

Therefore, using Hölder's inequality and (1.15), we have

$$
\begin{aligned}
|Tg(z)|^p &\le \left(\int_{\mathbb{D}} |G(z,\xi)|\,|g(\xi)|\,dA(\xi)\right)^p \\
&\le \int_{\mathbb{D}} |g(\xi)|^p\,\tau(\xi)^{p-1}|G(z,\xi)|\,dA(\xi)\left(\int_{\mathbb{D}} |G(z,\xi)|\,\frac{dA(\xi)}{\tau(\xi)}\right)^{p-1} \\
&\lesssim \int_{\mathbb{D}} |g(\xi)|^p\,\tau(\xi)^{p-1}|G(z,\xi)|\,dA(\xi).
\end{aligned}
$$

These and Fubini's theorem give

$$
\begin{aligned}
\int_{\mathbb{D}} |Tg(z)|^p dA(z) &\lesssim \int_{\mathbb{D}}\left(\int_{\mathbb{D}} |g(\xi)|^p\,\tau(\xi)^{p-1}|G(z,\xi)|\,dA(\xi)\right)dA(z) \\
&\lesssim \int_{\mathbb{D}} |g(\xi)|^p\,\tau(\xi)^{p-1}\left(\int_{\mathbb{D}} |G(z,\xi)|\,dA(z)\right)dA(\xi).
\end{aligned}
$$

Now, using the expression of the kernel $G(z,\xi)$ and the fact that χ_n are supported in $D(\delta_1\tau(z_n))$, we obtain

$$
\begin{aligned}
\int_{\mathbb{D}} |Tg(z)|^p dA(z) &\lesssim \int_{\mathbb{D}} |g(\xi)|^p\,\tau(\xi)^{p-1}\left(\int_{\mathbb{D}} \sum_{n=1}^{\infty} \frac{|F_{z_n}(z)|}{|\xi-z|}\,\frac{\chi_n(\xi)}{|F_{z_n}(\xi)|}\,\frac{\omega_*(z)^{1/2}}{\omega_*(\xi)^{1/2}}\,dA(z)\right)dA(\xi) \\
&\lesssim \sum_{n=1}^{\infty}\int_{\mathbb{D}} |g(\xi)|^p\,\tau(\xi)^{p-1}\left(\frac{\omega_*(\xi)^{-1/2}}{|F_{z_n}(\xi)|}\int_{\mathbb{D}} \frac{|F_{z_n}(z)|}{|\xi-z|}\,\omega_*(z)^{1/2}\,dA(z)\right)\chi_n(\xi)\,dA(\xi) \\
&\lesssim \sum_{n=1}^{\infty}\int_{D(\delta_1\tau(z_n))} |g(\xi)|^p\,\tau(\xi)^{p-1}\left(\frac{\omega_*(\xi)^{-1/2}}{|F_{z_n}(\xi)|}\int_{\mathbb{D}} \frac{|F_{z_n}(z)|}{|\xi-z|}\,\omega_*(z)^{1/2}\,dA(z)\right)dA(\xi).
\end{aligned}
$$

By using $|F_{z_n}(\xi)| \asymp \omega(\xi)^{-1/2}, \quad \xi \in D(\delta_1\tau(z_n))$, it follows that

$$
\begin{aligned}
&\int_{\mathbb{D}} |Tg(z)|^p dA(z) \\
&\quad\lesssim \sum_{n=1}^{\infty}\int_{D(\delta_1\tau(z_n))} |g(\xi)|^p\,\tau(\xi)^{p-1-\alpha/2}\left(\int_{\mathbb{D}} \frac{|F_{z_n}(z)|}{|\xi-z|}\,\omega_*(z)^{1/2}\,dA(z)\right)dA(\xi).
\end{aligned}
\tag{1.18}
$$

We claim that

$$
\int_{\mathbb{D}} \frac{|F_{z_n}(z)|}{|\xi-z|}\,\omega_*(z)^{1/2}\,dA(z) \lesssim \tau(\xi)^{1+\alpha/2}, \qquad \xi \in D(\delta_1\tau(z_n)). \tag{1.19}
$$

Assuming the claim, we can finish the proof. Indeed, since $\{D(\delta_1\tau(z_n))\}$ is a covering of $\mathbb{D}$ of finite multiplicity, using the claim we have

$$\int_{\mathbb{D}} |Tg(z)|^p dA(z) \lesssim \sum_{n=1}^{\infty} \int_{D(\delta_1\tau(z_n))} |g(\xi)|^p \, \tau(\xi)^p dA(\xi)$$

$$\lesssim \int_{\mathbb{D}} |g(\xi)|^p \, \tau(\xi)^p dA(\xi).$$

Therefore it remains to prove the inequality (1.19). We split this integral in two parts: one integrating over the disk $D(\delta_2\tau(\xi))$ and the other one over $\mathbb{D}\setminus D(\delta_2\tau(\xi))$. We compute the first integral, using (i) of (1.14), $\tau(\xi) \asymp \tau(z_n) \asymp \tau(z)$, $z \in D(\delta_2\tau(\xi))$ and by using polar coordinates, we obtain

$$\int_{D(\delta_2\tau(\xi))} \frac{|F_{z_n}(z)|}{|\xi - z|} \, \omega_*(z)^{1/2} \, dA(z) \lesssim \tau(\xi)^{\alpha/2} \int_{D(\delta_2\tau(\xi))} \frac{dA(z)}{|z - \xi|} \lesssim \tau(\xi)^{1+\alpha/2}. \tag{1.20}$$

Now we consider

$$I(\xi) := \int_{\mathbb{D}\setminus D(\delta_2\tau(\xi))} \frac{|F_{z_n}(z)|}{|\xi - z|} \, \omega_*(z)^{1/2} \, dA(z), \quad \xi \in D(\delta_1\tau(z_n)).$$

For $z \notin D(\delta_2\tau(\xi))$

$$|z - \xi| \le |z - z_n| + |z_n - \xi| \le |z - z_n| + \delta_1\tau(z_n)$$

$$\le |z - z_n| + \frac{2\delta_1}{\delta_2}|z - z_n| \le \big(1 + \frac{2\delta_1}{\delta_2}\big)|z - z_n|.$$

Then, again by using $\tau(\xi) \asymp \tau(z_n)$, we obtain

$$\frac{\min\big(\tau(z_n), \tau(z)\big)}{|z - z_n|} \le C \, \frac{\min\big(\tau(\xi), \tau(z)\big)}{|z - \xi|}$$

This together with (1.14) taking $M > \max(1, 2 + \alpha/2)$ give

$$I(\xi) \lesssim \int_{\mathbb{D}\setminus D(\delta_2\tau(\xi))} \frac{\tau(z)^{\alpha/2}}{|\xi - z|} \, \frac{\Big(\min\big(\tau(z_n), \tau(z)\big)\Big)^M}{|z - z_n|^M} \, dA(z)$$

$$\lesssim \int_{\mathbb{D}\setminus D(\delta_2\tau(\xi))} \tau(z)^{\alpha/2} \, \frac{\Big(\min\big(\tau(\xi), \tau(z)\big)\Big)^M}{|z - \xi|^{M+1}} \, dA(z).$$

• Suppose first that $\alpha \geq 0$. By $\tau(z) \leq C2^k\tau(\xi)$, for every $z \in R_k(\xi)$, $k = 0, 1, 2, \dots$, we get

$$I(\xi) \lesssim \sum_{k=1}^{\infty} \int_{R_k(\xi)} \frac{\big(2^k\tau(\xi)\big)^{\alpha/2}\tau(\xi)^M}{\big(2^k\tau(\xi)\big)^{M+1}} \, dA(z)$$

$$\lesssim \tau(\xi)^{1+\alpha/2} \sum_{k=1}^{\infty} \frac{1}{2^{k(M-2-\alpha/2)}} \lesssim \tau(\xi)^{1+\alpha/2}.$$

• If $\alpha < 0$, then

$$I(\xi) \lesssim \int_{\mathbb{D}\setminus D(\delta_2\tau(\xi))} \tau(z)^{\alpha/2} \frac{\Big(\min\big(\tau(\xi), \tau(z)\big)\Big)^M}{|z-\xi|^{M+1}} \, dA(z)$$

$$\lesssim \sum_{k=1}^{\infty} \int_{R_k(\xi)} \frac{\tau(\xi)^{M+\alpha/2}}{|z-\xi|^{M+1}} \, dA(z)$$

$$\lesssim \tau(\xi)^{1+\alpha/2} \sum_{k=1}^{\infty} \frac{1}{2^{k(M-1)}} \lesssim \tau(\xi)^{1+\alpha/2}.$$

This together with (1.20) establish (1.19). The proof is complete. □

1.4 Carleson type measures

Now we are going to recall some results on the boundedness of the embedding $i : X \to L^q(\mu)$ for some spaces X of analytic functions. We need first the notion of an s-Carleson measure. Given an arc I of the unit circle $\mathbb{T} := \partial\mathbb{D}$, the Carleson box associated to I is

$$S(I) = \Big\{z \in \mathbb{D} : 1 - \frac{|I|}{2\pi} < |z| < 1, \quad \frac{z}{|z|} \in I\Big\}.$$

For $\alpha > 0$, a positive Borel measure μ on $\mathbb{D}$ is called an α-Carleson measure if

$$\|\mu\|_{CM_\alpha} := \sup_{I \subset \mathbb{T}} \frac{\mu\big(S(I)\big)}{|I|^\alpha} < \infty.$$

The α-Carleson measures are the appropriate geometric object in order to describe the boundedness of $i : H^p \to L^q(\mu)$, as the following classical result of Carleson [10, 11] (case $q = p$) and Duren [20] (case $p < q$) shows.

Theorem B (Carleson-Duren)**.** *Let $0 < p \leq q < \infty$. Then the embedding $i : H^p \to L^q(\mu)$ is bounded if and only if μ is an q/p-Carleson measure.*

The corresponding result for $0 < q < p < \infty$ it seems that was considered a folklore theorem for a long time. It can be found stated in [65]. A detailed proof can be found in [40] or [69].

Theorem C. *Let $0 < q < p < \infty$. The following conditions are equivalent:*

(a) $i : H^p \to L^q(\mu)$ is bounded.

(b) The function $A_\mu 1(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|}$ is in $L^{\frac{p}{p-q}}(\mathbb{T})$.

(c) The sweep $\widetilde{\mu}$ of μ belongs to $L^{\frac{p}{p-q}}(\mathbb{T})$.

We recall that, for a measure μ on $\mathbb{D}$, the sweep of μ onto $\mathbb{T}$ is defined as

$$\widetilde{\mu}(\zeta) = \frac{1}{2\pi} \int_{\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2}\, d\mu(z).$$

A description of q-Carleson measures for $A^p(\omega^{p/2})$, $0 < p, q < \infty$, for weights ω in the class $\mathcal{W}$ was obtained in [51]. A positive measure μ is q-Carleson for $A^p(\omega^{p/2})$ when the inclusion $I_\mu : A^p(\omega^{p/2}) \longrightarrow L^q(\mathbb{D}, d\mu)$ is bounded. Next results were proved in [51, Theorem 1].

Theorem D. *Let $\omega \in \mathcal{W}$ and μ be a finite positive Borel measure on $\mathbb{D}$. Let $0 < p \le q < \infty$. Then $I_\mu : A^p(\omega^{p/2}) \longrightarrow L^q(\mathbb{D}, d\mu)$ is bounded if and only if for each sufficiently small $\delta > 0$ we have*

$$K_{\mu,\omega} := \sup_{a \in \mathbb{D}} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} \omega(\xi)^{-q/2}\, d\mu(\xi) < \infty. \tag{1.21}$$

Moreover, in that case, $K_{\mu,\omega} \asymp \|I_\mu\|^q_{A^p(\omega^{p/2}) \to L^q(\mathbb{D},d\mu)}$.

Theorem E. *Let $\omega \in \mathcal{W}$ and μ be a finite positive Borel measure on $\mathbb{D}$. Let $0 < p \le q < \infty$. Then, $I_\mu : A^p(\omega^{p/2}) \longrightarrow L^q(\mathbb{D}, d\mu)$ is compact if and only if for each sufficiently small $\delta > 0$ we have*

$$\lim_{r \to 1^-} \sup_{|a| > r} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} \omega(\xi)^{-q/2}\, d\mu(\xi) = 0. \tag{1.22}$$

Theorem F. *Let $\omega \in \mathcal{W}$ and let μ be a finite positive Borel measure on $\mathbb{D}$. Let $0 < q < p < \infty$. The following conditions are equivalent:*

(a) $I_\mu : A^p(\omega^{p/2}) \longrightarrow L^q(\mathbb{D}, d\mu)$ is compact.

(b) $I_\mu : A^p(\omega^{p/2}) \longrightarrow L^q(\mathbb{D}, d\mu)$ *is bounded.*

(c) For each sufficiently small $\delta > 0$, *the function*

$$F_\mu(z) = \frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \omega(\xi)^{-q/2}\, d\mu(\xi)$$

belongs to $L^{\frac{p}{p-q}}(\mathbb{D}, dA)$.

Moreover, one has

$$\|I_\mu\|^q_{A^p(\omega^{p/2}) \to L^q(\mu)} \asymp \|F_\mu\|_{L^{\frac{p}{p-q}}(\mathbb{D})}.$$

1.5 Atomic Decomposition

In this Section we are going to obtain an atomic decomposition for the large weighted Bergman space $A^2(\omega)$, that is, we show that every function in the Bergman spaces $A^2(\omega)$ with ω in the class $\mathcal{W}$ can be decomposed into a series of kernels functions. Atomic decomposition for standard Bergman spaces were obtained by Coifman and Rochberg [15] and has become a powerful tool in the study of such spaces. We refer to the books [53, 77, 76] for a modern proof of these results.

Proposition 1.5. *Let* $\omega \in \mathcal{W}$, *and* $\{z_k\}_{k\in\mathbb{N}} \subset \mathbb{D}$ *be the sequence that it is defined in Lemma B. The function given by*

$$F(z) := \sum_{k=0}^{\infty} \lambda_k\, \omega(z_k)^{1/2}\, \tau(z_k)\, K_{z_k}(z)$$

belongs to $A^2(\omega)$ *for every sequence* $\lambda = \{\lambda_k\} \in \ell^2$ *. Moreover,*

$$\|F\|_{A^2(\omega)} \lesssim \|\lambda\|_{\ell^2}.$$

Proof. First we show that the partial sums $F_N = \sum_{k=0}^N f_k$ converges uniformly on compact subsets of $\mathbb{D}$ which proves that F defines an analytic function on $\mathbb{D}$. Using Cauchy-Schwarz inequality, Lemma A and the norm estimate of $\|K_z\|_{A^2(\omega)}$, we have

$$\begin{aligned}
|F_N(z)| &\le \left(\sum_{k=0}^N |\lambda_k|^2\right)^{1/2} \left(\sum_{k=0}^N \tau(z_k)^2 |K_{z_k}(z)|^2\, \omega(z_k)\right)^{1/2} \\
&\lesssim \left(\sum_{k=0}^N |\lambda_k|^2\right)^{1/2} \left(\sum_{k=0}^N \int_{D(\delta\tau(z_k))} |K_z(\xi)|^2\, \omega(\xi)\, dA(\xi)\right)^{1/2} \\
&\lesssim \|\lambda\|_{\ell^2}\, \|K_z\|_{A^2(\omega)} \lesssim \frac{\omega(z)^{-1/2}}{\tau(z)}\, \|\lambda\|_{\ell^2},
\end{aligned}$$

which proves that the series defining F converges uniformly on compact subsets of $\mathbb{D}$.

To prove that $F \in A^2(\omega)$ with $\|F\|_{A^2(\omega)} \lesssim \|\lambda\|_{\ell^2}$, by duality we must show that $|\langle F, g\rangle_\omega| \lesssim \|\lambda\|_{\ell^2} \cdot \|g\|_{A^2(\omega)}$ for $g \in A^2(\omega)$. Clearly,

$$\begin{aligned}\big|\langle F, g\rangle_\omega\big| &\leq \sum_k |\lambda_k|\,\omega(z_k)^{1/2}\,\tau(z_k)\,|g(z_k)| \\ &\leq \|\lambda\|_{\ell^2}\left(\sum_k |g(z_k)|^2\,\omega(z_k)\,\tau(z_k)^2\right)^{1/2}.\end{aligned}$$

Finally, by Lemma , and since $\{z_n\}$ is a τ-lattice,

$$\sum_k |g(z_k)|^2\,\omega(z_k)\,\tau(z_k)^2 \lesssim \sum_k \int_{D(\delta\tau(z_k))} |g(\zeta)|^2\,\omega(z)\,dA(z) \lesssim \|g\|^2_{A^2(\omega)}.$$

This finishes the proof. □

Lemma 1.6. *Let $\omega \in \mathcal{E}$. There is a sequence $\{z_n\} \subset \mathbb{D}$ such that*

$$\sum_{n=0}^{\infty} |f(z_n)|^p \omega(z_n)^{p/2} \tau(z_n)^2 \gtrsim \|f\|^p_{A^p(\omega^{p/2})},$$

for all $f \in A^p(\omega^{p/2})$ and $p > 0$.

Proof. Let $\{z_n\}$ be an (ε, τ)-lattice on $\mathbb{D}$ (that exists by Lemma B) with $\varepsilon > 0$ small enough to be specified later. Let $f \in A^p(\omega^{p/2})$. We consider

$$I_f(n) := \sum_{n=0}^{\infty} |f(z_n)|^p\,\omega(z_n)^{p/2}\,\tau(z_n)^2.$$

We have

$$\begin{aligned}\|f\|^p_{A^p(\omega^{p/2})} &= \int_{\mathbb{D}} |f(z)|^p \omega(z)^{p/2} dA(z) \\ &\leq C\left[\sum_{n=0}^{\infty} \int_{D(\varepsilon\tau(z_n))} \Big(|f(z)|\omega(z)^{1/2} - |f(z_n)|\omega(z_n)^{1/2}\Big)^p dA(z) + C\varepsilon^2 I_f(n)\right].\end{aligned}$$

For $z \in D(\varepsilon\tau(z_n))$, there exists $\xi_{n,z} \in [z, z_n]$ such that

$$\begin{aligned}\Big(|f(z)|\omega(z)^{1/2} - |f(z_n)|\omega(z_n)^{1/2}\Big)^p &\leq \big|\nabla(|f|\,\omega^{1/2})(\xi_{n,z})\big|^p |z - z_n|^p \\ &\leq \varepsilon^p\,\tau(z_n)^p\,\big|\nabla(|f|\,\omega^{1/2})(\xi_{n,z})\big|^p.\end{aligned}$$

This together with Lemma 1.1, with $\delta_0 \in (0, m_\tau)$ fixed, yields

$$\int_{D(\varepsilon\tau(z_n))} \left(|f(z)|\omega(z)^{1/2} - |f(z_n)|\omega(z_n)^{1/2} \right)^p dA(z)$$

$$\leq C\,\varepsilon^p\,\tau(z_n)^p \int_{D(\varepsilon\tau(z_n))} \left(\frac{1}{\tau(\xi_{n,z})^{p+2}} \int_{D(\delta_0\tau(\xi_{n,z}))} |f(\eta)|^p \omega(\eta)^{p/2} dA(\eta) \right) dA(z).$$

Using that $\tau(\xi_{n,z}) \asymp \tau(z_n)$ and $D(\delta_0\tau(\xi_{n,z})) \subset D(3\delta_0\tau(z_n))$ for $z \in D(\varepsilon\tau(z_n))$, we obtain

$$\int_{D(\varepsilon\tau(z_n))} \left(|f(z)|\omega(z)^{1/2} - |f(z_n)|\omega(z_n)^{1/2} \right)^p dA(z)$$

$$\leq C\,\varepsilon^{p+2} \left(\int_{D(3\delta_0\tau(z_n))} |f(\eta)|^p \omega(\eta)^{p/2} dA(\eta) \right).$$

Therefore,

$$\|f\|^p_{A^p(\omega^{p/2})} \leq C\,\varepsilon^{p+2} \sum_{n=0}^{\infty} \int_{D(3\delta_0\tau(z_n))} |f(\eta)|^p \omega(\eta)^{p/2} dA(\eta) + C\varepsilon^2\, I_f(n).$$

By Lemma B every point $z \in \mathbb{D}$ belongs to at most $C\varepsilon^{-2}$ of the sets $D(3\delta_0\tau(z_n))$, and therefore

$$\|f\|^p_{A^p(\omega^{p/2})} \leq C\varepsilon^p \|f\|^p_{A^p(\omega^{p/2})} + C\,\varepsilon^2\, I_f(n).$$

Thus, taking $\varepsilon > 0$ so that $C\varepsilon^p < 1/2$, we get the desired result. $\square$

It is worth mentioning that, what actually Lemma 1.6 says, is that an (ε, τ)-lattice with $\varepsilon > 0$ small enough, is a sampling sequence for the Bergman space $A^p(\omega^{p/2})$. Recall that $\{z_n\} \subset \mathbb{D}$ is a sampling sequence for the Bergman space $A^p(\omega^{p/2})$ if

$$\|f\|^p_{A^p(\omega^{p/2})} \asymp \sum_n |f(z_n)|^p\, \omega(z_n)^{p/2}\, \tau(z_n)^2$$

for any $f \in A^p(\omega^{p/2})$. Just note that Lemma 1.6 gives one inequality, and the other follows by standard methods using Lemma A and the lattice properties. Sampling sequences on the classical Bergman space were characterized by K. Seip [59] (see also the monographs [22] and [60]). For sampling sequences on large weighted Bergman spaces we refer to [9].

Now we are ready to prove the result on the atomic decomposition of large weighted Bergman spaces $A^2(\omega)$. We use the notation k_z for the normalized reproducing kernels in $A^2(\omega)$, that is

$$k_z = \frac{K_z}{\|K_z\|_{A^2(\omega)}}.$$

Theorem 1.7. *Let $\omega \in \mathcal{W}$. There exists a lattice $\{z_n\} \subset \mathbb{D}$ such that:*

(i) For any $\lambda = \{\lambda_n\} \in \ell^2$, the function

$$f(z) = \sum_n \lambda_n \, k_{z_n}(z)$$

is in $A^2(\omega)$ with $\|f\|_{A^2(\omega)} \le C\|\lambda\|_{\ell^2}$.

(ii) For every $f \in A^2(\omega)$ exists $\lambda = \{\lambda_n\} \in \ell^2$ such that

$$f(z) = \sum_n \lambda_n \, k_{z_n}(z)$$

and $\|\lambda\|_{\ell^2} \le C \, \|f\|_{A^2(\omega)}$.

Proof. Because of the norm estimate $\|K_{z_n}\|^{-1}_{A^2(\omega)} \asymp \omega(z_n)^{1/2} \, \tau(z_n)$, part (i) is just Proposition 1.5. In order to prove (ii), we define a linear operator $S : \ell^2 \longrightarrow A^2(\omega)$ given by

$$S(\{\lambda_n\}) := \sum_{n=0}^{\infty} \lambda_n \, k_{z_n}.$$

By (i), the operator S is bounded. The adjoint operator $S^* : A^2(\omega) \to \ell^2$ is defined by

$$\langle Sx, f\rangle_\omega = \langle x, S^* f\rangle_\ell = \sum_n x_n \, \overline{(S^* f)_n}.$$

for every $x \in \ell^2$ and $f \in A^2(\omega)$. To compute S^*, let e_n denote the vector that equals 1 at the n-th coordinate and equals 0 at the other coordinates. Then $Se_n = k_{z_n}$, and using the reproducing formula we get

$$\overline{(S^* f)_n} = \langle e_n, S^* f\rangle_\ell = \langle Se_n, f\rangle_\omega = \langle k_{z_n}, f\rangle_\omega$$

$$= \|K_{z_n}\|^{-1}_{A^2(\omega)} \cdot \langle K_{z_n}, f\rangle_\omega = \frac{\overline{f(z_n)}}{\|K_{z_n}\|_{A^2(\omega)}}.$$

Hence, $S^* : A^2(\omega) \longrightarrow \ell^2$ is given by

$$S^* f = \{(S^* f)_n\} = \left\{ \frac{f(z_n)}{\|K_{z_n}\|_{A^2(\omega)}} \right\}_n.$$

We must prove that S is surjective in order to finish the proof of this case. By a classical result in functional analysis, it is enough to show that S^* is bounded below. By Lemma 1.6 and the estimate for the norm of K_z we obtain

$$\|S^* f\|_{\ell^2}^2 \asymp \sum_{n=0}^{\infty} |f(z_n)|^2 \, \omega(z_n) \, \tau(z_n)^2 \gtrsim \|f\|_{A^2(\omega)}^2,$$

which shows that S^* is bounded below. Finally, once the surjectivity is proved, the estimate $\|\lambda\|_{\ell^2} \le C \, \|f\|_{A^2(\omega)}$ is an standard application of the open mapping theorem. The proof is complete. $\square$

Chapter 2

Toeplitz operators

Throughout this chapter, μ will denote a finite positive Borel measure on $\mathbb{D}$ and ω would be a weight in the class $\mathcal{W}$ so that we can apply all the results of the previous chapter. The purpose of this chapter is to characterize the boundedness, compactness and membership in Schatten ideals of Toeplitz operators acting on $A^2(\omega)$.

Definition 2.1. *Let K_z be the reproducing kernel of $A^2(\omega)$. The Toeplitz operator T_μ^ω with symbol μ is given by*

$$T_\mu f(z) = T_\mu^\omega f(z) := \int_{\mathbb{D}} f(\xi)\, \overline{K_z(\xi)}\, \omega(\xi)\, d\mu(\xi).$$

Note that T_μ is very loosely defined here, because it is not clear when the integrals above will converge, even if the measure μ is finite. We suppose that μ is a finite positive Borel measure that satisfies the condition

$$\int_{\mathbb{D}} |K_z(\xi)|^2\, \omega(\xi)\, d\mu(\xi) < \infty. \tag{2.1}$$

Then, the Toeplitz operator T_μ is well-defined on a dense subset of $A^2(\omega)$. In fact, by Proposition 1.3, the set E of finite linear combinations of reproducing kernels is dense in $A^2(\omega)$. Therefore, it follows from condition (2.1) and the Cauchy-Schwartz inequality that $T_\mu(f)$ is well defined for any $f \in E$.

There is a lot of work on the study of Toeplitz operators acting on several spaces of holomorphic functions [13, 29, 39, 43, 57, 72], and the theory is especially well understood in the case of Hardy spaces or standard Bergman spaces (see [77] and the references therein). Luecking [39] was the first to study Toeplitz operators on the Bergman spaces with measures as symbols, and the study of Toeplitz operators acting on large weighted Bergman spaces was initiated by Lin and Rochberg [35]

2.1 Boundedness and compactness

In this section we describe the boundedness of Toeplitz operators on large Bergman spaces. For $\delta \in (0, m_\tau)$, we define a new function $\widehat{\mu}_\delta$ (the averaging function of μ) on $\mathbb{D}$ by

$$\widehat{\mu}_\delta(z) := \frac{\mu(D(\delta\tau(z))}{\tau(z)^2}, \quad z \in \mathbb{D}.$$

Theorem 2.1. *Let $\omega \in \mathcal{W}$. Then*

(i) $T_\mu : A^2(\omega) \to A^2(\omega)$ is bounded if and only if for each $\delta > 0$ small enough, one has

$$C_\mu := \sup_{z\in\mathbb{D}} \widehat{\mu}_\delta(z) < \infty. \tag{2.2}$$

Moreover, in that case, $\|T_\mu\| \asymp C_\mu$.

(ii) $T_\mu : A^2(\omega) \to A^2(\omega)$ is compact if and only if for each $\delta > 0$ small enough, one has

$$\lim_{r\to 1^-} \sup_{|z|>r} \widehat{\mu}_\delta(z) = 0. \tag{2.3}$$

For the proof, we need first the following lemma.

