A few things about hyperimaginaries and stable forking

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A FEW THINGS about HYPERIMAGINARIES and STABLE FORKING

(alias *"pas grand' chose sur presque rien"*)

PhD dissertation

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Pas Grand Chose sur Presque Rien

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To my Higher Self.

In memoriam Eric Jaligot — let your Soul resplend his full Light, over there.

Resumen en castellano

Este trabajo consta de varias partes: en la primera parte, se caracteriza una distancia de un espacio métrico. Dada una teoría (de primer orden) completa T, una relación de equivalencia tipo-definible es un tipo parcial sin parámetros $E(x, y)$, donde x e y son tuplas de variables de misma longitud α, tal que en cada modelo M de T, las realizaciones de E en $M^\alpha \times M^\alpha$ son una relación de equivalencia en M^α. Se dice además que E es acotada, si existe un cardinal κ tal que en cada modelo de T, E tiene a lo sumo κ clases. Un ejemplo típico son las relaciones de equivalencia "tener el mismo tipo completo sobre $\emptyset$" : el cardinal κ en este caso es $2^{|T|+|\alpha|}$.

Asociado a una relación de equivalencia tipo-definible acotada, está el conjunto X de sus clases en un modelo suficientemente saturado $\overline{M}$ de T ("suficientemente"=como mínimo, $|x|^+ + \aleph_0$-saturado, donde $|x|$ es la longitud de la tupla x). Resulta que, de la misma manera que el conjunto de los tipos completos $S(\emptyset)$ viene naturalmente con una topología profinita, tambien X viene con una topología, esta vez compacta Hausdorff.

Además, este espacio topológico no depende de la elección del modelo "suficientemente" saturado, es decir que dos tales modelos dan lugar a dos espacios homeomorfos. En otras palabras, el espacio topológico X es un invariante de la teoria T. Y si el lenguaje es numerable, esta topología tiene una base numerable de abiertos.

Ahora, es bien sabido que si un espacio topológico compacto Hausdorff tiene una base numerable de abiertos, entonces es metrizable. Damos de manera explícita esta distancia en un caso particular.

Lascar y Pillay en [4] han sacado provecho de un famoso teorema de estructura de los grupos compactos (el teorema de Peter-Weyl según el cual un grupo compacto Hausdorff es límite inversa de grupos compactos de Lie) para obtener un resultado de eliminación de hiperimaginarios acotados en cualquier teoría completa.

En la segunda parte, se pretende prescindir del teorema de Peter-Weil y probar directamente la eliminación de hiperimaginarios acotados por puros medios de teoría de modelos. No se consigue por completo, pero en el camino se introducen nociones tales como hiperimaginarios normales o DCC, y se prueban algunas de sus propiedades.

Una conjectura famosa dice que cada teoría *simple* (una clase de teorias del primer orden que extiende la de las teorías estables) tiene la *propiedad de la bifurcación estable*. Es decir, que si un tipo completo p sobre A bifurca sobre un subconjunto de B de parámetros, es por culpa de una instancia $\delta(x,a) \in p$ de una formula estable $\varphi(x,y)$.

En la tercera parte, mas técnica, se prueba que si T es simple, T tiene la propiedad de la bifurcación estable si y solo si T^{eq} la tiene.

La última parte es un esbozo de una posible relación entre la teoría de categorías y la teoría de modelos. Por desgracia, solo son especulaciones y la intuición de que deberia funcionar un posible enlace entre ambas, ya que no he conseguido ni siquiera tener los primeros resultados en esta dirección. Sin embargo, si resulta que funciona como lo intuyo, podria ser algo bastante prometedor.

Amen.

Preamble

I tried to keep the material of chapters 1 and 2 as much as possible self-consistent, in the sense that a non-specialist of the field could follow it without being bored. Maybe this is not a good idea: given a randomly chosen mathematician, the more likely is that either he is a specialist in model theory, or he knows nothing about it (even not the very basic things I suppose the reader should know, like compactness, types, etc..). Moreover, this has a cost: diluting the real content for the specialist. For him, let me say right from the start that he will find very few brand new results: the very core of the material presented here is concentrated in 1.2, 2.2, 2.3, 3.2, and maybe 1.4. If you are an expert in model theory and don't want to lose your time, go there. All the rest consists of (sometimes personal) presentation of already known stuff.

As for chapters 3 and 4, the technical prerequisites are more serious.

My heart bleeds not to have been able to present more consistent work. Really. Not because my ego would have been satisfied to be recognized and rewarded by the mathematician community. Rather because finding out something both beautifull and meaningfull would have been some sort of spiritual evolution: I would have known, for myself and by myself, that my Higher Self is able to create freely. Maybe the only thing that would have been more valuable and personal would have been a developpement of the germs presented in chapter 4. I tried, but I didn't succed. Limited brain power ? lack of ambition ? poor self-esteem leading to self-sabotage ? insensibility to the idea of being recognized by the community ? (ambition or need to be recognized by his peers being sometimes powerful and secret motors for great achievements)... I don't know but Oh Gosh ! how I wish I would ! ... maybe one day I'll know ... Anyway the wind blows ...

dans une régate, dans une voile ouverte
j'empiette sur les loques anciennes
régime de l'organisation
applaudit
hutte dans ma liberté de voir la mer
d'arranger mon socle sur l'onde
je révise la toile
je révise surtout l'ébauche de vivacité qu'il faut pour tout permettre allègrement
je musarde
la hutte de toile offre sa magnitude
exercice de sublimation
perturbation au régime
j'étouffe les allergies, je berne les insuffisances
je crée
je grimace, mais je crée
je crée et je me crée
digne poussin de la vengance des arbitres solennels
déguisé en humanoïde, découvre sa réalité
je dérape pour moudre cette arrogance
je crée, et je suis fidèle à la vie.

Preliminaries

My basic conventions and notations are :

I will not use the "bar" notation for tuples : a is meant to be a possibly infinite tuple. If sometimes I need to distinguish between elements and tuples, I will use the bar notation $\overline{a} = (a_i : i \in I)$ in that context. If I is an ordinal α, it is said that a is a tuple of length α, and we note $|a| = \alpha$.

I suppose some very basic knowledge of set theory (ZFC), essentially ordinal and cardinal numbers.

Recall that if one of cardinals κ and λ is infinite, $max\{\kappa, \lambda\} = \kappa + \lambda$, and the right hand side of the equality is the usual notation for it.

If $R(x, y)$ is a binary relation on a set A, an antichain (of length κ) for R is a tuple $(a_i : i \in \kappa)$, where κ is cardinal number, and $\neg R(a_i, a_j)$ for all $i < j < \kappa$. For example, if $R(x, y)$ is an equivalence relation with κ classes, R has an antichain of length κ, but no antichain of length $> \kappa$. If $R(x, y)$ is not reflexive, say $\neg R(a, a)$ for some $a \in A$, then it has antichains of any length for trivial reasons, so we avoid this uninteresting case, and always suppose R to be *reflexive*, when we talk of antichains. Henceforth, antichains are always injective maps as tuples (no repetitions).

Again, if $R(x, y)$ and $S(x, y)$ are binary relations on a set, we denote by $R \circ S$ their composition, i.e. $(R \circ S)(a, b)$ iff there exists c such that $R(a, c)$ and $S(c, b)$. As usual, $R \circ \cdots \circ R$ (n times) is denoted by R^n.

If E is an equivalence relation on a set, I will most often use the notation a_E for the E-class of a, and sometimes $[a]_E$ for typographical reasons.

I will use the same letter (M, N, etc ...) for a structure and its domain.

$|M|$ stands for the cardinal number of the set M.

I will suppose the reader has a minimum knowledge of model theory, by this I mean essentially : the notion of first-order formula, the compactness theorem, the notion of a complete theory, of an elementary extension of a structure, of an elementary map between sets of parameters, and of complete (and partial) types over some parameter set. Please refer to any introductory book in model theory if you are not familiar with those notions and concepts.

The symbols $\models$ and $\vdash$ have the usual meaning ($\Sigma \vdash \psi$ means that the sentence ψ is true in every model of Σ).

As usual, the basic objects we will be interested in are the models of a complete first-order theory T in a language L, with infinite models, along with the definable (or type-definable) sets in them.

$|T|$ is meant to be $|L| + \aleph_0$ (i.e. the cardinal number of the set of first-order formulas in L).

By very definition of a (partial) type and compactness theorem, every partial type over $A \subseteq M \models T$ has a realization in some elementary extension of M.

And by an elementary chain construction and compactness, if two tuples a and b from a model M have the same type, then there is an elementary extension N of M and an automorphism σ of N such that $\sigma(a) = b$.

Among the models of T it is convenient to distinguish some kind of "universal" ones, in the sense that there is no need to pass to an elementary extension to realize (complete) types, or to find an automorphism as above : everything reasonable enough to ask takes place inside the same model.

Obviously, the consistent set of formulas $\{x \neq a : a \in M\}$ has no realization in M, so that the best we can ask is that every complete type over some set of parameters from M *of size less than* $|M|$ is already realized in M: this is called *saturation*.

If for every elementary map f between sets $A, B \subseteq M$ of size less than $|M|$, there exists an automorphism of M that extends f, M is said to be *homogeneous*.

Now it turns out that saturation implies homogeneity, and moreover saturation implies that every model of T of size less than $|M|$ is elementary embedable in M. So clearly, saturated models are the kind of "universal" models we are looking for.

Unfortunately, saturated models of arbitrarily large size need not exist, unless you add axioms in ZFC, or you work with special kinds of theories (for example, ω-stable ones).

If you are willing to add axioms to ZFC, a possible choice is to add "there exist arbitrarily large strongly inaccessible cardinals", which is known to be consistent with ZFC. Indeed, one can see easily with an elementary chain construction that for any theory T and any such cardinal $\kappa > |T|$, there exists a saturated model of T of size κ.

If you want to stick with ZFC, and at the same time work with any kind of theory, you have to weaken a bit the notion of saturated model, and consider the notions of κ-saturation and κ-homogeneity (κ infinite cardinal) : a model M is κ-saturated if every complete type in finitely many variables over a set $A \subseteq M$ of size less than κ is (already) realized in M. And κ-homogeneous if every elementary map between two subsets $A, B \subseteq M$ of size less than κ extends to an automorphism of M. (beware that some authors call this *κ-strongly homogeneous*)

This time, κ-saturation does not implies κ-homogeneity, and we have to obtain both of them at the same time. This is possible in ZFC, for any theory T and any cardinal $\kappa \geq |T|$. The construction is based on the fact that if M is a model and κ is any infinite cardinal, there exists a κ^+-saturated elementary extension

$N \succ M$: to see this, just enumerate the set of all complete types over all subsets of M or size $\leq \kappa$, say $\{p_\alpha : \alpha < \lambda\}$, and construct inductively an elementary chain $(M_\alpha : \alpha < \lambda)$ such that $M_0 = M$, and p_α is realized in $M_{\alpha+1} \succ M_\alpha$ (taking union at limit ordinals). $N = \bigcup_{\alpha<\lambda} M_\alpha$ realizes all complete types over parameter sets $A \subseteq M$ of size $\leq \kappa$. This is the first step. Now using this pattern, again construct inductively an elementary chain $(N_\alpha : \alpha < \kappa^+)$ such that $N_0 = M$, $N_1 = N$, $N_{\alpha+1} \succ N_\alpha$ realizes all complete types over parameter sets $A \subseteq N_\alpha$ of size $\leq \kappa$ (taking union at limit ordinals). Clearly, by regularity of κ^+, $M' = \bigcup_{\alpha<\kappa^+} M_\alpha$ is κ^+-saturated, $M' \succ M$, and we are done.

M' need not be κ^+-homogeneous, so we have to perform another elementary chain kind of construction to get κ^+-homogeneity, without loosing κ^+-saturation. This is done as follows :

Construct inductively an elementary chain $(M_\alpha : \alpha < \kappa^+)$ such that $M_0 = M$, and $M_{\alpha+1} \succ M_\alpha$ is $|M_\alpha|^+$-saturated (taking union at limit ordinals). Again, because κ^+ is regular,

$$\overline{M} = \bigcup_{\alpha<\kappa^+} M_\alpha \qquad (*)$$

is κ^+-saturated. To see κ^+-homogeneity, read for example [8], lemma 2.1.1. (modulo the tipos in it ...). Since $\overline{M} \succ M$ is κ^+-saturated and κ^+ homogeneous, it is also obviously κ-saturated and κ-homogeneous, and we have proved the original claim.

A κ-saturated and κ-homogeneous model of T is traditionaly called a (κ)-*monster model*, and I will denote such model by $\overline{M}, \overline{N}$, etc...

Of course, since there are models of T of any arbitrarily large size, to fix a κ-monster model is not enough, and you have to deal with the family of κ-monster models, κ ranging through arbitrarily large cardinals : therefore, a sentence like "let T, with monster model $\overline{M}$", rather means "consider generically any one of those κ-monster models".

If you does not feel comfortable with this dependence on κ, and would rather like to have a unique "monster model" for T, there is something for you, but you have to accept of course that this "monster model" be not a set (since you want among other things that every model of T embeds in it, and there are always arbitrarily big models in T).

One possible framework to handle both sets and objects which are not sets (for example, collections of sets defined by a first-order formula in ZFC) is the well-known NBG set theory, where there are two sorts of objects : one sort for the "sets", and the other one for the so-called "proper classes".

In this context, if in $(*)$ you let run α over the (proper class) On of all ordinals, instead of bounding it by κ^+, you get in the end a proper class

$$\mathfrak{C} = \bigcup_{\alpha \in On} M_\alpha$$

with the following features : every complete type over any set $A \subseteq \mathfrak{C}$ of parameters is realized in $\mathfrak{C}$ (which implies that every model of T embeds in $\mathfrak{C}$), and every elementary map between sets $A, B \subseteq \mathfrak{C}$ extends to an automorphism of $\mathfrak{C}$: this is the unique "universal domain" you are looking for.

If you choose to strictly stay inside ZFC, and use κ-monster models $\overline{M}$, a subset of $\overline{M}$ of size less than κ is called a "small set" (likewise, a tuple of length less than κ is called a "small tuple").

If you prefer to use the proper class $\mathfrak{C}$, there is no need to introduce new vocabulary : we already have the words "set" and "proper class" at our disposal from the framework we work in.

I tend to be quite a bit schizophrenic in the very beginning of chapter 1, by not choosing any of these two options for the "monster model". This is to show the unexperimented reader how it works in both cases, and that these are essentially the same things, up to the difference of nature (set or proper class). Very soon I get better mental health, and keep my position with $\overline{M}$.

An induction on the length of tuples shows that in a κ-saturated model, not only complete types in finitely variables (over parameter sets of size less than κ) are realized, but also up to types in tuples of variables of length $\leq \kappa$ (over the same kind of parameters).

A standard notation for $\overline{M} \models \varphi(a_1, \ldots, a_n)$ or $\mathfrak{C} \models \varphi(a_1, \ldots, a_n)$ is simply $\models \varphi(a_1, \ldots, a_n)$.

If $M \models T$ and $\varphi(x_1, \ldots, x_n)$ is a formula with parameters from M, $\varphi(M)$ stands for $\{(a_1, \ldots, a_n) \in M^n : M \models \varphi(a_1, \ldots, a_n)\}$, and the same for a set of formulas instead of a single formula.

If the complete theory T is clear from the context, and $\Sigma(x), \Pi(x)$ are sets of formulas (with parameters A in some model of T), just write $\Sigma(x) \vdash \Pi(x)$ in place of $T(A) \cup \Sigma(x) \vdash \Pi(x)$. Adding to the original language L a constant $\bar{a}$ for each $a \in A$, along with a constant c_i for each variable x_i appearing in the tuple $x = (x_i : i \in I)$, this clearly means that in every model N of $T(A)$, $\Sigma(N) \subseteq \Pi(N)$.

One easily check that $T(A) \cup \Sigma(x) \vdash \Pi(x)$ iff $\Sigma(\overline{N}) \subseteq \Pi(\overline{N})$ for some κ-saturated model of $T(A)$, with $|x| < \kappa$.

If $A \subseteq M \models T$, a $(|A| + |x|)^+$-saturated extension of M (in the language L) is a $|x|^+$-saturated model of $T(A)$, so that we can use such model as a test for $\Sigma(x) \vdash \Pi(x)$ as above : this is another interesting feature of κ-saturated models, and the key to fill in the proofs for the first propositions of chapter 1 (making $A = \emptyset$).

Another interesting feature of a κ-saturated model is that the projection of a type-definable subset still is type-definable : specifically, if $\Phi(x, y)$ is a partial type with parameters A in a κ-saturated model $\overline{M}$, and if $|x|, |y|, |A| < \kappa$, then the projection set $X = \{x \in \overline{M}^{|x|} \mid$ there exists y $\in \overline{M}^{|y|}$ $s.t.$ $\models \Phi(x, y)\}$ is type-definable over A by the partial type $\Xi(x) = \{\exists y\big(\varphi_1(x, y) \wedge \cdots \wedge \varphi_n(x, y)\big) \mid \varphi_i(x, y) \in \Phi(x, y), 1 \leq i \leq n\}$. $\Xi(\overline{M}) \supseteq X$ is trivial, and true in any model, whereas $\Xi(\overline{M}) \subseteq X$ is easy to prove using κ-saturation.

Chapter 1

Bounded type-definable equivalence relations

1.1 Introduction

Here nothing is new, I just recall basic known facts and fix terminology and notations.

1.1.1 Type-definable equivalence relations

Let $E(x, y)$ be a partial type over $\emptyset$, with x, y tuples of variables of the same lenght α (possibly infinite). The following are equivalent :

1. E defines an equivalence relation in M^α, for every model M of T.
2. $\forall \lambda > |x| + \aleph_0$, E defines an equivalence relation in every λ-saturated model of T.
3. E defines an equivalence relation in some (any) $(|x|^+ + \aleph_0)$-saturated model of T.
4. E defines an equivalence relation in $\mathfrak{C}$

We say that E is a *type-definable equivalence relation* (in T), and call

$\pi : \mathfrak{C}^\alpha \longrightarrow \mathfrak{C}^\alpha/E$ the canonical projection.

Observe that if E is a 0-*definable* equivalence relation, 1. is equivalent to : E defines an equivalence relation in some (arbitrary) model of T (in contrast with 3.).

For example, having the same type over $\emptyset$ is a type-definable equivalence relation, letting $E(x, y) = \{\varphi(x) \leftrightarrow \varphi(y) : \varphi \in L\}$. Observe in this case that on each model, the number of classes is bounded by the number of types over $\emptyset$, which in turn is bounded by $2^{|T|+|x|}$. More generally, the following are equivalent for a type-definable equivalence relation E:

1. For some κ, the number of E-classes in every model of T is $\leq \kappa$.
2. There is some $\mu > |x| + \aleph_0$ s.t. for every$\lambda \geq \mu$, every λ-saturated model of T has $< \lambda$ classes.
3. There is some $\mu > |x| + \aleph_0$ s.t. some (any) μ-saturated model of T has $< \mu$ classes.
4. E has a small number (i.e. a set) of classes in $\mathfrak{C}$.

We say that E is a *bounded* type-definable equivalence relation.

These are immediate results. With some more work, one can show that there is a least bounded type-definable equivalence relation in $\mathfrak{C}^\alpha$, and it has $\leq 2^{|T|+|x|}$ classes; therefore, one can specify what μ in item 2. above is, and item 1. is also equivalent to :

2'. For every $\lambda \geq (2^{|T|+|x|})^+$, every λ-saturated model of T has $< \lambda$ classes.

3'. Any $(2^{|T|+|x|})^+$-saturated model of T has $\leq 2^{|T|+|x|}$ classes.

4'. E has $\leq 2^{|T|+|x|}$ classes in $\mathfrak{C}$.

The morality is : to check condition 1. above, just check that E has at most $2^{|T|+|x|}$ classes in some λ-saturated (and λ-homogeneous) model $\overline{M}$, $\lambda > 2^{|T|+|x|}$. Or, if you are willing to work with proper classes, check that E has at most $2^{|T|+|x|}$ classes in $\mathfrak{C}$.

Again, compare with the case of a 0-*definable* equivalence relation, where it is enough to check that it has a *finite* number of classes in some *arbitrary* model : then, it has the same finite number of classes in every model (just because having n classes is expressible by a first order sentence, and T is complete).

Notation 1.1.1 *The least type-definable equivalence relation (on tuples of a certain length) is denoted by E_{KP}. So there is one E_{KP} for each length of tuple from the monster model.*

Certainly if E is bounded, no $\varphi(x, y) \in E$ can have antichains of arbitrary finite lenght, for if not φ would have antichains of arbitrary infinite lenght by compactness, and so would have E, contradicting boundedness.

A reflexive formula $\varphi(x, y)$ is *thick* if for some $n < \omega$, φ has no antichain of lenght n, so that by the previous remark a bounded type definable equivalence relation is made of thick formulas. The converse is true, using Erdös-Rado theorem :

Let E be a type-definable equivalence relation; then E is bounded iff each $\varphi \in E$ is thick.

Observe that we can always assume that a type-definable equivalence relation is made of reflexive and symmetric formulas : if not, replace each $\varphi(x,y) \in E$ by $\varphi(x,y) \wedge \varphi(y,x)$.

It is not true that moreover the formulas can be assumed to be transitive, ie that every type-definable equivalence relation is an intersection of $\emptyset$-definable equivalence relations, but however we have the following approaching result :

Lemma 1.1.2 *Every type-definable equivalence relation $E(x,y)$ is the intersection of type-definable equivalence relations E_i of the following type :*

$E_i = \{\varphi_n^i(x,y) : n < \omega\}$*, where each* φ_n^i *is reflexive and symmetric, and* $(\varphi_{n+1}^i)^2 \vdash \varphi_n^i$.

Moreover, if E is made of a countable number of formulas, it is directly of the previous form.

Proof : Start from a representation of E by reflexive and symmetric formulas. Let $\varphi(x,y) \in E(x,y)$; using transitivity of E we get $E(x,y) \cup E(y,z) \vdash \varphi(x,z)$, and by compactness $\varphi_1(x,y) \wedge \varphi_1(y,z) \vdash \varphi(x,z)$, for some φ_1 finite conjunction of formulas of E. Repeating inductively ω times, we get a type-definable equivalence relation of the expected form, and the intersection of all of them, φ running through E, is E itself.

Now suppose we start with E countable, $E(x,y) = \{\theta_n(x,y) : n < \omega\}$; replacing θ_n by $\theta_0 \wedge \cdots \wedge \theta_n$, we can first of all assume $\theta_{n+1} \vdash \theta_n$, for all n; as before, and because a finite conjunction of formulas of E is one of them by the previous decreasing condition on θ_n', $\theta_{n_1}^2 \vdash \theta_0$ for some $n_1 \geq 0$; and we can assume $n_1 > 0$ because if $\theta_0^2 \vdash \theta_0$, then also $\theta_1^2 \vdash \theta_0$ thanks to $\theta_1 \vdash \theta_0$. Proceeding inductively, we get a strictly increasing sequence $n_i, i < \omega$, such that $\theta_{n_{i+1}}^2 \vdash \theta_{n_i}$, and thereby the result. ■

1.1.2 The logic topology

In a κ-saturated model, $\kappa \geq |x| + \aleph_0$, the set of classes of $E(x,y) =$ "$x \equiv y$" is in one-one correpondence with $S_x(\emptyset)$, the set of types over $\emptyset$ in variables x. Under this correspondence, the classical profinite topology of $S_x(\emptyset)$ translates as follows :

A set of classes Y is closed iff $\pi^{-1}(Y)$ is type-definable over $\emptyset$, and a base of clopen is given by $\{[\varphi] : \varphi \in L\}$, where $[\varphi] = \{a_E : \forall a' \ s.t. \ E(a,a'), \models \varphi(a')\}$.

More generally, one can endow the set of classes $X = \mathfrak{C}^\alpha/E$ of a bounded type-definable equivalence relation E with a compact Hausdorff topology, the so called *logic topology*, but if we want to get Hausdorffness we have to introduce parameters in the types :

by definition, a set of classes $Y \subseteq X$ is *closed* if $\pi^{-1}(Y)$ is type-definable *with (set) parameters* in $\mathfrak{C}$.

Since E is itself type-definable, a subset $Y \subseteq X$ is closed iff $Y = \pi(\Sigma(\mathfrak{C}))$, for some partial type with parameters $\Sigma(x)$.

Lemma 1.1.3 *This defines a compact Hausdorff topology on the set X of classes.*

Proof : The intersection of an arbitrary family of closed sets is obviously a closed set, so this defines a topology.

To check compactness we need to get the finite intersection property for closed sets, which is an immediate consequence of logical compactness.

Hausdorffness is as follows : let $a_E \neq b_E$; for some $\varphi \in E$, $\models \neg\varphi(a,b)$; as in the proof of previous lemma , $\varphi'^2 \vdash \varphi$ $(*)$ for some $\varphi' \in E$. Let $\mathcal{O}_1 = \{c_E : c_E \subseteq \varphi'(a, \mathfrak{C})\}$ and $\mathcal{O}_2 = \{c_E : c_E \subseteq \varphi'(b, \mathfrak{C})\}$; then, $\pi^{-1}(X \setminus \mathcal{O}_1)$ is the partial type $\exists y(E(x,y) \wedge \neg\varphi'(a,y))$, so that $\mathcal{O}_1$ is an open set, and so is $\mathcal{O}_2$ (here we see why we need types with parameters to get Hausdorffness); obviously, $a_E \in \mathcal{O}_1$, $b_E \in \mathcal{O}_2$, and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ thanks to $(*)$. ■

As in the case of $x \equiv y$, we have a base of open sets for the logic topology, given by formulas (with parameters) :

Lemma 1.1.4 *The family of sets of the form*

$$[\psi(x)] = \{a_E : \forall a' \ s.t.\ E(a,a'), \models \psi(a')\}$$

where $\psi(x)$ is a formula with parameters in $\mathfrak{C}$, is a base for the logic topology.

Proof : First of all, $X \setminus\ [\psi]$ is closed, as in the proof of previous lemma.

Let $\mathcal{O}$ be an open set, $\pi(x)$ a partial type defining $X \setminus \mathcal{O}$, and $a_E \in \mathcal{O}$; suppose for each $\chi(x) \in \pi(x)$, some representative c' of c_E satisfies $\models \chi(c')$; then, by (logical) compactness, some representative c' of c_E satisfies $\models \pi(c')$, but by definition of closed sets in X, $a_E \subseteq \pi(\mathfrak{C})$, in contradiction with $a_E \in \mathcal{O}$. Thus, let $\chi(x) \in \pi(x)$ such that $a_E \subseteq \neg\chi(\mathfrak{C})$; $a_E \in [\neg\chi]$, and we are done. ■

We can be even more precise about a base of open sets, and choose the formula $\psi(x)$ of the form $\varphi(x,a)$ for $\varphi(x,y) \in E(x,y)$, but before another definition :

For $\varphi(x,y) \in E(x,y)$, let

$$[[\varphi(x,a_E)]] = \{c_E : \forall a', c' \ s.t.\ E(a,a'), E(c,c'), \models \varphi(c',a')\}$$

(clearly an open set in X), so that $[[\varphi(x,a_E)]] \subseteq [\varphi(x,a)]$ $(**)$.

Lemma 1.1.5 *For each class $a_E \in X$, $\{[\varphi(x,a)] : \varphi \in E\}$ is a base of (open) neighbourhoods of a_E, and likewise $\{[[\varphi(x,a_E)]] : \varphi \in E\}$.*

Proof : Clearly $[\varphi(x,a)]$ and $[[\varphi(x,a_E)]]$ are open sets containing a_E. Let $\mathcal{O}$ be an open set containing a_E, and $[\psi] \subseteq \mathcal{O}$ containing a_E by lemma 1.1.4; by hypothesis, $E(x,a) \vdash \psi(x)$, so by (logical) compactness $\varphi(x,a) \vdash \psi(x)$, for some $\varphi \in E$, which means precisely $[\varphi(x,a)] \subseteq [\psi]$.

The corresponding result for $[[\varphi(x,a_E)]]$ is just because $(**)$. ■

We note respectively $\langle\varphi(x,a)\rangle$ and $\langle\langle\varphi(x,a_E)\rangle\rangle$ for the closed sets $X \setminus [\varphi(x,a)]$ and $X \setminus [[\varphi(x,a_E)]]$.

This topological space $X = \overline{M}/E$ is in fact an *invariant* of the complete theory T: if $\overline{N}$ is another λ-saturated model of T, $\lambda > 2^{|T|+|x|}$, $\overline{M}/E$ is homeomorphic to $\overline{N}/E$.

To see this, first note that it is enough to prove it when $\overline{N}$ is an elementary extension of $\overline{M}$; in this case, every E-class in $\overline{N}$ has a representative in $\overline{M}$: suppose not, and let $(a_i : i \in I)$ be a complete set of representatives of E-classes in $\overline{M}$; there would be $b \in \overline{N}^{|x|}$, and a familly $(\varphi_i(x,y) : i \in I)$ of formulas in $E(x,y)$ such that $\overline{N} \models \neg\varphi_i(b,a_i)$, for all $i \in I$.

Then, $\{\neg\varphi(x,a_i) : i \in I\}$ would be a partial type over a set of parameters of size less than the saturation of $\overline{M}$ (only a finite sub-tuple of each a_i appears in φ_i, and I is of size less than the saturation of $\overline{M}$ by hypothesis of boundedness) , and so would be realized in $\overline{M}$: contradiction with the fact that $(a_i : i \in I)$ is a complete set of representatives of E-classes in $\overline{M}$.

The conclusion is that the map $f\colon \overline{M}/E \to \overline{N}/E$, $a_E^{\overline{M}} \mapsto a_E^{\overline{N}}$, is a bijection.

This bijection is an homeomorphism : if $[a]_E \in \overline{M}/E$ belongs to the basic open set $[\psi(x)]$, $[a]_E \subseteq \psi(\overline{M})$, which means $E(x,a) \vdash \psi(x)$ because $\overline{M}$ is enough saturated, and so $[a]_E \subseteq \psi(\overline{N})$ also, ie $[a]_E$ also belongs to $[\psi(x)]$ as an element of $\overline{N}/E$.

The same argument using the saturation of $\overline{N}$ proves that $f([\psi(x)]) = [\psi(x)]$, so that f^{-1} is continuous, and even an homeomorphism since the spaces are compact Hausdorff.

Remark 1.1.6 *A slight modification of the arguments above gives a compact Hausdorff topology on the set of classes of a 0-type-definable equivalence relation $E(x,y)$ on the realizations of a partial type (without parameters) $\Sigma(x)$ in $\mathfrak{C}$.*

Lemmas 1.1.4 and 1.1.5 go through, as well as the fact that this topological space is an invariant of T in the sense given above.

1.1.3 The countable case

From now on (except in section 3), we assume moreover that $E(x, y)$ is made of a countable number of formulas (which is obviously the case if L and $|x|$ are countable). By Lemma 1.1.2, we can assume $E(x, y) = \{\theta_n(x, y) : n < \omega\}$, with θ_n reflexive and symmetric, and $\theta_{n+1}^2 \vdash \theta_n$ (in particular, $\theta_{n+1} \vdash \theta_n$).

In that case we know more about the logic topology :

Lemma 1.1.7 *The logic topology has a countable base of open sets.*

Proof : For fixed n, use topological compactness to get a finite covering $X = [\theta_n(x, a_0^n)] \cup \cdots \cup [\theta_n(x, a_{k_n}^n)]$ from the open covering $\bigcup_{a_E \in X} [\theta_n(x, a)]$.

Let's check that $\{[\theta_n(x, a_i^n)] : n < \omega, 0 \leq i \leq k_n\}$ is a base of open sets.

Let $\mathcal{O}$ be an open set, and $b_E \in \mathcal{O}$; by Lemma 1.1.5, there is some $n < \omega$ such that $b_E \in [\theta_n(x, b)] \subseteq \mathcal{O}$; let a_i^{n+1} such that $b_E \in [\theta_{n+1}(x, a_i^{n+1})]$ (finite covering); easily, $[\theta_{n+1}(x, a_i^{n+1})] \subseteq [\theta_n(x, b)]$, and we are done. ■

A well known result of topology says that a compact Hausdorff space is metrizable iff it has a countable base of open sets (an easy corollary of the so-called metrization theorem of Urysohn, see [6] for example).

I will caracterize such a distance model-theoretically, at least for some particular case.

1.2 The distance

The context is that of 1.1.3. Refer to preliminaries if you don't know what a thick formula is, and recall (c.f. discussion after 1.1.1) that a type-definable equivalence relation is bounded iff it consists of thick formulas.

In the sequel I will use the following abuse of notation : $\models \varphi(a_E, b_E)$ stands for $\models \varphi(a', b')$, for every representatives a', b' of a_E, b_E respectively. Other expressions like $\models \varphi(a, b_E)$ are self-explicit.

Lemma 1.2.1 *Let θ be a thick formula, and $k \geq 2$ such that there is an antichain for of length k for θ, but no antichain of length $k + 1$.*

Then the transitive closure of θ is θ^{2k+1}.

Proof : I will prove $\theta^{2(k+1)} \vdash \theta^{k+1}$, which is what we need to prove. Let $\models \theta^{2(k+1)}(a, b)$, and let l be the smallest integer such that $\theta^l(a, b)$.

Suppose $l > 2k + 1$, and $a_0, \ldots, a_{2k+2}, \ldots, a_l$ such that $\models \theta(a_i, a_{i+1})$ for all $0 \leq i \leq l - 1$. The property of k, applied to the sequence $a_0, a_2, \ldots, a_{2(k+1)}$, implies that for some $0 \leq i < j \leq k+1$, $\models \theta(a_{2i}, a_{2j})$, contradicting the minimality condition of l.

Therefore, $l \leq 2k+1$. ■

Observe that in the previous Lemma, if θ has no antichain at all, then $\models \forall x, y\ \theta(x,y)$, and θ is already an equivalence relation, so that the transitive closure of θ is θ itself.

Altogether, this means that if $\theta(x,y)$ is thick, some θ^i is an equivalence relation. Let $r \geq 1$ be the smallest integer j such that $\theta^j = \theta^{j+1}$. Call this integer r_n for θ_n.

We can assume the $r_n's$ are not bounded : if not, $r_n \leq r$ fore some r, and therefore by compactness $\bigcap_n \theta_n^r \subseteq E^r = E$. But obviously on the other hand $E \subseteq \theta_n \subseteq \theta_n^{r_n} = \theta_n^r$ for all n, so that $E \subseteq \bigcap_n \theta_n^r$. Therefore, $E = \bigcap_n \theta_n^r$, and E is the intersection of definable equivalence relations.

But that case is easily solved :

Lemma 1.2.2 *Let $E(x,y) = \bigwedge_n \chi_n(x,y)$, where each $\chi_n(x,y)$ is a 0-definable equivalence relation.*

Then, the map

$$\begin{cases} d(a_E, b_E) = 0 \text{ if } a_E = b_E \\ d(a_E, b_E) = 1/2^n \text{ if } a_E \neq b_E \text{ and } n \text{ is the least integer } k \text{ such that } \models \chi_k(a_E, b_E)\ (\min \emptyset = -1) \end{cases}$$

defines an ultrametric on $X = \overline{M}^\alpha / E$, whose topology is the logic topology.

Proof : Taking $\chi'_n = \chi_0 \wedge \cdots \wedge \chi_n$, we can assume the χ_n are definable equivalence relations with $\chi_{n+1} \vdash \chi_n$.

Symmetry of d is obvious, since each χ_n is symmetric.

Since by definition $d(a_E, b_E) = 0$ if $a_E = b_E$, it is clearly enough to prove the ultrametric inequality for pairwise distinct a_E, b_E, c_E.

Then, let $d(a_E, b_E) = \frac{1}{2^n}$ and $d(b_E, c_E) = \frac{1}{2^m}$. If $n = -1$ (i.e. $\models \neg\chi_0(a', b')$ for some representatives a', b' of a_E, b_E) or $m = -1$, clearly we have $d(a_E, c_E) \leq max\{d(a_E, b_E), d(b_E, c_E)\}$ since d is bounded by 2 by definition.

We can assume therefore that $\models \chi_n(a_E, b_E)$ and $\models \chi_m(a_E, b_E)$, with $n \geq m$, so that $\models \chi_n(a_E, b_E)$ and $\models \chi_n(b_E, c_E)$. Since χ is transitive, this implies $\chi_n(a_E, c_E)$, i.e. $d(a_E, c_E) \leq \frac{1}{2^n} = max\{d(a_E, b_E), d(b_E, c_E)\}$.

Finally, if $a_E \neq b_E$, $\models \neg\chi_n(a', b')$ for some representative a', b' of a_E, b_E, and therefore $d(a_E, b_E) \neq 0$.

Clearly we have for each $n \geq 0$ and a, b : $\models \chi_{n+1}(a_E, b_E)$ iff $d(a_E, b_E) < \frac{1}{2^n}$. Therefore, $[[\chi_{n+1}(x,a)]] = B_d(a_E, \frac{1}{2^n})$.

Since the open balls $\{B_d(a_E, \frac{1}{2^n}) : n \geq 0\}$ and the open sets $\{[[\chi_n(x,a)]] : n \geq 0\}$ form a neighborhood basis of a_E for respectively the topology of d and the logic topology, these two topologies agree. ■

The remaining of the section treats the case where the $r_n's$ are not bounded. In that case, some subsequence $(r_{f(n)})$ is strictly increasing, and clearly still we have $E = \bigcap_n \theta_{f(n)}$. Therefore, we can suppose that the $r_n's$ are strictly increasing.

