Universitat Politècnica de Catalunya
Departament de Matemàtica Aplicada I

PhD Thesis

Study of invariant manifolds in two different problems: the Hopf-zero singularity and neural synchrony

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Als meus pares

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Introduction

The main object of study of this thesis are invariant manifolds in the field of dynamical systems. We deal with two different and independent topics, namely, the study of exponentially small splitting of invariant manifolds in analytic unfoldings of the Hopf-zero singularity (in Part I) and the applications of the theory of invariant manifolds in problems inspired by neuroscience (in Part II). At the beginning of each part we introduce the corresponding problems properly, review the main open questions and summarize our results. Here we shall make only a short overview of each part.

On the one hand, in Part I we study analytic unfoldings of the Hopf-zero singularity and, more precisely, the splitting of certain invariant manifolds. One can see that, for an open set of parameters, the truncation of the normal form at any finite order of such unfoldings possesses two saddle-focus critical points and, when the parameters lie on a certain curve, they are connected by a one- and a two-dimensional heteroclinic manifolds. However, considering the whole vector field, one expects these heteroclinic connections to be destroyed. We study the case of generic unfoldings (the so-called singular case), but also non-generic ones (the regular case), where the perturbation is assumed to be smaller and Melnikov theory gives the correct size of the splitting. We find asymptotic formulas for the distance between these invariant manifolds, which turn out to be exponentially small in one of the perturbation parameters. In particular, with these formulas we make a significant step forward to giving a complete proof of the existence of infinitely many Shilnikov bifurcations in analytic unfoldings of the Hopf-zero singularity. This is an abstract and theoretical problem, and we use analytical tools, going from complex analysis to functional analysis.

On the other hand, in Part II we address questions of applied character. More precisely, our goal is to cross the existing borders between dynamical systems and neuroscience, trying to make dynamical systems theory closer to applied problems. In particular, we introduce and provide a deep analysis of the so-called Amplitude Response Functions for neuronal models. These can be used jointly with the Phase Response Functions to correctly predict the behavior of a neuron subject to external stimuli, in situations where the classical approach with Phase Response Curves fails. In the special case of a pulse-train periodic stimulus, the application of this theoretical frame leads to a 2D map, one variable controlling phase jumps and the other controlling amplitude jumps. We compare these maps to the classical 1D maps obtained via PRCs. In contrast with the theoretical framework of Part I, in this part we deal with an applied problem and use numerical methods to compute invariant manifolds of these maps.

In conclusion, this thesis provides a wide picture of interesting problems, ranging from theoretical to applied, where the role of the invariant manifolds is crucial to understand the dynamics, as well as the tools that can be used, which go from purely analytical to strictly computational.

Part I

Exponentially small splitting of invariant manifolds in analytic unfoldings of the Hopf-Zero singularity

Chapter 1

Introduction and main results

1.1 The Hopf-zero singularity and its unfoldings

The so-called Hopf-zero (or central) singularity consists in a vector field $X^* : \mathbb{R}^3 \to \mathbb{R}^3$, having the origin as a critical point, and such that the eigenvalues of the linear part at this point are 0, $\pm i\alpha^*$, for some $\alpha^* \neq 0$. Hence, after a linear change of variables, we can assume that the linear part of this vector field near the origin is:

$$DX^*(0,0,0) = \begin{pmatrix} 0 & \alpha^* & 0 \\ -\alpha^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this memoir, assuming analyticity and some generic conditions on X^*, we study some heteroclinic phenomena which appear in versal analytic unfoldings of this singularity in an open region of the parameter space. Note that, in the linear setting, it is clear that this singularity can be met by a generic family of linear vector fields depending on at least two parameters. Thus, it has codimension two. However, since $DX^*(0,0,0)$ has zero trace, it is reasonable to study it in the context of conservative vector fields. In this case, the singularity can be met by a generic linear family depending on one parameter, and so it has codimension one.

Here, we will work in the general setting (that is, with two parameters), since the conservative one is just a particular case of it. Hence, we will study generic analytic families $X_{\mu,\nu}$ of vector fields on $\mathbb{R}^3$ depending on two parameters $(\mu,\nu) \in \mathbb{R}^2$, such that $X_{0,0} = X^*$, the vector field described above. Following [Guc81] and [GH90], after some changes of variables we can write $X_{\mu,\nu}$ in its normal form of order two, namely:

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \bar{x}\,(\beta_0\nu - \beta_1\bar{z}) + \bar{y}\,(\alpha^* + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu),\\
\frac{d\bar{y}}{d\bar{t}} &= -\bar{x}\,(\alpha^* + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \bar{y}\,(\beta_0\nu - \beta_1\bar{z}) + \mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu),\\
\frac{d\bar{z}}{dt} &= -\gamma_0\mu + \gamma_1\bar{z}^2 + \gamma_2(\bar{x}^2+\bar{y}^2) + \gamma_3\mu^2 + \gamma_4\nu^2 + \gamma_5\mu\nu + \mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu).
\end{aligned} \tag{1.1}$$

Note that the coefficients β_1, γ_1, γ_2 and α_3 depend exclusively on the vector field X^*. We also observe that the conservative setting corresponds to taking $\nu = 0$, $\gamma_1 = \beta_1$ and imposing also that the higher order terms are divergence-free.

From now on, we will assume that X^* satisfies the following generic conditions:

$$\beta_1 \neq 0, \qquad \gamma_1 \neq 0. \tag{1.2}$$

Moreover, we will consider unfoldings satisfying the generic conditions:

$$\beta_0 \neq 0, \qquad \gamma_0 \neq 0.$$

Depending on the other coefficients α_i and γ_i, one obtains different qualitative behaviors for the orbits of the vector field $X_{\mu,\nu}$. The different versal unfoldings have been widely studied in the past, see for example [BV84, Gav78, GR83, Gav85, Guc81, GH90, Tak73, Tak74]. However, if (μ,ν) belongs to a particular open set of the parameter space, these unfoldings are still not completely understood. This set is defined by the following conditions:

$$\gamma_0\gamma_1\mu > 0, \qquad |\beta_0\nu| < |\beta_1|\sqrt{|\mu|}. \tag{1.3}$$

In this paper we will study the unfoldings $X_{\mu,\nu}$ with the parameters belonging to the open set defined by (1.3). In fact, redefining the parameters μ and ν and the variable $\bar{z}$, one can achieve:

$$\beta_0 = \gamma_0 = 1, \qquad \beta_1 > 0, \qquad \gamma_1 > 0, \tag{1.4}$$

and consequently the open set defined by (1.3) is now:

$$\mu > 0, \qquad |\nu| < \beta_1\sqrt{\mu}. \tag{1.5}$$

Moreover, dividing the variables $\bar{x}, \bar{y}$ and $\bar{z}$ by $\sqrt{\gamma_1}$, and rescaling time by $\sqrt{\gamma_1}$, redefining the coefficients and denoting $\alpha_0 = \alpha^*/\sqrt{\gamma_1}$, we can assume that $\gamma_1 = 1$, and therefore system (1.1) becomes:

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \bar{x}\,(\nu - \beta_1\bar{z}) + \bar{y}\,(\alpha_0 + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu),\\
\frac{d\bar{y}}{d\bar{t}} &= -\bar{x}\,(\alpha_0 + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \bar{y}\,(\nu - \beta_1\bar{z}) + \mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu),\\
\frac{d\bar{z}}{dt} &= -\mu + \bar{z}^2 + \gamma_2(\bar{x}^2+\bar{y}^2) + \gamma_3\mu^2 + \gamma_4\nu^2 + \gamma_5\mu\nu + \mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu).
\end{aligned} \tag{1.6}$$

We denote by $X^2_{\mu,\nu}$, usually called the truncation of the normal form of order two, the vector field obtained considering the terms of (1.6) up to order two. Therefore, one has:

$$X_{\mu,\nu} = X^2_{\mu,\nu} + F^2_{\mu,\nu}, \quad \text{where } F^2_{\mu,\nu}(\bar{x},\bar{y},\bar{z}) = \mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu).$$

Similarly, doing the normal form procedure up to any finite order n, one has:

$$X_{\mu,\nu} = X^n_{\mu,\nu} + F^n_{\mu,\nu} \qquad\qquad n \geq 2,$$

where $X^n_{\mu,\nu}(\bar{x},\bar{y},\bar{z})$ is a polynomial of degree n and:

$$F^n_{\mu,\nu}(\bar{x},\bar{y},\bar{z}) = \mathcal{O}_{n+1}(\bar{x},\bar{y},\bar{z},\mu,\nu).$$

Moreover, the truncation of the normal form $X^n_{\mu,\nu}$ has two saddle-focus critical points connected by a heteroclinic orbit and a two-dimensional heteroclinic surface (see Figure 1.1) for *any* finite order n. More precisely, one can show (see [Guc81], [GH90]) that if μ and ν are small enough, then for any $n \geq 2$:

1. $X^n_{\mu,\nu}$ has two critical points $\bar{S}^n_\pm(\mu,\nu) = (0,0,\bar{z}^n_\pm(\mu,\nu))$, with:

   $$\bar{z}^n_\pm(\mu,\nu) = \pm\sqrt{\mu} + \mathcal{O}((\mu^2+\nu^2)^{1/2}).$$

   The eigenvalues of $D_{\mu,\nu}(\bar{S}^n_\pm(\mu,\nu))$ are:

   $$\begin{aligned} \lambda_1^\pm &= \pm 2\sqrt{\mu} + \mathcal{O}((\mu^2+\nu^2)^{1/2}), \\ \lambda_2^\pm &= \nu \mp \beta_1\sqrt{\mu} + \sqrt{\mu}\,\mathcal{O}((\mu^2+\nu^2)^{1/2}) + i\left(\alpha_0 \pm \alpha_3\sqrt{\mu} + \mathcal{O}((\mu^2+\nu^2)^{1/2})\right), \\ \lambda_3^\pm &= \overline{\lambda_2^\pm}. \end{aligned}$$

   Hence, $\bar{S}^n_\pm(\mu,\nu)$ are both of saddle-focus type, $\bar{S}^n_+(\mu,\nu)$ having a one-dimensional unstable manifold and a two-dimensional stable one, and $\bar{S}^n_-(\mu,\nu)$ having a one-dimensional stable manifold and a two-dimensional unstable one.

2. The segment of the $\bar{z}$-axis between $\bar{S}^n_+(\mu,\nu)$ and $\bar{S}^n_-(\mu,\nu)$ is a heteroclinic connection (see Figure 1.1).

3. If $\gamma_2 > 0$ there exists a curve Γ_n in the (μ,ν)-plane of the form $\nu = m\sqrt{\mu} + \mathcal{O}(\mu^{3/2})$, such that for $(\mu,\nu) \in \Gamma_n$ the two-dimensional invariant manifolds of the points $\bar{S}^n_\pm(\mu,\nu)$ are coincident. In the conservative setting (where $\nu = 0$), the two-dimensional invariant manifolds of $\bar{S}^n_\pm(\mu)$ coincide for all values of μ (see Figure 1.1). The domain bounded by this heteroclinic surface has size $\mathcal{O}(\sqrt{\mu})$.

Item 1 yields that the whole vector field $X_{\mu,\nu} = X^n_{\mu,\nu} + F^n_{\mu,\nu}$ will have two critical points $\bar{S}_\pm(\mu,\nu)$ close to $\bar{S}^n_\pm(\mu,\nu)$, which will be also of saddle-focus type. However it is reasonable to expect that the heteroclinic connections described in items 2 and 3 (when when $\gamma_2 > 0$) will no longer persist in $X_{\mu,\nu}$. Moreover, for (μ,ν) close to Γ_n, what one might expect is that the breakdown of the heteroclinic connections causes the birth of a homoclinic orbit to the point $\bar{S}_+(\mu,\nu)$ (or $\bar{S}_-(\mu,\nu)$), giving rise to what is known as a Shilnikov bifurcation.

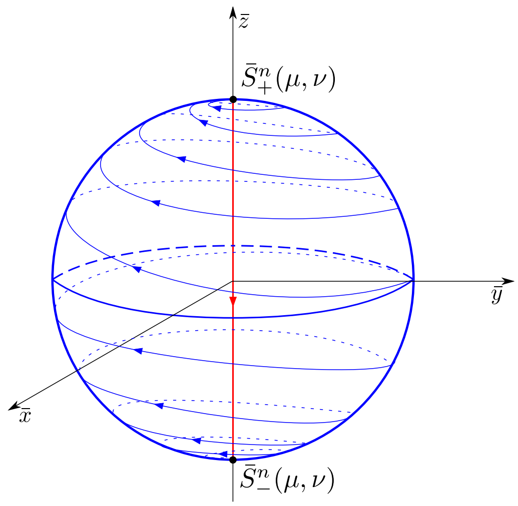


Figure 1.1: Phase portrait of the vector field $X^n_{\mu,\nu}$ for $(\mu,\nu)\in\Gamma_n$, for any $n\in\mathbb{N}$. In red and blue, the one- and two-dimensional heteroclinic connections respectively. The domain bounded by the two-dimensional heteroclinic connection has size $\mathcal{O}(\sqrt{\mu})$.

Definition. Let $X:\mathbb{R}^3\to\mathbb{R}^3$, and let $P\in\mathbb{R}^3$ be such that $X(P)=0$. We say that a Shilnikov Bifurcation occurs if (see Figure 1.2):

(i) The eigenvalues of $DX(P)$ are $-\rho\pm i\omega$ and λ, for certain $\rho,\omega,\lambda>0$,

(ii) $\lambda>\rho$,

(iii) There exists a homoclinic orbit $\gamma\subset W^{\mathrm{u}}(P)\cap W^{\mathrm{s}}(P)$.

In [Š65] (see also [Š67] for the analogous phenomenon for vector fields in $\mathbb{R}^4$), Shilnikov proved that if conditions (i)–(iii) above are satisfied, then there exist countably many periodic orbits in a neighborhood of the homoclinic orbit γ, giving rise to chaotic behavior. Clearly, $\bar{S}_+(\mu,\nu)$ satisfies the first condition. Condition (ii) is satisfied if $2\sqrt{\mu}>\beta_1\sqrt{\mu}-\nu$ (in particular, note that it is always satisfied in the conservative case, where $\beta_1=1$ and $\nu=0$). Thus, one only has to see whether the breakdown of the heteroclinic connections causes the birth of a homoclinic orbit γ to this point.

The existence of such Shilnikov bifurcations for $\mathcal{C}^\infty$ unfoldings of the Hopf-zero singularity is studied in [BV84]. In the first place, in that paper the authors show that, doing the normal form procedure up to order infinity and using Borel-Ritt's theorem, one can write $X_{\mu,\nu}=X^\infty_{\mu,\nu}+F^\infty_{\mu,\nu}$, where $X^\infty_{\mu,\nu}$ has the same properties 1, 2 and 3 as the vector fields $X^n_{\mu,\nu}$ described above, and $F^\infty_{\mu,\nu}=F^\infty_{\mu,\nu}(x,y,z)$ is a flat function at $(x,y,z,\mu,\nu)=(0,0,0,0,0)$. Their main result is that, given a family $X^\infty_{\mu,\nu}$, there exist flat perturbations $p^\infty_{\mu,\nu}$ such that the family:

$$X_{\mu,\nu}=X^\infty_{\mu,\nu}+p^\infty_{\mu,\nu} \tag{1.7}$$

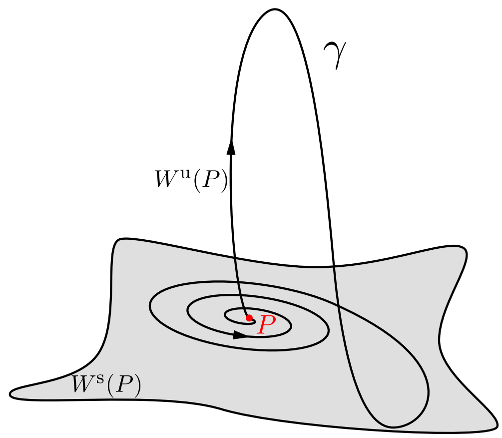


Figure 1.2: Homoclinic orbit γ of a saddle-focus critical point P.

possesses a sequence of Shilnikov bifurcations, occurring at parameter points (μ_l, ν_l), $l \in \mathbb{N}$, which accumulate at $(\mu, \nu) = (0, 0)$. Moreover, they prove that there is a dense subset of the unfoldings which do not have a Shilnikov bifurcation, but in the complement of this set this Shilnikov phenomenon occurs densely. We note that the authors give an existence theorem, but they do not provide conditions to check whether a concrete family $X_{\mu,\nu}$ possesses a Shilnikov bifurcation. Moreover, the fields of the family (1.7), for which they prove the existence of such bifurcations, are $\mathcal{C}^\infty$ but not analytic vector fields. In summary, their strategy consists in constructing suitable perturbations $p^\infty_{\mu,\nu}$ such that the heteroclinic connections of the family $X^\infty_{\mu,\nu}$ are destroyed.

The case of real analytic unfoldings of the singularity X^* has been open since then. It is possible that the strategy of Broer and Vegter can be adapted to the analytic case. Of course one cannot consider flat perturbations, but suitable perturbations could be constructed (although not straightforwardly) following [BT86] and [BT89]. However, another strategy must be followed if given *any* unfolding $X_{\mu,\nu}$ one wants to determine whether it will possess a sequence of Shilnikov bifurcations or not. The key point, as in the $\mathcal{C}^\infty$ case, is to determine if the unfolding $X_{\mu,\nu}$ possesses the aforementioned heteroclinic connections seen in the truncation of its normal form $X^n_{\mu,\nu}$.

Progress was made recently in [DIKS13], where the authors prove the equivalent result as [BV84] in the real analytic context assuming some upper and lower bounds of the distance between the invariant manifolds of $\bar{S}_+(\mu, \nu)$ and $\bar{S}_-(\mu, \nu)$ (see Section 1.4 for more details). In particular, the authors assume that the heteroclinic connections are destroyed. Our work computes asymptotic formulas of the splitting of these invariant manifolds which, as a consequence, allow to check if the assumptions in [DIKS13] are satisfied.

As we pointed out above, the truncation of the normal form $X^n_{\mu,\nu}$ possesses a one-and a two-dimensional heteroclinic connection (in the latter case, for a suitable choice of the parameters) for all finite n. In other words, the breakdown of these heteroclinic connections cannot be detected in the truncation of the normal form at any finite order and therefore, as it is usually called, it is a phenomenon *beyond all orders*. Moreover, since $X_{\mu,\nu} = X^n_{\mu,\nu} + F^n_{\mu,\nu}$, this breakdown must be caused by the remainder $F^n_{\mu,\nu}$, which satisfies:

$$F^n_{\mu,\nu}(\bar{x},\bar{y},\bar{z}) = \mathcal{O}_{n+1}(\bar{x},\bar{y},\bar{z},\mu,\nu).$$

We recall that the heteroclinic connections are inside a domain in $\mathbb{R}^3$ of size $\mathcal{O}(\sqrt{\mu})$, so that in this region we have:

$$F^n_{\mu,\nu}(\bar{x},\bar{y},\bar{z}) = \mathcal{O}_{n+1}(\sqrt{\mu},\nu).$$

Since this is valid for all n, the distance between the invariant manifolds should be smaller than any finite power of the perturbation parameters. For this reason, one expects this distance to be exponentially small in one of the perturbation parameters. Note that, since $X^n_{\mu,\nu}$ has a two-dimensional heteroclinic connection for all n but only when $(\mu,\nu) \in \Gamma_n = \{\nu = \nu_n(\mu)\}$, one expects that the distance between the two-dimensional invariant manifolds is exponentially small only when (μ,ν) is close to a certain curve $\Gamma_* = \{\nu = \nu_*(\mu)\}$, while for the one-dimensional case it should happen for all values of (μ,ν), if they are sufficiently small.

It is worth mentioning that, when studying exponentially small phenomena, one cannot use classical perturbation methods, since the truncation of the normal form gives insufficient information of the whole system. Therefore, more sophisticated techniques have to be introduced. Over the last decades these techniques have been developed, mainly for Hamiltonian systems and area preserving maps. We give a brief summary of these advances in next section.

1.2 Exponentially small splitting of invariant manifolds

The problem of exponentially small splitting of separatrices (or more generally, of invariant manifolds) was already considered the *fundamental problem of mechanics* by Poincaré in his famous work [Poi90]. There he studied Hamiltonian systems with two and a half degrees of freedom and realized that this phenomenon was responsible for the creation of chaotic behavior. He considered a model which, after reduction, became the perturbed pendulum:

$$\ddot{y} = 2\mu\sin y + 2\mu\varepsilon\cos y\cos t.$$

Using what later has become known as the Melnikov method (although Poincaré was actually the first one to use it, being rediscovered by Melnikov more than 70 years later) he proved that the splitting of the separatrices is exponentially small in μ, provided that ε is smaller than some exponentially small quantity. Of course, this latter assumption is enormously restrictive, but many years had to go by until it could be removed.

Indeed, this problem was not studied from a rigorous point of view until the end of the 80s and during the 90s. First, Neishtadt [Neĭ84] gave upper bounds for the splitting in Hamiltonian systems of one and a half and two degrees of freedom. Lazutkin [Laz] was the first to give an asymptotic expression of the splitting angle between the stable and unstable manifolds in the standard map. Although some details were left unfinished in this paper, it established the basis of the methods used later, for instance the analytic continuation of the invariant manifolds in the complex plane. It was not until years later that a complete proof of Lazutkin's result was given by Gelfreich in [Gel99].

After Lazutkin's paper, some works were published giving bounds of the splitting of the invariant manifolds. In [HMS88] Holmes, Marsden and Scheurle obtained both upper and lower bounds for rapidly forced systems. Later, Fontich and Simó [FS90a, FS90b] gave upper bounds of the splitting of separatrices for families of area-preserving diffeomorphisms close to the identity, in particular for Poincaré maps of non-autonomous Hamiltonian systems.

Later on, asymptotic formulas for several examples were obtained. Kruskal and Segur gave an exponentially small asymptotic formula for the breakdown of a heteroclinic connection in a third order differential equation which came from a model of crystal growth, see [KS91]. Delshams and Seara [DS92] gave an asymptotic expression of the splitting of separatrices in the rapidly forced pendulum (see also [Gel94] for a similar result, although assuming a fairly more restrictive condition on the size of the perturbation). These were the first asymptotic expressions for systems as the ones considered by Poincaré.

After these pioneering works, partial results for general Hamiltonian systems were given in [DS97, Gel97a, BF04, BF05]. A new approach that avoids some complications of Lazutin's method and that has had much influence in posterior studies of exponentially small splitting was introduced in [Sau01, LMS03]. It is important to note that, besides [Laz] and [KS91], all the examples cited above deal with the so-called regular case, in which some artificial condition about the smallness of the perturbation is required. In this case the Melnikov method gives the correct size of the splitting.

In the singular case (in which no artificial condition about the smallness of the perturbation is required), one often has to study a certain equation, usually called the *inner equation*. In [Gel97b], the corresponding inner equations was studied for several periodically perturbed second order equations. In [GS01] there is a rigorous study of the inner equation of the Hénon map using Resurgence Theory [Éca81a, Éca81b]. In [OSS03] there is a rigorous analysis of the inner equation for the Hamilton–Jacobi equation associated to a pendulum equation with a certain perturbation term, also using Resurgence Theory. In [Bal06] there is the only result which deals with the inner equation associated to very general type of polynomial Hamiltonians with a fast perturbation. In [BS08] the authors study the inner equation associated to the splitting of the one-dimensional heteroclinic connection of the Hopf-zero singularity in the conservative case, and we will use the results contained in this work in the present memoir. In [BM12], the inner equation for generalized standard maps is studied.

Besides the works of Lazutkin and Kruskal and Segur, there are very few works with rigorous proofs in the singular case for Hamiltonian systems or conservative maps. In [Tre97], Treschev gave an asymptotic formula for the splitting in the case of a pendulum with certain perturbations using a method called Continuous Averaging. In [Gel00] there is a detailed sketch of the proof for the splitting of separatrices of the equation of a pendulum with a particular perturbation and a complete rigorous proof, which also covers more general cases, is done in [GOS10]. Numerical results about the splitting for this problem can be found in [BO93, Gel97b]. In [GG11], the authors study the Hamiltonian-Hopf bifurcation (a Hamiltonian version of the singularity studied in this memoir) combining numerical and analytical techniques. The most general result dealing with Hamiltonian systems with a periodic perturbation in time is given in [BFGS12, Gua13]. In the case of two-dimensional symplectic maps see for instance [DRR99, GS08], where a detailed numerical study of the splitting is done, or [GB08] and [MSS11], where the splitting for the Hénon and McMillan maps are studied respectively.

All these works deal with either Hamiltonian systems or symplectic maps. To the best of our knowledge, the only works concerning exponentially small splitting of separatrices in a non-Hamiltonian setting are [Laz03] and [Lom00], where reversible systems are considered. Also, [Fon95] gives results for dissipative perturbations of Hamiltonian systems.

It is worth mentioning that the setting of this memoir is not similar to any of the works computing exponentially small splitting of invariant manifolds. Indeed, here we do not deal with a Hamiltonian system, the flow of the vector field might not be volume preserving (since we consider not only the conservative setting but also the dissipative one) and it is not a reversible system. For this reason, new methods had to be developed in order to prove the results found in this memoir, which we proceed to state now.

1.3 Main results

In this memoir we compute asymptotic formulas of the distance between the invariant manifolds of the critical points $\bar{S}_+(\mu,\nu)$ and $\bar{S}_-(\mu,\nu)$ introduced in Section 1.1. In the following, we denote by $\bar{d}^{\mathrm{u,s}}(\mu,\nu)$ the distance between the one-dimensional invariant manifolds (see Figure 1.3), and by $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu)$, $\theta\in[0,2\pi]$, the distance between the two-dimensional invariant manifolds (see Figures 1.4 and 1.5). Both distances are computed on the plane $\bar{z}=0$.

We now state the main results of this part of the memoir.

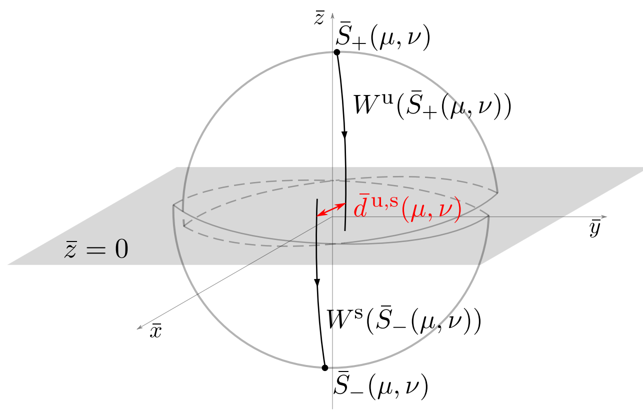


Figure 1.3: The distance $\bar{d}^{\mathrm{u,s}}(\mu,\nu)$ between the one-dimensional invariant manifolds of $\bar{S}_+(\mu,\nu)$ and $\bar{S}_-(\mu,\nu)$.

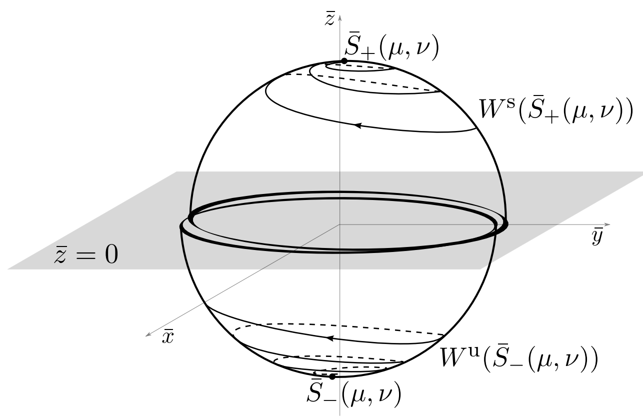


Figure 1.4: The two-dimensional invariant manifolds of $\bar{S}_+(\mu,\nu)$ and $\bar{S}_-(\mu,\nu)$ until they reach the plane $\bar{z}=0$.

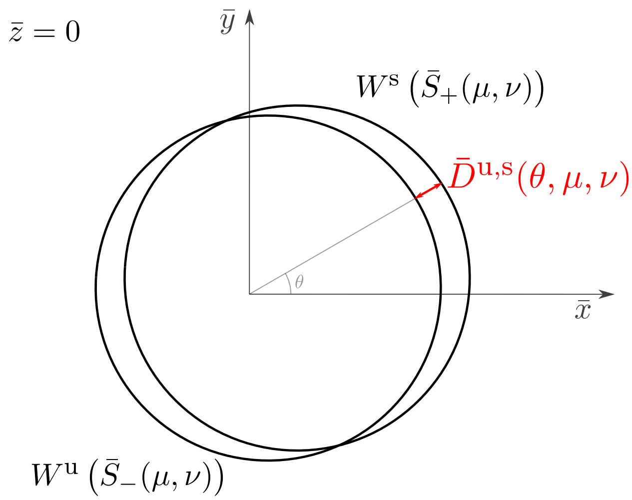


Figure 1.5: The intersection between the invariant manifolds and the plane $\bar{z}=0$, and the distance $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu)$ between them.

Theorem 1.1. *Consider system* (1.6)*, with* μ*,* $\beta_1>0$ *and* $|\nu|<\beta_1\sqrt{\mu}$*, which has two critical points* $\bar{S}_\pm(\mu,\nu)$ *of saddle-focus type. Let* $\bar{d}^{\mathrm{u,s}}(\mu,\nu)$ *(*$d^{\mathrm{u,s}}(\mu)$ *in the conservative case) be the distance between the one-dimensional stable manifold of* $\bar{S}_-(\mu,\nu)$ *and the one-dimensional unstable manifold of* $\bar{S}_+(\mu,\nu)$ *when they meet the plane* $\bar{z}=0$ *(see Figure 1.3). Let* $-h_0$ *be the coefficient of* $\bar{z}^3$ *in the third equation of system* (1.6)*. There exists a constant* $\mathcal{C}_0^*$*, depending on the full jet of* X^**, such that:*

1. *In the conservative case,* $d^{\mathrm{u,s}}(\mu)$ *is given asymptotically as* $\mu\to 0$ *by:*

$$\bar{d}^{\mathrm{u,s}}(\mu)=\mu^{-1/2}e^{-\frac{\alpha_0\pi}{2\sqrt{\mu}}}e^{\frac{\pi}{2}(\alpha_0h_0+\alpha_3)}\left(\mathcal{C}_0^*+\mathcal{O}\left(\frac{1}{|\log\mu|}\right)\right).$$

2. *In the dissipative case,* $d^{\mathrm{u,s}}(\mu,\nu)$ *is given asymptotically as* $\mu\to 0$ *by:*

$$\bar{d}^{\mathrm{u,s}}(\mu,\nu)=\mu^{-\frac{\beta_1}{2}}e^{-\frac{\alpha_0\pi}{2\sqrt{\mu}}}e^{\frac{\pi}{2}(\alpha_0h_0-\frac{\alpha_1\nu}{\sqrt{\mu}}+\alpha_3)}\left(\mathcal{C}_0^*+\mathcal{O}\left(\frac{1}{|\log\mu|}\right)\right).$$

This theorem is proved in Chapter 2.

Theorem 1.2. *Consider system* (1.6)*, with* μ*,* $\beta_1,\gamma_2>0$ *and* $|\nu|<\beta_1\sqrt{\mu}$*, which has two critical points* $\bar{S}_\pm(\mu,\nu)$ *of saddle-focus type. Let* $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu)$ *(*$\bar{D}^{\mathrm{u,s}}(\theta,\mu)$ *in the conservative case) be the distance between the two-dimensional unstable manifold of* $\bar{S}_-(\mu,\nu)$ *and the two-dimensional stable manifold of* $\bar{S}_+(\mu,\nu)$ *on the plane* $\bar{z}=0$ *(see Figure 1.5). There exist constants* $\mathcal{C}_1^*$*,* $\mathcal{C}_2^*$ *and* L*, such that:*

1. *In the conservative case, $\bar{D}^{\mathrm{u,s}}(\theta,\mu)$ is given asymptotically as $\mu \to 0$ by:*

$$\begin{aligned}\bar{D}^{\mathrm{u,s}}(\theta,\mu) &= \sqrt{\frac{\gamma_2}{2}}\frac{e^{-\frac{\alpha_0\pi}{2\sqrt{\mu}}}}{\mu^{\frac{3}{2}}}\Big[\mathcal{C}_1^*\cos(\theta - L\log\mu) + \mathcal{C}_2^*\sin(\theta - L\log\mu)\\ &\quad +\mathcal{O}\left(\frac{1}{|\log\mu|}\right)\Big].\end{aligned}$$

2. *In the dissipative case, given $a_1, a_2 \in \mathbb{R}$ and $a_3 > 0$, there exists a function $\nu = \nu(\mu)$ satisfying $\nu(0) = 0$, such that $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu(\mu))$ is given asymptotically as $\mu \to 0$ by:*

$$\begin{aligned}\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu(\mu)) &= a_1\mu^{a_2}e^{-\frac{a_3\pi}{2\beta_1\sqrt{\mu}}}(1+\mathcal{O}(\sqrt{\mu})) +\\ &\quad \sqrt{\frac{\gamma_2}{\beta_1+1}}\frac{e^{-\frac{\alpha_0\pi}{2\beta_1\sqrt{\mu}}}}{\mu^{\frac{1}{2}+\frac{1}{\beta_1}}}\Big[\mathcal{C}_1^*\cos(\theta - L\log\mu) + \mathcal{C}_2^*\sin(\theta - L\log\mu)\\ &\quad +\mathcal{O}\left(\frac{1}{|\log\mu|}\right)\Big].\end{aligned}$$

Moreover, the constants $\mathcal{C}_1^*$ and $\mathcal{C}_2^*$ depend on the full jet of X^*. The constant L is given by:

$$L = \frac{1}{2\beta_1}(\alpha_3 + \alpha_0 L_0),$$

for a certain constant L_0 which depends on a finite number of Taylor coefficients of X^*. A formula for the constant L_0 is given in Chapter 3, Lemma 3.5.17.

This theorem is proved mainly in Chapter 4, although many results of Chapter 3 are used during the proof.

Remark 1.3. The constants $\mathcal{C}_i^*$, which are usually called Stokes constants (see [Sto64, Sto02]), depend on the full jet of X^* and therefore, up to now, they can only be computed numerically. This computation is not trivial, and is not the goal of the present memoir. For the one-dimensional case, it has been done for particular examples in [LS09]. A detailed and accurate numerical computation in the one- and two-dimensional cases in many examples (in conservative and non-conservative settings) has been done in [DIKS13].

Remark 1.4. We note that in Theorem 1.2 we require that $\nu = \nu(\mu)$, while in Theorem 1.1 ν is a free parameter.

1.3.1 The regular vs. the singular case

As we shall see later on, in order to obtain the asymptotic formulas given in Theorems 1.1 and 1.2, one needs to study parameterizations of the invariant manifolds of $\bar{S}_+(\mu,\nu)$ and

$\bar{S}_-(\mu,\nu)$ not only on real domains, but also for complex ones. These domains need to be close to the singularities of the corresponding heteroclinic connection of the unperturbed system $X^2_{\mu,\nu}$. Far from these singularities, the invariant manifolds are well approximated by the unperturbed heteroclinic connection, but this is not the case near to them. This yields some technical difficulties.

A good way to start studying the invariant manifolds in these complex domains is considering smaller perturbations of the vector field $X^2_{\mu,\nu}$. One can introduce a new parameter $p \geq -2$ and consider the following system:

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \bar{x}\left(\nu - \beta_1\bar{z}\right) + \bar{y}\left(\alpha_0 + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}\right) + \mu^{\frac{p+2}{2}}\mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu),\\
\frac{d\bar{y}}{d\bar{t}} &= -\bar{x}\left(\alpha_0 + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}\right) + \bar{y}\left(\nu - \beta_1\bar{z}\right) + \mu^{\frac{p+2}{2}}\mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu),\\
\frac{d\bar{z}}{dt} &= -\mu + \bar{z}^2 + \gamma_2(\bar{x}^2+\bar{y}^2) + \mu^{\frac{p+2}{2}}\left(\gamma_3\mu^2 + \gamma_4\nu^2 + \gamma_5\mu\nu + \mathcal{O}_3(\bar{x},\bar{y},\bar{z},\mu,\nu)\right).
\end{aligned} \tag{1.8}$$

Clearly, for $p=-2$ we recover system (1.6), while for $p>-2$ the perturbative terms are smaller than those of system (1.6). We call the case $p>-2$ the *regular case*, while $p=-2$ is the *singular* one. The first case represents just a special subset of unfoldings of X^*, while the latter one represents a generic family of unfoldings of X^*.

As we mentioned in Section 1.2, in the singular case one needs to study the so-called *inner equation*, which is a parameter-free equation that gives good approximations of the invariant manifolds close to the singularities of the heteroclinic orbit of the unperturbed system. In addition, one needs to use complex matching techniques to compare the solutions of the inner equation and the solutions of the original system (1.6).

On the contrary, imposing the artificial condition $p>-2$, one can see that the heteroclinic connections of the unperturbed system $X^2_{\mu,\nu}$ give good approximations of the invariant manifolds, even close to their singularities. For this reason, this case is easier to deal with than the singular one, and regular perturbation techniques can be used to obtain the desired asymptotic formulas. These consist on suitable versions of the so-called Melnikov integrals (see [GH90, Mel63]). Thus, one can start studying this case to gain some intuition without getting lost with technical problems and, after that, one can proceed with the singular case. This is what we have done in the present memoir.

The regular case of the one-dimensional heteroclinic connection was studied (in the conservative setting) by Baldomá and Seara in [BS06]. We state here their result, which is analogous to item 1 of Theorem 1.1.

Theorem 1.1' ([BS06])**.** *Consider system* (1.8)*, with* $\mu, \beta_1 > 0$ *and* $|\nu| < \beta_1\sqrt{\mu}$*, which has two critical points* $\bar{S}_\pm(\mu,\nu)$ *of saddle-focus type. Let* $\bar{d}^{\mathrm{u,s}}(\mu,\nu)$ *(*$\bar{d}^{\mathrm{u,s}}(\mu)$ *in the conservative case) be the distance between the one-dimensional stable manifold of* $\bar{S}_-(\mu,\nu)$ *and the one-dimensional unstable manifold of* $\bar{S}_+(\mu,\nu)$ *when they meet the plane* $\bar{z}=0$*. In the conservative case, there exists a constant* $\mathcal{C}_0$ *such that* $\bar{d}^{\mathrm{u,s}}(\mu)$ *is given asymptotically* *as $\mu \to 0$ by:*

$$\bar{d}^{\mathrm{u,s}}(\mu) = \mu^{\frac{p+1}{2}} e^{-\frac{\alpha_0 \pi}{2\sqrt{\mu}}} e^{\frac{\pi}{2}\alpha_3} \left(\mathcal{C}_0 + \mathcal{O}\left(\mu^{\frac{p+2}{2}} |\log \mu| \right) \right).$$

The constant $\mathcal{C}_0$ is a suitable Melnikov integral, which can be written as the Borel transform of a known function.

Remark 1.5. In the dissipative case, the proof of Baldomá and Seara seems to work analogously (in particular, they do not use that the perturbative terms are divergence-free), and one would obtain that $\bar{d}^{\mathrm{u,s}}(\mu,\nu)$ is given asymptotically as $\mu \to 0$ by:

$$\bar{d}^{\mathrm{u,s}}(\mu,\nu) = \mu^{\frac{p+2-\beta_1}{2}} e^{-\frac{\alpha_0 \pi}{2\sqrt{\mu}}} e^{\frac{\pi}{2}(-\frac{\alpha_1 \nu}{\sqrt{\mu}}+\alpha_3)} \left(\mathcal{C}_0 + \mathcal{O}\left(\mu^{\frac{p+2}{2}} |\log \mu| \right) \right).$$

The regular case $p > -2$ of the two-dimensional heteroclinic connection is studied in Chapter 3. As we shall see, many of the results obtained in this chapter are still valid for $p = -2$, and we shall use them to prove Theorem 1.2. We obtain the following asymptotic formulas of the distance $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu)$, which are very similar to those of Theorem 1.2.

Theorem 1.2'. *Consider system (1.8), with μ, $\beta_1, \gamma_2 > 0$ and $|\nu| < \beta_1\sqrt{\mu}$, which has two critical points $\bar{S}_\pm(\mu,\nu)$ of saddle-focus type. Let $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu)$ ($\bar{D}^{\mathrm{u,s}}(\theta,\mu)$ in the conservative case) be the distance between the two-dimensional unstable manifold of $\bar{S}_-(\mu,\nu)$ and the two-dimensional stable manifold of $\bar{S}_+(\mu,\nu)$ on the plane $\bar{z} = 0$. For $p > -2$, there exist constants $\mathcal{C}_1$, $\mathcal{C}_2$ such that:*

1. *In the conservative case, $\bar{D}^{\mathrm{u,s}}(\theta,\mu)$ is given asymptotically as $\mu \to 0$ by:*

$$\begin{aligned} \bar{D}^{\mathrm{u,s}}(\theta,\mu) &= \sqrt{\frac{\gamma_2}{2}} \frac{e^{-\frac{\alpha_0\pi}{2\sqrt{\mu}}}}{\mu^{-\frac{p-1}{2}}} \left[\mathcal{C}_1 \cos\left(\theta - \frac{\alpha_3}{2}\log\mu\right) + \mathcal{C}_2 \sin\left(\theta - \frac{\alpha_3}{2}\log\mu\right) \right. \\ &\quad \left. + \mathcal{O}\left(\mu^{\frac{p+2}{2}} |\log\mu| + \mu^{3/2} \right) \right]. \end{aligned}$$

2. *In the dissipative case, given $a_1, a_2 \in \mathbb{R}$ and $a_3 > 0$, there exists a function $\nu = \nu(\mu)$ satisfying $\nu(0) = 0$, such that $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu(\mu))$ is given asymptotically as $\mu \to 0$ by:*

$$\begin{aligned} \bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu(\mu)) &= a_1 \mu^{a_2} e^{-\frac{a_3\pi}{2\beta_1\sqrt{\mu}}} \left(1 + \mathcal{O}\left(\mu^{\frac{p+3}{2}} \right) \right) + \\ &\quad \sqrt{\frac{\gamma_2}{\beta_1+1}} \frac{e^{-\frac{\alpha_0\pi}{2\beta_1\sqrt{\mu}}}}{\mu^{-\frac{p+1}{2}+\frac{1}{\beta_1}}} \left[\mathcal{C}_1 \cos\left(\theta - \frac{\alpha_3}{2\beta_1}\log\mu\right) \right. \\ &\quad \left. + \mathcal{C}_2 \sin\left(\theta - \frac{\alpha_3}{2\beta_1}\log\mu\right) + \mathcal{O}\left(\mu^{\frac{p+2}{2}} |\log\mu| + \mu^{3/2} \right) \right]. \end{aligned}$$

The constants $\mathcal{C}_1$ and $\mathcal{C}_2$ are suitable Melnikov integrals, which can be written respectively as the real and imaginary part of the Borel transform of a known function.

Remark 1.6. If we take $p = -2$ in Theorems 1.1' and 1.2', the leading terms have the same behavior as $\mu \to 0$ as the corresponding ones of Theorems 1.1 and 1.2. However, the error in Theorems 1.1' and 1.2' terms are not small anymore. One could think that these error terms are not sharp, and that the corresponding formulas are true even for $p = -2$. However, one has that $\mathcal{C}_i \neq \mathcal{C}_i^*$, so that these error terms contain some information that plays a role on the limit $p = -2$. Of course, note also that the constant L_0 appearing in Theorem 1.2 does not appear in Theorem 1.2'.

1.4 Future work: proving the existence of Shilnikov bifurcations

Note that if the constants $\mathcal{C}_i^*$ in Theorems 1.1 and 1.2 are such that:

$$\mathcal{C}_0^* \neq 0, \qquad (\mathcal{C}_1^*)^2 + (\mathcal{C}_2^*)^2 \neq 0, \tag{1.9}$$

then by the asymptotic formulas given in the same theorems we know that:

$$\bar{d}^{\mathrm{u,s}}(\mu,\nu) \neq 0, \qquad \bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu) \not\equiv 0.$$

For the regular case equivalent conditions for the constants $\mathcal{C}_i$ in Theorems 1.1' and 1.2' are obtained, namely:

$$\mathcal{C}_0 \neq 0, \qquad (\mathcal{C}_1)^2 + (\mathcal{C}_2)^2 \neq 0. \tag{1.10}$$

Thus, conditions (1.9) (respectively (1.10)) ensure that for every member of the family $X_{\mu,\nu}$ satisfying the open conditions $\gamma_2 > 0$, (1.4) and (1.5) the one-dimensional and the two-dimensional invariant manifolds of $\bar{S}_\pm(\mu,\nu)$ are not coincident. In other words, they ensure that the heteroclinic connections seen in the truncation of the normal form $X^n_{\mu,\nu}$ do not persist in $X_{\mu,\nu}$. As we pointed out in Section 1.1, this is a key fact for the existence of Shilnikov bifurcations. Moreover, the proof of [DIKS13] relies on some assumptions on the distances $\bar{d}^{\mathrm{u,s}}(\mu,\nu)$, and $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu)$ which our formulas allow to validate. For the sake of completeness, next we state the theorem by [DIKS13] with our notation. Before, however, we point out that the authors make an extra simplification of the unfoldings (which can be obtained by redefining parameters and reparametrizing time) that we do not make. With our notation, this simplification can be written in terms of some coefficients of system (1.6) as:

$$\begin{array}{rcll} \alpha_i & = & 0, & i = 1,\dots,3 \\ \gamma_i & = & 0, & i = 3,\dots,5. \end{array} \tag{1.11}$$

We stress that this simplification can be done without any restriction, but extra reparameterizations have to be done. Of course, our results are the same if (1.11) holds.

Theorem ([DIKS13]). *Let $X_{\mu,\nu}$ be an unfolding of X^* satisfying μ, $\beta_1, \gamma_2 > 0$, $|\nu| < \beta_1\sqrt{\mu}$ and (1.11). Let $\bar{d}^{\mathrm{u,s}}(\mu,\nu)$ denote the distance between the one-dimensional invariant manifolds of $\bar{S}_+(\mu,\nu)$ and $\bar{S}_-(\mu,\nu)$ on the plane $\{\bar{z}=0\}$ (see Figure 1.3). Let $\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu)$, $\theta \in [0,2\pi]$, denote the distance between the two-dimensional invariant manifolds on the plane $\{\bar{z}=0\}$ (see Figures 1.4 and 1.5). One can write:*

$$\bar{D}^{\mathrm{u,s}}(\theta,\mu,\nu) = \bar{D}^{\mathrm{u,s}}_{\mathrm{f}}(\mu,\nu) + \bar{D}^{\mathrm{u,s}}_{\mathrm{b}}(\theta,\mu,\nu),$$

where:

$$\bar{D}^{\mathrm{u,s}}_{\mathrm{f}}(\mu,\nu) = a\mu^2 + b\mu\nu + o(\|(\mu^2,\mu\nu)\|),$$

with $a \neq 0$, and:

$$\left|\frac{\bar{D}^{\mathrm{u,s}}_{\mathrm{b}}(\theta,\mu,\nu)}{\mu^m}\right| \to 0$$

as $\mu \to 0$ uniformly in θ and ν for all $m \in \mathbb{N}$.

Assume that there exist constants $C_1, C_2 > 0$, $0 < \tilde{C}_2 < C_2$, N_1, N_2 and $A_1, A_2 > 0$ and a curve $\nu = \nu(\mu)$, with $\nu(0) = 0$, such that:

(S1) $0 < \bar{d}^{\mathrm{u,s}}(\mu,\nu) < C_1 \mu^{N_1} e^{-\frac{A_1}{\sqrt{\mu}}}$.

(S2) *Let:*

$$\mathrm{M}(\mu,\nu) = \max_{\theta\in[0,2\pi]} \bar{D}^{\mathrm{u,s}}_{\mathrm{b}}(\theta,\mu,\nu).$$

Then:

$$\mathrm{M}(\mu,\nu) \geq C_2 \mu^{N_2} e^{-\frac{A_2}{\sqrt{\mu}}}.$$

(S3) $\left|\bar{D}^{\mathrm{u,s}}_{\mathrm{f}}(\mu,\nu(\mu))\right| < \tilde{C}_2 \mu^{N_2} e^{-\frac{A_2}{\sqrt{\mu}}}$.

(S4) $A_2/A_1 < 2/\beta_1$.

(S5) *Let $q_0(\mu,\nu)$ denote the first intersection point of $W^{\mathrm{u}}(\bar{S}_+(\mu,\nu))$ with $\bar{z}=0$. Let $\bar{q}_0(\mu,\nu)$ denote the first intersection point of the forward orbit of $q_0(\mu,\nu)$ with $\bar{z}=0$. Let $\theta_{q_0}(\mu,\nu)$ be its continuous argument when it is written in polar coordinates. There exists a function $\theta_0(\mu)$ such that $\bar{D}^{\mathrm{u,s}}_{\mathrm{b}}(\theta_0(\mu),\mu,\nu(\mu)) = 0$ for all $\mu < \mu_0$ such that as $\mu \to 0$:*

$$|\theta_{q_0}(\mu,\nu(\mu)) - \theta_0(\mu)| \to \infty.$$

Then, for $0 < \beta_1 < 2$, there exists a sequence of parameter values $\{\mu_n\}_{n\in\mathbb{N}}$ with $\mu_n \to 0$ as $n \to \infty$ such that the corresponding system at $(\mu_n, \nu(\mu_n))$ has a Shilnikov homoclinic orbit at $\bar{S}_+(\mu,\nu)$ which intersects $\bar{z}=0$ only at two points.

Note that, if $\mathcal{C}_0^* \neq 0$ and $(\mathcal{C}_1^*)^2 + (\mathcal{C}_2^*)^2 \neq 0$ (in the regular case, $\mathcal{C}_0 \neq 0$ and $(\mathcal{C}_1)^2 + (\mathcal{C}_2)^2 \neq 0$), then assumptions (S1)–(S4) are satisfied. Moreover, using the implicit function theorem it is easy to see that the function $\theta_0(\mu)$ such that $\bar{D}_{\mathrm{b}}^{\mathrm{u,s}}(\theta_0(\mu), \mu, \nu(\mu)) = 0$ in assumption (S5) is given by:

$$\theta_0(\mu) = L \log \mu - \arctan\left(\frac{\mathcal{C}_1^*}{\mathcal{C}_2^*}\right) + \mathcal{O}\left(\frac{1}{|\log \mu|}\right).$$

Thus, the last step to give a complete result on the existence of Shilnikov bifurcations in analytic unfoldings of the Hopf-zero singularity is to find an expression of the function $\theta_{q_0}(\mu, \nu)$ in assumption (S5), and check that as $\mu \to 0$:

$$|\theta_{q_0}(\mu, \nu) - \theta_0(\mu)| = \left|\theta_{q_0}(\mu, \nu) - \left(L \log \mu - \arctan\left(\frac{\mathcal{C}_1^*}{\mathcal{C}_2^*}\right) + \mathcal{O}\left(\frac{1}{|\log \mu|}\right)\right)\right| \to \infty.$$

We leave as future work to find such an expression of $\theta_{q_0}(\mu, \nu)$. In general terms one can proceed as follows. First, one should find an expression for $q_0(\mu, \nu)$, that is, the first intersection point of $W^{\mathrm{u}}(\bar{S}_+(\mu, \nu))$ with $\bar{z} = 0$. This expression can be obtained by regular perturbation methods and, in fact, can be deduced from our results, see Chapter 2. Next step is to study the passage near a saddle-focus point, which can be done using the so-called Shilnikov coordinates, see [Š65]. This yields to an expression of $\bar{q}_0(\mu, \nu)$ (the first intersection point of the forward orbit of $q_0(\mu, \nu)$ with $\bar{z} = 0$), and then we can easily obtain an expression of its angular variable $\theta_{q_0}(\mu, \nu)$.

Chapter 2

Breakdown of the 1D heteroclinic connection

In this chapter we prove Theorem 1.1. For completeness, we state the result again.

Theorem 2.1. *Consider system* (1.6)*, with* μ, $\beta_1 > 0$ *and* $|\nu| < \beta_1\sqrt{\mu}$*, which has two critical points* $\bar{S}_\pm(\mu,\nu)$ *of saddle-focus type. Let* $\bar{d}^{\mathrm{u,s}}(\mu,\nu)$ *(*$d^{\mathrm{u,s}}(\mu)$ *in the conservative case) be the distance between the one-dimensional stable manifold of* $\bar{S}_-(\mu,\nu)$ *and the one-dimensional unstable manifold of* $\bar{S}_+(\mu,\nu)$ *when they meet the plane* $\bar{z}=0$*. Let* $-h_0$ *be the coefficient of* $\bar{z}^3$ *in the third equation of system* (1.6)*. There exists a constant* $\mathcal{C}_0^*$*, depending on the full jet of* X^**, such that:*

1. *In the conservative case,* $d^{\mathrm{u,s}}(\mu)$ *is given asymptotically as* $\mu \to 0$ *by:*

$$\bar{d}^{\mathrm{u,s}}(\mu) = \mu^{-1/2} e^{-\frac{\alpha_0 \pi}{2\sqrt{\mu}}} e^{\frac{\pi}{2}(\alpha_0 h_0 + \alpha_3)} \left(\mathcal{C}_0^* + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right).$$

2. *In the dissipative case,* $d^{\mathrm{u,s}}(\mu,\nu)$ *is given asymptotically as* $\mu \to 0$ *by:*

$$\bar{d}^{\mathrm{u,s}}(\mu,\nu) = \mu^{-\frac{\beta_1}{2}} e^{-\frac{\alpha_0 \pi}{2\sqrt{\mu}}} e^{\frac{\pi}{2}\left(\alpha_0 h_0 - \frac{\alpha_1 \nu}{\sqrt{\mu}} + \alpha_3\right)} \left(\mathcal{C}_0^* + \mathcal{O}\left(\frac{1}{|\log \mu|} \right) \right).$$

In Section 2.1 we give the main ideas of the proof of Theorem 2.1. This section is organized as follows. First, in Subsection 2.1.1 we rescale variables and define new parameters which are more suitable for our purposes. We then state Theorem 2.1 with the new notation. After that we begin properly with the sketch of the proof. The first step consists in finding good parameterizations of the invariant manifolds defined in some complex domains, see Subsection 2.1.2. After that, in Subsection 2.1.3, we introduce and study the *inner equation*. Next, in Subsection 2.1.4, we study how well the solutions of the inner equation approximate the solutions of the original system. Finally, in Subsection

2.1.5, we sketch how the asymptotic formula of the splitting distance is found. After that, in the following sections of this chapter, we provide the proofs of the results previously stated.

2.1 Sketch of the proof

The aim of this section is to give the main ideas of how Theorem 2.1 is proved.

2.1.1 Set-up and notation

First of all we will rescale the variables and parameters so that the critical points are $\mathcal{O}(1)$, and not $\mathcal{O}(\sqrt{\mu})$ as we had in system (1.6). We define the new parameters $\delta = \sqrt{\mu}$, $\sigma = \delta^{-1}\nu$, and the new variables $x = \delta^{-1}\bar{x}$, $y = \delta^{-1}\bar{y}$, $z = \delta^{-1}\bar{z}$ and $t = \delta\bar{t}$. Then, renaming the coefficients $b = \gamma_2$, $c = \alpha_3$ and $\mathrm{d} = \beta_1$, system (1.6) becomes:

$$\begin{aligned}
\frac{dx}{dt} &= x\,(\sigma - dz) + \left(\frac{\alpha(\delta\sigma)}{\delta} + cz\right) y + \delta^{-2} f(\delta x, \delta y, \delta z, \delta, \delta\sigma),\\
\frac{dy}{dt} &= -\left(\frac{\alpha(\delta\sigma)}{\delta} + cz\right) x + y\,(\sigma - dz) + \delta^{-2} g(\delta x, \delta y, \delta z, \delta, \delta\sigma),\\
\frac{dz}{dt} &= -1 + b(x^2 + y^2) + z^2 + \delta^{-2} h(\delta x, \delta y, \delta z, \delta, \delta\sigma),
\end{aligned} \tag{2.1}$$

where $\mathrm{d} > 0$, f, g and h are real analytic functions of order three in all their variables, $\delta > 0$ is a small parameter and $|\sigma| < \mathrm{d}$. Moreover, $\alpha(\delta\sigma)$ is an analytic function such that $\alpha(0) = \alpha_0 \neq 0$ and $\alpha'(0) = \alpha_1$.

Remark 2.1.1. Without loss of generality, we can assume that α_0 and c are both positive constants. In particular, for δ small enough, $\alpha(\delta\sigma)$ will be also positive.

Remark 2.1.2. From now on, in order to shorten the notation, we will not write explicitly the dependence of α with respect to $\delta\sigma$. That is, we will write α instead of $\alpha(\delta\sigma)$. In fact, α will be treated as a parameter independent of δ and σ, since there exist two constants K_1 and K_2 such that for δ small enough:

$$0 < K_1 \leq \alpha(\delta\sigma) \leq K_2,$$

and both constants are independent of these two parameters.

Below we summarize some properties of the rescaled system (2.1), which can be deduced similarly as in [BS08].

Lemma 2.1.3. *For any value of $\delta > 0$, the unperturbed system (system (2.1) with $f = g = h = 0$) verifies:*

1. *It possesses two hyperbolic fixed points* $S^0_\pm = (0, 0, \pm 1)$ *which are of saddle-focus type with eigenvalues* $\sigma \mp \mathrm{d} + |\frac{\alpha}{\delta} \pm c| i$, $\sigma \mp \mathrm{d} - |\frac{\alpha}{\delta} \pm c| i$, *and* ± 2.
2. *The one-dimensional unstable manifold of* S^0_+ *and the one-dimensional stable manifold of* S^0_- *coincide along the heteroclinic connection* $\{(0, 0, z) : -1 < z < 1\}$. *This heteroclinic orbit can be parameterized by*

$$\Upsilon_0(t) = (0, 0, z_0(t)) = (0, 0, -\tanh t),$$

if we require $\Upsilon_0(0) = (0, 0, 0)$.

Lemma 2.1.4. *If* $\delta > 0$ *is small enough, system* (2.1) *has two fixed points* $S_\pm(\delta, \sigma)$ *of saddle-focus type:*

$$S_\pm(\delta, \sigma) = (x_\pm(\delta, \sigma), y_\pm(\delta, \sigma), z_\pm(\delta, \sigma)),$$

with:

$$x_\pm(\delta, \sigma) = \mathcal{O}(\delta^2, \delta^2\sigma^3) = \mathcal{O}(\delta^2), \quad y_\pm(\delta, \sigma) = \mathcal{O}(\delta^2, \delta^2\sigma^3) = \mathcal{O}(\delta^2),$$
$$z_\pm(\delta, \sigma) = \pm 1 + \mathcal{O}(\delta, \delta\sigma^3) = \pm 1 + \mathcal{O}(\delta).$$

$S_+(\delta, \sigma)$ *has a one-dimensional unstable manifold and a two-dimensional stable one. Conversely,* $S_-(\delta, \sigma)$ *has a one-dimensional stable manifold and a two-dimensional unstable one.*

Moreover, there are no other fixed points of (2.1) *in the closed ball* $B(\delta^{-1/3})$.

Remark 2.1.5. The fact that there are no other fixed points of the system inside the ball $B(\delta^{-1/3})$ allows us to look for the one-dimensional invariant manifolds of the critical points as bounded solutions (more precisely, solutions that stay in a ball centered at zero of radius independent of δ) for positive and negative time respectively.

Next theorem, which we will prove in the following, is the version of Theorem 2.1 in the new variables.

Theorem 2.1.6. *Consider system* (2.1), *with* $\delta, \mathrm{d} > 0$ *and* $|\sigma| < \mathrm{d}$. *Then there exists a constant* C^*, *such that the distance* $d^{\mathrm{u,s}}$ *between the one-dimensional stable manifold of* $S_-(\delta, \sigma)$ *and the one-dimensional unstable manifold of* $S_+(\delta, \sigma)$, *when they meet the plane* $z = 0$, *is given asymptotically by:*

$$d^{\mathrm{u,s}} = \delta^{-(1+\mathrm{d})} e^{-\frac{\alpha_0 \pi}{2\delta}} e^{\frac{\pi}{2}(\alpha_0 h_0 - \alpha_1 \sigma + c)} \left(C^* + \mathcal{O}\left(\frac{1}{\log(1/\delta)} \right) \right),$$

being $\alpha_0 = \alpha(0)$, $\alpha_1 = \alpha'(0)$ *and* $h_0 = -\lim_{z \to 0} z^{-3} h(0, 0, z, 0, 0)$.

Remark 2.1.7. The formula provided by Theorem 2.1.6 yields straightforwardly Theorem 2.1. Indeed, first one just needs to recall that $\delta = \sqrt{\mu}$, $\sigma = \nu/\sqrt{\mu}$ and that $\bar{d}^{\mathrm{u,s}}(\mu, \nu) = \sqrt{\mu} d^{\mathrm{u,s}}$ because $\bar{x} = \sqrt{\mu} x$ and $\bar{y} = \sqrt{\mu} y$. Then, recalling the change of notation $\mathrm{d} = \beta_1$, $c = \alpha_3$ and that in the conservative case $\nu = 0$ and $\beta_1 = 1$, one obtains the claim of Theorem 2.1.

Remark 2.1.8. The asymptotic formula provided in Theorem 2.1.6 for the distance $d^{\mathrm{u,s}}$ has the same qualitative behavior as the one proved in [BS06] in the conservative setting for the regular case. The main difference between both formulas is the constant C^*. While in Theorem 2.1.6 this constant depends on the full jet of f, g, h and (at the moment) can only be computed numerically, in the regular case C^* is completely determined by means of the Borel transform of some adequate analytic functions depending on $f(0,0,u,0)$ and $g(0,0,u,0)$.

Before we proceed, we introduce some notation that we will use for the rest of the chapter. On one hand, in $\mathbb{C}^n$ we will consider the norm $|.|$ as:

$$|(z_1,\dots,z_n)| = |z_1| + \cdots + |z_n|,$$

where $|z|$ stands for the ordinary modulus of a complex number. On the other hand, $B(r_0)$ will stand for the open ball of any vector space centered at zero and of radius r_0. Moreover, we will write $B^n(r_0)$ to denote $B(r_0)\times \overset{n)}{\cdots} \times B(r_0)$.

2.1.2 Existence of complex parameterizations in the outer domains

As it is usual in works where exponentially small phenomena must be detected, the first thing we have to do in order to prove Theorem 2.1.6 is to provide parameterizations of the one-dimensional invariant manifolds of the critical points $S_\pm(\delta,\sigma)$. Moreover, these have to be defined in some complex domains that are close to the singularities of the heteroclinic connection of the unperturbed system.

However, first we will introduce some changes of variables that will simplify the proof. The first one consists in performing a change that keeps the corresponding critical point constant with respect to the parameters. For instance, to prove the existence of a complex parameterization of the unstable manifold of $S_+(\delta,\sigma)$ we perform the $\mathcal{O}(\delta)$-close to the identity change C_1^{u} defined by:

$$(\tilde x,\tilde y,\tilde z) = C_1^{\mathrm{u}}(x,y,z,\delta,\delta\sigma) = (x - x_+(\delta,\sigma), y - y_+(\delta,\sigma), z - z_+(\delta,\sigma)+1), \tag{2.2}$$

obtaining a system of the form:

$$\begin{aligned}
\frac{d\tilde x}{dt} &= \tilde x\,(\sigma - d\tilde z) + \left(\frac{\alpha}{\delta} + c\tilde z\right)\tilde y + \delta^{-2} f^{\mathrm{u}}(\delta\tilde x,\delta\tilde y,\delta\tilde z,\delta,\delta\sigma),\\
\frac{d\tilde y}{dt} &= -\left(\frac{\alpha}{\delta} + c\tilde z\right)\tilde x + \tilde y\,(\sigma - d\tilde z) + \delta^{-2} g^{\mathrm{u}}(\delta\tilde x,\delta\tilde y,\delta\tilde z,\delta,\delta\sigma),\\
\frac{d\tilde z}{dt} &= -1 + b(\tilde x^2 + \tilde y^2) + \tilde z^2 + \delta^{-2} h^{\mathrm{u}}(\delta\tilde x,\delta\tilde y,\delta\tilde z,\delta,\delta\sigma),
\end{aligned} \tag{2.3}$$

where $f^{\mathrm{u}}(0,0,\delta,\delta,\delta\sigma) = g^{\mathrm{u}}(0,0,\delta,\delta,\delta\sigma) = h^{\mathrm{u}}(0,0,\delta,\delta,\delta\sigma) = 0$ for all δ, and hence has the critical point $S_+(\delta,\sigma)$ fixed at $(0,0,1)$. Moreover f^{u}, g^{u} and h^{u} are analytic and of order three in all their variables.

After that we do the change:

$$(\eta,\bar\eta,v) = C_2(\tilde x,\tilde y,\tilde z) = (\tilde x + i\tilde y, \tilde x - i\tilde y, z_0^{-1}(\tilde z)), \tag{2.4}$$

where $z_0(t) = -\tanh t$ is the third component of the heteroclinic connection $\Upsilon_0(t)$ of the unperturbed system. Then we obtain a system of the form:

$$\begin{aligned}
\frac{d\eta}{dt} &= -\left(\frac{\alpha}{\delta} + cz_0(v)\right) i\eta + \eta\,(\sigma - \mathrm{d}z_0(v)) + \delta^{-2}F_1^{\mathrm{u}}(\delta\eta,\delta\bar\eta,\delta z_0(v),\delta,\delta\sigma),\\
\frac{d\bar\eta}{dt} &= \left(\frac{\alpha}{\delta} + cz_0(v)\right) i\bar\eta + \bar\eta\,(\sigma - \mathrm{d}z_0(v)) + \delta^{-2}F_2^{\mathrm{u}}(\delta\eta,\delta\bar\eta,\delta z_0(v),\delta,\delta\sigma),\\
\frac{dv}{dt} &= 1 + \frac{b\eta\bar\eta + \delta^{-2}H^{\mathrm{u}}(\delta\eta,\delta\bar\eta,\delta z_0(v),\delta,\delta\sigma)}{-1 + z_0^2(v)},
\end{aligned} \tag{2.5}$$

where, again, $F_1^{\mathrm{u}}(0,0,\delta,\delta,\delta\sigma) = F_2^{\mathrm{u}}(0,0,\delta,\delta,\delta\sigma) = H^{\mathrm{u}}(0,0,\delta,\delta,\delta\sigma) = 0$ for all δ and are of order three, since:

$$\begin{aligned}
F_1^{\mathrm{u}}(\delta\eta,\delta\bar\eta,\delta z_0(v),\delta,\delta\sigma) &= f^{\mathrm{u}}\left(\frac{\delta(\eta+\bar\eta)}{2}, \frac{\delta(\eta-\bar\eta)}{2}, \delta z_0(v),\delta,\delta\sigma\right)\\
&\quad + ig^{\mathrm{u}}\left(\frac{\delta(\eta+\bar\eta)}{2}, \frac{\delta(\eta-\bar\eta)}{2}, \delta z_0(v),\delta,\delta\sigma\right),\\
F_2^{\mathrm{u}}(\delta\eta,\delta\bar\eta,\delta z_0(v),\delta,\delta\sigma) &= f^{\mathrm{u}}\left(\frac{\delta(\eta+\bar\eta)}{2}, \frac{\delta(\eta-\bar\eta)}{2}, \delta z_0(v),\delta,\delta\sigma\right)\\
&\quad - ig^{\mathrm{u}}\left(\frac{\delta(\eta+\bar\eta)}{2}, \frac{\delta(\eta-\bar\eta)}{2}, \delta z_0(v),\delta,\delta\sigma\right),\\
H^{\mathrm{u}}(\delta\eta,\delta\bar\eta,\delta z_0(v),\delta,\delta\sigma) &= h^{\mathrm{u}}\left(\frac{\delta(\eta+\bar\eta)}{2}, \frac{\delta(\eta-\bar\eta)}{2}, \delta z_0(v),\delta,\delta\sigma\right).
\end{aligned} \tag{2.6}$$

To prove the existence of the stable manifold of $S_-(\delta,\sigma)$, instead of the change C_1^{u} defined in (2.2), we do the change:

$$(\tilde x,\tilde y,\tilde z) = C_1^{\mathrm{s}}(x,y,z,\delta,\delta\sigma) = (x - x_-(\delta,\sigma), y - y_-(\delta,\sigma), z - z_-(\delta,\sigma) - 1),$$

and after that we do the change C_2. Then we obtain a system analogous to (2.5), where instead of F_i^{u} and H^{u} we have functions F_i^{s}, H^{s} such that $F_1^{\mathrm{s}}(0,0,-\delta,\delta,\delta\sigma) = F_2^{\mathrm{s}}(0,0,-\delta,\delta,\delta\sigma) = H^{\mathrm{s}}(0,0,-\delta,\delta,\delta\sigma) = 0$ for all δ.

We will denote:

$$\eta_\pm = \eta_\pm(\delta,\sigma) = x_\pm(\delta,\sigma) + iy_\pm(\delta,\sigma), \qquad \bar\eta_\pm = \bar\eta_\pm(\delta,\sigma) = \overline{\eta_\pm(\delta,\sigma)}, \qquad z_\pm = z_\pm(\delta,\sigma). \tag{2.7}$$

Remark 2.1.9. Note that as f, g and h are analytic functions, and since:

$$\delta\eta_\pm, \delta\bar{\eta}_\pm = \mathcal{O}(\delta^3), \quad \delta(z_\pm \mp 1) = \mathcal{O}(\delta^2),$$

there exist some $r_0^{\mathrm{u,s}}$, independent of δ and σ, such that for δ small enough $F_1^{\mathrm{u,s}}$, $F_2^{\mathrm{u,s}}$ and $H^{\mathrm{u,s}}$ are analytic whenever $(\delta\eta, \delta\bar{\eta}, \delta z, \delta, \delta\sigma) \in B^3(r_0^{\mathrm{u,s}}) \times B(\delta_0) \times B(\sigma_0)$ respectively. Moreover, using that they are of order three, it is easy to see that if $\phi = (\phi_1, \phi_2, \phi_3, \delta, \delta\sigma) \in B^3(r_0^{\mathrm{u,s}}) \times B(\delta_0) \times B(\sigma_0)$, then:

$$|F_1^{\mathrm{u,s}}(\phi)|, |F_2^{\mathrm{u,s}}(\phi)|, |H^{\mathrm{u,s}}(\phi)| \leq K|(\phi_1, \phi_2, \phi_3 \mp \delta, \delta, \delta\sigma)|^3,$$

respectively.

Finally, thinking of η and $\bar{\eta}$ as functions of v we get the following systems, respectively in the unstable and stable case:

$$\begin{aligned}
\frac{d\eta}{dv} &= \frac{-\left(\dfrac{\alpha}{\delta} + cz_0(v)\right) i\eta + \eta\,(\sigma - \mathrm{d}z_0(v)) + \delta^{-2} F_1^{\mathrm{u,s}}(\delta\eta, \delta\bar{\eta}, \delta z_0(v), \delta, \delta\sigma)}{1 + \dfrac{b\eta\bar{\eta} + \delta^{-2} H^{\mathrm{u,s}}(\delta\eta, \delta\bar{\eta}, \delta z_0(v), \delta, \delta\sigma)}{-1 + z_0^2(v)}}, \\
\frac{d\bar{\eta}}{dv} &= \frac{\left(\dfrac{\alpha}{\delta} + cz_0(v)\right) i\bar{\eta} + \bar{\eta}\,(\sigma - \mathrm{d}z_0(v)) + \delta^{-2} F_2^{\mathrm{u,s}}(\delta\eta, \delta\bar{\eta}, \delta z_0(v), \delta, \delta\sigma)}{1 + \dfrac{b\eta\bar{\eta} + \delta^{-2} H^{\mathrm{u,s}}(\delta\eta, \delta\bar{\eta}, \delta z_0(v), \delta, \delta\sigma)}{-1 + z_0^2(v)}}.
\end{aligned} \tag{2.8}$$

We will look for solutions $\zeta^{\mathrm{u,s}}(v) = (\eta^{\mathrm{u,s}}(v), \bar{\eta}^{\mathrm{u,s}}(v))$ of system (2.8) such that:

$$\lim_{v\to-\infty} \zeta^{\mathrm{u}}(v) = (0,0), \qquad \lim_{v\to+\infty} \zeta^{\mathrm{s}}(v) = (0,0). \tag{2.9}$$

After stating Theorem 2.1.10 we will justify that, indeed, $(\eta^{\mathrm{u,s}}(v), \bar{\eta}^{\mathrm{u,s}}(v), z_0(v))$ lead to parameterizations of the unstable and stable manifolds of the critical points $(0, 0, \pm 1)$ of system (2.3), respectively.

Once we have obtained a suitable system (2.8), the next step is to prove the existence of solutions verifying (2.9). The main idea is that system (2.8) has a linear part which is dominant. More precisely, we denote $\zeta = (\eta, \bar{\eta})^T$, $F^{\mathrm{u,s}} = (F_1^{\mathrm{u,s}}, F_2^{\mathrm{u,s}})^T$, and we define:

$$A(v) = \begin{pmatrix} -\left(\dfrac{\alpha}{\delta} + cz_0(v)\right) i + \sigma - \mathrm{d}z_0(v) & 0 \\ 0 & \left(\dfrac{\alpha}{\delta} + cz_0(v)\right) i + \sigma - \mathrm{d}z_0(v) \end{pmatrix}, \tag{2.10}$$

and:

$$R^{\mathrm{u,s}}(\zeta)(v) = \left(\frac{1}{1+\dfrac{b\eta\bar{\eta}+\delta^{-2}H^{\mathrm{u,s}}(\delta\zeta,\delta z_0(u),\delta,\delta\sigma)}{-1+z_0^2(u)}} - 1 \right) A(v)\zeta + \frac{\delta^{-2}F^{\mathrm{u,s}}(\delta\zeta,\delta z_0(v),\delta,\delta\sigma)}{1+\dfrac{b\eta\bar{\eta}+\delta^{-2}H^{\mathrm{u,s}}(\delta\zeta,\delta z_0(u),\delta,\delta\sigma)}{-1+z_0^2(u)}}. \tag{2.11}$$

Then, in the unstable case, system (2.8) joint with (2.9) can be rewritten as:

$$\frac{d\zeta}{dv} = A(v)\zeta + R^{\mathrm{u}}(\zeta)(v), \qquad \lim_{v\to-\infty}\zeta^{\mathrm{u}}(v) = (0,0) \tag{2.12}$$

and the corresponding for the stable one as:

$$\frac{d\zeta}{dv} = A(v)\zeta + R^{\mathrm{s}}(\zeta)(v), \qquad \lim_{v\to+\infty}\zeta^{\mathrm{s}}(v) = (0,0). \tag{2.13}$$

As we mentioned above, we will need to find parameterizations of the invariant manifolds defined not just for $v\in\mathbb{R}$, but in some complex domains that are close to the first singularities of the heteroclinic connection Υ_0 of the unperturbed system, which in this case are $v=\pm i\pi/2$. We will now proceed to introduce these complex domains. We define (see Figure 2.1):

$$D^{\mathrm{out,u}}_{\overline{\kappa},\beta} = \{v\in\mathbb{C} \,:\, |\mathrm{Im}\, v| \le \pi/2 - \overline{\kappa}\delta\log(1/\delta) - \tan\beta \mathrm{Re}\, v\}, \tag{2.14}$$

where $0<\beta<\pi/2$, $T>0$ and $\overline{\kappa}>0$ are constants independent of δ and σ. For technical reasons we will split the domain $D^{\mathrm{out,u}}_{\overline{\kappa},\beta}$ in two subsets (see Figure 2.1), namely:

$$D^{\mathrm{out,u}}_{\overline{\kappa},\beta,\infty} = \left\{v\in D^{\mathrm{out,u}}_{\overline{\kappa},\beta} \,:\, \mathrm{Re}\, v\le -T\right\}, \qquad D^{\mathrm{out,u}}_{\overline{\kappa},\beta,T} = \left\{v\in D^{\mathrm{out,u}}_{\overline{\kappa},\beta} \,:\, \mathrm{Re}\, v\ge -T\right\}. \tag{2.15}$$

Analogously, we define:

$$D^{\mathrm{out,s}}_{\overline{\kappa},\beta} = -D^{\mathrm{out,u}}_{\overline{\kappa},\beta}, \qquad D^{\mathrm{out,s}}_{\overline{\kappa},\beta,\infty} = -D^{\mathrm{out,u}}_{\overline{\kappa},\beta,\infty}, \qquad D^{\mathrm{out,s}}_{\overline{\kappa},\beta,T} = -D^{\mathrm{out,u}}_{\overline{\kappa},\beta,T}.$$

Theorem 2.1.10. *Let $\overline{\kappa}>0$ and $0<\beta<\pi/2$ be any fixed constants independent of δ and σ. Then, if $\delta>0$ is small enough, problem (2.12) has a solution $\zeta^{\mathrm{u}}(v)=(\eta^{\mathrm{u}}(v),\bar{\eta}^{\mathrm{u}}(v))$ defined for $v\in D^{\mathrm{out,u}}_{\overline{\kappa},\beta}$, and (2.13) has a solution $\zeta^{\mathrm{s}}(v)=(\eta^{\mathrm{s}}(v),\bar{\eta}^{\mathrm{s}}(v))$ defined for $v\in D^{\mathrm{out,s}}_{\overline{\kappa},\beta}$. Moreover there exists a constant K independent of δ and σ such that:*

$$|\zeta^{\mathrm{u}}(v)| \le \begin{cases} K\delta^2|z_0(v)-1| & \text{if } v\in D^{\mathrm{out,u}}_{\overline{\kappa},\beta,\infty}, \\ K\delta^2|z_0(v)-1|^3 & \text{if } v\in D^{\mathrm{out,u}}_{\overline{\kappa},\beta,T}, \end{cases}$$

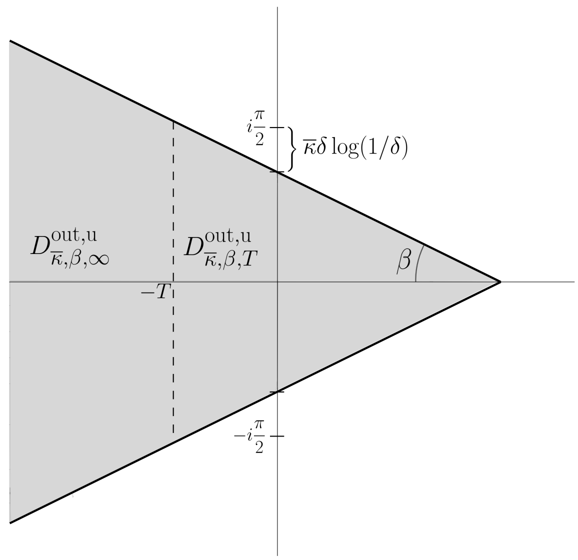


Figure 2.1: The outer domain $D^{\mathrm{out,u}}_{\overline{\kappa},\beta}$ for the unstable case with its subdomains $D^{\mathrm{out,u}}_{\overline{\kappa},\beta,T}$ and $D^{\mathrm{out,u}}_{\overline{\kappa},\beta,\infty}$.

$$|\zeta^{\mathrm{s}}(v)| \leq \begin{cases} K\delta^2|z_0(v)+1| & \textit{if } v \in D^{\mathrm{out,s}}_{\overline{\kappa},\beta,\infty}, \\ K\delta^2|z_0(v)+1|^3 & \textit{if } v \in D^{\mathrm{out,s}}_{\overline{\kappa},\beta,T}. \end{cases}$$

The proof of this result is postponed to Section 2.2. Now we enunciate the following corollary:

Corollary 2.1.11. Let $\overline{\kappa}$ and $0 < \beta < \pi/2$ be two fixed constants independent of δ and σ. Consider η^{u} and $\bar{\eta}^{\mathrm{u}}$ the functions given by Theorem 2.1.10, and let $v(t)$ be the solution of:

$$\frac{dv}{dt} = 1 + \frac{b\eta^{\mathrm{u}}(v)\bar{\eta}^{\mathrm{u}}(v) + \delta^{-2}H^{\mathrm{u}}(\delta\eta^{\mathrm{u}}(v), \delta\bar{\eta}^{\mathrm{u}}(v), \delta z_0(v), \delta, \delta\sigma)}{-1+z_0^2(v)} =: 1 + \mathcal{F}(v), \tag{2.16}$$

such that $v(0) = 0$. Then, $(\tilde{x}^{\mathrm{u}}(t), \tilde{y}^{\mathrm{u}}(t), \tilde{z}^{\mathrm{u}}(t))$ defined by:

$$\tilde{x}^{\mathrm{u}}(t) = \frac{\eta^{\mathrm{u}}(v(t)) + \bar{\eta}^{\mathrm{u}}(v(t))}{2}, \qquad \tilde{y}^{\mathrm{u}}(t) = \frac{\eta^{\mathrm{u}}(v(t)) - \bar{\eta}^{\mathrm{u}}(v(t))}{2}, \qquad \tilde{z}^{\mathrm{u}} = z_0(v(t)),$$

is a parameterization of the unstable manifold of the critical point $(0,0,1)$ of system (2.3). For the stable manifold of $(0,0,-1)$, one has an analogous result.

Proof. Indeed, it is clear that $(\tilde{x}^{\mathrm{u}}(t), \tilde{y}^{\mathrm{u}}(t), \tilde{z}^{\mathrm{u}}(t))$ is a solution of system (2.3), since it consists in performing the inverse change of C_2, defined in (2.4), for a particular solution of system (2.5). Hence, we just have to check that:

$$\lim_{t\to-\infty}(\tilde{x}^{\mathrm{u}}(t), \tilde{y}^{\mathrm{u}}(t), \tilde{z}^{\mathrm{u}}(t)) = (0,0,1).$$

Note that it is sufficient to prove that:

$$\lim_{t\to-\infty} v(t) = -\infty, \tag{2.17}$$

since, on the one hand $z_0(v) = -\tanh(v)$ goes to 1 as v goes to $-\infty$ and, on the other hand, from Theorem 2.1.10 we know that:

$$\lim_{v\to-\infty} (\eta^{\mathrm{u}}(v), \bar{\eta}^{\mathrm{u}}(v)) = (0, 0).$$

We will prove that (2.17) holds if $v(0) = 0$ as follows. Indeed, from (2.16) it is clear that:

$$t = \int_0^v \frac{1}{1+\mathcal{F}(w)} dw := \mathcal{G}(v).$$

Now, from Theorem 2.1.10 and the fact that $|z_0(v) - 1|$ is bounded for $v \in D^{\mathrm{out,u}}_{\overline{\kappa},\beta,T} \cap \mathbb{R}$, it is clear that for $v \in D^{\mathrm{out,u}}_{\overline{\kappa},\beta} \cap \mathbb{R}$:

$$|\eta^{\mathrm{u}}(v)|, |\bar{\eta}^{\mathrm{u}}(v)| \leq K\delta^2 |z_0(v) - 1|,$$

for some constant K. Using these bounds, the fact that e^v is bounded for $v \in D^{\mathrm{out,u}}_{\overline{\kappa},\beta} \cap \mathbb{R}$ and Remark 2.1.9, it can be easily seen that:

$$\begin{aligned} |\mathcal{F}(v)| &= \left| \frac{b\eta^{\mathrm{u}}(v)\bar{\eta}^{\mathrm{u}}(v) + \delta^{-2} H^{\mathrm{u}}(\delta\eta^{\mathrm{u}}(v), \delta\bar{\eta}^{\mathrm{u}}(v), \delta z_0(v), \delta, \delta\sigma)}{-1 + z_0^2(v)} \right| \\ &\leq \tilde{K} \left(\frac{\delta^4 |z_0(v) - 1|^2 + \delta |z_0(v) - 1|^3}{|-1 + z_0^2(v)|} \right) \leq \tilde{K} \left(\delta^4 e^{2v} + \delta e^{4v} \right) < \frac{1}{2}, \end{aligned}$$

if δ is small enough. Then it is clear that $\mathcal{G}'(v) = (1 + \mathcal{F}(v))^{-1}$ satisfies:

$$\mathcal{G}'(v) \geq \frac{1}{1 + 1/2} = \frac{2}{3} > 0. \tag{2.18}$$

On one hand, the fact that $\mathcal{G}'(v)$ is strictly positive implies that $\mathcal{G}(v)$ is strictly increasing. Then $\mathcal{G}$ is invertible in $D^{\mathrm{out,u}}_{\overline{\kappa},\beta} \cap \mathbb{R}$, and for $v \in D^{\mathrm{out,u}}_{\overline{\kappa},\beta} \cap \mathbb{R}$ we can write:

$$v = \mathcal{G}^{-1}(t). \tag{2.19}$$

Note that, as $\mathcal{G}$ is strictly increasing, so is $\mathcal{G}^{-1}$, and then we have that $v(t) \leq v(0) = 0$ for $t \leq 0$. Hence it is clear that $v(t) \in D^{\mathrm{out,u}}_{\overline{\kappa},\beta} \cap \mathbb{R}$ for all $t \leq 0$, and hence (2.19) has sense for all $t \leq 0$. On the other hand, we also have that:

$$\left(\mathcal{G}^{-1}\right)' = \frac{1}{\mathcal{G}'} \leq \frac{3}{2},$$

which implies that:

$$v = \int_0^t \left(\mathcal{G}^{-1}(s)\right)' ds \leq \frac{3}{2} t,$$

and hence we immediately obtain (2.17). □

Local parameterizations of the invariant manifolds

Theorem 2.1.10 provides us with complex parameterizations of the invariant manifolds, $\zeta^{\mathrm{u,s}} = (\eta^{\mathrm{u,s}}, \bar\eta^{\mathrm{u,s}})$, which are solutions of problems (2.12) and (2.13) respectively. However, in order to study their difference, it is very useful that both manifolds are given by functions that satisfy the same system in a common domain. We proceed to undo the changes C_1^{u} for ζ^{u} and C_1^{s} for ζ^{s}.

Consider:

$$V_\pm(u,\delta,\sigma) = z_0^{-1}(z_0(u) - z_\pm(\delta,\sigma) \pm 1) - u.$$

Let $(\eta^{\mathrm{u,s}}(v), \bar\eta^{\mathrm{u,s}}(v))$ be solutions of system (2.8) and:

$$\xi^{\mathrm{u,s}}(u) = \eta^{\mathrm{u,s}}(u + V_\pm(u,\delta,\sigma)) + \eta_\pm(\delta,\sigma), \qquad \bar\xi^{\mathrm{u,s}}(u) = \bar\eta^{\mathrm{u,s}}(u + V_\pm(u,\delta,\sigma)) + \bar\eta_\pm(\delta,\sigma). \tag{2.20}$$

Then, wherever $V_\pm(u,\delta,\sigma)$ is defined we have that $(\xi^{\mathrm{u,s}}, \bar\xi^{\mathrm{u,s}})$ are solutions of the following system:

$$\begin{aligned}
\frac{d\xi}{du} &= \frac{-\left(\dfrac{\alpha}{\delta} + cz_0(u)\right) i\xi + \xi\left(\sigma - \mathrm{d}z_0(u)\right) + \delta^{-2}F_1(\delta\xi, \delta\bar\xi, \delta z_0(u), \delta, \delta\sigma)}{1 + \dfrac{b\xi\bar\xi + \delta^{-2}H(\delta\xi, \delta\bar\xi, \delta z_0(u), \delta, \delta\sigma)}{-1 + z_0^2(u)}}, \\
\frac{d\bar\xi}{du} &= \frac{\left(\dfrac{\alpha}{\delta} + cz_0(u)\right) i\bar\xi + \bar\xi\left(\sigma - \mathrm{d}z_0(u)\right) + \delta^{-2}F_2(\delta\xi, \delta\bar\xi, \delta z_0(u), \delta, \delta\sigma)}{1 + \dfrac{b\xi\bar\xi + \delta^{-2}H(\delta\xi, \delta\bar\xi, \delta z_0(u), \delta, \delta\sigma)}{-1 + z_0^2(u)}},
\end{aligned} \tag{2.21}$$

where:

$$\begin{aligned}
F_1(\delta\xi, \delta\bar\xi, \delta z_0(v), \delta, \delta\sigma) &= f\left(\frac{\delta(\xi+\bar\xi)}{2}, \frac{\delta(\xi-\bar\xi)}{2}, \delta z_0(v), \delta, \delta\sigma\right) \\
&\quad + ig\left(\frac{\delta(\xi+\bar\xi)}{2}, \frac{\delta(\xi-\bar\xi)}{2}, \delta z_0(v), \delta, \delta\sigma\right), \\
F_2(\delta\xi, \delta\bar\xi, \delta z_0(v), \delta, \delta\sigma) &= f\left(\frac{\delta(\xi+\bar\xi)}{2}, \frac{\delta(\xi-\bar\xi)}{2}, \delta z_0(v), \delta, \delta\sigma\right) \\
&\quad - ig\left(\frac{\delta(\xi+\bar\xi)}{2}, \frac{\delta(\xi-\bar\xi)}{2}, \delta z_0(v), \delta, \delta\sigma\right), \\
H(\delta\xi, \delta\bar\xi, \delta z_0(v), \delta, \delta\sigma) &= h\left(\frac{\delta(\xi+\bar\xi)}{2}, \frac{\delta(\xi-\bar\xi)}{2}, \delta z_0(v), \delta, \delta\sigma\right).
\end{aligned} \tag{2.22}$$

Remark 2.1.12. From (2.22), it is clear that $F_1(\phi)$, $F_2(\phi)$ and $H(\phi)$ are of order three and analytic whenever $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in B^3(r_0) \times B(\delta_0) \times B(\sigma_0)$. Then we have that there exists some constant K, independent of δ and σ, such that:

$$|F_1(\phi)|, |F_2(\phi)|, |H(\phi)| \le K|(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)|^3. \tag{2.23}$$

Theorem 2.1.13. *Let $\kappa > 0$ and $0 < \beta < \pi/2$ be any constants independent of δ and σ. Then the one-dimensional invariant manifolds of $S_\pm(\delta,\sigma)$ can be parameterized respectively by:*

$$\xi = \xi^{\mathrm{u,s}}(u), \qquad \bar\xi = \bar\xi^{\mathrm{u,s}}(u), \qquad z = z_0(u), \qquad u \in D^{\mathrm{out},*}_{\kappa,\beta,T},$$

where $* = \mathrm{u,s}$ respectively, and $\varphi^{\mathrm{u,s}}(u) = (\xi^{\mathrm{u,s}}(u), \bar\xi^{\mathrm{u,s}}(u))$ are solutions of system (2.21). Moreover, there exists a constant K, independent of δ and σ, such that:

$$|\varphi^{\mathrm{u}}(u)| \le K\delta^2 |z_0(u) - 1|^3, \quad u \in D^{\mathrm{out,u}}_{\kappa,\beta,T},$$

$$|\varphi^{\mathrm{s}}(u)| \le K\delta^2 |z_0(u) + 1|^3, \quad u \in D^{\mathrm{out,s}}_{\kappa,\beta,T}.$$

The proof of this result can be found in Section 2.3.

2.1.3 The inner system

As we mentioned before, our study requires the knowledge of the asymptotics of the parameterizations $\varphi^{\mathrm{u,s}}(u)$, given in Theorem 2.1.13, for u near the singularities $\pm i\pi/2$. However, for $u \sim i\pi/2$ one has that $\varphi^{\mathrm{u,s}}(u) \sim \delta^{-1}$, so that they are no longer perturbative (recall that, in the variables $(\xi,\bar\xi)$, the heteroclinic orbit of the unperturbed system is $(\xi,\bar\xi) = (0,0)$). Hence, it is natural to look for good approximations of system (2.21) near $\pm i\pi/2$ in a different way. Here we will focus on the singularity $i\pi/2$, but similar results (which we will also state explicitly) can be proved near the singularity $-i\pi/2$.

To study the solutions of system (2.21) near $i\pi/2$, we define the new variables:

$$(\psi, \overline{\psi}, s) = C_3(\xi, \bar\xi, u, \delta)$$

by:

$$\psi = \delta\xi, \quad \overline{\psi} = \delta\bar\xi, \quad s = \frac{u - i\pi/2}{\delta}. \tag{2.24}$$

Recalling that $z_0(u) = -\tanh u$, we can write:

$$\begin{aligned} z_0(i\pi/2 + \delta s) &= \frac{-1}{\delta s} + l(\delta s), && \text{with } l(0) = 0, \\ (-1 + z_0^2(i\pi/2 + \delta s))^{-1} &= \delta^2 s^2 + \delta^3 s^3 m(\delta s), && \text{with } m(0) = 0. \end{aligned} \tag{2.25}$$

We note that both l and m are analytic if $|\delta s| < 1$. Then system (2.21) after performing the change C_3 becomes:

$$\begin{aligned} \frac{d\psi}{ds} &= \frac{-\left[\alpha + c(-s^{-1} + \delta l(\delta s))\right] i\psi - \psi(\delta\sigma - \mathrm{d}s^{-1} + \delta \mathrm{d} l(\delta s)) + F_1(\psi, \overline{\psi}, -s^{-1} + \delta l(\delta s), \delta, \delta\sigma)}{1 + \left[b\psi\overline{\psi} + H(\psi, \overline{\psi}, -s^{-1} + \delta l(\delta s), \delta, \delta\sigma)\right](s^2 + \delta s^3 m(\delta s))}, \\ \frac{d\overline{\psi}}{ds} &= \frac{\left[\alpha + c(-s^{-1} + \delta l(\delta s))\right] i\overline{\psi} - \overline{\psi}(\delta\sigma - \mathrm{d}s^{-1} + \delta \mathrm{d} l(\delta s)) + F_2(\psi, \overline{\psi}, -s^{-1} + \delta l(\delta s), \delta, \delta\sigma)}{1 + \left[b\psi\overline{\psi} + H(\psi, \overline{\psi}, -s^{-1} + \delta l(\delta s), \delta, \delta\sigma)\right](s^2 + \delta s^3 m(\delta s))}. \end{aligned} \tag{2.26}$$

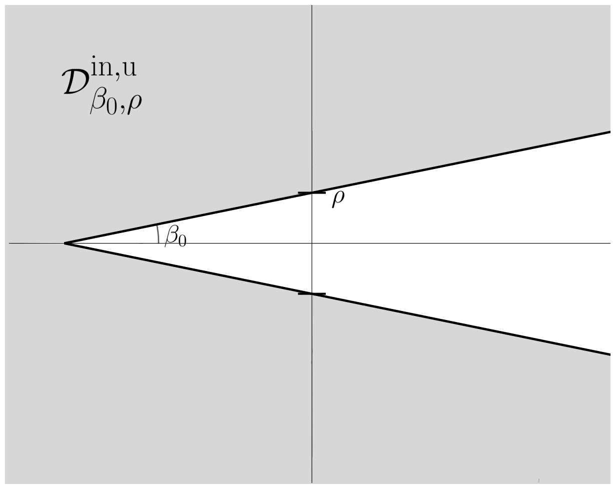


Figure 2.2: The inner domain, $\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\rho}$.

If we set $\delta = 0$ in this system, we obtain the *inner system*:

$$\begin{aligned} \frac{d\psi}{ds} &= \frac{-\left(\alpha - cs^{-1}\right) i\psi + \mathrm{d}\psi s^{-1} + F_1(\psi,\overline{\psi},-s^{-1},0,0)}{1+s^2\left[b\psi\overline{\psi} + H(\psi,\overline{\psi},-s^{-1},0,0)\right]}, \\ \frac{d\overline{\psi}}{ds} &= \frac{\left(\alpha - cs^{-1}\right) i\overline{\psi} + \mathrm{d}\overline{\psi} s^{-1} + F_2(\psi,\overline{\psi},-s^{-1},0,0)}{1+s^2\left[b\psi\overline{\psi} + H(\psi,\overline{\psi},-s^{-1},0,0)\right]}. \end{aligned} \tag{2.27}$$

Below, we will expose the results concerning the existence of two solutions $\Psi_0^{\mathrm{u,s}} = (\psi_0^{\mathrm{u,s}}, \overline{\psi}_0^{\mathrm{u,s}})$ of system (2.27) which, as we will see in Theorem 2.1.17, will give good approximations for the invariant manifolds for u near the singularity $i\pi/2$. Moreover, we will provide an asymptotic expression for the difference $\Psi_0^{\mathrm{u}} - \Psi_0^{\mathrm{s}}$, which will turn out to be very useful in Section 2.6.

Given $\beta_0, \rho > 0$, we define the following inner domains (see Figure 2.2):

$$\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\rho} = \{s \in \mathbb{C} \,:\, |\mathrm{Im}\, s| \geq \tan\beta_0 \mathrm{Re}\, s + \rho\}, \qquad \mathcal{D}^{\mathrm{in,s}}_{\beta_0,\rho} = -\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\rho}. \tag{2.28}$$

and:

$$E_{\beta_0,\rho} = \mathcal{D}^{\mathrm{in,u}}_{\beta_0,\rho} \cap \mathcal{D}^{\mathrm{in,s}}_{\beta_0,\rho} \cap \{s \in \mathbb{C} \,:\, \mathrm{Im}\, s < 0\}. \tag{2.29}$$

Remark 2.1.14. The inner domain $\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\rho}$ expressed in the outer variables is:

$$D^{\mathrm{in,u}}_{\beta_0,\rho} = \{u \in \mathbb{C} \,:\, |\mathrm{Im}\,(u - i\pi/2)| \geq \tan\beta_0 \mathrm{Re}\, u + \rho\delta\}.$$

It is easy to check that for all $0 < \beta_0, \beta < \pi/2$, if δ is small enough one has that $D^{\mathrm{out,u}}_{\kappa,\beta} \subset D^{\mathrm{in,u}}_{\beta_0,\rho}$. Analogously, we also have that $D^{\mathrm{out,s}}_{\kappa,\beta} \subset D^{\mathrm{in,s}}_{\beta_0,\rho}$, where $D^{\mathrm{in,s}}_{\beta_0,\rho} = -D^{\mathrm{in,u}}_{\beta_0,\rho}$.

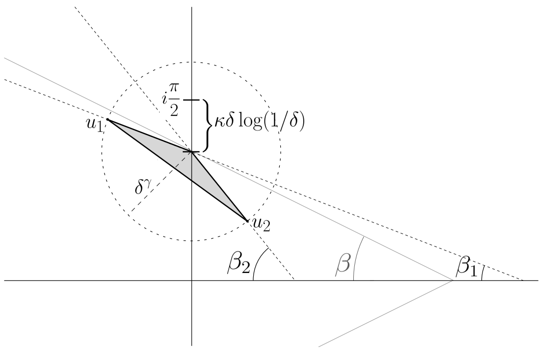
Theorem 2.1.15. *Let $\beta_0 > 0$ and ρ big enough. Then:*

1. *System (2.27) has two solutions $\Psi_0^{\mathrm{u,s}}(s) = (\psi_0^{\mathrm{u,s}}(s), \overline{\psi}_0^{\mathrm{u,s}}(s))$ defined for $s \in \mathcal{D}^{\mathrm{in},*}_{\beta_0,\rho}$, with $* = \mathrm{u}, \mathrm{s}$ respectively. Moreover there exists a constant K, such that:*
$$|\Psi_0^{\mathrm{u,s}}(s)| \leq K|s|^{-3}.$$

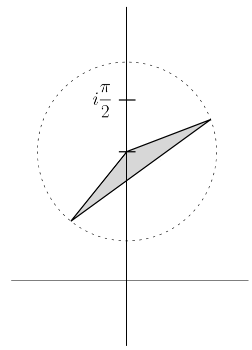
2. *Consider the difference:*
$$\Delta\Psi_0(s) = \Psi_0^{\mathrm{u}}(s) - \Psi_0^{\mathrm{s}}(s), \qquad s \in E_{\beta_0,\rho}.$$
There exists $C_{\mathrm{in}} \in \mathbb{C}$ and a function $\chi : E_{\beta_0,\rho} \to \mathbb{C}^2$ such that:
$$\Delta\Psi_0(s) = s^{\mathrm{d}} e^{-i(\alpha s - (c+\alpha h_0)\log s)} \left(\left(\begin{array}{c} C_{\mathrm{in}} \\ 0 \end{array} \right) + \chi(s) \right), \tag{2.30}$$
where $h_0 = \lim_{\mathrm{Re}\, s\to\infty} s^3 H(0,0,-s^{-1},0,0)$ and $\chi = (\chi_1, \chi_2)$ satisfies:
$$|\chi_1(s)| \leq K|s|^{-1}, \qquad |\chi_2(s)| \leq K|s|^{-2}.$$
Moreover, $C_{\mathrm{in}} \neq 0$ if and only if $\Delta\Psi_0 \neq 0$.

The inner system corresponding to system (2.21) with $\mathrm{d} = 1$ was exhaustively studied in [BS08]. Moreover, the authors used an extra parameter ε (not necessarily small) which we take $\varepsilon = 1$. Since the proof for the case where d is a free parameter and $\varepsilon = 1$ is completely analogous, we will give just the main ideas of how Theorem 2.1.15 can be proved for this case without going into details. These can be found in Section 2.4.

Remark 2.1.16. The change (2.24) allows us to study some approximations of the invariant manifolds and their difference near the singularity $i\pi/2$. However, if we want to approximate these manifolds and their difference near the singularity $-i\pi/2$, instead of change (2.24) one has to introduce the following change:
$$\psi = \delta\xi, \quad \overline{\psi} = \delta\bar{\xi}, \quad s = \frac{u + i\pi/2}{\delta}. \tag{2.31}$$
In this case, one can prove the existence of two solutions $\tilde{\Psi}_0^{\mathrm{u,s}}(s)$ of the inner system obtained after doing change (2.31), which are defined for $s \in \overline{\mathcal{D}^{\mathrm{in},*}_{\beta_0,\rho}}$, with $* = \mathrm{u}, \mathrm{s}$, where:
$$\overline{\mathcal{D}^{\mathrm{in},*}_{\beta_0,\rho}} = \{s \in \mathbb{C} \,:\, \overline{s} \in \mathcal{D}^{\mathrm{in},*}_{\beta_0,\rho}\}.$$
Moreover, for:
$$s \in \overline{E_{\beta_0,\rho}} := \overline{\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\rho}} \cap \overline{\mathcal{D}^{\mathrm{in,s}}_{\beta_0,\rho}} \cap \{s \in \mathbb{C} \,:\, \mathrm{Im}\, s > 0\},$$
the difference between these two solutions, $\Delta\tilde{\Psi}_0(s)$, is given asymptotically by:
$$\Delta\tilde{\Psi}_0(s) = s^{\mathrm{d}} e^{i(\alpha s - (c+\alpha h_0)\log s)} \left(\left(\begin{array}{c} 0 \\ \overline{C_{\mathrm{in}}} \end{array} \right) + \tilde{\chi}(s) \right),$$
where $\overline{C_{\mathrm{in}}}$ is the conjugate of the constant C_{in} in Theorem 2.1.15 and $\tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2)$ satisfies that $|\tilde{\chi}_1(s)| \leq |s|^{-2}$ and $|\tilde{\chi}_2(s)| \leq |s|^{-1}$.



(a) The matching domain $D^{\text{mch,u}}_{\kappa,\beta_1,\beta_2}$.



(b) The matching domain $D^{\text{mch,s}}_{\kappa,\beta_1,\beta_2}$.

Figure 2.3: The matching domains in the outer variables.

2.1.4 Study of the matching error

Let us recall the domains $D^{\text{out,u}}_{\kappa,\beta,T}$ and $D^{\text{out,s}}_{\kappa,\beta,T}$, defined in (2.15), where the parameterizations $\varphi^{\text{u,s}}$ of the invariant manifolds given by Theorem 2.1.13 are defined, for some fixed $\kappa > 0$ and $0 < \beta < \pi/2$. We also recall the domains $\mathcal{D}^{\text{in,u}}_{\beta_0,\rho}$ and $\mathcal{D}^{\text{in,s}}_{\beta_0,\rho}$, defined in (2.28), with $\rho > 0$ and $0 < \beta_0 < \pi/2$ fixed, where the solutions $\Psi_0^{\text{u,s}}$ given in Theorem 2.1.15 are defined. Now we take β_1, β_2 two constants independent of δ and σ, such that:

$$0 < \beta_1 < \beta < \beta_2 < \pi/2. \tag{2.32}$$

We define $u_j \in \mathbb{C}$, $j = 1, 2$ as the two points that satisfy (see Figure 2.3):

- $\operatorname{Im} u_j = -\tan\beta_j \operatorname{Re} u_j + \pi/2 - \kappa\delta\log(1/\delta)$,
- $|u_j - i(\pi/2 - \kappa\delta\log(1/\delta))| = \delta^\gamma$, where $\gamma \in (0,1)$ is a constant independent of δ and σ,
- $\operatorname{Re} u_1 < 0$, $\operatorname{Re} u_2 > 0$.

We also consider the following domains (see Figure 2.3):

$$\begin{aligned} D^{\text{mch,u}}_{\kappa,\beta_1,\beta_2} \;=\; \{ & u \in \mathbb{C} \,:\, \operatorname{Im} u \le -\tan\beta_1 \operatorname{Re} u + \pi/2 - \kappa\delta\log(1/\delta), \\ & \operatorname{Im} u \le -\tan\beta_2 \operatorname{Re} u + \pi/2 - \kappa\delta\log(1/\delta), \\ & \operatorname{Im} u \ge \operatorname{Im} u_1 - \tan\left(\tfrac{\beta_1+\beta_2}{2}\right)(\operatorname{Re} u - \operatorname{Re} u_1) \}, \end{aligned}$$

and:

$$D^{\text{mch,s}}_{\kappa,\beta_1,\beta_2} = \{u \in \mathbb{C} \,:\, -\overline{u} \in D^{\text{mch,u}}_{\kappa,\beta_1,\beta_2}\}.$$

We note that there exist two constants K_1 and K_2, independent of δ and σ, such that:

$$K_1\delta^\gamma \le |u_j - i\pi/2| \le K_2\delta^\gamma, \qquad j = 1, 2.$$

Moreover, for all $u \in D^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}$, $* = \mathrm{u},\mathrm{s}$, we have:

$$\kappa \cos\beta_1 \delta \log(1/\delta) \leq |u - i\pi/2| \leq K_2 \delta^\gamma. \tag{2.33}$$

Note that from (2.32) and (2.33) we have $D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2} \subset D^{\mathrm{out,u}}_{\kappa,\beta,T}$ and $D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2} \subset D^{\mathrm{out,s}}_{\kappa,\beta,T}$, if δ is small enough .

We also define the matching domains in the inner variables:

$$\mathcal{D}^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2} = \{s \in \mathbb{C} \,:\, i\pi/2 + s\delta \in D^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}\}, \qquad * = \mathrm{u},\mathrm{s} \tag{2.34}$$

and:

$$s_j = \frac{u_j - i\pi/2}{\delta}, \quad j = 1, 2. \tag{2.35}$$

It is clear that:

$$K_1 \delta^{\gamma-1} \leq |s_j| \leq K_2 \delta^{\gamma-1}, \qquad j = 1, 2, \tag{2.36}$$

and that for all $s \in \mathcal{D}^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}$, where $* = \mathrm{u},\mathrm{s}$, we have:

$$\kappa \cos\beta_1 \log(1/\delta) \leq |s| \leq K_2 \delta^{\gamma-1}.$$

Using that $D^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2} \subset D^{\mathrm{out},*}_{\kappa,\beta,T}$ and Remark 2.1.14 it is clear that $\mathcal{D}^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2} \subset \mathcal{D}^{\mathrm{in},*}_{\beta_0,\rho}$ if δ is small enough, $* = \mathrm{u},\mathrm{s}$.

The main result of this section is the following.

Theorem 2.1.17. *Let $\Psi^{\mathrm{u,s}}(s) = \delta\varphi^{\mathrm{u,s}}(\delta s + i\pi/2)$, where $\varphi^{\mathrm{u,s}}$ are the parameterizations given by Theorem 2.1.13. Then, if $s \in \mathcal{D}^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}$, for $* = \mathrm{u},\mathrm{s}$, one has $\Psi^{\mathrm{u,s}}(s) = \Psi_0^{\mathrm{u,s}}(s) + \Psi_1^{\mathrm{u,s}}(s)$, where $\Psi_0^{\mathrm{u,s}}(s)$ are the two solutions of the inner system* (2.27) *given by Theorem 2.1.15, and there exists a constant K, independent of δ and σ, such that:*

$$|\Psi_1^{\mathrm{u,s}}(s)| \leq K\delta^{1-\gamma}|s|^{-2}.$$

This Theorem is proved in Section 2.5. From this result, the following corollary is clear:

Corollary 2.1.18. For $u \in D^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}$, where $* = \mathrm{u},\mathrm{s}$, we have that:

$$\varphi^{\mathrm{u,s}}(u) = \frac{1}{\delta}\left(\Psi_0^{\mathrm{u,s}}\left(\frac{u - i\pi/2}{\delta}\right) + \Psi_1^{\mathrm{u,s}}\left(\frac{u - i\pi/2}{\delta}\right)\right),$$

where $\Psi_0^{\mathrm{u,s}}$ are the two solutions of the inner system (2.27) given by Theorem 2.1.15 and:

$$\left|\Psi_1^{\mathrm{u,s}}\left(\frac{u - i\pi/2}{\delta}\right)\right| \leq K\delta^{1-\gamma}\left|\frac{u - i\pi/2}{\delta}\right|^{-2},$$

for some constant K. Note that, as for $u \in D^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}$, $|u - i\pi/2| \geq K\delta\log(1/\delta)$, from this last inequality we obtain:

$$\left|\Psi_1^{\mathrm{u,s}}\left(\frac{u-i\pi/2}{\delta}\right)\right| \leq \frac{K\delta^{1-\gamma}}{\log^2(1/\delta)},$$

and since $\gamma \in (0,1)$ we obtain that $\Psi_0^{\mathrm{u,s}}$ are good approximations of $\varphi^{\mathrm{u,s}}$ in $D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$ and $D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$ respectively.

Remark 2.1.19. Theorem 2.1.17 and, more precisely, Corollary 2.1.18 provide us with a bound of the difference between the invariant manifolds $\varphi^{\mathrm{u,s}}(u)$ of Theorem 2.1.13 and the functions $\Psi_0^{\mathrm{u,s}}((u-i\pi/2)/\delta)$ given by Theorem 2.1.15, when u is near the singularity $i\pi/2$. One can proceed similarly to study this difference near the singularity $-i\pi/2$ as we pointed out in Remark 2.1.16. In this case, defining:

$$\overline{D^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}} = \{s \in \mathbb{C} \,:\, \overline{s} \in D^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}\}, \qquad\qquad \text{for } * = \mathrm{u,s}$$

we would obtain that for $u \in \overline{D^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}}$, one has:

$$\varphi^{\mathrm{u,s}}(u) = \frac{1}{\delta}\left(\tilde\Psi_0^{\mathrm{u,s}}\left(\frac{u+i\pi/2}{\delta}\right) + \tilde\Psi_1^{\mathrm{u,s}}\left(\frac{u+i\pi/2}{\delta}\right)\right),$$

where $\tilde\Psi_0^{\mathrm{u,s}}$ are the two solutions of the inner system derived from the change (2.31) in Remark 2.1.16, and:

$$\left|\tilde\Psi_1^{\mathrm{u,s}}\left(\frac{u+i\pi/2}{\delta}\right)\right| \leq K\delta^{1-\gamma}\left|\frac{u+i\pi/2}{\delta}\right|^{-2},$$

for some constant K.

2.1.5 Asymptotic formula for the splitting distance

Theorem 2.1.20. *Let φ^{u} and φ^{s} be the parameterizations given by Theorem 2.1.13. For $u \in D^{\mathrm{out,u}}_{\kappa,\beta,T} \cap D^{\mathrm{out,s}}_{\kappa,\beta,T}$, we define its difference:*

$$\Delta\varphi(u) = \varphi^{\mathrm{u}}(u) - \varphi^{\mathrm{s}}(u). \tag{2.37}$$

Let $C_{\mathrm{in}} \in \mathbb{C}$ be the constant in Theorem 2.1.15. If $C_{\mathrm{in}} \neq 0$, then:

$$\Delta\varphi(0) = \frac{e^{-\frac{\alpha_0\pi}{2\delta}}}{\delta^{1+\mathrm{d}}} e^{\frac{\pi}{2}(c+\alpha_0 h_0-\alpha_1\sigma)}\left(\begin{pmatrix} C_{\mathrm{in}} e^{-i\left(\frac{\sigma\pi}{2}+\frac{\alpha_0 h_0}{2}+(c+\alpha_0 h_0)\log\delta\right)} \\ \overline{C_{\mathrm{in}}} e^{i\left(\frac{\sigma\pi}{2}+\frac{\alpha_0 h_0}{2}+(c+\alpha_0 h_0)\log\delta\right)}\end{pmatrix} + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right),$$

where $h_0 = -\lim_{z\to 0} z^{-3} H(0,0,z,0,0)$, $\alpha_0 = \alpha(0)$ and $\alpha_1 = \alpha'(0)$.

Remark 2.1.21. Note that from Theorem 2.1.20, doing the inverse of change C_2 (defined in (2.4)) and taking norms, we obtain Theorem 2.1.6, with $C^* = |C_{\mathrm{in}}|$.

In this subsection we will give the main ideas of how Theorem 2.1.20 can be proved. The full proof can be found in Section 2.6.

First of all recall that both φ^{u} and φ^{s} satisfy equations (2.21). We will decompose system (2.21) in a more convenient form. For that we define:

$$\mathcal{A}(u) = \frac{1}{1-\frac{\delta h_0 z_0^3(u)}{-1+z_0^2(u)}}\begin{pmatrix} -\left(\frac{\alpha}{\delta}+cz_0(u)\right)i+\sigma-\mathrm{d}z_0(u) & 0 \\ 0 & \left(\frac{\alpha}{\delta}+cz_0(u)\right)i+\sigma-\mathrm{d}z_0(u)\end{pmatrix}, \tag{2.38}$$

$$\begin{aligned}\mathcal{R}(\varphi)(u) &= \frac{\delta^{-2}F(\delta\varphi,\delta z_0(u),\delta,\delta\sigma)}{1+\frac{b\xi\bar{\xi}+\delta^{-2}H(\delta\varphi,\delta z_0(u),\delta,\delta\sigma)}{-1+z_0^2(u)}} \\ &\quad + \left(\frac{1}{1+\frac{b\xi\bar{\xi}+\delta^{-2}H(\delta\varphi,\delta z_0(u),\delta,\delta\sigma)}{-1+z_0^2(u)}} - \frac{1}{1-\frac{\delta h_0 z_0^3(u)}{-1+z_0^2(u)}}\right)\mathcal{A}(u)\varphi.\end{aligned} \tag{2.39}$$

Then, system (2.21) can be written as:

$$\frac{d\varphi}{du} = \mathcal{A}(u)\varphi + \mathcal{R}(\varphi)(u). \tag{2.40}$$

Since φ^{u} and φ^{s} satisfy system (2.40), it is clear that its difference $\Delta\varphi = \varphi^{\mathrm{u}} - \varphi^{\mathrm{s}}$ satisfies:

$$\frac{d\Delta\varphi}{du} = \mathcal{A}(u)\Delta\varphi + \mathcal{R}(\varphi^{\mathrm{u}})(u) - \mathcal{R}(\varphi^{\mathrm{s}})(u).$$

Following [Sau01], using the mean value theorem we can still rewrite this equation as the following linear equation:

$$\frac{d\Delta\varphi}{du} = \mathcal{A}(u)\Delta\varphi + \mathcal{B}(u)\Delta\varphi, \tag{2.41}$$

with:

$$\mathcal{B}(u) = \int_0^1 D\mathcal{R}((1-\lambda)\varphi^{\mathrm{s}} - \lambda\varphi^{\mathrm{u}})(u)d\lambda. \tag{2.42}$$

We observe that we can think of the matrix $\mathcal{B}$ as just depending on u, because the existence of φ^{u} and φ^{s} has been already proved in Theorem 2.1.13.

The point of writing the system for $\Delta\varphi$ as (2.41) is that, as we shall see, we split it into a dominant part, the one corresponding to the matrix $\mathcal{A}(u)$, and a *small perturbation*, which corresponds to the the matrix $\mathcal{B}(u)$. This will allow us to find an asymptotic expression for $\Delta\varphi(u)$, with its dominant term given by the solution of the system:

$$\frac{d\Delta\varphi}{du} = \mathcal{A}(u)\Delta\varphi. \tag{2.43}$$

Lemma 2.1.22. *For $u \in D^{\text{out,u}}_{\kappa,\beta,T} \cap D^{\text{out,s}}_{\kappa,\beta,T}$, a fundamental matrix of the homogeneous system* (2.43) *is:*

$$\mathcal{M}(u) = \begin{pmatrix} m_1(u) & 0 \\ 0 & m_2(u) \end{pmatrix}, \tag{2.44}$$

with:

$$\begin{aligned} m_1(u) &= \cosh^{\mathrm{d}} u e^{-\alpha i u/\delta} e^{\sigma u} e^{\alpha h_0 i\left[-\frac{1}{2}\sinh^2 u + \log\cosh u\right]} e^{ic\log\cosh u}\left(1 + \mathcal{O}\left(\tfrac{1}{\log(1/\delta)}\right)\right), \\ m_2(u) &= \cosh^{\mathrm{d}} u e^{\alpha i u/\delta} e^{\sigma u} e^{-\alpha h_0 i\left[-\frac{1}{2}\sinh^2 u + \log\cosh u\right]} e^{-ic\log\cosh u}\left(1 + \mathcal{O}\left(\tfrac{1}{\log(1/\delta)}\right)\right). \end{aligned} \tag{2.45}$$

The proof of Lemma 2.1.22 will be given in Section 2.6.

In the following we shall give an heuristic idea of how the asymptotic formula given in Theorem 2.1.20 can be found. For simplicity, we will focus just on the first component of $\Delta\varphi$, that is $\Delta\xi$.

Let us omit the influence of $\mathcal{B}$, that is assume that $\mathcal{B}(u) \equiv 0$. Then, any solution Φ of (2.41) can be written as:

$$\Phi(u) = \mathcal{M}(u)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

for certain c_1, c_2, and its first component is $m_1(u)c_1$. Hence, $\Delta\xi(u) = m_1(u)c_1$ for a certain c_1. The main idea is that $\Delta\xi(u)$ is bounded when $u \in D^{\text{out,u}}_{\kappa,\beta,T} \cap D^{\text{out,s}}_{\kappa,\beta,T}$. The first thing we observe is that from the asymptotic expression of $m_1(u)$ in Lemma 2.1.22 we can already see that $\Delta\xi(u)$ has an exponentially small bound if $u \in \mathbb{R}$. Indeed, it is clear that when $u \sim i\pi/2$ we have that $m_1(u) \sim e^{\frac{\alpha\pi}{2\delta}}$, that is exponentially big. Then, c_1 has to be $\sim e^{-\frac{\alpha\pi}{2\delta}}$ for $\Delta\xi(u)$ to be bounded, i.e. it must be exponentially small. As a consequence, when $u \in \mathbb{R}$ we have that $\Delta\xi(u) = m_1(u)c_1$ is exponentially small.

However, we do not want a bound of $\Delta\xi$ but an asymptotic formula. Thus we have to find the constant c_1 that corresponds to $\Delta\xi$, or more concretely a good approximation c_1^0 of it. We recall that near the singularity $i\pi/2$ we have a good approximation $\Delta\psi_0$ of $\Delta\xi$ given by the study of the inner equation in Theorem 2.1.15. Then, if we consider the point:

$$u_+ = i\left(\frac{\pi}{2} - \kappa\delta\log(1/\delta)\right),$$

it is clear that the initial condition c_1 satisfies:

$$c_1 m_1(u_+) = \Delta\xi(u_+) \approx \delta^{-1}\Delta\psi_0\left(\frac{u_+ - i\pi/2}{\delta}\right).$$

From Theorem 2.1.15 we know that:

$$\delta^{-1}\Delta\psi_0\left(\frac{u_+ - i\pi/2}{\delta}\right) = \frac{(-i\lambda)^{\mathrm{d}}}{\delta} e^{-\lambda\alpha - (c+\alpha h_0)\log(-i\lambda)}(C_{\text{in}} + \chi_1(-i\lambda)),$$

where $\lambda = \kappa \log(1/\delta)$, and therefore:

$$\begin{aligned} c_1 &= m_1(u_+)^{-1}\Delta\xi(u_+) \approx m_1^{-1}(u_+)\delta^{-1}\Delta\psi_0\left(\frac{u_+ - i\pi/2}{\delta}\right) \\ &\approx m_1^{-1}(u_+)\frac{(-i\lambda)^{\mathrm{d}}}{\delta}e^{-\alpha\lambda+i(c+\alpha h_0)\log(-i\lambda)}C_{\mathrm{in}}, \end{aligned}$$

so that taking:

$$c_1^0 = m_1^{-1}(u_+)\frac{(-i\lambda)^{\mathrm{d}}}{\delta}e^{-\alpha\lambda+i(c+\alpha h_0)\log(-i\lambda)}C_{\mathrm{in}},$$

we obtain a good approximation $\Delta\xi_0(u)$ of $\Delta\xi(u)$ defined by:

$$\Delta\xi_0(u) := m_1(u)c_1^0.$$

Using the bound of the matching error given in Theorem 2.1.17, it can be proved that $\Delta\xi_0(u)$ is the dominant part of $\Delta\xi(u)$. Then, computing explicitly the asymptotic formula of $\Delta\xi_0(0)$ one obtains the first component of the dominant term of the formula given in Theorem 2.1.20. As we will see in Section 2.6, c_1^0 does not depend on κ.