Lemma 2.2. *Let $\omega \in \mathcal{W}$, and assume that $\widehat{\mu}_\delta$ is in $L^\infty(\mathbb{D})$ for some small $\delta > 0$. Then*

$$\langle T_\mu f, g\rangle_\omega = \int_{\mathbb{D}} f(\zeta)\, \overline{g(\zeta)}\, \omega(\zeta)\, d\mu(\zeta), \qquad f, g \in A^2(\omega).$$

Proof. Being ω a radial weight, the polynomials are dense in $A^2(\omega)$ and we may assume that g is an holomorphic polynomial. Because of Theorem D, the condition implies that $d\nu = \omega\, d\mu$ is a Carleson measure for $A^2(\omega)$. Then

$$\begin{aligned}
&\int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(\zeta)|\, |K_z(\zeta)|\, \omega(\zeta)\, d\mu(\zeta) \right) |g(z)|\, \omega(z)\, dA(z) \\
&\quad \le \|f\|_{L^2(\nu)} \int_{\mathbb{D}} \|K_z\|_{L^2(\nu)}\, |g(z)|\, \omega(z)\, dA(z) \\
&\quad \lesssim \|f\|_{A^2(\omega)} \cdot \|g\|_\infty \int_{\mathbb{D}} \|K_z\|_{A^2(\omega)}\, \omega(z)\, dA(z) \\
&\quad \lesssim \|f\|_{A^2(\omega)} \cdot \|g\|_\infty \int_{\mathbb{D}} \frac{\omega(z)^{1/2}}{\tau(z)}\, dA(z),
\end{aligned}$$

and this is finite (see [51, Lemma 2.3] for example). Thus, Fubini's theorem gives

$$\langle T_\mu f, g\rangle_\omega = \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(\zeta)\, \overline{K_z(\zeta)}\, \omega(\zeta)\, d\mu(\zeta) \right) \overline{g(z)}\, \omega(z)\, dA(z)$$

$$= \int_{\mathbb{D}} f(\zeta) \left(\int_{\mathbb{D}} \overline{g(z)}\, K_\zeta(z)\, \omega(z)\, dA(z) \right) \omega(\zeta)\, d\mu(\zeta)$$

$$= \int_{\mathbb{D}} f(\zeta)\, \overline{\langle g, K_\zeta\rangle_\omega}\, \omega(\zeta)\, d\mu(\zeta) = \int_{\mathbb{D}} f(\zeta)\, \overline{g(\zeta)}\, \omega(\zeta)\, d\mu(\zeta).$$

□

Proof of Theorem 2.1: boundedness

Assume first that T_μ is bounded on $A^2(\omega)$. For fixed $a \in \mathbb{D}$, one has

$$T_\mu K_a(a) = \int_{\mathbb{D}} |K_a(z)|^2\, \omega(z)\; d\mu(z).$$

By Lemma E, there is $\delta \in (0, m_\tau)$ such that $|K_a(z)| \asymp \|K_z\|_{A^2(\omega)} \cdot \|K_a\|_{A^2(\omega)}$, for every $z \in D(\delta\tau(a))$. This together with the norm estimate given in Lemma D and the fact that $\tau(z) \asymp \tau(a)$ for $z \in D(\delta\tau(a))$ gives

$$T_\mu K_a(a) \geq \int_{D(\delta\tau(a))} |K_a(z)|^2\, \omega(z)\, d\mu(z)$$

$$\gtrsim \int_{D(\delta\tau(a))} \|K_z\|^2_{A^2(\omega)}\, \|K_a\|^2_{A^2(\omega)}\, \omega(z)\, d\mu(z)$$

$$\asymp \frac{\mu(D(\delta\tau(a))}{\omega(a)\, \tau(a)^4} = \frac{\widehat{\mu}_\delta(a)}{\omega(a)\, \tau(a)^2}.$$

Therefore, by Lemma A and the estimate of the norm of the reproducing kernels, we obtain

$$\begin{aligned} \widehat{\mu}_\delta(a) &\lesssim \omega(a)\, \tau(a)^2\, |T_\mu K_a(a)| \leq \omega(a)^{1/2}\, \tau(a)\; \|T_\mu K_a\|_{A^2(\omega)} \\ &\leq \omega(a)^{1/2}\, \tau(a)\; \|T_\mu\| \cdot \|K_a\|_{A^2(\omega)} \lesssim \|T_\mu\|. \end{aligned} \tag{2.4}$$

Conversely, suppose that (2.2) holds. Let $f, g \in A^2(\omega)$. By Lemma 2.2 and because

$d\nu = \omega d\mu$ is a Carleson measure for $A^2(\omega)$ with $\|I_\nu\|^2 \asymp C_\mu := \sup_{z\in\mathbb{D}} \widehat{\mu}_\delta(z)$ (see Theorem D), we have

$$\left|\langle T_\mu f, g\rangle_\omega\right| \leq \int_{\mathbb{D}} |f(z)|\,|g(z)|\,d\nu(z) \leq \|f\|_{L^2(\nu)} \cdot \|g\|_{L^2(\nu)} \lesssim C_\mu \|f\|_{A^2(\omega)} \cdot \|g\|_{A^2(\omega)}.$$

This shows that T_μ is bounded on $A^2(\omega)$ with $\|T_\mu\| \lesssim C_\mu$ finishing the proof.

Proof of Theorem 2.1: compactness

Let k_z be the normalized reproducing kernels in $A^2(\omega)$. From (2.4) in the proof of the boundedness part, and the estimate for $\|K_z\|_{A^2(\omega)}$, we have

$$\widehat{\mu}_\delta(z) \lesssim \|T_\mu k_z\|_{A^2(\omega)}.$$

From Lemma A, it is easy to see that k_z converges to zero weakly as $|z| \to 1^-$. Thus, if T_μ is compact, from Theorem 1.14 in [77], we obtain (2.3).

Conversely, suppose (2.3) holds, and let $\{f_n\}$ be a sequence in $A^2(\omega)$ converging to zero weakly. To prove compactness, we must show that $\|T_\mu f_n\|_{A^2(\omega)} \to 0$. By the proof of the boundedness, we have

$$\|T_\mu f_n\|_{A^2(\omega)} \lesssim \|f_n\|_{L^2(\nu)},$$

with $d\nu := \omega d\mu$. By Theorem E, our assumption (2.3) implies that $I_\nu : A^2(\omega) \longrightarrow L^2(\mathbb{D}, d\nu)$ is compact, which implies that $\|f_n\|_{L^2(\nu)}$ tends to zero. Hence $\|T_\mu f_n\|_{A^2(\omega)} \to 0$ proving that T_μ is compact. The proof is complete.

2.2 Membership in Schatten classes

In this section we will provide a full characterization of the Schatten class membership of Toeplitz operators acting on large weighted Bergman spaces $A^2(\omega)$. The case of the classical weighted Bergman spaces A^2_α with $\alpha > -1$ was obtained by D. Luecking [39], and can be found also in Zhu's book [77].

Given $0 < p < \infty$, let $S_p(A^2(\omega))$ denote the Schatten p-class of operators from $A^2(\omega)$ to $A^2(\omega)$, it consists of those compact operators $T : A^2(\omega) \longrightarrow A^2(\omega)$ with its sequence of singular numbers λ_n belonging to ℓ^p, the p-summable sequence space. We recall that the singular numbers of a compact operator T are the square root of the eigenvalues of the positive operator T^*T, where T^* denotes the adjoint of T. One has $T \in S_p(A^2(\omega))$ if and only if $T^*T \in S_{p/2}(A^2(\omega))$. Also, the compact operator T admits a decomposition of the form

$$T = \sum_{n=1}^{\infty} \lambda_n \langle ., e_n\rangle_{A^2(\omega)} e_n,$$

where $\{\lambda_n\}$ are the singular numbers of T and $\{e_n\}$ is an orthonormal basis in $A^2(\omega)$. For $p \geq 1$, the class $S_p(A^2(\omega))$ is Banach space equipped with the norm

$$\|T\|_{S_p} := \left(\sum_{n=1}^{\infty} |\lambda_n|^p \right)^{1/p},$$

while for $0 < p < 1$ one has the inequality [45, Theorem 2.8]

$$\|S + T\|_{S_p}^p \leq \|S\|_{S_p}^p + \|T\|_{S_p}^p.$$

Two special cases are worth mentioning: S_1 is called the trace class, and S_2 is called the Hilbert-Schmidt class. For a positive operator T, let $tr(T)$ denote the usual trace defined by

$$tr(T) := \sum_{n=1}^{\infty} \langle Te_n, e_n \rangle_\omega = \|T\|_{S_1}.$$

More background information about the Schatten classes S_p can be found in [77, Chapter 1] for example.

The main result of this section is the following:

In this section, we are going to describe those positive Borel measures μ for which the Toeplitz operator T_μ belongs to the the Schatten ideal $S_p(A^2(\omega))$, for weights $\omega \in \mathcal{W}$. In order to obtain such a characterization, we need to introduce first some concepts.

We define the ω-Berezin transform $B_\omega \mu$ of the measure μ as

$$B_\omega \mu(z) := \int_{\mathbb{D}} |k_z(\xi)|^2 \, \omega(\xi) \, d\mu(\xi), \qquad z \in \mathbb{D},$$

where k_z are the normalized reproducing kernels in $A^2(\omega)$. We also recall that, for $\delta \in (0, m_\tau)$, the averaging function $\widehat{\mu}_\delta$ is given by

$$\widehat{\mu}_\delta(z) := \frac{\mu(D(\delta\tau(z))}{\tau(z)^2}, \qquad z \in \mathbb{D}.$$

We also consider the measure λ_τ given by

$$d\lambda_\tau(z) = \frac{dA(z)}{\tau(z)^2}, \qquad z \in \mathbb{D}.$$

Proposition 2.3. *Let $1 \leq p < \infty$, and $\omega \in \mathcal{W}$. The following conditions are equivalent :*

(*a*) *The function $B_\omega \mu$ is in $L^p(\mathbb{D}, d\lambda_\tau)$.*

(*b*) *The function $\widehat{\mu}_\delta$ is in $L^p(\mathbb{D}, d\lambda_\tau)$ for any $\delta \in (0, m_\tau)$ small enough.*

(c) *The sequence* $\{\widehat{\mu}_\delta(z_n)\}$ *is in* ℓ^p *for any* (δ,τ)*-lattice* $\{z_n\}$ *with* $\delta \in (0, \frac{m_\tau}{4})$ *sufficiently small.*

Proof. $(a) \Rightarrow (b)$. By Lemma E, for all $\delta \in (0, m_\tau)$ sufficiently small, one has

$$|K_z(\zeta)| \asymp \|K_z\|_{A^2(\omega)} \cdot \|K_\zeta\|_{A^2(\omega)}, \qquad \zeta \in D(\delta\tau(z))$$

Then

$$B_\omega\mu(z) = \int_{\mathbb{D}} |k_z(\zeta)|^2\, \omega(\zeta)\, d\mu(\zeta) \geq \|K_z\|_{A^2(\omega)}^{-2} \int_{D(\delta\tau(z))} |K_z(\zeta)|^2\, \omega(\zeta)\, d\mu(\zeta)$$

$$\asymp \int_{D(\delta\tau(z))} \|K_\zeta\|_{A^2(\omega)}^2\, \omega(\zeta)\, d\mu(\zeta) \asymp \widehat{\mu}_\delta(z).$$

Since $B_\omega\mu$ is in $L^p(\mathbb{D}, d\lambda_\tau)$, this gives (b).

$(b) \Rightarrow (c)$. Since $\widehat{\mu}_\delta(z_n) \lesssim \widehat{\mu}_{4\delta}(z)$, for $z \in D(\delta\tau(z_n))$, then

$$\sum_n \widehat{\mu}_\delta(z_n)^p \lesssim \sum_n \int_{D(\delta\tau(z_n))} \widehat{\mu}_{4\delta}(z)^p\, \frac{dA(z)}{\tau(z)^2} \lesssim \int_{\mathbb{D}} \widehat{\mu}_{4\delta}(z)^p\, d\lambda_\tau(z).$$

$(c) \Rightarrow (a)$. We have

$$B_\omega\mu(z) \leq \|K_z\|_{A^2(\omega)}^{-2} \sum_n \int_{D(\delta\tau(z_n))} |K_z(s)|^2\, \omega(s)\, d\mu(s).$$

We use Lemma A in order to obtain

$$\int_{D(\delta\tau(z_n))} |K_z(s)|^2\, \omega(s)\, d\mu(s) \lesssim \int_{D(\delta\tau(z_n))} \left(\frac{1}{\tau(s)^2} \int_{D(\delta\tau(s))} |K_z(\xi)|^2\, \omega(\xi)\, dA(\xi) \right) d\mu(s)$$

$$\lesssim \left(\int_{D(3\delta\tau(z_n))} |K_z(\xi)|^2\, \omega(\xi)\, dA(\xi) \right) \widehat{\mu}_\delta(z_n).$$

If $p > 1$, by Hölder's inequality,

$$\left(\sum_n \int_{D(\delta\tau(z_n))} |K_z(s)|^2\, \omega(s)\, d\mu(s) \right)^p$$

$$\lesssim \|K_z\|_{A^2(\omega)}^{2(p-1)} \sum_n \left(\int_{D(3\delta\tau(z_n))} |K_z(\xi)|^2\, \omega(\xi)\, dA(\xi) \right) \widehat{\mu}_\delta(z_n)^p.$$

This gives

$$\int_{\mathbb{D}} B_\omega\mu(z)^p \, d\lambda_\tau(z) \lesssim \sum_n \widehat{\mu}_\delta(z_n)^p \int_{D(3\delta\tau(z_n))} \left(\int_{\mathbb{D}} |K_\xi(z)|^2 \, \|K_z\|^{-2}_{A^2(\omega)} d\lambda_\tau(z) \right) \omega(\xi) \, dA(\xi).$$

Since $\|K_z\|^2_{A^2(\omega)} \asymp \tau(z)^{-2}\omega(z)^{-1}$, we have

$$\int_{\mathbb{D}} |K_\xi(z)|^2 \, \|K_z\|^{-2}_{A^2(\omega)} d\lambda_\tau(z) \asymp \|K_\xi\|^2_{A^2(\omega)} \asymp \tau(\xi)^{-2}\omega(\xi)^{-1}.$$

Putting this estimate in the previous inequality, we finally get

$$\int_{\mathbb{D}} B_\omega\mu(z)^p \, d\lambda_\tau(z) \lesssim \sum_n \widehat{\mu}_\delta(z_n)^p, \qquad p \geq 1.$$

This finishes the proof. □

Remark: *It should be observed that the equivalence of (b) and (c) in Proposition 2.3 continues to hold for $0 < p < 1$ as well as the implication (a) implies (b). In order to get equivalence with condition (a) even in the case $0 < p < 1$, it seems that one needs L^p-integral estimates for reproducing kernels, estimates that are not available nowadays.*

Next Lemma is the analogue to our setting of a well known result for standard Bergman spaces.

Lemma 2.4. *Let $\omega \in \mathcal{W}$, and T be a positive operator on $A^2(\omega)$. Let $\widetilde{T}$ be the Berezin transform of the operator T defined by*

$$\widetilde{T}(z) = \langle Tk_z, k_z \rangle_\omega, \quad z \in \mathbb{D}.$$

(a) Let $0 < p \leq 1$. If $\widetilde{T} \in L^p(\mathbb{D}, d\lambda_\tau)$ then T is in S_p.

(b) Let $p \geq 1$. If T is in S_p then $\widetilde{T} \in L^p(\mathbb{D}, d\lambda_\tau)$.

Proof. Let $p > 0$. The positive operator T is in S_p if and only if T^p is in the trace class S_1. Fix an orthonormal basis $\{e_k\}$ of $A^2(\omega)$. Since T^p is positive, it belongs to the trace class if and only if $\sum_k \langle T^p e_k, e_k \rangle_\omega < \infty$. Let $S = \sqrt{T^p}$. Then

$$\sum_k \langle T^p e_k, e_k \rangle_\omega = \sum_k \|Se_k\|^2_{A^2(\omega)}.$$

Now, by Fubini's theorem and Parseval's identity, we have

$$\sum_k \|Se_k\|^2_{A^2(\omega)} = \sum_k \int_{\mathbb{D}} |Se_k(z)|^2\,\omega(z)\,dA(z) = \sum_k \int_{\mathbb{D}} |\langle Se_k, K_z\rangle_\omega|^2\,\omega(z)\,dA(z)$$

$$= \int_{\mathbb{D}} \left(\sum_k |\langle e_k, SK_z\rangle_\omega|^2 \right) \omega(z)\,dA(z) = \int_{\mathbb{D}} \|SK_z\|^2_{A^2(\omega)}\,\omega(z)\,dA(z)$$

$$= \int_{\mathbb{D}} \langle T^p K_z, K_z\rangle_\omega\,\omega(z)\,dA(z) = \int_{\mathbb{D}} \langle T^p k_z, k_z\rangle_\omega \|K_z\|^2_{A^2(\omega)}\,\omega(z)\,dA(z)$$

$$\asymp \int_{\mathbb{D}} \langle T^p k_z, k_z\rangle_\omega\,d\lambda_\tau(z).$$

Hence, both (a) and (b) are consequences of the inequalities (see [77, Proposition 1.31])

$$\langle T^p k_z, k_z\rangle_\omega \le [\langle T k_z, k_z\rangle_\omega]^p = [\widetilde{T}(z)]^p, \qquad 0 < p \le 1$$

and

$$[\widetilde{T}(z)]^p = [\langle T k_z, k_z\rangle_\omega]^p \le \langle T^p k_z, k_z\rangle_\omega, \qquad p \ge 1.$$

This finishes the proof of the lemma. □

Proposition 2.5. *Let $\omega \in \mathcal{W}$. If $0 < p \le 1$ and $B_\omega\mu$ is in $L^p(\mathbb{D}, d\lambda_\tau)$, then T_μ belongs to $S_p(A^2(\omega))$. Conversely, if $p \ge 1$ and T_μ is in $S_p(A^2(\omega))$, then $B_\omega\mu \in L^p(\mathbb{D}, d\lambda_\tau)$.*

Proof. If $B_\omega\mu$ is in $L^p(\mathbb{D}, d\lambda_\tau)$, then it is easy to see that T_μ is bounded on $A^2(\omega)$ (just use the discrete version in Proposition 2.3 to see that the condition in Theorem 2.1 holds). Therefore, the result is a consequence of Lemma 2.4 since $\widetilde{T_\mu}(z) = B_\omega\mu(z)$. □

Now we are almost ready for the characterization of Schatten class Toeplitz operators, but we need first some technical lemmas on properties of lattices. We use the notation

$$d_\tau(z,\zeta) = \frac{|z-\zeta|}{\min(\tau(z), \tau(\zeta))}, \qquad z,\zeta \in \mathbb{D}.$$

Lemma 2.6. *Let $\tau \in \mathcal{L}$, and $\{z_j\}$ be a (δ, τ)-lattice on $\mathbb{D}$. For each $\zeta \in \mathbb{D}$, the set*

$$D_m(\zeta) = \big\{ z \in \mathbb{D} : d_\tau(z, \zeta) < 2^m \delta \big\}$$

contains at most K points of the lattice, where K depends on the positive integer m but not on the point ζ.

Proof. Let K be the number of points of the lattice contained in $D_m(\zeta)$. Due to the Lipschitz condition (B), we have

$$\tau(\zeta) \le \tau(z_j) + c_2 |\zeta - z_j| \le (1 + c_2 2^m \delta)\, \tau(z_j) = C_m\, \tau(z_j).$$

Then

$$K \cdot \tau(\zeta)^2 \le C_m^2 \sum_{z_j \in D_m(\zeta)} \tau(z_j)^2 \lesssim C_m^2 \cdot Area \left(\bigcup_{z_j \in D_m(\zeta)} D\Big(\frac{\delta}{4} \tau(z_j)\Big) \right).$$

As done before, we also have $\tau(z_j) \le C_m \tau(\zeta)$, if $z_j \in D_m(\zeta)$. From this we easily see that

$$D\Big(\frac{\delta}{4}\, \tau(z_j)\Big) \subset D\Big(c 2^m \delta \tau(\zeta)\Big)$$

for some constant c. Since the sets $\{D(\frac{\delta}{4}\, \tau(z_j))\}$ are pairwise disjoints, we have

$$\bigcup_{z_j \in D_m(\zeta)} D\Big(\frac{\delta}{4} \tau(z_j)\Big) \subset D\Big(c 2^m \delta \tau(\zeta)\Big).$$

Therefore, we get

$$K \cdot \tau(\zeta)^2 \le C_m^2 \cdot Area \left(D\Big(c 2^m \delta \tau(\zeta)\Big)\right) \lesssim C_m^2 2^{2m} \tau(\zeta)^2,$$

that implies $K \le C 2^{4m}$. □

Next, we use the result just proved to decompose any (δ, τ)-lattice into a finite number of "big" separated subsequences.

Lemma 2.7. *Let $\tau \in \mathcal{L}$ and $\delta \in (0, m_\tau)$. Let m be a positive integer. Any (δ, τ)-lattice $\{z_j\}$ on $\mathbb{D}$ can be partitioned into M subsequences such that, if a_j and a_k are different points in the same subsequence, then $d_\tau(a_j, a_k) \ge 2^m \delta$.*

Proof. Let K be the number given by Lemma 2.6. From the lattice $\{z_j\}$ extract a maximal $(2^m \delta)$-subsequence, that is, we select one point ξ_1 in our lattice, and then we continue selecting points ξ_n of the lattice so that $d_\tau(\xi_n, \xi) \ge 2^m \delta$ for all previous selected point ξ.

We stop once the subsequence is maximal, that is, when all the remaining points x of the lattice satisfy $d_\tau(x, \xi_x) < 2^m\delta$ for some ξ_x in the subsequence. With the remaining points of the lattice we extract another maximal $(2^m\delta)$-subsequence, and we repeat the process until we get $M = K + 1$ maximal $(2^m\delta)$-subsequences. If no point of the lattice is left, we are done. On the other hand, if a point ζ in the lattice is left, this means that there are $M = K + 1$ distinct points x_ζ (at least one for each subsequence) in the lattice with $d_\tau(\zeta, x_\zeta) < 2^m\delta$, in contradiction with the choice of K from Lemma 2.6. The proof is complete. □

Now we are ready for the main result of this Section, that characterizes the membership in the Schatten ideals of the Toeplitz operator acting on $A^2(\omega)$.

Theorem 2.8. *Let $\omega \in \mathcal{W}$ and $0 < p < \infty$. The following conditions are equivalent:*

- *(a) The Toeplitz operator T_μ is in $S_p(A^2(\omega))$.*
- *(b) The function $\widehat{\mu}_\delta$ is in $L^p(\mathbb{D}, d\lambda_\tau)$ for $\delta \in (0, m_\tau)$ sufficiently small.*
- *(c) The sequence $\{\widehat{\mu}_\delta(z_n)\}$ is in ℓ^p for any $\delta > 0$ small enough.*

Moreover, when $p \geq 1$, the previous conditions are also equivalent to:

- *(d) The function $B_\omega\mu$ is in $L^p(\mathbb{D}, d\lambda_\tau)$.*

Proof. By Proposition 2.3 and the remark following it, the statements (b) and (c) are equivalent, and when $p \geq 1$, they are also equivalent with condition (d). Hence, according to Proposition 2.5, the result is proved for $p = 1$ and we have the implication (a) implies (b) for $p > 1$. Moreover, the implication (c) implies (a) for $0 < p < 1$ is proved in [35] (the conditions on the weights are slightly different, but the same proof works for our class). Thus, it remains to prove that (b) implies (a) for $p > 1$, and that (a) implies (c) when $0 < p < 1$.

Let $1 < p < \infty$, and assume that $\widehat{\mu}_\delta \in L^p(\mathbb{D}, d\lambda_\tau)$ with $\delta \in (0, m_\tau)$ small enough. It is not difficult to see, using the equivalent discrete condition in (c) together with Theorem 2.1, that T_μ must be compact. For any orthonormal set $\{e_n\}$ of $A^2(\omega)$, we have

$$\sum_n \langle T_\mu e_n, e_n\rangle_\omega^p = \sum_n \left(\int_{\mathbb{D}} |e_n(z)|^2\, \omega(z)\, d\mu(z) \right)^p. \tag{2.5}$$

By Lemma A and Fubini's theorem,

$$\int_{\mathbb{D}} |e_n(z)|^2\, \omega(z)\, d\mu(z) \lesssim \int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} |e_n(\zeta)|^2\, \omega(\zeta)\, dA(\zeta) \right) d\mu(z)$$

$$\lesssim \int_{\mathbb{D}} |e_n(\zeta)|^2\, \omega(\zeta)\, \widehat{\mu}_\delta(\zeta)\, dA(\zeta).$$

Since $p > 1$ and $\|e_n\|_{A^2_\omega} = 1$, we can apply Hölder's inequality to get

$$\left(\int_{\mathbb{D}} |e_n(z)|^2\, \omega(z)\, d\mu(z)\right)^p \lesssim \int_{\mathbb{D}} |e_n(\zeta)|^2\, \omega(\zeta)\, \widehat{\mu}_\delta(\zeta)^p\, dA(\zeta).$$

Putting this into (2.5) and taking into account that $\|K_\zeta\|^2_{A^2(\omega)}\, \omega(\zeta) \asymp \tau(\zeta)^{-2}$, we see that

$$\begin{aligned}
\sum_n \langle T_\mu e_n, e_n\rangle_\omega^p &\lesssim \int_{\mathbb{D}} \left(\sum_n |e_n(\zeta)|^2\right) \omega(\zeta)\, \widehat{\mu}_\delta(\zeta)^p\, dA(\zeta) \\
&\leq \int_{\mathbb{D}} \|K_\zeta\|^2_{A^2_\omega}\, \omega(\zeta)\, \widehat{\mu}_\delta(\zeta)^p\, dA(\zeta) \\
&\asymp \int_{\mathbb{D}} \widehat{\mu}_\delta(\zeta)^p\, d\lambda_\tau(\zeta).
\end{aligned}$$

By [77, Theorem 1.27] this proves that T_μ is in S_p with $\|T_\mu\|_{S_p} \lesssim \|\widehat{\mu}_\delta\|_{L^p(\mathbb{D}, d\lambda_\tau)}$.

Next, let $0 < p < 1$, and suppose that $T_\mu \in S_p(A^2(\omega))$. We will prove that (c) holds. The method for this proof has his roots in previous work of S. Semmes [61] and D. Luecking [39]. Let $\{z_n\}$ be a (δ, τ)-lattice on $\mathbb{D}$. We want to show that $\{\widehat{\mu}_\delta(z_n)\}$ is in ℓ^p. To this end, we fix a large positive integer $m \geq 2$ and apply Lemma 2.7 to partition the lattice $\{z_n\}$ into M subsequences such that any two distinct points a_j and a_k in the same subsequence satisfy $d_\tau(a_j, a_k) \geq 2^m \delta$. Let $\{a_n\}$ be such a subsequence and consider the measure

$$\nu = \sum_n \mu \chi_n,$$

where χ_n denotes the characteristic function of $D(\delta\tau(a_n))$. Since $m \geq 2$, the disks $D(\delta\tau(a_n))$ are pairwise disjoints. Since T_μ is in S_p and $0 \leq \nu \leq \mu$, then $0 \leq T_\nu \leq T_\mu$ which implies that T_ν is also in S_p. Moreover, $\|T_\nu\|_{S_p} \leq \|T_\mu\|_{S_p}$. Fix an orthonormal basis $\{e_n\}$ for $A^2(\omega)$ and define an operator B on $A^2(\omega)$ by

$$B\left(\sum_n \lambda_n\, e_n\right) = \sum_n \lambda_n\, f_{a_n},$$

where $f_{a_n} = F_{a_n,N}/\tau(a_n)$ and $F_{a_n,N}$ are the functions appearing in Lemma C with N taken big enough so that $3Np - 4 > 2p$. By [51, Proposition 2], the operator B is bounded. Since $T_\nu \in S_p$, the operator $T = B^* T_\nu B$ is also in S_p, with

$$\|T\|_{S_p} \leq \|B\|^2 \cdot \|T_\nu\|_{S_p}.$$

We split the operator T as $T = D + E$, where D is the diagonal operator on $A^2(\omega)$ defined by

$$Df = \sum_{n=1}^{\infty} \langle Te_n, e_n\rangle_\omega \, \langle f, e_n\rangle_\omega \, e_n, \qquad f \in A^2(\omega),$$

and $E = T - D$. By the triangle inequality,

$$\|T\|_{S_p}^p \geq \|D\|_{S_p}^p - \|E\|_{S_p}^p. \tag{2.6}$$

Since D is positive diagonal operator, we have

$$\begin{aligned}
\|D\|_{S_p}^p &= \sum_n \langle Te_n, e_n\rangle_\omega^p = \sum_n \langle T_\nu f_{a_n}, f_{a_n}\rangle_\omega^p \\
&= \sum_n \left(\int_{\mathbb{D}} |f_{a_n}(z)|^2 \, \omega(z) \, d\nu(z) \right)^p \\
&\geq \sum_n \left(\int_{D(\delta\tau(a_n))} \frac{|F_{a_n,N}(z)|^2}{\tau(a_n)^2} \, \omega(z) \, d\mu(z) \right)^p .
\end{aligned}$$

Hence, by Lemma C, there is a positive constant C_1 such that

$$\|D\|_{S_p}^p \geq C_1 \sum_n \widehat{\mu}_\delta(a_n)^p. \tag{2.7}$$

On the other hand, since $0 < p < 1$, by [77, Proposition 1.29] and Lemma 2.2 we have

$$\begin{aligned}
\|E\|_{S_p}^p &\leq \sum_n \sum_k \langle Ee_n, e_k\rangle_\omega^p = \sum_{n,k:k\neq n} \langle T_\nu f_{a_n}, f_{a_k}\rangle_\omega^p \\
&\leq \sum_{n,k:k\neq n} \left(\int_{\mathbb{D}} |f_{a_n}(\xi)| \, |f_{a_k}(\xi)| \, \omega(\xi) \, d\nu(\xi) \right)^p \qquad (2.8) \\
&= \sum_{n.k:k\neq n} \left(\sum_j \int_{D(\delta\tau(a_j))} |f_{a_n}(\xi)| \, |f_{a_k}(\xi)| \, \omega(\xi) \, d\mu(\xi) \right)^p .
\end{aligned}$$

If $n \neq k$, then $d_\tau(a_n, a_k) \geq 2^m\delta$. Thus, for $\xi \in D(\delta\tau(a_j))$, is not difficult to see that either

$$d_\tau(\xi, a_n) \geq 2^{m-2}\delta \quad \text{or} \quad d_\tau(\xi, a_k) \geq 2^{m-2}\delta.$$