Our goal is to define in some way or another a distance (or distances) on the set of equivalence classes under E.

Observe first that if one defines a map $\delta : X \times X \to \mathbb{R}^+$ (X any set), that is symmetric and fullfils the triangular inequality, then the map :

$$\begin{cases} d(x,x) = 0 \\ d(x,y) = \delta(x,y) \text{ if x} \neq y \end{cases}$$

is a semi-distance on the set X.

Now let us fix our attention on some level n, and define δ_n as follows :

$$\begin{cases} \delta_n(a_E, b_E) = min\{i : \models \theta_n^i(a_E, b_E)\}, \text{ if this set is not empty} \\ \delta_n(a_E, b_E) = r_n + 1 \text{ if for some (all) representatives } a', b' \text{ of } a_E, b_E, \models \neg\theta_n^{r_n}(a', b') \end{cases}$$

The above equivalence between“for all” and “for some” is just because $\theta_n^{r_n}$ is an equivalence relation, and $E \subseteq \theta_n \subseteq \theta_n^{r_n}$.

Observe that the values of δ_n are in the finite set $\{1, 2, 3, ..., r_n + 1\}$.

Now we obviously have (**1**) :

– $\models \theta_n^i(a_E, b_E)$ implies $\delta_n(a_E, b_E) \leq i$.

– If $1 \leq i \leq r_n$: $\models \neg\theta_n^i(a', b')$ for some representaives a', b' of a_E, b_E implies $\delta_n(a_E, b_E) > i$

In particular, if $1 \leq i \leq r_n$: $\delta_n(a_E, b_E) \leq i \Longleftrightarrow \theta_n^i(a_E, b_E)$.

From this we get triangular inequality for δ_n :

Suppose first $\delta_n(a_E, b_E) = r$, $\delta_n(b_E, c_E) = s$, with both $r, s \leq r_n$, and let a', c' be representatives of a_E, c_E. Then $\models \theta_n^r(a,' b)$ and $\models \theta_n^s(b, c')$, hence $\models \theta_n^{r+s}(a', c')$, and therefore $\delta_n(a_E, c_E) \leq r + s$.

Now suppose $\delta_n(a_E, b_E) = r_n + 1$. Then $\delta_n(a_E, c_E) \leq r_n + 1 \leq (r_n + 1) + \delta_n(b_E, c_E) = \delta_n(a_E, b_E) + \delta_n(b_E, c_E)$.

Symmetry for δ_n is obvious, since θ_n, and therefore also each θ_n^i, is symmetric .

Let $\delta'_n = \dfrac{\delta_n}{2^n}$.

Note that for all a, $\delta'_n(a_E, a_E) = \frac{1}{2^n}$, but thanks to a previous observation, we can make δ'_n into a distance d_n.

Now what are the relations between the d_n's ?

To see this we use the fact that $\theta^2_{n+1} \vdash \theta_n$, which gives $\theta^{2i}_{n+1} \vdash \theta^i_n$, for all i.

Using (**1**), it implies immediately :

$$\text{If } 1 \leq 2i \leq r_{n+1} \text{ and } d_{n+1}(a_E, b_E) \leq \frac{i}{2^n} = \frac{2i}{2^{n+1}}, \text{ then } d_n(a_E, b_E) \leq \frac{i}{2^n} \qquad (\mathbf{2})$$

This in turn implies that, on a fixed couple (a_E, b_E), the $d_i's$ cannot decrease a lot when passing from level n to level $n+1$:

Lemma 1.2.3 *For every a, b and every integer n*

$$d_{n+1}(a_E, b_E) \geq d_n(a_E, b_E) - \frac{1}{2^{n+1}}$$

Proof : Since the $r_n's$ are strictly increasing, a subsequence $(r_{f(n)})$ can be extracted with the property $r_{f(n+1)} \geq 2r_{f(n)}$. Henceforth, since still $E = \bigcap_n \theta_{f(n)}$, we can assume that

$$r_{n+1} \geq 2r_n \qquad (\mathbf{3})$$

Suppose $d_n(a_E, b_E) = \frac{i}{2^n}$, with $i \geq 2$. Then $d_n(a_E, b_E) > \frac{i-1}{2^n}$, but thanks to (**3**), $1 \leq 2(i-1) \leq r_{n+1}$, so we can apply (**2**) to get $d_{n+1}(a_E, b_E) > \frac{i-1}{2^n}$.

But between $\frac{i-1}{2^n}$ and $\frac{i}{2^n}$ there is only one intermediate value of d_{n+1}, so that $d_{n+1}(a_E, b_E) \geq d_n(a_E, b_E) - \frac{1}{2^{n+1}}$.

The case $d_n(a_E, b_E) = \frac{1}{2^n}$ is obvious, since d_{n+1} takes its values in $\{\frac{1}{2^{n+1}}, \ldots, \frac{r_{n+1}+1}{2^{n+1}}\}$. ∎

Corollary 1.2.4 *For every a, b and every integers n, k*

$$d_{n+k}(a_E, b_E) \geq d_n(a_E, b_E) - \frac{1}{2^n}\big(1 - \frac{1}{2^k}\big)$$

Proof : An iteration of Lemma 1.2.3 gives

$$d_n(a_E, b_E) - d_{n+k}(a_E, b_E) \leq \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+k}} = \frac{1}{2^n}\big(1 - \frac{1}{2^k}\big)$$

. ∎

Proposition 1.2.5 *For fixed a, b, the sequence $(d_n(a_E, b_E))_{n<\omega}$ is convergent in $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.*

Proof : To simplify notation call the sequence $(d_n)_{n<\omega}$.

I will prove that two adherent values of (d_n) in $\overline{\mathbb{R}}$ must be equal, which will clearly prove that the sequence converges.

Suppose $c < c' \in \mathbb{R}$ are such adherent values, and let $(d_{f(n)})$ and $(d_{g(n)})$ be subsequences converging to c and c' respectively.

Let $\varepsilon = \frac{c'-c}{2} > 0$, and n_0 such that for all $n \geq n_o$, $d_{f(n)}$ belongs to the intervall of center c' and radius $\frac{\varepsilon}{2}$.

Let N be such that $\frac{1}{2^N} \leq \frac{\varepsilon}{2}$, and n_1 with $N - 1 \leq g(n_1)$, so that if $k = max\ (n_0, n_1)$, we get $d_{g(k)} > c' - \frac{\varepsilon}{2}$ and $N \leq g(k) + 1$.

Now we have $\dfrac{1}{2^{g(k)}} \leq \dfrac{1}{2^N} \leq \dfrac{\varepsilon}{2}$, and applying Corollary 1.2.4 we get

$$d_{g(k)+r} \geq d_{g(k)} - \frac{1}{2^{g(k)}}\left(1 - \frac{1}{2^r}\right)$$

and since $\dfrac{1}{2^{g(k)}}\left(1 - \dfrac{1}{2^r}\right) \leq \dfrac{\varepsilon}{2}$ and $d_{g(k)} > c' - \dfrac{\epsilon}{2}$, we obtain

$$d_{g(k)+r} > c' - \frac{\epsilon}{2} - \frac{\epsilon}{2} = c' - \varepsilon > c \text{ for all } r \geq 0$$

contradicting the fact that $(d_{f(n)})$ converges to c.

The remaining case $c' = +\infty$ is treated the same way without difficulties. ∎

Proposition 1.2.6 *$d'(a_E, b_E) = lim_n d_n(a_E, b_E)$ is a distance with values in $\mathbb{R} \cup \{+\infty\}$*

Proof : Since each d_n has symmetry and triangular inequality, passing to the limit gives symmetry and triangular inequality for d'.

If $a_E = b_E$, $d_n(a_E, b_E) = 0$ for all n, hence $d'(a_E, b_E) = 0$.

If $a_E \neq b_E$, $\models \neg\theta_n(a, b)$ for some integer n, so that $d_n(a_E, b_E) \geq \frac{2}{2^n}$. Applying Corollary 1.2.4, this implies $d_{n+k}(a_E, b_E) \geq \frac{1}{2^n}$ for all $k \geq 0$, and therefore $d'(a_E, b_E) = lim_n d_n(a_E, b_E) = lim_k d_{n+k}(a_E, b_E) \geq \frac{1}{2^n} > 0$. ∎

If we define $d = min\ (1, d')$, we get a real-valued distance, whose topology τ is the same than that of d'.

Proposition 1.2.7 *For every n and a, $B_d(a_E, \frac{1}{2^n}) \subseteq [\theta_n(x, a)]$. So the logic topology is coarser than τ.*

Proof : Let $b_E \in B_d(a_E, \frac{1}{2^n})$, i.e. $lim_n d_n(a_E, b_E) < \frac{1}{2^n}$. Then there exists some N such that $d_k(a_E, b_E) < \frac{1}{2^n}$ for all $k \geq N$ $(*)$.

Suppose $b_E \notin [\theta_n(x,a)]$, i.e. $\models \neg\theta_n(a,b')$ for some representative b' of b_E. This implies that $d_n(a_E,b_E) \geq \frac{2}{2^n} = \frac{1}{2^{n-1}}$, and applying again Corollary 1.2.4, we see that $d_k(a_E,b_E) > \frac{1}{2^n}$ for all $k \geq n$, contradicting $(*)$.

Since the open balls $\{B_d(a_E,\frac{1}{2^n})\ :\ n \geq 0\}$ form a neighborhood basis of each point a_E for τ, and the open sets $\{[\theta_n(x,a)]\ :\ n \geq 0\}$ also form a neighborhood basis of each point a_E for the logic topology (c.f. 1.1.2), we get that the logic toplogy is coarser then τ. ■

As we know, the type of any type-definable equivalence relation can be put into the form $\bigwedge_n \theta_n(x,y)$, where each θ_n is symmetric and reflexive, and $\theta_{n+1}^2 \vdash \theta_n$ for all $n \geq 0$.

If moreover $\theta_n \vdash \theta_{n+1}^2$ (i.e. $\theta_{n+1}^2 \equiv \theta_n$) for all n, then the topology τ of the distance d defined above is the logic topology :

Proposition 1.2.8 *Suppose $\theta_{n+1}^2 \equiv \theta_n$ for all $n \geq 0$.*

Then, for every n and a, $[[\theta_{n+1}(x,a)]] \subseteq B_d(a_E,\frac{1}{2^n})$. So the logic topology is finer than τ.

Proof : First I claim that for each a_E, b_E, the sequence $(d_n(a_E,b_E))$ is decreasing : indeed, this is obvious if $a_E = b_E$ (the sequence is constant $= 0$), and if $a_E \neq b_E$, let $d_n(a_E,b_E) = \frac{i}{2^n}$. In particular, $\models \theta_n^i(a_E,b_E)$, and by the hypothesis we get $\models \theta_{n+1}^{2i}(a_E,b_E)$, which implies $d_{n+1}(a_E,b_E) \leq \frac{2i}{2^{n+1}} = \frac{i}{2^n} = d_n(a_E,b_E)$.

Let $b_E \in [[\theta_{n+1}(x,a)]]$, i.e. $\models \theta_{n+1}(a_E,b_E)$, and $b_E \neq a_E$. Then, $d_n(a_E,b_E) = \frac{1}{2^{n+1}}$, and since $(d_n(a_E,b_E))$ is decreasing, $lim_n d_n(a_E,b_E) = d'(a_E,b_E) \leq \frac{1}{2^{n+1}} < \frac{1}{2^n}$. And since $d'(a_E,b_E) \leq 1$, $d(a_E,b_E) = inf(d'(a_E,b_E),1) = d'(a_E,b_E) < \frac{1}{2^n}$, and we are done. ■

All the examples of section 1.4 can be put into the form where the hypothesis $\theta_{n+1}^2 \equiv \theta_n$ holds.

Is it true for every bounded type-definable equivalence relation which is not an intersection of definable equivalence relations ?

1.3 Compact group associated with a bounded type-definable equivalence relation

Here I come back to known material, and drop the assumption of countable language.

Again, nothing is new, except maybe the alternative treatment of the so-called *Galois group* along with the Galois correspondence, avoiding the use of the (quasi compact) Lascar group $\mathrm{Aut}(\overline{M})/\mathrm{Autf}(\overline{M})$. However, I give in the Appendix B a motivation (among others) for the introduction of the Lascar group as a pure group-theoretic (regardless any topological structure on it) invariant of T.

In order to simplify notation, call Y the power $\overline{M}^\alpha$ on which the type-definable equivalence relation E is defined, where $\overline{M}$ is any sufficiently saturated model of T (ie λ-saturated, $\lambda > 2^{|T|+|x|}$). As noticed in 1.1.6, all the following results and arguments generalize very easily when Y is just a 0-type-definable set in $\overline{M}^\alpha$, and E is a 0-type-definable equivalence relation on Y.

As seen in section 1, with a bounded type-definable equivalence relation E in a theory T is associated a topological invariant of T, namely the homeomorphism class of the compact Hausdorff space $X = Y/E$.

One can go further, and associate to E another invariant which is a compact Hausdorff topological group with continuous action on X. Given the importance of compact groups (and compact group actions) throughout mathematics, and the extensive amount of knowledge about them (see for example [1]), this is far from beeing insignificant. In the next chapter we will refer to one possible way of taking advantage of it, making use of the structural result that every compact Hausdorff group is a projective limit of compact Lie groups.

First observe that $\mathrm{Aut}(\overline{M})$ acts on X in the more natural way : $\sigma \,.\, a_E = (\sigma a)_E$ (this definition is sound, because since E is type-definable, $E(a,b)$ iff $E(\sigma a, \sigma b)$).

So we can consider the permutation group of this action, namely the image of the group morphism associated with the action $\rho : \mathrm{Aut}(\overline{M}) \to \mathfrak{S}(X)$, where $\mathfrak{S}(X)$ is the group of permutations of the set X. Let's call G_X this image group.

G_X acts on X, and we want to endow it with a compact Hausdorff topology that makes this action continuous, as well as the group multiplication and the inverse function.

Still call $\rho\colon \mathrm{Aut}(\overline{M}) \twoheadrightarrow G_X = \rho(\mathrm{Aut}(\overline{M}))$.

Clearly, each element of G_X is an homeomorphism of the topological space X (the inverse image of the solution set of a type under an isomorphism of $\overline{M}$ is again the solution set of a type), so that

$$G_X \subseteq \mathrm{Homeo}(X) \subseteq \mathrm{Cont}(X,X) \subseteq X^X$$

(with transparent notations).

Now X is compact Hausdorff, so by Tychonov, so is X^X endowed with the product topology. In general, given a compact Hausdorff space X, there is no reason why $\mathrm{Cont}(X,X)$, or even $\mathrm{Homeo}(X)$, should be closed (or equivalently compact) sets in X^X with the product topology. But here the context makes possible the fact that G_X be a closed subset of X^X.

To see this, let us first characterize closed sets in X^I, for a small set I : let $\pi\colon Y \to X$ be the canonical projection, and $\delta = \pi^I : Y^I \to X^I$ be the application given by the universal property of the product X^I, namely $\delta((y_i : i \in I)) = ([y_i]_E : i \in I)$. Given an application $f \in X^I$, an application $\hat{f} \in Y^I$ such that $\delta(\hat{f}) = f$ is called a *lifting* of f.

In the sequel, the only topology I will consider on a set of the form X^I will be the product topology.

Lemma 1.3.1 *Let X be any topological space, I a set, and $C \subseteq X^I$. The following are equivalent :*

1. *C is closed*
2. *There exist a set J, a map $g : J \to I$ and a closed subset $F \subseteq X^J$ such that $C = \{f \in X^I : f \circ g \in F\}$*

Proof : $1 \Rightarrow 2$: take $J = I$, $g = \mathrm{Id_I}$, and $F = C$.

$2 \Rightarrow 1$: it is clearly enough to show that $X^I \to X^J : f \mapsto f \circ g$ is a continuous map for the product topologies : so let $U = \{h \in X^J : h(j_1) \in \mathcal{O}_1, \dots, h(j_k) \in \mathcal{O}_k\}$ be a basic open set of X^J, with $\mathcal{O}_1, \dots, \mathcal{O}_k$ open sets of X. Then, $f \circ g \in U$ iff $f \circ g(j_1) \in \mathcal{O}_1, \dots, f \circ g(j_k) \in \mathcal{O}_k$, iff $f \in V = \{\varphi \in X^I : \varphi(i_1) \in \mathcal{O}_1, \dots, \varphi(i_k) \in \mathcal{O}_k\}$, with $i_1 = g(j_1), \dots, i_k = g(j_k)$, showing that the preimage of U is the (basic) open set V. ■

Lemma 1.3.2 *Let I be a small set, and $C \subseteq X^I$. The following are equivalent :*

1. *C is a closed subset of X^I*
2. *$\delta^{-1}(C)$ is type-definable with small parameters*
3. *There exists a partial type $\Phi(y_i : i \in I)$ with small parameters such that $C = \{f \in X^I :$ for every (some) lifting $\hat{f}$ of f, $\models \Phi(\hat{f})\}$*
4. *There exist a small set J, a map $g \in I^J$, and a partial type with small parameters $\Phi(y_j : j \in J)$, s.t.*
   $C = \{f \in X^I :$ for every (some) lifting $\widehat{f \circ g}$ of $f \circ g$, $\models \Phi(\widehat{f \circ g})\}$

Proof :

$1 \Rightarrow 3$

Let $U = \{f \in X^I : f(i_1) \in \mathcal{O}_1, \dots, f(i_k) \in \mathcal{O}_k\}$ be a basic open set of X^I, with i_r's $\in X$, and $\mathcal{O}_r$'s open sets in X. Then, $X \setminus \mathcal{O}_r$ is a closed set in X, given say by the partial type $\Pi_r(y)$, with (small) parameters from $\overline{M}$ (cf 1.1.2). A closed set C is the intersection of closed sets of the form $X^I \setminus U$. To each such basic closed set corresponds the partial type (with small parameters) $\Pi_1(y_{i_1}) \vee \dots \vee \Pi_k(y_{i_k})$. Let $\Phi(y_i : i \in I)$ be the union of the partial types corresponding to the basic closed sets intervening in the intersection.

Now, using the fact that $a_E \in X \setminus \mathcal{O}_r$ iff $\models \Pi_r(a)$ (the very definition of a closed set in the logic topology), one sees easily that if f belongs to C, then every lifting $\hat{f}$ satisfies $\models \Phi(\hat{f})$; and that if f in X^I has some lifting $\hat{f}$ such that $\models \Phi(\hat{f})$, then f belongs to C. So we get $C = \{f \in X^I :$ for every (some) lifting $\hat{f}$ of f, $\models \Phi(\hat{f})\}$.

It remains to argue that $\Phi(y_i : i \in I)$ can be defined over a small set of parameters. If $\kappa < \lambda$ is the number of E-classes, then Φ is the union of at most 2^κ partial types with small parameters (there are at most 2^κ basic closed sets in X). So, if we choose the degree of saturation λ of $\overline{M}$ to be an inaccessible cardinal, we are done.

$3 \Rightarrow 2$

Since $\hat{f} \in \delta^{-1}(C)$ iff $\hat{f}$ is a lifting of some $f \in C$, $\delta^{-1}(C)$ is defined by the same partial type $\Phi(y_i : i \in I)$ as one easily checks.

$2 \Rightarrow 1$

An element of Y^I can be thougth of as an element of $\overline{M}^{I\times\alpha}$, i.e. a small tuple since both α and I have size less than λ. On these tuples, define a 0-type-definable equivalence relation E_I by $E_I((y_i : i \in I), (y'_i : i \in I))$ iff $\models \bigwedge_i E(y_i, y'_i)$. Since E has a small number of classes, so is E_I, and we can consider the compact Hausdorff topological space Y^I/E_I with the logic topology.

Let $\mu : Y^I \twoheadrightarrow Y^I/E_I$ be the canonical surjection. Clearly, $E_I(y, y')$ iff $\delta(y) = \delta(y')$, henceforth δ factorizes through μ by a bijection $\overline{\delta} : Y^I/E_I \to X^I$:

$$\begin{tikzcd}
Y^I \arrow[r, two heads, "\delta"] \arrow[d, "\mu"] & X^I \\
Y^I/E_I \arrow[ur, dashed, tail, two heads, "\sim", "\overline{\delta}"'] &
\end{tikzcd}$$

Transporting via $\overline{\delta}$ the logic topology, we get a compact Hausdorff topology τ on X^I. By the same argument as in $1 \Rightarrow 3$, one sees easily that each closed set for the logic topology on X^I is closed for τ, i.e. the logical topology is finer than τ. But both topologies are compact Hausdorff, so that they coincide.

Now let $C \subseteq X^I$ such that $\delta^{-1}(C)$ is type-definable with small parameters. Since $\delta = \overline{\delta} \circ \mu$, this means that C is the image of a closed set in Y^I/E_I, (namely $C = \overline{\delta}(\overline{\delta}^{-1}(C))$), so C is closed in τ, and C is closed in the logic topology since both topologies coincide.

Once we know that $1 \Leftrightarrow 3$, $3 \Leftrightarrow 4$ follows immediately from the previous Lemma.
■

Corollary 1.3.3 *G_X is closed (so compact) in X^X.*

Proof : We use item 3 of Lemma 1.3.2, with $I = X$. Fix $(b_x : x \in X)$ a tuple such that b_x is a representative of the class x, for every x in X. Let $\Phi(y_x : x \in X)$ be $\mathrm{tp}(b_x : x \in X)$.

If f belongs to G_X, $f = \rho(\sigma)$, for some $\sigma \in \mathrm{Aut}(\overline{M})$. Then, $(\sigma b_x : x \in X)$ is a lifting $\hat{f}$ of f, and $\models \Phi(\hat{f})$.

Conversely, suppose f in X^X has a lifting $\hat{f} = (c_x : x \in X)$ with $\models \Phi(\hat{f})$. Then, by homogeneity of $\overline{M}$, the elementary map $b_x \mapsto c_x$ extends to an element σ of

$\mathrm{Aut}(\overline{M})$, and clearly $f = \rho(\sigma)$. ■

So we have a compact Hausdorff topology on G_X, and we have to check that it makes G_X into a topological group, with continuous action on X. First, here is an intermediate weaker result :

Lemma 1.3.4 *Fix $a \in X$ and $g \in G_X$. The maps*

$$G_X \to X$$
$$f \mapsto f\ .\ a = f(a)$$

and

$$G_X \to G_X$$
$$f \mapsto f \circ g$$

are continuous.

Proof : The first map : Let C be a closed subset of X, given by the partial type $\Phi(y_a)$. Clearly, $\{f \in G_X : f(a) \in C\} = \{f \in G_X : \text{for every lifting } \hat{f}, \models \Phi(\hat{f}(a))\}$. Considering $\Phi(x)$ as a partial type in the variables $\Phi(y_i : i \in X)$, this set is also $\{f \in G_X : \text{for every lifting } \hat{f}, \models \Phi(\hat{f})\}$, which is a closed subset of G_X by lemma 1.3.2, implying that the map is continuous.

The second map : as in the proof of Lemma 1.3.1, the map $X^X \to X^X$: $f \mapsto f \circ g$ is continuous, and the map we are looking at is a restriction of it. ■

Corollary 1.3.5 *G_X endowed with the restriction of the product topology is a topological group, and the action on X is continuous.*

Proof :

<u>The product map</u> $G_X \times G_X \to G_X$ is continuous :

let $f, g \in G_X$ and U be a basic open neighborhood of $f \circ g$, $U = \{h \in G_X : h(b_1) \in \mathcal{O}_1, \dots, h(b_k) \in \mathcal{O}_k\}$. Since the open sets of the form $[[\psi(x,a)]]$, where $\psi(x,y) \in E(x,y)$, is a base of open sets in X, we can suppose each $\mathcal{O}_i$ is $[[\psi_i(x, f \circ g(b_i)]]$ for some $\psi_i(x,y) \in E(x,y)$.

Let $\psi'_i(x,y) \in E(x,y)$ be such that $(\psi'_i)^2 \vdash \theta_i$, $U' = \{h \in G_X : h(b_1) \in [[\psi'_1(x, f \circ g(b_1)]], \dots, h(b_k) \in [[\psi'_k(x, f \circ b(b_k)]]\}$, and $V = \{f' \in G_X : f' \circ g \in U'\}$. By Lemma 1.3.4, V is an open set, and by definition it contains f.

Let $W = \{g' \in G_X : g'(b_1) \in [[\psi'_1(x, g(b_1)]], \dots, g'(b_k) \in [[\psi'_k(x, g(b_k)]]\}$, a basic open neighborhood of g.

Now let $f' \in V$ and $g' \in W$. Since $f' = \rho(\sigma')$ for some $\sigma' \in \mathrm{Aut}(\overline{M})$, and $g'(b_i) \in [[\psi'_i(x, g(b_i))]]$ for all i, we get $f' \circ g(b_i) \in [[\psi'_i(x, f' \circ g'(b_i))]]$ for all i.

On the other hand, $f' \circ g(b_i) \in [[\psi_i'(x, f \circ g(b_i))]]$ for all i by definition of V.

Henceforth, $f' \circ g'(b_i) \in [[\psi_i(x, f \circ g(b_i))]]$ for all i, and we have proved that $V \times W$ is mapped into U.

The inverse map $G_X \to G_X$ is continuous :

As for the product, let $f \in G_X$, and $U = \{h \in G_X : h(b_1) \in [[\psi_1(x, f^{-1}(b_1))]], \ldots, h(b_k) \in [[\psi_k(x, f^{-1}(b_k))]]\}$ be a basic open neighborhood of f^{-1}, with $\psi_i \in E$.

Let $V = \{h \in G_X : h(f^{-1}(b_1)) \in [[\psi_1(x, b_1)]], \ldots, h(f^{-1}(b_k)) \in [[\psi_k(x, b_k)]]\}$ (an open neighborhood of f).

Then, V is mapped into U.

The action $G_X \times X \to X$ is continuous :

Let $f \in G_X$, $a \in X$, and (without lost of generality) $\mathcal{O} = [[\psi(x, f\,.\,a)]]$ a basic open neighborhood of $f\,.\,a = f(a)$ in X.

Let $\psi' \in E$ with $(\psi')^2 \vdash \psi$, $V = \{f' \in G_X : f'\,.\,a \in \mathcal{O}\}$ (an open set in G_X by Lemma 1.3.4, containing f by definition of $\mathcal{O}$), and $\mathcal{U} = [[\psi'(x, a)]]$ (an open neighborhood of a in X).

Then, $V \times \mathcal{U}$ is mapped into $\mathcal{O}$. ■

Now we go a little bit further, and characterize closed subsets of G_X. Recall the epimorphism ρ: $\mathrm{Aut}(\overline{M}) \to G_X$.

Corollary 1.3.6 *Let $C \subseteq G_X$, and $\overline{b} = (b_x : x \in X)$ be a tuple such that each b_x is a representative of x. The following are equivalent :*

1. *C is closed for the topology induced on G_X by the product topology on X^X*
2. *There exist a (small) tuple $(b_j : j \in J) \in Y^J$ and a partial type with small parameters $\Phi(y_j : j \in J)$ s.t.*
   $\rho^{-1}(C) = \{\sigma \in \mathrm{Aut}(\overline{M}) : \models \Phi(\sigma(b_j) : j \in J)\}$
3. *There exist a (small) tuple $\overline{a}$ in $\overline{M}$, and a partial type with small parameters $\Phi(\overline{x})$ s.t. $lg(\overline{a}) = lg(\overline{x})$ and*
   $\rho^{-1}(C) = \{\sigma \in \mathrm{Aut}(\overline{M}) : \models \Phi(\sigma(\overline{a}))\}$
4. *$\{\sigma(\overline{b}) : \sigma \in \rho^{-1}(C)\}$ is type-definable with small parameters*

Proof :

$1 \Leftrightarrow 2$ is a direct consequence of characterization 4. of closed subsets of X^X given by Lemma 1.3.2, and the fact that if $g \in X^J$ and $(b_j : i \in J)$ is a lifting of g in Y^J, then for every $\sigma \in \mathrm{Aut}(\overline{M})$, the tuple $(\sigma(b_j) : j \in J)$ is a lifting of $\rho(\sigma) \circ g$.

For $1 \Leftrightarrow 4$, let's use characterization 2. of a closed set of X^X given by Lemma 1.3.2.

First, suppose $\delta^{-1}(C)$ is type-definable by the partial type $\Phi(y_x : x \in X)$, and let $p(y_x : x \in X) = \mathrm{tp}(\bar{b})$. We will check that $A = \{\sigma(\bar{b}) : \sigma \in \rho^{-1}(C)\}$ is the solution set B of the type $\Phi(y_x : x \in X) \wedge p(y_x : x \in X)$.

$A \subseteq B$ is obvious, for if $\sigma \in \rho^{-1}(C)$, $\sigma(\bar{b})$ is a lifting of $\rho(\sigma) \in C$.

Suppose now $(c_x : x \in X)$ lies in B, i.e. $\models \Phi(c_x : x \in X) \wedge p(c_x : x \in X)$. Then, on one hand, there is some $\sigma \in \mathrm{Aut}(\overline{M})$ such that for all $x \in X$, $\sigma(b_x) = c_x$, and on the other hand there are some $f \in C$, and a lifting $\hat{f}$ of f, such that $c_x = \hat{f}(x)$ for every $x \in X$. But this means $\rho(\sigma) = f \in C$, and $B \subseteq A$.

Secondly , suppose $\{\sigma(\bar{b}) : \sigma \in \rho^{-1}(C)\}$ is the solution set of $\Psi(y_x : x \in X)$. It is then immediate to check that $\delta^{-1}(C)$ is the solution set of the partial type $\exists(z_x : x \in X)\ \Big(\Psi(z_x : x \in X) \bigwedge_x E(z_x, y_x)\Big)$.

$2 \Rightarrow 3$ is obvious.

$3 \Rightarrow 2$: Let $\bar{a} = (a_i : i \in I)$, $a_i \in \overline{M}$. For each i, choose any tuple $b_i = (b_i^r : r < \alpha) \in \overline{M}^\alpha$ such that $b_i^0 = a_i$, and complete the partial type $\Phi(\overline{x})$ with corresponding dummy variables. Then, you get 2. ■

Recall (c.f. discussion after Lemma 1.1.5) that $X = \overline{M}^\alpha/E$ is a topological invariant of T : if $\overline{N} \succ \overline{M}$ is a κ-saturated elementary extension, with $\kappa > |x| + \aleph_0$, then $a_E^{\overline{M}} \mapsto a_E^{\overline{N}}$ is an homeomorphism between $\overline{M}^\alpha/E$ and $\overline{N}^\alpha/E$.

With no surprise, G_X is also an invariant of T :

Proposition 1.3.7 *Let $E(x,y)$ be a type-definable equivalence relation.*

Let $\overline{M}$ [resp. $\overline{N}$] be a κ-saturated and κ-homogeneous [resp. λ-saturated and λ-homogeneous] model of T, with $\lambda > \kappa > |x| + \aleph_0$ and $\overline{M} \prec \overline{N}$.

Let $X \approx Z$ be the compact spaces $\overline{M}^{|x|}/E$ and $\overline{N}^{|x|}/E$.

Then G_X is isomorphic to G_Z as compact groups.

Proof : First of all, define a map $f\colon G_X \to G_Z$: let $g = \rho(\sigma) \in G_X$, with $\sigma \in \mathrm{Aut}(\overline{M})$. By λ-homogeneity of $\overline{N}$, extend σ to some $\tau \in \mathrm{Aut}(\overline{N})$.

Claim : $\rho(\tau) \in G_Z$ does not depend on the choice of σ or τ.
Proof : Suppose $g = \rho(\sigma) = \rho(\sigma')$, τ extends σ, and τ' extends σ'. Let $h = \rho(\tau), h' = \rho(\tau') \in G_Z$, and let $b_E \in Z$. Then, $b_E = a_E$, for some tuple a from $\overline{M}$. Therefore, we have $h(b_E) = h(a_E) = [\tau(a)]_E = [\sigma(a)]_E$, and likewise $h'(b_E) = h'(a_E) = [\tau'(a)]_E = [\sigma'(a)]_E$. But since σ and σ' define the same permutation of G_X, $\overline{M} \models E(\sigma(a), \sigma'(a))$, therefore also $\overline{N} \models E(\sigma(a), \sigma'(a))$, so that $h(b_E) = h'(b_E)$, and $h = h'$. ■

Define thus $f(\rho(\sigma)) = \rho(\tau)$, where τ is some extension of σ.

The fact that $f\colon G_X \to G_Z$ be a group morphism is an immediate consequence of the fact that, if τ [resp. τ'] is an extension of σ [resp. σ'], then $\tau\tau'$ is an extension of $\sigma\sigma'$.

- $\mathrm{Ker} f = 1$: an immediate consequence of the fact that every $b_E \in Z$ has a representative in $\overline{M}$.

- f is surjective : suppose $h = \rho(\tau) \in G_Z$, with $\tau \in \mathrm{Aut}(\overline{N})$. Let $\bar{b} = (b_x : x \in X)$ be (as in Corollary 1.3.6) a tuple of representatives of the E-classes in X. Since every E-class in Z has a representative in $\overline{M}$, τ induces a permutation $\widehat{\tau}$ of $\{b_x \mid x \in X\}$ defined as follows :

$$\overline{N} \models E(\tau(b_x), b_{\widehat{\tau}(x)})$$

  Then, the tuple $(\tau(b_x) : x \in X)$ satisfies in $\overline{N}$ the partial with small parameters in $\overline{M}$

$$\Psi(\overline{y}) = p(y_x : x \in X) \bigwedge_{x \in X} E(y_x, b_{\widehat{\tau}(x)})$$

  where $p(\overline{y})$ is the type of $\bar{b}$ over $\emptyset$.

  By κ-saturation of $\overline{M}$, $\Psi(\overline{y})$ is satisfied in $\overline{M}$, say by a tuple $\bar{c}$.

  Then, $\bar{b} \equiv \bar{c}$, and by κ-saturation of $\overline{M}$, there exists $\sigma \in \mathrm{Aut}(\overline{M})$ such that $\sigma(b_x) = c_x$ for every $x \in X$.

  Let $g = \rho(\sigma) \in G_X$, τ' an extension of σ to $\mathrm{Aut}(\overline{N})$, and $b_E = [b_x]_E \in Z$. Then

$$f(g)\big([b_x]_E\big) = [\tau'(b_x)]_E = [\sigma(b_x)]_E = [c_x]_E = [\tau(b_x)]_E = h([b_x])$$

  and therefore $h = f(g)$.

- f is an homeorphism : using item 4 of the characterization given by Corollary 1.3.6 of closed sets in G_X, it is immediate to see that the image of a closed set in G_X is a closed set in G_Z, so that f^{-1} is continuous, and even bi-continuous since the spaces are compact Hausdorff.

■

If we apply the previous results to the particular case of E_{KP} on $\overline{M}^{\omega}$, we get a one-one correspondence (“ à la Galois ”) between closed subgroups of G_{KP} and definably closed sets of bounded hyperimaginaries, where G_{KP} is G_X with $X = \overline{M}^{\omega}/E_{KP}$.

First of all, since E_{KP} is the least bounded type-definable equivalence relation on tuples of countable length, and every (bounded) hyperimaginary is equivalent to a tuple of (bounded) countable hyperimaginaries (c.f. 2.1 of the next chapter if you don't know about hyperimaginaries), fixing all the classes in $\overline{M}^{\omega}/E_{KP}$ is the same as fixing all bounded hyperimaginaries for an element of $\mathrm{Aut}(\overline{M})$.

Consequently, the action of $\mathrm{Aut}(\overline{M})$ on the set of bounded hyperimaginaries on one hand, and on $\overline{M}^{\omega}/E_{KP}$ on the other hand, have the same kernel Γ_1, so that G_{KP} acts faithfully on the set of bounded hyperimaginaries by $\rho(\sigma)\ .\ a_F = [\sigma a]_F$ ($\sigma \in \mathrm{Aut}(\overline{M})$ and a_F bounded hyperimaginary).

In order to show the "Galois correspondence" we need first to set some preliminary results :

Lemma 1.3.8 *Let $H \subseteq \mathrm{Aut}(\overline{M})$ be a subgroup, and a be a small tuple. The following are equivalent :*

1. *$H = \mathrm{Fix}(a_E)$ for some type-definable equivalence relation E*
2. *$\mathrm{Fix}(a) \subseteq H$ and the orbit of a under H is type-definable (with small parameters)*

Proof : See [4] Lemma 1.9. ■

Proposition 1.3.9 *Let $H \subseteq G_{KP}$ be a subgroup. Then H is closed iff $H = \mathrm{Fix}(e)$ for some bounded hyperimaginary e.*

Proof : Let $\bar{b}$ be a tuple as in Lemma 1.3.6. By Proposition 1.3.7, G_{KP} is an invariant of T, so let μ be its size, and $\overline{M}$ be a μ^+-saturated and μ^+-homogeneous model of T.

First suppose that $H = \mathrm{Fix}(e)$ for some bounded hyperimaginary e. Let $\sigma \in \rho^{-1}(H)$. Then, $\sigma(e) = e$, so that $\mathrm{tp}(\bar{b}/e) = \mathrm{tp}(\sigma(\bar{b})/e)$, and $\{\sigma(\bar{b}) : \sigma \in \rho^{-1}(H)\}$ is type-definable (over any representative of e). By lemma 1.3.6, H is closed in G_{KP}.