For the second component, $\Delta\bar{\xi}$, we can repeat the same arguments, but using the singularity $-i\pi/2$. Finally, this procedure can be adapted to the whole system (2.41) using the fact that, indeed, $\mathcal{B}(u)$ is small.

2.2 Proof of Theorem 2.1.10

In this section we will prove Theorem 2.1.10. In order to do that, first we need to define suitable Banach spaces in which we will work, which are the following:

$$\mathcal{X}^{\mathrm{out},*} = \left\{\phi : D^{\mathrm{out},*}_{\bar{\kappa},\beta} \to \mathbb{C} \,:\, \phi \text{ analytic}, \, \|\phi\|^*_{\mathrm{out}} < \infty\right\},$$

where $* = \mathrm{u},\mathrm{s}$, $D^{\mathrm{out},*}_{\bar{\kappa},\beta}$ are defined in (2.14) and the norm $\|.\|^{\mathrm{u,s}}_{\mathrm{out}}$ defined as:

$$\begin{aligned} \|\phi\|^{\mathrm{s}}_{\mathrm{out}} &= \sup_{v\in D^{\mathrm{out,s}}_{\bar{\kappa},\beta,\infty}} |(z_0(v)+1)^{-1}\phi(v)| + \sup_{v\in D^{\mathrm{out,s}}_{\bar{\kappa},\beta,T}} |(z_0(v)+1)^{-3}\phi(v)|, \\ \|\phi\|^{\mathrm{u}}_{\mathrm{out}} &= \sup_{v\in D^{\mathrm{out,u}}_{\bar{\kappa},\beta,\infty}} |(z_0(v)-1)^{-1}\phi(v)| + \sup_{v\in D^{\mathrm{out,u}}_{\bar{\kappa},\beta,T}} |(z_0(v)-1)^{-3}\phi(v)|. \end{aligned} \tag{2.46}$$

In the product space $\mathcal{X}^{\mathrm{out},*}\times\mathcal{X}^{\mathrm{out},*}$, with $* = \mathrm{u},\mathrm{s}$, we take the norm:

$$\|(\phi_1,\phi_2)\|^{\mathrm{u,s}}_{\mathrm{out},\times} = \|\phi_1\|^{\mathrm{u,s}}_{\mathrm{out}} + \|\phi_2\|^{\mathrm{u,s}}_{\mathrm{out}}, \qquad (\phi_1,\phi_2)\in\mathcal{X}^{\mathrm{out},*}\times\mathcal{X}^{\mathrm{out},*}.$$

Below we will introduce some notation that will allow us to see $\zeta^{\mathrm{u,s}}$ as fixed points of a certain operator. Given α and c, we define the linear operators acting on functions $\phi_1\in\mathcal{X}^{\mathrm{out},*}$, with $* = \mathrm{u},\mathrm{s}$ respectively:

$$L^{\mathrm{u,s}}_{\alpha,c}(\phi_1)(v) = \cosh^{\mathrm{d}} v\int_{\mp\infty}^0 \frac{1}{\cosh^{\mathrm{d}}(v+r)}e^{i\alpha r/\delta}e^{\sigma r}g^{\mathrm{u,s}}_c(v,r)\phi_1(v+r)dr, \tag{2.47}$$

where in the integral we take $-\infty$ for $L^{\mathrm{u}}_{\alpha,c}$ and $+\infty$ for $L^{\mathrm{s}}_{\alpha,c}$, and:

$$g_c^{\mathrm{u}}(v,r) = e^{ic(r+\log((1+e^{2v})/2)-\log((1+e^{2(v+r)})/2)},$$

$$g_c^{\mathrm{s}}(v,r) = e^{ic(-r+\log((1+e^{-2v})/2)-\log((1+e^{-2(v+r)})/2)},$$

Remark 2.2.1. One might think that instead of taking g_c, it would be more natural to take:

$$\hat{g}_c(v,r) = e^{ic(\log\cosh v - \log\cosh(v+r))}.$$

Although $g_c^{\mathrm{u}}(v,r) = g_c^{\mathrm{s}}(v,r) = \hat{g}_c(v,r)$ if $v,r\in\mathbb{R}$, this is not the case when $v,r\in\mathbb{C}$. In particular, one can see that if $v,r\in D^{\mathrm{out},*}_{\overline{\kappa},\beta}$, $*=\mathrm{u},\mathrm{s}$, the function $\hat{g}_c$ is not well defined. On the contrary, the function g_c^{u} is always well defined for $v,r\in D^{\mathrm{out,u}}_{\overline{\kappa},\beta}$ and g_c^{s} is well defined for $v,r\in D^{\mathrm{out,s}}_{\overline{\kappa},\beta}$.

Now, given a function $\phi=(\phi_1,\phi_2)\in\mathcal{X}^{\mathrm{out},*}\times\mathcal{X}^{\mathrm{out},*}$ we define the linear operator:

$$L^{\mathrm{u,s}}(\phi) = (L^{\mathrm{u,s}}_{\alpha,c}(\phi_1), L^{\mathrm{u,s}}_{-\alpha,-c}(\phi_2)). \tag{2.48}$$

The following lemma can be easily proved.

Lemma 2.2.2. *With the above notation, if a bounded and continuous function* $\zeta^{\mathrm{u,s}}: D^{\mathrm{out},*}_{\overline{\kappa},\beta}\to\mathbb{C}^3$, *with* $*=\mathrm{u},\mathrm{s}$ *respectively, satisfies the fixed point equation*

$$\zeta^{\mathrm{u,s}} = L^{\mathrm{u,s}}\circ R^{\mathrm{u,s}}(\zeta^{\mathrm{u,s}}), \tag{2.49}$$

then it is a solution of (2.12), (2.13) *respectively.*

In the rest of this section we will prove the following result, which implies straightforwardly Theorem 2.1.10.

Proposition 2.2.3. *Let* $\overline{\kappa}>0$ *and* $0<\beta<\pi/2$ *be any fixed constants independent of* δ *and* σ. *Then, if* $\delta>0$ *is small enough, problem* (2.12) *has a solution* ζ^{u} *defined in* $D^{\mathrm{out,u}}_{\overline{\kappa},\beta}$, *and* (2.13) *has a solution* ζ^{s} *defined in* $D^{\mathrm{out,s}}_{\overline{\kappa},\beta}$, *both satisfying that* $\zeta^{\mathrm{u,s}}=\zeta_0^{\mathrm{u,s}}+\zeta_1^{\mathrm{u,s}}$ *with the following properties:*

1. $\zeta_0^{\mathrm{u,s}} = L^{\mathrm{u,s}}\circ R^{\mathrm{u,s}}(0)\in\mathcal{X}^{\mathrm{out},*}\times\mathcal{X}^{\mathrm{out},*}$ *and there exists a constant* K *independent of* δ *and* σ *such that:*
$$\|\zeta_0^{\mathrm{u,s}}\|^{\mathrm{u,s}}_{\mathrm{out},\times}\le K\delta^2.$$

2. $\zeta_1^{\mathrm{u,s}}\in\mathcal{X}^{\mathrm{out},*}\times\mathcal{X}^{\mathrm{out},*}$, *and there exists a constant* K *independent of* δ *and* σ *such that:*
$$\|\zeta_1^{\mathrm{u,s}}\|^{\mathrm{u,s}}_{\mathrm{out},\times}\le\frac{K}{\log(1/\delta)}\|\zeta_0^{\mathrm{u,s}}\|^{\mathrm{u,s}}_{\mathrm{out},\times}$$

where $* = \mathrm{u}, \mathrm{s}$ *respectively.*

From Proposition 2.2.3 we obtain solutions ζ^{u} and ζ^{s} of systems given in (2.12) and (2.13) respectively. Note that by the definitions of $\mathcal{X}^{\mathrm{out,u}}$ and $\mathcal{X}^{\mathrm{out,s}}$, and since $\zeta^{\mathrm{u}} \in \mathcal{X}^{\mathrm{out,u}}$ and $\zeta^{\mathrm{s}} \in \mathcal{X}^{\mathrm{out,s}}$, we know that:

$$\lim_{\mathrm{Re}\, v \to -\infty} \zeta^{\mathrm{u}}(v) = (0,0), \qquad \lim_{\mathrm{Re}\, v \to +\infty} \zeta^{\mathrm{s}}(v) = (0,0).$$

From now on, we will focus just on the parameterization of the unstable manifold, ζ^{u}, being the proof for the stable one completely analogous. For this reason, if there is no danger of confusion, we will omit the superindices −u− of ζ, $\mathcal{X}^{\mathrm{out}}$, $D^{\mathrm{out}}_{\overline{\kappa},\beta}$, etc. Moreover, we will not write explicitly the dependence on v of ζ (or any function belonging to $\mathcal{X}^{\mathrm{out}}$). Finally, in the rest of the chapter, if no confusion is possible, we will denote by K any constant independent of δ and σ. Obviously, these constants K will depend on $\overline{\kappa}$ and β, which we will consider fixed.

Before proving Proposition 2.2.3 we will present some technical results.

Lemma 2.2.4. *Let* $\phi_1, \phi_2 \in \mathbb{C}^n$*, such that* $|\phi_1|, |\phi_2| < 1/2$*. Then:*

$$\left| \frac{1}{1+\phi_1} - \frac{1}{1+\phi_2} \right| \leq 4|\phi_1 - \phi_2|.$$

Lemma 2.2.5. *Let* $f : \mathbb{C}^n \to \mathbb{C}$ *be any function that is analytic in some open ball* $B(r_0^1) \times \cdots \times B(r_0^n) \subset \mathbb{C}^n$*, and assume that there exists some* $\phi^* \in \mathbb{C}^n$ *such that for all* $\phi \in B(r_0^1) \times \cdots \times B(r_0^n)$*:*

$$|f(\phi)| \leq K|\phi - \phi^*|^k, \tag{2.50}$$

for some constants $K > 0$ *and* $k \in \mathbb{N}$*. Take* $\phi \in B(r_0^1/2) \times \cdots \times B(r_0^n/2)$ *and assume that* $\phi - \phi^* \in B(r_0^1) \times \cdots \times B(r_0^n)$*. Then there exists a constant* $\tilde{K}$ *such that:*

$$|D_j f(\phi)| \leq \tilde{K}|\phi - \phi^*|^{k-1}, \tag{2.51}$$

where D_j *denotes the derivative with respect to the* $j-th$ *component* ϕ_j*.*

Corollary 2.2.6. Let $F_1^{\mathrm{u}}, F_2^{\mathrm{u}}$ and H^{u} the functions defined in (2.6). If $\zeta \in \mathcal{X}^{\mathrm{out}} \times \mathcal{X}^{\mathrm{out}}$ is such that $\|\zeta\|_{\mathrm{out},\times} \leq \delta^2 C$ for some constant C, we have that for δ small enough:

$$|D_j F_i^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)|, |D_j H^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)| \leq \begin{cases} K\delta^2 & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}, \\ K\delta^2 |z_0(v) - 1|^2 & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}. \end{cases} \tag{2.52}$$

Proof. We will prove this result just for $D_1F_1^{\mathrm{u}}$, being the other cases analogous. Note that for δ small enough $(\delta\zeta, \delta z_0(v)) \in B^3(r_0^{\mathrm{u}}/2)$ since by the fact that $\|\zeta\|_{\mathrm{out},\times} \leq \delta^2 C$ and the definition (2.46) of the norm $\|.\|_{\mathrm{out},\times}$ we have:

$$|\delta\zeta(v)| \leq \begin{cases} C\delta^3|z_0(v)-1| \leq \delta^3 C < r_0^{\mathrm{u}}/2 & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}, \\ C\delta^3|z_0(v)-1|^3 \leq \dfrac{CK}{\log^3(1/\delta)} < r_0^{\mathrm{u}}/2, & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}, \end{cases} \tag{2.53}$$

and:

$$|\delta z_0(v)| \leq \begin{cases} K\delta < r_0^{\mathrm{u}}/2 & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}, \\ \dfrac{K}{\log(1/\delta)} < r_0^{\mathrm{u}}/2, & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}, \end{cases}$$

Then we just have to take $\phi = (\delta\zeta, \delta z_0(v), \delta, \delta\sigma)$ and $\phi^* = (0,0,0,0,0)$ in Lemma 2.2.5. Indeed, it is clear that $\phi = \phi - \phi^* \in B^3(r_0^{\mathrm{u}}) \times B(\delta_0) \times B(\sigma_0)$. Then, since F_1^{u} is of order three and analytic in $B^3(r_0^{\mathrm{u}}) \times B(\delta_0) \times B(\sigma_0)$, we have:

$$|F_1^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)| \leq K|(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)|^3 = K|\phi - \phi^*|^3,$$

and then we have:

$$\begin{aligned} |D_1F_1^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)| &= |D_1F_1^{\mathrm{u}}(\phi)| \leq K|\phi-\phi^*|^2 \\ &= K|(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)|^2 \leq K|(\delta\zeta, \delta(z_0(v)-1), \delta, \delta\sigma)|^2. \end{aligned} \tag{2.54}$$

Moreover, since, by (2.53), $|\delta\zeta| \leq \delta^3 C|z_0(v)-1|$ if $v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}$ and $|\delta\zeta| \leq \delta^3 C|z_0(v)-1|^3$ if $v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}$, it is clear that:

$$|(\delta\zeta, \delta(z_0(v)-1), \delta, \delta\sigma)| \leq \begin{cases} K\delta & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}, \\ K\delta|z_0(v)-1| & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}, \end{cases}$$

for some constant K. With this bound and (2.54) we obtain immediately bound (2.52). □

Corollary 2.2.7. Let $F_1^{\mathrm{u}}, F_2^{\mathrm{u}}$ and H^{u} the functions defined in (2.6). If δ is small enough and $\|\zeta\|_{\mathrm{out},\times} \leq \delta^2 C$ for some constant C, there exists a constant K independent of δ and σ such that, for $i = 1, 2$:

$$|F_i^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)|, |H^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)| \leq \begin{cases} K\delta^3|z_0(v)-1| & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}, \\ K\delta^3|z_0(v)-1|^3, & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}. \end{cases}$$

Proof. Again, we will do the proof for F_1^{u}. Reasoning as in the proof of Corollary 2.2.6, we know that $(\delta\zeta, \delta z_0(v)) \in B^3(r_0^{\mathrm{u}})$ if δ is sufficiently small. Then, by the mean value theorem we have:

$$\begin{aligned} &|F_1^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)| = |F_1^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma) - F_1^{\mathrm{u}}(0,0,\delta,\delta,\delta\sigma)| \\ &\leq \int_0^1 |DF_1^{\mathrm{u}}(\lambda\delta\zeta, \delta + \lambda\delta(z_0(v)-1), \delta, \delta\sigma)| d\lambda \cdot |(\delta\zeta(v), \delta(z_0(v)-1))| \end{aligned} \tag{2.55}$$

provided that $F_1^{\mathrm{u}}(0,0,\delta,\delta,\delta\sigma) = 0$. By inequality (2.53) and the fact that, for $v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}$, $\delta|z_0(v)-1| \leq K$, one can easily deduce that $|\delta\zeta(v)| \leq K|\delta(z_0(v)-1)|$. Using this fact and reasoning as in the proof of Corollary 2.2.6 to bound $|DF_1^{\mathrm{u}}(\lambda\delta\zeta, \delta+\lambda\delta(z_0(v)-1), \delta, \delta\sigma)|$, inequality (2.55) yields:

$$|F_1^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)| \leq \begin{cases} K\delta^3|z_0(v)-1| & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}, \\ K\delta^3|z_0(v)-1|^3 & \text{if } v \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}, \end{cases}$$

and the claim is proved. □

Lemma 2.2.8. *Let $w \in D^{\mathrm{out}}_{\overline{\kappa},\beta}$. Then:*

1. *If $w \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}$, one has:*
$$|\cosh w| \geq \frac{e^{|\mathrm{Re}\, w|}}{4}.$$
2. *If $w \in D^{\mathrm{out}}_{\overline{\kappa},\beta}$, then:*
$$|e^{\pm ic\log((1+e^{2w})/2)}| < e^{c\pi}.$$

Lemma 2.2.9. *There exist constants K_1, K_2, K_3 and K_4, independent of δ and σ, such that*

1. *If $w \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}$ and $\mathrm{Im}\, w \geq 0$, then:*
   - *(a)* $K_1|w - i\pi/2| \leq |\cosh w| \leq K_2|w - i\pi/2|$,
   - *(b)* $K_3|w - i\pi/2| \leq |z_0(w)-1|^{-1} \leq K_4|w - i\pi/2|$,
2. *If $w \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}$ and $\mathrm{Im}\, w \leq 0$, then:*
   - *(a)* $K_1|w + i\pi/2| \leq |\cosh w| \leq K_2|w + i\pi/2|$,
   - *(b)* $K_3|w + i\pi/2| \leq |z_0(w)-1|^{-1} \leq K_4|w + i\pi/2|$,

Lemma 2.2.10. *If $v \in D^{\mathrm{out}}_{\overline{\kappa},\beta}$ and $w = v + re^{i(\pi-s)}$, with $r \in \mathbb{R}$, $r \geq 0$ and $s \in (0, \beta/2]$, then there exists a constant $K \neq 0$ independent of δ and σ such that:*

$$|w \pm i\pi/2| \geq K|v \pm i\pi/2|.$$

Lemma 2.2.11. *If $v \in D^{\text{out}}_{\overline{\kappa},\beta}$ and $w = v + re^{i(\pi-s)}$, with $r \in \mathbb{R}$, $r \geq 0$ and $s \in (0, \beta/2]$, then there exists a constant K independent of δ and σ such that:*

1. *(a)* $|\cosh v| \leq K|\cosh w|$.

   (b) Moreover, if $w \in D^{\text{out}}_{\overline{\kappa},\beta,\infty}$ then:

$$\frac{|\cosh v|}{|\cosh w|} \leq Ke^{-|r\cos(\pi-\beta/2)|}.$$

2. $|z_0(w) - 1| \leq K|z_0(v) - 1|$.

Lemma 2.2.12. *Let $R > 0$ be a constant big enough, and $v \in D^{\text{out}}_{\overline{\kappa},\beta}$. We define the complex path:*

$$\Gamma_1^R = \{w \in \mathbb{C} : \; w = re^{i(\pi-\beta/2)}, \, r \in [0, R]\}, \tag{2.56}$$

Then, if α, c, $\delta > 0$, the linear operator $L_{\alpha,c}$ defined in (2.47) *can be rewritten as:*

$$L_{\alpha,c}(\phi) = -\lim_{R\to+\infty} \int_{\Gamma_1^R} f_c(v,w)\phi(v+w)dw,$$

where $\phi \in \mathcal{X}^{\text{out}}$ and:

$$f_c(v,w) = \frac{\cosh^{\text{d}} v}{\cosh^{\text{d}}(v+w)} e^{i\alpha w/\delta} e^{\sigma w} e^{ic\left[w + \log((1+e^{2v})/2) - \log((1+e^{2(v+w)})/2)\right]} \tag{2.57}$$

Remark 2.2.13. For $L_{-\alpha,-c}(\phi)$ we get the same result but in curves of the form $\overline{\Gamma}_1^R := \left\{w \in \mathbb{C} \, : \, \overline{w} \in \Gamma_1^R\right\}$.

With these previous lemmas we can prove the following proposition, which characterizes how the operator $L = (L_{\alpha,c}, L_{-\alpha,-c})$, defined in (2.48), acts on $\mathcal{X}^{\text{out}} \times \mathcal{X}^{\text{out}}$.

Lemma 2.2.14. *The operator $L : \mathcal{X}^{\text{out}} \times \mathcal{X}^{\text{out}} \to \mathcal{X}^{\text{out}} \times \mathcal{X}^{\text{out}}$ is well defined and there exists a constant K independent of δ and σ such that for all $\phi \in \mathcal{X}^{\text{out}} \times \mathcal{X}^{\text{out}}$:*

$$\|L(\phi)\|_{\text{out},\times} \leq K\delta\|\phi\|_{\text{out},\times}.$$

Proof. We just need to bound $\|L_{\alpha,c}(\phi)\|_{\text{out}}$, since the case for $\|L_{-\alpha,-c}(\phi)\|_{\text{out}}$ is completely analogous. Note that by Lemma 2.2.12 we have that:

$$|L_{\alpha,c}(\phi)(v)| = \left|\lim_{R\to+\infty} \int_{\Gamma_1^R} f_c(v,w)\phi(v+w)dw\right|,$$

where Γ_1^R was defined in (2.56) and f_c was defined in (2.57). Now, parameterizing the curve Γ_1^R by $\gamma(r) = re^{i(\pi-\beta/2)}$, with $r \in [0, R]$, we get:

$$\begin{aligned} |L_{\alpha,c}(\phi)(v)| &= \left| \lim_{R\to+\infty} \int_0^R e^{i(\pi-\beta/2)} f_c(v, re^{i(\pi-\beta/2)}) \phi(v + re^{i(\pi-\beta/2)}) dr \right| \\ &= \left| \cosh^{\mathrm{d}} v \int_0^{+\infty} \frac{e^{i(\pi-\beta/2)} e^{\frac{i\alpha re^{i(\pi-\beta/2)}}{\delta}} e^{\sigma re^{i(\pi-\beta/2)}}}{\cosh^{\mathrm{d}}(v + re^{i(\pi-\beta/2)})} \tilde{g}_c(v, r) \phi(v + re^{i(\pi-\beta/2)}) dr \right|, \end{aligned}$$

where:

$$\tilde{g}_c(v, r) = g_c(v, re^{i(\pi-\beta/2)}) = e^{ic(re^{i(\pi-\beta/2)} + \log((1+e^{2v})/2) - \log((1+e^{2(v+re^{i(\pi-\beta/2)})})/2))}.$$

First we will see that there exists a constant K such that:

$$\left| \frac{\cosh^{\mathrm{d}} v \, e^{\sigma re^{i(\pi-\beta/2)}}}{\cosh^{\mathrm{d}}(v + re^{i(\pi-\beta/2)})} \right| \leq K. \tag{2.58}$$

On one hand, if $re^{i(\pi-\beta/2)} \in D^{\mathrm{out}}_{\overline{\kappa},\beta,\infty}$ then by part 1b of Lemma 2.2.11 we have that:

$$\left| \frac{\cosh^{\mathrm{d}} v \, e^{\sigma re^{i(\pi-\beta/2)}}}{\cosh^{\mathrm{d}}(v + re^{i(\pi-\beta/2)})} \right| \leq Ke^{-\mathrm{d}|r\cos(\pi-\beta/2)|} |e^{\sigma re^{i(\pi-\beta/2)}}| \leq Ke^{(|\sigma|-\mathrm{d})|r\cos(\pi-\beta/2)|} \leq K,$$

because $|\sigma| - \mathrm{d} < 0$. On the other hand, if $re^{i(\pi-\beta/2)} \in D^{\mathrm{out}}_{\overline{\kappa},\beta,T}$ it implies that $r \leq r^*$ for some $r^* < +\infty$ independent of δ and σ. Then, by part 1a of Lemma 2.2.11, we have that:

$$\begin{aligned} \left| \frac{\cosh^{\mathrm{d}} v \, e^{\sigma re^{i(\pi-\beta/2)}}}{\cosh^{\mathrm{d}}(v + re^{i(\pi-\beta/2)})} \right| &\leq K|e^{\sigma re^{i(\pi-\beta/2)}}| \leq Ke^{|\sigma||r\cos(\pi-\beta/2)|} \leq Ke^{|\sigma||r^*\cos(\pi-\beta/2)|} \\ &\leq Ke^{d|r^*\cos(\pi-\beta/2)|}. \end{aligned}$$

This finishes the proof of (2.58).

Now, to bound $\tilde{g}_c(v, r)$ we just use item 2 of Lemma 2.2.8:

$$\begin{aligned} |\tilde{g}_c(v, r)| &= |e^{ic(re^{i(\pi-\beta/2)} + \log(1+e^{2v}) - \log(1+e^{2(v+re^{i(\pi-\beta/2)})}))}| \leq e^{-c\mathrm{Im}\, re^{i(\pi-\beta/2)}} e^{2c\pi} \\ &= e^{-cr\sin(\beta/2)} e^{2c\pi} \leq e^{2c\pi}. \end{aligned} \tag{2.59}$$

Hence, using bounds (2.58) and (2.59) we have, for $v \in D^{\mathrm{out},}_{\overline{\kappa},\beta}$

$$|L_{\alpha,c}(\phi)(v)| \leq Ke^{2c\pi} \int_0^{+\infty} \left| e^{i\alpha re^{i(\pi-\beta/2)}/\delta} \right| |\phi(v + re^{i(\pi-\beta/2)})| dr. \tag{2.60}$$

Now we distinguish between the cases $v \in D^{\text{out}}_{\overline{\kappa},\beta,\infty}$ and $v \in D^{\text{out}}_{\overline{\kappa},\beta,T}$. On one hand, if $v \in D^{\text{out}}_{\overline{\kappa},\beta,\infty}$ we have $v + re^{i(\pi-\beta/2)} \in D^{\text{out}}_{\overline{\kappa},\beta,\infty}$ and then by part 2 of Lemma 2.2.11:

$$\begin{aligned} |L_{\alpha,c}(\phi)(v)| &\leq Ke^{2c\pi}\|\phi\|_{\text{out},\times}\int_0^{+\infty} e^{-\alpha r\sin(\pi-\beta/2)/\delta}|z_0(v+re^{i(\pi-\beta/2)})-1|dr \\ &\leq Ke^{2c\pi}\|\phi\|_{\text{out},\times}|z_0(v)-1|\int_0^{+\infty} e^{-\alpha r\sin(\pi-\beta/2)/\delta}dr. \end{aligned} \tag{2.61}$$

On the other hand, if $v \in D^{\text{out}}_{\overline{\kappa},\beta,T}$, let r^* be the value such that $v + re^{i(\pi-\beta/2)} \in D^{\text{out}}_{\overline{\kappa},\beta,\infty} \cap D^{\text{out}}_{\overline{\kappa},\beta,T}$. Then:

$$\begin{aligned} |L_{\alpha,c}(\phi)(v)| &\leq Ke^{2c\pi}\|\phi\|_{\text{out},\times}\left(\int_0^{r^*} e^{-\alpha r\sin(\pi-\beta/2)/\delta}|z_0(v+re^{i(\pi-\beta/2)})-1|^3dr\right. \\ &\quad\left.+\int_{r^*}^{+\infty} e^{-\alpha r\sin(\pi-\beta/2)/\delta}|z_0(v+re^{i(\pi-\beta/2)})-1|dr\right) \\ &\leq Ke^{2c\pi}\|\phi\|_{\text{out},\times}\left(\int_0^{r^*} e^{-\alpha r\sin(\pi-\beta/2)/\delta}|z_0(v)-1|^3dr\right. \\ &\quad\left.+\int_{r^*}^{+\infty} e^{-\alpha r\sin(\pi-\beta/2)/\delta}|z_0(v)-1|dr\right), \end{aligned}$$

where we have used part 2 of Lemma 2.2.11 again. Now, since for $v \in D^{\text{out}}_{\overline{\kappa},\beta,T}$ we have that $|z_0(v)-1| \leq K|z_0(v)-1|^3$, this last inequality yields:

$$|L_{\alpha,c}(\phi)(v)| \leq Ke^{2c\pi}\|\phi\|_{\text{out},\times}|z_0(v)-1|^3\int_0^{\infty} e^{-\alpha r\sin(\pi-\beta/2)/\delta}dr. \tag{2.62}$$

Hence, from (2.61) and (2.62) we can write:

$$|L_{\alpha,c}(\phi)(v)| \leq Ke^{2c\pi}\|\phi\|_{\text{out},\times}|z_0(v)-1|^\nu\int_0^{\infty} e^{-\alpha r\sin(\pi-\beta/2)/\delta}dr,$$

where $\nu = 1$ if $v \in D^{\text{out}}_{\overline{\kappa},\beta,\infty}$ and $\nu = 3$ otherwise.

If we compute the last integral explicitly we get that:

$$|L_{\alpha,c}(\phi)(v)| \leq \delta\frac{Ke^{2c\pi}}{\alpha\sin(\beta/2)}\|\phi\|_{\text{out},\times}|z_0(v)-1|^\nu,$$

and then, by definition (2.46) of the norm $\|.\|_{\text{out}}$, the result is clear. □

With Lemma 2.2.14, the first part of Proposition 2.2.3 will be proved. Concretely, we will prove the following:

Lemma 2.2.15. *The function* $\zeta_0 = L \circ R(0)$*, where* R *was defined in* (2.11) *and* L *in* (2.48)*, belongs to* $\mathcal{X}^{\text{out}} \times \mathcal{X}^{\text{out}}$*, and there exists a constant* K *independent of* δ *and* σ *such that:*

$$\|\zeta_0\|_{\text{out},\times} \leq K\delta^2.$$

Proof. By Lemma 2.2.14 it is clear that we just need to prove that $\|R(0)\|_{\text{out},\times} \leq K\delta$. Again, we will just bound the norm of the first component of $R(0)$, that is $R_1(0)$, being the second one analogous.

By (2.11) we have:

$$|R_1(0)(v)| = \frac{\delta^{-2}|F_1^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)|}{\left|1 + \frac{\delta^{-2}H^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right|} \leq \frac{\delta^{-2}|F_1^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)|}{\left|1 - \left|\frac{\delta^{-2}H^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right|\right|}.$$

First we will prove that, for $v \in D^{\text{out}}_{\overline{\kappa},\beta}$:

$$\frac{1}{\left|1 - \left|\frac{\delta^{-2}H^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right|\right|} \leq 2. \tag{2.63}$$

Indeed, if $v \in D^{\text{out}}_{\overline{\kappa},\beta,\infty}$, by Corollary 2.2.7:

$$\left|\frac{\delta^{-2}H^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right| \leq \frac{K\delta|z_0(v)-1|}{|-1+z_0^2(v)|} = 2K\delta|e^v\cosh v| \leq K\delta < \frac{1}{2}, \tag{2.64}$$

where we have used that $2e^v\cosh v = e^{2v}+1$ is bounded in $D^{\text{out}}_{\overline{\kappa},\beta,\infty}$ and that δ is sufficiently small. Otherwise, if $v \in D^{\text{out}}_{\overline{\kappa},\beta,T}$, again by Corollary 2.2.7 we have:

$$\left|\frac{\delta^{-2}H^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right| \leq \frac{K\delta|z_0(v)-1|^3}{|-1+z_0^2(v)|} = \frac{8K\delta e^{3v}}{|\cosh v|}.$$

Now, using Lemma 2.2.9 we have:

$$\frac{1}{|\cosh v|} \leq \frac{1}{K_1|v \mp i\pi/2|} \leq \frac{1}{K_1\delta\log(1/\delta)},$$

since $|v \mp i\pi/2| \geq K\delta\log(1/\delta)$ in $D^{\text{out}}_{\overline{\kappa},\beta}$. Moreover, for $v \in D^{\text{out}}_{\overline{\kappa},\beta,T}$ it is clear that e^{3v} is bounded. Therefore it is straightforward to see that:

$$\left|\frac{\delta^{-2}H^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right| \leq \frac{K}{\log(1/\delta)} < \frac{1}{2} \tag{2.65}$$

if δ is small enough. Then, from (2.64) and (2.65), bound (2.63) holds true.

Finally, from (2.63) and using again Corollary 2.2.7 it is clear that:

$$|R_1(0)(v)| \leq 2|\delta^{-2}F_1^{\text{u}}(0,\delta z_0(v),\delta,\delta\sigma)| \leq \begin{cases} K\delta|z_0(v)-1| & \text{if } v \in D^{\text{out}}_{\overline{\kappa},\beta,\infty}, \\ K\delta|z_0(v)-1|^3 & \text{if } v \in D^{\text{out}}_{\overline{\kappa},\beta,T}, \end{cases}$$

and then from the definition (2.46) of the norm $\|.\|_{\text{out}}$ we obtain the statement immediately. □

We enunciate the following technical lemma, due to Angenent [Ang93], which will simplify the proof of the second part of Proposition 2.2.3.

Lemma 2.2.16 ([Ang93])**.** *Let E be a complex Banach space, and let $f : B_r \to B_{\theta r}$ be a holomorphic mapping, where $B_\rho = \{x \in E : \|x\| < \rho\}$.*

If $\theta < 1/2$, then $f_{|B_{\theta r}}$ is a contraction, and hence has a unique fixed point in $B_{\theta r}$.

The following result will allow us to finish the proof of Proposition 2.2.3.

Lemma 2.2.17. *Let $\mathcal{F} := L \circ R$ and $B(r)$ be the ball of $\mathcal{X}^{\mathrm{out}} \times \mathcal{X}^{\mathrm{out}}$ centered at the origin of radius $r = 8\|\zeta_0\|_{\mathrm{out},\times}$. Then, $\mathcal{F} : B(r) \to B(r/4)$ is well defined. Moreover, there exists a constant K independent of δ and σ such that if $\zeta \in B(r)$:*

$$\|\mathcal{F}(\zeta) - \zeta_0\|_{\mathrm{out},\times} \leq \frac{1}{\log(1/\delta)} K \|\zeta\|_{\mathrm{out},\times}.$$

Proof. Note that it is sufficient to prove the inequality. Indeed, suppose that it holds, then taking $\zeta \in B(r)$ and δ sufficiently small we have:

$$\begin{aligned} \|\mathcal{F}(\zeta)\|_{\mathrm{out},\times} &\leq \|\mathcal{F}(\zeta) - \zeta_0\|_{\mathrm{out},\times} + \|\zeta_0\|_{\mathrm{out},\times} \leq \frac{1}{\log(1/\delta)} K \|\zeta\|_{\mathrm{out},\times} + \|\zeta_0\|_{\mathrm{out},\times} \\ &\leq \frac{1}{8} r + \frac{1}{8} r = \frac{1}{4} r, \end{aligned}$$

that is $\mathcal{F}(\zeta) \in B(r/4)$.

Now, recall that:

$$\mathcal{F}(\zeta) - \zeta_0 = L \circ R(\zeta) - L \circ R(0) = L \circ (R(\zeta) - R(0)),$$

where R was defined in (2.11). In order to make the proof clearer, we will decompose R as:

$$R(\zeta)(v) = S(\zeta)(v) + T(\zeta)(v) \cdot \zeta(v),$$

where:

$$S(\zeta)(v) = \frac{\delta^{-2} F^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)}{1 + \dfrac{b\eta\bar{\eta} + \delta^{-2} H^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)}{-1 + z_0^2(v)}},$$

$$T(\zeta)(v) = \left(\frac{1}{1 + \dfrac{b\eta\bar{\eta} + \delta^{-2} H^{\mathrm{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)}{-1 + z_0^2(v)}} - 1 \right) A(v).$$

Then we have that:

$$R(\zeta)(v) - R(0)(v) = S(\zeta)(v) - S(0)(v) + T(\zeta)(v) \cdot \zeta(v).$$

Now we shall bound these two last terms separately. We will begin by $S(\zeta)-S(0)$. We will prove that:

$$\|S(\zeta)-S(0)\|_{\mathrm{out},\times}\leq\frac{K}{\delta\log^2(1/\delta)}\|\zeta\|_{\mathrm{out},\times},\tag{2.66}$$

and we shall do it using the mean value theorem:

$$S(\zeta)(v)-S(0)(v)=\int_0^1 DS(\lambda\zeta)(v)d\lambda\cdot\zeta(v).$$

So let us bound $DS(\lambda\zeta)(v)$ with $\lambda\in[0,1]$. We claim that:

$$|DS(\lambda\zeta)(v)|\quad\leq\quad\left\{\begin{array}{ll}\delta K & \text{if } v\in D^{\mathrm{out}}_{\bar\kappa,\beta,\infty},\\ \delta K|z_0(v)-1|^2 & \text{if } v\in D^{\mathrm{out}}_{\bar\kappa,\beta,T}\end{array}\right\}\leq\frac{K}{\delta\log^2(1/\delta)}.\tag{2.67}$$

The second inequality is clear from the definition of $D^{\mathrm{out}}_{\bar\kappa,\beta}$, so we just have to check that the first one holds. For simplicity we will bound just one entry of the matrix DS, being the other three analogous. For instance we consider:

$$\begin{aligned}D_\eta S_1(\lambda\zeta)(v)&=\frac{\delta^{-1}D_\eta F_1^{\mathrm{u}}(\delta\lambda\zeta,\delta z_0(v),\delta,\delta\sigma)}{1+\frac{\lambda^2 b\eta\bar\eta+\delta^{-2}H^{\mathrm{u}}(\delta\lambda\zeta,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}}\\&-\frac{\delta^{-2}F_1^{\mathrm{u}}(\delta\lambda\zeta,\delta z_0(v),\delta,\delta\sigma)}{\left(1+\frac{\lambda^2 b\eta\bar\eta+\delta^{-2}H^{\mathrm{u}}(\delta\lambda\zeta,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right)^2}\left[\frac{\lambda b\bar\eta+\delta^{-1}D_\eta H^{\mathrm{u}}(\delta\lambda\zeta,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right].\end{aligned}$$

Our first claim is that, if $\zeta\in B(r)\subset\mathcal{X}^{\mathrm{out}}\times\mathcal{X}^{\mathrm{out}}$, then for δ sufficiently small:

$$\left|1+\tfrac{\lambda^2 b\eta\bar\eta+\delta^{-2}H^{\mathrm{u}}(\delta\lambda\zeta,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right|\geq\left|1-\left|\tfrac{\lambda^2 b\eta\bar\eta+\delta^{-2}H^{\mathrm{u}}(\delta\lambda\zeta,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right|\right|\geq\tfrac12.\tag{2.68}$$

Indeed, by definition (2.46) of $\|.\|_{\mathrm{out},\times}$ and Lemma 2.2.15 we have that:

$$\begin{aligned}|\lambda\zeta(v)|&\leq\lambda K\|\zeta\|_{\mathrm{out},\times}|z_0(v)-1|^\nu\leq Kr|z_0(v)-1|^\nu=8K\|\zeta_0\|_{\mathrm{out},\times}|z_0(v)-1|^\nu\\&\leq 8K\delta^2|z_0(v)-1|^\nu,\end{aligned}\tag{2.69}$$

with $\nu=1$ if $v\in D^{\mathrm{out}}_{\bar\kappa,\beta,\infty}$ and $\nu=3$ if $v\in D^{\mathrm{out}}_{\bar\kappa,\beta,T}$. Then it is easy to see that for δ sufficiently small:

$$\frac{\lambda^2|b||\eta||\bar\eta|}{|-1+z_0^2(v)|}\leq\frac{K}{\log(1/\delta)}\leq\frac14,\tag{2.70}$$

if $v\in D^{\mathrm{out}}_{\bar\kappa,\beta}$. Moreover, (2.69) implies that $\|\delta\lambda\zeta\|_{\mathrm{out},\times}\leq K\delta^3\leq K\delta^2$. Then, using Corollary 2.2.7 one can see that:

$$\left|\frac{\delta^{-2}H^{\mathrm{u}}(\delta\lambda\zeta,\delta z_0(v),\delta,\delta\sigma)}{-1+z_0^2(v)}\right|\leq\frac{K}{\log(1/\delta)}\leq\frac14\tag{2.71}$$

for $v \in D^{\text{out}}_{\bar{\kappa},\beta}$ and δ sufficiently small. Using bounds (2.70) and (2.71) one can straightforwardly see that (2.68) holds.

Our second claim, which is in fact Corollary 2.2.6, is that:

$$|\delta^{-1} D_\eta F_1^{\text{u}}(\delta\lambda\zeta(v), \delta z_0(v), \delta, \delta\sigma)| \leq \begin{cases} K\delta & \text{if } v \in D^{\text{out}}_{\bar{\kappa},\beta,\infty}, \\ K\delta|z_0(v)-1|^2 & \text{if } v \in D^{\text{out}}_{\bar{\kappa},\beta,T}. \end{cases} \tag{2.72}$$

Finally, it only remains to bound the last term of $D_\eta S_1$. We claim that:

$$\begin{aligned} &\left|\delta^{-2} F_1^{\text{u}}(\delta\lambda\zeta, \delta z_0(v), \delta, \delta\sigma)\left[\frac{\lambda b\bar{\eta} + \delta^{-1} D_\eta H^{\text{u}}(\delta\lambda\zeta, \delta z_0(v), \delta, \delta\sigma)}{-1+z_0^2(v)}\right]\right| \\ \leq & \begin{cases} K\delta & \text{if } v \in D^{\text{out}}_{\bar{\kappa},\beta,\infty}, \\ K\delta|z_0(v)-1|^2 & \text{if } v \in D^{\text{out}}_{\bar{\kappa},\beta,T}. \end{cases} \end{aligned} \tag{2.73}$$

This can be proved using Corollaries 2.2.6 and 2.2.7 to bound $D_\eta H^{\text{u}}$ and F_1^{u} respectively, and inequality (2.69) to bound $\bar{\eta}$.

In conclusion, from (2.68), (2.72) and (2.73) we obtain:

$$|D_\eta S_1(\lambda\zeta, v)| \leq \begin{cases} K\delta & \text{if } v \in D^{\text{out}}_{\bar{\kappa},\beta,\infty}, \\ K\delta|z_0(v)-1|^2 & \text{if } v \in D^{\text{out}}_{\bar{\kappa},\beta,T}, \end{cases}$$

and then, doing the same for the rest of the entries of the matrix DS, bound (2.67) is proved. Finally, (2.67) and the mean value theorem yield:

$$|S(\zeta)(v) - S(0)(v)| \leq \int_0^1 |DS(\lambda\zeta, v)| d\lambda \cdot |\zeta(v)| \leq \frac{K}{\delta \log^2(1/\delta)} |\zeta(v)|,$$

and that implies bound (2.66).

Now we shall proceed to bound $T(\zeta)(v) \cdot \zeta(v)$. We claim that:

$$\|T(\zeta) \cdot \zeta\|_{\text{out},\times} \leq \frac{K}{\delta \log(1/\delta)} \|\zeta\|_{\text{out},\times}. \tag{2.74}$$

Indeed, using (2.69) and the first inequalities in (2.70) and (2.71) with $\lambda = 1$, it can be seen that:

$$\left|\frac{b\eta\bar{\eta} + \delta^{-2} H^{\text{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)}{-1+z_0^2(v)}\right| \leq \frac{K}{\log(1/\delta)},$$

if δ is small enough, and consequently it is clear that:

$$\left|\frac{1}{1 + \frac{b\eta\bar{\eta} + \delta^{-2} H^{\text{u}}(\delta\zeta, \delta z_0(v), \delta, \delta\sigma)}{-1+z_0^2(v)}} - 1\right| \leq \frac{K}{\log(1/\delta)}. \tag{2.75}$$

Moreover, by the definition (2.10) of the matrix A it is straightforward to see that $|A(v)\zeta(v)| \leq K\delta^{-1}|\zeta(v)|$. Using this fact, bound (2.75) and the definition (2.46) of $\|.\|_{\mathrm{out},\times}$, it is clear that for $\nu = 1,3$ we have:

$$|T(\zeta)(v) \cdot \zeta(v)||z_0(v) - 1|^\nu \leq \frac{K}{\delta \log(1/\delta)} \|\zeta\|_{\mathrm{out},\times},$$

and then (2.74) is proved.

Finally, using (2.66) and (2.74) we have that:

$$\begin{aligned} \|R(\zeta) - R(0)\|_{\mathrm{out},\times} &= \|S(\zeta) - S(0) + T(\zeta) \cdot \zeta\|_{\mathrm{out},\times} \\ &\leq K\|\zeta\|_{\mathrm{out},\times} \left(\frac{1}{\delta \log^2(1/\delta)} + \frac{1}{\delta \log(1/\delta)} \right) \leq \frac{K}{\delta \log(1/\delta)} \|\zeta\|_{\mathrm{out},\times}, \end{aligned}$$

and then by Lemma 2.2.14 we obtain the desired bound:

$$\|L \circ (R(\zeta) - R(0))\|_{\mathrm{out},\times} \leq K\delta \|R(\zeta) - R(0)\|_{\mathrm{out},\times} \leq \frac{K}{\log(1/\delta)} \|\zeta\|_{\mathrm{out},\times}.$$

□

End of the proof of Proposition 2.2.3. As we already mentioned, the first part of Proposition 2.2.3 is proved in Lemma 2.2.15.

On the other hand, note that Lemma 2.2.16 and Lemma 2.2.17 imply that the operator $\mathcal{F} = L \circ R$ has a unique fixed point ζ^{u} in the ball of $\mathcal{X}^{\mathrm{out}} \times \mathcal{X}^{\mathrm{out}}$ of radius $8\|\zeta_0^{\mathrm{u}}\|_{\mathrm{out},\times}$. Then we just need to define $\zeta_1^{\mathrm{u}} = \zeta^{\mathrm{u}} - \zeta_0^{\mathrm{u}}$. It is clear that $\zeta^{\mathrm{u}} = \zeta_0 + \zeta_1$ and that by Lemma 2.2.17:

$$\|\zeta_1^{\mathrm{u}}\|_{\mathrm{out},\times} = \|\zeta^{\mathrm{u}} - \zeta_0^{\mathrm{u}}\|_{\mathrm{out},\times} = \|\mathcal{F}(\zeta^{\mathrm{u}}) - \mathcal{F}(0)\|_{\mathrm{out},\times} \leq \frac{K}{\log(1/\delta)} \|\zeta^{\mathrm{u}}\|_{\mathrm{out},\times} \leq \frac{K}{\log(1/\delta)} \|\zeta_0^{\mathrm{u}}\|_{\mathrm{out},\times},$$

and then the second part of Proposition 2.2.3 is clear. □

2.3 Proof of Theorem 2.1.13

Again, we will just focus on the proof for the unstable manifold, φ^{u}, being the one for the stable manifold analogous. We will also omit the superindices −u− whenever it does not lead to confusion.

Lemma 2.3.1. *Let $V_\pm(u,\delta,\sigma) = z_0^{-1}(z_0(u) - z_\pm(\delta,\sigma) \pm 1) - u$, where $z_\pm(\delta,\sigma)$ is the third component of the critical point $S_\pm(\delta,\sigma)$. Then, for all $u \in D^{\mathrm{out}}_{\kappa,\beta,T}$, there exists a constant C independent of δ and σ such that:*

$$|V_\pm(u,\delta,\sigma)| \leq \delta C.$$

Moreover, given any constant κ, if $u \in D^{\rm out}_{\kappa,\beta,T}$, then for $\delta > 0$ sufficiently small:

$$u + V_\pm(u,\delta,\sigma) \in D^{\rm out}_{\kappa/2,\beta}.$$

Proof. Consider the function $f(t) := z_0^{-1}(z_0(u) + t(-z_\pm(\delta,\sigma) \pm 1))$. It is clear that $V_\pm(u,\delta,\sigma) = f(1) - f(0)$. Moreover, for any $u \in D^{\rm out}_{\kappa,\beta,T}$ and $\delta > 0$, the function f is analytic. Using that:

$$\left|\frac{1}{-1+z_0^2(u)}\right| = \left|\cosh^2 u\right| \le M \qquad \text{if } u \in D^{\rm out}_{\kappa,\beta,T},$$

and that, by Lemma 2.1.4, $|-z_\pm(\delta,\sigma) \pm 1| \le K\delta$, one can easily see that $|f'(t)| \le \delta C$. Then, by the mean value theorem, the first part of the lemma is proved. Moreover, using the bound of $V_\pm(u,\delta,\sigma)$ it is straightforward to check that the second part of the lemma also holds.

□

End of the proof of Theorem 2.1.13. We just need to take $\overline{\kappa} = \kappa/2$. Then, by Lemma 2.3.1, if $u \in D^{\rm out}_{\kappa,\beta,T}$, $u+V_\pm(u,\delta,\sigma)$ belongs to $D^{\rm out}_{\kappa/2,\beta} = D^{\rm out}_{\overline{\kappa},\beta}$, where we know by Proposition 2.2.3 that the parameterizations $\zeta^{\rm u}$ and $\zeta^{\rm s}$ are defined. Then we just have to define $\varphi^{\rm s}$ and $\varphi^{\rm u}$ as:

$$\begin{array}{rcl} \varphi^{\rm s}(u) &=& \zeta^{\rm s}(u + V_-(u,\delta,\sigma)) + \zeta_-(\delta,\sigma), \\ \varphi^{\rm u}(u) &=& \zeta^{\rm u}(u + V_+(u,\delta,\sigma)) + \zeta_+(\delta,\sigma), \end{array} \qquad u \in D^{\rm out}_{\kappa,\beta,T} \tag{2.76}$$

where $\zeta_\pm(\delta,\sigma) = (\eta_\pm(\delta,\sigma), \bar\eta_\pm(\delta,\sigma))$, and $\eta_\pm$, $\bar\eta_\pm$ were defined in (2.7). As we pointed out in Subsection 2.1.2, both $\varphi^{\rm s}(u)$ and $\varphi^{\rm u}(u)$ satisfy system (2.21), and they are parameterizations of the stable and unstable manifolds of $S_-(\delta,\sigma)$ and $S_+(\delta,\sigma)$ respectively.

Finally, note that, for $u \in D^{\rm out}_{\kappa,\beta,T}$, one has:

$$|\varphi^{\rm u,s}(u)||z_0(u)-1|^3 \le |\zeta^{\rm u,s}(u + V_\pm(u,\delta,\sigma))||z_0(u)-1|^3 + |\zeta_\pm(\delta,\sigma)||z_0(u)-1|^3,$$

for some constant K. Now, on one hand, by Proposition 2.2.3 and using that for $u \in D^{\rm out}_{\kappa,\beta,T}$:

$$\left|\frac{z_0(u)-1}{z_0(u+V_\pm(u,\delta,\sigma))-1}\right| \le 1 + \frac{K}{\log(1/\delta)},$$

we have:

$$\begin{array}{rcl} |\zeta^{\rm u,s}(u+V_\pm(u,\delta,\sigma))||z_0(u)-1|^3 &\le& K|\zeta^{\rm u,s}(u+V_\pm(u,\delta,\sigma))||z_0(u+V_\pm(u,\delta,\sigma))-1|^3 \\ &\le& K\|\zeta^{\rm u,s}\|^{\rm u,s}_{\rm out} \le K\delta^2. \end{array}$$

On the other hand, recall that $\zeta_\pm(\delta,\sigma) = (\eta_\pm(\delta,\sigma), \bar\eta_\pm(\delta,\sigma))$, where $\eta_\pm(\delta,\sigma) = x_\pm(\delta,\sigma) + iy_\pm(\delta,\sigma)$, and then by Lemma 2.1.4, since $|\cosh u|$ is bounded in $D^{\rm out}_{\kappa,\beta,T}$, we obtain $|\zeta_\pm(\delta,\sigma)\cosh^3 u| \le K\delta^2$, and thus the last statement of Theorem 2.1.13 is clear. □

2.4 Sketch of the proof of Theorem 2.1.15

In this section we present the main ideas of how Theorem 2.1.15 is proved. As we already mentioned, the proof is analogous as the one found in [BS08], and hence for more details we refer the reader to that paper.

2.4.1 Existence of solutions $\Psi_0^{\mathrm{u,s}}$

First we will introduce the Banach spaces in which we will work. For $* = \mathrm{u}, \mathrm{s}$, we define:

$$\mathcal{X}_\nu^{\mathrm{in},*} = \{\phi : \mathcal{D}_{\beta_0,\rho}^{\mathrm{in},*} \to \mathbb{C},\ \phi \text{ analytic},\ \|\phi\|_{\mathrm{in},\nu}^{\mathrm{u,s}} := \sup_{s\in\mathcal{D}_{\beta_0,\rho}^{\mathrm{in},*}} |s^\nu \phi(s)| < \infty\},$$

where $\mathcal{D}_{\beta_0,\rho}^{\mathrm{in},*}$ are defined in (2.28). As usual, in the product space $\mathcal{X}_\nu^{\mathrm{in},*} \times \mathcal{X}_\nu^{\mathrm{in},*}$ we will take the norm:

$$\|(\phi_1,\phi_2)\|_{\mathrm{in},\nu,\times}^{\mathrm{u,s}} = \|\phi_1\|_{\mathrm{in},\nu}^{\mathrm{u,s}} + \|\phi_2\|_{\mathrm{in},\nu}^{\mathrm{u,s}}. \tag{2.77}$$

Now, if we call $\Psi = (\psi, \overline{\psi})$ the solutions of (2.27), $F = (F_1, F_2)$ and define:

$$h_0 = \lim_{\mathrm{Re}\, s\to\infty} s^3 H(0,0,-s^{-1},0,0), \tag{2.78}$$

$$\tilde{h}(\Psi,s) = s^2\left[b\psi\overline{\psi} + H(\Psi,-s^{-1},0,0)\right], \tag{2.79}$$

$$\tilde{\mathcal{A}}(s) = \begin{pmatrix} -(\alpha - cs^{-1})i + \mathrm{d}s^{-1} & 0 \\ 0 & (\alpha - cs^{-1})i + \mathrm{d}s^{-1} \end{pmatrix}, \tag{2.80}$$

and

$$\mathcal{R}(\Psi)(s) = \left(\frac{1}{1+\tilde{h}(\Psi,s)} - \frac{1}{1+h_0 s^{-1}}\right)\tilde{\mathcal{A}}(s)\Psi + \frac{F(\Psi,-s^{-1},0,0)}{1+\tilde{h}(\Psi,s)}, \tag{2.81}$$

then system (2.27) can be written as:

$$\frac{d\Psi}{ds} = \frac{1}{1+h_0 s^{-1}}\tilde{\mathcal{A}}(s)\Psi + \mathcal{R}(\Psi)(s). \tag{2.82}$$

One can easily prove the following lemma.

Lemma 2.4.1. *A fundamental matrix of the linear homogeneous system*

$$\frac{d\Psi}{ds} = \frac{1}{1+h_0 s^{-1}}\tilde{\mathcal{A}}(s)\Psi,$$

is:

$$\mathcal{M}(s) = \begin{pmatrix} m_1(s) & 0 \\ 0 & m_2(s) \end{pmatrix} = s^{\mathrm{d}}(1+h_0 s^{-1})^{\mathrm{d}} \begin{pmatrix} e^{-i(\alpha s+\beta(s))} & 0 \\ 0 & e^{i(\alpha s+\beta(s))} \end{pmatrix}, \tag{2.83}$$

where $\beta(s) = -(c+\alpha h_0)\log(s(1+h_0 s^{-1}))$.

Let us define $\Psi_0^{\mathrm{u,s}}$ implicitly by the following fixed point equation:

$$\Psi_0^{\mathrm{u,s}}(s) = \mathcal{M}(s) \int_{\mp\infty}^{0} \mathcal{M}(s+t)^{-1} \mathcal{R}(\Psi_0^{\mathrm{u,s}})(s+t)dt, \tag{2.84}$$

where $\mathcal{R}$ was defined in (2.81), and $+\infty$ corresponds to the stable case and $-\infty$ to the unstable one. Note that $\Psi_0^{\mathrm{u,s}}$ satisfy equation (2.82). For functions $\Phi \in \mathcal{X}_\nu^{\mathrm{in},*} \times \mathcal{X}_\nu^{\mathrm{in},*}$, we introduce the linear operators:

$$\mathcal{B}^{\mathrm{u,s}}(\Phi)(s) = \mathcal{M}(s) \int_{\mp\infty}^{0} \mathcal{M}(s+t)^{-1} \Phi(s+t)dt,$$

so that the fixed point equation (2.84) can be written as:

$$\Psi_0^{\mathrm{u,s}} = \mathcal{F}^{\mathrm{u,s}}(\Psi_0^{\mathrm{u,s}}) := \mathcal{B}^{\mathrm{u,s}} \circ \mathcal{R}(\Psi_0^{\mathrm{u,s}}). \tag{2.85}$$

The main result in this subsection, which is equivalent to item 1 of Theorem 2.1.15, is the following:

Proposition 2.4.2. *Given $\beta_0 > 0$, there exists $\rho > 0$ big enough such that system* (2.82) *has two solutions $\Psi_0^{\mathrm{u,s}}$ belonging to $\mathcal{X}_\nu^{\mathrm{in},*} \times \mathcal{X}_\nu^{\mathrm{in},*}$, $* = \mathrm{u}, \mathrm{s}$, of the form:*

$$\Psi_0^{\mathrm{u,s}} = \Psi_{0,0}^{\mathrm{u,s}} + \Psi_{0,1}^{\mathrm{u,s}},$$

with $\Psi_{0,0}^{\mathrm{u,s}} = \mathcal{B}^{\mathrm{u,s}} \circ \mathcal{R}(0) \in \mathcal{X}_3^{\mathrm{in},*} \times \mathcal{X}_3^{\mathrm{in},*}$, $\Psi_{0,1}^{\mathrm{u,s}} \in \mathcal{X}_4^{\mathrm{in},*} \times \mathcal{X}_4^{\mathrm{in},*}$, satisfying $\|\Psi_{0,1}^{\mathrm{u,s}}\|_{\mathrm{in},3,\times}^{\mathrm{u,s}} < \|\Psi_{0,0}^{\mathrm{u,s}}\|_{\mathrm{in},3,\times}^{\mathrm{u,s}}$.

Moreover, the functions $\Psi_0^{\mathrm{u,s}}$ are the unique solutions of system (2.82) *satisfying the asymptotic condition $\lim_{\mathrm{Re}\, s \to \mp\infty} \Psi_0^{\mathrm{u,s}}(s) = 0$, where $-$ corresponds to* u *and $+$ to* s.

This proposition is proved in [BS08] in the case $\mathrm{d} = 1$, and the case $\mathrm{d} \neq 1$ can be proved identically.

2.4.2 Asymptotic expression for the difference $\Delta\Psi_0$

Below we sketch how formula (2.30) can be found, which is an adaptation of the results of [BS08] for the case $\mathrm{d} \neq 1$. The first step is to realize that, since Ψ_0^{s} and Ψ_0^{u} satisfy equation (2.82), its difference $\Delta\Psi_0 = (\Delta\psi_0, \Delta\overline{\psi}_0)$ satisfies the following homogeneous linear equation:

$$\frac{d\Delta\Psi}{ds} = \left[\frac{1}{1 + h_0 s^{-1}} \tilde{\mathcal{A}}(s) + \tilde{\mathcal{R}}(s)\right] \Delta\Psi, \tag{2.86}$$

where $\tilde{\mathcal{R}}$ is the matrix defined by:

$$\tilde{\mathcal{R}}(s) = \int_0^1 D\mathcal{R}(\Psi_0^{\mathrm{s}}(s) + \lambda(\Psi_0^{\mathrm{u}}(s) - \Psi_0^{\mathrm{s}}(s)))d\lambda,$$

and $\mathcal{R}$ was defined in (2.81). As in [BS08], one deduces that any analytic solution of equation (2.86) that is bounded in the domain $E_{\beta_0,\rho}$, defined in (2.29), can be written as the following integral equation.

$$\Delta\psi_0(s) = s^{\mathrm{d}}(1+h_0s^{-1})^{\mathrm{d}}e^{-i(\alpha s+\beta(s))}\left[\kappa_0+\int_{-i\rho}^{s}\frac{e^{i(\alpha t+\beta(t))}}{t^{\mathrm{d}}(1+h_0t^{-1})^{\mathrm{d}}}\langle\tilde{\mathcal{R}}_1(t),\Delta\Psi_0(t)\rangle dt\right], \tag{2.87}$$

$$\Delta\overline{\psi}_0(s) = s^{\mathrm{d}}(1+h_0s^{-1})^{\mathrm{d}}e^{i(\alpha s+\beta(s))}\int_{-i\infty}^{s}\frac{e^{-i(\alpha t+\beta(t))}}{t^{\mathrm{d}}(1+h_0t^{-1})^{\mathrm{d}}}\langle\tilde{\mathcal{R}}_2(t),\Delta\Psi_0(t)\rangle dt, \tag{2.88}$$

where $\beta(s) = -(c+\alpha h_0)\log(s(1+h_0s^{-1}))$.

Now we define the linear operator $\mathcal{G}$ by the expression:

$$\mathcal{G}(\Phi)(s) = s^{\mathrm{d}}(1+h_0s^{-1})^{\mathrm{d}}\begin{pmatrix} e^{-i(\alpha s+\beta(s))}\displaystyle\int_{-i\rho}^{s}\frac{e^{i(\alpha t+\beta(t))}}{t^{\mathrm{d}}(1+h_0t^{-1})^{\mathrm{d}}}\langle\tilde{\mathcal{R}}_1(t),\Phi(t)\rangle dt \\ e^{i(\alpha s+\beta(s))}\displaystyle\int_{-i\infty}^{s}\frac{e^{-i(\alpha t+\beta(t))}}{t^{\mathrm{d}}(1+h_0t^{-1})^{\mathrm{d}}}\langle\tilde{\mathcal{R}}_2(t),\Phi(t)\rangle dt \end{pmatrix}$$

and the function:

$$\Delta\Psi_{0,0}(s) = s^{\mathrm{d}}(1+h_0s^{-1})^{\mathrm{d}}\begin{pmatrix}\kappa_0e^{-i(\alpha s+\beta(s))}\\ 0\end{pmatrix}.$$

Then we can rewrite (2.87) and (2.88) in the compact form:

$$\Delta\Psi_0(s) = \Delta\Psi_{0,0}(s)+\mathcal{G}(\Delta\Psi_0)(s). \tag{2.89}$$

Adapting the steps followed in [BS08], one can see that the operator $\mathrm{Id}-\mathcal{G}$ is invertible in a suitable Banach space, and therefore we can write:

$$\Delta\Psi_0 = (\mathrm{Id}-\mathcal{G})^{-1}(\Delta\Psi_{0,0}) = \sum_{n\geq 0}\mathcal{G}^n(\Delta\Psi_{0,0}). \tag{2.90}$$

The last step, once we know that $\Delta\Psi_0$ can be obtained form formula (2.90), is to study how the operator $\mathcal{G}$ and its iterates $\mathcal{G}^n$ act on $\Delta\Psi_{0,0}$. What one can prove is that there exists some constant $\overline{K}(\rho)$ such that:

$$\pi_1\mathcal{G}(\Delta\Psi_{0,0})(s) = s^de^{-i(\alpha s+\beta(s))}(\overline{K}(\rho)+\mathcal{O}(s^{-1})).$$

and that:

$$\pi_2\mathcal{G}(\Delta\Psi_{0,0})(s) = \mathcal{O}\left(s^{d-2}e^{-i(\alpha s+\beta(s))}\right).$$

Using standard functional analysis, formula (2.30) for $\Delta\Psi_0$ is found, finishing the proof of Theorem 2.1.15.

2.5 Proof of Theorem 2.1.17

Theorem 2.1.13 provides parameterizations of the invariant manifolds satisfying the same equation (2.21). Nevertheless, it does not give enough information about the behavior of these manifolds near the singularities $\pm i\pi/2$. To obtain this information we will use the solutions $\Psi_0^{\mathrm{u,s}}$ of the inner equation (2.27) given in Theorem 2.1.15. For this reason, in this section we will deal not with system (2.21) but with (2.26) (which comes from (2.21) after a change of variables). Moreover, we will restrict ourselves to the matching domains $D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$ and $D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$ (see Figure 2.3), or more precisely to these domains in the inner variables, that is $\mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$ and $\mathcal{D}^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$ (see the definition (2.34)).

Let us consider the Banach space:

$$\mathcal{X}^{\mathrm{mch},*} = \{\phi : \mathcal{D}^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2} \to \mathbb{C},\ \phi \text{ analytic}, \sup_{s\in\mathcal{D}^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}} |s|^2|\phi(s)| < \infty\}, \qquad * = \mathrm{u,s}$$

with the norm:

$$\|\phi\|^{\mathrm{u,s}}_{\mathrm{mch}} = \sup_{s\in\mathcal{D}^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}} |s|^2|\phi(s)|,$$

and we endow the product space $\mathcal{X}^{\mathrm{mch},*} \times \mathcal{X}^{\mathrm{mch},*}$ with the norm:

$$\|(\phi_1,\phi_2)\|^{\mathrm{u,s}}_{\mathrm{mch},\times} = \|\phi_1\|^{\mathrm{u,s}}_{\mathrm{mch}} + \|\phi_2\|^{\mathrm{u,s}}_{\mathrm{mch}}.$$

Now we present the main result of this section, which is equivalent to Theorem 2.1.17:

Proposition 2.5.1. *Let $\Psi^{\mathrm{u,s}}(s) = \delta\varphi^{\mathrm{u,s}}(\delta s + i\pi/2)$, where $\varphi^{\mathrm{u,s}}$ are the parameterizations given by Theorem 2.1.13. If $s \in \mathcal{D}^{\mathrm{mch},*}_{\kappa,\beta_1,\beta_2}$, for $* = \mathrm{u,s}$, one has $\Psi^{\mathrm{u,s}}(s) = \Psi_0^{\mathrm{u,s}}(s) + \Psi_1^{\mathrm{u,s}}(s)$, where $\Psi_0^{\mathrm{u,s}}$ are the two solutions of the inner system* (2.27) *given by Theorem 2.1.15 and:*

$$\|\Psi_1^{\mathrm{u,s}}\|^{\mathrm{u,s}}_{\mathrm{mch},\times} \le K\delta^{1-\gamma},$$

for some constant K.

Now we shall proceed to prove Proposition 2.5.1 for the unstable case. The stable case is analogous. As usual, we will omit the superindices −u− of the domain $\mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$, the Banach space $\mathcal{X}^{\mathrm{mch}}$ and the norm $\|.\|_{\mathrm{mch}}$, whenever there is no danger of confusion.

Before proceeding, we will explain the main steps to prove Proposition 2.5.1.

2.5.1 Notation and outline of the proof of Proposition 2.5.1

First of all, let us introduce some notation. We will call $\Psi = (\psi, \overline{\psi})$ the solutions of (2.26). Recalling the definitions (2.25) of $l(\delta s)$ and $m(\delta s)$, (2.79) of $\tilde{h}(\Psi, s)$ and (2.80) of $\tilde{\mathcal{A}}(s)$, we define:

$$
\begin{aligned}
h(\Psi,s,\delta,\sigma) &= \left[b\psi\overline{\psi}+H(\Psi,-s^{-1}+\delta l(\delta s),\delta,\delta\sigma)\right](s^2+\delta s^3 m(\delta s)), && (2.91)\\
\mathcal{A}(s,\delta,\sigma) &= \begin{pmatrix} a_1(s,\delta,\sigma) & 0\\ 0 & a_2(s,\delta,\sigma)\end{pmatrix} && (2.92)\\
X_0(\Psi,s) &= \frac{1}{1+\tilde{h}(\Psi,s)}\left[\tilde{\mathcal{A}}(s)\Psi+F(\Psi,-s^{-1},0,0)\right], && (2.93)\\
X_1(\Psi,s,\delta,\sigma) &= \frac{1}{1+h(\Psi,s,\delta,\sigma)}\left[\mathcal{A}(s,\delta,\sigma)\Psi+F(\Psi,-s^{-1}+\delta l(\delta s),\delta,\delta\sigma)\right]\\
&\quad -\frac{1}{1+\tilde{h}(\Psi,s)}\left[\tilde{\mathcal{A}}(s)\Psi+F(\Psi,-s^{-1},0,0)\right], && (2.94)
\end{aligned}
$$

where:

$$
\begin{aligned}
a_1(s,\delta,\sigma) &= -(\alpha+c(-s^{-1}+\delta l(\delta s))i-\delta\sigma+\mathrm{d}s^{-1}-\delta\mathrm{d}l(\delta s),\\
a_2(s,\delta,\sigma) &= (\alpha+c(-s^{-1}+\delta l(\delta s))i-\delta\sigma+\mathrm{d}s^{-1}-\delta\mathrm{d}l(\delta s)
\end{aligned} \qquad (2.95)
$$

Note that $\mathcal{A}(s,0,\sigma)=\tilde{\mathcal{A}}(s)$, and $h(\Psi,s,0,\sigma)=\tilde{h}(\Psi,s)$.

Then, the full system (2.26) can be written as:

$$
\frac{d\Psi}{ds}=X_0(\Psi,s)+X_1(\Psi,s,\delta,\sigma), \qquad (2.96)
$$

and the inner system (2.27) reads:

$$
\frac{d\Psi}{ds}=X_0(\Psi,s). \qquad (2.97)
$$

Let us consider Ψ^{u} defined as the parameterization of the one-dimensional unstable manifold of system (2.21) given by Theorem 2.1.13 in the inner variables, that is $\Psi^{\mathrm{u}}(s)=\delta\varphi^{\mathrm{u}}(\delta s+i\pi/2)$, which is a solution of (2.96). Moreover, consider the solution Ψ_0^{u} of the inner system (2.97) given by Theorem 2.1.15. Then, if we define their difference:

$$
\Psi_1^{\mathrm{u}}=\Psi^{\mathrm{u}}-\Psi_0^{\mathrm{u}}, \qquad (2.98)
$$

we have that Ψ_1^{u} satisfies:

$$
\begin{aligned}
\frac{d\Psi_1^{\mathrm{u}}}{ds} &= X_0(\Psi_0^{\mathrm{u}}+\Psi_1^{\mathrm{u}},s)+X_1(\Psi_0^{\mathrm{u}}+\Psi_1^{\mathrm{u}},s,\delta,\delta\sigma)-X_0(\Psi_0^{\mathrm{u}},s)\\
&= \frac{1}{1+h_0 s^{-1}}\tilde{\mathcal{A}}(s)\Psi_1^{\mathrm{u}}+\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)(s) && (2.99)
\end{aligned}
$$

where h_0 was defined in (2.78) and:

$$\begin{aligned}\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)(s) &= X_0(\Psi_0^{\mathrm{u}}+\Psi_1^{\mathrm{u}},s)-X_0(\Psi_0^{\mathrm{u}},s)-D_\Psi X_0(\Psi_0^{\mathrm{u}},s)\Psi_1^{\mathrm{u}}\\ &\quad +X_1(\Psi_0^{\mathrm{u}}+\Psi_1^{\mathrm{u}},s,\delta,\sigma)+\left[\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}},s)}-\frac{1}{1+h_0s^{-1}}\right]\tilde{\mathcal{A}}(s)\Psi_1^{\mathrm{u}}\\ &\quad +\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}},s)}D_\Psi F(\Psi_0^{\mathrm{u}},s^{-1},0,0)\Psi_1^{\mathrm{u}}\\ &\quad +D_\Psi\left[\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}},s)}\right](\tilde{\mathcal{A}}(s)\Psi_0^{\mathrm{u}}+F(\Psi_0^{\mathrm{u}},s^{-1},0,0))\Psi_1^{\mathrm{u}}. \qquad (2.100)\end{aligned}$$

Now consider the linear operator acting on functions $(\phi_1,\phi_2)\in\mathcal{X}^{\mathrm{mch}}\times\mathcal{X}^{\mathrm{mch}}$:

$$\mathcal{L}(\phi_1,\phi_2)(s)=\mathcal{M}(s)\begin{pmatrix}\displaystyle\int_{\Gamma(s_1,s)} m_1^{-1}(w)\phi_1(w)dw\\ \displaystyle\int_{\Gamma(s_2,s)} m_2^{-1}(w)\phi_2(w)dw\end{pmatrix}, \qquad (2.101)$$

where the matrix $\mathcal{M}(s)$ was defined in (2.83), s_i, $i=1,2$, were defined in (2.35) and $\Gamma(s_i,s)$ is any curve in $\mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$ going from s_i to s. Note that, since for $w\in\mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$ the functions $m_1(w)^{-1}\phi_1(w)$ and $m_2(w)^{-1}\phi_2(w)$ are analytic, by Cauchy's theorem the integrals in (2.101) do not depend on the choice of the curves $\Gamma(s_i,s)$.

Note that, since $\Psi_1^{\mathrm{u}}=\Psi^{\mathrm{u}}-\Psi_0^{\mathrm{u}}$, and the existence of Ψ^{u} and Ψ_0^{u} has already been proved, we can think of $\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)(s)$ as a function of s, that is:

$$\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)(s)=\tilde{\mathcal{R}}(s,\delta,\sigma).$$

Then, taking into account that the matrix $\tilde{\mathcal{A}}(s)$ is diagonal, system (2.99) can be written as the uncoupled system:

$$\frac{d\psi_1^{\mathrm{u}}}{ds}=\frac{1}{1+h_0s^{-1}}a_1(s,\delta,\sigma)\psi_1^{\mathrm{u}}+\tilde{\mathcal{R}}_1(s,\delta,\sigma),$$

$$\frac{d\overline{\psi}_1^{\mathrm{u}}}{ds}=\frac{1}{1+h_0s^{-1}}a_2(s,\delta,\sigma)\overline{\psi}_1^{\mathrm{u}}+\tilde{\mathcal{R}}_2(s,\delta,\sigma).$$

Then, ψ_1^{u} and $\overline{\psi}_1^{\mathrm{u}}$ are uniquely determined as solutions of the corresponding equation with an initial condition at a certain $s=s^*$. Since the equations of ψ_1 and $\overline{\psi}_1$ are uncoupled, we can take different s^* for each one. For ψ_1 we take $s^*=s_1$ and for $\overline{\psi}_1$ we take $s^*=s_2$. Then, using the notation (2.101), one can see that Ψ_1^{u} satisfies the fixed point equation:

$$\Psi_1^{\mathrm{u}}(s)=\mathcal{I}(c_1,c_2)(s)+\mathcal{L}\circ\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)(s), \qquad (2.102)$$

being:

$$\mathcal{I}(k_1,k_2)(s)=\mathcal{M}(s)\begin{pmatrix} k_1\\ k_2\end{pmatrix} \tag{2.103}$$

and

$$c_1=m_1^{-1}(s_1)\psi_1(s_1), \qquad c_2=m_2^{-1}(s_2)\overline{\psi}_1(s_2). \tag{2.104}$$

Now, let us explain the main steps to prove Proposition 2.5.1. First of all, we note that the fixed point equation (2.102) is equivalent to:

$$\Psi_1^{\mathrm{u}} = \mathcal{I}(c_1,c_2)+\mathcal{L}\circ\mathcal{R}(0,\delta,\sigma)+\mathcal{L}\circ[\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)-\mathcal{R}(0,\delta,\sigma)], \tag{2.105}$$

where:

$$\mathcal{L}\circ[\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)-\mathcal{R}(0,\delta,\sigma)]=\mathcal{M}(s)\begin{pmatrix} \int_{\Gamma(s_1,s)} m_1^{-1}(w)[\mathcal{R}_1(\Psi_1^{\mathrm{u}},\delta,\sigma)(w)-\mathcal{R}_1(0,\delta,\sigma)(w)]dw \\ \int_{\Gamma(s_2,s)} m_2^{-1}(w)[\mathcal{R}_2(\Psi_1^{\mathrm{u}},\delta,\sigma)(w)-\mathcal{R}_2(0,\delta,\sigma)(w)]dw \end{pmatrix}.$$

Note that:

$$\begin{aligned}\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)(w)-\mathcal{R}(0,\delta,\sigma)(w) &= \int_0^1 D_\Psi\mathcal{R}(\lambda\Psi_1^{\mathrm{u}},\delta,\sigma)(w)d\lambda\,\Psi_1^{\mathrm{u}}(w)\\ &= \int_0^1 D_\Psi\mathcal{R}(\lambda(\Psi^{\mathrm{u}}-\Psi_0^{\mathrm{u}}),\delta,\sigma)(w)d\lambda\,\Psi_1^{\mathrm{u}}(w).\end{aligned}$$

Now, since we already proved the existence of both parameterizations Ψ^{u} and Ψ_0^{u}, we can think of the integral term as independent of Ψ_1^{u}, that is:

$$\mathcal{R}(\Psi_1^{\mathrm{u}},\delta,\sigma)(w)-\mathcal{R}(0,\delta,\sigma)(w)=B(w)\Psi_1^{\mathrm{u}}(w),$$

where the matrix $B(w)$ is given by:

$$B(w)=\int_0^1 D_\Psi\mathcal{R}(\lambda(\Psi^{\mathrm{u}}-\Psi_0^{\mathrm{u}}),\delta,\sigma)(w)d\lambda.$$

Therefore, for $\Psi\in\mathcal{X}^{\mathrm{mch}}\times\mathcal{X}^{\mathrm{mch}}$, we can define the linear operators:

$$\mathcal{B}(\Psi)(w)=B(w)\Psi(w), \qquad \mathcal{G}(\Psi)(s)=\mathcal{L}\circ\mathcal{B}(\Psi)(s), \tag{2.106}$$

and then equation (2.102) can be rewritten as:

$$(\mathrm{Id}-\mathcal{G})\,\Psi_1^{\mathrm{u}}=\mathcal{I}(c_1,c_2)+\mathcal{L}\circ\mathcal{R}(0,\delta,\sigma). \tag{2.107}$$

We will proceed to study this equation as follows. First, in Subsections 2.5.2 and 2.5.3 we will study the linear operators $\mathcal{L}$ and $\mathcal{B}$ respectively. Then, in Subsection 2.5.4 we will study the independent term of (2.107), that is $\mathcal{I}(c_1,c_2)+\mathcal{L}\circ\mathcal{R}(0,\delta,\sigma)$. Finally, in Subsection 2.5.5 we will see that joining the results of the previous subsections allows us to guarantee that the operator $\mathrm{Id}-\mathcal{G}$ is invertible in $\mathcal{X}^{\mathrm{mch}}\times\mathcal{X}^{\mathrm{mch}}$ and to obtain the desired bound for the norm of Ψ_1^{u}.

2.5.2 The linear operator $\mathcal{L}$

As we already mentioned, in this subsection we will study the operator $\mathcal{L}$. However, before we present two technical lemmas. The first one is completely analogous to Lemma 2.2.10, and can be proved in the same way.

Lemma 2.5.2. *Let $s \in \mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$ and $w = s_1 + t(s - s_1)$, $\tilde{w} = s_2 + t(s - s_2)$, with $t \in [0,1]$. Then there exists $K \neq 0$ independent of δ and σ such that:*

$$|w|, |\tilde{w}| \geq K|s|.$$

Lemma 2.5.3. *Let $s \in \mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$ and $w = s_1 + t(s - s_1)$, $\tilde{w} = s_2 + t(s - s_2)$, with $t \in [0,1]$. Then there exists K independent of δ and σ such that:*

$$|m_1(s) m_1^{-1}(w)| \leq K e^{\alpha(1-t)\mathrm{Im}\,(s-s_1)}, \qquad |m_2(s) m_2^{-1}(\tilde{w})| \leq K e^{\alpha(1-t)\mathrm{Im}\,(s_2-s)},$$

where m_1 and m_2 are defined in (2.83).

Proof. We will do just the case for w. We have:

$$m_1(s) m_1^{-1}(w) = \frac{s^{\mathrm{d}}(1 + h_0 s^{-1})^{\mathrm{d}}}{w^{\mathrm{d}}(1 + h_0 w^{-1})^{\mathrm{d}}} e^{-i[\alpha(s-w) + \beta(s) - \beta(w)]}.$$

First of all note that by Lemma 2.5.2 we have that:

$$\left| \frac{s^{\mathrm{d}}(1 + h_0 s^{-1})^{\mathrm{d}}}{w^{\mathrm{d}}(1 + h_0 w^{-1})^{\mathrm{d}}} \right| \leq K \left| \frac{1 + h_0 s^{-1}}{1 + h_0 w^{-1}} \right|^{\mathrm{d}}.$$

Moreover, for δ small enough we have that $|s|, |w| \geq K \log(1/\delta) \geq 2h_0$ and hence:

$$\left| \frac{s^{\mathrm{d}}(1 + h_0 s^{-1})^{\mathrm{d}}}{w^{\mathrm{d}}(1 + h_0 w^{-1})^{\mathrm{d}}} \right| \leq K \frac{(1 + |h_0 s^{-1}|)^{\mathrm{d}}}{(1 - |h_0 w^{-1}|)^{\mathrm{d}}} \leq K. \tag{2.108}$$

On the other hand, we have that:

$$\left| e^{-i[\alpha(s-w) + \beta(s) - \beta(w)]} \right| \leq e^{\alpha \mathrm{Im}\,(s-w)} e^{|\mathrm{Im}\,\beta(s)| + |\mathrm{Im}\,\beta(w)|}.$$

Recall that $\beta(s) = -(c + \alpha h_0)\log(s + h_0)$ and therefore $\mathrm{Im}\,\beta(s) = -(c + \alpha h_0)\arg(s + h_0)$, obtaining for $\mathrm{Im}\,\beta(w)$ an analogous expression. It is clear that for $s \in \mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$ we have $\mathrm{Im}\, s \leq \mathrm{Im}\, s_1 < 0$, and then, since h_0 is real, we also have that $\mathrm{Im}\,(s + h_0) < 0$. Consequently, we have $\arg(s + h_0) \in (\pi, 2\pi)$ and hence:

$$|\mathrm{Im}\,\beta(s)|, |\mathrm{Im}\,\beta(w)| \leq (c + \alpha|h_0|)2\pi.$$

Then it is clear that:

$$\left| e^{-i[\alpha(s-w) + \beta(s) - \beta(w)]} \right| \leq e^{\alpha \mathrm{Im}\,(s-w)} e^{4\pi(c + \alpha|h_0|)} = e^{\alpha(1-t)\mathrm{Im}\,(s-s_1)} e^{4\pi(c + \alpha|h_0|)}. \tag{2.109}$$

In conclusion, from (2.108) and (2.109) we obtain the initial statement. □

The following lemma studies how the linear operator $\mathcal{L}$ acts on functions belonging to $\mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$.

Lemma 2.5.4. *The operator* $\mathcal{L} : \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}} \to \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$ *is well defined and there exists a constant* K *such that for any* $\phi \in \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$*, then:*

$$\|\mathcal{L} \circ \phi\|_{\mathrm{mch},\times} \leq K \|\phi\|_{\mathrm{mch},\times}.$$

Proof. We will check the bound for the first component. We have:

$$\pi^1 \mathcal{L} \circ \phi(s) = m_1(s) \int_{\Gamma(s_1,s)} m_1^{-1}(w)\phi_1(w)dw.$$

Taking $\Gamma(s_1, s)$ as the segment from s_1 to s and parameterizing it by $\gamma(t) = s_1 + t(s - s_1)$, $t \in [0, 1]$, we have:

$$\begin{aligned} |\pi^1 \mathcal{L} \circ \phi(s)| &\leq |s - s_1| \int_0^1 |m_1(s)m_1^{-1}(s_1 + t(s - s_1))\phi_1(s_1 + t(s - s_1))|dt \\ &\leq K|s_1 - s|\|\phi_1\|_{\mathrm{mch}} \int_0^1 |m_1(s)m_1^{-1}(s_1 + t(s - s_1))||s_1 + t(s - s_1)|^{-2}dt. \end{aligned}$$

Using Lemmas 2.5.2 and 2.5.3 it is clear that:

$$\begin{aligned} |\pi^1 \mathcal{L} \circ \phi(s)| &\leq K|s_1 - s|\|\phi_1\|_{\mathrm{mch}}|s|^{-2} \left| \int_0^1 e^{\alpha(1-t)\mathrm{Im}\,(s-s_1)}dt \right| \\ &= \frac{K|s - s_1|}{\alpha|\mathrm{Im}\,(s_1 - s)|}\|\phi_1\|_{\mathrm{mch}}|s|^{-2} \left|1 - e^{\alpha \mathrm{Im}\,(s-s_1)}\right|. \end{aligned}$$

Finally, we note that as $\mathrm{Im}\,(s - s_1) \leq 0$ we have that $\left|1 - e^{\alpha \mathrm{Im}\,(s-s_1)}\right| \leq 1$. Moreover, from the definition of $\mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$, using standard geometric arguments, it is easy to see that there exists a constant $C(\beta_1, \beta_2)$ such that $|s - s_1| \leq C(\beta_1, \beta_2)\,|\mathrm{Im}\, s - \mathrm{Im}\, s_1|$. Then it is clear that $|\pi^1 \mathcal{L} \circ \phi(s)| \leq K\|\phi_1\|_{\mathrm{mch}}|s|^{-2}$, and consequently $\|\pi^1 \mathcal{L} \circ \phi\|_{\mathrm{mch}} \leq K\|\phi_1\|_{\mathrm{mch}}$. □

2.5.3 The linear operator $\mathcal{B}$

Now we proceed to study the operator $\mathcal{B}$, defined in (2.106). However, before we will need to study the vector field X_1.

Lemma 2.5.5. *Consider the vector field* X_1 *defined in* (2.94)*, and let* $\Psi \in \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$*, such that* $\|\Psi\|_{\mathrm{mch},\times} \leq 1$*. Then there exists a constant* K *such that for all* $s \in \mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$*:*

$$|X_1(\Psi, s, \delta, \sigma)| \leq K\delta|s|^{-2}.$$

Proof. First of all we will rewrite X_1, which was defined in (2.94), in a more convenient way:

$$X_1(\Psi,s,\delta,\sigma) = \left[\frac{1}{1+h(\Psi,s,\delta,\sigma)} - \frac{1}{1+\tilde{h}(\Psi,s)}\right]\left[\mathcal{A}(s,\delta,\sigma)\Psi + F(\Psi,-s^{-1}+\delta l(\delta s),\delta,\sigma)\right]$$
$$+\frac{1}{1+\tilde{h}(\Psi,s)}\left[(\mathcal{A}(s,\delta,\sigma)-\tilde{\mathcal{A}}(s))\Psi + F(\Psi,-s^{-1}+\delta l(\delta s),\delta,\sigma) - F(\Psi,-s^{-1},0,0)\right],$$

where $l(\delta s)$ was defined in (2.25), $\tilde{h}(\Psi,s)$ in (2.79), $\tilde{\mathcal{A}}(s)$ in (2.80), $h(\Psi,s,\delta,\sigma)$ in (2.91) and $\mathcal{A}(s,\delta,\sigma)$ in (2.92). In the following we shall bound each term.

Our first claim is that:

$$\left|\left[\frac{1}{1+h(\Psi,s,\delta,\sigma)} - \frac{1}{1+\tilde{h}(\Psi,s)}\right]\right| \le K\delta. \tag{2.110}$$

First of all, note that by Remark 2.1.12 and the fact that $|\Psi(s)| \le \|\Psi\|_{\mathrm{mch},\times}|s|^{-2}$ we have:

$$\begin{aligned} |h(\Psi,s,\delta,\sigma)| &\le \left(b\|\Psi\|_{\mathrm{mch},\times}|s|^{-4} + K|s|^{-3}\right)\left(|s|^2 + K\delta^2|s^4|\right) \\ &\le K(|s|^{-2}+|s|^{-1}+\delta^2+\delta^2|s|) \\ &\le K\left(\frac{1}{\log^2(1/\delta)}+\frac{1}{\log(1/\delta)}+\delta^2+\delta^{1+\gamma}\right) \le \frac{1}{2}. \end{aligned} \tag{2.111}$$

Note that this bound is also valid for $\tilde{h}(\Psi,s) = h(\Psi,-s^{-1},0,0)$. Then, by Lemma 2.2.4 we obtain:

$$\begin{aligned} &\left|\left[\frac{1}{1+h(\Psi,s,\delta,\sigma)} - \frac{1}{1+\tilde{h}(\Psi,s)}\right]\right| \le 4|h(\Psi,s,\delta,\sigma)-\tilde{h}(\Psi,s)| \\ \le\ & 4\left|b\psi\overline{\psi} + H(\Psi,-s^{-1}+\delta l(\delta s),\delta,\delta\sigma)\right||\delta s^3 m(\delta s)| \\ &+4\left|H(\Psi,-s^{-1}+\delta l(\delta s),\delta,\delta\sigma) - H(\Psi,-s^{-1},0,0)\right||s|^2. \end{aligned} \tag{2.112}$$

Now, on one hand, we have:

$$\left|b\psi\overline{\psi} + H(\Psi,-s^{-1}+\delta l(\delta s),\delta,\delta\sigma)\right||\delta s^3 m(\delta s)| \le K\left(|s|^{-4}+|s^{-3}|\right)\delta^2|s|^4 \le K\delta. \tag{2.113}$$

On the other hand, note that:

$$\begin{aligned} &\left|H(\Psi,-s^{-1}+\delta l(\delta s),\delta,\delta\sigma) - H(\Psi,-s^{-1},0,0)\right| \\ &\le |(\delta l(\delta s),\delta,\delta\sigma)|\int_0^1 |\partial_{(z,\delta,\sigma)}H(\Psi,-s^{-1}+\lambda\delta l(\delta s),\lambda\delta,\lambda\delta\sigma)|d\lambda. \end{aligned}$$

Since for $\lambda\in[0,1]$ and for δ small enough one has $\phi = (\Psi,-s^{-1}+\lambda\delta l(\delta s),\lambda\delta,\lambda\delta\sigma) \in B^3(r_0/2)\times B(\delta_0/2)\times B(\sigma_0/2)$, from Remark 2.1.12 and applying again Lemma 2.2.5 (with

$\phi^* = 0$) we can bound all the derivatives of H by $K|\phi|^2$, and then it is straightforward to see that:

$$\begin{aligned}
&\left|H(\Psi, -s^{-1} + \delta l(\delta s), \delta, \delta\sigma) - H(\Psi, -s^{-1}, 0, 0)\right| \\
&\leq \int_0^1 K|(\Psi, -s^{-1} + \lambda\delta l(\delta s), \lambda\delta, \lambda\delta\sigma)|^2 d\lambda \cdot |(\delta l(\delta s), \delta, \delta\sigma)| \leq K|s|^{-2}|(\delta l(\delta s), \delta, \delta\sigma)| \\
&\leq K\delta|s|^{-2},
\end{aligned}$$

where we have used that $|\Psi(s)| \leq K|s|^{-2}$. Hence it is clear that:

$$\left|H(\Psi, -s^{-1} + \delta l(\delta s), \delta, \delta\sigma) - H(\Psi, -s^{-1}, 0, 0)\right| |s|^2 \leq K\delta \tag{2.114}$$

Substituting (2.113) and (2.114) in inequality (2.112), claim (2.110) is proved.

Our second claim is that:

$$|\mathcal{A}(s, \delta, \sigma)\Psi + F(\Psi, -s^{-1} + \delta l(\delta s), \delta, \sigma)| \leq K|s|^{-2}. \tag{2.115}$$

This is straightforward to check, since the matrix $\mathcal{A}(s, \delta, \sigma)$ is bounded for $s \in \mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$ (which is clear from (2.92) and (2.95)), $\Psi \in \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$ and $|F(\Psi, -s^{-1} + \delta l(\delta s), \delta, \sigma)| \leq K|s|^{-3}$.

Our third claim is that, since as we already mentioned $|\tilde{h}(\Psi, s)| \leq 1/2$, then:

$$\left|\frac{1}{1 + \tilde{h}(\Psi, s)}\right| \leq 2. \tag{2.116}$$

The last claim is that:

$$|(\mathcal{A}(s, \delta, \sigma) - \tilde{\mathcal{A}}(s))\Psi + F(\Psi, -s^{-1} + \delta l(\delta s), \delta, \sigma) - F(\Psi, -s^{-1}, 0, \sigma)| \leq K\delta|s|^{-2}. \tag{2.117}$$

First, we note that:

$$\mathcal{A}(s, \delta, \sigma) - \tilde{\mathcal{A}}(s) = \begin{pmatrix} (-1 - ic)\delta l(\delta s) - \delta\sigma & 0 \\ 0 & (-1 + ic)\delta l(\delta s) - \delta\sigma \end{pmatrix}$$

and since $|\delta l(\delta s)| = \mathcal{O}(\delta^{1+\gamma})$, it is clear that:

$$|(\mathcal{A}(s, \delta, \sigma) - \tilde{\mathcal{A}}(s))\Psi| \leq K\delta\|\Psi\|_{\mathrm{mch},\times}|s|^{-2}.$$

On the other hand, using the mean value theorem and Lemma 2.2.5 it is also easy to see that:

$$\begin{aligned}
&|F(\Psi, -s^{-1} + \delta l(\delta s), \delta, \sigma) - F(\Psi, -s^{-1}, 0, \sigma)| \\
\leq \;& \int_0^1 |D_{(z,\delta)}F(\Psi, -s^{-1} + t\delta l(\delta s), t\delta, \sigma)|dt|(\delta l(\delta s), \delta)| \leq K\delta|s|^{-2},
\end{aligned}$$

so inequality (2.117) is clear.

In conclusion, from bounds (2.110), (2.115), (2.116) and (2.117) we obtain:

$$|X_1(\Psi, s, \delta, \sigma)| \leq K\delta |s|^{-2}$$

□

Now we can proceed to study how the operator $\mathcal{B}$ acts on $\mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$.

Lemma 2.5.6. *If $\gamma \in (0,1)$ and $\Psi \in \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$, with $\|\Psi\|_{\mathrm{mch},\times} \leq 1$, then $\mathcal{B}(\Psi) \in \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$ and there exists a constant K such that:*

$$\|\mathcal{B}(\Psi)\|_{\mathrm{mch},\times} \leq \frac{K}{\log^2(1/\delta)}.$$

Proof. Recall that:

$$\begin{aligned} \mathcal{R}(\Psi, \delta, \sigma)(w) \;=\; & X_0(\Psi_0^{\mathrm{u}} + \Psi, w) - X_0(\Psi_0^{\mathrm{u}}, w) - D_\Psi X_0(\Psi_0^{\mathrm{u}}, w)\Psi \\ & + X_1(\Psi_0^{\mathrm{u}} + \Psi, w, \delta, \sigma) + \left[\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)} - \frac{1}{1+h_0 w}\right]\tilde{\mathcal{A}}(w)\Psi \\ & + \frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)} D_\Psi F(\Psi_0^{\mathrm{u}}, w^{-1}, 0, 0)\Psi \\ & + D_\Psi\left[\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)}\right](\tilde{\mathcal{A}}(w)\Psi_0^{\mathrm{u}} + F(\Psi_0^{\mathrm{u}}, w^{-1}, 0, 0))\Psi, \end{aligned}$$

and hence:

$$\begin{aligned} D_\Psi \mathcal{R}(\Psi, \delta, \sigma)(w) \;=\; & D_\Psi X_0(\Psi_0^{\mathrm{u}} + \Psi, w) - D_\Psi X_0(\Psi_0^{\mathrm{u}}, w) + D_\Psi X_1(\Psi_0^{\mathrm{u}} + \Psi, w, \delta, \sigma) \\ & + \left[\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)} - \frac{1}{1+h_0 w}\right]\tilde{\mathcal{A}}(w) \\ & + \frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)} D_\Psi F(\Psi_0^{\mathrm{u}}, w^{-1}, 0, 0) \\ & + D_\Psi\left[\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)}\right](\tilde{\mathcal{A}}(w)\Psi_0^{\mathrm{u}} + F(\Psi_0^{\mathrm{u}}, w^{-1}, 0, 0)). \end{aligned}$$

We will see that, for $w \in D^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$:

$$|D_\Psi \mathcal{R}(\Psi, \delta, \sigma)(w)| \leq \frac{K}{\log^2(1/\delta)}. \tag{2.118}$$

First of all we claim that:

$$|D_\Psi X_0(\Psi_0^{\mathrm{u}} + \Psi, w) - D_\Psi X_0(\Psi_0^{\mathrm{u}}, w)| \leq \frac{K}{\log^2(1/\delta)}. \tag{2.119}$$

This can be shown using the mean value theorem in each column of the matrix $D_\Psi X_0$. For example, we will prove the result for the first one, $D_\psi X_0$. Writing $\Psi_\lambda = \Psi_0^{\mathrm{u}} + \lambda\Psi$, the mean value theorem gives us the following bound:

$$\begin{aligned}
|D_\psi X_0(\Psi_0^{\mathrm{u}}+\Psi,w) - D_\psi X_0(\Psi_0^{\mathrm{u}},w)| &\leq \int_0^1 |D_\Psi D_\psi X_0(\Psi_\lambda,w)|\,d\lambda|\Psi(w)| \\
&\leq \int_0^1 |D_\Psi D_\psi X_0(\Psi_\lambda,w)|\,d\lambda\|\Psi\|_{\mathrm{mch},\times}|w|^{-2} \\
&\leq \int_0^1 |D_\Psi D_\psi X_0(\Psi_\lambda,w)|\,d\lambda\frac{K\|\Psi\|_{\mathrm{mch},\times}}{\log^2(1/\delta)}.
\end{aligned} \tag{2.120}$$

Then it is clear that in order to prove (2.119) it is only necessary to prove that the integral is bounded, or equivalently, that the integrand $D_\Psi D_\psi X_0(\Psi_\lambda,w)$ (which is a 2×2 matrix) is bounded for $\lambda\in[0,1]$. Note that, from definition (2.93) of X_0, fixing $w\in D^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}$ it is clear that $X_0(\phi,w)$ is bounded and analytic if:

$$\phi\in B^2(\tilde r_0)\subset B^2(r_0)\cap\{\phi\in\mathbb{C}^2\,:\,|\tilde h(\phi,w)|<1/2\},$$

for some $\tilde r_0$. Then, Cauchy's theorem implies that the derivatives of X_0 with respect to ψ and $\overline{\psi}$ are bounded. Hence, using again the same arguments, we prove that all the derivatives of order two are also bounded. Since for $\Psi\in\mathcal{X}^{\mathrm{mch},}\times\mathcal{X}^{\mathrm{mch},}$ we have that $\lambda\Psi(w)\in B^2(\tilde r_0/2)$ for δ small enough and $\lambda\in[0,1]$, we obtain that $D_\Psi D_\psi X_0(\Psi_\lambda,w)$ is bounded and thus (2.119) is proved.

Our next step will be to prove that:

$$|D_\Psi X_1(\Psi_0^{\mathrm{u}}+\Psi,w,\delta,\sigma)|\leq K\delta\leq\frac{K}{\log^2(1/\delta)}. \tag{2.121}$$

In fact, we will prove the result just for the derivative with respect to ψ, being the one with respect to $\overline{\psi}$ analogous. Note that if $\Psi+\Psi_0^{\mathrm{u}} = (\psi+\psi_0^{\mathrm{u}},\overline{\psi}+\overline{\psi}_0^{\mathrm{u}})\in\mathcal{X}^{\mathrm{mch}}\times\mathcal{X}^{\mathrm{mch}}$, then $(\psi+\psi_0^{\mathrm{u}}+|w|^{-2}e^{i\theta},\overline{\psi}+\overline{\psi}_0^{\mathrm{u}})\in\mathcal{X}^{\mathrm{mch}}\times\mathcal{X}^{\mathrm{mch}}$ too. Then first using Cauchy's theorem and later Lemma 2.5.5, we have:

$$\begin{aligned}
|D_\psi X_1(\Psi_0^{\mathrm{u}}+\Psi,w,\delta,\sigma)| &\leq \frac{1}{2\pi|w|^{-2}}\int_0^{2\pi}|X_1(\psi+\psi_0^{\mathrm{u}}+|w|^{-2}e^{i\theta},\overline{\psi}+\overline{\psi}_0^{\mathrm{u}},\delta,\sigma)|d\theta \\
&\leq \frac{1}{2\pi|w|^{-2}}\int_0^{2\pi}K\delta|w|^{-2}d\theta = K\delta,
\end{aligned}$$

and the claim is proved.

Now we claim that:

$$\left|\left[\frac{1}{1+\tilde h(\Psi_0^{\mathrm{u}},w)}-\frac{1}{1+h_0w^{-1}}\right]\tilde{\mathcal{A}}(w)\right|\leq K|w|^{-2}\leq\frac{K}{\log^2(1/\delta)}. \tag{2.122}$$

Indeed, on the one hand, note that $\tilde{\mathcal{A}}(w)$ is bounded. On the other hand, we observe that for δ small enough:

$$|\tilde{h}(\Psi_0^{\mathrm{u}}, w)| \leq \frac{1}{2}, \qquad |h_0 w|^{-1} \leq \frac{K}{\log(1/\delta)} \leq \frac{1}{2},$$

and then by Lemma 2.2.4 and definition (2.78) of h_0 we obtain that:

$$\left|\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)} - \frac{1}{1+h_0 w^{-1}}\right| \leq 4|\tilde{h}(\Psi_0^{\mathrm{u}}, w) - h_0 w^{-1}| \leq K|w|^{-2}$$

and then bound (2.122) is clear.

Our next claim, which can be easily proved using Lemma 2.2.5 and the fact that $|\Psi_0^{\mathrm{u}}(w)| \leq K|w|^{-3}$, is that:

$$\left|\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)} D_\Psi F(\Psi_0^{\mathrm{u}}, w^{-1}, 0)\right| \leq 2K|w|^{-2} \leq \frac{K}{\log^2(1/\delta)}. \tag{2.123}$$

Finally, we claim that:

$$\left|D_\Psi\left[\frac{1}{1+\tilde{h}(\Psi_0^{\mathrm{u}}, w)}\right](\tilde{\mathcal{A}}(w)\Psi_0^{\mathrm{u}} + F(\Psi_0^{\mathrm{u}}, w^{-1}, 0))\right| \leq K|w|^{-2} \leq \frac{K}{\log^2(1/\delta)}. \tag{2.124}$$

Indeed, we note that $D_\Psi(1+h(\Psi_0^{\mathrm{u}}, w))^{-1}$ is bounded. We have to use that $(1+h(\Psi_0^{\mathrm{u}}, w))^{-1}$ is bounded and analytic in a ball of radius $\tilde{r}_0$, and use Cauchy's theorem in a ball of radius $\tilde{r}_0/2$ (where Ψ_0 belongs to) to prove that the derivative with respect to Ψ is bounded. Finally, (2.124) follows from the following bounds:

$$|\tilde{\mathcal{A}}(w)\Psi_0^{\mathrm{u}}| \leq K\|\Psi_0^{\mathrm{u}}\|_{\mathrm{mch},\times}|w|^{-2}, \qquad |F(\Psi_0^{\mathrm{u}}, w^{-1}, 0, 0))| \leq K|w|^{-3}.$$

In the second bound we have used Remark 2.1.12.

With bounds (2.119), (2.121), (2.122), (2.123) and (2.124) we obtain that:

$$|B(w)\Psi(w)| \leq \int_0^1 |D_\Psi \mathcal{R}(\lambda\Psi, \delta, \sigma)(w)| d\lambda |\Psi(w)| \leq \frac{K}{\log^2(1/\delta)}|\Psi(w)|,$$

and then, since $\|\Psi\|_{\mathrm{mch},\times} \leq 1$, it is clear that:

$$\|\mathcal{B}(\Psi)\|_{\mathrm{mch},\times} \leq \frac{K}{\log^2(1/\delta)}\|\Psi\|_{\mathrm{mch},\times} \leq \frac{K}{\log^2(1/\delta)}.$$

□

2.5.4 The independent term

Finally, in this subsection we will study the independent term $\mathcal{I}(c_1, c_2) + \mathcal{L} \circ \mathcal{R}(0, \delta, \sigma)$. First we note that if in Lemma 2.5.5 we take $\Psi = \Psi_0^{\mathrm{u}} \in \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$, noting that $X_1(\Psi_0^{\mathrm{u}}, s, \delta, \sigma) = \mathcal{R}(0, \delta, \sigma)(s)$ (see the definition (2.100) of $\mathcal{R}$) and that for ρ big enough $\|\Psi_0\|_{\mathrm{mch},\times} \leq 1$, we obtain immediately the following corollary:

Corollary 2.5.7. $\mathcal{R}(0, \delta, \sigma) \in \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$ and there exists a constant K such that:

$$\|\mathcal{R}(0, \delta, \sigma)\|_{\mathrm{mch},\times} \leq K\delta.$$

Lemma 2.5.8. $\mathcal{I}(c_1, c_2) + \mathcal{L} \circ \mathcal{R}(0, \delta, \sigma) \in \mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$ *and:*

$$\|\mathcal{L} \circ \mathcal{R}(0, \delta, \sigma)\|_{\mathrm{mch},\times} \leq K\delta.$$

Moreover:

$$\|\mathcal{I}(c_1, c_2) + \mathcal{L} \circ \mathcal{R}(0, \delta, \sigma)\|_{\mathrm{mch},\times} \leq K\delta^{1-\gamma}.$$

Proof. The first part is a direct consequence of Lemma 2.5.4 and Corollary 2.5.7. To prove the second part, recall that:

$$\mathcal{I}(c_1, c_2)(s) = \mathcal{M}(s) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} m_1(s)c_1 \\ m_2(s)c_2 \end{pmatrix},$$

where $\mathcal{M}$ was defined in (2.83). Focusing on the first component, using (2.104), we have:

$$m_1(s)c_1 = \frac{m_1(s)}{m_1(s_1)} \psi_1^{\mathrm{u}}(s_1).$$

First we claim that:

$$|\psi_1^{\mathrm{u}}(s_1)| \leq K\delta^{3(1-\gamma)} \tag{2.125}$$

Indeed, we have $|\psi_1^{\mathrm{u}}(s_1)| \leq |\psi^{\mathrm{u}}(s_1)| + |\psi_0^{\mathrm{u}}(s_1)|$, so we just have to check that both terms satisfy the bound. On the one hand, for δ small enough $i\pi/2 + s_1\delta \in D^{\mathrm{out,u}}_{\kappa,\beta,T}$ (see (2.15) for the definition of $D^{\mathrm{out,u}}_{\kappa,\beta,T}$ and (2.35) for s_1). Then by Theorem 2.1.13:

$$|\psi^{\mathrm{u}}(s_1)| = |\delta\xi^{\mathrm{u}}(s_1\delta + i\pi/2)| \leq K\delta^3 |z_0(s_1\delta + i\pi/2) - 1|^3 \leq K\delta^3 |s_1\delta|^{-3}.$$

Then using that $|s_1\delta| \geq K_1\delta^\gamma$ (see (2.36)) we obtain immediately that:

$$|\psi^{\mathrm{u}}(s_1)| \leq K\delta^{3(1-\gamma)}.$$

On the other hand since, by Proposition 2.4.2, $\psi_0^{\mathrm{u}} \in \mathcal{X}_3^{\mathrm{in,u}}$, from definition (2.77) of the norm $\|.\|^{\mathrm{u}}_{\mathrm{in},3,\times}$ we know that:

$$|\psi_0^{\mathrm{u}}(s_1)| \leq \|\psi_0^{\mathrm{u}}\|^{\mathrm{u}}_{\mathrm{in},3,\times} |s_1|^{-3} \leq K\delta^{3(1-\gamma)},$$

where we have used (2.36) again, and then the claim is clear.

Then, by (2.125) and Lemma 2.5.3 we obtain:

$$|m_1(s)c_1| \leq Ke^{-\alpha \mathrm{Im}\,(s_1-s)}\delta^{3(1-\gamma)} \leq K\delta^{3(1-\gamma)}.$$

For the second component we obtain an analogous bound, and therefore it is clear that:

$$\|\mathcal{I}(c_1,c_2)\|_{\mathrm{mch},\times} \leq \sup_{s\in\mathcal{D}^{\mathrm{mch}}_{\kappa,\beta_1,\beta_2}} |s|^2 K\delta^{3(1-\gamma)} \leq K\delta^{1-\gamma},$$

and the lemma is proved, since $\delta < \delta^{1-\gamma}$. □

2.5.5 End of the proof of Proposition 2.5.1

From definition (2.106) of $\mathcal{G}$ and using Lemmas 2.5.4 and 2.5.6, we obtain that:

$$\|\mathcal{G}(\Psi)\|_{\mathrm{mch},\times} \leq K\log^{-2}(1/\delta)$$

if $\|\Psi\|_{\mathrm{mch},\times} \leq 1$. Hence, we have:

$$\|\mathcal{G}\| := \max_{\|\Psi\|_{\mathrm{mch},\times}\leq 1}\{\|\mathcal{G}(\Psi)\|_{\mathrm{mch},\times}\} \leq \frac{K}{\log^2(1/\delta)},$$

and then it is clear that for δ sufficiently small $\|\mathcal{G}\| < 1$. This fact implies that $\mathrm{Id}-\mathcal{G}$ is invertible in $\mathcal{X}^{\mathrm{mch}} \times \mathcal{X}^{\mathrm{mch}}$. Then equation (2.107) and Lemma 2.5.8 yield:

$$\begin{aligned}\|\Psi_1^{\mathrm{u}}\|_{\mathrm{mch},\times} &= \|(\mathrm{Id}-\mathcal{G})^{-1}(\mathcal{I}(c_1,c_2)+\mathcal{L}\circ\mathcal{R}(0,\delta,\sigma))\|_{\mathrm{mch},\times} \\ &\leq \|(\mathrm{Id}-\mathcal{G})^{-1}\|\|\mathcal{I}(c_1,c_2)+\mathcal{L}\circ\mathcal{R}(0,\delta,\sigma)\|_{\mathrm{mch},\times} \leq K\delta^{1-\gamma},\end{aligned}$$

proving thus Proposition 2.5.1.

2.6 Proof of Theorem 2.1.20

Let $\Delta\varphi$ (defined in (2.37)) be the difference between the parameterizations $\varphi^{\mathrm{u,s}}$. Our goal now is to provide a dominant term for this difference, as Theorem 2.1.20 enunciates. Note that $\Delta\varphi$, being a solution of (2.41). Moreover we note that $\mathcal{B}(u)\Delta\varphi(u)$ can be thought of simply as a function of u because $\Delta(u) = \varphi^{\mathrm{u}}(u) - \varphi^{\mathrm{s}}(u)$, and the existence of the functions $\varphi^{\mathrm{u,s}}$ has been already proved. Using this fact, system (2.41) can be reagarded as an uncoupled system (because the matrix $\mathcal{A}(u)$, defined in (2.38), is diagonal). Then it is easy to see that $\Delta\varphi$ satisfies:

$$\Delta\varphi(u) = \mathcal{M}(u)\left[\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \int_{u_+}^{\mathrm{u}} m_1^{-1}(w)\pi^1(\mathcal{B}(w)\Delta\varphi(w))dw \\ \int_{u_-}^{\mathrm{u}} m_2^{-1}(w)\pi^2(\mathcal{B}(w)\Delta\varphi(w))dw \end{pmatrix}\right], \tag{2.126}$$

where $\mathcal{B}$ was defined in (2.42), $\mathcal{M}$ in (2.44), m_1 and m_2 in (2.45), and c_1 and c_2 are some suitable constants.

As we did in the previous sections, we first need to introduce suitable complex domains and Banach spaces in which we will work. First of all, we define:

$$u_+ = i\left(\frac{\pi}{2} - \kappa\delta\log(1/\delta)\right) = it_+, \qquad u_- = i\left(-\frac{\pi}{2} + \kappa\delta\log(1/\delta)\right) = it_-.$$

Now, let $E = \{it \in \mathbb{C} \, : \, t \in (t_-, t_+)\}$. We consider the following Banach spaces:

$$\mathcal{X}^{\text{spl}} = \{\phi : E \to \mathbb{C} \, : \, \phi \text{ analytic}, \, \sup_{it\in E} |e^{\alpha(\pi/2-|t|)/\delta}\cos^{-\text{d}}(t)\phi(it)| < \infty\},$$

with the norm:

$$\|\phi\|_{\text{spl}} = \sup_{it\in E} |e^{\alpha(\pi/2-|t|)/\delta}\cos^{-\text{d}}(t)\phi(it)|. \tag{2.127}$$

As usual, in the product space $\mathcal{X}^{\text{spl}} \times \mathcal{X}^{\text{spl}}$ we will take the norm:

$$\|(\phi_1, \phi_2)\|_{\text{spl},\times} = \|\phi_1\|_{\text{spl}} + \|\phi_2\|_{\text{spl}}.$$

The main result of this section, which implies Theorem 2.1.20, is the following:

Proposition 2.6.1. *Let:*

$$\Delta\varphi_0(u) = \mathcal{M}(u)\begin{pmatrix} c_1^0 \\ c_2^0 \end{pmatrix},$$

where $\mathcal{M}$ *was defined in* (2.44)*, and:*

$$\begin{pmatrix} c_1^0 \\ c_2^0 \end{pmatrix} = \begin{pmatrix} m_1^{-1}(u_+)\frac{(-i\lambda)^{\text{d}}}{\delta}e^{-\alpha\lambda+i(c+\alpha h_0)\log(-i\lambda)}C_{\text{in}} \\ m_2^{-1}(u_-)\frac{(i\lambda)^{\text{d}}}{\delta}e^{-\alpha\lambda-i(c+\alpha h_0)\log(i\lambda)}\overline{C_{\text{in}}} \end{pmatrix}, \tag{2.128}$$

where $\lambda = \kappa\log(1/\delta)$ *and* C_{in} *is the constant defined in Theorem 2.1.15. Then, if* $C_{\text{in}} \neq 0$*, we have that* $\Delta\varphi = \Delta\varphi_0 + \Delta\varphi_1$*, where* $\Delta\varphi_1$ *is such that:*

$$\|\Delta\varphi_1\|_{\text{spl},\times} \leq \frac{K\|\Delta\varphi_0\|_{\text{spl},\times}}{\log(1/\delta)},$$

for some constant K *independent of* δ *and* σ*.*

Now we proceed to prove Lemma 2.1.22, which was stated in Subsection 2.1.5. To do that we will use the following technical lemma, which we do not prove here (see [DS97]).

Lemma 2.6.2. *Let* $\nu > 1$*. Then there exists a constant* K *such that, if* $u \in E$*, then:*

$$\left|\int_0^{\text{u}} \frac{1}{|\cosh w|^\nu}dw\right| \leq \frac{K}{\delta^{\nu-1}\log^{\nu-1}(1/\delta)}.$$

Proof of Lemma 2.1.22. Since $\mathcal{M}(u)$ is a fundamental matrix of $\dot{z} = \mathcal{A}(u)z$, with $\mathcal{A}(u) = \mathrm{diag}(a_1(u), a_2(u))$ defined in (2.38), we have that:

$$\mathcal{M}(u) = e^{\int_0^u \mathcal{A}(w)dw} = \begin{pmatrix} e^{\int_0^u a_1(w)dw} & 0 \\ 0 & e^{\int_0^u a_2(w)dw} \end{pmatrix}. \tag{2.129}$$

Let us compute just $m_1(u)$. We have:

$$\begin{aligned} \int_0^u a_1(w)dw &= -\frac{\alpha i}{\delta}\int_0^u \frac{1}{1-\frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}}dw + \sigma\int_0^u \frac{1}{1-\frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}}dw \\ &+ (-\mathrm{d}-ic)\int_0^u \frac{1}{1-\frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}}z_0(w)dw =: I_1(u)+I_2(u)+I_3(u). \end{aligned}$$

Now we shall give an asymptotic expression for each of these integrals separately. Note that:

$$\left|\frac{\delta h_0 z_0^3(u)}{-1+z_0^2(u)}\right| \leq \frac{K}{\log(1/\delta)} < 1,$$

if δ small enough, and hence we can write:

$$\frac{1}{1-\frac{\delta h_0 z_0^3(u)}{-1+z_0^2(u)}} = \sum_{k=0}^{\infty}\left(\frac{\delta h_0 z_0^3(u)}{-1+z_0^2(u)}\right)^k.$$

Hence, we can express I_1 as:

$$I_1(u) = -\frac{\alpha i}{\delta}\int_0^u \left(1+\frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}\right)dw - \frac{\alpha i}{\delta}\int_0^u \left(\frac{1}{1-\frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}} - 1 - \frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}\right)dw.$$

Now, note that:

$$\begin{aligned} \left|\frac{\alpha i}{\delta}\int_0^u \left(\frac{1}{1-\frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}} - 1 - \frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}\right)dw\right| &\leq \frac{K}{\delta}\int_0^u \delta^2 h_0^2 \left|\frac{z_0^3(w)}{-1+z_0^2(w)}\right|^2 dw \\ &\leq K\delta\int_0^u \frac{1}{|\cosh w|^2}dw \\ &\leq \frac{K}{\log(1/\delta)}, \end{aligned}$$

where in the last inequality we have used Lemma 2.6.2. Hence we have:

$$\begin{aligned} I_1(u) &= -\frac{\alpha i}{\delta}\int_0^u \left(1+\frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}\right)dw + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right) \\ &= -\frac{\alpha i u}{\delta} + \alpha h_0 i\left(-\frac{1}{2}\sinh^2 u + \log\cosh u\right) + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right). \end{aligned} \tag{2.130}$$

In the case of I_2, we can rewrite it in the following form:

$$I_2(u) = \sigma \int_0^u dw + \sigma \int_0^u \left(\frac{1}{1 - \frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}} - 1 \right) dw.$$

Now, we have:

$$\begin{aligned} \left| \int_0^u \left(\frac{1}{1 - \frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}} - 1 \right) dw \right| &\leq K \int_0^u \left| \frac{\delta h_0 z_0^3(w)}{-1 + z_0^2(w)} \right| dw \leq K\delta \int_0^u \frac{1}{|\cosh w|} dw \\ &\leq \frac{K}{\log(1/\delta)}, \end{aligned}$$

where we have used that for $u \in E$ one has $|\cosh^{-1} u| \leq \delta^{-1} \log^{-1}(1/\delta)$ and $|u| \leq \pi/2$. Then, it is clear that:

$$I_2(u) = \sigma \int_0^u dw + \mathcal{O}\left(\frac{1}{\log(1/\delta)} \right) = \sigma u + \mathcal{O}\left(\frac{1}{\log(1/\delta)} \right). \tag{2.131}$$

Finally, I_3 can be decomposed as:

$$I_3(u) = (-\mathrm{d} - ic) \int_0^u z_0(w)dw + (-\mathrm{d} - ic) \int_0^u \left(\frac{1}{1 - \frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}} - 1 \right) z_0(w)dw.$$

Again, we have:

$$\begin{aligned} \left| (-\mathrm{d} - ic) \int_0^u \left(\frac{1}{1 - \frac{\delta h_0 z_0^3(w)}{-1+z_0^2(w)}} - 1 \right) z_0(w)dw \right| &\leq K \int_0^u \left| \frac{\delta h_0 z_0^3(w)}{-1 + z_0^2(w)} \right| z_0(w)dw \\ &\leq K\delta \int_0^u \frac{1}{|\cosh^2 w|} dw \\ &\leq \frac{K}{\log(1/\delta)}, \end{aligned}$$

where in the last inequality we have used Lemma 2.6.2. Then:

$$I_3(u) = (-\mathrm{d} - ic) \int_0^u z_0(w)dw + \mathcal{O}\left(\frac{1}{\log(1/\delta)} \right) = (\mathrm{d} + ic) \log\cosh u + \mathcal{O}\left(\frac{1}{\log(1/\delta)} \right). \tag{2.132}$$

In conclusion, from (2.130), (2.131) and (2.132) and the fact that:

$$m_1(u) = e^{I_1(u)+I_2(u)+I_3(u)},$$

the asymptotic formula (2.45) is proved. $\square$

Lemma 2.6.3. *We have:*

$$\begin{aligned} m_1^{-1}(u_+) &= \frac{e^{-\frac{\alpha\pi}{2\delta}}}{\kappa^{\mathrm{d}}\delta^{\mathrm{d}+\kappa\alpha}\log^{\mathrm{d}}(1/\delta)} e^{-i[\frac{\sigma\pi}{2}+\frac{\alpha h_0}{2}+(c+\alpha h_0)\log\delta]} e^{-i(c+\alpha h_0)\log\lambda}\left(1+\mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right), \\ m_2^{-1}(u_-) &= \frac{e^{-\frac{\alpha\pi}{2\delta}}}{\kappa^{\mathrm{d}}\delta^{\mathrm{d}+\kappa\alpha}\log^{\mathrm{d}}(1/\delta)} e^{i[\frac{\sigma\pi}{2}+\frac{\alpha h_0}{2}+(c+\alpha h_0)\log\delta]} e^{i(c+\alpha h_0)\log\lambda}\left(1+\mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right), \end{aligned} \tag{2.133}$$

where $\lambda = \kappa\log(1/\delta)$.