Indeed, since $n \neq k$, then either $d_\tau(a_n, a_j) \geq 2^m\delta$ or $d_\tau(a_j, a_k) \geq 2^m\delta$. Suppose that $d_\tau(a_n, a_j) \geq 2^m\delta$. If $d_\tau(\xi, a_n) < 2^{m-2}\delta$, then

$$\begin{aligned} |a_n - a_j| &\leq |a_n - \xi| + |\xi - a_j| < 2^{m-2}\delta \min(\tau(a_n), \tau(\xi)) + \delta\tau(a_j) \\ &\leq 2^{m-1}\delta \min(\tau(a_n), \tau(a_j)) + \delta\tau(a_j). \end{aligned}$$

This directly gives a contradiction if $\min(\tau(a_n), \tau(a_j)) = \tau(a_j)$. In case that $\min(\tau(a_n), \tau(a_j)) = \tau(a_n)$, using the Lipschitz condition (B) we get

$$|a_n - a_j| < 2^{m-1}\delta \min(\tau(a_n), \tau(a_j)) + \delta\tau(a_n) + c_2\delta|a_n - a_j|$$

Since $c_2\delta \leq 1/4$, and $m \geq 2$, we see that this implies

$$d_\tau(a_n, a_j) < \frac{4}{3}(2^{m-1} + 1)\delta \leq 2^m\delta.$$

Thus, without loss of generality, we assume that $d_\tau(\xi, a_n) \geq 2^{m-2}\delta$. For any n and k we write

$$I_{nk}(\mu) = \sum_j \int_{D(\delta\tau(a_j))} |f_{a_n}(\xi)|\,|f_{a_k}(\xi)|\,\omega(\xi)\,d\mu(\xi).$$

With this notation and taking into account (2.8), we have

$$\|E\|_{S_p}^p \leq \sum_{n.k:k\neq n} I_{nk}(\mu)^p. \tag{2.9}$$

By Lemma C, we have

$$|F_{a_n,N}(\xi)| \lesssim d_\tau(\xi, a_n)^{-3N}.$$

Apply this inequality raised to the power $1/2$, together with the fact that $d_\tau(\xi, a_n) \geq 2^{m-2}\delta$ to get

$$|f_{a_n}(\xi)| = \frac{|F_{a_n,N}(\xi)|^{1/2}}{\tau(a_n)}\; |F_{a_n,N}(\xi)|^{1/2} \lesssim 2^{-\frac{3Nm}{2}}\, \frac{|f_{a_n}(\xi)|^{1/2}}{\tau(a_n)^{1/2}}. \tag{2.10}$$

We also have

$$|f_{a_k}(\xi)| \leq |f_{a_k}(\xi)|^{1/2} \cdot \|K_\xi\|_{A^2(\omega)}^{1/2} \tag{2.11}$$

Putting (2.10) and (2.11) into the definition of $I_{nk}(\mu)$, and using the norm estimate

$$\|K_\xi\|_{A^2(\omega)} \asymp \tau(\xi)^{-1}\,\omega(\xi)^{-1/2},$$

we obtain

$$I_{nk}(\mu) \lesssim \frac{2^{-\frac{3Nm}{2}}}{\tau(a_n)^{1/2}} \sum_j \frac{1}{\tau(a_j)^{1/2}} \int_{D(\delta\tau(a_j))} |f_{a_n}(\xi)|^{1/2} \, |f_{a_k}(\xi)|^{1/2} \, \omega(\xi)^{1/2} \, d\mu(\xi).$$

By Lemma A, for $\xi \in D(\delta\tau(a_j))$, one has

$$|f_{a_n}(\xi)|^{1/2} \, \omega(\xi)^{1/4} \lesssim \left(\frac{1}{\tau(\xi)^2} \int_{D(\delta\tau(\xi))} |f_{a_n}(z)|^{p/2} \, \omega(z)^{p/4} \, dA(z) \right)^{1/p}$$

$$\lesssim \tau(a_j)^{-2/p} \, S_n(a_j)^{1/p},$$

with

$$S_n(x) = \int_{D(3\delta\tau(x))} |f_{a_n}(z)|^{p/2} \omega(z)^{p/4} dA(z).$$

In the same manner we also have

$$|f_{a_k}(\xi)|^{1/2} \, \omega(\xi)^{1/4} \lesssim \tau(a_j)^{-2/p} \, S_k(a_j)^{1/p}.$$

Therefore, there is a positive constant C_2 such that

$$I_{nk}(\mu) \le C_2 \cdot \frac{2^{-\frac{3Nm}{2}}}{\tau(a_n)^{1/2}} \sum_j \frac{\tau(a_j)^{-4/p}}{\tau(a_j)^{1/2}} \; S_n(a_j)^{1/p} \cdot S_k(a_j)^{1/p} \; \mu\Big(D(\delta\tau(a_j))\Big)$$

$$= C_2 \cdot \frac{2^{-\frac{3Nm}{2}}}{\tau(a_n)^{1/2}} \sum_j \tau(a_j)^{3/2-4/p} \cdot S_n(a_j)^{1/p} \cdot S_k(a_j)^{1/p} \cdot \widehat{\mu}_\delta(a_j).$$

Since $0 < p < 1$ and $2^{-3Nm/2} \le 2^{-m}$, we get

$$I_{nk}(\mu)^p \le C_2^p \cdot \frac{2^{-mp}}{\tau(a_n)^{p/2}} \sum_j \tau(a_j)^{\frac{3p}{2}-4} \cdot S_n(a_j) \cdot S_k(a_j) \cdot \widehat{\mu}_\delta(a_j)^p.$$

Bearing in mind (2.9), this gives

$$\|E\|_{S_p}^p \le C_2^p \cdot 2^{-mp} \sum_j \tau(a_j)^{\frac{3p}{2}-4} \, \widehat{\mu}_\delta(a_j)^p \left(\sum_n \frac{S_n(a_j)}{\tau(a_n)^{p/2}} \right) \cdot \left(\sum_k S_k(a_j) \right). \tag{2.12}$$

On the other hand, we have

$$\begin{aligned}\sum_k S_k(a_j) &= \sum_k \int_{D(3\delta\tau(a_j))} |f_{a_k}(z)|^{p/2}\,\omega(z)^{p/4}\,dA(z)\\ &= \int_{D(3\delta\tau(a_j))} \Big(\sum_k |F_{a_k,N}(z)|^{p/2}\ \tau(a_k)^{-p/2}\Big)\,\omega(z)^{p/4}\,dA(z).\end{aligned}\tag{2.13}$$

Now, we claim that

$$\sum_k \tau(a_k)^{-p/2}\,|F_{a_k,N}(z)|^{p/2} \lesssim \tau(z)^{-p/2}\,\omega(z)^{-p/4}\,. \tag{2.14}$$

In order to prove (2.14), note first that using the estimate (1.8) in Lemma C, the estimate (1.2) and (iv) of Lemma B, we deduce that

$$\begin{aligned}&\sum_{\{a_k\in D(\delta_0\tau(z))\}} \tau(a_k)^{-p/2}\,|F_{a_k,N}(z)|^{p/2}\\ &\qquad\lesssim \omega(z)^{-p/4} \sum_{\{a_k\in D(\delta_0\tau(z))\}} \tau(a_k)^{-p/2} \lesssim \tau(z)^{-p/2}\,\omega(z)^{-p/4}\,.\end{aligned}\tag{2.15}$$

On the other hand, an application of the estimate (1.9) in Lemma C gives

$$\begin{aligned}&\sum_{\{a_k\notin D(\delta_0\tau(z))\}} \tau(a_k)^{-p/2}\,|F_{a_k,N}(z)|^{p/2}\\ &\qquad\lesssim \omega(z)^{-p/4}\tau(z)^{\frac{3Np}{2}-\frac{p}{2}-2} \sum_{\{a_k\notin D(\delta_0\tau(z))\}} \frac{\tau(a_k)^2}{|z-a_k|^{3Np/2}}\\ &\qquad= \omega(z)^{-p/4}\tau(z)^{\frac{3Np}{2}-\frac{p}{2}-2} \sum_{j=0}^{\infty}\sum_{a_k\in R_j(z)} \frac{\tau(a_k)^2}{|z-a_k|^{3Np/2}},\end{aligned}$$

where

$$R_j(z) = \big\{\zeta\in\mathbb{D} : 2^j\delta_0\tau(z) < |\zeta - z| \le 2^{j+1}\delta_0\tau(z)\big\}\,,\ j=0,1,2\dots$$

Now observe that, using condition (B) in the definition of the class $\mathcal{L}$, it is easy to see that, for $j=0,1,2,\dots,$

$$D(\delta_0\tau(a_k)) \subset D(5\delta_0 2^j\tau(z)) \quad \text{ if } \quad a_k\in D(2^{j+1}\delta_0\tau(z)).$$

This fact together with the finite multiplicity of the covering (see Lemma B) gives

$$\sum_{a_k\in R_j(z)} \tau(a_k)^2 \lesssim m\Big(D(5\delta_0 2^j\tau(z))\Big) \lesssim 2^{2j}\tau(z)^2.$$

Therefore, as $3Np-4>2p$,

$$\begin{aligned}\sum_{\{a_k\notin D(\delta_0\tau(z))\}}\tau(a_k)^{-p/2}\,|F_{a_k,N}(z)|^{p/2} &\lesssim \omega(z)^{-p/4}\tau(z)^{-\frac{p}{2}-2}\sum_{j=0}^{\infty}2^{-\frac{3Npj}{2}}\sum_{a_k\in R_j(z)}\tau(a_k)^2\\ &\lesssim \omega(z)^{-p/4}\tau(z)^{-p/2}\sum_{j=0}^{\infty}2^{\frac{(4-3Np)}{2}j}\\ &\lesssim \omega(z)^{-p/4}\tau(z)^{-p/2},\end{aligned}$$

which together with (2.15), proves the claim (2.14).

Putting (2.14) into (2.13) gives

$$\sum_k S_k(a_j)\lesssim \tau(a_j)^{2-p/2}.$$

Similarly,

$$\sum_n \frac{S_n(a_j)}{\tau(a_n)^{p/2}}\lesssim \tau(a_j)^{2-p}.$$

Putting these estimates into (2.12) we finally get

$$\|E\|_{S_p}^p\le C_2^p\cdot C_3\cdot 2^{-mp}\sum_j \widehat{\mu}_\delta(a_j)^p$$

for some positive constant C_3. Combining this with (2.6), (2.7) and choosing m large enough so that

$$C_2^p\cdot C_3\cdot 2^{-mp}\le C_1/2,$$

then we deduce that

$$\sum_j \widehat{\mu}_\delta(a_j)^p\le \frac{C_1}{2}\,\|T\|_{S_p}^p\le C_4\|T_\mu\|_{S_p}^p.$$

Since this holds for each one of the M subsequences of $\{z_n\}$, we obtain

$$\sum_n \widehat{\mu}_\delta(z_n)^p\le C_4\,M\,\|T_\mu\|_{S_p}^p \tag{2.16}$$

for all locally finite positive Borel measures μ such that

$$\sum_n \widehat{\mu}_\delta(z_n)^p<\infty.$$

Finally, an easy approximation argument then shows that (2.16) actually holds for all locally finite positive Borel measures μ. The proof is complete. □

Chapter 3

Area operators

Let μ be a finite positive Borel measure on the unit disk $\mathbb{D}$. The area operator A_μ is the sublinear operator defined by

$$A_\mu(f)(\zeta) = \int_{\Gamma(\zeta)} |f(z)| \frac{d\mu(z)}{1-|z|^2}, \qquad \zeta \in \mathbb{T} := \partial\mathbb{D},$$

where $\Gamma(\zeta)$ is a typical non-tangential approach region (Stolz region) with vertex $\zeta \in \mathbb{T}$.

The study of the area operator acting on the classical Hardy spaces H^p was initiated by W. Cohn [14]. He proved that, for $0 < p < \infty$, the area operator $A_\mu : H^p \to L^p(\mathbb{T})$ is bounded if and only if μ is a classical Carleson measure (that is, a 1-Carleson measure according to our definitions in Chapter 1). The area operator is useful in harmonic analysis, and is closely related to, for example, non-tangential maximal functions, Poisson integrals, Littlewood-Paley operators, tent spaces, etc. The study of A_μ acting on Hardy spaces was pursued later in [24], where a full description of the boundedness of $A_\mu : H^p \to L^q(\mathbb{T})$ for the case $0 < p < q < \infty$ and $1 \le q < p < \infty$ was obtained. In the setting of standard Bergman spaces, in [69] Z. Wu obtained a characterization of the boundedness of $A_\mu : A^p_\alpha \to L^q(\mathbb{T})$ for $1 \le p, q < \infty$. In this chapter we are going to extend these results to our large Bergman spaces $A^p(\omega)$ for weights ω in the class $\mathcal{W}$ and characterize those positive Borel measures μ for which the area operator $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$, $1 \le p, q < \infty$ is bounded.

The case $q = 1$ is also included in the other results of the chapter, but since it is easier and very instructive, it seems reasonable to look at it independently. For $\delta > 0$ we use the notation

$$\widehat{\mu}_{\delta,p}(z) = \frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \omega(\zeta)^{-1/p} \, d\mu(z).$$

Theorem 3.1. *Let $\omega \in \mathcal{W}$ and μ a finite positive Borel measure on $\mathbb{D}$. Then*

(i) Let $1 < p < \infty$. Then $A_\mu : A^p(\omega) \to L^1(\mathbb{T})$ is bounded if and only if $\widehat{\mu}_{\delta,p} \in L^{p/(p-1)}(\mathbb{D})$.

(ii) Let $0 < p \leq 1$. Then $A_\mu : A^p(\omega) \to L^1(\mathbb{T})$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{1}{\tau(z)^{2/p}} \int_{D(\delta(\tau(z))} \omega(\zeta)^{-1/p}\, d\mu(z) < \infty.$$

Proof. By Fubini's theorem,

$$\|A_\mu\|_{L^1(\mathbb{T})} = \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f(z)| \frac{d\mu(z)}{1-|z|^2}\, d\zeta = \int_{\mathbb{D}} |f(z)|\, d\mu(z).$$

Thus $A_\mu : A^p(\omega) \to L^1(\mathbb{T})$ is bounded if and only if μ is a 1-Carleson measure for $A^p(\omega)$, and by Theorem 1 of [51] (see also Theorems D and F) this is equivalent to

$$\sup_{z\in\mathbb{D}} \frac{1}{\tau(z)^{2/p}} \int_{D(\delta(\tau(z))} \omega(\zeta)^{-1/p}\, d\mu(z) < \infty \quad \text{if } 0 < p \leq 1,$$

and to $\widehat{\mu}_{\delta,p} \in L^{p/(p-1)}(\mathbb{D})$ if $p > 1$. □

3.1 The case $p \leq q$

For a positive function g defined on the unit circle $\mathbb{T}$, let

$$Tg(z) = \frac{1}{1-|z|} \int_{I(z)} g(\lambda)\, |d\lambda|,$$

where $I(0) = \mathbb{T}$, and for $z \neq 0$, $I(z)$ denotes the arc in $\mathbb{T}$ with center $z/|z|$ and length $2(1-|z|)$. We also use the notation Pg for the Poisson integral of g.

Lemma 3.2. *For all $z \in \mathbb{D}$, one has $Tg(z) \leq C\, Pg(z)$.*

Proof. Since $|\lambda - z| \leq C(1-|z|)$ for $\lambda \in I(z)$, one has

$$Pg(z) = \int_{\mathbb{T}} g(\lambda) \frac{1-|z|^2}{|\lambda - z|^2} |d\lambda| \geq \int_{I(z)} g(\lambda) \frac{1-|z|^2}{|\lambda - z|^2} |d\lambda| \geq C\, Tg(z).$$

□

Theorem 3.3. *Let $1 < p \leq q < \infty$, and $\omega \in \mathcal{W}$. Then $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$ is bounded if and only if for all $\delta > 0$ sufficiently small, the measure*

$$\widehat{\mu}_{\delta,p}(z)^{p'}\, dA(z)$$

is a (p'/q')-Carleson measure.

Here we recall that

$$\widehat{\mu}_{\delta,p}(z) = \frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} w(\zeta)^{-1/p} d\mu(\zeta).$$

Proof. If $1 < q < \infty$, by duality, the area operator $A_\mu : A^p(\omega) \to L^q(\mu)$ is bounded if and only if there is a positive constant C such that

$$\int_{\mathbb{T}} A_\mu(f)(\zeta)\, g(\zeta)\, |d\zeta| \le C\, \|g\|_{L^{q'}(\mathbb{T})} \|f\|_{A^p(\omega)}$$

for each positive function $g \in L^{q'}(\mathbb{T})$, where q' is the conjugate exponent of q. An application of Fubini's theorem yields

$$\begin{aligned}
\int_{\mathbb{T}} A_\mu(f)(\zeta)\, g(\zeta) |d\zeta| &= \int_{\mathbb{T}} \int_{\Gamma(\zeta)} |f(z)|\, \frac{d\mu(z)}{1-|z|^2}\, g(\zeta)\, |d\zeta| \\
&= \int_{\mathbb{T}} \int_{\mathbb{D}} \chi_{\Gamma(\zeta)}(z) |f(z)|\, \frac{d\mu(z)}{1-|z|^2}\, g(\zeta)\, |d\zeta| \\
&= \int_{\mathbb{D}} \left(\frac{1}{1-|z|^2} \int_{I(z)} g(\zeta)\, d\zeta \right) |f(z)|\, d\mu(z) \\
&= \int_{\mathbb{D}} Tg(z)\, |f(z)|\, d\mu(z).
\end{aligned}$$

By Theorem F, we have that $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$ is bounded if and only if for $\delta > 0$ sufficiently small, one has

$$\int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} |Tg(\zeta)|\, \omega(\zeta)^{-1/p}\, d\mu(\zeta) \right)^{p'} dA(z) \le C \|g\|^{p'}_{L^{q'}(\mathbb{T})}. \tag{3.1}$$

Now, in order to prove the sufficiency, we use Lemma 3.2, the fact that $Pg(\zeta) \le CPg(z)$ for $\zeta \in D(\delta\tau(z))$, and Carleson-Duren's theorem to obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} |Tg(\zeta)|\, \omega(\zeta)^{-1/p}\, d\mu(\zeta) \right)^{p'} dA(z) \\
&\qquad \le C \int_{\mathbb{D}} |Pg(z)|^{p'}\, \widehat{\mu}_{\delta,p}(z)^{p'}\, dA(z) \le C \|g\|^{p'}_{L^{q'}(\mathbb{T})}.
\end{aligned}$$

To prove the necessity, for a given arc $I \subset \mathbb{T}$, take $g = \chi_I$ in (3.1) to obtain

$$\int_{S(I)} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} |T\chi_I(\zeta)|\, \omega(\zeta)^{-1/p}\, d\mu(\zeta) \right)^{p'} dA(z) \le C|I|^{\frac{p'}{q'}},$$

and, since $T\chi_I(\zeta) \ge c > 0$ for $\zeta \in D(\delta\tau(z))$ and $z \in S(I)$, the result follows. □

Theorem 3.4. *Let $0 < p \le 1$, $p \le q < \infty$, and $\omega \in \mathcal{W}$. Then the area operator $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$ is bounded if and only if for all $\delta > 0$ sufficiently small,*

$$\sup_{a \in \mathbb{D}} \frac{(1-|a|)^{\frac{q-1}{q}}}{\tau(a)^{2/p}} \int_{D(\delta\tau(a))} \omega(z)^{-1/p}\, d\mu(z) < \infty. \tag{3.2}$$

Proof. Suppose first that $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$ is bounded. Fix $a \in \mathbb{D}$, and consider the test function $F_{a,p}(z)$ given in Lemma C. One has

$$\|A_\mu(F_{a,p})\|_{L^q(\mathbb{T})} \le C\|F_{a,p}\|_{A^p(\omega)} \le C\,\tau(a)^{2/p}.$$

On the other hand, there is a number $\delta > 0$ (independent of a and ζ) with $D(\delta\tau(a)) \subset \Gamma(\zeta)$ for $\zeta \in \frac{1}{2}I(a)$ (if I is an arc, then λI denotes the arc with the same center as I and length $\lambda|I|$). Therefore, using the fact that $|F_{a,p}(z)| \asymp \omega(z)^{-1/p}$ for $z \in D(\delta\tau(a))$ (see Lemma C) one has

$$\begin{aligned}
\|A_\mu(F_{a,p})\|^q_{L^q(\mathbb{T})} &\ge \int_{\frac{1}{2}I(a)} (A_\mu F_{a,p}(\zeta))^q\,|d\zeta| = \int_{\frac{1}{2}I(a)} \left(\int_{\Gamma(\zeta)} |F_{a,p}(z)|\,\frac{d\mu(z)}{1-|z|}\right)^q |d\zeta| \\
&\ge \int_{\frac{1}{2}I(a)} \left(\int_{D(\delta\tau(a))} |F_{a,p}(z)|\,\frac{d\mu(z)}{1-|z|}\right)^q |d\zeta| \\
&\ge C|I(a)| \left(\int_{D(\delta\tau(a))} \omega(z)^{-1/p}\,\frac{d\mu(z)}{1-|z|}\right)^q \\
&\ge C(1-|a|)^{1-q} \left(\int_{D(\delta\tau(a))} \omega(z)^{-1/p}\,d\mu(z)\right)^q.
\end{aligned}$$

Thus

$$\sup_{a\in\mathbb{D}} \frac{(1-|a|)^{1-\frac{1}{q}}}{\tau(a)^{2/p}} \int_{D(\delta\tau(a))} \omega(z)^{-1/p}\,d\mu(z) < \infty.$$

Conversely, suppose that (3.2) holds. We must show that the area operator $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$ is bounded. Let $\delta > 0$ be sufficiently small, and let $\{z_j\}$ be a (δ,τ)-lattice on $\mathbb{D}$. We use the notation $D_j = D(\delta\tau(z_j))$ and $\widetilde{D}_j = \widetilde{D}(\delta\tau(z_j))$. Using Lemma A we obtain

$$\begin{aligned}
A_\mu f(\zeta) &= \int_{\Gamma(\zeta)} |f(z)|\,\frac{d\mu(z)}{1-|z|} \\
&\le C \int_{\Gamma(\zeta)} \left(\frac{1}{\omega(z)\,\tau(z)^2} \int_{D(\delta\tau(z))} |f(\xi)|^p\,\omega(\xi)\,dA(\xi)\right)^{1/p} \frac{d\mu(z)}{1-|z|} \\
&\le C \sum_{j:D_j\cap\Gamma(\zeta)\neq\emptyset} \int_{D_j} \frac{\omega(z)^{-1/p}}{\tau(z)^{2/p}} \left(\int_{D(\delta\tau(z))} |f(\xi)|^p\,\omega(\xi)\,dA(\xi)\right)^{1/p} \frac{d\mu(z)}{1-|z|} \\
&\le C \sum_{j:D_j\cap\Gamma(\zeta)\neq\emptyset} \left(\int_{\widetilde{D}_j} |f(\xi)|^p\,\omega(\xi)\,dA(\xi)\right)^{1/p} \frac{(1-|z_j|)^{-1}}{\tau(z_j)^{2/p}} \int_{D_j} \omega(z)^{-1/p} d\mu(z)
\end{aligned}$$

Note that in the last inequality we used that $\tau(z) \asymp \tau(z_j)$ for $z \in D_j$. This, together with the assumption (3.2) yields

$$A_\mu f(\zeta) \le C \sum_{j:D_j\cap\Gamma(\zeta)\neq\emptyset} \left(\int_{\widetilde{D}_j} |f(\xi)|^p\,\omega(\xi)\,dA(\xi)\right)^{1/p} (1-|z_j|)^{-1/q}.$$

Since $0 < p \leq 1$, and $(1-|z_j|) \asymp (1-|\xi|)$ for $\xi \in \widetilde{D}_j$, we get

$$\begin{aligned} A_\mu f(\zeta)^p &\leq C \sum_{j: D_j \cap \Gamma(\zeta) \neq \emptyset} \int_{\widetilde{D}_j} |f(\xi)|^p (1-|\xi|)^{-p/q} \omega(\xi)\, dA(\xi) \\ &\leq C \int_{\widetilde{\Gamma}(\zeta)} |f(\xi)|^p (1-|\xi|)^{-p/q} \omega(\xi)\, dA(\xi), \end{aligned}$$

where $\widetilde{\Gamma}(\zeta)$ is some Stolz angle with vertex at ζ with bigger aperture than $\Gamma(\zeta)$. Thus, by Hölder's inequality and Fubini's theorem,

$$\begin{aligned} \|A_\mu f\|_{L^q}^q &= \int_{\mathbb{T}} \left(A_\mu f(\zeta)^p\right)^{q/p} |d\zeta| \\ &\leq C \int_{\mathbb{T}} \left(\int_{\widetilde{\Gamma}(\zeta)} |f(\xi)|^p (1-|\xi|)^{-p/q} \omega(\xi)\, da(\xi) \right)^{q/p} |d\zeta| \\ &\leq C \|f\|_{A^p(\omega)}^{q-p} \int_{\mathbb{T}} \int_{\widetilde{\Gamma}(\zeta)} |f(\xi)|^p \omega(\xi) \frac{dA(\xi)}{1-|\xi|} |d\zeta| \\ &= C \|f\|_{A^p(\omega)}^{q-p} \int_{\mathbb{D}} |f(\xi)|^p \omega(\xi) \left(\int_{\mathbb{T}} \chi_{\widetilde{\Gamma}(\zeta)}(\xi) |d\zeta| \right) \frac{dA(\xi)}{1-|\xi|} \\ &\leq C \|f\|_{A^p(\omega)}^q. \end{aligned}$$

Note that the last inequality is due to the fact that $\int_{\mathbb{T}} \chi_{\widetilde{\Gamma}(\zeta)}(\xi) |d\zeta| \asymp (1-|\xi|)$. Thus the proof is complete. □

3.2 The case $q < p$

Theorem 3.5. *Let $1 \leq q < p < \infty$, and $\omega \in \mathcal{W}$. Then the area operator $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$ is bounded if and only if for all $\delta > 0$ sufficiently small, the function*

$$F_\mu(\zeta) = \int_{\Gamma(\zeta)} \frac{\widehat{\mu}_{\delta,p}(z)^{p'}}{1-|z|} dA(z) \tag{3.3}$$

belongs to $L^{\frac{q(p-1)}{p-q}}(\mathbb{T})$.

Proof. Observe that, by Fubini's theorem, the condition $F_\mu \in L^1(\mathbb{T})$ is equivalent to the function $\widehat{\mu}_{\delta,p}$ being in $L^{p/(p-1)}(\mathbb{D})$. Thus, the case $q = 1$ is just part (i) of Theorem 3.1. Now, suppose that $1 < q < \infty$. In the same way as in the proof of Theorem 3.3, we see that $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$ is bounded if and only if for any positive function $g \in L^{q'}(\mathbb{T})$ we have

$$\int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} |Tg(\zeta)|\, \omega(\zeta)^{-1/p} d\mu(\zeta) \right)^{p'} dA(z) \leq C \|g\|_{L^{q'}(\mathbb{T})}^{p'}. \tag{3.4}$$

Observe that $\frac{q'}{q'-p'} = \frac{q(p-1)}{p-q}$. As $|Tg(\zeta)| \lesssim |Pg(\zeta)|$ and $|Pg(\zeta)| \asymp |Pg(z)|$ for $\zeta \in D(\delta\tau(z))$, if condition (3.3) holds, since now $p' < q'$, an application of Theorem C gives

$$\int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} |Tg(\zeta)|\, \omega(\zeta)^{-1/p}\, d\mu(\zeta) \right)^{p'} dA(z)$$

$$\leq C \int_{\mathbb{D}} |Pg(z)|^{p'} \widehat{\mu}_{\delta,p}(z)^{p'} dA(z) \leq C \|g\|^{p'}_{L^{q'}(\mathbb{T})}.$$

For the converse, we use that

$$Ph(z) \lesssim T\big((Ph)^*\big)(z) \tag{3.5}$$

where $u^*(\zeta) = \sup_{z\in\Gamma(\zeta)} |u(z)|$ is the non-tangential maximal function of u. To prove (3.5), simply use the obvious inequality

$$Ph(z) \leq (Ph)^*(\zeta), \qquad \zeta \in I(z)$$

and then integrate respect to ζ on $I(z)$. Hence, applying (3.4) with $g = (Ph)^*$ we obtain

$$\int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} |Ph(\zeta)|\, \omega(\zeta)^{-1/p}\, d\mu(\zeta) \right)^{p'} dA(z) \leq C \|h\|^{p'}_{L^{q'}(\mathbb{T})}.$$

As $|Ph(\zeta)| \asymp |Ph(z)|$ for $\zeta \in D(\delta\tau(z))$, this gives

$$\int_{\mathbb{D}} |Ph(z)|^{p'} \widehat{\mu}_{\delta,p}(z)^{p'} dA(z) \leq C \|h\|^{p'}_{L^{q'}(\mathbb{T})}.$$

By Theorem C we see that (3.3) is satisfied. □

Finally, in the case that $0 < q < 1$ we have a sufficient condition. We didn't know if this condition is also necessary, but when $q = 1$ it coincides with the one that characterizes the boundedness of the area operator.