Now suppose H is closed. Let $K = \rho^{-1}(H) \subseteq \mathrm{Aut}(\overline{M})$. If an automorphism of $\overline{M}$ fixes $\bar{b}$, it fixes every $x \in X$, so that $\mathrm{Fix}(\bar{b}) \subseteq K$. On the other hand, $\{\sigma(\bar{b}) : \sigma \in K\}$ is type-definable by Lemma 1.3.6. Using Lemma 1.3.8, we see that $K = \mathrm{Fix}(\bar{b}_F)$, for some type-definable equivalence relation F, and so also $H = \mathrm{Fix}(\bar{b}_F)$. The fact that $\bar{b}/F$ be bounded follows immediately from the fact that the quotient group $\mathrm{Aut}(\overline{M})/\Gamma_1 \approx G_{KP}$ has size less than the degree of saturation/homogeneity μ^+ of $\overline{M}$ (hence, so is the set of cosets $\mathrm{Aut}(\overline{M})/K$, since $\Gamma_1 \subseteq K \subseteq \mathrm{Aut}(\overline{M})$).

In fact, as seen in Appendix B, the size μ of G_{KP} is at most $2^{|T|}$, so that one can always take as monster models κ-saturated and κ-homogeneous models with $\kappa > 2^{|T|}$. ■

Call $\mathrm{bdd}(\emptyset)$ the set of bounded hyperimaginaries.

Equipped with this proposition, one sees immediately that

$$H \mapsto \mathrm{Fix}(H) = \{e \in \mathrm{bdd}(\emptyset) \mid g \cdot e = e \text{ for every } g \in H\}$$

and

$$A \mapsto \mathrm{Fix}(A) = \{g \in G_{KP} \mid \ g \cdot e = e \text{ for every } e \in A\}$$

are inverse isomorphisms between the lattice of closed subgroups of G_{KP} and the dual of the lattice of definably closed subsets of $\mathrm{bdd}(\emptyset)$.

1.4 Examples

Here I present some ways of representing the classic groups $O_n(\mathbb{R})$, $SO_n(\mathbb{R})$ and $U_n(\mathbb{C})$ as groups of the form G_X. Let me put those examples in perspective: it is already known (c.f. [10]) that *any* compact group is isomorphic to some G_X. But in those particular cases of compact (Lie) groups, I provide alternative and more direct constructions than in the general case.

1.4.1 Around the circle and orthogonal groups

$\mathbb{S}^1$, embedded in $\mathbb{R}^2$

The basic structure M_1 is $(\mathbb{S}^1; R_n : n < \omega)$, where $R_n(r,s)$ means $d(r,s) < \frac{1}{2^n}$, and d is the restriction of the Euclidian distance on $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Clearly R_n is symmetric and reflexive, and $R_{n+1}^2 \vdash R_n$ by triangular inequality; therefore, in $T_1 = Th(M_1)$ there is a type-definable equivalence relation $E(x,y)$ given by the partial type $\{R_n(x,y) : n < \omega\}$, meaning x and y are "infinitely close".

Moreover it is clear that each R_n is thick in T_1, so that E is bounded.

Observe that M_1 is 0-definable in $(\mathbb{R}; +, -, \, . \, , <, 0, 1)$, so we can take the monster model $\overline{M}$ of T_1 as the corresponding definable structure in a κ-saturated κ-homogeneous elementary extension $\mathbb{R}^*$ of $\mathbb{R}$.

One can define a type-definable equivalence relation F in the theory of $(\mathbb{R}; +, -, ., <, 0, 1)$ quite the same way as was done with $\mathbb{S}^1$: $F(x,y)$ iff $|x,y| < \frac{1}{2^n}$, for all $i < \omega$ (being infinitely close).

An element $a \in \mathbb{R}^*$ is said to be *bounded* if $m \leq a \leq n$ for some $m, n \in \mathbb{Z}$.

Lemma 1.4.1 *1. If $a \in \mathbb{R}^*$ is bounded, then a_F has a unique real representative.*

2. Let $a, b, a', b' \in \mathbb{R}^*$ s.t. $\models F(a,a')$ and $\models F(b,b')$; then $\models F(a+b, a'+b')$.

If moreover a, b are bounded, $\models F(aa', bb')$.

Proof :

1. Let $a^+ = \{q \in \mathbb{Q} : a < q\}$, and $a^- = \{q \in \mathbb{Q} : q \leq a\}$. These sets are non-void by hypothesis, so (a^+, a^-) is a rational cut; let $r \in \mathbb{R}$ be the associated real number, and suppose $|a - r| \geq \frac{1}{2^n}$ for some $n < \omega$ $(*)$.

Suppose moreover $r > a$ (the case $r < a$ is treated the same way); let q be a rational number such that $q < r$ and $s - q < \frac{1}{2^{n+1}}$, ie $q > s - \frac{1}{2^{n+1}}$. $(*)$ reads $-a \geq \frac{1}{2^n} - s$, and summing up the two inequalities leads to $q - a \geq \frac{1}{2^{n+1}} > 0$, so that $q \in a^+$, in contradiction with $q < r$.

Therefore, $|r - a| < \frac{1}{2^n}$ for all $n < \omega$, ie $\models F(r, a)$.

2. Let $n < \omega$, $M \in \mathbb{N}$ such that $|a|, |b| \leq M$, and $k < \omega$ with $\frac{M}{2^k} < \frac{1}{2^n}$.

   $|ab - a'b'| = |a(b-b') - b'(a'-a)| \leq |a||b-b'| + |b'||a'-a| \leq M(|b-b'| + |a'-a|) \leq \frac{M}{2^k}$, the last inequality since $|b - b'|, |a' - a| < \frac{1}{2^{k+1}}$ by hypothesis.

   The addition case is similar.

■

Lemma 1.4.2 *Each E-class in $\overline{M}$ contains a unique representative in $\mathbb{S}^1$.*

Proof :

If $r, s \in \mathbb{S}^1$ are such that $E(r, s)$, then clearly $r = s$, whence uniqueness.

Recall that the universe of $\overline{M}$ is $\{(a, b) \in (\mathbb{R}^*)^2 : a^2 + b^2 = 1\}$; since $\mathbb{R} \models \forall x \forall y\ (x^2 + y^2 = 1 \rightarrow (-1 \leq x \leq 1) \wedge (-1 \leq y \leq 1))$, every $(a, b) \in \mathfrak{C}$ has bounded components a and b.

Use Lemma 1.4.1 to get $r, s \in \mathbb{R}$ with $F(a, r)$ and $F(b, s)$.

By hypothesis, $a^2 + b^2 - 1 = 0$, and again Lemma 1.4.1 gives $F(r^2 + s^2 - 1, 0)$, ie $r^2 + b^2 - 1 = 0$, and $(r, s) \in \mathbb{S}^1$.

To finish we have to prove that $\models E((a, b), (r, s))$, but this is easily done observing that $F(a, r)$ and $F(b, s)$ imply respectively (applying again Lemma 1.4.1) $F(a - r, 0)$, $F(b - s, 0)$ and $F((a - r)^2, 0)$, $F((b - s)^2, 0)$.

Therefore, $d((a, b), (r, s)) = \sqrt{(a - r)^2 + (b - s)^2} < \frac{1}{2^n}$ for all $n < \omega$. ■

$\mathbb{S}^1$, with the intrinsic metric

Here the model is $M_2 = (\mathbb{S}^1; S_n : n < \omega)$, where $S_n(r, s)$ means $d'(r, s) < \frac{2\pi}{2^n}$, and d' is the distance on $\mathbb{S}^1$ given by the shortest arc lenght between a and b.

As before, the fact that d' is a metric translates into the facts that the S_n's are symmetric and reflexive, and $S_{n+1}^2 \vdash S_n$, and there is in $T_2 = Th(M_2)$ a bounded type definable equivalence relation $E'(x, y) = \bigwedge_{n<\omega} S_n(x, y)$.

But the particular form of the metric space $(\mathbb{S}^1, d')$ implies, in contrast with M_1 :

For all $m \geq 1, n \geq 0 : \mathbb{S}^1 \models S_n^m(r, s)$ iff $d'(r, s) < 2\pi . \frac{m}{2^n}$, $\quad (*)$

where S_n^m means the usual m-times composition $S_n \circ \cdots \circ S_n$.

In particular, for all $m, p \geq 1, n, q \geq 0$, the following are axioms for T_2 :

$$S_{n+1}^2 \equiv S_n$$

$$\forall x \forall y \forall z \; S_n^m(x,y) \wedge S_q^p(y,z) \rightarrow S_{n+q}^{2^q m + 2^n p}(x,z) \qquad (**)$$

Lemma 1.4.3 *The structure M_2 is 0-definable in* $(\mathbb{R}; +, -, \,.\,, <, 0, 1)$.

Proof : The universe is defined by the the formula $x^2 + y^2 = 1$, and the relations S_n by the following observation: if $A = (x, y)$, $A' = (x', y')$ are in $\mathbb{S}^1$, $d(A, A') < 2\pi/2^n$ iff there is σ in the special orthogonal group of $\mathbb{R}^2$, of measure less than $2\pi/2^n$ in the canonical (othonormal) base, such that $\sigma(A) = A'$, or $\sigma(A') = A$.

Writing s_n for the real number $\sin(2\pi/2^n)$, this amounts to say, for $n \geq 2$, and using the matrix representation $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ of σ in the canonical base :

there exists a, b such that $a^2+b^2 = 1$, $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ or $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, $0 \leq a$, $0 < b < s_n$,

which is obviously a formula (with parameters s_n) in the real field.

To see this formula is in fact without parameters, use the formulae

$\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$, $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$, to get s_{n+1} and $c_{n+1} := \cos(2\pi/2^{n+1})$ as an expression of s_n and c_n using only rational numbers and root extractions, and an induction on n from $s_2 = 1, c_2 = 0$.

The remaning cases S_0 and S_1 are obvious :

S_0 is always true, so definable, and S_1 is definable by $A \neq -A'$. ■

We can therefore take as a monster model $\overline{M}$ of T_2 the corresponding definable structure in a κ-saturated κ-homogeneous elementary extension of $\mathbb{R}^*$ of $\mathbb{R}$.

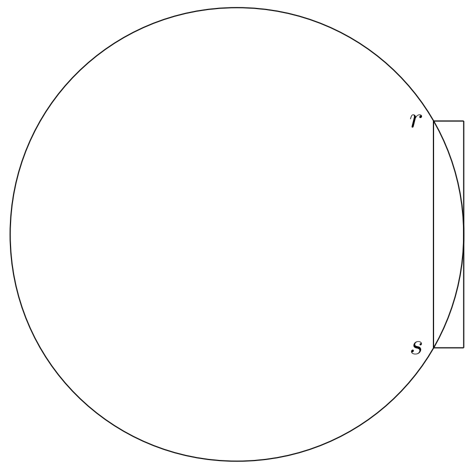
In the common universe for the monster models of T_1 and T_2, the relations R_n and S_n behave as follows :

$$\models S_n \rightarrow R_{n-3}, \text{ for } n \geq 3, \text{ and } \models R_n \rightarrow S_n$$

This is just because they behave so in $\mathbb{S}^1$, the reason being the following:

for $r, s \in \mathbb{S}^1$, $r \neq s$, $d(r,s) < d'(r,s) < 4d(r,s)$.

The first inequality is obvious, and the second is justified by the following drawing :



The conclusion is that in the common universe of the monster models of T_1 and T_2, $E = E'$, so that as in the case of T_1, there is in $\overline{M}$ a unique real representative for each E'-class.

This provides a bijection $f : \mathbb{S}^1 \to X = \overline{M}/E'$, $f(r) = r_{E'}$.

Observe that since the S_n's are reflexive, $S_n \vdash S_n^m$ for all $m \geq 1, n \geq 0$, and therefore E' is also given by the partial type $\{S_n^m(x,y) : m \geq 1, n \geq 0\}$.

Lemma 1.4.4 *Let $r, s \in \mathbb{S}^1$.*

1. *Suppose $\models S_n^m(r,s)$; then, $\models S_n^m(a,b)$ for all a, b s.t. $E'(r,a)$ and $E'(s,b)$.*
2. *Suppose $d'(r,s) > \frac{m}{2^n}$; then, $\models \neg S_n^m(a,b)$ for all a, b s.t. $E'(r,a)$ and $E'(s,b)$.*

Proof :

1. The hypothesis means $d'(r,s) < \frac{m}{2^n}$; let $\frac{i}{2^j}$ be such that $d'(r,s) < \frac{i}{2^j} < \frac{m}{2^n}$, and let $\frac{p}{2^q}$ be $\left(\frac{m}{2^n} - \frac{i}{2^j}\right)/2$.
   If $d'(x,r) < \frac{p}{2^q}$ and $d'(s,y) < \frac{p}{2^q}$, then
   $d'(x,y) \leq d'(x,r) + d'(r,s) + d'(s,y) < \frac{p}{2^q} + \frac{i}{2^j} + \frac{p}{2^q} = \frac{m}{2^n}$.
   This means (by $(*)$) : $\mathbb{S}^1 \models \forall x \forall y \quad S_q^p(x,r) \wedge S_q^p(y,s) \to S_n^m(x,y)$, and the same holds in the elementary extension $\overline{M}$, whence the result.
2. Let $\frac{p}{2^q}$ be such that $0 < \frac{p}{2^q} < \left(d'(r,s) - \frac{m}{2^n}\right)/2$. Let x and y be such that $d'(r,x), d'(y,s) < \frac{p}{2^q}$; then, $d'(r,s) \leq d'(r,x) + d'(x,y) + d'(y,r)$, $d'(x,y) \geq d'(r,s) - (d'(r,x) + d'(y,s))$, and $d'(x,y) > \frac{m}{2^n}$ by choice of $\frac{p}{2^q}$.
   Again, this means $\mathbb{S}^1 \models \forall x \forall y \quad S_q^p(x,r) \wedge S_q^p(y,s) \to \neg S_n^m(x,y)$, whence the result.

$\blacksquare$

Proposition 1.4.5 *The bijection $f : \mathbb{S}^1 \to X$ is an homeomorphism.*

Moreover, $f(B(r, \frac{1}{2^n})) = [S_n(x, r)]$, and $f(\overline{B}(r, \frac{1}{2^n})) =< S_n(x, r) >$.

Proof : Fisrt, check $f(B(r, \frac{1}{2^n})) = [S_n(x, r)]$ (this will ensure that f is an homeomorphism, since the sets $B(r, \frac{1}{2^n})$ and $[S_n(x, r)]$, $r \in \mathbb{S}^1$, $n < \omega$, are respectively bases of open sets of $\mathbb{S}^1$ and X) :

If $s_{E'} \in [S_n(x, r)]$, then $\models S_n(r, s)$, ie $d'(r, s) < \frac{1}{2^n}$, so $[S_n(x, r)] \subseteq f(B(r, \frac{1}{2^n}))$.

Conversely, suppose $d'(r, s) < \frac{1}{2^n}$, ie $\models S_n(r, s)$; then, by Lemma 1.4.4, $\models S_n(r, b)$ for every representative b of $s_{E'}$, so that $s_{E'} \in [S_n(x, r)]$, and $f(B(r, \frac{1}{2^n})) \subseteq [S_n(x, r)]$.

To prove $f(\overline{B}(r, \frac{1}{2^n})) = \langle S_n(x, r)\rangle$, first observe that since f is an homeorphism and $\overline{B}(r, \frac{1}{2^n})$ is the topological closure of $B(r, \frac{1}{2^n})$ (this is not true in every metric space), we have
$f(\overline{B}(r, \frac{1}{2^n})) = \overline{f(B(r, \frac{1}{2^n}))} = \overline{[S_n(x, a)]} \subseteq \langle S_n(x, r)\rangle$, since $[S_n(x, r)] \subseteq \langle S_n(x, r)\rangle$ and $\langle S_n(x, r)\rangle$ is closed.

The converse : let $s_{E'} \in \langle S_n(x, r)\rangle$, ie $\models S_n(r, b)$ for some representative b of $s_{E'}$; then, $d(r, s) \leq \frac{1}{2^n}$ by Lemma 1.4.4, and we are done. ■

Via the bijection f, every elementary permutation σ of X (ie a permutation induced by an automorphism of $\overline{M}$, or an element of G_X) induces a permutation $\overline{\sigma}$ of $\mathbb{S}^1$.

Proposition 1.4.6 *For every elementary permutation σ of X, $\overline{\sigma}$ is an isometry of $(\mathbb{S}^1, d')$.*

Proof : Let $r_{E'}$, $s_{E'} \in X$, where $r, s \in \mathbb{S}^1$, and let $u, v \in \mathbb{S}^1$ be representatives of $\sigma(r_{E'})$ and $\sigma(s_{E'})$ respectively (so that $u = \overline{\sigma}(r)$ and $v = \overline{\sigma}(s)$). If $d'(r, s) \geq \frac{m}{2^n}$, ie $\models \neg S_n^m(r, s)$, then also
$\models \neg S_n^m(\sigma(r), \sigma(s))$ because σ is an automophism of $\overline{M}$ (here we make the abuse of notation of writing σ for an automorphism, and also for the elementary permutation it induces on the set of classes X). By Lemma 1.4.4, $\models \neg S_n^m(u, v)$ also holds, ie $d'(u, v) \geq \frac{m}{2^n}$.

Applying the same argument to σ^{-1}, we get $d'(r, s) \geq \frac{m}{2^n}$ iff $d'(u, v) \geq \frac{m}{2^n}$.

Because the dyadic numbers are dense in $\mathbb{R}$, this implies that $d'(r, s)$ and $d'(u, v)$ have the same rational cut, whence $d'(\overline{\sigma}(r), \overline{\sigma}(s)) = d'(u, v) = d'(r, s)$. ■

The orthogonal group $O_2(\mathbb{R})$ acts on $\mathbb{S}^1$.

The kernel of this action is trivial (if $\sigma \in O_2(\mathbb{R})$ and $u \in \mathbb{R}^2$ are such that $\sigma(u) \neq u$, then also $\sigma(\dfrac{u}{\|u\|}) = \dfrac{1}{\|u\|}.\sigma(u) \neq \dfrac{u}{\|u\|}$, and $\dfrac{u}{\|u\|} \in \mathbb{S}^1$), and the image is the set of isometries of $(\mathbb{S}^1, d')$ (see appendix A).

Moreover, the isometries of $\mathbb{S}^1$ are also exactly the automophisms of the structure M_2.

Therefore, an isometry of $\mathbb{S}^1$ extends to an automorphism of $\overline{M}$ (and so to an elementary permutation of X), and we have just seen that conversely an elementary permutation of X induces an isometry of $\mathbb{S}^1$.

Resuming we have proved :

Proposition 1.4.7 *$\sigma \mapsto \overline{\sigma}$ is an isomorphism of groups :*
$G_X \to \mathrm{Isom}(\mathbb{S}^1, d') \cong O_2(\mathbb{R})$.

Moreover this isomorphism is continuous, making G_X and $O_2(\mathbb{R})$ isomorphic compact groups.

Here is a proof of continuity:

Recall there is a direct description of the topology of $O_2(\mathbb{R})$ independent of the choice of a basis, avoiding the passage to matrices : it is indeed well known that if $(E, ||.||)$ is a normed vectorial space, then $|||f||| = sup\{f(x) \ : \ ||x|| = 1\}$ is a norm on the vector space $\mathfrak{L}_c(E)$ of continuous endomorphisms of E.

If E is finite dimensional, then $\mathfrak{L}_c(E) = \mathfrak{L}(E)$.

Applying to $E = \mathbb{R}^2$ with $||.|| = ||.||_2$ (the Euclidian norm), we get a norm $|||.|||_2$ for $\mathfrak{L}(\mathbb{R}^2)$, and moreover the *sup* is here a *max* since the unit sphere for $||.||_2$ is compact in that case.

The toplogy on $O_2(\mathbb{R}) \subseteq \mathfrak{L}(\mathbb{R}^2)$ is then the restriction of the topology of the norm $|||.|||_2$ on $\mathfrak{L}(\mathbb{R}^2)$.

Specifically, using the definition of $|||.|||$ above, this topology is given by the following distance D on $O_2(\mathbb{R})$:

$$D(f,g) = max\ \{||f(r) - g(r)||_2 \ : \ r \in \mathbb{S}^1\}$$

Since $f \mapsto f_{\lceil \mathbb{S}^1}$ is an isomorphism of groups : $O_2(\mathbb{R}) \to \mathrm{Isom}(\mathbb{S}^1, d_2)$, one can transport the topology of $O_2(\mathbb{R})$ to $\mathrm{Isom}(\mathbb{S}^1, d_2) = \mathrm{Isom}(\mathbb{S}^1, d')$. Clearly this topology on $\mathrm{Isom}(\mathbb{S}^1, d')$ is given by the distance D' :

$$D'(f,g) = max\ \{d_2(f(r) - g(r))\ ;\ r \in \mathbb{S}^1\}$$

Also recall that $O_2(\mathbb{R})$ has two connected components: $SO_2(\mathbb{R})$ and $O_2(\mathbb{R}) \setminus SO_2(\mathbb{R})$. Therefore, each of these connected components is a clopen set, and it is enough to check the continuity on each one of these.

Let $f \in \mathrm{Isom}(\mathbb{S}^1, d')$. Recall the isomorphism in question is $f \mapsto \rho(\sigma)$, where $\sigma \in \mathrm{Aut}(\overline{M})$ is an extension of f and ρ: $\mathrm{Aut}(\overline{M}) \to G_X$ is the action of $\mathrm{Aut}(\overline{M})$ on X.

- Continuity on $SO_2(\mathbb{R})$: Let $f \in \mathrm{Isom}(\mathbb{S}^1, d')$ be a rotation, and let $n < \omega$. We want to prove that the image of $\mathcal{O}_f = B_{D'}(f, \frac{1}{2^n}) \cap \{$ rotations $\}$ is an open set in G_X. The crucial observation is that if g is another rotation, $||f(r) - g(r)||_2$

does not depend on $r \in S^1$. Fix $r_0 \in S^1$. Then $g \notin \mathcal{O}_f$ iff $\neg S_n(f(r_0), g(r_0))$. Call C_f the image of the complementary of $\mathcal{O}_f$ in the rotations by the isomorphism. Then $\rho^{-1}(C) = \{\sigma \in \mathrm{Aut}(\overline{M}) : \models \neg S_n(\sigma(r_0), f(r_0))\}$ (use Lemma 1.4.4). By Corollary 1.3.6, this is a closed set in G_X and we are done.

- Continuity on $O_2(\mathbb{R}) \setminus SO_2(\mathbb{R})$: This connected component consists precisely of the axial symmetries. The proof is exactly the same, using the same crucial fact as in the case of rotations (this is a bit less obvious than in the case of rotations, but make a drawing and you will be convinced !).

Generalization to $\mathbb{S}^n$ with the intrinsinc metric

All the results of the previous section generalize with very slight modification to $(\mathbb{S}^n, \delta)$, where δ is the intrinsinc metric. Here the structure is

$$M = (\mathbb{S}^n; S_n : n \in \omega)$$

where $S_n(r, s)$ means $\delta(r, s) < \frac{2\pi}{2^n}$. Again, M is definable without parameters in $(\mathbb{R}, +, -, \cdot, <, 0, 1)$, and we can take as monster of $T = Th(M)$ the correponding definable structure $\overline{M}$ in a κ-saturated and κ-homogeneous elementary extension $\mathbb{R}^*$ of $\mathbb{R}$ with $\kappa > |M| = 2^{\aleph_0}$. Let again $X = \overline{M}/E'$.

Still we have G_X isomorphic to $O_n(\mathbb{R})$ as compact groups, for the same reasons (using Appendix A, and the fact that $O_n(\mathbb{R})$ has $SO_n(\mathbb{R})$ and $O_n(\mathbb{R}) \setminus SO_n(\mathbb{R})$ as connected components for the continuity).

1.4.2 Extension of the previous examples and $SO_n(\mathbb{R})$

Recall that there is a notion of *oriented basis* in a $\mathbb{R}$-vector space E of finite dimension n: just fix an arbitrary basis $\mathbf{e} = (e_1, \ldots, e_n)$ of E and call another basis $(e'_1, \ldots, e'_n)$ *direct* or *directly oriented* if $\det_{\mathbf{e}}(e'_1, \ldots, e'_n) > 0$. This cuts the basis into two disjoint classes: the direct and the indirect ones.

Recall also that for any linear map $f \in L(E)$ and any tuple $(e'_1, \ldots, e'_n)$, a fundamental property of determinants gives $\det_{\mathbf{e}}(f(e'_1), \ldots, f(e'_n)) = \det_{\mathbf{e}}(e'_1, \ldots, e'_n).\det f$. Applied in the case where $f \in GL(E)$ and $(e'_1, \ldots, e'_n)$ it immediately implies that $f \in GL(E)$ sends direct basis to direct basis if and only if $\det f > 0$.

In particular, $f \in O_n(\mathbb{R})$ sends direct orthonormal basis to direct orthonormal ones if and only if $f \in SO_n(\mathbb{R})$. $(*)$

Add a new $n+1$-ary predicate P, and consider the structure

$$M = (\mathbb{S}^n; S_i : i < \omega, P)$$

where S_i as the same meaning as before, and $M \models P(r_1, \ldots, r_{n+1})$ iff $(r_1, \ldots, r_{n+1})$ is a direct orthonormal basis of $\mathbb{R}^{n+1}$.

It is clear that M is definable without parameters in $(\mathbb{R},+,-,\cdot,<,0,1)$, so that we can take as a monster model $\overline{M}$ for $T = Th(M)$ the corresponding definable structure in a κ-saturated and κ-homogeneous elementary extension model $\mathbb{R}^* \succ \mathbb{R}$.

The bounded type-definable equivalence relation E' in T is the same as before: $E'(x,y) = \bigwedge_{n<\omega} S_n(x,y)$, therefore the bijection $f\colon r \mapsto e_{E'}$ between $\mathbb{S}^n$ and $X = \overline{M}/E'$ is still valid.

As before, an automorphism σ of $\mathrm{Aut}(\overline{M})$ induces through f a permutation $\overline{\sigma}$ of $\mathbb{S}^n$. Proposition 1.4.6 tells us that such a permutation of $\mathbb{S}^n$ is in fact an isometry of $(\mathbb{S}^n,\delta)$.

But we also have the following:

Lemma 1.4.8 *Let $(r_1,\ldots,r_{n+1})$ be an orthonormal basis, seen as a tuple from $\mathbb{S}^n$. Let $(a_1,\ldots,a_{n+1})$ be a tuple from $\overline{M}$ such that $E'(a_i,r_i)$ for all $1 \le i \le n+1$.*

Then, $\models P(r_1,\ldots,r_{n+1})$ iff $\models P(a_1,\ldots,a_{n+1})$.

Proof : Since $\det\colon (\mathbb{R}^{n+1})^{n+1} \to \mathbb{R}$ is continuous, there exists N such that

$$M \models \forall x_1,\ldots,\forall x_{n+1}\big(\bigwedge_{1\le i\le n+1} S_N(r_i,x_i) \to (P(x_1,\ldots,x_{n+1}) \leftrightarrow P(r_1,\ldots,r_{n+1})\big)$$

The same formula is thus thus true in $\overline{M}$, whence the result. ■

We conclude that $\overline{\sigma}$ above is an isometry of $(\mathbb{S}^n,\delta)$ that respects P. Call such an isometry a *positive* isometry, and the set of such isometries $\mathrm{Isom}^+(\mathbb{S}^n,\delta)$.

The map $\sigma \mapsto \overline{\sigma}$ is consequently an isomorphism of groups $G_X \to \mathrm{Isom}^+(\mathbb{S}^n,\delta)$.

Lemma 1.4.9 *The map $g \mapsto g_{\lceil \mathbb{S}^n}$ is a group isomorphism $SO_n(\mathbb{R}) \to \mathrm{Isom}^+(\mathbb{S}^n,\delta)$.*

Proof : This is clear, using both Appendix A and $(*)$. ■

The outcome is that G_X is isomorphic to $SO_n(\mathbb{R})$. And it can be proved that moreover this isomorphism is continuous, so that G_X and $SO_n(\mathbb{R})$ are isomorphic as compact groups.

1.4.3 Unitary groups

Consider an n-dimensional vector space E over $\mathbb{C}$, equipped with an Hermitian scalar product $<\,,\,>$. Denote q the associated quadratic form $q(x) =< x,x >$, and $||.||$ the associated norm $||x|| = \sqrt{q(x)}$.

Denote $E_{\mathbb{R}}$ the restriction of scalars of E to $\mathbb{R}$. If $(e_1,\ldots e_n)$ is a basis for E, then $(e_1,ie_1,\ldots,e_n,ie_n)$ is a basis for $E_{\mathbb{R}}$, hence $\dim_{\mathbb{R}}(E_{\mathbb{R}}) = 2n$.

Observe that q is also a quadratic form of a scalar product on $E_{\mathbb{R}}$ (in the sense of real vector spaces): indeed, if the $\mathbb{C}$-basis $(e_1, \ldots, e_n)$ is orthonormal, then via this basis q on E is given by $(z_1, \ldots, z_n) \mapsto \sum_{1 \leq i \leq n} |z_i|^2$. Consequently, via the $\mathbb{R}$-basis $(e_1, ie_1, \ldots, e_n, ie_n)$, q on $E_{\mathbb{R}}$ is given by the map $(x_1, \ldots, x_{2n}) \mapsto \sum_{1 \leq i \leq 2n} x_i^2$, which is certainly the quadratic form of a scalar product on $E_{\mathbb{R}}$.

Henceforth, the Hermitian scalar product on E induces a scalar product on $E_{\mathbb{R}}$ and we can talk about $O(E_{\mathbb{R}})$ without ambiguity.

Since the quadratic forms for E and $E_{\mathbb{R}}$ are the same map q on the common underlying set, and since a $\mathbb{C}$-linear map is obviously $\mathbb{R}$-linear, every $f \in U(E)$ belongs to $O(E_{\mathbb{R}})$. And $U(E) \subseteq O(E_{\mathbb{R}})$ is characterized easily as follows:

Lemma 1.4.10 *Let $f \in O(E_{\mathbb{R}})$. The following are equivalent:*

1. *$f \in U(E)$*
2. *f is $\mathbb{C}$-linear*
3. *For every $x \in E$, $f(ix) = i.f(x)$*

Proof : 1. $\Leftrightarrow$ 2. is obvious, again having in mind that q is the quadratic map for both E and $E_{\mathbb{R}}$.

2. $\Rightarrow$ 3. is obvious.

3. $\Rightarrow$ 2. by hypothesis, f is $\mathbb{R}$-linear. Let $z = a + ib$, and $x \in E$. Then, $f(zx) = f(ax + ibx) = f(ax) + f(ibx) = f(ax) + i\ f(bx)$ by 3.

$= af(x) + ibf(x)$ by $\mathbb{R}$-linearity

$= z.f(x)$. ■

Apply the previous results to the case $E = \mathbb{C}^n$, equipped with the standard Hermitian scalar product $< (z_1, \ldots, z_n), (w_1, \ldots, w_n) >= \sum_i z_i \overline{w}_i$.

Then $E_{\mathbb{R}} = \mathbb{R}^{2n}$, and $\{\mathbf{x} \in \mathbb{C}^n \mid q(\mathbf{x}) = 1\}$ is clearly $\mathbb{S}^{2n-1} \subseteq \mathbb{R}^{2n}$.

Appendix A says that $f \mapsto f\lceil\mathbb{S}^{2n-1}$ is an isomorphism of groups $O_{2n}(\mathbb{R}) \to \mathrm{Isom}(\mathbb{S}^{2n-1}, \delta)$. The following result characterizes the image of $U_n(\mathbb{C})$ inside $\mathrm{Isom}(\mathbb{S}^{2n-1}, \delta)$. Denote $\mathrm{Isom}_{\mathbb{C}}(\mathbb{S}^{2n-1}, \delta)$ this image.

Observe that scalar multiplication by i in $\mathbb{C}^n$ induces a permutation $\mathfrak{g}$ on $\mathbb{S}^{2n-1} = \{\mathbf{x} \in \mathbb{C}^n \mid q(\mathbf{x}) = 1\}$, since $q(\mathbf{x}) = 1$ iff $q(i.\mathbf{x}) = |i|^2 q(\mathbf{x}) = 1$.

Lemma 1.4.11 *Let $f \in \mathrm{Isom}(\mathbb{S}^{2n-1}, \delta)$. Then, $f \in \mathrm{Isom}_{\mathbb{C}}(\mathbb{S}^{2n-1}, \delta)$ if and only if $f(\mathfrak{g}(r)) = \mathfrak{g}(f(r))$ for every $r \in \mathbb{S}^{2n-1}$.*

Proof : If $f = \widehat{f}\lceil\mathbb{S}^{2n-1}$ for some element $\widehat{f}$ of $U_n(\mathbb{C})$, then certainly $f(\mathfrak{g}(r)) = f(ir) = if(r) = \mathfrak{g}(f(r))$ since $\widehat{f}$ is $\mathbb{C}$-linear.

Conversely, if f is an isometry with this property, extend it by Appendix A to an element $\widehat{f}$ of $O_{2n}(\mathbb{R})$. Let $\mathbf{x} \neq 0 \in \mathbb{C}^n$. Then

$$\widehat{f}(i\mathbf{x}) = \widehat{f}(i||\mathbf{x}||\frac{\mathbf{x}}{||\mathbf{x}||}) = ||\mathbf{x}||\widehat{f}(i\frac{\mathbf{x}}{||\mathbf{x}||}) \quad \text{by } \mathbb{R}\text{-linearity of } \widehat{f}$$

$$= ||\mathbf{x}||.i\widehat{f}(\frac{\mathbf{x}}{||\mathbf{x}||}) \quad \text{by hypothesis on } f \text{ and since } \frac{\mathbf{x}}{||\mathbf{x}||} \in \mathbb{S}^{2n-1}$$

$$= \frac{||\mathbf{x}||}{||\mathbf{x}||}.i\widehat{f}(\mathbf{x}) = i\widehat{f}(\mathbf{x}) \quad \text{again by } \mathbb{R}\text{-linearity of } \widehat{f}.$$

By Lemma 1.4.10, $\widehat{f}$ belongs to $U_n(\mathbb{C})$. ■

Introduce the structure

$$M = (\mathbb{S}^{2n-1}; S_i : i < \omega, \mathfrak{g})$$

where $S_i(x, y)$ has the same meaning as in 1.4.1, and $\mathfrak{g}$ is as scalar multiplication by i as above.

It is clear that M is definable without parameters in $(\mathbb{R}, +, -, \cdot, <, 0, 1)$, so that we can take as a monster model $\overline{M}$ for $T = Th(M)$ the corresponding definable structure in a κ-saturated and κ-homogeneous elementary extension model $\mathbb{R}^* \succ \mathbb{R}$.

The bounded type-definable equivalence relation E' in T is the same as before: $E'(x, y) = \bigwedge_{n<\omega} S_n(x, y)$, therefore the bijection $f\colon r \mapsto e_{E'}$ between $\mathbb{S}^{2n-1}$ and $X = \overline{M}/E'$ is still valid.

As before, an automorphism σ of $\mathrm{Aut}(\overline{M})$ induces through f a permutation $\overline{\sigma}$ of $\mathbb{S}^{2n-1}$. Proposition 1.4.6 tells us that such a permutation of $\mathbb{S}^{2n-1}$ is in fact an isometry of $(\mathbb{S}^{2n-1}, \delta)$.

But we have the following

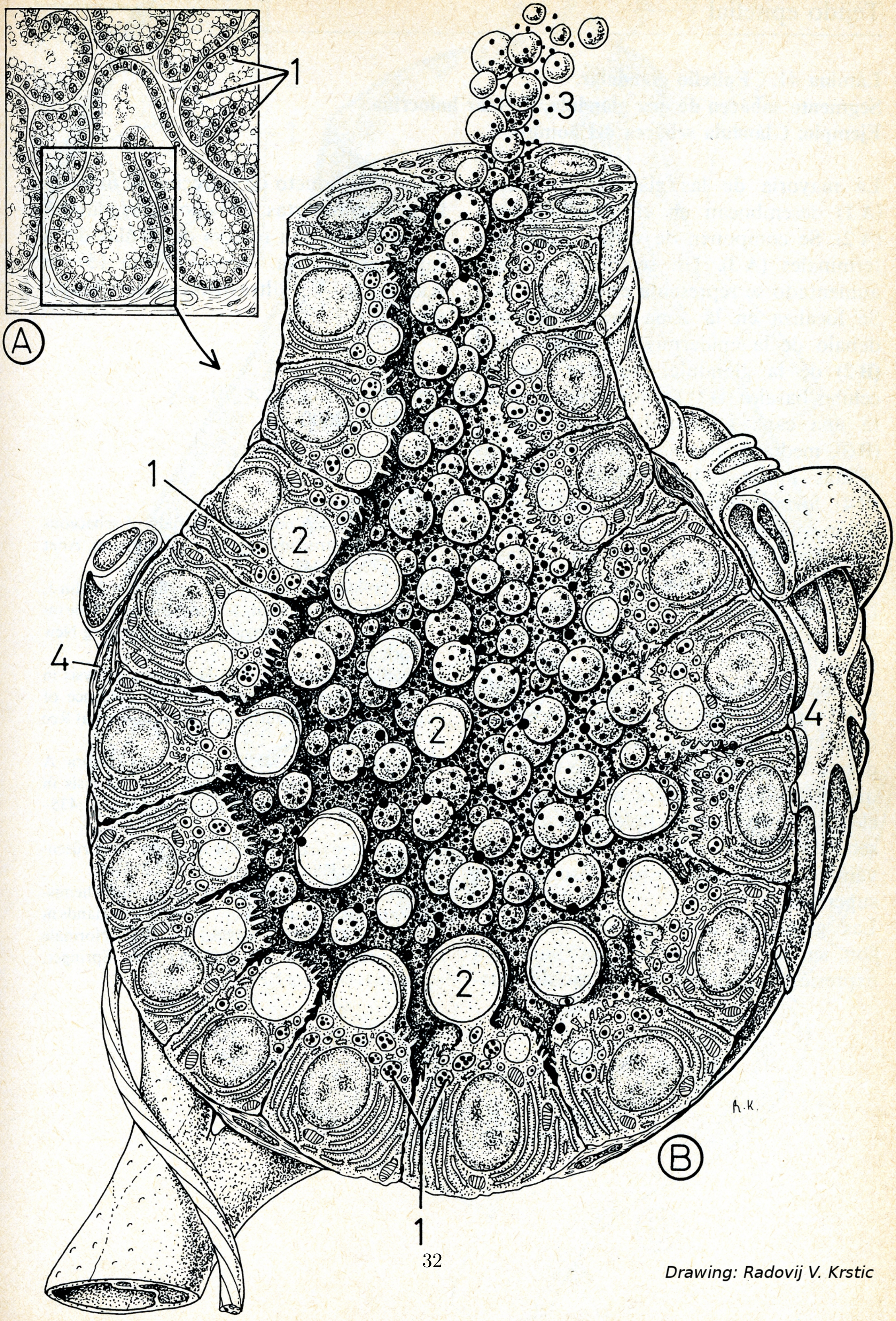
Lemma 1.4.12 *Let $r \in M$ and $a \in \overline{M}$. Then, $\overline{M} \models E'(r, a)$ iff $\overline{M} \models E'(\mathfrak{g}(r), \mathfrak{g}(a))$.*

Proof : Since $\mathfrak{g}$ is bicontinuous on $\mathbb{S}^{2n-1}$, for every $n < \omega$ there exists N such that $M \models \forall x \left(S_N(r, x) \to S_n(\mathfrak{g}(r), \mathfrak{g}(x))\right)$. Since the same holds in the elementary extension $\overline{M}$, we get the result. ■

As a consequence of this Lemma, $\overline{\sigma}$ above is in fact an isometry that respects $\mathfrak{g}$, i.e. an element of $\mathrm{Isom}_{\mathbb{C}}(\mathbb{S}^{2n-1}, \delta)$ (or equivalently an element of $\mathrm{Aut}(M)$).