Proof. Again, we will prove the asymptotic expression just for $m_1(u_+)$, being the other case analogous. First of all, from Lemma 2.1.22 we obtain:

$$\begin{aligned} & m_1^{-1}(u_+) \\ & = \cosh^{-d}(u_+) e^{\frac{\alpha i u_+}{\delta}} e^{-\sigma u_+} e^{-\alpha h_0 i\left[-\frac{1}{2}\sinh^2 u_+ + \log\cosh u_+\right]} e^{-ic\log\cosh u_+}\left(1+\mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right). \end{aligned} \tag{2.134}$$

Recall that $u_+ = i\pi/2 - i\kappa\delta\log(1/\delta)$. Then we have:

$$\cosh^{-d}(u_+) = \frac{1}{\kappa^{\mathrm{d}}\delta^{\mathrm{d}}\log^{\mathrm{d}}(1/\delta)}\left(1+\mathcal{O}(\delta\log(1/\delta))\right), \tag{2.135}$$

$$-\frac{1}{2}\sinh^2(u_+) = \frac{1}{2} + \mathcal{O}\left(\delta^2\log^2(1/\delta)\right), \tag{2.136}$$

$$\log\cosh(u_+) = \log\delta + \log(\kappa\log(1/\delta)) + \mathcal{O}\left(\delta^2\log^2(1/\delta)\right). \tag{2.137}$$

Moreover, it is clear that:

$$e^{i\alpha u_+/\delta} = e^{-\frac{\alpha\pi}{2\delta}} e^{\alpha\kappa\log(1/\delta)} = e^{-\frac{\alpha\pi}{2\delta}}\frac{1}{\delta^{\kappa\alpha}}. \tag{2.138}$$

Finally, we have:

$$e^{-\sigma u_+} = e^{-\frac{i\sigma\pi}{2}}(1+\mathcal{O}(\delta\log(1/\delta))). \tag{2.139}$$

Substituting (2.135), (2.136), (2.137), (2.138) and (2.139) in (2.134) the claim is proved. □

In the following we will proceed to prove Proposition 2.5.1, which will be possible with the lemmas below. In order to simplify the process, we will introduce the notation. For $k_1, k_2 \in \mathbb{C}$, we define:

$$\mathcal{I}(k_1,k_2)(u) = \mathcal{M}(u)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}, \tag{2.140}$$

where the matrix $\mathcal{M}(u)$ was defined in (2.129). Note that with this notation we have that $\Delta\varphi_0 = \mathcal{I}(c_1^0, c_2^0)$.

For functions $\phi \in \mathcal{X}^{\rm spl} \times \mathcal{X}^{\rm spl}$ we define the following operator:

$$\mathcal{G}(\phi)(u) = \begin{pmatrix} \mathcal{G}_1(\phi)(u) \\ \mathcal{G}_2(\phi)(u) \end{pmatrix} = \begin{pmatrix} m_1(u) \int_{u_+}^{\rm u} m_1^{-1}(w)\pi^1(\mathcal{B}(w)\phi(w))dw \\ m_2(u) \int_{u_-}^{\rm u} m_2^{-1}(w)\pi^2(\mathcal{B}(w)\phi(w))dw \end{pmatrix}, \tag{2.141}$$

where the matrix $\mathcal{B}(w)$ is defined in (2.42), $m_1(w)$ and $m_2(w)$ are defined in (2.45).

Lemma 2.6.4. $\Delta\varphi_1 = \Delta\varphi - \Delta\varphi_0$ *satisfies:*

$$\Delta\varphi_1(u) \quad = \quad \mathcal{I}(c_1 - c_1^0, c_2 - c_2^0)(u) + \mathcal{G}(\Delta\varphi_0)(u) + \mathcal{G}(\Delta\varphi_1)(u). \tag{2.142}$$

Moreover,

$$|c_1 - c_1^0|,\ |c_2 - c_2^0| \leq \frac{Ke^{-\frac{\alpha\pi}{2\delta}}}{\delta^{1+{\rm d}}\log(1/\delta)}. \tag{2.143}$$

Proof. To prove the first statement we just need to realize that the fixed point equation for $\Delta\varphi$ (2.126) can be written as:

$$\Delta\varphi = \mathcal{I}(c_1, c_2) + \mathcal{G}(\Delta\varphi). \tag{2.144}$$

Then, since $\Delta\varphi = \Delta\varphi_0 + \Delta\varphi_1$ and $\Delta\varphi_0 = \mathcal{I}(c_1^0, c_2^0)$, this last equality yields:

$$\Delta\varphi_1 = \mathcal{I}(c_1, c_2) - \mathcal{I}(c_1^0, c_2^0) + \mathcal{G}(\Delta\varphi_0 + \Delta\varphi_1),$$

and using that the operators $\mathcal{I}$ and $\mathcal{G}$ are linear we obtain equality (2.142).

Now we proceed to prove bound (2.143). We will just bound $c_1 - c_1^0$, the other component is analogous. We will write $\Delta\varphi = (\Delta\xi, \Delta\bar{\xi})$ and $\Delta\varphi_j = (\Delta\xi_j, \Delta\bar{\xi}_j)$, for $j = 0, 1$.

First note that, since $\mathcal{G}_1(\phi)(u_+) = 0$ for all $\phi \in \mathcal{X}^{\rm spl} \times \mathcal{X}^{\rm spl}$, equalities (2.140) and (2.144) yield:

$$c_1 = m_1^{-1}(u_+)\Delta\xi(u_+).$$

Moreover, since by definition $\Delta\varphi_0(u_+) = \mathcal{I}(c_1^0, c_2^0)(u_+)$, we also have:

$$c_1^0 = m_1^{-1}(u_+)\Delta\xi_0(u_+).$$

Then it is clear that:

$$|c_1 - c_1^0| = |m_1^{-1}(u_+)||\Delta\xi(u_+) - \Delta\xi_0(u_+)|. \tag{2.145}$$

Now, taking into account that $u_+ \in D^{\rm mch,u}_{\kappa,\beta_1,\beta_2} \cap D^{\rm mch,s}_{\kappa,\beta_1,\beta_2}$, from Corollary 2.1.18 we know that:

$$\Delta\xi(u_+) = \frac{1}{\delta}\left[\Delta\psi_0\left(\frac{u_+ - i\pi/2}{\delta}\right) + \Delta\psi_1\left(\frac{u_+ - i\pi/2}{\delta}\right)\right], \tag{2.146}$$

where we have written $\Delta\psi_j = \psi_j^{\mathrm{u}} - \psi_j^{\mathrm{s}}$, for $j = 0, 1$. Here $\psi_0^{\mathrm{u,s}}$ is the first component of the corresponding solution of the inner system (2.27), $\Psi_0^{\mathrm{u,s}}$, and $\psi_1^{\mathrm{u,s}}$ satisfy:

$$\frac{1}{\delta}\left|\psi_1^{\mathrm{u,s}}\left(\frac{u_+ - i\pi/2}{\delta}\right)\right| \leq \frac{K\delta^{-\gamma}}{\log^2(1/\delta)} \tag{2.147}$$

for some constant K. From (2.145) and (2.146) we have:

$$|c_1 - c_1^0| \leq |m_1^{-1}(u_+)| \left[\left|\frac{1}{\delta}\Delta\psi_0\left(\frac{u_+ - i\pi/2}{\delta}\right) - \Delta\xi_0(u_+)\right| + \left|\frac{1}{\delta}\Delta\psi_1\left(\frac{u_+ - i\pi/2}{\delta}\right)\right|\right]. \tag{2.148}$$

Now, on one hand, from Lemma 2.6.3 it is clear that:

$$|m_1^{-1}(u_+)| \leq Ke^{-\frac{\alpha\pi}{2\delta}}\frac{1}{\delta^{\mathrm{d}+\kappa\alpha}\log^{\mathrm{d}}(1/\delta)}, \tag{2.149}$$

where we have used that:

$$\left|e^{-i[\frac{\sigma\pi}{2}+\frac{\alpha h_0}{2}+(\alpha h_0+c)\log\delta]}e^{-i(\alpha h_0+c))\log\lambda}\right| = 1. \tag{2.150}$$

On the other hand, from definition (2.128) of c_1^0 and c_2^0 and the fact that $\Delta\xi_0(u_+) = m_1(u_+)c_1^0$ it is clear that:

$$\Delta\xi_0(u_+) = \frac{(-i\lambda)^{\mathrm{d}}}{\delta}e^{-\alpha\lambda+i(c+\alpha h_0)\log(-i\lambda)}C_{\mathrm{in}},$$

where $\lambda = \kappa\log(1/\delta)$. Then, from formula (2.30) of $\Delta\Psi_0$ in Theorem 2.1.15 and this last equality we have that:

$$\left|\frac{1}{\delta}\Delta\psi_0\left(\frac{u_+ - i\pi/2}{\delta}\right) - \Delta\xi_0(u_+)\right| = \left|\frac{(-i\lambda)^{\mathrm{d}}}{\delta}e^{-\alpha\lambda+i(c+\alpha h_0)\log(-i\lambda)}\chi_1(-i\lambda)\right|,$$

where χ_1 is the first component of the function χ in Theorem 2.1.15. Now, since by this proposition we know that $|\chi_1(s)| \leq K|s|^{-1}$, we have:

$$\left|\frac{1}{\delta}\Delta\psi_0\left(\frac{u_+ - i\pi/2}{\delta}\right) - \Delta\xi_0(u_+)\right| \leq K\frac{\lambda^{\mathrm{d}-1}}{\delta}\delta^{\kappa\alpha}|e^{i(c+\alpha h_0)\log(-i\lambda)}| \leq K\frac{\lambda^{\mathrm{d}-1}}{\delta}\delta^{\kappa\alpha}. \tag{2.151}$$

Then, bounds (2.147), (2.149) and (2.151) yield:

$$\begin{aligned}&|m_1^{-1}(u_+)|\left(\left|\frac{1}{\delta}\Delta\psi_0\left(\frac{u_+ - i\pi/2}{\delta}\right) - \Delta\xi_0(u_+)\right| + \left|\frac{1}{\delta}\Delta\psi_1\left(\frac{u_+ - i\pi/2}{\delta}\right)\right|\right)\\ &\leq Ke^{-\frac{\alpha\pi}{2\delta}}\left(\frac{1}{\delta^{1+\mathrm{d}}\log(1/\delta)} + \frac{1}{\delta^{\mathrm{d}+\kappa\alpha+\gamma}\log(1/\delta)}\right).\end{aligned} \tag{2.152}$$

Then, taking $\kappa > 0$ such that $0 < \kappa\alpha < 1-\gamma$, from (2.148) and (2.152) we obtain immediately:

$$|c_1 - c_1^0| \leq K e^{-\frac{\alpha\pi}{2\delta}} \frac{1}{\delta^{1+\mathrm{d}} \log(1/\delta)}.$$

□

Lemma 2.6.5. *Let k_1, $k_2 \in \mathbb{C}$. Then, $\mathcal{I}(k_1,k_2) \in \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$ and:*

$$\|\mathcal{I}(k_1,k_2)\|_{\mathrm{spl},\times} = (|k_1| + |k_2|) e^{\frac{\alpha\pi}{2\delta}} \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right).$$

Proof. We will bound just the norm of the first component of $\mathcal{I}(k_1,k_2)$, that is $\pi^1\mathcal{I}(k_1,k_2) = m_1(u)k_1$. For $it \in E$, from Lemma 2.1.22 we have:

$$\begin{aligned} & |m_1(it)k_1| |\cos^{-\mathrm{d}} t e^{\frac{\alpha(\pi/2-|t|)}{\delta}}| \\ = \ & |k_1| |e^{\frac{\alpha(\pi/2+t-|t|)}{\delta}}| |e^{i\sigma t}| |e^{\alpha h_0 i[\sin^2 t/2 + \log\cos t]}| |e^{ic\log\cos t}| \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right). \end{aligned}$$

Note that, since t is real and $|t| < \pi/2$, we have that $\log\cos t$ is real. Moreover, since $\sigma, \alpha, h_0 \in \mathbb{R}$, we have that $|e^{i\sigma t}| = |e^{\alpha h_0 i[\sin^2 t/2 + \log\cos t]}| = 1$, and then:

$$|m_1(it)k_1| |\cos^{-\mathrm{d}} t e^{\alpha(\pi/2-|t|)/\delta}| = |k_1| e^{\alpha(\pi/2+t-|t|)/\delta} \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right).$$

Then it is clear that:

$$\begin{aligned} \sup_{it\in E} |m_1(it)k_1| |\cos^{-\mathrm{d}} t e^{\alpha(\pi/2-|t|)/\delta}| &= |k_1| e^{\frac{\alpha\pi}{2\delta}} \sup_{it\in E} e^{\alpha(t-|t|)/\delta} \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right) \\ &= |k_1| e^{\frac{\alpha\pi}{2\delta}} \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right), \end{aligned}$$

and hence:

$$\|\pi^1\mathcal{I}(k_1,k_2)\|_{\mathrm{spl}} = |k_1| e^{\frac{\alpha\pi}{2\delta}} \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right),$$

obtaining the desired bound. □

Lemma 2.6.6.

$$\|\Delta\varphi_0\|_{\mathrm{spl},\times} = \frac{1}{\delta^{\mathrm{d}+1}} e^{\frac{\pi}{2}(c+\alpha h_0)} \left(|C_{\mathrm{in}}| + |\overline{C_{\mathrm{in}}}|\right) \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right).$$

Proof. Since $\Delta\varphi_0 = \mathcal{I}(c_1^0, c_2^0)$, we just have to bound c_1^0 and c_2^0 and then use Lemma 2.6.5. We will just bound c_1^0, being the other case analogous. Recall that:

$$c_1^0 = m_1^{-1}(u_+)\frac{(-i\lambda)^{\mathrm{d}}}{\delta}e^{-\alpha\lambda + i(c+\alpha h_0)\log(-i\lambda)}C_{\mathrm{in}},$$

with $\lambda = \kappa\log(1/\delta)$. On one hand, from formula (2.133) and (2.150) it is clear that:

$$|m_1^{-1}(u_+)| = \frac{1}{\kappa^{\mathrm{d}}\delta^{\mathrm{d}+\kappa\alpha}\log^{\mathrm{d}}(1/\delta)}e^{-\frac{\alpha\pi}{2\delta}}\left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right). \tag{2.153}$$

On the other hand, noting that $\log(-i\lambda) = \log\lambda - i\pi/2$ and $e^{-\alpha\lambda} = e^{-\alpha\kappa\log(1/\delta)} = \delta^{\kappa\alpha}$, we have:

$$\begin{aligned}\left|\frac{(-i\lambda)^{\mathrm{d}}}{\delta}e^{-\alpha\lambda+i(c+\alpha h_0)\log(-i\lambda)}\right| &= \frac{\kappa^{\mathrm{d}}\log^{\mathrm{d}}(1/\delta)}{\delta}\delta^{\kappa\alpha}e^{\frac{\pi}{2}(c+\alpha h_0)}\left|e^{i(c+\alpha h_0)\log\lambda}\right| \\ &= \frac{\kappa^{\mathrm{d}}\log^{\mathrm{d}}(1/\delta)}{\delta}\delta^{\kappa\alpha}e^{\frac{\pi}{2}(c+\alpha h_0)}. \end{aligned} \tag{2.154}$$

From (2.153) and (2.154) it is clear that:

$$|c_1^0| = \frac{1}{\delta^{1+\mathrm{d}}}e^{\frac{\pi}{2}(c+\alpha h_0)}e^{-\frac{\alpha\pi}{2\delta}}|C_{\mathrm{in}}|,$$

and then the initial claim is proved by Lemma 2.6.5. □

Lemma 2.6.7. *There exists a constant K such that, for all $it \in E$ and $l, j = 1, 2$, the matrix $\mathcal{B} = (b_{lj})$ satisfies:*

$$|b_{lj}(it)\cos^2(t)| \leq K\delta.$$

Proof. Recall that:

$$\mathcal{B}(u) = \int_0^1 D\mathcal{R}(\varphi_\lambda)(u)d\lambda,$$

where $\mathcal{R}$ was defined in (2.39). Note that $\varphi_\lambda = (1-\lambda)\varphi^{\mathrm{s}} - \lambda\varphi^{\mathrm{u}}$ and hence, by Theorem 2.1.13 it is clear that:

$$|\varphi_\lambda(u)| \leq K\delta^2|\cosh^{-3}u|. \tag{2.155}$$

We will prove that for $j = 1, 2$:

$$|\pi^j D\mathcal{R}(\varphi_\lambda)(u)| \leq \frac{K\delta}{|\cosh^2 u|}, \tag{2.156}$$

and then from the definition of $\mathcal{B}$ and the fact that $\cosh(it) = \cos t$, the statement will be clear. In fact, we will just do the proof for the first entry of the matrix $D\mathcal{R}$, since all the others are analogous. If we compute this entry, we get:

$$\begin{aligned}
D_\xi \mathcal{R}_1(\varphi_\lambda)(u) &= D_\xi \left[\frac{\delta^{-2} F_1(\delta\varphi_\lambda, \delta z_0(u), \delta, \delta\sigma)}{1 + \frac{b\xi_\lambda \bar{\xi}_\lambda + \delta^{-2} H(\delta\varphi_\lambda, \delta z_0(u), \delta, \delta\sigma)}{-1+z_0(u)^2}} \right] \\
&+ \left(\frac{1}{1 + \frac{b\xi_\lambda\bar{\xi}_\lambda + \delta^{-2} H(\delta\varphi_\lambda, \delta z_0(u), \delta, \delta\sigma)}{-1+z_0^2(u)}} - \frac{1}{1 - \frac{\delta h_0 z_0^3(u)}{-1+z_0^2(u)}} \right) a_1(u) \\
&+ \frac{-a_1(u)\xi_\lambda(u)}{\left(1 + \frac{b\xi_\lambda\bar{\xi}_\lambda + \delta^{-2} H(\delta\varphi_\lambda, \delta z_0(u), \delta, \delta\sigma)}{-1+z_0^2(u)}\right)^2} \frac{b\bar{\xi}_\lambda + \delta^{-1} D_\xi H(\delta\varphi_\lambda, \delta z_0(u), \delta, \delta\sigma)}{-1+z_0^2(u)} \\
&:= D_\xi \mathcal{R}_1^1(\varphi_\lambda)(u) + D_\xi \mathcal{R}_1^2(\varphi_\lambda)(u) + D_\xi \mathcal{R}_1^3(\varphi_\lambda)(u).
\end{aligned}$$

First we claim that that:

$$\left| D_\xi \mathcal{R}_1^1(\varphi_\lambda)(u) \right| \le \frac{K\delta}{|\cosh^2 u|}. \tag{2.157}$$

This can be proved computing the derivative explicitly and then using bound (2.155) and Corollaries 2.2.6 and 2.2.7. We skip the details since the proof is completely analogous as the one of bound (2.67).

Our next claim is that:

$$\left| D_\xi \mathcal{R}_1^2(\varphi_\lambda)(u) \right| \le \frac{K\delta^2}{|\cosh^3 u|}. \tag{2.158}$$

First of all note that for δ small enough:

$$\left| \frac{b\xi_\lambda\bar{\xi}_\lambda + \delta^{-2} H(\delta\varphi_\lambda, \delta z_0(u), \delta, \delta\sigma)}{-1+z_0^2(u)} \right|, \left| \frac{\delta h_0 z_0^3(u)}{-1+z_0^2(u)} \right| \le \frac{K}{\log(1/\delta)} < \frac{1}{2}, \tag{2.159}$$

and then by Lemma 2.2.4, we have:

$$\begin{aligned}
&\left| \left(\frac{1}{1 + \frac{b\xi_\lambda\bar{\xi}_\lambda + \delta^{-2} H(\delta\varphi_\lambda, \delta z_0(u), \delta, \delta\sigma)}{-1+z_0^2(u)}} - \frac{1}{1 - \frac{\delta h_0 z_0^3(u)}{-1+z_0^2(u)}} \right) \right| \\
\le\;& \frac{4}{|-1+z_0^2(u)|} \left| b\xi_\lambda\bar{\xi}_\lambda + \delta^{-2}(H(\delta\varphi_\lambda, \delta z_0(u), \delta, \delta\sigma) + \delta^3 h_0 z_0(u)) \right| \\
\le\;& \frac{K}{|-1+z_0^2(u)|} \left[b|\xi_\lambda||\bar{\xi}_\lambda| + K\delta(|\varphi_\lambda^3| + |\varphi_\lambda^2 z_0(u)| + |\varphi_\lambda z_0^2(u)|) \right]
\end{aligned} \tag{2.160}$$

where in the last inequality we have used the definition of h_0. It is easy to check that, since bound (2.155) holds, for $u \in E$, we have that:

$$|\delta\varphi_\lambda^3(u)|, |\delta\varphi_\lambda^2(u) z_0(u)|, |\delta\varphi_\lambda(u) z_0^2(u)| \le \frac{\delta^3 K}{|\cosh^5 u|}.$$

Moreover, we also have that:

$$|\xi_\lambda \bar{\xi}_\lambda| \leq \frac{\delta^4 K}{|\cosh^6 u|} \leq \frac{\delta^3 K}{|\cosh^5 u|},$$

and then (2.160) yields:

$$\left| D_\xi \mathcal{R}_1^2(\varphi_\lambda)(u) \right| \leq \frac{K|a_1(u)|}{|-1+z_0^2(u)|} \frac{\delta^3}{|\cosh^5 u|} \leq \frac{K\delta^3 |a_1(u)|}{|\cosh^3 u|}.$$

Finally we just need to note that $|a_1(u)| \leq K/\delta$ to obtain bound (2.158).

Our last claim is:

$$\left| D_\xi \mathcal{R}_1^3(\varphi_\lambda)(u) \right| \leq \frac{K\delta^2}{|\cosh^3 u|}. \tag{2.161}$$

This is quite straightforward to prove, using inequalities (2.155) and (2.159), Corollary 2.2.6 and that $|a_1(u)| \leq K/\delta$.

In conclusion, from bounds (2.157), (2.158) and (2.161) we have:

$$|D_\xi \mathcal{R}_1(u)| \leq K \left(\frac{\delta}{|\cosh^2 u|} + \frac{\delta^2}{|\cosh^3 u|} \right) \leq \frac{K\delta}{|\cosh^2 u|},$$

and thus (2.156) is proved. □

Lemma 2.6.8. *The operator* $\mathcal{G} : \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}} \to \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$ *is well defined, and for* $\phi \in \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$*:*

$$\|\mathcal{G}(\phi)\|_{\mathrm{spl},\times} \leq \frac{K\|\phi\|_{\mathrm{spl},\times}}{\log(1/\delta)}.$$

Proof. Again, we will bound just the first component:

$$|\mathcal{G}_1(\phi)(it)| = \left| m_1(it) \int_{t_+}^{t} m_1^{-1}(iw)\pi^1(\mathcal{B}(iw)(\phi(iw)))dw \right|.$$

Recalling the asymptotic formula (2.45) for $m_1(it)$ it is clear that:

$$|m_1(it)| \leq K \cos^{\mathrm{d}} t e^{\alpha t/\delta}, \qquad |m_1^{-1}(iw)| \leq K \cos^{-\mathrm{d}} w e^{-\alpha w/\delta}.$$

Using these bounds and Lemma 2.6.7 we can bound $\mathcal{G}_1(\phi)(it)$:

$$|\mathcal{G}_1(\phi)(it)| \leq K \cos^{\mathrm{d}} t e^{\alpha t/\delta} \int_t^{t_+} \cos^{-\mathrm{d}} w e^{-\alpha w/\delta} \frac{K\delta}{\cos^2 w} |\phi(iw)| dw.$$

Then, since $\phi \in \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$, recalling definition (2.127) of the norm $\|.\|_{\mathrm{spl},\times}$ we have:

$$|\phi(iw)| \leq \|\phi\|_{\mathrm{spl},\times} \cos^{\mathrm{d}} w e^{-\alpha(\pi/2-|w|)/\delta}$$

and therefore:

$$|\mathcal{G}_1(\phi)(it)| \leq K\delta\cos^{\mathrm{d}} t e^{\alpha t/\delta} e^{-\frac{\alpha\pi}{2\delta}} \|\phi\|_{\mathrm{spl},\times} \int_t^{t_+} e^{-\alpha(w-|w|)/\delta} \frac{1}{\cos^2 w} dw.$$

It is not difficult to check that for $t \in [t_-, t_+]$, there exists a constant C independent of δ and σ such that:

$$e^{\alpha t/\delta} \int_t^{t_+} e^{-\alpha(w-|w|)/\delta} \frac{1}{\cos^2 w} dw \leq C e^{\alpha|t|/\delta} \frac{1}{\kappa\delta\log(1/\delta)},$$

and then we obtain the desired bound:

$$\|\mathcal{G}_1(\phi)\|_{\mathrm{spl},\times} \leq \frac{K\|\phi\|_{\mathrm{spl},\times}}{\log(1/\delta)}.$$

□

End of the proof of Proposition 2.6.1. From Lemma 2.6.4 we can write:

$$(Id - \mathcal{G})\Delta\varphi_1 = \mathcal{I}(c_1 - c_1^0, c_2 - c_2^0) + \mathcal{G}(\Delta\varphi_0).$$

We note that, for $\delta > 0$, $\Delta\varphi_1 \in \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$ although *a priori* its norm is exponentially large with respect to δ. Indeed, we have $\Delta\varphi_1 = \Delta\varphi - \Delta\varphi_0$, and it is clear by Lemma 2.6.6 that $\Delta\varphi_0 \in \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$. Moreover, we have:

$$\begin{aligned} |\Delta\varphi(it)\cos^{-\mathrm{d}} t e^{\alpha(\pi/2-|t|)/\delta}| &\leq (\|\varphi^{\mathrm{u}}\|^{\mathrm{u}}_{\mathrm{out},\times} + \|\varphi^{\mathrm{s}}\|^{\mathrm{s}}_{\mathrm{out},\times})|z_0(it)-1|^3|\cos^{-\mathrm{d}} t|e^{\alpha(\pi/2-|t|)/\delta} \\ &\leq K\delta^2|\cos^{3-\mathrm{d}} t|e^{\alpha(\pi/2-|t|)/\delta} \leq \delta^{-(\mathrm{d}+1)} e^{\frac{\alpha\pi}{2\delta}} K < \infty, \quad (2.162) \end{aligned}$$

and thus it is clear that $\Delta\varphi \in \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$, and hence $\Delta\varphi_1 \in \mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$. Since from Lemma 2.6.8 we know that $\|\mathcal{G}\| < 1$ for δ small enough, the operator $Id - \mathcal{G}$ is invertible in $\mathcal{X}^{\mathrm{spl}} \times \mathcal{X}^{\mathrm{spl}}$. Therefore we can write:

$$\Delta\varphi_1 = (Id - \mathcal{G})^{-1}\left[\mathcal{I}(c_1 - c_1^0, c_2 - c_2^0) + \mathcal{G}(\Delta\varphi_0)\right],$$

and consequently we have:

$$\begin{aligned} \|\Delta\varphi_1\|_{\mathrm{spl},\times} &\leq \|Id - \mathcal{G}\|^{-1}_{\mathrm{spl},\times}\left[\|\mathcal{I}(c_1 - c_1^0, c_2 - c_2^0)\|_{\mathrm{spl},\times} + \|\mathcal{G}(\Delta\varphi_0)\|_{\mathrm{spl},\times}\right] \\ &\leq K\left(\|\mathcal{I}(c_1 - c_1^0, c_2 - c_2^0)\|_{\mathrm{spl},\times} + \|\mathcal{G}(\Delta\varphi_0)\|_{\mathrm{spl},\times}\right). \quad (2.163) \end{aligned}$$

Now, from (2.163) we will be able to improve bound (2.162), realizing that in fact it is not exponentially large with respect to δ. On one hand, using first Lemma 2.6.5 and after Lemma 2.6.4, we have:

$$\|\mathcal{I}(c_1 - c_1^0, c_2 - c_2^0)\|_{\mathrm{spl},\times} \leq K(|c_1 - c_1^0| + |c_2 - c_2^0|)e^{\frac{\alpha\pi}{2\delta}} \leq \frac{K}{\delta^{\mathrm{d}+1}\log(1/\delta)}.$$

Then, from Lemma 2.6.6 it is clear that, if $\|\Delta\varphi_0\|_{\text{spl},\times} \neq 0$ (which is equivalent to $C_{\text{in}} \neq 0$), we have:

$$\|\mathcal{I}(c_1 - c_1^0, c_2 - c_2^0)\|_{\text{spl},\times} \leq \frac{K\|\Delta\varphi_0\|_{\text{spl},\times}}{\log(1/\delta)}. \tag{2.164}$$

On the other hand, from Lemma 2.6.8 we have:

$$\|\mathcal{G}(\Delta\varphi_0)\|_{\text{spl},\times} \leq \frac{K\|\Delta\varphi_0\|_{\text{spl},\times}}{\log(1/\delta)}. \tag{2.165}$$

Substituting (2.164) and (2.165) in (2.163) we obtain the desired bound:

$$\|\Delta\varphi_1\|_{\text{spl},\times} \leq \frac{K\|\Delta\varphi_0\|_{\text{spl},\times}}{\log(1/\delta)}.$$

□

End of the Proof of Theorem 2.1.20. From Proposition 2.6.1, we know that $\Delta\varphi = \Delta\varphi_0 + \Delta\varphi_1$, with:

$$|\Delta\varphi_1(it)| \leq K\frac{\|\Delta\varphi_0\|_{\text{spl},\times}}{\log(1/\delta)}e^{-\alpha(\pi/2-|t|)/\delta}|\cos^{\text{d}} t|,$$

and hence by Lemma 2.6.6 we obtain:

$$|\Delta\varphi_1(it)| \leq \frac{K}{\delta^{\text{d}+1}\log(1/\delta)}e^{\frac{\pi}{2}(c+\alpha h_0)}\left(|C_{\text{in}}| + |\overline{C_{\text{in}}}|\right)e^{-\alpha(\pi/2-|t|)/\delta}|\cos^{\text{d}} t|.$$

For $t = 0$ this formula gives the bound:

$$|\Delta\varphi_1(0)| \leq \frac{K}{\delta^{\text{d}+1}\log(1/\delta)}e^{\frac{\pi}{2}(c+\alpha h_0)}\left(|C_{\text{in}}| + |\overline{C_{\text{in}}}|\right)e^{-\frac{\alpha\pi}{2\delta}}. \tag{2.166}$$

Moreover, by definition of $\Delta\varphi_0$ it is clear that $\Delta\varphi_0(0) = (c_1^0, c_2^0)$. Then by definition (2.128) of c_1^0 and c_2^0, Lemma 2.6.3 and formula (2.154) we obtain:

$$c_1^0 = \frac{1}{\delta^{\text{d}+1}}e^{-\frac{\alpha\pi}{2\delta}}C_{\text{in}}e^{\frac{\pi}{2}(c+\alpha h_0)-i\left(\frac{\sigma\pi}{2}+\frac{\alpha h_0}{2}+(c+\alpha h_0)\log\delta\right)}\left(1+\mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right), \tag{2.167}$$

and $c_2^0 = \overline{c_1^0}$. Finally, we just need to realize that, since $\alpha = \alpha_0 + \alpha_1\delta\sigma + \mathcal{O}(\delta^2)$, we have:

$$e^{-\frac{\alpha\pi}{2\delta}} = e^{-\frac{\alpha_0\pi}{2\delta}-\frac{\alpha_1\sigma\pi}{2}}(1+\mathcal{O}(\delta)),$$

and:

$$e^{\frac{\pi}{2}(c+\alpha h_0)-i\left(\frac{\sigma\pi}{2}+\frac{\alpha h_0}{2}+(c+\alpha h_0)\log\delta\right)} = e^{\frac{\pi}{2}(c+\alpha_0 h_0)-i\left(\frac{\sigma\pi}{2}+\frac{\alpha_0 h_0}{2}+(c+\alpha_0 h_0)\log\delta\right)}(1+\mathcal{O}(\delta)),$$

so that (2.167) becomes:

$$c_1^0 = \frac{1}{\delta^{\mathrm{d}+1}} e^{-\frac{\alpha_0 \pi}{2\delta}} C_{\mathrm{in}} e^{\frac{\pi}{2}(c+\alpha_0 h_0 - \sigma\alpha_1) - i\left(\frac{\sigma\pi}{2} + \frac{\alpha_0 h_0}{2} + (c+\alpha_0 h_0)\log\delta\right)} \left(1 + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right). \quad (2.168)$$

Then, from (2.166) and (2.168) and the fact that $\Delta\varphi(0) = (c_1^0, c_2^0) + \Delta\varphi_1(0)$ we obtain:

$$\Delta\varphi(0) = \frac{1}{\delta^{\mathrm{d}+1}} e^{-\frac{\alpha_0 \pi}{2\delta}} e^{\frac{\pi}{2}(c+\alpha_0 h_0 - \sigma\alpha_1)} \left(\begin{pmatrix} C_{\mathrm{in}} e^{-i\left(\frac{\sigma\pi}{2} + \frac{\alpha_0 h_0}{2} + (c+\alpha_0 h_0)\log\delta\right)} \\ \overline{C_{\mathrm{in}}} e^{i\left(\frac{\sigma\pi}{2} + \frac{\alpha_0 h_0}{2} + (c+\alpha_0 h_0)\log\delta\right)} \end{pmatrix} + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\right).$$

and the theorem is proved. □

Chapter 3

Breakdown of the 2D heteroclinic connection: the regular case

In this chapter we prove the following result, which has Theorem 1.2' as a corollary.

Theorem 3.1. *Consider system* (1.8)*, with* μ*,* $\beta_1, \gamma_2 > 0$ *and* $|\nu| < \beta_1\sqrt{\mu}$*, which has two critical points* $\bar{S}_{\pm}(\mu,\nu)$ *of saddle-focus type. Let* $\bar{D}^{\mathrm{u,s}}(u,\theta,\mu,\nu)$ *(*$\bar{D}^{\mathrm{u,s}}(u,\theta,\mu)$ *in the conservative case) be the distance between the two-dimensional unstable manifold of* $\bar{S}_{-}(\mu,\nu)$ *and the two-dimensional stable manifold of* $\bar{S}_{+}(\mu,\nu)$ *on the plane* $\bar{z} = \tanh(u)$*. Let:*

$$\bar{\vartheta}(u,\mu) = \frac{\alpha_0 u}{\sqrt{\mu}} + \frac{\alpha_3}{\beta_1}\left[\log\cosh(\beta_1 u) - \frac{1}{2}\log\mu\right].$$

For $p > -2$*, there exist constants* $\mathcal{C}_1$*,* $\mathcal{C}_2$ *and* $T_0 > 0$ *such that for all* $u \in [-T_0, T_0]$ *and* $\theta \in \mathbb{S}^1$*, the following holds:*

1. *In the conservative case,* $\bar{D}^{\mathrm{u,s}}(u,\theta,\mu)$ *is given asymptotically, as* $\mu \to 0$*, by:*

$$\begin{aligned}\bar{D}^{\mathrm{u,s}}(u,\theta,\mu) &= \sqrt{\frac{\gamma_2}{2}}\frac{e^{-\frac{\alpha_0\pi}{2\sqrt{\mu}}}}{\mu^{-\frac{p-1}{2}}}\cosh^3(u)\Bigg[\mathcal{C}_1\cos\left(\theta+\bar{\vartheta}(u,\mu)\right) \\ &\quad +\mathcal{C}_2\sin\left(\theta+\bar{\vartheta}(u,\mu)\right)+\mathcal{O}\left(\mu^{\frac{p+2}{2}}|\log(\mu)|+\mu^{3/2}\right)\Bigg].\end{aligned}$$

2. *In the dissipative case, given* $a_1, a_2 \in \mathbb{R}$ *and* $a_3 > 0$*, there exists a function* $\nu = \nu(\mu)$ *satisfying* $\nu(0) = 0$*, such that* $\bar{D}^{\mathrm{u,s}}(u,\theta,\mu,\nu(\mu))$ *is given asymptotically, as* $\mu \to 0$*,*

by:

$$\begin{aligned}\bar{D}^{\mathrm{u,s}}(u,\theta,\mu,\nu(\mu)) &= \cosh^{1+\frac{2}{\beta_1}}(\beta_1 u)a_1\mu^{a_2}e^{-\frac{a_3\pi}{2\beta_1\sqrt{\mu}}}\left(1+\mathcal{O}\left(\mu^{\frac{p+3}{2}}\right)\right)\\ &+\sqrt{\frac{\gamma_2}{\beta_1+1}}\cosh^{1+\frac{2}{\beta_1}}(\beta_1 u)\frac{e^{-\frac{\alpha_0\pi}{2\beta_1\sqrt{\mu}}}}{\mu^{-\frac{p+1}{2}+\frac{1}{\beta_1}}}\Bigg[\mathcal{C}_1\cos\left(\theta+\bar{\vartheta}(u,\mu)\right)\\ &+\mathcal{C}_2\sin\left(\theta+\bar{\vartheta}(u,\mu)\right)+\mathcal{O}\left(\mu^{\frac{p+2}{2}}|\log(\mu)|+\mu^{3/2}\right)\Bigg].\end{aligned}$$

In Section 3.1 we give the main ideas of the proof of Theorem 3.1. The rest of the sections of this chapter are devoted to proving all the results stated in that section. We now summarize all the subsections that can be found in Section 3.1, each one consisting in one step of the proof of Theorem 3.1. The first step consists on performing some additional changes of variables to simplify system (1.8). These changes are close to the identity, so that they not modify the asymptotic formula of the difference between the invariant manifolds, but allow to simplify the proof of the theorem. After that, we rescale variables and introduce the new parameters δ and σ as we did in Chapter 2. This is explained in detail in Subsection 3.1.1. In Subsection 3.1.2 we briefly give a parameterization of the heteroclinic connection of the unperturbed system. Then, in Subsection refsubsecintroparams2D , we find parameterizations of the 2-dimensional invariant manifolds defined in suitable complex domains. In Subsection 3.1.4 we give some new parameterizations of these manifolds that are more adequate to our purposes. Next, in Subsection 3.1.5, we introduce and study the Melnikov function adapted to this problem. After that, in Subsection 3.1.6, we give some properties of the difference between the invariant manifolds. Finally, in Subsection 3.1.7, we state a result equivalent to Theorem 3.1, and give a proof using the partial results that have been stated in the previous subsections.

We point out that the results of Subsections 3.1.1–3.1.4 and 3.1.6 are valid for $p \geq -2$, that is, in both the regular and singular cases. Consequently, the results contained there will also be used in Chapter 4, where the singular case $p=-2$ is studied.

3.1 Set-up and heuristics of the proof

3.1.1 Preliminary considerations

Before dealing with the splitting of the 2-dimensional invariant manifolds, we make some simplifications. These simplifications were not needed in Chapter 2, but of course the results of that chapter also hold after making them.

Performing an additional step of the normal form procedure, we can write (1.6) in its normal form of order three:

$$\begin{aligned}
\frac{d\bar{x}}{d\bar{t}} &= \bar{x}\,(\nu - \beta_1 \bar{z}) + \bar{y}\,(\alpha_0 + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) \\
&\quad + \bar{x}A(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu) + \bar{y}B(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu) + \mathcal{O}_4(\bar{x}, \bar{y}, \bar{z}, \mu, \nu), \\
\frac{d\bar{y}}{d\bar{t}} &= -\bar{x}\,(\alpha_0 + \alpha_1\nu + \alpha_2\mu + \alpha_3\bar{z}) + \bar{y}\,(\nu - \beta_1\bar{z}) \\
&\quad + \bar{y}A(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu) - \bar{x}B(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu) + \mathcal{O}_4(\bar{x}, \bar{y}, \bar{z}, \mu, \nu), \\
\frac{d\bar{z}}{dt} &= -\mu + \bar{z}^2 + \gamma_2(\bar{x}^2+\bar{y}^2) + \gamma_3\mu^2 + \gamma_4\nu^2 + \gamma_5\mu\nu + C(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu) \\
&\quad + \mathcal{O}_4(\bar{x}, \bar{y}, \bar{z}, \mu, \nu).
\end{aligned} \tag{3.1}$$

where A, B and C are some functions satisfying:

$$\begin{aligned}
xA(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu),\ xB(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu) &= \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu), \\
yA(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu),\ yB(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu) &= \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu), \\
C(\bar{x}^2+\bar{y}^2, \bar{z}, \mu, \nu) &= \mathcal{O}_3(\bar{x}, \bar{y}, \bar{z}, \mu, \nu).
\end{aligned} \tag{3.2}$$

Now we rescale system (3.1) as we did in the beginning of Chapter 2. We define the new parameters $\delta = \sqrt{\mu}$, $\sigma = \delta^{-1}\nu$, and the new variables $x = \delta^{-1}\bar{x}$, $y = \delta^{-1}\bar{y}$, $z = \delta^{-1}\bar{z}$ and $t = \delta\bar{t}$. Then, renaming the coefficients $b = \gamma_2$, $c = \alpha_3$ and $\mathrm{d} = \beta_1$, system (3.1) becomes:

$$\begin{aligned}
\frac{dx}{dt} &= x\,(\sigma - \mathrm{d}z) + \left(\frac{\alpha(\delta^2, \delta\sigma)}{\delta} + cz\right) y + \delta^{-2} f(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\
\frac{dy}{dt} &= -\left(\frac{\alpha(\delta^2, \delta\sigma)}{\delta} + cz\right) x + y\,(\sigma - \mathrm{d}z) + \delta^{-2} g(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\
\frac{dz}{dt} &= -1 + b(x^2+y^2) + z^2 + \delta^{-2} h(\delta x, \delta y, \delta z, \delta, \delta\sigma),
\end{aligned} \tag{3.3}$$

where:

$$\alpha(\delta^2, \delta\sigma) = \alpha_0 + \alpha_1\delta\sigma + \alpha_2\delta^2,$$

and:

$$\begin{aligned}
f(\delta x, \delta y, \delta z, \delta, \delta\sigma) &= \delta x A(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) + \delta y B(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) \\
&\quad + \mathcal{O}_4(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \\
g(\delta x, \delta y, \delta z, \delta, \delta\sigma) &= \delta y A(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) - \delta x B(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) \\
&\quad + \mathcal{O}_4(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \\
h(\delta x, \delta y, \delta z, \delta, \delta\sigma) &= \gamma_3\delta^4 + \gamma_4\delta^2\sigma^2 + \gamma_5\delta^3\sigma + C(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) \\
&\quad + \mathcal{O}_4(\delta x, \delta y, \delta z, \delta^2, \delta\sigma).
\end{aligned} \tag{3.4}$$

We can introduce an artificial parameter $p \geq -2$ and consider the following system:

$$\begin{aligned}
\frac{dx}{dt} &= x(\sigma - \mathrm{d}z) + \left(\frac{\alpha(\delta^2, \delta\sigma)}{\delta} + cz\right) y + \delta^p f(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\
\frac{dy}{dt} &= -\left(\frac{\alpha(\delta^2, \delta\sigma)}{\delta} + cz\right) x + y(\sigma - \mathrm{d}z) + \delta^p g(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\
\frac{dz}{dt} &= -1 + b(x^2 + y^2) + z^2 + \delta^p h(\delta x, \delta y, \delta z, \delta, \delta\sigma),
\end{aligned} \tag{3.5}$$

If $p = -2$ we recover system (3.3), and if $p > -2$ we obtain a system with a smaller perturbation. The case $p > -2$ is referred to as the *regular case*, and it is simpler to study than the case $p = -2$, which we shall designate as the *singular case*.

Remark 3.1.1. Using classical perturbation methods, one can easily see that if σ is not of order δ^{p+3}, then the difference between the 2-dimensional invariant manifolds is not exponentially small. Therefore, in what follows we will assume always that $|\sigma| \leq \sigma^* \delta^{p+3}$, for some constant σ^*, since the exponentially small case is the only one where the Shilnikov phenomenon can occur. In Chapter 2 it has been proved that, in the 1-dimensional case, the distance is exponentially small even if σ is larger than $\mathcal{O}(\delta^{p+3})$.

Let us make some remarks on the functions f, g and h. Since $|\sigma| \leq \sigma^* \delta^{p+3}$, for bounded (x, y, z) one has:

$$\delta x A(\delta^2(x^2 + y^2), \delta z, \delta^2, \delta\sigma) = x\mathcal{O}(\delta^3),$$

and:

$$\delta y B(\delta^2(x^2 + y^2), \delta z, \delta^2, \delta\sigma) = y\mathcal{O}(\delta^3).$$

Then it is clear from the definition (3.4) of f that:

$$\lim_{(x,y,z,\delta)\to(0,0,\pm1,0)} \delta^{-3} f(\delta x, \delta y, \delta z, \delta, \delta\sigma) = 0. \tag{3.6}$$

An analogous argument yields that:

$$\lim_{(x,y,z,\delta)\to(0,0,\pm1,0)} \delta^{-3} g(\delta x, \delta y, \delta z, \delta, \delta\sigma) = 0. \tag{3.7}$$

Now, let us define:

$$\begin{aligned}
h_3 &= \lim_{(x,y,z,\delta)\to(0,0,1,0)} \delta^{-3} h(\delta x, \delta y, \delta z, \delta, \delta\sigma) \\
&= \lim_{(x,y,z,\delta)\to(0,0,1,0)} \delta^{-3} C(\delta^2(x^2 + y^2), \delta z, \delta^2, \delta\sigma),
\end{aligned} \tag{3.8}$$

where we have taken into account the definition (3.4) of h and that $|\sigma| \leq \sigma^* \delta^{p+3}$. Recall that by (3.2):

$$C(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) = \mathcal{O}_3(\delta x, \delta y, \delta z, \delta^2, \delta\sigma),$$

so that h_3 is well defined. Moreover, this fact also yields:

$$\lim_{(x,y,z,\delta)\to(0,0,-1,0)} \delta^{-3} h(\delta x, \delta y, \delta z, \delta, \delta\sigma) = -h_3.$$

Now let us perform the following change:

$$\tilde{z} = z + \frac{h_3}{2}\delta^{p+3}. \tag{3.9}$$

Then system (3.5) writes out as:

$$\begin{aligned} \frac{dx}{dt} &= x\,(\sigma - \mathrm{d}\tilde{z}) + \left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} + c\tilde{z}\right) y + \delta^p \tilde{f}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma), \\ \frac{dy}{dt} &= -\left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} + c\tilde{z}\right) x + y\,(\sigma - \mathrm{d}\tilde{z}) + \delta^p \tilde{g}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma), \\ \frac{d\tilde{z}}{dt} &= -1 + b(x^2+y^2) + \tilde{z}^2 + \delta^p \tilde{h}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma), \end{aligned} \tag{3.10}$$

where:

$$\begin{aligned} \tilde{f}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) &= \delta x \tilde{A}(\delta^2(x^2+y^2), \delta\tilde{z}, \delta^2, \delta\sigma) + \delta y \tilde{B}(\delta^2(x^2+y^2), \delta\tilde{z}, \delta^2, \delta\sigma) \\ &\quad + \mathcal{O}_4(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \\ \tilde{g}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) &= \delta y \tilde{A}(\delta^2(x^2+y^2), \delta\tilde{z}, \delta^2, \delta\sigma) - \delta x \tilde{B}(\delta^2(x^2+y^2), \delta\tilde{z}, \delta^2, \delta\sigma) \\ &\quad + \mathcal{O}_4(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \\ \tilde{h}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) &= -h_3 \tilde{z}\delta^3 + \frac{h_3^2}{4}\delta^{6+p} + \gamma_3\delta^4 + \gamma_4\delta^2\sigma^2 + \gamma_5\delta^3\sigma \\ &\quad + \tilde{C}(\delta^2(x^2+y^2), \delta\tilde{z}, \delta^2, \delta\sigma) + \mathcal{O}_4(\delta x, \delta y, \delta\tilde{z}, \delta^2, \delta\sigma), \end{aligned} \tag{3.11}$$

and:

$$\tilde{A}(\delta^2(x^2+y^2), \delta\tilde{z}, \delta^2, \delta\sigma) = A\left(\delta^2(x^2+y^2), \delta\left(\tilde{z} - \frac{h_3}{2}\delta^{p+3}\right), \delta^2, \delta\sigma\right) + \frac{h_3}{2}\mathrm{d}\delta^2,$$

$$\tilde{B}(\delta^2(x^2+y^2), \delta\tilde{z}, \delta^2, \delta\sigma) = B\left(\delta^2(x^2+y^2), \delta\left(\tilde{z} - \frac{h_3}{2}\delta^{p+3}\right), \delta^2, \delta\sigma\right) - \frac{h_3}{2}c\delta^2,$$

and:

$$\tilde{C}(\delta^2(x^2+y^2), \delta\tilde{z}, \delta^2, \delta\sigma) = C\left(\delta^2(x^2+y^2), \delta\left(\tilde{z} - \frac{h_3}{2}\delta^{p+3}\right), \delta^2, \delta\sigma\right).$$

From properties (3.6) and (3.7), the formulas for $\tilde{f}$ and $\tilde{g}$ and recalling again that we assume $|\sigma| \leq \sigma^* \delta^{p+3}$, it is clear that one has:

$$\begin{aligned} & \lim_{(x,y,\tilde{z},\delta)\to(0,0,\pm1,0)} \delta^{-3}\tilde{f}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) \\ = & \lim_{(x,y,\tilde{z},\delta)\to(0,0,\pm1,0)} \delta^{-3}\tilde{g}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) = 0. \end{aligned} \tag{3.12}$$

Moreover, by definition (3.8) of h_3 we also have:

$$\lim_{(x,y,\tilde{z},\delta)\to(0,0,\pm1,0)} \delta^{-3}\tilde{h}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) = 0. \tag{3.13}$$

Lemma 3.1.2. *Let $|\sigma| \leq \delta^{p+3}\sigma^*$ for some constant σ^*. Then, the vector field X defined by* (3.10) *has two critical points $\tilde{S}_\pm(\delta,\sigma) = (x_\pm(\delta,\sigma), y_\pm(\delta,\sigma), \tilde{z}_\pm(\delta,\sigma))$ of the form:*

$$x_\pm(\delta,\sigma) = \mathcal{O}(\delta^{p+5}), \qquad y_\pm(\delta,\sigma) = \mathcal{O}(\delta^{p+5}),$$

and:

$$\tilde{z}_\pm(\delta,\sigma) = \pm1 + \mathcal{O}(\delta^{p+4}).$$

Define:

$$\begin{aligned} \tilde{A}_\pm &= \delta^{-2}\tilde{A}(0,0,\pm\delta,\delta^2,\delta\sigma), \\ \tilde{B}_\pm &= \delta^{-2}\tilde{B}(0,0,\pm\delta,\delta^2,\delta\sigma), \\ \partial_z\tilde{C}_\pm &= \delta^{-3}\partial_z\tilde{C}(0,0,\pm\delta,\delta^2,\delta\sigma). \end{aligned} \tag{3.14}$$

One can easily see that $\tilde{A}_\pm$, $\tilde{B}_\pm$, $\partial_z\tilde{C}_\pm = \mathcal{O}(1)$ and are are well defined as $\delta \to 0$. The critical points $\tilde{S}_\pm(\delta,\sigma)$ are of saddle-focus type, and the corresponding eigenvalues of the differential matrix $DX(\tilde{S}_\pm(\delta,\sigma))$ are given by:

$$\begin{aligned} \lambda_1^\pm &= \sigma \mp \mathrm{d} + \delta^{p+3}\tilde{A}_\pm + i\left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} \pm c + \delta^{p+3}\tilde{B}_\pm\right) + \mathcal{O}(\delta^{p+4}), \\ \lambda_2^\pm &= \overline{\lambda_1^\pm} = \sigma \mp \mathrm{d} + \delta^{p+3}\tilde{A}_\pm - i\left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} \pm c + \delta^{p+3}\tilde{B}_\pm\right) + \mathcal{O}(\delta^{p+4}), \\ \lambda_3^\pm &= \pm2 + \delta^{p+3}\left(-h_3 + \partial_z\tilde{C}_\pm\right) + \mathcal{O}(\delta^{p+4}). \end{aligned}$$

Moreover, there exist two matrices $M_+(\delta,\sigma)$ and $M_-(\delta,\sigma)$ of the form:

$$M_\pm(\delta,\sigma) = \mathrm{Id} + \delta^{p+5}M_\pm^1(\delta,\sigma)$$

with $M_\pm^1(\delta,\sigma)$ having bounded entries for δ and σ sufficiently small, such that:

$$M_\pm^{-1}(\delta,\sigma)DX(\tilde{S}_\pm(\delta,\sigma))M_\pm(\delta,\sigma) = \begin{pmatrix} \mathrm{Re}\,\lambda_1^\pm & -\mathrm{Im}\,\lambda_1^\pm & 0 \\ \mathrm{Im}\,\lambda_1^\pm & \mathrm{Re}\,\lambda_1^\pm & 0 \\ 0 & 0 & \lambda_3^\pm \end{pmatrix}.$$

Proof. We note that the existence of these critical points was stated in Chapter 2. We start computing their asymptotic size in the setting of system (3.10). Clearly, $(x_\pm, y_\pm, \tilde{z}_\pm)$ are critical points of system (3.10) if $\hat{X}(x_\pm, y_\pm, \tilde{z}_\pm, \delta, \sigma) = 0$, where:

$$\hat{X}(x,y,\tilde{z},\delta,\sigma) = \begin{pmatrix} \delta x\,(\sigma - \mathrm{d}\tilde{z}) + (\alpha(\delta^2,\delta\sigma) + \delta c\tilde{z})\, y + \delta^{p+1}\tilde{f}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) \\ -\,(\alpha(\delta^2,\delta\sigma) + \delta c\tilde{z})\, x + \delta y\,(\sigma - \mathrm{d}\tilde{z}) + \delta^{p+1}\tilde{g}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) \\ -1 + b(x^2+y^2) + \tilde{z}^2 + \delta^p\tilde{h}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) \end{pmatrix}$$

For convenience, we consider the function:

$$\begin{aligned} &\mathcal{H}(x,y,\tilde{z},\delta,\sigma,\varepsilon_1,\varepsilon_2) \\ &= \begin{pmatrix} \delta x\,(\sigma - \mathrm{d}\tilde{z}) + (\alpha(\delta^2,\delta\sigma) + \delta c\tilde{z})\, y + \varepsilon_1\delta^{-3}\tilde{f}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) \\ -\,(\alpha(\delta^2,\delta\sigma) + \delta c\tilde{z})\, x + \delta y\,(\sigma - \mathrm{d}\tilde{z}) + \varepsilon_1\delta^{-3}\tilde{g}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) \\ -1 + b(x^2+y^2) + \tilde{z}^2 + \varepsilon_2\delta^{-3}\tilde{h}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma) \end{pmatrix} \end{aligned} \tag{3.15}$$

Note that $\mathcal{H}(x,y,\tilde{z},\delta,\sigma,\delta^{p+4},\delta^{p+3}) = \hat{X}(x,y,\tilde{z},\delta,\sigma)$.

Now, it is clear that:

$$\mathcal{H}(0,0,\pm1,0,0,0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad D_{xy\tilde{z}}\mathcal{H}(0,0,\pm1,0,0,0) = \begin{pmatrix} 0 & \alpha_0 & 0 \\ -\alpha_0 & 0 & 0 \\ 0 & 0 & \pm2 \end{pmatrix}.$$

Clearly $\det D_{xy\tilde{z}}\mathcal{H}(0,0,\pm1,0,0,0) \neq 0$, and thus the implicit function theorem gives us the existence of zeros of $\mathcal{H}$ of the form $(x_\pm(\delta,\sigma,\varepsilon_1,\varepsilon_2), y_\pm(\delta,\sigma,\varepsilon_1,\varepsilon_2), z_\pm(\delta,\sigma,\varepsilon_1,\varepsilon_2))$, for sufficiently small δ, σ, ε_1 and ε_2, of the form:

$$\begin{aligned} &\begin{pmatrix} x_\pm(\delta,\sigma,\varepsilon_1,\varepsilon_2) \\ y_\pm(\delta,\sigma,\varepsilon_1,\varepsilon_2) \\ z_\pm(\delta,\sigma,\varepsilon_1,\varepsilon_2) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \pm1 \end{pmatrix} - \left[D_{xyz}\mathcal{H}(0,0,\pm1,0,0,0)\right]^{-1} D_{\delta\sigma\varepsilon_1\varepsilon_2}\mathcal{H}(0,0,\pm1,0,0,0) \begin{pmatrix} \delta \\ \sigma \\ \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \\ &+\mathcal{O}_2(\delta,\sigma,\varepsilon_1,\varepsilon_2). \end{aligned}$$

One has:

$$D_{\delta\sigma\varepsilon_1\varepsilon_2}\mathcal{H}(0,0,\pm1,0,0,0) = \begin{pmatrix} 0 & 0 & \tilde{f}_3^\pm & 0 \\ 0 & 0 & \tilde{g}_3^\pm & 0 \\ 0 & 0 & 0 & \tilde{h}_3^\pm \end{pmatrix},$$

where:

$$\tilde{f}_3^\pm = \lim_{(x,y,\tilde{z},\delta)\to(0,0,\pm1,0)} \delta^{-3}\tilde{f}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma),$$

$$\tilde{g}_3^{\pm} = \lim_{(x,y,\tilde{z},\delta)\to(0,0,\pm1,0)} \delta^{-3}\tilde{g}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma),$$

$$\tilde{h}_3^{\pm} = \lim_{(x,y,\tilde{z},\delta)\to(0,0,\pm1,0)} \delta^{-3}\tilde{h}(\delta x, \delta y, \delta\tilde{z}, \delta, \delta\sigma).$$

As we pointed out in (3.12) and (3.13), $\tilde{f}_3^{\pm} = \tilde{g}_3^{\pm} = \tilde{h}_3^{\pm} = 0$, so that $(x_{\pm}, y_{\pm}, \tilde{z}_{\pm} \mp 1)$ begin with terms of order two in $(\delta, \sigma, \varepsilon_1, \varepsilon_2)$. Moreover, expanding $x_{\pm}$, $y_{\pm}$ and $\tilde{z}_{\pm}$ in their Taylor series, substituting them in (3.15) and equating all the terms of the corresponding order, one can see by induction that $x_{\pm}$ and $y_{\pm}$ do not have terms of the form $\delta^l\sigma^m\varepsilon_2^n$ for any $l, m, n \geq 0$. Similarly, one can see that $\tilde{z}_{\pm}$ does not have terms of the form $\delta^l\sigma^m$ for any $l, m \geq 0$. In other words:

$$\begin{pmatrix} x_{\pm}(\delta,\sigma,\varepsilon_1,\varepsilon_2) \\ y_{\pm}(\delta,\sigma,\varepsilon_1,\varepsilon_2) \\ \tilde{z}_{\pm}(\delta,\sigma,\varepsilon_1,\varepsilon_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm1 \end{pmatrix} + \begin{pmatrix} \varepsilon_1\mathcal{O}(\delta,\sigma,\varepsilon_1,\varepsilon_2) \\ \varepsilon_1\mathcal{O}(\delta,\sigma,\varepsilon_1,\varepsilon_2) \\ \varepsilon_1\mathcal{O}(\delta,\sigma,\varepsilon_1\varepsilon_2) + \varepsilon_2\mathcal{O}(\delta,\sigma,\varepsilon_1\varepsilon_2) \end{pmatrix}.$$

Finally, taking $\varepsilon_1 = \delta^{p+4}$, $\varepsilon_2 = \delta^{p+3}$, and recalling that $|\sigma| \leq \sigma^*\delta^{p+3}$ we obtain:

$$\begin{pmatrix} x_{\pm}(\delta,\sigma,\delta^{p+4},\delta^{p+3}) \\ y_{\pm}(\delta,\sigma,\delta^{p+4},\delta^{p+3}) \\ \tilde{z}_{\pm}(\delta,\sigma,\delta^{p+4},\delta^{p+3}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \pm1 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\delta^{p+5}) \\ \mathcal{O}(\delta^{p+5}) \\ \mathcal{O}(\delta^{p+4}) \end{pmatrix}.$$

In order to shorten the notation, for the rest of the proof we shall omit the dependence of $(x_{\pm}, y_{\pm}, \tilde{z}_{\pm})$ with respect to the parameters δ and σ.

Now we prove the given asymptotic formula of the eigenvalues. Taking into account the form (3.11) of $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$, one can check that:

$$\begin{aligned} &DX(x_{\pm}, y_{\pm}, z_{\pm}) = \\ &\begin{pmatrix} \sigma \mp \mathrm{d} + \delta^{p+3}\tilde{A}_{\pm} & \frac{\alpha(\delta^2,\delta\sigma)}{\delta} \pm c + \delta^{p+3}\tilde{B}_{\pm} & 0 \\ -\left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} \pm c + \delta^{p+3}\tilde{B}_{\pm}\right) & \sigma \mp \mathrm{d} + \delta^{p+3}\tilde{A}_{\pm} & 0 \\ 0 & 0 & \pm2 + \delta^{p+3}\left(-h_3 + \partial_z\tilde{C}_{\pm}\right) \end{pmatrix} \\ &+\mathcal{O}(\delta^{p+4}), \end{aligned} \tag{3.16}$$

where $\tilde{A}_{\pm}$, $\tilde{B}_{\pm}$ and $\partial_z\tilde{C}_{\pm}$ are defined in (3.14). Clearly, the matrix:

$$\begin{pmatrix} \sigma \mp \mathrm{d} + \delta^{p+3}\tilde{A}_{\pm} & \frac{\alpha(\delta^2,\delta\sigma)}{\delta} \pm c + \delta^{p+3}\tilde{B}_{\pm} & 0 \\ -\left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} \pm c + \delta^{p+3}\tilde{B}_{\pm}\right) & \sigma \mp \mathrm{d} + \delta^{p+3}\tilde{A}_{\pm} & 0 \\ 0 & 0 & \pm2 + \delta^{p+3}\left(-h_3 + \partial_z\tilde{C}_{\pm}\right) \end{pmatrix}$$

has eigenvalues:

$$\begin{aligned}
\hat{\lambda}_1^{\pm} &= \sigma \mp \mathrm{d} + \delta^{p+3}\tilde{A}_{\pm} + i\left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} \pm c + \delta^{p+3}\tilde{B}_{\pm}\right), \\
\hat{\lambda}_2^{\pm} &= = \sigma \mp \mathrm{d} + \delta^{p+3}\tilde{A}_{\pm} - i\left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} \pm c + \delta^{p+3}\tilde{B}_{\pm}\right), \\
\hat{\lambda}_3^{\pm} &= \pm 2 + \delta^{p+3}\left(-h_3 + \partial_z \tilde{C}_{\pm}\right).
\end{aligned} \tag{3.17}$$

By continuity of the eigenvalues, the eigenvalues of the matrix $DX(x_\pm, y_\pm, \tilde{z}_\pm)$ are $\lambda_j^{\pm} = \hat{\lambda}_j^{\pm} + \mathcal{O}(\delta^{p+4})$, $j = 1, 2, 3$.

Finally, we prove the existence of the matrices $M_\pm(\delta,\sigma)$ and their asymptotic form. First we look for the eigenvectors corresponding to each eigenvalue. We start with $\lambda_1^{\pm}$. Using (3.16), the fact that $\lambda_1^{\pm} = \hat{\lambda}_1^{\pm} + \mathcal{O}(\delta^{p+4})$ and definition (3.17) of $\hat{\lambda}_1^{\pm}$, one has:

$$DX(x_\pm, y_\pm, \tilde{z}_\pm) - \lambda_1^{\pm}\mathrm{Id} = \begin{pmatrix} -i\mathrm{Im}\,\hat{\lambda}_1^{\pm} & \mathrm{Im}\,\hat{\lambda}_1^{\pm} & 0 \\ -\mathrm{Im}\,\hat{\lambda}_1^{\pm} & -i\mathrm{Im}\,\hat{\lambda}_1^{\pm} & 0 \\ 0 & 0 & \pm 2 - \hat{\lambda}_1^{\pm} + \delta^{p+3}\left(-h_3 + \partial_z\tilde{C}_{\pm}\right) \end{pmatrix} + \mathcal{O}(\delta^{p+4})$$

Then, to find an eigenvector $u = (u_1, u_2, u_3)$ of $DX(x_\pm, y_\pm, \tilde{z}_\pm)$ of eigenvalue $\lambda_1^{\pm}$, it is enough to find a zero of:

$$\delta\left(DX(x_\pm, y_\pm, \tilde{z}_\pm) - \lambda_1^{\pm}\mathrm{Id}\right)u = \left[\begin{pmatrix} -i\delta\mathrm{Im}\,\hat{\lambda}_1^{\pm} & \delta\mathrm{Im}\,\hat{\lambda}_1^{\pm} & 0 \\ -\delta\mathrm{Im}\,\hat{\lambda}_1^{\pm} & -i\delta\mathrm{Im}\,\hat{\lambda}_1^{\pm} & 0 \\ 0 & 0 & \pm 2\delta - \delta\hat{\lambda}_1^{\pm} + \delta^{p+4}\left(-h_3 + \partial_z\tilde{C}_{\pm}\right) \end{pmatrix} + \mathcal{O}(\delta^{p+5})\right]\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

One can easily see that if $u_1 = 0$, then the only solution is $u_2 = u_3 = 0$, so that $u_1 \neq 0$ necessarily. In particular we can take $u_1 = 1$. Now we consider the function:

$$\mathcal{I}(u_2, u_3, \delta, \sigma, \varepsilon) = \left[\begin{pmatrix} -\delta\mathrm{Im}\,\hat{\lambda}_1^{\pm} & -i\delta\mathrm{Im}\,\hat{\lambda}_1^{\pm} & 0 \\ 0 & 0 & \pm 2\delta - \delta\hat{\lambda}_1^{\pm} + \delta^{p+4}\left(-h_3 + \partial_z\tilde{C}_{\pm}\right) \end{pmatrix} + \mathcal{O}(\varepsilon)\right]\begin{pmatrix} 1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{3.18}$$

We point out that if $\mathcal{I}(u_2, u_3, \delta, \sigma, \delta^{p+5}) = 0$, then $(1, u_2, u_3)$ is an eigenvector of the matrix $DX(x_\pm, y_\pm, \tilde{z}_\pm)$ of eigenvalue $\lambda_1^{\pm}$. Now, noting that $\delta\hat{\lambda}_1^{\pm} = i\alpha_0 + \mathcal{O}(\delta)$, one has:

$$\mathcal{I}(i, 0, 0, 0, 0) = 0, \qquad \det D_{u_2 u_3}\mathcal{I}(i, 0, 0, 0, 0) \neq 0,$$

so that again the implicit function theorem ensures the existence of functions $u_2(\delta,\sigma,\varepsilon)$ and $u_3(\delta,\sigma,\varepsilon)$ such that $\mathcal{I}(u_2(\delta,\sigma,\varepsilon),u_3(\delta,\sigma,\varepsilon),\delta,\sigma,\varepsilon)=0$. Moreover, expanding u_2 and u_3 in their Taylor series and using them in expression (3.18), one can easily see that all the terms of u_2 and u_3 of order larger or equal than one in $(\delta,\sigma,\varepsilon)$ are divisible by ε. Then we obtain that:

$$\begin{pmatrix} u_2(\delta,\sigma,\varepsilon) \\ u_3(\delta,\sigma,\varepsilon) \end{pmatrix} = \begin{pmatrix} i \\ 0 \end{pmatrix} + \mathcal{O}(\varepsilon).$$

Then one just needs to take $\varepsilon=\delta^{p+5}$, to obtain:

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 \\ i+\mathcal{O}(\delta^{p+5}) \\ \mathcal{O}(\delta^{p+5}) \end{pmatrix}$$

is an eigenvector of $DX(x_\pm,y_\pm,z_\pm)$ of eigenvalue $\lambda_1^\pm$. Analogously:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} i+\mathcal{O}(\delta^{p+5}) \\ 1 \\ \mathcal{O}(\delta^{p+5}) \end{pmatrix}$$

is an eigenvector of eigenvalue $\lambda_2^\pm$. Finally, to find an eigenvector $w=(w_1,w_2,w_3)$ of eigenvalue $\lambda_3^\pm$ we proceed similarly. The vector w must satisfy:

$$\begin{aligned} &\delta\left(DX(x_\pm,y_\pm,\tilde z_\pm)-\lambda_3^\pm \mathrm{Id}\right)w \\ &= \left[\begin{pmatrix} -\delta\hat\lambda_3^\pm+\delta(\sigma\pm \mathrm{d}) & \alpha(\delta^2,\delta\sigma)\pm\delta c & 0 \\ -\left(\alpha(\delta^2,\delta\sigma)\pm\delta c\right) & -\delta\hat\lambda_3^\pm+\delta(\sigma\pm \mathrm{d}) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathcal{O}(\delta^{p+5})\right]\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \end{aligned}$$

One can see that $w_3\neq 0$ necessarily, otherwise the only solution is $w_1=w_2=0$. Thus we can take $w_3=1$. Then one just needs to consider the function:

$$\mathcal{J}(w_1,w_2,\delta,\sigma,\varepsilon) = \left[\begin{pmatrix} -\delta\hat\lambda_3^\pm+\delta(\sigma\pm \mathrm{d}) & \alpha(\delta^2,\delta\sigma)\pm\delta c & 0 \\ -\left(\alpha(\delta^2,\delta\sigma)\pm\delta c\right) & -\delta\hat\lambda_3^\pm+\delta(\sigma\pm \mathrm{d}) & 0 \end{pmatrix} + \mathcal{O}(\varepsilon)\right]\begin{pmatrix} w_1 \\ w_2 \\ 1 \end{pmatrix},$$

Again, if $\mathcal{J}(w_1,w_2,\delta,\sigma,\delta^{p+5})=0$, then $(w_1,w_2,1)$ is an eigenvector of $DX(x_\pm,y_\pm,\tilde z_\pm)$ of eigenvalue $\lambda_3^\pm$. One has:

$$\mathcal{J}(0,0,0,0,0)=0, \qquad \det D_{w_1w_2}\mathcal{J}(0,0,0,0,0)\neq 0,$$

so that we can use the implicit function theorem. Reasoning as above yields:

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \mathcal{O}(\delta^{p+5}) \\ \mathcal{O}(\delta^{p+5}) \\ 1 \end{pmatrix}.$$

In conclusion, the matrix:

$$P(\delta,\sigma) = \begin{pmatrix} u, & v, & w \end{pmatrix} = \begin{pmatrix} 1 & i & 0 \\ i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\delta^{p+5}),$$

puts $DX(x_\pm, y_\pm, \tilde{z}_\pm)$ in its (complex) Jordan form. One can easily check that defining:

$$Q = \begin{pmatrix} 1/2 & -i/2 & 0 \\ -i/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then the matrix $M(\delta,\sigma) = P(\delta,\sigma)Q$ puts $DX(x_\pm, y_\pm, \tilde{z}_\pm)$ in its real Jordan form. A straightforward computation also gives that $M(\delta,\sigma) = \mathrm{Id} + \mathcal{O}(\delta^{p+5})$. □

3.1.2 Set-up

From now on we shall work with system (3.10) and drop tildes. That is, we consider the system:

$$\begin{aligned} \frac{dx}{dt} &= x\,(\sigma - \mathrm{d}z) + \left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} + cz\right) y + \delta^p f(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\ \frac{dy}{dt} &= -\left(\frac{\alpha(\delta^2,\delta\sigma)}{\delta} + cz\right) x + y\,(\sigma - \mathrm{d}z) + \delta^p g(\delta x, \delta y, \delta z, \delta, \delta\sigma), \\ \frac{dz}{dt} &= -1 + b(x^2+y^2) + z^2 + \delta^p h(\delta x, \delta y, \delta z, \delta, \delta\sigma), \end{aligned} \tag{3.19}$$

where $b > 0$, $\mathrm{d} > 0$, $\delta > 0$ is a small parameter and $|\sigma| < \mathrm{d}$. Moreover, $\alpha(\delta^2,\delta\sigma)$ is a function such that $\alpha(0,0) = \alpha_0 \neq 0$. The functions f, g and h are real analytic in $B^3(r_0) \times B(\delta_0) \times B(\sigma_0) \subset \mathbb{C}^3 \times \mathbb{R}^2$, and we assume r_0 to be large enough. They are of the form:

$$\begin{aligned} f(\delta x, \delta y, \delta z, \delta, \delta\sigma) &= \delta x A(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) + \delta y B(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) \\ &\quad + \mathcal{O}_4(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \\ g(\delta x, \delta y, \delta z, \delta, \delta\sigma) &= \delta y A(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) - \delta x B(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) \\ &\quad + \mathcal{O}_4(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \\ h(\delta x, \delta y, \delta z, \delta, \delta\sigma) &= -h_3 z\delta^3 + \frac{h_3^2}{4}\delta^{6+p} + \gamma_3\delta^4 + \gamma_4\delta^2\sigma^2 + \gamma_5\delta^3\sigma \\ &\quad + C(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) + \mathcal{O}_4(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \end{aligned} \tag{3.20}$$

where $h_3, \gamma_3, \gamma_4, \gamma_5$ are some fixed constants and A, B and C are functions satisfying:

$$\begin{aligned} \delta x A(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma),\ \delta x B(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) &= \mathcal{O}_3(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \\ \delta y A(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma),\ \delta y B(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) &= \mathcal{O}_3(\delta x, \delta y, \delta z, \delta^2, \delta\sigma), \\ C(\delta^2(x^2+y^2), \delta z, \delta^2, \delta\sigma) &= \mathcal{O}_3(\delta x, \delta y, \delta z, \delta^2, \delta\sigma). \end{aligned}$$

Moreover, f, g and h satisfy:

$$\lim_{(x,y,z,\delta)\to(0,0,\pm1,0)} \delta^{-3} f(\delta x, \delta y, \delta z, \delta, \delta\sigma) = 0,$$

$$\lim_{(x,y,z,\delta)\to(0,0,\pm1,0)} \delta^{-3} g(\delta x, \delta y, \delta z, \delta, \delta\sigma) = 0,$$

$$\lim_{(x,y,z,\delta)\to(0,0,\pm1,0)} \delta^{-3} h(\delta x, \delta y, \delta z, \delta, \delta\sigma) = 0.$$

Note that, in particular, $f,\ g,\ h = \mathcal{O}_3(\delta x, \delta y, \delta z, \delta, \delta\sigma)$. Of course, this is a much weaker property than the actual functions f, g and h satisfy, but for most of our purposes this will suffice. Since it is much more simple than the actual properties of f, g and h, whenever there is no need to use specific form (3.20) of f, g and h we shall simply say that $f,\ g,\ h = \mathcal{O}_3(\delta x, \delta y, \delta z, \delta, \delta\sigma)$.

Remark 3.1.3. Without loss of generality, we can assume that both α_0 and c are positive constants. In particular, for δ small enough, $\alpha(\delta^2, \delta\sigma)$ will be also positive.

From Lemma 3.1.2 we know that (3.19) has two critical points $S_+(\delta,\sigma)$ and $S_-(\delta,\sigma)$ of saddle-focus type. The aim of this chapter is to find suitable parameterizations of the two-dimensional invariant manifolds of these critical points and then provide an asymptotic formula of the difference between them.

For our purposes it will be very useful to write system (3.19) in "symplectic" cylindric coordinates:

$$x = \sqrt{2r}\cos\theta, \qquad y = \sqrt{2r}\sin\theta, \qquad z = z. \tag{3.21}$$

After this change system (3.19) writes out as:

$$\begin{aligned} \frac{dr}{dt} &= 2r(\sigma - \mathrm{d}z) + \delta^p F(\delta r, \theta, \delta z, \delta, \delta\sigma), \\ \frac{d\theta}{dt} &= -\frac{\alpha}{\delta} - cz + \delta^p G(\delta r, \theta, \delta z, \delta, \delta\sigma), \\ \frac{dz}{dt} &= -1 + 2br + z^2 + \delta^p H(\delta r, \theta, \delta z, \delta, \delta\sigma), \end{aligned} \tag{3.22}$$

where:

$$\begin{aligned} F(\delta r, \theta, \delta z, \delta, \delta\sigma) &= \sqrt{2r}\cos\theta f(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma) \\ &\quad + \sqrt{2r}\sin\theta g(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma), \\ G(\delta r, \theta, \delta z, \delta, \delta\sigma) &= \frac{1}{\sqrt{2r}}\cos\theta g(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma) \\ &\quad - \frac{1}{\sqrt{2r}}\sin\theta f(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma), \\ H(\delta r, \theta, \delta z, \delta, \delta\sigma) &= h(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma). \end{aligned} \tag{3.23}$$

Since $b > 0$, the unperturbed system (that is, system (3.22) with $F = G = H = 0$ and $\sigma = 0$), has a 2-dimensional heteroclinic manifold connecting $S_+(\delta, 0) = (0, 0, 1)$ and $S_-(\delta, 0) = (0, 0, -1)$ given by:

$$\Gamma := \left\{ (r, z) \in \mathbb{R}^2 \; : \; -1 + \frac{2br}{\mathrm{d}+1} + z^2 = 0 \right\}.$$

This manifold can be parameterized by the solutions of the unperturbed system starting at time $t = 0$ on the plane $z = 0$ and with angular variable $\theta = \theta_0$ by:

$$r = R_0(t) := \frac{(\mathrm{d}+1)}{2b} \frac{1}{\cosh^2(\mathrm{d}t)}, \tag{3.24}$$

$$\theta = \Theta_0(t, \theta_0) := \theta_0 - \frac{\alpha}{\delta} t - \frac{c}{\mathrm{d}} \log\cosh(\mathrm{d}t), \tag{3.25}$$

$$z = Z_0(t) := \tanh(\mathrm{d}t), \tag{3.26}$$

with $t \in \mathbb{R}$, and $\theta_0 \in [0, 2\pi)$. Our aim is to study the 2-dimensional invariant manifolds of $S_+(\delta, \sigma)$ and $S_-(\delta, \sigma)$ with $F, G, H \neq 0$.

3.1.3 Parameterizations of the 2-dimensional manifolds

We would like to write the two-dimensional invariant manifolds of $S_+(\delta, \sigma)$ and $S_-(\delta, \sigma)$ of system (3.22) as graphs over z and the angular variable θ:

$$\begin{aligned} x &= x^{\mathrm{u,s}}(z, \theta), \\ y &= y^{\mathrm{u,s}}(z, \theta), \\ z &= z. \end{aligned}$$

However, we will not do exactly that, but instead we will introduce a new variable v defined by:

$$v = Z_0^{-1}(z) = \mathrm{d}^{-1}\mathrm{atanh}(z),$$

or equivalently:

$$z = Z_0(v).$$

Then the invariant manifolds in symplectic polar coordinates will be parameterized by:

$$r = R^{\mathrm{u,s}}(v, \theta), \quad z = Z_0(v), \qquad v \in \mathbb{R}, \, \theta \in \mathbb{S}^1 \tag{3.27}$$

or in Cartesian coordinates:

$$x = \sqrt{2R^{\mathrm{u,s}}(v,\theta)} \cos\theta, \qquad y = \sqrt{2R^{\mathrm{u,s}}(v,\theta)} \sin\theta, \qquad z = Z_0(v).$$

This method, being very useful for our purposes, has some drawbacks. For example, it is obvious that when $v \to \pm\infty$ then $z = Z_0(v) \to \pm 1$. Thus, if the z-component of the critical points is not equal to ± 1 respectively, the method will not work.

To avoid such kind of problems, we will perform certain changes of variables. For example, for the unstable manifold of $S_-(\delta,\sigma) = (x_-(\delta,\sigma), y_-(\delta,\sigma), z_-(\delta,\sigma))$, first we do the following change:

$$(\tilde{x},\tilde{y},\tilde{z}) = C_1^{\mathrm{u}}(x,y,z,\delta,\sigma) = (x - x_-(\delta,\sigma), y - y_-(\delta,\sigma), z - z_-(\delta,\sigma)). \tag{3.28}$$

After this change, the critical point is $C_1^{\mathrm{u}}(S_-(\delta,\sigma),\delta,\delta\sigma) = (0,0,0)$. After that, we perform an additional change C_2^{u}:

$$\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = C_2^{\mathrm{u}}(\tilde{x},\tilde{y},\tilde{z},\delta,\sigma) = M_-(\delta,\sigma)\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \tag{3.29}$$

where $M_-(\delta,\sigma) = \mathrm{Id} + \delta^{p+5}M_-^1(\delta,\sigma)$ is the matrix given in Lemma 3.1.2, that puts the $DX(S_-(\delta,\sigma))$ in real Jordan form, where DX denotes the differential matrix of the vector field given by (3.19). We can write the change $C_2^{\mathrm{u}} \circ C_1^{\mathrm{u}}$ explicitly as:

$$\begin{aligned} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} &= C_2^{\mathrm{u}} \circ C_1^{\mathrm{u}}(x,y,z,\delta,\sigma) \\ &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x_-(\delta,\sigma) \\ y_-(\delta,\sigma) \\ z_-(\delta,\sigma)+1 \end{pmatrix} + \delta^{p+5}M_-^1(\delta,\sigma)\begin{pmatrix} x - x_-(\delta,\sigma) \\ y - y_-(\delta,\sigma) \\ z - z_-(\delta,\sigma) \end{pmatrix}. \end{aligned} \tag{3.30}$$

Clearly, by Lemma 3.1.2, $C_2^{\mathrm{u}} \circ C_1^{\mathrm{u}}$ is $\mathcal{O}(\delta^{p+4})-$close to the identity and puts the critical point at $\hat{S}_- = (0,0,-1)$. For the stable manifold of $S_+(\delta,\sigma)$ one performs changes C_1^{s} and C_2^{s} analogous to C_1^{u} and C_2^{u}, and in this case one has that the critical point becomes the point $\hat{S}_+ = (0,0,1)$.

After changes C_1^{u} and C_2^{u} (C_1^{s} and C_2^{s} respectively) and dropping hats, system (3.19) becomes:

$$\begin{aligned} \frac{dx}{dt} &= x\,(\sigma - \mathrm{d}z) + \left(\frac{\alpha}{\delta} + cz\right) y + \delta^p f^{\mathrm{u,s}}(\delta x,\delta y,\delta z,\delta,\delta\sigma), \\ \frac{dy}{dt} &= -\left(\frac{\alpha}{\delta} + cz\right) x + y\,(\sigma - \mathrm{d}z) + \delta^p g^{\mathrm{u,s}}(\delta x,\delta y,\delta z,\delta,\delta\sigma), \\ \frac{dz}{dt} &= -1 + b(x^2+y^2) + z^2 + \delta^p h^{\mathrm{u,s}}(\delta x,\delta y,\delta z,\delta,\delta\sigma), \end{aligned} \tag{3.31}$$

where we have omitted the dependence of α with respect to δ and σ. Here, $f^{\mathrm{u,s}}$, $g^{\mathrm{u,s}}$ and $h^{\mathrm{u,s}}$ are real analytic functions in $B^3(r_0) \times B(\delta_0) \times B(\sigma_0) \subset \mathbb{C}^3 \times \mathbb{R}^2$, with $f^{\mathrm{u,s}}$, $g^{\mathrm{u,s}}$, $h^{\mathrm{u,s}} = O_3(\delta x,\delta y,\delta z,\delta,\delta\sigma)$, and moreover:

$$\begin{aligned} f^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) &= g^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) = h^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) = 0, \\ f^{\mathrm{s}}(0,0,\delta,\delta,\delta\sigma) &= g^{\mathrm{s}}(0,0,\delta,\delta,\delta\sigma) = h^{\mathrm{s}}(0,0,\delta,\delta,\delta\sigma) = 0, \end{aligned} \tag{3.32}$$

and:

$$\begin{array}{rcccl} \partial_z f^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) & = & \partial_z g^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) & = & 0, \\ \partial_x h^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) & = & \partial_y h^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) & = & 0, \\ \partial_z f^{\mathrm{s}}(0,0,\delta,\delta,\delta\sigma) & = & \partial_z g^{\mathrm{s}}(0,0,\delta,\delta,\delta\sigma) & = & 0, \\ \partial_x h^{\mathrm{s}}(0,0,\delta,\delta,\delta\sigma) & = & \partial_y h^{\mathrm{s}}(0,0,\delta,\delta,\delta\sigma) & = & 0. \end{array} \tag{3.33}$$

System (3.31) in symplectic cylindric coordinates writes out as:

$$\begin{aligned} \frac{dr}{dt} &= 2r(\sigma - \mathrm{d}z) + \delta^p F^{\mathrm{u,s}}(\delta r, \theta, \delta z, \delta, \delta\sigma), \\ \frac{d\theta}{dt} &= -\frac{\alpha}{\delta} - cz + \delta^p G^{\mathrm{u,s}}(\delta r, \theta, \delta z, \delta, \delta\sigma), \\ \frac{dz}{dt} &= -1 + 2br + z^2 + \delta^p H^{\mathrm{u,s}}(\delta r, \theta, \delta z, \delta, \delta\sigma), \end{aligned} \tag{3.34}$$

where $F^{\mathrm{u,s}}$, $G^{\mathrm{u,s}}$ and $H^{\mathrm{u,s}}$ are defined analogously as F, G and H in (3.23) using $f^{\mathrm{u,s}}$, $g^{\mathrm{u,s}}$ and $h^{\mathrm{u,s}}$:

$$\begin{aligned} F^{\mathrm{u,s}}(\delta r, \theta, \delta z, \delta, \delta\sigma) &= \sqrt{2r}\cos\theta f^{\mathrm{u,s}}(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma) \\ &\quad + \sqrt{2r}\sin\theta g^{\mathrm{u,s}}(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma), \\ G^{\mathrm{u,s}}(\delta r, \theta, \delta z, \delta, \delta\sigma) &= \frac{1}{\sqrt{2r}}\cos\theta g^{\mathrm{u,s}}(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma) \\ &\quad - \frac{1}{\sqrt{2r}}\sin\theta f^{\mathrm{u,s}}(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma), \\ H^{\mathrm{u,s}}(\delta r, \theta, \delta z, \delta, \delta\sigma) &= h^{\mathrm{u,s}}(\delta\sqrt{2r}\cos\theta, \delta\sqrt{2r}\sin\theta, \delta z, \delta, \delta\sigma). \end{aligned} \tag{3.35}$$

We note that the unperturbed system remains the same after these changes, and hence it has the same heteroclinic connection defined in (3.24), (3.25) and (3.26).

Now we can proceed to look for parameterizations of the form (3.27). It is clear that the functions $R^{\mathrm{u,s}}$ have to satisfy, respectively, the invariance equation defined by the following PDE:

$$\frac{d\theta}{dt}\partial_\theta R^{\mathrm{u,s}} + \frac{dv}{dt}\partial_v R^{\mathrm{u,s}} = 2(\sigma - \mathrm{d}Z_0(v))R^{\mathrm{u,s}} + \delta^p F^{\mathrm{u,s}}(\delta R^{\mathrm{u,s}}, \theta, \delta Z_0(v), \delta, \delta\sigma),$$

that is, using (3.34) and that $\frac{dv}{dt} = \mathrm{d}^{-1}(1 - Z_0^2(v))^{-1}\frac{dz}{dt}$:

$$\begin{aligned} &\left(-\delta^{-1}\alpha - cZ_0(v) + \delta^p G^{\mathrm{u,s}}(\delta R^{\mathrm{u,s}}, \theta, \delta Z_0(v), \delta, \delta\sigma)\right)\partial_\theta R^{\mathrm{u,s}} \\ &+ \left(\frac{-1 + 2bR^{\mathrm{u,s}} + Z_0^2(v) + \delta^p H^{\mathrm{u,s}}(\delta R^{\mathrm{u,s}}, \theta, Z_0(v), \delta, \delta\sigma)}{\mathrm{d}(1 - Z_0^2(v))}\right)\partial_v R^{\mathrm{u,s}} \\ = \;& 2(\sigma - \mathrm{d}Z_0(v))R^{\mathrm{u,s}} + \delta^p F^{\mathrm{u,s}}(\delta R^{\mathrm{u,s}}, \theta, \delta Z_0(v), \delta, \delta\sigma). \end{aligned} \tag{3.36}$$

Now we impose that $R^{\mathrm{u,s}}(v,\theta) = R_0(v) + R_1^{\mathrm{u,s}}(v,\theta)$, where R_0 is given in (3.24). For the moment we will abuse notation and write:

$$\begin{aligned} F^{\mathrm{u,s}}(R_1^{\mathrm{u,s}}) &= F^{\mathrm{u,s}}(\delta(R_0(v) + R_1^{\mathrm{u,s}}(v,\theta)), \theta, \delta Z_0(v), \delta, \delta\sigma), \\ G^{\mathrm{u,s}}(R_1^{\mathrm{u,s}}) &= G^{\mathrm{u,s}}(\delta(R_0(v) + R_1^{\mathrm{u,s}}(v,\theta)), \theta, \delta Z_0(v), \delta, \delta\sigma), \\ H^{\mathrm{u,s}}(R_1^{\mathrm{u,s}}) &= H^{\mathrm{u,s}}(\delta(R_0(v) + R_1^{\mathrm{u,s}}(v,\theta)), \theta, \delta Z_0(v), \delta, \delta\sigma). \end{aligned} \tag{3.37}$$

Using that:

$$R_0'(v) = -2\mathrm{d}R_0(v)Z_0(v), \tag{3.38}$$

and:

$$-1 + 2bR_0(v) + Z_0^2(v) = \mathrm{d}(1 - Z_0^2(v)), \tag{3.39}$$

equation (3.36) writes out as:

$$\begin{aligned} & \left(-\delta^{-1}\alpha - cZ_0(v) + \delta^p G^{\mathrm{u,s}}(R_1^{\mathrm{u,s}})\right)\partial_\theta R_1^{\mathrm{u,s}} \\ & + \left(1 + \frac{2bR_1^{\mathrm{u,s}} + \delta^p H^{\mathrm{u,s}}(R_1^{\mathrm{u,s}})}{\mathrm{d}(1 - Z_0^2(v))}\right)(-2\mathrm{d}R_0(v)Z_0(v) + \partial_v R_1^{\mathrm{u,s}}) \\ = \; & 2(\sigma - \mathrm{d}Z_0(v))\left(R_0(v) + R_1^{\mathrm{u,s}}\right) + \delta^p F^{\mathrm{u,s}}(R_1^{\mathrm{u,s}}). \end{aligned} \tag{3.40}$$

Finally, noting that:

$$\frac{2bR_0(v)}{1 - Z_0^2(v)} = \mathrm{d} + 1,$$

and putting all terms which are either perturbative or non-linear in $R_1^{\mathrm{u,s}}$ in the right-hand side of the equality and the remaining terms in the left, (3.40) can be written as:

$$\begin{aligned} & \left(-\delta^{-1}\alpha - cZ_0(v)\right)\partial_\theta R_1^{\mathrm{u,s}} + \partial_v R_1^{\mathrm{u,s}} - 2Z_0(v)R_1^{\mathrm{u,s}} \\ = \; & 2\sigma(R_0(v) + R_1^{\mathrm{u,s}}) + \delta^p F^{\mathrm{u,s}}(R_1^{\mathrm{u,s}}) + \delta^p \frac{\mathrm{d}+1}{b} Z_0(v) H^{\mathrm{u,s}}(R_1^{\mathrm{u,s}}) - \delta^p G^{\mathrm{u,s}}(R_1^{\mathrm{u,s}})\partial_\theta R_1^{\mathrm{u,s}} \\ & - \left(\frac{2bR_1^{\mathrm{u,s}} + \delta^p H^{\mathrm{u,s}}(R_1^{\mathrm{u,s}})}{\mathrm{d}(1 - Z_0^2(v))}\right)\partial_v R_1^{\mathrm{u,s}}. \end{aligned} \tag{3.41}$$

For clarity, we will use the following notation for equation (3.41):

$$\mathcal{L}(R_1^{\mathrm{u,s}}) = \mathcal{F}^{\mathrm{u,s}}(R_1^{\mathrm{u,s}}), \tag{3.42}$$

where $\mathcal{L}$ is the linear operator defined by:

$$\mathcal{L}(R) := \left(-\delta^{-1}\alpha - cZ_0(v)\right)\partial_\theta R + \partial_v R - 2Z_0(v)R \tag{3.43}$$

and $\mathcal{F}^{\mathrm{u}}$, respectively $\mathcal{F}^{\mathrm{s}}$, by:

$$\begin{aligned} \mathcal{F}^{\mathrm{u,s}}(R) \;:=\; & 2\sigma(R_0(v) + R) + \delta^p F^{\mathrm{u,s}}(R) + \delta^p \frac{\mathrm{d}+1}{b} Z_0(v) H^{\mathrm{u,s}}(R) \\ & -\delta^p G^{\mathrm{u,s}}(R)\partial_\theta R - \left(\frac{2bR + \delta^p H^{\mathrm{u,s}}(R)}{\mathrm{d}(1 - Z_0^2(v))}\right)\partial_v R. \end{aligned} \tag{3.44}$$

The functions $R^{\mathrm{u}} = R_0 + R_1^{\mathrm{u}}$ and $R^{\mathrm{s}} = R_0 + R_1^{\mathrm{s}}$ lead to parameterizations of the invariant manifolds if $R_1^{\mathrm{u,s}}$ satisfy respectively:

$$\mathcal{L}(R_1^{\mathrm{u}}) = \mathcal{F}^{\mathrm{u}}(R_1^{\mathrm{u}}), \qquad \lim_{v\to-\infty} R_1^{\mathrm{u}}(v,\theta) = 0, \tag{3.45}$$

$$\mathcal{L}(R_1^{\mathrm{s}}) = \mathcal{F}^{\mathrm{s}}(R_1^{\mathrm{s}}), \qquad \lim_{v\to+\infty} R_1^{\mathrm{s}}(v,\theta) = 0. \tag{3.46}$$

Problems (3.45) and (3.46) can be written as fixed point equations using suitable right inverses of the operator $\mathcal{L}$. These right inverses can be found easily solving the ordinary differential equations satisfied by the Fourier coefficients $R^{[l]}(v)$ of any function $R(v,\theta)$ that is a solution of $\mathcal{L}(R) = \phi$, for a given function ϕ. Indeed, given $\phi(v,\theta)$, we define:

$$\mathcal{G}^{\mathrm{u}}(\phi)(v,\theta) := \sum_{l\in\mathbb{Z}} \mathcal{G}^{\mathrm{u}[l]}(\phi)(v)e^{il\theta}, \tag{3.47}$$

and $\mathcal{G}^{\mathrm{u}[l]}$ as:

$$\mathcal{G}^{\mathrm{u}[l]}(\phi)(v) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\int_{-\infty}^{0} \frac{e^{-il\left(\delta^{-1}\alpha s - cs + \frac{c}{\mathrm{d}}\log\left(1+e^{2\mathrm{d}(v+s)}\right) - \frac{c}{\mathrm{d}}\log\left(1+e^{2\mathrm{d}v}\right)\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+s))}\phi^{[l]}(v+s)ds. \tag{3.48}$$

In the stable case, instead, we will consider the operator $\mathcal{G}^{\mathrm{s}}$ defined analogously as $\mathcal{G}^{\mathrm{u}}$:

$$\mathcal{G}^{\mathrm{s}}(\phi)(v,\theta) := \sum_{l\in\mathbb{Z}} \mathcal{G}^{\mathrm{s}[l]}(\phi)(v)e^{il\theta}, \tag{3.49}$$

with coefficients:

$$\mathcal{G}^{\mathrm{s}[l]}(\phi)(v) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\int_{+\infty}^{0} \frac{e^{-il\left(\delta^{-1}\alpha s + cs + \frac{c}{\mathrm{d}}\log\left(1+e^{-2\mathrm{d}(v+s)}\right) - \frac{c}{\mathrm{d}}\log\left(1+e^{-2\mathrm{d}v}\right)\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+s))}\phi^{[l]}(v+s)ds. \tag{3.50}$$

Remark 3.1.4. We note that it would be more natural to consider the following Fourier coefficients:

$$\mathcal{G}^{\mathrm{u}[l]}(\phi)(v) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\int_{-\infty}^{0} \frac{e^{-il\left(\delta^{-1}\alpha s + \frac{c}{\mathrm{d}}\log\frac{\cosh(\mathrm{d}(v+s))}{\cosh(\mathrm{d}v)}\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+s))}\phi^{[l]}(v+s)ds. \tag{3.51}$$

and:

$$\mathcal{G}^{\mathrm{s}[l]}(\phi)(v) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\int_{+\infty}^{0} \frac{e^{-il\left(\delta^{-1}\alpha s + \frac{c}{\mathrm{d}}\log\frac{\cosh(\mathrm{d}(v+s))}{\cosh(\mathrm{d}v)}\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+s))}\phi^{[l]}(v+s)ds, \tag{3.52}$$

respectively in the unstable and stable case. However, expressions (3.51) and (3.52) are not well defined when we take complex values of v. For this reason we take definitions (3.48) and (3.50), which for real values of v coincide with (3.51) and (3.52), and are well defined when we take v in some complex domains which we will define below.

Lemma 3.1.5. *One has:*

$$\mathcal{L} \circ \mathcal{G}^{\mathrm{u,s}} = \mathrm{Id}.$$

Moreover, if we define the operators:

$$\tilde{\mathcal{F}}^{\mathrm{u,s}} := \mathcal{G}^{\mathrm{u,s}} \circ \mathcal{F}^{\mathrm{u,s}}, \tag{3.53}$$

with $\mathcal{F}^{\mathrm{u,s}}$ given in (3.44)*, we have that if R_1^{u} and R_1^{s} satisfy the fixed point equations:*

$$R_1^{\mathrm{u}} = \tilde{\mathcal{F}}^{\mathrm{u}}(R_1^{\mathrm{u}}), \qquad R_1^{\mathrm{s}} = \tilde{\mathcal{F}}^{\mathrm{s}}(R_1^{\mathrm{s}}), \tag{3.54}$$

then they are solutions of problems (3.45) *and* (3.46) *respectively.*

We now define the complex domains in which $R^{\mathrm{u,s}}$ will be defined. For the sake of clarity, we first define these domains for the unstable case.

We want these domains to be close to the singularities of the heteroclinic connection of the unperturbed system closest to the real line. From (3.24), (3.25) and (3.26) it is clear that these singularities are $\pm\frac{i\pi}{2\mathrm{d}}$. Moreover, for technical reasons, it will be convenient that these domains have a triangular shape. To this aim, let $0 < \beta < \pi/2$ and $\kappa^* > 0$ be two constants independent of δ and σ. Let $\kappa = \kappa(\delta)$ be any function satisfying that for $0 < \delta < 1$:

$$\kappa^*\delta \leq \kappa\delta \leq \frac{\pi}{4\mathrm{d}}. \tag{3.55}$$

Then we define the domain (see Figure 3.1a):

$$D^{\mathrm{u}}_{\kappa,\beta} = \left\{ v \in \mathbb{C} \,:\, |\mathrm{Im}\, v| \leq \frac{\pi}{2\mathrm{d}} - \kappa\delta - \tan\beta \mathrm{Re}\, v \right\}. \tag{3.56}$$

For technical reasons we will split the domain $D^{\mathrm{u}}_{\kappa,\beta}$ in two subsets. Let $T > 0$ be any constant independent of β, κ^*, δ and σ. Then we define (see Figure 3.1a):

$$D^{\mathrm{u}}_{\kappa,\beta,\infty} = \left\{ v \in D^{\mathrm{u}}_{\kappa,\beta} \,:\, \mathrm{Re}\, v \leq -T \right\}, \qquad D^{\mathrm{u}}_{\kappa,\beta,T} = \left\{ v \in D^{\mathrm{u}}_{\kappa,\beta} \,:\, \mathrm{Re}\, v \geq -T \right\}. \tag{3.57}$$

Analogously, for the stable case we define (see Figure 3.1b):

$$D^{\mathrm{s}}_{\kappa,\beta} = -D^{\mathrm{u}}_{\kappa,\beta}, \qquad D^{\mathrm{s}}_{\kappa,\beta,\infty} = -D^{\mathrm{u}}_{\kappa,\beta,\infty}, \qquad D^{\mathrm{s}}_{\kappa,\beta,T} = -D^{\mathrm{u}}_{\kappa,\beta,T}.$$

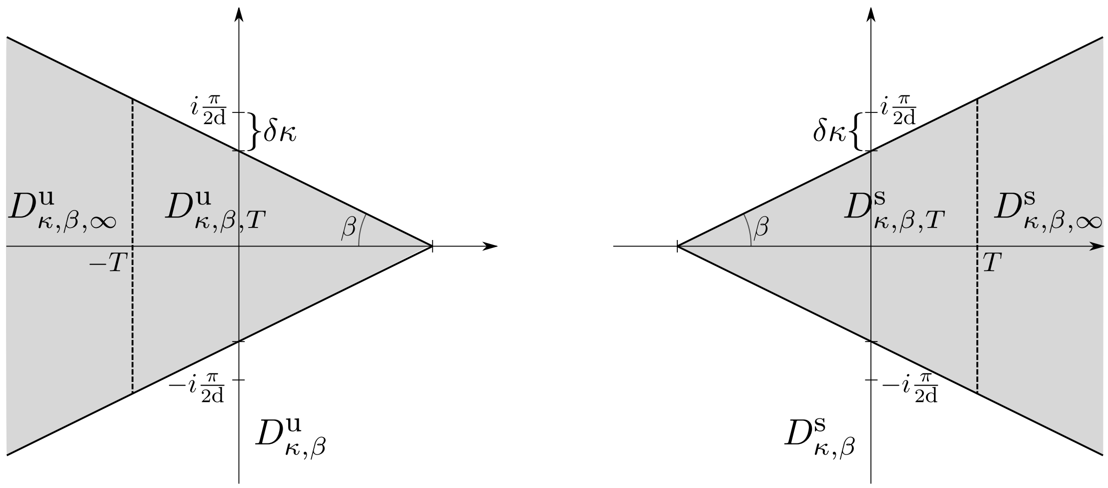
For any fixed real $\omega > 0$, we also define the complex domains (see Figure 3.2):

$$\mathbb{T}_\omega := \{\theta \in \mathbb{C}/(2\pi\mathbb{Z}) \,:\, |\mathrm{Im}\,\theta| \leq \omega\}. \tag{3.58}$$

If we want to follow an iterative scheme to find solutions of equations (3.54), we need to use a good first approximation. The candidates are:

$$R^{\mathrm{u}}_{10} := \tilde{\mathcal{F}}^{\mathrm{u}}(0), \qquad R^{\mathrm{s}}_{10} := \tilde{\mathcal{F}}^{\mathrm{s}}(0). \tag{3.59}$$

Indeed, as the following theorem shows, these will be good approximations of R_1^{u} and R_1^{s} for all $(v,\theta) \in D^{\mathrm{u,s}}_{\kappa,\beta} \times \mathbb{T}_\omega$ respectively. Moreover, this result gives also the asymptotic behavior of these functions when v tends to infinity or approaches the singularities $\pm i\pi/(2\mathrm{d})$.



(a) The outer domain $D^{u}_{\kappa,\beta}$ for the unstable case with its subdomains $D^{u}_{\kappa,\beta,T}$ and $D^{u}_{\kappa,\beta,\infty}$.

(b) The outer domain $D^{s}_{\kappa,\beta}$ for the stable case with its subdomains $D^{s}_{\kappa,\beta,T}$ and $D^{s}_{\kappa,\beta,\infty}$.

Figure 3.1: The outer domains $D^{u}_{\kappa,\beta}$ and $D^{s}_{\kappa,\beta}$.

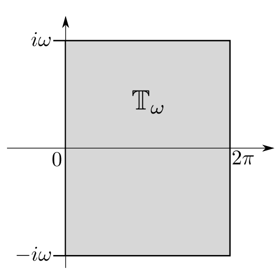


Figure 3.2: The domain $\mathbb{T}_{\omega}$.

Theorem 3.1.6 (Outer Theorem). *Let $p \geq -2$, $0 < \beta < \pi/2$ and $\omega > 0$ be any constants. There exist constants $\kappa^* \geq 1$, $\sigma^* > 0$ and $\delta^* > 0$, such that if $\kappa = \kappa(\delta)$ satisfies condition* (3.55) *then for all $0 < \delta \leq \delta^*$ and $|\sigma| \leq \sigma^* \delta^{p+3}$, problems* (3.45) *and* (3.46) *have solutions $R_1^{\mathrm{u}}(v,\theta)$ and $R_1^{\mathrm{s}}(v,\theta)$ defined for $(v,\theta) \in D^{\mathrm{u}}_{\kappa,\beta} \times \mathbb{T}_\omega$ and $(v,\theta) \in D^{\mathrm{s}}_{\kappa,\beta} \times \mathbb{T}_\omega$ respectively.*

Moreover $R_1^{\mathrm{u,s}} = R_{10}^{\mathrm{u,s}} + R_{11}^{\mathrm{u,s}}$, and there exists a constant M, independent of δ, σ and κ, such that:

$$|R_{10}^{\mathrm{u,s}}(v,\theta)| \leq \begin{cases} M\delta^{p+3}|\cosh(\mathrm{d}v)|^{-3} & \text{if } v \in D^{\mathrm{u,s}}_{\kappa,\beta,T}, \quad \theta \in \mathbb{T}_\omega, \\ M\delta^{p+3}|\cosh(\mathrm{d}v)|^{-2} & \text{if } v \in D^{\mathrm{u,s}}_{\kappa,\beta,\infty}, \quad \theta \in \mathbb{T}_\omega, \end{cases}$$

and:

$$|R_{11}^{\mathrm{u,s}}(v,\theta)| \leq \begin{cases} M\delta^{2p+6}|\cosh(\mathrm{d}v)|^{-4} & \text{if } v \in D^{\mathrm{u,s}}_{\kappa,\beta,T}, \quad \theta \in \mathbb{T}_\omega, \\ M\delta^{2p+6}|\cosh(\mathrm{d}v)|^{-2} & \text{if } v \in D^{\mathrm{u,s}}_{\kappa,\beta,\infty}, \quad \theta \in \mathbb{T}_\omega, \end{cases}$$

and for $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$ (respectively, $v \in D^{\mathrm{s}}_{\kappa,\beta,T}$) and for all $\theta \in \mathbb{T}_\omega$ one has:

$$|\partial_v R_1^{\mathrm{u,s}}(v,\theta)| \leq M\delta^{p+3}|\cosh(\mathrm{d}v)|^{-4}, \qquad |\partial_\theta R_1^{\mathrm{u,s}}(v,\theta)| \leq M\delta^{p+4}|\cosh(\mathrm{d}v)|^{-4}.$$

The proof of this result is postponed to Section 3.2.

3.1.4 Local parameterizations of the invariant manifolds

We point out that the fact that $R_1^{\mathrm{u}}(v,\theta)$ and $R_1^{\mathrm{s}}(v,\theta)$ satisfy different equations is not adequate for our purposes. We will now proceed to obtain new parameterizations $r_1^{\mathrm{u}}(v,\theta)$ and $r_1^{\mathrm{s}}(v,\theta)$ of the invariant manifolds. These new parameterizations will be solutions of the same functional equations and therefore suitable for our purposes of comparing them.

For $* = \mathrm{u},\mathrm{s}$, we define:

$$X^*(v,\theta) := \sqrt{2(R_0(v) + R_1^*(v,\theta))}\cos\theta, \qquad Y^*(v,\theta) := \sqrt{2(R_0(v) + R_1^*(v,\theta))}\sin\theta.$$

We note that $(X^*(v,\theta), Y^*(v,\theta), Z_0(v))$ is a parameterization of the unstable (respectively stable) manifold of the equilibrium point $\hat{S}_\mp = (0,0,\mp 1)$ of system (3.31). Then, considering the inverse of the changes C_1^* and C_2^* (defined in (3.28), (3.29)), we define:

$$(x^*(v,\theta), y^*(v,\theta), z^*(v,\theta)) := (C_1^*)^{-1} \circ (C_2^*)^{-1}(X^*(v,\theta), Y^*(v,\theta), Z_0(v)).$$

Now $(x^*(v,\theta), y^*(v,\theta), z^*(v,\theta))$ is a parameterization of the unstable (respectively stable) manifold of the equilibrium point $S_\mp(\delta,\sigma)$ (that is, the critical point of the original system (3.19)).

To compare both $(x^{\mathrm{u}}(v,\theta), y^{\mathrm{u}}(v,\theta))$ and $(x^{\mathrm{s}}(v,\theta), y^{\mathrm{s}}(v,\theta))$ on the $z-$plane, we consider new parameterizations depending on a parameter u defined by:

$$Z_0(u) = z^*(v,\theta),$$

or analogously:

$$u = u^*(v,\theta) := Z_0^{-1}(z^*(v,\theta)).$$

The functions $u^*(v,\theta)$ are defined for v belonging in the bounded domains $D^*_{\kappa,\beta,T}$, and they are a diffeomorphism. Therefore we can write $v = v^*(u,\theta)$ for $(u,\theta) \in D^*_{\bar\kappa,\beta,\bar T} \times \mathbb{T}_\omega$ for suitable $\bar\kappa > \kappa$ and $0 < \bar T < T$ (see Lemma 3.3.6 for more details). With this notation we have:

$$z^*(v(u,\theta),\theta) = Z_0(u).$$

Then $(x^*(v(u,\theta),\theta), y^*(v(u,\theta),\theta), Z_0(u))$ are other parameterizations of the unstable and stable manifolds of $S_\mp(\delta,\sigma)$ respectively.

Now we define $r^*(u,\theta)$ as:

$$r^*(u,\theta) = \frac{1}{2}\left[(x^*(v(u,\theta),\theta))^2 + (y^*(v(u,\theta),\theta))^2\right],$$

or equivalently:

$$x^*(v(u,\theta),\theta) = \sqrt{2r^*(u,\theta)}\cos\theta, \qquad y^*(v(u,\theta),\theta) = \sqrt{2r^*(u,\theta)}\sin\theta.$$

Note that $(x^*(v(u,\theta),\theta), y^*(v(u,\theta), Z_0(u))$ are a parameterization of the unstable and stable manifolds of the equilibrium points $S_\mp(\delta,\sigma)$ of system (3.19). Then, similarly as we deduced equation (3.36), but using equation (3.22) instead of (3.34), one has that r^* are solutions of the invariance equation:

$$\begin{aligned} &\left(-\delta^{-1}\alpha - cZ_0(u) + \delta^p G(\delta r,\theta,\delta Z_0(u),\delta,\delta\sigma)\right)\partial_\theta r \\ &+\left(\frac{-1+2br+Z_0^2(u)+\delta^p H(\delta r,\theta,Z_0(u),\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(u))}\right)\partial_u r \\ =\;& 2r(\sigma - \mathrm{d}Z_0(u)) + \delta^p F(\delta r,\theta,\delta Z_0(u),\delta,\delta\sigma). \end{aligned} \tag{3.60}$$

We note that now the functions F, G and H are the original ones defined in (3.23).

Finally we write:

$$r^{\mathrm{u,s}}(u,\theta) = R_0(u) + r_1^{\mathrm{u,s}}(u,\theta). \tag{3.61}$$

Then, from (3.60), one can see that r_1^{u} and r_1^{s} are solutions of:

$$\begin{aligned} &\left(-\delta^{-1}\alpha - cZ_0(u)\right)\partial_\theta r_1 + \partial_u r_1 - 2Z_0(u)r_1 = 2\sigma(R_0(u)+r_1) + \delta^p F(r_1) \\ &+\delta^p\frac{\mathrm{d}+1}{b}Z_0(u)H(r_1) - \delta^p G(r_1)\partial_\theta r_1 - \left(\frac{2br_1+\delta^p H(r_1)}{\mathrm{d}(1-Z_0^2(u))}\right)\partial_u r_1, \end{aligned} \tag{3.62}$$

where we have used the same abuse of notation for F, G and H as in (3.37). For clarity, we will use the following notation for equation (3.62):

$$\mathcal{L}(r_1^{\mathrm{u,s}}) = \mathcal{F}(r_1^{\mathrm{u,s}}), \tag{3.63}$$

where $\mathcal{L}$ is the operator defined in (3.43) and $\mathcal{F}$ is the operator:

$$\begin{aligned}\mathcal{F}(r) \quad := \quad & 2\sigma(R_0(u)+r)+\delta^p F(r)+\delta^p\frac{\mathrm{d}+1}{b}Z_0(u)H(r)\\ & -\delta^p G(r)\partial_\theta r-\left(\frac{2br+\delta^p H(r)}{\mathrm{d}(1-Z_0^2(u))}\right)\partial_u r. \end{aligned} \tag{3.64}$$

Next result summarizes these ideas and gives the main properties of the functions $r_1^{\mathrm{u}}(u,\theta)$ and $r_1^{\mathrm{s}}(u,\theta)$. We point out that we abuse notation and use the parameters κ, β T and ω of the domains $D^{\mathrm{u}}_{\kappa,\beta}\times\mathbb{T}_\omega$ and $D^{\mathrm{s}}_{\kappa,\beta}\times\mathbb{T}_\omega$, being understood that they are not the same κ, β, T and ω appearing in Theorem 3.1.6.

Theorem 3.1.7. *Let $p\geq -2$ and $0<\beta<\pi/2$ be any constants. There exist constants $\kappa^*\geq 1$, $\sigma^*>0$ and $\delta^*>0$, such that if $\kappa=\kappa(\delta)$ satisfies condition* (3.55)*, then for all $0<\delta\leq\delta^*$ and $|\sigma|\leq\sigma^*\delta^{p+3}$, the unstable manifold of $S_-(\delta,\sigma)$ and the stable manifold of $S_+(\delta,\sigma)$ are given respectively by:*

$$W^{\mathrm{u}}(u,\theta)=(r^{\mathrm{u}}(u,\theta)\cos\theta, r^{\mathrm{u}}(u,\theta)\sin\theta, Z_0(u)), \qquad (u,\theta)\in D^{\mathrm{u}}_{\kappa,\beta,T}\times\mathbb{T}_\omega,$$

$$W^{\mathrm{s}}(u,\theta)=(r^{\mathrm{s}}(u,\theta)\cos\theta, r^{\mathrm{s}}(u,\theta)\sin\theta, Z_0(u)), \qquad (u,\theta)\in D^{\mathrm{s}}_{\kappa,\beta,T}\times\mathbb{T}_\omega,$$

with:

$$r^{\mathrm{u}}(u,\theta)=R_0(u)+r_1^{\mathrm{u}}(u,\theta), \qquad r^{\mathrm{s}}(u,\theta)=R_0(u)+r_1^{\mathrm{s}}(u,\theta),$$

where r_1^{u} and r_1^{s} satisfy the same equation (3.62)*.*

One has:

$$r_1^{\mathrm{u}}=r_{10}^{\mathrm{u}}+r_{11}^{\mathrm{u}}, \qquad r_1^{\mathrm{s}}=r_{10}^{\mathrm{s}}+r_{11}^{\mathrm{s}}$$

with:

$$r_{10}^{\mathrm{u}}:=\mathcal{G}^{\mathrm{u}}\circ\mathcal{F}(0), \qquad r_{10}^{\mathrm{s}}:=\mathcal{G}^{\mathrm{s}}\circ\mathcal{F}(0),$$

where $\mathcal{G}^{\mathrm{u}}$ is defined in (3.47)*–*(3.48)*, $\mathcal{G}^{\mathrm{s}}$ is defined in* (3.49)*–*(3.50) *and $\mathcal{F}$ is defined in* (3.64)*. Moreover, $r_{10}^{\mathrm{u,s}}(u,\theta)$ is defined for $u\in D^{\mathrm{u,s}}_{\kappa,\beta,\infty}\cup D^{\mathrm{u,s}}_{\kappa,\beta,T}$ and $\theta\in\mathbb{T}_\omega$, and there exists a constant M such that for all $(u,\theta)\in D^{\mathrm{u,s}}_{\kappa,\beta,T}\times\mathbb{T}_\omega$:*

$$|r_{10}^{\mathrm{u,s}}(u,\theta)|\leq M\delta^{p+3}|\cosh(\mathrm{d}u)|^{-3}$$

$$|r_{11}^{\mathrm{u,s}}(u,\theta)|\leq M\left(\delta^{2p+6}|\cosh(\mathrm{d}u)|^{-4}+\delta^{p+4}|\cosh(\mathrm{d}u)|^{-1}\right),$$

and:

$$|\partial_u r_1^{\mathrm{u,s}}(u,\theta)|\leq M\delta^{p+3}|\cosh(\mathrm{d}u)|^{-4}, \qquad |\partial_\theta r_1^{\mathrm{u,s}}(u,\theta)|\leq M\delta^{p+4}|\cosh(\mathrm{d}u)|^{-4}.$$

The proof of this Theorem can be found in Section 3.3.

3.1.5 The Melnikov function

Our final aim is to find an asymptotic formula of the difference $\Delta(u,\theta) = r^{\mathrm{u}}(u,\theta) - r^{\mathrm{s}}(u,\theta)$. Recall that by Theorem 3.1.7 we have:

$$\Delta(u,\theta) = r^{\mathrm{u}}_{10}(u,\theta) - r^{\mathrm{s}}_{10}(u,\theta) + r^{\mathrm{u}}_{11}(u,\theta) - r^{\mathrm{s}}_{11}(u,\theta). \tag{3.65}$$

Also by Theorem 3.1.7, we know that r^{u}_{10} and r^{s}_{10} are larger than r^{u}_{11} and r^{s}_{11}. Hence, it is natural to think that the first order of the difference is given by the difference of these dominant terms. That is, we expect that:

$$\Delta(u,\theta) = r^{\mathrm{u}}_{10}(u,\theta) - r^{\mathrm{s}}_{10}(u,\theta) + \text{h.o.t.}$$

In the following, we will see that this is true for $p > -2$, that is, for non-generic unfoldings. We deal with the case $p = -2$ in Chapter 4, and in fact we prove that this is not true.

Let us consider the difference between r^{u}_{10} and r^{s}_{10}:

$$M(u,\theta) := r^{\mathrm{u}}_{10}(u,\theta) - r^{\mathrm{s}}_{10}(u,\theta) = \mathcal{G}^{\mathrm{u}}(\mathcal{F}(0)) - \mathcal{G}^{\mathrm{s}}(\mathcal{F}(0)). \tag{3.66}$$

Since $M(u,\theta)$ is $2\pi-$periodic in θ, its second variable, we can write its Fourier series:

$$M(u,\theta) = \sum_{l\in\mathbb{Z}} M^{[l]}(u)e^{il\theta}. \tag{3.67}$$

By Theorem 3.1.7 and expressions (3.48) and (3.50) of the Fourier coefficients $\mathcal{G}^{\mathrm{u}[l]}$ and $\mathcal{G}^{\mathrm{s}[l]}$ we can find formulas for the coefficients $M^{[l]}(u)$. However, since we just need to deal with $u \in \mathbb{R}$ we can consider expressions (3.51) and (3.52) (that are equivalent to (3.48) and (3.50) for $u \in \mathbb{R}$), and then one sees that:

$$\begin{aligned} &M^{[l]}(u) = \\ &\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)e^{il(\delta^{-1}\alpha u+\mathrm{cd}^{-1}\log\cosh(\mathrm{d}u))}\int_{-\infty}^{+\infty}\frac{e^{-il(\delta^{-1}\alpha w+\mathrm{cd}^{-1}\log\cosh(\mathrm{d}w))}\mathcal{F}^{[l]}(0)(w)}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw. \end{aligned} \tag{3.68}$$

In fact, one can easily see that:

$$M(u,\theta) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_{-\infty}^{+\infty}\frac{\mathcal{F}(0)\left(w,\theta-\delta^{-1}\alpha(w-u)-\mathrm{cd}^{-1}\log\left(\frac{\cosh(\mathrm{d}w)}{\cosh(\mathrm{d}u)}\right)\right)}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw, \tag{3.69}$$

which is the well known Melnikov function adapted to this problem. Moreover, from (3.68) it is clear that we can write series (3.67) as:

$$M(u,\theta) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\sum_{l\in\mathbb{Z}}\Upsilon_0^{[l]}e^{il(\theta+\delta^{-1}\alpha u+\mathrm{cd}^{-1}\log\cosh(\mathrm{d}u))}, \tag{3.70}$$

where $\Upsilon_0^{[l]}$ are the coefficients:

$$\Upsilon_0^{[l]} = \int_{-\infty}^{+\infty} \frac{e^{-il(\delta^{-1}\alpha w + \mathrm{c}\mathrm{d}^{-1}\log\cosh(\mathrm{d}w))}\mathcal{F}^{[l]}(0)(w)}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}w)} dw. \tag{3.71}$$

Note that $\Upsilon_0^{[l]}$ are independent of u. In addition:

$$M^{[l]}(u) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u) e^{il(\delta^{-1}\alpha u + \mathrm{c}\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))}\Upsilon_0^{[l]}.$$

In the following lemma we provide closed formulas for $\Upsilon_0^{[1]}$ and $\Upsilon_0^{[-1]}$ in terms of Borel transforms of some functions depending on the perturbative terms. We also prove that (besides the average $\Upsilon_0^{[0]}$) they are the dominant coefficients. To this purpose, we recall that given a function $m(w,\theta) = \sum_{n\geq 0} m_n(\theta) w^{n+1+ik}$, periodic in θ, we define its Borel transform $\hat{m}(\zeta,\theta)$ as:

$$\hat{m}(\zeta,\theta) = \sum_{n\geq 0} m_n(\theta) \frac{\zeta^{n+ik}}{\Gamma(n+1+ik)}.$$

Theorem 3.1.8. *Consider the function:*

$$m(w,\theta) = w^{\frac{2}{\mathrm{d}}+i\frac{\mathrm{c}}{\mathrm{d}}} F\left(\frac{\mathrm{d}+1}{2b}w^2, \theta, -iw, 0, 0\right) - i\frac{\mathrm{d}+1}{b} w^{1+\frac{2}{\mathrm{d}}+i\frac{\mathrm{c}}{\mathrm{d}}} H\left(\frac{\mathrm{d}+1}{2b}w^2, \theta, -iw, 0, 0\right), \tag{3.72}$$

where F and H are the functions defined in (3.23)*. Let $\hat{m}^{[l]}(\zeta)$ be the l-th Fourier coefficient of its Borel transform $\hat{m}(\zeta,\theta) = \sum_{l\in\mathbb{Z}} \hat{m}^{[l]}(\zeta)e^{il\theta}$. Then:*

$$\begin{aligned}
\Upsilon_0^{[1]} &= \frac{2\pi}{\mathrm{d}} \delta^{p-\frac{2}{\mathrm{d}}-i\frac{\mathrm{c}}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \left(\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right) + \mathcal{O}(\delta)\right), \\
\Upsilon_0^{[-1]} &= \overline{\Upsilon_0^{[1]}} = \frac{2\pi}{\mathrm{d}} \delta^{p-\frac{2}{\mathrm{d}}+i\frac{\mathrm{c}}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \left(\overline{\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right)} + \mathcal{O}(\delta)\right),
\end{aligned}$$

where $\overline{z}$ denotes the complex conjugate of z.