Theorem 3.6. *Let $q < p < \infty$ with $0 < q \leq 1$. Suppose that the function*

$$F^{\mu}_{p,q}(z) = \frac{(1-|z|)^{\frac{1-q}{q}}}{\tau(z)^{2/q}} \int_{D(\delta\tau(z))} \omega(\zeta)^{-1/p}\, d\mu(\zeta)$$

belongs to $L^{\frac{pq}{p-q}}(\mathbb{D})$. Then the area operator $A_\mu : A^p(\omega) \to L^q(\mathbb{T})$ is bounded.

Proof. As can be seen from the proof of Theorem 3.4, we have

$$A_\mu f(\zeta) \lesssim \sum_{D_j \cap \Gamma(\zeta) \neq \emptyset} \left(\int_{\widetilde{D}_j} |f(z)|^p \omega(z)\, dA(z) \right)^{1/p} (1-|z_j|^2)^{-\frac{1}{q}}\, \tau(z_j)^{-2\frac{(q-p)}{pq}} F^{\mu}_{p,q}(z_j).$$

Since $0 < q \leq 1$, this gives

$$A_\mu f(\zeta)^q \lesssim \sum_{D_j \cap \Gamma(\zeta) \neq \emptyset} \left(\int_{\widetilde{D}_j} |f(z)|^p \omega(z)\, dA(z) \right)^{q/p} (1 - |z_j|^2)^{-1}\, \tau(z_j)^{-2\frac{(q-p)}{p}} F^\mu_{p,q}(z_j)^q.$$

At this point we use Hölder's inequality in order to obtain

$$A_\mu f(\zeta)^q \lesssim (I)^{q/p} \cdot (II)^{(p-q)/p}, \tag{3.6}$$

with

$$\begin{aligned} (I) &:= \sum_{D_j \cap \Gamma(\zeta) \neq \emptyset} \int_{\widetilde{D}_j} |f(z)|^p\, \omega(z)\, dA(z)\, (1 - |z_j|)^{-1} \\ &\asymp \sum_{D_j \cap \Gamma(\zeta) \neq \emptyset} \int_{\widetilde{D}_j} |f(z)|^p\, \omega(z)\, \frac{dA(z)}{1 - |z|} \\ &\lesssim \int_{\widetilde{\Gamma}(\zeta)} |f(z)|^p\, \omega(z)\, \frac{dA(z)}{1 - |z|}, \end{aligned} \tag{3.7}$$

where $\widetilde{\Gamma}(\zeta)$ is a Stolz region with bigger aperture, and

$$(II) := \sum_{D_j \cap \Gamma(\zeta) \neq \emptyset} (1 - |z_j|^2)^{-1}\, \tau(z_j)^2 F^\mu_{p,q}(z_j)^{\frac{qp}{p-q}}.$$

Now, it is clear that $F^\mu_{p,q}(z_j) \asymp F^\mu_{p,q}(z)$ if $z \in D_j$. Thus

$$\tau(z_j)^2 F^\mu_{p,q}(z_j)^{\frac{qp}{p-q}} \asymp \int_{D_j} F^\mu_{p,q}(z)^{\frac{qp}{p-q}}\, dA(z).$$

This gives

$$(II) \asymp \sum_{D_j \cap \Gamma(\zeta) \neq \emptyset} \int_{D_j} F^\mu_{p,q}(z)^{\frac{qp}{p-q}}\, \frac{dA(z)}{1 - |z|} \lesssim \int_{\widetilde{\Gamma}(\zeta)} F^\mu_{p,q}(z)^{\frac{qp}{p-q}}\, \frac{dA(z)}{1 - |z|}. \tag{3.8}$$

Finally, bearing in mind (3.6), (3.7) and (3.8), we get

$$\begin{aligned} \|A_\mu f\|^q_{L^q(\mathbb{T})} &= \int_{\mathbb{T}} A_\mu f(\zeta)^q |d\zeta| \\ &\lesssim \int_{\mathbb{T}} \left(\int_{\widetilde{\Gamma}(\zeta)} |f(z)|^p\, \omega(z)\, \frac{dA(z)}{1 - |z|} \right)^{q/p} \left(\int_{\widetilde{\Gamma}(\zeta)} F^\mu_{p,q}(z)^{\frac{qp}{p-q}}\, \frac{dA(z)}{1 - |z|} \right)^{\frac{p-q}{p}} |d\zeta| \\ &\leq \left(\int_{\mathbb{T}} \int_{\widetilde{\Gamma}(\zeta)} |f(z)|^p\, \omega(z)\, \frac{dA(z)}{1 - |z|}\, |d\zeta| \right)^{q/p} \left(\int_{\mathbb{T}} \int_{\widetilde{\Gamma}(\zeta)} F^\mu_{p,q}(z)^{\frac{qp}{p-q}}\, \frac{dA(z)}{1 - |z|}\, |d\zeta| \right)^{\frac{p-q}{p}} \\ &\asymp \left(\int_{\mathbb{D}} |f(z)|^p\, \omega(z)\, dA(z) \right)^{q/p} \left(\int_{\mathbb{D}} F^\mu_{p,q}(z)^{\frac{qp}{p-q}}\, dA(z) \right)^{\frac{p-q}{p}} \end{aligned}$$

This finishes the proof. □

Chapter 4

The exponential weight

For the case of the exponential weight

$$\omega(z) = \exp\Big(-\frac{A}{1-|z|^2}\Big), \quad A > 0$$

a further study of the corresponding weighted Bergman space can be done, as it has been recently proved in [16] that the corresponding reproducing kernel K_z satisfies the estimate

$$\int_{\mathbb{D}} |K_z(\xi)|\, \omega(\xi)^{1/2}\, dA(\xi) \lesssim \omega(z)^{-1/2}. \tag{4.1}$$

This estimate allows to study the non-Hilbert space case. It turns out that when studying properties or operators such as the Bergman projection, Toeplitz operators or Hankel operators, where the reproducing kernels are involved, the most convenient settings are the spaces $A^p(\omega^{p/2})$. We consider the class $\mathcal{E}$ to consist of those weights in $\mathcal{W}$ satisfying condition (4.1). In the last section of this chapter, we will prove that the family of exponential type weights ω_σ given by (1.1) satisfy condition (4.1) for $0 < \sigma < \infty$.

4.1 Integral estimates for reproducing kernels

For a given weight v we introduce the growth space $L^\infty(v)$ that consists of those measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^\infty(v)} := ess \sup_{z\in\mathbb{D}} |f(z)| v(z) < \infty,$$

and let $A^\infty(v) := L^\infty(v) \cap H(\mathbb{D})$.

Lemma 4.1. *Let $\omega \in \mathcal{E}$. For each $z \in \mathbb{D}$, we have*

$$\|K_z\|_{A^\infty(\omega^{1/2})} \lesssim \omega(z)^{-1/2}\, \tau(z)^{-2}.$$

Proof. By Lemma A and condition (4.1), we have

$$\omega(\xi)^{1/2}|K_z(\xi)| = \omega(\xi)^{1/2}|K_\xi(z)| \lesssim \frac{\omega(\xi)^{1/2}}{\omega(z)^{1/2}\,\tau(z)^2}\int_{D(\delta\tau(z))} |K_\xi(s)|\,\omega(s)^{1/2}\,dA(s)$$
$$\lesssim \omega(z)^{-1/2}\,\tau(z)^{-2}.$$

□

This Lemma together with condition (4.1) allows us to obtain the following estimate for the norm of the reproducing kernel in $A^p(\omega^{p/2})$.

Lemma 4.2. *Let $1 \le p < \infty$, $\omega \in \mathcal{E}$ and $z \in \mathbb{D}$. Then*

$$\|K_z\|_{A^p(\omega^{p/2})} \asymp \omega(z)^{-1/2}\,\tau(z)^{-2(p-1)/p}.$$

Proof. By (4.1) and Lemma 4.1, we have

$$\|K_z\|^p_{A^p(\omega^{p/2})} = \int_{\mathbb{D}} |K_z(\xi)|^p\,\omega(\xi)^{p/2}\,dA(\xi) = \int_{\mathbb{D}} |K_z(\xi)|\,\omega(\xi)^{1/2}\left(|K_z(\xi)|\,\omega(\xi)^{1/2}\right)^{p-1}\,dA(\xi)$$

$$\le \|K_z\|^{p-1}_{A^\infty(\omega^{1/2})}\int_{\mathbb{D}} |K_z(\xi)|\,\omega(\xi)^{1/2}\,dA(\xi) \lesssim \|K_z\|^{p-1}_{A^\infty(\omega^{1/2})}\,\omega(z)^{-1/2}$$

$$\lesssim \omega(z)^{-p/2}\tau(z)^{-2(p-1)}.$$

On the other hand, by

$$\|K_z\|^p_{A^p(\omega^{p/2})} \ge \int_{D(\delta\tau(z))} |K_z(\xi)|^p\,\omega(\xi)^{p/2}\,dA(\xi)$$

$$\ge C\|K_z\|^p_{A^2(\omega)}\int_{D(\delta\tau(z))} \|K_\xi\|^p_{A^2(\omega)}\,\omega(\xi)^{1/2}\,dA(\xi)$$

$$\ge C\,\omega(z)^{-p/2}\tau(z)^{-2(p-1)}.$$

□

4.2 Bounded projections

The boundedness of Bergman projection is a fact of fundamental importance. In the case of the unit disc, the boundedness of Bergman projections is studied in [27], [77] and it immediately gives the duality between the Bergman spaces. The orthogonal Bergman projection P_ω from $L^2(\mathbb{D}, \omega dA)$ to $A^2(\omega)$ is given by

$$P_\omega f(z) = \int_{\mathbb{D}} f(\xi)\overline{K_z(\xi)}\,\omega(\xi)\; dA(\xi).$$

The natural Bergman projection is not necessarily bounded on $L^p(\mathbb{D}, \omega dA)$ unless $p = 2$, (see [18] and [71] for more details). However, we are going to see next that P_ω is bounded on $L^p(\omega^{p/2}) := L^p(\mathbb{D}, \omega^{p/2}\, dA)$. We first prove the boundedness of the operator P_ω^+ defined as

$$P_\omega^+ f(z) = \int_{\mathbb{D}} f(\xi)\,|K_z(\xi)|\,\omega(\xi)\; dA(\xi).$$

We mention here that, for the case of the exponential weight with $\sigma = 1$, the results of this section has been obtained recently in [16].

Theorem 4.3. *Let $1 \le p < \infty$ and $\omega \in \mathcal{E}$. The operator P_ω^+ is bounded on $L^p(\omega^{p/2})$ and on $L^\infty(\omega^{1/2})$.*

Proof. We first consider the easiest case $p = 1$. By Fubini's theorem and condition (4.1) we obtain

$$\begin{aligned}
\|P_\omega^+ f\|_{A^1(\omega^{1/2})} &= \int_{\mathbb{D}} |P_\omega^+ f(z)|\,\omega(z)^{1/2}\; dA(z)\\
&\le \int_{\mathbb{D}}\int_{\mathbb{D}} |f(z)||K_z(\xi)|\,\omega(\xi)\,\omega(z)^{1/2}\; dA(\xi)\; dA(z)\\
&= \int_{\mathbb{D}} |f(\xi)|\,\omega(\xi)\left(\int_{\mathbb{D}} |K_\xi(z)|\,\omega(z)^{1/2} dA(z)\right) dA(\xi)\\
&\lesssim \int_{\mathbb{D}} |f(\xi)|\,\omega(\xi)^{1/2} dA(\xi) = \|f\|_{L^1(\omega^{1/2})}.
\end{aligned}$$

Next, we consider the case $1 < p < \infty$. Let p' denote the conjugate exponent of p. By Hölder's inequality and (4.1) we get

$$\begin{aligned}
|P_\omega^+ f(z)|^p &\le \left(\int_{\mathbb{D}} |f(\xi)|^p\,|K_z(\xi)|\,\omega(\xi)^{\frac{p+1}{2}} dA(\xi)\right)\left(\int_{\mathbb{D}} |K_z(\xi)|\omega(\xi)^{1/2}\, dA(\xi)\right)^{p-1}\\
&\lesssim \left(\int_{\mathbb{D}} |f(\xi)|^p|K_z(\xi)|\omega(\xi)^{\frac{p+1}{2}} dA(\xi)\right)\omega(z)^{-\frac{(p-1)}{2}}.
\end{aligned}$$

This together with Fubini's theorem and another application of (4.1) gives

$$\begin{aligned}
\|P_\omega^+ f\|_{A^p(\omega^{p/2})}^p &= \int_{\mathbb{D}} |P_\omega^+ f(z)|^p\, \omega(z)^{p/2} dA(z) \\
&\lesssim \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(\xi)|^p |K_z(\xi)|\, \omega(\xi)^{\frac{p+1}{2}}\; dA(\xi) \right) \omega(z)^{1/2} dA(z) \\
&= \int_{\mathbb{D}} |f(\xi)|^p\, \omega(\xi)^{\frac{p+1}{2}} \left(\int_{\mathbb{D}} |K_\xi(z)|\, \omega(z)^{1/2} dA(z) \right) dA(\xi) \\
&\lesssim \|f\|_{L^p(\omega^{p/2})}^p.
\end{aligned}$$

Finally, if $f \in L^\infty(\omega^{1/2})$, by condition (4.1) we get

$$\begin{aligned}
\omega(z)^{1/2} |P_\omega^+(f)(z)| &\le \omega(z)^{1/2} \int_{\mathbb{D}} |f(\xi)|\, |K_z(\xi)|\, \omega(\xi) dA(\xi) \\
&\le \|f\|_{L^\infty(\omega^{1/2})}\, \omega(z)^{1/2} \int_{\mathbb{D}} |K_z(\xi)|\, \omega(\xi)^{1/2}\, dA(\xi) \\
&\lesssim \|f\|_{L^\infty(\omega^{1/2})}.
\end{aligned}$$

This shows that P_ω^+ is bounded on $L^\infty(\omega^{1/2})$. The proof is complete. □

Theorem 4.4. *Let* $1 \le p < \infty$ *and* $\omega \in \mathcal{E}$. *The Bergman projection* $P_\omega : L^p(\omega^{p/2}) \longrightarrow A^p(\omega^{p/2})$ *is bounded. Moreover,* $P_\omega : L^\infty(\omega^{1/2}) \to A^\infty(\omega^{1/2})$ *is also bounded.*

Proof. In view of Theorem 4.3, it remains to see that $P_\omega f$ defines an analytic function on $\mathbb{D}$. This follows easily by density, the boundedness of P_ω^+ and the completeness of $A^p(\omega^{p/2})$. □

Corollary 4.5. *Let* $\omega \in \mathcal{E}$. *The reproducing formula* $f = P_\omega f$ *holds for each* $f \in A^1(\omega^{1/2})$.

Proof. This is an immediate consequence of the boundedness of the Bergman projection and the density of the polynomials. □

4.3 Complex interpolation and duality

An elementary introduction to the basic theory of complex interpolation, including the complex interpolation of L^p spaces can be found in Chapter 2 of the book [77]. We assume in this section that the reader is familiar with that theory. First of all, we recall the following well-known interpolation theorem of Stein and Weiss [63].

Theorem 4.6. *Suppose that ω, ω_0 and ω_1 are weight functions on $\mathbb{D}$. If $1 \le p_0 \le p_1 \le \infty$ and $0 \le \theta \le 1$, then*

$$\big[L^{p_0}(\mathbb{D}, \omega_0 dA), L^{p_1}(\mathbb{D}, \omega_1 dA)\big]_\theta = L^p(\mathbb{D}, \omega dA)$$

with equal norms, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \omega^{1/p} = \omega_0^{\frac{1-\theta}{p_0}}\,\omega_1^{\frac{\theta}{p_1}}.$$

With this and the result on bounded projections we can obtain the following result on complex interpolation of large weighted Bergman spaces.

Theorem 4.7. *Let ω be a weight in the class $\mathcal{E}$. If $1 \le p_0 \le p_1 \le \infty$ and $0 \le \theta \le 1$, then*

$$[A^{p_0}(\omega^{p_0/2}), A^{p_1}(\omega^{p_1/2})] = A^p(\omega^{p/2}),$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Proof. The inclusion $[A^{p_0}(\omega^{p_0/2}), A^{p_1}(\omega^{p_1/2})] \subset A^p(\omega^{p/2})$ is a consequence of the definition of complex interpolation, the fact that each $A^{p_k}(\omega^{p_k/2})$ is a closed subspace of $L^{p_k}(\omega^{p_k/2})$ and $\big[L^{p_0}(\omega^{p_0/2}), L^{p_1}(\omega^{p_1/2})\big]_\theta = L^p(\omega^{p/2})$. This last assertion follows from Theorem 4.6.

On the other hand, if $f \in A^p(\omega^{p/2}) \subset L^p(\omega^{p/2})$, it follows from Theorem 4.6 that

$$\big[L^{p_0}(\omega^{p_0/2}), L^{p_1}(\omega^{p_1/2})\big]_\theta = L^p(\omega^{p/2}).$$

Thus, there exists a function $F_\zeta(z)$ ($z \in \mathbb{D}$ and $0 \le \operatorname{Re}\zeta \le 1$) and a positive constant C such that:

(a) $F_\theta(z) = f(z)$ for all $z \in \mathbb{D}$.

(b) $\|F_\zeta\|_{L^{p_0}(\omega^{p_0/2})} \le C$ for all $\operatorname{Re}\zeta = 0$.

(c) $\|F_\zeta\|_{L^{p_1}(\omega^{p_1/2})} \le C$ for all $\operatorname{Re}\zeta = 1$.

Define a function G_ζ by $G_\zeta(z) = P_\omega F_\zeta(z)$. Due to the reproducing formula in Corollary 4.5 and Theorem 4.4 we have:

(a) $G_\theta(z) = f(z)$ for all $z \in \mathbb{D}$.

(b) $\|G_\zeta\|_{L^{p_0}(\omega^{p_0/2})} \le C$ for all $\operatorname{Re}\zeta = 0$.

(c) $\|G_\zeta\|_{L^{p_1}(\omega^{p_1/2})} \le C$ for all $\operatorname{Re}\zeta = 1$.

Since each function G_ζ is analytic on $\mathbb{D}$, we conclude that f belongs to $[A^{p_0}(\omega^{p_0/2}), A^{p_1}(\omega^{p_1/2})]$. This completes the proof of the theorem. $\square$

As in the case of the standard Bergman spaces, one can use the result just proved on the boundedness of the Bergman projection P_ω in $L^p(\omega^{p/2})$ to identify the dual spaces of $A^p(\omega^{p/2})$. As usual, if X is a Banach space, we denote its dual by X^*. Next two results (Theorems 4.8 and 4.9) on the duality of Bergman spaces with exponential type weights appears also on [16].

Theorem 4.8. *Let $\omega \in \mathcal{E}$ and $1 < p < \infty$. The dual space of $A^p(\omega^{p/2})$ can be identified (with equivalent norms) with $A^{p'}(\omega^{p'/2})$ under the integral pairing*

$$\langle f, g\rangle_\omega = \int_{\mathbb{D}} f(z)\,\overline{g(z)}\,\omega(z)\,dA(z).$$

Here p' denotes the conjugate exponent of p, that is, $p' = p/(p-1)$.

Proof. Let $1 < p < \infty$ and let $p' = p/(p-1)$ be its dual exponent. Given a function $g \in A^{p'}(\omega^{p'/2})$, Hölder's inequality implies that the linear functional $\psi_g : A^p(\omega^{p/2}) \longrightarrow \mathbb{C}$ defined by

$$\psi_g(f) := \int_{\mathbb{D}} f(\xi)\ \overline{g(\xi)}\,\omega(\xi)\,dA(\xi), \quad f \in A^p(\omega^{p/2}),$$

is bounded with $\|\psi_g\| \le \|g\|_{A^{p'}(\omega^{p'/2})}$.
Conversely, let $T \in \big(A^p(\omega^{p/2})\big)^*$. By Hahn-Banach we can extend T to an element $\widetilde{T} \in \big(L^p(\mathbb{D}, \omega^{p/2} dA)\big)^*$ such that $\|\widetilde{T}\| = \|T\|$. By the Riesz representation Theorem there exists $H \in L^{p'}(\mathbb{D}, \omega^{p/2} dA)$ with $\|H\|_{L^{p'}(\omega^{p/2})} = \|\widetilde{T}\| = \|T\|$ such that

$$\widetilde{T}(f) = \int_{\mathbb{D}} f(\xi)\,\overline{H(\xi)}\,\omega(\xi)^{p/2}\,dA(\xi),$$

for every $f \in A^p(\omega^{p/2})$. Consider the function $h(\xi) = H(\xi)\,\omega(\xi)^{\frac{p}{2}-1}$. Then $h \in L^{p'}(\omega^{p'/2})$ with

$$\|h\|_{L^{p'}(\omega^{p'/2})} = \|H\|_{L^{p'}(\omega^{p/2})} = \|T\|,$$

and

$$T(f) = \widetilde{T}(f) = \int_{\mathbb{D}} f(\xi)\ \overline{h(\xi)}\,\omega(\xi)\,dA(\xi), \qquad f \in A^p(\omega^{p/2}).$$

Let $g = P_\omega h$. By Theorem 4.4, the Bergman projection $P_\omega : L^{p'}(\omega^{p'/2}) \longrightarrow A^{p'}(\omega^{p'/2})$ is bounded. Thus $g \in A^{p'}(\omega^{p'/2})$ with

$$\|g\|_{A^{p'}(\omega^{p'/2})} = \|P_\omega h\|_{A^{p'}(\omega^{p'/2})} \lesssim \|h\|_{L^{p'}(\omega^{p'/2})} = \|T\|.$$

From Fubini's theorem it is easy to see that P_ω is self-adjoint. Indeed,

$$\begin{aligned}
\langle P_\omega f, g\rangle_\omega &= \int_{\mathbb{D}} P_\omega f(\xi)\, \overline{g(\xi)}\, \omega(\xi)\, dA(\xi) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(s)\, \overline{K_\xi(s)}\, \omega(s)\, dA(s) \right) \overline{g(\xi)}\, \omega(\xi)\, dA(\xi) \\
&= \int_{\mathbb{D}} \overline{\left(\int_{\mathbb{D}} g(\xi)\, \overline{K_s(\xi)}\, \omega(\xi)\, dA(\xi) \right)}\, f(s)\, \omega(s)\, dA(s) \\
&= \int_{\mathbb{D}} \overline{P_\omega g(s)}\, f(s)\, \omega(s)\, dA(s) = \langle f, P_\omega g\rangle_\omega.
\end{aligned}$$

The interchange of the order of integration is well justified, because of the boundedness of the operator P_ω^+ (see Theorem 4.3) given by

$$P_\omega^+ f(z) = \int_{\mathbb{D}} f(\zeta)\, |K_z(\zeta)|\, \omega(\zeta)\, dA(\zeta).$$

Therefore, since $f = P_\omega f$ for every $f \in A^p(\omega^{p/2})$ according to Corollary 4.5, we get

$$\begin{aligned}
T(f) &= \int_{\mathbb{D}} f(\xi)\, \overline{h(\xi)}\, \omega(\xi)\, dA(\xi) \\
&= \langle f, P_\omega h\rangle_\omega = \langle f, g\rangle_\omega = \psi_g(f).
\end{aligned}$$

Finally, the function g is unique. Indeed, if there is another function $\widetilde{g} \in A^{p'}(\omega^{p'/2})$ with $T(f) = \psi_g(f) = \psi_{\widetilde{g}}(f)$ for every $f \in A^p(\omega^{p/2})$, then by taking $f = K_a$ for each $a \in \mathbb{D}$ (that belongs to $A^p(\omega^{p/2})$ due to Lemma 4.2), and using the reproducing formula, we obtain

$$g(a) = \overline{\psi_g(K_a)} = \overline{\psi_{\widetilde{g}}(K_a)} = \widetilde{g}(a), \;\; a \in \mathbb{D}.$$

Thus, any bounded linear functional T is of the form $T = \psi_g$ for some unique $g \in A^{p'}(\omega^{p'/2})$ and, furthermore

$$\|T\| \asymp \|g\|_{A^{p'}(\omega^{p'/2})}.$$

The proof is complete. □

Theorem 4.9. *Let $\omega \in \mathcal{E}$. The dual space of $A^1(\omega^{1/2})$ can be identified (with equivalent norms) with $A^\infty(\omega^{1/2})$ under the integral pairing $\langle f, g\rangle_\omega$.*

Proof. Let $g \in A^\infty(\omega^{1/2})$. The linear functional $\psi_g : A^1(\omega^{1/2}) \longrightarrow \mathbb{C}$ defined by $\psi_g(f) := \langle f, g\rangle_\omega$ is bounded with $\|\psi_g\| \leq \|g\|_{L^\infty(\omega^{1/2})}$, since for every $f \in A^1(\omega^{1/2})$

$$|\psi_g(f)| \leq \|g\|_{L^\infty(\omega^{1/2})} \|f\|_{A^1(\omega^{1/2})}.$$

Conversely, let $T \in \left(A^1(\omega^{1/2})\right)^*$. Consider the space X that consists of the functions of the form $h = f\omega^{1/2}$ with $f \in A^1(\omega^{1/2})$. Clearly X is a subspace of $L^1(\mathbb{D}, dA)$ and $F(h) := T(h\omega^{-1/2}) = T(f)$ defines a bounded linear functional on X with $\|F\| = \|T\|$. By Hahn- Banach, F has an extension $\widetilde{F} \in (L^1(\mathbb{D}, dA))^*$ with $\|\widetilde{F}\| = \|F\|$. Hence, there is a function $G \in L^\infty(\mathbb{D}, dA)$ with $\|G\|_{L^\infty(\mathbb{D},dA)} = \|F\|$ such that

$$F(h) = \widetilde{F}(h) = \int_{\mathbb{D}} h(\xi)\, \overline{G(\xi)}\, dA(\xi), \qquad h \in X,$$

or

$$T(f) = \int_{\mathbb{D}} f(\xi)\overline{G(\xi)}\, \omega(\xi)^{1/2}\, dA(\xi), \qquad f \in A^1(\omega^{1/2}).$$

Consider the function $H(z) = \omega(z)^{-1/2}G(z)$. Then $H \in L^\infty(\omega^{1/2})$ with

$$\|H\|_{L^\infty(\omega^{1/2})} = \|G\|_{L^\infty(\mathbb{D},dA)} = \|F\| = \|T\|.$$

By Theorem 4.4, the function $g = P_\omega H$ is in $A^\infty(\omega^{1/2})$ with

$$\|g\|_{A^\infty(\omega^{1/2})} \lesssim \|H\|_{L^\infty(\omega^{1/2})} = \|T\|.$$

Also, for $f \in A^1(\omega^{1/2})$, by the reproducing formula, we have

$$T(f) = \int_{\mathbb{D}} f(\xi)\overline{H(\xi)}\, \omega(\xi)\, dA(\xi) = \langle P_\omega f, H\rangle_\omega = \langle f, P_\omega H\rangle_\omega = \psi_g(f).$$

Finally, as in the proof of Theorem 4.8 the function g is unique. □

Corollary 4.10. *Let $\omega \in \mathcal{E}$. The set E of finite linear combinations of reproducing kernels is dense in $A^p(\omega^{p/2})$, $1 \leq p < \infty$.*

Proof. Since E is a linear subspace of $A^p(\omega^{p/2})$, by standard functional analysis and the duality results in Theorems 4.8 and 4.9, it is enough to prove that $g \equiv 0$ if $g \in A^{p'}(\omega^{p'/2})$ satisfies $\langle f, g\rangle_\omega = 0$ for each f in E (with p' being the conjugate exponent of p, and $g \in A^\infty(\omega^{1/2})$ if $p = 1$). But, taking $f = K_z$ for each $z \in \mathbb{D}$ and using the reproducing formula, we get $g(z) = P_\omega g(z) = \langle g, K_z\rangle_\omega = 0$, for each $z \in \mathbb{D}$. This finishes the proof. □

Our next goal is to identify the predual of $A^1(\omega^{1/2})$. For a given weight v, we introduce the space $A_0(v)$ consisting of those functions $f \in A^\infty(v)$ with $\lim_{|z|\to 1^-} v(z)|f(z)| = 0$. Clearly, $A_0(v)$ is a closed subspace of $A^\infty(v)$.