Jointing all together, the conclusion is that $\sigma \mapsto \widehat{\overline{\sigma}}$ is an isomorphism between G_X and $U_n(\mathbb{C})$. Again, it can be shown that moreover it is continuous, i.e. an isomorphism of compact groups.

La machine à vapeur à remonter le temps



Drawing: Radovij V. Krstic

Chapter 2

Normal Hyperimaginaries

This chapter is an attempt to completely avoid the use of knowledge about compact Hausdorff groups in the proof of elimination of bounded hyperimaginaries as given in [4]. I didn't succeed in it, but I reduced the proof of elimination of hyperimaginaries to a model-theoretic condition called Peter-Weyl condition. Moreover, I fixed an error in the proof of the original paper [4].

Another approach of these issues would be sort of an opposite one: take account of the use of knowledge on compact groups to prove some result of elimination of hyperimaginaries, but go further, trying to see the appearance of compact objets as a particular case of a vaster framework, and take advantage of that wider context to prove other results of elimination of hyperimaginaries. This is the spirit of Chapter 4.

2.1 Background : hyperimaginaries, bounded hyperimaginaries

I will present here rapidly some facts, mainly without proofs. For detailed background on hyperimaginaries, please refer to [5]

2.1.1 Imaginaries and algebraic imaginaries

Fix a 0-definable equivalence relation in some complete theory T in language L, defined by a formula without parameters $E(x, y)$ with x, y finite tuples of variables of the same length n. An *imaginary* is a class of E inside some arbitrary model M of T, i.e. an object of the form a_E, where a is a tuple from M.

Can such object be considered as an element of some first-order structure in some language (which has a "reasonable" relation with the original language L) ? the answer is yes, and this is the famous T^{eq} construction of S.Shelah. Please refer to Appendix C for an exposition of that construction and motivations for considering definable equivalence relations if you are not used to it.

Now if instead of fixing a definable equivalence relation, you fix an arbitrary model M of T, and let vary the definable equivalence relations of T, to each such relation E corresponds an interpretation S_E^M of the sort S_E in M, namely the quotient set M^n/E, so that the disjoint union of the S_E^M's as E vary is the (multi-sorted) structure M^{eq} in the language L^{eq}, model of the complete theory T^{eq}. And every model of T^{eq} is isomorphic to some M^{eq} with $M \models T$ (again c.f. Appendix C).

Moreover, $\overline{M} \models T$ is κ-saturated and κ-homogeneous iff $\overline{M}^{eq} \models T^{eq}$ is so, which is a direct consequence of the fact that L^{eq} keeps a "reasonable" relation with L. Henceforth, the monster models of T^{eq} are very well described in terms of those of T, and inside such a model one can take advantage of the general machinery (the fact that one works within multi-sorted logic does not change almost nothing compared with the usual "one-sorted" logic). For example (and essentially) :

- if $A \subseteq \overline{M}^{eq}$ has size less than κ, then the orbit of a small tuple a from $\overline{M}^{eq}$ under $\mathrm{Aut}(\overline{M}^{eq}/A)$ (isomorphic to $\mathrm{Aut}(\overline{M}/A)$, c.f. Appendix) is exactly the set of realizations of $\mathrm{tp}(a/A)$ in $\overline{M}^{eq}$ (which is κ-homogeneity).

- Again if $A \subseteq \overline{M}^{eq}$ has size $< \kappa$, an element $a \in \overline{M}^{eq}$ satisfies a formula (in L^{eq}) with parameters from A with just one [respectively finitely many] solution in $\overline{M}^{eq}$ iff the orbit of a under $\mathrm{Aut}(\overline{M}^{eq}/A) = \mathrm{Aut}(\overline{M}/A)$ is a singleton [resp. finite] (which is a direct consequence of κ-homogeneity and κ-saturation).

Again, let me insist that those properties are not specific to the eq-construction: they are true in any κ-saturated and κ-homogeneous structure $\overline{M}$ in any (one-sorted or multi-sorted) language. They allow, inside such a structure, to express model-theoretic notions (making intervening formulas of the language) in terms of the action of groups of the type $\mathrm{Aut}(\overline{M}/A)$ on some $\overline{M}^{\alpha}$ (avoiding any reference to formulas).

The properties the second item above characterizes are called *definability [resp algebraicity] over* A. More precisely, we say in that case that a is *definable [resp algebraic]* over A (in the case of the language of rings, think of an element of an extension L/K of fields which is algebraic over K in the classical sense of theory of fields to understand the terminology). And that a formula (with parameters) is *algebraic* if the definable set it defines is finite.

Inside a structure which is not κ-saturated and κ-homogeneous (or if the structure is κ-saturated and κ-homogeneous, but the subset A has size $\geq \kappa$), we can also talk of an element a algebraic over a subset A. But in that case, we just have to come back to the model-theoretic characterization in terms of formulas over A. Let $\mathrm{acl}_M(A)$ be the set of elements of M algebraic over A [resp. $\mathrm{dcl}_M(A)$ the elements definable over A]. acl^{eq} and dcl^{eq} are often used in the particular case of the language L^{eq}.Using the definition of algebraicity in terms of algebraic formulas, it is easy to see that for every $A \subseteq M$, $\mathrm{acl}_M(\mathrm{acl}_M(A)) = \mathrm{acl}_M(A)$ [and likewise for dcl_M]. This is a crucial property of acl_M and dcl_M, which along with the trivial properties $A \subseteq \mathrm{acl}_M(A)$, $A \subseteq B$ implies $\mathrm{acl}_M(A) \subseteq \mathrm{acl}_M(B)$ [and likewise for dcl_M], says that acl_M [resp. dcl_M] is a *closure* operator on M.

Moreover, still using the definition with algebraic formulas, we see immediately that $|\mathrm{acl}_M(A)| \leq |A| + |T|$. Consequently, if $\kappa > |T|$, $\overline{M}$ is a κ-saturated and κ-homogeneous model, and $A \subseteq \overline{M}$ has size $< \kappa$, then still $\mathrm{acl}_{\overline{M}}(A)$ has size $< \kappa$. As we will see in the next subsection, this will not be the case for $\mathrm{bdd}_{\overline{M}}$.

Suppose $M \prec N$ (any language, arbitrary structures), $a \in M$ and $A \subseteq M$. Then clearly (using the only possible definition of algebraicity in the general case, i.e. with algebraic formulas) a is algebraic over A in M iff a is algebraic over A in N. Also, if $b \in N$ is algebraic over A, then $b \in M$, so that $\mathrm{acl}_M(A) = \mathrm{acl}_N(A) \quad (*)$, and we can drop the dependence on the model, writing simply acl and dcl in place of acl_M and dcl_M. Observe that $\mathrm{acl}(A) = \mathrm{acl}_M(A)$ is a sub-structure of M (not elementary sub-structure in general however !).

Henceforth, one can always use, if he or she wishes, the characterization of algebraicity by automorphisms, taking N as a sufficiently saturated and homogeneous elementary extension of M (specifically $|A|^+$-saturated and -homogeneous), which is always possible as well known.

Another consequence of $(*)$ is that in a complete theory T, since every two models always elementary embed in a third one, and since the image of $\mathrm{acl}_M(A)$ by an isomorphism $f\colon M \to N$ is clearly $\mathrm{acl}_N(f(A))$, two sub-structures $\mathrm{acl}_M(\emptyset)$ and $\mathrm{acl}_N(\emptyset)$ from two arbitrary models M, N of T are isomorphic. The outcome is that $\mathrm{acl}(\emptyset)$ is an invariant of T (and more generally $\mathrm{acl}(A)$ is an invariant of T_A).

Observe that if $A \subseteq M$, then

$$|\mathrm{acl}(A)| \leq |T| + |A| \quad (**)$$

This is just because an algebraic formula over A has by definition only a finite number of solutions in M, and there is at most $|T| + |A|$ formulas over A.

In particular, $(**)$ is true for acl^{eq}, but as we will see below, this is not true anymore for bdd, although quite.

An *algebraic imaginary* is by definition an imaginary which is algebraic over $\emptyset$.

Observe that imaginaries and algebraic imaginaries can be defined as equivalence classes inside *arbitrary* models of T, although it is certainly very usefull to consider only those defined inside a monster model of T for all the reasons discussed above.

This will *not* be the case for hyperimaginaries : these objects *have* to be defined as equivalence classes inside certain saturated and homogeneous models, and it makes no sense to define them in arbitrary models.

2.1.2 Hyperimaginaries and bounded hyperimaginaries

If, instead of considering just 0-definable equivalence relations, you also want to consider 0-type definable equivalence relations (see Appendix C.3. for motivations to do that), what happens ? you could add, for each such equivalence relation $E(x,y)$, a new sort S_E of a multi-sorted language, as well as a symbol p_E, aimed to be interpreted as the projection $M^{|x|} \to M^{|x|}/E$ for an arbitrary model $M \models T$ (just as was done in the case of imaginaries).

The problem here is that you lose the "reasonable" relationship with the original language L. For example, there is no way of expressing in the original language that two elements a_E, b_E in the "sort" S_E^M are distinct, since the negation of a type is no longer a type.

Consequently, you have to forget to introduce a new language, and have to stick with the original language L in that case. But recall that in a "big" saturated and homogeneous model, most of the model-theoretic notions can be rephrased without the use of the language, just in terms of the action of the automorphism group on the tuples.

Henceforth, the only possibility we are left with is to hope, if we are lucky, that if M is not an arbitrary model, but a κ-saturated and κ- homogeneous model $\overline{M}$, then the action of $\mathrm{Aut}(\overline{M})$ on the disjoint union of the $S_E^{\overline{M}}$'s have some model-theoretic significance of one kind or another. But certainly we have to restrict to those type-definable equivalence relations of tuples less than κ if we have to expect anything in that direction, so that we are forced to consider the disjoint union of the $S_E^{\overline{M}}$'s, where $E(x,y)$ is a type-definable equivalence relation with $|x| < \kappa$. Call this union $\overline{M}^{heq}$. An *hyperimaginary* is an element belonging to some $\overline{M}^{heq}$. Another way of saying it : an *hyperimaginary* is an equivalence class a_E of some type-definable equivalence relation $E(x,y)$ in T, where a is a tuple in some κ-saturated and κ-homogeneous model and $|x| < \kappa$.

Observe the difference with M^{eq} : first of all, we defined M^{eq} for an abitrary model M, while here we define $\overline{M}^{heq}$ only for "monster" models $\overline{M}$. And second, the whole set of definable equivalence relations where taking into account in each M^{eq}, while only some of the type-definable equivalence relations are taking into account to define $\overline{M}^{heq}$; a type-definable equivalence relation $E(x,y)$ with $|x| \geq \kappa$ will be taken into account, but in another monster model, specifically any λ-monster model, with $|x| < \lambda$.

Clearly $\mathrm{Aut}(\overline{M})$ acts on $\overline{M}^{heq}$. Now given $h, e = a_E \in \overline{M}^{heq}$, the hope I refered to above is that the orbit of e under $\mathrm{Aut}(\overline{M}/h)$ would have some model-theoretic meaning in the language L. It turns out that it is true: there exists a partial type $\Phi(x)$ (in L) over some representative of h such that for every $b_E \in S_E^{\overline{M}}$: $b_E = \sigma(e)$ for some $\sigma \in \mathrm{Aut}(\overline{M}/h)$ if and only if $\models \Phi(b)$. This is an easy exercise using compactness, see This partial type in L is called for obvious reasons the type of e over h (in $\overline{M}$), $\mathrm{tp}^{\overline{\mathrm{M}}}(e/h)$. Using the explicit expression of Φ, it is immediate to see that if $\overline{N} \succ \overline{M}$ is another monster model, then $\mathrm{tp}^{\overline{\mathrm{M}}}(e/h) = \mathrm{tp}^{\overline{\mathrm{N}}}(e/h)$, which is not a surprise of course, since it is the case for ordinary complete types. The notation for this type is $\mathrm{tp}(e/h)$, and we note, as usual, $e \equiv_h e'$ when $\mathrm{tp}(e/h) = \mathrm{tp}(e'/h)$ (as sets of L-formulas over the same representative of h).

Of course, in the case of imaginaries $\overline{M}^{eq}$, and since we have at our disposal the language L^{eq} and $\overline{M}^{eq}$ is κ-saturated and κ-homogeneous as an L^{eq}-structure, $\Phi(x)$ is just $\mathrm{tp}(e/h)$, a complete type over h in the language L^{eq}.

Why having considered only orbits under $\mathrm{Aut}(\overline{M}/h)$ for $h \in \overline{M}^{heq}$, and not under $\mathrm{Aut}(\overline{M}/A)$, for $A \subseteq \overline{M}^{heq}$ with $|A| < \kappa$, as would have been expected ? This is because the latter can be reduced to the former: we can indeed assume, since regular cardinals are ubiquitous in ZFC, that κ is regular. In that case, enumerate A as a small tuple $(e_i : i < \mu)$ $(\mu < \kappa)$. Each e_i is of the form $[a_i]_{E_i}$, where a_i is a tuple $(a_i^j : j \in \mu_i)$ of length $\mu_i < \kappa$. Let $I = \bigcup_{i<\mu} \{i\} \times \mu_i$, a set of size $< \kappa$ by regularity of κ. Let $E((x_i : i \in I), (y_i : i \in I))$ be the 0-type definable equivalence relation $\bigwedge_{i<\mu} E_i((x_j : j \in \{i\} \times \mu_i), (y_j : j \in \{i\} \times \mu_i))$. Let $b = (b_i : i \in I)$ be the "concatenation" of the tuples a_i, namely $b_{(i,j)} = a_i^j$. Then clearly, $\mathrm{Aut}(\overline{M}/A) = \mathrm{Aut}(\overline{M}/b_E)$.

In the next definitions, I will also only use a single hyperimaginary of $\overline{M}^{heq}$ as parameter set, but keep in mind that it also takes into account parameter sets of hyperimaginaries of size less than κ as the discussion above explains.

Now recall that in a κ-monster model $\overline{M}$, an element a is definable [resp algebraic] over A ($|A| < \kappa$), iff it has a one-element [resp a finite] orbit under $\mathrm{Aut}(\overline{M}/A)$. In particular, an imaginary $e \in \overline{M}^{eq}$ is definable [resp algebraic] over another imaginary h iff it has a one-element [resp finite] orbit under $\mathrm{Aut}(\overline{M}/h)$. And by κ-saturation, if e is not algebraic over h, the orbit of e under $\mathrm{Aut}(\overline{M}/h)$ has size $\geq \kappa$.

However, if an hyperimaginary $e \in \overline{M}^{heq}$ does not have a finite orbit under $\mathrm{Aut}(\overline{M}/h)$ ($h \in \overline{M}^{heq}$), it does not have necessarily an orbit $\geq \kappa$: for example in 1.4.1, for any two elements $r, s \in M_1$ there exists $f \in \mathrm{Aut}(M_1)$ such that $f(r) = s$ (for example a rotation of $\mathbb{S}^1$) so that $r \equiv s$. Therefore, in any monster model $\overline{M} \succ M_1$, the orbit of an E-class under $\mathrm{Aut}(\overline{M})$ is $\{r_E : r \in M_1\}$, of size $2^{\aleph_0}$ (recall that every E-class in $\overline{M}$ has a representative in M_1). This proves that if $\overline{M}$ is κ-saturated with $\kappa > 2^{\aleph_0}$, this orbit has size less than κ, but is not finite.

This phenomenon guides us to define the following :

Let $\overline{M}$ be κ-saturated κ-homogeneous and $e = a_E, h = b_F \in \overline{M}^{heq}$ $(E(x,y), F(x',y'))$. The hyperimaginary e is *bounded* over h if its orbit under $\mathrm{Aut}(\overline{M}/h)$ has size $< \kappa$.

We have to check however that this definition is model-theoretic in nature, i.e. that it does not depend on the choice of $\overline{M}$. By this I mean that if $\overline{N} \succ \overline{M}$ is another λ-saturated and λ-homogeneous model with $|x|, |x'| < \lambda$, then a_E (in $\overline{N}$) still has an orbit under $\mathrm{Aut}(\overline{N}/h)$ of size less than λ. We will see indeed that the map $x_E^{\overline{M}} \mapsto x_E^{\overline{N}}$ is a one-one correspondence between the orbit of $a_E^{\overline{M}}$ under $\mathrm{Aut}(\overline{M}/b_F^{\overline{M}})$ and the orbit of $a_E^{\overline{N}}$ under $\mathrm{Aut}(\overline{N}/b_F^{\overline{N}})$. Let $\Phi(x) = \mathrm{tp}^{\overline{M}}(a_E^{\overline{M}}/b_F^{\overline{M}}) = \mathrm{tp}^{\overline{N}}(a_E^{\overline{N}}/b_F^{\overline{N}})$.

Enumerate a set of representatives $(c_i : i < \mu)$ of the elements of the orbit of e under $\mathrm{Aut}(\overline{M}/h)$ (so by hypothesis, $\mu < \kappa$). Suppose there exists $d_E^{\overline{N}}$ in the orbit of $a_E^{\overline{N}}$ under $\mathrm{Aut}(\overline{N}/b_F^{\overline{N}})$, distinct of all the $[c_i^{\overline{N}}]_E$'s. Then, $\overline{N} \models \Phi(d) \wedge \bigwedge_{i<\mu} \neg\varphi_i(c_i, d)$ for some $\varphi_i(x, y) \in E(x, y)$, so that $\Phi(x) \wedge \bigwedge_{i<\mu} \neg\varphi_i(c_i, x)$ is a partial type over $(c_i : i < \mu)$. By κ-saturation of $\overline{M}$, and since $\mu < \kappa$, this type would be realized in $\overline{M}$, in contradiction with the fact that $(c_i : i < \mu)$ is a complete set of representatives of the elements of the orbit of e under $\mathrm{Aut}(\overline{M}/h)$.

The same kind of result holds for the notion of being *definable* over h instead of being bounded in the following sense: $e \in \overline{M}^{heq}$ is *definable* over $h \in \overline{M}^{heq}$ if e has a one-element orbit under $\mathrm{Aut}(\overline{M}/h)$, i.e. $\mathrm{Aut}(\overline{M}/h) \subseteq \mathrm{Aut}(\overline{M}/e)$.

For a fixed $\overline{M}$ and $h \in \overline{M}^{heq}$, call $\mathrm{bdd}_{\overline{M}}(h)$ the set of all $e \in \overline{M}^{heq}$ which are bounded over h, and $\mathrm{dcl}_{\overline{M}}^{heq}(h)$ the set of all $e \in \overline{M}^{heq}$ which are definable over h.

As we saw above, a fundamental property of acl and dcl (in particular acl^{eq} and dcl^{eq}) is that, in any structure M, it defines a closure operator on $\mathfrak{P}(M)$, essentially meaning that $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$. The problem with $\mathrm{bdd}_{\overline{M}}$ is that it cannot be composed twice inside $\overline{M}^{heq}$: the formal reason is that $\mathrm{bdd}_{\overline{M}}(e) \subseteq \overline{M}^{heq}$ always has size $\geq \kappa$ (for every $\alpha < \kappa$, the unique class of the trivial equivalence relation on tuples of length α given by $E((x_i : i < \alpha), (y_i : i < \alpha))$ iff $x_0 = x_0$, belongs to $\mathrm{bdd}_{\overline{M}}(e)$), so that a tuple enumerating $\mathrm{bdd}_{\overline{M}}(e)$ is not an element of $\overline{M}^{heq}$, and cannot be applied $\mathrm{bdd}_{\overline{M}}$ again. However, the essential feature of the property is preserved:

$$e \in \mathrm{bdd}_{\overline{M}}(h) \text{ implies that } \mathrm{bdd}_{\overline{M}}(e) \subseteq \mathrm{bdd}_{\overline{M}}(h) \text{ (and the same for } \mathrm{dcl}_{\overline{M}}^{heq}) \quad (**)$$

Two elements e, f of $\overline{M}^{heq}$ are said to be *interdefinable* or *equivalent*, noted $e \sim f$, if e is definable over h and h is definable over e, i.e. $\mathrm{Aut}(\overline{M}/e) = \mathrm{Aut}(\overline{M}/f)$.

An hyperimaginary e is said to be *finitary* if $e \sim a_E$ with a finite tuple, and *countable* if $e \sim a_E$ with a countable tuple. Equivalently (see [4] Lemma 1.9.), an hyperimaginary e is finitary if there exists a finite tuple a such that $e \in \mathrm{dcl}^{heq}(a)$.

If $\overline{N} \succ \overline{M}$ is a λ-saturated and λ-homogeneous model with $\kappa < \lambda$ and $h = b_F \in \overline{M}^{heq}$, the discussion above says that $e = a_E \in \overline{M}^{heq}$ belongs to $\mathrm{bdd}_{\overline{M}}(b_F)$ implies that $a_E^{\overline{N}} \in \overline{N}^{heq}$ belongs to $\mathrm{bdd}_{\overline{N}}(b_F^{\overline{N}})$. Consequently, there is a map $a_E^{\overline{M}} \mapsto a_E^{\overline{N}}$ from $\mathrm{bdd}_{\overline{M}}(h)$ to $\mathrm{bdd}_{\overline{N}}(h)$, obviously injective. But some elements of $\mathrm{bdd}_{\overline{N}}(h)$ does not belong to the image (elements of $\overline{N}^{heq}$ of the form $d_G \in \mathrm{bdd}_{\overline{N}}(h)$, with $\kappa \leq |d| < \lambda$).

This contrasts with the case of imaginaries, where $a_E^M \mapsto a_E^N$ is a one-one correspondence between $\mathrm{acl}_M^{eq}(b_F^M) \subseteq M^{eq}$ and $\mathrm{acl}_N^{eq}(b_F^N) \subseteq N^{eq}$, for every $M \prec N$ and b tuple from M (The reason is that in that case we have the language L^{eq} at our disposal, that $M \prec N$ implies $a_E^M \mapsto a_E^N$ is an elementary embedding from M^{eq} into N^{eq} in the language L^{eq}, and that in any language, an elementary embedding f sends $\mathrm{acl}(A)$ onto $\mathrm{acl}(f(A))$).

But Proposition 1.1.2 immediately implies that every hyperimaginary $e \in \overline{M}^{heq}$

is equivalent to a family $(e_i : i \in I)$ of *countable* hyperimaginaries (observe that in each equivalence relation E_i only appear a countable set X_i of variables, and that $\models E(a, \sigma(a))$ iff $\models \bigwedge_i E_i(a_{\lceil X_i}, \sigma(a_{\lceil X_i}))$ for every automorphism σ) , and $(**)$ implies that if moreover e belongs to $\mathrm{bdd}_{\overline{M}}(h)$, so does the e_i's.

The outcome is that in a sense, all the information contained in $\mathrm{bdd}_{\overline{M}}(h)$ is already present in $\mathrm{bdd}_{\overline{M}}(h) \cap A$, where A is the set of countable hyperimaginaries, and A is contained in every $\overline{M}^{heq}$ for $\overline{M}$ κ-saturated and κ-homogeneous model such that $\kappa > \aleph_0$.

Consequently, we will drop the dependence of $\overline{M}$, and simply write $\mathrm{bdd}(h)$ [resp. $\mathrm{dcl}^{heq}(h)$] in place of $\mathrm{bdd}_{\overline{M}}(h)$ [resp. $\mathrm{dcl}^{heq}_{\overline{M}}$] for $h \in \overline{M}^{heq}$.

As observed a few line above, $\mathrm{bdd}(a_E)$ has always size $\geq \kappa$ in $\overline{M}^{heq}$ although a is a small tuple, in contrast with the case of acl^{eq}, where $|A| < \kappa$ implies $|\mathrm{acl}^{eq}(A)| < \kappa$ (c.f. 2.1.1). But using again countable hyperimaginaries allows to show that, in a certain λ-saturated and λ-homogeneous elementary extension $\overline{N} \succ \overline{M}$, there exist some $e \in \overline{N}^{heq}$ such that every automorphism of $\overline{N}$ that fixes e fixes $\mathrm{bdd}_{\overline{N}}(a_E^{\overline{N}})$ (provided that λ satisfies some strong conditions, adapt proof of 15.18 in[5] to the present context).

Thus, this feature of acl is in a sense also preserved in the context of bdd.

A last word: there is a more elegant presentation of hyperimaginaries, bdd and dcl^{heq}, avoiding this apparent dependence on the model $\overline{M}$, and taking into account just one automorphism group of a certain structure, and not the various $\mathrm{Aut}(\overline{M})$. Indeed, consider the monster model $\mathfrak{C} \models T$ as a proper class in BNG instead of sets of ZFC (c.f. Preliminaries). Define a hyperimaginary as an object of the form a_E, where a is a tuple in $\mathfrak{C}$ indexed by a *set*, and E is a 0-type-definable equivalence relation on T, and $\mathfrak{C}^{heq} = \bigcup_E \mathfrak{C}^\alpha/E$, as E runs through such equivalence relations (a proper class). In this context, define "e bounded over h" if the orbit of e under $\mathrm{Aut}(\mathfrak{C}/h)$ is a *set*. It is easy to see that this presentation is equivalent to the other one, basically because every tuple a from $\mathfrak{C}$ indexed by a set lies into an elementary κ-saturated and κ-homogeneous (some κ) sub-structure $\overline{M} \prec \mathfrak{C}$ which is indeed a *set*. And vice versa. But $\mathfrak{C}$ is not a canonical object associated to T, while the class of models of T in ZFC, as a whole, is a such a canonical object. This is why the important thing, in the end, is what happens at the level of models of T in ZFC, and that the use of $\mathfrak{C}$ could fool a beginner who is not fully aware of the whole picture. However, although $\mathfrak{C}$ is not a canonical object associated to T, it can be proved that two such proper classes are isomorphic (just as two saturated structures in ZFC are isomorphic), so that it is not so bad in the end, and certainly a much more elegant framework to present all those notions, again at the condition of being aware of the whole picture. The only cost of it is, in order to avoid any problem of foundation, to know about the axiomatic of BNG, and to check that all the constructions involved are valid in this context. But nothing is free.

In the sequel we fix a κ-saturated and κ-homogeneous model of a complete theory T, with κ regular $> 2^{|T|}$.

For a hyperimaginary e, let $\mathrm{Fix}(e) = \mathrm{Aut}(\overline{M}/e)$ be the group of automorphisms of the monster model $\overline{M}$ fixing e. Thus, a hyperimaginary d is *definable over* e if $f(d) = d$ for all $f \in \mathrm{Fix}(e)$. If A is a set of hyperimaginaries of size less that κ, $e \sim A$ means that $e \sim d$ for a sequence d enumerating A as explained above. In some cases we will be interested in automorphisms fixing A set-wise. We write $e \overset{\mathrm{sw}}{\sim} A$ to mean that $\mathrm{Fix}(e)$ is the set of all automorphisms f such that $f(A) = A$ (set-wise). If $(A_i : i \in I)$ is a sequence of sets, we write $e \overset{\mathrm{sw}}{\sim} (A_i : i \in I)$ meaning that $\mathrm{Fix}(e)$ is the set of all automorphisms f such that $f(A_i) = A_i$ for all $i \in I$.

As explained above, there is a single hyperimaginary e which is interdefinable with $\mathrm{bdd}(\emptyset)$, in the sense that $\mathrm{dcl}(e) = \mathrm{bdd}(\emptyset)$. For any index set I, the relation $\equiv_{\mathrm{bdd}(\emptyset)}$ of having the same type over $\mathrm{bdd}(\emptyset)$ restricted to I-tuples is the smallest bounded (i.e., with a number of classes less than κ) 0-type-definable equivalence relation on I-tuples. It is also called the *Kim-Pillay* equivalence relation and its classes are called KP-strong types. The set of all KP-classes of α-tuples is $\overline{M}^{\alpha}/\mathrm{KP}$.

Lascar and Pillay proved in [4] that every bounded hyperimaginary is equivalent to a sequence of finitary hyperimaginaries. Their proof rely on an application of the Peter-Weyl theorem on the structure of compact Hausdorff groups according to which each such group is an inverse limit of compact Lie groups. We seek a purely model-theoretical proof of the same result, avoiding the use of the Peter-Weyl theorem. There are particular cases where the existence of such a sequence of finitary hyperimaginaries is easy to guarantee: normal hyperimaginaries and KP-classes (see Proposition 2.2.9 and Lemma 2.3.4 below)

2.2 Normal hyperimaginaries

Recall from the discussion before Lemma 1.3.8 that $\mathrm{Aut}(\overline{M})$ acts on $X = \overline{M}^{\omega}/E_{KP}$ as well as on $\mathrm{bdd}(\emptyset)$

$$\begin{tikzcd} \mathrm{Aut}(\overline{M}) \arrow[r, two heads, "\zeta"] \arrow[d, two heads, "\rho"] & \mathrm{Aut}(\mathrm{bdd}(\emptyset)) \\ G_X \arrow[ur, tail, "\sim"'] & \end{tikzcd}$$

and that those actions have a common kernel $\Gamma_1 = \ker\rho = \ker\zeta$.

Consequently the group $G = G_X$ acts faithfully on $\mathrm{bdd}(\emptyset)$.

Also recall from 1.3 that G is made a compact Hausdorff group, whose closed subgroups are precisely of the form $\mathrm{Fix}_G(e)$ under this action on $\mathrm{bdd}(\emptyset)$, namely $\mathrm{Fix}_G(e) = \{g \in G \mid g \cdot e = e\}$, for some $e \in \mathrm{bdd}(\emptyset)$. According to the Peter-Weyl theorem, there is a family $(G_i : i \in I)$ of normal closed subgroups G_i of G such that $\bigcap_{i\in I} G_i = \{1\}$ and each G/G_i is a compact Lie group. A compact group is a Lie group if and only if it has the descending chain condition (DCC) on closed subgroups. Each G_i is of the form $\mathrm{Fix}_G(e_i)$ for some $e_i \in \mathrm{bdd}(\emptyset)$. Let $\mathrm{Fix}(e_i) = \rho^{-1}(\mathrm{Fix}_G(e_i))$ be the corresponding subgroup of $\mathrm{Aut}(\overline{M})$. Note that $\mathrm{Fix}_G(e)$ is a normal subgroup of G if and only if $\mathrm{Fix}(e)$ is a normal subgroup of $\mathrm{Aut}(\overline{M})$. Moreover $\bigcap_{i \in I} \mathrm{Fix}(e_i) = \mathrm{Aut}(\overline{/}\mathrm{bdd}(\emptyset))$ and therefore $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$. The DCC of G/G_i translates as follows: there is no strictly descending chain $(G_{i,j} : j < \omega)$ of closed subgroups $G_{i,j+1} \leq G_{i,j}$ of G extending G_i. A descending chain of subgroups of G of the form $\mathrm{Fix}_G(e_j)$ is strict if and only if the corresponding descending chain of subgroups $\mathrm{Fix}(e_j)$ of $\mathrm{Aut}(\overline{M})$ is strict. This explains the following definitions:

Definition 2.2.1 A hyperimaginary e is *normal* if $\mathrm{Fix}(e)$ is a normal subgroup of $\mathrm{Aut}(\overline{M})$. A hyperimaginary e is *DCC* if there is no sequence $(e_n : n < \omega)$ of hyperimaginaries $e_n \in \mathrm{dcl}(e)$ such that $e_n \in \mathrm{dcl}(e_{n+1})$ and $e_{n+1} \notin \mathrm{dcl}(e_n)$ for each $n < \omega$.

The Peter-Weyl theorem provides a sequence $(e_i : i \in I)$ of normal DCC hyperimaginaries $e_i \in \mathrm{bdd}(\emptyset)$ such that $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$. We will see that normal hyperimaginaries are bounded and that normal DCC hyperimaginaries are finitary. We will show that in order to prove the Lascar-Pillay theorem it is in fact enough to find a sequence $(e_i : i \in I)$ of finitary normal hyperimaginaries e_i such that $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$.

Definition 2.2.2 Call *Peter-Weyl condition* the statement that there is a sequence $(e_i : i \in I)$ of finitary normal hyperimaginaries e_i such that $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$.

I have not found a proof of the Peter-Weyl condition avoiding the use of the Peter-Weyl theorem, but I can offer an easy-to-follow proof of the Lascar-Pillay theorem assuming this condition.

Lemma 2.2.3 *Let G a group acting on a set X, and $a \in X$. The following are equivalent:*

1. *The isotropy subgroup* $\mathrm{Fix}(a)$ *is normal in G*
2. *For any a' in the orbit $G \cdot a$,* $\mathrm{Fix}(a') = \mathrm{Fix}(a)$
3. *For any a' in the orbit $G \cdot a$,* $\mathrm{Fix}(a') \supseteq \mathrm{Fix}(a)$
4. $\mathrm{Fix}(a) = \mathrm{Fix}(G \cdot a)$
5. $\mathrm{Fix}(a) = \mathrm{Fix}(G \cdot b)$ *for some $b \in X$*

Proof : Immediate, keeping in mind that for every $g \in G$ and $a \in X$, $\mathrm{Fix}(g \cdot a) = g\ \mathrm{Fix}(a)\ g^{-1}$, and that for every subgroup $H \subseteq G$, $\bigcap_{g \in G} gHg^{-1}$ is a normal subgroup in G (the largest normal subgroup in G contained in H). ■

Proposition 2.2.4 *The following are equivalent for any hyperimaginary e:*

1. *e is normal.*
2. *For any $e' \equiv e$, $e' \in \mathrm{dcl}(e)$.*
3. *$e \sim (f(e) : f \in \mathrm{Aut}(\overline{M}))$*
4. *e is equivalent to a sequence enumerating an orbit of a hyperimaginary.*

Proof : A direct application of the previous Lemma. ■

Remark 2.2.5 *Normal hyperimaginaries are bounded.*

Proof : Let e be normal. If $(e_i : i < \kappa)$ is a long enough sequence of different conjugates of e, then we can find $i < j < \kappa$ with $e_i \equiv_e e_j$. Since e_i, e_j are definable over e, $e_i = e_j$, a contradiction. ■

Proposition 2.2.6 *A hyperimaginary e is normal if and only if for any index set I, the equivalence relation $\equiv_e$ on I-tuples is 0-type-definable.*

Proof : Let $(e_j : j \in J)$ be a (bounded) orbit equivalent to e. Then $\equiv_e = \equiv_{(e_j : j \in J)}$, which is clearly invariant and type-definable, hence 0-type-definable.

If $\equiv_e$ is 0-type definable, then also $\equiv_e$ as a relation between hyperimaginaries is 0-type-definable. Let $f \in \mathrm{Fix}(e)$ and $g \in \mathrm{Aut}(\overline{M})$ such that $g(e) = e'$. Then $e' \equiv_e f(e')$. If we apply g^{-1} we see that $e \equiv_e g^{-1}f(e')$ and hence $g^{-1}f(e') = e$. If we apply g we conclude that $f(e') = g(e) = e'$. Therefore $e' \in \mathrm{dcl}(e)$. ■

Remark 2.2.7 *If each e_i is normal, then $(e_i : i \in I)$ is normal.*

Lemma 2.2.8 *Let $e = a_E$ be normal.*

1. *$e \sim a_{\equiv_e}$.*
2. *For any tuple m enumerating a model, $e \sim m_{\equiv_e}$.*

Proof : *1.* If e is normal, then $\equiv_e$ is 0-type-definable and $a_{\equiv_e}$ is a hyperimaginary. Assume first $f \in \mathrm{Fix}(e)$. Then $a \equiv_e f(a)$ and therefore $f(a_{\equiv_e}) = a_{\equiv_e}$. For the other direction, assume now $f(a_{\equiv_e}) = a_{\equiv_e}$. Then $f(a) \equiv_e a$. Since $a_E = e$, $f(a_E) = e$, that is, $f(e) = e$.