Moreover, there exists a constant K such that for all $|l| \geq 2$:

$$\left|\Upsilon_0^{[l]}\right| \leq K\delta^{p-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}\frac{3|l|}{4}}. \tag{3.73}$$

In conclusion, defining:

$$\vartheta(u,\delta) = \delta^{-1}\alpha u + \mathrm{c}\mathrm{d}^{-1}\left[\log\cosh(\mathrm{d}u) - \log\delta\right]$$

for $u \in \mathbb{R}$ and $\theta \in \mathbb{S}^1$ one has that:

$$\begin{aligned}
M(u,\theta) &= \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\Bigg[\Upsilon_0^{[0]} + \\
&\frac{2\pi}{\mathrm{d}}\delta^{p-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \Big(C_1\cos(\theta+\vartheta(u,\delta)) + C_2\sin(\theta+\vartheta(u,\delta)) + \mathcal{O}(\delta)\Big)\Bigg],
\end{aligned} \tag{3.74}$$

where:

$$C_1 = 2\mathrm{Re}\left(\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right)\right), \qquad C_2 = -2\mathrm{Im}\left(\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right)\right).$$

The proof of this result can be found in Section 3.4.

We point out that, due to the exponential smallness of $\Upsilon_0^{[l]}$, $|l| \geq 1$, the dominant term of the Melnikov function for real values of u is its average $\Upsilon_0^{[0]}$. However, we are interested in the case where this difference is exponentially small, because it is the case when the Shilnikov phenomenon is expected to occur. We will give more details about this coefficient in Subsection 3.1.7, Theorem 3.1.10.

3.1.6 The difference

In this section we shall study the difference $\Delta(u,\theta) = r_1^{\mathrm{u}}(u,\theta) - r_1^{\mathrm{s}}(u,\theta)$. Here we will give only the main result and some intuitive ideas of how it is proved. For all the details we refer the reader to Section 3.5.

The first thing we have to do is to find an equation for the difference Δ. To this aim, we subtract the PDEs (3.62) for r_1^{u} and r_1^{s}, and then using the mean value theorem we obtain an equation of the following form:

$$\begin{aligned} & \left(-\delta^{-1}\alpha - cZ_0(u)\right)\partial_\theta\Delta + \partial_u\Delta - 2Z_0(u)\Delta \\ = \; & (2\sigma + l_1(u,\theta))\Delta + l_2(u,\theta)\partial_u\Delta + l_3(u,\theta)\partial_\theta\Delta. \end{aligned} \tag{3.75}$$

Here the functions $l_i(u,\theta)$, $i = 1,\ldots 3$, are some functions which are "small" in the appropriate sense. The exact expression of these functions and the precise meaning of "small" will be given in Section 3.5.

Recall that r_1^{u} and r_1^{s} are defined respectively in the domains $D^{\mathrm{u}}_{\kappa,\beta,T} \times \mathbb{T}_\omega$ and $D^{\mathrm{s}}_{\kappa,\beta,T} \times \mathbb{T}_\omega$. Thus, their difference will be defined in the intersection of these two domains. So from now on we will consider $(u,\theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$, where we define $D_{\kappa,\beta}$ as (see Figure 3.3):

$$D_{\kappa,\beta} = D^{\mathrm{u}}_{\kappa,\beta,T} \cap D^{\mathrm{s}}_{\kappa,\beta,T}. \tag{3.76}$$

Now we present the heuristic ideas of the results of Section 3.5. We will study *all* the solutions of equation (3.75). More precisely, we shall see that all of them can be written in a special form. By the so-called method of variation of constants, every solution $\Delta(u,\theta)$ of (3.75) can be written as:

$$\Delta(u,\theta) = P(u,\theta)k(u,\theta), \tag{3.77}$$

where $P(u,\theta)$ is a particular solution of this same equation satisfying $P(u,\theta) \neq 0$, and $k(u,\theta)$ satisfies the associated homogeneous PDE:

$$\left(-\delta^{-1}\alpha - cZ_0(u)\right)\partial_\theta k + \partial_u k = l_2(u,\theta)\partial_u k + l_3(u,\theta)\partial_\theta k. \tag{3.78}$$

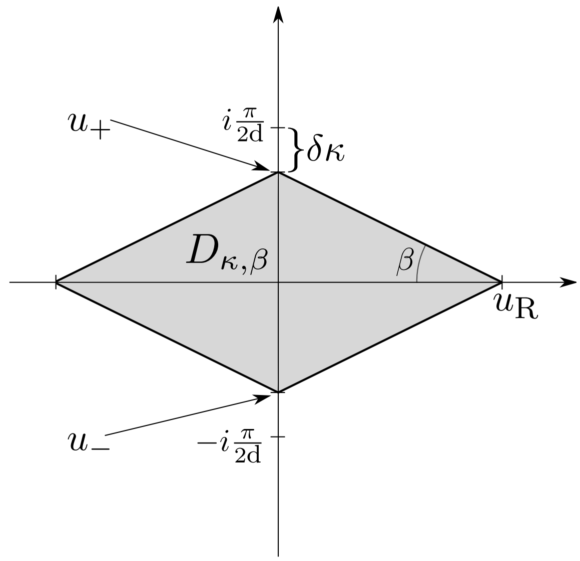


Figure 3.3: The domain $D_{\kappa,\beta}$.

This result is stated more precisely in Lemma 3.5.3. Let us now mention some properties of these functions $k(u,\theta)$ and $P(u,\theta)$.

To study the function $k(u,\theta)$ we shall rely on the form of equation (3.78). These kind of equations have been broadly studied over the past. One of its main features is that if $\xi(u,\theta)$ is a particular solution of (3.78) such that $(\xi(u,\theta),\theta)$ is injective in $D_{\kappa,\beta}\times\mathbb{T}_\omega$, then *any* solution $k(u,\theta)$ of (3.78) can be written as:

$$k(u,\theta)=\tilde{k}(\xi(u,\theta)),$$

for some function $\tilde{k}(\tau)$. Thus, we have to find a suitable particular solution of equation (3.78). Since the functions l_i are "small", equation (3.78) is a small perturbation of:

$$\left(-\delta^{-1}\alpha-cZ_0(u)\right)\partial_\theta k+\partial_u k=0.$$

A solution of this equation is given by $\xi_0(u,\theta)=\theta+\delta^{-1}\alpha u+c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u)$. Then, we shall look for a particular solution of (3.78) of the following form:

$$\xi(u,\theta)=\theta+\delta^{-1}\alpha u+c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u)+C(u,\theta),$$

where, as we expect, $C(u,\theta)$ will be a "small" function. These results are contained in a rigorous way in Propositions 3.5.4 and 3.5.5.

To study the particular solution $P(u,\theta)$ of (3.75) we note that, being $\sigma=\mathcal{O}(\delta^{p+3})$ and l_i "small", equation (3.75) is a "small" perturbation of:

$$\left(-\delta^{-1}\alpha-cZ_0(u)\right)\partial_\theta\Delta+\partial_u\Delta-2Z_0(u)\Delta=0.$$

A solution of this equation is given by $P_0(u) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)$. Therefore, we shall look for a particular solution of (3.75) of the form:

$$P(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1 + P_1(u,\theta)),$$

where $P_1(u,\theta)$ will be "small". The rigorous statement of these result is given in Proposition 3.5.7.

As a conclusion of all the previous considerations, one obtains the following result, which characterizes the form of the difference Δ as well as the sizes of the functions $P_1(u,\theta)$ and $C(u,\theta)$ described above.

Theorem 3.1.9. *Let $p \geq -2$ and $|\sigma| \leq \delta^{p+3}\sigma^*$. The difference $\Delta(u,\theta)$ can be written as:*

$$\Delta(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1 + P_1(u,\theta))\tilde{k}(\xi(u,\theta)), \tag{3.79}$$

where $\tilde{k}(\tau)$ is a 2π−periodic function, the function $\xi(u,\theta)$ is defined as:

$$\xi(u,\theta) = \theta + \delta^{-1}\alpha u + \mathrm{c}\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta),$$

and $P_1(u,\theta)$ and $C(u,\theta)$ are real analytic functions, defined for all $(u,\theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$ and such that:

1. *There exist a constant $L_0 \in \mathbb{R}$ and functions $L(u)$ and $\chi(u,\theta)$ such that:*

$$C(u,\theta) = \delta^{p+2}\mathrm{d}^{-1}\alpha L_0 \log\cosh(\mathrm{d}u) + \alpha L(u) + \chi(u,\theta),$$

   where, for all $(u,\theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$:

$$|L(u)| \leq M\delta^{p+2}, \qquad |L'(u)| \leq M\delta^{p+2}, \qquad |\chi(u,\theta)| \leq \frac{M\delta^{p+3}}{|\cosh(\mathrm{d}u)|}, \tag{3.80}$$

   for some constant M. L_0 and $L(u)$ are determined by a finite number of Taylor coefficients of the functions f, g and h appearing in (3.19)*. A formula for L_0 is given in Lemma 3.5.17, and a formula for $L(u)$ is given in Remark 3.5.18. Moreover, $L(0) = 0$ and $L(u)$ is defined on the limit $u \to i\pi/(2\mathrm{d})$.*

2. *For all $(u,\theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$:*

$$|P_1(u,\theta)| \leq \frac{M\delta^{p+3}}{|\cosh(\mathrm{d}u)|}, \tag{3.81}$$

   for some constant M.

   Moreover, in the conservative case P_1 can be chosen as:

$$P_1(u,\theta) = \frac{\partial_u C(u,\theta) - l_3(u,\theta)}{\delta^{-1}\alpha + cZ_0(u) + l_3(u,\theta)},$$

   where $l_3(u,\theta)$ is given by (3.75)*.*

The proof of this result and the explicit expressions of L_0, $L(u)$ and $l_3(u,\theta)$ can be found in Section 3.5.

3.1.7 First order of the difference

In the last section we have seen what form the difference $\Delta(u,\theta)$ has. Now we shall find a first order of this difference, which will allow us to find the desired asymptotic formula for Δ as $\delta \to 0$. Let us denote by $\Upsilon^{[0]}$ the average of the function $\tilde{k}(\tau)$ of Theorem 3.1.9, and define:

$$\tilde{k}_0(\tau) := \Upsilon^{[0]} + \sum_{l\neq 0} \Upsilon_0^{[l]} e^{2\pi i\tau}, \tag{3.82}$$

where $\Upsilon_0^{[l]}$ are the constants appearing in the Fourier coefficients of the Melnikov function, defined in (3.71). Then we shall define our candidate to be the first order of the difference as:

$$\Delta_0(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\tilde{k}_0(\theta+\delta^{-1}\alpha u+\mathrm{d}^{-1}\log\cosh(\mathrm{d}u)+C(u,\theta)). \tag{3.83}$$

We note that we have not chosen the average of $\tilde{k}_0$ to be the coefficient $\Upsilon_0^{[0]}$ appearing in the average of the Melnikov function (as one might expect) but $\Upsilon^{[0]}$, the average of $\tilde{k}(\tau)$ in Theorem 3.1.9. We also point out that this coefficient is unknown, unlike the coefficients $\Upsilon_0^{[l]}$, $l \neq 0$, which have an explicit formula and have been computed and bounded in Theorem 3.1.8. This coefficient $\Upsilon^{[0]}$ plays a crucial role, because *a priori* it might not be exponentially small and thus it would be the dominant term of $\tilde{k}_0$, since by Theorem 3.1.8 all other coefficients $\Upsilon_0^{[l]}$, $l \neq 0$, are exponentially small.

Next result deals with the coefficient $\Upsilon^{[0]} = \Upsilon^{[0]}(\delta,\sigma)$, and also with the coefficient $\Upsilon_0^{[0]}$ of the Melnikov function. We distinguish between the conservative and dissipative cases, since they are qualitatively different concerning these coefficients.

Theorem 3.1.10. *Let $p \geq -2$. Let $\Upsilon^{[0]}$ be the average of the function $\tilde{k}(\tau)$, given in Theorem 3.1.9, and $\Upsilon_0^{[0]}$ be the constant defined in* (3.71).

1. *In the conservative case, for all $0 \leq \delta \leq \delta_0$ one has:*

$$\Upsilon^{[0]} = 0, \qquad \Upsilon_0^{[0]} = 0.$$

2. *In the dissipative case, there exists a curve $\sigma = \sigma_*^0(\delta) = \mathcal{O}(\delta^{p+3})$ such that for all $0 \leq \delta \leq \delta_0$ one has:*

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\delta,\sigma_*^0(\delta)) = 0.$$

   In addition, given constants a_1, $a_2 \in \mathbb{R}$ and $a_3 > 0$, there exists a curve $\sigma = \sigma_*(\delta) = \mathcal{O}(\delta^{p+3})$ such that for all $0 \leq \delta \leq \delta_0$ one has:

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\delta,\sigma_*(\delta)) = a_1\delta^{a_2}e^{-\frac{a_3\pi}{2\mathrm{d}\delta}}.$$

   Along these curves one has:

$$\Upsilon_0^{[0]} = \Upsilon_0^{[0]}(\delta,\sigma_*(\delta)) = \mathcal{O}(\delta^{p+4}).$$

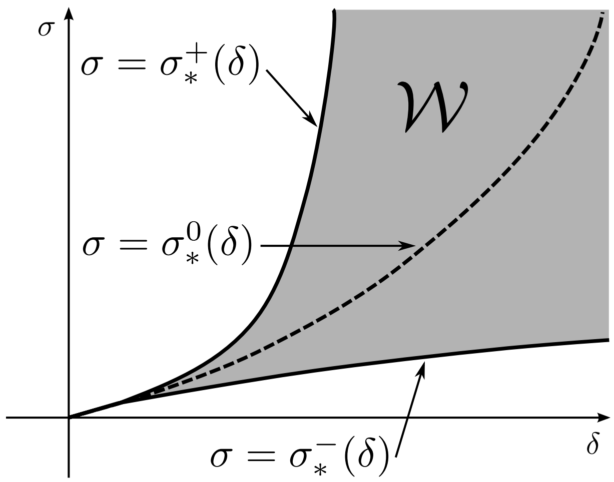


Figure 3.4: The curve $\sigma = \sigma_*^0(\delta)$ and a wedge-shaped domain $\mathcal{W}$ around it. Inside this domain, the coefficient $\Upsilon^{[0]}$ is exponentially small.

For the proof of this theorem we refer the reader to Section 3.6.

Remark 3.1.11. In the dissipative case one can see that, given a_1, $a_2 \in \mathbb{R}$ and $a_3 > 0$, the curve $\sigma_*(\delta)$ in Theorem 3.1.10 satisfies:

$$\sigma_*(\delta) - \sigma_*^0(\delta) = a_1 \delta^{a_2} e^{-\frac{a_3 \pi}{2 \mathrm{d} \delta}} \left(1 + \mathcal{O}(\delta)\right). \tag{3.84}$$

Now let us fix some constants $a_1^+ > 0$ and $a_1^- < 0$. Fix also $a_2^+, a_2^- \in \mathbb{R}$ and $a_3^+, a_3^- > 0$. Define $\sigma_*^+(\delta)$ as the curve in of Theorem 3.1.10 corresponding to the constants a_1^+, a_2^+ and a_3^+, and $\sigma_*^-(\delta)$ as the curve in of Theorem 3.1.10 corresponding to the constants a_1^-, a_2^- and a_3^-. By (3.84) one has that $\sigma_*^-(\delta) \leq \sigma_*^+(\delta)$ for δ sufficiently small. Define the domain:

$$\mathcal{W} := \{(\delta, \sigma) \in \mathbb{R}^2 \,:\, \sigma_*^-(\delta) \leq \sigma \leq \sigma_*^+(\delta)\}.$$

This domain is a wedge-shaped domain around $\sigma_*^0(\delta)$ (see Figure 3.4). Moreover there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ and $(\delta, \sigma) \in \mathcal{W}$, the coefficient $\Upsilon^{[0]}(\delta, \sigma)$ is exponentially small. More precisely, let us denote $\bar{a}_3 = \min\{a_3^+, a_3^-\}$. Define $\bar{a}_1 = a_1^+$ and $\bar{a}_2 = a_2^+$ if the minimum is achieved in a_3^+, otherwise we take $\bar{a}_1 = a_1^-$ and $\bar{a}_2 = a_2^-$. Then:

$$|\Upsilon^{[0]}(\delta, \sigma)| \leq |\bar{a}_1| \delta^{\bar{a}_2} e^{-\frac{\bar{a}_3 \pi}{2 \mathrm{d} \delta}}, \qquad \text{if} \quad 0 < \delta \leq \delta_0, \quad (\delta, \sigma) \in \mathcal{W}.$$

Next result shows that, in the regular case $p > -2$, the remaining Fourier coefficients of $\tilde{k}$, the function which appears in Theorem 3.1.9, are well approximated by the Fourier coefficients of $\tilde{k}_0$ defined in (3.82). The proof of this Proposition is done in Section 3.7.

Proposition 3.1.12. *Let us denote by $\Upsilon^{[l]}$ the Fourier coefficients of the function $\tilde{k}(\tau)$ of Theorem 3.1.9. If $p > -2$, there exist two constants M_1, M_2 such that for all $l \neq 0$:*

1. *In the conservative case one has:*
$$\left|\Upsilon^{[l]}-\Upsilon_0^{[l]}\right|\leq M_1\left(\frac{\delta^{2p}}{\kappa^5}|\log(\delta\kappa)|+\frac{\delta^{p+1}}{\kappa^3}\right)e^{-\frac{\alpha}{\delta}\left(\frac{\pi}{2}-(\kappa+M_2)\delta\right)|l|}.$$

2. *In the dissipative case, if $\sigma=\sigma_*(\delta)$ one has:*
$$\left|\Upsilon^{[l]}-\Upsilon_0^{[l]}\right|\leq M_1\left(\frac{1}{\kappa^{3+\frac{2}{\mathrm{d}}}}\delta^{2\left(p+1-\frac{1}{\mathrm{d}}\right)}|\log(\delta\kappa)|+\frac{\delta^{p+3-\frac{2}{\mathrm{d}}}}{\kappa^{1+\frac{2}{\mathrm{d}}}}\right)e^{-\frac{\alpha}{\delta}\left(\frac{\pi}{2\mathrm{d}}-(\kappa+M_2)\delta\right)|l|}.$$

Now we can state the main theorem of this chapter:

Theorem 3.1.13. *Let $p>-2$ and κ a sufficiently large constant. Consider the function:*
$$m(w,\theta)=w^{\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}F\left(\frac{\mathrm{d}+1}{2b}w^2,\theta,-iw,0,0\right)-i\frac{\mathrm{d}+1}{b}w^{1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}H\left(\frac{\mathrm{d}+1}{2b}w^2,\theta,-iw,0,0\right),$$
where F and H are the functions defined in (3.23). *Let $\hat{m}^{[l]}(\zeta)$ be the l-th Fourier coefficient of its Borel transform $\hat{m}(\zeta,\theta)=\sum_{l\in\mathbb{Z}}\hat{m}^{[l]}(\zeta)e^{il\theta}$, and define:*
$$C_1=2\mathrm{Re}\left(\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right)\right),\qquad C_2=-2\mathrm{Im}\left(\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right)\right).$$
Let:
$$\vartheta(u,\delta)=\delta^{-1}\alpha u+c\mathrm{d}^{-1}\left[\log\cosh(\mathrm{d}u)-\log\delta\right].$$
There exists $T_0>0$ such that for all $u\in[-T_0,T_0]$ and $\theta\in\mathbb{S}^1$, the following holds:

1. *In the conservative case (where* $\mathrm{d}=1$*):*
$$\begin{aligned}\Delta(u,\theta) &= 2\pi\delta^{p-2}\cosh^2(u)e^{-\frac{\alpha\pi}{2\delta}}\Big[C_1\cos\Big(\theta+\vartheta(u,\delta)\Big)\\ &\quad+C_2\sin\Big(\theta+\vartheta(u,\delta)\Big)+\mathcal{O}\left(\delta^{p+2}|\log(\delta)|+\delta^3\right)\Big],\end{aligned}$$

2. *In the dissipative case, if $\sigma=\sigma_*(\delta)$ is one of the curves defined in Theorem 3.1.10, one has:*
$$\begin{aligned}\Delta(u,\theta) &= \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\Upsilon^{[0]}(\delta,\sigma_*(\delta))\Big(1+\mathcal{O}\left(\delta^{p+3}\right)\Big)\\ &\quad+\frac{2\pi}{\mathrm{d}}\delta^{p-\frac{2}{\mathrm{d}}}\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\Big[C_1\cos\Big(\theta+\vartheta(u,\delta)\Big)\\ &\quad+C_2\sin\Big(\theta+\vartheta(u,\delta)\Big)+\mathcal{O}\left(\delta^{p+2}|\log(\delta)|+\delta^3\right)\Big].\end{aligned}$$

Proof. Recalling the definition (3.83) of Δ_0 and the form (3.79) of Δ given in Theorem 3.1.9, we can write:

$$\Delta(u,\theta) = \Delta_0(u,\theta) + \Delta_1(u,\theta), \tag{3.85}$$

where:

$$\Delta_1(u,\theta) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)(1 + P_1(u,\theta)) \sum_{l\neq 0} \left(\Upsilon^{[l]} - \Upsilon_0^{[l]}\right) e^{il(\theta+\delta^{-1}\alpha u + \mathrm{c}\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta))}.$$

The proofs in the conservative case and the dissipative one are completely analogous. However, since the bounds in each case are slightly different, we will do them separately to make the arguments less cumbersome.

Let us start with the conservative case. In this case, whenever the coefficient "d" appears in a formula, we shall substitute it directly by d = 1. First of all we note that since we are taking $u \in [-T_0, T_0]$ and $\theta \in \mathbb{S}^1$ we have that $C(u,\theta)$ is real. Then, using Theorem 3.1.9 to bound $|1 + P_1(u,\theta)|$ and Proposition 3.1.12 to bound $|\Upsilon^{[l]} - \Upsilon_0^{[l]}|$ we obtain:

$$|\Delta_1(u,\theta)| \leq K\left(\delta^{2p}|\log(\delta)| + \delta^{p+1}\right) e^{-\frac{\alpha\pi}{2\delta}}. \tag{3.86}$$

We point out that we have omitted the explicit dependence on κ since we have taken it to be a constant independent of δ. Next step is to bound $\tilde{k}_0(\tau)$, defined in (3.82), as well as its derivative $\tilde{k}_0'(\tau)$ for $\tau \in \mathbb{S}^1$. Theorem 3.1.10 states that $\Upsilon^{[0]} = 0$. In Theorem 3.1.8 we have explicit expressions for $\Upsilon_0^{[1]}$ and $\Upsilon_0^{[-1]}$, and bound for $\Upsilon_0^{[l]}$, $|l| \geq 2$. Putting all this together, we obtain:

$$\sup_{\tau\in\mathbb{S}^1} |\tilde{k}_0(\tau)| \leq K\delta^{p-2} e^{-\frac{\alpha\pi}{2\delta}}, \qquad \sup_{\tau\in\mathbb{S}^1} |\tilde{k}_0'(\tau)| \leq K\delta^{p-2} e^{-\frac{\alpha\pi}{2\delta}}. \tag{3.87}$$

Then, from expression (3.83) of Δ_0, using bound (3.87), the bound for P_1 and the expression of C obtained in Theorem 3.1.9, and recalling that $u \in [-T_0, T_0]$ so that $|\cosh(\mathrm{d}u)| \geq K > 0$, we obtain:

$$\Delta_0(u,\theta) = \cosh^2(u)\tilde{k}_0(\theta + \delta^{-1}\alpha u + c\log\cosh(u)) + \mathcal{O}\left(\delta^{2p} e^{-\frac{\alpha\pi}{2\delta}}\right).$$

Again, by definition (3.82) of $\tilde{k}_0$ and expression (3.70) of $M(u,\theta)$ it is clear that:

$$\cosh^2(u)\tilde{k}_0(\theta + \delta^{-1}\alpha u + c\log\cosh(u)) = M(u,\theta),$$

where we have used that $\Upsilon^{[0]} = \Upsilon_0^{[0]} = 0$ by Theorem 3.1.10. Then:

$$\Delta_0(u,\theta) = M(u,\theta) + \mathcal{O}\left(\delta^{2p} e^{-\frac{\alpha\pi}{2\delta}}\right). \tag{3.88}$$

Finally, we just need to use (3.86) and (3.88) in (3.85), and then substitute $M(u,\theta)$ by it asymptotic expression (3.74) given in Theorem 3.1.8, and we get the claim in item 1 above.

To prove the result in the dissipative setting, one just has to follow the same steps. On the first place, using Theorem 3.1.9 and Proposition 3.1.12, in this case we have:

$$|\Delta_1(u,\theta)| \leq K\left(\delta^{2\left(p+1-\frac{1}{\mathrm{d}}\right)}|\log(\delta)| + \delta^{p+3-\frac{2}{\mathrm{d}}}\right)e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}. \tag{3.89}$$

Again, we omit the explicit dependence on κ since it is a constant independent of δ. Now, by definition (3.82) of $\tilde{k}_0$ and the bounds obtained in Theorem 3.1.8 for the coefficients $\Upsilon_0^{[l]}$, $l \neq 0$, and taking $\sigma = \sigma_*(\delta)$ defined in Theorem 3.1.10, we obtain that:

$$\sup_{\tau\in\mathbb{S}^1} |\tilde{k}_0(\tau)| \leq |\Upsilon^{[0]}| + K\delta^{p-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}, \tag{3.90}$$

and similarly:

$$\sup_{\tau\in\mathbb{S}^1} |\tilde{k}_0'(\tau)| \leq K\delta^{p-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}. \tag{3.91}$$

Then, from expression (3.83) of Δ_0, using bounds (3.90) and (3.91), the bound for P_1 and the expression for C obtained in Theorem 3.1.9 one has:

$$\begin{aligned}\Delta_0(u,\theta) &= \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\tilde{k}_0(\theta + \delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u)) \\ &\quad + \mathcal{O}\left(\delta^{2\left(p+1-\frac{1}{\mathrm{d}}\right)}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} + \delta^{p+3}|\Upsilon^{[0]}|\right).\end{aligned}$$

By the definition (3.82) of $\tilde{k}_0$ it is clear that:

$$\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\tilde{k}_0(\theta + \delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u)) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\Upsilon^{[0]} + M(u,\theta) - \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\Upsilon_0^{[0]},$$

and then:

$$\begin{aligned}\Delta_0(u,\theta) &= \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\Upsilon^{[0]} + M(u,\theta) - \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\Upsilon_0^{[0]} \\ &\quad + \mathcal{O}\left(\delta^{2\left(p+1-\frac{1}{\mathrm{d}}\right)}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} + \delta^{p+3}|\Upsilon^{[0]}|\right).\end{aligned} \tag{3.92}$$

To finish, we use (3.89) and (3.92) in (3.85), and substitute $M(u,\theta)$ by its asymptotic expression given in Theorem 3.1.8. Then we obtain the claim stated in item 2 above. □

Theorem 3.1.13 easily yields Theorem 3.1.

Proof of Theorem 3.1. First of all we recall that in Subsection 3.1.1 we performed the change (3.9), namely:

$$\tilde{z} = z + \frac{h_3}{2}\delta^{p+3}.$$

This change is $\mathcal{O}(\delta^{p+3})$-close to the identity, so that the asymptotic formula after performing the inverse change is exactly the same as in Theorem 3.1.13. Now, we point out that

$\Delta(u,\theta)$ is not the actual distance between the invariant manifolds, since we computed the difference in "symplectic" cylindric coordinates. The actual distance is given by:

$$D(u,\theta)=\sqrt{2(R_0(u)+r_1^{\mathrm{u}}(u,\theta))}-\sqrt{2(R_0(u)+r_1^{\mathrm{s}}(u,\theta))}=\frac{1}{\sqrt{2R_0(u)}}\Delta(u,\theta)+\mathcal{O}_2(\Delta(u,\theta)).$$

Using the definition (3.24) of $R_0(u)$ one obtains:

$$D(u,\theta)=\sqrt{\frac{b}{\mathrm{d}+1}}\cosh(\mathrm{d}u)\Delta(u,\theta)+\mathcal{O}_2(\Delta(u,\theta)).$$

To obtain the formulas given in Theorem 3.1 first we recall that $\bar{x}=\delta x$ and $\bar{y}=\delta y$, so that:

$$\bar{D}^{\mathrm{u,s}}(u,\theta)=\delta D(u,\theta)=\delta\sqrt{\frac{b}{\mathrm{d}+1}}\cosh(\mathrm{d}u)\Delta(u,\theta)+\delta\mathcal{O}_2(\Delta(u,\theta)).$$

Finally, one just needs to recall recall that $\delta=\sqrt{\mu}$, $\sigma=\delta^{-1}\nu=\nu/\sqrt{\mu}$, take into account the notation $b=\gamma_2$, $\mathrm{d}=\beta_1$ and $c=\alpha_3$ and redefine the coefficients a_1 and a_2. □

3.2 Proof of Theorem 3.1.6

All the constants that appear in the statements of the following results might depend on δ^*, σ^* and κ^*, but never on δ, σ and κ. Moreover, we assume that δ^* and σ^* are sufficiently small, and κ^* is sufficiently large. More precisely, in the following proofs we will have to assume that a finite number of inequalities are satisfied, which can be achieved by taking δ^* and σ^* sufficiently small, and κ^* sufficiently large. Thus, we will do that without explicitly stating it anymore. Finally, to make formulas shorter and avoid keeping track of constants that do not play any role in the proofs, we will use K to denote *any* constant independent of the parameters δ, σ and κ. These conventions are valid for all the sections of this work.

3.2.1 Banach spaces and technical lemmas

In this subsection we will introduce the Banach spaces in which we will solve the fixed point equations (3.54), and some technical results we will use later.

We will consider functions $\phi:D^{\mathrm{u}}_{\kappa,\beta}\times\mathbb{T}_\omega\to\mathbb{C}$, where the domain $D^{\mathrm{u}}_{\kappa,\beta}$ is defined in (3.56) and $\mathbb{T}_\omega$ is defined in (3.58) (see also Figures 3.1 and 3.2). They can be written in their Fourier series:

$$\phi(v,\theta)=\sum_{l\in\mathbb{Z}}\phi^{[l]}(v)e^{il\theta}.$$

Then, for the unstable case, we define the norms:

$$\|\phi\|^{\mathrm{u}}_{n,m,\omega}:=\sum_{l\in\mathbb{Z}}\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}e^{|l|\omega},\tag{3.93}$$

where:

$$\left\|\phi^{[l]}\right\|_{n,m}^{\mathrm{u}} = \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,T}} \left|\cosh^n(\mathrm{d}v)\phi^{[l]}(v)\right| + \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}} \left|\cosh^m(\mathrm{d}v)\phi^{[l]}(v)\right|. \tag{3.94}$$

We also define the following norms:

$$\llbracket\phi\rrbracket^{\mathrm{u}}_{n,m,\omega} := \|\phi\|^{\mathrm{u}}_{n,m,\omega} + \|\partial_v\phi\|^{\mathrm{u}}_{n+1,m,\omega} + \delta^{-1}\|\partial_\theta\phi\|^{\mathrm{u}}_{n+1,m,\omega}. \tag{3.95}$$

Finally, we consider the Banach spaces:

$$\mathcal{X}^{\mathrm{u}}_{n,m,\omega} := \left\{\phi : D^{\mathrm{u}}_{\kappa,\beta}\times\mathbb{T}_\omega \to \mathbb{C} \,:\, \phi(v,\theta) \text{ is analytic and } \|\phi\|^{\mathrm{u}}_{n,m,\omega} < +\infty\right\}, \tag{3.96}$$

and:

$$\tilde{\mathcal{X}}^{\mathrm{u}}_{n,m,\omega} := \left\{\phi : D^{\mathrm{u}}_{\kappa,\beta}\times\mathbb{T}_\omega \to \mathbb{C} \,:\, \phi(v,\theta) \text{ is analytic and } \llbracket\phi\rrbracket^{\mathrm{u}}_{n,m,\omega} < +\infty\right\}. \tag{3.97}$$

For functions $\Phi = (\phi_1,\phi_2,\phi_3)$ in the product space:

$$\mathcal{X}^{\mathrm{u},\times}_{n,m,\omega} := \mathcal{X}^{\mathrm{u}}_{n,m,\omega}\times\mathcal{X}^{\mathrm{u}}_{n,m,\omega}\times\mathcal{X}^{\mathrm{u}}_{n,m,\omega},$$

we will take the norm:

$$\|\Phi\|^{\mathrm{u},\times}_{n,m,\omega} := \max\left\{\|\phi_1\|^{\mathrm{u}}_{n,m,\omega},\, \|\phi_2\|^{\mathrm{u}}_{n,m,\omega},\, \|\phi_3\|^{\mathrm{u}}_{n,m,\omega}\right\}. \tag{3.98}$$

For the stable case, we consider norms and Banach spaces analogously defined in the corresponding domains $D^{\mathrm{s}}_{\kappa,\beta}$.

For functions $\phi : \mathbb{T}_\omega \to \mathbb{C}$, we will take the norm:

$$\|\phi\|_\omega := \sum_{l\in\mathbb{Z}} \left|\phi^{[l]}\right| e^{|l|\omega}.$$

Now we can state a result which will have as corollary Theorem 3.1.6. We will devote the rest of the section to prove it.

Proposition 3.2.1. *Let $p \geq -2$ and $0<\beta<\pi/2$ be any constants. There exist constants $\kappa^*\geq 1$, $\sigma^*>0$ and $\delta^*>0$, such that if $\kappa=\kappa(\delta)$ satisfies condition* (3.55) *then for all $0<\delta\leq\delta^*$ and σ satisfying:*

$$|\sigma| \leq \sigma^*\delta^{p+3}, \tag{3.99}$$

the fixed point equations in (3.54) *have solutions $R^{\mathrm{u}}_1(v,\theta)$ and $R^{\mathrm{s}}_1(v,\theta)$ defined respectively for $(v,\theta)\in D^{\mathrm{u}}_{\kappa,\beta}\times\mathbb{T}_\omega$ and $(v,\theta)\in D^{\mathrm{s}}_{\kappa,\beta}\times\mathbb{T}_\omega$.*

Moreover, they satisfy that $R^{\mathrm{u,s}}_1 = R^{\mathrm{u,s}}_{10} + R^{\mathrm{u,s}}_{11}$ with the following properties:

1. *$R^{\mathrm{u,s}}_{10} := \tilde{\mathcal{F}}^{\mathrm{u,s}}(0) \in \tilde{\mathcal{X}}^{\mathrm{u,s}}_{3,2,\omega}$, where $\tilde{\mathcal{F}}^{\mathrm{u,s}}$ are defined in* (3.53)*, and there exists a constant M such that:*

$$\llbracket R^{\mathrm{u,s}}_{10}\rrbracket^{\mathrm{u,s}}_{3,2,\omega} \leq M\delta^{p+3}.$$

2. $R_{11}^{\mathrm{u,s}} \in \tilde{\mathcal{X}}_{4,2,\omega}^{\mathrm{u,s}}$, *and there exists a constant* M *such that:*

$$\llbracket R_{11}^{\mathrm{u,s}} \rrbracket_{4,2,\omega}^{\mathrm{u,s}} \leq M\delta^{p+3} \llbracket R_{10}^{\mathrm{u,s}} \rrbracket_{3,2,\omega}^{\mathrm{u,s}}.$$

In the following we will prove Proposition 3.2.1, but just in the unstable case. The proofs for the stable one are completely analogous. First of all we present some technical results dealing with the norms $\|.\|_{n,m}$, $\|.\|_{n,m,\omega}$ and $\|.\|_{\omega}$, which will be needed throughout the chapter and summarize the basic properties of these norms.

Lemma 3.2.2. *Let* $\phi \in \mathcal{X}_{n,m,\omega}^{\mathrm{u}}$. *Then:*

1. *For all* $n_+ > n$, $\phi \in \mathcal{X}_{n_+,m,\omega}^{\mathrm{u}}$ *and there exists a constant* M *independent of* δ, σ *and* κ *such that:*

$$\|\phi\|_{n_+,m,\omega}^{\mathrm{u}} \leq M \|\phi\|_{n,m,\omega}^{\mathrm{u}}.$$

2. *For all* $n_- < n$, $\phi \in \mathcal{X}_{n_-,m,\omega}^{\mathrm{u}}$ *and there exists a constant* M *independent of* δ, σ *and* κ *such that:*

$$\|\phi\|_{n_-,m,\omega}^{\mathrm{u}} \leq \frac{M}{(\delta\kappa)^{n-n_-}} \|\phi\|_{n,m,\omega}^{\mathrm{u}}.$$

3. *For all* $m_- < m$, $\phi \in \mathcal{X}_{n,m_-,\omega}^{\mathrm{u}}$ *and there exists a constant* M *independent of* δ, σ *and* κ *such that:*

$$\|\phi\|_{n,m_-,\omega}^{\mathrm{u}} \leq M \|\phi\|_{n,m,\omega}^{\mathrm{u}}.$$

4. *For all* $\omega_- < \omega$, $\phi \in \mathcal{X}_{n,m,\omega_-}^{\mathrm{u}}$ *and there exists a constant* M *independent of* δ, σ *and* κ *such that:*

$$\|\phi\|_{n,m,\omega_-}^{\mathrm{u}} \leq M \|\phi\|_{n,m,\omega}^{\mathrm{u}}.$$

Lemma 3.2.3. *The following properties are satisfied:*

1. *Let* ϕ_1, $\phi_2 : D_{\kappa,\beta}^{\mathrm{u}} \to \mathbb{C}$ *such that* $\|\phi_i\|_{n_i,m_i}^{\mathrm{u}} < \infty$ *for* $i = 1, 2$. *Then, for all* $m \leq m_1 + m_2$, *there exists a constant* M *independent of* δ, σ *and* κ *such that:*

$$\|\phi_1\phi_2\|_{n_1+n_2,m}^{\mathrm{u}} \leq M \|\phi_1\|_{n_1,m_1}^{\mathrm{u}} \|\phi_2\|_{n_2,m_2}^{\mathrm{u}}.$$

2. *Let* $\phi_1 \in \mathcal{X}_{n_1,m_1,\omega}^{\mathrm{u}}$ *and* $\phi_2 \in \mathcal{X}_{n_2,m_2,\omega}^{\mathrm{u}}$. *Then, for all* $m \leq m_1 + m_2$, *the product* $\phi_1\phi_2 \in \mathcal{X}_{n_1+n_2,m,\omega}^{\mathrm{u}}$, *and there exists a constant* M *independent of* δ, σ *and* κ *such that:*

$$\|\phi_1\phi_2\|_{n_1+n_2,m,\omega}^{\mathrm{u}} \leq M \|\phi_1\|_{n_1,m_1,\omega}^{\mathrm{u}} \|\phi_2\|_{n_2,m_2,\omega}^{\mathrm{u}}.$$

Proof. The first item can be proved straightforwardly from the definition (3.93) of the norm $\|.\|^{\mathrm{u}}_{n_1+n_2,m}$:

$$\begin{aligned}
&\|\phi_1\phi_2\|_{n_1+n_2,m} = \sup_{u\in D^{\mathrm{u}}_{\kappa,\beta,T}} \left|\cosh^{n_1+n_2}(\mathrm{d}u)\phi_1(u)\phi_2(u)\right| + \sup_{u\in D^{\mathrm{u}}_{\kappa,\beta,\infty}} \left|\cosh^{m}(\mathrm{d}u)\phi_1(u)\phi_2(u)\right| \\
&\leq \sup_{u\in D^{\mathrm{u}}_{\kappa,\beta,T}} \left|\cosh^{n_1}(\mathrm{d}u)\phi_1(u)\right| \sup_{u\in D^{\mathrm{u}}_{\kappa,\beta,T}} \left|\cosh^{n_2}(\mathrm{d}u)\phi_2(u)\right| \\
&+ \sup_{u\in D^{\mathrm{u}}_{\kappa,\beta,\infty}} \left|\cosh^{m_1}(\mathrm{d}u)\phi_1(u)\right| \sup_{u\in D^{\mathrm{u}}_{\kappa,\beta,\infty}} \left|\cosh^{m_2}(\mathrm{d}u)\phi_2(u)\right| \sup_{u\in D^{\mathrm{u}}_{\kappa,\beta,\infty}} \left|\frac{1}{\cosh^{m_1+m_2-m}(\mathrm{d}u)}\right| \\
&\leq M\|\phi_1\|^{\mathrm{u}}_{n_1,m_1}\|\phi_2\|^{\mathrm{u}}_{n_2,m_2},
\end{aligned}$$

for a suitable constant M (that depends just on m_1+m_2-m), where we have used that $m_1+m_2-m\geq 0$ and that $\left|\cosh^{-1}(\mathrm{d}u)\right|\leq K$ for $u\in D^{\mathrm{u}}_{\kappa,\beta,\infty}$.

For the second statement, using the definition (3.94) of the norm $\|.\|^{\mathrm{u}}_{n_1+n_2,m,\omega}$ and item 1 of this lemma (and noting that the constant M is independent of the functions ϕ_1 and ϕ_2) we obtain:

$$\|\phi_1\phi_2\|_{n_1+n_2,m} = \sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \left\|\phi_1^{[j]}\phi_2^{[l-j]}\right\|^{\mathrm{u}}_{n_1+n_2,m} e^{|l|\omega} \leq M\sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \left\|\phi_1^{[j]}\right\|^{\mathrm{u}}_{n_1,m_1} \left\|\phi_2^{[l-j]}\right\|^{\mathrm{u}}_{n_2,m_2} e^{|l|\omega}.$$

Now, since $|l|\leq |j|+|l-j|$, the last inequality yields:

$$\begin{aligned}
\|\phi_1\phi_2\|_{n_1+n_2,m} &\leq M\sum_{l\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} \left\|\phi_1^{[j]}\right\|^{\mathrm{u}}_{n_1,m_1} e^{|j|\omega} \left\|\phi_2^{[l-j]}\right\|^{\mathrm{u}}_{n_2,m_2} e^{|l-j|\omega} \\
&= M\left(\sum_{l\in\mathbb{Z}} \left\|\phi_1^{[l]}\right\|^{\mathrm{u}}_{n_1,m_1} e^{i|l|\omega}\right)\left(\sum_{l\in\mathbb{Z}} \left\|\phi_2^{[l]}\right\|^{\mathrm{u}}_{n_2,m_2} e^{i|l|\omega}\right) \\
&= M\|\phi_1\|_{n_1,m_1,\omega}\|\phi_2\|_{n_2,m_2,\omega},
\end{aligned}$$

and thus the proof is finished. □

Lemma 3.2.4. *Let $\omega_0>\omega$.*

1. *Let $\phi:\mathbb{T}_{\omega_0}\to\mathbb{C}$ be analytic. Then there exists a constant M such that:*

$$\|\phi\|_\omega \leq M\sup_{\theta\in\mathbb{T}_{\omega_0}}|\phi(\theta)|.$$

2. *Let $\phi\in\mathcal{X}^{\mathrm{u}}_{n,m,\omega_0}$. Then there exists a constant M such that:*

$$\|\phi\|^{\mathrm{u}}_{n,m,\omega}\leq M\left(\sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\ v\in D^{\mathrm{u}}_{\kappa,\beta,T}}}|\cosh^n(\mathrm{d}v)\phi(v,\theta)| + \sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\ v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}}|\cosh^m(\mathrm{d}v)\phi(v,\theta)|\right)$$

Proof. We just prove item 2, since item 1 is a particular case with $\phi(v,\theta) = \phi(\theta)$ and $n, m = 0$. Recalling definition (3.93) of the norm, we have:

$$\|\phi\|^{\mathrm{u}}_{n,m,\omega} = \sum_{l\in\mathbb{Z}} \left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m} e^{|l|\omega} = \sum_{l\in\mathbb{Z}} \left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m} e^{|l|\omega_0} e^{|l|(\omega-\omega_0)}. \tag{3.100}$$

On the other hand, we have:

$$\phi^{[l]}(v) = \frac{1}{2\pi}\int_0^{2\pi} \phi(v,\theta)e^{-il\theta}d\theta.$$

Now, for $l \neq 0$, since $\phi(v,\cdot)$ is analytic in $\mathbb{T}_{\omega_0}$, we have that the integral along the boundary of the rectangle $[0,2\pi]\times[0,-i\omega_0|l|/l]$ is zero, and then it can be easily shown that:

$$\begin{aligned}
\phi^{[l]}(v) &= \frac{1}{2\pi}\int_0^{2\pi} \phi\left(v,\theta - i\frac{|l|}{l}\omega_0\right) e^{-il\left(\theta - i\frac{|l|}{l}\omega_0\right)}d\theta \\
&= \frac{1}{2\pi}e^{-|l|\omega_0}\int_0^{2\pi} \phi\left(v,\theta - i\frac{|l|}{l}\omega_0\right) e^{-il\theta}d\theta,
\end{aligned}$$

and hence for any $k \geq 0$:

$$\begin{aligned}
|\cosh^k(\mathrm{d}v)\phi^{[l]}(v)| &\leq \frac{1}{2\pi}e^{-|l|\omega_0}\int_0^{2\pi} \left|\cosh^k(\mathrm{d}v)\phi\left(v,\theta - i\frac{|l|}{l}\omega_0\right)\right| \left|e^{-il\theta}\right| d\theta \\
&\leq e^{-|l|\omega_0}\sup_{\theta\in\mathbb{T}_{\omega_0}} |\cosh^k(\mathrm{d}v)\phi(v,\theta)|.
\end{aligned}$$

Then, for $l \neq 0$ it is clear that:

$$\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m} \leq e^{-|l|\omega_0}\left(\sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\ v\in D^{\mathrm{u}}_{\kappa,\beta,T}}} |\cosh^n(\mathrm{d}v)\phi(v,\theta)| + \sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\ v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}} |\cosh^m(\mathrm{d}v)\phi(v,\theta)|\right).$$

Moreover, this inequality is satisfied trivially also by $l = 0$. Then, substituting it in (3.100) we obtain:

$$\begin{aligned}
\|\phi\|^{\mathrm{u}}_{n,m,\omega} &\leq \sum_{l\in\mathbb{Z}}\left(\sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\ v\in D^{\mathrm{u}}_{\kappa,\beta,T}}} |\cosh^n(\mathrm{d}v)\phi(v,\theta)| + \sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\ v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}} |\cosh^m(\mathrm{d}v)\phi(v,\theta)|\right) e^{|l|(\omega-\omega_0)} \\
&\leq M\left(\sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\ v\in D^{\mathrm{u}}_{\kappa,\beta,T}}} |\cosh^n(\mathrm{d}v)\phi(v,\theta)| + \sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\ v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}} |\cosh^m(\mathrm{d}v)\phi(v,\theta)|\right),
\end{aligned}$$

where we have used that, since $\omega - \omega_0 \leq 0$, we have $\sum_{l\in\mathbb{Z}} e^{|l|(\omega-\omega_0)} = M < \infty$.

□

Lemma 3.2.5. *Consider a family of functions* $\phi_\lambda \in \mathcal{X}^{\mathrm{u}}_{n,m,\omega}$*, with* $\lambda \in [0,1]$*, such that* $\|\phi_\lambda\|^{\mathrm{u}}_{n,m,\omega} \leq M$*. Then:*

$$\left\|\int_0^1 \phi_\lambda d\lambda\right\|^{\mathrm{u}}_{n,m,\omega} \leq \int_0^1 \|\phi_\lambda\|^{\mathrm{u}}_{n,m,\omega} d\lambda.$$

Proof. First we claim that:

$$\left(\int_0^1 \phi_\lambda(v,\theta)d\lambda\right)^{[l]} = \int_0^1 \phi_\lambda^{[l]}(v)d\lambda. \tag{3.101}$$

This is obvious, since:

$$\begin{aligned}\left(\int_0^1 \phi_\lambda(v,\theta)d\lambda\right)^{[l]} &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \phi_\lambda(v,\theta)d\lambda e^{-il\theta}d\theta = \int_0^1 \frac{1}{2\pi}\int_0^{2\pi} \phi_\lambda(v,\theta)e^{-il\theta}d\theta d\lambda\\ &= \int_0^1 \phi_\lambda^{[l]}(v)d\lambda.\end{aligned}$$

Now we claim that:

$$\begin{aligned}\sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,T}}\left|\int_0^1 \phi_\lambda^{[l]}(v)d\lambda\cosh^n v\right| &\leq \int_0^1 \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,T}}\left|\phi_\lambda^{[l]}(v)\cosh^n v\right| d\lambda,\\ \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}\left|\int_0^1 \phi_\lambda^{[l]}(v)d\lambda\cosh^m v\right| &\leq \int_0^1 \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}\left|\phi_\lambda^{[l]}(v)\cosh^m v\right| d\lambda.\end{aligned} \tag{3.102}$$

This is easily proved. This yields:

$$\left\|\left(\int_0^1 \phi_\lambda d\lambda\right)^{[l]}\right\|^{\mathrm{u}}_{n,m} \leq \int_0^1 \|\phi_\lambda^{[l]}\|^{\mathrm{u}}_{n,m} d\lambda. \tag{3.103}$$

Indeed, from the definition (3.93) of the norm $\|.\|^{\mathrm{u}}_{m,m}$, and using first (3.101) and later (3.102) we have:

$$\begin{aligned}&\left\|\left(\int_0^1 \phi_\lambda d\lambda\right)^{[l]}\right\|^{\mathrm{u}}_{n,m}\\ &= \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,T}}\left|\left(\int_0^1 \phi_\lambda(v,\theta)d\lambda\right)^{[l]}\cosh^n v\right| + \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}\left|\left(\int_0^1 \phi_\lambda(v,\theta)d\lambda\right)^{[l]}\cosh^m v\right|\\ &= \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,T}}\left|\int_0^1 \phi_\lambda^{[l]}(v)d\lambda\cosh^n v\right| + \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}\left|\int_0^1 \phi_\lambda^{[l]}(v)d\lambda\cosh^m v\right|\\ &\leq \int_0^1 \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,T}}\left|\phi_\lambda^{[l]}(v)\cosh^n v\right| d\lambda + \int_0^1 \sup_{v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}\left|\phi_\lambda^{[l]}(v)\cosh^m v\right| d\lambda\\ &= \int_0^1 \|\phi_\lambda^{[l]}\|^{\mathrm{u}}_{n,m} d\lambda.\end{aligned}$$

We finish the proof of the lemma using definition (3.94) of the norm $\|.\|^{\mathrm{u}}_{n,m,\omega}$ and (3.103):

$$\begin{aligned}\left\|\int_0^1 \phi_\lambda d\lambda\right\|^{\mathrm{u}}_{n,m,\omega} &= \sum_{l\in\mathbb{Z}}\left\|\left(\int_0^1 \phi_\lambda d\lambda\right)^{[l]}\right\|^{\mathrm{u}}_{n,m} e^{|l|\omega} \leq \sum_{l\in\mathbb{Z}}\int_0^1 \|\phi_\lambda^{[l]}\|^{\mathrm{u}}_{n,m} e^{|l|\omega} d\lambda \\ &= \int_0^1 \sum_{l\in\mathbb{Z}} \|\phi_\lambda^{[l]}\|^{\mathrm{u}}_{n,m} e^{|l|\omega} d\lambda = \int_0^1 \|\phi_\lambda\|^{\mathrm{u}}_{n,m,\omega} d\lambda,\end{aligned}$$

where we have used the monotone convergence theorem to exchange the sum and the integral. □

Lemma 3.2.6. *Let $\Phi_0 \in \mathcal{X}^{\mathrm{u},\times}_{1,1,\omega}$ such that $\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega} \leq \delta C_0$, for some constant C_0. Let $\omega_0 > \omega$ and $F : \mathbb{C}^3 \times \mathbb{T}_{\omega_0} \to \mathbb{C}$ be analytic for $|x| \leq \rho_0$ for some $\rho_0 > 0$, $x \in \mathbb{C}^3$, and $\theta \in \mathbb{T}_{\omega_0}$. Moreover, let us assume that in this domain one has:*

$$|F(x,\theta)| \leq C_F |x|^n$$

for some $n \geq 1$ and some constant C_F. Let $m \leq n$. Then, if κ^* is large enough and verifies condition (3.55) *and δ^* is small enough, there exists a constant M such that for all $0 < \delta \leq \delta^*$:*

1. *$F \circ \Phi_0 \in \mathcal{X}^{\mathrm{u}}_{n,m,\omega}$ and:*

$$\|F(\Phi_0(v,\theta),\theta)\|^{\mathrm{u}}_{n,m,\omega} \leq M C_F (\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega})^n.$$

2. *If $\Phi_1,\ \Phi_2 \in \mathcal{X}^{\mathrm{u},\times}_{2,1,\omega}$ are such that $\|\Phi_i\|^{\mathrm{u},\times}_{2,1,\omega} \leq \delta C_0^i$ for some constants C_0^i, $i = 1, 2$, then:*

$$\begin{aligned}&\|F(\Phi_0(v,\theta) + \delta\Phi_1(v,\theta),\theta) - F(\Phi_0(v,\theta) + \delta\Phi_2(v,\theta),\theta)\|^{\mathrm{u}}_{n+1,m,\omega} \\ &\leq M C_F \delta^n \|\Phi_1(v,\theta) - \Phi_2(v,\theta))\|^{\mathrm{u},\times}_{2,1,\omega}.\end{aligned}$$

Proof. To prove the first item, let us write F in its power series:

$$F(x,\theta) = \sum_{\substack{k\in\mathbb{N}^3\\|k|\geq n}} a_k(\theta) x^k,$$

which is convergent for $|x| < \rho_0$. First we proceed to bound $\|a_k\|^{\mathrm{u}}_{\omega}$, where $k = (k_1,k_2,k_3) \in \mathbb{N}^3$. Since $F(x,\theta)$ is analytic for $x \in B^3(\rho_0)$ and $\theta \in \mathbb{T}_{\omega_0}$, by Cauchy's integral formula we have that for all $\theta \in \mathbb{T}_{\omega_0}$:

$$\begin{aligned}|a_k(\theta)| &\leq \frac{1}{(2\pi)^3}\int_{\partial B^3(\rho_0)} \frac{|F(z,\theta)|}{|z_1|^{k_1}|z_2|^{k_2}|z_3|^{k_3}} dz_1 dz_2 dz_3 \leq \rho_0^{-|k|} \sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\|z|\leq\rho_0}} |F(z,\theta)| \\ &\leq C_F \rho_0^{-|k|} \sup_{\substack{\theta\in\mathbb{T}_{\omega_0}\\|z|\leq\rho_0}} |z|^n \leq C_F \rho_0^{n-|k|}.\end{aligned}$$

Then by item 1 of Lemma 3.2.4 we obtain that:

$$\|a_k\|_\omega \leq C_F \rho_0^{n-|k|}. \tag{3.104}$$

Now, on one hand, we observe that for $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$:

$$|\Phi_0(v,\theta)| \leq \frac{\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega}}{|\cosh(\mathrm{d}v)|} \leq \frac{K\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega}}{\delta\kappa} \leq \frac{K}{\kappa}.$$

On the other hand, it is clear that for $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$:

$$|\Phi_0(v,\theta)| \leq \frac{\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega}}{|\cosh(\mathrm{d}v)|} \leq \delta K.$$

Thus, taking κ^* sufficiently large and δ^* sufficiently small we obtain that for all $(v,\theta) \in D^{\mathrm{u}}_{\kappa,\beta} \times \mathbb{T}_\omega$:

$$|\Phi_0(v,\theta)| \leq \rho_0.$$

Then, writing $\Phi_0 = (\phi_1,\phi_2,\phi_3)$, we can expand $F(\Phi_0(v,\theta),\theta)$ in the power series:

$$F(\Phi_0(v,\theta),\theta) = \sum_{\substack{k\in\mathbb{N}^3 \\ |k|\geq n}} a_k(\theta)\phi_1^{k_1}(v,\theta)\phi_2^{k_2}(v,\theta)\phi_3^{k_3}(v,\theta).$$

and by Lemma 3.2.3 we have that:

$$\|F(\Phi_0(v,\theta),\theta)\|^{\mathrm{u}}_{n,m,\omega} \leq M \sum_{\substack{k\in\mathbb{N}^3 \\ |k|\geq n}} \|a_k\|_\omega \left\|\phi_1^{k_1}\phi_2^{k_2}\phi_3^{k_3}\right\|^{\mathrm{u}}_{n,m,\omega}. \tag{3.105}$$

First, we shall bound the terms $\left\|\phi_1^{k_1}\phi_2^{k_2}\phi_3^{k_3}\right\|^{\mathrm{u}}_{n,m,\omega}$. Using item 3 of Lemma 3.2.2 (noting that $m \leq n \leq |k|$) and again Lemma 3.2.3, it is easy to see that:

$$\left\|\phi_1^{k_1}\phi_2^{k_2}\phi_3^{k_3}\right\|^{\mathrm{u}}_{n,m,\omega} \leq M\left\|\phi_1^{k_1}\phi_2^{k_2}\phi_3^{k_3}\right\|^{\mathrm{u}}_{n,|k|,\omega} \leq K\left(\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega}\right)^n \left(\|\Phi_0\|^{\mathrm{u},\times}_{0,1,\omega}\right)^{|k|-n}. \tag{3.106}$$

Now, by item 2 of Lemma 3.2.2 and for κ^* large enough:

$$\|\phi_i\|^{\mathrm{u}}_{0,1,\omega} \leq \frac{M}{\delta\kappa}\|\phi_i\|^{\mathrm{u}}_{1,1,\omega} \leq \frac{\rho_0}{2C_0\delta}\|\phi_i\|^{\mathrm{u}}_{1,1,\omega}. \tag{3.107}$$

By definition (3.98) of $\|.\|^{\mathrm{u},\times}_{n,m,\omega}$, it is clear that (3.107) yields:

$$\|\Phi_0\|^{\mathrm{u},\times}_{0,1,\omega} \leq \frac{\rho_0}{2C_0\delta}\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega},$$

and then (3.106) writes as:

$$\left\|\phi_1^{k_1}\phi_2^{k_2}\phi_3^{k_3}\right\|_{n,m,\omega}^{\mathrm{u}} \leq K\left(\|\Phi_0\|_{1,1,\omega}^{\mathrm{u},\times}\right)^{|k|}\left(\frac{\rho_0}{2C_0\delta}\right)^{|k|-n}. \tag{3.108}$$

In conclusion, using inequalities (3.108) and (3.104) in equation (3.105) we have that:

$$\|F(\Phi_0(v,\theta),\theta)\|_{n,m,\omega}^{\mathrm{u}} \leq KC\sum_{\substack{k\in\mathbb{N}^3\\|k|\geq n}}\left(\|\Phi_0\|_{1,1,\omega}^{\mathrm{u},\times}\right)^{|k|}\left(\frac{1}{2C_0\delta}\right)^{|k|-n},$$

and since $\|\Phi_0\|_{1,1,\omega}^{\mathrm{u},\times} \leq C_0\delta$, we obtain:

$$\|F(\Phi_0(v,\theta),\theta)\|_{n,m,\omega}^{\mathrm{u}} \leq KC\left(\|\Phi_0\|_{1,1,\omega}^{\mathrm{u},\times}\right)^{n}\sum_{\substack{k\in\mathbb{N}^3\\|k|\geq n}}\left(\frac{1}{2}\right)^{|k|-n},$$

and putting $M = K\sum_{\substack{k\in\mathbb{N}^3\\|k|\geq n}}\left(\frac{1}{2}\right)^{|k|-n} < \infty$ we obtain the claim of the first item of the lemma.

Now we shall proceed to prove item 2 of the lemma. Denoting $\Phi_\lambda = \Phi_0+(1-\lambda)\delta\Phi_1+\lambda\delta\Phi_2$, using Lemma 3.2.2 we obtain that for $\lambda\in[0,1]$:

$$\|\Phi_\lambda\|_{1,1,\omega}^{\mathrm{u},\times} \leq \|\Phi_0\|_{1,1,\omega}^{\mathrm{u},\times} + \frac{M}{\kappa}\|\Phi_1\|_{2,1,\omega}^{\mathrm{u},\times} + \frac{M}{\kappa}\|\Phi_2\|_{2,1,\omega}^{\mathrm{u},\times} \leq K\left(\delta+\frac{\delta}{\kappa}\right) \leq K\delta, \tag{3.109}$$

for some constant K. Then, using first the mean value theorem and after Lemma 3.2.3 we have that:

$$\begin{aligned}
&\|F(\Phi_0(v,\theta)+\delta\Phi_1(v,\theta),\theta) - F(\Phi_0(v,\theta)+\delta\Phi_2(v,\theta),\theta)\|_{n+1,m,\omega}^{\mathrm{u}} &(3.110)\\
&= \delta\left\|\int_0^1 D_xF(\Phi_\lambda,\theta)d\lambda\cdot(\Phi_1-\Phi_2)\right\|_{n+1,m,\omega}^{\mathrm{u}}\\
&\leq M\delta\left\|\int_0^1 D_xF(\Phi_\lambda,\theta)d\lambda\right\|_{n-1,m-1,\omega}^{\mathrm{u},\times}\|\Phi_1-\Phi_2\|_{2,1,\omega}^{\mathrm{u},\times}. &(3.111)
\end{aligned}$$

Using Lemma 3.2.5 in each entry of the matrix $\int_0^1 D_xF(\Phi_\lambda,\theta)d\lambda$, we have that:

$$\begin{aligned}
\left\|\int_0^1 D_xF(\Phi_\lambda,\theta)d\lambda\right\|_{n-1,m-1,\omega}^{\mathrm{u},\times} &\leq \int_0^1 \|D_xF(\Phi_\lambda,\theta)\|_{n-1,m-1,\omega}^{\mathrm{u},\times}\,d\lambda\\
&\leq \sup_{\lambda\in[0,1]}\|D_xF(\Phi_\lambda,\theta)\|_{n-1,m-1,\omega}^{\mathrm{u},\times}\\
&\leq \sup_{\|\Phi\|_{1,1,\omega}^{\mathrm{u}}\leq K\delta}\|D_xF(\Phi,\theta)\|_{n-1,m-1,\omega}^{\mathrm{u},\times}\,, \qquad (3.112)
\end{aligned}$$

where in the last step we have used (3.109). Moreover, since $|F(x,\theta)| \leq C_F|x|^n$, it can be easily seen that $|D_xF(x,\theta)| \leq MC_F|x|^{n-1}$. Then, using item 1 of this same lemma, from (3.112) we obtain that:

$$\left\| \int_0^1 D_xF(\Phi_\lambda,\theta)d\lambda \right\|^{\mathrm{u},\times}_{n-1,m-1,\omega} \leq MC_F \sup_{\|\Phi\|^{\mathrm{u}}_{1,1,\omega}\leq K\delta} \left(\|\Phi\|^{\mathrm{u}}_{1,1,\omega}\right)^{n-1} \leq MC_FK^{n-1}\delta^{n-1}. \tag{3.113}$$

Finally, using bound (3.113) in equation (3.110), and renaming $M = \frac{MC_FK^{n-1}}{\kappa^*}$ the claim of the second item is proved. □

3.2.2 The operator $\mathcal{G}^{\mathrm{u}}$

In this section we will study the properties of the operator $\mathcal{G}^{\mathrm{u}}$ defined in (3.47). For the sake of convenience we recall that $\mathcal{G}^{\mathrm{u}}$ is the operator acting on functions ϕ defined by:

$$\mathcal{G}^{\mathrm{u}}(\phi)(v,\theta) := \sum_{l\in\mathbb{Z}} \mathcal{G}^{\mathrm{u}[l]}(\phi)(v)e^{il\theta}, \tag{3.114}$$

with Fourier coefficients:

$$\begin{aligned} &\mathcal{G}^{\mathrm{u}[l]}(\phi)(v) \\ = \ &\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\int_{-\infty}^0 \frac{e^{-il\left(\delta^{-1}\alpha s - cs + \frac{c}{\mathrm{d}}\log\left(1+e^{2\mathrm{d}(v+s)}\right) - \frac{c}{\mathrm{d}}\log\left(1+e^{2\mathrm{d}v}\right)\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+s))}\phi^{[l]}(v+s)ds, \end{aligned} \tag{3.115}$$

where, as usual, $\phi^{[l]}(v)$ denotes the $l-$th Fourier coefficient of $\phi(v,\theta)$.

Lemma 3.2.7. *Let $v \in D^{\mathrm{u}}_{\kappa,\beta}$ be fixed, and consider $s = s_\pm(t,v)$ defined implicitly by (see Figure 3.5):*

$$s_\pm - \frac{c\delta}{\alpha}s_\pm + \frac{c\delta}{\mathrm{d}\alpha}\left(\log\left(1+e^{2\mathrm{d}(v+s_\pm)}\right) - \log\left(1+e^{2\mathrm{d}v}\right)\right) = -te^{\pm i\frac{\beta}{2}}.$$

The function $s_\pm(t,v)$ is well-defined for all $t \in [0,+\infty)$ and $v \in D^{\mathrm{u}}_{\kappa,\beta}$, and can be written as:

$$s_\pm(t,v) = \frac{t}{1-c\delta\alpha^{-1}}\left(-e^{\pm i\frac{\beta}{2}} + \frac{s^1_\pm(t,v)}{t}\right).$$

Moreover, there exists a constant M such that for all $t \in [0,\infty)$ and $v \in D^{\mathrm{u}}_{\kappa,\beta}$:

1. $|s^1_\pm(t,v)| \leq M\delta\log(\delta\kappa)$.

2. $s^1_\pm(0,v) = 0$.

3. $\left|\frac{s^1_\pm(t,v)}{t}\right| \leq \frac{M}{\kappa}$.

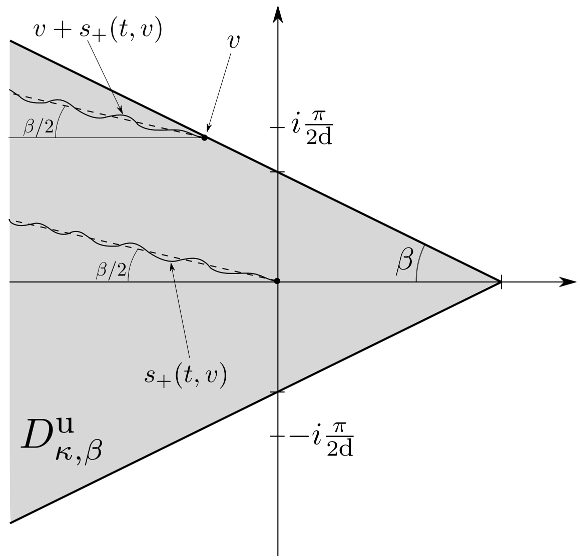


Figure 3.5: The domain $D^{\mathrm{u}}_{\kappa,\beta}$ with an example of the curves $s_+(t,v)$ and $v+s_+(t,v)$. The discontinuous lines are $-te^{i\beta/2}$ and $v-te^{i\beta/2}$ respectively.

4. *$v+s_\pm(t,v)\in D^{\mathrm{u}}_{\kappa,\beta}$ for all $t\in[0,+\infty)$ (see Figure 3.5), and we can write:*

$$s_\pm(t,v) = -\rho(t,v)e^{i\tilde{\beta}_\pm(t,v)},$$

with:

$$|\rho(t,v)-t| \leq \frac{M}{\kappa}, \qquad \left|\tilde{\beta}_\pm(t,v) \mp \frac{\beta}{2}\right| \leq \bar{\beta} < \beta/3.$$

Proof. Denoting $Q=Q(\delta)=(1-c\delta\alpha^{-1})^{-1}$, the function $s^1_\pm$ must satisfy the following equation:

$$s^1_\pm + \frac{c\delta}{\mathrm{d}\alpha}\left(\log\left(1+e^{2\mathrm{d}(v+Q(-te^{\pm i\frac{\beta}{2}}+s^1_\pm))}\right) - \log\left(1+e^{2\mathrm{d}v}\right)\right) = 0,$$

which can be rewritten as the following fixed point equation:

$$s^1_\pm = \mathcal{F}(s^1_\pm) := \frac{-c\delta}{\mathrm{d}\alpha}\left(\log\left(1+e^{2\mathrm{d}(v+Q(-te^{\pm i\frac{\beta}{2}}+s^1_\pm))}\right) - \log\left(1+e^{2\mathrm{d}v}\right)\right). \tag{3.116}$$

We will prove that $\mathcal{F}$ has indeed a fixed point in a given ball. To that end, let us define:

$$s^1_0 := \mathcal{F}(0) = \frac{-c\delta}{\mathrm{d}\alpha}\left(\log\left(1+e^{2\mathrm{d}(v-Qte^{\pm i\frac{\beta}{2}})}\right) - \log\left(1+e^{2\mathrm{d}v}\right)\right).$$

Note that we just write s_0^1 without specifying whether we are in the + or − case to avoid cumbersome notation. We claim that there exists a constant K such that:

$$|s_0^1| \leq K\delta\log(\delta\kappa). \tag{3.117}$$

Indeed, we have that:

$$|s_0^1| \leq \frac{c\delta}{\mathrm{d}\alpha}\left(\left|\log\left(1+e^{2\mathrm{d}(v-Qte^{\pm i\frac{\beta}{2}})}\right)\right| + \left|\log\left(1+e^{2\mathrm{d}v}\right)\right|\right). \tag{3.118}$$

Now, on one hand it is easy to see that for $w \in D^{\mathrm{u}}_{\kappa,\beta}$, one has that $\left|\mathrm{Im}\,\log\left(1+e^{2\mathrm{d}w}\right)\right| = \left|\arg\left(1+e^{2\mathrm{d}w}\right)\right| < \pi$. On the other hand, one has that:

$$\left|\mathrm{Re}\,\log\left(1+e^{2\mathrm{d}w}\right)\right| = \left|\log\left|1+e^{2\mathrm{d}w}\right|\right|.$$

It is clear that there exist constants $K_1 \neq 0$ and K_2 such that for $w \in D^{\mathrm{u}}_{\kappa,\beta}$:

$$K_1\delta\kappa \leq \left|1+e^{2\mathrm{d}w}\right|, \qquad \left|e^{2\mathrm{d}w}\right| \leq K_2,$$

and hence, it is easy to see that:

$$\left|\log\left|1+e^{2\mathrm{d}w}\right|\right| \leq K|\log(\delta\kappa)|,$$

for some constant K. Thus:

$$\begin{aligned}\left|\log\left(1+e^{2\mathrm{d}w}\right)\right| &\leq \left|\mathrm{Re}\,\log\left(1+e^{2\mathrm{d}w}\right)\right| + \left|\mathrm{Im}\,\log\left(1+e^{2\mathrm{d}w}\right)\right| \\ &\leq K|\log(\delta\kappa)|\left(1+\frac{\pi}{K|\log(\delta\kappa)|}\right) \leq K|\log(\delta\kappa)|, \end{aligned} \tag{3.119}$$

for a certain K. In particular, bound (3.119) is valid for $w = v \in D^{\mathrm{u}}_{\kappa,\beta}$ and $w = v - Qte^{\pm i\frac{\beta}{2}} \in D^{\mathrm{u}}_{\kappa,\beta}$ for all $t \in [0,+\infty)$, since $Q > 0$ for δ small enough. Using (3.119) in (3.118), the claim (3.117) is proved straightforwardly.

Now consider $w_1, w_2 \in B(2|s_0^1|) \subset \mathbb{C}$, the ball of center 0 and radius $2|s_0^1|$. Our next claim is that for κ large enough:

$$|\mathcal{F}(w_1) - \mathcal{F}(w_2)| \leq \frac{1}{2}|w_1 - w_2|. \tag{3.120}$$

This can be proved easily using the mean value theorem and the fact that, for all $v \in D^{\mathrm{u}}_{\kappa,\beta}$, $t \in [0,+\infty)$ and $w \in B(2|s_0^1|)$, one has that:

$$|\mathcal{F}'(w)| \leq \frac{K}{\kappa},$$

for some constant K. Taking κ sufficiently large, so that $K/\kappa < 1/2$, we obtain (3.120). This implies that $\mathcal{F}$ has a unique fixed point in $B(2|s_0^1|)$. Indeed, take $w \in B(2|s_0^1|)$, then by (3.120):

$$|\mathcal{F}(w)| \leq |\mathcal{F}(w) - \mathcal{F}(0)| + |\mathcal{F}(0)| \leq \frac{1}{2}|w| + |s_0^1| \leq 2|s_0^1|,$$

and hence $\mathcal{F} : B(2|s_0^1|) \to B(2|s_0^1|)$. The fact that $\mathcal{F}$ is contractive is obvious from (3.120), and hence it has a unique fixed point $s_\pm^1 = s_\pm^1(v,t)$. Moreover, since $s_\pm^1 \in B(2|s_0^1|)$, by (3.117) it is clear that item 1 of the lemma holds. To see that item 2 also holds, one just has to take $t = 0$ in equation (3.116). Clearly, in this case $s_\pm^1 = 0$ is a fixed point. Finally, we prove that item 3 is also true. Differentiating implicitly (3.116) with respect to t, it is easy to see that there exists a constant K such that:

$$\left|\partial_t s_\pm^1(t,v)\right| \leq \frac{K}{\kappa}, \tag{3.121}$$

for all $t \in [0,+\infty)$ and $v \in D^{\mathrm{u}}_{\kappa,\beta}$. Then, using item 2 above and the mean value theorem we have:

$$\left|\frac{s_\pm^1(t,v)}{t}\right| = \left|\frac{s_\pm^1(t,v) - s_\pm^1(0,v)}{t}\right| \leq \sup_{\xi\in[0,t]} \left|\partial_t s_\pm^1(\xi,v)\right| \leq \frac{K}{\kappa},$$

and the claim is proved.

Items 1, 2, and 3 yield easily item 4, and hence the proof is finished.

□

We do not prove the following two lemmas, since they can be proved using standard geometric arguments.

Lemma 3.2.8. *Let $v \in D^{\mathrm{u}}_{\kappa,\beta}$.*

1. *If $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$, $t \in [0,\infty)$. Then there exists a constant $M \neq 0$ such that:*

$$\left|v + s_\pm(t,v) - \frac{i\pi}{2\mathrm{d}}\right| \geq M\left|v - \frac{i\pi}{2\mathrm{d}}\right|,$$

and:

$$|\cosh(\mathrm{d}(v + s_\pm(t,v)))| \geq M|\cosh(\mathrm{d}v)|,$$

where $s_\pm(t,v)$ are the functions given in Lemma 3.2.7.

2. *If $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$, then:*

   (a) $|\cosh(\mathrm{d}v)| \geq \dfrac{e^{\mathrm{d}|\mathrm{Re}\,(v)|}}{4}$.

(b) *There exist two constants $A > 0$ and M such that if $s_\pm(t,v)$ are the functions defined in Lemma 3.2.7, then:*

$$\frac{|\cosh(\mathrm{d}v)|}{|\cosh(\mathrm{d}(v+s_\pm(t,v)))|} \le Me^{-A|t|},$$

Lemma 3.2.9. *Consider $s_\pm(t,v)$ the functions defined in Lemma 3.2.7 and let $n > 1$. Let t^* be such that $v + s_\pm(t,v) \in D^{\mathrm{u}}_{\kappa,\beta,T}$ for all $0 \le t \le t^*$. For all $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$, there exists a constant M such that:*

$$\int_0^{t^*} \frac{1}{|\cosh(\mathrm{d}(v+s_\pm(t,v)))|^n} dt \le \frac{M}{|\cosh(\mathrm{d}v)|^{n-1}}$$

The following lemma allows us to express the integrals $\mathcal{G}^{\mathrm{u}[l]}(\phi)(v)$ (see (3.115)) in a more suitable way, changing the integration path by means of $s_\pm(t,v)$.

Lemma 3.2.10. *Consider the curve (see Figure 3.5):*

$$\Gamma^R_\pm := \{z \in \mathbb{C} \,:\, z = s_\pm(t,v),\, t \in [0,R]\},$$

where $s_\pm(t,v)$ are the functions given in Lemma 3.2.7.

Then, if $m > 0$ and $\phi \in \mathcal{X}^{\mathrm{u}}_{n,m,\omega}$ one has:

$$\mathcal{G}^{\mathrm{u}[l]}(\phi)(v) = -\lim_{R\to+\infty} \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v) \int_{\Gamma^R_\pm} \frac{e^{-il\left(\delta^{-1}\alpha z - cz + \frac{c}{d}\log\left(1+e^{2\mathrm{d}(v+z)}\right) - \frac{c}{d}\log\left(1+e^{2\mathrm{d}v}\right)\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+z))} \phi^{[l]}(v+z)dz,$$

where the coefficients $\mathcal{G}^{\mathrm{u}[l]}$ where defined in (3.115)*, and we take the integral over Γ^R_+ for $l \ge 0$ and over Γ^R_- otherwise.*

Proof. Recall that, by item 4 of Lemma 3.2.7, $s_\pm(t,v) = -\rho(t,v)e^{i\tilde\beta_\pm(t,v)}$. We will do the proof just for $l \ge 0$, since the proof for $l < 0$ is completely analogous replacing Γ^R_+ by Γ^R_-. Moreover, to shorten the notation, we shall omit the subscript $+$ in the notation.

Let us consider the following curves:

$$\begin{aligned}\Gamma^R_1 &= \{z \in \mathbb{C} \,:\, z = t,\, t \in [-\rho(R,v), 0]\} \\ \Gamma^R_2 &= \left\{z \in \mathbb{C} \,:\, z = -\rho(R,v)e^{it},\, t \in [\tilde\beta_-(R,v), 0]\right\}\end{aligned}$$

On the one hand note that, since $\rho(t,v) \to +\infty$ as $t \to +\infty$, we have that:

$$\mathcal{G}^{\mathrm{u}[l]}(\phi)(v) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v) \lim_{R\to+\infty} \int_{\Gamma^R_1} \frac{e^{-il\left(\delta^{-1}\alpha z - cz + \frac{c}{d}\log\left(1+e^{2\mathrm{d}(v+z)}\right) - \frac{c}{d}\log\left(1+e^{2\mathrm{d}v}\right)\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+z))} \phi^{[l]}(v+z)dz.$$

On the other hand, since the closed curve $\Gamma^R \cup \Gamma_1^R \cup \Gamma_2^R \subset D^{\mathrm{u}}_{\kappa,\beta}$, the integrand is analytic in the interior of this curve. Then we conclude that the integral along this closed curve is equal to zero. Hence, in order to prove the lemma we just have to check that the integral along the curve Γ_2^R goes to zero as $t \to +\infty$.

We note that, for R large enough, $\Gamma_2^R \subset D^{\mathrm{u}}_{\kappa,\beta,\infty}$ and also $v+z \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$ for all $v \in D^{\mathrm{u}}_{\kappa,\beta}$ and $z \in \Gamma_2^R$. Then the following bounds hold:

$$
\begin{aligned}
\left|\phi^{[l]}(v+z)\right| &\leq \frac{\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m,\omega}}{\left|\cosh(\mathrm{d}(v+z))\right|^m},\\
\left|e^{-il\left(\delta^{-1}\alpha z - cz\right)}\right| &= e^{\delta^{-1}\alpha l \mathrm{Im}\, z\left(1-\frac{\delta c}{\alpha}\right)},\\
\left|e^{-il\left(\frac{c}{d}\log\left(1+e^{2\mathrm{d}(v+z)}\right)-\frac{c}{d}\log\left(1+e^{2\mathrm{d}v}\right)\right)}\right| &\leq e^{\frac{2c}{d}\pi|l|}.
\end{aligned}
$$

Then, parameterizing Γ_2^R as $z = -\rho(R,v)e^{it}$, we have that $\mathrm{Im}\, z = -\rho(R,v)\sin t$ and hence:

$$
\begin{aligned}
&\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\int_{\Gamma_2^R}\frac{e^{-il\left(\delta^{-1}\alpha z - cz+\frac{c}{d}\log\left(1+e^{2\mathrm{d}(v+z)}\right)-\frac{c}{d}\log\left(1+e^{2\mathrm{d}v}\right)\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+z))}\phi^{[l]}(v+z)dz\right|\\
&\leq \left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right|\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m,\omega} e^{\frac{2c}{d}\pi|l|}\rho(R,v)\int_{\tilde\beta_+(R,v)}^0 \frac{e^{-\delta^{-1}\alpha l\rho(R,v)\left(1-\frac{\delta c}{\alpha}\right)\sin t}}{\left|\cosh(\mathrm{d}(v-\rho(R,v)e^{it}))\right|^{\frac{2}{\mathrm{d}}+m}}dt\\
&\leq K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m,\omega} e^{\frac{2c}{d}\pi|l|}\rho(R,v)\int_{\tilde\beta_+(R,v)}^0 \frac{e^{-\delta^{-1}\alpha l\rho(R,v)\left(1-\frac{\delta c}{\alpha}\right)\sin t}}{\left|\cosh(\mathrm{d}(v-\rho(R,v)e^{it}))\right|^{m}}dt, \qquad (3.122)
\end{aligned}
$$

where in the last step we have used Lemma 3.2.8.

Observe that for R large enough $\rho(R,v)\geq 0$. Moreover, note that $l\sin t \geq 0$. Then we have that:

$$
e^{-\delta^{-1}\alpha l\rho(R,v)\left(1-\frac{\delta c}{\alpha}\right)\sin t} \leq 1. \qquad (3.123)
$$

Moreover, using (a) of item 2 of Lemma 3.2.8 for $w = v-\rho(R,v)e^{it}$, with $t\in[0,\tilde\beta_+(R,v)]$, and (3.123) in (3.122) we obtain:

$$
\begin{aligned}
&\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\int_{\Gamma_2^R}\frac{e^{-il\left(\delta^{-1}\alpha z - cz+\frac{c}{d}\log\left(1+e^{2\mathrm{d}(v+z)}\right)-\frac{c}{d}\log\left(1+e^{2\mathrm{d}v}\right)\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+z))}\phi^{[l]}(v+z)dz\right|\\
&\leq K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m,\omega} e^{\frac{2c}{d}\pi|l|}\rho(R,v)\int_{\tilde\beta_+(R,v)}^0 e^{-m\mathrm{d}|\mathrm{Re}\,(v-\rho(R,v)e^{it})|}dt. \qquad (3.124)
\end{aligned}
$$

Finally, we note that for $v\in D^{\mathrm{u}}_{\kappa,\beta,\infty}$, $t\in[0,\tilde\beta_+(R,v)]$ and R large enough :

$$
|\mathrm{Re}\,(v-\rho(R,v)e^{it})| \geq \rho(R,v)\cos t - T \geq KR,
$$

where we have used that, by item 4 of Lemma 3.2.7, $\rho(R,v)-R$ is bounded. Then (3.124) yields:

$$\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\int_{\Gamma_2^R}\frac{e^{-il\left(\delta^{-1}\alpha z-cz+\frac{c}{d}\log\left(1+e^{2\mathrm{d}(v+z)}\right)-\frac{c}{d}\log\left(1+e^{2\mathrm{d}v}\right)\right)}}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+z))}\phi^{[l]}(v+z)dz\right|$$
$$\leq K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m,\omega}e^{\frac{2c}{d}\pi|l|}Re^{-m\mathrm{d}KR},$$

which clearly goes to zero as R goes to infinity, and thus the proof is finished. ☐

In the following lemma we summarize the main properties of the operator $\mathcal{G}^{\mathrm{u}}$ and its Fourier coefficients.

Lemma 3.2.11. *Let $n\geq 0$, $m\geq 0$ and $\phi\in\mathcal{X}^{\mathrm{u}}_{n,m,\omega}$. There exists a constant M such that for all $l\in\mathbb{Z}$:*

1. *If $n\geq 1$, then $\|\mathcal{G}^{\mathrm{u}[l]}(\phi)\|^{\mathrm{u}}_{n-1,m}\leq M\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}$.*
2. *If $l\neq 0$ and $n\geq 0$, then $\|\mathcal{G}^{\mathrm{u}[l]}(\phi)\|^{\mathrm{u}}_{n,m}\leq\dfrac{\delta M\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}}{|l|}$.*
3. *As a consequence we have that if $n\geq 1$:*

$$\|\mathcal{G}^{\mathrm{u}}(\phi)\|^{\mathrm{u}}_{n-1,m,\omega}\leq M\|\phi\|^{\mathrm{u}}_{n,m,\omega}.$$

   Moreover, if $\phi^{[0]}(v)=0$, then for all $n\geq 0$:

$$\|\mathcal{G}^{\mathrm{u}}(\phi)\|^{\mathrm{u}}_{n,m,\omega}\leq M\delta\|\phi\|^{\mathrm{u}}_{n,m,\omega}.$$

4. *If $n\geq 0$, $\|\partial_\theta\mathcal{G}^{\mathrm{u}}(\phi)\|^{\mathrm{u}}_{n,m,\omega}\leq M\delta\|\phi\|^{\mathrm{u}}_{n,m,\omega}$.*
5. *If $n\geq 1$, $\|\partial_v\mathcal{G}^{\mathrm{u}}(\phi)\|^{\mathrm{u}}_{n,m,\omega}\leq M\|\phi\|^{\mathrm{u}}_{n,m,\omega}$.*

In conclusion, from the previous items it is straightforward to see that if $\phi\in\mathcal{X}^{\mathrm{u}}_{n,m,\omega}$, $n\geq 1$, then $\mathcal{G}^{\mathrm{u}}(\phi)\in\tilde{\mathcal{X}}^{\mathrm{u}}_{n-1,m,\omega}$ and:

$$[\![\mathcal{G}^{\mathrm{u}}(\phi)]\!]^{\mathrm{u}}_{n-1,m,\omega}\leq M\|\phi\|^{\mathrm{u}}_{n,m,\omega}.$$

Proof. To prove this result we will use Lemma 3.2.10, that is, we will consider the integrals along the curve $\Gamma^R_\pm$ and take the limit for $R\to+\infty$. We parameterize this curve as $z=s_\pm(t,v)$, $t\in[0,R]$, where $s_\pm(t,v)$ is the function defined in Lemma 3.2.7. Recall that, by definition, $s_\pm(t,v)$ satisfies:

$$s_\pm(t,v)-\frac{c\delta}{\alpha}s_\pm(t,v)+\frac{c\delta}{\mathrm{d}\alpha}\left(\log\left(1+e^{2\mathrm{d}(v+s_\pm(t,v))}\right)-\log\left(1+e^{2\mathrm{d}v}\right)\right)=-te^{\pm i\frac{\beta}{2}}.$$

In the following, whenever we write $s_\pm(t,v)$, we take $s_+(t,v)$ if $l \geq 0$ and $s_-(t,v)$ otherwise. We point out that, with this choice:

$$\mathrm{Re}\left(il\delta^{-1}\alpha t e^{\pm i\beta/2}\right) = -|l|\sin(\beta/2)\delta^{-1}\alpha t < 0. \tag{3.125}$$

Moreover, recall that $s_\pm(t,v) = \frac{t}{1-c\delta\alpha^{-1}}\left(-e^{\pm i\frac{\beta}{2}} + \frac{s^1_\pm(t,v)}{t}\right)$, and hence by (3.121) there exists a constant K such that:

$$|\partial_t s_\pm(t,v)| \leq K(|e^{\pm i\beta/2}| + |\partial_t s^1_\pm(t,v)|) \leq K. \tag{3.126}$$

Then, for all $v \in D^{\mathrm{u}}_{\kappa,\beta}$ we have:

$$\begin{aligned} \left|\mathcal{G}^{[l]}(\phi)(v)\right| &\leq \left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right| \int_0^\infty \frac{|\partial_t s_\pm(t,v)||\phi^{[l]}(v+s_\pm(t,v))e^{il\delta^{-1}\alpha t e^{\pm i\beta/2}}|}{\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+s_\pm(t,v)))\right|} dt \\ &\leq K\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right| \int_0^\infty \frac{\left|\phi^{[l]}(v+s_\pm(t,v))e^{il\delta^{-1}\alpha t e^{\pm i\beta/2}}\right|}{\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+s_\pm(t,v)))\right|} dt \end{aligned} \tag{3.127}$$

Let us prove item 1. We note that, by (3.125), for all $l \in \mathbb{Z}$ one has:

$$\left|e^{il\delta^{-1}\alpha t e^{\pm\beta/2}}\right| = e^{-|l|\sin(\beta/2)\delta^{-1}\alpha t} \leq 1. \tag{3.128}$$

Take $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$. Then $v + s_\pm(t,v) \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$, and equation (3.127) with (3.128) yield:

$$\begin{aligned} \left|\mathcal{G}^{[l]}(\phi)(v)\right| &\leq K\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right| \int_0^\infty \frac{|\phi^{[l]}(v+s_\pm(t,v))|}{\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}(v+s_\pm(t,v)))\right|} dt \\ &\leq K\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right| \left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m} \int_0^\infty \frac{1}{|\cosh(\mathrm{d}(v+s_\pm(t,v)))|^{m+\frac{2}{\mathrm{d}}}} dt. \end{aligned} \tag{3.129}$$

Using item 2b of Lemma 3.2.8 and noting that $m + 2/\mathrm{d} > 0$ for $m \geq 0$, from (3.129) we obtain for $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$:

$$\left|\mathcal{G}^{[l]}(\phi)(v)\right| \leq \frac{K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}}{|\cosh(\mathrm{d}v)|^m} \int_0^\infty e^{-\left(m+\frac{2}{\mathrm{d}}\right)At} dt \leq \frac{K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}}{|\cosh(\mathrm{d}v)|^m}, \tag{3.130}$$

for some constant K, which clearly does not depend on l.

Now take $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$. Let $t^* \in \mathbb{R}$, $t^* \geq 0$, be such that $v + s_\pm(t^*,v) \in D^{\mathrm{u}}_{\kappa,\beta,T} \cap D^{\mathrm{u}}_{\kappa,\beta,\infty}$. Then, reasoning analogously as in the case for $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$, one can see that:

$$\begin{aligned}\left|\mathcal{G}^{[l]}(\phi)(v)\right| &\leq K\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right| \left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}\left(\int_0^{t^*}\frac{dt}{\left|\cosh(\mathrm{d}(v+s_\pm(t,v)))\right|^{n+\frac{2}{\mathrm{d}}}}\right.\\ &\quad\left.+\int_{t^*}^{\infty}\frac{dt}{\left|\cosh(\mathrm{d}(v+s_\pm(t,v)))\right|^{m+\frac{2}{\mathrm{d}}}}\right)\\ &\leq K\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right| \left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}\left(\int_0^{t^*}\frac{1}{\left|\cosh(\mathrm{d}(v+s_\pm(t,v)))\right|^{n+\frac{2}{\mathrm{d}}}}dt+K\right),\end{aligned} \tag{3.131}$$

where the fact that the second integral is bounded is clear by Lemma 3.2.8. Now, by Lemma 3.2.9 (taking into account that $n+2/\mathrm{d} > 1$ for $n \geq 1$) and the fact that $|\cosh(\mathrm{d}v)|$ is bounded for $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$, it is clear that (3.131) yields:

$$\left|\mathcal{G}^{[l]}(\phi)(v)\right| \leq \frac{K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}}{|\cosh(\mathrm{d}v)|^{n-1}}, \qquad \text{for } v \in D^{\mathrm{u}}_{\kappa,\beta,T}, \tag{3.132}$$

for some constant K, which again does not depend on l. Bounds (3.130) and (3.132) yield straightforwardly item 1.

Now we shall prove item 2. On the one hand, if we take $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$, using that $\phi \in \mathcal{X}^{\mathrm{u}}_{n,m,\omega}$ and the fact that if $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$ then $v + s_\pm(t,v) \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$ for all $t \geq 0$, (3.127) yields:

$$\begin{aligned}\left|\mathcal{G}^{[l]}(\phi)(v)\right| &\leq K\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right| \left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}\int_0^{\infty}\frac{\left|e^{il\delta^{-1}\alpha t e^{\pm i\beta/2}}\right|dt}{|\cosh(v+s_\pm(t,v))|^{m+\frac{2}{\mathrm{d}}}}\\ &\leq \frac{K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}}{|\cosh(\mathrm{d}v)|^{m}}\int_0^{\infty}\left|e^{il\delta^{-1}\alpha t e^{\pm i\beta/2}}\right|dt,\end{aligned}$$

where we have used item 2b of Lemma 3.2.8 again, and that $e^{-A\left(m+\frac{2}{\mathrm{d}}\right)|t|} \leq 1$. Using (3.125) in this last expression we obtain that if $l \neq 0$, then for $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$:

$$\left|\mathcal{G}^{[l]}(\phi)(v)\right| \leq \frac{K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}}{|\cosh(\mathrm{d}v)|^{m}}\int_0^{\infty}e^{-|l|\sin(\beta/2)\delta^{-1}\alpha t}dt \leq \frac{K\delta\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}}{|l|\,|\cosh(\mathrm{d}v)|^{m}}, \tag{3.133}$$

for some constant K, which does not depend on l. On the other hand, if $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$, let again $t^* \in \mathbb{R}$, $0 \leq t^* \leq M$, be such that $v + s_\pm(t^*,v) \in D^{\mathrm{u}}_{\kappa,\beta,T} \cap D^{\mathrm{u}}_{\kappa,\beta,\infty}$. Then, reasoning analogously as in the case for $v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$, one can see that:

$$\begin{aligned}\left|\mathcal{G}^{[l]}(\phi)(v)\right| &\leq K\left|\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}v)\right| \left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}\left(\int_0^{t^*}\frac{e^{-|l|\sin(\beta/2)\delta^{-1}\alpha t}dt}{\left|\cosh(\mathrm{d}(v+s_\pm(t,v)))\right|^{n+\frac{2}{\mathrm{d}}}}\right.\\ &\quad\left.+\int_{t^*}^{\infty}\frac{e^{-|l|\sin(\beta/2)\delta^{-1}\alpha t}dt}{\left|\cosh(\mathrm{d}(v+s_\pm(t,v)))\right|^{m+\frac{2}{\mathrm{d}}}}\right)\\ &\leq K\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}\left(\frac{1}{\left|\cosh(\mathrm{d}v)\right|^n}\int_0^{t^*}e^{-|l|\sin(\beta/2)\delta^{-1}\alpha t}dt+\int_{t^*}^{\infty}e^{-|l|\sin(\beta/2)\delta^{-1}\alpha t}dt\right)\end{aligned}$$

where to bound the first integral we have used item 1 of Lemma 3.2.8 and to bound the second one we have used that if $w \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$, then $\left|\cosh(\mathrm{d}w)\right|^{-1}$ is bounded. Now, since $|\cosh(\mathrm{d}v)|$ is bounded for $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$, it is clear that this last equation yields, for $l \neq 0$:

$$\left|\mathcal{G}^{[l]}(\phi)(v)\right| \leq \frac{K\delta\left\|\phi^{[l]}\right\|^{\mathrm{u}}_{n,m}}{|l|\left|\cosh(\mathrm{d}v)\right|^n} \tag{3.134}$$

for some constant K, which again does not depend on l. Thus, using (3.133) and (3.134), we prove straightforwardly item 2.

Item 3 is a direct consequence of items 2 and 1, where we use that the constants K do not depend on the index l of the Fourier coefficient.

Item 4 is easily proved, noting that:

$$\partial_\theta\mathcal{G}(\phi)(v,\theta) = \sum_{\substack{l\in\mathbb{Z}\\ l\neq 0}} il\mathcal{G}^{[l]}(\phi)(v)e^{il\theta},$$

and using item 2.

Finally, to prove item 5, we just need to use that since $\mathcal{G}$ is a right inverse of the operator $\mathcal{L}$, we have:

$$(-\delta^{-1}\alpha - cZ_0(v))\partial_\theta\mathcal{G}(\phi) + \partial_v\mathcal{G}(\phi) - 2Z_0(v)\mathcal{G}(\phi) = \phi,$$

and then using Lemma 3.2.3 we obtain that:

$$\|\partial_v\mathcal{G}(\phi)\|^{\mathrm{u}}_{n,m,\omega} \leq \|\phi\|^{\mathrm{u}}_{n,m,\omega} + K\|Z_0\|^{\mathrm{u}}_{1,0}\|\mathcal{G}(\phi)\|^{\mathrm{u}}_{n-1,m,\omega} + K\left(\delta^{-1}+\|Z_0\|^{\mathrm{u}}_{0,0}\right)\|\partial_\theta\mathcal{G}(\phi)\|^{\mathrm{u}}_{n,m,\omega}.$$

Using items 3 and 4 and the fact that $\|Z_0\|^{\mathrm{u}}_{1,0} \leq K$, and $\|Z_0\|^{\mathrm{u}}_{0,0} \leq K\delta^{-1}$, item 5 is proved. □

3.2.3 The independent term $\tilde{\mathcal{F}}^{\mathrm{u}}(0)$

In this section we will proceed to bound the independent term of the fixed point equation (3.54) which is $\tilde{\mathcal{F}}^{\mathrm{u}}(0) = \mathcal{G}^{\mathrm{u}} \circ \mathcal{F}^{\mathrm{u}}(0)$, where:

$$\begin{aligned} \mathcal{F}^{\mathrm{u}}(0)(v,\theta) &= 2\sigma R_0(v) + \delta^p \frac{\mathrm{d}+1}{b} Z_0(v) H^{\mathrm{u}}(\delta R_0(v), \theta, \delta Z_0(v), \delta, \delta\sigma) \\ &\quad + \delta^p F^{\mathrm{u}}(\delta R_0(v), \theta, \delta Z_0(v), \delta, \delta\sigma). \end{aligned} \tag{3.135}$$

Lemma 3.2.12. *Let C_R be some constant, and $R(v,\theta)$ such that $\|R\|_{2,2,\omega} \leq C_R$. There exists a constant M such that:*

1. $\|F^{\mathrm{u}}(\delta R(v,\theta), \theta, \delta Z_0(v), \delta, \delta\sigma)\|^{\mathrm{u}}_{4,2,\omega} \leq M\delta^3$,
2. $\|G^{\mathrm{u}}(\delta R(v,\theta), \theta, \delta Z_0(v), \delta, \delta\sigma)\|^{\mathrm{u}}_{2,0,\omega} \leq M\delta^3$,
3. $\|H^{\mathrm{u}}(\delta R(v,\theta), \theta, \delta Z_0(v), \delta, \delta\sigma)\|^{\mathrm{u}}_{3,2,\omega} \leq M\delta^3$,

where F^{u}, G^{u} and H^{u} are defined in (3.35)*. In particular, this holds for $R = R_0$.*

Proof. For this proof we will use properties (3.32) and (3.33) of the functions f^{u}, g^{u} and h^{u}. First we will prove item 1. Recall that:

$$\begin{aligned} & F^{\mathrm{u}}(\delta R(v,\theta), \theta, \delta Z_0(v), \delta, \delta\sigma) \\ &= \sqrt{2R(v,\theta)} \cos\theta f^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta, \delta\sqrt{2R(v,\theta)}\sin\theta, \delta Z_0(v), \delta, \delta\sigma) \\ &+ \sqrt{2R(v,\theta)} \sin\theta g^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta, \delta\sqrt{2R(v,\theta)}\sin\theta, \delta Z_0(v), \delta, \delta\sigma). \end{aligned}$$

Since f^{u} is of order three in all their variables and $f^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) = 0$ for all δ small enough and σ, it can be easily seen that in its domain of analyticity:

$$f^{\mathrm{u}}(z_1, z_2, z_3, \delta, \delta\sigma) = \sum_{n=1}^{3} \delta^{3-n} f^{\mathrm{u}}_n(z_1, z_2, z_3 + \delta, \delta, \delta\sigma), \tag{3.136}$$

where:

$$|f^{\mathrm{u}}_n(z_1, z_2, z_3 + \delta, \delta, \delta\sigma)| \leq K|(z_1, z_2, z_3 + \delta)|^n.$$

Now we define $\Phi_0(v,\theta) = (\delta\sqrt{2R(v,\theta)}\cos\theta, \delta\sqrt{2R(v,\theta)}\sin\theta, \delta(Z_0(v)+1))$. Note that if $\|R\|_{2,2,\omega} \leq C_R$ then $\|\sqrt{2R}\|_{1,1,\omega} \leq K$ for some constant K. Then:

$$\|\delta\sqrt{2R(v,\theta)}\cos\theta\|_{1,1,\omega} \leq K\delta.$$

$$\|\delta\sqrt{2R(v,\theta)}\sin\theta\|_{1,1,\omega} \leq K\delta.$$

Then, recalling that $Z_0(u) = \tanh(\mathrm{d}u)$ and using that:

$$|\delta(Z_0(v)+1)| \leq \begin{cases} K\delta\,|\cosh(\mathrm{d}v)|^{-1} & \text{if } v \in D^{\mathrm{u}}_{\kappa,\beta,T}, \\ K\delta\,|\cosh(\mathrm{d}v)|^{-2} \leq K\delta\,|\cosh(\mathrm{d}v)|^{-1} & \text{if } v \in D^{\mathrm{u}}_{\kappa,\beta,\infty}, \end{cases}$$

it is clear that $\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega} \leq K\delta$. Hence, from item 1 of Lemma 3.2.2 and item 1 of Lemma 3.2.6 we readily see that since $1 \leq n \leq 3$:

$$\|f^{\mathrm{u}}_n(\Phi_0(v,\theta),\delta,\delta\sigma)\|^{\mathrm{u}}_{3,1,\omega} \leq \|f^{\mathrm{u}}_n(\Phi_0(v,\theta),\delta,\delta\sigma)\|^{\mathrm{u}}_{n,1,\omega} \leq K\delta^n.$$

In conclusion, substituting this bound for each n in expression (3.136) we have:

$$\|f^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta, \delta\sqrt{2R(v,\theta)}\sin\theta, \delta Z_0(v),\delta,\delta\sigma)\|^{\mathrm{u}}_{3,1,\omega} \leq K\delta^3. \tag{3.137}$$

Reasoning analogously, we obtain the same bound for g:

$$\|g^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta, \delta\sqrt{2R(v,\theta)}\sin\theta, \delta Z_0(v),\delta,\delta\sigma)\|^{\mathrm{u}}_{3,1,\omega} \leq K\delta^3. \tag{3.138}$$

Finally it is clear that:

$$\begin{aligned} & \|F^{\mathrm{u}}(\delta R(v,\theta),\theta,\delta Z_0(v),\delta,\delta\sigma)\|^{\mathrm{u}}_{4,2,\omega} \\ \leq\ & \|\sqrt{2R}\cos\theta\|_{1,1,\omega}\|f^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta,\delta\sqrt{2R(v,\theta)}\sin\theta,\delta Z_0(v),\delta,\delta\sigma)\|^{\mathrm{u}}_{3,1,\omega} \\ & +\|\sqrt{2R}\sin\theta\|_{1,1,\omega}\|g^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta,\delta\sqrt{2R(v,\theta)}\sin\theta,\delta Z_0(v),\delta,\delta\sigma)\|^{\mathrm{u}}_{3,1,\omega} \\ \leq\ & K\delta^3. \end{aligned}$$

To prove item 2, first we recall that:

$$\begin{aligned} & G^{\mathrm{u}}(\delta R(v,\theta),\theta,\delta Z_0(v),\delta,\delta\sigma) \\ & = \frac{1}{\sqrt{2R(v,\theta)}}\cos\theta g^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta,\delta\sqrt{2R(v,\theta)}\sin\theta,\delta Z_0(v),\delta,\delta\sigma) \\ & -\frac{1}{\sqrt{2R(v,\theta)}}\sin\theta f^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta,\delta\sqrt{2R(v,\theta)}\sin\theta,\delta Z_0(v),\delta,\delta\sigma). \end{aligned}$$

Then we just have note that:

$$\left\|\frac{1}{\sqrt{2R(v,\theta)}}\right\|_{-1,-1,\omega} \leq K,$$

and then using bounds (3.137) and (3.138) we obtain:

$$\begin{aligned} & \|G^{\mathrm{u}}(\delta R(v,\theta),\theta,\delta Z_0(v),\delta,\delta\sigma)\|^{\mathrm{u}}_{2,0,\omega} \\ \leq\ & \left\|\frac{\cos\theta}{\sqrt{2R(v,\theta)}}\right\|_{-1,-1,\omega}\|f^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta,\delta\sqrt{2R(v,\theta)}\sin\theta,\delta Z_0(v),\delta,\delta\sigma)\|^{\mathrm{u}}_{3,1,\omega} \\ & +\left\|\frac{\sin\theta}{\sqrt{2R(v,\theta)}}\right\|_{-1,-1,\omega}\|g^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta,\delta\sqrt{2R(v,\theta)}\sin\theta,\delta Z_0(v),\delta,\delta\sigma)\|^{\mathrm{u}}_{3,1,\omega} \\ \leq\ & K\delta^3. \end{aligned}$$

Finally we proceed to prove item 3. In this case we have:

$$H^{\mathrm{u}}(\delta R(v,\theta),\theta,\delta Z_0(v),\delta,\delta\sigma) = h^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta,\delta\sqrt{2R(v,\theta)}\sin\theta,\delta Z_0(v),\delta,\delta\sigma).$$

Again, we have that h^{u} is of order three in all their variables, that $h^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) = 0$ and moreover $\partial_x h^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) = \partial_y h^{\mathrm{u}}(0,0,-\delta,\delta,\delta\sigma) = 0$. For this reason, we can write h^{u} as:

$$h^{\mathrm{u}}(z_1,z_2,z_3,\delta,\delta\sigma) = \delta^2(a_1 + a_2\sigma^2)(z_3+\delta) + \sum_{n=2}^{3}\delta^{3-n}h_n^{\mathrm{u}}(z_1,z_2,z_3+\delta,\delta,\delta\sigma), \qquad (3.139)$$

for some coefficients a_1, a_2, and where $|h_n^{\mathrm{u}}(z_1,z_2,z_3+\delta,\delta,\delta\sigma)| \leq K|(z_1,z_2,z_3+\delta)|^n$. Reasoning as above, and using that now $n \geq 2$, by Lemma 3.2.6 we obtain that:

$$\|h_n^{\mathrm{u}}(\Phi_0(v,\theta),\delta,\delta\sigma)\|_{n,2,\omega}^{\mathrm{u}} \leq K\left(\|\Phi_0\|_{1,1,\omega}^{\mathrm{u},\times}\right)^n \leq K\delta^n,$$

and then by Lemma 3.2.2, since $n \leq 3$, we obtain:

$$\|h_n^{\mathrm{u}}(\Phi_0(v,\theta),\delta,\delta\sigma)\|_{3,2,\omega}^{\mathrm{u}} \leq K\delta^n. \qquad (3.140)$$

Now we just need to note that $|\delta(Z_0(v)+1)| \leq K\delta|\cosh(\mathrm{d}v)|^{-2}$, and then using this fact and (3.140), equation (3.139) yields:

$$\|h^{\mathrm{u}}(\delta\sqrt{2R(v,\theta)}\cos\theta,\delta\sqrt{2R(v,\theta)}\sin\theta,\delta Z_0(v),\delta,\delta\sigma)\|_{3,2,\omega}^{\mathrm{u}} \leq K\delta^3,$$

and the claim is proved. □

Lemma 3.2.13. *Let $|\sigma| \leq \sigma^*\delta^{p+3}$. Consider the function $\mathcal{F}^{\mathrm{u}}(0)$ (see (3.135)) and let $\tilde{\mathcal{F}}^{\mathrm{u}} = \mathcal{G}^{\mathrm{u}} \circ \mathcal{F}^{\mathrm{u}}$, where $\mathcal{G}^{\mathrm{u}}$ was defined in (3.114). There exists a constant M such that:*

$$[\![\tilde{\mathcal{F}}^{\mathrm{u}}(0)]\!]_{3,2,\omega}^{\mathrm{u}} \leq M\delta^{p+3}.$$

Proof. Since $\tilde{\mathcal{F}}^{\mathrm{u}} = \mathcal{G}^{\mathrm{u}} \circ \mathcal{F}^{\mathrm{u}}$, by Lemma 3.2.11 it is enough to prove that:

$$\|\mathcal{F}^{\mathrm{u}}(0)\|_{4,2,\omega}^{\mathrm{u}} \leq M\delta^{p+3}.$$

This is clear, since using Lemmas 3.2.2 and 3.2.12:

$$\begin{aligned}
\|\mathcal{F}^{\mathrm{u}}(0)\|_{4,2,\omega}^{\mathrm{u}} &\leq \|2\sigma R_0(v)\|_{4,2,\omega}^{\mathrm{u}} + \delta^p\frac{\mathrm{d}+1}{b}\|Z_0(v)H^{\mathrm{u}}(\delta R_0(v),\theta,\delta Z_0(v),\delta,\delta\sigma)\|_{4,2,\omega}^{\mathrm{u}} \\
&\quad + \delta^p\|F^{\mathrm{u}}(\delta R_0(v),\theta,\delta Z_0(v),\delta,\delta\sigma)\|_{4,2,\omega}^{\mathrm{u}} \\
&\leq K\big(2\sigma + \delta^p\|Z_0\|_{1,0}^{\mathrm{u}}\|H^{\mathrm{u}}(\delta R_0(v),\theta,\delta Z_0(v),\delta,\delta\sigma)\|_{3,2,\omega}^{\mathrm{u}} \\
&\quad + \delta^p\|F^{\mathrm{u}}(\delta R_0(v),\theta,\delta Z_0(v),\delta,\delta\sigma)\|_{4,2,\omega}^{\mathrm{u}}\big) \\
&\leq K(\sigma + \delta^{p+3}) \leq K\delta^{p+3}.
\end{aligned}$$

□

3.2.4 The fixed point

Lemma 3.2.14. *Let* $\phi \in \mathcal{X}^{\mathrm{u}}_{3,2,\omega}$ *such that* $\|\phi\|^{\mathrm{u}}_{3,2,\omega} \leq C\delta$*, for some constant* C*. Then there exists a constant* M *such that:*

$$\|F^{\mathrm{u}}\left(\delta(R_0+\phi),\theta,\delta Z_0,\delta,\delta\sigma\right)\|^{\mathrm{u}}_{4,2,\omega} \leq M\delta^3,$$

$$\|G^{\mathrm{u}}\left(\delta(R_0+\phi),\theta,\delta Z_0,\delta,\delta\sigma\right)\|^{\mathrm{u}}_{2,0,\omega} \leq M\delta^3,$$

and:

$$\|H^{\mathrm{u}}\left(\delta(R_0+\phi),\theta,\delta Z_0,\delta,\delta\sigma\right)\|^{\mathrm{u}}_{3,2,\omega} \leq M\delta^3.$$

Proof. This lemma is a corollary of Lemma 3.2.12, noting that since $\|\phi\|^{\mathrm{u}}_{3,2,\omega} \leq C\delta$:

$$\|R_0+\phi\|^{\mathrm{u}}_{2,2,\omega} \leq \|R_0\|^{\mathrm{u}}_{2,2,\omega} + \frac{K}{\delta\kappa}\|\phi\|^{\mathrm{u}}_{3,2,\omega} \leq K.$$

□

Lemma 3.2.15. *Let* ϕ_1*,* $\phi_2 \in \mathcal{X}^{\mathrm{u}}_{3,2,\omega}$ *be such that* $\|\phi_i\|^{\mathrm{u}}_{3,2,\omega} \leq C\delta$*, for some constant* C*,* $i=1,2$*. Then there exists a constant* M *such that:*

1. $\|F^{\mathrm{u}}\left(\delta(R_0+\phi_1),\theta,\delta Z_0,\delta,\delta\sigma\right) - F^{\mathrm{u}}\left(\delta(R_0+\phi_2),\theta,\delta Z_0,\delta,\delta\sigma\right)\|^{\mathrm{u}}_{5,2,\omega} \leq M\delta^3\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}$,
2. $\|G^{\mathrm{u}}\left(\delta(R_0+\phi_1),\theta,\delta Z_0,\delta,\delta\sigma\right) - G^{\mathrm{u}}\left(\delta(R_0+\phi_2),\theta,\delta Z_0,\delta,\delta\sigma\right)\|^{\mathrm{u}}_{3,0,\omega} \leq M\delta^3\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}$,
3. $\|H^{\mathrm{u}}\left(\delta(R_0+\phi_1),\theta,\delta Z_0,\delta,\delta\sigma\right) - H^{\mathrm{u}}\left(\delta(R_0+\phi_2),\theta,\delta Z_0,\delta,\delta\sigma\right)\|^{\mathrm{u}}_{4,2,\omega} \leq \kappa\delta^3\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}$.

Proof. We will proceed as in the proof of Lemma 3.2.12. To prove the first item, recall that:

$$\begin{aligned}
&F^{\mathrm{u}}(\delta(R_0+\phi_i),\theta,\delta Z_0,\delta,\delta\sigma)\\
&= \sqrt{2(R_0+\phi_i)}\cos\theta f^{\mathrm{u}}(\delta\sqrt{2(R_0+\phi_i)}\cos\theta,\delta\sqrt{2(R_0+\phi_i)}\sin\theta,\delta Z_0,\delta,\delta\sigma)\\
&+\sqrt{2(R_0+\phi_i)}\sin\theta g^{\mathrm{u}}(\delta\sqrt{2(R_0+\phi_i)}\cos\theta,\delta\sqrt{2(R_0+\phi_i)}\sin\theta,\delta Z_0,\delta,\delta\sigma).
\end{aligned}$$

We shall use formula (3.136) for f^{u}. Again, we define:

$$\Phi_0(v,\theta) = (\delta\sqrt{2R_0(v)}\cos\theta,\delta\sqrt{2R_0(v)}\sin\theta,\delta(Z_0(v)+1))$$

and, for $i=1,2$:

$$\Phi_i(v,\theta) = ((\sqrt{2(R_0(v)+\phi_i(v,\theta))}-\sqrt{2R_0(v)})\cos\theta,(\sqrt{2(R_0(v)+\phi_i(v,\theta))}-\sqrt{2R_0(v)})\sin\theta,0).$$

Then it is easy to see that $\|\Phi_0\|^{\mathrm{u},\times}_{1,1,\omega} \leq K\delta$ and, by hypothesis and the mean value theorem, also $\|\Phi_i\|^{\mathrm{u},\times}_{2,1,\omega} \leq K\delta$. Hence, by Lemma 3.2.2 and item 2 of Lemma 3.2.6:

$$\begin{aligned}
&\|f^{\mathrm{u}}_n(\Phi_0(v,\theta)+\delta\Phi_1(v,\theta),\delta,\delta\sigma) - f^{\mathrm{u}}_n(\Phi_0(v,\theta)+\delta\Phi_2(v,\theta),\delta,\delta\sigma)\|^{\mathrm{u}}_{4,1,\omega}\\
\leq\ &\|f^{\mathrm{u}}_n(\Phi_0(v,\theta)+\delta\Phi_1(v,\theta),\delta,\delta\sigma) - f^{\mathrm{u}}_n(\Phi_0(v,\theta)+\delta\Phi_2(v,\theta),\delta,\delta\sigma)\|^{\mathrm{u}}_{n+1,1,\omega}\\
\leq\ &K\delta^n\|\Phi_1-\Phi_2\|^{\mathrm{u},\times}_{2,1,\omega}.
\end{aligned}$$

In conclusion, substituting this bound for each n in expression (3.136) we have:

$$\begin{aligned} & \|f^{\mathrm{u}}(\Phi_0(v,\theta)+\delta\Phi_1(v,\theta),\delta,\delta\sigma)-f^{\mathrm{u}}(\Phi_0(v,\theta)+\delta\Phi_2(v,\theta),\delta,\delta\sigma)\|^{\mathrm{u}}_{4,1,\omega} \\ \leq\ & K\delta^3\|\Phi_1-\Phi_2\|^{\mathrm{u},\times}_{2,1,\omega}. \end{aligned} \tag{3.141}$$

Reasoning in a similar way, the same bound for the difference between $g^{\mathrm{u}}(\Phi_0(v,\theta)+\delta\Phi_i(v,\theta),\delta,\delta\sigma)$, $i=1,2$, is obtained. Moreover, taking into account the definition of Φ_i, it is easy to see that:

$$\|\Phi_1-\Phi_2\|^{\mathrm{u},\times}_{2,1,\omega}\leq K\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}. \tag{3.142}$$

To shorten the formulas, for $i=1,2$ we will denote:

$$\hat{F}_i(v,\theta)=\cos\theta f^{\mathrm{u}}(\Phi_0(v,\theta)+\delta\Phi_i(v,\theta),\delta,\delta\sigma)+\sin\theta g^{\mathrm{u}}(\Phi_0(v,\theta)+\delta\Phi_i(v,\theta),\delta,\delta\sigma).$$

Then we can write:

$$\begin{aligned} & F^{\mathrm{u}}(\delta(R_0+\phi_1),\theta,\delta Z_0,\delta,\delta\sigma)-F^{\mathrm{u}}(\delta(R_0+\phi_2),\theta,\delta Z_0,\delta,\delta\sigma) \\ =\ & \left(\sqrt{2(R_0(v)+\phi_1(v,\theta))}-\sqrt{2(R_0(v)+\phi_2(v,\theta))}\right)\hat{F}_1(v,\theta) \\ & +\sqrt{2(R_0(v)+\phi_2(v,\theta))}\left(\hat{F}_1(v,\theta)-\hat{F}_2(v,\theta)\right). \end{aligned}$$

On the one hand, using that $\|\sqrt{2(R_0+\phi_i)}\|^{\mathrm{u}}_{1,1,\omega}\leq K$, $i=1,2$, it is easy to see that:

$$\begin{aligned} & \|\sqrt{2(R_0+\phi_1)}-\sqrt{2(R_0+\phi_2)}\|^{\mathrm{u}}_{2,1,\omega} \\ \leq\ & \|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}\left\|\frac{1}{\sqrt{2(R_0+\phi_1)}+\sqrt{2(R_0+\phi_2)}}\right\|^{\mathrm{u}}_{-1,-1} \\ \leq\ & K\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}. \end{aligned} \tag{3.143}$$

On the other hand, using bounds (3.137) and (3.138) with $R=R_0+\phi_i$, $i=1,2$ one obtains:

$$\|\hat{F}_1\|^{\mathrm{u}}_{3,1,\omega}\leq K\delta^3. \tag{3.144}$$

Finally using (3.141) and (3.142) we straightforwardly obtain that:

$$\|\hat{F}_1-\hat{F}_2\|^{\mathrm{u}}_{4,1,\omega}\leq K\delta^3\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}. \tag{3.145}$$

Then, using first Lemma 3.2.3, after Lemma 3.2.2 and finally (3.143), (3.144) and (3.145), we obtain:

$$\begin{aligned} & \|F^{\mathrm{u}}(\delta(R_0+\phi_1),\theta,\delta Z_0,\delta,\delta\sigma)-F^{\mathrm{u}}(\delta(R_0+\phi_2),\theta,\delta Z_0,\delta,\delta\sigma)\|^{\mathrm{u}}_{5,2,\omega} \\ & \leq K\left(\|\sqrt{2(R_0+\phi_1)}-\sqrt{2(R_0+\phi_2)}\|^{\mathrm{u}}_{2,1,\omega}\|\hat{F}_1\|^{\mathrm{u}}_{3,1,\omega}+\|\sqrt{2(R_0+\phi_2)}\|^{\mathrm{u}}_{1,1,\omega}\|\hat{F}_1-\hat{F}_2\|^{\mathrm{u}}_{4,1,\omega}\right) \\ & \leq K\delta^3\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}, \end{aligned}$$

and the first item of the lemma is proved.