Theorem 4.11. *Let $\omega \in \mathcal{E}$. Under the integral pairing $\langle f, g\rangle_\omega$, the dual space of $A_0(\omega^{1/2})$ can be identified (with equivalent norms) with $A^1(\omega^{1/2})$.*

Proof. If $g \in A^1(\omega^{1/2})$, clearly $\Lambda_g(f) = \langle f, g\rangle_\omega$ defines a bounded linear functional in $A_0(\omega^{1/2})$ with $\|\Lambda_g\| \le \|g\|_{A^1(\omega^{1/2})}$. Conversely, assume that $\Lambda \in \big(A_0(\omega^{1/2})\big)^*$. Consider the space X that consists of functions of the form $h = f\omega^{1/2}$ with $f \in A_0(\omega^{1/2})$. Clearly X is a subspace of $C_0(\mathbb{D})$ (the space of all continuous functions vanishing at the boundary) and $T(h) = \Lambda(\omega^{-1/2}h) = \Lambda(f)$ defines a bounded linear functional on X with $\|T\| = \|\Lambda\|$. By Hahn-Banach, T has an extension $\widetilde{T} \in \big(C_0(\mathbb{D})\big)^*$ with $\|\widetilde{T}\| = \|T\|$. Hence, by Riesz representation theorem, there is a measure $\mu \in \mathcal{M}(\mathbb{D})$ (the Banach space of all complex Borel measures μ equipped with the variation norm $\|\mu\|_{\mathcal{M}}$) with $\|\mu\|_{\mathcal{M}} = \|T\|$ such that

$$T(h) = \widetilde{T}(h) = \int_{\mathbb{D}} h(\zeta)\, d\mu(\zeta), \qquad h \in X,$$

or

$$\Lambda(f) = \int_{\mathbb{D}} f(\zeta)\, \omega(\zeta)^{1/2}\, d\mu(\zeta), \qquad f \in A_0(\omega^{1/2}).$$

Consider the function g defined on the unit disk by

$$\overline{g(z)} = \int_{\mathbb{D}} K_z(\zeta)\, \omega(\zeta)^{1/2}\, d\mu(\zeta), \qquad z \in \mathbb{D}.$$

Clearly g is analytic on $\mathbb{D}$ and, by Fubini's theorem and condition (4.1), we have

$$\begin{aligned}
\|g\|_{A^1(\omega^{1/2})} &\le \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |K_z(\zeta)|\, \omega(\zeta)^{1/2}\, d|\mu|(\zeta) \right) \omega(z)^{1/2} dA(z) \\
&= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |K_\zeta(z)|\, \omega(z)^{1/2}\, dA(z) \right) \omega(\zeta)^{1/2}\, d|\mu|(\zeta) \\
&\lesssim |\mu|(\mathbb{D}) = \|\mu\|_{\mathcal{M}} = \|\Lambda\|,
\end{aligned}$$

proving that g belongs to $A^1(\omega^{1/2})$. Now, since $A_0(\omega^{1/2}) \subset A^2(\omega)$, the reproducing formula

$f(\zeta) = \langle f, K_\zeta \rangle_\omega$ holds for all $f \in A_0(\omega^{1/2})$. This and Fubini's theorem yields

$$\begin{aligned}\Lambda_g(f) = \langle f, g \rangle_\omega &= \int_{\mathbb{D}} f(z) \left(\int_{\mathbb{D}} K_z(\zeta)\, \omega(\zeta)^{1/2}\, d\mu(\zeta) \right) \omega(z)\, dA(z) \\ &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} f(z)\, \overline{K_\zeta(z)}\, \omega(z)\, dA(z) \right) \omega(\zeta)^{1/2}\, d\mu(\zeta) \\ &= \int_{\mathbb{D}} f(\zeta)\, \omega(\zeta)^{1/2}\, d\mu(\zeta) = \Lambda(f).\end{aligned}$$

By the reproducing formula, the function g is uniquely determined by the identity $g(z) = \overline{\Lambda(K_z)}$. This completes the proof. □

For the case of normal weights, the analogues of Theorems 4.9 and 4.11 were obtained by Shields and Williams in [62]. They also asked what happens with the exponential weights, problem that is solved in the present work.

4.4 Atomic decomposition

With the help of the duality results and the estimates for the p-norm of the reproducing kernels K_z, we can obtain the atomic decomposition of Bergman spaces with exponential weights in the case that $p \geq 1$. We use the notation $k_{p,z}$ for the normalized reproducing kernels in $A^p(\omega^{p/2})$, that is

$$k_{p,z} = \frac{K_z}{\|K_z\|_{A^p(\omega^{p/2})}}.$$

Theorem 4.12. *Let $\omega \in \mathcal{E}$ and $1 \leq p < \infty$. There exists a lattice $\{z_n\} \subset \mathbb{D}$ such that:*

(i) For any $\lambda = \{\lambda_n\} \in \ell^p$, the function

$$f(z) = \sum_n \lambda_n\, k_{p,z_n}(z)$$

is in $A^p(\omega^{p/2})$ with $\|f\|_{A^p(\omega^{p/2})} \leq C \|\lambda\|_{\ell^p}$.

(ii) For every $f \in A^p(\omega^{p/2})$ exists $\lambda = \{\lambda_n\} \in \ell^p$ such that

$$f(z) = \sum_n \lambda_n\, k_{p,z_n}(z)$$

and $\|\lambda\|_{\ell^p} \leq C\, \|f\|_{A^p(\omega^{p/2})}$.

Proof. (i) As done in Proposition 1.5, the function f defines an analytic function on $\mathbb{D}$. Set

$$M(z) := \sum_{k=0}^{\infty} \tau(z_k)^2 \omega(z_k)^{1/2} |K_{z_k}(z)|.$$

By Hölder's inequality we have

$$\|F\|^p_{A^p(\omega^{p/2})} \leq \int_{\mathbb{D}} \left(\sum_{k=0}^{\infty} |\lambda_k| \omega(z_k)^{1/2} \, \tau(z_k)^{2(\frac{p-1}{p})} |K_{z_k}(z)| \right)^p \omega(z)^{p/2} \, dA(z).$$

$$\lesssim \int_{\mathbb{D}} \Big(\sum_{k=0}^{\infty} |\lambda_k|^p \omega(z_k)^{1/2} |K_{z_k}(z)| \Big) \, M(z)^{p-1} \omega(z)^{p/2} \, dA(z).$$

On the other hand, using Lemma A, Lemma B and condition (4.1) we have

$$\begin{aligned} M(z) : &= \sum_{k=0}^{\infty} \tau(z_k)^2 \, \omega(z_k)^{1/2} \, |K_{z_k}(z)| \\ &\lesssim \sum_{k=0}^{\infty} \int_{D(\delta\tau(z_k))} |K_z(\xi)| \, \omega(\xi)^{1/2} dA(\xi) \\ &\lesssim \int_{\mathbb{D}} |K_z(\xi)| \, \omega(\xi)^{1/2} \, dA(\xi) \lesssim \omega(z)^{-1/2}. \end{aligned}$$

Therefore, applying condition (4.1) again, we obtain

$$\begin{aligned} \|F\|^p_{A^p(\omega^{p/2})} &\lesssim \int_{\mathbb{D}} \left(\sum_{k=0}^{\infty} |\lambda_k|^p \omega(z_k)^{1/2} \, |K_{z_k}(z)| \right) \omega(z)^{1/2} \, dA(z) \\ &\lesssim \sum_{k=0}^{\infty} |\lambda_k|^p \omega(z_k)^{1/2} \int_{\mathbb{D}} |K_{z_k}(z)| \omega(z)^{1/2} \, dA(z) \\ &\lesssim \sum_{k=0}^{\infty} |\lambda_k|^p. \end{aligned}$$

In order to prove (ii), we define a linear operator $S : \ell^p \longrightarrow A^p(\omega^{p/2})$ given by

$$S(\{\lambda_n\}) := \sum_{n=0}^{\infty} \lambda_n \, k_{p,z_n}.$$

By (i), the operator S is bounded. By the duality results obtained in the previous Chapter, when $1 < p < \infty$, the adjoint operator $S^* : A^{p'}(\omega^{p'/2}) \to \ell^{p'}$, where p' is the conjugate exponent of p, is defined by

$$\langle Sx, f \rangle_\omega = \ \langle x, S^* f \rangle_\ell = \sum_n x_n \, \overline{(S^* f)_n}.$$

for every $x \in \ell^p$ and $f \in A^{p'}(\omega^{p'/2})$. From here, the proof follows the same lines as in Theorem 1.7

When $p = 1$, then $S^* : A^\infty(\omega^{1/2}) \longrightarrow \ell^\infty$ is given by

$$\{(S^*f)_n\} = \left\{ \frac{f(z_n)}{\|K_{z_n}\|_{A^1(\omega^{1/2})}} \right\}_n.$$

Hence we must show that

$$\sup_{z\in\mathbb{D}} \omega(z)^{1/2}|f(z)| = \|f\|_{A^\infty(\omega^{1/2})} \lesssim \|S^*f\|_{\ell^\infty} \asymp \sup_n \omega(z_n)^{1/2}|f(z_n)|,$$

for $f \in A^\infty(\omega^{1/2})$. However this can be proved with the same method of the proof of Lemma 1.6. Indeed, let $z \in \mathbb{D}$. Then there is a point z_k with $z \in D(\varepsilon\tau(z_k))$. By Lemma A, we have

$$\omega(z)^{1/2}|f(z)| \le \frac{C_1}{\varepsilon^2\,\tau(z)^2} \int_{D(\varepsilon\tau(z))} \Big(|f(\zeta)|\,\omega(\zeta)^{1/2} - |f(z_k)|\,\omega(z_k)^{1/2}\Big) dA(\zeta)$$

$$+ C_1\,|f(z_k)|\,\omega(z_k)^{1/2}.$$

As done in the proof of Lemma 1.6, we have

$$\Big||f(\zeta)|\,\omega(\zeta)^{1/2} - |f(z_k)|\,\omega(z_k)^{1/2}\Big| \le C_2\,\varepsilon\,\frac{1}{\tau(z_k)^2}\int_{D(3\delta_0\tau(z_k))} |f(\xi)|\,\omega(\xi)^{1/2}\,dA(\xi)$$

$$\le C_3\,\varepsilon\,\|f\|_{A^\infty(\omega^{1/2})}.$$

Thus, putting this in the previous estimate, we obtain

$$\omega(z)^{1/2}|f(z)| \le C_4\,\varepsilon\,\|f\|_{A^\infty(\omega^{1/2})} + C_1 \sup_n \omega(z_n)^{1/2}|f(z_n)|.$$

Finally, taking the supremum on z and $\varepsilon > 0$ small enough so that $C_4\varepsilon \le 1/2$, we have

$$\|f\|_{A^\infty(\omega^{1/2})} \lesssim \sup_n \omega(z_n)^{1/2}|f(z_n)|.$$

The proof is complete.

□

4.5 Toeplitz operators

In this section we are going to extend the results in Chapter 2 to the non-Hilbert space setting, when the weight ω is in the class $\mathcal{E}$. Concretely, we describe the boundedness of the Toeplitz operators $T_\mu : A^p(\omega^{p/2}) \to A^q(\omega^{q/2})$ for $1 \le p, q < \infty$. Recall that the Toeplitz operator T_μ is defined by

$$T_\mu f(z) = \int_{\mathbb{D}} f(\xi)\, \overline{K_z(\xi)}\, \omega(\xi)\, d\mu(\xi).$$

Also, recall that, for $\delta \in (0, m_\tau)$, the averaging function of μ on $\mathbb{D}$ is given by

$$\widehat{\mu}_\delta(z) := \frac{\mu(D(\delta\tau(z)))}{\tau(z)^2}, \quad z \in \mathbb{D}.$$

Theorem 4.13. *Let $\omega \in \mathcal{E}$, $1 \le p \le q < \infty$ and μ be a finite positive Borel measure on $\mathbb{D}$ satisfying (2.1). Then $T_\mu : A^p(\omega^{p/2}) \to A^q(\omega^{q/2})$ is bounded if and only if for each $\delta \in (0, m_\tau)$ sufficiently small*

$$E(\mu) = \sup_{z \in \mathbb{D}} \tau(z)^{2(\frac{1}{q} - \frac{1}{p})}\, \widehat{\mu}_\delta(z) < \infty. \tag{4.2}$$

Moreover,

$$\|T_\mu\|_{A^p(\omega^{p/2}) \to A^q(\omega^{q/2})} \asymp E(\mu).$$

Proof. Since we have the estimate $\|K_z\|_{A^p(\omega^{p/2})} \asymp \omega(z)^{-1/2}\, \tau(z)^{-2(p-1)/p}$, if we assume that the Toeplitz operator $T_\mu : A^p(\omega^{p/2}) \longrightarrow A^q(\omega^{q/2})$ is bounded, then we obtain (4.2) with the same argument as in the proof of Theorem 2.1.

Conversely, we suppose that (4.2) holds. We first prove that

$$\int_{\mathbb{D}} |K_z(\xi)|\, \omega(\xi)^{1/2}\; \tau(\xi)^{-2(\frac{1}{p} - \frac{1}{q})}\, d\mu(\xi) \lesssim E(\mu)\, \omega(z)^{-1/2}. \tag{4.3}$$

Indeed, by Lemma A, we have

$$|K_z(\xi)|\, \omega(\xi)^{1/2} \lesssim \frac{1}{\tau(\xi)^2} \int_{D(\frac{\delta}{2}\tau(\xi))} |K_z(s)|\; \omega(s)^{1/2}\, dA(s).$$

Then, by Fubini's theorem, the fact that $\tau(s) \asymp \tau(\xi)$ for $s \in D(\delta\tau(\xi))$, and condition (4.1), we get

$$\begin{aligned}
\int_{\mathbb{D}} |K_z(\xi)|\, \omega(\xi)^{1/2}\, \tau(\xi)^{-2(\frac{1}{p} - \frac{1}{q})} d\mu(\xi) &\lesssim \int_{\mathbb{D}} |K_z(s)|\, \omega(s)^{1/2}\, \tau(s)^{2(\frac{1}{q} - \frac{1}{p})}\, \widehat{\mu}_\delta(s)\; dA(s) \\
&\lesssim E(\mu) \int_{\mathbb{D}} |K_z(s)|\; \omega(s)^{1/2}\, dA(s) \\
&\lesssim E(\mu)\; \omega(z)^{-1/2}.
\end{aligned}$$

This establishes (4.3). Now we proceed to prove that T_μ is bounded. If $q>1$, by Hölder's inequality, we obtain

$$|T_\mu f(z)|^q \leq \left(\int_{\mathbb{D}}|f(\xi)|\,|K_z(\xi)|\;\omega(\xi)\,d\mu(\xi)\right)^q$$

$$\leq \left(\int_{\mathbb{D}}|f(\xi)|^q\,\omega(\xi)^{\frac{q+1}{2}}|K_z(\xi)|\,\tau(\xi)^{2(\frac{1}{p}-\frac{1}{q})(q-1)}d\mu(\xi)\right)\left(\int_{\mathbb{D}}|K_z(\xi)|\,\omega(\xi)^{1/2}\tau(\xi)^{-2(\frac{1}{p}-\frac{1}{q})}\,d\mu(\xi)\right)^{q-1}.$$

Using (4.3), we have

$$|T_\mu f(z)|^q \lesssim E(\mu)^{q-1}\left(\int_{\mathbb{D}}|f(\xi)|^q\,\omega(\xi)^{\frac{q+1}{2}}\,|K_z(\xi)|\,\tau(\xi)^{2(\frac{1}{p}-\frac{1}{q})(q-1)}d\mu(\xi)\right)\,\omega(z)^{-\frac{(q-1)}{2}}.$$

If $q=1$, this holds directly. By Fubini's theorem and condition (4.1), we obtain

$$\|T_\mu f\|^q_{A^q(\omega^{q/2})} = \int_{\mathbb{D}}|T_\mu f(z)|^q\,\omega(z)^{q/2}\,dA(z)$$

$$\lesssim E(\mu)^{q-1}\int_{\mathbb{D}}|f(\xi)|^q\,\omega(\xi)^{\frac{(q+1)}{2}}\,\tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})}\left(\int_{\mathbb{D}}|K_\xi(z)|\,\omega(z)^{1/2}dA(z)\right)d\mu(\xi)$$

$$\lesssim E(\mu)^{q-1}\int_{\mathbb{D}}|f(\xi)|^q\,\omega(\xi)^{q/2}\,\tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})}d\mu(\xi).$$

Consider the measure ν given by

$$d\nu(\xi) := \omega(\xi)^{q/2}\tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})}d\mu(\xi).$$

Since (4.2) holds, then by Theorem D, the identity $I_\nu : A^p(\omega^{p/2}) \longrightarrow L^q(\mathbb{D}, d\nu)$ is bounded. Moreover, $\|I_\nu\| \lesssim E(\mu)^{1/q}$. Therefore,

$$\|T_\mu f\|^q_{A^q(\omega^{q/2})} \lesssim E(\mu)^{q-1}\int_{\mathbb{D}}|f(z)|^q\,d\nu(z) \lesssim E(\mu)^q\cdot\|f\|^q_{A^p(\omega^{p/2})}. \tag{4.4}$$

This finishes the proof. □

In order to describe the boundedness of $T_\mu : A^p(\omega^{p/2}) \to A^q(\omega^{q/2})$ when $1\leq q<p<\infty$, we need first an auxiliary result.

Proposition 4.14. *Let $\omega\in\mathcal{E}$ and $1<q<p<\infty$. If $\widehat{\mu}_\delta \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA)$, then*

$$J_{\delta,q} := \int_{\mathbb{D}}\left(\frac{1}{\tau(z)^2}\int_{D(\delta\tau(z))}|f(\xi)|\,\omega(\xi)^{1/2}d\mu(\xi)\right)^q dA(z) \lesssim \|\widehat{\mu}_\delta\|^q_{L^{\frac{pq}{p-q}}(\mathbb{D})}\cdot\|f\|^q_{A^p(\omega^{p/2})},$$

for any $f\in A^p(\omega^{p/2})$.

Proof. For $z \in \mathbb{D}$ and $\xi \in D(\delta\tau(z))$, by Lemma A, Lemma B and (1.2), we obtain

$$
\begin{aligned}
|f(\xi)|\,\omega(\xi)^{1/2} &\lesssim \left(\frac{1}{\tau(\xi)^2}\int_{D(\frac{\delta}{3}\tau(\xi))} |f(s)|^p\,\omega(s)^{p/2}dA(s)\right)^{1/p} \\
&\lesssim \left(\frac{1}{\tau(z)^2}\int_{D(\delta\tau(z))} |f(s)|^p\,\omega(s)^{p/2}dA(s)\right)^{1/p}.
\end{aligned}
$$

This gives,

$$
\frac{1}{\tau(z)^2}\int_{D(\delta\tau(z))} |f(\xi)|\,\omega(\xi)^{1/2}d\mu(\xi) \lesssim \widehat{\mu}_\delta(z)\left(\frac{1}{\tau(z)^2}\int_{D(\delta\tau(z))} |f(s)|^p\,\omega(s)^{p/2}dA(s)\right)^{1/p}.
$$

Therefore,

$$
J_{\delta,q} \lesssim \int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2}\int_{D(\delta\tau(z))} |f(s)|^p\,\omega(s)^{p/2}dA(s)\right)^{q/p} \widehat{\mu}_\delta(z)^q \; dA(z).
$$

Applying Hölder's inequality

$$
J_{\delta,q} \lesssim \left(\int_{\mathbb{D}} \frac{1}{\tau(z)^2}\int_{D(\delta\tau(z))} |f(s)|^p\,\omega(s)^{p/2}dA(s)dA(z)\right)^{q/p} \|\widehat{\mu}_\delta\|^q_{L^{\frac{pq}{p-q}}(\mathbb{D})}. \tag{4.5}
$$

On the other hand, by Fubini's theorem and $\tau(z) \asymp \tau(s)$, for $s \in D(\delta\tau(z))$, we have

$$
\int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2}\int_{D(\delta\tau(z))} |f(s)|^p\,\omega(s)^{p/2}dA(s)\right)dA(z) \lesssim \|f\|^p_{A^p(\omega^{p/2})}.
$$

Combining this with (4.5), we get

$$
J_{\delta,q} \lesssim \|f\|^q_{A^p(\omega^{p/2})} \cdot \|\widehat{\mu}_\delta\|^q_{L^{\frac{pq}{p-q}}(\mathbb{D})}.
$$

The proof is complete. $\square$

Theorem 4.15. *Let $\omega \in \mathcal{E}$, $1 \leq q < p < \infty$ and μ be a finite positive Borel measure on $\mathbb{D}$ satisfying* (2.1). *The following conditions are equivalent:*

(i) The Toeplitz operator $T_\mu : A^p(\omega^{p/2}) \longrightarrow A^q(\omega^{q/2})$ is bounded.

(ii) For each sufficiently small $\delta > 0$, $\widehat{\mu}_\delta \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA)$.

Moreover, we have

$$
\|T_\mu\|_{A^p(\omega^{p/2}) \to A^q(\omega^{q/2})} \asymp \|\widehat{\mu}_\delta\|_{L^{\frac{pq}{p-q}}(\mathbb{D})}.
$$

Proof. (*i*) $\Longrightarrow$ (*ii*) For an arbitrary sequence $\lambda = \{\lambda_k\} \in \ell^p$, we consider the function

$$G_t(z) = \sum_{k=0}^{\infty} \lambda_k \; r_k(t) \; \omega(z_k)^{1/2} \, \tau(z_k)^{2(\frac{p-1}{p})} \, K_{z_k}(z) \quad , 0 < t < 1,$$

where $r_k(t)$ is a sequence of Rademacher functions (see [41] or Appendix A of [21]) and $\{z_k\}$ is the sequence given in Lemma B. Because of the norm estimate

$$\|K_z\|_{A^p(\omega^{p/2})} \asymp \omega(z)^{-1/2} \, \tau(z)^{-2(p-1)/p}$$

given in Lemma 4.2, by part (i) of Theorem 4.12 we obtain

$$\|G_t\|_{A^p(\omega^{p/2})} \lesssim \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p}.$$

Thus, the boundedness of $T_\mu : A^p(\omega^{p/2}) \longrightarrow A^q(\omega^{q/2})$ gives

$$\|T_\mu G_t\|^q_{A^q(\omega^{q/2})} \lesssim \|T_\mu\|^q \cdot \|\lambda\|^q_{\ell^p}.$$

In other words, we have

$$\int_{\mathbb{D}} \Big| \sum_{k=0}^{\infty} \lambda_k \; r_k(t) \; \omega(z_k)^{1/2} \, \tau(z_k)^{2(\frac{p-1}{p})} \, T_\mu K_{z_k}(z) \Big|^q \omega(z)^{q/2} \, dA(z) \lesssim \|T_\mu\|^q \cdot \|\lambda\|^q_{\ell^p}.$$

Integrating with respect to t from 0 to 1, applying Fubini's theorem and invoking Khinchine's inequality (see [41]), we obtain

$$B := \int_{\mathbb{D}} \left(\sum_{k=0}^{\infty} |\lambda_k|^2 \, \omega(z_k) \, \tau(z_k)^{4(\frac{p-1}{p})} \, |T_\mu K_{z_k}(z)|^2 \right)^{q/2} \omega(z)^{q/2} \, dA(z) \lesssim \|T_\mu\|^q \cdot \|\lambda\|^q_{\ell^p}.$$

Let χ_k denote the characteristic function of the set $D(3\delta\tau(z_k))$. Since the covering $\{D(3\delta\tau(z_k))\}$ of $\mathbb{D}$ has finite multiplicity N, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} |\lambda_k|^q \, \omega(z_k)^{q/2} \, \tau(z_k)^{2q(\frac{p-1}{p})} \int_{D(3\delta\tau(z_k))} |T_\mu K_{z_k}(z)|^q \, \omega(z)^{q/2} \, dA(z) \\
&= \int_{\mathbb{D}} \sum_{k=0}^{\infty} |\lambda_k|^q \, \omega(z_k)^{q/2} \, \tau(z_k)^{2q(\frac{p-1}{p})} \, |T_\mu K_{z_k}(z)|^q \, \chi_k(z) \, \omega(z)^{q/2} \, dA(z) \\
&\leq \max\{1, N^{1-\frac{q}{2}}\} \; B.
\end{aligned} \tag{4.6}$$

Now, using Lemma A, yields

$$\sum_{k=0}^{\infty} |\lambda_k|^q \, \omega(z_k)^{q/2} \, \tau(z_k)^{2q(\frac{p-1}{p})+2} \, |T_\mu K_{z_k}(z_k)|^q \, \omega(z_k)^{q/2} \lesssim \|T_\mu\|^q \cdot \|\lambda\|^q_{\ell^p}.$$

On the other hand, since for small $\delta > 0$, we have $|K_{z_k}(z)| \asymp \|K_{z_k}\|_{A^2(\omega)} \|K_z\|_{A^2(\omega)}$ for every $z \in D(\delta\tau(z_k))$, applying Lemma D and (1.2), we have

$$\begin{aligned} |T_\mu K_{z_k}(z_k)| &\geq \int_{D(\delta\tau(z_k))} |K_{z_k}(z)|^2 \, \omega(z) \, d\mu(z) \\ &\asymp \|K_{z_k}\|^2_{A^2(\omega)} \int_{D(\delta\tau(z_k))} \|K_z\|^2_{A^2(\omega)} \, \omega(z) \, d\mu(z) \\ &\asymp \frac{\omega(z_k)^{-1} \, \widehat{\mu}_\delta(z_k)}{\tau(z_k)^2}. \end{aligned}$$

That is,

$$|T_\mu K_{z_k}(z_k)|^q \, \omega(z_k)^{q/2} \gtrsim \frac{\omega(z_k)^{-\frac{q}{2}} \, \widehat{\mu}_\delta(z_k)^q}{\tau(z_k)^{2q}}.$$

Therefore,

$$\sum_{k=0}^{\infty} |\lambda_k|^q \, \tau(z_k)^{2q(\frac{1}{q}-\frac{1}{p})} \, \widehat{\mu}_\delta(z_k)^q \lesssim \|T_\mu\|^q \cdot \|\lambda\|^q_{\ell^p}.$$

Then, using the duality between $\ell^{p/q}$ and $\ell^{\frac{p}{p-q}}$ we conclude that

$$\sum_{k=0}^{\infty} \left(\tau(z_k)^{2q(\frac{1}{q}-\frac{1}{p})} \, \widehat{\mu}_\delta(z_k)^q \right)^{\frac{p}{p-q}} \lesssim \|T_\mu\|^{\frac{pq}{p-q}},$$

that means

$$\sum_{k=0}^{\infty} \tau(z_k)^2 \; \widehat{\mu}_\delta(z_k)^{\frac{pq}{p-q}} \lesssim \|T_\mu\|^{\frac{pq}{p-q}}.$$

This is the discrete version of our condition. To obtain the continuous version, simply note that

$$\widehat{\mu}_\delta(z) \lesssim \widehat{\mu}_{4\delta}(z_k), \quad z \in D(\delta\tau(z_k)).$$

Then,

$$\int_{\mathbb{D}} \widehat{\mu}_\delta(z)^{\frac{pq}{p-q}} \, dA(z) \leq \sum_{k=0}^{\infty} \int_{D(\delta\tau(z_k))} \widehat{\mu}_\delta(z)^{\frac{pq}{p-q}} \, dA(z) \lesssim \sum_{k=0}^{\infty} \tau(z_k)^2 \; \widehat{\mu}_{4\delta}(z_k)^{\frac{pq}{p-q}}.$$

This finishes the proof of this implication.

$(ii) \implies (i)$ First we begin with the easiest case $q = 1$. By Fubini's theorem and condition (4.1), we have

$$
\begin{aligned}
\|T_\mu f\|_{A^1(\omega^{1/2})} &= \int_{\mathbb{D}} |T_\mu f(z)|\, \omega(z)^{1/2}\, dA(z) \\
&\leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} |f(\xi)|\, |K_z(\xi)|\, \omega(\xi)\, d\mu(\xi) \right) \omega(z)^{1/2}\, dA(z) \\
&= \int_{\mathbb{D}} |f(\xi)| \left(\int_{\mathbb{D}} |K_\xi(z)|\, \omega(z)^{1/2} dA(z) \right) \omega(\xi)\, d\mu(\xi) \\
&\lesssim \int_{\mathbb{D}} |f(\xi)|\, \omega(\xi)^{1/2}\, d\mu(\xi).
\end{aligned}
\tag{4.7}
$$

Now, by using Theorem F with the measure given by

$$
d\nu(\xi) := \omega(\xi)^{1/2}\, d\mu(\xi),
$$

it gives the desired result.