2. Assume m enumerates a model. Clearly, $m_{\equiv_e} \in \mathrm{dcl}(e)$. On the other hand, if f fixes $m_{\equiv_e}$ then $m \equiv_e f(m)$ and there is some $g \in \mathrm{Fix}(e)$ such that $g(m) = f(m)$. It follows that fg^{-1} fixes point-wise a model and it is a strong automorphism, which implies it fixes every element of $\mathrm{bdd}(\emptyset)$. Hence $f(e) = fg^{-1}g(e) = fg^{-1}(e) = e$. ■

Proposition 2.2.9 *Every normal hyperimaginary is equivalent to a sequence of finitary hyperimaginaries.*

Proof : Let e be normal. By the previous lemma, $\equiv_e$ is type-definable over $\emptyset$ and $e \sim a_{\equiv_e}$ for some tuple a. Let $a = (a_i : i < \kappa)$ and for each finite $X \subseteq \kappa$ let E^X be defined for κ-tuples b, c by

$$E^X(b,c) \Leftrightarrow b \restriction X \equiv_e c \restriction X.$$

If $e^X = a_{E^X}$, then each e^X is finitary and $e \sim (e^X : X \subseteq \kappa \text{ finite })$. ■

Lemma 2.2.10 *Every normal DCC hyperimaginary is finitary.*

Proof : Let e be normal DCC. Choose, like in the proof of Proposition 2.2.9, a tuple $a = (a_i : i < \kappa)$ such that $e \sim a_{\equiv_e}$ and define E^X and e^X as in that proof. Clearly, $e^X \in \mathrm{dcl}(e)$ and if $X \subseteq Y$, then $e^X \in \mathrm{dcl}(e^Y)$. Since e is DCC, there is some finite X such that for all finite $Y \supseteq X$, $e^Y \in \mathrm{dcl}(e^X)$. It follows that $e \sim e^X$ and hence e is finitary. ■

Proposition 2.2.11 *1. For any* 0*-type-definable equivalence relation on κ-tuples F, for any hyperimaginary e, if $E = \equiv_e$, then the relational product $E \circ F = F \circ E = E \circ F \circ E$ is an equivalence relation.*

2. Given normal e and $d \in \mathrm{bdd}(\emptyset)$, there are a κ-tuple m and a 0*-type-definable equivalence relation F on κ-tuples such that, if E is the* 0*-type-definable equivalence relation $\equiv_e$ on κ-tuples, then $m_E \sim e$, $m_F \sim d$ and $m_{E \circ F} \sim \inf(e, d)$.*

Proof : *1.* We must check symmetry and transitivity of $E \circ F$. For symmetry, assume $a \equiv_e bFc$ and choose an automorphism f such that $f(e) = e$ and $f(a) = b$. Let c' be such that $f(c') = c$. Then $ac' \equiv bc$ and therefore $F(a, c')$. Hence $c \equiv_e c'Fa$. Using now symmetry, for transitivity it is enough to prove that if $a \equiv_e bFc \equiv_e d$, then $aE \circ Fd$. Choose $f \in \mathrm{Fix}(e)$ such that $f(c) = d$. Then $a \equiv_e f(b)Fd$.

2. Let $d = a_G$ for a tuple a, and extend a to a tuple $m = (m_i : i < \kappa)$ enumerating a model. Let $I \subseteq \kappa$ be such that $a = (m_i : i \in I)$ and define F by

$$F(x,y) \leftrightarrow G(x \restriction I, y \restriction I).$$

It is a 0-type-definable equivalence relation and $m_F \sim d$. Let $E = \equiv_e$. By Lemma 2.2.8, $m_E \sim e$. It is clear that $m_{E\circ F} \in \mathrm{dcl}(m_E) \cap \mathrm{dcl}(m_F)$. Now we assume $e' \in \mathrm{dcl}(m_E) \cap \mathrm{dcl}(m_F)$ and we check that $e' \in \mathrm{dcl}(m_{E\circ F})$. For this purpose, let f be an automorphism fixing $m_{E\circ F}$. Then $E \circ F(m, f(m))$ and by symmetry $F \circ E(m, f(m))$. Let b be such that $F(m, b) \wedge E(b, f(m))$. Since $b \equiv_e f(m)$, there is an automorphism $g \in \mathrm{Fix}(e)$ such that $g(b) = f(m)$. Then $F(g(m), g(b))$, that is $F(m, g^{-1}f(m))$. Let $h = g^{-1}f$. Since h fixes m_F, $h(e') = e'$. Since $g \in \mathrm{Fix}(e)$, $m \equiv_e g(m)$ and hence g fixes m_E and $g(e') = e'$. Therefore $f(e') = gh(e') = g(e') = e'$. ■

Remark 2.2.12 *To prove the Peter-Weyl condition it is enough to prove that for every finitary bounded hyperimaginary e there is a family $(e_i : i \in I)$ of finitary normal hyperimaginaries e_i such that $e \in \mathrm{dcl}(e_i : i \in I)$.*

Proof : There is a normal e such that $e \sim \mathrm{bdd}(\emptyset)$. Since e is equivalent to a family of finitary bounded hyperimaginaries and each finitary bounded hyperimaginary is definable over a family of finitary normal hyperimaginary, we conclude that e is definable over a family $(e_i : i \in I)$ of finitary normal hyperimaginaries. It follows that $e \sim (e_i : i \in I)$. ∎

Corollary 2.2.13 (Lascar-Pillay) *Every bounded hyperimaginary is equivalent to a sequence of finitary hyperimaginaries.*

Proof : (Assuming the Peter-Weyl condition) Let d be a bounded hyperimaginary and choose a family $(e_i : i \in I)$ of finitary normal hyperimaginaries such that $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$. Let $\kappa \geq |I|, |d|, |T|$, and for each $i \in I$ let E_i be the equivalence relation $\equiv_{e_i}$ on κ-tuples. Let E be the Kim-Pillay equivalence relation $\equiv_{\mathrm{bdd}(\emptyset)}$ on κ-tuples. We may assume that the family is closed under finite composition (that is, for any $i, j \in I$ there is some $k \in I$ such that $e_k \sim e_i e_j$), which implies $E = \bigcap_{i \in I} E_i$. Choose with Proposition 2.2.11 a 0-type-definable bounded equivalence relation F on κ-tuples and some κ-tuple m such that $d \sim m_F$, $e_i \sim m_{E_i}$ and $\inf(e_i, d) \sim m_{E_i \circ F}$. Since e_i is finitary, $\inf(e_i, d)$ is finitary too. We claim that $d \sim (\inf(e_i, d) : i \in I)$. Notice that $F = E \circ F$. Hence $d \sim m_{E \circ F}$ and it is enough to check that $m_{E \circ F} \in \mathrm{dcl}(m_{E_i \circ F} : i \in I)$. Let f be an automorphism fixing each $m_{E_i \circ F}$. Then for each $i \in I$ there is some a_i such that

$$E_i(m, a_i) \wedge F(a_i, f(m)).$$

By compactness there is some a such that $E(m, a) \wedge F(a, f(m))$. Hence f fixes $m_{E \circ F}$. ∎

Remark 2.2.14 *The Galois correspondence provides another proof of Corollary 2.2.13 in terms of groups. Let d be a bounded hyperimaginary and let $(e_i : i \in I)$ be a family of finitary normal hyperimaginaries such that $(e_i : i \in I) \sim \mathrm{bdd}(\emptyset)$. As above, we may assume that the family is closed under finite composition. Let $H_i = \mathrm{Fix}(e_i)$, a closed normal subgroup of the Galois group of T. Under the Galois correspondence, the conditions on the e_i's means that $\bigcap_i H_i = \{1\}$, and for each i, j there is some k such that $H_i \cap H_j = H_k$. Let $H = \mathrm{Fix}(d)$, and consider $L_i = H.H_i$, a closed subgroup of the Galois group. Again, the Galois correspondence tells us that $L_i = \mathrm{Fix}(h_i)$ for some bounded hyperimaginary h_i, and certainly h_i is finitary since e_i is. Now $\bigcap_i L_i = \bigcap_i H.H_i = H.\bigcap_i H_i = H$ (c.f. Lemma 2.2.15 below), which means that $d \sim (h_i : i \in I)$.*

This has at least the virtue to fix an error in the proof of 2.2.13 in the original paper [4]: at line 11 *of the proof of* 4.15*, it is claimed that*

$$\bigcap\{D_n : n \in \omega\} \subseteq L_i \quad (*)$$

If you note $D'_n = \mathrm{Aut}(\overline{M}/a^n) \subseteq \mathrm{Aut}(\overline{M})$ (so that $D_n = \mu(D'_n)$), the hypothesis on the a^n's says that $\mu(\bigcap_n D'n) \subseteq L_i$. Henceforth $(*)$ is equivalent to $\bigcap_n \mu(D'_n) \subseteq \mu(\bigcap_n D'_n)$. This in turn is equivalent (applying μ^{-1}) to see that $\bigcap_n \Gamma_1 \cdot D'_n \subseteq$

$\Gamma_1 \cdot \bigcap_n D'_n$ *in* $\mathrm{Aut}(\overline{M})$. *This inclusion is certainly true for closed subgroups in a compact group according to Lemma 2.2.15 , but* $\mathrm{Aut}(\overline{M})$ *is not such a group, so that there is no chance this inclusion holds.*

Lemma 2.2.15 *Let G be a compact group, H be a closed subgroup, and $\mathcal{H} = \{H_i : i \in I\}$ be a family of closed subgroups such that $K, L \in \mathcal{H}$ implies that there is some $M \in \mathcal{H}$ with $M \subseteq K \cap L$.*

Then

$$\bigcap_i H.H_i = H.\bigcap_i H_i$$

Proof : The only non obvious inclusion is $\bigcap_i H.H_i \subseteq H.\bigcap_i H_i$. So let $g \in \bigcap_i H.H_i$, and $i_1, \dots, i_n \in I$. By induction on the hypothesis, there is some $H_j \in \mathcal{H}$ such that $H_j \subseteq H_{i_1} \cap \dots \cap H_{i_n}$. Then $g \in H.H_j$, i.e. $g = a_j.h_j$ for some $a_j \in H$ and $h_j \in H_j$. Therefore $g.h_j^{-1} = a_j$, and $g.H_j \cap H \neq \emptyset$, which implies that $(g.H_{i_1} \cap H) \cap \dots \cap (g.H_{i_n} \cap H) \neq \emptyset$. Since the family $\{g.H_i \cap H : i \in I\}$ of closed subsets of G has the finite intersection property and G is compact, it has a non-void intersection. Let $h \in \bigcap_i g.H_i \cap H = \left(\bigcap_i g.H_i\right) \cap H$. Then $h \in H$ and for every i, $h = g.h_i$ for some $h_i \in H_i$, or equivalently $h^{-1}.g \in \bigcap_i H_i$, i.e. $g \in H.\bigcap_i H_i$. ■

In the particular case where H is normal closed, or the family $\mathcal{H}$ consists of normal closed subgroups, then the products $H.H_i$ also are normal closed subgroup. In the general case of closed subgroups, the $H.H_i$'s are just closed subset of G.

2.3 Local types of hyperimaginaries

Definition 2.3.1 Let e, d be hyperimaginaries. The *orbit* of e over d is the set $\mathcal{O}(e/d)$ of all hyperimaginaries e' such that $e \equiv_d e'$.

Remark 2.3.2 *Notice that for an automorphism f, the condition $f(\mathcal{O}(e/d)) = \mathcal{O}(e/d)$ is equivalent to the conjunction of $e \equiv_d f(e)$ and $e \equiv_d f^{-1}(e)$.*

The next lemma is due to Buechler, Pillay and Wagner (Lemma 2.18 in [15]). It basically says that we can consider $\mathcal{O}(e/d)$ as a hyperimaginary if $e \in \mathrm{bdd}(d)$. In Proposition 2.3.6 below I have generalized this fact to any closed set in a Kim-Pillay space. I apply this to some closed sets $\mathcal{O}_\varphi(e/d)$ obtaining thus some hyperimaginaries $h_{p,\varphi,d}$. For $d \in \mathrm{bdd}(\emptyset)$ and $p(x) = \mathrm{tp}(e/\emptyset)$ one can understand $\mathrm{tp}(e/h_{p,\varphi,d})$ as an approximation to the φ-type of e over d.

Remark 2.3.3 *If $e \in \mathrm{bdd}(d)$, then $\mathcal{O}(e/d)$ is $\overset{\mathrm{sw}}{\sim}$-equivalent to some hyperimaginary h, in the sense that the automorphisms of the monster model fixing h is the set of automorphisms fixing set-wise $\mathcal{O}(e/d)$.*

Lemma 2.3.4 *If $e = (a_i : i < \omega)_{\mathrm{KP}}$ and $e_n = (a_i : i \leq n)_{\mathrm{KP}}$, then $e \sim (e_n : n < \omega)$.*

Proof : For every automorphism f, $f \in \mathrm{Fix}(e)$ iff $(a_i : i < \omega) \equiv_{\mathrm{bdd}(\emptyset)} (f(a_i) : i < \omega)$ iff $(a_i : i \leq n) \equiv_{\mathrm{bdd}(\emptyset)} (f(a_i) : i \leq n)$ for all $n < \omega$ iff $f(e_n) = e_n$ for all $n < \omega$. ■

Proposition 2.3.5 *If $d \in \mathrm{bdd}(\emptyset)$, then $d \overset{\mathrm{sw}}{\sim} (\mathcal{O}(e/d) : e \in \overline{M}^{\omega}/\mathrm{KP})$ and $d \overset{\mathrm{sw}}{\sim}$ $(\mathcal{O}(e/d) : e \in \overline{M}^{n}/\mathrm{KP}, n < \omega)$*

Proof : If $f \in \mathrm{Fix}(d)$, then f permutes every orbit $\mathcal{O}(e/d)$.

Assume f permutes every $\mathcal{O}(e/d)$ for every countable KP-class $e \in \overline{M}^{\omega}/\mathrm{KP}$. It is well known that each hyperimaginary is equivalent to a sequence of countable hyperimaginaries. Hence $d \sim (d_i : i \in I)$, where every d_i is a countable hyperimaginary. Choose an ω-tuple a_i and a bounded 0-type-definable equivalence relation E_i such that $d_i = a_{iE_i}$. By hypothesis, $f(a_{i\mathrm{KP}}) \in \mathcal{O}(a_{i\mathrm{KP}}/d)$ and therefore $f(a_{i\mathrm{KP}}) = g_i(a_{i\mathrm{KP}})$ for some $g_i \in \mathrm{Fix}(d)$. Note that d_i is a union of KP-classes of ω-tuples. Since g_i fixes d_i, f permutes these KP-classes and then $f(d_i) = d_i$. Since f fixes each d_i, $f(d) = d$.

Assume now f permutes every $\mathcal{O}(e/d)$ for every finitary KP-class $e \in \overline{M}^{n}/\mathrm{KP}$. We show that f permutes $\mathcal{O}(e/d)$ for every countable KP-class $e \in \overline{M}^{\omega}/\mathrm{KP}$. Let $e = (a_i : i < \omega)_{\mathrm{KP}}$ and let $e_n = (a_i : i \leq n)_{\mathrm{KP}}$. Since $e_n \equiv_d f(e_n)$ for all $n < \omega$, $(e_n : n < \omega) \equiv_d (f(e_n) : n < \omega)$. Choose $g \in \mathrm{Fix}(d)$ such that $g(e_n : n < \omega) = (f(e_n) : n < \omega)$. Then $f^{-1}g(e_n) = e_n$ for all $n < \omega$ and by Lemma 2.3.4 $f^{-1}g(e) = e$. It follows that $e \equiv_d f(e)$ and hence f permutes $\mathcal{O}(e/d)$. ■

Proposition 2.3.6 *Every closed set C in a Kim-Pillay space is $\overset{\mathrm{sw}}{\sim}$-equivalent to a hyperimaginary h_C, that is, the automorphisms of the monster model fixing set-wise C are the automorphisms fixing h_C.*

Proof : Let E be a bounded 0-type-definable equivalence relation on α-tuples and let $X = \overline{M}^{\alpha}/E$ be the corresponding Kim-Pillay space. If $C \subseteq X$ is closed, then for some partial type $\pi(x, z)$, for some tuple b, $\pi(\overline{M}, b) = \{a : a_E \in C\}$. For each formula $\theta(x, y) \in E(x, y)$ there is a maximal length $n = n_\theta < \omega$ of a sequence of tuples $(a_i : i < n)$ such that $a_{iE} \in C$ and $\models \neg\theta(a_i, a_j)$ for all $i < j < n$. Let $(a_i^\theta : i < n_\theta)$ witness it, let $\Sigma_\theta(z, z')$ be the partial type

$$\exists(x_i : i < n_\theta)(\bigwedge_{i<j<n_\theta} \neg\theta(x_i, x_j) \wedge \bigwedge_{i<n_\theta} \pi(x_i, z) \wedge \bigwedge_{i<n_\theta} \pi(x_i, z'))$$

and

$$F(z, z') = \bigwedge_{\theta \in E} \Sigma_\theta(z, z')$$

Claim: For every automorphism f, $f(C) = C$ if and only if $\models F(b, f(b))$.

Proof : From left to right it is straightforward. For the other direction, assume $\models F(b, f(b))$ and choose $(c_i^\theta : i < n_\theta, \theta \in E)$ witnessing it. Let $a_E \in C$. Then $\models \pi(a, b)$, and hence $\models \pi(f(a), f(b))$. By maximality of n_θ, for every $\theta \in E$ there is some $i < n_\theta$ such that $\models \theta(f(a), c_i^\theta)$. By compactness, $E(f(a), x) \cup \pi(x, b)$

is consistent and therefore $f(a_E) \in C$. This shows that $f(C) \subseteq C$. By the same reason, $f^{-1}(C) \subseteq C$, that is, $C \subseteq f(C)$. ■

It follows from the claim that F defines a 0-type-definable equivalence relation on realizations of $p(x) = \mathrm{tp}(b)$. By standard arguments it can be extended to a 0-type-definable equivalence relation defined for all tuples of the length of b. The hyperimaginary b_F satisfies the requirements. ■

Definition 2.3.7 Let e, d be hyperimaginaries. If $\varphi(x, y) \in L$, $\models \varphi(e, d)$ means that $\models \varphi(a, b)$ for some representatives a, b of e, d respectively. Notice that $e \equiv_d e'$ iff $\models \varphi(e, d) \Leftrightarrow \models \varphi(e', d)$ for all $\varphi(x, y) \in L$. Let $\mathcal{O}_\varphi(e/d) = \{e' : e' \equiv e \text{ and } \models \varphi(e', d)\}$. Let $p(x) = \mathrm{tp}(e)$ and assume $e \in \mathrm{bdd}(\emptyset)$. Then $e = a_E$ for some tuple a and some bounded 0-type-definable equivalence relation E. The set of all E-classes is a Kim-Pillay space and $\mathcal{O}_\varphi(e/d)$ defines a closed subset. By Lemma 2.3.6 there is some hyperimaginary $h_{p,\varphi,d}$ such that

$$h_{p,\varphi,d} \overset{\mathrm{sw}}{\sim} \mathcal{O}_\varphi(e/d).$$

The equivalence relation $E(e, e')$ defined by $\models \varphi(e, d) \Leftrightarrow \models \varphi(e', d)$ is not, in general, type-definable. This is the reason why an adequate treatment of local types (or φ-types) is missing in the model theory of hyperimaginaries. The following results show that the types $\mathrm{tp}(e/h_{\mathrm{tp}(e),\varphi,d})$ are (for e bounded) a substitute for the φ-type of e over d and we apply this in Corollary 2.3.10 to obtain a new decomposition of a bounded hyperimaginary in terms of orbits.

Remark 2.3.8 *Let d be a hyperimaginary, $e \in \mathrm{bdd}(\emptyset)$, $p(x) = \mathrm{tp}(e)$ and $\varphi(x, y) \in L$.*

1. *If $e' \equiv_{h_{p,\varphi,d}} e$ then $\models \varphi(e, d) \Leftrightarrow \models \varphi(e', d)$.*
2. *$h_{p,\varphi,d} \in \mathrm{dcl}(d)$.*

Proof : Clear. ■

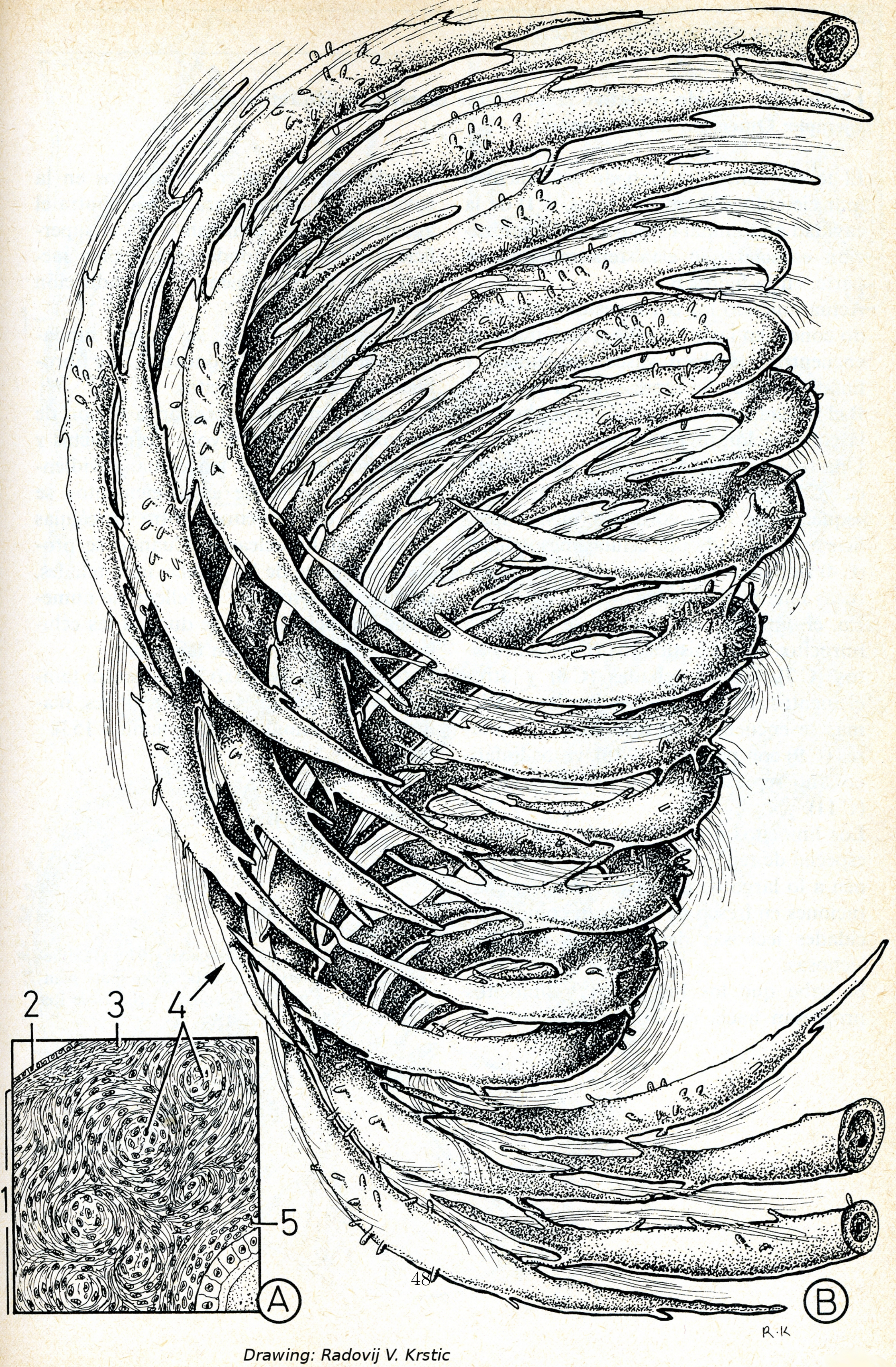
Proposition 2.3.9 *Let d be a hyperimaginary, $e \in \mathrm{bdd}(\emptyset)$ and $p(x) = \mathrm{tp}(e)$. For any $e' \models p$:*

$$e' \equiv_d e \text{ if and only if } e' \equiv_{h_{p,\varphi,d}} e \text{ for every } \varphi(x, y) \in L$$

Proof : By Remark 2.3.8. ■

Corollary 2.3.10 *If $d \in \mathrm{bdd}(\emptyset)$, then $d \overset{\mathrm{sw}}{\sim} (\mathcal{O}(e/h_{\mathrm{tp}(e),\varphi,d}) : e \in \overline{M}^n/\mathrm{KP}, n < \omega, \varphi \in L)$.*

Proof : Let $d \in \mathrm{bdd}(\emptyset)$. If $f \in \mathrm{Fix}(d)$, then f fixes $h_{\mathrm{tp}(e),\varphi,d}$ and permutes $\mathcal{O}(e/h_{\mathrm{tp}(e),\varphi,d})$. On the other hand, if $e \in \overline{M}^n/\mathrm{KP}$ and f permutes all the orbits $(\mathcal{O}(e/h_{\mathrm{tp}(e),\varphi,d})$, then $e \equiv_{h_{\mathrm{tp}(e),\varphi,d}} f(e)$ for all φ and by Proposition 2.3.9 $e \equiv_d f(e)$. Similarly, $e \equiv_d f^{-1}(e)$. It follows that f permutes $\mathcal{O}(e/d)$. By Proposition 2.3.5, $f(d) = d$. ■



Drawing: Radovij V. Krstic

Chapter 3

Stable Forking in T^{eq}

The main result of this chapter is that a simple theory T has the stable forking property if and only if T^{eq} has.

As usual we fix a monster model $\mathfrak{C}$ of T (or a κ-saturated and κ-homogeneous model $\overline{M}$ if you prefer, my schizophrenia is coming back...), and consider only *sets* – not proper classes – of parameters (in particular, models) inside $\mathfrak{C}$ (resp. only subsets of $\overline{M}$ of size less than κ).

3.1 Introduction : stable formulas, local types and canonical bases

A formula $\varphi(x, y)$ (x and y disjoint tuples of variables) has the *order property* in some complete theory T if for some model $M \models T$, and some tuples $(a_i : i < \omega)$, $(b_i : i < \omega)$ from M, holds

$$M \models \varphi(a_i, b_j) \quad \text{if and only if} \quad i < j$$

Or equivalently, if this property holds for some tuples from $\mathfrak{C}$. Such a tuple $(a_i, b_i : i < \omega)$ is called an ω-ladder for $\varphi(x, y)$. n-ladders for $n < \omega$ are defined the same way.

For a formula $\varphi(x, y)$, denote $\varphi^{-1}(y, x)$ the formula $\varphi(x, y)$ with variables y in the first place, and variables x in second place.

A formula is *stable* if it does not have the order property in T. T is stable if every formula is stable in T.

Lemma 3.1.1 *Let $\varphi(x, y), \psi(x, y) \in L$ be formulas, and T be a complete theory.*

1. *Let z be a tuple of variables containing y, and disjoint from x. Then, $\varphi(x, y)$ is stable if and only if $\varphi(x, z)$ is stable.*

2. *$\varphi(x,y)$ is stable in T if and only if for some $n<\omega$, and some (any) model $M \models T$, there is no n-ladder for φ in M.*

3. *$\varphi(x,y)$ is stable if and only if $\varphi^{-1}(y,x)$ is stable.*

4. *$\varphi(x,y)$ is stable if and only if $\neg\varphi(x,y)$ is stable.*

5. *If $\varphi(x,y)$ and $\psi(x,y)$ are stable, then $\varphi \wedge \psi$ and $\varphi \vee \psi$ are stable.*

6. *Any boolean combination $\chi(x; y_1, \ldots, y_n)$ of stables formulas of the form $\varphi_i(x, y_i)$ where y_i are pairwise disjoint tuples of variables, is stable.*

Proof : 1. is obvious by the very definition of the order property.

2. is by compactness.

3. is using 2., since if $(a_0b_0, \ldots, a_{n-1}b_{n-1})$ is an n-ladder for $\varphi^{-1}(y,x)$, then $(a_{n-i}b_{n-i} : 0 \leq i \leq n-1)$ is an n-ladder for $\varphi(x,y)$.

4. is using 2. again, since if $(a_0b_0, \ldots, a_{n-1}b_{n-1})$ is an n-ladder for $\neg\varphi(x,y)$, then $(a_{n-i}b_{n-i-1} : 0 \leq i \leq n-2\}$ is an $(n-1)$-ladder for $\varphi(x,y)$.

5. Suppose $(a_i, b_i : i < \omega)$ is an ω-ladder for $\varphi(x,y) \wedge \psi(x,y)$. Then, by Ramsey's theorem, there exists an infinite $I \subseteq \omega$ such that $(a_i, b_i : i \in I)$ is an ω-ladder for $\varphi(x,y)$ or $\psi(x,y)$. The case of disjunction is likewise.

6. is immediate using 1., 4. and 5. ■

As discussed in Appendix C.3., the characterization of forking in terms of definability of types does not hold anymore in simple theories, but a substitute for it is given by canonical bases:

$$a \mathop{\smile\!\!\!\!\!\!\;|}_{A} B \quad \text{if and only if} \quad Cb(a/B) \subseteq \mathrm{bdd}(A) \quad (*)$$

However, at the local level of δ-types (local types) for $\delta(x,y)$ stable formula, the characterization of forking in terms of definability of types holds in *any* theory, in particular in a simple theory. As forking behave particularly well in a simple theory, this interplay between global properties of forking (for example $(*)$) and local properties of forking (at the level of δ-types for δ stable) is crucial.

This is the bottom line of the results in the next section. To be more specific, I will explain briefly the treatment of local types for stable formulas.

The definition of definability of types in Appendix C.3. has a “local” flavor. Indeed, define a *local φ-type* over A ($\varphi(x,y) \in L$) to be a maximal consistent set of formulas of the form $\varphi(x,a)$ or $\neg\varphi(x,a)$ (a tuple from A).

Define such a φ-type q to be *definable* over B if $\{a \in A^m : \varphi(x,a) \in q\}$ is definable over B $(*)$.

If $p(x) \in S(A)$ and $\varphi(x,y) \in L$, define $p_{\lceil\varphi}$ to be the φ-type over A

$$\{\varphi(x,a) : \varphi(x,a) \in p\} \cup \{\neg\varphi(x,a) : \neg\varphi(x,a) \in p\}$$

Then you can see immediately that $p = p(x) \in S(M)$ is definable over $A \subseteq M$ (in the sense of Appendix C.3.) if and only if $p_{\lceil\varphi}$ is definable over A for every $\varphi(x,y) \in L$.

Appendix C.3. also says that if every formula $\varphi(x,y)$ is stable in T, then every type over a model is definable. But you could wonder if this remains true at the local level of δ-types, for $\delta(x,y)$ stable, for an arbitrary theory T. Again, the astonishing answer is yes (in fact, the properties of types in a stable theory presented in C.3. can be deduced from the local properties of δ-types in any theory as below, c.f.[12] Chapter 1 section 2):

Fact 1 : Let T be an arbitrary complete theory, and let $\delta(x,y)$ be a stable formula in T. Then, any δ-type over a model M is definable (over M).

Moreover, a definition can be taken as a positive boolean combination of instances of δ^{-1}, i.e. formulas of the form $\delta(m,y)$ for some tuple $m \in M$. The notation for this formula $\psi(y)$ defining p is $d_p x \delta(x,y)$.

Again, see [12], Lemma 2.2. for a proof, or [5] Lemma 2.10. for a more transparent proof of this fundamental fact.

But beware: although the above definition of a φ-type is very natural, it does not fit all the needs of a local treatment. Specifically, such a local type over an algebraically closed set A in T^{eq} can have various A-definable extensions over a model $M^{eq} \supseteq A$. This is certainly not desirable, since in a stable theory, every complete type $p \in S(A)$ has a unique A-definable extension to any model $M^{eq} \supseteq A$ (c.f. C.3.).

In order to recover this essential feature in the context of local types, we have to enlarge a little bit the φ-types, and take as definition the following:

Definition 3.1.2 *Let $A \subseteq \mathfrak{C}$ be a set of parameters in an arbitrary theory T, and $\varphi(x,y) \in L$. A φ-type is a maximal consistent set of A-formulas of the form $\psi(x,a)$, where a is a tuple from A, and $\psi(x,a)$ is equivalent to a boolean combination of formulas of the form $\varphi(x,b)$, b tuple from $\mathfrak{C}$.*

Note $S_\varphi(A)$ for the set of φ-types over A.

First of all, observe that the apparent dependence of $\mathfrak{C}$ in this definition (by the use of tuples b from $\mathfrak{C}$) is a false one, since if $A \subseteq M \prec \mathfrak{C}$, taking parameters b from M instead of $\mathfrak{C}$ gives the same set of A-formulas.

Secondly, a φ-type according to this new definition clearly contains a φ-type according to the first definition, but those two partial types need not to be equivalent in general.

However, the two definitions give rise to equivalent partial types if $A = M$ a model.

Also, still is true that $p \in S(M)$ is definable over $A \subseteq M$ iff $p\lceil_\varphi$ is definable over A for every φ, where $p\lceil_\varphi = \{\psi(x,a) : \psi(x,a) \in p$ and $\psi(x,a)$ as in 3.1.2$\}$, and definability is defined exactly as in $(*)$.

But now, the essential feature remains true in the local treatment with the new definition of local types:

Fact 2 : Let T be a theory, $\delta(x,y)$ be a stable formula in T, and A be an algebraically closed set in T^{eq}. Then, for any δ-type p over A, and every model $M^{eq} \supseteq A$, p has a unique extension to a δ-type over M^{eq} that is definable over A.

Moreover, the same formula over A can serve for all models containing A to define these extensions. Again, denote this formula by $d_p x\delta(x,y)$.

Refer to [12], Corollary 2.9. for detailed proof.

For a $p \in S_\delta(A)$, A algebraically closed in T^{eq}, call $Cb(p)$ a canonical parameter for his definition $d_p x\delta(x,y)$ (c.f. Appendix C.2. if you don't know about canonical parameters).

For δ-types over arbitrary sets of parameters, define a notion $\sqsubseteq$ as follows:

Definition 3.1.3 *Let T be an arbitrary theory and $\delta(x,y)$ be a stable formula in T. Let $A \subseteq B$, $p \in S_\delta(A)$ and $q \in S_\delta(B)$.*

Then $p \sqsubseteq q$ if

1. *$p \subseteq q$*

2. *For some (every) extension q' of q to $S_\delta(\mathrm{acl}^{eq}(B))$, q' is definable over $\mathrm{acl}^{eq}(A)$.*

The equivalence between "some" and "every" in the definition above is an immediate consequence of a general and easily proved fact: namely, if $p \in S(A)$, two extensions q, q' of p to complete types over $\mathrm{acl}(A)$ are conjugate over A (i.e. $\sigma(q) = q'$ for some $\sigma \in \mathrm{Aut}(\mathfrak{C}/A)$).

An easy consequence of Fact 2 is that $p \sqsubseteq q$ and $q \sqsubseteq r$ implies $p \sqsubseteq r$ (transitivity of $\sqsubseteq$).

Two types $p \in S_\delta(A)$ and $q \in S_\delta(B)$ with A, B algebraically closed sets in T^{eq} are said to be *parallel* ($p \mathbin{//} q$) if for some (every) model $M^{eq} \supseteq A, B$, there exists $r \in S_\delta(M^{eq})$ such that $p \sqsubseteq r$ and $q \sqsubseteq r$. Again using Fact 2, one sees immediately that $p \mathbin{//} q$ iff p and q have equivalent definitions.

The key result which is used in the sequel is the characterization of this notion $\sqsubseteq$ in terms of the notion of dividing formulas:

Proposition 3.1.4 *Let T be a theory, and $\delta(x,y)$ be a stable formula in T. Let $A \subseteq B$, $p \in S_\delta(A)$, $q \in S_\delta(B)$. Let also $\overline{a} \models q$. The following are equivalent:*

1. $p \sqsubseteq q$
2. $Cb(\overline{a}/\mathrm{acl}^{eq}(B)) \in \mathrm{acl}^{eq}(A)$.
3. *For every $\psi(x,b) \in q$, and every A-indiscernable sequence $(b_i : i < \omega)$ in $\mathrm{tp}(b/A)$, $\{\psi(x,b_i) : i < \omega\}$ is consistent. (i.e. $\psi(x,b)$ does not divide over A)*

Proof : $1 \Leftrightarrow 2$ is immediate using the property of canonical parameters in terms of minimal definably closed set of definition (c.f. C.2.).

$1 \Leftrightarrow 3$: see [12], Lemma 2.16. ■

For technical reasons, the notion of *dividing* is not convenient for all purposes, and we rather work with a looser notion called *forking*: if a formula divides over some parameter set A, then it forks over A. In general the two notions are not equivalent, but in a simple theory they turn out to be equivalent. In a simple theory, this notion of forking (dividing) gives rise to a relation $\downarrow = \downarrow^{div} = \downarrow^{fork}$ with very nice properties (which I will not explicit here, c.f. [5] Chapter 12 for example).

In particular, if you assume $B = M$ is a model of a simple theory in Definition 3.1.3, you get $p \sqsubseteq q$ iff $Cb(p) \in \mathrm{acl}^{eq}(A)$ iff q does not fork over A (i.e. no formula in q forks over A).