To prove item 2 we proceed similarly. Now we have:

$$
\begin{aligned}
&G^{\mathrm{u}}(\delta(R_0+\phi_i),\theta,\delta Z_0,\delta,\delta\sigma)\\
&=\frac{1}{\sqrt{2(R_0+\phi_i)}}\cos\theta g^{\mathrm{u}}(\delta\sqrt{2(R_0+\phi_i)}\cos\theta,\delta\sqrt{2(R_0+\phi_i)}\sin\theta,\delta Z_0,\delta,\delta\sigma)\\
&-\frac{1}{\sqrt{2(R_0+\phi_i)}}\sin\theta f^{\mathrm{u}}(\delta\sqrt{2(R_0+\phi_i)}\cos\theta,\delta\sqrt{2(R_0+\phi_i)}\sin\theta,\delta Z_0,\delta,\delta\sigma).
\end{aligned}
$$

Denoting:

$$
\hat{G}_i(v,\theta)=\cos\theta g^{\mathrm{u}}(\Phi_0(v,\theta)+\delta\Phi_i(v,\theta),\delta,\delta\sigma)-\sin\theta f^{\mathrm{u}}(\Phi_0(v,\theta)+\delta\Phi_i(v,\theta),\delta,\delta\sigma),
$$

for $i=1,2$, we can write:

$$
\begin{aligned}
&G^{\mathrm{u}}(\delta(R_0+\phi_1),\theta,\delta Z_0,\delta,\delta\sigma)-G^{\mathrm{u}}(\delta(R_0+\phi_2),\theta,\delta Z_0,\delta,\delta\sigma)\\
=\;&\left(\frac{1}{\sqrt{2(R_0(v)+\phi_1(v,\theta))}}-\frac{1}{\sqrt{2(R_0(v)+\phi_2(v,\theta))}}\right)\hat{G}_1(v,\theta)\\
&+\frac{1}{\sqrt{2(R_0(v)+\phi_2(v,\theta))}}\left(\hat{G}_1(v,\theta)-\hat{G}_2(v,\theta)\right).
\end{aligned}
$$

In this case, using that $\left\|\frac{1}{\sqrt{2(R_0+\phi_i)}}\right\|^{\mathrm{u}}_{-1,-1,\omega}\leq K$, $i=1,2$, we have that:

$$
\begin{aligned}
&\left\|\frac{1}{\sqrt{2(R_0+\phi_1)}}-\frac{1}{\sqrt{2(R_0+\phi_2)}}\right\|^{\mathrm{u}}_{0,-1,\omega}\\
&\leq\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}\left\|\frac{1}{\sqrt{2(R_0+\phi_1)}\sqrt{2(R_0+\phi_2)}\left(\sqrt{2(R_0+\phi_1)}+\sqrt{2(R_0+\phi_2)}\right)}\right\|^{\mathrm{u}}_{-3,-3}\\
&\leq K\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}.
\end{aligned}
\tag{3.146}
$$

Moreover, similarly as in (3.144) and (3.145) we have:

$$
\|\hat{G}_1\|^{\mathrm{u}}_{3,1,\omega}\leq K\delta^3, \tag{3.147}
$$

and:

$$
\|\hat{G}_1-\hat{G}_2\|^{\mathrm{u}}_{4,1,\omega}\leq K\delta^3\|\phi_1-\phi_2\|^{\mathrm{u}}_{3,2,\omega}. \tag{3.148}
$$

Thus, again using first Lemma 3.2.3, after Lemma 3.2.2 and finally (3.146), (3.147) and (3.148), we obtain:

$$\begin{aligned}
&\|G^{\rm u}(\delta(R_0+\phi_1),\theta,\delta Z_0,\delta,\delta\sigma) - G^{\rm u}(\delta(R_0+\phi_2),\theta,\delta Z_0,\delta,\delta\sigma)\|^{\rm u}_{3,0,\omega}\\
\le\ & K\left(\left\|\frac{1}{\sqrt{2(R_0+\phi_1)}}-\frac{1}{\sqrt{2(R_0+\phi_2)}}\right\|^{\rm u}_{0,-1,\omega}\|\hat G_1\|^{\rm u}_{3,1,\omega}+\right.\\
&\left.\left\|\frac{1}{\sqrt{2(R_0+\phi_2)}}\right\|^{\rm u}_{-1,-1,\omega}\|\hat G_1-\hat G_2\|^{\rm u}_{4,1,\omega}\right)\\
\le\ & K\delta^3\|\phi_1-\phi_2\|^{\rm u}_{3,2,\omega}.
\end{aligned}$$

Finally, to prove the last item, recall that:

$$H^{\rm u}(\delta(R_0+\phi),\theta,\delta Z_0,\delta,\delta\sigma) = h^{\rm u}(\delta\sqrt{2(R_0+\phi)}\cos\theta,\delta\sqrt{2(R_0+\phi)}\sin\theta,\delta Z_0,\delta,\delta\sigma).$$

Using formula (3.139) for $h^{\rm u}$ and reasoning as above we obtain:

$$\begin{aligned}
&\|h^{\rm u}_n(\Phi_0(v,\theta)+\delta\Phi_1(v,\theta),\delta,\delta\sigma) - h^{\rm u}_n(\Phi_0(v,\theta)+\delta\Phi_2(v,\theta),\delta,\delta\sigma)\|^{\rm u}_{4,2,\omega}\\
&\le K\delta^n\|\Phi_1-\Phi_2\|^{{\rm u},\times}_{2,1,\omega}.
\end{aligned} \tag{3.149}$$

Now we note that the linear term in (3.139) cancels when subtracting:

$$h^{\rm u}(\Phi_0(v,\theta)+\delta\Phi_i(v,\theta),\delta,\delta\sigma) \qquad i=1,2,$$

since in both cases $\phi_3 = \delta Z_0(v)$. Hence, using (3.149), equation (3.139) yields:

$$\|h^{\rm u}(\Phi_0(v,\theta)+\delta\Phi_1(v,\theta),\delta,\delta\sigma) - h^{\rm u}(\Phi_0(v,\theta)+\delta\Phi_2(v,\theta),\delta,\delta\sigma)\|^{\rm u}_{4,2,\omega} \le K\delta^3\|\Phi_1-\Phi_2\|^{{\rm u},\times}_{2,1,\omega},$$

so that item 3 of the lemma is clear using bound (3.142). □

Lemma 3.2.16. *Let $|\sigma|\le\sigma^*\delta^{p+3}$. Let $\tilde\phi_1$, $\tilde\phi_2\in\tilde{\mathcal{X}}^{\rm u}_{3,2,\omega}$ (see* (3.97) *for the definition of $\tilde{\mathcal{X}}^{\rm u}_{n,m,\omega}$) such that for some constant C and $i=1,2$, $\llbracket\tilde\phi_i\rrbracket^{\rm u}_{3,2,\omega}\le C\delta^{p+3}$ (see* (3.95) *for the definition of the norm). Then there exists a constant M such that:*

$$\llbracket\tilde{\mathcal{F}}^{\rm u}(\tilde\phi_1)-\tilde{\mathcal{F}}^{\rm u}(\tilde\phi_2)\rrbracket^{\rm u}_{4,2,\omega}\le M\delta^{p+3}\llbracket\tilde\phi_1-\tilde\phi_2\rrbracket^{\rm u}_{3,2,\omega},$$

where the norm $\llbracket\cdot\rrbracket_{n,m,\omega}$ was defined in (3.95).

Proof. Again, since $\tilde{\mathcal{F}}^{\rm u}=\mathcal{G}^{\rm u}\circ\mathcal{F}^{\rm u}$, and using now that $\mathcal{G}^{\rm u}$ is linear, by Lemma 3.2.11 (which allows us to relate $\llbracket\mathcal{G}^{\rm u}(\phi)\rrbracket^{\rm u}_{n-1,m,\omega}$ with $\|\phi\|^{\rm u}_{n,m,\omega}$) it is only necessary to prove that if $\llbracket\tilde\phi_i\rrbracket^{\rm u}_{3,2,\omega}\le C\delta^{p+3}$, then:

$$\|\mathcal{F}^{\rm u}(\tilde\phi_1)-\mathcal{F}^{\rm u}(\tilde\phi_2)\|^{\rm u}_{5,2,\omega}\le K\delta^{p+3}\llbracket\tilde\phi_1-\tilde\phi_2\rrbracket^{\rm u}_{3,2,\omega}.$$

For clarity, we shall bound the different terms of $\mathcal{F}^{\mathrm{u}}(\tilde{\phi}_1) - \mathcal{F}^{\mathrm{u}}(\tilde{\phi}_2)$ (see (3.44) for the definition of $\mathcal{F}^{\mathrm{u}}$). On one hand, we note that using Lemmas 3.2.2 and 3.2.15 we have:

$$\begin{aligned} & \|2\sigma(R_0+\tilde{\phi}_1) + \delta^p F^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,\delta Z_0,\delta,\delta\sigma) - 2\sigma(R_0+\tilde{\phi}_2) \\ & +\delta^p F^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,\delta Z_0,\delta,\delta\sigma)\|^{\mathrm{u}}_{5,2,\omega} \\ \leq\ & 2|\sigma|\|\tilde{\phi}_1-\tilde{\phi}_2\|^{\mathrm{u}}_{5,2,\omega} \\ & +\delta^p\|F^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,\delta Z_0,\delta,\delta\sigma) - F^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,\delta Z_0,\delta,\delta\sigma)\|^{\mathrm{u}}_{5,2,\omega} \\ \leq\ & K|\sigma|\|\tilde{\phi}_1-\tilde{\phi}_2\|^{\mathrm{u}}_{3,2,\omega} + K\delta^{p+3}\|\tilde{\phi}_1-\tilde{\phi}_2\|^{\mathrm{u}}_{3,2,\omega} \leq K\delta^{p+3}[\![\tilde{\phi}_1-\tilde{\phi}_2]\!]^{\mathrm{u}}_{3,2,\omega}, \end{aligned} \tag{3.150}$$

where in the last step we have used that $|\sigma| \leq \delta^{p+3}\sigma^*$.

On the other hand, using Lemmas 3.2.3 and 3.2.15 again and the fact that $\|Z_0\|_{1,0} \leq K$, we have:

$$\begin{aligned} & \left\| \frac{\mathrm{d}+1}{b} Z_0(v) \left(\delta^p H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,Z_0,\delta,\delta\sigma) - \delta^p H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,Z_0,\delta,\delta\sigma)\right)\right\|^{\mathrm{u}}_{5,2,\omega} \\ & K\delta^p\|Z_0\|^{\mathrm{u}}_{1,0}\|H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,Z_0,\delta,\delta\sigma) - H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,Z_0,\delta,\delta\sigma)\|^{\mathrm{u}}_{4,2,\omega} \\ & \leq K\delta^{p+3}\|\tilde{\phi}_1-\tilde{\phi}_2\|^{\mathrm{u}}_{3,2,\omega} \leq K\delta^{p+3}[\![\tilde{\phi}_1-\tilde{\phi}_2]\!]^{\mathrm{u}}_{3,2,\omega}. \end{aligned} \tag{3.151}$$

Now we claim that:

$$\begin{aligned} & \delta^p \left\| G^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,\delta Z_0,\delta,\delta\sigma)\partial_\theta\tilde{\phi}_1 - G^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,\delta Z_0,\delta,\delta\sigma)\partial_\theta\tilde{\phi}_2\right\|^{\mathrm{u}}_{5,2,\omega} \\ \leq\ & K\delta^{p+3}[\![\tilde{\phi}_1-\tilde{\phi}_2]\!]^{\mathrm{u}}_{3,2,\omega}. \end{aligned} \tag{3.152}$$

Indeed, we can write:

$$\begin{aligned} & \delta^p \left\| G^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,\delta Z_0,\delta,\delta\sigma)\partial_\theta\tilde{\phi}_1 - G^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,\delta Z_0,\delta,\delta\sigma)\partial_\theta\tilde{\phi}_2\right\|^{\mathrm{u}}_{5,2,\omega} \\ & \leq \delta^p\|G^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,\delta Z_0,\delta,\delta\sigma) - G^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,\delta Z_0,\delta,\delta\sigma)\|^{\mathrm{u}}_{3,0,\omega}\|\partial_\theta\tilde{\phi}_1\|^{\mathrm{u}}_{2,2,\omega} \\ & +\delta^p\|G^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,\delta Z_0,\delta,\delta\sigma)\|^{\mathrm{u}}_{2,0,\omega}\|\partial_\theta(\tilde{\phi}_1-\tilde{\phi}_2)\|^{\mathrm{u}}_{3,2,\omega}. \end{aligned} \tag{3.153}$$

Now we note that:

$$\|\partial_\theta\tilde{\phi}_1\|^{\mathrm{u}}_{2,2,\omega} \leq K\frac{\left\|\partial_\theta\tilde{\phi}_1\right\|^{\mathrm{u}}_{4,2,\omega}}{\delta^2\kappa^2} \leq K\frac{[\![\tilde{\phi}_1]\!]^{\mathrm{u}}_{3,2,\omega}}{\delta\kappa^2} \leq K\frac{\delta^{p+2}}{\kappa^2} \leq K. \tag{3.154}$$

and:

$$\|\partial_\theta(\tilde{\phi}_1-\tilde{\phi}_2)\|^{\mathrm{u}}_{3,2,\omega} \leq \frac{K}{\delta\kappa}\|\partial_\theta(\tilde{\phi}_1-\tilde{\phi}_2)\|^{\mathrm{u}}_{4,2,\omega} \leq \frac{K}{\kappa}[\![\tilde{\phi}_1-\tilde{\phi}_2]\!]^{\mathrm{u}}_{3,2,\omega}. \tag{3.155}$$

Thus, using bounds (3.154) and (3.155) and Lemmas 3.2.14 and 3.2.15 in equation (3.153), we obtain bound (3.152).

Our last claim is:

$$\begin{aligned}
&\left\| \frac{2b\tilde{\phi}_1 + \delta^p H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,Z_0,\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(v))}\partial_v\tilde{\phi}_1 - \frac{2b\tilde{\phi}_2 + \delta^p H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,Z_0,\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(v))}\partial_v\tilde{\phi}_2 \right\|_{5,2,\omega}^{\mathrm{u}} \\
\leq \;& K\delta^{p+3}\llbracket\tilde{\phi}_1-\tilde{\phi}_2\rrbracket_{3,2,\omega}^{\mathrm{u}}. \qquad (3.156)
\end{aligned}$$

We have:

$$\begin{aligned}
&\left\| \frac{2b\tilde{\phi}_1 + \delta^p H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,Z_0,\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(v))}\partial_v\tilde{\phi}_1 - \frac{2b\tilde{\phi}_2 + \delta^p H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,Z_0,\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(v))}\partial_v\tilde{\phi}_2 \right\|_{5,2,\omega}^{\mathrm{u}} \leq \\
&\left\| \frac{2b(\tilde{\phi}_1-\tilde{\phi}_2)}{\mathrm{d}(1-Z_0^2(v))}\partial_v\tilde{\phi}_1 \right\|_{5,2,\omega}^{\mathrm{u}} + \delta^p \left\| \frac{H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,Z_0,\delta,\delta\sigma) - H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,Z_0,\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(v))}\partial_v\tilde{\phi}_1 \right\|_{5,2,\omega}^{\mathrm{u}} \\
&+ \left\| \frac{2b\tilde{\phi}_2}{\mathrm{d}(1-Z_0^2(v))}\partial_v(\tilde{\phi}_1-\tilde{\phi}_2) \right\|_{5,2,\omega}^{\mathrm{u}} + \delta^p \left\| \frac{H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,Z_0,\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(v))}\partial_v(\tilde{\phi}_1-\tilde{\phi}_2) \right\|_{5,2,\omega}^{\mathrm{u}}. \qquad (3.157)
\end{aligned}$$

We will see that each term in (3.157) satisfies bound (3.156), and so the claim will be proved. For the first two terms we will use the fact that $\left\|\partial_v\tilde{\phi}_1\right\|_{4,2,\omega}^{\mathrm{u}} \leq K\delta^{p+3}$. Indeed, for the first term we have:

$$\begin{aligned}
\left\| \frac{2b(\tilde{\phi}_1-\tilde{\phi}_2)}{\mathrm{d}(1-Z_0^2(v))}\partial_v\tilde{\phi}_1 \right\|_{5,2,\omega}^{\mathrm{u}} &\leq K\|\tilde{\phi}_1-\tilde{\phi}_2\|_{3,2,\omega}^{\mathrm{u}} \left\| \frac{1}{1-Z_0^2(v)} \right\|_{-2,-2,\omega}^{\mathrm{u}} \|\partial_v\tilde{\phi}_1\|_{4,2,\omega}^{\mathrm{u}} \\
&\leq K\delta^{p+3}\llbracket\tilde{\phi}_1-\tilde{\phi}_2\rrbracket_{3,2,\omega}^{\mathrm{u}}.
\end{aligned}$$

For the second term we also use Lemma 3.2.15 and that $p\geq -2$, and we obtain:

$$\begin{aligned}
&\delta^p \left\| \frac{H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,Z_0,\delta,\delta\sigma) - H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,Z_0,\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(v))}\partial_v\tilde{\phi}_1 \right\|_{5,2,\omega}^{\mathrm{u}} \\
&\leq K\delta^p \|H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_1),\theta,Z_0,\delta,\delta\sigma) - H^{\mathrm{u}}(\delta(R_0+\tilde{\phi}_2),\theta,Z_0,\delta,\delta\sigma)\|_{4,2,\omega}^{\mathrm{u}} \left\| \frac{\partial_v\tilde{\phi}_1}{1-Z_0^2(v)} \right\|_{1,0,\omega}^{\mathrm{u}} \\
&\leq K\delta^{p+3}\|\tilde{\phi}_1-\tilde{\phi}_2\|_{3,2,\omega}^{\mathrm{u}} \left\| \frac{1}{1-Z_0^2(v)} \right\|_{-2,-2,\omega}^{\mathrm{u}} \|\partial_v\tilde{\phi}_1\|_{3,2,\omega}^{\mathrm{u}} \leq K\delta^{p+3}\|\tilde{\phi}_1-\tilde{\phi}_2\|_{3,2,\omega}^{\mathrm{u}} \frac{\|\partial_v\tilde{\phi}_1\|_{4,2,\omega}^{\mathrm{u}}}{\delta\kappa} \\
&\leq K\delta^{p+3}\llbracket\tilde{\phi}_1-\tilde{\phi}_2\rrbracket_{3,2,\omega}^{\mathrm{u}}.
\end{aligned}$$

To bound the last two terms of (3.157) we will use that $\|\tilde{\phi}_2\|_{3,2,\omega}^{\mathrm{u}} \leq K\delta^{p+3}$. Indeed, on the one hand we have:

$$\begin{aligned}
\left\| \frac{2b\tilde{\phi}_2}{\mathrm{d}(1-Z_0^2(v))}\partial_v(\tilde{\phi}_1-\tilde{\phi}_2) \right\|_{5,2,\omega}^{\mathrm{u}} &\leq K\|\tilde{\phi}_2\|_{3,2,\omega}^{\mathrm{u}} \left\| \frac{1}{1-Z_0^2(v)} \right\|_{-2,-2,\omega}^{\mathrm{u}} \|\partial_v(\tilde{\phi}_1-\tilde{\phi}_2)\|_{4,2,\omega}^{\mathrm{u}} \\
&\leq K\delta^{p+3}\llbracket\tilde{\phi}_1-\tilde{\phi}_2\rrbracket_{3,2,\omega}^{\mathrm{u}}.
\end{aligned}$$

On the other hand, to bound the last term in (3.157), using also Lemma 3.2.14 we obtain:

$$
\begin{aligned}
&\delta^p \left\| \frac{H^{\mathrm{u}}(\delta(R_0+\tilde\phi_2),\theta,Z_0,\delta,\delta\sigma)}{\mathrm{d}(1-Z_0^2(v))} \partial_v(\tilde\phi_1-\tilde\phi_2) \right\|^{\mathrm{u}}_{5,2,\omega} \\
&\leq K\delta^p \|H^{\mathrm{u}}(\delta(R_0+\tilde\phi_2),\theta,Z_0,\delta,\delta\sigma)\|^{\mathrm{u}}_{3,2,\omega} \left\| \frac{1}{1-Z_0^2(v)} \right\|^{\mathrm{u}}_{-2,-2,\omega} \|\partial_v(\tilde\phi_1-\tilde\phi_2)\|^{\mathrm{u}}_{4,2,\omega} \\
&\leq K\delta^{p+3} \llbracket \tilde\phi_1-\tilde\phi_2 \rrbracket^{\mathrm{u}}_{3,2,\omega}.
\end{aligned}
$$

This finishes the proof of bound (3.156), and ends the proof of the Lemma. □

The following lemma is the last step before finishing the proof of Proposition 3.2.1.

Lemma 3.2.17. *Let $p \geq -2$ and $|\sigma| \leq \sigma^*\delta^{p+3}$. Define $r := 2\llbracket \tilde{\mathcal{F}}^{\mathrm{u}}(0) \rrbracket^{\mathrm{u}}_{3,2,\omega}$ and $B(r) \subset \tilde{\mathcal{X}}^{\mathrm{u}}_{3,2,\omega}$ the ball of radius r centered at zero. If $\kappa \geq \kappa^*$ and κ^* is large enough, then $\tilde{\mathcal{F}}^{\mathrm{u}}$ has a unique fixed point in $B(r)$.*

Proof. First of all, we point out that $r \leq K\delta^{p+3}$ by Lemma 3.2.13. By the properties of the norm $\llbracket . \rrbracket^{\mathrm{u}}_{n,m,\omega}$ and Lemma 3.2.16, it is clear that that for $\phi_1, \phi_2 \in B(r)$:

$$
\llbracket \tilde{\mathcal{F}}^{\mathrm{u}}(\tilde\phi_1) - \tilde{\mathcal{F}}^{\mathrm{u}}(\tilde\phi_2) \rrbracket^{\mathrm{u}}_{3,2,\omega} \leq \frac{K}{\delta\kappa} \llbracket \tilde{\mathcal{F}}^{\mathrm{u}}(\tilde\phi_1) - \tilde{\mathcal{F}}^{\mathrm{u}}(\tilde\phi_2) \rrbracket^{\mathrm{u}}_{4,2,\omega} \leq \frac{K\delta^{p+2}}{\kappa} \llbracket \tilde\phi_1 - \tilde\phi_2 \rrbracket^{\mathrm{u}}_{3,2,\omega}.
$$

Clearly, if $p \geq -2$ and κ^* is large enough, $\tilde{\mathcal{F}}^{\mathrm{u}}$ is contractive in $B(r)$. Thus, we just have to check that $\tilde{\mathcal{F}}^{\mathrm{u}} : B(r) \to B(r)$, since then by the fixed point theorem the claim of the lemma is clear.

Then, if $\tilde\phi \in B(r)$, by Lemma 3.2.16 we have that:

$$
\llbracket \tilde{\mathcal{F}}^{\mathrm{u}}(\tilde\phi) \rrbracket^{\mathrm{u}}_{3,2,\omega} \leq \llbracket \tilde{\mathcal{F}}^{\mathrm{u}}(\tilde\phi) - \tilde{\mathcal{F}}^{\mathrm{u}}(0) \rrbracket^{\mathrm{u}}_{3,2,\omega} + \llbracket \tilde{\mathcal{F}}^{\mathrm{u}}(0) \rrbracket^{\mathrm{u}}_{3,2,\omega} \leq K\frac{\delta^{p+2}}{\kappa} \llbracket \tilde\phi \rrbracket^{\mathrm{u}}_{3,2,\omega} + \frac{1}{2} r < r,
$$

That is, $\tilde{\mathcal{F}}^{\mathrm{u}} : B(r) \to B(r)$. □

End of the proof of Proposition 3.2.1. It is clear that $R_1^{\mathrm{u}} \in B(r)$ is the fixed point of $\tilde{\mathcal{F}}^{\mathrm{u}}$ obtained in Lemma 3.2.17. Then, item 1 of Proposition 3.2.1 is a direct consequence of Lemma 3.2.13. To prove item 2, we just need to note that:

$$
R_{11}^{\mathrm{u}} = R_1^{\mathrm{u}} - R_{10}^{\mathrm{u}} = \tilde{\mathcal{F}}^{\mathrm{u}}(R_1^{\mathrm{u}}) - \tilde{\mathcal{F}}^{\mathrm{u}}(0).
$$

Using Lemma 3.2.16 in the last equality, we obtain that:

$$
\llbracket R_{11}^{\mathrm{u}} \rrbracket^{\mathrm{u}}_{4,2,\omega} \leq K\delta^{p+3} \llbracket R_1^{\mathrm{u}} \rrbracket^{\mathrm{u}}_{3,2,\omega}
$$

for some constant K, and then the bound follows from the fact that, by Lemma 3.2.17, $\llbracket R_1^{\mathrm{u}} \rrbracket^{\mathrm{u}}_{3,2,\omega} \leq K \llbracket R_{10}^{\mathrm{u}} \rrbracket^{\mathrm{u}}_{3,2,\omega}$. □

3.3 Proof of Theorem 3.1.7

In this section we shall prove Theorem 3.1.7 concerning the functions r_1^{u} and r_1^{s}. We shall make use of the same norms $\|.\|_{n,m,\omega}^{\mathrm{u,s}}$ and $[\![.]\!]_{n,m,\omega}^{\mathrm{u,s}}$ introduced in Section 3.2, see (3.94) and (3.95) for their definitions. However, the functions r_1^{u} and r_1^{s} will be only defined in domains of the type $D_{\kappa,\beta,T}^{\mathrm{u}} \times \mathbb{T}_\omega$ (respectively $D_{\kappa,\beta,T}^{\mathrm{s}} \times \mathbb{T}_\omega$), so that we shall also use the norms:

$$\|\phi\|_{n,\omega}^{\mathrm{u}} := \sum_{l\in\mathbb{Z}} \left\|\phi^{[l]}\right\|_n^{\mathrm{u}} e^{|l|\omega}$$

where:

$$\left\|\phi^{[l]}\right\|_n^{\mathrm{u}} = \sup_{v\in D_{\kappa,\beta,T}^{\mathrm{u}}} \left|\cosh^n(\mathrm{d}v)\phi^{[l]}(v)\right|,$$

and:

$$[\![\phi]\!]_{n,\omega}^{\mathrm{u}} := \|\phi\|_{n,\omega}^{\mathrm{u}} + \|\partial_v\phi\|_{n+1,\omega}^{\mathrm{u}} + \delta^{-1}\|\partial_\theta\phi\|_{n+1,\omega}^{\mathrm{u}}.$$

Definitions for $\|.\|_{n,\omega}^{\mathrm{s}}$ and $[\![.]\!]_{n,\omega}^{\mathrm{s}}$ are analogous, replacing $D_{\kappa,\beta,T}^{\mathrm{u}}$ by $D_{\kappa,\beta,T}^{\mathrm{s}}$. Of course, the norms $\|.\|_{n,\omega}^{\mathrm{u,s}}$ and $[\![.]\!]_{n,\omega}^{\mathrm{u,s}}$ have analogous properties as $\|.\|_{n,m,\omega}^{\mathrm{u,s}}$ and $[\![.]\!]_{n,m,\omega}^{\mathrm{u,s}}$, summarized in Lemmas 3.2.2, 3.2.3 and 3.2.4. We point out that in the notation of the norms above, there is no explicit reference to the constants κ and T of the domains $D_{\kappa,\beta,T}^{\mathrm{u}}$ or $D_{\kappa,\beta,T}^{\mathrm{s}}$. However, in the following, if a function is defined in $D_{\bar\kappa,\beta,\bar T}^{\mathrm{u,s}}$ for $\bar\kappa \neq \kappa$ and $\bar T \neq T$, it is understood that the suprema are taken in $D_{\bar\kappa,\beta,\bar T}^{\mathrm{u,s}}$.

Let us recall some definitions given in Section 3.1.4. Let $R_1^{\mathrm{u,s}}(v,\theta)$ be the functions given by Theorem 3.1.6, then we consider:

$$\begin{aligned} X^{\mathrm{u,s}}(v,\theta) &:= \sqrt{2(R_0(v)+R_1^{\mathrm{u,s}}(v,\theta))}\cos\theta, \\ Y^{\mathrm{u,s}}(v,\theta) &:= \sqrt{2(R_0(v)+R_1^{\mathrm{u,s}}(v,\theta))}\sin\theta. \end{aligned} \tag{3.158}$$

Define:

$$(x^{\mathrm{u,s}}(v,\theta), y^{\mathrm{u,s}}(v,\theta), z^{\mathrm{u,s}}(v,\theta)) := (C_1^{\mathrm{u,s}})^{-1}\circ(C_2^{\mathrm{u,s}})^{-1}(X^{\mathrm{u,s}}(v,\theta), Y^{\mathrm{u,s}}(v,\theta), Z_0(v))), \tag{3.159}$$

where C_1^{u} and C_2^{u} are given respectively in (3.28) and (3.29) (and definitions for C_1^{s} and C_2^{s} are analogous). Finally, we define the functions $v^{\mathrm{u,s}}(u,\theta)$ as the functions satisfying respectively:

$$Z_0(u) = z^{\mathrm{u}}(v^{\mathrm{u}}(u,\theta),\theta), \qquad Z_0(u) = z^{\mathrm{s}}(v^{\mathrm{s}}(u,\theta),\theta). \tag{3.160}$$

We shall prove the following Proposition, which has Theorem 3.1.7 as an obvious corollary.

Proposition 3.3.1. *Let $\kappa = \kappa(\delta)$ and the constants β, T, ω be fixed in Theorem 3.1.6. Fix $m > 0$ a constant independent of δ and σ. Let $\bar\kappa = \bar\kappa(\delta)$ satisfying condition* (3.55) *and such that $\bar\kappa - \kappa > m$, and let $\bar\beta$, $\bar T$, $\bar\omega$ be constants such that $0 < \bar\beta < \beta$, $0 < \bar T < T$ and $0 < \bar\omega < \omega$. For $u \in D_{\bar\kappa,\bar\beta,\bar T}^{\mathrm{u}}$ (respectively $u \in D_{\bar\kappa,\bar\beta,\bar T}^{\mathrm{s}}$) and $\theta \in \mathbb{T}_{\bar\omega}$ the functions $v^{\mathrm{u,s}}(u,\theta)$ given in* (3.160) *are well defined.*

Moreover, let $x^{\mathrm{u,s}}(v,\theta)$, $y^{\mathrm{u,s}}(v,\theta)$ *be the functions defined in* (3.159). *Define:*

$$r^{\mathrm{u,s}}(u,\theta)=\frac{1}{2}\left[(x^{\mathrm{u,s}}(v^{\mathrm{u,s}}(u,\theta),\theta))^2+(y^{\mathrm{u,s}}(v^{\mathrm{u,s}}(u,\theta),\theta))^2\right],$$

and:

$$r_1^{\mathrm{u,s}}(u,\theta)=r^{\mathrm{u,s}}(u,\theta)-R_0(u). \tag{3.161}$$

Then, if $|\sigma|\le\sigma^*\delta^{p+3}$ *there exists a constant* M *such that:*

1. $r_1^{\mathrm{u}}(u,\theta)$ *and* $r_1^{\mathrm{s}}(u,\theta)$ *satisfy equation* (3.62) *and:*

$$\llbracket r_1^{\mathrm{u,s}}\rrbracket_{3,\bar\omega}^{\mathrm{u,s}}\le M\delta^{p+3}.$$

2. *Let:*

$$r_{10}^{\mathrm{u,s}}=\mathcal{G}^{\mathrm{u,s}}\circ\mathcal{F}(0),$$

where $\mathcal{G}^{\mathrm{u}}$ *and* $\mathcal{G}^{\mathrm{u}}$ *are the operators defined respectively in* (3.47) *and* (3.49), *and* $\mathcal{F}$ *is the operator defined in* (3.64). *Define:*

$$r_{11}^{\mathrm{u,s}}(u,\theta):=r_1^{\mathrm{u,s}}(u,\theta)-r_{10}^{\mathrm{u,s}}(u,\theta).$$

Then, for all $u\in D^{\mathrm{u}}_{\bar\kappa,\beta,\bar T}$ *(respectively* $u\in D^{\mathrm{s}}_{\bar\kappa,\beta,\bar T}$*) and* $\theta\in\mathbb{T}_{\bar\omega}$ *one has:*

$$|r_{10}^{\mathrm{u,s}}(u,\theta)|\le M\delta^{p+3}|\cosh(\mathrm{d}u)|^{-3},$$

and:

$$|r_{11}^{\mathrm{u,s}}(u,\theta)|\le M\left(\frac{\delta^{2p+6}}{|\cosh(\mathrm{d}u)|^4}+\frac{\delta^{p+4}}{|\cosh(\mathrm{d}u)|}\right).$$

Let us explain briefly how we proceed in this section. First, in Subsection 3.3.1, we state a technical lemma providing a useful property of the norm $\|.\|^{\mathrm{u}}_{n,\omega}$. Then we give some further technical lemmas concerning the norms of the functions f, g and h introduced in system (3.19), and the difference between these functions and the functions $f^{\mathrm{u,s}}$, $g^{\mathrm{u,s}}$ and $h^{\mathrm{u,s}}$ (see Lemmas 3.3.3 and 3.3.4). As a consequence, we can easily prove Lemma 3.3.5 concerning the norms of functions $\mathcal{F}^{\mathrm{u,s}}(0)$ and $\mathcal{F}(0)$ (see (3.44)and (3.64) for their definition). After that, in Lemma 3.3.6 we state the existence of the functions $v^{\mathrm{u,s}}(u,\theta)$ and we see that $v^{\mathrm{u,s}}(u,\theta)\approx u$. Finally, in Subsection 3.3.2 we use all these results to prove Proposition 3.3.1.

3.3.1 Technical lemmas

The following lemma can be proved in a similar way as [Bal06], Lemma 4.3, item (iii).

Lemma 3.3.2. *Let* $\phi : D^{\mathrm{u,s}}_{\kappa,\beta,T} \times \mathbb{T}_\omega \to \mathbb{C}$*, and a constant* C *such that:*

$$\|\phi\|^{\mathrm{u,s}}_{n,\omega} \leq C.$$

Let $\kappa = \kappa(\delta)$ *and the constants* β*,* T*,* ω *be fixed in Theorem 3.1.6. Fix* $m > 0$ *a constant independent of* δ *and* σ*. Let* $\bar{\kappa} = \bar{\kappa}(\delta)$ *satisfying condition* (3.55) *and such that* $\bar{\kappa} - \kappa > m$*, and let* $\bar{\beta}$*,* $\bar{T}$*,* $\bar{\omega}$ *be constants such that* $0 < \bar{\beta} < \beta$*,* $0 < \bar{T} < T$ *and* $0 < \bar{\omega} < \omega$*. Then there exists a constant* M *such that:*

$$\|\partial_u \phi\|^{\mathrm{u,s}}_{n+1,\bar{\omega}} \leq MC,$$

where the suprema of the norm are taken in $D^{\mathrm{u,s}}_{\bar{\kappa},\bar{\beta},\bar{T}}$.

Lemma 3.3.3. *Let:*

$$\eta(u, \theta) = (x(u, \theta), y(u, \theta), z(u, \theta))$$

any function such that:

$$\|\eta(u, \theta)\|^{\mathrm{u,s}}_{1,0,\omega'} \leq K,$$

for some $\omega' > \omega$*. Let* f*,* g *and* h *the functions appearing in system* (3.19)*. Then, there exists a constant* M *such that:*

1. *One has:*

$$\|f(\delta\eta(u, \theta), \delta, \delta\sigma)\|^{\mathrm{u,s}}_{3,0,\omega} \leq M\delta^3,$$

$$\|g(\delta\eta(u, \theta), \delta, \delta\sigma)\|^{\mathrm{u,s}}_{3,0,\omega} \leq M\delta^3,$$

$$\|h(\delta\eta(u, \theta), \delta, \delta\sigma)\|^{\mathrm{u,s}}_{3,0,\omega} \leq M\delta^3.$$

2. *Similarly, one has:*

$$\|D_\eta f(\delta\eta(u, \theta), \delta, \delta\sigma)\|^{\mathrm{u,s}}_{2,0,\omega} \leq M\delta^3,$$

$$\|D_\eta g(\delta\eta(u, \theta), \delta, \delta\sigma)\|^{\mathrm{u,s}}_{2,0,\omega} \leq M\delta^3,$$

$$\|D_\eta h(\delta\eta(u, \theta), \delta, \delta\sigma)\|^{\mathrm{u,s}}_{2,0,\omega} \leq M\delta^3.$$

Proof. We only prove item 1 for the function f and the unstable case. All the other cases and item 2 are proved analogously.

Since f is of order three in all their variables, one has:

$$|f(\delta\eta(u, \theta), \delta, \delta\sigma)| \leq |(\delta\eta(u, \theta), \delta, \delta\sigma)|^3.$$

On the one hand, using that $\|\eta(u, \theta)\|^{\mathrm{u,s}}_{1,0,\omega'} \leq K$, if $u \in D^{\mathrm{u}}_{\kappa,\beta,\infty}$ we have:

$$|\delta\eta(u, \theta)| \leq \delta K.$$

Then, it is clear that:

$$\sup_{\substack{\theta\in\omega' \\ u\in D^{\mathrm{u}}_{\kappa,\beta,\infty}}} |f(\delta\eta(u,\theta),\delta,\delta\sigma)| \leq K\delta^3. \tag{3.162}$$

On the other hand, if $u\in D^{\mathrm{u}}_{\kappa,\beta,T}$, since $\|\eta(u,\theta)\|^{\mathrm{u,s}}_{1,0,\omega'}\leq K$ it is clear that:

$$|\delta\cosh(\mathrm{d}u)\eta(u,\theta)|\leq \delta K.$$

Then we easily obtain:

$$\sup_{\substack{\theta\in\omega' \\ u\in D^{\mathrm{u}}_{\kappa,\beta,T}}} |\cosh^3(\mathrm{d}u) f(\delta\eta(u,\theta),\delta,\delta\sigma)| \leq K\delta^3. \tag{3.163}$$

Finally, one only has to use Lemma 3.2.4 and bounds (3.162) and (3.163) to obtain straightforwardly:

$$\|f(\delta\eta(u,\theta),\delta,\delta\sigma)\|^{\mathrm{u,s}}_{3,0,\omega}\leq K\delta^3.$$

□

Lemma 3.3.4. *Let f, g and h be the functions appearing in system* (3.19) *and $f^{\mathrm{u,s}}$, $g^{\mathrm{u,s}}$ and $h^{\mathrm{u,s}}$ the functions appearing in system* (3.31)*. Denote:*

$$\eta_0(u,\theta)=(\sqrt{2R_0(u)}\cos\theta,\sqrt{2R_0(u)}\sin\theta,Z_0(u)),$$

where R_0 and Z_0 are defined in (3.24) *and* (3.26) *respectively. Then there exists a constant M such that:*

$$\|f(\delta\eta_0(u,\theta),\delta,\delta\sigma)-f^{\mathrm{u,s}}(\delta\eta_0(u,\theta),\delta,\delta\sigma)\|^{\mathrm{u,s}}_{1,0,\omega}\leq M\delta^4,$$

$$\|g(\delta\eta_0(u,\theta),\delta,\delta\sigma)-g^{\mathrm{u,s}}(\delta\eta_0(u,\theta),\delta,\delta\sigma)\|^{\mathrm{u,s}}_{1,0,\omega}\leq M\delta^4,$$

$$\|h(\delta\eta_0(u,\theta),\delta,\delta\sigma)-h^{\mathrm{u,s}}(\delta\eta_0(u,\theta),\delta,\delta\sigma)\|^{\mathrm{u,s}}_{1,0,\omega}\leq M\delta^4.$$

Proof. We shall prove only the unstable case, the stable one is analogous. We begin by introducing some notation. Let:

$$\eta=\begin{pmatrix} x\\ y\\ z\end{pmatrix},\qquad \eta_-=\begin{pmatrix} x_-(\delta,\sigma)\\ y_-(\delta,\sigma)\\ z_-(\delta,\sigma)\end{pmatrix}\qquad \tilde\eta_-=\begin{pmatrix} x_-(\delta,\sigma)\\ y_-(\delta,\sigma)\\ z_-(\delta,\sigma)+1\end{pmatrix}.$$

We point out that by Lemma 3.1.2:

$$\tilde\eta_-=\begin{pmatrix}\mathcal{O}(\delta^{p+5})\\ \mathcal{O}(\delta^{p+5})\\ \mathcal{O}(\delta^{p+4})\end{pmatrix}. \tag{3.164}$$

Let $\Phi = C_2^{\mathrm{u}} \circ C_1^{\mathrm{u}}$, where C_1^{u} is the change defined in (3.28) and C_2^{u} is the change defined in (3.29). Then we define:

$$\xi = \Phi(\eta). \tag{3.165}$$

With this notation, expression (3.30) of the change $\Phi = C_2^{\mathrm{u}} \circ C_1^{\mathrm{u}}$ writes out as:

$$\Phi(\eta) = \eta - \tilde{\eta}_- + \delta^{p+5} M_-^1(\delta,\sigma)(\eta - \eta_-). \tag{3.166}$$

In particular, it is clear that there exists a matrix $\hat{M}_-^1(\delta,\sigma)$ such that:

$$\Phi^{-1}(\xi) = \xi + \tilde{\eta}_- + \delta^{p+5} \hat{M}_-^1(\delta,\sigma) \left(\xi + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right). \tag{3.167}$$

We denote system (3.19) as:

$$\dot{\eta} = X_0(\eta) + \delta^p X_1(\eta),$$

and (3.31) as:

$$\dot{\xi} = X_0(\xi) + \delta^p Y_1(\xi).$$

Note that:

$$X_1 = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \qquad Y_1 = \begin{pmatrix} f^{\mathrm{u}} \\ g^{\mathrm{u}} \\ h^{\mathrm{u}} \end{pmatrix}.$$

Differentiating (3.165) with respect to time, we obtain:

$$\dot{\xi} = D\Phi(\Phi^{-1}(\xi))\dot{\eta} = D\Phi(\Phi^{-1}(\xi)) \left(X_0(\Phi^{-1}(\xi)) + \delta^p X_1(\Phi^{-1}(\xi)) \right).$$

Thus one has:

$$Y_1(\xi) = D\Phi(\Phi^{-1}(\xi)) \left(\delta^{-p} X_0(\Phi^{-1}(\xi)) + X_1(\Phi^{-1}(\xi)) \right) - \delta^{-p} X_0(\xi).$$

Now that we have introduced all the necessary notation, we can restate the claim of the lemma as:

$$\| X_1(\eta_0) - Y_1(\eta_0) \|_{1,0,\omega}^{\mathrm{u}} \leq K\delta^4, \tag{3.168}$$

where we take the norm in each component of the vector. We can write:

$$\begin{aligned} X_1(\eta_0) - Y_1(\eta_0) &= \left[\mathrm{Id} - D\Phi(\Phi^{-1}(\eta_0)) \right] \left(\delta^{-p} X_0(\Phi^{-1}(\eta_0)) + X_1(\Phi^{-1}(\eta_0)) \right) \\ &+ \delta^{-p} \left(X_0(\eta_0) - X_0(\Phi^{-1}(\eta_0)) \right) + X_1(\eta_0) - X_1(\Phi^{-1}(\eta_0)). \end{aligned} \tag{3.169}$$

On the one hand, we note that from (3.166) one has:

$$D\Phi(\Phi^{-1}(\eta_0)) = \mathrm{Id} + \delta^{p+5} M_-^1(\delta,\sigma).$$

On the other hand, we point out that since $\|\eta_0\|^{\mathrm{u}}_{1,0,\omega} \leq K$, expression (3.167) implies that $\|\Phi^{-1}(\eta_0)\|^{\mathrm{u}}_{1,0,\omega} \leq K$. Then, using the exact expression of X_0, one can easily check that:

$$\|X_0(\Phi^{-1}(\eta_0))\|^{\mathrm{u}}_{1,0,\omega} \leq \frac{K}{\delta}.$$

Moreover, we note that $\|\Phi^{-1}(\eta_0)\|^{\mathrm{u}}_{1,0,\omega'} \leq K$ for any $\omega'\mathbb{R}$. Then using item 1 of Lemma 3.3.3 one has:

$$\|X_1(\Phi^{-1}(\eta_0))\|^{\mathrm{u}}_{1,0,\omega} \leq \frac{K}{\delta^2}\|X_1(\Phi^{-1}(\eta_0))\|^{\mathrm{u}}_{3,0,\omega} \leq K\delta.$$

Thus we readily obtain:

$$\left\|\left[\mathrm{Id} - D\Phi(\Phi^{-1}(\eta_0))\right]\left(\delta^{-p}X_0(\Phi^{-1}(\eta_0)) + X_1(\Phi^{-1}(\eta_0))\right)\right\|^{\mathrm{u}}_{1,0,\omega} \leq K\delta^4. \tag{3.170}$$

Now, we have:

$$\left|X_0(\eta_0) - X_0(\Phi^{-1}(\eta_0))\right| \leq \sup_{t\in[0,1]} \left|DX_0(\eta_0 + t(\Phi^{-1}(\eta_0) - \eta_0)) \cdot (\Phi^{-1}(\eta_0) - \eta_0)\right|.$$

First we note that:

$$\|\eta_0 + t(\Phi^{-1}(\eta_0) - \eta_0)\|^{\mathrm{u}}_{1,0,\omega} \leq K. \tag{3.171}$$

Then, using (3.164), (3.167) and the exact expression of DX_0 one can easily see that:

$$\left\|DX_0(\eta_0 + t(\Phi^{-1}(\eta_0) - \eta_0)) \cdot (\Phi^{-1}(\eta_0) - \eta_0)\right\|^{\mathrm{u}}_{1,0,\omega} \leq K\delta^{p+4},$$

and then:

$$\delta^{-p}\left\|X_0(\eta_0) - X_0(\Phi^{-1}(\eta_0))\right\|^{\mathrm{u}}_{1,0,\omega} \leq K\delta^4. \tag{3.172}$$

Finally, one has:

$$\left|X_1(\eta_0) - X_1(\Phi^{-1}(\eta_0))\right| \leq \sup_{t\in[0,1]} \left|DX_1(\eta_0 + t(\Phi^{-1}(\eta_0) - \eta_0)) \cdot (\Phi^{-1}(\eta_0) - \eta_0)\right|.$$

By bound (3.171) we can use item 2 of Lemma 3.3.3, which yields:

$$\|DX_1(\eta_0 + t(\Phi^{-1}(\eta_0) - \eta_0))\|^{\mathrm{u}}_{1,0,\omega} \leq \frac{K}{\delta}\|DX_1(\eta_0 + t(\Phi^{-1}(\eta_0) - \eta_0))\|^{\mathrm{u}}_{2,0,\omega} \leq K\delta^2.$$

Using again (3.167) to bound $|\Phi^{-1}(\eta_0) - \eta_0|$, we obtain:

$$\left\|X_1(\eta_0) - X_1(\Phi^{-1}(\eta_0))\right\|^{\mathrm{u}}_{1,0,\omega} \leq K\delta^{p+6}. \tag{3.173}$$

Using bounds (3.170), (3.172) and (3.173) in (3.169) and the fact that $p \geq -2$, we obtain claim (3.168), and the lemma is proved. $\square$

Lemma 3.3.5. *Let $\mathcal{F}$ be the operator defined in* (3.64) *and $\mathcal{F}^{\mathrm{u,s}}$ the operator defined in* (3.44)*. Let $|\sigma| \leq \sigma^* \delta^{p+3}$. There exists a constant M such that:*

1. $\|\mathcal{F}(0)\|^{\mathrm{u,s}}_{4,0,\omega} \leq M\delta^{p+3}$.

2. $\|\mathcal{F}(0) - \mathcal{F}^{\mathrm{u,s}}(0)\|^{\mathrm{u,s}}_{2,0,\omega} \leq M\delta^{p+4}$.

Proof. To prove item 1, we use the definition (3.64) of $\mathcal{F}$ and we obtain:

$$\mathcal{F}(0) = 2\sigma R_0 + \delta^p F(0) + \delta^p \frac{\mathrm{d}+1}{b} Z_0(u) H(0), \tag{3.174}$$

where F and H are the functions defined in (3.23) (and we make the abuse of notation pointed out in (3.37)). Using the definitions (3.23) of F and H, the fact that $\|\sqrt{2R_0}\|^{\mathrm{u}}_{1,1,\omega} \leq K$, item 1 of Lemma 3.3.3 and the properties of the norms $\|.\|^{\mathrm{u}}_{n,m,\omega}$ summarized in Lemma 3.2.2, one easily obtains that:

$$\|F(0)\|^{\mathrm{u}}_{4,1,\omega} \leq K\delta^3, \qquad \|H(0)\|^{\mathrm{u}}_{3,0,\omega} \leq K\delta^3.$$

Using these bounds in (3.174) and taking into account that $\|R_0\|^{\mathrm{u}}_{4,0,\omega} \leq \|R_0\|^{\mathrm{u}}_{2,2,\omega} \leq K$, $\|Z_0\|^{\mathrm{u}}_{1,0,\omega} \leq K$, $|\sigma| \leq \sigma^*\delta^{p+3}$ one obtains the claim of item 1.

Now we prove item 2 in a similar way. Using the definitions (3.64) of $\mathcal{F}$ and (3.44) of $\mathcal{F}^{\mathrm{u,s}}$ one has:

$$\mathcal{F}(0) - \mathcal{F}^{\mathrm{u,s}}(0) = \delta^p \left(F(0) - F^{\mathrm{u,s}}(0)\right) + \delta^p \frac{\mathrm{d}+1}{b} Z_0(u) \left(H(0) - H^{\mathrm{u,s}}(0)\right), \tag{3.175}$$

We claim that:

$$\begin{aligned} \|F(0) - F^{\mathrm{u,s}}(0)\|^{\mathrm{u}}_{2,1,\omega} &\leq K\delta^4, \\ \|H(0) - H^{\mathrm{u,s}}(0)\|^{\mathrm{u}}_{1,0,\omega} &\leq K\delta^4. \end{aligned}$$

This is straightforward to prove, again using the definitions (3.23) of F and H and (3.35) of $F^{\mathrm{u,s}}$ and $H^{\mathrm{u,s}}$, the fact that $\|\sqrt{2R_0}\|^{\mathrm{u}}_{1,1,\omega} \leq K$ and Lemma 3.3.4. Finally, substituting these bounds in (3.175) and recalling that $\|Z_0\|^{\mathrm{u}}_{1,0,\omega} \leq K$ finishes the proof the lemma. □

Lemma 3.3.6. *Let $\kappa = \kappa(\delta)$ and the constant T be fixed in Theorem 3.1.6. Fix $m > 0$ a constant independent of δ and σ. Let $\bar\kappa = \bar\kappa(\delta)$ satisfying condition* (3.55) *and such that $\bar\kappa - \kappa > m$, and let $\bar T$ be a constant such that $0 < \bar T < T$. Let $z^{\mathrm{u}}(v,\theta)$ be the function defined in* (3.159)*. If δ is sufficiently small, the function $v^{\mathrm{u,s}}(u,\theta)$ defined implicitly by:*

$$Z_0(u) = z^{\mathrm{u,s}}(v^{\mathrm{u,s}}(u,\theta),\theta) \tag{3.176}$$

is well defined for all $u \in D^{\mathrm{u}}_{\bar\kappa,\beta,\bar T}$ (respectively $u \in D^{\mathrm{s}}_{\bar\kappa,\beta,\bar T}$) and $\theta \in \mathbb{T}_\omega$, and there exists a constant M such that:

$$|v^{\mathrm{u,s}}(u,\theta) - u| \leq M\delta^{p+4} |\cosh(\mathrm{d}u)|^2.$$

Proof. Again, we shall do the proof for the unstable case only. Moreover, to avoid cumbersome notation we shall write $v(u,\theta)$ instead of $v^{\mathrm{u}}(u,\theta)$.

First, recalling expression (3.167) of $\Phi^{-1} = (C_2^{\mathrm{u}} \circ C_1^{\mathrm{u}})^{-1}$ and that, by Lemma 3.1.2, $z_-(\delta,\sigma) - 1 = \mathcal{O}(\delta^{p+4})$, we can write:

$$z^{\mathrm{u}}(v,\theta) = Z_0(v) + a\delta^{p+4} + \delta^{p+5}\left(m_{31}X^{\mathrm{u}}(v,\theta) + m_{32}Y^{\mathrm{u}}(v,\theta) + m_{33}(Z_0(v)+1)\right),$$

for some coefficient a (depending on δ and σ), and where the coefficients m_{3j} (which also depend on δ and σ) denote the j-th component of the third row of matrix $\hat{M}_-^1(\delta,\sigma)$ introduced in (3.167). We introduce the following function:

$$\chi(v,\theta) := \delta^{-p-4}\left(z^{\mathrm{u}}(v,\theta) - Z_0(v)\right) = a + \delta\left(m_{31}X^{\mathrm{u}}(v,\theta) + m_{32}Y^{\mathrm{u}}(v,\theta) + m_{33}(Z_0(v)+1)\right),$$

so that:

$$z^{\mathrm{u}}(v,\theta) = Z_0(v) + \delta^{p+4}\chi(v,\theta).$$

Now we give some bounds of χ that will be used later on. Using the definition (3.158) of X^{u}, Y^{u} and Theorem 3.1.6, one can see that for all $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$ and $\theta \in \mathbb{T}_\omega$:

$$|\chi(v,\theta)| \leq K. \tag{3.177}$$

Similarly, for all $v \in D^{\mathrm{u}}_{\kappa,\beta,T}$ and $\theta \in \mathbb{T}_\omega$ one has:

$$|\partial_v \chi(v,\theta)| \leq \frac{K}{\delta}. \tag{3.178}$$

Now, with this notation, we can write the implicit definition of v (3.176) as the fixed point equation:

$$v = Z_0^{-1}(Z_0(u) - \delta^{p+4}\chi(v,\theta)). \tag{3.179}$$

We shall look for a solution $v = v(u,\theta)$ of (3.179) of the form:

$$v(u,\theta) = u + V(u,\theta),$$

where $V(u,\theta)$ is some function to be determined, which we expect that will be small. Using this expression of $v(u,\theta)$ in equation (3.179), we obtain the following fixed point equation for V:

$$V(u,\theta) = Z_0^{-1}(Z_0(u) - \delta^{p+4}\chi(u + V(u,\theta),\theta)) - u := \varphi(V)(u,\theta). \tag{3.180}$$

We shall see that the operator φ has a fixed point in the Banach space $\mathcal{B}$ defined as:

$$\mathcal{B} = \{\phi : D^{\mathrm{u}}_{\bar\kappa,\beta,\bar T} \times \mathbb{T}_\omega \to \mathbb{C} \,:\, \phi \text{ is analytic, } \|\phi\|_{-2} < +\infty\},$$

where:

$$\|\phi\|_{-2} := \sup_{(u,\theta)\in D^{\mathrm{u}}_{\bar\kappa,\beta,\bar T}\times\mathbb{T}_\omega} |\cosh^{-2}(\mathrm{d}u)\phi(u,\theta)|.$$

More precisely, let $B(2\|\varphi(0)\|_{-2}) \subset \mathcal{B}$ denote the ball centered at the origin and radius $2\|\varphi(0)\|_{-2}$. Then we shall see that φ has a fixed point in $B(2\|\varphi(0)\|_{-2})$ using the fixed point theorem. To shorten the notation, we shall omit the dependence of φ with respect to u and θ, writing $\varphi(V)$ instead of $\varphi(V)(u,\theta)$.

First we claim that $\varphi(0) \in \mathcal{B}$ and:

$$\|\varphi(0)\|_{-2} \le K\delta^{p+4}. \tag{3.181}$$

Indeed, using the definition (3.180) of φ and the mean value theorem one has:

$$|\varphi(0)| = \left|Z_0^{-1}\left(Z_0(u) - \delta^{p+4}\chi(u,\theta)\right) - u\right| \le \sup_{\lambda\in[0,1]} \left|\frac{\delta^{p+4}\chi(u,\theta)}{1-\left(Z_0(u)-\lambda\delta^{p+4}\chi(u,\theta)\right)^2}\right|. \tag{3.182}$$

Using bound (3.177) of $\chi(u,\theta)$ (and noting that $D^{\mathrm{u}}_{\bar\kappa,\beta,\bar T} \subset D^{\mathrm{u}}_{\kappa,\beta,T}$ because $\bar\kappa > \kappa$ and $\bar T < T$) it is easy to see that for all $\lambda \in [0,1]$:

$$\left|\frac{1}{1-\left(Z_0(u)-\lambda\delta^{p+4}\chi(u,\theta)\right)^2}\right| \le \left|\frac{K}{1-Z_0^2(u)}\right| = K|\cosh(\mathrm{d}u)|^2,$$

and then by (3.182) we obtain:

$$|\varphi(0)| \le K\delta^{p+4}|\cosh(\mathrm{d}u)|^2,$$

which yields claim (3.181).

Now, let V_1, $V_2 \in B(2\|\varphi(0)\|_{-2})$. We claim that:

$$\|\varphi(V_1)-\varphi(V_2)\|_{-2} \le K\delta^{p+3}\|V_1-V_2\|_{-2}. \tag{3.183}$$

Again, using the definition (3.180) of φ and the mean value theorem one has:

$$\begin{aligned}|\varphi(V_1)-\varphi(V_2)| &= \left|Z_0^{-1}\big(Z_0(u)-\delta^{p+4}\chi(u+V_1,\theta)\big) - Z_0^{-1}\big(Z_0(u)-\delta^{p+4}\chi(u+V_2,\theta)\big)\right| \\ &\le \sup_{\lambda\in[0,1]}\left|\frac{\delta^{p+4}\partial_v\chi(u+V_1+\lambda(V_2-V_1),\theta)}{1-\left(Z_0(u)-\delta^{p+4}\chi(u+V_1+\lambda(V_2-V_1),\theta)\right)^2}(V_1-V_2)\right|.\end{aligned}$$

Since V_1, $V_2 \in B(2\|\varphi(0)\|_{-2})$, using bound (3.181) of $\|\varphi(0)\|_{-2}$ in particular we have that for all $\lambda\in[0,1]$:

$$|V_1+\lambda(V_2-V_1)| \le K\delta^{p+4}.$$

This implies that, for δ sufficiently small and $u \in D^{\mathrm{u}}_{\bar\kappa,\beta,\bar T}$, one has $u+V_1+\lambda(V_2-V_1) \in D^{\mathrm{u}}_{\kappa,\beta,T}$ and then using bound (3.177) of χ it is easy to see that:

$$\left|\frac{1}{1-\left(Z_0(u)-\delta^{p+4}\chi(u+V_1+\lambda(V_2-V_1),\theta)\right)^2}\right| \le K|\cosh(\mathrm{d}u)|^2.$$

Using this fact and bound (3.178) of $\partial_v \chi$ we obtain:

$$|\varphi(V_1) - \varphi(V_2)| \leq K\delta^{p+3}|\cosh^2(\mathrm{d}u)||V_1 - V_2|.$$

Now we just have to note that $|V_1(u,\theta) - V_2(u,\theta)| \leq |\cosh(\mathrm{d}u)|^2 \|V_1 - V_2\|_{-2}$, and since $\cosh(\mathrm{d}u)$ is bounded in $D^{\mathrm{u}}_{\bar{\kappa},\beta,\bar{T}}$, we obtain easily claim (3.183).

To finish, we simply have to note that by bound (3.183) we know that φ is contractive in $B(2\|\varphi(0)\|_{-2})$ if δ is sufficiently small, so that the fixed point theorem ensures that there exists a unique fixed point V of φ. Finally, defining $v(u,\theta) = u + V(u,\theta)$ and using bound (3.181) we obtain:

$$\begin{aligned} |v(u,\theta) - u| &= |V(u,\theta)| \leq |\cosh^2(\mathrm{d}u)| \|V\|_{-2} \leq 2|\cosh^2(\mathrm{d}u)| \|\varphi(0)\|_{-2} \\ &\leq K\delta^{p+4}|\cosh^2(\mathrm{d}u)|. \end{aligned}$$

□

3.3.2 Proof of Proposition 3.3.1

With all the previous results we are ready to prove Proposition 3.3.1.

Proof of Proposition 3.3.1. Again, we prove only the unstable case and omit the superindex of $v^{\mathrm{u}}(u,\theta)$, writing simply $v(u,\theta)$. The fact that this function is well defined for $(u,\theta) \in D^{\mathrm{u}}_{\bar{\kappa},\beta,\bar{T}} \times \mathbb{T}_{\bar{\omega}}$ is a straight consequence of Lemma 3.3.6 (and the fact that $\mathbb{T}_{\bar{\omega}} \subset \mathbb{T}_{\omega}$ because $\bar{\omega} < \omega$).

We start with some previous considerations. From the definition (3.159) of $x^{\mathrm{u}}(v,\theta)$ and $y^{\mathrm{u}}(v,\theta)$, and since $x_-(\delta,\sigma), y_-(\delta,\sigma) = \mathcal{O}(\delta^{p+5})$ by Lemma 3.1.2, it is easy to see that:

$$\begin{aligned} x^{\mathrm{u}}(v,\theta) &= X^{\mathrm{u}}(v,\theta) + b\delta^{p+5} + \delta^{p+5}\left(m_{11}X^{\mathrm{u}}(v,\theta) + m_{12}Y^{\mathrm{u}}(v,\theta) + m_{13}(Z_0(v)+1)\right), \\ y^{\mathrm{u}}(v,\theta) &= Y^{\mathrm{u}}(v,\theta) + c\delta^{p+5} + \delta^{p+5}\left(m_{21}X^{\mathrm{u}}(v,\theta) + m_{22}Y^{\mathrm{u}}(v,\theta) + m_{23}(Z_0(v)+1)\right), \end{aligned} \tag{3.184}$$

for some coefficients b and c (depending on δ and σ), and where m_{ij} denotes the entry (i,j) of the matrix $\hat{M}_-^{-1}(\delta,\sigma)$ introduced in (3.167). We point out that, from the definition (3.158) of $X^{\mathrm{u}}(v,\theta)$ and $Y^{\mathrm{u}}(v,\theta)$, and using Theorem 3.1.6, it is easy to see that for $(v,\theta) \in D^{\mathrm{out,u}}_{\kappa,\beta,T} \times \mathbb{T}_{\omega}$:

$$|X^{\mathrm{u}}(v,\theta)| \leq \frac{K}{|\cosh(\mathrm{d}v)|}, \qquad |Y^{\mathrm{u}}(v,\theta)| \leq \frac{K}{|\cosh(\mathrm{d}v)|}, \tag{3.185}$$

and:

$$|\partial_v X^{\mathrm{u}}(v,\theta)| \leq \frac{K}{|\cosh(\mathrm{d}v)|^2}, \qquad |\partial_v Y^{\mathrm{u}}(v,\theta)| \leq \frac{K}{|\cosh(\mathrm{d}v)|^2}. \tag{3.186}$$

In particular, using (3.185) in (3.184) we get that for $(v,\theta) \in D^{\mathrm{out,u}}_{\kappa,\beta,T} \times \mathbb{T}_{\omega}$:

$$x^{\mathrm{u}}(v,\theta) = X^{\mathrm{u}}(v,\theta) + \mathcal{O}(\delta^{p+4}), \qquad y^{\mathrm{u}}(v,\theta) = Y^{\mathrm{u}}(v,\theta) + \mathcal{O}(\delta^{p+4}). \tag{3.187}$$

Finally, we point out that by the mean value theorem:

$$|X^{\mathrm{u}}(v(u,\theta),\theta)-X^{\mathrm{u}}(u,\theta)|\leq \sup_{\lambda\in[0,1]}|\partial_v X(v(u,\theta)+\lambda(v(u,\theta)-u))||v(u,\theta)-u|.$$

By Lemma 3.3.6, taking δ sufficiently we can ensure that for $u\in D^{\mathrm{u}}_{\bar\kappa,\beta,\bar T}$ and $\lambda\in[0,1]$ one has $v(u,\theta)+\lambda(v(u,\theta)-u)\in D^{\mathrm{u}}_{\kappa,\beta,T}$. Then, bound (3.186) and Lemma 3.3.6 yield:

$$|X^{\mathrm{u}}(v(u,\theta),\theta)-X^{\mathrm{u}}(u,\theta)|\leq K\delta^{p+4}\sup_{\lambda\in[0,1]}\frac{|\cosh(\mathrm{d}u)|^2}{|\cosh(\mathrm{d}(v(u,\theta)+\lambda(v(u,\theta)-u)))|^2}.$$

Using again Lemma 3.3.6 one can easily see that:

$$\frac{|\cosh(\mathrm{d}u)|^2}{|\cosh(\mathrm{d}(v(u,\theta)+\lambda(v(u,\theta)-u)))|^2}\leq K,$$

so that:

$$|X^{\mathrm{u}}(v(u,\theta),\theta)-X^{\mathrm{u}}(u,\theta)|\leq K\delta^{p+4}. \tag{3.188}$$

Similarly one obtains:

$$|Y^{\mathrm{u}}(v(u,\theta),\theta)-Y^{\mathrm{u}}(u,\theta)|\leq K\delta^{p+4}. \tag{3.189}$$

Using these two facts in (3.187) one obtains:

$$x^{\mathrm{u}}(v(u,\theta),\theta)=X^{\mathrm{u}}(u,\theta)+\mathcal{O}(\delta^{p+4}),\qquad y^{\mathrm{u}}(v(u,\theta),\theta)=Y^{\mathrm{u}}(u,\theta)+\mathcal{O}(\delta^{p+4}). \tag{3.190}$$

Then, taking into account also (3.185), we obtain:

$$\begin{aligned}
r^{\mathrm{u}}(u,\theta) &= \frac{1}{2}\left[(x^{\mathrm{u}}(v(u,\theta),\theta))^2+(y^{\mathrm{u}}(v(u,\theta),\theta))^2\right]\\
&= \frac{1}{2}\left[(X^{\mathrm{u}}(u,\theta))^2+(Y^{\mathrm{u}}(u,\theta))^2\right]+\mathcal{O}\left(\frac{\delta^{p+4}}{\cosh(\mathrm{d}u)}\right)\\
&= R_0(u)+R_1^{\mathrm{u}}(u,\theta)+\mathcal{O}\left(\frac{\delta^{p+4}}{\cosh(\mathrm{d}u)}\right),
\end{aligned}$$

where in the las step we have used the definition of X^{u} and Y^{u}. Then, recalling definition (3.161) of r_1^{u}, we have that:

$$r_1^{\mathrm{u}}(u,\theta)=R_1^{\mathrm{u}}(u,\theta)+\mathcal{O}\left(\frac{\delta^{p+4}}{\cosh(\mathrm{d}u)}\right). \tag{3.191}$$

In the following we shall denote:

$$r_2^{\mathrm{u}}(u,\theta)=r_1^{\mathrm{u}}(u,\theta)-R_1^{\mathrm{u}}(u,\theta)=\mathcal{O}\left(\frac{\delta^{p+4}}{\cosh(\mathrm{d}u)}\right). \tag{3.192}$$

Now we proceed to prove item 1. The fact that r_1^{u} satisfies equation (3.62) was already justified in Section 3.1.4. Clearly, by Proposition 3.2.1, R_1^{u} satisfies:

$$\llbracket R_1 \rrbracket_{3,\omega}^{\mathrm{u}} \leq K\delta^{p+3}. \tag{3.193}$$

On the other hand, by (3.192) we know that $r_2^{\mathrm{u}}(u,\theta)$ satisfies:

$$\sup_{(u,\theta)\in D^{\mathrm{u}}_{\bar\kappa,\beta,\bar T}\times\mathbb{T}_\omega} |\cosh(\mathrm{d}u) r_2^{\mathrm{u}}(u,\theta)| \leq K\delta^{p+4}.$$

Since $0<\bar\omega<\omega$, Lemma 3.2.4 gives that:

$$\|r_2^{\mathrm{u}}\|_{1,\bar\omega}^{\mathrm{u}} \leq K\delta^{p+4}. \tag{3.194}$$

In particular, this also implies trivially:

$$\|r_2^{\mathrm{u}}\|_{3,\bar\omega}^{\mathrm{u}} \leq K\delta^{p+4}. \tag{3.195}$$

Moreover, taking $\bar\beta<\beta$, increasing $\bar\kappa$ and decreasing $\bar T$, by Lemma 3.3.2 we have:

$$\|\partial_u r_2^{\mathrm{u}}\|_{4,\bar\omega}^{\mathrm{u}} \leq \|r_2^{\mathrm{u}}\|_{3,\bar\omega}^{\mathrm{u}} \leq K\delta^{p+4}. \tag{3.196}$$

We point out that we have abused notation, using the same $\bar\kappa$ and $\bar T$ although they are different from the previous ones. However, they still satisfy $\bar\kappa-\kappa>m$ and $0<\bar T<T$. Bounds (3.193), (3.195) and (3.196) yield:

$$\|r_1^{\mathrm{u}}\|_{3,\bar\omega}^{\mathrm{u}} \leq K\delta^{p+3}, \qquad \|\partial_u r_1^{\mathrm{u}}\|_{4,\bar\omega}^{\mathrm{u}} \leq K\delta^{p+3}.$$

Finally, using these bounds and the fact that r_1^{u} satisfies equation (3.62), we easily obtain:

$$\|\partial_\theta r_1\|_{4,\bar\omega}^{\mathrm{u}} \leq K\delta^{p+4},$$

so that item 1 is clear.

To prove item 2, we first recall that by Proposition 3.2.1 we have $R_1^{\mathrm{u}}(u,\theta)=\mathcal{G}^{\mathrm{u}}\circ\mathcal{F}^{\mathrm{u}}(0)+R_{11}^{\mathrm{u}}(u,\theta)$, with:

$$\|R_{11}^{\mathrm{u}}\|_{4,\omega}^{\mathrm{u}} \leq K\delta^{2p+6}. \tag{3.197}$$

Then, expression (3.191) of r_1^{u} yields:

$$r_1^{\mathrm{u}}(u,\theta)=\mathcal{G}^{\mathrm{u}}\circ\mathcal{F}^{\mathrm{u}}(0)+R_{11}^{\mathrm{u}}(u,\theta)+r_2^{\mathrm{u}}(u,\theta).$$

Since $\mathcal{G}^{\mathrm{u}}$ is linear, we can write:

$$\mathcal{G}^{\mathrm{u}}\circ\mathcal{F}^{\mathrm{u}}(0)=\mathcal{G}^{\mathrm{u}}\circ\mathcal{F}(0)+\mathcal{G}^{\mathrm{u}}\circ\left(\mathcal{F}^{\mathrm{u}}(0)-\mathcal{F}(0)\right),$$

and then denoting $r_{10}^{\mathrm{u}} = \mathcal{G}^{\mathrm{u}} \circ \mathcal{F}^{\mathrm{u}}(0)$ we can write:

$$r_1^{\mathrm{u}}(u,\theta) = r_{10}^{\mathrm{u}}(u,\theta) + \mathcal{G}^{\mathrm{u}} \circ (\mathcal{F}^{\mathrm{u}}(0) - \mathcal{F}(0)) + R_{11}^{\mathrm{u}}(u,\theta) + r_2^{\mathrm{u}}(u,\theta).$$

On the one hand, by item 1 of Lemma 3.3.5 we know that $\|\mathcal{F}(0)\|_{4,0,\omega}^{\mathrm{u}} \leq K\delta^{p+3}$. Then, using the properties of the operator $\mathcal{G}^{\mathrm{u}}$ given in Lemma 3.2.11, we obtain:

$$\|\mathcal{G}^{\mathrm{u}} \circ \mathcal{F}(0)\|_{3,0,\omega}^{\mathrm{u}} \leq K\delta^{p+3},$$

that is, for $u \in D_{\bar{\kappa},\bar{\beta},\bar{T}}^{\mathrm{u}}$:

$$|r_{10}^{\mathrm{u}}(u,\theta)| = |\mathcal{G}^{\mathrm{u}} \circ \mathcal{F}(0)(u,\theta)| \leq \frac{K\delta^{p+3}}{|\cosh(\mathrm{d}u)|^3}.$$

On the other hand, using that $\|\mathcal{F}^{\mathrm{u}}(0) - \mathcal{F}(0)\|_{2,0,\omega}^{\mathrm{u}} \leq K\delta^{p+4}$ by item 2 of Lemma 3.3.5 and the properties of the operator $\mathcal{G}^{\mathrm{u}}$ given in Lemma 3.2.11, we obtain:

$$\|\mathcal{G}^{\mathrm{u}} \circ (\mathcal{F}^{\mathrm{u}}(0) - \mathcal{F}(0))\,\|_{1,\omega}^{\mathrm{u}} \leq K\delta^{p+4}. \tag{3.198}$$

Naming:

$$r_{11}^{\mathrm{u}}(u,\theta) = \mathcal{G}^{\mathrm{u}} \circ (\mathcal{F}^{\mathrm{u}}(0) - \mathcal{F}(0)) + R_{11}^{\mathrm{u}}(u,\theta) + r_2^{\mathrm{u}}(u,\theta),$$

and using bounds (3.194), (3.197) and (3.198) it is clear that for all $(u,\theta) \in D_{\bar{\kappa},\beta,\bar{T}}^{\mathrm{u}} \times \mathbb{T}_{\bar{\omega}}$:

$$|r_{11}^{\mathrm{u}}(u,\theta)| \leq K\left(\frac{\delta^{2p+6}}{|\cosh(\mathrm{d}u)|^4} + \frac{\delta^{p+4}}{|\cosh(\mathrm{d}u)|}\right).$$

□

3.4 Proof of Theorem 3.1.8

For convenience, we recall the definition of the coefficients $\Upsilon_0^{[l]}$ given in (3.71):

$$\Upsilon_0^{[l]} = \int_{-\infty}^{+\infty} \frac{e^{-il(\delta^{-1}\alpha w + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}w))}\mathcal{F}^{[l]}(0)(w)}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw, \tag{3.199}$$

where $\mathcal{F}^{[l]}(0)$ is the l-th Fourier coefficient of $\mathcal{F}(0)$ (see (3.64)). First of all let us note that, since for real values of (u,θ) the Melnikov function $M(u,\theta) \in \mathbb{R}$ (see (3.69) for its definition), one immediately has that:

$$\Upsilon_0^{[-l]} = \overline{\Upsilon_0^{[l]}}. \tag{3.200}$$

Hence, we just have to compute $\Upsilon_0^{[l]}$ with $l > 0$.

For $C \in \mathbb{R}$ and $l, n, Q \in \mathbb{N}$, we define the following integrals:

$$I_{n,Q}^{l,C} = \int_{-\infty}^{+\infty} \frac{e^{-\delta^{-1}\alpha i|l|s}\sinh^n(\mathrm{d}s)}{\cosh^{Q+1+iC|l|}(\mathrm{d}s)}ds, \qquad Q+1>n. \tag{3.201}$$

Let us denote by f_{qkmn}, g_{qkmn} and h_{qkmn} the Taylor coefficients of f, g and h respectively, namely:

$$f(\delta x, \delta y, \delta z, \delta, \delta\sigma) = \sum_{q=3}^{\infty} \delta^q \sum_{k+m+n\leq q} f_{qkmn}(\sigma) x^k y^m z^n, \tag{3.202}$$

and analogously for g and h. In the following we shall write f_{qkmn} instead of $f_{qkmn}(\sigma)$, but of course these coefficients still depend on σ. Note that one has $f_{qkmn} = f_{qkmn}(0) + \mathcal{O}(\sigma) = f_{qkmn}(0) + \mathcal{O}(\delta^{p+3})$, since we just consider the case $|\sigma| \leq \sigma^*\delta^{p+3}$.

Now, recalling the definition (3.64) of $\mathcal{F}$, the notation (3.37) and the definition (3.23) of F and H, we have:

$$\begin{aligned}\mathcal{F}(0) \;=\;& 2\sigma R_0(u) + \delta^p \Big[\sqrt{2R_0(u)}\cos\theta f(\delta\sqrt{2R_0(u)}\cos\theta, \delta\sqrt{2R_0(u)}\sin\theta, \delta Z_0(u), \delta, \delta\sigma) \\ & + \sqrt{2R_0(u)}\sin\theta g(\delta\sqrt{2R_0(u)}\cos\theta, \delta\sqrt{2R_0(u)}\sin\theta, \delta Z_0(u)\delta, \delta\sigma) \\ & + \frac{\mathrm{d}+1}{b} Z_0(u) h(\delta\sqrt{2R_0(u)}\cos\theta, \delta\sqrt{2R_0(u)}\sin\theta, \delta Z_0(u)\delta, \delta\sigma)\Big],\end{aligned} \tag{3.203}$$

where:

$$R_0(u) = \frac{(\mathrm{d}+1)}{2b}\frac{1}{\cosh^2(\mathrm{d}u)}, \qquad Z_0(u) = \tanh(\mathrm{d}u).$$

Denote by $a_{k,m}^{[l]}$ the l-th Fourier coefficient of the function $\cos^k\theta\sin^m\theta$. Then, substituting f, g and h for its Taylor series in (3.203), it can be seen that, for $l > 0$, $\Upsilon_0^{[l]}$ writes out as:

$$\begin{aligned}\Upsilon_0^{[l]} \;=\;& \delta^p \sum_{q=3}^{\infty}\sum_{k+m+n\leq q} \delta^q f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k+1,m}^{[l]} I_{n,k+m+n+2\mathrm{d}^{-1}}^{l,\mathrm{cd}^{-1}} \\ & + \delta^p \sum_{q=3}^{\infty}\sum_{k+m+n\leq q} \delta^q g_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k,m+1}^{[l]} I_{n,k+m+n+2\mathrm{d}^{-1}}^{l,\mathrm{cd}^{-1}} \\ & + \delta^p \sum_{q=3}^{\infty}\sum_{k+m+n\leq q} \delta^q h_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+2} a_{k,m}^{[l]} I_{n+1,k+m+n+2\mathrm{d}^{-1}}^{l,\mathrm{cd}^{-1}},\end{aligned} \tag{3.204}$$

with $I_{n,Q}^{l,C}$ defined in (3.201). A bound of these integrals for $|l| \geq 2$ can be easily found:

Lemma 3.4.1. *Let C be fixed. There exists a constant M such that for all $|l| \geq 2$, $Q \geq 1$ and n such that $Q+1>n$, if δ sufficiently small then:*

$$\left|I_{n,Q}^{l,C}\right| \leq M^{Q+1}\delta^{-Q} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}\frac{3|l|}{4}}.$$

Proof. Using Cauchy's theorem, one can easily see that the integration path of the integrals $I_{n,Q}^{l,C}$ (see (3.201)) can be changed to:

$$s = s(t) := -\frac{i}{\mathrm{d}}\left(\frac{\pi}{2} - \delta\right) + t, \qquad t \in (-\infty, \infty).$$

Then one obtains:

$$I_{n,Q}^{l,C} = e^{-\frac{\alpha}{\mathrm{d}}|l|\left(\frac{\pi}{2\delta}-1\right)} \int_{-\infty}^{+\infty} \frac{e^{-\delta^{-1}\alpha i|l|t} \sinh^n(\mathrm{d}s(t))}{\cosh^{Q+1+iC|l|}(\mathrm{d}s(t))} dt. \tag{3.205}$$

One can easily check that $\mathrm{Re}\cosh(\mathrm{d}s(t)) \geq 0$, and so $|\arg\cosh(s(t))| \leq \pi/2$. Using this and the fact that for $z \in \mathbb{C}$:

$$|z|^{Q+1+iC|l|} = |z|^{Q+1} e^{-C|l|\arg z} \geq |z|^{Q+1} e^{-|Cl\arg z|},$$

we obtain:

$$|\cosh^{Q+1+iC|l|}(\mathrm{d}s(t))| \geq |\cosh^{Q+1}(\mathrm{d}s(t))| e^{-|Cl|\frac{\pi}{2}}.$$

Using this bound in expression (3.205) of $I_{n,Q}^{l,C}$, and taking into account that $|e^{-\delta^{-1}\alpha i|l|t}| = 1$, one readily obtains:

$$\begin{aligned} \left|I_{n,Q}^{l,C}\right| &\leq e^{-\frac{\alpha}{\mathrm{d}}|l|\left(\frac{\pi}{2\delta}-1-|C|\frac{\pi}{2}\right)} \int_{-\infty}^{+\infty} \frac{|\sinh(\mathrm{d}s(t))|^n}{|\cosh(\mathrm{d}s(t))|^{Q+1}} dt \\ &= e^{-\frac{\alpha}{\mathrm{d}}|l|\left(\frac{\pi}{2\delta}-1-|C|\frac{\pi}{2}\right)} \left(\int_{-\infty}^{-1} \frac{|\sinh(\mathrm{d}s(t))|^n}{|\cosh(\mathrm{d}s(t))|^{Q+1}} dt + \int_{1}^{+\infty} \frac{|\sinh(\mathrm{d}s(t))|^n}{|\cosh(\mathrm{d}s(t))|^{Q+1}} dt \right) \\ &\quad + e^{-\frac{\alpha}{\mathrm{d}}|l|\left(\frac{\pi}{2\delta}-1-|C|\frac{\pi}{2}\right)} \int_{-1}^{1} \frac{|\sinh(\mathrm{d}s(t))|^n}{|\cosh(\mathrm{d}s(t))|^{Q+1}} dt. \end{aligned} \tag{3.206}$$

Now, on the one hand it is easy to see that:

$$|\sinh(\mathrm{d}s(t))|^n \leq e^{\mathrm{d}|\mathrm{Re}\, s(t)|n} = e^{\mathrm{d}|t|n}. \tag{3.207}$$

Moreover, for $|t| \geq 1$ one has $|\cosh(\mathrm{d}s(t))| \geq K e^{\mathrm{d}|\mathrm{Re}\, s(t)|} = K e^{\mathrm{d}|t|}$ for some constant K. This fact and bound (3.207) yield:

$$\int_{-\infty}^{-1} \frac{|\sinh(\mathrm{d}s(t))|^n}{|\cosh(\mathrm{d}s(t))|^{Q+1}} dt \leq K^{Q+1} \int_{-\infty}^{-1} e^{-\mathrm{d}|t|(Q+1-n)} dt = \frac{K^{Q+1}}{\mathrm{d}(Q+1-n)}, \tag{3.208}$$

where we have used that $Q+1-n > 0$ so that the integral converges. Analogously:

$$\int_{1}^{+\infty} \frac{|\sinh(\mathrm{d}s(t))|^n}{|\cosh(\mathrm{d}s(t))|^{Q+1}} dt \leq \frac{K^{Q+1}}{\mathrm{d}(Q+1-n)}. \tag{3.209}$$

Finally, for $t \in [-1,1]$ is easy to see that:

$$\frac{1}{|\cosh(\mathrm{d}s(t))|} \le K \frac{1}{\left|\mathrm{d}s(t) - i\frac{\pi}{2}\right|},$$

and:

$$\frac{|\sinh(\mathrm{d}s(t))|}{|\cosh(\mathrm{d}s(t))|} \le K \frac{1}{\left|\mathrm{d}s(t) - i\frac{\pi}{2}\right|}.$$

for some constant K. Then:

$$\int_{-1}^{1} \frac{|\sinh(\mathrm{d}s(t))|^n}{|\cosh(\mathrm{d}s(t))|^{Q+1}} dt \le K^{Q+1} \int_{-1}^{1} \frac{dt}{\left|\mathrm{d}s(t) - i\frac{\pi}{2}\right|^{Q+1}},$$

where again we have used that $Q+1 > n$. Since $Q \ge 1$, this yields:

$$\int_{-1}^{1} \frac{|\sinh(\mathrm{d}s(t))|^n}{|\cosh(\mathrm{d}s(t))|^{Q+1}} dt \le K^{Q+1}\delta^{-Q}. \tag{3.210}$$

Using bounds (3.208), (3.209) and (3.210) in equation (3.206) we obtain:

$$\left| I^{l,C}_{n,Q} \right| \le K^{Q+1}\delta^{-Q} e^{-\frac{\alpha}{\mathrm{d}}|l|\left(\frac{\pi}{2\delta} - 1 - |C|\frac{\pi}{2}\right)}.$$

The proof is finished taking δ sufficiently small so that:

$$\frac{\pi}{2\delta} - 1 - |C|\frac{\pi}{2} > \frac{3\pi}{8\delta}.$$

□

Our goal now will be to find an asymptotic formula for the integrals $I^{l,C}_{n,Q}$ with $l = 1$, which will dominate over the integrals with $|l| \ge 2$. First of all, we give a recurrence that is valid for all $l \ne 0$.

Lemma 3.4.2. *Let C be fixed. Then, for all $l \ne 0$, $n \ge 1$ and $Q > 0$ such that $Q+1 > n$, the following recurrence holds:*

$$I^{l,C}_{n,Q} = \frac{-|l|\alpha i}{\mathrm{d}\delta(Q + iC|l|)} I^{l,C}_{n-1,Q-1} + \frac{n-1}{Q+iC|l|} I^{l,C}_{n-2,Q-2}.$$

Proof. We just need to integrate by parts $I^{l,C}_{n-1,Q-1}$:

$$\begin{aligned}
I^{l,C}_{n-1,Q-1} &= \int_{-\infty}^{\infty} \frac{e^{-i\delta^{-1}\alpha|l|s}\sinh^{n-1}(\mathrm{d}s)}{\cosh^{Q+iC|l|}(\mathrm{d}s)} ds = \left[\frac{-1}{i\delta^{-1}\alpha|l|} e^{-i\delta^{-1}\alpha|l|s} \frac{\sinh^{n-1}(\mathrm{d}s)}{\cosh^{Q+iC|l|}(\mathrm{d}s)} \right]_{-\infty}^{\infty} \\
&\quad + \frac{\mathrm{d}(n-1)}{|l|\delta^{-1}\alpha i} \int_{-\infty}^{\infty} \frac{e^{-i\delta^{-1}\alpha|l|s}\sinh^{n-2}(\mathrm{d}s)}{\cosh^{Q-1+iC|l|}(\mathrm{d}s)} ds \\
&\quad - \frac{\mathrm{d}(Q+iC|l|)}{|l|\delta^{-1}\alpha i} \int_{-\infty}^{\infty} \frac{e^{-i\delta^{-1}\alpha|l|s}\sinh^{n}(\mathrm{d}s)}{\cosh^{Q+1+iC|l|}(\mathrm{d}s)} ds \\
&= \frac{\mathrm{d}(n-1)}{|l|\delta^{-1}\alpha i} I^{l,C}_{n-2,Q-2} - \frac{\mathrm{d}(Q+iC|l|)}{|l|\delta^{-1}\alpha i} I^{l,C}_{n,Q},
\end{aligned}$$

where we have used that, since $Q+1>n$, the first summand is zero. Now one just has to isolate $I^{l,C}_{n,Q}$ in the last equation to finish the proof. □

Now we summarize some properties of the Gamma function that will be needed later on.

Lemma 3.4.3. *Let $z, A \in \mathbb{C}$. Then:*

1. $\Gamma(z)\Gamma(\overline{z}) = |\Gamma(z)|^2$.
2. *(Stirling Formula) If $|\arg z| < \pi$, then:*
$$\Gamma(z) = e^{-z} e^{\left(z-\frac{1}{2}\right)\log z}(2\pi)^{\frac{1}{2}}(1+\mathcal{O}(z^{-1})).$$
3. *If $z = iy$, $y \in \mathbb{R}$, then:*
$$|\Gamma(iy)| = \frac{\sqrt{\pi}}{|y\sinh(\pi y)|^{1/2}}.$$
4. *If $|\arg z| < \pi$ and $|A| \le A^*$ for some constant A^*, then:*
$$\Gamma(z+A) = \Gamma(z) z^A (1+\mathcal{O}(z^{-1})).$$
5. *There exists a constant $M \ge 3/2$ and a function $J(z,A)$ such that for all $z \in \mathbb{C}$ with $|z| \ge 3$, $|\arg z| < \pi$, and all $A \in \mathbb{R}$ with $A \ge 1$, one has:*
$$\Gamma(z+A) = \Gamma(z) z^A (1 + z^{-1} J(z,A)),$$
and $|J(z,A)| \le M\Gamma(A)$.

Proof. All of the items above except item 5 are standard facts, see for instance in [AS72]. To prove item 5, fix $A^* \in \mathbb{R}$, $A^* \ge 3$. Then, for all $1 \le A \le A^*$ and $z \in \mathbb{C}$ with $|\arg z| < \pi$, item 5 is a consequence of item 4. For $A \ge A^*$, we proceed by induction over the integer part of A (clearly all $A \ge 1$ can be written as $A = \tilde{A} + n$, with $1 \le \tilde{A} \le A^*$, $n \in \mathbb{N}$). Assume that item 5 holds for a given A. Then:

$$\begin{aligned}\Gamma(z+A+1) &= (z+A)\Gamma(z+A) = (z+A)\Gamma(z)z^A\left(1+\frac{1}{z}J(z,A)\right) \\ &= \Gamma(z)z^{A+1}\left(1+\frac{1}{z}J(z,A)+\frac{A}{z}+\frac{A}{z^2}J(z,A)\right). \end{aligned} \tag{3.211}$$

By hypothesis of induction $|J(z,A)| \le M\Gamma(A)$. Then, since $\Gamma(A+1) = A\Gamma(A)$, one has:

$$\left|\frac{1}{z}J(z,A)+\frac{A}{z}+\frac{A}{z^2}J(z,A)\right| \le \frac{M}{|z|}\Gamma(A+1)\left(\frac{1}{A}+\frac{A}{M\Gamma(A+1)}+\frac{1}{|z|}\right).$$

Clearly, since $A \geq A^* \geq 3$ and $|z| \geq 3$, we just need to prove that:

$$\frac{A}{M\Gamma(A+1)} \leq \frac{1}{3} \tag{3.212}$$

and then (3.211) yields item 5. To check that (3.212) holds we just need to note that, since $A \geq 3$, then $\Gamma(A) \geq \Gamma(3) = 2$, and one has $\Gamma(A+1) = A\Gamma(A) \geq 2A$. This implies (3.212) since $M \geq 3/2$. □

Next, we find an asymptotic formula for $I_{0,Q}^{1,C}$.

Lemma 3.4.4. *Let C be fixed. For all $Q \geq 1$ one has:*

$$I_{0,Q}^{1,C} = \frac{2\pi}{\mathrm{d}}\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q+iC}\frac{e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}}{\Gamma(Q+1+iC)} + \mathcal{O}\left(\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q-1}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\right).$$

Proof. Performing the change of variables $w = \tanh(\mathrm{d}s)$, and using that $\cosh^2(\mathrm{d}s) = 1-\tanh^2(\mathrm{d}s)$ and $e^{2\mathrm{d}s} = (1+\tanh(\mathrm{d}s))/(1-\tanh(\mathrm{d}s))$, one has that:

$$I_{0,Q}^{1,C} = \frac{1}{\mathrm{d}}\int_{-1}^{1}(1+w)^{\frac{\mathrm{d}(Q-1+iC)-i\delta^{-1}\alpha}{2\mathrm{d}}}(1-w)^{\frac{\mathrm{d}(Q-1+iC)+i\delta^{-1}\alpha}{2\mathrm{d}}}dw.$$

Naming:

$$\begin{aligned} a &= \frac{\mathrm{d}(Q+1+iC)+i\delta^{-1}\alpha}{2\mathrm{d}}, \\ b &= Q+1+iC, \end{aligned}$$

we can rewrite the last equation as:

$$I_{0,Q}^{1,C} = \frac{1}{\mathrm{d}}\int_{-1}^{1}(1+w)^{b-a-1}(1-w)^{a-1}dw = 2^{b-1}\mathrm{d}^{-1}\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)}M(a,b,0),$$

where we have used a well known formula involving the Gamma function and the confluent hypergeometric function $M(a,b,z)$ (see for instance [AS72]). Moreover we have that for all a and b, $M(a,b,0) = 1$, so that we can write:

$$I_{0,Q}^{1,C} = 2^{Q+iC}\mathrm{d}^{-1}\frac{\Gamma_Q^C}{\Gamma(Q+1+iC)}. \tag{3.213}$$

where:

$$\Gamma_Q^C := \Gamma(b-a)\Gamma(a). \tag{3.214}$$

We now shall find an asymptotic expression for Γ_Q^C. Let:

$$A = \frac{Q+1}{2} \geq 1, \qquad z_{\pm} = i\frac{\mathrm{d}C \pm \delta^{-1}\alpha}{2\mathrm{d}},$$

so that $b-a=A+z_-$ and $a=A+z_+$. We note that $|\arg z_\pm|=\pi/2<\pi$ and that for sufficiently small δ one has $|z_\pm|\geq 3$. Then, by item 5 of Lemma 3.4.3 we have:

$$\begin{aligned}\Gamma_Q^C &= \Gamma(A+z_-)\Gamma(A+z_+)\\ &= z_+^A z_-^A\Gamma(z_-)\Gamma(z_+)\left(1+\frac{1}{z_+}J(z_+,A)\right)\left(1+\frac{1}{z_-}J(z_-,A)\right),\end{aligned}\tag{3.215}$$

where $|J(z_\pm,A)|\leq M\Gamma(A)$.

First we note that:

$$\begin{aligned}z_+^A z_-^A &= \left(i\frac{\mathrm{d}C+\delta^{-1}\alpha}{2\mathrm{d}}\right)^{\frac{Q+1}{2}}\left(i\frac{\mathrm{d}C-\delta^{-1}\alpha}{2\mathrm{d}}\right)^{\frac{Q+1}{2}}\\ &= \left(\frac{\alpha}{2\mathrm{d}\delta}\right)^{Q+1}\left(1-\frac{\mathrm{d}^2C^2\delta^2}{\alpha^2}\right)^{\frac{Q+1}{2}}.\end{aligned}\tag{3.216}$$

Now we compute the product $\Gamma(z_-)\Gamma(z_+)$. We shall prove that:

$$\Gamma(z_-)\Gamma(z_+)=2\pi\left(\frac{\alpha}{2\mathrm{d}\delta}\right)^{iC-1}e^{-\frac{\pi\alpha}{2\mathrm{d}\delta}}(1+\mathcal{O}(\delta)).\tag{3.217}$$

To prove (3.217), we first use item 3 of Lemma 3.4.3 for $\Gamma(z_-)$ and we get:

$$\begin{aligned}\Gamma(z_-)\Gamma(z_+) &= |\Gamma(z_-)|^2\frac{\Gamma(z_+)}{\Gamma(\overline{z_-})}=\frac{\pi}{|z_-\sinh(\pi\mathrm{Im}\,z_-)|}\frac{\Gamma(z_+)}{\Gamma(\overline{z_-})}\\ &= 4\pi\frac{\mathrm{d}\delta}{\alpha}e^{-\frac{\pi\alpha}{2\mathrm{d}\delta}}e^{\frac{\pi|C|}{2}}\frac{\Gamma(z_+)}{\Gamma(\overline{z_-})}(1+\mathcal{O}(\delta)).\end{aligned}\tag{3.218}$$

Using item 2 of Lemma 3.4.3 we have:

$$\frac{\Gamma(z_+)}{\Gamma(\overline{z_-})}=e^{-z_++\overline{z_-}}e^{(z_+-\frac12)\log(z_+)-(\overline{z_-}-\frac12)\log(\overline{z_-})}\left(1+\mathcal{O}\left(|\overline{z_-}|^{-1}+|z_+|^{-1}\right)\right)\tag{3.219}$$

We note that $\overline{z_-}=z_+-iC$, and then:

$$\left(\overline{z_-}-\frac12\right)\log(\overline{z_-})=\left(z_+-\frac12-iC\right)\log\left(z_+(1-iCz_+^{-1})\right).$$

We note that, since z_+ is purely imaginary, $1-iCz_+^{-1}=1-C|z_+|^{-1}$ is real. Then, using that for $\beta_1,\beta_2>0$, one has $\log(i\beta_1\beta_2)=\log(i\beta_1)+\log(\beta_2)$, the last equation writes out as:

$$\left(\overline{z_-}-\frac12\right)\log(\overline{z_-})=\left(z_+-\frac12-iC\right)\left(\log(z_+)+\log\left(1-C|z_+|^{-1}\right)\right)\tag{3.220}$$

Since $|z_+|^{-1} = \mathcal{O}(\delta)$ one has:

$$\log\left(1 - C|z_+|^{-1}\right) = -C|z_+|^{-1}(1+\mathcal{O}(\delta)).$$

Then, using also that $z_+/|z_+| = i$, we can write (3.220) as:

$$\left(\overline{z_-} - \frac{1}{2}\right)\log(\overline{z_-}) = \left(z_+ - \frac{1}{2}\right)\log(z_+) - iC\log(z_+) - iC + \mathcal{O}(\delta). \tag{3.221}$$

Using (3.221) in (3.219), and recalling that $-z_+ + \overline{z_-} = -iC$ one obtains:

$$\begin{aligned}\frac{\Gamma(z_+)}{\Gamma(\overline{z_-})} &= e^{-iC}e^{iC\log(z_+)+iC+\mathcal{O}(\delta)}\left(1+\mathcal{O}\left(\delta\right)\right)\\ &= \left(i\frac{\mathrm{d}C+\delta^{-1}\alpha}{2\mathrm{d}}\right)^{iC}\left(1+\mathcal{O}\left(\delta\right)\right)\\ &= \left(\frac{i\alpha}{2\mathrm{d}\delta}\right)^{iC}\left(1+\mathcal{O}\left(\delta\right)\right).\end{aligned} \tag{3.222}$$

Substituting (3.222) in (3.218) and noting that $i^{iC} = e^{-\frac{\pi C}{2}}$ we obtain (3.217).

Now, we claim that:

$$\left(1+\frac{1}{z_+}J(z_+,A)\right)\left(1+\frac{1}{z_-}J(z_-,A)\right) = 1 + |\Gamma(Q+1+iC)|e^{\frac{\pi|C|}{2}}\mathcal{O}\left(\delta\right). \tag{3.223}$$

Indeed, recalling that $z_\pm^{-1} = \mathcal{O}(\delta)$, $|J(z_\pm,A)| \leq M|\Gamma(A)|$ and that $A = (Q+1)/2$, we have:

$$\begin{aligned}&\left|\left(1+\frac{1}{z_+}J(z_+,A)\right)\left(1+\frac{1}{z_-}J(z_-,A)\right)-1\right|\\ \leq\ & K\left(\delta\Gamma\left(\frac{Q+1}{2}\right)+\delta^2\Gamma^2\left(\frac{Q+1}{2}\right)\right),\end{aligned} \tag{3.224}$$

for some constant K. We note that if $Q \geq 1$ then:

$$\Gamma\left(\frac{Q+1}{2}\right) \leq \left[\Gamma(Q+1)\right]^{1/2} \leq \Gamma(Q+1),$$

and so (3.224) writes out as:

$$\left|\left(1+\frac{1}{z_+}J(z_+,A)\right)\left(1+\frac{1}{z_-}J(z_-,A)\right)-1\right| \leq K\delta\Gamma(Q+1). \tag{3.225}$$

On the one hand, for $C = 0$ it is clear that (3.225) yields (3.223). On the other hand, for $C \neq 0$, using the property that $\Gamma(z+1) = z\Gamma(z)$ it is easy to see that:

$$\Gamma(Q+1) \leq \frac{|\Gamma(Q+1+iC)|}{|C\Gamma(iC)|}. \tag{3.226}$$

Thus, using item 3 of Lemma 3.4.3 we obtain:

$$\Gamma(Q+1) \le \frac{|\Gamma(Q+1+iC)||\sinh(\pi C)|^{1/2}}{(\pi|C|)^{1/2}} \le K\Gamma(Q+1+iC)e^{\frac{\pi|C|}{2}}. \qquad (3.227)$$

Equations (3.225) and (3.227) yield (3.223).

Finally, substituting (3.216), (3.217) and (3.223) in equation (3.215) we obtain:

$$\Gamma_Q^C = 2\pi\left(\frac{\alpha}{2\mathrm{d}\delta}\right)^{Q+iC} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\left(1+|\Gamma(Q+1+iC)|e^{\frac{\pi|C|}{2}}\mathcal{O}(\delta)\right)(1+\mathcal{O}(\delta)),$$

and then (3.213) yields:

$$I_{0,Q}^{1,C} = \frac{2\pi}{\mathrm{d}}\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q+iC}\frac{e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}}{|\Gamma(Q+1+iC)|}\left(1+|\Gamma(Q+1+iC)|e^{\frac{\pi|C|}{2}}\mathcal{O}(\delta)\right)(1+\mathcal{O}(\delta)). \qquad (3.228)$$

Since $|\Gamma(Q+1+iC)| \ge K > 0$ for some constant K, formula (3.228) yields the claim of the lemma. □

Finally, we can give an asymptotic formula of $I_{n,Q}^{1,C}$ for all $n \ge 0$.

Lemma 3.4.5. *Let C be fixed. Then for all $Q \ge 1$ and $n \ge 0$ such that $Q+1 > n$ one has:*

$$I_{n,Q}^{1,C} = \frac{2\pi}{\mathrm{d}}\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q+iC}\frac{(-i)^n}{\Gamma(Q+1+iC)}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} + \mathcal{O}\left(\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q-1}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\right).$$

Proof. First we point out that if $n = 0$ the statement is proved in Lemma 3.4.4. For $n \ge 1$ and $Q+1 > n$ we proceed by induction, using the recurrence of Lemma 3.4.2.

Let us assume that the lemma holds for $I_{n-1,Q-1}^{1,C}$ and $I_{n-2,Q-2}^{1,C}$ (note that in the case $n = 1$, the recurrence of Lemma 3.4.2 just involves the integral $I_{0,Q-1}^{l,C}$, so we can proceed in the same way as for $n \ge 2$). Using the recurrence of Lemma 3.4.2 we obtain:

$$\begin{aligned} I_{n,Q}^{1,C} &= \frac{-\alpha i}{\delta\mathrm{d}(Q+iC)}I_{n-1,Q-1}^{1,C} + \frac{n-1}{Q+iC}I_{n-2,Q-2}^{1,C} \\ &= -\frac{\alpha i}{\delta\mathrm{d}(Q+iC)}\frac{2\pi}{\mathrm{d}}\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q-1+iC}\frac{(-i)^{n-1}}{\Gamma(Q+iC)}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \\ &\quad +\frac{n-1}{Q+iC}\frac{2\pi}{\mathrm{d}}\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q-2+iC}\frac{(-i)^{n-2}}{\Gamma(Q-1+iC)}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} + \mathcal{O}\left(\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q-1}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\right). \end{aligned}$$

Since $z\Gamma(z) = \Gamma(z+1)$, it is clear that the first term in the sum coincides with the dominant term of the asymptotic expression in item 1 above. Thus, we just have to see that the size of the second term coincides with the error terms. This is clear, since for $Q+1 > n$ one has:

$$\left|\frac{n-1}{Q+iC}\right| \le K, \qquad \left|\frac{1}{\Gamma(Q-1+iC)}\right| \le K$$

for some constant K. Thus, the second term in the sum can be bounded by:

$$\left|\frac{n-1}{Q+iC}\frac{2\pi}{\mathrm{d}}\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q-2+iC}\frac{(-i)^{n-2}}{\Gamma(Q-1+iC)}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\right|\leq K\left(\frac{\alpha}{\mathrm{d}\delta}\right)^{Q-2}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}},$$

which is smaller than the error terms. □

End of the proof of Theorem 3.1.8. First we focus on $\Upsilon_0^{[1]}$. We shall study the first sum appearing in formula (3.204) of $\Upsilon_0^{[l]}$ taking $l=1$, the other two are done analogously. We can write this first sum as:

$$\begin{aligned}
&\delta^p \sum_{q=3}^{\infty}\sum_{k+m+n\leq q}\delta^q f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k+1,m}^{[1]} I_{n,k+m+n+2\mathrm{d}^{-1}}^{1,c\mathrm{d}^{-1}}\\
&= \delta^p \sum_{q=3}^{\infty}\sum_{k+m+n=q}\delta^q f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k+1,m}^{[1]} I_{n,q+2\mathrm{d}^{-1}}^{1,c\mathrm{d}^{-1}}\\
&\quad +\delta^p \sum_{q=3}^{\infty}\sum_{k+m+n<q}\delta^q f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k+1,m}^{[1]} I_{n,k+m+n+2\mathrm{d}^{-1}}^{1,c\mathrm{d}^{-1}}
\end{aligned} \tag{3.229}$$

On the one hand, using Lemma 3.4.5 with $C=c\mathrm{d}^{-1}$ and $Q=q+2\mathrm{d}^{-1}$, the first term in (3.229) writes out as:

$$\begin{aligned}
&\delta^p \sum_{q=3}^{\infty}\sum_{k+m+n=q}\delta^q f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k+1,m}^{[1]} I_{n,q+2\mathrm{d}^{-1}}^{1,c\mathrm{d}^{-1}}\\
&= \frac{2\pi}{\mathrm{d}}\delta^{p-\frac{2}{\mathrm{d}}-i\frac{c}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\sum_{q=3}^{\infty}\sum_{k+m+n=q}\frac{f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}(-i)^n a_{k+1,m}^{[1]}\alpha^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\mathrm{d}^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)}\\
&\quad +\delta^{p-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\sum_{q=3}^{\infty}\sum_{k+m+n=q}\frac{f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k+1,m}^{[1]}\alpha^{q-1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\mathrm{d}^{q-1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}\mathcal{O}(\delta)\\
&= \frac{2\pi}{\mathrm{d}}\delta^{p-\frac{2}{\mathrm{d}}-i\frac{c}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\sum_{q=3}^{\infty}\sum_{k+m+n=q}\frac{f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}(-i)^n a_{k+1,m}^{[1]}\alpha^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\mathrm{d}^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)}\\
&\quad +\mathcal{O}\left(\delta^{p+1-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\right),
\end{aligned} \tag{3.230}$$

where in the last step we have used that $|a_{k,m}^{[1]}|\leq 1$ for all k and m (because $a_{k,m}^{[1]}$ are Fourier coefficients of the functions $\cos^k\theta\sin^m\theta$), and we have assumed that the radius of convergence of f is sufficiently large and thus second sum converges. Similarly, using again Lemma 3.4.5 with $C = c\mathrm{d}^{-1}$ and $Q = k+m+n+2\mathrm{d}^{-1}$ it is easy to see that:

$$\begin{aligned} \delta^q \left| I^{1,c\mathrm{d}^{-1}}_{n,k+m+n+2\mathrm{d}^{-1}} \right| &\leq \delta^{q-(k+m+n+\frac{2}{\mathrm{d}})} \left(\frac{\alpha}{\mathrm{d}}\right)^{k+m+n+\frac{2}{\mathrm{d}}} \frac{e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}}{\left|\Gamma\left(k+m+n+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)\right|} \\ &\leq \delta^{q-(k+m+n+\frac{2}{\mathrm{d}})} \left(\frac{\alpha}{\mathrm{d}}\right)^{k+m+n+\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}, \end{aligned}$$

where in the second step we have used that for all $k+m+n \geq 3$ one has:

$$\left|\Gamma\left(k+m+n+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)\right| \geq K > 0$$

for some constant K. Then, the second term in (3.229) can be bounded by:

$$\begin{aligned} &\left| \delta^p \sum_{q=3}^{\infty} \sum_{k+m+n<q} \delta^q f_{qkmn} \left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a^{[1]}_{k+1,m} I^{1,c\mathrm{d}^{-1}}_{n,k+m+n+2\mathrm{d}^{-1}} \right| \\ \leq\ & K\delta^{p-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \sum_{q=3}^{\infty} \sum_{k+m+n<q} \delta^{q-(k+m+n)} \left| \frac{f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a^{[1]}_{k+1,m}\alpha^{q+\frac{2}{\mathrm{d}}}}{\mathrm{d}^{q+\frac{2}{\mathrm{d}}}} \right| \\ \leq\ & K\delta^{p+1-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}, \end{aligned} \tag{3.231}$$

where we have taken into account that $q-(k+m+n) \geq 1$ and that the radius of convergence of f is sufficiently large so that the sum converges. Using (3.230) and (3.231) in (3.229) we obtain the following expression for the first sum in formula (3.204) (with $l = 1$):

$$\begin{aligned} &\delta^p \sum_{q=3}^{\infty} \sum_{k+m+n\leq q} \delta^q f_{qkmn} \left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a^{[1]}_{k+1,m} I^{1,c\mathrm{d}^{-1}}_{n,k+m+n+2\mathrm{d}^{-1}} \\ =\ & \frac{2\pi}{\mathrm{d}} \delta^{p-\frac{2}{\mathrm{d}}-i\frac{c}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \sum_{q=3}^{\infty} \sum_{k+m+n=q} \frac{f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} (-i)^n a^{[1]}_{k+1,m} \alpha^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\mathrm{d}^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)} \\ &+\mathcal{O}\left(\delta^{p+1-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\right). \end{aligned}$$

Reasoning analogously for the other sums appearing in formula (3.204) of $\Upsilon_0^{[1]}$ we obtain:

$$\begin{aligned}
\Upsilon_0^{[1]} &= \frac{2\pi}{\mathrm{d}}\delta^{p-\frac{2}{\mathrm{d}}-i\frac{c}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\left[\sum_{q=3}^{\infty}\sum_{k+m+n=q}\frac{f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}(-i)^n a_{k+1,m}^{[1]}\alpha^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\mathrm{d}^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)}\right.\\
&\quad+\sum_{q=3}^{\infty}\sum_{k+m+n=q}\frac{g_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}(-i)^n a_{k,m+1}^{[1]}\alpha^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\mathrm{d}^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)}\\
&\quad\left.+\sum_{q=3}^{\infty}\sum_{k+m+n=q}\frac{h_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+2}(-i)^{n+1} a_{k,m}^{[1]}\alpha^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\mathrm{d}^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)}\right]\\
&\quad+\mathcal{O}\left(\delta^{p+1-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\right). \qquad (3.232)
\end{aligned}$$

Now we shall see that expression (3.232) of $\Upsilon_0^{[1]}$ can be written in terms of the Borel transform of the function:

$$\begin{aligned}
m(w,\theta) &= w^{\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}F\left(\tfrac{\mathrm{d}+1}{2b}w^2,\theta,-iw,0,0\right)-i\tfrac{\mathrm{d}+1}{b}w^{1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}H\left(\tfrac{\mathrm{d}+1}{2b}w^2,\theta,-iw,0,0\right)\\
&= \sqrt{\frac{\mathrm{d}+1}{b}}w^{1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}\cos\theta f\left(\sqrt{\frac{\mathrm{d}+1}{b}}w\cos\theta,\sqrt{\frac{\mathrm{d}+1}{b}}w\sin\theta,-iw,0,0)\right)\\
&\quad+\sqrt{\frac{\mathrm{d}+1}{b}}w^{1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}\sin\theta g\left(\sqrt{\frac{\mathrm{d}+1}{b}}w\cos\theta,\sqrt{\frac{\mathrm{d}+1}{b}}w\sin\theta,-iw,0,0)\right)\\
&\quad-i\frac{\mathrm{d}+1}{b}w^{1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}h\left(\sqrt{\frac{\mathrm{d}+1}{b}}w\cos\theta,\sqrt{\frac{\mathrm{d}+1}{b}}w\sin\theta,-iw,0,0)\right).
\end{aligned}$$

Here, in the last equality we have just used the definitions of F and H given in (3.23). Now, we substitute f, g and h by its Taylor series (see (3.202) for the expression of f, the ones for g and h are equivalent). From (3.202) one can see that, taking the two last variables in f equal to zero implies that the second sum is done only over the terms

$k+m+n=q$. Then we obtain:

$$\begin{aligned} m(w,\theta) \;=\; & \sum_{q=3}^{\infty}\sum_{k+m+n=q} f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}(-i)^n\cos^{k+1}\theta\sin^m\theta w^{q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}} \\ & +\sum_{q=3}^{\infty}\sum_{k+m+n=q} g_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}(-i)^n\cos^{k}\theta\sin^{m+1}\theta w^{q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}} \\ & +\sum_{q=3}^{\infty}\sum_{k+m+n=q} h_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+2}(-i)^{n+1}\cos^{k}\theta\sin^{m}\theta w^{q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}. \end{aligned} \tag{3.233}$$

Now we recall that for a function $m(w,\theta)$ of the form:

$$m(w,\theta)=\sum_{n\geq 0} m_n(\theta)w^{n+1+ik}$$

its Borel transform $\hat{m}(\zeta,\theta)$ is defined as:

$$\hat{m}(\zeta,\theta)=\sum_{n\geq 0} m_n(\theta)\frac{\zeta^{n+ik}}{\Gamma(n+1+ik)},$$

and its Fourier coefficients $\hat{m}^{[l]}(\zeta)$ are then given by:

$$\hat{m}^{[l]}(\zeta)=\sum_{n\geq 0} m_n^{[l]}\frac{\zeta^{n+ik}}{\Gamma(n+1+ik)}.$$

Thus, the Fourier coefficient $\hat{m}^{[1]}(\zeta)$ of the Borel transform of the function $m(w,\theta)$ defined in (3.233) is:

$$\begin{aligned} \hat{m}^{[1]}(\zeta) \;=\; & \sum_{q=3}^{\infty}\sum_{k+m+n=q} \frac{f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}(-i)^n a_{k+1,m}^{[1]}\zeta^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)} \\ & +\sum_{q=3}^{\infty}\sum_{k+m+n=q} \frac{g_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}(-i)^n a_{k,m+1}^{[1]}\zeta^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)} \\ & +\sum_{q=3}^{\infty}\sum_{k+m+n=q} \frac{h_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+2}(-i)^{n+1} a_{k,m}^{[1]}\zeta^{q+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}}}{\Gamma\left(q+1+\frac{2}{\mathrm{d}}+i\frac{c}{\mathrm{d}}\right)}, \end{aligned} \tag{3.234}$$

where we recall that $a_{k,m}^{[1]}$ denotes the first Fourier coefficient of the function $\cos^k\theta\sin^m\theta$. Comparing expressions (3.232) and (3.234) it is clear that: it is clear that:

$$\Upsilon_0^{[1]} = \frac{2\pi}{\mathrm{d}}\delta^{p-\frac{2}{\mathrm{d}}-i\frac{c}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\left(\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right)+\mathcal{O}(\delta)\right).$$

To finish the first part of the theorem we just need to use that $\Upsilon_0^{[-1]} = \overline{\Upsilon_0^{[-1]}}$.

Now, in order to prove bound (3.73), we take formula (3.204) of $\Upsilon_0^{[l]}$ and use the bounds provided by Lemma 3.4.1, with $C = c\mathrm{d}^{-1}$ and $Q = k+m+n+2\mathrm{d}^{-1}$. Then, for $|l|\geq 2$ one obtains:

$$\begin{aligned}
&\left|\Upsilon_0^{[l]}\right| \\
\leq\ & K\delta^{p-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}\frac{3|l|}{4}}\left(\sum_{q=3}^{\infty}\sum_{k+m+n\leq q}M^{k+m+n+\frac{2}{\mathrm{d}}}\delta^{q-(k+m+n)}\left|f_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}a_{k+1,m}^{[l]}\right|\right.\\
&+\sum_{q=3}^{\infty}\sum_{k+m+n\leq q}M^{k+m+n+\frac{2}{\mathrm{d}}}\delta^{q-(k+m+n)}\left|g_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1}a_{k,m+1}^{[l]}\right|\\
&\left.+\sum_{q=3}^{\infty}\sum_{k+m+n\leq q}M^{k+m+n+\frac{2}{\mathrm{d}}}\delta^{q-(k+m+n)}\left|h_{qkmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+2}a_{k,m}^{[l]}\right|\right)\\
\leq\ & K\delta^{p-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}\frac{3|l|}{4}},
\end{aligned}$$

where in the last inequality we have used that $q-(k+m+n)\geq 0$ and that f, g and h are analytic in a ball of radius sufficiently large, and therefore the series converge. Thus, bound (3.73) is proved.

Finally, to prove the asymptotic expression (3.74) of $M(u,\theta)$, we first take expression (3.70) and use bounds (3.73) of $\Upsilon_0^{[l]}$ with $|l|\geq 2$. Then, for $u\in\mathbb{R}$ and $\theta\in\mathbb{S}^1$, one has that:

$$\begin{aligned}
M(u,\theta) &= \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\left[\Upsilon_0^{[0]}+\Upsilon_0^{[1]}e^{i(\theta+\delta^{-1}\alpha u+c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))}\right.\\
&\left.+\Upsilon_0^{[-1]}e^{-i(\theta+\delta^{-1}\alpha u+c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))}+\mathcal{O}\left(\delta^{p-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}\frac{3}{2}}\right)\right]. \qquad (3.235)
\end{aligned}$$

Using the asymptotic formulas for $\Upsilon_0^{[1]}$ and $\Upsilon_0^{[-1]}$, the fact that $\delta^{-i\frac{c}{\mathrm{d}}} = e^{-i\frac{c}{\mathrm{d}}\log\delta}$ and denoting:

$$\vartheta(u,\delta) = \delta^{-1}\alpha u + c\mathrm{d}^{-1}[\log\cosh(\mathrm{d}u)-\log\delta]$$

one has:

$$\Upsilon_0^{[1]}e^{i(\theta+\delta^{-1}\alpha u+c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))} = \frac{2\pi}{\mathrm{d}}\delta^{p-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}e^{i(\theta+\vartheta(u,\delta))}\left(\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right)+\mathcal{O}(\delta)\right),$$

and:

$$\Upsilon_0^{[-1]} e^{-i(\theta+\delta^{-1}\alpha u + \mathrm{c}\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))} = \frac{2\pi}{\mathrm{d}}\delta^{p-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} e^{-i(\theta+\vartheta(u,\delta))}\left(\overline{\hat{m}^{[1]}\left(\frac{\alpha}{\mathrm{d}}\right)} + \mathcal{O}(\delta)\right).$$

Using these expressions in (3.235) and the fact that:

$$ze^{ix} + \overline{z}e^{-ix} = 2\mathrm{Re}\, z\cos x - 2\mathrm{Im}\, z \sin x$$

we obtain directly expression (3.74). □

3.5 Proof of Theorem 3.1.9

In this section we will prove Theorem 3.1.9. As we already mentioned in Section 3.1.6, the main idea is to find a suitable PDE for $\Delta(u,\theta) = r_1^{\mathrm{u}}(u,\theta) - r_1^{\mathrm{s}}(u,\theta)$, and study all the solutions of this equation. We recall that we will consider $(u,\theta)\in D_{\kappa,\beta}\times\mathbb{T}_\omega$, where (see Figure 3.3):

$$D_{\kappa,\beta} = D^{\mathrm{u}}_{\kappa,\beta,T}\cap D^{\mathrm{s}}_{\kappa,\beta,T}. \tag{3.236}$$

To obtain a PDE for Δ, we subtract the PDEs for r_1^{u} and r_1^{s}, given in (3.62). Then, making use of the mean value theorem, one can easily see that Δ satisfies:

$$\begin{aligned} &\left(-\delta^{-1}\alpha - cZ_0(u)\right)\partial_\theta\Delta + \partial_u\Delta - 2Z_0(u)\Delta \\ =\; &(2\sigma + l_1(u,\theta))\Delta + l_2(u,\theta)\partial_u\Delta + l_3(u,\theta)\partial_\theta\Delta, \end{aligned} \tag{3.237}$$

where, denoting $r_\lambda = (r_1^{\mathrm{u}} + r_1^{\mathrm{s}})/2 + \lambda(r_1^{\mathrm{u}} - r_1^{\mathrm{s}})/2$, the functions l_i are defined as:

$$\begin{aligned} l_1(u,\theta) &= \frac{\delta^p}{2}\int_{-1}^1 \partial_r F(r_\lambda)d\lambda + \frac{\delta^p(\mathrm{d}+1)}{2b}Z_0(u)\int_{-1}^1 \partial_r H(r_\lambda)d\lambda - \frac{\delta^p}{2}\int_{-1}^1\partial_r G(r_\lambda)\partial_\theta r_\lambda d\lambda \\ &\quad -\frac{b}{\mathrm{d}(1-Z_0^2(u))}(\partial_u r_1^{\mathrm{u}} + \partial_u r_1^{\mathrm{s}}) - \frac{\delta^p}{2\mathrm{d}(1-Z_0^2(u))}\int_{-1}^1 \partial_r H(r_\lambda)\partial_u r_\lambda d\lambda, \end{aligned} \tag{3.238}$$

$$l_2(u,\theta) = -\frac{b}{\mathrm{d}(1-Z_0^2(u))}(r_1^{\mathrm{u}} + r_1^{\mathrm{s}}) - \frac{\delta^p}{2\mathrm{d}(1-Z_0^2(u))}\int_{-1}^1 H(r_\lambda)d\lambda, \tag{3.239}$$

$$l_3(u,\theta) = -\frac{\delta^p}{2}\int_{-1}^1 G(r_\lambda)d\lambda. \tag{3.240}$$

First of all we introduce the Banach spaces in which we will solve equation (3.237), and give some bounds of the functions l_i which will be needed later on.

3.5.1 Banach spaces and technical lemmas

We begin by defining the Banach spaces in which we will work. We will consider functions $\phi : D_{\kappa,\beta} \times \mathbb{T}_\omega \to \mathbb{C}$. Again, they can be written in their Fourier series:

$$\phi(v,\theta) = \sum_{l\in\mathbb{Z}} \phi^{[l]}(v) e^{il\theta}.$$

Then we define the norms:

$$\|\phi\|_{n,\omega} := \sum_{l\in\mathbb{Z}} \left\|\phi^{[l]}\right\|_n e^{|l|\omega},$$

where:

$$\left\|\phi^{[l]}\right\|_n = \sup_{v\in D_{\kappa,\beta}} \left|\cosh^n(\mathrm{d}v)\phi^{[l]}(v)\right|.$$

As before, we also define the norms:

$$\llbracket\phi\rrbracket_{n,\omega} := \|\phi\|_{n,\omega} + \|\partial_v\phi\|_{n+1,\omega} + \delta^{-1}\|\partial_\theta\phi\|_{n+1,\omega}. \tag{3.241}$$

We will consider the Banach spaces endowed with this norms:

$$\mathcal{X}_{n,\omega} := \left\{\phi : D_{\kappa,\beta} \times \mathbb{T}_\omega \to \mathbb{C} \,:\, \phi(v,\theta) \text{ is analytic, such that } \|\phi\|_{n,\omega} < +\infty\right\},$$

and:

$$\tilde{\mathcal{X}}_{n,\omega} := \left\{\phi : D_{\kappa,\beta} \times \mathbb{T}_\omega \to \mathbb{C} \,:\, \phi(v,\theta) \text{ is analytic, such that } \llbracket\phi\rrbracket_{n,\omega} < +\infty\right\}.$$

Remark 3.5.1. From Theorem 3.1.7, it is straightforward to see that $r_1^{\mathrm{u,s}}$ satisfy $r_1^{\mathrm{u,s}} \in \tilde{\mathcal{X}}_{3,\omega}$ and that there exists a constant M such that:

$$\llbracket r_1^{\mathrm{u,s}}\rrbracket_{3,\omega} \le M\delta^{p+3}.$$

In the following subsections we will need to have bounds of the perturbative terms l_i appearing in equation (3.237). The next lemma provides such bounds.