Finally, we study the case $1 < q < \infty$. Let $\{z_j\}$ be the sequence given in Lemma B. Applying Lemma B and Lemma A, we obtain

$$
\begin{aligned}
|T_\mu f(z)| &\leq \sum_{j=0}^{\infty} \int_{D(\delta\tau(z_j)} |f(\xi)|\, |K_z(\xi)|\, \omega(\xi)\, d\mu(\xi) \\
&\lesssim \sum_{j=0}^{\infty} \int_{D(\delta\tau(z_j))} |f(\xi)|\, \omega(\xi)^{1/2} \left(\frac{1}{\tau(\xi)^2} \int_{D(\delta\tau(\xi))} |K_z(s)|\, \omega(s)^{1/2}\, dA(s) \right) d\mu(\xi) \\
&\lesssim \sum_{j=0}^{\infty} \left(\int_{D(\delta\tau(z_j))} |f(\xi)|\, \omega(\xi)^{1/2}\, \frac{d\mu(\xi)}{\tau(\xi)^2} \right) \int_{D(3\delta\tau(z_j))} |K_z(s)|\, \omega(s)^{1/2}\, dA(s).
\end{aligned}
$$

Applying Hölder's inequality, we get

$$
|T_\mu f(z)|^q \lesssim M(z) \times N(z),
$$

where

$$
M(z) := \sum_{j=0}^{\infty} \left(\int_{D(\delta\tau(z_j))} |f(\xi)|\, \omega(\xi)^{1/2}\, \frac{d\mu(\xi)}{\tau(\xi)^2} \right)^q \int_{D(3\delta\tau(z_j))} |K_z(s)|\, \omega(s)^{1/2}\, dA(s),
$$

and

$$N(z) := \left(\sum_{j=0}^{\infty} \int_{D(3\delta\tau(z_j))} |K_z(s)|\, \omega(s)^{1/2}\, dA(s) \right)^{q-1}.$$

Furthermore, by Lemma B and condition (4.1), we have

$$N(z) \lesssim \left(\int_{\mathbb{D}} |K_z(s)|\, \omega(s)^{1/2} dA(s) \right)^{q-1} \lesssim \omega(z)^{\frac{1-q}{2}}.$$

Thus

$$|T_\mu f(z)|^q\, \omega(z)^{q/2} \lesssim M(z)\, \omega(z)^{1/2}.$$

This gives

$$\|T_\mu f\|^q_{A^q(\omega^{q/2})} \lesssim \sum_{j=0}^{\infty} \left(\int_{D(\delta\tau(z_j))} |f(\xi)|\, \omega(\xi)^{1/2}\, \frac{d\mu(\xi)}{\tau(\xi)^2} \right)^q K(j),$$

where

$$K(j) := \int_{\mathbb{D}} \left(\int_{D(3\delta\tau(z_j))} |K_z(s)|\, \omega(s)^{1/2} dA(s) \right) \omega(z)^{1/2}\, dA(z),$$

which by Fubini's theorem and condition (4.1)

$$K(j) \lesssim \tau(z_j)^2.$$

Combining this with using (1.2) and Proposition 4.14, it shows that

$$\begin{aligned}
\|T_\mu f\|^q_{A^q(\omega^{q/2})} &\lesssim \sum_{j=0}^{\infty} \tau(z_j)^2 \left(\frac{1}{\tau(z_j)^2} \int_{D(\delta\tau(z_j))} |f(\xi)|\, \omega(\xi)^{1/2}\, d\mu(\xi) \right)^q \\
&\lesssim \int_{\mathbb{D}} \left(\frac{1}{\tau(z)^2} \int_{D(4\delta\tau(z))} |f(\xi)|\, \omega(\xi)^{1/2} d\mu(\xi) \right)^q dA(z) \\
&\lesssim \|\widehat{\mu}_{4\delta}\|^q_{L^{\frac{pq}{p-q}}(\mathbb{D})} \cdot \|f\|^q_{A^p(\omega^{p/2})}.
\end{aligned}$$

This proves the desired result. $\square$

Next we characterize compact Toeplitz operators on weighted Bergman spaces $A^p(\omega^{p/2})$ for weights ω in the class $\mathcal{E}$. We need first a lemma.

Lemma 4.16. *Let $1 < p < \infty$, and let $k_{p,z}$ the normalized reproducing kernels in $A^p(\omega^{p/2})$. Then $k_{p,z} \to 0$ weakly as $|z| \to 1^-$.*

Proof. By duality and the reproducing kernel properties, we must show that $|g(z)|/\|K_z\|_{A^p(\omega^{p/2})}$ goes to zero as $|z| \to 1^-$ whenever g is in $A^{p'}(\omega^{p'/2})$, where p' denotes the conjugate exponent of p, but this follows easily by the density of the polynomials and Lemma A. □

Theorem 4.17. *Let $\omega \in \mathcal{E}$, $1 < p \le q < \infty$ and μ be a finite positive Borel measure on $\mathbb{D}$ satisfying* (2.1). *Then the Toeplitz operator $T_\mu : A^p(\omega^{p/2}) \longrightarrow A^q(\omega^{q/2})$ is compact if and only if, for each $\delta \in (0, m_\tau)$ small enough, one has*

$$\lim_{r\to 1^-} \sup_{|a|>r} \tau(a)^{2(\frac{1}{q}-\frac{1}{p})}\, \widehat{\mu}_\delta(a) = 0. \tag{4.8}$$

Proof. First we assume that T_μ is compact. Following the proof of the boundedness part and the fact that $\|K_a\|^{-1}_{A^p(\omega^{p/2})} \asymp \omega(a)^{1/2}\, \tau(a)^{\frac{2(p-1)}{p}}$, we get the estimate

$$\tau(a)^{2(\frac{1}{q}-\frac{1}{p})}\, \widehat{\mu}_\delta(a) \lesssim \frac{\omega(a)^{1/2}}{\tau(a)^{2(\frac{1}{p}-1)}}\, \|T_\mu K_a\|_{A^q(\omega^{q/2})} \lesssim \|T_\mu k_{p,a}\|_{A^q(\omega^{q/2})}, \tag{4.9}$$

where $k_{p,a}$ are the normalized reproducing kernels in $A^p(\omega^{p/2})$. Since, by Lemma 4.16, $k_{p,a}$ tends to zero weakly, and T_μ is compact, the result follows.

Conversely, we suppose that (4.8) holds. Let $\{f_n\} \subset A^p(\omega^{p/2})$ be a bounded sequence converging to zero uniformly on compact subsets of $\mathbb{D}$. By (4.4), we have

$$\|T_\mu f_n\|_{A^q(\omega^{q/2})} \lesssim \int_{\mathbb{D}} |f_n(z)|^q\, d\nu(z) = \|I_\nu f_n\|_{L^q(\mathbb{D},d\nu)}, \tag{4.10}$$

where $I_\nu : A^p(\omega^{p/2}) \longrightarrow L^q(\mathbb{D}, d\nu)$ with $d\nu(\xi) = \omega(\xi)^{q/2}\, \tau(\xi)^{2(q-1)(\frac{1}{p}-\frac{1}{q})}\, d\mu(\xi)$. By using $\tau(a) \asymp \tau(\xi)$, for $\xi \in D(\delta\tau(a))$, we have

$$\sup_{|a|>r} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} \omega(\xi)^{-q/2}\, d\nu(\xi) \lesssim \sup_{|a|>r} \tau(a)^{2(\frac{1}{q}-\frac{1}{p})}\, \widehat{\mu}_\delta(a).$$

By Theorem E, I_ν is compact, and in view of (4.10), T_μ is compact. □

Theorem 4.18. *Let $\omega \in \mathcal{E}$, $1 \le q < p < \infty$ and μ be a finite positive Borel measure on $\mathbb{D}$ satisfy* (2.1). *The following conditions are equivalent:*

(i) The Toeplitz operator $T_\mu : A^p(\omega^{p/2}) \longrightarrow A^q(\omega^{q/2})$ is compact.

(ii) For each sufficiently small $\delta > 0$

$$\widehat{\mu}_\delta \in L^{\frac{pq}{p-q}}(\mathbb{D}, dA). \tag{4.11}$$

Proof. If T_μ is compact, then it is bounded, and by Theorem 4.15 we get the desired result. Conversely, if (4.11) holds then, by Theorem 4.15, T_μ is bounded. Since, by Theorem 4.12 the spaces $A^p(\omega^{p/2})$ and $A^q(\omega^{q/2})$ are isomorphic to ℓ^p, the result is a consequence of a general result of Banach space theory: it is known that, for $1 \leq q < p < \infty$, every bounded operator from ℓ^p to ℓ^q is compact (see [37, Theorem I.2.7]). □

4.6 Hankel operators

One of the most important classes of operators acting on spaces of analytic functions are the Hankel operators. When acting on the classical Hardy spaces, their study presents [50, 54] a broad range of applications such as to control theory, approximation theory, prediction theory, perturbation theory and interpolation problems. Furthermore, one can find an extensive literature on Hankel operators acting on other classical function spaces in one or several complex variables, such as Bergman spaces [2, 3, 32, 73, 74], Fock spaces [55] or Dirichlet spaces [67, 68]. In this section, we are going to study big Hankel operators acting on our large weighted Bergman spaces.

Definition 4.1. *Let M_g denote the multiplication operator induced by a function g, and P_ω be the Bergman projection, where ω is a weight in the class $\mathcal{E}$. The Hankel operator H_g is given by*

$$H_g = H_g^\omega := (I - P_\omega)M_g.$$

We assume that the function g satisfies

$$gK_z \in L^1(\omega^{1/2}), \qquad z \in \mathbb{D}. \tag{4.12}$$

Under this assumption, the Hankel operator H_g^ω is well-defined on the set E of all finite linear combinations of reproducing kernels and therefore, is densely defined in the weighted Bergman space $A^p(\omega^{p/2})$, $1 \leq p < \infty$. Also, for $f \in E$, one has

$$H_g^\omega f(z) = \int_{\mathbb{D}} \big(g(z) - g(s)\big)\, f(s)\, \overline{K_z(s)}\, \omega(s)\, dA(s).$$

We are going to study the boundedness and compactness when the symbol is conjugate analytic. In the Hilbert space case $A^2(\omega)$, and for weights in the class $\mathcal{W}$, a characterization of the boundedness, compactness and membership in Schatten classes of the Hankel operator $H_{\bar{g}} : A^2(\omega) \to L^2(\omega)$ was obtained in [23]. In order to extend such results to the non-Hilbert space setting, we need estimates for the p-norm of the reproducing kernels, and it is here when the condition (4.1) and the exponential type class $\mathcal{E}$ enters in action. Before going to study the boundedness of the Hankel operator on $A^p(\omega^{p/2})$ with conjugate analytic symbols we need the following Lemma.

Lemma F. *Let $1 \le p < \infty$, $g \in H(\mathbb{D})$ and $a \in \mathbb{D}$. Then*

$$\tau(a)|g'(a)| \le C\left(\frac{1}{\tau(a)^2}\int_{D(\delta\tau(a))} |g(z)-g(a)|^p\, dA(z)\right)^{1/p}.$$

Proof. See for example [23]. □

Now we are ready to characterize the boundedness of the Hankel operator with conjugate analytic symbols acting on large weighted Bergman spaces in term of the growth of the maximum modulus of g'. We begin with the case $1 \le p \le q < \infty$.

Theorem 4.19. *Let $\omega \in \mathcal{E}$, $1 \le p \le q < \infty$ and $g \in H(\mathbb{D})$ satisfying (4.12). The Hankel operator $H_{\overline{g}} : A^p(\omega^{p/2}) \longrightarrow L^q(\omega^{q/2})$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \tau(z)^{1+2(\frac{1}{q}-\frac{1}{p})}|g'(z)| < \infty. \tag{4.13}$$

Proof. Suppose first that $H_{\overline{g}} : A^p(\omega^{p/2}) \longrightarrow L^q(\omega^{q/2})$ is bounded. Thus

$$\|H_{\overline{g}}K_a\|_{L^q(\omega^{q/2})} \le \|H_{\overline{g}}\|\, \|K_a\|_{A^p(\omega^{p/2})}.$$

For each $z \in \mathbb{D}$, consider the function $g_z(\xi) := \big(g(z) - g(\xi)\big)K_z(\xi)$. The condition (4.12) ensures that $g_z \in A^1(\omega^{1/2})$, and by the reproducing formula in Corollary 4.5, one has

$$\begin{aligned} H_{\overline{g}}K_a(z) &= \int_{\mathbb{D}} \overline{\big(g(z)-g(\xi)\big)}\, K_a(\xi)\, \overline{K_z(\xi)}\, \omega(\xi)\, dA(\xi) \\ &= \overline{\langle g_z, K_a\rangle_\omega} = \overline{g_z(a)}. \end{aligned}$$

Now, for δ small enough we have $|K_z(a)| \asymp \|K_z\|_{A^2(\omega)}\|K_a\|_{A^2(\omega)}$ for $z \in D(\delta\tau(a))$. Hence, by Lemma D and (1.2), we have

$$\begin{aligned} \|H_{\overline{g}}K_a\|^q_{L^q(\omega^{q/2})} &= \int_{\mathbb{D}} |g(z)-g(a)|^q\, |K_z(a)|^q\, \omega(z)^{q/2}\, dA(z) \\ &\ge \int_{D(\delta\tau(a))} |g(z)-g(a)|^q\, |K_z(a)|^q\, \omega(z)^{q/2}\, dA(z) \\ &\asymp \int_{D(\delta\tau(a))} |g(z)-g(a)|^q\, \|K_z\|^q_{A^2(\omega)}\, \|K_a\|^q_{A^2(\omega)}\, \omega(z)^{q/2}\, dA(z) \\ &\asymp \frac{\|K_a\|^q_{A^2(\omega)}}{\tau(a)^q}\int_{D(\delta\tau(a))} |g(z)-g(a)|^q\, dA(z). \end{aligned}$$

Because of the boundedness of the Hankel operator $H_{\overline{g}}$, we have

$$\begin{aligned} \|H_{\overline{g}}\|^q &\ge \frac{\|H_{\overline{g}}K_a\|^q_{L^q(\omega^{q/2})}}{\|K_a\|^q_{A^p(\omega^{p/2})}} \\ &\gtrsim \|K_a\|^{-q}_{A^p(\omega^{p/2})}\frac{\|K_a\|^q_{A^2(\omega)}}{\tau(a)^q}\int_{D(\delta\tau(a))} |g(z)-g(a)|^q\, dA(z). \end{aligned}$$

Finally, by the estimates on the norm of K_z in Lemma 4.2 and Lemma D, we obtain

$$\|H_{\overline{g}}\| \gtrsim \tau(a)^{2(\frac{1}{q}-\frac{1}{p})}\left(\frac{1}{\tau(a)^2}\int_{D(\delta\tau(a))}|g(z)-g(a)|^q\,dA(z)\right)^{1/q}.$$

By Lemma F, this completes the proof of this implication.

Conversely, assume that (4.13) holds and let $1 \le p \le q < \infty$. By Theorem 1.4, there exists a solution u of the equation $\overline{\partial}u = f\,\overline{g'}$ in $L^q(\omega^{q/2})$ such that

$$\|u\|^q_{L^q(\omega^{q/2})} \lesssim \int_{\mathbb{D}} |\overline{\partial}u(z)|^q\,\omega(z)^{q/2}\,\tau(z)^q\,dA(z).$$

Since any solution v of the $\overline{\partial}$-equation has the form $v = u-h$, with $h \in H(\mathbb{D})$, and because $H_{\overline{g}}f$ is also a solution of $\overline{\partial}$-equation, there is a function $h \in H(\mathbb{D})$ such that $H_{\overline{g}}f = u-h$. As a result of $P_\omega\big(H_{\overline{g}}f\big) = 0$, we have $H_{\overline{g}}f = \big(I - P_\omega\big)u$, where I is the identity operator. Therefore, by the boundedness of P_ω on $L^q(\omega^{q/2})$ (see Theorem 4.4), we obtain

$$\begin{aligned}\|H_{\overline{g}}f\|^q_{L^q(\omega^{q/2})} &\le \|I-P_\omega\|^q\,\|u\|^q_{L^q(\omega^{q/2})} \lesssim \|u\|^q_{L^q(\omega^{q/2})}\\ &\lesssim \int_{\mathbb{D}}|f(z)|^q\,|g'(z)|^q\,\omega(z)^{q/2}\,\tau(z)^q\,dA(z).\end{aligned} \tag{4.14}$$

By our assumption (4.13), we have

$$\|H_{\overline{g}}f\|^q_{L^q(\omega^{q/2})} \lesssim \int_{\mathbb{D}}|f(z)|^q\,\omega(z)^{q/2}\,\tau(z)^{2q(\frac{1}{p}-\frac{1}{q})}\,dA(z).$$

On the other hand, by Lemma A

$$|f(z)|\,\omega(z)^{1/2} \lesssim \tau(z)^{-2/p}\,\|f\|_{A^p(\omega^{p/2})}.$$

Using the last pointwise estimate, we have

$$\|H_{\overline{g}}f\|^q_{L^q(\omega^{q/2})} \lesssim \|f\|^{q-p}_{A^p(\omega^{p/2})}\int_{\mathbb{D}}|f(z)|^p\,\omega(z)^{p/2}dA(z) = \|f\|^q_{A^p(\omega^{p/2})}.$$

This completes the proof. □

Next we are going to characterize the boundedness of the Hankel operator with conjugate analytic symbols when $1 \le q < p < \infty$. Before that we prove the following Lemma:

Lemma 4.20. *Let $\delta_0 \in (0, m_\tau)$ and $0 < r < \infty$. Then*

$$|f'(z)|^r \lesssim \frac{1}{\tau(z)^{r+2}} \int_{D(\delta_0\tau(z))} |f(s)|^r \, dA(s),$$

for $f \in H(\mathbb{D})$.

Proof. By Cauchy's integral formula and Lemma A, we get

$$\begin{aligned} |f'(z)| &\lesssim \left| \int_{|\eta - z| = \frac{\delta_0\tau(z)}{4}} \frac{f(\eta)}{(\eta - z)^2} \, d\eta \right| \lesssim \frac{1}{\tau(z)^2} \int_{|\eta - z| = \frac{\delta_0\tau(z)}{4}} |f(\eta)| \, |d\eta| \\ &\lesssim \frac{1}{\tau(z)^2} \int_{|\eta - z| = \frac{\delta_0\tau(z)}{4}} \left(\frac{1}{\tau(\eta)^2} \int_{D(\delta_0\tau(\eta)/4)} |f(s)|^r dA(s) \right)^{1/r} |d\eta|. \end{aligned}$$

An application of $\tau(\eta) \asymp \tau(z)$, for $\eta \in D(\delta_0\tau(z)/2)$, gives

$$\begin{aligned} |f'(z)|^r &\lesssim \frac{1}{\tau(z)^2} \int_{|\eta - z| = \frac{\delta_0\tau(z)}{4}} \left(\frac{1}{\tau(z)^2} \int_{D(\delta_0\tau(z)/2)} |f(s)|^r dA(s) \right)^{1/r} |d\eta| \\ &\lesssim \frac{1}{\tau(z)^{1+\frac{2}{r}}} \left(\int_{D(\delta_0\tau(z)/2)} |f(s)|^r dA(s) \right)^{1/r}, \end{aligned}$$

which proves the desired result. □

The following result gives the characterization of the boundedness of the Hankel operator going from $A^p(\omega^{p/2})$ into $L^q(\omega^{q/2})$ when $1 \le q < p < \infty$.

Theorem 4.21. *Let $\omega \in \mathcal{E}$, $1 \le q < p < \infty$ and $g \in H(\mathbb{D})$ satisfying (4.12). Then the following statement are equivalent:*

(a) The Hankel operator $H_{\overline{g}} : A^p(\omega^{p/2}) \longrightarrow L^q(\omega^{q/2})$ is bounded.

(b) The function $\tau g'$ belongs to $L^r(\mathbb{D}, dA)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Proof. Suppose that $\tau\, g' \in L^r(\mathbb{D}, dA)$. By (4.14), since $p/q > 1$ a simple application of Hölder's inequality, we get

$$\|H_{\overline{g}} f\|^q_{L^q(\omega^{q/2})} \lesssim \int_{\mathbb{D}} |f(z)|^q \, |g'(z)|^q \, \omega(z)^{q/2} \, \tau(z)^q \, dA(z) \le \|f\|^q_{A^p(\omega^{p/2})} \|\tau\, g'\|^q_{L^r(\mathbb{D}, dA)}.$$

This proves the boundedness of $H_{\overline{g}}$.

Conversely, pick $\varepsilon > 0$ and let $\{z_k\}$ be an (ε, τ)- lattice on $\mathbb{D}$. For a sequence $\lambda = \{\lambda_k\} \in \ell^p$, we consider the function

$$G_t(z) = \sum_{k=0}^{\infty} \lambda_k \; r_k(t) \; \omega(z_k)^{1/2} \, \tau(z_k)^{2(\frac{p-1}{p})} \, K_{z_k}(z) \quad , 0 < t < 1,$$

where $r_k(t)$ is a sequence of Rademacher functions. Because of the norm estimate for reproducing kernels given in Lemma 4.2, by part (i) of Theorem 4.12, we obtain

$$\|G_t\|_{A^p(\omega^{p/2})} \lesssim \|\lambda\|_{\ell^p}.$$

Thus, the boundedness of $H_{\overline{g}}$ gives

$$\|H_{\overline{g}} \, G_t\|^q_{L^q(\omega^{q/2})} \lesssim \|\lambda\|^q_{\ell^p}.$$

Therefore,

$$\int_{\mathbb{D}} \Big| \sum_{k=0}^{\infty} \lambda_k \; r_k(t) \; \omega(z_k)^{1/2} \, \tau(z_k)^{2(\frac{p-1}{p})} \, H_{\overline{g}} K_{z_k}(z) \Big|^q \omega(z)^{q/2} \, dA(z) \lesssim \|\lambda\|^q_{\ell^p}.$$

Using the same method in (4.6), we obtain

$$\int_{\mathbb{D}} \sum_{k=0}^{\infty} |\lambda_k|^q \, \omega(z_k)^{q/2} \, \tau(z_k)^{2q(\frac{p-1}{p})} \left| H_{\overline{g}} K_{z_k}(z) \right|^q \chi_k(z) \, \omega(z)^{q/2} \, dA(z) \lesssim \|\lambda\|^q_{\ell^p}, \tag{4.15}$$

which χ_k is the characteristic function of the set $D(3 \, \varepsilon \, \tau(z_k))$. Additionally, by Lemma E, Lemma D and (1.2), we get

$$\left| H_{\overline{g}} K_{z_k}(z) \right|^q \omega(z)^{q/2} = \left| g(z) - g(z_k) \right|^q |K_{z_k}(z)|^q \, \omega(z)^{q/2} \gtrsim \frac{\omega(z_k)^{-q/2}}{\tau(z_k)^{2q}} \left| g(z) - g(z_k) \right|^q.$$

Putting this in (4.15), it gives

$$\sum_{k=0}^{\infty} |\lambda_k|^q \tau(z_k)^{-2q/p} \int_{D(3 \, \varepsilon \, \tau(z_k))} \left| g(z) - g(z_k) \right|^q dA(z) \lesssim \|\lambda\|^q_{\ell^p}.$$

Furthermore, by Lemma F we obtain

$$\sum_{k=0}^{\infty} |\lambda_k|^q \tau(z_k)^{-2q/p+2} |\tau(z_k) g'(z_k)|^q \lesssim \|\lambda\|^q_{\ell^p}.$$

Moreover, by the duality between $\ell^{p/q}$ and $\ell^{\frac{p}{p-q}}$, it follows

$$I_{g'} := \sum_{k=0}^{\infty} \tau(z_k)^2 \big(\tau(z_k) |g'(z_k)| \big)^r < \infty.$$

In order to finish the proof we will justify that $\|\tau\, g'\|^r_{L^r(\mathbb{D},dA)} \lesssim I_{g'}$. For that, on the one hand by Lemma 4.20 applied to g', we get

$$\big|g''(\xi)\big|^r \lesssim \frac{1}{\tau(\xi)^{r+2}} \int_{D(\delta_0\tau(\xi))} |g'(s)|^r\, dA(s). \tag{4.16}$$

On the other hand, by Cauchy estimates there exists $\xi \in [z, z_k]$ such that

$$\big|g'(z) - g'(z_k)\big| \le |g''(\xi)||z - z_k|.$$

Using $\tau(\xi) \asymp \tau(z) \asymp \tau(z_k)$, for $\xi, z \in D(\varepsilon\,\tau(z_k))$, we have

$$\begin{aligned}
\|\tau\, g'\|^r_{L^r(\mathbb{D},dA)} &\le \sum_k \int_{D(\varepsilon\,\tau(z_k))} \tau(z)^r |g'(z)|^r dA(z) \\
&\le C \sum_k \int_{D(\varepsilon\,\tau(z_k))} \tau(z_k)^r \big|g'(z) - g'(z_k)\big|^r\, dA(z) + C\varepsilon^2 I_{g'} \\
&\le C\varepsilon^r \sum_k \int_{D(\varepsilon\,\tau(z_k))} \tau(z_k)^{2r} \big|g''(\xi)\big|^r\, dA(z) + C\varepsilon^2 I_{g'}.
\end{aligned}$$

By (4.16) and using again (1.2), we obtain

$$\begin{aligned}
\|\tau\, g'\|^r_{L^r(\mathbb{D},dA)} &\le C\,\varepsilon^r \sum_k \int_{D(\varepsilon\,\tau(z_k))} \frac{\tau(z_k)^{2r}}{\tau(\xi)^{r+2}} \int_{D(\delta_0\tau(\xi))} |g'(s)|^r\, dA(s)\, dA(z) + \varepsilon^2\, I_{g'} \\
&\le C\,\varepsilon^{r+2} \sum_k \int_{D(3\delta_0\tau(z_k))} \tau(s)^r\, |g'(s)|^r\, dA(s) + \varepsilon^2\, I_{g'}.
\end{aligned}$$

By Lemma B, every point $z \in \mathbb{D}$ belongs to at the most $C\varepsilon^{-2}$ of the sets $D(3\delta_0\tau(z_k))$. Hence

$$\big(1 - C\,\varepsilon^r\big)\|\tau\, g'\|^r_{L^r(\mathbb{D},dA)} \lesssim I_{g'}.$$

Thus, taking ε so that $C\,\varepsilon^r < 1/2$, we get the desired result. □

Next we characterize the compactness of the Hankel operator with conjugate analytic symbol acting from $A^p(\omega^{p/2})$ into $L^q(\omega^{q/2})$, $1 \le p, q < \infty$. This characterization will be given in two theorems depending on the order of p and q. We begin with the case $1 \le p \le q < \infty$.

Theorem 4.22. *Let $\omega \in \mathcal{E}$, $1 < p \le q < \infty$ and $g \in H(\mathbb{D})$ satisfying (4.12). Then, the Hankel operator $H_{\overline{g}} : A^p(\omega^{p/2}) \longrightarrow L^q(\omega^{q/2})$ is compact if and only if*

$$\lim_{|a|\to 1^-} \tau(a)^{1+2(\frac{1}{q}-\frac{1}{p})}|g'(a)| = 0.$$

Proof. Let $a \in \mathbb{D}$ and $1 < p \leq q < \infty$. Recall that $k_{a,p}$ is the normalized reproducing kernel in $A^p(\omega^{p/2})$. By Lemma 4.16, $k_{a,p} \to 0$ weakly. Thus, if $H_{\overline{g}}$ is compact, then

$$\lim_{|a|\to 1^-} \|H_{\overline{g}}\, k_{a,p}\|_{L^q(\omega^{q/2})} = 0.$$

Let δ be small enough such that $|K_a(z)| \asymp \|K_a\|_{A^2(\omega)}\, \|K_z\|_{A^2(\omega)}$, for $z \in D(\delta\tau(a))$. By Lemma D, using (1.2) and the norm estimate $\|K_a\|_{A^p(\omega^{p/2})} \asymp \omega(a)^{-1/2}\tau(a)^{2(\frac{1-p}{p})}$, we have

$$\begin{aligned}\|H_{\overline{g}}\, k_{a,p}\|^q_{L^q(\omega^{q/2})} &\geq \int_{D(\delta\tau(a))} \big|g(a) - g(z)\big|^q \,\frac{|K_a(z)|^q\, \omega(z)^{q/2}}{\|K_a\|^q_{A^p(\omega^{p/2})}}\, dA(z)\\ &\gtrsim \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} \big|g(a) - g(z)\big|^q dA(z).\end{aligned}$$

It follows from Lemma F,

$$\|H_{\overline{g}}\, k_{a,p}\|_{L^q(\omega^{q/2})} \gtrsim \tau(a)^{1+2(\frac{1}{q}-\frac{1}{p})}|g'(a)|.$$

This implies that

$$\lim_{|a|\to 1^-} \tau(a)^{1+2(\frac{1}{q}-\frac{1}{p})}|g'(a)| = 0,$$

which completes the proof of this implication.

Conversely, let $\{f_n\}$ be a bounded sequence in $A^p(\omega^{p/2})$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. To show compactness, it is standard to see that it is enough to prove that $\|H_{\overline{g}} f_n\|_{L^q(\omega^{q/2})} \to 0$. By the assumption, given any $\varepsilon > 0$, there is $0 < r_0 < 1$ such that

$$\tau(z)^{1+2(\frac{1}{q}-\frac{1}{p})}|g'(z)| < \varepsilon, \qquad\qquad r_0 < |z| < 1.$$

Since $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, exists an integer n_0 such that

$$|f_n(z)| < \varepsilon \quad \text{for} \quad |z| \leq r_0 \quad \text{and} \quad n \geq n_0.$$

According to (4.14), we have

$$\begin{aligned}\|H_{\overline{g}} f_n\|^q_{L^q(\omega^{q/2})} &\lesssim \int_{\mathbb{D}} |f_n(z)|^q\, |g'(z)|^q\, \omega(z)^{q/2}\, \tau(z)^q\, dA(z)\\ &= \left(\int_{|z|\leq r_0} + \int_{r_0<|z|<1}\right)|f_n(z)|^q\, |g'(z)|^q\, \omega(z)^{q/2}\, \tau(z)^q\, dA(z).\end{aligned}$$

On the one hand, it is easy to see that

$$\int_{|z|\leq r_0} |f_n(z)|^q\, |g'(z)|^q\, \omega(z)^{q/2}\, \tau(z)^q\, dA(z) \lesssim \varepsilon^q. \tag{4.17}$$

On the other hand, by Lemma A, we have the pointwise estimate

$$|f_n(z)| \lesssim \omega(z)^{-1/2}\tau(z)^{-2/p}\|f_n\|_{A^p(\omega^{p/2})}.$$

Applying this together with our assumption, we get

$$\begin{aligned}
&\int_{r_0<|z|<1} |f_n(z)|^q\,|g'(z)|^q\,\omega(z)^{q/2}\,\tau(z)^q\,dA(z)\\
&\qquad < \varepsilon^q \int_{\mathbb{D}} |f_n(z)|^p\,|f_n(z)|^{q-p}\,\omega(z)^{q/2}\,\tau(z)^{2q\left(\frac{1}{p}-\frac{1}{q}\right)}\,dA(z)\\
&\qquad \lesssim \varepsilon^q\,\|f_n\|^{q-p}_{A^p(\omega^{p/2})} \int_{\mathbb{D}} |f_n(z)|^p\,\omega(z)^{p/2}\,dA(z) = \varepsilon^q\,\|f_n\|^q_{A^p(\omega^{p/2})}.
\end{aligned}$$

Combining this with (4.17) gives us that $\lim_{n\to\infty} \|H_{\overline{g}} f_n\|_{L^q(\omega^{q/2})} = 0$. This shows that the Hankel operator $H_{\overline{g}} : A^p(\omega^{p/2}) \longrightarrow L^q(\omega^{q/2})$ is compact. $\square$

The next Theorem give us the compactness characterization of the Hankel operator when $1 \le q < p < \infty$.