And since the notion of non-forking for types is expressed formula by formula, if $a \downarrow_A B$ (i.e. $\mathrm{tp}(a/B)$ does not fork over A), then certainly $\mathrm{tp}_\varphi(a/B) = \mathrm{tp}(a/B)\lceil\varphi$ does not fork over A either.

3.2 Weak stable forking, stable forking property in T and T^{eq}

In a *stable* theory, all formulas $\varphi(x,y)$ are stable by definition, and therefore holds the following property of types :

Definition 3.2.1 *T is said to have* stable forking property *if whenever $A \subseteq B$ and $p(x) \in S(B)$ forks over A, then there exists an instance of a* stable *formula $\delta(x,b) \in p(x)$ which forks over A.*

A famous conjecture says that it holds also for *simple* theories.

If we allow the stable formula $\delta(x,y)$ in the previous definition to have parameters in the base set of parameters A, we get the following :

Definition 3.2.2 *T is said to have* weak stable forking property *if whenever $A \subseteq B$ and $p(x) \in S(B)$ forks over A, then there exists an instance $\delta(x,b)$ of a* stable *formula $\delta(x,y) \in L(A)$, such that $\delta(x,b) \in p(x)$ and which forks over A.*

An obvious obstruction to the equivalence between "stable forking" and "weak stable forking" is that if $\delta(x,y;z) \in L$ and $\delta(x,y;a)$ is stable, then $\delta(x,yz)$ is not stable in general.

But even if one tries to be a little bit finer, there is no reason why there should exists a formula $\nu(z) \in \mathrm{t}p(a)$ such that $\delta(x,yz) \wedge \nu(z)$ be stable :

For example, consider the following structure M: 2 unary relations E, P, one ternary relation $R(x,y,z)$. The universe is the disjoint union $\omega^\omega \coprod \omega$, $M^E = \omega^\omega$, $M^P = \omega$. If $f, g \in M^E$ and $n \in M^P$, then $R(f,g,n)$ iff $f(n) = g(n)$ so that for fixed z, $R(x,y,z)$ defines and equivalence relation on M^E. Take

$$\delta(x,yz) = E(x) \wedge E(y) \wedge P(z) \wedge R(x,y,z)$$

Observe that every two elements $z, z' \in M^P$ have the same type over $\emptyset$ since the identity map on M^E and any permutation on P^M which sends z to z' clearly define an automorphism of M. Henceforth, for every $a \in P^M$ and every $\nu(z) \in \mathrm{tp}(a)$, $\delta(x,yz) \wedge \nu(z) \equiv \delta(x,yz)$. Now fix $a \in M^P$. Since $\delta(x,ya)$ is an equivalence relation on M^E, it is stable (no 3-ladder by transitivity). But $\delta(x,yz)$ is not stable, as the following ω-ladder $(c_i\ ,\ b_i a_i)_{i<\omega}$ testify it:

$$c_i(j) = \begin{cases} 1 & \text{if } 0 \leq j \leq i \\ j+1 & \text{if } j > i \end{cases}$$

$$b_i(j) = \begin{cases} 0 & \text{if } j \neq i \\ i+1 & \text{if } j = i \end{cases}$$

$$a_i = i$$

Question (weak stable forking conjecture):

Does every simple theory has *weak stable forking* property ?

Lemma 3.2.3 *Let $\delta(x,y)$ be a* stable *formula, $\chi(x,z)$, $\theta(v,x)$ be arbitrary formulas, and n an integer.*

1. *If $\chi(x,a) \equiv \delta(x,b)$, then for some $\nu(z) \in \mathrm{t}p(a)$, $\chi(x,z) \wedge \nu(z)$ is stable.*
2. *$\delta'(v,y) = \exists x \Big[\big(\theta(v,x) \wedge \exists^{=n} x\ \theta(v,x)\big) \wedge \delta(x,y) \Big]$ is a stable formula.*

Proof :

1. Suppose for every $\nu(z) \in \mathrm{t}p(z)$, $\chi(x,z) \wedge \nu(z)$ is unstable. By compactness, there is some $(c_i a_i : i < \omega)$ such that $a_i \equiv a$ for all i, and $\models \chi(c_i, a_j)$ if and only if $i < j$. Let b_i be such that $b_i a_i \equiv ba$; then, $\chi(x, a_i)$ is equivalent to $\delta(x, b_i)$ for all i, so that we have : $i < j$ if and only if $\models \chi(c_i, a_j)$ if and only if $\models \delta(c_i, b_j)$, contradicting $\delta(x,y)$ beeing stable.

2. Suppose $\delta'(v,y)$ unstable, i.e. there exists $(d_i b_i : i < \omega)$ such that $\models \delta'(d_i, b_j)$ if and only if $i < j$. For each i, choose an enumeration $(a_i^1, \ldots, a_i^n)$ of the solution set of $\theta(d_i, x)$. By hypothesis, for each couple (i,j) with $i < j$, there is some $1 \leq k \leq n$ such that $\models \delta(a_i^k, b_j)$, which gives an application from the 2-elements subsets of ω on $\{1, \ldots, n\}$. By Ramsey's theorem, there is an infinite subset $I \subseteq \omega$ such that all 2-elements subsets of I are sended to the same element k_0 by this application.

   In particular, if $i, j \in I$ we have : $i < j$ if and only if $\models \delta(a_i^{k_0}, b_j)$, a contradiction with $\delta(x,y)$ stable.

■

It is enough to check the stable forking property for types over models :

Lemma 3.2.4 *Suppose for every model M, every $A \subseteq M$, and every $p(x) \in S(M)$ which forks over B, there exists an instance of a stable formula $\varphi(x,b) \in p(x)$such that $\varphi(x,b)$ forks over A.*

Then, T has stable forking property.

Proof :

Let $A \subseteq B \subseteq M$, and $p(x) \in S(B)$ that forks over A. Let $p'(x) \in S(M)$ be a non forking extension of p to M. Then p' forks over A, and by hypothesis there exists a stable formula $\delta(x,y)$ and a tuple $b \in M$ such that $\delta(x,b) \in p'(x)$ and $\delta(x,b)$ forks over A. Therefore, $q(x) = p'_{\lceil \delta} \in S_\delta(M)$ forks over A, and by hypothesis q does not fork over B (since p' does not fork over B).

By transitivity of $\sqsubseteq$ (c.f. 3.1), $q_{\lceil B}$ forks over A, so that there exists $\chi(x,c) \in q_{\lceil B}$ that forks over A. But $\chi(x,c)$ is equivalent to a boolean combination of instances of stable formulas, and by Lemma 3.1.1 such a boolean combination is itself an instance of a stable formula. By Lemma 3.2.3, $\chi'(x,y) = \chi(x,y) \wedge \nu(y)$ is stable, for some $\nu(y) \in \mathrm{tp}(c)$. Now $\chi'(x,c) \equiv \chi(x,c)$, and obviously $q_{\lceil B} = p_{\lceil \delta}$, so that $\chi'(x,c) \in p(x)$, and we are done. ■

Proposition 3.2.5 *If T has stable forking property, then T^{eq} has stable forking for complete types over real parameters.*

Proof : Let $c_E \not\downarrow_A B$ be a forking relation, with $A \subseteq B \subseteq \mathfrak{C}$, and E a definable equivalence relation.

Choose a $(|A|^+ + \aleph_0)$-saturated model M containing B and c, and a representative c' of c_E such that

$$c' \underset{c_E}{\downarrow} M \qquad (*)$$

Then since $c_E \in \mathrm{dcl}^{eq}(c')$, $c' \not\downarrow_A B$, and by hypothesis there exists some stable formula $\delta(x, y)$ such that for some tuple b in B, $\models \delta(c', b)$, and $\delta(x, b)$ forks over A.

Consider $p(x) = tp_\delta(c'/M)$, and his $\delta-$definition $d_p x\delta(x, y)$. Being a boolean combination of instances (in M) of δ^{-1}, $d_p x\delta(x, y)$ is an instance of a stable formula.

Now $(*)$ implies that p does not fork over c_E, i.e. $d_p x\delta(x, y)$ is definable over $\mathrm{acl}^{eq}(c_E)$, so that there exists $d_F \in \mathrm{acl}^{eq}(c_E)$ and an $L^{eq}-$formula $\chi(w, y)$ such that $d_p x\delta(x, y) \equiv \chi(d_F, y)$.

By lemma 3.2.3 , $\chi(w, y) \wedge \nu(w)$ is a stable formula, for some $\nu(w) \in tp(d_F)$, so that without lost of generality, $\chi(w, y)$ is stable.

Since $\models \delta(c', b)$ and $\chi(d_F, y)$ is a definition for p, $\models \chi(d_F, b)$.

Let $q(w) = \mathrm{t}p(d_F)$.

<u>*Claim 1*</u> : $q(w) \cup \{\chi(w, b)\}$ divides over A.

Proof :

Suppose $q(w) \cup \{\chi(w, b)\}$ does not divide over A, and let $(\widehat{b}_i : i \in \omega)$ be an A-indiscernable sequence in $\mathrm{t}p(b/A)$. By saturation hypothesis of M, the sequence can be choosen in M. Then, $q(w) \cup \{\chi(w, \widehat{b}_i) : i \in \omega\}$ is consistent, and by the saturation hypothesis of M, is satisfied by $\widehat{d}_F \in \mathrm{dcl}^{\mathrm{eq}}(M)$ let's say. Now we have $\widehat{d}_F \equiv d_F$, and $\models \bigwedge_i \chi(\widehat{d}_F, \widehat{b}_i)$. If $(b_i : i \in \omega)$ is such that $\widehat{d}_F(\widehat{b}_i : i \in \omega) \equiv d_F(b_i : i \in \omega)$, we have $\models \bigwedge_i \chi(d_F, b_i)$. Again by saturation of M, and since $d_F \in \mathrm{dcl}^{\mathrm{eq}}(M)$, the sequence $(b_i : i < \omega)$ can be choosen in M.

But since $\chi(d_F, y)$ is a definition of p, and $(b_i : i \in \omega)$ is a sequence in M, this means $\models \bigwedge_i \varphi(c', b_i)$, or again by conjugation : $\models \bigwedge_i \varphi(\widehat{c}, \widehat{b}_i)$, where $c'(b_i :\in \omega) \equiv \widehat{c}(\widehat{b}_i : i \in \omega)$.

Since $(\widehat{b}_i : i \in \omega)$ is an arbitrary A-indiscernable sequence in $\mathrm{t}p(b/A)$, this means $\varphi(x, b)$ does not divide over A, and since T is simple, $\varphi(x, b)$ does not fork over A: a contradiction. ■

Therefore, for some $\mu(w) \in q(w)$ and some integer k, $\chi'(w, b) = \mu(w) \wedge \chi(w, b)$ k-divides over A. Obviously, $\chi'(w, y)$ still is stable.

Choose an algebraic formula $\theta_0(c_E, w)$ with n solutions, such that d_F is one of them, and let $\theta(v, w) = \theta_0(v, w) \wedge \exists^{=n} w\ \theta_0(v, w)$. Let $\chi''(v, y) = \exists w\ \big(\theta(v, w) \wedge \chi(w, y)\big)$, so that in particular $\models \chi''(c_E, b)$. By lemma 3.2.3, $\chi''(v, y)$ is stable.

<u>*Claim 2*</u> : $\chi''(v, b)$ $(n(k-1)+1)$-divides over A.

Proof : Suppose not, and let $l = n(k-1)+1$: there exists g_E such that

$$\models \bigwedge_{0 \leq i \leq l-1} \chi'(g_E, b_i) \qquad (**)$$

Let Y be the set consisting of the n solutions of $\theta(g_E, w)$. For each $i < l$, we can choose by $(**)$ $a^i_F \in Y$ such that $\models \chi(a^i_F, b_i)$

But the application $i \mapsto a^i_F$ from l to Y has a fiber of size at least k (if not, $l \leq n(k-1)$), so there exist $a_F \in Y$, and $I \subseteq l$, $\#I \geq k$, such that

$$\models \bigwedge_{i \in I} \chi(a_F, b_i)$$

a contradiction with the hypothesis. ■

This completes the proof of the proposition. ■

Proposition 3.2.6 *If T^{eq} has stable forking property for complete types with real parameters, then T^{eq} has stable forking property.*

Proof :

By Lemma 3.2.4, it is enough to check stable forking in T^{eq} for types over models.

So let $c \not\perp_A M$, with $A \subseteq \mathrm{dcl}^{\mathrm{eq}} M = M^{eq}$ and $c \in \mathfrak{C}^{eq}$. Let $A' \subseteq \mathfrak{C}$ be a set of representatives of the elements of A such that

$$A' \underset{A}{\perp} cM \quad (*)$$

Then clearly $A \subseteq \mathrm{dcl}^{\mathrm{eq}}(A')$ $(**)$ and $c \not\perp_{A'} M$.

By hypothesis, there exists a stable $\delta(v; yz)$ and tuples m, a' respectively from M and A' such that $\models \delta(c; ma')$ and $\delta(v; ma')$ forks over A'. Now let N be a model containing MA' such that

$$c \underset{MA'}{\perp} N \quad (***)$$

Then $\delta(v, ma') \in p(v) = tp_\delta(c/N)$, so that $p(v)$ forks over A', hence the canonical base c_δ of p does not belong to $\mathrm{acl}^{\mathrm{eq}}(A')$. By $(**)$, $c_\delta \notin \mathrm{acl}^{\mathrm{eq}}(A)$.

Note that $(*)$ implies $c \perp_{AM} A'$ and so $c \perp_M A'$ because clearly $A \perp_M cA'$. Hence, thanks to $(***)$, $c \perp_M N$, and $q(v) := tp_\delta(c/M)$ is parallel to p, so that both have the same canonical base c_δ, and $q(v)$ forks over A since $c_\delta \notin \mathrm{acl}^{eq}(A)$.

Let $\psi(v, a)$ a formula in q responsable for forking over A. Then by Lemma 3.2.3, an equivalent formula $\psi'(v, a)$ is an instance of a stable formula and we are done. ■

Combining the two previous propositions gives the following :

Theorem 3.2.7 *For T simple, T has stable forking property if and only if T^{eq} has.*

The only thing of this theorem that remains to be proved in that T^{eq} has stable forking implies T has, but this is obvious : if $B \subseteq \mathfrak{C}$, a complete type p over B in L extends to a unique complete type $\hat{p}$ over B in L^{eq}; now if p forks over A, so does $\hat{p}$, and applying the hypothesis we find an instance $\delta(x, b)$ of a stable formula $\delta(x, y)$ in L^{eq}, which belongs to $\hat{p}$, and forks over A.

By a well known fact (see Appendix C) , and since x and y are real tuples of variables, there exists an L-formula $\delta^*(x, y)$ such that $\mathfrak{C}^{eq} \models \delta(c, d)$ if and only if $\mathfrak{C} \models \delta^*(c, d)$.

Therefore, $\delta^*(x, b) \in p$, $\delta^*(x, y)$ is stable, and $\delta^*(x, b)$ forks over A, and we are done.

The proof of this theorem might appear unnessesarily intricated, and I would have liked to find a more direct one.

For example, the obvious thing that comes in mind when trying to prove 3.2.5 is, starting from $\delta(x, y)$ (same notation as in the proof), to consider the L^{eq}-formula

$$\phi(v, y) = \exists x \left(v = \pi_E(x) \wedge \delta(x, y)\right)$$

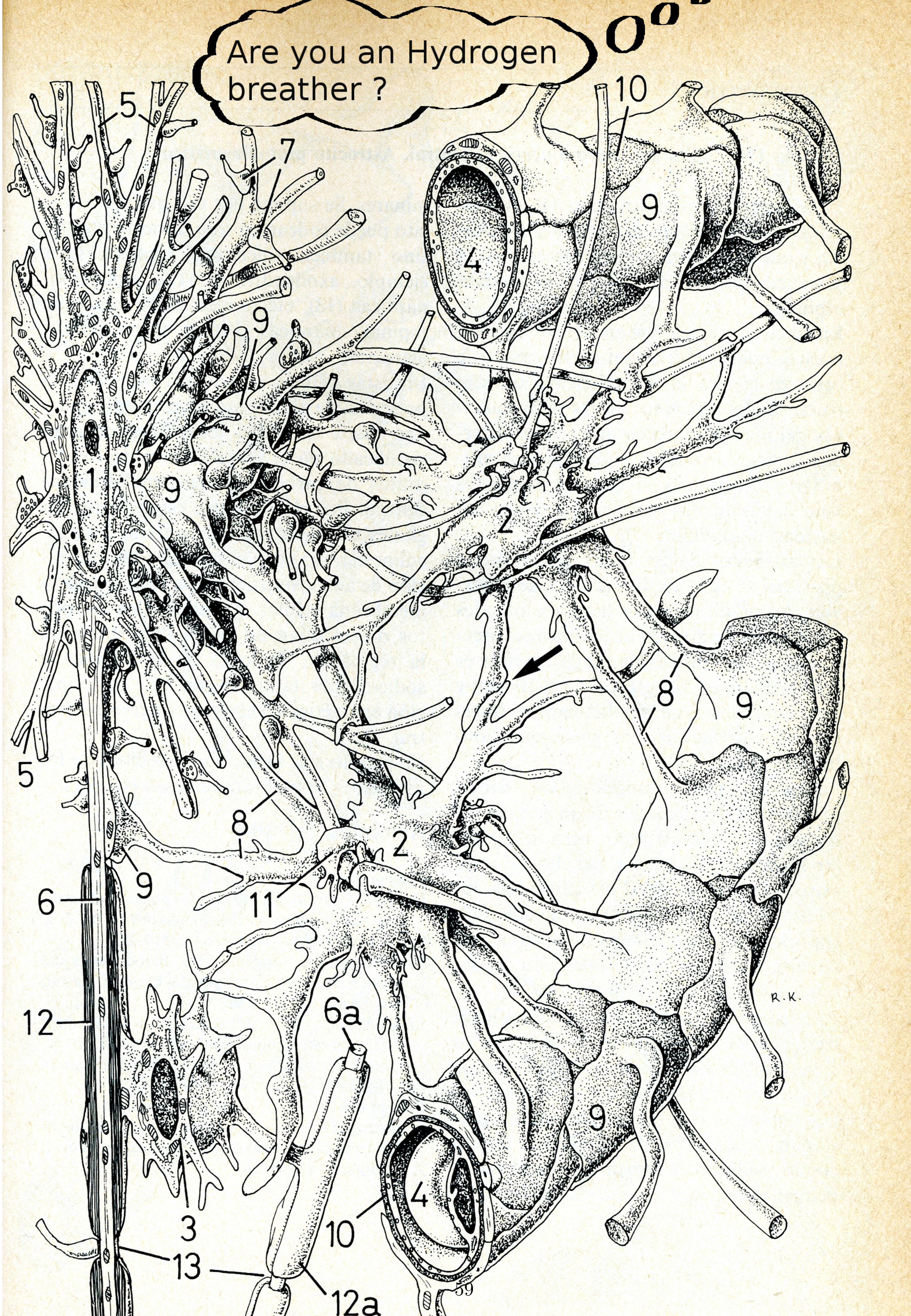
Then, we do have $\models \phi(c_E, b)$, as well as $\phi(v, b)$ forks over A, but the stability of $\phi(v, y)$ fails for quite obvious reasons. For those who doubt of it, here is a counterexample :

Example 3.2.8 *The universe is $\omega \times \omega$. The language has two binary relations E, F, which interpret as equivalence relations : E has ω clases, which are presicely the sections $\omega \times \{n\}$, $n < \omega$. F also has ω classes $F_0, \dots, F_n, \dots, F_\omega$, which are: $F_0 = \{(0,0)\}, F_1 = \{(1,0),(1,1)\}, \dots, F_n = \{(n,0), \dots, (n,n)\}, \dots, F_\omega =$ the rest of it.*

Being an equivalence relation, $F(x, y)$ is a stable formula (it has no ladder of height 3 by transitivity).

Nethertheless, the L^{eq}-formula $\phi(v, y) = \exists x \left(v = \pi_E(x) \wedge F(x, y)\right)$ is unstable, as the sequences $([(0, i)]_E : i < \omega)$ and $((i, 0) : i < \omega)$ testify it.

The same kind of problems arise with $\phi'(v, y) = \forall x \left(v = \pi_E(x) \rightarrow \delta(x, y)\right)$ in place of $\phi(v, y)$.



Drawing: Radovij V. Krstic

Chapter 4

A Dream ...

I had a dream: and if the "Galois correspondence" presented in Chapter 1 (which was the tool allowing elimination of bounded hyperimaginaries in favor of finitary ones, along with some structural result of compact groups) were just a local piece of a vaster picture ?

Indeed, in ACF_0, the general theory specializes as the classical Galois correspondence between the closed subgroups of $\mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$ and the subfields of $\overline{\mathbb{Q}}$. But, as A.Grothendieck observed in the 60's, this is a fragment of a more general frame.

Let's see why first of all.

4.1 Seeing the classical Galois correspondence as part of a more general result

Let $K \subseteq L$ be a Galois extension of fields. The classical Galois correspondence is an isomorphism between the lattice of closed subgroups of $\mathrm{Gal}[L : K]$ (with the profinite Krull topology), and the dual of the lattice of subfields of $[L : K]$.

As well known, a preorder is made a (small) category, where the objects are the elements of the preorder, and there is at most one arrow between two objects a and b, iff $a \leq b$. And a functor between two such categories is just a monotone map between the preorders.

Interpreting the previous lattices as small categories, the Galois correspondence just says that these two categories are isomorphic.

Let's look at the precise form of this isomorphism. It sends an intermediate field $K \subseteq M \subseteq L$ to the (closed) subgroup $\mathrm{Gal}[L : M] \leqslant \mathrm{Gal}[L : K]$. This very definition of this subgroup needs the fact that M be a subfield of L : what if M were a Galois extension of K, not included in L ? It is not clear at first sight how would it be possible to define an object similar to $\mathrm{Gal}[L : M]$ in this case.

The trick is the following : change your perspective, and consider the bijection between the subgroups of a group G and the quotients of the G-set G (acting as usual on the left by left translation), or equivalently the equivalence relations on G compatible with the structure of G-set of G.

This bijection is just $H \leqslant G \mapsto (g \sim g'$ iff $g^{-1}g' \in H)$ in one way, and $\sim\, \mapsto [1]_\sim$ in the other.

Now call $G = \mathrm{Gal}[L : K]$, and look at the object $H = \mathrm{Gal}[L : M] \leqslant G$ associated with M, as a quotient of the G-set G (i.e. the set G/H of left cosets gH).

Observe that $\mathrm{Hom}_K(M, L)$ is a G-set, the action being $(\sigma, f) \mapsto \sigma \circ f$ ($\sigma \in G$, $f \in \mathrm{Hom}_K(M, L)$). Moreover, there is an epimorphism of G-sets

$$\begin{tikzcd} G \arrow[r, two heads] & \mathrm{Hom}_K(M, L) \end{tikzcd}$$

given by restriction to M, that factors through the quotient $G \to G/H$

$$\begin{tikzcd} G \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Hom}_K(M, L) \\ G/H \arrow[ur, dashed, tail, two heads, "\sim"] & \end{tikzcd}$$

by an isomorphism of G-sets : $G/H \longrightarrow \mathrm{Hom}_K(M, L)$.

But the object $\mathrm{Hom}_K(M, L)$ is now defined without any reference of M being included in L, which is the key of the following generalisation :

Definition 4.1.1 *Let $K \subseteq L$ be a Galois extension of fields.*

*A (commutative with unit) K-algebra A is **split by L** if*

1. *A is algebraic over K*
2. *For every $a \in A$, the minimal polynomial of a over K splits in L, with simple roots.*

Of course, the minimal polynomial of an element of A need not be irreducible as in the case of fields, but it is well defined because $K[X]$ is a principal ideal domain.

In the sequel, I will state a few results without proof. Refer for example to [6] for details (although it might be not the best reference for that first algebraic part), and [2] for the categorical background.

Proposition 4.1.2 *Let $K \subseteq L$ be a Galois extension of fields.*

If A is a K-algebra split by L, then the set $\mathrm{Hom}_K(A, L)$ is the inverse limit of all $\mathrm{Hom}_K(B, L)$, B running through the finite dimensional sub-algebras of A. Therefore, $\mathrm{Hom}_K(A, L)$ is naturally equipped with a profinite topology.

In the previous discussion, it is easy to check that everything took place not only in the category of G-sets, but in the category of G-profinite spaces (i.e. profinite spaces with a continuous action of the profinite group G).

This is the natural category to consider also in the generalized frame :

Proposition 4.1.3 *Let $K \subseteq L$ be a Galois extension of fields and A a K-algebra split by L.*

The action of $G = \mathrm{Gal}[L:K]$ on $\mathrm{Hom}_K(A, L)$ given by $(\sigma, f) \mapsto \sigma \circ f$ is continuous, so that $\mathrm{Hom}_K(A, L)$ is a G-profinite space.

Notation 4.1.4 *Let $K \subseteq L$ a Galois extension of fields. Note $\mathrm{Split_K(L)}$ for the category of K-algebras split by L (a full sub-category of the category of K-algebras), and $\mathcal{P}^{\mathrm{Gal}[L:K]}$ the category of profinite $\mathrm{Gal}[L:K]$-sets.*

The generalized theorem is then :

Theorem 4.1.5 *Let $K \subseteq L$ be a Galois extension of fields.*

The functor $\mathcal{F} : A \mapsto \mathrm{Hom}_K(A, L)$ from $\mathrm{Split_K(L)}$ to $\mathcal{P}^{\mathrm{Gal}[L:K]}$, is a contravariant equivalence of categories (the arrow-part of this functor being the obvious one).

One easily sees that restricting this functor to the sub-category of $\mathrm{Split_K(L)}$ given by the intermediate field extensions $K \subseteq M \subseteq L$, with the inclusion maps (isomorphic to the small category "lattice of intermediate fields"), gives back (up to isomorphism) the classical Galois correspondence, which in turn can be seen as a "local piece" of the functor $\mathcal{F}$.

4.2 Going on extending the algebraic context : Galois descent morphisms and internal groupoids

Now, what was so special about a Galois extension of fields $K \subseteq L$? could it would be replaced by certain kinds of ring morphisms $\sigma \colon R \to S$? This kind of morphisms are called *Galois descent*. Without entering the technicalities of the definition, let me say that the inclusion $K \subseteq L$ of a Galois extension of fields is such a morphism, along with other relevant classes of ring-morphisms.

4.2.1 Morphisms of Galois descent

The important thing to keep in mind for further generalization is that the definitions of an "R-algebra split by S" and of a Galois descent morphism lie on an adjunction $\mathcal{S} \dashv \mathcal{C}$ between the dual category R^{op} of commutative rings and the category $\mathcal{P}$ of profinite spaces.

The object part of these functors are as follows :

$\mathcal{S}(A)$ is the space of ultrafilters of the Boolean algebra of idempotents of the ring A.

$\mathcal{C}(X)$ is the ring of continuous functions : $X \to \mathbb{Z}$, where $\mathbb{Z}$ is given the discrete topology.

With some work, it can be seen that $\mathcal{S}$ is right adjoint to $\mathcal{C}$ and induces, for every ring R, an adjunction $\mathcal{S}_R \dashv \mathcal{C}_R$ from the dual of the category of rings below R (i.e. the dual of the category of R-algebras), to the category $\mathcal{P}/\mathcal{S}(R)$ of profinite spaces above $\mathcal{S}(R)$.

Now the definitions are as follows (recall that if $\sigma\colon R \to S$ is a morphism of rings and A is an R-algebra, $S \otimes_R A$ is an S-algebra, the "extension of scalars to S") :

Definition 4.2.1 *Let $\sigma\colon R \to S$ be an arbitrary ring morphism. Let η be the unit of the adjunction*

$$\begin{tikzcd} (S\text{-}Alg)^{op} \arrow[r, shift right, "\mathcal{S}_S"'] & \mathcal{P}/\mathcal{S}(S) \arrow[l, shift right, "\mathcal{C}_S"'] \end{tikzcd}$$

discribed above.

*An R-algebra A is **split by** σ if $\eta_{S\otimes_R A}\colon \mathcal{C}_S\mathcal{S}_S(S \otimes_R A) \longrightarrow S \otimes_R A$ is an isomorphism (for the sake of commodity arrows in **Ring**^{op} are written as arrows in **Ring**).*

The definition of a morphism of Galois descent is in term of the functor $S \otimes_R -$ of extension of scalars, along with the adjunction $\mathcal{S}_S \dashv \mathcal{C}_S$:

Definition 4.2.2 *A morphism of rings $\sigma\colon R \to S$ is of **Galois descent** if*

1. *The functor $S \otimes_R -$ reflects isomorphisms*
2. *For every pair* $\begin{tikzcd} A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \end{tikzcd}$ *such that* $\begin{tikzcd} S\otimes_R A \arrow[r, shift left, "1\otimes_R u"] \arrow[r, shift right, "1\otimes_R v"'] & S\otimes_R B \end{tikzcd}$ *is a split equalizer, there exists an equalizer of the pair (u, v) that is is preserved by the functor $S \otimes_R -$*
3. *For every object (X, φ) of $\mathcal{P}/\mathcal{S}(S)$, the R-algebra $\mathcal{C}_S(X, \varphi)$ is split by σ*

Observe that the first two items of the preceding definition can be easily formulated for an arrow $f\colon X \to Y$ in any category $\mathbf{C}$ with pullback as follows :

The functor "pullback along f" : $f^*\colon \mathbf{C}/Y \to \mathbf{C}/X$ given by the following pullback diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, dashed] & A \arrow[d] \\ \mathrm{X} \arrow[r] & \mathrm{Y} \end{tikzcd}$$

has the properties :

1. f^* reflects isomorphisms
2. Every parallel pair (u, v) of arrows of $\mathbf{C}/Y$ such that (f^*u, f^*v) has a split coequalizer in $\mathbf{C}/X$, has a coequalizer that is preserved by f^*

Indeed, the category **Ring** of commutative rings with unit has pushouts defined by

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ \mathrm{C} \arrow[r] & B \otimes_A C \end{tikzcd}$$

so that the dual category $\mathbf{Ring}^{op}$ has pullback, and if we specialize these conditions to $\mathbf{Ring}^{op}$, we get the two first items of the previous definition. This remark is made in order to prepare a further generalization (outside the algebraic context) in the next section, which could have (as I have the feeling) a possible link to model theory.

4.2.2 Internal groupoids, internal presheaves

Still there is a last bunch of definitions before stating the Galois theorem for commutative rings.

Recall that the notion of a category can be seen as a generalization of that of a monoid. Indeed, a monoid is nothing more than a (small) category with one object.

As a group is a monoid in which all elements are invertible, a *groupoid* is a category with all arrows invertible, and a group is nothing but a (small) groupoid with one object.

Also recall that among the various equivalent formal presentations of the notion of a category, one is as follows (in first-order context — think of the first-order definition of ZFC —) :

A category consists of a 2-sorted universe $(\mathbf{O}, \mathbf{A})$, $\mathbf{O}$ being the "object sort", and $\mathbf{A}$ the "arrow sort". The language consists of unary function symbols

$$\begin{tikzcd} \mathbf{A} \arrow[r, shift left=8, "d_0"] \arrow[r, shift right=8, "d_1"'] & \mathbf{O} \arrow[l, "n"'] \end{tikzcd}$$

d_0 and d_1 stand for the domain and codomain of an arrow f, and we write as usual $f\colon\ d_0(f) \to d_1(f)$.

n stands for the neutral arrow (identity arrow) of an object a, and we write as usual $n(a)$ as $\mathbf{1}_a$.

The composable arrows $\mathbf{A} \times_{\mathbf{O}} \mathbf{A}$ is the subcollection of $\mathbf{A} \times \mathbf{A}$ consisting of those pairs (f, g) such that $d_1(f) = d_0(g)$.

And the language contains a binary function symbol $m\colon \mathbf{A} \times_{\mathbf{O}} \mathbf{A} \to \mathbf{A}$, standing for composition (multiplication) of composable arrows.

As usual, we write $g \circ f$ for $m(f, g)$.

It is then straightforward to express as first order sentences $T_{\mathbf{Cat}}$ in this language the classical axioms for a category.

Observe that a small category is nothing but a model of $T_{\mathbf{Cat}}$ inside ZFC, i.e. with $\mathbf{A}, \mathbf{O}$ *sets*, and d_0, d_1, n, m *functions*.

More remarkably, the first order sentences of $T_{\mathbf{Cat}}$ can be entirely replaced by diagrammatic schemes in the category **Set**, so that a small category can also be seen as an "internal category" inside the category **Set**.

Here is how this "arrow-fashion" translation is done :

- $\mathbf{A}$ and $\mathbf{O}$ are objects of **Set**, d_0, d_1, n are arrows in **Set**.
- Observe that $\mathbf{A} \times_{\mathbf{O}} \mathbf{A}$ can be defined in the category **Set** (up to isomorphism, as always with universal constructions in a category) as the pullback

$$\begin{tikzcd} \mathbf{A} \times_{\mathbf{O}} \mathbf{A} \arrow[r, "p_1"] \arrow[d, "p_0"'] & \mathbf{A} \arrow[d, "d_0"] \\ \mathbf{A} \arrow[r, "d_1"'] & \mathbf{O} \end{tikzcd}$$

  where $\mathbf{A} \times_{\mathbf{O}} \mathbf{A} = \{(a, a') \mid d_1(a) = d_0(a')\}$, and p_0, p_1 are the restrictions of the projections to, respectively, the first and second coordinate. (this data is well known to be a pullback of (d_1, d_0) in the category **Set**).

- If

$$\begin{tikzcd}
D \arrow[dr, dashed] \arrow[drr, "\alpha"] \arrow[ddr, "\beta"'] & & \\
& A \times_C B \arrow[r] \arrow[d] & A \arrow[d, "\psi"] \\
& B \arrow[r, "\varphi"'] & C
\end{tikzcd}$$

where (in any category) $A \times_C B$ is a pullback of (φ, ψ), call the (unique) dashed arrow $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

- Then, considering the objects and arrows in **Set**

$$\begin{tikzcd}
C_2 \arrow[r, shift left=8, "p_0"] \arrow[r, "m"] \arrow[r, shift right=8, "p_1"'] & C_1 \arrow[r, shift left=8, "d_0"] \arrow[r, shift right=8, "d_1"'] & C_0 \arrow[l, "n"']
\end{tikzcd}$$

where C_0 is the set $\mathbf{O}$, C_1 is $\mathbf{A}$, C_2 is $\mathbf{A} \times_{\mathbf{O}} \mathbf{A}$, the following properties in **Set** express the axioms that $(\mathbf{O}, \mathbf{A})$ is a category :

(C1) The square

$$\begin{tikzcd}
C_2 \arrow[r, "p_0"] \arrow[d, "p_1"'] & C_1 \arrow[d, "d_1"] \\
C_1 \arrow[r, "d_0"'] & C_0
\end{tikzcd}$$

is a pullback.

(C2) The triangles

$$\begin{tikzcd}
C_0 \arrow[r, "n"] \arrow[dr, equal] & C_1 \arrow[d, "d_0"] \\
& C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_0 \arrow[r, "n"] \arrow[dr, equal] & C_1 \arrow[d, "d_1"] \\
& C_0
\end{tikzcd}$$

are commutative.

(C3) The squares

$$\begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "p_0"'] & C_1 \arrow[d, "d_0"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "p_1"'] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

are commutative.

(C4) The triangles

$$\begin{tikzcd} C_1 \arrow[r, "\begin{pmatrix} n \circ d_0 \\ \mathbf{1}_{C_1} \end{pmatrix}"] \arrow[dr, equal] & C_2 \arrow[d, "m"] \\ & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\begin{pmatrix} \mathbf{1}_{C_1} \\ n \circ d_1 \end{pmatrix}"] \arrow[dr, equal] & C_2 \arrow[d, "m"] \\ & C_1 \end{tikzcd}$$

are commutative.

(C5) Considering further the pullback

$$\begin{tikzcd} C_3 \arrow[r, "g_0"] \arrow[d, "g_1"'] & C_2 \arrow[d, "p_1"] \\ C_2 \arrow[r, "p_0"'] & C_1 \end{tikzcd}$$

the diagram

$$\begin{tikzcd} C_3 \arrow[r, "\begin{pmatrix} p_0 \circ g_0 \\ m \circ g_1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} m \circ g_0 \\ p_1 \circ g_1 \end{pmatrix}"'] & C_2 \arrow[d, "m"] \\ C_2 \arrow[r, "m"'] & C_1 \end{tikzcd}$$

is commutative.

(C1) is just the definition of $\mathbf{A} \times_{\mathbf{O}} \mathbf{A}$.

(C2) says that $\mathbf{1}_A$ is an arrow from A to A.

(C3) says that if $\alpha \in \mathrm{Hom}(A, B)$ and $\beta \in \mathrm{Hom}(B, C)$, then $\beta \circ \alpha \in \mathrm{Hom}(A, C)$.

(C4) says that each $\mathbf{1}_A$ is neutral (left and right)

(C5) says that the composition law is associative (observe that C_3 is the set "composable triples").

'

Likewise, a small groupoid can be defined entirely in terms of commutative diagrams in **Set** : just add an arrow $\tau : C_1 \to C_1$, intended to mean the function that inverts arrows, along with a pair of diagrammatic conditions (**C6**) and (**C7**) that I won't write here not to be too long.