Lemma 3.5.2. *Let $l_i(u,\theta)$, $i = 1,2,3$, be the functions defined respectively in* (3.238), (3.239) *and* (3.240). *Then there exists a constant M such that:*

$$\|l_1\|_{2,\omega} \le M\delta^{p+3}, \qquad \|l_2\|_{1,\omega} \le M\delta^{p+3}, \qquad \|l_3\|_{2,\omega} \le M\delta^{p+3}.$$

Proof. First of all, recall that Remark 3.5.1 gives:

$$\|r_1^{\mathrm{u,s}}\|_{3,\omega} \le M\delta^{p+3}, \quad \|\partial_u r_1^{\mathrm{u,s}}\|_{4,\omega} \le M\delta^{p+3}, \quad \|\partial_\theta r_1^{\mathrm{u,s}}\|_{4,\omega} \le M\delta^{p+4}.$$

Consequently, $r_\lambda = (r_1^{\mathrm{u}}+r_1^{\mathrm{s}})/2+\lambda(r_1^{\mathrm{u}}-r_1^{\mathrm{s}})/2$ will satisfy the same bounds for all $\lambda \in [0,1]$.

Now, proceeding as in the proof of Lemmas 3.2.14 and 3.2.15 (using Lemma 3.2.6) one can see that:

$$\|\partial_r F(r_\lambda)\|_{2,\omega} \leq K\delta^3,$$

$$\|G(r_\lambda)\|_{2,\omega} \leq K\delta^3, \quad \|\partial_r G(r_\lambda)\|_{0,\omega} \leq K\delta^3,$$

and:

$$\|H(r_\lambda)\|_{3,\omega} \leq K\delta^3, \quad \|\partial_r H(r_\lambda)\|_{1,\omega} \leq K\delta^3.$$

Using the properties of the norms $\|.\|_{n,\omega}$ we obtain the bounds for l_1, l_2 and l_3. □

3.5.2 Structure of Δ

In this subsection we state rigorously the ideas introduced in Section 3.1.6, which will allow us to prove Theorem 3.1.9. We begin by stating a trivial result regarding the form of the solutions of (3.237).

Lemma 3.5.3 (Variation of constants)**.** *Let $P(u,\theta)$ be a particular solution of* (3.237) *such that it is 2π-periodic in θ and satisfying $P(u,\theta) \neq 0$ for all $(u,\theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$. Then, every solution $\Delta(u,\theta)$ of equation* (3.237) *defined for $(u,\theta) \in D_{\kappa,\beta} \times \mathbb{T}_\omega$ can be written as:*

$$\Delta(u,\theta) = P(u,\theta)k(u,\theta),$$

where $k(u,\theta)$ is a solution of the homogeneous PDE:

$$\left(-\frac{\alpha}{\delta} - cZ_0(u)\right)\partial_\theta k + \partial_u k = l_2(u,\theta)\partial_u k + l_3(u,\theta)\partial_\theta k, \tag{3.242}$$

which is 2π-periodic in θ.

Next proposition, whose proof is straightforward, will allow to write the function $k(u,\theta)$ in terms of a particular solution of (3.242).

Proposition 3.5.4. *Let $\xi(u,\theta)$ be a particular solution of* (3.242) *such that $(\xi(u,\theta),\theta)$ is injective in $D_{\kappa,\beta} \times \mathbb{T}_\omega$. Then any solution $k(u,\theta)$ of* (3.242) *defined in $D_{\kappa,\beta} \times \mathbb{T}_\omega$ can be written as:*

$$k(u,\theta) = \tilde{k}(\xi(u,\theta)),$$

for some function $\tilde{k}$.

Thus, we need to find particular solutions of equations (3.242) and (3.237) suitable for our purposes. These solutions are found respectively in Propositions 3.5.5 and 3.5.7. The corresponding proofs of these propositions are deferred to Sections 3.5.4 and 3.5.5.

Proposition 3.5.5. *Let $C(u,\theta)$ be a solution of the equation:*

$$(-\delta^{-1}\alpha - cZ_0(u))\partial_\theta C + \partial_u C = l_2(u,\theta)(\delta^{-1}\alpha + cZ_0(u) + \partial_u C) + l_3(u,\theta)(1+\partial_\theta C), \quad (3.243)$$

defined in $D_{\kappa,\beta}\times\mathbb{T}_\omega$. Then:

$$\xi(u,\theta) = \theta + \delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta) \quad (3.244)$$

is a solution of equation (3.242).

Moreover, there exists a particular solution $C(u,\theta)$ of (3.243) *of the form:*

$$C(u,\theta) = \delta^{-1}\alpha\int_0^u l_2^{[0]}(w)dw + C_1(u,\theta),$$

where $l_2^{[0]}(u)$ denotes the average of the function $l_2(u,\theta)$ defined in (3.239)*. The following holds:*

1. *Let $|\sigma|\le\delta^{p+3}\sigma^*$. There exist constants L_0 and M and functions $L(u)$ and $\Lambda(u)$ such that:*

$$\int_0^u l_2^{[0]}(w)dw = \delta^{p+3}\mathrm{d}^{-1}L_0\log\cosh(\mathrm{d}u) + \delta L(u) + \delta\Lambda(u), \quad (3.245)$$

and:

$$\|L\|_0 \le M\delta^{p+2}, \qquad \|L'\|_0 \le M\delta^{p+2}, \qquad \|\Lambda\|_1 \le M\delta^{p+3}.$$

A formula for L_0 is given in Lemma 3.5.17, and a formula for $L(u)$ is given in Remark 3.5.18. Moreover, $L_0\in\mathbb{R}$, $L(0)=0$ and $L(u)$ is defined on the limit $u\to i\pi/(2\mathrm{d})$.

2. *There exists a constant M such that for $p\ge -2$ one has:*

$$\|C_1\|_{1,\omega} \le M\delta^{p+3}, \quad (3.246)$$

$$\|\partial_u C\|_{1,\omega} \le M\delta^{p+2}, \qquad \|\partial_\theta C\|_{1,\omega} \le M\delta^{p+3}, \quad (3.247)$$

and such that $(\xi(u,\theta),\theta)$, with $\xi(u,\theta)$ given by (3.244)*, is injective in $D_{\kappa,\beta}\times\mathbb{T}_\omega$.*

Remark 3.5.6. Note that using $\xi(u,\theta)$ (the function defined in (3.244)) in Proposition 3.5.4 we have:

$$k(u,\theta) = \tilde{k}(\theta + \delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta)).$$

Since $k(u,\theta)$ and $C(u,\theta)$ are $2\pi-$periodic in θ, we have that $\tilde{k}(\tau)$ is $2\pi-$periodic.

Proposition 3.5.7. *Let $P_1(u,\theta)$ be a solution of the equation:*

$$\begin{aligned}\left(-\delta^{-1}\alpha - cZ_0(u)\right)\partial_\theta P_1 + \partial_u P_1 &= (2\sigma + l_1(u,\theta) + 2Z_0(u)l_2(u,\theta))(1+P_1) \\ &\quad + l_2(u,\theta)\partial_u P_1 + l_3(u,\theta)\partial_\theta P_1,\end{aligned} \tag{3.248}$$

which is $2\pi-$periodic in θ. Then:

$$P(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta)) \tag{3.249}$$

is a solution of equation (3.237)*.*

Moreover, there exists a particular solution $P_1(u,\theta)$ of (3.248) *defined in $D_{\kappa,\beta}\times\mathbb{T}_\omega$ and a constant M such that for $p\geq -2$ one has:*

$$\llbracket P_1(u,\theta)\rrbracket_{1,\omega} \leq M\delta^{p+3}. \tag{3.250}$$

As a consequence, $P(u,\theta)$ given by (3.249) *satisfies that $P(u,\theta)\neq 0$ for all $(u,\theta)\in D_{\kappa,\beta}\times\mathbb{T}_\omega$ and $p\geq -2$, if κ is large enough.*

In the conservative case, this particular solution can be taken as:

$$P_1(u,\theta) = \frac{\partial_u C(u,\theta) - l_3(u,\theta)}{\delta^{-1}\alpha + cZ_0(u) + l_3(u,\theta)},$$

where $C(u,\theta)$ is the function given in Proposition 3.5.5 satisfying (3.243)*, and $l_3(u,\theta)$ is defined in* (3.240)*.*

All these results yield the proof of Theorem 3.1.9.

Proof of Theorem 3.1.9. Using Lemma 3.5.3 and Propositions 3.5.4, 3.5.5 and 3.5.7 we obtain straightforwardly the claim of Theorem 3.1.9, namely that $\Delta(u,\theta)$ can be written as:

$$\Delta(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\tilde{k}(\theta+\delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta))$$

for some $2\pi-$periodic function $\tilde{k}(\tau)$, and some functions $C(u,\theta)$, $P_1(u,\theta)$ $2\pi-$periodic in θ. Moreover, it is also clear that:

$$C(u,\theta) = \delta^{p+2}\mathrm{d}^{-1}\alpha L_0\log\cosh(\mathrm{d}u) + \alpha L(u) + \chi(u,\theta),$$

with $\chi(u,\theta) = \alpha\Lambda(u) + C_1(u,\theta)$. Using Propositions 3.5.5 and 3.5.7 it is straightforward to see that bounds (3.80) and (3.81) hold. □

The goal of the remaining part of this section will be to prove Propositions 3.5.5 and 3.5.7. To this aim, first we point out that the linear operator on the left hand side of equations (3.243) and (3.248) is in both cases:

$$\hat{\mathcal{L}}(\phi) = (-\delta^{-1}\alpha - cZ_0(u))\partial_\theta\phi + \partial_u\phi. \tag{3.251}$$

Let us define the operators:

$$\mathcal{A}(\phi) = l_2(u,\theta)(\delta^{-1}\alpha + cZ_0(u) + \partial_u\phi) + l_3(u,\theta)(1+\partial_\theta\phi). \tag{3.252}$$

and:

$$\mathcal{B}(\phi) = (2\sigma + l_1(u,\theta) + 2Z_0(u)l_2(u,\theta))(1+\phi) + l_2(u,\theta)\partial_u\phi + l_3(u,\theta)\partial_\theta\phi, \tag{3.253}$$

Then equation (3.243) can be written as:

$$\hat{\mathcal{L}}(C) = \mathcal{A}(C), \tag{3.254}$$

and (3.248) can be written as:

$$\hat{\mathcal{L}}(P_1) = \mathcal{B}(P_1). \tag{3.255}$$

The proof of both propositions basically relies on finding a suitable particular solution of the corresponding equation. In order to do that, we will use a right inverse of the operator $\hat{\mathcal{L}}$, which we will call $\hat{\mathcal{G}}$. Then we will be able to write equations (3.254) and (3.255) as fixed point equations, and solve them using an iterative scheme.

We define $\hat{\mathcal{G}}$ as the operator acting on functions $\phi(u,\theta)$ defined for $(u,\theta)\in D_{\kappa,\beta}\times\mathbb{T}_\omega$ as:

$$\hat{\mathcal{G}}(\phi)(u,\theta) = \sum_{l\in\mathbb{Z}} \hat{\mathcal{G}}^{[l]}(\phi)(u)e^{il\theta}, \tag{3.256}$$

where:

$$\begin{aligned}
\hat{\mathcal{G}}^{[l]}(\phi)(u) &= \int_{u_+}^{u} e^{-il\delta^{-1}\alpha(w-u)-ilc\mathrm{d}^{-1}\log\left(\frac{\cosh(\mathrm{d}w)}{\cosh(\mathrm{d}u)}\right)}\phi^{[l]}(w)dw, && \text{if } l<0,\\
\hat{\mathcal{G}}^{[0]}(\phi)(u) &= \int_{u_\mathrm{R}}^{u} \phi^{[0]}(w)dw, \\
\hat{\mathcal{G}}^{[l]}(\phi)(u) &= \int_{u_-}^{u} e^{-il\delta^{-1}\alpha(w-u)-ilc\mathrm{d}^{-1}\log\left(\frac{\cosh(\mathrm{d}w)}{\cosh(\mathrm{d}u)}\right)}\phi^{[l]}(w)dw, && \text{if } l>0,
\end{aligned} \tag{3.257}$$

and $u_\pm = \pm i(\pi/(2\mathrm{d}) - \delta\kappa)$ and $u_\mathrm{R}\in\mathbb{R}$ is the point of $D_{\kappa,\beta}$ with largest real part (see Figure 3.3 in Subsection 3.1.6).

3.5.3 The operator $\hat{\mathcal{G}}$

In this subsection we will study the properties of the operator $\hat{\mathcal{G}}$. We begin by introducing some technical lemmas.

Lemma 3.5.8. *Let $n\geq 0$. There exists a constant M such that for all $l\neq 0$ and $u\in D_{\kappa,\beta}$:*

$$I(u) := \left|\cosh^n(\mathrm{d}u)\int_{u_\pm}^{u}\frac{e^{-\delta^{-1}\alpha il(w-u)-ilc\mathrm{d}^{-1}(\log\cosh(\mathrm{d}w)-\log\cosh(\mathrm{d}u))}dw}{\cosh^n(\mathrm{d}w)}\right| \leq \frac{M\delta}{|l|},$$

where $u_\pm = \pm i(\pi/(2\mathrm{d})-\kappa\delta)$, and we take u_+ if $l<0$ and u_- if $l>0$, so that $l\,\mathrm{Im}\,(u_\pm - u) = -|l||\mathrm{Im}\,(u_\pm - u)|$ for all $u\in D_{\kappa,\beta}$.

Proof. Since the integrand is holomorphic inside $D_{\kappa,\beta}$, we can take any curve between u and $u_\pm$. Thus for $l<0$ we take the curve $w_+(t,u)=u_+ + t(u-u_+)$ and for $l>0$ we take the curve $w_-(t,u)=u_- + t(u-u_-)$, $t\in[0,1]$. Then we have:

$$I(u)=\left|\cosh^n(\mathrm{d}u)\int_0^1 \frac{e^{-\delta^{-1}\alpha(1-t)il(u_\pm-u)-ilc\mathrm{d}^{-1}(\log\cosh(\mathrm{d}w_\pm(t,u))-\log\cosh(\mathrm{d}u))}}{\cosh^n(\mathrm{d}w_\pm(t,u))}(u-u_\pm)dt\right|. \tag{3.258}$$

We note that since with our choice we always have $l\mathrm{Im}\,(u_\pm-u)=-|l||\mathrm{Im}\,(u_\pm-u)|$ we can write for all $l\neq 0$:

$$|e^{-\delta^{-1}\alpha(1-t)il(u_\pm-u)}|=e^{\delta^{-1}\alpha(1-t)l\mathrm{Im}\,(u_\pm-u)}=e^{-\delta^{-1}\alpha(1-t)|l||\mathrm{Im}\,(u_\pm-u)|}. \tag{3.259}$$

Moreover, using the mean value theorem it is easy to see that:

$$c\mathrm{d}^{-1}\left|\log\cosh(\mathrm{d}w_\pm(t,u))-\log\cosh(\mathrm{d}u)\right|\le\frac{K}{\delta\kappa}|u-u_\pm|\le\frac{K}{\delta\kappa}|\mathrm{Im}\,(u-u_\pm)|,$$

where in the last step we have used that for $w\in D_{\kappa,\beta}$ one has that $|\mathrm{Re}\,(w)|\le K|\mathrm{Im}\,(w)|$. This yields:

$$\begin{aligned}\left|e^{-ilc\mathrm{d}^{-1}(\log\cosh(\mathrm{d}w_\pm(t,u))-\log\cosh(\mathrm{d}u))}\right| &\le e^{|lc\mathrm{d}^{-1}(\log\cosh(\mathrm{d}w_\pm(t,u))-\log\cosh(\mathrm{d}u))|}\\ &\le e^{|l|\frac{K}{\delta\kappa}|\mathrm{Im}\,(u-u_\pm)|}. \end{aligned}\tag{3.260}$$

Equations (3.259) and (3.260) imply that for κ sufficiently large:

$$\begin{aligned}\left|e^{-\delta^{-1}\alpha il(1-t)(u_\pm-u)-ilc\mathrm{d}^{-1}(\log\cosh(\mathrm{d}w_\pm(t,u))-\log\cosh(\mathrm{d}u))}\right| &\le e^{-\delta^{-1}\alpha(1-t)|l||\mathrm{Im}\,(u-u_\pm)|\left(1-\frac{K}{\kappa}\right)}\\ &\le e^{-\frac{1}{2}\delta^{-1}\alpha(1-t)|l||\mathrm{Im}\,(u-u_\pm)|}. \end{aligned}\tag{3.261}$$

From this expression, the case $n=0$ is straightforward to check. For $n>0$, let us assume that $\mathrm{Im}\,u\ge 0$ (the case $\mathrm{Im}\,u\le 0$ is completely analogous). We just prove the case $l<0$ (i.e., taking u_+ in the integrals), since the case $l>0$ is easier to check (note that assuming $\mathrm{Im}\,u\ge 0$, we have $|\mathrm{Im}\,(u-u_-)|\ge K>0$, so that expression (3.261) is exponentially small for $t<1$).

Thus, let us assume that $l>0$. Using (3.261) in (3.258) and using that in this case $K_1|w-i\pi/(2\mathrm{d})|\le|\cosh(\mathrm{d}w)|\le K_2|w-i\pi/(2\mathrm{d})|$ as usual, we have:

$$I(u)\le K|u-u_+|\left|u-\frac{i\pi}{2\mathrm{d}}\right|^n\left|\int_0^1\frac{e^{-\frac{1}{2}\delta^{-1}\alpha|l|(1-t)\mathrm{Im}\,(u-u_+)}dt}{\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|^n}\right| =: J(u). \tag{3.262}$$

Performing the integral in (3.262) by parts, we obtain:

$$\begin{aligned} J(u) \leq\ & K|u-u_+|\left|u-\frac{i\pi}{2\mathrm{d}}\right|^n \left|\left[\frac{2\delta e^{-\frac{1}{2}\delta^{-1}\alpha(1-t)|l||\mathrm{Im}\,(u-u_+)}}{\alpha|l||\mathrm{Im}\,(u-u_+)|\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|^n}\right]_0^1\right| \\ & +K|u-u_+|\left|u-\frac{i\pi}{2\mathrm{d}}\right|^n \left|\frac{2\delta n}{\alpha|l||\mathrm{Im}\,(u-u_+)|}\int_0^1 \frac{e^{-\frac{1}{2}\delta^{-1}\alpha|l|(1-t)\mathrm{Im}\,(u-u_+)}}{\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|^{n+2}}A(t,u)dt\right|, \end{aligned} \tag{3.263}$$

where:

$$A(t,u)=\mathrm{Re}\,w_+(t,u)\frac{\partial \mathrm{Re}\,w_+}{\partial t}(t,u)+\left(\mathrm{Im}\,w_+(t,u)-\frac{\pi}{2\mathrm{d}}\right)\frac{\partial \mathrm{Im}\,w_+}{\partial t}(t,u).$$

Now, on the one hand it can be easily seen that:

$$\begin{aligned} &\left|\left[\frac{2\delta e^{-\frac{1}{2}\delta^{-1}\alpha(1-t)|l||\mathrm{Im}\,(u-u_+)}}{\alpha|l||\mathrm{Im}\,(u-u_+)|\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|^n}\right]_0^1\right| \\ &\leq \frac{2\delta}{\alpha|l||\mathrm{Im}\,(u-u_+)|\left|u-\frac{i\pi}{2\mathrm{d}}\right|^n}\left(1+\frac{\left|u-\frac{i\pi}{2\mathrm{d}}\right|^n}{\left|u_+-\frac{i\pi}{2\mathrm{d}}\right|^n}e^{-\frac{1}{2}\delta^{-1}\alpha|l||\mathrm{Im}\,(u-u_+)}\right) \\ &\leq \frac{K\delta}{|l||\mathrm{Im}\,(u-u_+)|\left|u-\frac{i\pi}{2\mathrm{d}}\right|^n}. \end{aligned} \tag{3.264}$$

On the other hand, we note that:

$$|A(t,u)|\leq K\left(|\mathrm{Re}\,w_+(t,u)|+\left|\mathrm{Im}\,w_+(t,u)-\frac{\pi}{2\mathrm{d}}\right|\right)|\mathrm{Im}\,(u-u_+)|,$$

and:

$$\frac{1}{\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|^2}\leq\frac{K}{\delta\kappa}\frac{1}{\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|},$$

and hence writing $x=\mathrm{Re}\,(w_+(t,u)-i\pi/(2\mathrm{d}))$ and $y=\mathrm{Im}\,(w_+(t,u)-i\pi/(2\mathrm{d}))$ we have:

$$\frac{|A(t,u)|}{\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|^2}\leq\frac{K(|x|+|y|)}{\delta\kappa\sqrt{x^2+y^2}}|\mathrm{Im}\,(u-u_+)|\leq\frac{K}{\delta\kappa}|\mathrm{Im}\,(u-u_+)|. \tag{3.265}$$

Thus, substituting bound (3.265) in the second summand of (3.263) we obtain:

$$\begin{aligned} &\frac{2\delta n}{\alpha|l||\mathrm{Im}\,(u-u_+)|}\left|\int_0^1\frac{e^{-\frac{1}{2}\delta^{-1}\alpha|l|(1-t)\mathrm{Im}\,(u-u_+)}dw}{\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|^{n+2}}A(t,u)dt\right| \\ &\leq\frac{K}{\kappa}\left|\int_0^1\frac{e^{-\frac{1}{2}\delta^{-1}\alpha|l|(1-t)\mathrm{Im}\,(u-u_+)}dw}{\left|w_+(t,u)-\frac{i\pi}{2\mathrm{d}}\right|^n}dt\right|=\frac{K}{\kappa|u-u_+|\left|u-\frac{i\pi}{2\mathrm{d}}\right|^n}J(u). \end{aligned} \tag{3.266}$$

In conclusion, substituting (3.264) and (3.266) in (3.263), we obtain:

$$J(u) \leq \frac{K\delta}{|l|} + \frac{K}{\kappa} J(u),$$

and thus, if κ is sufficiently large we have:

$$J(u) \leq \frac{K\delta}{|l| \left(1 - \frac{K}{\kappa}\right)} \leq \frac{M\delta}{|l|},$$

for some suitable constant M, and then by (3.262) the lemma is proved. $\square$

Lemma 3.5.9. *For any $n \geq 2$ there exists a constant M such that for all $u \in D_{\kappa,\beta}$:*

$$\left| \cosh^{n-1}(\mathrm{d}u) \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^n(\mathrm{d}w)} dw \right| \leq M,$$

where $u_{\mathrm{R}} \in \mathbb{R}$ is the point of $D_{\kappa,\beta}$ with largest real part.

Proof. We will do the proof with $\mathrm{d} = 1$, since with a trivial change of variables we obtain the result for any $\mathrm{d} \neq 0$.

We will distinguish the case $n = 2$, $n = 3$ and $n > 3$. In the first case, by simply integrating one obtains:

$$\cosh u \int_{u_R}^{u} \frac{1}{\cosh^2 w} dw = \sinh u - \cosh u \tanh u_R.$$

The fact that this is bounded independently of δ is clear, since for u bounded $\sinh u$ and $\cosh u$ are bounded, and since $u_R \in \mathbb{R}$ and is also bounded, so is $\tanh u_R$.

In the second case, we integrate by parts and obtain:

$$\begin{aligned}
\cosh^2 u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^3 w} dw &= \cosh^2 u \left[\frac{\sinh u}{\cosh^2 u} - \frac{\sinh u_R}{\cosh^2 u_R} \right] + \cosh^2 u \int_{u_{\mathrm{R}}}^{u} \frac{\sinh^2 u}{\cosh^3 w} dw \\
&= \cosh^2 u \left[\frac{\sinh u}{\cosh^2 u} - \frac{\sinh u_R}{\cosh^2 u_R} \right] + \cosh^2 u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh w} dw \\
&\quad - \cosh^2 u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^3 w} dw,
\end{aligned}$$

and hence:

$$\begin{aligned}
&\left| \cosh^2 u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^3 w} dw \right| \\
&\leq \frac{1}{2} \left[|\sinh u| - |\cosh^2 u| \left| \frac{\sinh u_R}{\cosh^2 u_R} \right| \right] + \frac{|\cosh^2 u|}{2} \int_{u_{\mathrm{R}}}^{u} \left| \frac{1}{\cosh w} \right| dw. \quad (3.267)
\end{aligned}$$

We note that the integration can be done along any curve joining u and u_R, in particular a straight segment joining these two points. For all w in this segment we have:

$$\frac{1}{|\cosh w|} \leq \frac{K}{|\cosh u|},$$

so that from (3.267) and the fact that $|u - u_{\mathrm{R}}|$ is bounded we have:

$$\left|\cosh^2 u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^3 w} dw\right| \leq \frac{1}{2}\left[|\sinh u| - |\cosh^2 u| \left|\frac{\sinh u_R}{\cosh^2 u_R}\right|\right] + \frac{K}{2}|\cosh u| \leq K.$$

The case $n > 3$ can be proved by induction. Indeed, assume that the inequality holds for $n-2$, then integrating by parts:

$$\begin{aligned}
\cosh^{n-1} u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^n w} dw &= \cosh^{n-1} u \left[\frac{\sinh u}{\cosh^{n-1} u} - \frac{\sinh u_R}{\cosh^{n-1} u_R}\right] \\
&\quad +(n-2)\cosh^{n-1} u \int_{u_{\mathrm{R}}}^{u} \frac{\sinh^2 u}{\cosh^n w} dw \\
&= \cosh^{n-1} u \left[\frac{\sinh u}{\cosh^{n-1} u} - \frac{\sinh u_R}{\cosh^{n-1} u_R}\right] \\
&\quad +(n-2)\cosh^{n-1} u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^{n-2} w} dw \\
&\quad -(n-2)\cosh^{n-1} u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^{n} w} dw.
\end{aligned}$$

Thus, one has:

$$\begin{aligned}
\left|\cosh^{n-1} u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^n w} dw\right| &\leq \frac{1}{n-1}\left[|\sinh u| - |\cosh^{n-1} u| \left|\frac{\sinh u_R}{\cosh^{n-1} u_R}\right|\right] \\
&\quad + \frac{|\cosh^2 u|}{n-1}\left|\cosh^{n-3} u \int_{u_{\mathrm{R}}}^{u} \frac{1}{\cosh^{n-2} u}\right| \leq K,
\end{aligned}$$

where in the last step we have used the induction hypothesis. □

With these previous lemmas one can prove the following result, reasoning analogously as in the proof of Lemma 3.2.11.

Lemma 3.5.10. *Let $l \in \mathbb{Z}$, $n \geq 1$ and $\phi \in \mathcal{X}_{n,\omega}$. There exists a constant M such that:*

1. *If $n > 1$, then $\|\hat{\mathcal{G}}^{[l]}(\phi)\|_{n-1} \leq M \left\|\phi^{[l]}\right\|_n$.*

2. *If $l \neq 0$, then $\|\hat{\mathcal{G}}^{[l]}(\phi)\|_n \leq \dfrac{\delta M \left\|\phi^{[l]}\right\|_n}{|l|}$.*

3. *As a consequence we have that if* $n > 1$*:*

$$\|\hat{\mathcal{G}}(\phi)\|_{n-1,\omega} \leq M\|\phi\|_{n,\omega}.$$

Moreover, if $\phi^{[0]}(v) = 0$*, then for all* $n \geq 1$*:*

$$\|\hat{\mathcal{G}}(\phi)\|_{n,\omega} \leq \delta M\|\phi\|_{n,\omega}.$$

4. $\|\partial_\theta \hat{\mathcal{G}}(\phi)\|_{n,\omega} \leq \delta M\|\phi\|_{n,\omega}$.

5. $\|\partial_u \hat{\mathcal{G}}(\phi)\|_{n,\omega} \leq M\|\phi\|_{n,\omega}$.

6. *In conclusion, from the previous items it is straightforward to see that if* $n > 1$ *and* $\phi \in \mathcal{X}_{n,\omega}$*, then* $\hat{\mathcal{G}}(\phi) \in \tilde{\mathcal{X}}_{n-1,\omega}$ *and there exists a constant* M *such that:*

$$[\![\hat{\mathcal{G}}(\phi)]\!]_{n-1,\omega} \leq M\|\phi\|_{n,\omega}.$$

3.5.4 Proof of Proposition 3.5.5

In this subsection we will prove Proposition 3.5.5. More precisely, we will find C_1 such that the function:

$$C(u,\theta) = \delta^{-1}\alpha \int_0^u l_2^{[0]}(w)dw + C_1(u,\theta) \tag{3.268}$$

satisfies equation (3.243). To that aim, let us define:

$$\hat{l}_2(u,\theta) = l_2(u,\theta) - l_2^{[0]}(u).$$

It is easy to see that in order that $C(u,\theta)$ defined in (3.268) satisfies (3.243) it is enough that C_1 satisfies the following equation:

$$\begin{aligned}(-\delta^{-1}\alpha - cZ_0(u))\partial_\theta C_1 + \partial_u C_1 &= \delta^{-1}\alpha\hat{l}_2(u,\theta) + l_2(u,\theta)(cZ_0(u) + \delta^{-1}\alpha l_2^{[0]}(u) + \partial_u C_1)\\ &\quad + l_3(u,\theta)(1 + \partial_\theta C_1). \end{aligned} \tag{3.269}$$

We define the operator $\mathcal{A}_1$ as:

$$\mathcal{A}_1(\phi) = \delta^{-1}\alpha\hat{l}_2(u,\theta) + l_2(u,\theta)(cZ_0(u) + \delta^{-1}\alpha l_2^{[0]}(u) + \partial_u\phi) + l_3(u,\theta)(1 + \partial_\theta\phi). \tag{3.270}$$

Then equation (3.269) can be rewritten as:

$$\hat{\mathcal{L}}(C_1) = \mathcal{A}_1(C_1), \tag{3.271}$$

where $\hat{\mathcal{L}}$ was defined in (3.251). Clearly, it is enough to solve the fixed point equation:

$$C_1 = \tilde{\mathcal{A}}_1(C_1), \tag{3.272}$$

where $\tilde{\mathcal{A}} = \hat{\mathcal{G}} \circ \mathcal{A}_1$, and $\hat{\mathcal{G}}$ is the operator defined in (3.256).

Lemma 3.5.11. *For κ big enough and $p \geq -2$, the operator $\tilde{\mathcal{A}}_1 : \tilde{\mathcal{X}}_{0,\omega} \to \tilde{\mathcal{X}}_{0,\omega}$. Moreover, there exists a constant M such that $[\![\tilde{\mathcal{A}}_1(0)]\!]_{0,\omega} \leq M\delta^{p+2}$, and $\tilde{\mathcal{A}}_1$ has a unique fixed point in the ball $B\left(2[\![\tilde{\mathcal{A}}_1(0)]\!]_{0,\omega}\right) \subset \tilde{\mathcal{X}}_{0,\omega}$.*

Proof. First of all we shall prove that:

$$[\![\tilde{\mathcal{A}}_1(0)]\!]_{0,\omega} \leq K\delta^{p+2}. \tag{3.273}$$

We have:

$$\mathcal{A}_1(0) = \delta^{-1}\alpha\hat{l}_2(u,\theta) + l_2(u,\theta)(cZ_0(u) + \delta^{-1}\alpha l_2^{[0]}(u)) + l_3(u,\theta).$$

To prove (3.273) we shall bound the Fourier coefficients of $\mathcal{A}_1(0)$ and then use Lemma 3.5.10. We shall bound the zeroth Fourier coefficient in a different way as the other ones. Indeed, on the one hand, since by definition $\hat{l}_2$ has zero average, one has:

$$\mathcal{A}_1^{[0]}(0) = l_2^{[0]}(u)(cZ_0(u) + \delta^{-1}\alpha l_2^{[0]}(u)) + l_3^{[0]}(u).$$

Using Lemma 3.5.2 and the properties of the norm, it is straightforward to see that:

$$\|\mathcal{A}_1^{[0]}(0)\|_2 \leq \|l_2^{[0]}\|_1(c\|Z_0\|_1 + \delta^{-1}\alpha\|l_2^{[0]}\|_1) + \|l_3^{[0]}\|_2 \leq K\delta^{p+3}. \tag{3.274}$$

Then, by item 1 of Lemma 3.5.10 one has:

$$\|\hat{\mathcal{G}}^{[0]}(\mathcal{A}_1(0))\|_1 \leq K\|\mathcal{A}_1^{[0]}(0)\|_2 \leq K\delta^{p+3}. \tag{3.275}$$

On the other hand, for the remaining Fourier coefficients one has:

$$\mathcal{A}_1^{[l]}(0) = l_2^{[l]}(u)(\delta^{-1}\alpha + cZ_0(u) + \delta^{-1}\alpha l_2^{[0]}(u)) + l_3^{[l]}(u) \qquad l \neq 0.$$

Again, using Lemma 3.5.2 and the properties of the norm, we obtain:

$$\|\mathcal{A}_1^{[l]}(0)\|_1 \leq \|l_2^{[l]}\|_1(\delta^{-1}\alpha + c\|Z_0\|_0 + \delta^{-1}\alpha\|l_2^{[0]}\|_0) + \|l_3^{[l]}\|_1 \leq K\delta^{p+2}. \tag{3.276}$$

Then by item 2 of Lemma 3.5.10 and taking into account that $l \neq 0$, we have:

$$\|\hat{\mathcal{G}}^{[l]}(\mathcal{A}_1(0))\|_1 \leq \frac{K\delta\|\mathcal{A}_1^{[l]}(0)\|_1}{|l|} \leq \frac{K\delta^{p+3}}{|l|}. \tag{3.277}$$

From (3.275) and (3.277) we obtain:

$$\|\tilde{\mathcal{A}}_1(0)\|_{1,\omega} \leq K\delta^{p+3}, \tag{3.278}$$

and as a consequence:

$$\|\tilde{\mathcal{A}}_1(0)\|_{0,\omega} \leq K\frac{\delta^{p+2}}{\kappa} \leq K\delta^{p+2}. \tag{3.279}$$

We note that from bounds (3.274) and (3.276) we also have:

$$\|\mathcal{A}_1(0)\|_{1,\omega} \leq K\delta^{p+2},$$

and then from items 4 and 5 of Lemma 3.5.10 we obtain directly:

$$\|\partial_u \tilde{\mathcal{A}}_1(0)\|_{1,\omega} \leq K\|\mathcal{A}_1(0)\|_{1,\omega} \leq K\delta^{p+2}, \tag{3.280}$$

and:

$$\|\partial_\theta \tilde{\mathcal{A}}_1(0)\|_{1,\omega} \leq K\delta\|\mathcal{A}_1(0)\|_{1,\omega} \leq K\delta^{p+3}. \tag{3.281}$$

From bounds (3.279), (3.280) and (3.281) one obtains bound (3.273).

The next step is to find the Lipschitz constant of $\tilde{\mathcal{A}}_1$. We claim that given two functions $\phi_1, \phi_2 \in \tilde{\mathcal{X}}_{0,\omega}$:

$$[\![\tilde{\mathcal{A}}_1(\phi_1) - \tilde{\mathcal{A}}_1(\phi_2)]\!]_{0,\omega} \leq \frac{K}{\kappa}\delta^{p+2}[\![\phi_1 - \phi_2]\!]_{0,\omega}. \tag{3.282}$$

We have:

$$\mathcal{A}_1(\phi_1) - \mathcal{A}_1(\phi_2) = l_2(u,\theta)\partial_u(\phi_1 - \phi_2) + l_3(u,\theta)\partial_\theta(\phi_1 - \phi_2).$$

By Lemma 3.5.2 and the properties of the norm, it is clear that:

$$\begin{aligned} \|\mathcal{A}_1(\phi_1) - \mathcal{A}_1(\phi_2)\|_{2,\omega} &\leq \|l_2\|_{1,\omega}\|\partial_u(\phi_1 - \phi_2)\|_{1,\omega} + \|l_3\|_{1,\omega}\|\partial_\theta(\phi_1 - \phi_2)\|_{1,\omega} \\ &\leq K\delta^{p+3}[\![\phi_1 - \phi_2]\!]_{0,\omega}. \end{aligned}$$

By item 6 of Lemma 3.5.10 this implies:

$$[\![\tilde{\mathcal{A}}_1(\phi_1) - \tilde{\mathcal{A}}_1(\phi_2)]\!]_{1,\omega} \leq K\delta^{p+3}[\![\phi_1 - \phi_2]\!]_{0,\omega},$$

and using the properties of the norm we obtain bound (3.282).

To finish the proof, we take κ sufficiently large such that the Lipschitz constant in (3.282) is smaller than 1. Then clearly $\tilde{\mathcal{A}}_1 : B\left(2[\![\tilde{\mathcal{A}}_1(0)]\!]_{0,\omega}\right) \to B\left(2[\![\tilde{\mathcal{A}}_1(0)]\!]_{0,\omega}\right)$ and since it is contractive, it has a unique fixed point in this ball. □

On the integral $\int_0^u l_2^{[0]}(w)dw$

Now we make some further considerations on the integral $\int_0^u l_2^{[0]}(w)dw$. First of all, we point out that using Lemma 3.5.2 and the fact that for $w \in D_{\kappa,\beta}$ one has $|\cosh(\mathrm{d}w)| \geq K|w^2 - \pi^2/(2\mathrm{d})^2|$, one obtains:

$$\left|\delta^{-1}\alpha\int_0^u l_2^{[0]}(w)dw\right| \leq K\delta^{p+2}\int_0^u |\cosh(\mathrm{d}w)|^{-1}dw \leq K\delta^{p+2}|\log(\delta\kappa)|.$$

Hence, in the regular case $p > -2$ this integral is small and one can avoid to take into account its contribution to the function $C(u,\theta)$ defined in (3.268). However, in the singular case $p=-2$, one needs to have some more precise knowledge of its behavior. This is what now we shall proceed to do. We recall that, from the definition (3.239) of $l_2(u,\theta)$, one has:

$$l_2^{[0]}(u) = -\frac{b}{\mathrm{d}(1-Z_0^2(u))}(r_1^{\mathrm{u}\,[0]}(u)+r_1^{\mathrm{s}\,[0]}(u)) - \frac{\delta^p}{2\mathrm{d}(1-Z_0^2(u))}\int_{-1}^{1}(H(r_\lambda))^{[0]}d\lambda, \quad (3.283)$$

where $r_\lambda = (r_1^{\mathrm{u}}+r_1^{\mathrm{s}})/2+\lambda(r_1^{\mathrm{u}}-r_1^{\mathrm{s}})/2$.

In the next Lemma we give the specific form of the averages r_1^{u} and r_1^{s}.

Lemma 3.5.12. *Let $|\sigma|\leq\delta^{p+3}\sigma^*$. There exist two constants ρ_0^{u}, $\rho_0^{\mathrm{s}}\in\mathbb{R}$ and functions $\rho_1^{\mathrm{u,s}}(u)$ and $\rho_2^{\mathrm{u,s}}(u)$ such that:*

$$r_1^{\mathrm{u}\,[0]}(u) = \delta^{p+3}\rho_0^{\mathrm{u}}\frac{\tanh(\mathrm{d}u)}{\cosh^2(\mathrm{d}u)}+\delta^{p+3}\rho_1^{\mathrm{u}}(u)+\delta^{p+4}\rho_2^{\mathrm{u}}(u),$$

$$r_1^{\mathrm{s}\,[0]}(u) = \delta^{p+3}\rho_0^{\mathrm{s}}\frac{\tanh(\mathrm{d}u)}{\cosh^2(\mathrm{d}u)}+\delta^{p+3}\rho_1^{\mathrm{s}}(u)+\delta^{p+4}\rho_2^{\mathrm{s}}(u).$$

Moreover, there exists a constant M such that for $i=1,2$:

$$\|\rho_1^{\mathrm{u}}\|_2\leq M, \qquad \|\rho_2^{\mathrm{u}}\|_4\leq M,$$

$$\|\rho_1^{\mathrm{s}}\|_2\leq M, \qquad \|\rho_2^{\mathrm{s}}\|_4\leq M,$$

and the functions $\rho_1^{\mathrm{u}}(u)\cosh^2(\mathrm{d}u)$ and $\rho_1^{\mathrm{s}}(u)\cosh^2(\mathrm{d}u)$ are defined on the limit $u\to i\pi/(2\mathrm{d})$.

Proof. As usual, we do the proof just for the unstable case, since the stable case is analogous.

Recall that from Theorem 3.1.7 we have:

$$r_1^{\mathrm{u}\,[0]}(u) = {r_{10}^{\mathrm{u}}}^{[0]}(u)+{r_{11}^{\mathrm{u}}}^{[0]}(u), \quad (3.284)$$

where:

$${r_{10}^{\mathrm{u}}}^{[0]}(u) = \mathcal{G}^{\mathrm{u}[0]}(\mathcal{F}(0))(u),$$

and:

$$\left\|{r_{11}^{\mathrm{u}}}^{[0]}\right\|_4 \leq K\left(\delta^{2p+6}+\delta^{p+4}\right). \quad (3.285)$$

First we deal with ${r_{10}^{\mathrm{u}}}^{[0]}(u)$. Recalling the definition (3.48) of $\mathcal{G}^{\mathrm{u}[l]}$ we have:

$${r_{10}^{\mathrm{u}}}^{[0]}(u) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_{-\infty}^{u}\frac{\mathcal{F}^{[0]}(0)(s)}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}s)}ds.$$

Now, from the definition (3.64) of $\mathcal{F}$ we obtain that:

$$\mathcal{F}^{[0]}(0)(u) = 2\sigma R_0(u) + \delta^p \left(F(0)\right)^{[0]} + \delta^p \frac{\mathrm{d}+1}{b} Z_0(u) \left(H(0)\right)^{[0]}.$$

We recall that $F(0)$ and $H(0)$ are an abuse of notation for:

$$F(0) = F(\delta R_0(u), \theta, \delta Z_0(u), \delta, \delta\sigma),$$

$$H(0) = H(\delta R_0(u), \theta, \delta Z_0(u), \delta, \delta\sigma).$$

Since $R_0(u)$ and $Z_0(u)$ are independent of θ, it is clear that:

$$(F(\delta R_0(u), \theta, \delta Z_0(u), \delta, \delta\sigma))^{[0]} = F^{[0]}(\delta R_0(u), \delta Z_0(u), \delta, \delta\sigma),$$

$$(H(\delta R_0(u), \theta, \delta Z_0(u), \delta, \delta\sigma))^{[0]} = H^{[0]}(\delta R_0(u), \delta Z_0(u), \delta, \delta\sigma).$$

From the definition (3.23) of F and H and taking into account that the functions f, g and h are of order three in all their variables, it is easy to see that:

$$\mathcal{F}^{[0]}(0)(u) = 2\sigma R_0(u) + \frac{\delta^{p+3}\mathcal{F}_1(u)}{\cosh^4(\mathrm{d}u)} + \frac{\delta^{p+4}\mathcal{F}_2(u,\delta,\sigma)}{\cosh^5(\mathrm{d}u)},$$

where:

$$\begin{aligned}\mathcal{F}_1(u) \quad = \quad & \cosh^4(\mathrm{d}u) \lim_{\delta\to 0} \delta^{-3} \Bigg(F^{[0]}(\delta R_0(u), \delta Z_0(u), \delta, \delta\sigma) \\ & + \frac{\mathrm{d}+1}{b} Z_0(u) H^{[0]}(\delta R_0(u), \delta Z_0(u), \delta, \delta\sigma) \Bigg). \end{aligned} \tag{3.286}$$

Moreover, $\mathcal{F}_i$, $i = 1, 2$, are functions such that for all $u \in D_{\kappa,\beta}$:

$$|\mathcal{F}_1(u)| \leq K, \qquad |\mathcal{F}_2(u,\delta,\sigma)| \leq K.$$

In fact, one can see that $\mathcal{F}_1(u)$ is defined in the limit $u \to i\pi/(2\mathrm{d})$ (see Lemma 3.5.14 for an explicit formula for $\mathcal{F}_1(u)$). Now, defining:

$$\tilde{\rho}_1^{\mathrm{u}}(u) = \mathcal{G}^{\mathrm{u}\,[0]} \left(\frac{\mathcal{F}_1(u)}{\cosh^4(\mathrm{d}u)} \right), \tag{3.287}$$

and:

$$\hat{\rho}_1^{\mathrm{u}}(u) = \mathcal{G}^{\mathrm{u}\,[0]}(2R_0(u)), \qquad \tilde{\rho}_2^{\mathrm{u}}(u) = \mathcal{G}^{\mathrm{u}\,[0]} \left(\frac{\mathcal{F}_2(u,\delta,\sigma)}{\cosh^5(\mathrm{d}u)} \right),$$

we can write:

$${r_{10}^{\mathrm{u}}}^{[0]}(u) = \mathcal{G}^{\mathrm{u}\,[0]}(\mathcal{F}(0))(u) = \sigma\hat{\rho}_1^{\mathrm{u}}(u) + \delta^{p+3}\tilde{\rho}_1^{\mathrm{u}}(u) + \delta^{p+4}\tilde{\rho}_2^{\mathrm{u}}(u). \tag{3.288}$$

By Lemma 3.2.11 it is clear that:

$$\|\hat{\rho}_1^{\mathrm{u}}\|_1 \leq K, \quad \|\tilde{\rho}_1^{\mathrm{u}}\|_3 \leq K, \quad \|\tilde{\rho}_2^{\mathrm{u}}\|_4 \leq K. \tag{3.289}$$

Moreover, one can easily see that $\tilde{\rho}_1^{\mathrm{u}}(u)\cosh^3(\mathrm{d}u)$ is defined as $u \to i\pi/(2\mathrm{d})$. Then we just need to define ρ_0^{u} and $\rho_1^{\mathrm{u}}(u)$ as:

$$\rho_0^{\mathrm{u}} = \lim_{u \to \frac{i\pi}{2\mathrm{d}}} \tilde{\rho}_1^{\mathrm{u}}(u) \frac{\cosh^2(\mathrm{d}u)}{\tanh(\mathrm{d}u)} = -i \lim_{u \to \frac{i\pi}{2\mathrm{d}}} \tilde{\rho}_1^{\mathrm{u}}(u)\cosh^3(\mathrm{d}u), \tag{3.290}$$

and:

$$\rho_1^{\mathrm{u}}(u) = \delta^{-(p+3)}\sigma\hat{\rho}_1^{\mathrm{u}}(u) + \tilde{\rho}_1^{\mathrm{u}}(u) - \rho_0^{\mathrm{u}}\frac{\tanh(\mathrm{d}u)}{\cosh^2(\mathrm{d}u)},$$

so that (3.288) writes out as:

$$r_{10}^{\mathrm{u}\,[0]}(u) = \delta^{p+3}\rho_0^{\mathrm{u}}\frac{\tanh(\mathrm{d}u)}{\cosh^2(\mathrm{d}u)} + \delta^{p+3}\rho_1^{\mathrm{u}}(u) + \delta^{p+4}\tilde{\rho}_2^{\mathrm{u}}(u). \tag{3.291}$$

Recalling that $|\sigma| \leq \delta^{p+3}\sigma^*$, since $\|\rho_1^{\mathrm{u}}\|_3$ is bounded and $\cosh^3(\mathrm{d}u)\rho_1^{\mathrm{u}}(u) \to 0$ as $u \to i\pi/(2\mathrm{d})$ it is easy to see that:

$$\|\rho_1^{\mathrm{u}}\|_2 \leq K.$$

One can also see that $\rho_1^{\mathrm{u}}(u)\cosh^2(\mathrm{d}u)$ is defined as $u \to i\pi/(2\mathrm{d})$. Finally, defining:

$$\rho_2^{\mathrm{u}}(u) = \tilde{\rho}_2^{\mathrm{u}}(u) + \delta^{-(p+4)}r_{11}^{\mathrm{u}\,[0]}(u)$$

(3.284) and (3.291) yield:

$$r_1^{\mathrm{u}[0]}(u) = \delta^{p+3}\rho_0^{\mathrm{u}}\frac{\tanh(\mathrm{d}u)}{\cosh^2(\mathrm{d}u)} + \delta^{p+3}\rho_1^{\mathrm{u}}(u) + \delta^{p+4}\rho_2^{\mathrm{u}}(u).$$

From (3.285) and (3.289) it is clear that for $p \geq -2$:

$$\|\rho_2^{\mathrm{u}}\|_4 \leq K.$$

The fact that $\rho_0^{\mathrm{u}} \in \mathbb{R}$, which is not obvious from its definition (3.290), is a consequence of the explicit formula of ρ_0^{u} given in Lemma 3.5.15. □

Lemma 3.5.13. *Let $|\sigma| \leq \delta^{p+3}\sigma^*$, and:*

$$\tilde{l}_2^{[0]}(u) = \lim_{\delta \to 0} \delta^{-p-3} l_2^{[0]}(u)\tanh^{-1}(\mathrm{d}u). \tag{3.292}$$

Define L_0 as the following limit, that is well defined:

$$L_0 = \lim_{u \to i\frac{\pi}{2\mathrm{d}}} \tilde{l}_2^{[0]}(u). \tag{3.293}$$

Then, there exist functions $L(u)$ and $\Lambda(u)$ and a constant M such that for all $u \in D_{\kappa,\beta}$:

$$\int_0^u l_2^{[0]}(w)dw = \delta^{p+3}\mathrm{d}^{-1}L_0 \log\cosh(\mathrm{d}u) + \delta L(u) + \delta\Lambda(u),$$

and:

$$\|L\|_0 \leq M\delta^{p+2} \qquad \|L'\|_0 \leq M\delta^{p+2}, \qquad \|\Lambda\|_1 \leq M\delta^{p+3}.$$

Moreover, $L_0 \in \mathbb{R}$, $L(0) = 0$ and $L(u)$ is defined on the limit $u \to i\pi/(2\mathrm{d})$.

Proof. Let $r_\lambda = (r_1^{\mathrm{u}} + r_1^{\mathrm{s}})/2 + \lambda(r_1^{\mathrm{u}} - r_1^{\mathrm{s}})/2$. Similarly as in the proof of Lemma 3.5.12, recalling the definition (3.23) of H, using that the function h is of order 3 in all their variables and that $|r_\lambda| \leq K\delta^{p+3}|\cosh(\mathrm{d}u)|^{-3}$ by Theorem 3.1.7, using Taylor expansions one can prove that:

$$\begin{aligned}\frac{1}{2}\int_{-1}^{1}(H(r_\lambda))^{[0]}d\lambda &= (H(0))^{[0]} + \frac{\delta^{p+6}\tilde{H}_2(u,\delta,\sigma)}{\cosh^4(\mathrm{d}u)} \\ &= \frac{\delta^3\tilde{H}_1(u)}{\cosh^3(\mathrm{d}u)} + \frac{\delta^4 H_2(u,\delta,\sigma)}{\cosh^4(\mathrm{d}u)},\end{aligned} \tag{3.294}$$

for some bounded functions $\tilde{H}_1$, $\tilde{H}_2$ and H_2. More precisely, $\tilde{H}_1$ is:

$$\begin{aligned}\tilde{H}_1(u) &= \cosh^3(\mathrm{d}u)\lim_{\delta\to 0}\delta^{-3}(H(0))^{[0]} \\ &= \cosh^3(\mathrm{d}u)\lim_{\delta\to 0}\delta^{-3}H^{[0]}(\delta R_0(u), \delta Z_0(u), \delta, \delta\sigma).\end{aligned} \tag{3.295}$$

One can see that it is defined on the limit $u \to \frac{i\pi}{2\mathrm{d}}$ (see Lemma 3.5.16 for an explicit formula of $\tilde{H}_1(u)$). Then, defining:

$$H_0 = \lim_{u\to i\frac{\pi}{2\mathrm{d}}}\frac{\tilde{H}_1(u)}{\sinh(\mathrm{d}u)} = -i\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\tilde{H}_1(u), \tag{3.296}$$

and:

$$H_1(u) = \frac{\tilde{H}_1(u) - H_0\sinh(\mathrm{d}u)}{\cosh(\mathrm{d}u)},$$

expression (3.294) writes out as:

$$\int_{-1}^{1}(H(r_\lambda))^{[0]}d\lambda = \delta^3 H_0\frac{\tanh(\mathrm{d}u)}{\cosh^2(\mathrm{d}u)} + \frac{\delta^3 H_1(u)}{\cosh^2(\mathrm{d}u)} + \frac{\delta^4 H_2(u,\delta,\sigma)}{\cosh^4(\mathrm{d}u)}.$$

The functions $H_1(u)$ and $H_2(u,\delta,\sigma)$ satisfy that for all $u \in D_{\kappa,\beta}$:

$$|H_1(u)| \leq K, \qquad |H_2(u,\delta,\sigma)| \leq K. \tag{3.297}$$

Moreover, one can see that $H_0 \in \mathbb{R}$ (see Lemma 3.5.16).

Then, from expression (3.283) of $l_2^{[0]}(u)$, using Lemma 3.5.12 and recalling that $(1 - Z_0^2(u))^{-1} = \cosh^2(\mathrm{d}u)$, we can write:

$$l_2^{[0]}(u) = \delta^{p+3} L_0 \tanh(\mathrm{d}u) + \delta^{p+3} L_1(u) + \delta^{p+4} L_2^{\mathrm{u}}(u), \tag{3.298}$$

with:

$$L_0 = -\frac{b}{\mathrm{d}}(\rho_0^{\mathrm{u}} + \rho_0^{\mathrm{s}}) - \frac{1}{\mathrm{d}} H_0, \tag{3.299}$$

$$L_1(u) = -\frac{b}{\mathrm{d}}(\rho_1^{\mathrm{u}}(u) + \rho_1^{\mathrm{s}}(u)) \cosh^2(\mathrm{d}u) - \frac{1}{\mathrm{d}} H_1(u),$$

$$L_2(u) = -\frac{b}{\mathrm{d}}(\rho_2^{\mathrm{u}}(u) + \rho_2^{\mathrm{s}}(u)) \cosh^2(\mathrm{d}u) - \frac{H_2(u,\delta,\sigma)}{\mathrm{d}\cosh^2(\mathrm{d}u)}.$$

Clearly $L_0 \in \mathbb{R}$. Using (3.297) and Lemma 3.5.12, we obtain:

$$\|L_1\|_0 \le K \qquad \|L_2\|_2 \le K.$$

Moreover, $L_1(u)$ is defined on the limit $u \to i\pi/(2\mathrm{d})$. From (3.298), it is also clear that:

$$\tilde{l}_2^{[0]}(u) = \lim_{\delta \to 0} \delta^{-p-3} l_2^{[0]}(u) \tanh^{-1}(\mathrm{d}u) = L_0 + L_1(u) \tanh^{-1}(\mathrm{d}u),$$

and then:

$$\lim_{u \to i\frac{\pi}{2\mathrm{d}}} \tilde{l}_2^{[0]}(u) = L_0 + \lim_{u \to i\frac{\pi}{2\mathrm{d}}} L_1(u) \tanh^{-1}(\mathrm{d}u) = L_0.$$

Finally we define:

$$L(u) = \delta^{p+2} \int_0^u L_1(w) dw, \tag{3.300}$$

and:

$$\Lambda(u) = \delta^{p+3} \int_0^u L_2(w) dw.$$

Then, integrating (3.298) one obtains:

$$\int_0^u l_2^{[0]}(w) dw = \delta^{p+3} \mathrm{d}^{-1} L_0 \log\cosh(\mathrm{d}u) + \delta L(u) + \delta \Lambda(u). \tag{3.301}$$

Clearly, since $\|L_1\|_0 \le K$, one immediately obtains $\|L\|_0 \le K\delta^{p+2}$ and $\|L'\|_0 \le K\delta^{p+2}$. The fact that $L(0) = 0$ is also obvious, and $L(u)$ is defined on the limit $u \to i\pi/(2\mathrm{d})$ because $L_1(u)$ is. To finish we note that, since $\|L_2\|_2 \le K$, using Lemma 3.5.9 one can easily see that $\|\Lambda\|_1 \le K\delta^{p+3}$. $\square$

End of the proof of Proposition 3.5.5. Let us define $C_1(u,\theta)$ as the unique fixed point of the operator $\tilde{\mathcal{A}}_1$ in the ball $B\left(2\llbracket\tilde{\mathcal{A}}_1(0)\rrbracket_{0,\omega}\right)$, which exists by Lemma 3.5.11. Let $\xi(u,\theta)$ be the function defined as:

$$\xi(\theta,u)=\theta+\delta^{-1}\alpha u+c\mathrm{d}^{-1}\cosh(\mathrm{d}u)+\delta^{-1}\alpha\int_0^u l_2^{[0]}(w)dw+C_1(u,\theta).$$

Since $C_1(u,\theta)$ satisfies the fixed point equation (3.272) (and thus it also satisfies the PDE (3.269)), it is easy to see that $\xi(u,\theta)$ satisfies the homogeneous PDE (3.242). Moreover, formula (3.245) is proved in Lemma 3.5.13 (see (3.301)). It remains to check that bounds (3.246) and (3.247) hold and that $(\xi(\theta,u),\theta)$ is injective.

First we shall see that $C_1(u,\theta)$ satisfies bound (3.246). We point out that this is not given directly by Lemma 3.5.11, but it can be obtained *a posteriori*. Indeed, by definition C_1 satisfies:

$$C_1=\hat{\mathcal{G}}(\mathcal{A}_1(C_1)).$$

By the definition of (3.270) of the operator $\mathcal{A}_1$, and since $\hat{\mathcal{G}}$ is linear, we can write:

$$C_1=\hat{\mathcal{G}}(\mathcal{A}_1(0))+\hat{\mathcal{G}}(l_2(u,\theta)\partial_u C_1+l_3(u,\theta)\partial_\theta C_1). \tag{3.302}$$

On the one hand, we recall bound (3.278) which stated:

$$\|\hat{\mathcal{G}}(\mathcal{A}_1(0))\|_{1,\omega}\le K\delta^{p+3}. \tag{3.303}$$

On the other hand, since $C_1\in B\left(2\llbracket\tilde{\mathcal{A}}_1(0)\rrbracket_{0,\omega}\right)$, by the definition of the norm $\llbracket.\rrbracket_{0,\omega}$ and the bound of $\llbracket\tilde{\mathcal{A}}_1(0)\rrbracket_{0,\omega}$ provided by Lemma 3.5.11, one has:

$$\|\partial_u C_1\|_{1,\omega}\le K\delta^{p+2},\qquad \|\partial_\theta C_1\|_{1,\omega}\le K\delta^{p+3}. \tag{3.304}$$

Then, using Lemma 3.5.2 and bounds (3.304) it is easy to see that:

$$\|l_2(u,\theta)\partial_u C_1+l_3(u,\theta)\partial_\theta C_1\|_{2,\omega}\le K\delta^{2p+5},$$

so that by item 3 of Lemma 3.5.10 we obtain:

$$\|\hat{\mathcal{G}}(l_2(u,\theta)\partial_u C_1+l_3(u,\theta)\partial_\theta C_1)\|_{1,\omega}\le K\delta^{2p+5}. \tag{3.305}$$

Using bounds (3.303) and (3.305) in equation (3.302), and recalling that $p\ge -2$, we obtain:

$$\|C_1\|_{1,\omega}\le K\delta^{p+3},$$

and then bound (3.246) is obtained.

Now we prove that $C(u,\theta)=\delta^{-1}\alpha\int_0^u l_2^{[0]}(w)dw+C_1(u,\theta)$ satisfies bounds (3.247). On the one hand, we have:

$$\partial_u C(u,\theta)=\delta^{-1}\alpha l_2^{[0]}(u)+\partial_u C_1(u,\theta),$$

so that using Lemma 3.5.2 and (3.304) it is clear that $\|\partial_u C\|_{1,\omega} \leq K\delta^{p+2}$. On the other hand, we have:

$$\partial_\theta C(u,\theta) = \partial_\theta C_1(u,\theta),$$

so that $\|\partial_\theta C\|_{1,\omega} \leq K\delta^{p+3}$ is a straight consequence of (3.304).

It only remains to prove that $(\xi(\theta,u),\theta)$ is injective. Let us assume $\xi(u_1,\theta) = \xi(u_2,\theta)$. This means:

$$u_1 - u_2 = \delta \mathrm{d}^{-1}\alpha^{-1}c(\log\cosh(\mathrm{d}u_1) - \log\cosh(\mathrm{d}u_2)) + \delta\alpha^{-1}(C(u_1,\theta) - C(u_2,\theta)). \quad (3.306)$$

On the one hand, for $u_1, u_2 \in D_{\kappa,\beta}$ we have:

$$\begin{aligned} & \delta \mathrm{d}^{-1}\alpha^{-1}c|\log\cosh(\mathrm{d}u_1) - \log\cosh(\mathrm{d}u_2)| \\ \leq\ & \delta \mathrm{d}^{-1}\alpha^{-1}c\int_0^1 |\tanh(u_2 + \lambda(u_1 - u_2))|d\lambda|u_1 - u_2| \\ \leq\ & \frac{K}{\kappa}|u_1 - u_2|. \end{aligned} \quad (3.307)$$

On the other hand, using the mean value theorem and bound (3.247):

$$\begin{aligned} \delta|C(u_1,\theta) - C(u_2,\theta)| &\leq \delta\int_0^1 |\partial_u C(u_2 + \lambda(u_1 - u_2),\theta)|d\lambda|u_1 - u_2| \\ &\leq \delta|u_1 - u_2| \sup_{(u,\theta)\in D_{\kappa,\beta}\times\mathbb{T}_\omega} |\partial_u C(u,\theta)| \\ &\leq \frac{K\delta^{p+2}}{\kappa}|u_1 - u_2|, \end{aligned} \quad (3.308)$$

Thus, using bounds (3.307) and (3.308) in (3.306), since $p \geq -2$, we know that there exists a constant K such that:

$$|u_1 - u_2| \leq \frac{K}{\kappa}|u_1 - u_2|.$$

Taking κ sufficiently large such that $K/\kappa < 1$ yields $u_1 = u_2$. □

Formulas for L_0 and $L(u)$

As we shall see, the constant L_0 and the function $L(u)$, introduced in Lemma 3.5.13, will appear in the leading term of the asymptotic expression of the distance between the two-dimensional invariant manifolds. Therefore, it is useful to have explicit formulas of L_0 and $L(u)$. That is what we proceed to do now. To that aim, we shall use formula (3.299) of L_0 and (3.300) of $L(u)$. More precisely, we shall rewrite them in terms of the

Taylor coefficients of the functions f, g and h. We use the following notation for these Taylor series (see (3.202)):

$$\begin{aligned} f(\delta x,\delta y,\delta z,\delta,\delta\sigma) &= \sum_{q=3}^{\infty}\delta^q \sum_{k+m+n\leq q} f_{qkmn}(\sigma)x^k y^m z^n, \\ g(\delta x,\delta y,\delta z,\delta,\delta\sigma) &= \sum_{q=3}^{\infty}\delta^q \sum_{k+m+n\leq q} g_{qkmn}(\sigma)x^k y^m z^n, \\ h(\delta x,\delta y,\delta z,\delta,\delta\sigma) &= \sum_{q=3}^{\infty}\delta^q \sum_{k+m+n\leq q} h_{qkmn}(\sigma)x^k y^m z^n. \end{aligned} \tag{3.309}$$

In the following we shall not write explicitly the dependence of f_{qkmn}, g_{qkmn} and h_{qkmn} with respect to σ.

Lemma 3.5.14. *Let us denote:*

$$a_{k,m}^{[0]} = \frac{1}{2\pi}\int_0^{2\pi} \cos^k\theta \sin^m\theta d\theta.$$

The function $\mathcal{F}_1(u)$ *defined in* (3.286) *is given by:*

$$\begin{aligned} \mathcal{F}_1(u) \;\; = & \sum_{k+m+n\leq 3} f_{3kmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k+1,m}^{[0]}\sinh^n(\mathrm{d}u)\cosh^{4-(k+m+n+1)}(\mathrm{d}u) \\ & + \sum_{k+m+n\leq 3} g_{3kmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k,m+1}^{[0]}\sinh^n(\mathrm{d}u)\cosh^{4-(k+m+n+1)}(\mathrm{d}u) \\ & + \sum_{k+m+n\leq 3} h_{3kmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+2} a_{k,m}^{[0]}\sinh^{n+1}(\mathrm{d}u)\cosh^{4-(k+m+n+1)}(\mathrm{d}u), \end{aligned}$$

where f_{qkmn}*,* g_{qkmn} *and* h_{qkmn} *are the Taylor coefficients of* f*,* g *and* h *introduced in* (3.309)*.*

Proof. To obtain this formula, one just has to take definition (3.286) of $\mathcal{F}_1(u)$, recall definitions (3.23) of F and H, substitute f, g and h by their Taylor series (3.309), take averages and note that as $\delta \to 0$ the only surviving terms in the sums are those with $q = 3$. Finally we only need to recall that:

$$R_0(u) = \frac{\mathrm{d}+1}{2b}\frac{1}{\cosh^2(\mathrm{d}u)}, \qquad\qquad Z_0(u) = \tanh(\mathrm{d}u),$$

and the desired formula is obtained. □

Lemma 3.5.15. *The constants ρ_0^{u} and ρ_0^{s} introduced in Lemma 3.5.12 are given by:*

$$\begin{aligned}\rho_0^{\mathrm{u}} = \rho_0^{\mathrm{s}} &= \frac{\mathrm{d}+1}{2b(3\mathrm{d}+2)}\left[\frac{(\mathrm{d}+1)}{4b}\left(f_{3120}+g_{3210}+3f_{3300}+3g_{3030}\right)-\left(f_{3102}+g_{3012}\right)\right.\\ &\left. -\frac{\mathrm{d}+1}{b}\left(h_{3201}+h_{3021}\right)+2h_{3003}\right],\end{aligned}$$

where f_{qkmn}, g_{qkmn} and h_{qkmn} are the Taylor coefficients of f, g and h introduced in (3.309)*. In particular, $\rho_0^{\mathrm{u}}, \rho_0^{\mathrm{s}} \in \mathbb{R}$.*

Proof. From the definition (3.290) of ρ_0^{u} one has:

$$\rho_0^{\mathrm{u}} = -i \lim_{u\to i\frac{\pi}{2\mathrm{d}}} \cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_{-\infty}^{u}\frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw,$$

where we have used definitions (3.287) of $\tilde{\rho}_1^{\mathrm{u}}$ and (3.48) of $\mathcal{G}^{\mathrm{u}[0]}$. Similarly, for ρ_0^{s} one has:

$$\rho_0^{\mathrm{s}} = -i \lim_{u\to i\frac{\pi}{2\mathrm{d}}} \cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_{+\infty}^{u}\frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw.$$

Recall that $\mathcal{F}_1(u)$ is defined in (3.286). Since $\mathcal{F}_1(w)$ is bounded, one has:

$$\left|\int_{\pm\infty}^{0}\frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw\right| \leq K.$$

Then, using that $\cosh(\mathrm{d}u)\to 0$ as $u\to i\pi/(2\mathrm{d})$, we have:

$$\begin{aligned}\rho_0^{\mathrm{u}} &= -i \lim_{u\to i\frac{\pi}{2\mathrm{d}}} \cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\left(\int_{-\infty}^{0}\frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw+\int_{0}^{u}\frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw\right)\\ &= -i \lim_{u\to i\frac{\pi}{2\mathrm{d}}} \cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_{0}^{u}\frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw,\end{aligned}$$

Similarly, we obtain:

$$\rho_0^{\mathrm{s}} = -i \lim_{u\to i\frac{\pi}{2\mathrm{d}}} \cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_{0}^{u}\frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw = \rho_0^{\mathrm{u}}.$$

Now, from the formula of $\mathcal{F}_1(u)$ given in Lemma 3.5.14, we can write:

$$\mathcal{F}_1(u) = \mathcal{F}_1^3(u) + \cosh(\mathrm{d}u)\mathcal{F}_1^{<3}(u),$$

where:

$$\begin{aligned}\mathcal{F}_1^3(u) &= \sum_{k+m+n=3} f_{3kmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k+1,m}^{[0]}\sinh^n(\mathrm{d}u)\\ &\quad+\sum_{k+m+n=3} g_{3kmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+1} a_{k,m+1}^{[0]}\sinh^n(\mathrm{d}u)\\ &\quad+\sum_{k+m+n=3} h_{3kmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m+2} a_{k,m}^{[0]}\sinh^{n+1}(\mathrm{d}u), \qquad (3.310)\end{aligned}$$

and $\mathcal{F}_1^{<3}(u)$ is a function (which corresponds to the terms in the sums with $k+m+n<3$) such that $\|\mathcal{F}_1^{<3}\|_0 \leq K$. Using this notation, we have:

$$\begin{aligned}\rho_0^{\mathrm{u}} &= -i\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\left(\int_0^u\frac{\mathcal{F}_1^3(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw+\int_0^u\frac{\mathcal{F}_1^{<3}(w)}{\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw\right)\\ &= -i\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_0^u\frac{\mathcal{F}_1^3(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw, \qquad (3.311)\end{aligned}$$

where we have used that (as a consequence of Lemma 3.5.9):

$$\left|\int_0^u\frac{\mathcal{F}_1^{<3}(w)}{\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw\right| \leq \frac{K}{|\cosh^{2+\frac{2}{\mathrm{d}}}(\mathrm{d}u)|},$$

and hence the second term vanishes. Now, we focus on the definition (3.310) of $\mathcal{F}_1^3(u)$. We recall that $a_{k,m}^{[0]}$ denotes the average of the function $\cos^k\theta\sin^m\theta$. Clearly, these coefficients are nonzero if and only if k and m are even. The only nonzero terms in (3.310) are:

$$a_{0,0}^{[0]}=1, \qquad a_{2,0}^{[0]}=a_{0,2}^{[0]}=\frac{1}{2}, \qquad a_{2,2}^{[0]}=\frac{1}{8}, \qquad a_{4,0}^{[0]}=a_{0,4}^{[0]}=\frac{3}{8}.$$

Substituting these values in (3.310) one obtains:

$$\begin{aligned}\mathcal{F}_1^3(u) &= \frac{1}{8}\left(\frac{\mathrm{d}+1}{b}\right)^2\left[f_{3120}+g_{3210}+3f_{3300}+3g_{3030}\right]\\ &\quad+\frac{\mathrm{d}+1}{2b}\sinh^2(\mathrm{d}u)\left[f_{3102}+g_{3012}+\frac{\mathrm{d}+1}{b}\left(h_{3201}+h_{3021}\right)\right]\\ &\quad+\frac{\mathrm{d}+1}{b}\sinh^4(\mathrm{d}u)h_{3003}. \qquad (3.312)\end{aligned}$$

Finally, we just need to note that:

$$\begin{aligned}&\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_0^u\frac{\sinh^2(\mathrm{d}w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw\\ &=\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\left(\int_0^u\frac{-1}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw+\int_0^u\frac{1}{\cosh^{2+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw\right)\\ &=\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_0^u\frac{-1}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw,\end{aligned}$$

and reasoning analogously:

$$\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_0^u\frac{\sinh^4(\mathrm{d}w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw=\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_0^u\frac{1}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw.$$

Having this in mind, we just need to substitute formula (3.312) of $\mathcal{F}_1^3(u)$ in (3.311) to obtain:

$$\begin{aligned}\rho_0^{\mathrm{u}} &= -i\frac{\mathrm{d}+1}{2b}\left[\frac{(\mathrm{d}+1)}{4b}\left(f_{3120}+g_{3210}+3f_{3300}+3g_{3030}\right)-\left(f_{3102}+g_{3012}\right)\right.\\ &\left.-\frac{\mathrm{d}+1}{b}\left(h_{3201}+h_{3021}\right)+2h_{3003}\right]\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_0^u\frac{1}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw.\end{aligned}$$

Finally, using L'Hôpital's rule, one can see that:

$$\lim_{u\to i\frac{\pi}{2\mathrm{d}}}\cosh^{3+\frac{2}{\mathrm{d}}}(\mathrm{d}u)\int_0^u\frac{1}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw=\frac{i}{3\mathrm{d}+2},$$

and thus we obtain the desired formula of ρ_0^{u}. □

Lemma 3.5.16. *The function $\tilde{H}_1(u)$ defined in* (3.295) *is given by:*

$$\tilde{H}_1(u)=\sum_{k+m+n\leq 3}h_{3kmn}\left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m}a_{k,m}^{[0]}\sinh^n(\mathrm{d}u)\cosh^{3-(k+m+n)}(\mathrm{d}u),$$

and the constant H_0 defined in (3.296) *is given by:*

$$H_0=-h_{3003}+\frac{\mathrm{d}+1}{2b}\left(h_{3021}+h_{3201}\right).$$

In particular, $H_0\in\mathbb{R}$.

Proof. To obtain the formula for $\tilde{H}_1(u)$ one has to do the same as in the proof of Lemma 3.5.14. That is, one takes definition (3.295) of $\tilde{H}_1(u)$, recalls definition (3.23) of H, substitutes h by its Taylor series (3.309), takes averages and takes into account that as $\delta \to 0$ the only surviving terms in the sum are those with $q = 3$. Finally one substitutes $R_0(u)$ and $Z_0(u)$ by their definitions.

Now we shall obtain the formula for H_0. We recall its definition, given in (3.296):

$$H_0 = -i \lim_{u \to i\frac{\pi}{2\mathrm{d}}} \tilde{H}_1(u).$$

We first point out that we can rewrite $\tilde{H}_1(u)$ as:

$$\tilde{H}_1(u) = \tilde{H}_1^3(u) + \cosh(\mathrm{d}u)\tilde{H}_1^{<3}(u),$$

where:

$$\tilde{H}_1^3(u) = \sum_{k+m+n=3} h_{3kmn} \left(\sqrt{\frac{\mathrm{d}+1}{b}}\right)^{k+m} a_{k,m}^{[0]} \sinh^n(\mathrm{d}u), \tag{3.313}$$

and $\tilde{H}_1^{<3}(u)$ is a function (that corresponds to the terms in the sum with $k+m+n < 3$) satisfying $\|\tilde{H}_1^{<3}\|_0 \leq K$. Clearly, one has:

$$H_0 = -i \lim_{u \to i\frac{\pi}{2\mathrm{d}}} \left(\tilde{H}_1^3(u) + \cosh(\mathrm{d}u)\tilde{H}_1^{<3}(u)\right) = -i \lim_{u \to i\frac{\pi}{2\mathrm{d}}} \tilde{H}_1^3(u).$$

Again, the only non-zero coefficients $a_{k,m}^{[0]}$ in (3.313) are those with even k and m, which are:

$$a_{0,0}^{[0]} = 1, \qquad a_{0,2}^{[0]} = a_{2,0}^{[0]} = \frac{1}{2}.$$

Then:

$$\tilde{H}_1^3(u) = h_{3003} \sinh^3(\mathrm{d}u) + \frac{\mathrm{d}+1}{2b} \sinh(\mathrm{d}u) \left(h_{3021} + h_{3201}\right).$$

Taking this into account, we obtain:

$$H_0 = -i \lim_{u \to i\frac{\pi}{2\mathrm{d}}} \tilde{H}_1^3(u) = -h_{3003} + \frac{\mathrm{d}+1}{2b} \left(h_{3021} + h_{3201}\right).$$

□

After obtaining explicit formulas for ρ_0^{u}, ρ_0^{s} and H_0, we can give an explicit formula for L_0.

Lemma 3.5.17. *The constant L_0 introduced in Lemma 3.5.13 is given by:*

$$\begin{aligned} L_0 &= \frac{-(\mathrm{d}+1)}{\mathrm{d}(3\mathrm{d}+2)} \left[\frac{(\mathrm{d}+1)}{4b} \left(f_{3120} + g_{3210} + 3f_{3300} + 3g_{3030}\right) - \left(f_{3102} + g_{3012}\right)\right. \\ &\quad \left. -\frac{\mathrm{d}+1}{b} \left(h_{3201} + h_{3021}\right) + 2h_{3003}\right] - \frac{1}{\mathrm{d}} \left[-h_{3003} + \frac{\mathrm{d}+1}{2b} \left(h_{3021} + h_{3201}\right)\right]. \end{aligned}$$

In the conservative case, one has:

$$L_0 = -h_{3003}.$$

Proof. One just needs to recall the definition (3.299) of L_0:

$$L_0 = \frac{-b}{\mathrm{d}}(\rho_0^{\mathrm{u}} + \rho_0^{\mathrm{s}}) - \frac{1}{\mathrm{d}}H_0,$$

and then use the formulas of ρ_0^{u} and ρ_0^{s} given in Lemma 3.5.15 and the formula of H_0 given in Lemma 3.5.16.

In the conservative case, we have that $\mathrm{d} = 1$. Moreover, since $\partial_x f + \partial_y g + \partial_z h = 0$, we obtain:

$$\begin{aligned} 3f_{3300} + g_{3210} + h_{3201} &= 0, \\ f_{3120} + 3g_{3030} + h_{3021} &= 0, \\ f_{3102} + g_{3012} + 3h_{3003} &= 0. \end{aligned}$$

Using these facts in the formula of L_0 one readily obtains that:

$$L_0 = -h_{3003}.$$

□

Remark 3.5.18. From Lemmas 3.5.14, 3.5.15 and 3.5.16 one can obtain an explicit formula for the function $L(u)$, recalling that by definition:

$$L(u) = \delta^{p+2} \int_0^u L_1(w)dw,$$

where:

$$L_1(u) = -\frac{b}{\mathrm{d}} \cosh^2(\mathrm{d}u)(\rho_1^{\mathrm{u}}(u) + \rho_1^{\mathrm{s}}(u)) - \frac{1}{\mathrm{d}} H_1(u),$$

and:

$$\rho_1^{\mathrm{u}}(u) = \delta^{-(p+3)}\sigma \int_{-\infty}^u \frac{R_0(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw + \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u) \int_{-\infty}^u \frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw - \rho_0^{\mathrm{u}} \frac{\tanh(\mathrm{d}u)}{\cosh^2(\mathrm{d}u)},$$

$$\rho_1^{\mathrm{s}}(u) = \delta^{-(p+3)}\sigma \int_{+\infty}^u \frac{R_0(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw + \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u) \int_{+\infty}^u \frac{\mathcal{F}_1(w)}{\cosh^{4+\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw - \rho_0^{\mathrm{s}} \frac{\tanh(\mathrm{d}u)}{\cosh^2(\mathrm{d}u)},$$

$$H_1(u) = \frac{\tilde{H}_1(u) - H_0 \sinh(\mathrm{d}u)}{\cosh(\mathrm{d}u)}.$$

3.5.5 Proof of Proposition 3.5.7

We distinguish between the dissipative and the conservative case, since in the first case P_1 will be found using a fixed point equation, while in the latter it will be defined in terms of the function $C(u,\theta)$ of Proposition 3.5.5.

The dissipative case

In this subsection we will follow the same exact steps as in Subsection 3.5.4 in order to prove Proposition 3.5.7. Again, we will find a particular solution of equation (3.255):

$$\hat{\mathcal{L}}(P_1) = \mathcal{B}(P_1),$$

where $\hat{\mathcal{L}}$ was defined in (3.251) and $\mathcal{B}$ was defined in (3.253). We shall do it by solving the fixed point equation:

$$P_1 = \tilde{\mathcal{B}}(P_1), \tag{3.314}$$

where $\tilde{\mathcal{B}} = \hat{\mathcal{G}} \circ \mathcal{B}$, and $\hat{\mathcal{G}}$ is the operator defined by (3.256) and (3.257), and $\mathcal{B}$ is defined in (3.253).

Lemma 3.5.19. *For κ big enough and $p \geq -2$, the operator $\tilde{\mathcal{B}} : \tilde{\mathcal{X}}_{1,\omega} \to \tilde{\mathcal{X}}_{1,\omega}$, and it has a unique fixed point in the ball $B\left(2\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega}\right) \subset \tilde{\mathcal{X}}_{1,\omega}$. Moreover, there exists a constant M such that $\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega} \leq K\delta^{p+3}$.*

Proof. First of all, we shall prove the bound:

$$\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega} \leq K\delta^{p+3}. \tag{3.315}$$

Indeed, we have:

$$\mathcal{B}(0)(u,\theta) = 2\sigma + l_1(u,\theta) + 2Z_0(u) l_2(u,\theta).$$

Using Lemma 3.5.2 and that $|\sigma| \leq \sigma^* \delta^{p+3}$, it is straightforward to prove that there exists a constant K such that:

$$\|\mathcal{B}(0)\|_{2,\omega} \leq K\delta^{p+3}.$$

Then item 6 of Lemma 3.5.10 yields bound (3.315).

Next step is to find the Lipschitz constant of the operator $\tilde{\mathcal{B}}$. We claim that if $\phi_1, \phi_2 \in B\left(2\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega}\right)$, then there exists a constant K such that:

$$\llbracket \tilde{\mathcal{B}}(\phi_1) - \tilde{\mathcal{B}}(\phi_2) \rrbracket_{1,\omega} \leq K \frac{\delta^{p+2}}{\kappa} \llbracket \phi_1 - \phi_2 \rrbracket_{1,\omega}. \tag{3.316}$$

Again, by item 6 of Lemma 3.5.10 it is enough to prove:

$$\|\mathcal{B}(\phi_1) - \mathcal{B}(\phi_2)\|_{2,\omega} \leq K \frac{\delta^{p+2}}{\kappa} \llbracket \phi_1 - \phi_2 \rrbracket_{1,\omega}. \tag{3.317}$$

We have:

$$\begin{aligned}\|\mathcal{B}(\phi_1)-\mathcal{B}(\phi_2)\|_{2,\omega} \;\leq\;& 2|\sigma|\|\phi_1-\phi_2\|_{2,\omega}+\|l_1\|_{1,\omega}\|\phi_1-\phi_2\|_{1,\omega}\\ &+2\|Z_0\|_{1,\omega}\|l_2\|_{0,\omega}\|\phi_1-\phi_2\|_{1,\omega}+\|l_2\|_{0,\omega}\|\partial_u(\phi_1-\phi_2)\|_{2,\omega}\\ &+\|l_3\|_{1,\omega}\|\partial_\theta(\phi_1-\phi_2)\|_{1,\omega}. \end{aligned} \tag{3.318}$$

First we note that, since $\sigma=\mathcal{O}(\delta^{p+3})$:

$$|\sigma|\|\phi_1-\phi_2\|_{2,\omega}\leq K\frac{\delta^{p+2}}{\kappa}\|\phi_1-\phi_2\|_{1,\omega}\leq K\frac{\delta^{p+2}}{\kappa}[\![\phi_1-\phi_2]\!]_{1,\omega}. \tag{3.319}$$

Similarly, by Lemma 3.5.2:

$$\|l_1\|_{1,\omega}\leq K\frac{\delta^{p+2}}{\kappa},\qquad \|l_2\|_{0,\omega}\leq K\frac{\delta^{p+2}}{\kappa},\qquad \|l_3\|_{1,\omega}\leq K\frac{\delta^{p+2}}{\kappa}. \tag{3.320}$$

Finally, we just need to note that:

$$\|\partial_\theta(\phi_1-\phi_2)\|_{1,\omega}\leq\frac{K}{\delta\kappa}\|\partial_\theta(\phi_1-\phi_2)\|_{2,\omega}\leq K[\![\phi_1-\phi_1)]\!]_{1,\omega}. \tag{3.321}$$

Using the definition of the norm $[\![.]\!]_{1,\omega}$, the fact that $\|Z_0\|_{1,\omega}\leq K$ and bounds (3.319), (3.320) and (3.321) in equation (3.318) we obtain immediately bound (3.317).

To finish the proof, we just need to take κ large enough such that the Lipschitz constant in (3.316) is smaller than 1. Then $\tilde{\mathcal{B}}:B\left(2[\![\tilde{\mathcal{B}}(0)]\!]_{1,\omega}\right)\to B\left(2[\![\tilde{\mathcal{B}}(0)]\!]_{1,\omega}\right)$ and since it is contractive, it has a unique fixed point in this ball. □

End of the proof of Proposition 3.5.7. We define P_1 as the unique fixed point of $\mathcal{B}$ in $B\left(2[\![\tilde{\mathcal{B}}(0)]\!]_{1,\omega}\right)$, whose existence is given by Lemma 3.5.19. Clearly, using bound (3.315), one has:

$$[\![P_1]\!]_{1,\omega}\leq 2[\![\tilde{\mathcal{B}}(0)]\!]_{1,\omega}\leq K\delta^{p+3}$$

The fact P_1 satisfies equation (3.248) is clear since it is a solution of equation (3.314). Moreover, one can easily check that $P(u,\theta)=\cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))$ satisfies equation (3.237).

Finally, since:

$$\sup_{(u,\theta)\in D_{\kappa,\beta}\times\mathbb{T}_\omega}|P_1(u,\theta)|\leq[\![P_1]\!]_{1,\omega}\sup_{(u,\theta)\in D_{\kappa,\beta}\times\mathbb{T}_\omega}|\cosh^{-1}(\mathrm{d}u)|\leq K\frac{\delta^{p+2}}{\kappa},$$

taking κ sufficiently large we obtain:

$$|1+P_1(u,\theta)|\geq 1-K\frac{\delta^{p+2}}{\kappa}\neq 0.$$

Since $\cosh^{2/\mathrm{d}}(\mathrm{d}u)\neq 0$ for $u\in D_{\kappa,\beta}$ we can ensure that, for $(u,\theta)\in D_{\kappa,\beta}\times\mathbb{T}_\omega$, $P(u,\theta)=\cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\neq 0$. □

The conservative case

We recall that in the conservative case we have d $=1$ and $\sigma=0$. In the following, whenever these parameters appear they will be automatically substituted for these values.

Proof of Proposition 3.5.7 (conservative case). Let:

$$P_1(u,\theta)=\frac{\partial_u C(u,\theta)-l_3(u,\theta)}{\delta^{-1}\alpha+cZ_0(u)+l_3(u,\theta)}, \tag{3.322}$$

where $C(u,\theta)$ is the function given by Proposition 3.5.5 and $l_3(u,\theta)$ is defined in (3.240). First let us check that it satisfies bound (3.250), that is:

$$|P_1(u,\theta)|\le\frac{K}{\kappa}\delta^{p+2}, \tag{3.323}$$

for all $(u,\theta)\in D_{\kappa,\beta}\times\mathbb{T}_\omega$. On the one hand, note that by Proposition 3.5.5 and Lemma 3.5.2 we have :

$$|\partial_u C(u,\theta)|\le\frac{K}{\kappa}\delta^{p+1},\qquad |l_3(u,\theta)|\le\frac{K}{\kappa^2}\delta^{p+1}. \tag{3.324}$$

On the other hand, taking κ sufficiently large, we also have:

$$\left|\frac{1}{\delta^{-1}\alpha+cZ_0(u)+l_3(u,\theta)}\right|\le K\delta. \tag{3.325}$$

Then (3.323) follows directly from using (3.324) and (3.325) in (3.322).

It only remains to prove that P_1 defined as in (3.322) satisfies equation (3.248). For clarity, we recall that equation (3.248) is:

$$\begin{aligned}\left(-\delta^{-1}\alpha-cZ_0(u)\right)\partial_\theta P_1+\partial_u P_1 &= (l_1(u,\theta)+2Z_0(u)l_2(u,\theta))(1+P_1)\\ &\quad+l_2(u,\theta)\partial_u P_1+l_3(u,\theta)\partial_\theta P_1.\end{aligned} \tag{3.326}$$

We also recall the equation satisfied by $C(u,\theta)$ (given in (3.243)), since we shall use it in the following:

$$(-\delta^{-1}\alpha-cZ_0(u))\partial_\theta C+\partial_u C=l_2(u,\theta)(\delta^{-1}\alpha+cZ_0(u)+\partial_u C)+l_3(u,\theta)(1+\partial_\theta C). \tag{3.327}$$

First of all, from definition (3.322) of P_1 we obtain the following equation for P_1:

$$\begin{aligned}&\left[-\delta^{-1}\alpha-cZ_0(u)-l_3(u,\theta)\right]\partial_\theta P_1+\partial_u P_1\\ &\quad=\frac{\left[-\delta^{-1}\alpha-cZ_0(u)-l_3(u,\theta)\right]\partial_\theta(\partial_u C)+\partial_u(\partial_u C)}{\delta^{-1}\alpha+cZ_0(u)+l_3(u,\theta)}+\partial_\theta l_3(u,\theta)(1+P_1)\\ &\qquad-\frac{\partial_u l_3(u,\theta)+\left[c(1-Z_0^2(u))+\partial_u l_3(u,\theta)\right]P_1}{\delta^{-1}\alpha+cZ_0(u)+l_3(u,\theta)}\end{aligned} \tag{3.328}$$

If we differentiate both sides of equation (3.327) with respect to u, we obtain the following PDE for $\partial_u C$:

$$\begin{aligned}\left[-\delta^{-1}\alpha - cZ_0(u) - l_3(u,\theta)\right]\partial_\theta(\partial_u C) - \left[c(1-Z_0^2(u)) + \partial_u l_3(u,\theta)\right]\partial_\theta C + \partial_u(\partial_u C)\\ =\partial_u l_2(u,\theta)\left[\delta^{-1}\alpha + cZ_0(u) + \partial_u C\right] + l_2(u,\theta)\left[c(1-Z_0^2(u)) + \partial_u(\partial_u C)\right]\\ + \partial_u l_3(u,\theta). \qquad (3.329)\end{aligned}$$

Now, on the one hand, from (3.327) and keeping in mind the definition of P_1 one can see that:

$$\partial_\theta C = P_1 - l_2(u,\theta)(1+P_1). \qquad (3.330)$$

On the other hand, differentiating (3.322) with respect to u gives:

$$\begin{aligned}\partial_u(\partial_u C) \;=\; &\left[\delta^{-1}\alpha + cZ_0(u) + l_3(u,\theta)\right]\partial_u P_1 + \left[c(1-Z_0^2(u)) + \partial_u l_3(u,\theta)\right]P_1\\ &+\partial_u l_3(u,\theta). \qquad (3.331)\end{aligned}$$

Using equalities (3.330) and (3.331) in equation (3.329) yields the following PDE for $\partial_u C$:

$$\begin{aligned}\left[-\delta^{-1}\alpha - cZ_0(u) - l_3(u,\theta)\right]\partial_\theta(\partial_u C) + \partial_u(\partial_u C)\\ =c(1-Z_0^2(u))P_1 + \partial_u l_3(u,\theta)P_1\\ + \partial_u l_2(u,\theta)\left[\delta^{-1}\alpha + cZ_0(u) + l_3(u,\theta)\right](1+P_1(u,\theta))\\ + l_2(u,\theta)\left[\delta^{-1}\alpha + cZ_0(u) + l_3(u,\theta)\right]\partial_u P_1 + \partial_u l_3(u,\theta). \qquad (3.332)\end{aligned}$$

Then substituting (3.332) in (3.328) one obtains:

$$\begin{aligned}(-\delta^{-1}\alpha - cZ_0(u) - l_3(u,\theta))\partial_\theta P_1 + \partial_u P_1\\ =(\partial_u l_2(u,\theta) + \partial_\theta l_3(u,\theta))(1+P_1) + l_2(u,\theta)\partial_u P_1. \qquad (3.333)\end{aligned}$$

In order to see that equations (3.333) and (3.326) are the same, one just needs that:

$$l_1(u,\theta) + 2Z_0(u)l_2(u,\theta) = \partial_u l_2(u,\theta) + \partial_\theta l_3(u,\theta). \qquad (3.334)$$

Let us write the right hand side of the equality explicitly. Using an analogous notation for H as introduced in (3.37), recalling expression (3.39) for $Z_0'(u)$, definitions (3.239) of l_2 and (3.240) of l_3 and that:

$$\frac{d}{du}\left(\frac{1}{1-Z_0^2(u)}\right) = \frac{2Z_0(u)}{1-Z_0^2(u)},$$

one has:

$$\begin{aligned}\partial_u l_2(u,\theta)+\partial_\theta l_3(u,\theta) &= 2Z_0(u)l_2(u,\theta)-\frac{b}{1-Z_0^2(u)}(\partial_u r_1^{\mathrm{u}}+\partial_u r_1^{\mathrm{s}})\\ &\quad -\frac{\delta^p}{2}\int_{-1}^{1}(\partial_z H(r_\lambda)+\partial_\theta G(r_\lambda))d\lambda\\ &\quad -\frac{\delta^p}{2(1-Z_0^2(u))}\int_{-1}^{1}\partial_r H(r_\lambda)(-2R_0(u)Z_0(u)+\partial_u r_\lambda)d\lambda\\ &\quad -\frac{\delta^p}{2}\int_{-1}^{1}\partial_r G(r_\lambda)\partial_\theta r_\lambda d\lambda. \end{aligned} \tag{3.335}$$

Finally, one just has to note that since the divergence of the vector field is zero (also in the coordinates (r,θ,z), since the change (3.21) is symplectic) one has:

$$\partial_z H(r_\lambda)+\partial_\theta G(r_\lambda)=-\partial_r F(r_\lambda),$$

and using the fact that:

$$\frac{R_0(u)Z_0(u)}{1-Z_0^2(u)}=\frac{1}{b}Z_0(u)$$

it is clear from the definitions (3.238) and (3.239) of l_1 and l_2 that (3.335) is:

$$\partial_u l_2(u,\theta)+\partial_\theta l_3(u,\theta)=l_1(u,\theta)+2Z_0(u)l_2(u,\theta),$$

so that (3.334) is proved, and hence P_1 defined in (3.322) satisfies equation (3.326). □

3.6 Proof of Theorem 3.1.10

3.6.1 The conservative case

In this subsection we shall study the coefficients $\Upsilon^{[0]}$ and $\Upsilon_0^{[0]}$ in the conservative case. We recall that in this setting we have $\mathrm{d}=1$ and $\sigma=0$. Whenever we refer to previous formulas and expressions where these parameters appear, we shall substitute them for these values directly.

Proposition 3.6.1. *If the vector field* (3.22) *is conservative, the Melnikov function* $M(u,\theta)$ *has zero average. More precisely, recalling definition* (3.71) *of the coefficients* $\Upsilon_0^{[l]}$*, one has:*

$$\Upsilon_0^{[0]}=0.$$

Proof. We begin by averaging in system (3.22). Recalling that $\sigma = 0$, $\mathrm{d} = 1$, the averaged system is:

$$\begin{aligned} \frac{dr}{dt} &= -2rz + \delta^p F^{[0]}(\delta r, \delta z, \delta,), \\ \frac{d\theta}{dt} &= -\frac{\alpha}{\delta} - cz + \delta^p G^{[0]}(\delta r, \delta z, \delta), \\ \frac{dz}{dt} &= -1 + 2br + z^2 + \delta^p H^{[0]}(\delta r, \delta z, \delta). \end{aligned} \tag{3.336}$$

Since this system is still conservative (taking averages does not change this fact), one has:

$$\partial_r F^{[0]}(\delta r, \delta z, \delta) = -\partial_z H^{[0]}(\delta r, \delta z, \delta). \tag{3.337}$$

Using (3.337) one can easily see that system (3.336) has the following first integral:

$$\mathcal{U}(r,z) = -r + br^2 + rz^2 + \delta^p \int_0^r H^{[0]}(\delta s, \delta z, \delta) ds. \tag{3.338}$$

Now, let us recall that by definition (3.71):

$$\Upsilon_0^{[0]} = \int_{-\infty}^{+\infty} \frac{\mathcal{F}^{[0]}(0)(w)}{\cosh^2 w} dw. \tag{3.339}$$

Now we just need to note that:

$$\begin{aligned} \frac{d}{dw}\left(\mathcal{U}(R_0(w), Z_0(w))\right) &= \frac{-\delta^p F^{[0]}(\delta R_0(w), \delta Z_0(w), \delta) - \delta^p \frac{2}{b} Z_0(w) H^{[0]}(\delta R_0(w), \delta Z_0(w), \delta)}{\cosh^2 w} \\ &= -\frac{\mathcal{F}^{[0]}(0)(w)}{\cosh^2 w}, \end{aligned}$$

where again we have used (3.337), the fact that $F^{[0]}(0, \delta Z_0(w), \delta) = 0$ (this is clear from definition (3.23) of F) and that since the heteroclinic of the unperturbed system does not depend on θ, one has:

$$\begin{aligned} \left(F(\delta R_0(w), \theta, \delta Z_0(w), \delta)\right)^{[0]} &= F^{[0]}(\delta R_0(w), \delta Z_0(w), \delta), \\ \left(H(\delta R_0(w), \theta, \delta Z_0(w), \delta)\right)^{[0]} &= H^{[0]}(\delta R_0(w), \delta Z_0(w), \delta). \end{aligned}$$

We point out that $\frac{d}{dw}\left(\mathcal{U}(R_0(w), Z_0(w))\right)$ is not zero since $(R_0(w), Z_0(w))$ is a solution of the unperturbed system but not of the whole system (3.336), and thus $\mathcal{U}$ is not constant along $(R_0(w), Z_0(w))$. Then we have:

$$\Upsilon_0^{[0]} = -\int_{-\infty}^{+\infty} \frac{d}{dw}\left(\mathcal{U}(R_0(w), Z_0(w))\right) dw = -\lim_{t\to\infty} \left[\mathcal{U}(R_0(t), Z_0(t)) - \mathcal{U}(R_0(-t), Z_0(-t))\right].$$

Noting that $(R_0(\pm t), Z_0(\pm t)) \to (0, \pm 1)$ as $t \to \pm\infty$ and that $\mathcal{U}(0, \pm 1) = 0$, we obtain $\Upsilon_0^{[0]} = 0$. □

Now we will prove that $\Upsilon^{[0]} = 0$. This proof is more involved and requires some previous considerations. We shall use the fact that, in the conservative setting, the 2-dimensional invariant manifolds of S_+ and S_- always intersect. This can be seen using standard arguments of volume preservation. Let us introduce some notation concerning this intersection. First of all we fix $\theta_0 \in [0, 2\pi)$, which is arbitrary but will remain the same for the rest of this subsection. We consider the following plane:

$$\Sigma_{\theta_0} = \{(x, y, z) \in \mathbb{R}^3 \, : \, x \sin\theta_0 - y\cos\theta_0 = 0\}.$$

We define p_1 as the first intersection of the $2-$dimensional invariant manifolds of S_+ and S_- contained in the section Σ_{θ_0}. This point p_1 is $\mathcal{O}(\delta^{p+3})$-close to $(\frac{1}{b}\cos\theta_0, \frac{1}{b}\sin\theta_0, 0)$, which is the first intersection in the unperturbed case. The orbit of p_1, namely:

$$\Gamma_{p_1} := \{\varphi_t(p_1), \, t \in \mathbb{R}\}, \tag{3.340}$$

where φ_t stands for the flow the vector field (3.19), is a heteroclinic orbit and for small δ it intersects many times the section Σ_{θ_0}. We define:

$$t_2 = \min\{t > 0 \, : \, \varphi_t(p_1) \in \Sigma_{\theta_0}\}, \quad p_2 = \varphi_{t_2}(p_1),$$

and:

$$t_3 = \min\{t > t_2 \, : \, \varphi_t(p_1) \in \Sigma_{\theta_0}\}, \quad p_3 = \varphi_{t_3}(p_1).$$

Remark 3.6.2. Note that, since $\dot{\theta} < 0$ (this can be easily seen in equation (3.22), provided that δ is sufficiently small), p_2 has angular variable $\theta_0 - \pi$ and p_3 has angular variable $\theta_0 - 2\pi$.

Let $\pi_z : \mathbb{R}^3 \to \mathbb{R}$ denote the projection to the third component. Then we define:

$$z_i = \pi_z(p_i) \tag{3.341}$$

and we define u_i as:

$$u_i = Z_0^{-1}(z_i) = \operatorname{atanh}(z_i), \tag{3.342}$$

See Figure 3.6a.

We point out that with this notation we can write:

$$\Delta(u_i, \theta_0) = 0, \qquad i = 1, \ldots, 3,$$

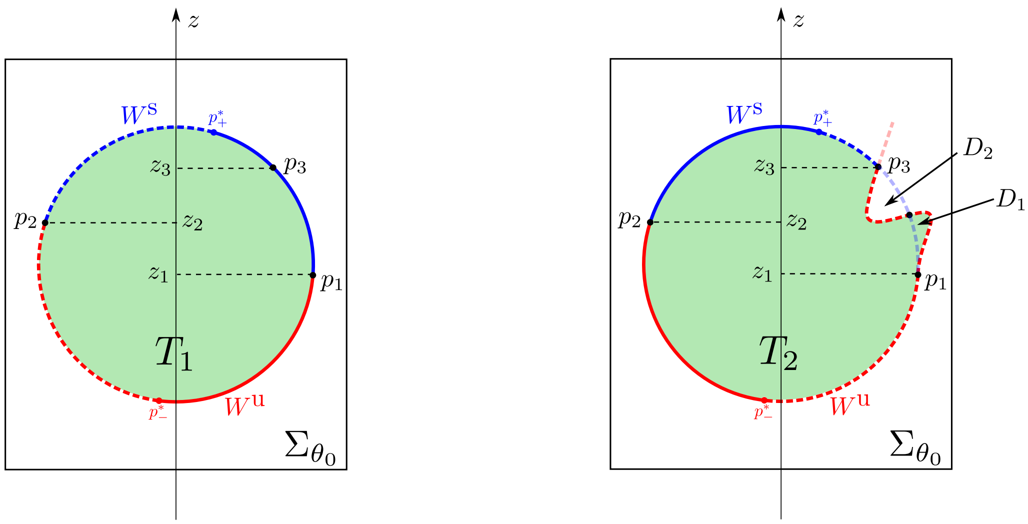
where as usual $\Delta(u, \theta) = r^{\mathrm{u}}(u, \theta) - r^{\mathrm{s}}(u, \theta)$.

Lemma 3.6.3. *Let u_1 and u_3 be defined as in* (3.342)*. Define:*

$$\tau^* = \xi(u_1, \theta_0) = \theta_0 + \delta^{-1}\alpha u_1 + c\log\cosh u_1 + C(u_1, \theta_0),$$

where $\xi(u, \theta)$ and $C(u, \theta)$ are the functions given in Theorem 3.1.9. Then:

$$\xi(u_3, \theta_0) = \theta_0 + \delta^{-1}\alpha u_3 + c\log\cosh u_3 + C(u_3, \theta_0) = \tau^* + 2\pi.$$



(a) The domain T_1 on the section Σ_{θ_0}. (b) The domain T_2 on the section Σ_{θ_0}.

Figure 3.6: The domains T_1 and T_2. In red, the unstable manifold of S_-, and in blue the stable manifold of S_+. The continuous (respectively, discontinuous) lines on the left are mapped to the continuous (discontinuous) lines on the right with the same color via the flow ϕ.

Proof. Let $s_0 > 1$. For any $s \in [-s_0, s_0]$, we define $u = u(s)$ as the (unique) solution of:

$$\xi(u(s), \theta_0 - 2\pi s) = \theta_0 - 2\pi s + \delta^{-1}\alpha u(s) + c \log\cosh u(s) + C(u(s), \theta_0 - 2\pi s) = \tau^*. \quad (3.343)$$

The fact that equation (3.343) has a unique solution for all $s \in [-s_0, s_0]$ if δ is sufficiently small can bee seen, for instance, by the implicit function theorem. By definition of τ^*, the unique solution at $s = 0$ is $u(0) = u_1$.

Now, since $\Delta(u_1, \theta_0) = 0$ and $\Delta(u, \theta) = \cosh^2(u)(1 + P_1(u, \theta))\tilde{k}(\xi(u, \theta))$ by Theorem 3.1.9, using that $\cosh(u_1) \neq 0$ and that P_1 is small we have:

$$0 = \tilde{k}(\xi(u_1, \theta_0)) = \tilde{k}(\tau^*) = \tilde{k}(\xi(u(s), \theta_0 - 2\pi s)).$$

Consequently, $\Delta(u(s), \theta_0 - 2\pi s) = 0$. Hence defining $r(s) := r^{\mathrm{u}}(u(s), \theta_0 - 2\pi s)$, we have that the curve:

$$\gamma(s) := (r(s), \theta_0 - 2\pi s, Z_0(u(s))), \qquad s \in [-s_0, s_0],$$

is part of a heteroclinic orbit expressed in the symplectic polar coordinates. Since $u(0) = u_1$ and p_1 in these coordinates is $(r^{\mathrm{u}}(u_1, \theta_0), \theta_0, Z_0(u_1)) = \gamma(0)$, clearly $\gamma(s)$ is a part of the heteroclinic orbit Γ_{p_1}, defined in (3.340).

Taking $s = 1$, we obtain the point in Γ_{p_1} with angular variable $\theta_0 - 2\pi$. By Remark 3.6.2, this point is precisely p_3. This implies that $u(1) = u_3$, and then equation (3.343) yields.

$$\theta_0 - 2\pi + \delta^{-1}\alpha u_3 + c\log\cosh u_3 + C(u_3, \theta_0 - 2\pi) = \tau^*,$$

and since $C(u, \theta)$ is 2π periodic in θ we obtain:

$$\theta_0 + \delta^{-1}\alpha u_3 + c\log\cosh u_3 + C(u_3, \theta_0) = \tau^* + 2\pi.$$

□

Lemma 3.6.4. *Let u_1 and u_3 be the $u-$coordinate of the heteroclinic points $p_1, p_3 \in \Sigma_{\theta_0}$ respectively, defined in* (3.342). *Let $\Upsilon^{[0]}$ be the average of the function $\tilde{k}(\tau)$ given in Theorem 3.1.9. Then one has:*

$$\Upsilon^{[0]} = \frac{1}{2\pi}\int_{u_1}^{u_3} \frac{\Delta(u, \theta_0)}{\cosh^2(u)(1 + P_1(u, \theta_0))}\left(\delta^{-1}\alpha + cZ_0(u) + \partial_u C(u, \theta_0)\right) du.$$

Proof. It can be obtained straightforwardly from the fact that:

$$\Upsilon^{[0]} = \frac{1}{2\pi}\int_{\tau^*}^{\tau^*+2\pi} \tilde{k}(\tau)d\tau,$$

where $\tau^* = \theta_0 + \delta^{-1}\alpha u_1 + c\log\cosh u_1 + C(u_1, \theta_0)$. Indeed, one just has to perform the change $\tau = \theta_0 + \delta^{-1}\alpha u + c\log\cosh u + C(u, \theta_0)$. Then, recalling that by Theorem 3.1.9:

$$\Delta(u, \theta_0) = \cosh^2(u)(1 + P_1(u, \theta_0))\tilde{k}(\theta_0 + \delta^{-1}\alpha u + c\log\cosh(u) + C(u, \theta_0)),$$

and that by Lemma 3.6.3:

$$\theta_0 + \delta^{-1}\alpha u_3 + c\log\cosh u_3 + C(u_3, \theta_0) = \tau^* + 2\pi,$$

one obtains the claim of the lemma. □

Proposition 3.6.5. *One has:*

$$\int_{u_1}^{u_3} \frac{\Delta(u, \theta_0)}{\cosh^2(u)}\left(\delta^{-1}\alpha + cZ_0(u) + l_3(u, \theta_0)\right) du = 0. \tag{3.344}$$

Proof. Recall the definition (3.341) of z_1 and z_3. Let us denote by $\tilde{r}^{\mathrm{u}}(z, \theta) := r^{\mathrm{u}}(Z_0^{-1}(z), \theta)$ the $r-$component of the unstable manifold of S_- as a function of z and θ, and similarly $\tilde{r}^{\mathrm{s}}(z, \theta)$ for the stable manifold of S_+. We denote:

$$\tilde{G}(r, z) = G(\delta r, \theta_0, \delta z, \delta),$$

where G is the function defined in (3.23) (recall that in the conservative case there is no dependence on the parameter σ). We shall prove the following:

$$\int_{z_1}^{z_3}\int_{\tilde{r}^{\mathrm{s}}(z,\theta_0)}^{\tilde{r}^{\mathrm{u}}(z,\theta_0)} (\delta^{-1}\alpha + cz - \delta^p \tilde{G}(r,z))drdz = 0. \qquad (3.345)$$

This yields claim (3.344). Indeed, assume (3.345) is true. Then we make the change:

$$r = \tilde{r}_\lambda := \frac{1}{2}(\tilde{r}^{\mathrm{u}}(z,\theta_0) + \tilde{r}^{\mathrm{s}}(z,\theta_0)) + \frac{\lambda}{2}(\tilde{r}^{\mathrm{u}}(z,\theta_0) - \tilde{r}^{\mathrm{s}}(z,\theta_0)), \quad \lambda \in [-1,1],$$

and, denoting $\tilde{\Delta}(z,\theta) = \tilde{r}^{\mathrm{u}}(z,\theta_0) - \tilde{r}^{\mathrm{s}}(z,\theta_0)$, equation (3.345) becomes:

$$\int_{z_1}^{z_3}\left(\delta^{-1}\alpha + cz - \frac{1}{2}\int_{-1}^{1}\delta^p\tilde{G}(\tilde{r}_\lambda, z)d\lambda\right)\tilde{\Delta}(z,\theta_0)dz = 0.$$

Then, we perform the change $z = Z_0(u)$ and recalling the definition (3.342) of u_1 and u_3, and the definition (3.240) of l_3 we obtain (3.344).

To prove (3.345) we shall use basically that the system is divergence-free, and apply the divergence theorem in a suitable 3−dimensional domain. However, we first need to introduce some notation. Consider the intersection of the 2−dimensional unstable manifold of S_- and Σ_{θ_0}. The lower part of this intersection is a curve that joins p_1 and p_2, having a shape close to an arch of ellipse. Similarly, if we consider the intersection of the 2−dimensional stable manifold of S_+ and Σ_{θ_0}, its upper part is a curve that also joins p_1 and p_2, with a similar shape. We define $T_1 \subset \Sigma_{\theta_0}$ as de domain bounded by these two curves (see Figure 3.6a).

In the following we shall denote $X(x,y,z) : \mathbb{R}^3 \to \mathbb{R}^3$ the vector field defining our system, given by (3.19), and X_1, X_2 and X_3 each of its components. We note that if $p \in \partial T_1$ and:

$$X_1(p)\sin\theta_0 - X_2(p)\cos\theta_0 \neq 0$$

then there exists a unique $\tau(p) > 0$ such that $\varphi_{\tau(p)}(p)$ is the next intersection of the orbit going through p and Σ_{θ_0}. This is clear from the fact that the orbits inside $W^{\mathrm{u}}(S_-)$ are $\mathcal{O}(\delta)$−close to the orbits of the heteroclinic connection of the unperturbed system for $t \in (-\infty, T]$, for some constant T, and the same happens for the orbits inside $W^{\mathrm{s}}(S_+)$ and $t \in [T, \infty)$. We also point out that there are just two points $p_-^*, p_+^* \in \Sigma_{\theta_0}$ (close to S_- and S_+ respectively) such that:

$$X_1(p_\pm^*)\sin\theta_0 - X_2(p_\pm^*)\cos\theta_0 = 0.$$

See Figure 3.6a. For such points we can define $\tau(p_\pm^*) = 0$. With this definition, the function $\varphi_{\tau(p)}(p)$ is continuous for $p \in \partial T_1$. Then we define $T_2 \subset \Sigma_{\theta_0}$ (see Figure 3.6b) as the domain bounded by ∂T_2, where:

$$\partial T_2 = \{\varphi_{\tau(p)}(p) \,:\, p \in \partial T_1\}.$$

Finally we define:

$$T_3 = \{\varphi_t(p) \,:\, p \in \partial T_1,\, t \in (0, \tau(p))\}.$$

We point out that T_3 is tangent to the flow of X. Moreover, T_1, T_2 and T_3 are the boundary of a closed 3−dimensional domain. That is, there exists a closed domain $V \subset \mathbb{R}^3$ such that:

$$T_1 \cup T_2 \cup T_3 = \partial V.$$

Now we shall use the divergence theorem in this domain. Since $\mathrm{div} X \equiv 0$ we have:

$$\begin{aligned} 0 &= \iiint_V \mathrm{div} X dV = \iint_{\partial V} X \cdot \vec{n}_{\partial V} dS \\ &= \iint_{T_1} X \cdot \vec{n}_{T_1} dS + \iint_{T_2} X \cdot \vec{n}_{T_2} dS + \iint_{T_3} X \cdot \vec{n}_{T_3} dS, \end{aligned} \tag{3.346}$$

where $\vec{n}_{\partial V}$ denotes the unitary normal vector to ∂V pointing outside V, and the same with $\vec{n}_{T_i}$, $i = 1, 2, 3$. On the one hand, recall T_3 is tangent to the flow, and hence:

$$X \cdot \vec{n}_{T_3} = 0.$$

On the other hand, one has that:

$$\vec{n}_{T_1} = (-\sin\theta_0, \cos\theta_0, 0) = -\vec{n}_{T_2}.$$

Thus (3.346) becomes:

$$\begin{aligned} 0 &= \iint_{T_2} (X_1 \sin\theta_0 - X_2 \cos\theta_0) dS - \iint_{T_1} (X_1 \sin\theta_0 - X_2 \cos\theta_0) dS \\ &= \iint_{D_1} (X_1 \sin\theta_0 - X_2 \cos\theta_0) dS - \iint_{D_2} (X_1 \sin\theta_0 - X_2 \cos\theta_0) dS, \end{aligned} \tag{3.347}$$

where $D_1 = T_2 \setminus T_1$ and $D_2 = T_1 \setminus T_2$ (see Figure 3.6b). Taking the parameterization:

$$x = \sqrt{2r}\cos\theta_0, \quad y = \sqrt{2r}\sin\theta_0, \quad z = z,$$

it is easy to see that (3.347) writes out as:

$$0 = \int_{z_1}^{z_3} \int_{\tilde{r}^{\mathrm{s}}(z,\theta_0)}^{\tilde{r}^{\mathrm{u}}(z,\theta_0)} \frac{X_1(\sqrt{2r}\cos\theta_0, \sqrt{2r}\sin\theta_0, z)\sin\theta_0 - X_2(\sqrt{2r}\cos\theta_0, \sqrt{2r}\sin\theta_0, z)\cos\theta_0}{\sqrt{2r}} dr dz. \tag{3.348}$$

Finally, we just need to note that:

$$\begin{aligned} & X_1(\sqrt{2r}\cos\theta_0, \sqrt{2r}\sin\theta_0, z)\sin\theta_0 - X_2(\sqrt{2r}\cos\theta_0, \sqrt{2r}\sin\theta_0, z)\cos\theta_0 \\ = & \sqrt{2r}\left(\delta^{-1}\alpha + cz - \delta^p \tilde{G}(r,z)\right), \end{aligned}$$

and then (3.348) yields (3.345). □

End of the proof of Theorem 3.1.10 (conservative case). On one hand, Prop. 3.6.1 states directly that $\Upsilon_0^{[0]} = 0$. On the other hand, to see that $\Upsilon^{[0]} = 0$ we note that from Proposition 3.5.7, we choose P_1 such that:

$$\frac{\delta^{-1}\alpha + cZ_0(u) + \partial_u C(u,\theta_0)}{1 + P_1(u,\theta_0)} = \delta^{-1}\alpha + cZ_0(u) + l_3(u,\theta_0).$$

Then, substituting this in the equality of Lemma 3.6.4 we get:

$$\Upsilon^{[0]} = \frac{1}{2\pi}\int_{u_1}^{u_3} \frac{\Delta(u,\theta_0)}{\cosh^2(u)} \left(\delta^{-1}\alpha + cZ_0(u) + l_3(u,\theta_0)\right) du.$$

Finally, Proposition 3.6.5 yields that $\Upsilon^{[0]} = 0$, and the proof is finished. □

3.6.2 The dissipative case

The dissipative case is relatively easier than the conservative one, since it follows from the implicit function theorem. For clarity next proposition states again the result contained in Theorem 3.1.10 concerning this case.

Proposition 3.6.6. *Let $p \geq -2$. Let a_1, $a_2 \in \mathbb{R}$ and $a_3 > 0$. Then there exists $\delta_0 = \delta_0(a_1, a_2, a_3)$ and a curve $\sigma = \sigma_*(\delta) = \mathcal{O}(\delta^{p+3})$ such that for all $0 \leq \delta \leq \delta_0$ one has:*

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\delta, \sigma_*(\delta)) = a_1 \delta^{a_2} e^{-\frac{a_3\pi}{2\mathrm{d}\delta}},$$

and:

$$\Upsilon_0^{[0]} = \Upsilon_0^{[0]}(\delta, \sigma_*(\delta)) = \mathcal{O}(\delta^{p+4}).$$

Proof. We have:

$$\Upsilon^{[0]} = \frac{1}{2\pi}\int_0^{2\pi} \tilde{k}(\tau) d\tau. \tag{3.349}$$

We perform the change $\tau = \theta + C(0,\theta)$ in the previous integral, where $C(u,\theta)$ is the function in Theorem 3.1.9. Since also by Theorem 3.1.9 we have:

$$\tilde{k}(\theta + \delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta)) = \frac{\Delta(u,\theta)}{\cosh^{2/\mathrm{d}}(\mathrm{d}u)(1 + P_1(u,\theta))},$$

after this change (3.349) becomes:

$$\Upsilon^{[0]} = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \frac{\Delta(0,\theta)}{1 + P_1(0,\theta)}(1 + \partial_\theta C(0,\theta)) d\theta. \tag{3.350}$$

Here:

$$\theta_1 = 0 + \mathcal{O}\left(\delta^{p+3}\right) \qquad \theta_2 = 2\pi + \mathcal{O}\left(\delta^{p+3}\right), \tag{3.351}$$

where we have used the bounds for $C(u,\theta)$ obtained in Theorem 3.1.9 for $u=0$. Now, on the one hand, by Theorem 3.1.7 we have:

$$|\Delta(0,\theta)| \leq |r_1^{\mathrm{u}}(0,\theta)| + |r_1^{\mathrm{s}}(0,\theta)| \leq K\delta^{p+3}. \tag{3.352}$$

and recalling the notation $M(u,\theta) = r_{10}^{\mathrm{u}}(u,\theta) - r_{10}^{\mathrm{s}}(u,\theta)$:

$$|\Delta(0,\theta) - M(0,\theta)| \leq |r_{11}^{\mathrm{u}}(0,\theta)| + |r_{11}^{\mathrm{s}}(0,\theta)| \leq K\left(\delta^{2p+6} + \delta^{p+4}\right), \tag{3.353}$$

where we have used the bounds of $r_{11}^{\mathrm{u,s}}(u,\theta)$ given in Theorem 3.1.7. On the other hand, by Proposition 3.5.5:

$$|\partial_\theta C(0,\theta)| \leq K\delta^{p+3}, \tag{3.354}$$

and Proposition 3.5.7 gives:

$$\left|\frac{1}{1+P_1(0,\theta)} - 1\right| \leq K|P_1(0,\theta)| \leq K\delta^{p+3}. \tag{3.355}$$

Thus, using bounds (3.351), (3.352), (3.354) and (3.355) in equation (3.350) we obtain:

$$\Upsilon^{[0]} = \frac{1}{2\pi}\int_0^{2\pi} M(0,\theta)d\theta + \mathcal{O}(\delta^{p+4}) = M^{[0]}(0) + \mathcal{O}(\delta^{p+4}), \tag{3.356}$$

where we have used that $p \geq -2$. Now, by formula (3.68) of $M^{[l]}(u)$, the Fourier coefficients of the Melnikov function, and recalling the definitions (3.64) of $\mathcal{F}$ and (3.24) of R_0, we get:

$$\begin{aligned} M^{[0]}(0) &= \int_{-\infty}^{+\infty} \frac{\mathcal{F}^{[0]}(0)(w)}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw \\ &= \sigma\frac{\mathrm{d}+1}{b}\int_{-\infty}^{+\infty}\frac{1}{\cosh^{\frac{2}{\mathrm{d}}+2}(\mathrm{d}w)}dw + \delta^p\int_{-\infty}^{+\infty}\frac{F^{[0]}(0)+\frac{\mathrm{d}+1}{b}Z_0(w)H^{[0]}(0)}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw. \end{aligned}$$

We introduce the following notation:

$$I = \frac{\mathrm{d}+1}{b}\int_{-\infty}^{+\infty}\frac{1}{\cosh^{\frac{2}{\mathrm{d}}+2}(\mathrm{d}w)}dw,$$

$$J = \delta^{-3}\int_{-\infty}^{+\infty}\frac{F^{[0]}(0)+\frac{\mathrm{d}+1}{b}Z_0(w)H^{[0]}(0)}{\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}w)}dw.$$

We recall the notation (3.37) of F and H, and observe that for all $w \in \mathbb{R}$:

$$|F(0)| = |F(\delta R_0(w), \theta, \delta Z_0(w), \delta, \delta\sigma)| \leq K\delta^3,$$

$$|H(0)| = |H(\delta R_0(w), \theta, \delta Z_0(w), \delta, \delta\sigma)| \leq K\delta^3,$$

so that J is bounded as $\delta \to 0$. Thus, we can rewrite (3.356) as:

$$\Upsilon^{[0]} = \sigma I + \delta^{p+3} J + \mathcal{O}(\delta^{p+4}).$$

Then, putting $\sigma = \hat{\sigma}\delta^{p+3}$, we have that $\Upsilon^{[0]} = a_1 \delta^{a_2} e^{-\frac{a_3\pi}{2\mathrm{d}\delta}}$ if:

$$f(\hat{\sigma}, \delta) := \hat{\sigma} I + J + \mathcal{O}(\delta) - a_1 \delta^{a_2 - p - 3} e^{-\frac{a_3\pi}{2\mathrm{d}\delta}} = 0.$$

It is clear that $I \neq 0$, and thus:

$$f\left(-\frac{J}{I}, 0\right) = 0, \qquad \frac{\partial f}{\partial \hat{\sigma}}\left(-\frac{J}{I}, 0\right) = I \neq 0,$$

where we have used that $a_3 > 0$ so that the last term and all its derivatives vanish at $\delta = 0$. Then we can apply the implicit function theorem, so that there exists δ_0 and a curve $\hat{\sigma}_*(\delta) = -J/I + \mathcal{O}(\delta)$ such that $f(\hat{\sigma}_*(\delta), \delta) = 0$ for all $0 \leq \delta \leq \delta_0$. The curve $\sigma_*(\delta) := \hat{\sigma}_*(\delta)\delta^{p+3}$ is the one in the statement of the lemma.