Theorem 4.23. *Let $\omega \in \mathcal{E}$, $1 \le q < p < \infty$ and $g \in H(\mathbb{D})$ satisfying* (4.12)*. The following conditions are equivalent:*

(a) The Hankel operator $H_{\overline{g}} : A^p(\omega^{p/2}) \longrightarrow L^q(\omega^{q/2})$ is compact.

(b) The function $\tau g'$ belongs to $L^r(\mathbb{D}, dA)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Proof. (*a*) $\Longrightarrow$ (*b*) Assume that $H_{\overline{g}}$ is compact. Then $H_{\overline{g}}$ is bounded. Hence, by applying Theorem 4.21 we get the desired result.
(*b*) $\Longrightarrow$ (*a*) Suppose that $\tau g'$ belongs to $L^r(\mathbb{D}, dA)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. By Theorem 4.21, the Hankel operator $H_{\overline{g}}$ is bounded, and as a result of Theorem 4.12 the space $A^p(\omega^{p/2})$ is isomorphic to ℓ^p. In this case, $H_{\overline{g}}$ is also compact, due to a general result of Banach space theory: it is known that, for $1 \le q < p < \infty$, every bounded operator from ℓ^p to ℓ^q is compact (see [37, Theorem I.2.7]). This finishes the proof. $\square$

4.7 Examples of weights in the class $\mathcal{E}$

In this Section, we are going to show that the family of exponential type weights ω_σ given by

$$\omega_\sigma(z) = \exp\left(\frac{-A}{(1-|z|^2)^\sigma}\right), \quad \sigma > 0,\ A > 0. \tag{4.18}$$

satisfy condition (4.1), and therefore they are in the class $\mathcal{E}$. Notice that the case $\sigma = 1$ was obtained in [16], and we follow his method with appropriate modifications.

For $\lambda > 0$, let

$$v_\lambda = \int_0^1 r^\lambda \, \omega_\sigma(r)\, dr$$

Because the functions

$$e_n(z) = \frac{z^n}{\sqrt{2v_{2n+1}}}$$

form an orthonormal basis of $A^2(\omega_\sigma)$, the reproducing kernel K_z of $A^2(\omega_\sigma)$ is given by

$$K_z(w) = \sum_n \frac{(\bar{z}w)^n}{2v_{2n+1}}.$$

We let $K(z) = \sum_n \frac{z^n}{2v_{2n+1}}$. We need the following result.

Proposition 4.24. *Let $0 < \sigma < \infty$. Then*

$$M_1(r, K) := \int_0^{2\pi} |K(re^{i\theta})| \frac{d\theta}{2\pi} \lesssim \left(\frac{1}{1-r}\right)^{1+\frac{\sigma}{2}} \exp\left(\frac{A}{(1-r)^\sigma}\right)$$

We postpone the proof of the proposition for a moment, and we first use it in order to show that the exponential weights are in the class $\mathcal{E}$.

Theorem 4.25. *For $0 < \sigma < \infty$, let ω_σ the exponential type weight given by* (4.18)*. Then*

$$\int_{\mathbb{D}} |K_z(\zeta)| \, \omega_\sigma(\zeta)^{1/2} \, dA(\zeta) \lesssim \omega_\sigma(z)^{-1/2}.$$

Proof. Passing to polar coordinates, we have

$$\int_{\mathbb{D}} |K_z(\zeta)| \, \omega_\sigma(\zeta)^{1/2} \, dA(\zeta) = \int_0^1 \left(\int_0^{2\pi} |K_z(se^{i\theta})| \frac{d\theta}{\pi} \right) s \, \omega_\sigma(s)^{1/2} \, ds.$$

Set $z = re^{i\varphi}$. As

$$\int_0^{2\pi} |K_z(se^{i\theta})| \, d\theta = \int_0^{2\pi} |K(rse^{i(\theta-\varphi)})| \, d\theta = \int_0^{2\pi} |K(rse^{i\theta})| \, d\theta,$$

we get

$$\int_{\mathbb{D}} |K_z(\zeta)| \, \omega_\sigma(\zeta)^{1/2} \, dA(\zeta) = 2 \int_0^1 M_1(rs, K) \, s \, \omega_\sigma(s)^{1/2} \, ds.$$

Therefore, applying Proposition 4.24, we obtain

$$\int_{\mathbb{D}} |K_z(\zeta)| \, \omega_\sigma(\zeta)^{1/2} \omega_\sigma(\zeta)^{1/2} \, dA(\zeta) \lesssim \int_0^1 \exp\left(Ag_r(s)\right) \frac{1}{(1-rs)^{1+\frac{\sigma}{2}}} ds,$$

with

$$g_r(s) := \frac{1}{(1-rs)^\sigma} - \frac{1}{2}\left(\frac{1}{(1-r^2)^\sigma} + \frac{1}{(1-s^2)^\sigma}\right).$$

Lemma 4.26. *For $0 < \sigma < \infty$. There is a positive constant C depending on σ such that*

$$g_r(s) \leq C\left(\frac{s-r}{1-sr}\right)\left(\frac{r}{(1-r^2)^\sigma} - \frac{s}{(1-s^2)^\sigma}\right).$$

Proof. We may assume that $r \leq s$, because changing the order of r and s does not affect the result. Observe that

$$g_r(s) = \frac{1}{2}\left(\frac{1}{(1-rs)^\beta} - \frac{1}{(1-r^2)^\beta}\right) + \frac{1}{2}\left(\frac{1}{(1-rs)^\beta} - \frac{1}{(1-s^2)^\beta}\right). \tag{4.19}$$

By the mean value theorem of differential calculus, there exists $x_1 \in (r, s)$ such that

$$(1-r^2)^\sigma - (1-rs)^\sigma = \sigma r\,(1-rx_1)^{\sigma-1}(s-r).$$

We first consider the case $0 < \sigma \leq 1$. As $\sigma \leq 1$ and $x_1 \leq s$, we have $(1-rx_1)^{\sigma-1} \leq (1-rs)^{\sigma-1}$. Then

$$\begin{aligned}\frac{1}{(1-rs)^\sigma} - \frac{1}{(1-r^2)^\sigma} &= \frac{(1-r^2)^\sigma - (1-rs)^\sigma}{(1-r^2)^\sigma(1-rs)^\sigma} \\ &\leq \frac{\sigma r\,(s-r)}{(1-r^2)^\sigma(1-rs)}.\end{aligned} \tag{4.20}$$

Similarly, there exists $x_2 \in (r, s)$ such that

$$\frac{1}{(1-rs)^\sigma} - \frac{1}{(1-s^2)^\sigma} = \frac{-\sigma s\,(1-sx_2)^{\sigma-1}(s-r)}{(1-s^2)^\sigma(1-rs)^\sigma}.$$

Since $(1-sx_2) < (1-rs)$ and $\sigma \leq 1$, we get

$$\frac{1}{(1-rs)^\sigma} - \frac{1}{(1-s^2)^\sigma} \leq -\sigma\frac{s\,(s-r)}{(1-s^2)^\sigma(1-rs)}.$$

Combining this one with (4.19) and bearing in mind (4.20), we get the desired result for $0 < \sigma \leq 1$ with $C = \sigma/2$.

Next, we consider the case $1 < \sigma \leq 2$. In this case, as $\sigma/2 \leq 1$, arguing as before we get

$$\begin{aligned}(1-r^2)^\sigma - (1-rs)^\sigma &= ((1-r^2)^{\sigma/2} - (1-rs)^{\sigma/2})\cdot((1-r^2)^{\sigma/2} + (1-rs)^{\sigma/2}) \\ &\leq \frac{\sigma}{2}\,r\,(1-rs)^{\sigma/2-1}(s-r)\Big((1-r^2)^{\sigma/2} + (1-rs)^{\sigma/2}\Big).\end{aligned}$$

This gives

$$\frac{1}{(1-rs)^\sigma}-\frac{1}{(1-r^2)^\sigma}\le\frac{\sigma}{2}\frac{r(s-r)}{(1-r^2)^\sigma(1-rs)}+\frac{\sigma}{2}\frac{r(s-r)}{(1-r^2)^{\sigma/2}(1-rs)^{\sigma/2+1}}. \tag{4.21}$$

Similarly,

$$(1-s^2)^\sigma-(1-rs)^\sigma\le-\frac{\sigma}{2}s(1-rs)^{\sigma/2-1}(s-r)\Big((1-s^2)^{\sigma/2}+(1-rs)^{\sigma/2}\Big),$$

and we obtain

$$\frac{1}{(1-rs)^\sigma}-\frac{1}{(1-s^2)^\sigma}\le-\frac{\sigma}{2}\frac{s(s-r)}{(1-s^2)^\sigma(1-rs)}-\frac{\sigma}{2}\frac{s(s-r)}{(1-s^2)^{\sigma/2}(1-rs)^{\sigma/2+1}}. \tag{4.22}$$

Inserting (4.21) and (4.22) into (4.19) we get

$$\begin{aligned}g_r(s)&\le\frac{\sigma}{4}\left(\frac{s-r}{1-rs}\right)\left(\frac{r}{(1-r^2)^\sigma}-\frac{s}{(1-s^2)^\sigma}\right)\\&\quad+\frac{\sigma}{4}\frac{s-r}{(1-rs)^{\frac{\sigma}{2}+1}}\left(\frac{r}{(1-r^2)^{\sigma/2}}-\frac{s}{(1-s^2)^{\sigma/2}}\right).\end{aligned}$$

Hence the result follows in this case with $C=\sigma/4$, because every summand is negative.

Next, we show by induction that, for $2^{m-1}<\sigma\le 2^m$ with $m\ge 1$, we have

$$g_r^\sigma(s)=g_r(s)\le\frac{\sigma}{2^m}\left(\frac{s-r}{1-rs}\right)\left(\frac{r}{(1-r^2)^\sigma}-\frac{s}{(1-s^2)^\sigma}\right).$$

We have just proved the case $m=1$. So assume $m\ge 2$ and that the result is true for $m-1$ and proceed to prove the case m. Using the identities

$$(1-r^2)^\sigma-(1-rs)^\sigma=\Big((1-r^2)^{\sigma/2}-(1-rs)^{\sigma/2}\Big)\cdot\Big((1-r^2)^{\sigma/2}+(1-rs)^{\sigma/2}\Big)$$

and

$$(1-s^2)^\sigma-(1-rs)^\sigma=\Big((1-s^2)^{\sigma/2}-(1-rs)^{\sigma/2}\Big)\cdot\Big((1-s^2)^{\sigma/2}+(1-rs)^{\sigma/2}\Big)$$

we get

$$\begin{aligned}\frac{1}{(1-rs)^\sigma}-\frac{1}{(1-r^2)^\sigma}&=(1-r^2)^{-\sigma/2}\left(\frac{1}{(1-rs)^{\sigma/2}}-\frac{1}{(1-r^2)^{\sigma/2}}\right)\\&\quad+(1-rs)^{-\sigma/2}\left(\frac{1}{(1-rs)^{\sigma/2}}-\frac{1}{(1-r^2)^{\sigma/2}}\right)\end{aligned} \tag{4.23}$$

and

$$\frac{1}{(1-rs)^{\sigma}} - \frac{1}{(1-s^2)^{\sigma}} = (1-s^2)^{-\sigma/2}\left(\frac{1}{(1-rs)^{\sigma/2}} - \frac{1}{(1-s^2)^{\sigma/2}}\right) \\ + (1-rs)^{-\sigma/2}\left(\frac{1}{(1-rs)^{\sigma/2}} - \frac{1}{(1-s^2)^{\sigma/2}}\right). \tag{4.24}$$

Adding the two identities, we obtain

$$2g_r^{\sigma}(s) = (1-r^2)^{-\sigma/2}\left(\frac{1}{(1-rs)^{\sigma/2}} - \frac{1}{(1-r^2)^{\sigma/2}}\right) \\ + (1-s^2)^{-\sigma/2}\left(\frac{1}{(1-rs)^{\sigma/2}} - \frac{1}{(1-s^2)^{\sigma/2}}\right) + (1-rs)^{-\sigma/2}\, 2g_r^{\sigma/2}(s).$$

As $g_r^{\sigma/2}(s) \leq 0$, this gives

$$2g_r^{\sigma}(s) \leq (1-r^2)^{-\sigma/2}\left(\frac{1}{(1-rs)^{\sigma/2}} - \frac{1}{(1-r^2)^{\sigma/2}}\right) + (1-s^2)^{-\sigma/2}\left(\frac{1}{(1-rs)^{\sigma/2}} - \frac{1}{(1-s^2)^{\sigma/2}}\right).$$

Use again the identities (4.23) and (4.24), but with σ replaced by $\sigma/2$ in order to get

$$2g_r^{\sigma}(s) \leq (I) + (II) + (III) + (IV),$$

with

$$(I) := (1-r^2)^{-\sigma/2}(1-r^2)^{-\sigma/4}\left(\frac{1}{(1-rs)^{\sigma/4}} - \frac{1}{(1-r^2)^{\sigma/4}}\right),$$

$$(II) := (1-r^2)^{-\sigma/2}(1-rs)^{-\sigma/4}\left(\frac{1}{(1-rs)^{\sigma/4}} - \frac{1}{(1-r^2)^{\sigma/4}}\right)$$

$$(III) := (1-s^2)^{-\sigma/2}(1-s^2)^{-\sigma/4}\left(\frac{1}{(1-rs)^{\sigma/4}} - \frac{1}{(1-s^2)^{\sigma/4}}\right)$$

and

$$(IV) := (1-s^2)^{-\sigma/2}(1-rs)^{-\sigma/4}\left(\frac{1}{(1-rs)^{\sigma/4}} - \frac{1}{(1-s^2)^{\sigma/4}}\right).$$

Note that, as $r \leq s$, the term in brackets in (II) is positive, and also we have $(1-r^2)^{-\sigma/2} \leq (1-s^2)^{-\sigma/2}$. Therefore

$$(II) + (IV) \leq 2(1-s^2)^{-\sigma/2}(1-rs)^{-\sigma/4}\, g_r^{\sigma/4}(s) \leq 0.$$

This gives

$$2g_r^\sigma(s) \leq (I) + (III) = (1-r^2)^{-\sigma/2}(1-r^2)^{-\sigma/4}\left(\frac{1}{(1-rs)^{\sigma/4}} - \frac{1}{(1-r^2)^{\sigma/4}}\right)$$

$$+ (1-s^2)^{-\sigma/2}(1-s^2)^{-\sigma/4}\left(\frac{1}{(1-rs)^{\sigma/4}} - \frac{1}{(1-s^2)^{\sigma/4}}\right)$$

$$= \left(\frac{(1-r^2)^{\sigma/4} - (1-rs)^{\sigma/4}}{(1-rs)^{\sigma/4}\,(1-r^2)^{\sigma}}\right) + \left(\frac{(1-s^2)^{\sigma/4} - (1-rs)^{\sigma/4}}{(1-rs)^{\sigma/4}\,(1-s^2)^{\sigma}}\right).$$

Iterating this process, we arrive at

$$2g_r^\sigma(s) \leq \left(\frac{(1-r^2)^{\sigma/2^m} - (1-rs)^{\sigma/2^m}}{(1-rs)^{\sigma/2^m}\,(1-r^2)^{\sigma}}\right) + \left(\frac{(1-s^2)^{\sigma/2^m} - (1-rs)^{\sigma/2^m}}{(1-rs)^{\sigma/2^m}\,(1-s^2)^{\sigma}}\right).$$

As $\sigma/2^m \leq 1$, using the meanvalue theorem as in the case $\sigma \leq 1$, we have

$$(1-r^2)^{\sigma/2^m} - (1-rs)^{\sigma/2^m} \leq \big(\sigma/2^m\big) r(1-rs)^{\sigma/2^m-1}\,(s-r),$$

and

$$(1-s^2)^{\sigma/2^m} - (1-rs)^{\sigma/2^m} \leq -\big(\sigma/2^m\big) s(1-rs)^{\sigma/2^m-1}\,(s-r).$$

This finally gives

$$2g_r^\sigma(s) \leq \big(\frac{\sigma}{2^m}\big)\left(\frac{s-r}{1-rs}\right)\left(\frac{r}{(1-r^2)^\sigma} - \frac{s}{(1-s^2)^\sigma}\right).$$

□

Now, we can continue with the proof. Set

$$f_r(s) = \exp\big(Ag_r(s)\big)\,\frac{1}{(1-rs)^{1+\frac{\sigma}{2}}}.$$

We must show that

$$\int_0^1 f_r(s)\,ds \leq C \tag{4.25}$$

for some constant C not depending on r. Without loss of generality, we may assume that $r > 1/2$. From Lemma 4.26, we see that $g_r(s) \leq 0$. Hence,

$$\int_0^{1/2} f_r(s)\,ds \leq \int_0^{1/2} \frac{ds}{(1-rs)^{1+\frac{\sigma}{2}}} \leq 2^{\sigma/2}. \tag{4.26}$$

Also,

$$\int_{r-(1-r)^{1+\frac{\sigma}{2}}}^{r+(1-r)^{1+\frac{\sigma}{2}}} f_r(s)\,ds \le \int_{r-(1-r)^{1+\frac{\sigma}{2}}}^{r+(1-r)^{1+\frac{\sigma}{2}}} \frac{ds}{(1-rs)^{1+\frac{\sigma}{2}}} \le 2^{\sigma/2}$$
$$\le \int_{r-(1-r)^{1+\frac{\sigma}{2}}}^{r+(1-r)^{1+\frac{\sigma}{2}}} \frac{ds}{(1-r)^{1+\frac{\sigma}{2}}} \le 2. \tag{4.27}$$

In view of (4.26) and (4.27), to see that (4.25) holds, it remains to prove that

$$\int_{r+(1-r)^{1+\frac{\sigma}{2}}}^{1} f_r(s)\,ds \le C_1 \tag{4.28}$$

and

$$\int_{1/2}^{r-(1-r)^{1+\frac{\sigma}{2}}} f_r(s)\,ds \le C_2. \tag{4.29}$$

We begin with the proof of (4.28). Let $N = N(r)$ be the largest positive integer such that

$$2^N < \frac{1}{(1-r)^{\sigma/2}}.$$

Observe that

$$r + 2^k(1-r)^{1+\frac{\sigma}{2}} \ge 1 \quad \Leftrightarrow \quad 2^k \ge (1-r)^{-\sigma/2}.$$

Then

$$\int_{r+(1-r)^{1+\sigma/2}}^{1} f_r(s)\,ds = \sum_{k=0}^{N} \int_{J_k(r)\cap[0,1]} f_r(s)\,ds,$$

with

$$J_k(r) = \big(r + 2^k(1-r)^{1+\frac{\sigma}{2}}, r + 2^{k+1}(1-r)^{1+\frac{\sigma}{2}}\big).$$

As $s > r$, from Lemma 4.26 we have

$$g_r(s) \le -C\,A_r(s)\cdot B_r(s) \tag{4.30}$$

with

$$A_r(s) := \frac{s-r}{1-sr}$$

and

$$B_r(s) := \frac{s}{(1-s^2)^\sigma} - \frac{r}{(1-r^2)^\sigma}.$$

For $s \in J_k(r)$, we have

$$A_r(s) \ge \frac{s-r}{2(1-r)} \ge 2^{k-1}(1-r)^{\sigma/2}. \tag{4.31}$$

Let $h(x) = x/(1-x^2)^\sigma$. Then

$$h'(x) = \frac{1 - x^2 + 2\sigma x^2}{(1-x^2)^{\sigma+1}}.$$

By the mean value theorem, there is $x \in (r, s)$ with

$$B_r(s) = \frac{1 + (2\sigma - 1)x^2}{(1-x^2)^{\sigma+1}} (s - r).$$

If $2\sigma \geq 1$, then $1+(2\sigma-1)x^2 \geq 1$ and, if $2\sigma < 1$, because $x < 1$, we have $1+(2\sigma-1)x^2 > 2\sigma$. Therefore,

$$1 + (2\sigma - 1)x^2 \geq C_\sigma := \min(1, 2\sigma).$$

Hence, as $x > r$, we obtain

$$B_r(s) \geq C_\sigma \frac{s - r}{(1-r^2)^{\sigma+1}}.$$

Since

$$s - r \geq 2^k (1-r)^{1+\frac{\sigma}{2}}, \qquad s \in J_k(r),$$

we have

$$B_r(s) \geq C(\sigma) 2^k (1-r)^{-\sigma/2}.$$

Putting this estimate together with (4.31) into (4.30) we obtain

$$g_r(s) \leq -C(\sigma) 2^{2k}, \qquad s \in J_k(r).$$

Hence,

$$\int_{r+(1-r)^{1+\sigma/2}}^1 f_r(s)\, ds = \sum_{k=0}^N \int_{J_k(r)\cap[0,1]} e^{A g_r(s)} \, (1 - sr)^{-1-\frac{\sigma}{2}} \, ds$$

$$\leq \sum_{k=0}^N e^{-C2^{2k}} \int_{J_k(r)} \frac{ds}{(1-sr)^{1+\frac{\sigma}{2}}}$$

$$\leq \sum_{k=0}^N e^{-C2^{2k}} \frac{|J_k(r)|}{(1-r)^{1+\frac{\sigma}{2}}} = \sum_{k=0}^N 2^k e^{-C2^{2k}} = C_1 < \infty.$$

This proves (4.28). Therefore, in order to finish the proof of this case, it remains to prove that (4.29) holds, that is, we must show that

$$\int_{1/2}^{r-(1-r)^{1+\frac{\sigma}{2}}} f_r(s)\, ds \leq C_2.$$

Let $M = M(r)$ be the largest positive integer such that

$$2^M < \frac{r - 1/2}{(1-r)^{1+\beta/2}}.$$

Then

$$\int_{1/2}^{r-(1-r)^{1+\sigma/2}} f_r(s)\,ds \le \sum_{k=0}^{M} \int_{J_k(r)\cap[1/2,1]} f_r(s)\,ds \tag{4.32}$$

with

$$J_k(r) = \big(r - 2^{k+1}(1-r)^{1+\sigma/2}, r - 2^k(1-r)^{1+\sigma/2}\big).$$

Now $r > s$, so that from Lemma 4.26 we get

$$g_r(s) \le -C(\sigma)\,C_r(s)\cdot D_r(s) \tag{4.33}$$

with

$$C_r(s) := \frac{r-s}{1-sr}, \qquad \text{and} \qquad D_r(s) := \frac{r}{(1-r^2)^\sigma} - \frac{s}{(1-s^2)^\sigma}.$$

We have

$$r - s \ge 2^k(1-r)^{1+\sigma/2}, \qquad s \in J_k(r),$$

and

$$1 - sr \le 1 - (r - 2^{k+1}(1-r)^{1+\frac{\sigma}{2}})r \le 2(1-r)\big(1 + 2^k(1-r)^{\frac{\sigma}{2}}\big), \qquad s \in J_k(r).$$

This gives

$$C_r(s) \ \ge \ \frac{2^{k-1}(1-r)^{\sigma/2}}{1 + 2^k(1-r)^{\sigma/2}}, \qquad s \in J_k(r). \tag{4.34}$$

In order to estimate the term $D_r(s)$, take a non-negative integer m with $2^m\sigma \ge 1$. Then we claim that

$$D_r(s) \ge 2^{-m} \quad \frac{r(1-s^2)^{2^m\sigma} - s(1-r^2)^{2^m\sigma}}{(1-r^2)^\sigma(1-s^2)^{2^m\sigma}}. \tag{4.35}$$

We prove the claim by induction on m. The result is obvious for $m = 0$. Assume the claim is true for m and proceed to show that the case $m+1$ also holds. By induction, we have

$$\begin{aligned} D_r(s) &\ge 2^{-m} \quad \frac{r(1-s^2)^{2^m\sigma} - s(1-r^2)^{2^m\sigma}}{(1-r^2)^\sigma(1-s^2)^{2^m\sigma}} \\ &= 2^{-m} \quad \frac{r^2(1-s^2)^{2^{m+1}\sigma} - s^2(1-r^2)^{2^{m+1}\sigma}}{(1-r^2)^\sigma(1-s^2)^{2^m\sigma}\big(r(1-s^2)^{2^m\sigma} + s(1-r^2)^{2^m\sigma}\big)}. \end{aligned}$$

Because $s \leq r$, then $(1-r^2) \leq (1-s^2)$ and

$$D_r(s) \geq \frac{2^{-m}\, r}{r+s} \cdot \frac{r(1-s^2)^{2^{m+1}\sigma} - s(1-r^2)^{2^{m+1}\sigma}}{(1-r^2)^{\sigma}(1-s^2)^{2^{m+1}\sigma}}$$

$$\geq 2^{-(m+1)} \cdot \frac{r(1-s^2)^{2^{m+1}\sigma} - s(1-r^2)^{2^{m+1}\sigma}}{(1-r^2)^{\sigma}(1-s^2)^{2^{m+1}\sigma}}.$$

Hence the claim is proved.