Observing that all we needed to know about **Set** to express conditions $(\mathbf{C_1}), \ldots, (\mathbf{C_7})$ was to have pullbacks, this allows to define more generally :

Definition 4.2.3 *Let* $\mathbf{C}$ *be a category with pullbacks.*

An internal groupoid *in* $\mathbf{C}$ *consists of a data of the form :*

$$\begin{tikzcd} C_2 \arrow[r, shift left=6, "p_0"] \arrow[r, "m"] \arrow[r, shift right=6, "p_1"'] & C_1 \arrow[r, shift left=6, "d_0"] \arrow[r, shift right=6, "d_1"'] & C_0 \arrow[l, "n"'] \end{tikzcd}$$

fulfilling conditions $(\mathbf{C_1})$ *to* (**C7**) *above.*

The object C_0 *is called the "objects of the internal groupoid", and the object* C_1 *the "arrows".*

Remark 4.2.4 *In the category* **Top** *of topological spaces [resp.* **CH** *of compact Hausdorff topological spaces,* $\mathcal{P}$ *of profinite spaces] , an internal groupoid with* $C_0 = \{*\}$ *is just a topological group [resp a compact group, a profinite group] (the reason is that a map* $\cdot \to \{*\}$ *or* $\{*\} \to \cdot$ *is automatically continuous, so that you don't have to care about the continuity of* d_0, d_1, n *in this case).*

But beware that an internal groupoid in **Top** *[resp* **CH**, $\mathcal{P}$*] is more than just a family* $(G_i)_{i \in C_0}$ *of topological groups [resp compact groups, profinite groups] with a family of homeomorphisms between them satisfying certain diagrammatic conditions (namely, if* $f : i \to j$*, where* $f \in C_1$*,* $i, j \in C_0$*, the homeomorphism* $G_i \to G_j$ $g \mapsto f \circ g \circ f^{-1}$*). Because one you've said that (which is true a can be checked easily), you've said nothing about the continuity of the maps* d_0, d_1, n*.*

Remark 4.2.5 *Let* G *be a group, and* $\mathbf{G}$ *the one-oject category associated to* G *(denote* $*$ *this unique object)*

If X *is a* G*-set with external law* $(g, x) \mapsto g{\cdot}x$*, associate the following presheaf* $\mathcal{F}$ *over* $\mathbf{G}$ *(i.e. the functor* $\mathcal{F} : \mathbf{G} \to \mathbf{Set}$*) :*

- $\mathcal{F}(*) = X$
- $\mathcal{F}(g) = x \mapsto g \cdot x \in X^X$

The fact that X be a G-set ensures that $\mathcal{F}$ is a functor.

Conversely, associate to a presheaf $\mathcal{F}\colon \mathbf{G} \to \mathbf{Set}$ the G-set $X = \mathcal{F}(*)$ with the law $g \cdot x = \mathcal{F}(g)(x)$.

These constructions are the object-part of isomorphisms between the category of G-sets with morphisms of G-sets, and the category $\mathbf{Set}^{\mathbf{G}}$ of presheaves over $\mathbf{G}$ with natural transformations between functors.

Coming back to **Set**, one can go further and define diagrammatically the notion of a functor from a small category to **Set** (in general, a functor from a small category $\mathbf{C}$ to **Set** is also called a *presheaf* over $\mathbf{C}$).

Again, this allows to define more generally the notion of an *internal presheaf over an internal category* in any category $\mathbf{C}$ with pullbacks.

Finally, fixing an internal category $\mathfrak{d}$ in a category $\mathbf{C}$ with pullbacks, we can also define diagrammatically the notion of a natural transformation between two internal presheaves over $\mathfrak{d}$ (the so-called "internal natural transformations"), leading to the category $\mathbf{C}^{\mathfrak{d}}$ of the internal presheaves over the internal category $\mathfrak{d}$.

This is not difficult (although sometimes tricky) to list the diagrams in $\mathbf{C}$ needed to express all the notions above (begin to express all that in **Set**, and just copy *mutus mutandis* to an abitrary category with pullbacks, just as we did with the notion of an internal category at the beginning), but tiresome enough to avoid writing it down here.

Remark 4.2.6 *If G is not just a group, but a topological [resp compact, resp profinite] group, associate with G the internal groupoid $\mathfrak{g}$ in* **Top** *[resp* **CH***, $\mathcal{P}$] as in Remark 4.2.4.*

Then, it can be checked in the same spirit as Remark 4.2.5 that the category of G-spaces [resp G-compact spaces, G-profinite spaces] with morphisms of G-spaces, is isomorphic to the category $\mathbf{Top}^{\mathfrak{g}}$ [resp $\mathbf{CH}^{\mathfrak{g}}$, $\mathcal{P}^{\mathfrak{g}}$] of internal presheaves over the internal groupoid $\mathfrak{g}$ in the category **Top** *[resp* **CH***, $\mathcal{P}$] with internal natural transformations.*

A key fact in the sequel is the following, that allows to construct very special kinds of internal groupoids in any category with pullbacks. Recall that in a category, a *kernel pair* (p_1, p_2) for an arrow σ is (if it exists) a pullback

$$\begin{tikzcd}
{} \arrow[r, "p_2"] \arrow[d, "p_1"'] & {} \arrow[d, "\sigma"] \\
{} \arrow[r, "\sigma"'] & {}
\end{tikzcd}$$

Lemma 4.2.7 *Let* $\mathbf{C}$ *be a category with pullbacks, and* $\sigma\colon S \to R$ *be an arrow. Then, the following data is an internal groupoid*

$$\begin{tikzcd}
(S \times_R S) \times_S (S \times_R S) \arrow[r, shift left=4, "\pi_2"] \arrow[r, "m"] \arrow[r, shift right=4, "\pi_1"'] & S \times_R S \arrow[loop above, "\tau"] \arrow[r, shift left=4, "p_1"] \arrow[r, shift right=4, "p_2"'] & S \arrow[l, "\Delta"']
\end{tikzcd}$$

where

- (p_1, p_2) *is a kernel pair of* σ*.*
- Δ *is the "diagonal map"* $\begin{pmatrix} \mathbf{1}_S \\ \mathbf{1}_S \end{pmatrix}$ $\Big(s \mapsto (s,s)$ *in the case* $\mathbf{C} = \mathbf{Set}\Big)$.
- τ *is the "twisting map"* $\begin{pmatrix} p_2 \\ p_1 \end{pmatrix}$ $\Big((s,t) \mapsto (t,s)$ *in the case of* $\mathbf{Set}\Big)$.
- *If*

$$\begin{tikzcd}
(S \times_R S) \times_S (S \times_R S) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S \times_R S \arrow[d, "p_1"] \\
S \times_R S \arrow[r, "p_2"'] & S
\end{tikzcd}$$

is a pullback, $m = \begin{pmatrix} p_2 \circ \pi_2 \\ p_1 \circ \pi_1 \end{pmatrix}$. $\Big(((s,t),(s',t')) \mapsto (s,t')$ *in the case* $\mathbf{Set}\Big)$.

Remark 4.2.8 *As recalled at the end of* 2.1.*, the category* ***Ring***^{op} *has pullbacks, and if* $\sigma R \to S$ *is an arrow in* ***Ring****, the internal groupoid of the previous Lemma is (writing arrows in* **Ring** *instead of* ***Ring***^{op} *for the sake of commodity) :*

$$\begin{tikzcd}
(S \otimes_R S) \otimes_S (S \otimes_R S) & S \otimes_R S \arrow[loop above, "\tau"] \arrow[l, shift right=4, "\pi_2"'] \arrow[l, "m"'] \arrow[l, shift left=4, "\pi_1"] \arrow[r, "\mu"] & S \arrow[l, shift right=4, "s_1"'] \arrow[l, shift left=4, "s_2"]
\end{tikzcd}$$

where $s_1(a) = a \otimes 1$*,* $s_2(a) = 1 \otimes a$*,* $\mu(a \otimes b) = ab$*,* $\tau(a \otimes b) = b \otimes a$*,* $m(a \otimes b) = (a \otimes 1) \otimes (1 \otimes b)$*.*

This is the internal co-groupoid of the cokernel pair of σ *(in* **Ring***).*

4.2.3 Galois theorem for morphisms of Galois descent

Equipped with that machinery, it can be proved the following

Proposition 4.2.9 *Let* $\sigma : R \to S$ *a Galois descent morphism of rings.*

Then, the functor $\mathcal{S}$ *transforms the internal cogroupoid of the kernel pair of* σ *(c.f. Remark 4.2.8) into an internal groupoid* $\mathrm{Gal}[\sigma]$ *in the category* $\mathcal{P}$ *of profinite spaces.*

Theorem 4.2.10 (Galois theorem) *Let* $\sigma : R \to S$ *be a Galois descent morphism of rings and* $\mathrm{Gal}[\sigma]$ *the corresponding Galois groupoid in the category of profinite spaces. There exists an equivalence of categories*

$$\big(\mathrm{Split}_R(\sigma)\big)^{op} \approx \mathcal{P}^{\mathrm{Gal}[\sigma]}$$

between the dual of the category of R*-algebras split by* σ *and the category of internal presheaves on* $\mathrm{Gal}[\sigma]$ *in the category of profinite spaces.*

Remark 4.2.11 *In the particular case where* σ *is the inclusion map of a Galois extension* $K \subseteq L$ *of fields,* $\mathcal{S}(L) = \{*\}$ *since the boolean algebra of the idempotents is reduced to* $\{0, 1\}$*. Therefore,* $\mathrm{Gal}[\sigma]$ *is an internal groupoid in* $\mathcal{P}$ *with* $C_0 = \{*\}$*, i.e. a profinite group.*

It can be proved that the profinite group associated to $\mathrm{Gal}[\sigma]$ *is isomorphic to the classical profinite group* $\mathrm{Gal}[L : K]$*, and Remark 4.2.6 provides an isomorphism between the category of* $\mathrm{Gal}[L : K]$*-profinite spaces and the category* $\mathcal{P}^{\mathrm{Gal}[\sigma]}$ *of internal presheaves over* $\mathrm{Gal}[\sigma]$*.*

Keeping in mind that in that particular case, $\mathrm{Split}_K(\sigma)$ *is precisely the* K*-algebras split by* L *as in 4.1.1, we recover the (generalised) Galois theorem for fields 4.1.5.*

4.3 Extending out of the algebraic context : categorical Galois theorem and possible link with model theory

The reason of the long and tedious previous section is to make transparent and meaningful the generalization out of the strict algebraic realm.

It turns out that, transcribing *mutus mutandis* the definitions of the previous section in the case of general categories **A** and **P** instead of $\mathbf{Ring}^{op}$ and $\mathcal{P}$, the main result still is true.

Specifically, here is the context :

$\mathbf{A}, \mathbf{P}$ are categories with pullbacks.

$\mathcal{S} \dashv \mathcal{P}$ is an adjunction:

$$\begin{tikzcd} \mathbf{A} \arrow[r, shift right, "\mathcal{S}"'] & \mathbf{P} \arrow[l, shift right, "\mathcal{C}"'] \end{tikzcd}$$

The fact that $\mathbf{A}$ has pullbacks allows to restrict $\mathcal{S}$ to an adjunction

$$\begin{tikzcd} \mathbf{A}/A \arrow[r, shift right, "\mathcal{S}_A"'] & \mathbf{P}/\mathcal{S}(A) \arrow[l, shift right, "\mathcal{C}_A"'] \end{tikzcd}$$

for any object A of $\mathbf{A}$.

Again the fact that $\mathbf{A}$ has pullbacks allows to exhibit, for every arrow $\sigma\colon S \to R$ in $\mathbf{A}$, an adjunction $\Sigma_\sigma \dashv \sigma^*$:

$$\begin{tikzcd} \mathbf{A}/S \arrow[r, shift right, "\Sigma_\sigma"'] & \mathbf{A}/R \arrow[l, shift right, "\sigma^*"'] \end{tikzcd}$$

where σ^* is the functor "pullback along σ" (c.f. discussion after Definition 4.2.2), and Σ_σ is the obvious functor which maps (X, f) to $(X, \sigma \circ f)$.

For $\sigma\colon S \to R$, the involved functors are displayed as follows :

$$\begin{tikzcd} \mathbf{A}/S \arrow[r, shift right, "\mathcal{S}_S"'] \arrow[d, shift right, "\Sigma_\sigma"'] & \mathbf{P}/\mathcal{S}(S) \arrow[l, shift right, "\mathcal{C}_S"'] \arrow[d, shift right, "\Sigma_{\mathcal{S}(\sigma)}"'] \\ \mathbf{A}/R \arrow[r, shift right, "\mathcal{S}_R"'] \arrow[u, shift right, "\sigma^*"'] & \mathbf{P}/\mathcal{S}(R) \arrow[l, shift right, "\mathcal{C}_R"'] \arrow[u, shift right, "\mathcal{S}(\sigma^*)"'] \end{tikzcd}$$

Now two definitions :

Definition 4.3.1 *Let $\mathcal{S} \dashv \mathcal{C}$ be an adjunction*

$$\begin{tikzcd} \mathbf{A} \arrow[r, shift right, "\mathcal{S}"'] & \mathbf{P} \arrow[l, shift right, "\mathcal{C}"'] \end{tikzcd}$$

between categories $\mathbf{A}, \mathbf{P}$ *with pullback, and* $\sigma\colon S \to R$ *an arrow in* $\mathbf{A}$*.*

An object (A, a) *of* $\boldsymbol{A}/R$ *is* ***split by*** σ *if the unit*

$$\eta^S_{\sigma^*(A,a)}\colon \sigma^*(A,a) \longrightarrow \mathcal{C}_S\mathcal{S}_S\sigma^*(A,a)$$

of the adjunction $\mathcal{S}_S \dashv \mathcal{C}_S$ *is an isomorphism at the object* $\sigma^*(A,a)$*.*

Definition 4.3.2 *Let* $\mathcal{S} \dashv \mathcal{C}$ *be an adjunction*

$$\begin{tikzcd} \mathbf{A} \arrow[r, shift right, "\mathcal{S}"'] & \mathbf{P} \arrow[l, shift right, "\mathcal{C}"'] \end{tikzcd}$$

between categories $\mathbf{A}, \mathbf{P}$ *with pullback.*

An arrow $\sigma\colon S \to R$ *is of* ***relative Galois descent*** *(with respext to these data) if*

1. *The functor* σ^* *reflects isomorphisms.*

2. *Every parallel pair* (u, v) *of arrows of* $\boldsymbol{A}/R$ *such that* (σ^*u, σ^*v) *has a split coequalizer in* $\boldsymbol{A}/S$*, has a coequalizer that is preserved by* σ^**.*

3. *The counit* ε^S *of the adjunction* $\mathcal{S}_S \dashv \mathcal{C}_S$ *is an isomorphism of functors.*

4. *For every object* $(X, f) \in \boldsymbol{P}/\mathcal{S}(S)$*, the object* $(\Sigma_\sigma \circ \mathcal{C}_S)(X, f) \in \boldsymbol{A}/R$ *is split by* σ*.*

Proposition 4.2.9 goes through in this context :

Proposition 4.3.3 *Let* $\sigma\colon S \to R$ *be a Galois descent morphism, relative to an adjunction*

$$\begin{tikzcd} \mathbf{A} \arrow[r, shift right, "\mathcal{S}"'] & \mathbf{P} \arrow[l, shift right, "\mathcal{C}"'] \end{tikzcd}$$

between categories with pullback.

Then the internal groupoid of the kernel pair of σ *(c.f. Lemma 4.2.7) is transformed by the functor* $\mathcal{S}$ *into an internal groupoid* Gal$[\sigma]$ *in* $\mathbf{P}$*. This internal groupoid is called the* Galois groupoid of σ*.*

Finally, the categorical Galois theorem :

Theorem 4.3.4 *Let $\mathcal{S} \dashv \mathcal{C}$ be an adjunction*

$$\mathbf{A} \underset{\mathcal{S}}{\overset{\mathcal{C}}{\rightleftarrows}} \mathbf{P}$$

between categories $\mathbf{A}, \mathbf{P}$ with pullback, $\sigma \colon S \to R$ be a relative Galois descent morphism in $\mathbf{A}$ and $\mathrm{Gal}[\sigma]$ the corresponding Galois groupoid in $\mathbf{P}$. There exists an equivalence of categories

$$\mathrm{Split}_R(\sigma) \approx \boldsymbol{P}^{\mathrm{Gal}[\sigma]}$$

between the category of objects of $\boldsymbol{A}/A$ split by σ, and the category of internal presheaves on the internal groupoid $\mathrm{Gal}[\sigma]$ in $\mathbf{P}$.

My plan of attack was the following :

(I) Beginning with $T = ACF_0$, where

- G_{KP} is isomorphic to $\mathrm{Gal}[\overline{\mathbb{Q}}/\mathbb{Q}]$ as compact (profinite) groups.
- The lattice of definably closed subsets of $\mathrm{bdd}(\emptyset)$ is isomorphic to the lattice of subfields of $\overline{\mathbb{Q}}$.
- Through those isomorphisms, the anti-isomorphism discussed after 1.3.9 is the classical Galois correspondence between closed subgroups of $\mathrm{Gal}[\overline{\mathbb{Q}}/\mathbb{Q}]$ and subfields of $\overline{\mathbb{Q}}$.

try and find

- A category $\mathbf{A}$ (defined in model-theorectic terms from the theory T, and aiming to replace the category of $\mathbb{Q}$-algebras split by $\overline{\mathbb{Q}}$).
- An adjunction

  $$\mathbf{A} \underset{\mathcal{S}}{\overset{\mathcal{C}}{\rightleftarrows}} \mathcal{P}$$

  where $\mathcal{P}$ is the category of profinite spaces.
- A morphism of Galois descent in $\mathbf{A}$ (relative to this adjunction) $\sigma \colon S \to R$

with the following properties :

- $\mathcal{S}(S) = \{*\}$ (so that the internal groupoid $\mathrm{Gal}[\sigma]$ is a profinite group, c.f. Remark 4.2.4).
- The profinite group $\mathrm{Gal}[\sigma]$ is isomorphic to $\mathrm{Gal}[\overline{\mathbb{Q}}/\mathbb{Q}]$.
- The lattice of closed subgroups of $\mathrm{Gal}[\overline{\mathbb{Q}}/\mathbb{Q}]$ is (isomorphic to) a subcategory of the subcategory $\mathrm{Split}_R(\sigma)$ of the objects of $\mathbf{A}/R$ split by σ.

- The equivalence of categories given by 4.3.4, when restricted to $\mathrm{Split}_R(\sigma)$, gives back the classical Galois correspondence.

(II) Inspired by what should be the category **A** in (II), try and find for a general theory T

- A category **B** defined from the theory T.
- An adjunction

$$\begin{tikzcd} \mathbf{B} \arrow[r, shift right, "\mathcal{S}"'] & \mathbf{CH} \arrow[l, shift right, "\mathcal{C}"'] \end{tikzcd}$$

  where **CH** is the category of compact spaces.
- A morphism of Galois descent in **B** (relative to these data) $\sigma\colon S \to R$.

with the following properties :

- $\mathcal{S}(S) = \{*\}$ (so that the internal groupoid $\mathrm{Gal}[\sigma]$ is a compact group, c.f. Remark 4.2.4).
- The compact group $\mathrm{Gal}[\sigma]$ is isomorphic to G_{KP}.
- The lattice of closed subgroups of G_{KP} is (isomorphic to) a subcategory of the category $\mathrm{Split}_R(\sigma)$ of the objects of $\mathbf{B}/R$ split by σ.
- The equivalence of categories given by 4.3.4, when restricted to $\mathrm{Split}_R(\sigma)$, gives back the correspondence discussed after 1.3.9.

(III) Recover the proof of elimination of bounded hyperimaginaries using the equivalence of categories in (II).

(IV) For some kind of simple theories, try and find some similar data ($\mathrm{Gal}[\sigma]$ might well be an internal groupoid, not necessarily a group like in (I) and (II)), such that the equivalence of categories allows to prove some kind of elimination of hyperimaginaries.

So far I failed, but still I believe something is to be discovered in that direction.

Chapter 5

Conclusión

No tiene sentido ninguno hacer una parte de conclusiones para una tesis de matemáticas, pero me conformaré a los requisitos de la comisión.

En este trabajo se han obetenido los siguientes resultados:

1. Describir la distancia de un espacio métrico en puros terminos de teoría de modelos, por lo menos en una clase de teorías.

2. Construir ejemplos explícitos de relaciones de equivalencia tipo-definibles y acotadas, en determinadas teorías, tales que su grupo de permutationes elementales sea isomorfo a los grupos compactos $O_n(\mathbb{R})$, $SO_n(\mathbb{R})$, y $U_n(\mathbb{C})$.

3. Definir nociones tales como hiperimaginarios *normales* y *DCC* y probar algunas de sus propiedades.

4. Probar que en una teoría simple T, la propiedad de la bifurcación estable es equivalente en T y en T^{eq}.

Appendices

Appendix A

Isometries of $\mathbb{S}^n$

$\mathbb{R}^m$ is given the canonical Euclidian structure with the standard scalar product $\langle (x_1 \dots x_m), (y_1 \dots y_m) \rangle = \sum_{i=1}^{m} x_i y_i$. The quadratic form corresponding to the scalar product is $q(x) = \langle x, x \rangle$, and the norm derived from it is $\|x\| = \sqrt{q(x)}$.

I will use repeatidely the polar identity $q(x-y) = q(x) + q(y) - 2\langle x, y \rangle$ (which is true of any symmetric bilinear form over any field).

The sphere $\mathbb{S}^n$ is as usual the subset of $\mathbb{R}^{n+1}$ given by $\{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

The aim of this appendix is to provide a proof that every isometry of the sphere $\mathbb{S}^n$ with the intrinsic metric δ can be extended to an element of the orthogonal group $O_{n+1}(\mathbb{R})$.

Recall that if $x, y \in \mathbb{S}^n$, $\delta(x, y)$ is the shortest arc length between x and y. Since for every $x, y \in \mathbb{R}^{n+1}$, $\langle x, y \rangle = \|x\|.\|y\|.cos(\widehat{x,y})$, specializing this to elements of $\mathbb{S}^n$ immediately provides $\delta(x, y) = \arccos(\langle x, y \rangle)$.

There is another metric on $\mathbb{S}^n$, namely the restriction to $\mathbb{S}^n$ of the metric d of the norm $\|.\|$ in $\mathbb{R}^{n+1}$: $d(x, y) = \|x - y\| = \sqrt{q(x-y)}$.

Lemma A.0.5 *Let f be a map $\mathbb{R}^m \to \mathbb{R}^m$. The following are equivalent :*

1. *$\langle f(x), f(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^m$ (i.e. f respects the scalar product)*
2. *f is linear and $q(f(x)) = q(x)$ for all $x \in \mathbb{R}^m$ (i.e. f is linear and respects the quadratic form)*
3. *$f \in O_m(\mathbb{R})$*

Proof : $1 \Rightarrow 2$ Take an orthonormal basis $(e_i)_i$ for $\mathbb{R}^m$. By the hypothesis, $(f(e_i))_i$ is an orthonormal family, so is linearly independent, henceforth an orthonormal basis.

Decomposing $f(\sum_i \alpha_i e_i)$ in the basis $(f(e_i))_i$ we get $f(\sum_i \alpha_i e_i) = \sum_i \beta_i f(e_i)$. But since $(f(e_i))_i$ is orthonormal, the β_i's are given by $\beta_j = \langle f(\sum_i \alpha_i e_i), f(e_j)\rangle = \langle \sum_i \alpha_i e_i, e_j\rangle = \alpha_j$. We conclude that for all $\alpha_i \in \mathbb{R}$, $f(\sum_i \alpha_i e_i) = \sum_i \alpha_i f(e_i)$, so f is linear (to check linearity of a map between vectorial spaces, it is enough to check linearity on linear combinations of a generating family).

$2 \Rightarrow 3$ We only have to check that f is a linear automorphism, since by definition an element of $O_m(\mathbb{R})$ is a linear automorphism that respects the quadratic form. Since $\mathbb{R}^m$ has finite dimension, it is enough to see that $\mathrm{Ker}(f) = 0$. Let x with $f(x) = 0$. Then, $0 = q(f(x)) = q(x)$, so that $x = 0$ since q is definite.

$3 \Rightarrow 1$ Since the base field $\mathbb{R}$ has characteristic $\neq 2$, the polar identity also reads $\langle x, y\rangle = \frac{1}{2}\Big(q(x) + q(y) - q(x-y)\Big)$.

If $f \in O_m(\mathbb{R})$ and $x, y \in \mathbb{R}^m$, we have $\langle f(x), f(y)\rangle = \frac{1}{2}\Big(q(f(x)) + q(f(y)) - q(f(x) - f(y))\Big) = \frac{1}{2}\Big(q(f(x)) + q(f(y)) - q(f(x-y))\Big) = \frac{1}{2}\Big(q(x) + q(y) - q(x-y)\Big) = \langle x, y\rangle$. ■

Lemma A.0.6 *Let f be a permutation of the set $\mathbb{S}^n$. Then, f is an isometry for d iff f respects the scalar product.*

Proof : If $\langle f(x), f(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{S}^n$ (i.e. f respects the scalar product), then $q(f(x) - f(y)) = q(f(x)) + q(f(y)) - 2\langle f(x), f(y)\rangle = q(x) + q(y) - 2q(x, y) = q(x - y)$, and taking square roots implies that f is an isometry for d.

Conversely, suppose f is an isometry for d, i.e. $q(f(x) - f(y)) = q(x - y)$, for all $x, y \in \mathbb{S}^n$. The polar identity gives $\langle f(x), f(y)\rangle = \frac{1}{2}\Big(q(f(x)) + q(f(y)) - q(f(x) - f(y))\Big)$. Since $x, y, f(x), f(y)$ belong to $\mathbb{S}^n$, they all have norm one, and applying the hypothesis, the right hand term of the last equality is equal to $\frac{1}{2}\Big(q(x) + q(y) - q(x - y)\Big) = \langle x, y\rangle$. ■

The isometries of $\mathbb{S}^n$ for d and δ are the same :

Corollary A.0.7 *Let f be a map $\mathbb{S}^n \to \mathbb{S}^n$. Then, f is an isometry for d iff f is an isometry for δ.*

Proof : Suppose f is an isometry for δ, and let $x, y \in \mathbb{S}^n$. By hypothesis, $\delta(f(x), f(y)) = \delta(x, y)$, i.e. $\arccos(\langle f(x), f(y)\rangle) = \arccos(\langle x, y\rangle)$. Therefore, $\langle f(x), f(y)\rangle = \langle x, y\rangle$, and the previous Lemma tells us that f is an isometry for d.

Suppose f is an isometry for d, and let $x, y \in \mathbb{S}^n$ Again by the previous Lemma, f preserves the scalar product, and $\delta(f(x), f(y)) = \arccos(\langle f(x), f(y)\rangle) = \arccos(\langle x, y\rangle) = \delta(x, y)$. ■

Proposition A.0.8 *If f is an isometry of $\mathbb{S}^n$ for δ, then f extends to an element of $O_{n+1}(\mathbb{R})$.*

Proof : First of all, f is an isometry for d by Corollary A.0.7, so f respects the scalar product by A.0.6

Now let $\widehat{f}\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the map defined as follows :

$\widehat{f}(0) = 0$, and $\widehat{f}(x) = \|x\| f\Big(\dfrac{x}{\|x\|}\Big)$ if $x \neq 0$.

Clearly, $\widehat{f}$ extends f.

Since f respects the scalar product, so does $\widehat{f}$: if $x, y \in \mathbb{R}^{n+1}$ are both non-zero, $\langle \widehat{f}(x), \widehat{f}(y)\rangle = \langle\ \|x\| f\Big(\dfrac{x}{\|x\|}\Big), \|y\| f\Big(\dfrac{y}{\|y\|}\Big)\ \rangle = \|x\|\|y\| \langle\ f\Big(\dfrac{x}{\|x\|}\Big), f\Big(\dfrac{y}{\|y\|}\Big)\ \rangle = \|x\|\|y\| \langle \dfrac{x}{\|x\|}, \dfrac{y}{\|y\|}\rangle = \langle x, y\rangle$. And if $x = 0$, $\langle f(x), f(y)\rangle = \langle 0, f(y)\rangle = 0 = \langle 0, y\rangle = \langle x, y\rangle$.

By A.0.5, $\widehat{f}$ is an element of $O_{n+1}(\mathbb{R})$. ∎

Appendix B

The Lascar group

Among the automorphism groups of the models of a complete first order theory T, if you were to choose one which is an invariant of T, certainly you would begin to try some $\mathrm{Aut}(\overline{M})$, where $\overline{M}$ is a λ-saturated and λ-homogeneous model (λ big enough).

But if $\overline{N} \succ \overline{M}$ is a κ-saturated κ-homogeneous model with $\kappa > \lambda$, there is no chance that $\mathrm{Aut}(\overline{N})$ be isomorphic to $\mathrm{Aut}(\overline{M})$: if $\sigma \in \mathrm{Aut}(\overline{M})$, there exists (by κ-homogeneity of $\overline{N}$) some $\tau \in \mathrm{Aut}(\overline{N})$ which extends σ, but such an extension is not unique; so which one to choose in order to make this assignment a group morphism ? this is far from being clear, and even if it were possible, there is no reason why this should be one-to-one.

Nevertheless, two such extensions τ_1 and τ_2 clearly have the property that $\tau_1\tau_2^{-1}$ fixes $\overline{M}$ pointwise, and therefore one is forced to consider cosets of automorphism groups instead of the full automorphism group.

The new natural question is now : is there exist a quotient of $\mathrm{Aut}(\overline{M})$ which is an invariant of T ?

According to what we've just observed, a good canditate would be to quotient $\mathrm{Aut}(\overline{M})$ by the subgroup generated by the subgroups of the form $\mathrm{Fix}(M)$, with $M \prec \overline{M}$ and M small (meaning, recall it, of sizs less than κ). Call this subgroup $\mathrm{Autf}(\overline{M})$ (for "strong automoprhisms"). An element of that subgroup can be written as $\sigma_1 \ldots \sigma_n$, with $\sigma_i \in \mathrm{Fix}(M_i)$.

First of all, if $M \prec \overline{M}$ and $\sigma \in \mathrm{Aut}(M)$, then $\sigma(M) \prec \overline{M}$, and has the same size as M. Henceforth, $\mathrm{Autf}(\overline{M})$ is *normal* in $\mathrm{Aut}(\overline{M})$ (since $\sigma \in \mathrm{Fix}(M)$ and $\tau \in \mathrm{Aut}(\overline{M})$ implies $\tau\sigma\tau^{-1} \in \mathrm{Fix}(\tau M)$), so that we can consider the quotient group $\mathrm{Aut}(\overline{M})/\mathrm{Autf}(\overline{M})$.

If $\overline{N} \succ \overline{M}$ is a κ-saturated, κ-homogeneous model of T with $\kappa > |M|$, we want to define a canonical morphism from $\mathrm{Aut}(\overline{M})/\mathrm{Autf}(\overline{M})$ to $\mathrm{Aut}(\overline{N})/\mathrm{Autf}(\overline{N})$.

As above, every $\sigma \in \mathrm{Aut}(\overline{M})$ extends to some $\tau \in \mathrm{Aut}(\overline{N})$, and two such extensions are in the same coset modulo $\mathrm{Fix}(M)$. Since by hypothesis $|M| < \kappa$, they are also in the same coset modulo $\mathrm{Autf}(\overline{N})$, and we can define the map $\sigma \mapsto [\tau] = f(\sigma)$, where τ is any extension of σ to $\overline{N}$, and $[\tau]$ is the coset of τ modulo $\mathrm{Autf}(\overline{N})$.

If σ extends to τ and σ' extends to τ', then $\tau\tau'$ is an extension of $\sigma\sigma'$, and since $[\tau\tau'] = [\tau][\tau']$, this map is a group morphism $\mathrm{Aut}(\overline{M}) \to \mathrm{Aut}(\overline{N})/\mathrm{Autf}(\overline{N})$.

If $\sigma \in \mathrm{Autf}(\overline{M})$, then clearly any extension τ belongs to $\mathrm{Autf}(\overline{N})$ (the reason being that $\lambda \leq |M| < \kappa$), so that the canonical morphism f factors through the projection map $\mathrm{Aut}(\overline{M}) \to \mathrm{Aut}(\overline{M})/\mathrm{Autf}(\overline{M})$ by a morphism $\overline{f}\colon \mathrm{Aut}(\overline{M})/\mathrm{Autf}(\overline{M}) \to \mathrm{Aut}(\overline{N})/\mathrm{Autf}(\overline{N})$, $\overline{f}([\sigma]) = f(\sigma)$.

It remains to show that $\overline{f}$ is an isomorphism. This will be a consequence of a few easy lemmas :

Lemma B.0.9 *Let $M, N \prec \overline{M}$ be small submodels, and $\sigma, \sigma' \in \mathrm{Aut}(\overline{M})$.
If $\sigma(M) \equiv_N \sigma'(M)$, then σ and σ' have the same coset modulo* $\mathrm{Autf}(\overline{M})$.

Proof : By $\mathrm{tp}(M/N)$ I mean $\mathrm{tp}(m/N)$, where m is a tuple enumerating M. Because M, N are small, $\sigma(m) \equiv_N \sigma'(m)$ implies there exists $\tau \in \mathrm{Aut}(\overline{M}/N)$ such that $\tau\sigma(m) = \sigma'(m)$ (by λ-homogeneity), i.e. $\sigma'^{-1}\tau\sigma = \rho$ fixes M pointwise.

Now $\sigma' = \tau\sigma\rho^{-1}$, and since $\tau \in \mathrm{Autf}(\overline{M})$ and $\mathrm{Autf}(\overline{M})$ is normal in $\mathrm{Aut}(\overline{M})$, $\tau\sigma = \sigma\tau'$ for some $\tau' \in \mathrm{Autf}(\overline{M})$. Therefore, $\sigma' = \sigma\tau'\rho$, and we are done. ■

The group $\mathrm{Autf}(\overline{M})$, as a subgroup of $\mathrm{Aut}(\overline{M})$, acts on any set of the form $\overline{M}^{\alpha}$ (α ordinal).

Definition B.0.10 *Two tuples $a, b \in \overline{M}^{\alpha}$ are said to have the same* Lascar strong type *if they are conjugated under the action of* $\mathrm{Autf}(\overline{M})$*, i.e. if there exists $\sigma \in \mathrm{Autf}(\overline{M})$ such that $\sigma(a) = b$.*

If $\overline{N} \succ \overline{M}$ is a κ-monster model ($\kappa > |\overline{M}|$), there are a priori two notions of "having the same Lascar strong type" for tuples from $\overline{M}$: one for the action of $\mathrm{Autf}(\overline{M})$, and the other for the action of $\mathrm{Autf}(\overline{N})$. The use of the term "type" in "Lascar strong type" suggests that these two notions coincide since $\overline{N} \succ \overline{M}$, and it is indeed the case, at least for small tuples from $\overline{M}$:

Lemma B.0.11 *Let a, b be small tuples from $\overline{M}$. Then a and b have the same Lascar strong type in $\overline{M}$ iff they have the same Lascar strong type in $\overline{N}$.*

Proof : Suppose a and b have the same Lascar strong type in $\overline{N}$. Then by definition, there exist $a_o = a, \ldots, a_n = b$ and small submodels $N_1 \ldots, N_n$ of $\overline{N}$ such that

$$a_i \equiv_{N_{i+1}} a_{i+1} \quad \text{for every } 0 \leq i \leq n-1$$

"Small" in $\overline{N}$ means of size less than κ, but we can always find $P_i \prec N_i$ of size less than λ, and still we have

$$a_i \equiv_{P_{i+1}} a_{i+1} \quad \text{for every } 0 \leq i \leq n-1$$

Since the tuples and models involved have size less than λ, we can find by λ-saturation of $\overline{M}$ a sequence $a'_0, \dots, P'_n$ in $\overline{M}$ with the same type over ab as $a_0, \dots, P_n$. This sequence shows that a and b have the same Lascar strong type in $\overline{M}$ since by λ-homogeneity of $\overline{M}$, having the same type over a small set P is equivalent to be in the same orbit under $\mathrm{Aut}(\overline{M}/P)$. ■

Proposition B.0.12 *The canonical morphism* $\overline{f}\colon \mathrm{Aut}(\overline{M})/\mathrm{Autf}(\overline{M}) \to \mathrm{Aut}(\overline{N})/\mathrm{Autf}(\overline{N})$ *is an isomorphism.*

Proof : Surjectivity : fix two small submodels $M, N \prec \overline{M}$, and let $\tau \in \mathrm{Aut}(\overline{M})$. τ might send M outside $\overline{M}$, but by λ-saturation of $\overline{M}$, there is some $M' \prec \overline{M}$ such that $M' \equiv_N \tau(M)$. Clearly $M' \equiv M$, so by λ-homogeneity of $\overline{M}$, let $\sigma \in \mathrm{Aut}(\overline{M})$ such that $\sigma(M) = M'$. Now by κ-homogeneity of $\overline{N}$ and because $|M| < \kappa$, extend σ to some $\tau' \in \mathrm{Aut}(\overline{N})$. Then $\tau(M) \equiv_N \tau(M)$, and by Lemma B.0.9 $[\tau] = [\tau']$. But by construction, $f(\sigma) = [\tau']$, whence $\overline{f}([\sigma]) = f(\sigma) = [\tau'] = [\tau]$.