Clearly, since $\Upsilon_0^{[0]} = M^{[0]}(0) = \sigma I + \delta^{p+3} J$, one has:

$$\Upsilon_0^{[0]} = \Upsilon_0^{[0]}(\delta, \sigma_*(\delta)) = \Upsilon_0^{[0]}\left(\delta, -\frac{J}{I}\delta^{p+3} + \mathcal{O}(\delta^{p+4})\right) = \mathcal{O}(\delta^{p+4}).$$

□

3.7 Proof of Proposition 3.1.12

Proof of Proposition 3.1.12. Let us define $\Delta_1(u, \theta) = \Delta(u, \theta) - \Delta_0(u, \theta)$. For the sake of clarity, we recall definition (3.83) of $\Delta_0(u, \theta)$:

$$\Delta_0(u, \theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1 + P_1(u, \theta))\tilde{k}_0(\theta + \delta^{-1}\alpha u + \mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u, \theta)),$$

where:

$$\tilde{k}_0(\tau) = \Upsilon^{[0]} + \sum_{l \neq 0} \Upsilon_0^{[l]} e^{2\pi i \tau}.$$

Note that by Theorem 3.1.9 and the definition Δ_0 we have:

$$\Delta_1(u, \theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1 + P_1(u, \theta)) \sum_{l \neq 0} \left(\Upsilon^{[l]} - \Upsilon_0^{[l]}\right) e^{il(\theta + \delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u, \theta))}.$$

We point out that in order to obtain sharp bounds for $\Upsilon^{[l]} - \Upsilon_0^{[l]}$ we need to take $u \in D_{\kappa,\beta} \subset \mathbb{C}$, but θ can be taken real. Thus, in this proof, instead of considering $\theta \in \mathbb{T}_\omega$ we will take $\theta \in \mathbb{S}^1$.

Now we consider:

$$\tilde{\Delta}_1(w,\theta) = \sum_{l\neq 0}\left(\Upsilon^{[l]} - \Upsilon_0^{[l]}\right)e^{il(\theta+\delta^{-1}\alpha w)}.$$

Let us define the function:

$$F(u,\theta) = u + \delta\alpha^{-1}\left[\mathrm{c}\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta)\right].$$

It can be easily seen that for all $(u,\theta) \in D_{\kappa,\beta}\times\mathbb{S}^1$ the change $(w,\theta) = (F(u,\theta),\theta)$ is a diffeomorphism between $D_{\kappa,\beta}\times\mathbb{S}^1$ and its image $\tilde{D}_{\kappa,\beta}\times\mathbb{S}^1$, with inverse $(u,\theta) = (G(w,\theta),\theta)$. In other words, the function $G(w,\theta)$ satisfies:

$$\tilde{\Delta}_1(w,\theta) = \frac{\Delta_1(G(w,\theta),\theta)}{\cosh^{2/\mathrm{d}}(\mathrm{d}G(w,\theta))(1+P_1(G(w,\theta),\theta))}. \tag{3.357}$$

Note that $\tilde{\Delta}_1(w,\theta)$ is $2\pi-$periodic in θ, and its $l-$th Fourier coefficient is:

$$\tilde{\Delta}_1^{[l]}(w) = \left(\Upsilon^{[l]} - \Upsilon_0^{[l]}\right)e^{i\delta^{-1}\alpha w}.$$

Hence we know that for all $w\in\tilde{D}_{\kappa,\beta}$:

$$\left|\Upsilon^{[l]} - \Upsilon_0^{[l]}\right| = \frac{1}{2\pi}\left|e^{-i\delta^{-1}\alpha wl}\int_0^{2\pi}\tilde{\Delta}_1(w,\theta)e^{-il\theta}d\theta\right| \leq \left|e^{-i\delta^{-1}\alpha wl}\right|\sup_{\theta\in\mathbb{S}^1}\left|\tilde{\Delta}_1(w,\theta)\right|. \tag{3.358}$$

We note that this is valid for all $w\in\tilde{D}_{\kappa,\beta}$. Let us denote $u_\pm = \pm i\left(\frac{\pi}{2\mathrm{d}} - \kappa\delta\right)$. We shall take w in (3.358) as $w = w_+ := F(u_+,\theta)\in\tilde{D}_{\kappa,\beta}$ for $l<0$, and $w = w_- := F(u_-,\theta)\in\tilde{D}_{\kappa,\beta}$ for $l>0$. One has:

$$\begin{aligned}\left|e^{il\delta^{-1}\alpha w_\pm}\right| &= e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta}-\alpha\kappa\right)|l| - |l|\mathrm{c}\mathrm{d}^{-1}\mathrm{Im}\,\log\cosh(\mathrm{d}u_\pm) - |l|\mathrm{Im}\,C(u_\pm,\theta)} \\ &\leq e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta}-\alpha\kappa\right)|l| + (\mathrm{c}\mathrm{d}^{-1}|\mathrm{Im}\,\log\cosh(\mathrm{d}u_\pm)| + |\mathrm{Im}\,C(u_\pm,\theta)|)|l|}.\end{aligned} \tag{3.359}$$

Now, on the one hand we have:

$$\mathrm{Im}\,\log\cosh(\mathrm{d}u_\pm) = \arg(\cosh(\mathrm{d}u_\pm)).$$

Note that $\cosh(\mathrm{d}u_\pm) = \sin(\delta\kappa)\in\mathbb{R}_+$, so that:

$$\mathrm{Im}\,\log\cosh(\mathrm{d}u_\pm) = 0.$$

On the other hand we have:

$$|\mathrm{Im}\,C(u_\pm,\theta)| \leq |C(u_\pm,\theta)| \leq K\delta^{p+2}|\log(\delta\kappa)| \leq M_2,$$

for a certain constant M_2, where we have used Theorem 3.1.9 and that $p > -2$. Substituting the previous identity and bound in (3.359) we get:

$$\left| e^{il\delta^{-1}\alpha w_{\pm}} \right| \leq e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta} - \alpha(\kappa+M_2)\right)|l|}. \tag{3.360}$$

Then taking $w = w_{\pm}$ in (3.358) and using (3.360) we get:

$$\left| \Upsilon^{[l]} - \Upsilon_0^{[l]} \right| \leq K e^{-\frac{\alpha}{\delta}\left(\frac{\pi}{2\mathrm{d}} - (\kappa+M_2)\delta\right)|l|} \sup_{\theta\in\mathbb{S}^1} \left| \tilde{\Delta}_1(w_{\pm},\theta) \right|. \tag{3.361}$$

So it only remains to bound $\tilde{\Delta}_1$. On the one hand, it is clear by (3.357) that:

$$\begin{aligned} \sup_{\theta\in\mathbb{S}^1} \left| \tilde{\Delta}_1(w_{\pm},\theta) \right| &= \sup_{\theta\in\mathbb{S}^1} \left| \frac{\Delta_1(G(w_{\pm},\theta),\theta)}{\cosh^{2/\mathrm{d}}(\mathrm{d}G(w_{\pm},\theta))(1+P_1(G(w_{\pm},\theta),\theta))} \right| \\ &\leq K(\delta\kappa)^{-\frac{2}{\mathrm{d}}} \sup_{\theta\in\mathbb{S}^1} \left| \Delta_1(u_{\pm},\theta) \right|. \end{aligned} \tag{3.362}$$

We claim that exists a constant K such that for all $\theta \in \mathbb{S}^1$:

$$|\Delta_1(u_{\pm},\theta)| \leq K \left(\frac{\delta^{2(p+1)}}{\kappa^3} |\log(\delta\kappa)| + \frac{\delta^{p+3}}{\kappa} \right). \tag{3.363}$$

We shall write Δ_1 in a more adequate way to prove bound (3.363). Recall that $\Delta_1(u,\theta) = \Delta(u,\theta) - \Delta_0(u,\theta)$. Now, note that, from the definition (3.83) of Δ_0 and the definition (3.82) of $\tilde{k}_0$, we can rewrite Δ_0 as:

$$\begin{aligned} \Delta_0(u,\theta) &= \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\Upsilon^{[0]} && (3.364) \\ &\quad + \cosh^{2/\mathrm{d}}(\mathrm{d}u) \sum_{l\neq 0} \Upsilon_0^{[l]} e^{2\pi il(\theta+\delta^{-1}\alpha + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta))} \\ &\quad + \cosh^{2/\mathrm{d}}(\mathrm{d}u) P_1(u,\theta) \sum_{l\neq 0} \Upsilon_0^{[l]} e^{2\pi il(\theta+\delta^{-1}\alpha + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta))}. && (3.365) \end{aligned}$$

For the sake of clarity, let us recall equality (3.70) involving the Melnikov function $M(u,\theta)$ defined in (3.66). This equality states:

$$M(u,\theta) = \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u) \sum_{l\in\mathbb{Z}} \Upsilon_0^{[l]} e^{il(\theta+\delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))}. \tag{3.366}$$

Using this, it is easy to see that (3.364) can be rewritten as:

$$\begin{aligned} \Delta_0(u,\theta) &= \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\Upsilon^{[0]} + M(u,\theta) - \cosh^{2/\mathrm{d}}(\mathrm{d}u)\Upsilon_0^{[0]} \\ &\quad + \cosh^{2/\mathrm{d}}(\mathrm{d}u) \sum_{l\neq 0} \Upsilon_0^{[l]} e^{2\pi il(\theta+\delta^{-1}\alpha + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))} \left(e^{ilC(u,\theta)} - 1\right) \\ &\quad + \cosh^{2/\mathrm{d}}(\mathrm{d}u) P_1(u,\theta) \sum_{l\neq 0} \Upsilon_0^{[l]} e^{2\pi il(\theta+\delta^{-1}\alpha + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u) + C(u,\theta))}. \end{aligned} \tag{3.367}$$

Taking into account that by (3.65) and (3.66):

$$\Delta(u,\theta) = M(u,\theta) + r_{11}^{\mathrm{u}}(u,\theta) - r_{11}^{\mathrm{s}}(u,\theta) \tag{3.368}$$

subtracting (3.368) and (3.367), we can rewrite $\Delta_1 = \Delta - \Delta_0$ as:

$$\begin{aligned}\Delta_1(u,\theta) &= r_{11}^{\mathrm{u}}(u,\theta) - r_{11}^{\mathrm{s}}(u,\theta) - \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\Upsilon^{[0]} + \cosh^{2/\mathrm{d}}(\mathrm{d}u)\Upsilon_0^{[0]} \\ &\quad - \cosh^{2/\mathrm{d}}(\mathrm{d}u)\sum_{l\neq 0}\Upsilon_0^{[l]} e^{il(\theta+\delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))}\left(e^{ilC(u,\theta)}-1\right) \\ &\quad - \cosh^{2/\mathrm{d}}(\mathrm{d}u)P_1(u,\theta)\sum_{l\neq 0}\Upsilon_0^{[l]} e^{il(\theta+\delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u)+C(u,\theta))}. \end{aligned} \tag{3.369}$$

Recalling that the Melnikov function $M(u,\theta)$ is periodic in θ, and that from (3.366) its Fourier coefficients are:

$$M^{[l]}(u) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)\Upsilon_0^{[l]} e^{il(\delta^{-1}\alpha u + c\mathrm{d}^{-1}\log\cosh(\mathrm{d}u))}.$$

Thus (3.369) can be written as:

$$\begin{aligned}\Delta_1(u,\theta) &= r_{11}^{\mathrm{u}}(u,\theta) - r_{11}^{\mathrm{s}}(u,\theta) - \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\Upsilon^{[0]} + \cosh^{2/\mathrm{d}}(\mathrm{d}u)\Upsilon_0^{[0]} \\ &\quad - \sum_{l\neq 0} M^{[l]}(u)e^{il\theta}\left(e^{ilC(u,\theta)}-1\right) - P_1(u,\theta)\sum_{l\neq 0} M^{[l]}(u)e^{il(\theta+C(u,\theta))}. \end{aligned} \tag{3.370}$$

We shall prove bound (3.363) by bounding each term in the decomposition (3.370) of Δ_1, with $u = u_\pm$. First of all, from Theorem 3.1.7 we can easily bound the first term in (3.370):

$$\begin{aligned}|r_{11}^{\mathrm{u}}(u_\pm,\theta) - r_{11}^{\mathrm{s}}(u_\pm,\theta)| &\leq K\left(\delta^{2p+6}|\cosh(\mathrm{d}u_\pm)|^{-4} + \delta^{p+4}|\cosh(\mathrm{d}u_\pm)|^{-1}\right) \\ &\leq K\left(\frac{\delta^{2(p+1)}}{\kappa^4} + \frac{\delta^{p+3}}{\kappa}\right). \end{aligned} \tag{3.371}$$

By Theorem 3.1.10, the second term in (3.370) is zero in the conservative case, since $\Upsilon^{[0]} = 0$. In the dissipative case, since we take $\sigma = \sigma_*(\delta)$, this term satisfies:

$$|\cosh^{2/\mathrm{d}}(\mathrm{d}u_\pm)(1+P_1(u_\pm,\theta))\Upsilon^{[0]}| \leq K\delta^{a_2}e^{-\frac{a_3\pi}{2\mathrm{d}\delta}}, \tag{3.372}$$

where we have used that by Theorem 3.1.9:

$$|P_1(u_\pm,\theta)| \leq K\frac{\delta^{p+2}}{\kappa}. \tag{3.373}$$

To bound the third term in (3.370) we just need to use again Theorem 3.1.10 to obtain the following bound in the dissipative case (since we take $\sigma = \sigma_*(\delta)$):

$$\left|\cosh^{2/\mathrm{d}}(\mathrm{d}u_\pm)\Upsilon_0^{[0]}\right| \leq K\delta^{p+4}. \tag{3.374}$$

In the conservative case we have $\Upsilon_0^{[0]} = 0$, so that this term does not appear.

Now, using that $|e^z - 1| \le |z|e^{|z|}$, by Theorem 3.1.9 we know that:

$$\left|e^{ilC(u_\pm,\theta)} - 1\right| \le |lC(u_\pm,\theta)|e^{|lC(u_\pm,\theta)|} \le K|l|\delta^{p+2}|\log(\delta\kappa)|e^{K|l|\delta^{p+2}|\log(\delta\kappa)|}. \tag{3.375}$$

We also point out that the Fourier coefficients of the Melnikov function, $M^{[l]}(u)$, satisfy:

$$|M^{[l]}(u_\pm)| \le K\frac{\delta^p}{\kappa^3}e^{-|l|\omega}. \tag{3.376}$$

To prove (3.376), first one has to take into account that $M(u,\theta) = r_{10}^{\mathrm{u}}(u,\theta) - r_{10}^{\mathrm{s}}(u,\theta)$ by (3.66). We point out that for $\theta \in \mathbb{S}^1$ we have $\theta - i\frac{|l|}{l}\omega \in \mathbb{T}_\omega$. Since for any u, $r_{10}^{\mathrm{u,s}}(u,.)$ are analytic in $\mathbb{T}_\omega$, then one has that:

$$M^{[l]}(u) = \frac{e^{-|l|\omega}}{2\pi}\int_0^{2\pi} M\left(u, \theta - i\frac{|l|}{l}\omega\right)e^{-il\theta}d\theta. \tag{3.377}$$

Then, we just need to note that by Theorem 3.1.7:

$$\begin{aligned}
\sup_{\theta\in\mathbb{S}^1}\left|M\left(u_\pm, \theta - i\frac{|l|}{l}\omega\right)\right| &\le \sup_{\theta\in\mathbb{T}_\omega}\left(|r_{10}^{\mathrm{u}}(u_\pm,\theta)| + |r_{10}^{\mathrm{s}}(u_\pm,\theta)|\right) \\
&\le K\frac{\delta^{p+3}}{|\cosh(\mathrm{d}u_\pm)|^3} \le K\frac{\delta^p}{\kappa^3}.
\end{aligned}$$

Using this bound in (3.377) one obtains (3.376).

Now, using bounds (3.375) and (3.376) (and taking into account that θ is real) we obtain the bound for the fourth term in (3.370):

$$\begin{aligned}
\left|\sum_{l\neq 0} M^{[l]}(u_\pm)e^{il\theta}\left(e^{ilC(u_\pm,\theta)} - 1\right)\right| &\le K\frac{\delta^{2(p+1)}}{\kappa^3}|\log(\delta\kappa)|\sum_{l\neq 0}|l|e^{-|l|(\omega - K\delta^{p+2}|\log(\delta\kappa)|)} \\
&\le K\frac{\delta^{2(p+1)}}{\kappa^3}|\log(\delta\kappa)|,
\end{aligned} \tag{3.378}$$

where we have used that $p > -2$ and thus the sum converges for δ small enough. If we use bounds (3.373) and (3.376) can bound the fifth term in (3.370) by:

$$\begin{aligned}
\left|P_1(u_\pm,\theta)\sum_{l\neq 0} M^{[l]}(u_\pm)e^{il(\theta + C(u_\pm,\theta))}\right| &\le K\frac{\delta^{2(p+1)}}{\kappa^4}\sum_{l\neq 0} e^{-|l|(\omega - |C(u_\pm,\theta)|)} \\
&\le K\frac{\delta^{2(p+1)}}{\kappa^4},
\end{aligned} \tag{3.379}$$

where again we have used the bound of $C(u,\theta)$ provided by Theorem 3.1.9, so that for $p > -2$ and the sum converges for δ small enough.

Clearly, substituting bounds (3.371), (3.372), (3.374), (3.378) and (3.379) in expression (3.370) yields bound (3.363).

Finally, for the conservative case we use bound (3.363) in (3.362) and, recalling that in this case $\mathrm{d} = 1$, we obtain:

$$\sup_{\theta \in \mathbb{S}^1} \left| \tilde{\Delta}_1(w_\pm, \theta) \right| \leq K \left(\frac{\delta^{2p}}{\kappa^5} |\log(\delta\kappa)| + \frac{\delta^{p+1}}{\kappa^3} \right).$$

In the dissipative case we use bound (3.363) in (3.362) and we get:

$$\sup_{\theta \in \times \mathbb{S}^1} \left| \tilde{\Delta}_1(w_\pm, \theta) \right| \leq K \left(\frac{\delta^{2\left(p+1-\frac{1}{\mathrm{d}}\right)}}{\kappa^{3+2/\mathrm{d}}} |\log(\delta\kappa)| + \frac{\delta^{p+3-\frac{2}{\mathrm{d}}}}{\kappa^{1+2/\mathrm{d}}} \right).$$

Using these bounds in (3.361), both claims of the proposition are proved. □

Chapter 4

Breakdown of the 2D heteroclinic connection: the singular case

In this chapter we prove the following result, which has Theorem 1.2 as a corollary.

Theorem 4.1. *Consider system* (1.6)*, with* μ*,* $\beta_1, \gamma_2 > 0$ *and* $|\nu| < \beta_1\sqrt{\mu}$*, which has two critical points* $\bar{S}_\pm(\mu,\nu)$ *of saddle-focus type. Let* $\bar{D}^{\mathrm{u,s}}(u,\theta,\mu,\nu)$ *(*$\bar{D}^{\mathrm{u,s}}(u,\theta,\mu)$ *in the conservative case) be the distance between the two-dimensional unstable manifold of* $\bar{S}_-(\mu,\nu)$ *and the two-dimensional stable manifold of* $\bar{S}_+(\mu,\nu)$ *on the plane* $\bar{z} = \tanh(u)$*. There exist constants* $\mathcal{C}_1^*$*,* $\mathcal{C}_2^*$ L_0 *and* $T_0 > 0$ *and a bounded function* $L(u)$*, with* $L(0) = 0$*, such that for all* $u \in [-T_0, T_0]$ *and* $\theta \in \mathbb{S}^1$*, defining:*

$$\bar{\vartheta}(u,\mu) = \frac{\alpha_0 u}{\sqrt{\mu}} + \frac{1}{\beta_1}(\alpha_3 + \alpha_0 L_0)\left[\log\cosh(\beta_1 u) - \frac{1}{2}\log\mu\right] + \alpha_0 L(u),$$

the following holds:

1. *In the conservative case,* $\bar{D}^{\mathrm{u,s}}(u,\theta,\mu)$ *is given asymptotically, as* $\mu \to 0$*, by:*

$$\begin{aligned}\bar{D}^{\mathrm{u,s}}(u,\theta,\mu) &= \sqrt{\frac{\gamma_2}{2}}\frac{e^{-\frac{\alpha_0\pi}{2\sqrt{\mu}}}}{\mu^{\frac{3}{2}}}\cosh^3(u)\Big[\mathcal{C}_1^*\cos\Big(\theta + \bar{\vartheta}(u,\mu)\Big) \\ &\quad + \mathcal{C}_2^*\sin\Big(\theta + \bar{\vartheta}(u,\mu)\Big) + \mathcal{O}\left(\frac{1}{|\log\mu|}\right)\Big].\end{aligned}$$

2. *In the dissipative case, given* $a_1, a_2 \in \mathbb{R}$ *and* $a_3 > 0$*, there exists a function* $\nu = \nu(\mu)$ *satisfying* $\nu(0) = 0$*, such that* $\bar{D}^{\mathrm{u,s}}(u,\theta,\mu,\nu(\mu))$ *is given asymptotically, as* $\mu \to 0$*,*

by:

$$\begin{aligned}\bar{D}^{\mathrm{u,s}}(u,\theta,\mu,\nu(\mu)) &= \cosh^{1+\frac{2}{\beta_1}}(\beta_1 u) a_1 \mu^{a_2} e^{-\frac{a_3\pi}{2\beta_1\sqrt{\mu}}}\left(1+\mathcal{O}\left(\mu^{\frac{p+3}{2}}\right)\right)\\ &+\sqrt{\frac{\gamma_2}{\beta_1+1}}\cosh^{1+\frac{2}{\beta_1}}(\beta_1 u)\frac{e^{-\frac{\alpha_0\pi}{2\beta_1\sqrt{\mu}}}}{\mu^{\frac{1}{2}+\frac{1}{\beta_1}}}\left[\mathcal{C}_1^*\cos\left(\theta+\bar{\vartheta}(u,\mu)\right)\right.\\ &\left.+\mathcal{C}_2^*\sin\left(\theta+\bar{\vartheta}(u,\mu)\right)+\mathcal{O}\left(\frac{1}{|\log(\mu)|}\right)\right].\end{aligned}$$

Formulas for the constant L_0 and the function $L(u)$ are given in Chapter 3, Lemma 3.5.17 and Remark 3.5.18 respectively.

The main ideas of the proof of Theorem 4.1 are given in Section 4.1.

4.1 Set-up and heuristics of the proof

In the previous chapter, we studied system (3.19), which corresponds to an unfolding of the Hopf-zero singularity after rescaling the variables in a suitable way and renaming the parameters. From this system, we were able to compute an asymptotic formula of the distance between the 2-dimensional invariant manifolds of the critical points $S_\pm(\mu,\nu)$ only for the case $p>-2$. This restriction is not natural, and corresponds to non-generic unfoldings for which the perturbative terms are smaller than in the generic case. In this chapter we shall study the generic case, that is, system (3.19) with $p=-2$:

$$\begin{aligned}\frac{dx}{dt} &= x(\sigma-\mathrm{d}z)+\left(\frac{\alpha(\delta\sigma)}{\delta}+cz\right)y+\delta^{-2}f(\delta x,\delta y,\delta z,\delta,\delta\sigma),\\ \frac{dy}{dt} &= -\left(\frac{\alpha(\delta\sigma)}{\delta}+cz\right)x+y(\sigma-\mathrm{d}z)+\delta^{-2}g(\delta x,\delta y,\delta z,\delta,\delta\sigma),\\ \frac{dz}{dt} &= -1+b(x^2+y^2)+z^2+\delta^{-2}h(\delta x,\delta y,\delta z,\delta,\delta\sigma).\end{aligned}\tag{4.1}$$

The main difference between this chapter and Chapter 3 is the study of the so-called *inner equation*, which is an equation indpendent of the parameters δ and σ, whose solutions approximate the solutions of system (4.1) close to the singularities of the heteroclinic connection of the unperturbed system (see (3.24)–(3.26) for a parameterization of this heteroclinic connection). However, we shall use Theorem 3.1.7 also in this chapter, since it was proved even for $p=-2$. For completeness, we state it here substituting $p=-2$.

Theorem 4.1.1. *Consider the following PDE:*

$$\left(-\delta^{-1}\alpha - cZ_0(u)\right)\partial_\theta r_1^{\mathrm{u,s}} + \partial_u r_1^{\mathrm{u,s}} - 2Z_0(u) r_1^{\mathrm{u,s}} = 2\sigma(R_0(u) + r_1^{\mathrm{u,s}}) + \delta^{-2}F(r_1^{\mathrm{u,s}})$$
$$+\delta - 2\frac{\mathrm{d}+1}{b} Z_0(u) H(r_1^{\mathrm{u,s}}) - \delta^{-2}G(r_1^{\mathrm{u,s}})\partial_\theta r_1^{\mathrm{u,s}} - \left(\frac{2br_1^{\mathrm{u,s}} + \delta^{-2}H(r_1^{\mathrm{u,s}})}{\mathrm{d}(1-Z_0^2(u))}\right)\partial_u r_1^{\mathrm{u,s}}, \quad (4.2)$$

where F, G and H are defined in (3.23) *and we abuse notation as in* (3.37)*. Let $0 < \beta < \pi/2$ be any constants. There exist constants $\bar\kappa^* \geq 1$, $\sigma^* > 0$ and $\delta^* > 0$, such that if $\bar\kappa = \bar\kappa(\delta)$ satisfies:*

$$\kappa^*\delta \leq \kappa\delta \leq \frac{\pi}{4\mathrm{d}}, \quad (4.3)$$

then for all $0 < \delta \leq \delta^*$ and $|\sigma| \leq \sigma^*\delta$, the unstable manifold of $S_-(\delta,\sigma)$ and the stable manifold of $S_+(\delta,\sigma)$ are given respectively by:

$$r = r^{\mathrm{u}}(u,\theta) = R_0(u) + r_1^{\mathrm{u}}(u,\theta), \qquad z = Z_0(u) \qquad (u,\theta) \in D^{\mathrm{u}}_{\kappa,\beta,T} \times \mathbb{T}_\omega,$$

$$r = r^{\mathrm{s}}(u,\theta) = R_0(u) + r_1^{\mathrm{s}}(u,\theta), \qquad z = Z_0(u) \qquad (u,\theta) \in D^{\mathrm{s}}_{\kappa,\beta,T} \times \mathbb{T}_\omega,$$

where r_1^{u} and r_1^{s} satisfy the same equation (4.2)*.*

Moreover, there exists a constant M such that for all $(u,\theta) \in D^{\mathrm{u,s}}_{\kappa,\beta,T} \times \mathbb{T}_\omega$:

$$|r_1^{\mathrm{u,s}}(u,\theta)| \leq M\delta|\cosh(\mathrm{d}u)|^{-3}.$$

4.1.1 Derivation of the inner equation

In this subsection we shall explain what is the inner equation and how it is obtained. After that, we shall enumerate the steps of the proof of Theorem 4.1, which are explained heuristically in the following subsections. The rest of the chapter will consist in proving the results that we state below.

Let us consider a function r_1 satisfying the equation (4.2), namely:

$$\left(-\delta^{-1}\alpha - cZ_0(u)\right)\partial_\theta r_1 + \partial_u r_1 - 2Z_0(u) r_1 = 2\sigma(R_0(u) + r_1) + \delta^{-2}F(r_1)$$
$$+\delta^{-2}\frac{\mathrm{d}+1}{b} Z_0(u) H(r_1) - \delta^{-2}G(r_1)\partial_\theta r_1 - \left(\frac{2br_1 + \delta^{-2}H(r_1)}{\mathrm{d}(1-Z_0^2(u))}\right)\partial_u r_1. \quad (4.4)$$

Now we perform the following change of variables:

$$s = s(u) = \frac{1}{\delta Z_0(u)} = \frac{1}{\delta\tanh(\mathrm{d}u)}, \quad (4.5)$$

that is:

$$u = Z_0^{-1}\left(\frac{1}{\delta s}\right) = \mathrm{d}^{-1}\mathrm{arctanh}\left(\frac{1}{\delta s}\right).$$

We note that, since:

$$\frac{1}{\cosh^2(\mathrm{d}u)} = 1 - \tanh^2(\mathrm{d}u) = 1 - \frac{1}{\delta^2 s^2},$$

we can write:

$$R_0(u) = \frac{(\mathrm{d}+1)}{2b}\frac{1}{\cosh^2(\mathrm{d}u)} = \frac{\mathrm{d}+1}{2b}\left(1 - \frac{1}{\delta^2 s^2}\right).$$

Now we define the new functions:

$$\psi(s,\theta) = \delta^2 r_1\left(Z_0^{-1}\left(\frac{1}{\delta s}\right),\theta\right). \tag{4.6}$$

We do this rescaling because, as one can see from Theorem 4.1.1, when $s \sim \mathcal{O}(1)$ (that is, when $u - i\pi/(2\mathrm{d}) \sim \mathcal{O}(\delta)$) then $\psi(s,\theta) \sim \mathcal{O}(1)$.

Remark 4.1.2. We observe that change (4.5) is well-defined for u belonging to some sufficiently small neighborhoods of $\pm i\pi/(2\mathrm{d})$. Hence, through (4.6), we will be able to study the parameterizations $r_1^{\mathrm{u}}(u,\theta)$ and $r_1^{\mathrm{s}}(u,\theta)$ for u in these neighborhoods.

Using changes (4.5) and (4.6) we can restate Theorem 4.1.1 in the new variables. Clearly, the domain of definition of the resulting functions is $(s,\theta) \in \mathcal{D}^{\mathrm{u}}_{\kappa,\beta,T} \times \mathbb{T}_\omega$, where:

$$\begin{aligned}
\mathcal{D}^{\mathrm{u}}_{\kappa,\beta,T} &= \left\{ s \in \mathbb{C} \,:\, s = \frac{1}{\delta\tanh(\mathrm{d}u)},\ u \in D^{\mathrm{u}}_{\kappa,\beta,T} \cap \left\{ |\mathrm{Im}\, u| \geq \frac{\pi}{4\mathrm{d}} \right\} \right\}, \qquad (4.7)\\
\mathcal{D}^{\mathrm{s}}_{\kappa,\beta,T} &= \left\{ s \in \mathbb{C} \,:\, s = \frac{1}{\delta\tanh(\mathrm{d}u)},\ u \in D^{\mathrm{s}}_{\kappa,\beta,T} \cap \left\{ |\mathrm{Im}\, u| \geq \frac{\pi}{4\mathrm{d}} \right\} \right\}.
\end{aligned}$$

Theorem 4.1.3. *Consider the functions $r_1^{\mathrm{u}}(u,\theta)$ and $r_1^{\mathrm{u}}(u,\theta)$ given by Theorem 4.1.1, and define:*

$$\psi^{\mathrm{u}}(s,\theta) = \delta^2 r_1^{\mathrm{u}}\left(Z_0^{-1}\left(\frac{1}{\delta s}\right),\theta\right), \qquad (s,\theta) \in \mathcal{D}^{\mathrm{u}}_{\kappa,\beta,T} \times \mathbb{T}_\omega,$$

$$\psi^{\mathrm{s}}(s,\theta) = \delta^2 r_1^{\mathrm{s}}\left(Z_0^{-1}\left(\frac{1}{\delta s}\right),\theta\right), \qquad (s,\theta) \in \mathcal{D}^{\mathrm{s}}_{\kappa,\beta,T} \times \mathbb{T}_\omega.$$

Then there exists a constant M such that:

$$|\psi^{\mathrm{u}}(s,\theta)| \leq M|s|^{-3}, \qquad (s,\theta) \in \mathcal{D}^{\mathrm{u}}_{\kappa,\beta,T} \times \mathbb{T}_\omega,$$

$$|\psi^{\mathrm{s}}(s,\theta)| \leq M|s|^{-3}, \qquad (s,\theta) \in \mathcal{D}^{\mathrm{s}}_{\kappa,\beta,T} \times \mathbb{T}_\omega.$$

Now, noting that:

$$\begin{aligned}\partial_s\psi(s,\theta) &= \frac{\delta^3}{\mathrm{d}(1-\delta^2 s^2)}\partial_u r_1\left(Z_0^{-1}\left(\frac{1}{\delta s}\right),\theta\right)\\ \partial_\theta\psi(s,\theta) &= \delta^2\partial_\theta r_1\left(Z_0^{-1}\left(\frac{1}{\delta s}\right),\theta\right),\end{aligned} \tag{4.8}$$

equation (4.4) yields the following PDE for ψ:

$$\begin{aligned}&\delta^{-3}(-\alpha-cs^{-1})\partial_\theta\psi+\delta^{-3}\mathrm{d}(1-\delta^2 s^2)\partial_s\psi-2\delta^{-3}s^{-1}\psi\\ &\qquad=2\sigma\left(\frac{\mathrm{d}+1}{2b}\left(1-\frac{1}{\delta^2 s^2}\right)+\delta^{-2}\psi\right)+\delta^{-2}F(\delta^{-2}\psi)+\delta^{-3}\frac{\mathrm{d}+1}{b}s^{-1}H(\delta^{-2}\psi)\\ &\qquad\quad-\delta^{-4}G(\delta^{-2}\psi)\partial_\theta\psi+\delta^{-3}s^2\left(2b\psi+H(\delta^{-2}\psi)\right)\partial_s\psi,\end{aligned} \tag{4.9}$$

with F, G and H defined in (3.23).

Let us define:

$$\rho(\psi,s,\delta)=\sqrt{\frac{\mathrm{d}+1}{b}\left(-s^{-2}+\delta^2\right)+2\psi}. \tag{4.10}$$

Note that $\rho(\psi,s,\theta)$ is just $\delta\sqrt{2(R_0(u)+r_1)}$ in the new coordinates. We also define the functions $\hat F$, $\hat G$ and $\hat H$ as:

$$\begin{aligned}\hat F(\psi,s,\theta,\delta,\delta\sigma) &= \rho(\psi,s,\delta)\cos\theta f(\rho(\psi,s,\delta)\cos\theta,\rho(\psi,s,\delta)\sin\theta,s^{-1},\delta,\delta\sigma)\\ &\quad+\rho(\psi,s,\delta)\sin\theta g(\rho(\psi,s,\delta)\cos\theta,\rho(\psi,s,\delta)\sin\theta,s^{-1},\delta,\delta\sigma),\\ \hat G(\psi,s,\theta,\delta,\delta\sigma) &= \frac{1}{\rho(\psi,s,\delta)}\cos\theta g(\rho(\psi,s,\delta)\cos\theta,\rho(\psi,s,\delta)\sin\theta,s^{-1},\delta,\delta\sigma)\\ &\quad-\frac{1}{\rho(\psi,s,\delta)}\sin\theta f(\rho(\psi,s,\delta)\cos\theta,\rho(\psi,s,\delta)\sin\theta,s^{-1},\delta,\delta\sigma),\\ \hat H(\psi,s,\theta,\delta,\delta\sigma) &= h(\rho(\psi,s,\delta)\cos\theta,\rho(\psi,s,\delta)\sin\theta,s^{-1},\delta,\delta\sigma).\end{aligned} \tag{4.11}$$

We point out that $\hat F$, $\hat G$ and $\hat H$ are well defined at $\delta=0$. Then, recalling the notation (3.37) and the definition (3.23) of F, G and H, we can write:

$$\begin{aligned}F(\delta^{-2}\psi) &= \delta^{-1}\hat F(\psi,s,\theta,\delta,\delta\sigma),\\ G(\delta^{-2}\psi) &= \delta\hat G(\psi,s,\theta,\delta,\delta\sigma),\\ H(\delta^{-2}\psi) &= \hat H(\psi,s,\theta,\delta,\delta\sigma).\end{aligned} \tag{4.12}$$

Using this notation in (4.9), multiplying both sides of the equality by δ^3 and putting the terms $-cs^{-1}\partial_\theta\psi$ and $\mathrm{d}\delta^2 s^2\partial_s\psi$ on the right hand side of the equality, we obtain the following equation:

$$
\begin{aligned}
-\alpha\partial_\theta\psi + \mathrm{d}\partial_s\psi - 2s^{-1}\psi = {} & cs^{-1}\partial_\theta\psi + \mathrm{d}\delta^2 s^2\partial_s\psi + 2\sigma\left(\frac{\mathrm{d}+1}{2b}\left(\delta^3 - \frac{\delta}{s^2}\right) + \delta\psi\right) \\
& + \hat{F}(\psi, s, \theta, \delta, \delta\sigma) + \frac{\mathrm{d}+1}{b}s^{-1}\hat{H}(\psi, s, \theta, \delta, \delta\sigma) \\
& - \hat{G}(\psi, s, \theta, \delta, \delta\sigma)\partial_\theta\psi + s^2\left(2b\psi + \hat{H}(\psi, s, \theta, \delta, \delta\sigma)\right)\partial_s\psi. \qquad (4.13)
\end{aligned}
$$

If we set $\delta = 0$ and call ψ_{in} the solutions of the resulting equation, we have:

$$
\begin{aligned}
-\alpha\partial_\theta\psi_{\mathrm{in}} + \mathrm{d}\partial_s\psi_{\mathrm{in}} - 2s^{-1}\psi_{\mathrm{in}} \;=\; & cs^{-1}\partial_\theta\psi_{\mathrm{in}} + \hat{F}(\psi_{\mathrm{in}}, s, \theta, 0, 0) \\
& + \frac{\mathrm{d}+1}{b}s^{-1}\hat{H}(\psi_{\mathrm{in}}, s, \theta, 0, 0) - \hat{G}(\psi_{\mathrm{in}}, s, \theta, 0, 0)\partial_\theta\psi_{\mathrm{in}} \\
& + s^2\left(2b\psi_{\mathrm{in}} + \hat{H}(\psi_{\mathrm{in}}, s, \theta, 0, 0)\right)\partial_s\psi_{\mathrm{in}}. \qquad (4.14)
\end{aligned}
$$

This is the so-called inner equation. Defining the operators:

$$\mathcal{L}(\psi) = -\alpha\partial_\theta\psi + \mathrm{d}\partial_s\psi - 2s^{-1}\psi, \qquad (4.15)$$

and:

$$
\begin{aligned}
\mathcal{M}(\psi, \delta) \;=\; & cs^{-1}\partial_\theta\psi + \mathrm{d}\delta^2 s^2\partial_s\psi + 2\sigma\left(\frac{\mathrm{d}+1}{2b}\left(\delta^3 - \frac{\delta}{s^2}\right) + \delta\psi\right) + \hat{F}(\psi, s, \theta, \delta, \delta\sigma) \\
& + \frac{\mathrm{d}+1}{b}s^{-1}\hat{H}(\psi, s, \theta, \delta, \delta\sigma) - \hat{G}(\psi, s, \theta, \delta, \delta\sigma)\partial_\theta\psi \\
& + s^2\left(2b\psi + \hat{H}(\psi, s, \theta, \delta, \delta\sigma)\right)\partial_s\psi, \qquad (4.16)
\end{aligned}
$$

equation (4.13) can be written as:

$$\mathcal{L}(\psi) = \mathcal{M}(\psi, \delta), \qquad (4.17)$$

and the inner equation (4.14) as:

$$\mathcal{L}(\psi_{\mathrm{in}}) = \mathcal{M}(\psi_{\mathrm{in}}, 0). \qquad (4.18)$$

We point out that, although in general we avoid writing explicitly the dependence of functions and operators with respect to δ to simplify the notation, in this case we write $\mathcal{M}(\psi, \delta)$ in order to make clear the difference between the case $\delta \neq 0$ and $\delta = 0$.

The purpose of this chapter is to study two solutions $\psi_{\mathrm{in}}^{\mathrm{u}}$ and $\psi_{\mathrm{in}}^{\mathrm{s}}$ of the inner equation (4.18). Moreover, we shall see that the first order of the difference $\Delta(u, \theta) := r_1^{\mathrm{u}}(u, \theta) - r_1^{\mathrm{s}}(u, \theta)$ with respect to δ is given by $\psi_{\mathrm{in}}^{\mathrm{u}} - \psi_{\mathrm{in}}^{\mathrm{s}}$. Now we will briefly explain the steps we will follow to reach our goal. All these steps will be explained more in detail, although still without technicalities, in the following subsections. We shall indicate in each case the corresponding subsection where these general details can be found. These steps are:

1. First, we shall find two particular solutions of equation (4.18), denoted by $\psi^{\rm u}_{\rm in}(s,\theta)$ and $\psi^{\rm s}_{\rm in}(s,\theta)$, satisfying some asymptotic conditions. We also study some properties of these solutions (see Subsection 4.1.2).

2. After that, we shall see that if s belongs to a suitable complex domain, then $\psi^{\rm u}_{\rm in}(s,\theta)$ and $\psi^{\rm s}_{\rm in}(s,\theta)$ are respectively good approximations of the functions $\psi^{\rm u}(u,\theta)$ and $\psi^{\rm s}(u,\theta)$ defined in Theorem 4.1.1. To prove this fact, we shall study the difference $\psi^{\rm u,s}_1(s,\theta) := \psi^{\rm u,s}(s,\theta) - \psi^{\rm u,s}_{\rm in}(s,\theta)$. We will refer to $\psi^{\rm u}_1$ and $\psi^{\rm s}_1$ as the matching error. We shall prove that $\psi^{\rm u}_1$ and $\psi^{\rm s}_1$ are small in an adequate sense. The reader can find the main ideas in Subsection 4.1.3.

3. Next, we shall find an asymptotic formula for the difference:
$$\Delta\psi_{\rm in}(s,\theta) := \psi^{\rm u}_{\rm in}(s,\theta) - \psi^{\rm s}_{\rm in}(s,\theta).$$
This is summarized in Subsection 4.1.4.

4. Finally, using the previous results, one can use $\Delta\psi_{\rm in}(s,\theta)$ to find an asymptotic formula of $\Delta(u,\theta)$. This is explained in more detail in Subsection 4.1.5.

4.1.2 Study of the inner equation

We shall start by studying the inner equation (4.18) and finding two suitable solutions. However, let us first introduce the complex domains in which the solutions of (4.18) will be defined.

Given $\beta_0, \bar\kappa > 0$, we define (see Figure 4.1):
$$\mathcal{D}^{\rm in,u}_{\beta_0,\bar\kappa} = \{s\in\mathbb{C} \,:\, |{\rm Im}\, s| \geq \tan\beta_0 {\rm Re}\, s + \bar\kappa\}, \qquad \mathcal{D}^{\rm in,s}_{\beta_0,\bar\kappa} = -\mathcal{D}^{\rm in,u}_{\beta_0,\bar\kappa}. \tag{4.19}$$

We also define the domains in terms of the outer variables u:
$$D^{\rm in,u}_{\beta_0,\bar\kappa} = \{u\in\mathbb{C} \,:\, |{\rm Im}\,(u-i\pi/2)| \geq \tan\beta_0 {\rm Re}\, u + \bar\kappa\delta\}, \qquad D^{\rm in,s}_{\beta_0,\bar\kappa} = -D^{\rm in,u}_{\beta_0,\bar\kappa}. \tag{4.20}$$

It is easy to check that taking $\bar\kappa = \kappa/2$, where κ is the parameter defining the domains $D^{\rm u}_{\kappa,\beta}$ introduced in (3.56), and choosing an adequate $T>0$, then for all $0<\beta_0,\beta<\pi/2$ and for δ small enough one has that $D^{\rm u}_{\kappa,\beta,T} \subset D^{\rm in,u}_{\beta_0,\bar\kappa}$ (see Figure 4.2). Analogously, we also have that $D^{\rm s}_{\kappa,\beta,T} \subset D^{\rm in,s}_{\beta_0,\bar\kappa}$.

We will look for particular solutions $\psi^{\rm u}_{\rm in}$ and $\psi^{\rm s}_{\rm in}$ of equation (4.18) satisfying respectively:
$$\lim_{{\rm Re}\,(s)\to-\infty} |\psi^{\rm u}_{\rm in}(s,\theta)| \to 0, \qquad (s,\theta)\in \mathcal{D}^{\rm in,u}_{\beta_0,\bar\kappa}\times\mathbb{T}_\omega, \tag{4.21}$$
and:
$$\lim_{{\rm Re}\,(s)\to+\infty} |\psi^{\rm s}_{\rm in}(s,\theta)| \to 0, \qquad (s,\theta)\in \mathcal{D}^{\rm in,s}_{\beta_0,\bar\kappa}\times\mathbb{T}_\omega. \tag{4.22}$$

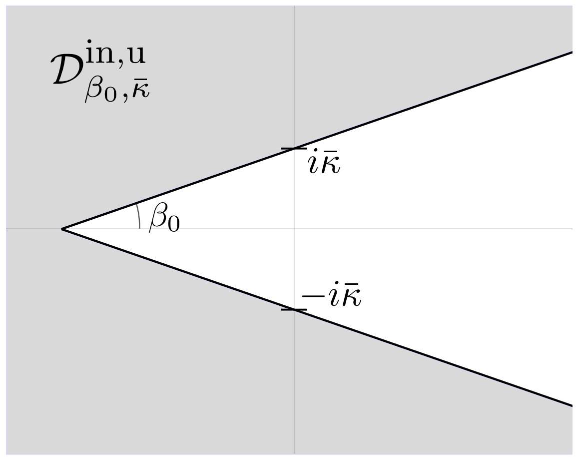


Figure 4.1: The domain $\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa}$.

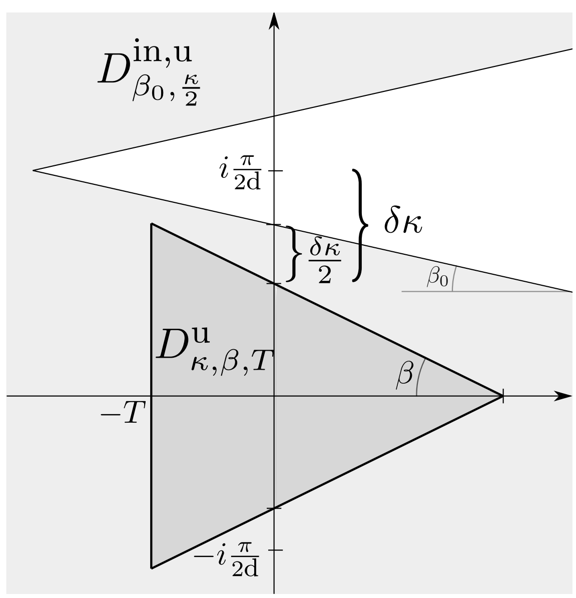


Figure 4.2: The domains $D^{\mathrm{u}}_{\kappa,\beta,T}$ and $D^{\mathrm{in,u}}_{\beta_0,\bar\kappa}$ with $\bar\kappa = \kappa/2$.

We will find these solutions by means of a suitable right inverse of the operator $\mathcal{L}$ defined in (4.15). More precisely, assume $\mathcal{G}$ is such that $\mathcal{L}\circ\mathcal{G}=\mathrm{Id}$. If ψ_{in} satisfies the implicit equation:

$$\psi_{\mathrm{in}}=\mathcal{G}(\mathcal{M}(\psi_{\mathrm{in}},0)), \tag{4.23}$$

then clearly ψ_{in} is a solution of equation (4.18). In other words, $\mathcal{G}$ allows us to write (4.18) as the fixed point equation (4.23). We now introduce the right inverse used in each case: $\mathcal{G}^{\mathrm{u}}$, which will allow us to prove the existence of the function $\psi_{\mathrm{in}}^{\mathrm{u}}$ satisfying (4.21), and $\mathcal{G}^{\mathrm{s}}$, corresponding to the function $\psi_{\mathrm{in}}^{\mathrm{s}}$ satisfying (4.22). We shall refer to each case as the "unstable" one and the "stable" one, because we shall see that each one approximates respectively the unstable or stable manifold (more precisely, ψ^{u} and ψ^{s} defined in Theorem 4.1.1) in some bounded domains.

Given a function $\phi(s,\theta)$, $2\pi-$periodic in θ, we define $\mathcal{G}^{\mathrm{u}}$ as:

$$\mathcal{G}^{\mathrm{u}}(\phi)(s,\theta)=\sum_{l\in\mathbb{Z}}\mathcal{G}^{\mathrm{u}[l]}(\phi)(s)e^{il\theta}, \tag{4.24}$$

where the Fourier coefficients $\mathcal{G}^{\mathrm{u}[l]}(\phi)$ are defined as:

$$\mathcal{G}^{\mathrm{u}[l]}(\phi)(s)=s^{\frac{2}{\mathrm{d}}}\int_{-\infty}^{s}\frac{e^{-\frac{il\alpha}{\mathrm{d}}(w-s)}}{w^{\frac{2}{\mathrm{d}}}}\phi^{[l]}(w)dw. \tag{4.25}$$

Here $\phi^{[l]}$ stands for the $l-$th Fourier coefficient of ϕ, and $\int_{-\infty}^{s}$ means the integral over any path such that $\mathrm{Re}\, s\to-\infty$.

For the stable case, the definitions are analogous: given a function $\phi(s,\theta)$, $\mathcal{G}^{\mathrm{s}}$ is defined as:

$$\mathcal{G}^{\mathrm{s}}(\phi)(s,\theta)=\sum_{l\in\mathbb{Z}}\mathcal{G}^{\mathrm{s}[l]}(\phi)(s)e^{il\theta}, \tag{4.26}$$

where $\mathcal{G}^{\mathrm{s}[l]}(\phi)$ are defined as:

$$\mathcal{G}^{\mathrm{s}[l]}(\phi)(s)=s^{\frac{2}{\mathrm{d}}}\int_{+\infty}^{s}\frac{e^{-\frac{il\alpha}{\mathrm{d}}(w-s)}}{w^{\frac{2}{\mathrm{d}}}}\phi^{[l]}(w)dw. \tag{4.27}$$

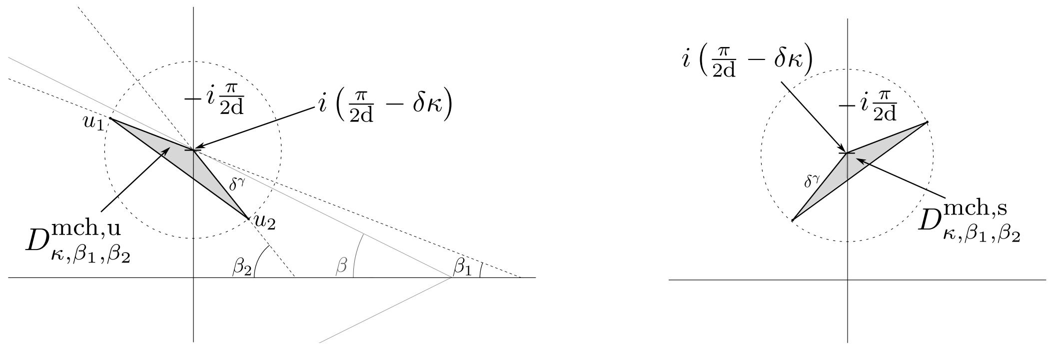
One can check easily that:

$$\mathcal{L}\circ\mathcal{G}^{\mathrm{u}}=\mathcal{L}\circ\mathcal{G}^{\mathrm{s}}=\mathrm{Id}. \tag{4.28}$$

Now we can state the following theorem, stating the existence of both functions $\psi_{\mathrm{in}}^{\mathrm{u}}$ and $\psi_{\mathrm{in}}^{\mathrm{s}}$.

Theorem 4.1.4. *Let $\beta_0>0$ and $\bar\kappa$ be large enough. Then equation (4.18) has two solutions $\psi_{\mathrm{in}}^{\mathrm{u}}(s,\theta)$ and $\psi_{\mathrm{in}}^{\mathrm{s}}(s,\theta)$, defined respectively for $(s,\theta)\in\mathcal{D}_{\beta_0,\bar\kappa}^{\mathrm{in,u}}$ and $(s,\theta)\in\mathcal{D}_{\beta_0,\bar\kappa}^{\mathrm{in,s}}$, such that for some constant M:*

$$|\psi_{\mathrm{in}}^{\mathrm{u}}(s,\theta)|\le M|s|^{-3},\qquad (s,\theta)\in\mathcal{D}_{\beta_0,\bar\kappa}^{\mathrm{in,u}},$$



(a) The domain $D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$. (b) The domain $D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$.

Figure 4.3: The domains $D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$ and $D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$.

$$|\psi^{\mathrm{s}}_{\mathrm{in}}(s,\theta)| \leq M|s|^{-3}, \qquad (s,\theta)\in\mathcal{D}^{\mathrm{in,s}}_{\beta_0,\bar\kappa}.$$

Moreover:

$$|\psi^{\mathrm{u}}_{\mathrm{in}}(s,\theta)-\mathcal{G}^{\mathrm{u}}(\mathcal{M}(0,0))(s,\theta)| \leq M|s|^{-4}, \qquad (s,\theta)\in\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa},$$

$$|\psi^{\mathrm{s}}_{\mathrm{in}}(s,\theta)-\mathcal{G}^{\mathrm{s}}(\mathcal{M}(0,0))(s,\theta)| \leq M|s|^{-4}, \qquad (s,\theta)\in\mathcal{D}^{\mathrm{in,s}}_{\beta_0,\bar\kappa}.$$

The proof of Theorem 4.1.4 can be found in Section 4.2.

4.1.3 Study of the matching errors ψ_1^{u} and ψ_1^{s}

In this section we shall show that the functions $\psi^{\mathrm{u}}_{\mathrm{in}}$ and $\psi^{\mathrm{s}}_{\mathrm{in}}$ found in Theorem 4.1.4 approximate the functions ψ^{u} and ψ^{s}, defined in Theorem 4.1.3, in some complex domains which, when written in the (u,θ) variables, correspond to domains close to the singularities $\pm i\pi/(2\mathrm{d})$. Let us first define these domains. Recall that $\psi^{\mathrm{us}}(s,\theta)=\delta^2 r_1^{\mathrm{u,s}}(Z_0^{-1}(\delta^{-1}s^{-1}),\theta)$ and that r_1^{u} and r_1^{s} are defined in the domains $D^{\mathrm{u}}_{\kappa,\beta,T}\times\mathbb{T}_\omega$ and $D^{\mathrm{s}}_{\kappa,\beta,T}\times\mathbb{T}_\omega$ (see (3.57)), where κ satisfies condition (4.3) and β is some fixed constant. Take β_1,β_2 two constants independent of δ and σ, such that:

$$0<\beta_1<\beta<\beta_2<\pi/2. \tag{4.29}$$

Fix also a constant $\gamma\in(0,1)$. We define the points $u_j\in\mathbb{C}$, $j=1,2$ to be those satisfying (see Figure 4.3):

- $\mathrm{Im}\,u_j=-\tan\beta_j\mathrm{Re}\,u_j+\frac{\pi}{2\mathrm{d}}-\delta\kappa,$
- $\left|u_j-i(\frac{\pi}{2\mathrm{d}}-\delta\kappa)\right|=\delta^\gamma,$
- $\mathrm{Re}\,u_1<0$, $\mathrm{Re}\,u_2>0$.

Then we define the following domain (see Figure 4.3):

$$D^{\rm mch,u}_{\kappa,\beta_1,\beta_2} = \Big\{u\in\mathbb{C} \,:\, \mathrm{Im}\, u \le -\tan\beta_1 \mathrm{Re}\, u + \frac{\pi}{2\mathrm{d}} - \delta\kappa,\ \mathrm{Im}\, u \le -\tan\beta_2 \mathrm{Re}\, u + \frac{\pi}{2\mathrm{d}} - \delta\kappa, \\ \mathrm{Im}\, u \ge \mathrm{Im}\, u_1 - \tan\left(\tfrac{\beta_1+\beta_2}{2}\right)(\mathrm{Re}\, u - \mathrm{Re}\, u_1)\Big\}.$$

Note that $D^{\rm mch,u}_{\kappa,\beta_1,\beta_2}$ is a triangular domain, with vertices u_1, u_2 and $i(\pi/(2\mathrm{d}) - \delta\kappa)$. We also define:

$$D^{\rm mch,s}_{\kappa,\beta_1,\beta_2} = \{u\in\mathbb{C} \,:\, -\overline{u} \in D^{\rm mch,u}_{\kappa,\beta_1,\beta_2}\}.$$

One can easily see that taking $\bar\kappa = \kappa/2$ one has (see Figures 4.2 and 4.3):

$$D^{\rm mch,u}_{\kappa,\beta_1,\beta_2} \subset D^{\rm u}_{\kappa,\beta,T} \subset D^{\rm in,u}_{\beta_0,\bar\kappa},$$

$$D^{\rm mch,s}_{\kappa,\beta_1,\beta_2} \subset D^{\rm s}_{\kappa,\beta,T} \subset D^{\rm in,s}_{\beta_0,\bar\kappa},$$

where $D^{\rm u}_{\kappa,\beta,T}$ was defined in (3.57), $D^{\rm s}_{\kappa,\beta,T} = -D^{\rm u}_{\kappa,\beta,T}$ and $D^{\rm in,u}_{\beta_0,\bar\kappa}$ and $D^{\rm in,s}_{\beta_0,\bar\kappa}$ were defined in (4.20).

Finally we define the following domains in terms of the inner variables s:

$$\begin{aligned} \mathcal{D}^{\rm mch,u}_{\kappa,\beta_1,\beta_2} &= \{s\in\mathbb{C} \,:\, i\frac{\pi}{2\mathrm{d}} + s\delta \in D^{\rm mch,u}_{\kappa,\beta_1,\beta_2}\}, \\ \mathcal{D}^{\rm mch,s}_{\kappa,\beta_1,\beta_2} &= \{s\in\mathbb{C} \,:\, i\frac{\pi}{2\mathrm{d}} + s\delta \in D^{\rm mch,s}_{\kappa,\beta_1,\beta_2}\}. \end{aligned} \tag{4.30}$$

One also has that for $\bar\kappa = \kappa/2$ and taking δ sufficiently small:

$$\mathcal{D}^{\rm mch,u}_{\kappa,\beta_1,\beta_2} \subset \mathcal{D}^{\rm in,u}_{\beta_0,\bar\kappa}, \qquad \mathcal{D}^{\rm mch,s}_{\kappa,\beta_1,\beta_2} \subset \mathcal{D}^{\rm in,s}_{\beta_0,\bar\kappa},$$

where $\mathcal{D}^{\rm in,u}_{\beta_0,\bar\kappa}$ and $\mathcal{D}^{\rm in,s}_{\beta_0,\bar\kappa}$ were defined in (4.19).

We will also denote:

$$s_j = \frac{u_j - i\frac{\pi}{2\mathrm{d}}}{\delta}, \quad j = 1,2. \tag{4.31}$$

It is clear that:

$$K_1\delta^{\gamma-1} \le |s_j| \le K_2\delta^{\gamma-1}, \qquad j=1,2, \tag{4.32}$$

and that for all $s\in\mathcal{D}^{\rm mch,u}_{\kappa,\beta_1,\beta_2}$ we have:

$$\kappa\cos\beta_1 \le |s| \le K_2\delta^{\gamma-1}, \tag{4.33}$$

and the same happens for $s\in\mathcal{D}^{\rm mch,s}_{\kappa,\beta_1,\beta_2}$.

The goal is to see how well the function $\psi^{\rm u}_{\rm in}(s,\theta)$ approximates $\psi^{\rm u}(s,\theta)$ for $(s,\theta)\in\mathcal{D}^{\rm mch,u}_{\kappa,\beta_1,\beta_2}\times\mathbb{T}_\omega$, and the same for $\psi^{\rm s}_{\rm in}(s,\theta)$ and $\psi^{\rm s}(s,\theta)$ when $(s,\theta)\in\mathcal{D}^{\rm mch,s}_{\kappa,\beta_1,\beta_2}\times\mathbb{T}_\omega$. To that aim, we recall the definition of the matching error:

$$\psi^{\rm u}_1(s,\theta) := \psi^{\rm u}(s,\theta) - \psi^{\rm u}_{\rm in}(s,\theta), \qquad \psi^{\rm s}_1(s,\theta) := \psi^{\rm s}(s,\theta) - \psi^{\rm s}_{\rm in}(s,\theta). \tag{4.34}$$

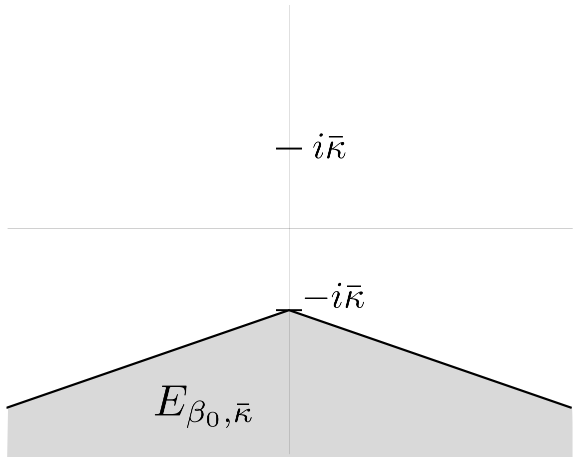


Figure 4.4: The domain $E_{\beta_0,\bar\kappa}$.

We stress that Theorems 4.1.3 and 4.1.4 yield directly the existence of ψ_1^{u} and ψ_1^{s}. These theorems also provide us with a non-sharp upper bound for these functions. In the following result we prove that, restricting ψ_1^{u} and ψ_1^{s} to the smaller domains $\mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$ and $\mathcal{D}^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$ respectively, we can get better upper bounds.

Theorem 4.1.5. *Consider ψ_1^{u} and ψ_1^{s} defined in* (4.34). *There exists a constant M such that:*

- *For $s \in \mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$, one has $|\psi_1^{\mathrm{u}}(s,\theta)| \leq M\delta^{1-\gamma}|s|^{-2}$.*
- *For $s \in \mathcal{D}^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$, one has $|\psi_1^{\mathrm{s}}(s,\theta)| \leq M\delta^{1-\gamma}|s|^{-2}$.*

The proof of this Theorem can be found in Section 4.3.

4.1.4 Study of the difference $\Delta\psi_{\mathrm{in}} = \psi_{\mathrm{in}}^{\mathrm{u}} - \psi_{\mathrm{in}}^{\mathrm{s}}$

Once the existence of these two particular solutions is established, one can proceed to look for an asymptotic expression of their difference $\Delta\psi_{\mathrm{in}} = \psi_{\mathrm{in}}^{\mathrm{u}} - \psi_{\mathrm{in}}^{\mathrm{s}}$. We will study this difference for $(s,\theta) \in E_{\beta_0,\bar\kappa} \times \mathbb{T}_\omega$, where $E_{\beta_0,\bar\kappa}$ is the following domain (see Figure 4.4):

$$E_{\beta_0,\bar\kappa} = \mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa} \cap \mathcal{D}^{\mathrm{in,s}}_{\beta_0,\bar\kappa} \cap \{s \in \mathbb{C} \,:\, \mathrm{Im}\, s < 0\}. \tag{4.35}$$

Subtracting equations (4.14) for $\psi_{\mathrm{in}}^{\mathrm{u}}$ and $\psi_{\mathrm{in}}^{\mathrm{s}}$ and using the mean value theorem, one obtains a linear equation for $\Delta\psi_{\mathrm{in}}$ that has the following form:

$$\begin{aligned} -\alpha\partial_\theta\Delta\psi_{\mathrm{in}} + \mathrm{d}\partial_s\Delta\psi_{\mathrm{in}} - 2s^{-1}\Delta\psi_{\mathrm{in}} \\ = a_1(s,\theta)\Delta\psi_{\mathrm{in}} + a_2(s,\theta)\partial_s\Delta\psi_{\mathrm{in}} + (cs^{-1} + a_3(s,\theta))\partial_\theta\Delta\psi_{\mathrm{in}}, \end{aligned} \tag{4.36}$$

for certain "small" (in the appropriate sense) functions $a_1(s,\theta)$, $a_2(s,\theta)$ and $a_3(s,\theta)$, which we will specify in Section 4.4. Of course, $a_i(s,\theta)$, $i=1,2,3$, depend on $\psi_{\text{in}}^{\text{u}}$ and $\psi_{\text{in}}^{\text{s}}$, which now are known functions. Since $\Delta\psi_{\text{in}}$ is a solution of (4.36), we first study the form that all solutions of this equation have. Next we give the main ideas of how this can be done, which basically are the same as in Chapter 3, Section 3.1.6.

Let $P^{\text{in}}(s,\theta)$ be a particular solution of (4.36) such that $P^{\text{in}}(s,\theta)\neq 0$ for $(s,\theta)\in E_{\beta_0,\bar{\kappa}}\times\mathbb{T}_\omega$. Then, using the method of variation of constants, one can easily see that every solution of (4.36) can be written as:

$$\Delta\psi_{\text{in}}(s,\theta) = P^{\text{in}}(s,\theta)k^{\text{in}}(s,\theta),$$

where $k^{\text{in}}(s,\theta)$ is a solution the homogeneous equation:

$$-\alpha\partial_\theta k^{\text{in}} + \mathrm{d}\partial_s k^{\text{in}} = a_2(s,\theta)\partial_s k^{\text{in}} + (cs^{-1}+a_3(s,\theta))\partial_\theta k^{\text{in}}. \tag{4.37}$$

First let us describe how we can find a suitable particular solution $P^{\text{in}}(s,\theta)$ of equation (4.36). Since the functions $a_1(s,\theta)$, $a_2(s,\theta)$ and $a_3(s,\theta)$ are "small" and cs^{-1} is also small if we take s to be sufficiently large, equation (4.36) can be regarded as a small perturbation of:

$$-\alpha\partial_\theta\Delta\psi_{\text{in}} + \mathrm{d}\partial_s\Delta\psi_{\text{in}} - 2s^{-1}\Delta\psi_{\text{in}} = 0.$$

This equation has a trivial solution given by $P_0^{\text{in}}(s,\theta) = s^{2/\mathrm{d}}$. Thus, we will look for a solution of the form:

$$P^{\text{in}}(s,\theta) = s^{2/\mathrm{d}}(1+P_1^{\text{in}}(s,\theta)),$$

where $P_1^{\text{in}}(s,\theta)$ will be a "small" function. Note that, being $P_1^{\text{in}}(s,\theta)$ small, we will ensure that $P^{\text{in}}(s,\theta)\neq 0$ for $(s,\theta)\in E_{\beta_0,\bar{\kappa}}\times\mathbb{T}_\omega$. The rigorous statement of this result can be found in Proposition 4.4.7 in Section 4.4.

Now we shall sketch the study of equation (4.37). In fact, equations of the form (4.37) have been studied in previous works. One of its main features is that if $\xi(s,\theta)$ is a solution of (4.37) such that $(\xi(s,\theta),\theta)$ is injective in $E_{\beta_0,\bar{\kappa}}\times\mathbb{T}_\omega$, then any other solution $k^{\text{in}}(s,\theta)$ of this equation can be written as:

$$k^{\text{in}}(s,\theta) = \tilde{k}^{\text{in}}(\xi(s,\theta)),$$

for some function $\tilde{k}^{\text{in}}(\tau)$. Thus, we need to find a suitable particular solution ξ of (4.37). To find such a suitable particular solution of (4.37), one could to proceed as we did with $P^{\text{in}}(s,\theta)$. Indeed, since the functions $a_2(s,\theta)$ and $cs^{-1}+a_3(s,\theta)$ are "small", one could derive that the dominant part of equation (4.37) is given by:

$$-\alpha\partial_\theta k^{\text{in}} + \mathrm{d}\partial_s k^{\text{in}} = 0. \tag{4.38}$$

A trivial solution of (4.38) is given by:

$$\xi_0(s,\theta) = \theta + \mathrm{d}^{-1}\alpha s,$$

and thus we could expect to find a suitable solution of (4.37) given by $\xi(s,\theta) = \xi_0(s,\theta) + \xi_1(s,\theta)$, where $\xi_1(s,\theta)$ is supposed to be a "small" function.

However, as we shall see later on in Section 4.4, this is not quite accurate. Nevertheless, to some extent it summarizes the underlying idea of the proof. In fact, we will see that the dominant part of equation (4.37) is:

$$-\alpha\partial_\theta k^{\mathrm{in}} + \mathrm{d}\partial_s k^{\mathrm{in}} = \mathrm{d}L_0 s^{-1}\partial_s k^{\mathrm{in}} + cs^{-1}\partial_\theta k^{\mathrm{in}}, \tag{4.39}$$

where L_0 is the constant given in Theorem 3.1.9. As we shall see, this constant is closely related to the function $a_2(s,\theta)$. More precisely, we will see in Lemma 4.4.3:

$$a_2(s,\theta) = \frac{\tilde{a}_2(\theta)}{s} + \mathcal{O}(s^{-2}),$$

and $\mathrm{d}L_0$ is the average of $\tilde{a}_2(\theta)$. Note that the function $\xi_0(s,\theta)$ defined as:

$$\xi_0(s,\theta) = \theta + \mathrm{d}^{-1}\alpha s + \mathrm{d}^{-1}(c+\alpha L_0)\log s,$$

solves (4.39) up to terms of order s^{-2}. Thus, the particular solution $\xi(s,\theta)$ that we will use is:

$$\xi(s,\theta) = \theta + \mathrm{d}^{-1}\alpha s + \mathrm{d}^{-1}(c+\alpha L_0)\log s + \varphi(s,\theta),$$

where $\varphi(s,\theta)$ is a function that is "small" in the appropriate sense. This result is contained in Proposition 4.4.5 in Section 4.4.

All these considerations lead to the following result.

Theorem 4.1.6. *Consider the difference:*

$$\Delta\psi_{\mathrm{in}}(s,\theta) = \psi_{\mathrm{in}}^{\mathrm{u}}(s,\theta) - \psi_{\mathrm{in}}^{\mathrm{s}}(s,\theta), \qquad (s,\theta) \in E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega.$$

Let L_0 be the constant given in Theorem 3.1.9. Then there exists a $2\pi-$periodic function $\tilde{k}^{\mathrm{in}}(\tau)$ and two functions $\varphi(s,\theta)$ and $P_1^{\mathrm{in}}(s,\theta)$ that are $2\pi-$periodic in θ such that:

$$\Delta\psi_{\mathrm{in}}(s,\theta) = s^{2/\mathrm{d}}(1 + P_1^{\mathrm{in}}(s,\theta))\tilde{k}^{\mathrm{in}}(\theta + \mathrm{d}^{-1}\alpha s + \mathrm{d}^{-1}(c+\alpha L_0)\log s + \varphi(s,\theta)).$$

The function $\tilde{k}^{\mathrm{in}}(\tau)$ is such that:

$$\tilde{k}^{\mathrm{in}}(\tau) = \sum_{l<0} \Upsilon_{\mathrm{in}}^{[l]} e^{il\tau}.$$

Moreover, there exist a constant M such that the Fourier coefficients of $\tilde{k}^{\mathrm{in}}$, $\Upsilon_{\mathrm{in}}^{[l]}$, satisfy:

$$\left|\Upsilon_{\mathrm{in}}^{[l]}\right| \leq M$$

and such that for all $(s,\theta) \in E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$ one has:

$$|\varphi(s,\theta)| \leq \frac{M}{|s|}, \qquad |P_1^{\mathrm{in}}(s,\theta)| \leq \frac{M}{|s|}.$$

The proof of this Theorem can be found in Section 4.4.

4.1.5 An asymptotic formula for the difference $\Delta = r_1^{\mathrm{u}} - r_1^{\mathrm{s}}$

Finally, we shall use the information obtained in the previous subsections to find an asymptotic formula for $\Delta(u,\theta) = r_1^{\mathrm{u}}(u,\theta) - r_1^{\mathrm{s}}(u,\theta)$. By Theorem 3.1.9 in Chapter 3 we know that:

$$\begin{aligned}\Delta(u,\theta) &= r_1^{\mathrm{u}}(u,\theta) - r_1^{\mathrm{s}}(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\tilde{k}(\xi(u,\theta)) \\ &= \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\sum_{l\in\mathbb{Z}} \Upsilon^{[l]} e^{il\xi(u,\theta)},\end{aligned} \tag{4.40}$$

where $P_1(u,\theta)$ and $\xi(u,\theta)$ are given in the same Theorem 3.1.9 and $\Upsilon^{[l]}$, that are the Fourier coefficients of the function $\tilde{k}(\tau)$, are unknown. Of course, they depend on δ and σ although we do not write it explicitly. We recall that:

$$|P_1(u,\theta)| \le \frac{K\delta}{\cosh(\mathrm{d}u)} \le \frac{K}{\kappa}, \tag{4.41}$$

for all $u \in D_{\kappa,\beta}$ and $\theta \in \mathbb{T}_\omega$, and that:

$$\xi(u,\theta) = \theta + \delta^{-1}\alpha u + \mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(\mathrm{d}u) + \alpha L(u) + \chi(u,\theta), \tag{4.42}$$

for some constant L_0 and functions $L(u)$ and $\chi(u,\theta)$. Moreover, these functions satisfy that for all $u \in D_{\kappa,\beta}$ and $\theta \in \mathbb{T}_\omega$:

$$|L(u)| \le K, \qquad |\chi(u,\theta)| \le \frac{K\delta}{\cosh(\mathrm{d}u)} \le \frac{K}{\kappa}. \tag{4.43}$$

We also point out that in the conservative case $\Upsilon^{[0]} = 0$ by Theorem 3.1.10. In the dissipative case, we assume that σ lies on one of the curves $\sigma_*(\delta)$ given by Theorem 3.1.10, that is:

$$\Upsilon^{[0]} = \Upsilon^{[0]}(\delta,\sigma_*(\delta)) = a_1\delta^{a_2}e^{-\frac{a_3\pi}{2\mathrm{d}\delta}}. \tag{4.44}$$

for some $a_1, a_2 \in \mathbb{R}$ and $a_3 > 0$.

From expression (4.40) one can already see that the coefficients $\Upsilon^{[l]}$ are exponentially small with respect to δ. Indeed, as a first exploration, we can consider the case $P_1(u,\theta) \equiv \chi(u,\theta) \equiv 0$. This case can give some insight since, as one can see from (4.41) and (4.43), they are "small" functions when we take large κ. If we make this simplification, $\xi(u,\theta)$ has the form $\xi(u,\theta) = \theta + \tilde{\xi}(u)$, with $\tilde{\xi}(u) = \delta^{-1}\alpha u + \mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(\mathrm{d}u) + \alpha L(u)$ (see (4.42)). Then $\Upsilon^{[l]}e^{il\tilde{\xi}(u)}$ are the Fourier coefficients of the function $\Delta(u,\theta)\cosh^{-2/\mathrm{d}}(\mathrm{d}u)$. In other words:

$$\begin{aligned}\left|\Upsilon^{[l]}\right| &= \left|\frac{e^{-il(\delta^{-1}\alpha u + \mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(\mathrm{d}u)+\alpha L(u))}}{2\pi}\int_0^{2\pi}\frac{\Delta(u,\theta)e^{-il\theta}}{\cosh^{2/\mathrm{d}}(\mathrm{d}u)}d\theta\right| \\ &\le \left|e^{-il(\delta^{-1}\alpha u + \mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(\mathrm{d}u)+\alpha L(u))}\right| \sup_{\theta\in[0,2\pi]}\left|\frac{\Delta(u,\theta)}{\cosh^{2/\mathrm{d}}(\mathrm{d}u)}\right|.\end{aligned}$$

We note that this inequality is valid for all $u \in D^{\mathrm{u}}_{\kappa,\beta,T} \cap D^{\mathrm{s}}_{\kappa,\beta,T}$. In particular, taking $u = u_+ := i(\pi/(2\mathrm{d}) - \kappa\delta)$ for $l < 0$ and $u = u_- := -u_+$ for $l > 0$, and noting that:

$$\operatorname{Im} \log\cosh(\mathrm{d}u_\pm) = 0, \qquad \frac{1}{|\cosh^{2/\mathrm{d}}(\mathrm{d}u_+)|} \le K(\kappa\delta)^{-2/\mathrm{d}},$$

one obtains:

$$\left|\Upsilon^{[l]}\right| \le K(\kappa\delta)^{-2/\mathrm{d}} e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta} - \alpha\kappa - \alpha|\operatorname{Im} L(u_\pm)|\right)|l|} \sup_{\theta\in[0,2\pi]} |\Delta(u_\pm,\theta)|\,.$$

Recalling that $\Delta(u,\theta) = r_1^{\mathrm{u}}(u,\theta) - r_1^{\mathrm{s}}(u,\theta)$ and using that $|r_1^{\mathrm{u,s}}(u_\pm,\theta)| \le M\delta^{-2}\kappa^{-3}$ by Theorem 4.1.1, we obtain readily (renaming K):

$$\left|\Upsilon^{[l]}\right| \le K\frac{\delta^{-2-2/\mathrm{d}}}{\kappa^{3+2/\mathrm{d}}} e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta} - \alpha\kappa - \alpha|\operatorname{Im} L(u_\pm)|\right)|l|}.$$

In particular, for $l = \pm 1$ we obtain:

$$\left|\Upsilon^{[\pm 1]}\right| \le K\frac{\delta^{-2-2/\mathrm{d}}}{\kappa^{3+2/\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta} + \alpha\kappa},$$

and for $|l| \ge 2$:

$$\left|\Upsilon^{[l]}\right| \le K\frac{\delta^{-2-2/\mathrm{d}}}{\kappa^{3+2/\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}\frac{3|l|}{4}},$$

where we have used that $\delta\kappa$ and $\delta|\operatorname{Im} L(u_\pm)|$ are arbitrarily small, by condition (4.3) and bound (4.43) respectively. Putting all these bounds together and using expression (4.44) of $\Upsilon^{[0]}$, we obtain that for real values of u and θ:

$$|\Delta(u,\theta)| \le \cosh^{2/\mathrm{d}}(\mathrm{d}u) a_1 \delta^{a_2} e^{-\frac{a_3\pi}{2\mathrm{d}\delta}} + 4K\frac{\delta^{-2-2/\mathrm{d}}}{\kappa^{3+2/\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta} + \alpha\kappa}.$$

The following result states that the same exponentially small bounds hold when $P_1(u,\theta) \not\equiv 0$ and $\chi(u,\theta) \not\equiv 0$.

Lemma 4.1.7. *Let $\Upsilon^{[l]}$, $l \in \mathbb{Z}$, $l \ne 0$, be the coefficients appearing in* (4.40)*. There exists a constant M such that:*

$$\left|\Upsilon^{[\pm 1]}\right| \le M\frac{\delta^{-2-2/\mathrm{d}}}{\kappa^{3+2/\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta} + \alpha\kappa},$$

and for $|l| \ge 2$:

$$\left|\Upsilon^{[l]}\right| \le M\frac{\delta^{-2-2/\mathrm{d}}}{\kappa^{3+2/\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}\frac{3|l|}{4}}.$$

The proof of this Lemma is postponed to Section 4.5.

Lemma 4.1.7 hints that, disregarding the coefficient $\Upsilon^{[0]}$, the dominant term of $\Delta(u,\theta)$ is determined essentially by the coefficients $\Upsilon^{[1]}$ and $\Upsilon^{[-1]}$ in expression (4.40). Thus, we shall look for a first asymptotic order of $\Delta(u,\theta)$ of the following form:

$$\Delta_0(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\left(\Upsilon^{[0]} + \Upsilon_0^{[1]}e^{i\xi(u,\theta)} + \Upsilon_0^{[-1]}e^{-i\xi(u,\theta)}\right), \tag{4.45}$$

for certain $\Upsilon_0^{[1]}$ and $\Upsilon_0^{[-1]}$. Of course, we expect $\Upsilon_0^{[\pm 1]} = \mathcal{O}\left(\delta^{-2-2/\mathrm{d}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\right)$. We stress that $\Upsilon^{[0]}$ is the same coefficient (depending on δ and σ) appearing in (4.40). From Theorem 3.1.10 we know that in the conservative case the coefficient $\Upsilon^{[0]}$ is zero, while in the dissipative case it can be made zero (or exponentially small) with the right choice of the parameter σ (see expression (4.44)). Thus, we just have to guess how to choose $\Upsilon_0^{[\pm 1]}$ in order that $\Delta_0(u,\theta)$ yields a good approximation of the difference $\Delta(u,\theta)$ indeed. We now proceed to give an intuitive idea of how this can be done.

As we have just seen, a key point to find the exponentially small bound of $\Upsilon^{[l]}$ is to evaluate the difference $\Delta(u,\theta)$ at the points $u_\pm = \pm i(\pi/(2\mathrm{d}) - \kappa\delta)$. The same happens if, instead of just obtaining an upper bound, one wants to obtain an asymptotic expression. Note that $u_+ \in D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2} \cap D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$. In this region, as a consequence of Theorem 4.1.5, the functions r_1^{u} and r_1^{s} are well approximated by the solutions of the inner equation (4.14) given by Theorem 4.1.4, $\psi_{\mathrm{in}}^{\mathrm{u}}$ and $\psi_{\mathrm{in}}^{\mathrm{s}}$. More precisely for $u_+ \in D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2} \cap D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$:

$$\begin{aligned} r_1^{\mathrm{u}}(u,\theta) &\approx \delta^{-2}\psi_{\mathrm{in}}^{\mathrm{u}}(s(u),\theta), \\ r_1^{\mathrm{s}}(u,\theta) &\approx \delta^{-2}\psi_{\mathrm{in}}^{\mathrm{s}}(s(u),\theta), \end{aligned}$$

where:

$$s(u) = \frac{1}{\delta Z_0(u)} = \frac{\cosh(\mathrm{d}u)}{\delta\sinh(\mathrm{d}u)}. \tag{4.46}$$

Hence, for $u \in D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2} \cap D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$, we can see that:

$$\Delta(u,\theta) \approx \Delta_{\mathrm{in}}(u,\theta) := \delta^{-2}\psi_{\mathrm{in}}^{\mathrm{u}}(s(u),\theta) - \delta^{-2}\psi_{\mathrm{in}}^{\mathrm{s}}(s(u),\theta) = \delta^{-2}\Delta\psi_{\mathrm{in}}(s(u),\theta). \tag{4.47}$$

Now, by Theorem 4.1.6, for $u \in D^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2} \cap D^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$ we have:

$$\begin{aligned} \Delta_{\mathrm{in}}(u,\theta) &= \delta^{-2}\Delta\psi_{\mathrm{in}}(s(u),\theta) = \delta^{-2}s^{2/\mathrm{d}}(u)\left(1+P_1^{\mathrm{in}}(s(u),\theta)\right)\tilde{k}^{\mathrm{in}}(\xi_{\mathrm{in}}(s(u),\theta)) \\ &= \delta^{-2}s^{2/\mathrm{d}}(u)\left(1+P_1^{\mathrm{in}}(s(u),\theta)\right)\sum_{l<0}\Upsilon_{\mathrm{in}}^{[l]}e^{il\xi_{\mathrm{in}}(s(u),\theta)}, \end{aligned} \tag{4.48}$$

where:

$$\xi_{\mathrm{in}}(s(u),\theta) = \theta + \mathrm{d}^{-1}\alpha s(u) + \mathrm{d}^{-1}(c+\alpha L_0)\log s(u) + \varphi(s(u),\theta), \tag{4.49}$$

and $\Upsilon_{\rm in}$ are constants independent of δ and σ. Taking into account the definition (4.46) of $s(u)$ we can write:

$$\log s(u) = \log\cosh(\mathrm{d}u) - \log\delta - \log\sinh(\mathrm{d}u),$$

and:

$$s(u) = \frac{\mathrm{d}u - i\pi/2}{\delta} + \frac{1}{\delta}\mathcal{O}\left((\mathrm{d}u - i\pi/2)^3\right).$$

Then (4.49) writes out as:

$$\begin{aligned}\xi_{\rm in}(s(u),\theta) &= \theta + \alpha\frac{u - i\pi/(2\mathrm{d})}{\delta} \\ &\quad + \mathrm{d}^{-1}(c+\alpha L_0)\left(\log\cosh(\mathrm{d}u) - \log\delta - \log\sinh(\mathrm{d}u)\right) \\ &\quad + \varphi\left(s(u),\theta\right) + \delta^{-1}\mathcal{O}\left((\mathrm{d}u - i\pi/2)^3\right),\end{aligned} \tag{4.50}$$

We want to evaluate this expression at $u = u_+$. We note that:

$$(\mathrm{d}u_+ - i\pi/2) = -i\delta\kappa,$$

and:

$$\log\sinh(\mathrm{d}u_+) = \log|\sinh(\mathrm{d}u_+)| + i\frac{\pi}{2} = i\frac{\pi}{2} + \mathcal{O}\left(\delta^2\kappa^2\right).$$

Then, equation (4.50) with $u = u_+$ writes out as:

$$\begin{aligned}\xi_{\rm in}(s(u_+),\theta) &= \theta - i\alpha\kappa + \mathrm{d}^{-1}(c+\alpha L_0)\left(\log\cosh(\mathrm{d}u_+) - \log\delta - i\frac{\pi}{2}\right) + \varphi\left(s(u_+),\theta\right) \\ &\quad + \mathcal{O}\left(\delta^2\kappa^3\right),\end{aligned} \tag{4.51}$$

Substituting (4.51) in expression (4.48), taking into account that $P_1^{\rm in}$ and φ are "small" by Theorem 4.1.6 and that:

$$s^{2/\mathrm{d}}(u_+) = \left(\frac{\cosh(\mathrm{d}u_+)}{\delta\sinh(\mathrm{d}u_+)}\right)^{2/\mathrm{d}} = \delta^{-2/\mathrm{d}}(-i)^{2/\mathrm{d}}\cosh^{2/\mathrm{d}}(\mathrm{d}u_+)\left(1 + \mathcal{O}\left(\delta^2\kappa^2\right)\right),$$

we obtain:

$$\begin{aligned}&\Delta_{\rm in}(u_+,\theta) \\ &\quad \approx \delta^{-2-\frac{2}{\mathrm{d}}}(-i)^{\frac{2}{\mathrm{d}}}\cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u_+)\sum_{l<0}\Upsilon_{\rm in}^{[l]}e^{il\left(\theta - i\alpha\kappa + \mathrm{d}^{-1}(c+\alpha L_0)\left(\log\cosh(\mathrm{d}u_+) - \log\delta - i\frac{\pi}{2}\right)\right)}.\end{aligned} \tag{4.52}$$

Now we turn again to $\Delta(u,\theta)$. Using that, by Theorem 3.1.9, the functions $\chi(u,\theta)$ and $P_1(u,\theta)$ are "small" and the following limit is well-defined:

$$L_+ := \lim_{u\to i\frac{\pi}{2\mathrm{d}}} L(u), \tag{4.53}$$

from (4.40) we have:

$$\Delta(u_+,\theta) \approx \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u_+)\sum_{l\in\mathbb{Z}}\Upsilon^{[l]}e^{il\left(\theta+i\frac{\pi}{2\mathrm{d}\delta}-i\alpha\kappa+\mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(u_+)+\alpha L_+\right)}$$

$$\approx \cosh^{\frac{2}{\mathrm{d}}}(\mathrm{d}u_+)\left(\Upsilon^{[0]}+\sum_{l<0}\Upsilon^{[l]}e^{\frac{\alpha\pi}{2\mathrm{d}\delta}|l|}e^{il\left(\theta-i\alpha\kappa+\mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(u_+)+\alpha L_+\right)}\right). \tag{4.54}$$

Note that the contribution of the terms in the sum with $l>0$ is exponentially small, so there is no need to take them into account.

Finally, we recall that, as we pointed out in (4.47), $\Delta(u_+,\theta)\approx\Delta_{\mathrm{in}}(u_+,\theta)$. As a consequence, it seems reasonable that each term in the sum (4.54) should be approximated by the corresponding term in (4.52), in particular $\Upsilon^{[-1]}$. In other words, we have:

$$\Upsilon^{[-1]} \approx \frac{\delta^{-2-\frac{2}{\mathrm{d}}}(-i)^{\frac{2}{\mathrm{d}}}\Upsilon_{\mathrm{in}}^{[-1]}e^{-i\left(\theta-i\alpha\kappa+\mathrm{d}^{-1}(c+\alpha L_0)\left(\log\cosh(\mathrm{d}u_+)-\log\delta-i\frac{\pi}{2}\right)\right)}}{e^{\frac{\alpha\pi}{2\mathrm{d}\delta}}e^{-i\left(\theta-i\alpha\kappa+\mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(u_+)+\alpha L_+\right)}}. \tag{4.55}$$

Simplifying the terms that appear in both the numerator and the denominator, equation (4.55) yields:

$$\Upsilon^{[-1]} \approx \delta^{-2-\frac{2}{\mathrm{d}}}(-i)^{\frac{2}{\mathrm{d}}}\Upsilon_{\mathrm{in}}^{[-1]}e^{\mathrm{d}^{-1}(c+\alpha L_0)\left(i\log\delta-\frac{\pi}{2}\right)+i\alpha L_+}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}.$$

For this reason we choose $\Upsilon_0^{[-1]}$ in (4.45) to be precisely:

$$\Upsilon_0^{[-1]} = \delta^{-2-\frac{2}{\mathrm{d}}}(-i)^{\frac{2}{\mathrm{d}}}\Upsilon_{\mathrm{in}}^{[-1]}e^{\mathrm{d}^{-1}(c+\alpha L_0)\left(i\log\delta-\frac{\pi}{2}\right)+i\alpha L_+}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}. \tag{4.56}$$

Since $\Delta(u,\theta)$ is real analytic, then $\Upsilon^{[1]}=\overline{\Upsilon^{[-1]}}$. Thus, we define:

$$\Upsilon_0^{[1]} := \overline{\Upsilon_0^{[-1]}} = \delta^{-2-\frac{2}{\mathrm{d}}}i^{\frac{2}{\mathrm{d}}}\overline{\Upsilon_{\mathrm{in}}^{[-1]}}e^{\mathrm{d}^{-1}(c+\alpha L_0)\left(-i\log\delta-\frac{\pi}{2}\right)-i\alpha L_+}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}. \tag{4.57}$$

Of course, to prove that $\Delta_0(u,\theta)$ is the first order of $\Delta(u,\theta)$, we need to see that:

$$\begin{aligned}\Delta_1(u,\theta) &:= \Delta(u,\theta)-\Delta_0(u,\theta)\\ &= \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\Bigg[\left(\Upsilon^{[1]}-\Upsilon_0^{[1]}\right)e^{i\xi(u,\theta)}+\left(\Upsilon^{[-1]}-\Upsilon_0^{[-1]}\right)e^{-i\xi(u,\theta)}\\ &\quad+\sum_{|l|\geq 2}\Upsilon^{[l]}e^{il\xi(u,\theta)}\Bigg].\end{aligned} \tag{4.58}$$

is smaller than $\Delta_0(u,\theta)$. By Lemma 4.1.7 it is clear that the terms involving $\Upsilon^{[l]}$ with $|l|\geq 2$ are smaller than Δ_0. The following result states that the terms $\Upsilon^{[\pm1]}-\Upsilon_0^{[\pm1]}$ are also small. Its proof can be found in Section 4.5.

Proposition 4.1.8. *Let $\kappa = \kappa_0 \log(1/\delta)$, with $\kappa_0 > 0$ any constant such that $1-\gamma > \alpha\kappa_0$. Let $\Upsilon_0^{[\pm 1]}$ be defined as (4.56) and (4.57). There exists a constant M such that:*

1. *In the conservative case, one has:*
$$\left|\Upsilon^{[\pm 1]} - \Upsilon_0^{[\pm 1]}\right| \leq \frac{M}{\kappa}\delta^{-4} e^{-\frac{\alpha\pi}{2\delta}}.$$

2. *In the dissipative case, let $\sigma_*(\delta)$ be one of the curves defined in Theorem 3.1.10. Then, for $\sigma = \sigma_*(\delta)$ one has:*
$$\left|\Upsilon^{[\pm 1]} - \Upsilon_0^{[\pm 1]}\right| \leq \frac{M}{\kappa}\delta^{-2-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}.$$

Using this result and formulas (4.56) and (4.57) of $\Upsilon_0^{[-1]}$ and $\Upsilon_0^{[1]}$ we can prove the main theorem of this chapter:

Theorem 4.1.9. *Let $\kappa = \kappa_0 \log(1/\delta)$, with $\kappa_0 > 0$ any constant such that $1-\gamma > \alpha\kappa_0$. Define:*
$$\vartheta(u,\delta) = \delta^{-1}\alpha u + \mathrm{d}^{-1}(c + \alpha L_0)\left[\log\cosh(\mathrm{d}u) - \log\delta\right] + \alpha L(u),$$
where L_0 and $L(u)$ are given in Theorem 3.1.9. Define:
$$\mathcal{C}^* = \mathcal{C}_1^* + i\mathcal{C}_2^* := 2(-i)^{\frac{2}{\mathrm{d}}}\Upsilon_{\mathrm{in}}^{[-1]} e^{-\mathrm{d}^{-1}(c+\alpha L_0)\frac{\pi}{2} + i\alpha L_+}.$$
where $\Upsilon_{\mathrm{in}}^{[-1]}$ appears in Theorem 4.1.6 and $L_+ = \lim_{u\to i\frac{\pi}{2\mathrm{d}}} L(u)$. Then there exist $T_0 > 0$ and $\delta_0 > 0$ such that for all $u \in [-T_0, T_0]$, $\theta \in \mathbb{S}^1$ and $0 < \delta < \delta_0$ the following holds:

1. *In the conservative case (where $\mathrm{d} = 1$):*
$$\begin{aligned}\Delta(u,\theta) &= \delta^{-4}\cosh^2(u) e^{-\frac{\alpha\pi}{2\delta}}\Bigg[\mathcal{C}_1^*\cos\Big(\theta + \vartheta(u,\delta)\Big) + \mathcal{C}_2^*\sin\Big(\theta + \vartheta(u,\delta)\Big)\\ &\quad + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\Bigg].\end{aligned}$$

2. *In the dissipative case, let $\sigma = \sigma_*(\delta)$ be one of the curves defined in Theorem 3.1.10. Then:*
$$\begin{aligned}\Delta(u,\theta) &= \cosh^{2/\mathrm{d}}(\mathrm{d}u)\Upsilon^{[0]}(\delta,\sigma_*(\delta))\left(1 + \mathcal{O}(\delta)\right)\\ &\quad + \delta^{-2-\frac{2}{\mathrm{d}}}\cosh^{2/\mathrm{d}}(\mathrm{d}u) e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\Bigg[\mathcal{C}_1^*\cos\Big(\theta + \vartheta(u,\delta)\Big) + \mathcal{C}_2^*\sin\Big(\theta + \vartheta(u,\delta)\Big)\\ &\quad + \mathcal{O}\left(\frac{1}{\log(1/\delta)}\right)\Bigg].\end{aligned}$$

Proof. Recalling the definition (4.45) of $\Delta_0(u,\theta)$ and the form of $\Delta(u,\theta)$ given by Theorem 3.1.9, we can write:

$$\Delta(u,\theta) = \Delta_0(u,\theta) + \Delta_1(u,\theta), \tag{4.59}$$

where:

$$\begin{aligned}\Delta_1(u,\theta) &= \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\Bigg(\left(\Upsilon^{[1]} - \Upsilon_0^{[1]}\right)e^{i\xi(u,\theta)} \\ &\quad + \left(\Upsilon^{[-1]} - \Upsilon_0^{[-1]}\right)e^{-i\xi(u,\theta)} + \sum_{|l|\geq 2}\Upsilon^{[l]}e^{il\xi(u,\theta)}\Bigg).\end{aligned}$$

On the one hand, using Lemma 4.1.7 to bound $|\Upsilon^{[l]}|$ for $l \geq 2$, Proposition 4.1.8 to bound $|\Upsilon^{[\pm 1]} - \Upsilon_0^{[\pm 1]}|$ and the fact that $\xi(u,\theta) \in \mathbb{R}$ for $(u,\theta) \in [-T_0, T_0] \times \mathbb{S}^1$, it is easy to see that:

$$|\Delta_1(u,\theta)| \leq K\cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\frac{\delta^{-2-2/\mathrm{d}}}{\kappa}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}.$$

Since, by (4.41), for $u \in [-T_0, T_0]$ we have:

$$|P_1(u,\theta)| \leq \frac{K\delta}{|\cosh(\mathrm{d}u)|} \leq K\delta, \tag{4.60}$$

and $\kappa = \kappa_0 \log(1/\delta)$, this yields (renaming K):

$$|\Delta_1(u,\theta)| \leq K\cosh^{2/\mathrm{d}}(\mathrm{d}u)\frac{\delta^{-2-2/\mathrm{d}}}{\log(1/\delta)}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}. \tag{4.61}$$

On the other hand, let us recall the definition (4.45) of $\Delta_0(u,\theta)$:

$$\Delta_0(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1+P_1(u,\theta))\left(\Upsilon^{[0]} + \Upsilon_0^{[1]}e^{i\xi(u,\theta)} + \Upsilon_0^{[-1]}e^{-i\xi(u,\theta)}\right),$$

where $\Upsilon_0^{[-1]}$ and $\Upsilon_0^{[1]}$ are given in (4.56) and (4.57) and $\xi(u,\theta)$ is given in (4.42). Then one has:

$$\Upsilon_0^{[-1]}e^{-i\xi(u,\theta)} = \delta^{-2-\frac{2}{\mathrm{d}}}e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}\frac{\mathcal{C}^*}{2}e^{-i(\theta+\vartheta(u,\delta)+\varphi(u,\theta))},$$

with:

$$\mathcal{C}^* = 2(-i)^{\frac{2}{\mathrm{d}}}\Upsilon_{\mathrm{in}}^{[-1]}e^{-\mathrm{d}^{-1}(c+\alpha L_0)\frac{\pi}{2}+i\alpha L_+}.$$

By Theorem 4.1.6, for $u \in [-T_0, T_0]$ one has:

$$|\varphi(u,\theta)| \leq \frac{K\delta}{|\cosh(\mathrm{d}u)|} \leq K\delta,$$

so that:

$$\Upsilon_0^{[-1]} e^{-i\xi(u,\theta)} = \delta^{-2-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \frac{\mathcal{C}^*}{2} e^{-i(\theta+\vartheta(u,\delta))}(1+\mathcal{O}(\delta)).$$

Then, using also bound (4.60) and the fact that $\Upsilon^{[1]} = \overline{\Upsilon^{-[1]}}$, we obtain:

$$\begin{aligned}\Delta_0(u,\theta) &= \cosh^{2/\mathrm{d}}(\mathrm{d}u)\Upsilon^{[0]}(1+\mathcal{O}(\delta)) \\ &\quad + \cosh^{2/\mathrm{d}}(\mathrm{d}u)\delta^{-2-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \left[\frac{\overline{\mathcal{C}^*}}{2} e^{i(\theta+\vartheta(u,\delta))} + \frac{\mathcal{C}^*}{2} e^{-i(\theta+\vartheta(u,\delta))}\right](1+\mathcal{O}(\delta)).\end{aligned} \tag{4.62}$$

Then we only have to note that:

$$\frac{\overline{\mathcal{C}^*}}{2} e^{i(\theta+\vartheta(u,\delta))} + \frac{\mathcal{C}^*}{2} e^{-i(\theta+\vartheta(u,\delta))} = \operatorname{Re}\mathcal{C}^* \cos(\theta+\vartheta(u,\delta)) + \operatorname{Im}\mathcal{C}^* \sin(\theta+\vartheta(u,\delta)),$$

so that using (4.61) and (4.62) in (4.59) we obtain the claim of the lemma. In the conservative case we take into account that $\mathrm{d}=1$ and $\Upsilon^{[0]}=0$ by Theorem 3.1.10. □

Remark 4.1.10. Theorem 4.1.9 yields straightforwardly Theorem 4.1 just following the same steps as in the proof of Theorem 3.1, see the end of Section 3.1 in Chapter 3.

Remark 4.1.11. As in the previous chapters, to make formulas shorter and avoid keeping track of constants that do not play any role in the proofs, throughout all this chapter we will use K to denote *any* constant independent of the parameters δ, σ and κ.

4.2 Proof of Theorem 4.1.4. The inner equation

As we pointed out in (4.28), the operator $\mathcal{G}^{\mathrm{u}}$ defined in (4.24) is a right inverse of the linear operator $\mathcal{L}$ (see (4.15) for its definition). Thus, the inner equation (4.18) can be written as the following fixed point equation:

$$\psi_{\mathrm{in}}^{\mathrm{u}} = \tilde{\mathcal{M}}^{\mathrm{u}}(\psi_{\mathrm{in}}^{\mathrm{u}}), \tag{4.63}$$

where:

$$\tilde{\mathcal{M}}^{\mathrm{u}}(\phi) = \mathcal{G}^{\mathrm{u}} \circ \mathcal{M}(\phi, 0), \tag{4.64}$$

and $\mathcal{M}$ is defined in (4.16). The proof of Theorem 4.1.4 relies on proving that the operator $\tilde{\mathcal{M}}^{\mathrm{u}}$ has a fixed point in a suitable Banach space. Thus we start, in the following subsection, by defining such Banach spaces.

4.2.1 Banach spaces

Let $\phi : \mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa} \to \mathbb{C}$. Then, for any $n \in \mathbb{R}$, we define the norm $\|.\|^{\mathrm{u}}_n$ as:

$$\|\phi\|^{\mathrm{u}}_n := \sup_{s\in\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa}} |s^n\phi(s)|. \tag{4.65}$$

For $\phi : \mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa} \times \mathbb{T}_\omega \to \mathbb{C}$, writing $\phi(s,\theta) = \sum_{l\in\mathbb{Z}} \phi^{[l]}(s)e^{il\theta}$, we define the norm $\|.\|^{\mathrm{u}}_{n,\omega}$ as:

$$\|\phi\|^{\mathrm{u}}_{n,\omega} := \sum_{l\in\mathbb{Z}} \|\phi^{[l]}\|^{\mathrm{u}}_n e^{|l|\omega}. \tag{4.66}$$

Then we define the Banach space $\mathcal{X}^{\mathrm{u}}_{n,\omega}$ as the space of analytic functions having finite norm $\|.\|^{\mathrm{u}}_{n,\omega}$:

$$\mathcal{X}^{\mathrm{u}}_{n,\omega} := \{\phi : \mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa} \times \mathbb{T}_\omega \to \mathbb{C} \,:\, \phi \text{ is analytic}, \|\phi\|^{\mathrm{u}}_{n,\omega} < \infty\}. \tag{4.67}$$

Since we are dealing with a PDE that involves the derivatives with respect the variables s and θ, we also consider the following norm:

$$\llbracket\phi\rrbracket^{\mathrm{u}}_{n,\omega} := \|\phi\|^{\mathrm{u}}_{n,\omega} + \|\partial_s\phi\|^{\mathrm{u}}_{n+1,\omega} + \|\partial_\theta\phi\|^{\mathrm{u}}_{n+1,\omega}, \tag{4.68}$$

and the corresponding Banach space:

$$\tilde{\mathcal{X}}^{\mathrm{u}}_{n,\omega} := \{\phi : \mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa} \times \mathbb{T}_\omega \to \mathbb{C} \,:\, \phi \text{ is analytic}, \llbracket\phi\rrbracket^{\mathrm{u}}_{n,\omega} < \infty\}. \tag{4.69}$$

For the stable case, we define analog norms $\|.\|^{\mathrm{s}}_{n,\omega}$ and $\llbracket.\rrbracket^{\mathrm{s}}_{n,\omega}$ and Banach Spaces $\mathcal{X}^{\mathrm{s}}_{n,\omega}$ and $\tilde{\mathcal{X}}^{\mathrm{s}}_{n,\omega}$, just replacing the domain $\mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar\kappa}$ by $\mathcal{D}^{\mathrm{in,s}}_{\beta_0,\bar\kappa}$.

In the following lemma we summarize some known properties of these Banach Spaces and norms.

Lemma 4.2.1. *Let $n_1, n_2 \in \mathbb{R}$, $n_1, n_2 \geq 0$.*

1. *If $n_1 \leq n_2$, then $\mathcal{X}^{\mathrm{u,s}}_{n_2,\omega} \subset \mathcal{X}^{\mathrm{u,s}}_{n_1,\omega}$, and there exists a constant M such that for all $\phi \in \mathcal{X}^{\mathrm{u,s}}_{n_2,\omega}$:*
$$\|\phi\|^{\mathrm{u,s}}_{n_1,\omega} \leq \frac{M}{\bar\kappa^{n_2-n_1}}\|\phi\|^{\mathrm{u,s}}_{n_2,\omega}.$$

2. *If $\phi_1 \in \mathcal{X}^{\mathrm{u,s}}_{n_1,\omega}$, $\phi_2 \in \mathcal{X}^{\mathrm{u,s}}_{n_2,\omega}$, then $\phi_1\phi_2 \in \mathcal{X}^{\mathrm{u,s}}_{n_1+n_2,\omega}$ and there exists a constant M such that:*
$$\|\phi_1\phi_2\|^{\mathrm{u,s}}_{n_1+n_2,\omega} \leq M\|\phi_1\|^{\mathrm{u,s}}_{n_1,\omega}\|\phi_2\|^{\mathrm{u,s}}_{n_2,\omega}.$$

Now we enunciate the following result, that has Theorem 4.1.4 as an obvious corollary.

Proposition 4.2.2. *Let $\beta_0 > 0$. Let $\bar{\kappa} > 0$ be large enough. Then equation* (4.63) *has two solutions $\psi_{\rm in}^{\rm u}(s,\theta)$ and $\psi_{\rm in}^{\rm s}(s,\theta)$, such that $\psi_{\rm in}^{\rm u} \in \tilde{\mathcal{X}}_{3,\omega}^{\rm u}$ and $\psi_{\rm in}^{\rm s} \in \tilde{\mathcal{X}}_{3,\omega}^{\rm s}$, and there exists a constant M such that:*

$$\llbracket \psi_{\rm in}^{\rm u} \rrbracket_{3,\omega}^{\rm u} \leq M, \qquad \llbracket \psi_{\rm in}^{\rm s} \rrbracket_{3,\omega}^{\rm s} \leq M.$$

Moreover:

$$\llbracket \psi_{\rm in}^{\rm u} - \tilde{\mathcal{M}}^{\rm u}(0) \rrbracket_{4,\omega}^{\rm u} \leq M, \qquad \llbracket \psi_{\rm in}^{\rm s} - \tilde{\mathcal{M}}^{\rm s}(0) \rrbracket_{4,\omega}^{\rm s} \leq M.$$

The rest of the section is devoted to proving this proposition for the unstable case. The proof for the stable one is completely analogous.

4.2.2 Technical lemmas

First of all, we start giving some technical properties of the functions $\hat{F}$, $\hat{G}$, $\hat{H}$ (defined in (4.11)) and their derivatives. These will be useful not just in this section, but also in Subsections 4.3.3 and 4.4.2.

To avoid tedious computations, we shall skip the proofs of the following results. They can be proved in a very similar way as Lemmas 3.2.14 and 3.2.15 in Chapter 3.

Lemma 4.2.3. *Let C be any constant, and $\phi \in \mathcal{X}_{3,\omega}^{\rm u}$ with $\|\phi\|_{3,\omega}^{\rm u} \leq C$. There exists a constant M such that:*

1. $\|\hat{F}(\phi, s, \theta, 0, 0)\|_{4,\omega}^{\rm u} \leq M$,
2. $\|\hat{G}(\phi, s, \theta, 0, 0)\|_{2,\omega}^{\rm u} \leq M$,
3. $\|\hat{H}(\phi, s, \theta, 0, 0)\|_{3,\omega}^{\rm u} \leq M$,

where $\hat{F}$, $\hat{G}$, $\hat{H}$ are defined in (4.11).

Lemma 4.2.4. *Let C be any fixed constant, and $\phi \in \mathcal{X}_{3,\omega}^{\rm u}$ with $\|\phi\|_{3,\omega}^{\rm u} \leq C$. If $\bar{\kappa}$ is sufficiently large, there exists a constant M such that:*

1. $\|D_\phi \hat{F}(\phi, s, \theta, 0, 0)\|_{2,\omega}^{\rm u} \leq M$,
2. $\|D_\phi \hat{G}(\phi, s, \theta, 0, 0)\|_{0,\omega}^{\rm u} \leq M$,
3. $\|D_\phi \hat{H}(\phi, s, \theta, 0, 0)\|_{1,\omega}^{\rm u} \leq M$,

where $\hat{F}$, $\hat{G}$, $\hat{H}$ are defined in (4.11).

Lemma 4.2.5. *Let C be any fixed constant, and $\phi_1, \phi_2 \in \mathcal{X}_{3,\omega}^{\rm u}$ such that $\|\phi_i\|_{3,\omega}^{\rm u} \leq C$, for $i = 1, 2$. There exists a constant M such that:*

1. $\|\hat{F}(\phi_1, s, \theta, 0, 0) - \hat{F}(\phi_2, s, \theta, 0, 0)\|^{\mathrm{u}}_{5,\omega} \le M\|\phi_1 - \phi_2\|^{\mathrm{u}}_{3,\omega}$,
2. $\|\hat{G}(\phi_1, s, \theta, 0, 0) - \hat{G}(\phi_2, s, \theta, 0, 0)\|^{\mathrm{u}}_{3,\omega} \le M\|\phi_1 - \phi_2\|^{\mathrm{u}}_{3,\omega}$,
3. $\|\hat{H}(\phi_1, s, \theta, 0, 0) - \hat{H}(\phi_2, s, \theta, 0, 0)\|^{\mathrm{u}}_{4,\omega} \le M\|\phi_1 - \phi_2\|^{\mathrm{u}}_{3,\omega}$,

where $\hat{F}$, $\hat{G}$, $\hat{H}$ *are defined in* (4.11).

Now we summarize the main properties of the operator $\mathcal{G}^{\mathrm{u}}$.

Lemma 4.2.6. *Let* $n \ge 1$ *and* $\phi \in \mathcal{X}^{\mathrm{u}}_{n,\omega}$. *There exists a constant* M *such that:*

1. $\|\mathcal{G}^{\mathrm{u}}(\phi)\|^{\mathrm{u}}_{n-1,\omega} \le M\|\phi\|^{\mathrm{u}}_{n,\omega}$.
2. *If* $\phi^{[0]}(s) = 0$, *then* $\|\mathcal{G}^{\mathrm{u}}(\phi)\|^{\mathrm{u}}_{n,\omega} \le M\|\phi\|^{\mathrm{u}}_{n,\omega}$.
3. *In addition, one has that* $[\![\mathcal{G}^{\mathrm{u}}(\phi)]\!]^{\mathrm{u}}_{n-1,\omega} \le M\|\phi\|^{\mathrm{u}}_{n,\omega}$.

4.2.3 The fixed point

Finally we can proceed to prove the existence of a fixed point of the operator $\tilde{\mathcal{M}}^{\mathrm{u}}$, given in (4.64), in a suitable ball of the Banach space $\mathcal{X}^{\mathrm{u}}_{n,\omega}$ for a particular n. We first begin by studying the *independent term* $\tilde{\mathcal{M}}^{\mathrm{u}}(0)$, which will give us the size of the aforementioned ball.

Lemma 4.2.7. *There exists a constant* M *such that:*

$$[\![\tilde{\mathcal{M}}^{\mathrm{u}}(0)]\!]^{\mathrm{u}}_{3,\omega} \le M,$$

where $\tilde{\mathcal{M}}^{\mathrm{u}}$ *is defined in* (4.64).

Proof. This is straightforward. Noting that:

$$\mathcal{M}(0,0) = \hat{F}(0, s, \theta, 0, 0) + \frac{\mathrm{d}+1}{b} s^{-1}\hat{H}(0, s, \theta, 0, 0),$$

by Lemma 4.2.3 it is clear that $\|\mathcal{M}(0,0)\|^{\mathrm{u}}_{4,\omega} \le K$ for some constant K. Then, since $\tilde{\mathcal{M}}^{\mathrm{u}}(0) = \mathcal{G}^{\mathrm{u}} \circ \mathcal{M}(0,0)$, one just needs to use item 3 of Lemma 4.2.6 to obtain the claim of the lemma. □

The next step is to find a Lipschitz constant of the operator $\tilde{\mathcal{M}}^{\mathrm{u}}$.

Lemma 4.2.8. *Let* $\phi_1, \phi_2 \in \tilde{\mathcal{X}}^{\mathrm{u}}_{3,\omega}$ *such that* $[\![\phi_i]\!]^{\mathrm{u}}_{3,\omega} \le C$ *for some constant* C. *Then, there exists a constant* M *such that:*

$$[\![\tilde{\mathcal{M}}^{\mathrm{u}}(\phi_1) - \tilde{\mathcal{M}}^{\mathrm{u}}(\phi_2)]\!]^{\mathrm{u}}_{4,\omega} \le M[\![\phi_1 - \phi_2]\!]^{\mathrm{u}}_{3,\omega}.$$

Proof. First we note that since $\mathcal{G}^{\mathrm{u}}$ is linear:

$$\tilde{\mathcal{M}}^{\mathrm{u}}(\phi_1) - \tilde{\mathcal{M}}^{\mathrm{u}}(\phi_2) = \mathcal{G}^{\mathrm{u}}(\mathcal{M}(\phi_1, 0) - \mathcal{M}(\phi_2, 0)).$$

Hence, by item 3 of Lemma 4.2.6, we just need to prove:

$$\|\mathcal{M}(\phi_1, 0) - \mathcal{M}(\phi_2, 0)\|^{\mathrm{u}}_{5,\omega} \le K[\![\phi_1 - \phi_2]\!]^{\mathrm{u}}_{3,\omega}. \tag{4.70}$$

Now, by definition (4.16) of $\mathcal{M}$, we have:

$$\begin{aligned}
\mathcal{M}(\phi_1, 0) - \mathcal{M}(\phi_2, 0) \;=\;& cs^{-1}\partial_\theta(\phi_1 - \phi_2) + \hat{F}(\phi_1, s, \theta, 0, 0) - \hat{F}(\phi_2, s, \theta, 0, 0) \\
&+ \frac{\mathrm{d}+1}{b} s^{-1} \left[\hat{H}(\phi_1, s, \theta, 0, 0) - \hat{H}(\phi_2, s, \theta, 0, 0)\right] \\
&- \left[\hat{G}(\phi_1, s, \theta, 0, 0) - \hat{G}(\phi_2, s, \theta, 0, 0)\right] \partial_\theta \phi_1 \\
&- \hat{G}(\phi_2, s, \theta, 0, 0)\partial_\theta(\phi_1 - \phi_2) \\
&+ s^2 \left[2b(\phi_1 - \phi_2) + \hat{H}(\phi_1, s, \theta, 0, 0) - \hat{H}(\phi_2, s, \theta, 0, 0)\right] \partial_s \phi_1 \\
&+ s^2 \left[2b\phi_2 + \hat{H}(\phi_2, s, \theta, 0, 0)\right] \partial_s(\phi_1 - \phi_2).
\end{aligned}$$

One just needs to use the properties of the norm summarized in Lemma 4.2.1, the bounds of Lemmas 4.2.3 and 4.2.5, and take into account that:

$$\|\phi_1 - \phi_2\|^{\mathrm{u}}_{3,\omega} \le [\![\phi_1 - \phi_2]\!]^{\mathrm{u}}_{3,\omega}, \qquad \|\partial_\theta(\phi_1 - \phi_2)\|^{\mathrm{u}}_{4,\omega} \le [\![\phi_1 - \phi_2]\!]^{\mathrm{u}}_{3,\omega},$$
$$\|\partial_s(\phi_1 - \phi_2)\|^{\mathrm{u}}_{4,\omega} \le [\![\phi_1 - \phi_2]\!]^{\mathrm{u}}_{3,\omega},$$

and then (4.70) is obtained easily. □

End of the proof of Proposition 4.2.2. Let $\phi_1, \phi_2 \in B(2[\![\tilde{\mathcal{M}}^{\mathrm{u}}(0)]\!]^{\mathrm{u}}_{3,\omega}) \subset \tilde{\mathcal{X}}^{\mathrm{u}}_{3,\omega}$. Using Lemmas 4.2.1 and 4.2.8 we obtain:

$$[\![\tilde{\mathcal{M}}^{\mathrm{u}}(\phi_1) - \tilde{\mathcal{M}}^{\mathrm{u}}(\phi_2)]\!]^{\mathrm{u}}_{3,\omega} \le \frac{K}{\bar{\kappa}}[\![\tilde{\mathcal{M}}^{\mathrm{u}}(\phi_1) - \tilde{\mathcal{M}}^{\mathrm{u}}(\phi_2)]\!]^{\mathrm{u}}_{4,\omega} \le \frac{K}{\bar{\kappa}}[\![\phi_1 - \phi_2]\!]^{\mathrm{u}}_{3,\omega}.$$

Hence, for $\bar{\kappa}$ sufficiently large, $\mathcal{M}^{\mathrm{u}}$ is contractive and:

$$\tilde{\mathcal{M}}^{\mathrm{u}} : B(2[\![\tilde{\mathcal{M}}^{\mathrm{u}}(0)]\!]^{\mathrm{u}}_{3,\omega}) \to B(2[\![\tilde{\mathcal{M}}^{\mathrm{u}}(0)]\!]^{\mathrm{u}}_{3,\omega}),$$

so that it has a unique fixed point $\psi^{\mathrm{u}}_{\mathrm{in}} \in B(2[\![\tilde{\mathcal{M}}^{\mathrm{u}}(0)]\!]^{\mathrm{u}}_{3,\omega})$. In other words, $\psi^{\mathrm{u}}_{\mathrm{in}}$ satisfies equation (4.63), and $[\![\psi^{\mathrm{u}}_{\mathrm{in}}]\!]^{\mathrm{u}}_{3,\omega} \le 2[\![\tilde{\mathcal{M}}^{\mathrm{u}}(0)]\!]^{\mathrm{u}}_{3,\omega} \le K$ by Lemma 4.2.7. Finally, using Lemma 4.2.8 again, we obtain:

$$[\![\psi^{\mathrm{u}}_{\mathrm{in}} - \tilde{\mathcal{M}}^{\mathrm{u}}(0)]\!]^{\mathrm{u}}_{4,\omega} = [\![\tilde{\mathcal{M}}^{\mathrm{u}}(\psi^{\mathrm{u}}_{\mathrm{in}}) - \tilde{\mathcal{M}}^{\mathrm{u}}(0)]\!]^{\mathrm{u}}_{4,\omega} \le K[\![\psi^{\mathrm{u}}_{\mathrm{in}}]\!]^{\mathrm{u}}_{3,\omega} \le K,$$

and Proposition 4.2.2 is proved. □

4.3 Proof of Theorem 4.1.5. The matching errors

Let us recall that the main object of study of this section are the so-called matching errors, defined in (4.34) as:

$$\psi_1^{\mathrm{u}}(s,\theta) = \psi^{\mathrm{u}}(s,\theta) - \psi_{\mathrm{in}}^{\mathrm{u}}(s,\theta), \qquad \psi_1^{\mathrm{s}}(s,\theta) = \psi^{\mathrm{s}}(s,\theta) - \psi_{\mathrm{in}}^{\mathrm{s}}(s,\theta), \tag{4.71}$$

where ψ^{u} and ψ^{s} are given in Theorem 4.1.3 and $\psi_{\mathrm{in}}^{\mathrm{u}}$ and $\psi_{\mathrm{in}}^{\mathrm{s}}$ are given in Theorem 4.1.4.

We will prove the following proposition, which is equivalent to Theorem 4.1.5. We abuse notation and use the same norms and Banach spaces as in Section 4.2, although here all the functions and suprema are taken in $\mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$ (respectively $\mathcal{D}^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$).