As $2^m\sigma \geq 1$ and $s \leq r$, then $(1-r^2)^{2^m\sigma-1} \leq (1-s^2)^{2^m\sigma-1}$, and using the inequality (4.35), we obtain

$$D_r(s) \geq 2^{-m}\, \frac{r(1-s^2)^{2^m\sigma} - s(1-r^2)^{2^m\sigma-1}(1-r^2)}{(1-r^2)^{\sigma}(1-s^2)^{2^m\sigma}}$$

$$\geq 2^{-m}\, \frac{r(1-s^2) - s(1-r^2)}{(1-r^2)^{\sigma}(1-s^2)} \geq 2^{-m} \frac{r-s}{(1-r^2)^{\sigma}(1-s^2)}.$$

Now, for $s \in J_k(r)$, we have

$$r - s \geq 2^k(1-r)^{1+\sigma/2},$$

and

$$1-s^2 \leq 2(1-s) \leq 2\big(1-(r-2^{k+1}(1-r)^{1+\sigma/2})\big) = 2(1-r)(1+2^{k+1}(1-r)^{\sigma/2}).$$

This gives

$$D_r(s) \geq \frac{2^{-(m+1)}\, 2^k(1-r)^{-\sigma/2}}{(1+r)^{\sigma}(1+2^{k+1}(1-r)^{\sigma/2})} \geq 2^{-(m+1+\sigma)} \cdot \frac{2^k(1-r)^{-\sigma/2}}{(1+2^{k+1}(1-r)^{\sigma/2})}, \qquad s \in J_k(r).$$

Putting this together with (4.34) into (4.33) we obtain

$$g_r(s) \leq -C(\sigma) \cdot \frac{2^{2k}}{(1+2^{k+1}(1-r)^{\sigma/2})^2}, \qquad s \in J_k(r); \quad 1/2 \leq s < 1.$$

Then

$$\int_{1/2}^{r-(1-r)^{1+\sigma/2}} f_r(s)\, ds \leq \sum_{k=0}^{M} \int_{J_k(r)\cap[1/2,1]} e^{Ag_r(s)}\, (1-sr)^{-1-\frac{\sigma}{2}}\, ds$$
$$\leq \sum_{k=0}^{M} \exp\Big(-C\, \frac{2^{2k}}{\big(1+2^{k+1}(1-r)^{\sigma/2}\big)^2}\Big) \int_{J_k(r)\cap[1/2,1]} \frac{ds}{(1-sr)^{1+\frac{\sigma}{2}}}\, ds.$$

Since

$$1-sr \geq 1-s \geq 1-(r-2^k(1-r)^{1+\sigma/2}) = (1-r)(1+2^k(1-r)^{\sigma/2}), \qquad s \in J_k(r).$$

and $|J_k(r)| = 2^k(1-r)^{1+\sigma/2}$ we get

$$\int_{1/2}^{r-(1-r)^{1+\sigma/2}} f_r(s)\,ds \le \sum_{k=0}^{M} \exp\Big(-C\,\frac{2^{2k}}{\big(1+2^{k+1}(1-r)^{\sigma/2}\big)^2}\Big)\frac{2^k}{\big(1+2^k(1-r)^{\frac{\sigma}{2}}\big)^{1+\frac{\sigma}{2}}}. \quad (4.36)$$

Because $h_r(x) = (1+x(1-r)^{\sigma/2})^{-(1+\sigma/2)}$ is a decreasing function, then

$$\frac{2^k}{(1+2^k(1-r)^{\frac{\sigma}{2}})^{1+\sigma/2}} \le 2\int_{2^{k-1}}^{2^k} \frac{dx}{(1+x(1-r)^{\frac{\sigma}{2}})^{1+\sigma/2}}.$$

Also, for $x \in [2^{k-1}, 2^k]$, we have

$$\frac{2^{2k}}{(1+2^{k+1}(1-r)^{\sigma/2})^2} \ge \frac{2^{2k}}{16(1+2^{k-1}(1-r)^{\sigma/2})^2} \ge \frac{x^2}{16(1+x(1-r)^{\sigma/2})^2}.$$

Therefore, we obtain

$$\sum_{k=0}^{M} \exp\Big(-C\,\frac{2^{2k}}{(1+2^{k+1}(1-r)^{\sigma/2})^2}\Big)\frac{2^k}{\big(1+2^k(1-r)^{\sigma/2}\big)^{1+\sigma/2}}$$

$$\le 2\int_{1/2}^{(1-r)^{-(1+\frac{\sigma}{2})}} \exp\Bigg(\frac{-Cx^2}{\big(1+x(1-r)^{\sigma/2}\big)^2}\Bigg)\frac{dx}{(1+x(1-r)^{\sigma/2})^{1+\sigma/2}}.$$

Bearing in mind (4.36), we arrive at

$$\int_{1/2}^{r-(1-r)^{1+\sigma/2}} f_r(s)\,ds \le 2\int_{1/2}^{(1-r)^{-(1+\frac{\sigma}{2})}} \exp\Bigg(\frac{-Cx^2}{\big(1+x(1-r)^{\sigma/2}\big)^2}\Bigg)\frac{dx}{(1+x(1-r)^{\sigma/2})^{1+\sigma/2}}. \quad (4.37)$$

Consider the function

$$g(x) = \exp\Bigg(\frac{-Cx^2}{(1+x(1-r)^{\sigma/2})^2}\Bigg)\big(1+x(1-r)^{\frac{\sigma}{2}}\big)^{-(1+\sigma/2)}.$$

To finish the proof it remains to see that

$$\int_{1/2}^{(1-r)^{-\frac{\sigma}{2}}} g(x)\,dx \le K_1 \quad \text{and} \quad \int_{(1-r)^{-\frac{\sigma}{2}}}^{(1-r)^{-(1+\frac{\sigma}{2})}} g(x)\,dx \le K_2$$

for some positive constants K_1 and K_2 not depending on r. It is clear that

$$\int_{1/2}^{(1-r)^{-\frac{\sigma}{2}}} g(x)\,dx \le \int_{1/2}^{\infty} e^{-Cx^2/4}\,dx = K_1 < \infty.$$

Finally, using the change of variables $t = (1-r)^{\sigma/2}x$ and the fact that the function $k(t) = -t(1+t)^{-1}$ decreases, we get

$$\int_{(1-r)^{-\frac{\sigma}{2}}}^{(1-r)^{-(1+\frac{\sigma}{2})}} g(x)\,dx = \int_1^{(1-r)^{-1}} \exp\left(\frac{-Ct^2}{(1-r)^\sigma(1+t)^2}\right)\frac{dt}{(1-r)^{\sigma/2}(1+t)^{1+\sigma/2}}$$

$$\leq (1-r)^{-\sigma/2}\exp\left(\frac{-C}{4(1-r)^\sigma}\right)\int_1^{(1-r)^{-1}}\frac{dt}{(1+t)^{1+\sigma/2}} \leq K_2$$

for some constant K_2 not depending on r. This completes the proof. □

The rest of the section is devoted to the proof of Proposition 4.24. We need first some lemmas, whose proof uses results of [33] involving the Legendre-Fenschel transform.

Lemma 4.27. *Let $A, \sigma > 0$ and let $c_0 = A^{\frac{1}{\sigma+1}}\left(\sigma^{\frac{1}{\sigma+1}} + \sigma^{-\frac{\sigma}{\sigma+1}}\right)$. Then*

$$\sum_{n=1}^\infty \exp\left(c_0 n^{\frac{\sigma}{\sigma+1}}\right)\rho^n \lesssim \left(\frac{1}{1-\rho}\right)^{1+\sigma/2}\exp\left(\frac{A}{(1-\rho)^\sigma}\right).$$

Proof. We have

$$\sum_{n=1}^\infty \exp\left(c_0 n^{\frac{\sigma}{\sigma+1}}\right)\rho^n = \sum_{n=1}^\infty e^{N(n)}\,\rho^n$$

with

$$N(x) = c_0\,x^{\sigma/(\sigma+1)}, \qquad x > 0.$$

A calculation shows that its inverse Legendre-Fenschel transform is given by

$$u(t) = \sup_{x>0}\big(N(x) - xt\big) = A\,t^{-\sigma}.$$

Since $u''(t) = A\sigma(\sigma+1)t^{-(2+\sigma)}$, by Corollary 1 in [33] we get

$$\sum_{n=1}^\infty e^{N(n)}\,\rho^n \asymp \sqrt{u''(\log 1/\rho)}\; e^{u(\log 1/\rho)}$$

$$= \sqrt{A\sigma(\sigma+1)}\left(\frac{1}{\log 1/\rho}\right)^{1+\sigma/2}\exp\left(\frac{A}{(\log 1/\rho)^\sigma}\right)$$

$$\leq \sqrt{A\sigma(\sigma+1)}\left(\frac{1}{1-\rho}\right)^{1+\sigma/2}\exp\left(\frac{A}{(1-\rho)^\sigma}\right).$$

□

Lemma 4.28. *Let* $0 < \sigma < \infty$ *and*

$$I(\lambda) = \int_0^1 r^\lambda \exp\Big(- \frac{A}{(\log \frac{1}{r})^\sigma}\Big)\, dr, \quad A > 0.$$

Then

$$I(\lambda) \asymp \lambda^{-\frac{(2+\sigma)}{2(\sigma+1)}} \exp\big(- c_0 \lambda^{\frac{\sigma}{\sigma+1}}\big), \qquad \lambda \to \infty,$$

where $c_0 = A^{\frac{1}{\sigma+1}}\big(\sigma^{\frac{1}{\sigma+1}} + \sigma^{-\frac{\sigma}{\sigma+1}}\big)$.

Proof. This is proved for $0 < \sigma \le 1$ in [18, 19], but since we need the explicit expression of the constant c_0 we give the proof here. After the change of variables $t = -\log r$, we have

$$I(\lambda) = \int_0^\infty e^{-At^{-\sigma}-(\lambda+1)t} dt = \int_0^\infty e^{-v(t)-(\lambda+1)t} dt,$$

with $v(t) = At^{-\sigma}$. Consider its Legendre-Fenschel transform

$$L(x) = \inf_{t>0} \big(v(t) + xt \big).$$

A simple calculation shows that $L(x) = c_0 x^{\sigma/(\sigma+1)}$. Since $L''(x) = -c_0 \frac{\sigma}{(\sigma+1)^2} x^{-\frac{(2+\sigma)}{\sigma+1}}$, by [33, Theorem 1], we have

$$I(\lambda) \asymp \sqrt{-L''(\lambda)}\, e^{-L(\lambda)} \asymp \lambda^{-\frac{(2+\sigma)}{2(\sigma+1)}} \exp\big(- c_0 \lambda^{\frac{\sigma}{\sigma+1}}\big), \qquad \lambda \to \infty.$$

$\square$

Proof of Proposition 4.24

Step 1. Following the same argument as in [16] we get

$$M_1(\rho, K) \le C + \sum_{n=3}^{\infty} M_n,$$

with

$$M_n := \max_{x \in [2^{n-1}, 2^{n+1}]} \frac{\rho^x}{v_{2x+1}}.$$

We recall that

$$v_\lambda = \int_0^1 s^\lambda\, \omega_\sigma(s)\, ds.$$

Step 2. Changing variables and then using Lemma 4.28 we have

$$v_{2x+1} \asymp x^{\frac{-(2+\sigma)}{2\sigma+2}} \exp\big(- c_0\, x^{\frac{\sigma}{\sigma+1}}\big), \qquad x \to \infty,$$

Therefore,

$$\frac{\rho^x}{v_{2x+1}} \asymp \rho^x\, x^{\frac{2+\sigma}{2\sigma+2}} \exp\left(c_0\, x^{\frac{\sigma}{\sigma+1}}\right) = e^{h(x)}, \qquad x \to \infty,$$

where

$$h(x) = x \log \rho + \left(\frac{2+\sigma}{2\sigma+2}\right) \log x + c_0 x^{\frac{\sigma}{\sigma+1}}.$$

Since $h''(x) = -\frac{(2+\sigma)}{2(\sigma+1)x^2} - \frac{\sigma}{(\sigma+1)^2} x^{-\frac{(\sigma+2)}{\sigma+1}} < 0$, then h can only have a critical point x_ρ, and in case it exists, it must be a maximum. Next we are going to show that it has a unique maximum x_ρ which is comparable with $y_\rho = (\frac{1}{\log\frac{1}{\rho}})^{\sigma+1}$. Indeed, for $n \geq \left(\frac{\sigma+1}{\sigma c_0}\right)^{\sigma+1}$ we have

$$h'\Big(\frac{1}{n}\, y_\rho\Big) = \log \rho + \frac{(2+\sigma)n}{2(\sigma+1)} \left(\log \frac{1}{\rho}\right)^{\sigma+1} + \frac{\sigma\, c_0}{\sigma+1}\, n^{\frac{1}{\sigma+1}} \log \frac{1}{\rho}$$

$$= \left(\log \frac{1}{\rho}\right) \cdot \left(\frac{\sigma\, c_0}{\sigma+1}\, n^{\frac{1}{\sigma+1}} - 1\right) + \frac{(2+\sigma)n}{2(\sigma+1)} \left(\log \frac{1}{\rho}\right)^{\sigma+1} > 0.$$

Also, for $m \geq \max\left((\log \frac{1}{\rho})^{\sigma+1};\, \frac{2+\sigma(1+2c_0)}{2(\sigma+1)}\right)$, (note that we can take m independent of ρ, since $(\log\frac{1}{\rho})^{\sigma+1}$ is bounded for $1/2 \leq \rho < 1$)

$$h'(m\, y_\rho) \leq \log \rho + \frac{2+\sigma(1+2c_0)}{2(\sigma+1)} (m\, y_\rho)^{\frac{-1}{\sigma+1}}$$

$$= \left(\log \frac{1}{\rho}\right) \left(\frac{2+\sigma(1+2c_0)}{2(\sigma+1) m^{\frac{1}{\sigma+1}}} - 1\right) < 0.$$

In the last inequality we used that $x^{\frac{1}{\beta+1}} \leq x$ for $x \geq 1$. Therefore, there exists $x_\rho \in (\frac{y_\rho}{n}, m y_\rho)$ such that $h'(x_\rho) = 0$. This gives us

$$\sup_{x\in(0,\infty)} e^{h(x)} = e^{h(x_\rho)} \asymp \left(\frac{1}{1-\rho}\right)^{1+\sigma/2} \exp\big(g(x_\rho)\big), \tag{4.38}$$

with

$$g(x) = x \log \rho + c_0 x^{\frac{\sigma}{\sigma+1}}.$$

now, a calculation shows that g has a unique maximum at the point

$$z_\rho = \left(\frac{\sigma c_0}{(\sigma+1)} \frac{1}{\log\frac{1}{\rho}}\right)^{\sigma+1}.$$

We have

$$g(z_\rho) = -\left(\frac{\sigma c_0}{\sigma+1}\right)^{\sigma+1} \frac{1}{(\log\frac{1}{\rho})^\sigma} + c_0 \left(\frac{\sigma c_0}{(\sigma+1)} \frac{1}{\log\frac{1}{\rho}}\right)^\sigma$$

$$= \left(\frac{\sigma c_0}{\sigma+1}\right)^{\sigma} \frac{1}{(\log\frac{1}{\rho})^\sigma} \left(c_0 - \frac{\sigma c_0}{\sigma+1}\right)$$

$$= \sigma^\sigma \left(\frac{c_0}{\sigma+1}\right)^{\sigma+1} \frac{1}{(\log\frac{1}{\rho})^\sigma} = \frac{A}{(\log\frac{1}{\rho})^\sigma},$$

because, as $c_0 = A^{\frac{1}{\sigma+1}}\left(\sigma^{\frac{1}{\sigma+1}} + \sigma^{-\frac{\sigma}{\sigma+1}}\right)$, then

$$\sigma^\sigma \left(\frac{c_0}{\sigma+1}\right)^{\sigma+1} = \frac{A\,\sigma^\sigma}{(\sigma+1)^{\sigma+1}}\left(\sigma^{\frac{1}{\sigma+1}} + \sigma^{-\frac{\sigma}{\sigma+1}}\right)^{\sigma+1} = A.$$

This together with (4.38) yields

$$\begin{aligned} \sup_{x\in(0,\infty)} e^{h(x)} &\lesssim \left(\frac{1}{1-\rho}\right)^{1+\sigma/2} \exp\big(g(z_\rho)\big) \\ &= \left(\frac{1}{1-\rho}\right)^{1+\sigma/2} \exp\left(\frac{A}{(\log\frac{1}{\rho})^\sigma}\right) \\ &\le \left(\frac{1}{1-\rho}\right)^{1+\sigma/2} \exp\left(\frac{A}{(1-\rho)^\sigma}\right). \end{aligned} \tag{4.39}$$

Step 3. Choose $n_0 \in \mathbb{N}$ such that $2^{n_0} \le x_\rho < 2^{n_0+1}$ and split the above sum as follows:

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{n_0-2} M_n + \sum_{n=n_0-1}^{n_0+1} M_n + \sum_{n=n_0+2}^{\infty} M_n.$$

For $1 \le n \le n_0 - 2$, because of the monotonicity of h

$$M_n = \max_{x\in[2^{n-1},2^{n+1}]} \frac{\rho^x}{v_{2x+1}} = \max_{x\in[2^{n-1},2^{n+1}]} e^{h(x)} = e^{h(2^{n+1})}.$$

And it follows from (4.39) that

$$\sum_{n=n_0-1}^{n_0+1} M_n \lesssim \left(\frac{1}{1-\rho}\right)^{1+\sigma/2} \exp\left(\frac{A}{(1-\rho)^\sigma}\right).$$

Using again the monotonicity of h for $n \geq n_0 + 2$,

$$M_n = \max_{x \in [2^{n-1}, 2^{n+1}]} \frac{\rho^x}{v_{2x+1}} = \max_{x \in [2^{n-1}, 2^{n+1}]} e^{h(x)} = e^{h(2^{n-1})}.$$

So,

$$\sum_{n=1}^{\infty} M_n \leq \sum_{n=1}^{n_0-2} e^{h(2^{n+1})} + \sum_{n=n_0+2}^{\infty} e^{h(2^{n-1})} + \left(\frac{1}{1-\rho}\right)^{1+\sigma/2} \exp\left(\frac{A}{(1-\rho)^{\sigma}}\right). \tag{4.40}$$

Next, for $n \leq n_0 - 2$ and since h increases,

$$e^{h(2^{n+1})} = \frac{1}{2^{n+1}} \sum_{k=2^{n+1}}^{2^{n+2}-1} e^{h(2^{n+1})} \leq \frac{1}{2^{n+1}} \sum_{k=2^{n+1}}^{2^{n+2}-1} e^{h(k)} \leq 2 \sum_{k=2^{n+1}}^{2^{n+2}-1} \frac{e^{h(k)}}{k}.$$

And for $n \geq n_0 + 2$ and since h decreases,

$$e^{h(2^{n-1})} = \frac{1}{2^{n-2}} \sum_{k=2^{n-2}}^{2^{n-1}-1} e^{h(2^{n-1})} \leq \frac{4}{2^n} \sum_{k=2^{n-2}}^{2^{n-1}-1} e^{h(k)} \leq 2 \sum_{k=2^{n-2}}^{2^{n-1}-1} \frac{e^{h(k)}}{k}.$$

Using this one in (4.40), we have

$$\begin{aligned} M_1(r, K) &\lesssim \left(\frac{1}{1-\rho}\right)^{1+\sigma/2} \exp\left(\frac{A}{(1-\rho)^{\sigma}}\right) + \sum_{k=1}^{\infty} \frac{e^{h(k)}}{k} \\ &\asymp \left(\frac{1}{1-\rho}\right)^{1+\sigma/2} \exp\left(\frac{A}{(1-\rho)^{\sigma}}\right) + \sum_{k=1}^{\infty} k^{-\frac{\sigma}{2\sigma+2}} \exp\left(c_0\, k^{\frac{\sigma}{\sigma+1}}\right) \rho^k. \end{aligned}$$

Finally, as

$$\sum_{k=1}^{\infty} k^{-\frac{\sigma}{2\sigma+2}} \exp\left(c_0\, k^{\frac{\sigma}{\sigma+1}}\right) \rho^k \leq \sum_{k=1}^{\infty} \exp\left(c_0\, k^{\frac{\sigma}{\sigma+1}}\right) \rho^k,$$

applying Lemma 4.27, we get the result. The proof is complete.

Chapter 5

Open questions and future research

We believe that we have done a satisfactory work in order to get a better understanding of the function properties of large weighted Bergman spaces and the operators acting on them. We hope that this work is going to attract many other researchers to this area, and expect that the study of this function spaces is going to experience a period of intensive research in the next years. However, we have not been able to solve all the problems we had in mind, and in this last chapter we discuss some open problems we left, as well as some other problems we think it can be interesting to look on the future.

5.1 Extension of the results to $p \neq 2$

One can look if it is possible to extend the results obtained for the class $\mathcal{W}$ to the case $p \neq 2$, in the same way as the results obtained in Chapter for the class $\mathcal{E}$. Of course, one way to do that is to prove the estimate

$$\int_{\mathbb{D}} |K_z(\zeta)|\, \omega(\zeta)^{1/2}\, dA(\zeta) \lesssim \omega(z)^{-1/2}. \tag{5.1}$$

This condition will follow if one is able to show the following pointwise estimate for reproducing kernels: for each $M \geq 1$, there exists a constant $C > 0$ (depending on M) such that, for each $z, \xi \in \mathbb{D}$ one has

$$|K_z(\xi)| \leq C \frac{1}{\tau(z)} \frac{1}{\tau(\xi)} \omega(z)^{-1/2} \omega(\xi)^{-1/2} \left(\frac{\min(\tau(z), \tau(\xi))}{|z - \xi|} \right)^M. \tag{5.2}$$

The obtention of this estimate leads to the estimate (5.1), and according to the results of Chapter 4 to estimate the norm of the reproducing kernels in $A^p(\omega^{p/2})$ for $p \geq 1$. It also leads to the same estimate even for $0 < p < 1$ and even to estimate the p-norms of K_z on the associated Bergman spaces $A^p(\omega_*^{p/2})$ for associated weights ω_* of the form $\omega_*(z) = \omega(z)\, \tau(z)^\alpha$ with $\alpha \in \mathbb{R}$. This will allow to study the case $0 < p < 1$ and the associated weighted Bergman spaces.

In view of the estimates for the test functions $F_{a,p}$ given in Lemma C, it seems reasonable to think that the pointwise estimate (5.2) must be true for weights in the class $\mathcal{W}$, but we have not succeeded on that task (we have not been able to find a reasonable condition for $\tau(z)$ in order to apply the methods developed by Marzo and Ortega-Cerdà in [44] for weighted Fock spaces).

5.2 Localization and Compactness

In the recent years, a great effort has been done on looking for general conditions for describing the compactness of an operator acting either on standard Bergman spaces or in the classical Fock spaces [30, 31, 46, 66, 70]. In a recent paper, J. Xia and D. Zheng [70] introduced a class of "sufficiently localized" operators on the Fock space including the algebraic closure of the Toeplitz operators. They proved that every bounded operator T from the C^*-algebra generated by the sufficiently localized operators whose Berezin transform vanishes at infinity is compact in the Fock space. The concept of sufficiently localized operators was extended to a larger class of operators (either in the Fock space setting or in the Bergman space setting) by J. Isralowitz, M. Mitkowski and B. Wick [31] introducing what they called "weakly localized operators". The analogue of this last concept is what we are going to introduce next on the setting of large weighted Bergman spaces, when the weight ω is in the class $\mathcal{E}$ considered in Chapter 4. For the class $\mathcal{W}$, we need to restrict to the case $p = 2$ because of the lack of estimates for $\|K_z\|_{A^p(\omega^{p/2})}$.

For $1 < p < \infty$, let p' denote its conjugate exponent. Let $k_{p,z}$ denote the normalized reproducing kernels on $A^p(\omega^{p/2})$, that is, $k_{p,z} = K_z/\|K_z\|_{A^p(\omega^{p/2})}$. Given an operator T acting on $A^p(\omega^{p/2})$, we define the quantities

$$W_1(T) := \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |\langle T k_{p,z}, k_{p',\xi}\rangle_\omega| \,\frac{dA(\xi)}{\tau(\xi)^2},$$

and

$$W_2(T) := \sup_{\xi\in\mathbb{D}} \int_{\mathbb{D}} |\langle T k_{p,z}, k_{p',\xi}\rangle_\omega| \,\frac{dA(z)}{\tau(z)^2}.$$

Definition 5.1. *Let $1 < p < \infty$. A linear operator T acting on $A^p(\omega^{p/2})$ is said to be* ***weakly localized*** *if*

$$W(T) = \max\big(W_1(T), W_2(T)\big) < \infty$$

and

$$V_1^m(T) := \sup_{z\in\mathbb{D}} \int_{D_m(z)^c} |\langle T k_{p,z}, k_{p',\xi}\rangle_\omega| \,\frac{dA(\xi)}{\tau(\xi)^2} \to 0,$$

$$V_2^m(T) := \sup_{\xi\in\mathbb{D}} \int_{D_m(\xi)^c} |\langle T k_{p,z}, k_{p',\xi}\rangle_\omega| \,\frac{dA(z)}{\tau(z)^2} \to 0,$$

as $m \to \infty$, where $D_m(z) := \left\{\xi \in \mathbb{D} :\ d_\tau(z,\xi) = \frac{|z-\xi|}{\min(\tau(z),\tau(\xi))} < 2^m\delta\right\}$.

It is not difficult to see that if $W(T) < \infty$, then T is bounded on $A^p(\omega^{p/2})$. Thus the class of weakly localized operators is included in the class all bounded operators.

Lemma 5.1. *Let $\omega \in \mathcal{E}$ and $1 < p < \infty$. If $W(T) < \infty$, then T is bounded on $A^p(\omega^{p/2})$.*

Proof. Let E be the linear span of the reproducing kernels of $A^2(\omega)$, that is dense in $A^p(\omega^{p/2})$. If $f \in E$, by the reproducing formula, we have

$$Tf(z) = \langle Tf, K_z \rangle_\omega = \langle f, T^* K_z \rangle_\omega = \int_{\mathbb{D}} f(\xi)\, \overline{T^* K_z(\xi)}\, \omega(\xi)\, dA(\xi).$$

Here T^* denotes the adjoint operator, acting now on $A^{p'}(\omega^{p'/2})$ (see the duality result in Theorem 4.8). Since

$$\overline{T^* K_z(\xi)} = \|K_\xi\|_{A^p(\omega^{p/2})} \cdot \|K_z\|_{A^{p'}(\omega^{p'/2})} \langle T k_{p,\xi}, k_{p',z} \rangle_\omega$$

and, because of Lemma 4.2,

$$\|K_\xi\|_{A^p(\omega^{p/2})} \cdot \|K_z\|_{A^{p'}(\omega^{p'/2})} \asymp \omega(z)^{-1/2}\, \omega(\xi)^{-1/2}\, \tau(z)^{-2/p}\, \tau(\xi)^{-2/p'},$$

we get

$$|Tf(z)|\, \omega(z)^{1/2} \lesssim \tau(z)^{-2/p} \int_{\mathbb{D}} \big|f(\xi)\big|\, \big|\langle T k_{p,\xi}, k_{p',z} \rangle_\omega\big|\, \omega(\xi)^{1/2}\, \frac{dA(\xi)}{\tau(\xi)^{2/p'}}.$$

Thus, Hölder's inequality yields

$$|Tf(z)|^p\, \omega(z)^{p/2} \lesssim W_2(T)^{p-1}\, \tau(z)^{-2} \int_{\mathbb{D}} |f(\xi)|^p |\langle T k_{p,\xi}, k_{p',z} \rangle_\omega|\, \omega(\xi)^{p/2}\, dA(\xi).$$

Therefore, we have

$$\begin{aligned}
\|Tf\|^p_{A^p(\omega^{p/2})} &= \int_{\mathbb{D}} |Tf(z)|^p\, \omega(z)^{p/2}\, dA(z) \\
&\lesssim W_2(T)^{p-1} \int_{\mathbb{D}} |f(\xi)|^p\, \omega(\xi)^{p/2} \left(\int_{\mathbb{D}} |\langle T k_{p,\xi}, k_{p',z} \rangle_\omega| \frac{dA(z)}{\tau(z)^2} \right) dA(\xi) \\
&\lesssim W_1(T) \cdot W_2(T)^{p-1} \|f\|^p_{A^p(\omega^{p/2})}.
\end{aligned}$$

This finishes the proof. $\square$

Denote by $\mathcal{A}_{\omega,p}(\mathbb{D})$ the set of all weakly localized operators acting on $A^p(\omega^{p/2})$.

The main goal of this project is to extend to our large Bergman spaces, the results proved in [31] for standard Bergman spaces or classical Fock spaces. That is, we want to show that if $T \in \mathcal{A}_{\omega,p}(\mathbb{D})$, then T is compact on $A^p(\omega^{p/2})$ if and only if $B_\omega T(z) \to 0$ as $|z| \to 1^-$, where B_ω denotes the ω-Berezin transform of the operator T given by

$$B_\omega T(z) = \frac{TK_z(z)}{\|K_z\|_2^2}, \qquad z \in \mathbb{D}.$$

We have just begin to look at that problem, and following the lines of [31] we have proved that for $\omega \in \mathcal{E}$ and $1 \leq p < \infty$, the class $\mathcal{A}_{\omega,p}(\mathbb{D})$ forms an algebra, and that for $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ is in $\mathcal{A}_{\omega,p}(\mathbb{D})$. However, there is still plenty of work to do before getting the result we are looking for.

5.3 Big Hankel operators

Another interesting problem is to study, for weights ω in the class $\mathcal{E}$, the simultaneous boundedness and compactness of the big Hankel operators H_f and $H_{\overline{f}}$ acting on $A^p(\omega^{p/2})$ for general symbols f satisfying $fK_z \in L^1(\omega^{1/2})$ for each $z \in \mathbb{D}$. We have been able to prove that, for $1 \leq p < \infty$, if $f \in BMO^p(\tau)$ then H_f and $H_{\overline{f}}$ are both bounded on $A^p(\omega^{p/2})$.

For $1 \leq p < \infty$, and $\delta \in (0, m_\tau)$, let $BMO^p_\delta(\tau)$ denote the space of L^p_{loc}-integrable functions g on $\mathbb{D}$ such that

$$\|g\|_{BMO^p_\delta(\tau)} := \sup_{z\in\mathbb{D}} \left(\frac{1}{\delta^2\tau(z)^2} \int_{D(\delta\tau(z))} |g(s) - \widehat{g}_\delta(z)|^p \, dA(s) \right)^{1/p},$$

where

$$\widehat{g}_\delta(z) := \frac{1}{\delta^2\tau(z)^2} \int_{D(\delta\tau(z))} g(s)\, dA(s).$$

We have been able to study the structure of these spaces, and in particular, they are independent of δ, so that we can fix some $\delta \in (0, m_\tau)$ and put $BMO^p(\tau) = BMO^p_\delta(\tau)$.

Conversely, we conjecture that if H_f and $H_{\overline{f}}$ are both bounded, then $f \in BMO^p(\tau)$. There are several things supporting this conjecture: the analogue problem in the case of standard weighted Bergman spaces is true [77]; in case that the symbol g is analytic, then it is not difficult to see that g being in $BMO^p(\tau)$ is equivalent to $\tau g'$ being a bounded function and this is just the condition characterizing the boundedness of $H_{\overline{g}}$ obtained in Theorem 5.3. Finally, as we commented before, we know how to prove the result in case that the reproducing kernels K_z does not have any zero in $\mathbb{D}$, but it is unlikely that this in going to happen in general.

5.4 Small Hankel operators

Another interesting operator is the small Hankel operator $h_{\overline{g}}$ for a given holomorphic symbol g, defined as

$$h_{\overline{g}}f(z) = \int_{\mathbb{D}} (\overline{g}f)(\zeta)\, K_z(\zeta)\, \omega(\zeta)\, dA(\zeta).$$

A good problem for the future is to look at the boundedness of $h_{\overline{g}} : A^2(\omega) \to \overline{A^2(\omega)}$. This problem is equivalent to characterizing the boundedness of the bilinear form $B_g(f,h) = \langle fh, g\rangle_\omega$ acting on $A^2(\omega) \times A^2(\omega)$. It looks that new ideas must be developed in order to attach this problem. In case that one succeeds on finding such a characterization, then one can try to extend the results to $A^p(\omega^{p/2})$.

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