Injectivity : suppose $\tau \in \mathrm{Aut}(\overline{M})$ extends to a strong automorphism $\tau' \in \mathrm{Autf}(\overline{N})$, i.e. $\overline{f}([\tau]) = 1$. Fix a small submodel $M \prec \overline{M}$. Then M and $\tau(M)$ have the same Lascar strong type in $\overline{N}$, whence also in $\overline{M}$ by B.0.11. Let $g \in \mathrm{Autf}(\overline{M})$ such that $g(M) = \tau(M)$. Then $g^{-1}\tau$ fixes M point-wise, so that $g^{-1}\tau = h \in \mathrm{Autf}(\overline{M})$. Therefore, $\tau = gf$ is a strong automorphism, which means that $\overline{f}$ is injective. ■

Once we know that $\overline{f}$ is an isomorphism, we get immediately that for any κ-monster model $\overline{N}$ of T, $\mathrm{Aut}(\overline{N})/\mathrm{Autf}(\overline{M})$ is isomorphic to $\mathrm{Aut}(\overline{M})/\mathrm{Autf}(\overline{M})$: this is just because $\overline{N}$ elementarilly embeds into an elementary extension of $\overline{M}$.

The conclusion is that the quotient group $\mathrm{Aut}(\overline{M})/\mathrm{Autf}(\overline{M})$ does not depend on the choice of the monster model $\overline{M}$, so is an invariant of T. Call it $\mathrm{G_L(T)}$ (L for Lascar).

Morover, the size of this quotient group is controlled by $|T|$:

Proposition B.0.13 $\mathrm{G_L(T)}$ *has size at most* $2^{|T|}$.

Proof : Let M, N be small submodels of $\overline{M}$. Let $m = (m_i : i \in I)$ be a tuple enumerating M. Denote by $S_m(N) \subseteq S_I(N)$ the (closed) subset of $S_I(N)$ defined by the type $\mathrm{tp}(m/\emptyset)$. Take $p \in S_m(N)$, and realize it in $\overline{M}$ by λ-saturation: $a \models p$. By λ-homogeneity and since by hypothesis $m \equiv a$, choose $\sigma \in \mathrm{Aut}(\overline{M})$ such that $\sigma(m) = a$.Then, Lemma B.0.9 precisely says that $[\sigma]$ does not depend on the choice of the realization of p, so that there is a well defined map

$$S_m(N) \to G_L(T)$$

This map is obviously surjective, so that $G_L(T)$ has size at most that of $S_m(N)$. But if you choose M, N to be submodels of size $|T|$, then $|S_m(N)| \leq 2^{|T|}$. ■

The link with the group G_{KP} defined in Chapter 1 is the following:

It can be showed that if an automorphism of $\overline{M}$ fixes a small submodel, then it fixes every bounded hyperimaginary (in particular every E_{KP}-class in $\overline{M}^{\omega}$). Therefore, $\mathrm{Autf}(\overline{M}) \subseteq \Gamma_1$, and G_{KP} is a quotient of $G_L(T)$. See [10] or [4] for a way of equipping $G_L(T)$ with a compact (not necessarily Hausdorff) topology such that the compact Hausdorff topology on G_{KP} is the quotient topology of that of $G_L(T)$.

Appendix C

The *eq* construction

C.1 Some motivations for considering definable equivalence relations

C.1.1 Smallest definably closed sets of definition

The definable relations in a structure M are the relations in some M^n of the form $\varphi(M) = \{\overline{x} \in M^n \ : \ M \models \varphi(\overline{x})\}$, for some formula $\varphi(x_1, \ldots, x_n)$ with parameters from M.

The automorphism group $\mathrm{Aut}(M)$ acts on the definable relations in M, since for every $\sigma \in \mathrm{Aut}(M)$, $M \models \varphi(\overline{x}, \overline{a})$ iff $M \models \varphi(\sigma(\overline{x}), \sigma(\overline{a}))$, i.e. σ transforms the definable relation of $\varphi(\overline{x}, \overline{a})$ into the definable relation of $\varphi(\overline{x}, \sigma(\overline{a}))$.

As usual, call $\mathrm{Fix}(R)$ the isotropy group of a definable relation R under this action, i.e. the set of $\sigma \in \mathrm{Aut}(M)$ that fixes R set-wise.

Recall that in a κ-saturated structure $\overline{M}$, the definable relations have a nice characterization in terms of automorphisms in the following sense :

Let $A \subseteq \overline{M}$ with $|A| < \kappa$, and $\varphi(x_1, \ldots, x_n)$ be a formulas with parameters from $\overline{M}$. The following are equivalent :

1. There exists a formula $\psi(x_1, \ldots, x_n)$ with parameters in A such that

$$\overline{M} \models \forall x_1 \ldots \forall x_n \ \ \varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)$$

   (i.e. the definable relation given by φ is definable over A).

2. $\varphi(\overline{M})$ is set-wise invariant under $\mathrm{Aut}(\overline{M}/A)$ (in other words, $\mathrm{Aut}(\overline{M}/A) \subseteq \mathrm{Fix}(\varphi(\overline{M}))$.

Recall also that in a κ-saturated structure $\overline{M}$, the property of being definable over $A \subseteq \overline{M}$ of size $< \kappa$ can be reformulated in terms of the action of the automorphism group :

3. $a \in M$ is definable over A (i.e. $\overline{M} \models \chi(a)$, and $|\chi(\overline{M})| = 1$, for some formula $\chi(x)$ over A) if and only if

4. a has a one-element orbit under $\mathrm{Aut}(\overline{M}/A)$ (in other words, $\mathrm{Aut}(\overline{M}/A) \subseteq \mathrm{Fix}(a)$).

Using those properties of definability over A of relations and elements in a κ-saturated structure, we get the following result :

Lemma C.1.1 *Let $\overline{M}$ be a κ-saturated structure, $B \subseteq \overline{M}$ of size less than κ, and $R(x_1, \dots, x_n)$ a definable relation in $\overline{M}$.*

If $\mathrm{Aut}(\overline{M}/B) = \mathrm{Fix}(R)$, then $\mathrm{dcl}(B)$ is the smallest definably closed set of definition of R.

Proof : Since $\mathrm{Aut}(\overline{M}/B) \subseteq \mathrm{Fix}(R)$ and $|B| < \kappa$, the equivalence 1. $\Leftrightarrow$ 2. above implies that R is definable over B, hence over $\mathrm{dcl}(B)$.

Let A with $\mathrm{dcl}(A) = A$ such that R is definable over A, and $\sigma \in \mathrm{Aut}(\overline{M}/A)$. Then, $\sigma \in \mathrm{Fix}(R) = \mathrm{Aut}(\overline{M}/B)$, whence $B \subseteq \mathrm{dcl}(A)$ using the equivalence 3. $\Leftrightarrow$ 4.. Henceforth $\mathrm{dcl}(B) \subseteq \mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A) = A$, and $\mathrm{dcl}(B)$ is the smallest definably closed set of definition for R. ■

Not every definable relation in M has a smallest definably closed set of definition in M, but if you are willing to handle equivalence classes of 0-definable equivalence relations, the following holds :

Lemma C.1.2 *Let M be a structure, and $\varphi(\overline{x}, \overline{a})$ be a formula with parameters $\overline{a}$ from M. Then, there exist a 0-definable equivalence relation $E(\overline{y}, \overline{y}')$ such that $\mathrm{Aut}(M/\overline{a}_E) = \mathrm{Fix}(\varphi(M, \overline{a}))$.*

Proof : First of all, observe that given a 0-definable equivalence relation $E(y_1, \dots, y_m; y'_1, \dots, y'_m)$, $\mathrm{Aut}(M)$ acts on the quotient set M^m/E. This is just because for every $\overline{y}, \overline{y}'$ and every $\sigma \in \mathrm{Aut}(M)$, $M \models E(\overline{y}, \overline{y}')$ iff $M \models E(\sigma(\overline{y}), \sigma(\overline{y}'))$. The action is thus $\sigma \cdot \overline{a}_E = [\sigma(\overline{a})]_E$, and talking of $\mathrm{Aut}(M/\overline{a}_E) = \{\sigma \in \mathrm{Aut}(M) \mid \sigma(\overline{a}_E) = \overline{a}_E\}$ makes sense.

Define $E(\overline{y}, \overline{y}') = \forall \overline{x}\big(\varphi(\overline{x}, \overline{y}) \leftrightarrow \varphi(\overline{x}, \overline{y}')\big)$. This is a formula without parameters, and this is clearly an equivalence relation since $E(\overline{b}, \overline{b}')$ means $f(\overline{b}) = f(\overline{b}')$, where $f \colon M^m \to \mathfrak{P}(M^n)$, $\overline{b} \mapsto \varphi(M, \overline{b})$.

Now the following lines are clearly equivalent for $\sigma \in \mathrm{Aut}(M)$:

$\sigma \in \mathrm{Fix}(\varphi(M, \overline{a}))$

For every $\overline{x} \in M^n$, $M \models \varphi(\overline{x}, \overline{a})$ iff $M \models \varphi(\sigma(\overline{x}), \overline{a})$

For every $\overline{x} \in M^n$, $M \models \varphi(\overline{x}, \overline{a})$ iff $M \models \varphi(\sigma^{-1}(\overline{x}), \overline{a})$ (if σ fixes R set-wise, so does σ^{-1}).

For every $\overline{x} \in M^n$, $M \models \varphi(\overline{x}, \overline{a})$ iff $M \models \varphi(\overline{x}, \sigma(\overline{a}))$ (applying σ to the previous line).

$M \models \forall \overline{x}\big(\varphi(\overline{x}, \overline{a}) \leftrightarrow \varphi(\overline{x}, \sigma(\overline{a})\big)$

$M \models E(\overline{a}, \sigma(\overline{a}))$

$\sigma \cdot \overline{a}_E = \overline{a}_E$

$\sigma \in \mathrm{Aut}(M/\overline{a}_E)$. ■

The class $\overline{a}_E$ in the previous Lemma is called a *canonical parameter* for $\varphi(\overline{x}, \overline{a})$ (or for the definable relation associated to it). More generally, every class $\overline{b}_F$ such that F is a 0-definable equivalence relation and $\mathrm{Fix}(\varphi(M)) = \mathrm{Aut}(M/\overline{b}_F)$ is called a canonical parameter for $\varphi(\overline{x}, \overline{a})$.

Observe that M need not have any saturation feature in the previous Lemma. But comparing with Lemma C.1.1, if $M = \overline{M}$ does have some saturation feature, could we say that "$\mathrm{dcl}(\overline{a}_E)$ is the smallest definably closed set of definition for $\varphi(\overline{M})$" ? what would be the meaning of $\mathrm{dcl}(\overline{a}_E)$? The introduction of the language L^{eq} in the next section will be an answer to that.

But before, another motivation for considering definable equivalence relations.

C.1.2 Interpreting one structure in another

Here are two relevant examples showing how definable equivalence relations appear naturally throughout mathematics:

1. Starting with a skew field D, a standard construction is that of the projective plane $\mathbb{P}^2(D)$ over D: the points are the vectorial lines of the D-vector space D^3, and the lines are the lines belonging to planes of D^3. These sets of points P and lines L, together with the incidence relation I between them defined by $I(p, l)$ iff $p \in l$ (p in P, l in L), is well known to satisfy the famous axioms (first order sentences in the 2-sorted language $\mathfrak{L}$ with sorts P and L, and binary relation $I \subseteq P \times L$):

    - "by two distinct points pass a unique line" (i.e. $\forall x^P \forall y^P \big(\neg(x^P =_P y^P) \rightarrow \exists\,!\, z^L I(x^P, z^L) \wedge I(y^P, z^L)\big)\big)$.
    - " two distinct lines intersect at a point" (necessarily unique by previous axiom)
    - "each line pass through at least 3 points"
    - " there exist 3 non-colinear points"
    - theorem of Desargues.

    Now the sort the 2-sorted structure $\mathbb{P}^2(D)$ in the language $\mathfrak{L}$ can be seen as a "quotient" inside the structure $(D, +, -, \cdot, 0, 1)$ in the language $\mathfrak{R}$ of rings:

    - Consider the (definable without parameters in $\mathfrak{R}$) equivalence relation on $D^3 - \{0\}$ given by $\overline{x} \sim \overline{y}$ if $\overline{x}$ and $\overline{y}$ are colinear. The sort P is then clearly in bijection with $\left(D^3 - \{0\}\right)/\sim$, and $D^3 - \{0\}$ is a definable set of the structure $(D, +, -, \cdot, 0, 1)$.

- It is well known that the planes of D^3 are in one-one correspondence with the lines of the dual space of D^3. Henceforth, introduce the (definable without parameters in $\mathfrak{R}$) equivalence relation on $D^3 - \{0\}$

$$E((a,b,c),(a',b',c')) = \exists t \forall x \forall y \forall z \ \left(ax + by + cz = t(a'x + b'y + c'z)\right)$$

  expressing (through the coordinates in the dual of the canonical basis of D^3) that two non-zero linear forms on D^3 are co-linear.

  Then, the set of planes in D^3 is in one-one correspondence with the set of classes $\left(D^3 - \{0\}\right)/F$ by $[(a,b,c)]_F \mapsto \{(x,y,z) \in D^3 \mid ax + by + cz = 0\}$ (the kernel of a non-zero linear form on D^3).

  Clearly, the sort L of $\mathbb{P}^2(D)$ can be seen as this quotient set.
- The incidence relation I can be also seen as a definable relation in $\mathfrak{R}$ between
  $\left(D^3 - \{0\}\right)/\sim$ and $\left(D^3 - \{0\}\right)/F$: $I([x,y,z]_\sim, [a,b,c]_F)$ iff $ax + by + cz = 0$ (which clearly does not depend on the choice of representatives for $\sim$- and E-classes).

2. Conversely, it is well known that from a 2-sorted structure $\mathcal{P}$ in the language $\mathfrak{L}$ satisfying the 5 axioms above, one can construct a skew field D such that $\mathcal{P}$ is isomorphic to $\mathbb{P}^2(D)$. See for example [12], or [13].

   If you enter the details of that construction, you can see that again the set D is a set of classes of a definable (in $\mathfrak{L}$) equivalence relation on $\mathcal{P}$, the operations on it $+, -, \cdot$ being definable relations between those classes, and the constants $0, 1$ being definable classes.

From those illuminating examples, we can define more generally the following:

Definition C.1.3 *Let M be a structure in a language L, and M' a structure in a language L'.*

We say that M' is interpretable (with parameters) *in M is there exist:*

- *An M- definable (in L) relation $R \subseteq M^n$.*
- *An M-definable (in L) equivalence relation $E(\overline{x}, \overline{x}')$ on R.*
- *For each p-ary relation symbol S of L', an L-formula $\varphi_S(\overline{x}_1, \ldots, \overline{x}_p)$ with parameters from M (each $\overline{x}_i$ an n-tuple).*
- *For each p-ary function symbol F of L', and L-formula $\varphi_F(\overline{x}_1, \ldots, \overline{x}_p, \overline{x}_{p+1})$ with parameters from M (each $\overline{x}_i$ an n-tuple).*

Such that:

- *The satisfaction of each φ_S inside R is compatible with the equivalence relation E: If $E(\overline{x}_1, \overline{x}'_1), \ldots, E(\overline{x}_p, \overline{x}'_p)$, then $M \models \varphi_S(\overline{x}_1, \ldots, \overline{x}_p)$ iff $M \models \varphi_S(\overline{x}'_1, \ldots, \overline{x}'_p)$. This allows to define a p-ary relation $S_{R/E}$ on the set of classes R/E.*

- *For each function symbol F of L', φ_F defines a map $F_{R/E}\colon (R/E)^p \to R/E$: If $E(\overline{x}_1, \overline{x}'_1), \ldots, E(\overline{x}_p, \overline{x}'_p)$, then there exists some $\overline{x}_{p+1} \in R$ such that $M \models \varphi_F(\overline{x}_1, \ldots, \overline{x}_p, \overline{x}_{p+1})$, and $M \models \varphi_F(\overline{x}'_1, \ldots, \overline{x}'_p, \overline{x}'_{p+1})$ implies $E(\overline{x}_{p+1}, \overline{x}'_{p+1})$.*

- *The L'-structure $(R/E, S_{R/E}, \ldots, F_{R/E}, \ldots)$ is isomorphic to M'.*

If moreover, the equivalence relation E on R is not needed (just the definable relation R, and the definable relations φ_S and φ_F on some powers of R), we say that M' is *definable (with parameters)* in M. This is clearly a particular case of interpretation, where $E(\overline{x}, \overline{x}')$ is the trivial (definable) equivalence relation $\overline{x} = \overline{x}'$.

In the next section, we will see that a structure M' in a language L' is interpretable with parameters in another structure M in a language L if and only if M' is definable with parameters in M^{eq} in the language L^{eq}. Consequently, the M^{eq} construction also has the virtue to unify the two notions of interpretability and definability of one structure into another, and provides a much shorter definition of interpretability than that of Definition C.1.3.

C.2 The language L^{eq} and the complete theory T^{eq}

Let T be a complete theory in the language L.

In order to link Lemma C.1.1 and Lemma C.1.2, we need to introduce a new language L^{eq} that allows classes under definable equivalence relations to be considered as elements of some structure.

First observe that if $M.N \models T$, then the set of 0-definable equivalence relation on some power of M = the set of 0-definable equivalence relations on some power of N (being an equivalence relation is a first-order sentence, and T is complete). Call this set $\mathcal{E}$.

A first attempt would be to introduce, for each such 0-definable equivalence relation E, a unary predicate S_E intented to be interpreted in the new structure M^{eq} as the quotient set M^n/E, and to put $M^{eq} = \coprod_{E \in \mathcal{E}} M^n/E = \coprod_{E \in \mathcal{E}} S_E^{M^{eq}}$.

Note that the trivial equivalence relation on M given by $x = y$ allows to see the underlying set of the structure M (in one-one correspondence with $M/_=$) as part of M^{eq}. Consequently, it is also reasonable to introduce new function symbols p_E intended to be interpreted as the canonical projection $M^n = (M/_=)^n \to M^n/E = S_E^{M^{eq}}$.

The new language will therefore be $L^{eq} = L \cup \{S_E, p_E : E \in \mathcal{E}\}$, and from a model $M \models T$ one constructs the structure M^{eq} in the language L^{eq} as was done above.

The only problem with that construction is that, since there are infinitely many $E \in \mathcal{E}$, some elementary extensions $N \succ M^{eq}$ contain elements out of every S_E^N ($\bigwedge_{E\in\mathcal{E}} \neg S_E(x)$ is finitely consistent, so realize it in an elementary extension). This is not really a problem in fact, but there is a simple way of avoiding these extra elements: the use of a multi-sorted language instead of one-sorted. Specifically, the set of sorts is $\{S_E : E \in \mathcal{E}\}$. The sort $S_=$ is for the equivalence relation $x = y$ on M referred to above, and will called the *home sort*, in one-one correspondence with the underlying set of M. As always for many-sorted languages, there are for each sort S_E a countable set of variables $\{x_n^{S_E} : n < \omega\}$, as well as an equality symbol $=_{S_E}$. And of course we introduce the function symbols p_E intended to be interpreted as the projection map as above. Note that in a multi-sorted language, each relation of function symbol must come labelled with the sorts on which it is intended to be interpreted, so that p_E should be written rather $p_E^{S_=,\ldots,S_=;S_E}$, but we will call it p_E to avoid too much notation. Also recall that multi-sorted first order logic basically works the same as standard one-sorted first order logic, in particular the compactness theorem still is true.

L^{eq} is the multi-sorted language with set of sorts $\{S_E : E \in \mathcal{E}\}$, function symbols $\{p_E = p_E^{S_=,\ldots,S_=;S_E} : E \in \mathcal{E}$ defined on n-tuples, and $S_=$ repeated n times above $p_E\}$, and all the symbols already appearing in L (more precisely, the equality symbol $=$ of L is replaced by the equality symbol $=_=$ of the home sort $S_=$, each n-relation symbol R is replaced by $R^{S_=,\ldots,S_=}$, each function symbol F by $F^{S_=,\ldots,S_=;S_=}$) $(*)$.

With no surprise, starting from a model $M \models T$, we construct an L^{eq}-structure M^{eq}: the sort S_E in M^{eq} is M^n/E, and all the symbols of L^{eq} are interpreted in M^{eq} as they are intended to. In particular, the home sort can be seen as an L-structure isomorphic to M.

The intuition one has to grasp behind the technicalities is that "nothing essential has been added when passing from M to M^{eq}". The precise meaning of this vague assertion is that M^{eq} (as an L^{eq}-structure) is interpretable (without parameters) in M, in the sense of Definition C.1.3.

A consequence of this is that L^{eq} keeps a "good" relationship with the original language L in the following sense:

Lemma C.2.1 *Let M be an L-structure. Let $\varphi(x_1^{S_{E_1}}, \ldots, x_n^{S_{E_n}})$ be a formula of L^{eq}. Then, there exists a formula $\varphi^*(\overline{y}_1, \ldots, \overline{y}_n)$ of L such that*

$$M^{eq} \models \varphi(\overline{a}^1_{E_1}, \ldots, \overline{a}^n_{E_n}) \text{ if and only if } M \models \varphi^*(\overline{a}^1, \ldots, \overline{a}^n)$$

.

Proof : This is in fact a general fact about one structure interpretable in another: there exists such a dictionary between formulas in the new language and formulas in the original one.

The proof is done with induction on the complexity of formulas.

For example in that particular case of L^{eq} and L, if $\varphi(x^{S_E}, x_1^{S_=}, \ldots, x_n^{S_=})$ is the atomic formula $x^{S_E} = p_E(x_1^{S_=}, \ldots, x_n^{S_=})$, then we can clearly take $E(y_1, \ldots, y_n; x_1, \ldots, x_n)$

as $\varphi^*(\overline{y}, \overline{x}) = \varphi^*(y_1, \dots, y_n; x_1, \dots, x_n)$. The case of the other atomic formulas is similar.

Assuming $\psi^* \in L$ is already constructed from $\psi \in L^{eq}$, if $\varphi(x_1^{S_{E_1}}, \dots, x_n^{S_{E_n}})$ is the formula $\forall z^{S_E} \psi(z^{S_E}, x_1^{S_{E_1}}, \dots, x_n^{S_{E_n}})$, we can clearly take $\forall \overline{u} \psi^*(\overline{u}, \overline{y}_1, \dots, \overline{y}_n)$ as $\varphi^*(\overline{y}_1, \dots, \overline{y}_n)$. The other cases of the induction are similar. ■

This simple but key result allows to very easily check the following facts:

- If M is κ-saturated and κ-homogeneous, so is M^{eq}.
- If $M, N \models T$ then $M^{eq} \equiv N^{eq}$.
- If $M \prec N$ are models of T, then the map $a_E^M \mapsto a_E^N$ is an elementary embedding from M^{eq} into N^{eq}.

Now obviously each M^{eq} with $M \models T$ is a model of the following theory T^{eq} in L^{eq}:

1. All sentences of T rewritten in $S_=$ as in $(*)$.
2. For each sort S_E: $\forall y^{S_E} \exists x_1^{S_=} \dots \exists x_n^{S_=}\, p_E(x_1^{S_=}, \dots x_n^{S_=}) = y^{S_E}$.
3. For each sort S_E: $\forall x_1^{S_=}, \dots \forall x_n^{S_=} \forall y_1^{S_=}, \dots \forall y_n^{S_=} \big(p_E(x_1^{S_=}, \dots, x_n^{S_=}) =_{S_E} p_E(y_1^{S_=}, \dots, y_n^{S_=}) \leftrightarrow E(x_1^{S_=}, \dots x_n^{S_=}; y_1^{S_=}, \dots y_n^{S_=})\big)$

Conversely, it is immediate that any model P of T^{eq} is isomorphic to an M^{eq}, for some $M \models T$ (specifically, take M the home sort $P_=$).

The conclusion is that T^{eq} is a *complete* theory in L^{eq}, whose models are very well described in terms of models of T.

On the other hand, it is clear that every automorphism of M^{eq} induces (restricting to the home sort) an automorphism of M. Conversely, one easily check that every $\sigma \in \mathrm{Aut}(M)$ extends uniquely to $\widehat{\sigma} \in \mathrm{Aut}(M^{eq})$: just take $\widehat{\sigma}(a_E) = [\sigma(a)]_E$ (well defined map since $M \models E(\overline{x}, \overline{y})$ iff $M \models E(\sigma(\overline{x}), \sigma(\overline{y}))$).

Therefore, the map $\sigma \mapsto \widehat{\sigma}$ is an isomorphism of groups between $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M^{eq})$.

Call dcl^{eq} and acl^{eq} the definable and algebraic closure in the models of T^{eq}.

In this context, and putting all together, the link between Lemma C.1.1 and Lemma C.1.2 appears to be the following:

Proposition C.2.2 *Let M be a structure in language L, $\varphi(\overline{x}, \overline{a})$ be a formula in L with parameters from M, and $e \in M^{eq}$ be a canonical parameter of φ as in C.1.2.*

Then, $\mathrm{dcl}^{eq}(e) \subseteq M^{eq}$ *is the smallest definably closed subset of definition for φ.*

Proof : Let $\overline{M} \succ M$ be an $\aleph_0$-saturated and $\aleph_0$-homogeneous extension. As remarked above, M^{eq} elementary embeds into $\overline{M}^{eq}$, so that we can assume $M^{eq} \prec \overline{M}^{eq}$. Also, $\overline{M}^{eq}$ is $\aleph_0$-saturated and $\aleph_0$-homogeneous. The construction of e as an equivalence class $\overline{a}_E$ in M shows that the class e still is a canonical parameter for φ in $\overline{M}$, i.e.

$$\mathrm{Fix}(\varphi(\overline{M})) = \mathrm{Aut}(\overline{M}/e)$$

.

But as observed above, $\mathrm{Aut}(\overline{M}/e) = \mathrm{Aut}(\overline{M}^{eq}/e)$, and Lemma C.1.1 tells us that $\mathrm{dcl}^{eq}(e)$ is the smallest definably closed subset of definition in $\overline{M}^{eq}$ for φ.

Since $e \in M^{eq}$ and $M^{eq} \prec \overline{M}^{eq}$, $\mathrm{dcl}^{eq}(e) \subseteq M^{eq}$, so that $\mathrm{dcl}^{eq}(e)$ is also the smallest definably closed set of definition for φ in M^{eq}. ■

Now the link of L^{eq} with interpretability as defined in C.1.2:

Clearly if the equivalence relation $E(\overline{x}, \overline{x}')$ of Definition C.1.3 is defined without parameters in L, the structure M' is definable with parameters (the eventual parameters coming from R, φ_R or φ_S) in the sort S_E of M^{eq}. And vice versa, if a structure M' is definable in M^{eq}, it is interpretable in M. This is just an application of the fundamental fact C.2.1.

But what happens if E comes with parameters ? the following Lemma says that still M' is definable in M^{eq}:

Lemma C.2.3 *Let R be a definable set in M, and $E(\overline{x}, \overline{x}'; \overline{a})$ be a formula with parameters $\overline{a}$ from M defining an equivalence relation on R.*

Then there exists a 0-definable equivalence relation $F(\overline{xy}, \overline{x}'\overline{y}')$ such that R/E is in one-one correspondence with a definable set in the sort S_E.

Proof : Defining an equivalence relation on a set $\mathcal{Z}$ is equivalent to give a family of disjoint subsets of $\mathcal{Z}$ whose union is $\mathcal{Z}$.

In the case of a cartesian product $X \times Y$, if you define an equivalence relation E_y on each section $X \times \{y\}$, and since those sections are pairwise disjoint, you clearly define an equivalence relation on the whole $X \times Y$ by jointing together all the classes. The resulting equivalence relation is expressed saying that (x, y) and (x', y') are related iff $y = y'$ and $E_y(x, x')$.

Let $\overline{y}$ be a tuple of variables of same length m as $\overline{a}$, and let n be the length of $\overline{x}$.

We want to define an equivalence relation without parameters on each section of $M^n \times M^m$.

A candidate for $M^n \times \{\overline{a}\}$ is $E(\overline{x}, \overline{x}'; \overline{a})$ itself. More generally, let $\chi(\overline{y})$ be the formula (without parameters !) that says "$E(\overline{x}, \overline{x}'; \overline{y})$ is an equivalence relation in $(\overline{x}, \overline{x}')$". If $M \models \chi(\overline{b})$, put $E(\overline{x}, \overline{x}'; \overline{b})$ as an equivalence relation on the section

$M^n \times \{\bar{b}\}$. And if $M \models \neg\chi(\bar{b})$, put the trivial equivalence relation $\overline{x} = \overline{x}'$ on the section $M^n \times \{\bar{b}\}$.

Obviously, the resulting equivalence relation $F(\overline{xy}, \overline{x}'\overline{y}')$ on $M^n \times M^m$ is definable without parameters, and M^n/E is in one-one correspondence with $M^n \times \{\overline{a}\}/F$, a definable (in L^{eq}) subset in the sort S_F.

If instead of starting from E defined on the whole M^n you start with E defined on R, first extend it to E' on M^n by saying that outside R the classes are singletons (again a definable equivalence relation over $\overline{a}$), and then restrict the previous one-one correspondence to $R/E \subseteq M^n/E'$ to get a one-one correspondence between R/E and a definable subset in the sort S_F. ■

Therefore, we have seen that:

Proposition C.2.4 *A structure M' is interpretable with parameters in another structure M if and only if M' is definable in M^{eq}.*

C.3 Some motivations for considering type-definable equivalence relations

In C.1.1 we considered canonical parameters for definable relations. Let's push a step further, and consider complete types over $A \subseteq M$ (which are after all nothing more than ultrafilters of definable sets over A).

For fixed $\sigma \in \mathrm{Aut}(M)$, the map $\varphi(\overline{x}, \overline{a}) \mapsto \varphi(\overline{x}, \sigma(\overline{a}))$ induces a map $S_n(A) \to S_n(\sigma(A))$, $p \mapsto \sigma \cdot p$. If $p \in S_n(M)$, $\sigma \cdot p \in S_n(M)$, and this obviously defines an action of $\mathrm{Aut}(M)$ on the set $S_n(M)$.

What does it mean that $\sigma \cdot p = p$ (i.e. $\sigma \in \mathrm{Fix}(p)$) for this action ? simply that for every formula $\varphi(\overline{x}, \overline{y})$ and every tuple $\overline{a}$ from M of the same length as $\overline{y}$:

$$\varphi(\overline{x}, \overline{a}) \in p \quad \text{if and only if} \quad \varphi(\overline{x}, \sigma(\overline{a})) \in p \quad (*)$$

For fixed $\varphi(\overline{x}, \overline{y})$, the set of tuples $\overline{a}$ from M such that $\varphi(\overline{x}, \overline{a}) \in p$ is not necessarily definable of course. But if it does, we have at our disposal a canonical parameter e^p_φ for it, and Lemma C.1.1 says that $(*)$ is equivalent to $\sigma(e^p_\varphi) = e^p_\varphi$ in M^{eq}.

And if it is the case for all $\varphi(\overline{x}, \overline{y}) \in L$, then clearly $\sigma \in \mathrm{Fix}(p)$ if and only if $\sigma(e^p_\varphi) = e^p_\varphi$ for every φ.

This motivates the following definition:

Definition C.3.1 *A type $p \in S(M)$ is said to be* definable *over some parameter set $B \subseteq M$ if for every $\varphi(\overline{x}, \overline{y}) \in L$,*

$$\{\overline{a} \in M^m \mid \varphi(\overline{x}, \overline{a}) \in p\} \text{ is definable over } B$$

If a type $p \in S(M)$ is definable over M, call the set of canonical parameters $\{e^p_\varphi : \varphi \in L\}$ as described above a *canonical base* for p, noted $Cb(p)$, so that

$$\sigma \in \mathrm{Fix}(p) \text{ iff } \sigma \in \mathrm{Aut}(M^{eq}/Cb(p))$$

a generalization of what happens for definable relations.

And one sees easily that $\mathrm{dcl}^{eq}(Cb(p)) \subseteq M^{eq}$ is the smallest definably closed set of definition for p $\quad(**)$.

Now suppose T is a theory in which every over a model M is definable in M. Define in such a theory a relation $\overline{a} \downarrow_M N$, meaning that $M \prec N$, and $p = \mathrm{tp}(\overline{a}/N)$ is definable over M (not just over N). The property $(**)$ above shows that (since $M^{eq} \prec N^{eq}$ is a definably closed set)

$$\overline{a} \underset{M}{\downarrow} N \quad \text{if and only if} \quad Cb(p) \subseteq M^{eq}$$

It can be shown (c.f. [12] Remark 2.3.) the amazing fact that a complete theory T has all types over models definable if and only if T is *stable* (i.e. no formula has the *order property* in T, c.f. again [12] Chapter 1). Using C.2.1 and the definition of a stable theory in terms of formulas with the order property, it is immediate that T is stable if and only if T^{eq} is stable.

Another remarkable fact of a stable theory (see [12] Corollary 2.9.) is that if $A \subseteq M^{eq}$ is an algebraically closed set in T^{eq} (i.e. $\mathrm{acl}^{eq}(A) = A$), and $p \in S(A)$, then for every model $N^{eq} \supseteq A$, p has a *unique* extension $q \in S(N^{eq})$ that is definable over A. Moreover, the same formulas over A can be chosen to define p in every such model N^{eq}. This is another motivation for considering definable equivalence relations, since this property is not true in general for algebraically closed sets in T: if $A \subseteq M$ is such that $\mathrm{acl}(A) = A$ and $p \in S(A)$, then p can have two distinct extensions $q, q' \in S(M)$ that are definable over A.

If $p \in S(A)$ with $\mathrm{acl}^{eq}(A) = A$ in a stable theory, we say that p is definable over $B \subseteq A$ if the formulas that define the unique extension to a model are all definable over B.

This definition allows to define $\overline{a} \downarrow_A B$ for every $A \subseteq B$ and not just for elementary extensions. The meaning is that some (every) extension of $\mathrm{tp}(\overline{a}/B)$ to $\mathrm{acl}^{eq}(A)$ is definable over $\mathrm{acl}^{eq}(A)$. If $\overline{a} \models p$, write $Cb(\overline{a}/A)$ for $Cb(tp(\overline{a}/\mathrm{acl}^{eq}(A))$.

Again using $(**)$, we have immediately (since $\mathrm{acl}^{eq}(A)$ is a definably closed set)

$$\overline{a} \underset{A}{\downarrow} B \quad \text{if and only if} \quad Cb(\overline{a}/B) \subseteq \mathrm{acl}^{eq}(A) \quad (***)$$

The class of stable theories extends to that of *simple* theories. Like a stable theory, a simple theory is one in which certain "wild" configurations are prohibited.

In the case of stable theories, the order property is prohibited, while in the case of simple theories the so called *tree property* (c.f. [5]) is prohibited. And since the presence of the tree property implies that of the order property, stable implies simple. In both cases, the absence of those configurations allows the theories to have some sort of "tamed" behavior. Mainly, this tamed behavior is expressed by the existence of a nice relation of "independence" $\downarrow$.

For stables theories, this relation $\downarrow$ is the one defined above in terms of definability of types. It turns out that in stable theories, definability of types is related to another notion called *forking* (c.f. [5]) in the following sense: a type $p \in S(A)$ is definable over $B \subseteq A$ iff no formula in p *forks* over B (A, B algebraically closed in T^{eq}). The consequence is that the relation $\overline{a} \downarrow_A B$ in a stable theory can be characterized by $\mathrm{tp}(\overline{a}/A$ *does not fork* over B.

Since stable theories are exactly those for which every type over a model is definable, definability of types is not a meaningful notion for simple non-stable theories. However, the notion of forking go on behaving very well in simple theories, so that we *define* $\downarrow$ by means of forking in that context.

Since one has to forgot about definability of types in simple theories, and since $(***)$ in stable theories was obtained precisely by means of definability of types, it seems that no such characterization holds for a simple theory.

This is where 0-type definable equivalence relations pop up, and here is roughly how it works:

In a simple theory, some particular complete types are ubiquitous: the so-called *amalgamation bases*. In particular, every type over a model is an amalgamation base. Associated to an amalgamation base p is the so called *amalgamation class* $\mathcal{P}_p$ of p. For a given amalgamation base p, a technical construction (c.f.[5], chapter 17) allows to exhibit a 0-type definable equivalence relation E and a tuple a such that ($\overline{M}$ a monster model)

$$\mathrm{Fix}(\mathcal{P}_p) = \mathrm{Aut}(\overline{M}/a_E)$$

The class a_E is called a *canonical base* for p. We recognize the same pattern over and over. In that case, this allows to show again that (in a simple theory, c.f. [5] chapter 17)

$$\overline{a} \underset{A}{\downarrow} B \quad \text{if and only if} \quad Cb(\overline{a}/B) \subseteq \mathrm{bdd}(A)$$

This characterization of $\downarrow$ can be thought as a substitute for definability in simple theories. (refer to 2.1 for the meaning of bdd, which is a generalization of acl^{eq}).

The motivation for considering type-definable equivalence relations in a simple theory are at least two-fold:

1. The need to consider more general kinds of amalgamation bases as just types

over models. This is because, starting with a complete type $p \in S(A)$, A arbitrary, one needs to find $B \supseteq A$ such that every extension of p to $S(B)$ does not fork over A, and models containing A does not have this property. In a stable theory the solution is $B = \mathrm{acl}^{eq}(A)$, but in a simple theory a type an algebraically closed set in T^{eq} does not need to be an amalgamation base. Here the solution is rather $B = \mathrm{bdd}(A)$, making appear type definable equivalence relations.

2. Once one knows that a general amalgamation base in a simple theory has to be taken as a type over an hyperimaginary (i.e. a class under a type definable equivalence relation), the construction of its canonical base, in the sense specified above, also makes appear type-definable equivalence relations.

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