Proposition 4.3.1. *Consider the functions $\psi_1^{\mathrm{u}}(s,\theta)$ and $\psi_1^{\mathrm{u}}(s,\theta)$ defined in* (4.71)*. Then, one has $\psi_1^{\mathrm{u}} \in \tilde{\mathcal{X}}^{\mathrm{u}}_{2,\omega}$ and $\psi_1^{\mathrm{s}} \in \tilde{\mathcal{X}}^{\mathrm{s}}_{2,\omega}$. Moreover there exists a constant M such that:*

$$[\![\psi_1^{\mathrm{u}}]\!]^{\mathrm{u}}_{2,\omega} \leq M\delta^{1-\gamma}, \qquad [\![\psi_1^{\mathrm{s}}]\!]^{\mathrm{s}}_{2,\omega} \leq M\delta^{1-\gamma}.$$

The rest of this section is devoted to proving this result. In the following we shall focus just on the unstable case, that is on ψ_1^{u}, but the argument can be analogously done for the stable case.

4.3.1 Decomposition of ψ_1^{u}

In this subsection we shall see that ψ_1^{u} can be written in a particular way, which will be very convenient to find the desired bounds.

Note that ψ^{u} and $\psi_{\mathrm{in}}^{\mathrm{u}}$ are some specific functions, the existence of which we already know by Theorems 4.1.3 and 4.1.4. These theorems provide us with an *a priori* bound of the matching errors ψ_1^{u} and ψ_1^{s}. Our goal is to improve these bounds as stated in Proposition 4.3.1.

Recall that $\psi^{\mathrm{u}}(s,\theta)$ is defined for $s \in \mathcal{D}^{\mathrm{u}}_{\kappa,\beta,T}$ (see (4.7)) and $\psi_{\mathrm{in}}^{\mathrm{u}}(s,\theta)$ is defined for $s \in \mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar{\kappa}}$ (see (2.28)). Then, since $\mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2} \subset \mathcal{D}^{\mathrm{u}}_{\kappa,\beta,T} \subset \mathcal{D}^{\mathrm{in,u}}_{\beta_0,\bar{\kappa}}$, one has that $\psi_1^{\mathrm{u}} = \psi^{\mathrm{u}} - \psi_{\mathrm{in}}^{\mathrm{u}}$ is defined in $\mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$. Finally, we recall that ψ^{u} satisfies:

$$\mathcal{L}(\psi^{\mathrm{u}}) = \mathcal{M}(\psi^{\mathrm{u}},\delta),$$

and $\psi_{\mathrm{in}}^{\mathrm{u}}$ satisfies:

$$\mathcal{L}(\psi_{\mathrm{in}}^{\mathrm{u}}) = \mathcal{M}(\psi_{\mathrm{in}}^{\mathrm{u}},0),$$

where $\mathcal{L}$ is the linear operator defined in (4.15) and $\mathcal{M}$ is the operator defined in (4.16). Then, clearly ψ_1^{u} satisfies the PDE obtained by subtracting the previous ones. In other words, defining the operator $\mathcal{M}_1^{\mathrm{u}}$ as:

$$\mathcal{M}_1^{\mathrm{u}}(\psi_1^{\mathrm{u}}) = \mathcal{M}(\psi_{\mathrm{in}}^{\mathrm{u}} + \psi_1^{\mathrm{u}},\delta) - \mathcal{M}(\psi_{\mathrm{in}}^{\mathrm{u}},0), \tag{4.72}$$

then $\psi_1^{\rm u}$ satisfies:

$$\mathcal{L}(\psi_1^{\rm u}) = \mathcal{M}_1^{\rm u}(\psi_1^{\rm u}). \tag{4.73}$$

Note that, for convenience, we avoid writing explicitly the dependence of $\mathcal{M}_1^{\rm u}$ with respect to δ.

Next Lemma characterizes $\psi_1^{\rm u}$ by means of the initial conditions of its Fourier coefficients. We recall that $\psi_1^{\rm u}(s,\theta)$ is $2\pi-$periodic in θ.

Lemma 4.3.2. *Let s_1 and s_2 be the points defined in* (4.31)*. The function $\psi_1^{\rm u}(u,\theta)$ defined in* (4.71) *is the unique function satisfying equation* (4.73) *whose Fourier coefficients $\psi_1^{{\rm u}[l]}(s)$ satisfy:*

$$\begin{array}{rcll} \psi_1^{{\rm u}[l]}(s_1) &=& \psi^{{\rm u}[l]}(s_1) - \psi_{\rm in}^{{\rm u}\,[l]}(s_1) & \textit{if } l<0, \\ \psi_1^{{\rm u}[l]}(s_2) &=& \psi^{{\rm u}[l]}(s_2) - \psi_{\rm in}^{{\rm u}\,[l]}(s_2) & \textit{if } l\geq 0. \end{array} \tag{4.74}$$

Proof. Since $\psi_1^{\rm u}$ is $2\pi-$periodic in θ, it is uniquely determined by its Fourier coefficients. Writing equation (4.73) in terms of these Fourier coefficients, one easily obtains that each Fourier coefficient $\psi_1^{{\rm u}[l]}(s)$ satisfies a given ODE. Moreover, solutions of ODEs are uniquely determined by an initial condition at a given time. We choose this initial time to be $s=s_1$ for $l<0$ and $s=s_2$ for $l\geq 0$. Since by definition $\psi_1^{{\rm u}[l]}(s) = \psi^{{\rm u}[l]}(s) - \psi_{\rm in}^{{\rm u}\,[l]}(s)$, we obtain precisely (4.74). □

The values $\psi_1^{{\rm u}[l]}(s_1)$ and $\psi_1^{{\rm u}[l]}(s_2)$ will be bounded later on using Theorem 4.1.3 and Theorem 4.1.4. For convenience, from now on we will denote them by $C_l^{\rm u}$:

$$C_l^{\rm u} := \left\{ \begin{array}{ll} \psi^{{\rm u}[l]}(s_1) - \psi_{\rm in}^{{\rm u}\,[l]}(s_1) & \text{if } l<0, \\ \psi^{{\rm u}[l]}(s_2) - \psi_{\rm in}^{{\rm u}\,[l]}(s_2) & \text{if } l\geq 0. \end{array}\right. \tag{4.75}$$

Recall that our ultimate goal is to find a sharp bound of $\psi_1^{\rm u}(s,\theta)$, for $(s,\theta)\in\mathcal{D}^{\rm mch,u}_{\kappa,\beta_1,\beta_2}$. To that aim we write $\psi_1^{\rm u}$ as a function that satisfies (4.73) and (4.74) by means of a solution of the homogeneous equation $\mathcal{L}(\psi)=0$ with initial conditions (4.74) and a suitable solution of a fixed point equation. More precisely, we shall follow three steps:

1. First, we construct a function $\Phi^{\rm u}(s,\theta)$ that satisfies:

   (a) $\mathcal{L}\circ\Phi^{\rm u} = 0$,

   (b) $\Phi^{{\rm u}[l]}(s_i) = C_l^{\rm u}$, where we take $i=1$ if $l<0$ and $i=2$ otherwise.

   This can be trivially done, defining $\Phi^{\rm u}$ as the function:

$$\Phi^{\rm u}(s,\theta) = \sum_{k\in\mathbb{Z}} \Phi^{{\rm u}[l]}(s) e^{il\theta}, \tag{4.76}$$

where:

$$\begin{aligned}\Phi^{\mathrm{u}[l]}(s) &= \frac{C_l^{\mathrm{u}}}{s_1^{2/\mathrm{d}}} s^{2/\mathrm{d}} e^{\mathrm{d}^{-1}\alpha(s-s_1)il} && \text{if } l < 0,\\ \Phi^{\mathrm{u}[l]}(s) &= \frac{C_l^{\mathrm{u}}}{s_2^{2/\mathrm{d}}} s^{2/\mathrm{d}} e^{\mathrm{d}^{-1}\alpha(s-s_2)il} && \text{if } l \geq 0.\end{aligned} \tag{4.77}$$

2. The second step consists in finding a right inverse $\mathcal{G}_0^{\mathrm{u}}$ of the operator $\mathcal{L}$. We can define it via its Fourier coefficients $\mathcal{G}_0^{\mathrm{u}[l]}$. That is, given a function $\phi(s,\theta)$ we consider:

$$\mathcal{G}_0^{\mathrm{u}}(\phi)(s,\theta) = \sum_{k\in\mathbb{Z}} \mathcal{G}_0^{\mathrm{u}[l]}(\phi)(s) e^{il\theta}, \tag{4.78}$$

and we choose $\mathcal{G}_0^{\mathrm{u}[l]}$ so that for all functions $\phi(s,\theta)$ the following holds:

(c) $\mathcal{G}_0^{\mathrm{u}[l]}(\phi)(s_1) = 0$, if $l < 0$,

(d) $\mathcal{G}_0^{\mathrm{u}[l]}(\phi)(s_2) = 0$, if $l \geq 0$.

One can easily see that if we define:

$$\mathcal{G}_0^{\mathrm{u}[l]}(\phi)(s) = \mathrm{d}^{-1} s^{\frac{2}{\mathrm{d}}} \int_{s_1}^{s} \frac{e^{-\frac{il\alpha}{\mathrm{d}}(w-s)}}{w^{\frac{2}{\mathrm{d}}}} \phi^{[l]}(w)dw \qquad \text{if } l < 0, \tag{4.79}$$

$$\mathcal{G}_0^{\mathrm{u}[l]}(\phi)(s) = \mathrm{d}^{-1} s^{\frac{2}{\mathrm{d}}} \int_{s_2}^{s} \frac{e^{-\frac{il\alpha}{\mathrm{d}}(w-s)}}{w^{\frac{2}{\mathrm{d}}}} \phi^{[l]}(w)dw \qquad \text{if } l \geq 0, \tag{4.80}$$

then $\mathcal{G}_0^{\mathrm{u}}$ defined as in (4.78) satisfies conditions (c) and (d).

3. Now we point out that items (a)–(d) above imply that the function ϕ defined implicitly by:

$$\phi = \Phi^{\mathrm{u}} + \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(\phi)),$$

satisfies (4.73) and (4.74). Since by Lemma 4.3.2 ψ_1^{u} is the only function satisfying (4.73) and (4.74), we can write:

$$\psi_1^{\mathrm{u}} = \Phi^{\mathrm{u}} + \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(\psi_1^{\mathrm{u}})). \tag{4.81}$$

This yields to the third and final step. We define the operator:

$$\tilde{\mathcal{M}}_1^{\mathrm{u}}(\phi) := \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(\phi)) - \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0)). \tag{4.82}$$

Then we can rewrite (4.81) as:

$$(\mathrm{Id} - \tilde{\mathcal{M}}_1^{\mathrm{u}})(\psi_1^{\mathrm{u}}) = \Phi^{\mathrm{u}} + \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0)). \tag{4.83}$$

Thus, we just need to see that the operator $\mathrm{Id} - \tilde{\mathcal{M}}_1^{\mathrm{u}}$ is invertible (which basically consists in proving that $\tilde{\mathcal{M}}_1^{\mathrm{u}}$ has "small" norm). We point out that, unlike ψ_1^{u}, we have an explicit formula for functions Φ^{u} and $\mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0))$, so that these functions can be bounded easily. Then (4.83) will allow us to bound ψ_1^{u} using bounds of the functions Φ^{u} and $\mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0))$.

We shall proceed as follows. First, we summarize the main properties of the operator $\mathcal{G}_0^{\mathrm{u}}$. After that, we find bounds of Φ^{u} and $\mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0))$. This is done in Subsection 4.3.2. Finally, in Subsection 4.3.3 we study the operator $\tilde{\mathcal{M}}_1^{\mathrm{u}}$ to see that $\mathrm{Id} - \tilde{\mathcal{M}}_1^{\mathrm{u}}$ is invertible, which yields the proof of Proposition 4.3.1.

4.3.2 The functions Φ^{u} and $\mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0))$

First we summarize some standard properties of the operator $\mathcal{G}_0^{\mathrm{u}}$ defined in (4.78).

Lemma 4.3.3. *Let $n \geq 1$ and $\phi \in \mathcal{X}_{n,\omega}^{\mathrm{u}}$. There exists a constant M such that:*

1. $\|\mathcal{G}_0^{\mathrm{u}}(\phi)\|_{n-1,\omega}^{\mathrm{u}} \leq M\|\phi\|_{n,\omega}^{\mathrm{u}}$.
2. *If $\phi^{[0]}(s) = 0$, then $\|\mathcal{G}_0^{\mathrm{u}}(\phi)\|_{n,\omega}^{\mathrm{u}} \leq M\|\phi\|_{n,\omega}^{\mathrm{u}}$.*
3. *In addition, one has that $\llbracket\mathcal{G}_0^{\mathrm{u}}(\phi)\rrbracket_{n-1,\omega}^{\mathrm{u}} \leq M\|\phi\|_{n,\omega}^{\mathrm{u}}$.*

Lemma 4.3.4. *The function Φ^{u} defined in (4.76)–(4.77) satisfies $\Phi^{\mathrm{u}} \in \tilde{\mathcal{X}}_{2,\omega}^{\mathrm{u}}$. Moreover, there exists a constant M such that:*

$$\llbracket\Phi^{\mathrm{u}}\rrbracket_{2,\omega}^{\mathrm{u}} \leq M\delta^{1-\gamma}.$$

Proof. Let us recall the definition (4.77) of the Fourier coefficients $\Phi^{\mathrm{u}}(s,\theta)$:

$$\Phi^{\mathrm{u}[l]}(s) = \frac{C_l^{\mathrm{u}}}{s_j^{2/\mathrm{d}}} s^{2/\mathrm{d}} e^{\mathrm{d}^{-1}\alpha(s-s_j)il}, \tag{4.84}$$

where $j = 1$ if $l < 0$ and $j = 2$ if $l \geq 0$. From the definition (4.75) of C_l^{u} and using Theorems 4.1.3 and 4.1.4, it is clear that:

$$|C_l^{\mathrm{u}}| \leq \left(\|\psi^{\mathrm{u}[l]}\|_3^{\mathrm{u}} + \|\psi_{\mathrm{in}}^{\mathrm{u}\,[l]}\|_3^{\mathrm{u}}\right) |s_j|^{-3}. \tag{4.85}$$

Moreover, since $\mathrm{Im}\,(s - s_j)l > 0$, we have $|e^{\mathrm{d}^{-1}\alpha(s-s_j)il}| = e^{-\mathrm{d}^{-1}\alpha \mathrm{Im}\,(s-s_j)l} < 1$. Then, from (4.84) and (4.85), it is clear that:

$$\left|\Phi^{\mathrm{u}[l]}(s)s^2\right| \leq |s_j|^{-3-2/\mathrm{d}}|s|^{2+2/\mathrm{d}}\left(\|\psi^{\mathrm{u}[l]}\|_3^{\mathrm{u}} + \|\psi_{\mathrm{in}}^{\mathrm{u}\,[l]}\|_3^{\mathrm{u}}\right).$$

Using (4.32) and (4.33), for all $s \in \mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$ one has:

$$\frac{|s|}{|s_j|} \leq K,$$

and then we obtain:

$$\left|\Phi^{\mathrm{u}[l]}(s)s^2\right| \leq K|s_j|^{-1}\left(\|\psi^{\mathrm{u}[l]}\|^{\mathrm{u}}_3 + \|\psi^{\mathrm{u}}_{\mathrm{in}}{}^{[l]}\|^{\mathrm{u}}_3\right).$$

Since by (4.32) $|s_j|^{-1} \leq K\delta^{1-\gamma}$ and recalling the definition (4.66) of the norm $\|.\|^{\mathrm{u}}_{2,\omega}$ this last inequality yields:

$$\|\Phi^{\mathrm{u}}\|^{\mathrm{u}}_{2,\omega} \leq K\delta^{1-\gamma}\left(\|\psi^{\mathrm{u}}\|^{\mathrm{u}}_{3,\omega} + \|\psi^{\mathrm{u}}_{\mathrm{in}}\|^{\mathrm{u}}_{3,\omega}\right) \leq K\delta^{1-\gamma}\left(\llbracket\psi^{\mathrm{u}}\rrbracket^{\mathrm{u}}_{3,\omega} + \llbracket\psi^{\mathrm{u}}_{\mathrm{in}}\rrbracket^{\mathrm{u}}_{3,\omega}\right) \leq K\delta^{1-\gamma}, \quad (4.86)$$

where in the last step we have used that $\llbracket\psi^{\mathrm{u}}\rrbracket^{\mathrm{u}}_{3,\omega} + \llbracket\psi^{\mathrm{u}}_{\mathrm{in}}\rrbracket^{\mathrm{u}}_{3,\omega} \leq K$ by Theorems 4.1.1 and 4.1.4.

Now we proceed to bound $\|\partial_\theta\Phi^{\mathrm{u}}\|_{3,\omega}$. We note that the Fourier coefficients of $\partial_\theta\Phi^{\mathrm{u}}$ are given by:

$$\left(\partial_\theta\Phi^{\mathrm{u}}\right)^{[l]}(s) = \frac{ilC^{\mathrm{u}}_l}{s_j^{2/\mathrm{d}}}s^{2/\mathrm{d}}e^{\mathrm{d}^{-1}\alpha(s-s_j)il}.$$

We note that:

$$|ilC^{\mathrm{u}}_l| \leq \left(\|il\psi^{\mathrm{u}[l]}\|^{\mathrm{u}}_4 + \|il\psi^{\mathrm{u}}_{\mathrm{in}}{}^{[l]}\|^{\mathrm{u}}_4\right)|s_j|^{-4} = \left(\|\left(\partial_\theta\psi^{\mathrm{u}}\right)^{[l]}\|^{\mathrm{u}}_4 + \|\left(\partial_\theta\psi^{\mathrm{u}}_{\mathrm{in}}\right)^{[l]}\|^{\mathrm{u}}_4\right)|s_j|^{-4}, \quad (4.87)$$

so that reasoning analogously as in the previous case we reach the conclusion that:

$$\begin{aligned}\|\partial_\theta\Phi^{\mathrm{u}}\|^{\mathrm{u}}_{3,\omega} &\leq K\delta^{1-\gamma}\left(\|\partial_\theta\psi^{\mathrm{u}}\|^{\mathrm{u}}_{4,\omega} + \|\partial_\theta\psi^{\mathrm{u}}_{\mathrm{in}}\|^{\mathrm{u}}_{4,\omega}\right) \leq K\delta^{1-\gamma}\left(\llbracket\psi^{\mathrm{u}}\rrbracket^{\mathrm{u}}_{3,\omega} + \llbracket\psi^{\mathrm{u}}_{\mathrm{in}}\rrbracket^{\mathrm{u}}_{3,\omega}\right)\\ &\leq K\delta^{1-\gamma}. \end{aligned} \quad (4.88)$$

Finally we bound $\|\partial_s\Phi^{\mathrm{u}}\|_{3,\omega}$. Differentiating the Fourier coefficients of $\Phi^{\mathrm{u}[l]}$ defined in (4.84) with respect to s we obtain:

$$\frac{d}{ds}\Phi^{\mathrm{u}[l]}(s) = \frac{2C^{\mathrm{u}}_l}{\mathrm{d}s_j^{2/\mathrm{d}}}s^{2/\mathrm{d}-1}e^{\mathrm{d}^{-1}\alpha(s-s_j)il} + \frac{C^{\mathrm{u}}_l}{s_j^{2/\mathrm{d}}}s^{2/\mathrm{d}}\mathrm{d}^{-1}\alpha il e^{\mathrm{d}^{-1}\alpha(s-s_j)il}.$$

Reasoning analogously as in the previous cases, using bound (4.85) for the first term in the sum and bound (4.87) for the second term, we obtain:

$$\|\partial_s\Phi^{\mathrm{u}}\|^{\mathrm{u}}_{3,\omega} \leq K\delta^{1-\gamma}\left(\llbracket\psi^{\mathrm{u}}\rrbracket^{\mathrm{u}}_{3,\omega} + \llbracket\psi^{\mathrm{u}}_{\mathrm{in}}\rrbracket^{\mathrm{u}}_{3,\omega}\right) \leq K\delta^{1-\gamma}. \quad (4.89)$$

Bounds (4.86), (4.88) and (4.89) yield directly the claim of the lemma. □

We state the following technical lemmas and avoid to write their tedious proofs. They can be proved following the same ideas used in Lemmas 3.2.14 and 3.2.15 in Chapter 3, taking into account that for $s \in \mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2}$ one has:

$$K_1\kappa \le |s| \le K_2\delta^{\gamma-1} \qquad \text{and} \qquad \delta < |s|^{-1}. \tag{4.90}$$

Lemma 4.3.5. *Let C be any constant, and $\phi \in \mathcal{X}^{\mathrm{u}}_{3,\omega}$ with $\|\phi\|^{\mathrm{u}}_{3,\omega} \le C$. There exists a constant M such that:*

1. $\|\hat{F}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{4,\omega} \le M$,
2. $\|\hat{G}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{2,\omega} \le M$,
3. $\|\hat{H}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{3,\omega} \le M$,

where $\hat{F}$, $\hat{G}$, $\hat{H}$ are defined in (4.11).

Note that, in the definition (4.11) of $\hat{F}$, $\hat{G}$ and $\hat{H}$, the variable ϕ always appears inside the function $\rho(\phi, s, \delta)$. In the following lemma one needs that $\rho(\phi, s, \delta) \neq 0$, so that we shall assume that $\|\phi\|^{\mathrm{u}}_{2,\omega}$ is small. One can see from definition (4.10) that this is enough to ensure that $\rho(\phi, s, \delta) \neq 0$.

Lemma 4.3.6. *Let C be any fixed constant, and $\phi \in \mathcal{X}^{\mathrm{u}}_{2,\omega}$ with $\|\phi\|^{\mathrm{u}}_{2,\omega} \le C/\bar{\kappa}$. If $\bar{\kappa}$ is sufficiently large, there exists a constant M such that:*

1. $\|D_\phi\hat{F}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{2,\omega} \le M$,
2. $\|D_\phi\hat{G}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{0,\omega} \le M$,
3. $\|D_\phi\hat{H}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{1,\omega} \le M$,

where $\hat{F}$, $\hat{G}$, $\hat{H}$ are defined in (4.11). *In particular, if $\phi \in \mathcal{X}^{\mathrm{u}}_{3,\omega}$ and $\|\phi\|^{\mathrm{u}}_{3,\omega} \le C$, items 1–3 above hold.*

Lemma 4.3.7. *Let C be any fixed constant, and $\phi \in \mathcal{X}^{\mathrm{u}}_{3,\omega}$ with $\|\phi\|^{\mathrm{u}}_{3,\omega} \le C$. There exists a constant M such that:*

1. $\|D_\delta\hat{F}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{3,\omega} \le M$,
2. $\|D_\delta\hat{G}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{1,\omega} \le M$,
3. $\|D_\delta\hat{H}(\phi, s, \theta, \delta, \delta\sigma)\|^{\mathrm{u}}_{2,\omega} \le M$,

where $\hat{F}$, $\hat{G}$ and $\hat{H}$ are defined in (4.11).

The following result is a straightforward consequence of the previous lemma and the mean value theorem.

Lemma 4.3.8. *Let C be any fixed constant, and $\phi \in \mathcal{X}_{3,\omega}^{\mathrm{u}}$ such that $\|\phi\|_{3,\omega}^{\mathrm{u}} \leq C$. There exists a constant M such that:*

1. $\|\hat{F}(\phi, s, \theta, \delta, \delta\sigma) - \hat{F}(\phi, s, \theta, 0, 0)\|_{3,\omega}^{\mathrm{u}} \leq M\delta,$
2. $\|\hat{G}(\phi, s, \theta, \delta, \delta\sigma) - \hat{G}(\phi, s, \theta, 0, 0)\|_{1,\omega}^{\mathrm{u}} \leq M\delta,$
3. $\|\hat{H}(\phi, s, \theta, \delta, \delta\sigma) - \hat{H}(\phi, s, \theta, 0, 0)\|_{2,\omega}^{\mathrm{u}} \leq M\delta,$

where $\hat{F}$, $\hat{G}$ and $\hat{H}$ are defined in (4.11)*.*

Lemma 4.3.9. *Let $\sigma = \mathcal{O}(\delta)$. Then the function $\mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0)) \in \tilde{\mathcal{X}}_{2,\omega}^{\mathrm{u}}$, where $\mathcal{M}_1^{\mathrm{u}}$ is defined in* (4.72) *and $\mathcal{G}_0^{\mathrm{u}}$ is defined in* (4.78)*. Moreover, there exists a constant M such that:*

$$\llbracket \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0)) \rrbracket_{2,\omega}^{\mathrm{u}} \leq M\delta.$$

Proof. By item 3 of Lemma 4.3.3 it is enough to prove that:

$$\|\mathcal{M}_1^{\mathrm{u}}(0)\|_{3,\omega}^{\mathrm{u}} \leq K\delta. \tag{4.91}$$

Recall that:

$$\mathcal{M}_1^{\mathrm{u}}(0) = \mathcal{M}(\psi_{\mathrm{in}}^{\mathrm{u}}, \delta) - \mathcal{M}(\psi_{\mathrm{in}}^{\mathrm{u}}, 0).$$

Thus, from definition (4.16) of $\mathcal{M}$ we obtain:

$$\begin{aligned}
\mathcal{M}_1^{\mathrm{u}}(0) \;=\;& \mathrm{d}\delta^2 s^2 \partial_s \psi_{\mathrm{in}}^{\mathrm{u}} + 2\sigma \frac{\mathrm{d}+1}{2b}\left(\left(\delta^3 - \frac{\delta}{s^2}\right) + \delta\psi_{\mathrm{in}}^{\mathrm{u}}\right) \\
&+ \hat{F}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma) - \hat{F}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, 0, 0) \\
&+ \frac{\mathrm{d}+1}{b} s^{-1}\left(\hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma) - \hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, 0, 0)\right) \\
&- \left(\hat{G}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma) - \hat{G}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, 0, 0)\right)\partial_\theta \psi_{\mathrm{in}}^{\mathrm{u}} \\
&+ s^2\left(\hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma) - \hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, 0, 0)\right)\partial_s \psi_{\mathrm{in}}^{\mathrm{u}}.
\end{aligned}$$

Now, using that for $s \in \mathcal{D}_{\kappa,\beta_1,\beta_2}^{\mathrm{mch,u}}$ (see (4.90)):

$$K_1\kappa \leq |s| \leq K_2\delta^{\gamma-1}, \tag{4.92}$$

and the fact that $\llbracket \psi_{\mathrm{in}}^{\mathrm{u}} \rrbracket_{3,\omega}^{\mathrm{u}} \leq K$, it is easy to check that:

$$\left\|\delta^2 s^2 \partial_s \psi_{\mathrm{in}}^{\mathrm{u}}\right\|_{3,\omega}^{\mathrm{u}} \leq K\delta^{1+\gamma},$$

and since $\sigma = \mathcal{O}(\delta)$:

$$\left\| 2\sigma \frac{\mathrm{d}+1}{2b}\left(\left(\delta^3 - \frac{\delta}{s^2}\right) + \delta\psi_{\mathrm{in}}^{\mathrm{u}}\right)\right\|_{3,\omega}^{\mathrm{u}} \leq K\delta^{1+\gamma}.$$

These facts and Lemma 4.3.8, jointly with the properties of the norm $\|.\|_{n,\omega}^{\mathrm{u}}$, yield directly bound (4.91). □

4.3.3 The operator $\tilde{\mathcal{M}}_1^{\mathrm{u}}$

The following Lemma is a direct consequence of Lemma 4.3.6.

Lemma 4.3.10. *Let $\phi \in \mathcal{X}_{2,\omega}^{\mathrm{u}}$ such that $\|\phi\|_{2,\omega}^{\mathrm{u}} \leq C/\kappa$. There exists a constant M such that:*

1. $\|\hat{F}(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi, s, \theta, \delta, \delta\sigma) - \hat{F}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma)\|_{4,\omega}^{\mathrm{u}} \leq M\|\phi\|_{2,\omega}^{\mathrm{u}}$,
2. $\|\hat{G}(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi, s, \theta, \delta, \delta\sigma) - \hat{G}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, , \delta, \delta\sigma)\|_{2,\omega}^{\mathrm{u}} \leq M\|\phi\|_{2,\omega}^{\mathrm{u}}$,
3. $\|\hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi, s, \theta, \delta, \delta\sigma) - \hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, , \delta, \delta\sigma)\|_{3,\omega}^{\mathrm{u}} \leq M\|\phi\|_{2,\omega}^{\mathrm{u}}$.

Lemma 4.3.11. *Fix $C > 0$ and let $\phi \in \tilde{\mathcal{X}}_{2,\omega}^{\mathrm{u}}$ such that $\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}} \leq C/\kappa$. Let $\sigma = \mathcal{O}(\delta)$. Then $\tilde{\mathcal{M}}_1^{\mathrm{u}}(\phi) \in \tilde{\mathcal{X}}_{2,\omega}^{\mathrm{u}}$ and there exists a constant M such that:*

$$\llbracket\tilde{\mathcal{M}}_1^{\mathrm{u}}(\phi)\rrbracket_{2,\omega}^{\mathrm{u}} \leq \frac{M}{\kappa}\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}}.$$

In particular, for sufficiently large κ:

$$\|\tilde{\mathcal{M}}_1^{\mathrm{u}}\| := \sup_{\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}} \leq \frac{C}{\kappa}} \left\{ \frac{\llbracket\tilde{\mathcal{M}}_1(\phi)\rrbracket_{2,\omega}^{\mathrm{u}}}{\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}}} \right\} < 1.$$

Proof. Recall that $\tilde{\mathcal{M}}_1^{\mathrm{u}}(\phi) = \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(\phi)) - \mathcal{G}_0(\mathcal{M}_1^{\mathrm{u}}(0)) = \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(\phi) - \mathcal{M}_1^{\mathrm{u}}(0))$. Thus, by item 3 of Lemma 4.3.3 it is sufficient to prove that:

$$\|\mathcal{M}_1^{\mathrm{u}}(\phi) - \mathcal{M}_1^{\mathrm{u}}(0)\|_{3,\omega}^{\mathrm{u}} \leq \frac{K}{\kappa}\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}}. \tag{4.93}$$

By definition (4.72) of $\mathcal{M}_1^{\mathrm{u}}$, one has:

$$\mathcal{M}_1^{\mathrm{u}}(\phi) - \mathcal{M}_1^{\mathrm{u}}(0) = \mathcal{M}(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi, \delta) - \mathcal{M}(\psi_{\mathrm{in}}^{\mathrm{u}}, \delta).$$

Using definition (4.16) of $\mathcal{M}$, one obtains:

$$\begin{aligned}
\mathcal{M}_1^{\mathrm{u}}(\phi) - \mathcal{M}_1^{\mathrm{u}}(0) \quad = \quad & cs^{-1}\partial_\theta\phi + \mathrm{d}\delta^2 s^2\partial_s\phi + 2\sigma\delta\phi \\
& +\hat{F}(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi, s, \theta, \delta, \delta\sigma) - \hat{F}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma) \\
& +\frac{\mathrm{d}+1}{b}s^{-1}\left(\hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi, s, \theta, \delta, \delta\sigma) - \hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma)\right) \\
& -\left(\hat{G}(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi, s, \theta, \delta, \delta\sigma) - \hat{G}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma)\right)\partial_\theta(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi) \\
& -\hat{G}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma)\partial_\theta\phi \\
& +s^2\left(2b\phi + \hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi, s, \theta, \delta, \delta\sigma) - \hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma)\right)\partial_s(\psi_{\mathrm{in}}^{\mathrm{u}} + \phi) \\
& +s^2\left(2b\psi_{\mathrm{in}}^{\mathrm{u}} + \hat{H}(\psi_{\mathrm{in}}^{\mathrm{u}}, s, \theta, \delta, \delta\sigma)\right)\partial_s\phi.
\end{aligned}$$

Let us denote:

$$\mathcal{R}(\phi) := \mathcal{M}_1^{\mathrm{u}}(\phi) - \mathcal{M}_1^{\mathrm{u}}(0) - \mathrm{d}\delta^2 s^2\partial_s\phi - 2bs^2\phi\partial_s\phi.$$

First we claim that:

$$\|\mathcal{R}(\phi)\|_{3,\omega}^{\mathrm{u}} \leq \frac{K}{\kappa}\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}}. \tag{4.94}$$

Indeed, using Lemmas 4.3.5 and 4.3.10, the properties of the norm $\|.\|_{n,\omega}^{\mathrm{u}}$, the fact that $\llbracket\psi_{\mathrm{in}}^{\mathrm{u}}\rrbracket_{3,\omega}^{\mathrm{u}} \leq K$ and that $\delta \leq K|s|^{-1}$ for $s \in \mathcal{D}_{\kappa,\beta_1,\beta_2}^{\mathrm{mch,u}}$, one obtains easily that:

$$\|\mathcal{R}(\phi)\|_{4,\omega}^{\mathrm{u}} \leq K\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}}.$$

Then we just need to note that:

$$\|\mathcal{R}(\phi)\|_{3,\omega}^{\mathrm{u}} \leq \frac{K}{\kappa}\|\mathcal{R}(\phi)\|_{4,\omega}^{\mathrm{u}},$$

and so bound (4.94) is obtained.

Now we just need to note that since $|s| \leq K\delta^{\gamma-1}$:

$$\|\mathrm{d}\delta^2 s^2\partial_s\phi\|_{3,\omega}^{\mathrm{u}} \leq K\delta^{2\gamma}\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}}. \tag{4.95}$$

Finally, since by assumption $\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}} \leq C/\kappa$, then:

$$\|2bs^2\phi\partial_s\phi\|_{3,\omega}^{\mathrm{u}} \leq K\left(\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}}\right)^2 \leq \frac{K}{\kappa}\llbracket\phi\rrbracket_{2,\omega}^{\mathrm{u}}. \tag{4.96}$$

Bounds (4.94), (4.95) and (4.96) yield bound (4.93), and so the proof is finished. □

End of the proof of Proposition 4.3.1. By Lemma 4.3.11 we have that $\|\tilde{\mathcal{M}}_1^{\mathrm{u}}\| < 1$ if κ is large enough, so that the operator $\mathrm{Id} - \tilde{\mathcal{M}}_1^{\mathrm{u}}$ is invertible in the ball $B(C/\kappa) \subset \tilde{\mathcal{X}}_{2,\omega}^{\mathrm{u}}$. Since by Lemmas 4.3.4 and 4.3.9 we have that:

$$\llbracket\Phi^{\mathrm{u}} + \mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0))\rrbracket_{2,\omega}^{\mathrm{u}} \leq \|\Phi^{\mathrm{u}}\|_{2,\omega}^{\mathrm{u}} + \|\mathcal{G}_0^{\mathrm{u}}(\mathcal{M}_1^{\mathrm{u}}(0))\|_{2,\omega}^{\mathrm{u}} \leq K\left(\delta^{1-\gamma} + \delta\right) \leq C/\kappa,$$

from equation (4.83) we can write:

$$\psi_1^{\rm u} = (\mathrm{Id} - \tilde{\mathcal{M}}_1^{\rm u})^{-1}(\Phi^{\rm u} + \mathcal{G}_0^{\rm u}(\mathcal{M}_1^{\rm u}(0))).$$

Then:

$$[\![\psi_1^{\rm u}]\!]_{2,\omega}^{\rm u} \leq [\![(\mathrm{Id} - \tilde{\mathcal{M}}_1^{\rm u})^{-1}]\!]_{2,\omega}^{\rm u}[\![\Phi^{\rm u} + \mathcal{G}_0^{\rm u}(\mathcal{M}_1^{\rm u}(0))]\!]_{2,\omega}^{\rm u} \leq K\delta^{1-\gamma},$$

so that Proposition 4.3.1 is proved. □

4.4 Proof of Theorem 4.1.6. The difference $\Delta\psi_{\rm in}$

Since $\psi_{\rm in}^{\rm u}$ and $\psi_{\rm in}^{\rm s}$ are solutions of the same equation (4.14), subtracting $\psi_{\rm in}^{\rm u}$ and $\psi_{\rm in}^{\rm s}$ and using the mean value theorem one obtains the following equation for $\Delta\psi_{\rm in} = \psi_{\rm in}^{\rm u} - \psi_{\rm in}^{\rm s}$:

$$\begin{aligned}&-\alpha\partial_\theta\Delta\psi_{\rm in} + \mathrm{d}\partial_s\Delta\psi_{\rm in} - 2s^{-1}\Delta\psi_{\rm in}\\ &\qquad = a_1(s,\theta)\Delta\psi_{\rm in} + a_2(s,\theta)\partial_s\Delta\psi_{\rm in} + (cs^{-1} + a_3(s,\theta))\partial_\theta\Delta\psi_{\rm in},\end{aligned} \tag{4.97}$$

where, denoting $\psi_\lambda = (\psi_{\rm in}^{\rm u} + \psi_{\rm in}^{\rm s})/2 + \lambda(\psi_{\rm in}^{\rm u} - \psi_{\rm in}^{\rm s})/2$, we define:

$$\begin{aligned} a_1(s,\theta) &= \frac{1}{2}\int_{-1}^{1}\partial_\psi\hat{F}(\psi_\lambda,s,\theta,0,0)d\lambda + \frac{\mathrm{d}+1}{2b}s^{-1}\int_{-1}^{1}\partial_\psi\hat{H}(\psi_\lambda,s,\theta,0,0)d\lambda\\ &\quad -\frac{1}{2}\int_{-1}^{1}\partial_\psi\hat{G}(\psi_\lambda,s,\theta,0,0)\partial_\theta\psi_\lambda d\lambda + bs^2(\partial_s\psi_{\rm in}^{\rm u} + \partial_s\psi_{\rm in}^{\rm s})\\ &\quad +\frac{1}{2}s^2\int_{-1}^{1}\partial_\psi\hat{H}(\psi_\lambda,s,\theta,0,0)\partial_s\psi_\lambda d\lambda, \end{aligned} \tag{4.98}$$

$$a_2(s,\theta) = bs^2(\psi_{\rm in}^{\rm u} + \psi_{\rm in}^{\rm s}) + \frac{1}{2}s^2\int_{-1}^{1}\hat{H}(\psi_\lambda,s,\theta,0,0)d\lambda \tag{4.99}$$

$$a_3(s,\theta) = -\frac{1}{2}\int_{-1}^{1}\hat{G}(\psi_\lambda,s,\theta,0,0)d\lambda. \tag{4.100}$$

We recall that $\hat{F}$, $\hat{G}$ and $\hat{H}$ are defined in (4.11) and that the difference $\Delta\psi_{\rm in}$ is defined for $s \in E_{\beta_0,\bar\kappa} = \mathcal{D}^{\rm in,u}_{\beta_0,\bar\kappa} \cap \mathcal{D}^{\rm in,s}_{\beta_0,\bar\kappa}$ and $\theta \in \mathbb{T}_\omega$.

Now we shall introduce the Banach spaces in which we will solve equation (4.97). After that, we shall state rigorously the ideas introduced in the introductory Subsection 4.1.4.

4.4.1 Banach Spaces

The spaces and norms are basically the same as in Subsection 4.2.1, but restricted to $E_{\beta_0,\bar\kappa} = \mathcal{D}^{\rm in,u}_{\beta_0,\bar\kappa} \cap \mathcal{D}^{\rm in,s}_{\beta_0,\bar\kappa}$.

Let $\phi : E_{\beta_0,\bar{\kappa}} \to \mathbb{C}$. Then, for any $n \in \mathbb{R}$, we define the norm $\|.\|_n$ as:

$$\|\phi\|_n := \sup_{s\in E_{\beta_0,\bar{\kappa}}} |s^n \phi(s)|. \tag{4.101}$$

For $\phi : E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega \to \mathbb{C}$, writing $\phi(s,\theta) = \sum_{l\in\mathbb{Z}} \phi^{[l]}(s)e^{il\theta}$, we define the norm $\|.\|_{n,\omega}$ as:

$$\|\phi\|_{n,\omega} := \sum_{l\in\mathbb{Z}} \|\phi^{[l]}\|_n e^{|l|\omega}. \tag{4.102}$$

Then we define the Banach space $\mathcal{X}_{\omega,n}$ as the space of analytic functions having finite norm $\|.\|_{n,\omega}$:

$$\mathcal{X}_{\omega,n} := \{\phi : E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega \to \mathbb{C} \,:\, \phi \text{ is analytic}, \|\phi\|_{n,\omega} < \infty\}. \tag{4.103}$$

Again, we also consider the following norm:

$$\llbracket\phi\rrbracket_{n,\omega} := \|\phi\|_{n,\omega} + \|\partial_s\phi\|_{n+1,\omega} + \|\partial_\theta\phi\|_{n+1,\omega}, \tag{4.104}$$

and the corresponding Banach space:

$$\tilde{\mathcal{X}}_{n,\omega} := \{\phi : E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega \to \mathbb{C} \,:\, \phi \text{ is analytic}, \llbracket\phi\rrbracket_{n,\omega} < \infty\}. \tag{4.105}$$

4.4.2 Statement of results

Lemma 4.4.1 (Variation of constants)**.** *Let $P^{\text{in}}(s,\theta)$ be a particular solution of* (4.97) *such that it is 2π-periodic in θ and satisfying $P^{\text{in}}(s,\theta) \neq 0$ for all $(s,\theta) \in E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$. Then, every solution $\Delta\psi_{\text{in}}(s,\theta)$ of equation* (4.97) *defined for $(s,\theta) \in E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$ can be written as:*

$$\Delta\psi_{\text{in}}(s,\theta) = P^{\text{in}}(s,\theta)k^{\text{in}}(s,\theta),$$

where $k^{\text{in}}(s,\theta)$ is a solution of:

$$-\alpha\partial_\theta k^{\text{in}} + \mathrm{d}\partial_s k^{\text{in}} = a_2(s,\theta)\partial_s k^{\text{in}} + (cs^{-1} + a_3(s,\theta))\partial_\theta k^{\text{in}}. \tag{4.106}$$

which is 2π-periodic in θ.

Proposition 4.4.2. *Let $\xi_{\text{in}}(s,\theta)$ be a particular solution of* (4.106) *such that $(\xi_{\text{in}}(s,\theta),\theta)$ is injective in $E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$. Then any solution $k^{\text{in}}(s,\theta)$ of* (4.106) *defined in $E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$ can be written as:*

$$k^{\text{in}}(s,\theta) = \tilde{k}^{\text{in}}(\xi_{\text{in}}(s,\theta)),$$

for some function $\tilde{k}$.

Before proceeding, we shall make some definitions. Let $a_2^{[0]}(s)$ denote the average of $a_2(s,\theta)$. Let us denote:

$$a_0 = \lim_{\substack{s\in E_{\beta_0,\bar\kappa}\\ |s|\to\infty}} s a_2^{[0]}(s). \tag{4.107}$$

Lemma 4.4.3. *The limit* (4.107) *exists, and there exists a constant M such that:*

$$|a_2^{[0]}(s) - s^{-1}a_0| \leq \frac{M}{|s|^2}.$$

Moreover, one has:

$$a_0 = \mathrm{d}L_0, \tag{4.108}$$

where L_0 is the constant given in Theorem 3.1.9.

Proof. From the definition (4.99) of $a_2(s,\theta)$ and Theorem 4.1.4 (which gives some properties of the functions $\psi_{\rm in}^{\rm u}(s,\theta)$ and $\psi_{\rm in}^{\rm s}(s,\theta)$) one obtains that:

$$a_2(s,\theta) = bs^2\left[\mathcal{G}^{\rm u}(\mathcal{M}(0,0))(s,\theta) + \mathcal{G}^{\rm s}(\mathcal{M}(0,0))(s,\theta)\right] + s^2\hat H(0,s,\theta,0,0) + \mathcal{O}(s^{-2}),$$

where $\mathcal{G}^{\rm u}$ is defined in (4.24), $\mathcal{G}^{\rm s}$ in (4.26) and $\mathcal{M}$ in (4.16). From this expression, and recalling that:

$$\mathcal{M}(0,0) = \hat F(0,s,\theta,0,0) + \frac{\mathrm{d}+1}{b}s^{-1}\hat H(0,s,\theta,0,0),$$

one can see that the limit:

$$\lim_{\substack{s\in E_{\beta_0,\bar\kappa}\\ |s|\to\infty}} s a_2^{[0]}(s)$$

exists, and then one just needs to take averages to obtain a_0. From definitions (4.11) of $\hat F$ and $\hat H$ it is also easy to see that:

$$|a_2^{[0]}(s) - s^{-1}a_0| \leq \frac{K}{|s|^2}$$

for some constant K. To prove (4.108), we recall that L_0 is defined in (3.292)–(3.293) as:

$$L_0 = \lim_{u\to i\frac{\pi}{2\mathrm{d}}} \lim_{\delta\to 0} \delta^{-1} l_2^{[0]}(u)\tanh^{-1}(\mathrm{d}u),$$

where $l_2(u,\theta)$ is defined in (3.239). Let us consider:

$$u(s) = \mathrm{d}^{-1}\mathrm{arctanh}\left(\frac{1}{\delta s}\right).$$

From the definitions (3.239) of $l_2(u,\theta)$ and (4.99) of $a_2(s,\theta)$, recalling that:

$$\psi^{\rm u,s}(s,\theta) = \delta^2 r_1^{\rm u,s}(u(s),\theta),$$

the fact that $\psi^{\mathrm{u,s}} = \psi_{\mathrm{in}}^{\mathrm{u,s}} + \psi_1^{\mathrm{u,s}}$, using the bounds provided in Theorems 4.1.3 and 4.1.5 for $\psi^{\mathrm{u,s}}$ and $\psi_1^{\mathrm{u,s}}$ respectively, and using formula (4.12) (which relates H and $\hat{H}$) one can see that for $s \in \mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2} \cap \mathcal{D}^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$:

$$l_2(u(s),\theta) = \mathrm{d}^{-1} a_2(s,\theta) + \mathcal{O}(\delta^{1-\gamma}).$$

In particular, taking $|s| \leq K\delta^{(\gamma-1)/2}$, $s \in \mathcal{D}^{\mathrm{mch,u}}_{\kappa,\beta_1,\beta_2} \cap \mathcal{D}^{\mathrm{mch,s}}_{\kappa,\beta_1,\beta_2}$, one has:

$$l_2(u(s),\theta) = \mathrm{d}^{-1} a_2(s,\theta) + \mathcal{O}(s^{-2}),$$

so that:

$$l_2^{[0]}(u(s)) = \mathrm{d}^{-1} a_2^{[0]}(s) + \mathcal{O}(s^{-2}).$$

Since:

$$|a_2^{[0]}(s) - s^{-1}a_0| \leq \frac{K}{|s|^2}$$

this yields:

$$l_2^{[0]}(u(s)) = \mathrm{d}^{-1} a_0 s^{-1} + \mathcal{O}(s^{-2}).$$

Taking $s = 1/(\delta \tanh(\mathrm{d}u))$ (with $|u - i\pi/(2\mathrm{d})| \leq \delta^{(1+\gamma)/2}$ so that $|s| \leq K\delta^{(\gamma-1)/2}$) we obtain:

$$l_2^{[0]}(u) = \delta \mathrm{d}^{-1} a_0 \tanh(\mathrm{d}u) + \mathcal{O}(\delta^2 \tanh^2(\mathrm{d}u)).$$

Thus:

$$L_0 = \lim_{u \to i\frac{\pi}{2\mathrm{d}}} \lim_{\delta \to 0} \delta^{-1} l_2^{[0]}(u) \tanh^{-1}(\mathrm{d}u) = \mathrm{d}^{-1} a_0,$$

and the claim is proved. □

Now we define:

$$\bar{a}_2(s,\theta) = a_2(s,\theta) - \mathrm{d}s^{-1}L_0. \tag{4.109}$$

Recalling that by Proposition 4.2.2 we have that $\psi_{\mathrm{in}}^{\mathrm{u}} \in \tilde{\mathcal{X}}^{\mathrm{u}}_{3,\omega}$ and using Lemmas 4.2.3, 4.2.4 and 4.4.3, it is straightforward to prove the following result.

Lemma 4.4.4. *Consider the functions $a_i(s,\theta)$, $i = 1,2,3$ defined respectively in* (4.98), (4.99) *and* (4.100)*, and the function $\bar{a}_2(s,\theta)$ defined in* (4.109)*. There exists a constant M such that:*

1. $\|a_1\|_{2,\omega} \leq M$,
2. $\|a_2\|_{1,\omega} \leq M$, *and* $\|\bar{a}_2^{[0]}\|_2 \leq M$,
3. $\|a_3\|_{2,\omega} \leq M$.

Proposition 4.4.5. *Let $\varphi(u,\theta)$ be a $2\pi-$periodic in θ solution of the equation:*

$$\begin{aligned} -\alpha\partial_\theta\varphi + \mathrm{d}\partial_s\varphi &= \mathrm{d}^{-1}\alpha\bar{a}_2(s,\theta) + \mathrm{d}^{-1}(c+\alpha L_0)s^{-1}a_2(s,\theta) + a_3(s,\theta) + a_2(s,\theta)\partial_s\varphi \\ &\quad +(cs^{-1} + a_3(s,\theta))\partial_\theta\varphi. \end{aligned} \tag{4.110}$$

Then:

$$\xi_{\rm in}(s,\theta) = \theta + \mathrm{d}^{-1}\alpha s + \mathrm{d}^{-1}(c+\alpha L_0)\log s + \varphi(s,\theta) \tag{4.111}$$

is a solution of equation (4.106).

Moreover, there exists a function $\varphi(s,\theta)$ satisfying (4.110) *and a constant M such that:*

$$\|\varphi\|_{1,\omega} \leq M, \tag{4.112}$$

$$\|\partial_s\varphi\|_{1,\omega} \leq M, \qquad \|\partial_\theta\varphi\|_{1,\omega} \leq M, \tag{4.113}$$

and such that $(\xi_{\rm in}(s,\theta),\theta)$, with $\xi_{\rm in}(s,\theta)$ given by (4.111)*, is injective in $E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$.*

Remark 4.4.6. Note that using this function $\xi_{\rm in}(s,\theta)$ in Proposition 4.4.5 we have:

$$k^{\rm in}(s,\theta) = \tilde{k}^{\rm in}(\theta + \mathrm{d}^{-1}\alpha s + \mathrm{d}^{-1}(c+\alpha L_0)\log s + \varphi(s,\theta)).$$

Since $k^{\rm in}(s,\theta)$ and $\varphi(s,\theta)$ are $2\pi-$periodic in θ, we have that $\tilde{k}^{\rm in}(\tau)$ is $2\pi-$periodic.

Proposition 4.4.7. *Let $P_1^{\rm in}(u,\theta)$ be a $2\pi-$periodic in θ solution of the equation:*

$$\begin{aligned} -\alpha\partial_\theta P_1^{\rm in} + \mathrm{d}\partial_s P_1^{\rm in} &= (a_1(s,\theta) + 2\mathrm{d}^{-1}s^{-1}a_2(s,\theta))(1+P_1^{\rm in}) + a_2(s,\theta)\partial_s P_1^{\rm in} \\ &\quad +(cs^{-1} + a_3(s,\theta))\partial_\theta P_1^{\rm in}. \end{aligned} \tag{4.114}$$

Then:

$$P^{\rm in}(s,\theta) = s^{2/\mathrm{d}}(1 + P_1^{\rm in}(s,\theta)) \tag{4.115}$$

is a solution of equation (4.97).

Moreover, there exists a particular solution $P_1^{\rm in}(s,\theta)$ of (4.114) *and a constant M such that:*

$$[\![P_1^{\rm in}]\!]_{1,\omega} \leq M. \tag{4.116}$$

As a consequence, $P^{\rm in}(s,\theta)$ given by (4.115) *satisfies that $P^{\rm in}(s,\theta) \neq 0$ for all $(s,\theta) \in E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$.*

Proof of Theorem 4.1.6. We use Lemma 4.4.1 and Propositions 4.4.2, 4.4.5 and 4.4.7, and we obtain straightforwardly the first claim of Theorem 4.1.6. That is, the difference $\Delta\psi_{\rm in}(s,\theta)$ can be written as:

$$\Delta\psi_{\rm in}(s,\theta) = s^{2/\mathrm{d}}(1 + P_1^{\rm in}(s,\theta))\tilde{k}^{\rm in}(\theta + \mathrm{d}^{-1}\alpha s + \mathrm{d}^{-1}(c+\alpha L_0)\log s + \varphi(s,\theta)) \tag{4.117}$$

for some $2\pi-$periodic function $\tilde{k}^{\rm in}(\tau)$.

Since $\tilde{k}^{\mathrm{in}}$ is 2π-periodic, we can write it in its Fourier series:

$$\tilde{k}^{\mathrm{in}}(\tau) = \sum_{l\in\mathbb{Z}} \Upsilon_{\mathrm{in}}^{[l]} e^{il\tau}.$$

Now we note that by the definition of $\Delta\psi_{\mathrm{in}}$ and Theorem 4.1.4 we have for all $s \in E_{\beta_0,\bar{\kappa}}$:

$$|\Delta\psi_{\mathrm{in}}(s,\theta)| \le |\psi_{\mathrm{in}}^{\mathrm{u}}(s,\theta)| + |\psi_{\mathrm{in}}^{\mathrm{u}}(s,\theta)| \le \frac{K}{|s|^3}.$$

In particular:

$$\lim_{\mathrm{Im}\, s\to -\infty} \Delta\psi_{\mathrm{in}}(s,\theta) = 0.$$

Since $\Delta\psi_{\mathrm{in}}(s,\theta)$ is defined for $\mathrm{Im}\, s \to -\infty$, expression (4.117) of $\Delta\psi_{\mathrm{in}}$ implies that $\tilde{k}^{\mathrm{in}}$ is defined for $\mathrm{Im}\,\tau \to -\infty$, and moreover:

$$\lim_{\mathrm{Im}\,\tau\to -\infty} \tilde{k}^{\mathrm{in}}(\tau) = 0.$$

In particular, $|\tilde{k}^{\mathrm{in}}(\tau)| \le M$ and so $|\Upsilon_{\mathrm{in}}^{[l]}| \le M$ for all $l \in \mathbb{Z}$. Moreover, there exists $t^* > 0$ such that $\tilde{k}^{\mathrm{in}}(\tau - it)$ is defined for all $\tau \in [0, 2\pi]$ and $t \ge t^*$. Then one has:

$$\Upsilon_{\mathrm{in}}^{[l]} = \frac{1}{2\pi}\int_0^{2\pi} \tilde{k}^{\mathrm{in}}(\tau - it)e^{-il(\tau - it)}d\tau = \frac{e^{-lt}}{2\pi}\int_0^{2\pi} \tilde{k}^{\mathrm{in}}(\tau - it)e^{-il\tau}d\tau.$$

Since:

$$\left|\int_0^{2\pi} \tilde{k}^{\mathrm{in}}(\tau - it)e^{-il\tau}d\tau\right| \le K,$$

taking the limit $t \to +\infty$ one obtains that $\Upsilon_{\mathrm{in}}^{[l]} = 0$ for $l > 0$. As a consequence, one also obtains that:

$$0 = \lim_{\mathrm{Im}\,\tau\to -\infty} \tilde{k}^{\mathrm{in}}(\tau) = \lim_{\mathrm{Im}\,\tau\to -\infty} \sum_{l\ge 0} \Upsilon_{\mathrm{in}}^{[l]} e^{il\tau} = \Upsilon_{\mathrm{in}}^{[0]}.$$

Finally, the bound for $\varphi(s,\theta)$ is a consequence of Proposition 4.4.5, formula (4.112), and the bound for $P_1^{\mathrm{in}}(s,\theta)$ is a consequence of Proposition 4.4.7, formula (4.116). □

Next we shall focus on Propositions 4.4.5 and 4.4.7. We shall prove both of them in a very similar way. First, note that in the left-hand side of equations (4.110) and (4.114) we have the same linear operator, namely:

$$\hat{\mathcal{L}}(\phi) = -\alpha\partial_\theta\phi + \mathrm{d}\partial_s\phi. \tag{4.118}$$

Moreover, $\varphi(s,\theta)$ and $P_1^{\mathrm{in}}(s,\theta)$ are defined in the same domain $E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$. Now, to solve equation (4.110) we consider the operator:

$$\begin{aligned}\mathcal{A}(\phi) \;=\;& \mathrm{d}^{-1}\alpha\bar{a}_2(s,\theta) + \mathrm{d}^{-1}(c+\alpha L_0)s^{-1}a_2(s,\theta) + a_3(s,\theta) + a_2(s,\theta)\partial_s\phi \\ &+(cs^{-1} + a_3(s,\theta))\partial_\theta\phi,\end{aligned} \tag{4.119}$$

and to solve equation (4.114):

$$\mathcal{B}(\phi) = (a_1(s,\theta) + 2\mathrm{d}^{-1}s^{-1}a_2(s,\theta))(1+\phi) + a_2(s,\theta)\partial_s\phi + (cs^{-1} + a_3(s,\theta))\partial_\theta\phi. \quad (4.120)$$

Then equation (4.110) can be written as:

$$\hat{\mathcal{L}}(\varphi) = \mathcal{A}(\varphi), \quad (4.121)$$

and (4.114) can be written as:

$$\hat{\mathcal{L}}(P_1^{\rm in}) = \mathcal{B}(P_1^{\rm in}). \quad (4.122)$$

Note that equations (4.121) and (4.122) can be rewritten as fixed point equations using a right inverse of the operator $\hat{\mathcal{L}}$. The key point to prove Propositions 4.4.5 and 4.4.7 will be to do so with a suitable right inverse that ensures that the solutions φ and $P_1^{\rm in}$ obtained satisfy the properties contained in these propositions.

Let us denote:

$$s_0 = -i\bar{\kappa}. \quad (4.123)$$

Then we define the following right inverse of $\hat{\mathcal{L}}$, which shall denote by $\hat{\mathcal{G}}$, as the operator acting on functions $\phi(s,\theta)$ given by:

$$\hat{\mathcal{G}}(\phi)(s,\theta) = \sum_{l\in\mathbb{Z}} \hat{\mathcal{G}}^{[l]}(\phi)(s)e^{il\theta}, \quad (4.124)$$

where:

$$\hat{\mathcal{G}}^{[l]}(\phi)(s) = \int_{s_0}^{s} e^{-il\alpha(w-s)}\phi^{[l]}(w)dw, \qquad \text{if } l<0, \quad (4.125)$$

$$\hat{\mathcal{G}}^{[l]}(\phi)(s) = \int_{-i\infty}^{s} e^{-il\alpha(w-s)}\phi^{[l]}(w)dw, \qquad \text{if } l\geq 0. \quad (4.126)$$

One can easily see that $\mathcal{L}\circ\hat{\mathcal{G}} = \mathrm{Id}$.

Next Lemma gives some properties of of the operator $\hat{\mathcal{G}}$ and its Fourier coefficients.

Lemma 4.4.8. *Let $n\geq 1$ and $\phi\in\mathcal{X}^{\rm u}_{n,\omega}$. There exists a constant M such that:*

1. *For all $l\neq 0$, $\|\hat{\mathcal{G}}(\phi)^{[l]}\|_n \leq \dfrac{M}{|l|}\|\phi^{[l]}\|_n$.*
2. *If $n>1$, then $\|\hat{\mathcal{G}}(\phi)^{[0]}\|_{n-1} \leq M\|\phi^{[0]}\|_n$.*
3. *If $n>1$, then $\|\hat{\mathcal{G}}(\phi)\|_{n-1,\omega} \leq M\|\phi\|_{n,\omega}$.*
4. *$\|\partial_\theta\hat{\mathcal{G}}(\phi)\|_{n,\omega} \leq M\|\phi\|_n$, and $\|\partial_s\hat{\mathcal{G}}(\phi)\|_{n,\omega} \leq M\|\phi\|_n$.*
5. *Moreover, if $n>1$, then $\llbracket\hat{\mathcal{G}}(\phi)\rrbracket_{n-1,\omega} \leq M\|\phi\|_{n,\omega}$.*

4.4.3 Proof of Proposition 4.4.5

By means of $\hat{\mathcal{G}}$, we can rewrite equation (4.121) as the following fixed point equation:

$$\varphi = \hat{\mathcal{G}} \circ \mathcal{A}(\varphi) =: \tilde{\mathcal{A}}(\varphi). \tag{4.127}$$

Thus, we proceed to prove that the operator $\tilde{\mathcal{A}}$ has a unique fixed point in a certain ball.

Lemma 4.4.9. *There exists a constant M such that:*

$$\|\tilde{\mathcal{A}}(0)\|_{1,\omega} \leq M, \qquad \|\partial_\theta \tilde{\mathcal{A}}(0)\|_{1,\omega} \leq M, \qquad \|\partial_s \tilde{\mathcal{A}}(0)\|_{1,\omega} \leq M,$$

and hence:

$$[\![\tilde{\mathcal{A}}(0)]\!]_{0,\omega} \leq M.$$

Remark 4.4.10. Note that, although we can bound $\tilde{\mathcal{A}}(0)$ using the norm $\|.\|_{1,\omega}$, we have to take the norm $[\![.]\!]_{0,\omega}$ since the bounds of the derivatives are with the norm $\|.\|_{1,\omega}$ too.

Proof. By Lemma 4.4.4 it is straightforward to see that:

$$\|\mathcal{A}(0)\|_{1,\omega} \leq K. \tag{4.128}$$

In particular, this implies that for all $l \in \mathbb{Z}$:

$$\|\mathcal{A}^{[l]}(0)\|_1 \leq K. \tag{4.129}$$

However, for the zeroth Fourier coefficient we need to improve this bound. We claim that:

$$\|\mathcal{A}^{[0]}(0)\|_2 \leq K. \tag{4.130}$$

Indeed, from the definition (4.119) of $\mathcal{A}$ it is clear that:

$$\mathcal{A}^{[0]}(0) = \mathrm{d}^{-1}\alpha\bar{a}_2^{[0]}(s) + \mathrm{d}^{-1}(c + \alpha L_0)s^{-1}a_2^{[0]}(s) + a_3^{[0]}(s).$$

We note that by Lemma 4.4.4 one has $\|a_2^{[0]}(s)\|_1 \leq K$ (so that $\|s^{-1}a_2^{[0]}(s)\|_2 \leq K$), $\|a_3^{[0]}(s)\|_2 \leq K$ and $\|\bar{a}_2^{[0]}\|_2 \leq K$. These bounds yield (4.130).

Now, on the one hand, using item 2 of Lemma 4.4.8 with bound (4.130) we obtain that:

$$\|\hat{\mathcal{G}}^{[0]}(\mathcal{A}(0))\|_1 \leq K\|\mathcal{A}^{[0]}(0)\|_2. \tag{4.131}$$

On the other hand, using item 1 of Lemma 4.4.8 with bound (4.129) yields:

$$\|\hat{\mathcal{G}}^{[l]}(\mathcal{A}(0))\|_1 \leq \frac{K}{|l|}\|\mathcal{A}^{[l]}(0)\|_1. \tag{4.132}$$

Clearly, since $\tilde{\mathcal{A}} = \hat{\mathcal{G}} \circ \mathcal{A}$, bounds (4.131) and (4.132) imply that

$$\|\tilde{\mathcal{A}}(0)\|_{1,\omega} \leq K(\|\mathcal{A}^{[0]}(0)\|_2 + \|\mathcal{A}(0)\|_{1,\omega}) \leq K,$$

where in the last inequality we have used (4.128) and (4.130).

Finally using item 4 of Lemma 4.4.8 with bounds (4.132) we obtain that:

$$\|\partial_\theta \tilde{\mathcal{A}}(0)\|_{1,\omega} \leq K, \qquad \|\partial_s \tilde{\mathcal{A}}(0)\|_{1,\omega} \leq K.$$

In particular:

$$\begin{aligned}
\llbracket \tilde{\mathcal{A}}(0) \rrbracket_{0,\omega} &= \|\tilde{\mathcal{A}}(0)\|_{0,\omega} + \|\partial_\theta \tilde{\mathcal{A}}(0)\|_{1,\omega} + \|\partial_s \tilde{\mathcal{A}}(0)\|_{1,\omega} \\
&\leq \frac{K}{\bar{\kappa}} \|\tilde{\mathcal{A}}(0)\|_{1,\omega} + \|\partial_\theta \tilde{\mathcal{A}}(0)\|_{1,\omega} + \|\partial_s \tilde{\mathcal{A}}(0)\|_{1,\omega} \leq K.
\end{aligned}$$

□

Lemma 4.4.11. *Let $\phi_1, \phi_2 \in \tilde{\mathcal{X}}_{0,\omega}$ such that $\llbracket \phi_i \rrbracket_{0,\omega} \leq C$ for some constant C. Then, there exists a constant M such that:*

$$\llbracket \tilde{\mathcal{A}}(\phi_1) - \tilde{\mathcal{A}}(\phi_2) \rrbracket_{1,\omega} \leq M \llbracket \phi_1 - \phi_2 \rrbracket_{0,\omega}.$$

In particular:

$$\llbracket \tilde{\mathcal{A}}(\phi_1) - \tilde{\mathcal{A}}(\phi_2) \rrbracket_{0,\omega} \leq \frac{M}{\bar{\kappa}} \llbracket \phi_1 - \phi_2 \rrbracket_{0,\omega}.$$

Proof. Using the definition (4.119) of $\mathcal{A}$, it is clear that:

$$\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2) = a_2(s,\theta)\partial_s(\phi_1 - \phi_1) + (cs^{-1} + a_3(s,\theta))\partial_\theta(\phi_1 - \phi_2).$$

Then by Lemma 4.4.4, the properties of the norm $\|.\|_{n,\omega}$ and the definition of the norm $\llbracket . \rrbracket_{n,\omega}$ we have:

$$\begin{aligned}
\|\mathcal{A}(\phi_1) - \mathcal{A}(\phi_2)\|_{2,\omega} &\leq \|a_2(s,\theta)\|_{1,\omega} \|\partial_s(\phi_1 - \phi_1)\|_{1,\omega} \\
&\quad + \|cs^{-1} + a_3(s,\theta)\|_{1,\omega} \|\partial_\theta(\phi_1 - \phi_2)\|_{1,\omega} \\
&\leq K \llbracket \phi_1 - \phi_2 \rrbracket_{0,\omega}. \qquad (4.133)
\end{aligned}$$

Thus, by item 5 of Lemma 4.4.8 we obtain:

$$\llbracket \tilde{\mathcal{A}}(\phi_1) - \tilde{\mathcal{A}}(\phi_2) \rrbracket_{1,\omega} \leq K \llbracket \phi_1 - \phi_2 \rrbracket_{0,\omega}.$$

By the properties of the norm $\|.\|$ it is straightforward to check that this implies:

$$\llbracket \tilde{\mathcal{A}}(\phi_1) - \tilde{\mathcal{A}}(\phi_2) \rrbracket_{0,\omega} \leq \frac{K}{\bar{\kappa}} \llbracket \phi_1 - \phi_2 \rrbracket_{0,\omega},$$

and the claim is proved. □

End of the proof of Proposition 4.4.5. Lemma 4.4.11 implies that taking $\bar{\kappa}$ is sufficiently large, then $\tilde{\mathcal{A}} : B(2\llbracket\tilde{\mathcal{A}}(0)\rrbracket_{0,\omega}) \to B(2\llbracket\tilde{\mathcal{A}}(0)\rrbracket_{0,\omega})$, and it has a unique fixed point:

$$\varphi \in B(2\llbracket\tilde{\mathcal{A}}(0)\rrbracket_{0,\omega}) \subset \tilde{\mathcal{X}}_{0,\omega}.$$

By construction, φ satisfies equation (4.110). Then, one can easily check that $\xi_{\text{in}}(s,\theta)$ defined as:

$$\xi_{\text{in}}(s,\theta) = \theta + \mathrm{d}^{-1}\alpha s + \mathrm{d}^{-1}(c+\alpha L_0)\log s + \varphi(s,\theta) \tag{4.134}$$

satisfies equation (4.106).

By the definition of the norm $\llbracket . \rrbracket_{n,\omega}$ and since $\llbracket\varphi\rrbracket_{0,\omega} \le 2\llbracket\tilde{\mathcal{A}}(0)\rrbracket_{0,\omega} \le K$ by Lemma 4.4.9, one obtains bound (4.113). We point out that this does not imply (4.112) directly, but it can be used to prove this bound *a posteriori* with the following argument. Since φ is the unique fixed point of $\tilde{\mathcal{A}}$, we can write:

$$\varphi = \tilde{\mathcal{A}}(\varphi) = \tilde{\mathcal{A}}(0) + \tilde{\mathcal{A}}(\varphi) - \tilde{\mathcal{A}}(0). \tag{4.135}$$

On the one hand, from Lemma 4.4.9 we know that:

$$\|\tilde{\mathcal{A}}(0)\|_{1,\omega} \le K.$$

On the other hand, $\llbracket\varphi\rrbracket_{0,\omega} \le 2\llbracket\tilde{\mathcal{A}}(0)\rrbracket_{0,\omega} \le K$ by Lemma 4.4.11, we have:

$$\|\tilde{\mathcal{A}}(\varphi) - \tilde{\mathcal{A}}(0)\|_{1,\omega} \le \llbracket\tilde{\mathcal{A}}(\varphi) - \tilde{\mathcal{A}}(0)\rrbracket_{1,\omega} \le K\llbracket\varphi\rrbracket_{0,\omega} \le K.$$

Then, from (4.135) it is clear that:

$$\llbracket\varphi\rrbracket_{1,\omega} \le K.$$

Now we shall prove that $(\xi_{\text{in}}(s,\theta),\theta)$, with ξ_{in} as in (4.134), is injective in $E_{\beta_0,\bar{\kappa}} \times \mathbb{T}_\omega$. Assume that $\xi_{\text{in}}(s_1,\theta) = \xi_{\text{in}}(s_2,\theta)$ for some $s_1, s_2 \in E_{\beta_0,\bar{\kappa}}$. This implies:

$$\mathrm{d}^{-1}\alpha(s_1 - s_2) + \mathrm{d}^{-1}(c+\alpha L_0)\left(\log s_1 - \log s_2\right) + \varphi(s_1,\theta) - \varphi(s_2,\theta) = 0. \tag{4.136}$$

Using the mean value theorem and denoting $s_\lambda = s_1 + \lambda(s_2 - s_1)$ one has that:

$$\begin{aligned}\mathrm{d}^{-1}(c+\alpha L_0)&\left(\log s_1 - \log s_2\right) + \varphi(s_1,\theta) - \varphi(s_2,\theta)\\ &= \int_0^1 \left[\frac{\mathrm{d}^{-1}(c+\alpha L_0)}{s_\lambda} + \partial_s\varphi(s_\lambda,\theta)\right] d\lambda (s_1 - s_2),\end{aligned}$$

and thus (4.136) yields:

$$\left(\mathrm{d}^{-1}\alpha + \int_0^1 \left[\frac{\mathrm{d}^{-1}(c+\alpha L_0)}{s_\lambda} + \partial_s\varphi(s_\lambda,\theta)\right] d\lambda\right)(s_1 - s_2) = 0. \tag{4.137}$$

We note that if $s_1,s_2 \in E_{\beta_0,\bar\kappa}$, then $s_\lambda \in E_{\beta_0,\bar\kappa}$. Thus, $|s_\lambda| \geq \bar\kappa$ and then:

$$\left|\frac{\mathrm{d}^{-1}(c+\alpha L_0)}{s_\lambda}\right| \leq \frac{K}{\bar\kappa}, \qquad |\partial_s\varphi(s_\lambda,\theta)| \leq \frac{K}{\bar\kappa},$$

where in the second inequality we have used that $\|\partial_s\varphi\|_1 \leq K$. Hence we obtain:

$$\left|\mathrm{d}^{-1}\alpha + \int_0^1 \left[\frac{\mathrm{d}^{-1}(c+\alpha L_0)}{s_\lambda} + \partial_s\varphi(s_\lambda,\theta)\right] d\lambda\right| \geq \mathrm{d}^{-1}\alpha - \frac{K}{\bar\kappa} > 0,$$

if $\bar\kappa$ is large enough. Thus, from (4.137) we get that $s_1 - s_2 = 0$, so that $s_1 = s_2$. □

4.4.4 Proof of Proposition 4.4.7

Similarly as in the previous subsection, first we rewrite equation (4.122) as a fixed point equation:

$$P_1^{\rm in} = \hat{\mathcal{G}} \circ \mathcal{B}(P_1^{\rm in}) := \tilde{\mathcal{B}}(P_1^{\rm in}). \tag{4.138}$$

Again, we prove that the operator $\tilde{\mathcal{B}}$ has a unique fixed point in a certain ball.

Lemma 4.4.12. *There exists a constant M such that:*

$$\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega} \leq M.$$

Proof. We just need to see that:

$$\|\mathcal{B}(0)\|_{2,\omega} \leq K, \tag{4.139}$$

and then the claim of the lemma follows using item 5 of Lemma 4.4.8. From definition (4.120) of $\mathcal{B}$ it is clear that:

$$\mathcal{B}(0) = a_1(s,\theta) + 2\mathrm{d}^{-1}s^{-1}a_2(s,\theta).$$

Bound (4.139) is deduced straightforwardly from Lemma 4.4.4. □

Lemma 4.4.13. *Let $\phi_1,\phi_2 \in \tilde{\mathcal{X}}_{1,\omega}$ such that $\llbracket \phi_i \rrbracket_{1,\omega} \leq C$ for some constant C. Then, there exists a constant M such that:*

$$\llbracket \tilde{\mathcal{B}}(\phi_1) - \tilde{\mathcal{B}}(\phi_2) \rrbracket_{2,\omega} \leq M \llbracket \phi_1 - \phi_2 \rrbracket_{1,\omega}.$$

In particular:

$$\llbracket \tilde{\mathcal{B}}(\phi_1) - \tilde{\mathcal{B}}(\phi_2) \rrbracket_{1,\omega} \leq \frac{M}{\bar\kappa} \llbracket \phi_1 - \phi_2 \rrbracket_{1,\omega}.$$

Proof. Again, we just need to prove that:

$$\|\mathcal{B}(\phi_1) - \mathcal{B}(\phi_2)\|_{3,\omega} \leq K \llbracket \phi_1 - \phi_2 \rrbracket_{1,\omega}, \tag{4.140}$$

and use item 5 of Lemma 4.4.8. From definition (4.120) of $\mathcal{B}$ we have:

$$\begin{aligned} \mathcal{B}(\phi_1) - \mathcal{B}(\phi_2) &= \left[a_1(s,\theta) + 2\mathrm{d}^{-1}s^{-1}a_2(s,\theta)\right](\phi_1 - \phi_2) + a_2(s,\theta)\partial_s(\phi_1 - \phi_2) \\ &\quad + \left[cs^{-1} + a_3(s,\theta)\right]\partial_\theta(\phi_1 - \phi_2). \end{aligned}$$

Using Lemma 4.4.4 it is clear that:

$$\begin{aligned} \|\mathcal{B}(\phi_1) - \mathcal{B}(\phi_2)\|_{3,\omega} &\leq K\left(\|\phi_1 - \phi_2\|_{1,\omega} + \|\partial_s(\phi_1 - \phi_2)\|_{2,\omega} + \|\partial_\theta(\phi_1 - \phi_2)\|_{2,\omega}\right) \\ &\leq K \llbracket \phi_1 - \phi_2 \rrbracket_{1,\omega}, \end{aligned}$$

and so we obtain bound (4.140). Finally we just need to note that:

$$\llbracket \tilde{\mathcal{B}}(\phi_1) - \tilde{\mathcal{B}}(\phi_2) \rrbracket_{1,\omega} \leq \frac{K}{\bar{\kappa}} \llbracket \tilde{\mathcal{B}}(\phi_1) - \tilde{\mathcal{B}}(\phi_2) \rrbracket_{2,\omega} \leq \frac{K}{\bar{\kappa}} \llbracket \phi_1 - \phi_2 \rrbracket_{1,\omega}.$$

□

End of the proof of Proposition 4.4.7. Lemma 4.4.13 yields that, if $\bar{\kappa}$ sufficiently large, then $\tilde{\mathcal{B}} : B(2\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega}) \to B(2\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega})$ and it has a unique fixed point:

$$P_1^{\mathrm{in}} \in B(2\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega}) \subset \tilde{\mathcal{X}}_{1,\omega}.$$

It is easy to see that if P_1^{in} satisfies the fixed point equation $P_1^{\mathrm{in}} = \tilde{\mathcal{B}}(P_1^{\mathrm{in}})$ then it satisfies (4.114), and $P^{\mathrm{in}}(s,\theta) = s^{2/\mathrm{d}}(1 + P_1^{\mathrm{in}}(s,\theta))$ satisfies equation (4.97).

Bound (4.116) follows from the fact that:

$$\llbracket P_1^{\mathrm{in}} \rrbracket_{1,\omega} \leq 2\llbracket \tilde{\mathcal{B}}(0) \rrbracket_{1,\omega} \leq K$$

by Lemma 4.4.12. □

4.5 Proofs of results of Subsection 4.1.5

We begin with the proof of Lemma 4.1.7.

Proof of Lemma 4.1.7. We recall that by Theorem 3.1.9 we have:

$$\Delta(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1 + P_1(u,\theta)) \sum_{l \in \mathbb{Z}} \Upsilon^{[l]} e^{il\xi(u,\theta)},$$

with:

$$\xi(u,\theta) = \theta + \delta^{-1}\alpha u + \mathrm{d}^{-1}(c + \alpha L_0)\log\cosh(\mathrm{d}u) + \alpha L(u) + \chi(u,\theta).$$

To shorten the notation, we shall denote:

$$C(u,\theta) = \mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(\mathrm{d}u) + \alpha L(u) + \chi(u,\theta).$$

Let us define the function:

$$F(u,\theta) = u + \delta\alpha^{-1}C(u,\theta).$$

Then $\xi(u,\theta)$ writes out as:

$$\xi(u,\theta) = \theta + \frac{\alpha}{\delta}F(u,\theta).$$

Since $(\xi(u,\theta),\theta)$ is injective in $D_{\kappa,\beta}\times\mathbb{T}_\omega$ by Theorem 3.1.9, $(F(u,\theta),\theta)$ is also injective in the same domain. In particular, for all $(u,\theta)\in D_{\kappa,\beta}\times\mathbb{S}^1$, the change $(w,\theta) = (F(u,\theta),\theta)$ is a diffeomorphism between $D_{\kappa,\beta}\times\mathbb{S}^1$ and its image $\tilde{D}_{\kappa,\beta}\times\mathbb{S}^1$, with inverse $(u,\theta) = (G(w,\theta),\theta)$. Then, if we define the function:

$$\tilde{\Delta}(w,\theta) = \sum_{l\in\mathbb{Z}}\Upsilon^{[l]}e^{il(\theta+\delta^{-1}\alpha w)}.$$

one has that $G(w,\theta)$ satisfies:

$$\tilde{\Delta}(w,\theta) = \frac{\Delta(G(w,\theta),\theta)}{\cosh^{2/\mathrm{d}}(\mathrm{d}G(w,\theta))(1+P_1(G(w,\theta),\theta))}. \tag{4.141}$$

Note that $\tilde{\Delta}(w,\theta)$ is $2\pi-$periodic in θ, and its $l-$th Fourier coefficient is:

$$\tilde{\Delta}^{[l]}(w) = \Upsilon^{[l]}e^{il\delta^{-1}\alpha w}.$$

Hence we know that for all $w\in\tilde{D}_{\kappa,\beta}$:

$$\left|\Upsilon^{[l]}\right| = \frac{1}{2\pi}\left|e^{-i\delta^{-1}\alpha wl}\int_0^{2\pi}\tilde{\Delta}(w,\theta)e^{-il\theta}d\theta\right| \le \left|e^{-i\delta^{-1}\alpha wl}\right|\sup_{\theta\in\mathbb{S}^1}\left|\tilde{\Delta}(w,\theta)\right|. \tag{4.142}$$

This inequality is valid for all $w\in\tilde{D}_{\kappa,\beta}$. Let us denote $u_\pm = \pm i\left(\frac{\pi}{2\mathrm{d}}-\kappa\delta\right)$. Then, if in (4.142) we take $w = w_+ := F(u_+,\theta)\in\tilde{D}_{\kappa,\beta}$ for $l<0$ and $w = w_- := F(u_-,\theta)\in\tilde{D}_{\kappa,\beta}$ for $l>0$, one obtains:

$$\left|\Upsilon^{[l]}\right| \le e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta}-\alpha\kappa-|\mathrm{Im}\,C(u_\pm,\theta)|\right)|l|}\sup_{\theta\in\mathbb{S}^1}\left|\tilde{\Delta}(w_\pm,\theta)\right|. \tag{4.143}$$

Recall that F is the inverse of G, so that from (4.141) we obtain:

$$\tilde{\Delta}(w_\pm,\theta) = \frac{\Delta(u_\pm,\theta)}{\cosh^{2/\mathrm{d}}(\mathrm{d}u_\pm)(1+P_1(u_\pm,\theta))}. \tag{4.144}$$

Thus, using that:

$$\left|\frac{1}{\cosh(\mathrm{d}u_\pm)}\right| = \frac{1}{\delta\kappa} + \mathcal{O}(\delta\kappa),$$

and that by Theorem 3.1.9:

$$\left|\frac{1}{1+P_1(u_\pm,\theta)}\right| \leq \frac{1}{1-\frac{K\delta}{\cosh(\mathrm{d}u_\pm)}} \leq \frac{1}{1-\frac{K}{\kappa}} \leq K,$$

where we have taken κ sufficiently large, equation (4.143) writes out as:

$$\left|\Upsilon^{[l]}\right| \leq \frac{K}{\delta^{\frac{2}{\mathrm{d}}}\kappa^{\frac{2}{\mathrm{d}}}} e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta}-\alpha\kappa-|\mathrm{Im}\, C(u_\pm,\theta)|\right)|l|} \sup_{\theta\in\mathbb{S}^1} |\Delta(u_\pm,\theta)| \,. \tag{4.145}$$

Now, on the one hand, taking into account that the constant L_0, given in Theorem 3.1.9, satisfies $L_0 \in \mathbb{R}$, we have:

$$|\mathrm{Im}\, C(u_\pm,\theta)| \leq \mathrm{d}^{-1}(c+\alpha L_0)|\mathrm{Im}\,\log\cosh(\mathrm{d}u_\pm)| + \alpha|L(u_\pm)| + |\chi(u_\pm,\theta)|.$$

Since:

$$\mathrm{Im}\,\log\cosh(\mathrm{d}u_\pm) = \arg(\cosh(\mathrm{d}u_\pm)).$$

and $\cosh(\mathrm{d}u_\pm) = \sin(\delta\kappa) \in \mathbb{R}_+$ we have:

$$\mathrm{Im}\,\log\cosh(\mathrm{d}u_\pm) = 0.$$

Then, using that $|L(u_\pm)| \leq K$ and $|\chi(u_\pm,\theta)| \leq K$ by Theorem 3.1.9, we obtain:

$$|\mathrm{Im}\, C(u_\pm,\theta)| \leq K.$$

Then for $|l| = 1$ we obtain:

$$\left|e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta}-\alpha\kappa-|\mathrm{Im}\, C(u_\pm,\theta)|\right)}\right| \leq K e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}+\alpha\kappa}. \tag{4.146}$$

For $l \geq 2$ we take δ sufficiently small so that:

$$1 - \frac{2\mathrm{d}\delta}{\alpha\pi}\left(\alpha\kappa + |\mathrm{Im}\, C(u_\pm,\theta)|\right) \geq \frac{3}{4}.$$

Then one has:

$$\left|e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta}-\alpha\kappa-|\mathrm{Im}\, C(u_\pm,\theta)|\right)|l|}\right| \leq e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}\frac{3|l|}{4}}. \tag{4.147}$$

On the other hand, by Theorem 4.1.1 we have:

$$|\Delta(u_\pm,\theta)| \leq |r_1^{\mathrm{u}}(u_\pm,\theta)| + |r_1^{\mathrm{s}}(u_\pm,\theta)| \leq \frac{K\delta}{|\cosh(\mathrm{d}u_\pm)|^3} \leq \frac{K}{\delta^2\kappa^3}. \tag{4.148}$$

To obtain the claim of the lemma for $|l| = 1$, we just need to use bounds (4.146) and (4.148) in equation (4.145). Similarly, for $|l| \geq 2$ we use bounds (4.147) and (4.148) in equation (4.145). □

The following result relates the functions $\xi(u,\theta)$ and $\xi_{\text{in}}(s(u),\theta)$, given in Theorems 3.1.9 and 4.1.6 respectively, when u is close to the singularity $i\pi/(2\mathrm{d})$.

Lemma 4.5.1. *Let $\kappa = \kappa_0 \log(1/\delta)$ for some constant $\kappa_0 > 0$. Let $L(u)$ be the function given in Theorem 3.1.9 and define:*

$$L_+ = \lim_{u \to i\frac{\pi}{2\mathrm{d}}} L(u)$$

as in (4.53)*. Let:*

$$u_+ = i\left(\frac{\pi}{2\mathrm{d}} - \delta\kappa\right),$$

and $s(u) = \frac{1}{\delta}\tanh(\mathrm{d}u)$ the function defined in (4.46)*. There exists a function $\eta(\theta)$ and a constant M satisfying:*

$$\sup_{\theta \in \mathbb{S}^1} |\eta(\theta)| \leq \frac{M}{\kappa},$$

such that the functions $\xi(u,\theta)$ and $\xi_{\text{in}}(s(u),\theta)$, given in Theorems 3.1.9 and 4.1.6 respectively, are related by:

$$\xi_{\text{in}}(s(u_+),\theta) = \xi(u_+,\theta) - i\frac{\alpha\pi}{2\mathrm{d}\delta} - \mathrm{d}^{-1}(c + \alpha L_0)\left(\log\delta + i\frac{\pi}{2}\right) - \alpha L_+ + \eta(\theta).$$

In particular one has:

$$\Upsilon_{\text{in}}^{[-1]} e^{-i\xi_{\text{in}}(s(u_+),\theta)} = \frac{\delta^{2+\frac{2}{\mathrm{d}}}}{(-i)^{\frac{2}{\mathrm{d}}}} \Upsilon_0^{[-1]} e^{-i(\xi(u_+,\theta)+\eta(\theta))}, \tag{4.149}$$

where $\Upsilon_{\text{in}}^{[-1]}$ is given in Theorem 4.1.6 and $\Upsilon_0^{[-1]}$ is defined in (4.56)*.*

Proof. On the one hand, we recall expression (4.51) of $\xi_{\text{in}}(s(u_+),\theta))$ given in Subsection 4.1.5:

$$\begin{aligned} \xi_{\text{in}}(s(u_+),\theta) \;&=\; \theta - i\alpha\kappa + \mathrm{d}^{-1}(c+\alpha L_0)\left(\log\cosh(\mathrm{d}u_+) - \log\delta - i\frac{\pi}{2}\right) + \varphi\left(s(u_+),\theta\right) \\ &\quad + \mathcal{O}\left(\delta^2\kappa^3\right), \end{aligned} \tag{4.150}$$

where $\varphi(s,\theta)$ is given in Theorem 4.1.6. On the other hand, recalling the definition of $\xi(u,\theta)$, given in Theorem 3.1.9, and using the definition of u_+ we have that:

$$\xi(u_+,\theta) = \theta + i\frac{\alpha\pi}{2\mathrm{d}\delta} - i\alpha\kappa + \mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(\mathrm{d}u_+) + \alpha L(u_+) + \chi(u_+,\theta). \tag{4.151}$$

Recalling that, by Theorem 3.1.9, for all $u \in D_{\kappa,\beta}$:

$$|L'(u)| \leq K,$$

and definition (4.53) of L_+, it is easy to see that:

$$|L(u_+) - L_+| \leq K\delta\kappa.$$

Moreover, by Theorems 3.1.9 and 4.1.6 we have:

$$|\chi(u_+,\theta)| \leq \frac{K\delta}{|\cosh(\mathrm{d}u_+)|} \leq \frac{K}{\kappa}, \qquad |\varphi\left(s(u_+),\theta\right)| \leq \frac{K}{|s(u_+)|} \leq \frac{K}{\kappa}.$$

Then, comparing expressions (4.150) and (4.151) we obtain readily:

$$\xi_{\mathrm{in}}(s(u_+),\theta) = \xi(u_+,\theta) - i\frac{\alpha\pi}{2\mathrm{d}\delta} - \mathrm{d}^{-1}(c+\alpha L_0)\left(\log\delta + i\frac{\pi}{2}\right) - \alpha L_+ + \eta(\theta),$$

with:

$$\eta(\theta) = \varphi\left(s(u_+),\theta\right) + L(u_+) - L_+ - \chi(u_+,\theta) + \mathcal{O}\left(\delta^2\kappa^3\right).$$

Clearly, $|\eta(\theta)| \leq K/\kappa$ for some constant K. Finally, to obtain formula (4.149) one just needs to use this expression of $\xi_{\mathrm{in}}(s(u_+),\theta)$ and the definition (4.56) of $\Upsilon_0^{[-1]}$. □

Proof of Proposition 4.1.8. First of all we point out that we just need to prove the result for $\Upsilon^{[-1]}$, since $\Delta(u,\theta)$, $\Delta_0(u,\theta)$ are real analytic, and then:

$$\Upsilon^{[1]} - \Upsilon_0^{[1]} = \overline{\Upsilon^{[-1]} - \Upsilon_0^{[-1]}}.$$

Recalling expression (4.58) of $\Delta_1(u,\theta) = \Delta(u,\theta) - \Delta_0(u,\theta)$ and repeating the same argument used to get bound (4.145) in the proof of Lemma 4.1.7, substituting $\Delta(u,\theta)$ by $\Delta_1(u,\theta)$, yields:

$$\left|\Upsilon^{[-1]} - \Upsilon_0^{[-1]}\right| \leq \frac{K}{\delta^{\frac{2}{\mathrm{d}}}\kappa^{\frac{2}{\mathrm{d}}}} e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta} - \alpha\kappa - |\mathrm{Im}\, C(u_+,\theta)|\right)} \sup_{\theta\in\mathbb{S}^1} |\Delta_1(u_+,\theta)|, \tag{4.152}$$

where $u_+ = i(\pi/(2\mathrm{d}) - \delta\kappa)$ and:

$$C(u,\theta) = \mathrm{d}^{-1}(c+\alpha L_0)\log\cosh(\mathrm{d}u) + \alpha L(u) + \chi(u,\theta),$$

with L_0, $L(u)$ and $\chi(u,\theta)$ given in Theorem 3.1.9. Taking into account that we have $\kappa = \kappa_0\log(1/\delta)$, equation (4.152) writes out as:

$$\left|\Upsilon^{[-1]} - \Upsilon_0^{[-1]}\right| \leq \frac{K}{\delta^{\frac{2}{\mathrm{d}}+\alpha\kappa_0}\log^{\frac{2}{\mathrm{d}}}(1/\delta)} e^{-\left(\frac{\alpha\pi}{2\mathrm{d}\delta} - |\mathrm{Im}\, C(u_+,\theta)|\right)} \sup_{\theta\in\mathbb{S}^1} |\Delta_1(u_+,\theta)|.$$

Using Theorem 3.1.9 one can easily see that:

$$C(u_+,\theta) = \mathrm{d}^{-1}\alpha L_0\log\cosh(\mathrm{d}u_+) + \mathcal{O}(1).$$

Moreover, we recall that $L_0 \in \mathbb{R}$ and $\operatorname{Im} \log\cosh(u_+) = 0$. Then:

$$\left|\Upsilon^{[-1]} - \Upsilon_0^{[-1]}\right| \leq \frac{K}{\delta^{\frac{2}{\mathrm{d}}+\alpha\kappa_0} \log^{\frac{2}{\mathrm{d}}}(1/\delta)} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}} \sup_{\theta\in\mathbb{S}^1} |\Delta_1(u_+,\theta)|\,, \tag{4.153}$$

for some constant K. Now we claim that there exists a constant K such that for all $\theta \in \mathbb{S}^1$:

$$|\Delta_1(u_+,\theta)| \leq K\frac{\delta^{-2+\alpha\kappa_0}}{\kappa^{1-\frac{2}{\mathrm{d}}}}. \tag{4.154}$$

Clearly, using (4.154) in (4.153) and recalling that $\kappa = \kappa_0 \log(1/\delta)$ we obtain the claim of the proposition (in the conservative case we just need to take $\mathrm{d} = 1$). Hence, the rest of the proof is devoted to proving bound (4.154).

To prove (4.154) we first rewrite $\Delta_1(u_+,\theta) = \Delta(u_+,\theta) - \Delta_0(u_+,\theta)$ in the following way:

$$\Delta_1(u_+,\theta) = \Delta(u_+,\theta) - \Delta_{\mathrm{in}}(u_+,\theta) + \Delta_{\mathrm{in}}(u_+,\theta) - \Delta_0(u_+,\theta). \tag{4.155}$$

First we bound $\Delta(u_+,\theta) - \Delta_{\mathrm{in}}(u_+,\theta)$. We have:

$$\begin{aligned}\Delta(u,\theta) - \Delta_{\mathrm{in}}(u,\theta) &= r_1^{\mathrm{u}}(u,\theta) - r_1^{\mathrm{s}}(u,\theta) - \delta^{-2}\left[\psi_{\mathrm{in}}^{\mathrm{u}}(s(u),\theta) - \psi_{\mathrm{in}}^{\mathrm{s}}(s(u),\theta)\right] \\ &= \delta^{-2}\left[\psi^{\mathrm{u}}(s(u),\theta) - \psi_{\mathrm{in}}^{\mathrm{u}}(s(u),\theta))\right] - \delta^{-2}\left[\psi^{\mathrm{s}}(s(u),\theta) - \psi_{\mathrm{in}}^{\mathrm{s}}(s(u),\theta))\right] \\ &= \delta^{-2}\left[\psi_1^{\mathrm{u}}(s(u),\theta) - \psi_1^{\mathrm{s}}(s(u),\theta)\right],\end{aligned}$$

where we have used that by definition $r_1^{\mathrm{u,s}}(u,\theta) = \delta^{-2}\psi^{\mathrm{u,s}}(s(u),\theta)$ and $\psi_1^{\mathrm{u,s}} = \psi^{\mathrm{u,s}} - \psi_{\mathrm{in}}^{\mathrm{u,s}}$. Thus, using that $|\psi_1^{\mathrm{u,s}}(s,\theta)| \leq K\delta^{1-\gamma}|s|^{-2}$ by Theorem 4.1.5, it is clear that:

$$|\Delta(u_+,\theta) - \Delta_{\mathrm{in}}(u_+,\theta)| \leq \frac{K}{|s(u_+)|^2}\delta^{-1-\gamma} \leq \frac{K}{\kappa^2}\delta^{-1-\gamma}, \tag{4.156}$$

where we have used that:

$$|s(u_+)| = \left|\frac{\cosh(\mathrm{d}u_+)}{\delta\sinh(\mathrm{d}u_+)}\right| = \mathrm{d}\kappa + \mathcal{O}(\delta^2\kappa^3). \tag{4.157}$$

Now we shall bound $\Delta_{\mathrm{in}}(u_+,\theta) - \Delta_0(u_+,\theta)$. First let us point out the following. On the one hand, we recall that by definition $\Delta_{\mathrm{in}}(u,\theta) = \delta^{-2}\Delta\psi_{\mathrm{in}}(s(u),\theta)$, where $\Delta\psi_{\mathrm{in}}(s,\theta)$ is given in Theorem 4.1.6. From the expression of $\Delta\psi_{\mathrm{in}}$ found in this theorem we have:

$$\Delta_{\mathrm{in}}(u,\theta) = \delta^{-2}s^{2/\mathrm{d}}(u)(1 + P_1^{\mathrm{in}}(s(u),\theta))\sum_{l<0}\Upsilon_{\mathrm{in}}^{[l]}e^{il\xi_{\mathrm{in}}(s(u),\theta)}.$$

On the other hand, we recall definition (4.45) of Δ_0:

$$\Delta_0(u,\theta) = \cosh^{2/\mathrm{d}}(\mathrm{d}u)(1 + P_1(u,\theta))\left(\Upsilon^{[0]} + \Upsilon_0^{[1]}e^{i\xi(u,\theta)} + \Upsilon_0^{[-1]}e^{-i\xi(u,\theta)}\right).$$

Thus, we can write $\Delta_{\rm in}(u_+,\theta)-\Delta_0(u_+,\theta)$ as:

$$\begin{aligned}\Delta_{\rm in}(u_+,\theta)-\Delta_0(u_+,\theta) &= \delta^{-2}s^{2/{\rm d}}(u_+)(1+P_1^{\rm in}(s(u_+),\theta))\Upsilon_{\rm in}^{[-1]}e^{-i\xi_{\rm in}(s(u_+),\theta)}\\ &\quad-\cosh^{2/{\rm d}}({\rm d}u_+)(1+P_1(u_+,\theta))\Upsilon_0^{[-1]}e^{-i\xi(u_+,\theta)}\\ &\quad+\delta^{-2}s^{2/{\rm d}}(u_+)(1+P_1^{\rm in}(s(u_+),\theta))\sum_{l\le -2}\Upsilon_{\rm in}^{[l]}e^{il\xi_{\rm in}(s(u_+),\theta)}\\ &\quad-\cosh^{2/{\rm d}}({\rm d}u_+)(1+P_1(u_+,\theta))\left(\Upsilon^{[0]}+\Upsilon_0^{[1]}e^{i\xi(u_+,\theta)}\right).\end{aligned}\tag{4.158}$$

First we shall prove that:

$$\begin{aligned}\Big|\delta^{-2}s^{2/{\rm d}}(u_+)\ &(1+P_1^{\rm in}(s(u_+),\theta))\Upsilon_{\rm in}^{[-1]}e^{-i\xi_{\rm in}(s(u_+),\theta)}\\ &-\cosh^{2/{\rm d}}({\rm d}u_+)(1+P_1(u_+,\theta))\Upsilon_0^{[-1]}e^{-i\xi(u_+,\theta)}\Big|\le K\frac{\delta^{-2+\alpha\kappa_0}}{\kappa^{1-\frac{2}{\rm d}}}.\end{aligned}\tag{4.159}$$

Indeed, recalling that $s(u)=1/(\delta\tanh({\rm d}u))$ and the relation between $\Upsilon_{\rm in}^{[-1]}$ and $\Upsilon_0^{[-1]}$ given in Lemma 4.5.1, formula (4.149), one has:

$$\begin{aligned}\delta^{-2}s^{2/{\rm d}}(u_+)&(1+P_1^{\rm in}(s(u_+),\theta))\Upsilon_{\rm in}^{[-1]}e^{-i\xi_{\rm in}(s(u_+),\theta)}\\ &=\frac{1}{(-i\tanh({\rm d}u_+))^{\frac{2}{\rm d}}}(1+P_1^{\rm in}(s(u_+),\theta))\Upsilon_0^{[-1]}e^{-i(\xi(u_+,\theta)+\eta(\theta))}.\end{aligned}$$

Hence we can write:

$$\begin{aligned}\Big|\delta^{-2}s^{2/{\rm d}}(u_+)\ &(1+P_1^{\rm in}(s(u_+),\theta))\Upsilon_{\rm in}^{[-1]}e^{-i\xi_{\rm in}(s(u_+),\theta)}\\ &-\cosh^{2/{\rm d}}({\rm d}u_+)(1+P_1(u_+,\theta))\Upsilon_0^{[-1]}e^{-i\xi(u_+,\theta)}\Big|\\ =&\Bigg|\frac{1}{(-i\tanh({\rm d}u_+))^{\frac{2}{\rm d}}}(1+P_1^{\rm in}(s(u_+),\theta))\Upsilon_0^{[-1]}e^{-i(\xi(u_+,\theta)+\eta(\theta))}\\ &-\cosh^{2/{\rm d}}({\rm d}u_+)(1+P_1(u_+,\theta))\Upsilon_0^{[-1]}e^{-i\xi(u_+,\theta)}\Bigg|\\ \le&\left|\left[\frac{1}{(-i\tanh({\rm d}u_+))^{\frac{2}{\rm d}}}-\cosh^{\frac{2}{\rm d}}({\rm d}u_+)\right](1+P_1^{\rm in}(s(u_+),\theta))\Upsilon_0^{[-1]}e^{-(i\xi(u_+,\theta)+\eta(\theta))}\right|\\ &+\left|\cosh^{\frac{2}{\rm d}}({\rm d}u_+)\left[P_1^{\rm in}(s(u_+),\theta)-P_1(u_+,\theta)\right]\Upsilon_0^{[-1]}e^{-i(\xi(u_+,\theta)+\eta(\theta))}\right|\\ &+\left|\cosh^{\frac{2}{\rm d}}({\rm d}u_+)(1+P_1(u_+,\theta))\Upsilon_0^{[-1]}e^{-i\xi(u_+,\theta)}\left(e^{-i\eta(\theta)}-1\right)\right|\end{aligned}\tag{4.160}$$

One can easily check that:

$$\left|\frac{1}{(-i\tanh({\rm d}u_+))^{\frac{2}{\rm d}}}-\cosh^{2/{\rm d}}({\rm d}u_+)\right|\le K(\delta\kappa)^{2+\frac{2}{\rm d}}.\tag{4.161}$$

Moreover, from Theorem 4.1.6 and using expression (4.157) of $s(u_+)$ we obtain:

$$\left|P_1^{\mathrm{in}}(s(u_+),\theta)\right| \leq \frac{K}{\kappa}, \qquad \left|1+P_1^{\mathrm{in}}(s(u_+),\theta)\right| \leq K. \tag{4.162}$$

Similarly, by Theorem 3.1.9 in Chapter 3 and taking into account that:

$$\cosh(\mathrm{d}u_+) = \mathrm{d}\delta\kappa + \mathcal{O}(\delta^3\kappa^3), \tag{4.163}$$

we obtain:

$$|P_1(u_+,\theta)| \leq \frac{K}{\kappa}, \qquad |1+P_1(u_+,\theta)| \leq K. \tag{4.164}$$

From the definition (4.56) of $\Upsilon_0^{[l]}$ it is also trivial to check that:

$$\left|\Upsilon_0^{[-1]}\right| \leq K\delta^{-2-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}. \tag{4.165}$$

From Theorem 3.1.9 and taking into account that $\operatorname{Im}\log\cosh(\mathrm{d}u_+) = 0$, we also have (see (4.151) for an expression of $\xi(u_+,\theta)$):

$$\operatorname{Im}\xi(u_+,\theta) = \frac{\alpha\pi}{2\mathrm{d}\delta} - \alpha\kappa + \operatorname{Im}C(u_+,\theta) = \frac{\alpha\pi}{2\mathrm{d}\delta} - \alpha\kappa + \mathcal{O}\left(1\right), \tag{4.166}$$

so that:

$$\left|e^{-i\xi(u_+,\theta)}\right| \leq K e^{\frac{\alpha\pi}{2\mathrm{d}\delta}-\alpha\kappa} = K\delta^{\alpha\kappa_0} e^{\frac{\alpha\pi}{2\mathrm{d}\delta}}, \tag{4.167}$$

where in the last step we have used that $\kappa = \kappa_0\log(1/\delta)$. Similarly, using also that $|\eta(\theta)| \leq K/\kappa$ by Lemma 4.5.1, we obtain:

$$\left|e^{-i(\xi(u_+,\theta)+\eta(\theta))}\right| \leq K\delta^{\alpha\kappa_0} e^{\frac{\alpha\pi}{2\mathrm{d}\delta}}, \tag{4.168}$$

and:

$$\left|e^{-i\eta(\theta)}-1\right| \leq |\eta(\theta)| e^{|\eta(\theta)|} \leq \frac{K}{\kappa}. \tag{4.169}$$

In conclusion, using bounds (4.161)–(4.169) in equation (4.160) we obtain bound (4.159).

Now we shall prove that:

$$\left|\delta^{-2}s^{2/\mathrm{d}}(u_+)(1+P_1^{\mathrm{in}}(s(u_+),\theta))\sum_{l\leq -2}\Upsilon_{\mathrm{in}}^{[l]} e^{il\xi_{\mathrm{in}}(s(u_+),\theta)}\right| \leq K\frac{\delta^{-2+\alpha\kappa_0}}{\kappa^{1-\frac{2}{\mathrm{d}}}}. \tag{4.170}$$

We note that by expression (4.150) of $\xi_{\mathrm{in}}(s(u_+),\theta)$ and recalling that we take θ real, and that $L_0\in\mathbb{R}$ and $\operatorname{Im}\log\cosh(\mathrm{d}u_+) = 0$, we have:

$$\operatorname{Im}\xi_{\mathrm{in}}(s(u_+),\theta) = -\alpha\kappa - \mathrm{d}^{-1}(c+\alpha L_0)\frac{\pi}{2} + \mathcal{O}\left(\frac{1}{\kappa}\right),$$

so that for $l \leq -2$ and assuming that $\kappa = \kappa_0 \log(1/\delta)$ is sufficiently large:

$$\left|e^{il\xi_{\mathrm{in}}(s(u_+),\theta)}\right| \leq e^{-|l|(\alpha\kappa - \mathcal{O}(1))} \leq e^{-\frac{3|l|}{4}\alpha\kappa} = \delta^{\frac{3|l|}{4}\alpha\kappa_0}.$$

Moreover from Theorem 4.1.6 we know that $|\Upsilon_{\mathrm{in}}^{[l]}| \leq K$ for all $l < 0$. Thus we obtain:

$$\left|\sum_{l\leq -2} \Upsilon_{\mathrm{in}}^{[l]} e^{il\xi_{\mathrm{in}}(s(u_+),\theta)}\right| \leq K \sum_{l\leq -2} \delta^{\alpha\kappa_0\frac{3|l|}{4}} \leq K\delta^{\frac{3}{2}\alpha\kappa_0}.$$

Then, using that $|s^{2/\mathrm{d}}(u_+)| \leq K\kappa^{\frac{2}{\mathrm{d}}}$ and bound (4.162) of $1 + P_1^{\mathrm{in}}(s(u_+,),\theta)$ we obtain:

$$\left|\delta^{-2} s^{2/\mathrm{d}}(u)(1 + P_1^{\mathrm{in}}(s(u_+),\theta)) \sum_{l\leq -2} \Upsilon_{\mathrm{in}}^{[l]} e^{il\xi_{\mathrm{in}}(s(u_+),\theta)}\right| \leq K\delta^{-2+\frac{3}{2}\alpha\kappa_0}\kappa^{\frac{2}{\mathrm{d}}}.$$

Since $\delta^{\frac{\alpha\kappa_0}{2}} \leq 1/\kappa$ for δ sufficiently small, this yields bound (4.170).

Finally we shall prove that:

$$\left|\cosh^{2/\mathrm{d}}(\mathrm{d}u_+)(1 + P_1(u_+,\theta))\left(\Upsilon^{[0]} + \Upsilon_0^{[1]} e^{i\xi(u_+,\theta)}\right)\right| \leq K\frac{\delta^{-2+\alpha\kappa_0}}{\kappa^{1-\frac{2}{\mathrm{d}}}}. \tag{4.171}$$

On the one hand, we recall that in the conservative case $\Upsilon^{[0]} = 0$, and in the dissipative one we take $\sigma = \sigma_*(\delta)$, so that:

$$|\Upsilon^{[0]}| = |a_1|\delta^{a_2} e^{-\frac{a_3\pi}{2\mathrm{d}\delta}},$$

for some a_1, $a_2 \in \mathbb{R}$ and $a_3 > 0$. On the other hand, from the definition (4.57) of $\Upsilon_0^{[1]}$ it is clear that:

$$\left|\Upsilon_0^{[1]}\right| \leq K\delta^{-2-\frac{2}{\mathrm{d}}} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}.$$

Using expression (4.166) of $\operatorname{Im}\xi(u_+,\theta)$ it is clear that:

$$|e^{i\xi(u_+,\theta)}| \leq K\delta^{-\alpha\kappa_0} e^{-\frac{\alpha\pi}{2\mathrm{d}\delta}}.$$

Hence, using also that $|\cosh(\mathrm{d}u_+)| \leq K\delta\kappa$ (by (4.163)) and bound (4.164) of $1+P_1(u_+,\theta)$ we obtain:

$$\begin{aligned}&\left|\cosh^{2/\mathrm{d}}(\mathrm{d}u_+)(1 + P_1(u_+,\theta))\left(\Upsilon^{[0]} + \Upsilon_0^{[1]} e^{i\xi(u_+,\theta)}\right)\right| \\ &\qquad \leq K(\delta\kappa)^{\frac{2}{\mathrm{d}}}\left(|a_1|\delta^{a_2} e^{-\frac{a_3\pi}{2\mathrm{d}\delta}} + \delta^{-2-\frac{2}{\mathrm{d}}-\alpha\kappa_0} e^{-\frac{\alpha\pi}{\mathrm{d}\delta}}\right).\end{aligned}$$

Clearly (noting that $a_3 > 0$), this is yields (4.171).

In conclusion, using bounds (4.159), (4.170) and (4.171) in (4.158) we obtain:

$$|\Delta_{\rm in}(u_+,\theta)-\Delta_0(u_+,\theta)|\le K\frac{\delta^{-2+\alpha\kappa_0}}{\kappa^{1-\frac{2}{\rm d}}}.$$

Using this bound and bound (4.156) in (4.155) we obtain:

$$|\Delta_1(u_+,\theta)|\le \frac{K}{\kappa^2}\delta^{-1-\gamma}+K\frac{\delta^{-2+\alpha\kappa_0}}{\kappa^{1-\frac{2}{\rm d}}}.$$

Then we just need to recall that $1-\gamma>\alpha\kappa_0$ by hypothesis, so that $\delta^{-1-\gamma}<\delta^{-2+\alpha\kappa_0}$, and we obtain bound (4.154). $\square$

Part II

Invariant manifolds in neuroscience

Chapter 5

Introduction

The theory of oscillators has drawn much interest in biology and, more specifically, neuroscience, since they can be used to model and describe many phenomena, as a neuron that spikes regularly. Moreover, many oscillators can be described by their phase of oscillation, which reduces the complexity of the study.

In particular, many interesting questions arise when dealing with *coupled* oscillators. In these problems, one is interested in determining the effects when an oscillator receives an input coming from other oscillators. For instance, it is important to know whether it will tend to a synchronization state or not. One way to study this problem is to determine the behavior of the phase of the oscillator after receiving a given input. A paradigmatic example of this situation are two neurons that are coupled via synapses.

The phase response (or *resetting*) curve (PRC) is a profusely used tool in neuroscience to study the effect of a perturbation on the phase of a neuron with oscillatory dynamics (see excellent surveys in [Izh07, ET10, SPB12]). Experimentalists find the PRC of a given neuron stimulating it briefly, and measuring the phase-shift compared to the unperturbed neuron (see Figure 5.1). Theoretical neuroscientists have developed methods to determine the PRC from mathematical models, in which an oscillating neuron can be thought of as a limit cycle in an n-dimensional space (see Figure 5.2). The most used method is the so-called Adjoint method, where a first order approximation of the PRC is found solving a variational equation. Of course, this method can only be applied in a perturbative setting, and provides a correct approximation of the PRC only if the perturbation is sufficiently small.

In other words, several conditions are required to apply this method, for instance weak perturbations, long time in between them, strong convergence to the limit cycle, etc. These ensure that the system relaxes back to the limit cycle before the next perturbation/kick is received. In this case, one can reduce the study to the phase dynamics on the oscillatory solution (namely, a limit cycle). However, in realistic situations, we may not be able to determine whether the system is on an asymptotic stationary state (limit cycle); on the other hand, the system may not show indeed regular spiking, specially

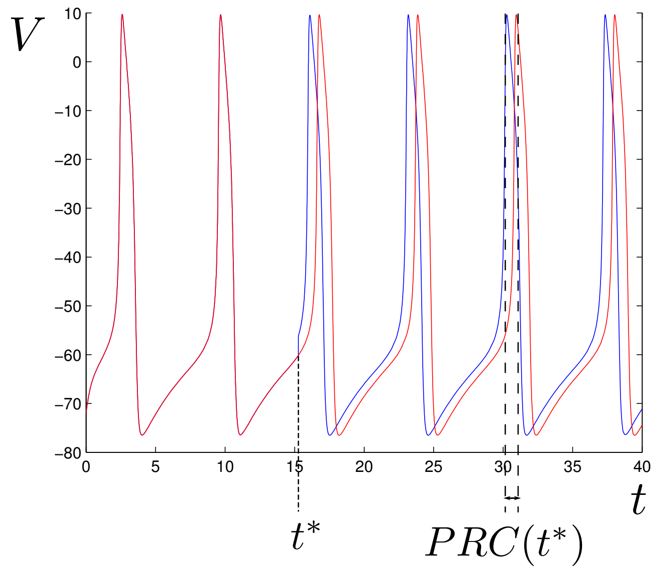
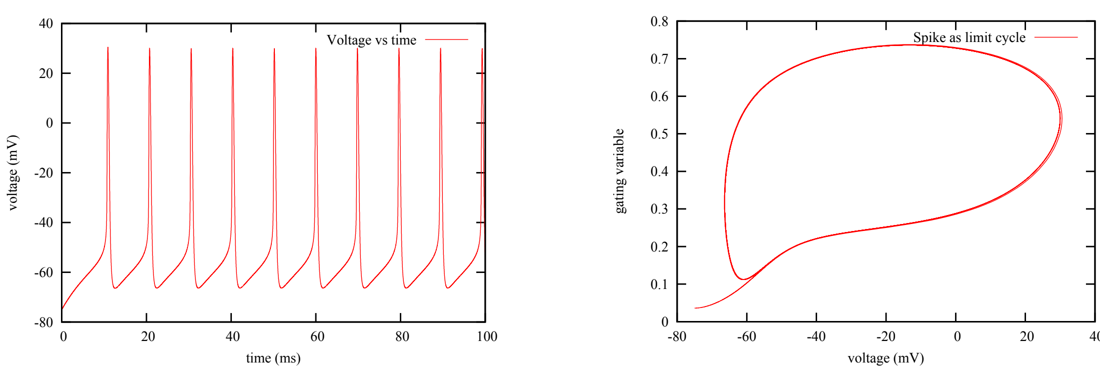


Figure 5.1: Experimental computation of PRCs. A brief stimulation at time $t = t^*$ is produced to a neuron. The PRC at this phase is then computed as the difference between the time of the spike of the perturbed neuron (in blue) and the unperturbed one (in red).



(a) (b)

Figure 5.2: Spiking neuron (a) seen as a limit cycle in (b), from a reduced Hodgkin-Huxley model.

because of noise, see for instance [GER05, EBTN11]. Moreover, even in the absence of noise, strong forcing may send the dynamics away from the asymptotic state, eventually close to other nearby invariant manifolds [OM11]; thus, the rate of convergence to the attractor plays an important role as well as the input frequency, which can be relatively high, for instance under bursting-like stimuli. All these factors may prevent the trajectories to relax back to the limit cycle before the next stimulus arrives and raise the question of how is the phase variation outside an attractor (that is, in transient states) and how far can we rely on the phase reduction (PRC).

Recently, tools to study the phase variation outside an attracting limit cycle have been developed. They rely on the concept of isochrons (manifolds transversal to the limit cycle and invariant under time maps given by the flow, see Figure 5.3), introduced by Winfree (see [Win75]) in biological problems, from which one can extend the definition of phase in a neighborhood of the limit cycle. In [GH09], the authors showed how to compute a parameterization of the isochrons (see also [OM09, SG09, MI12] for other computational methods) as well as the change in phase due to the kicks received when the system is approaching the limit cycle but not yet on it. This approach allows to control the phase advancement outside the limit cycle (that is, in the transient states) and build up the *Phase Response Functions* (PRF), a generalization of the PRCs.

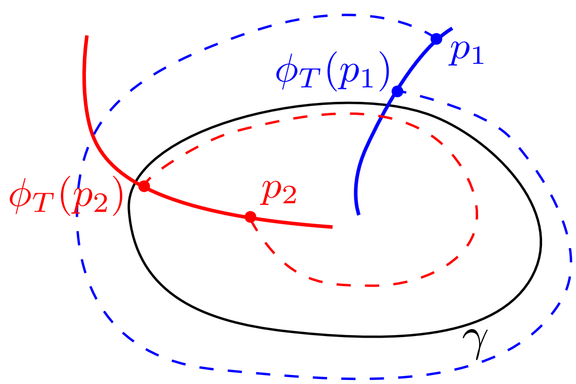


Figure 5.3: Example of isochrons. In black, a limit cycle γ of period T. In solid lines (red and blue), two isochrons. The points p_1 and p_2 return to the same isochron after time T.

In Chapter 6 we complete the extension of advancement functions to the transient states by defining the *Amplitude Response Function* (ARF), and we provide methods to compute it by controlling the changes induced by perturbations in a *transversal variable*, which represents some distance to the limit cycle. The knowledge of both the PRF and the ARF allows us to consider special problems in which these functions can forecast the asymptotic phase of an oscillator under pulsatile repetitive stimuli. Namely, we are interested in testing the differences in situations where the required hypotheses for the PRC approach fail, for instance high frequency stimuly, slow convergence to the limit cycle or strong perturbations. Then, we focus on pulse-train stimuli. In this case, the variations of the extended phase and the amplitude can be controlled by means of a 2D map; this 2D map extends the classical 1D map used when the dynamics is restricted to the limit cycle or phase-reduction is assumed, see for instance [Izh07, Ch. 10]. We then consider consider two examples, and try to determine by a fairly naive inspection the differences between the corresponding 1D and 2D maps. This part has been published in [CGH13].

After this first exploration, in Chapter 7 we use some existing numerical methods to investigate further the 2D map. In particular, we are interested in computing the invariant curves of the 2D map and determining their internal dynamics. This is done in the first place with a Newton-like method (see [HCF$^+$14]). After that, following [BST98], we compute Taylor series of the invariant curves and internal dynamics. This method is also useful to determine the so-called Arnold Tongues. However, we can only compute these invariant curves for very weak stimulus amplitude, being unable to reach values close to their breakdown.

In general, the aim of this part is to set up a paradigm to study phase and amplitude changes under more general conditions than those allowed by the classical approach using PRCs. For this purpose, we have constructed PRFs and ARFs which eventually could be applied to general situations. However, the type of the stimuli is also relevant and the theory cannot be applied straightforwardly for any kind of stimulus (for instance, continuous stimuli). Thus we exemplify the above paradigm in the frame of pulse-train periodic stimuli, in which this theory can be straightforwardly applied. Our hope is to illustrate the underlying dynamics in these situations and shed light on the importance of considering transient effects in synchronization problems.

Chapter 6

Phase-Amplitude Response Functions

6.1 Introduction

In this chapter we mainly introduce and make a first exploration of a context in which the extension of the phase response curves (PRCs), that is, the so-called phase response functions (PRFs) can be applied. In Section 6.2 we review the main ideas that will be used throughout this chapter, namely, the concept of isochrons, PRCs and the PRFs, which are the extension of PRCs in a neighborhood of the limit cycle introduced in [GH09].

In Section 6.3, we complete the extension of advancement functions to the transient states by defining the *Amplitude Response Function* (ARF), and we provide methods to compute it by controlling the changes induced by perturbations in a *transversal variable*, which represents some distance to the limit cycle. One of the methods presented here to compute the ARFs is an extension of the well-known *adjoint method* for the computation of PRCs, see for instance [EK91, BHM04], [Izh07, Ch. 10] or [ET10, Ch. 8].

In Section 6.4, we consider a one-dimensional map that models pulse-train stimulus, which is classically defined through the PRC, see for instance [Izh07, Ch. 10]. We then introduce the map obtained considering the knowledge provided by the PRF and ARF, which is two-dimensional. As an illustration, we consider a canonical model for which we compute the PRFs and ARFs thanks to the exact knowledge of the isochrons. In this "canonical" example, we apply a two parametric periodic forcing (varying the stimulus strength and frequency) and make predictions both with our 2D map and the classical 1D map; we use rotation numbers to illustrate the differences between the two predictions and we observe differences up to two orders of magnitude in favor of the 2D predictions, specially when the stimulation frequency is high or the strength of the stimulus is large. We also use this example to shed light to the role of hyperbolicity of the limit cycle as well as geometric aspects of the isochrons (see also [LWCY12] for a related study of the effect of isochrons' shear). Finally, we also present the comparison of the two approaches in a conductance-based neuron model, where we do not know the isochrons analytically.

Summing up, we aim at enlightening the contribution of transient effects in predicting the phase response, focusing on the importance of the "level" of hyperbolicity of the limit cycle, but also on the relative positions of the isochrons with respect to the limit cycle. Since PRCs are used for predicting synchronization properties, see [Erm96], [Izh07, Ch. 10] or [ET10, Ch. 8], our final goal is to show the limits of the phase reduction approach to prevent wrong predictions in synchronization problems. A deeper and more technical insight of this issue is the focus of Chapter 7.

6.2 Set-up of the problem: Isochrons and Phase Response Functions (PRF)

In this section we set up the problem and we review some of the results in [GH09] that serve as a starting point of the study that we present in this paper.

Consider an autonomous system of ODEs in the plane

$$\dot{\mathbf{x}} = X(\mathbf{x}), \ \mathbf{x} \in U \subseteq \mathbb{R}^2, \tag{6.1}$$

and denote by ϕ_t the flow associated to (6.1). Assume that X is an analytic vector field and that (6.1) has a hyperbolic limit cycle Γ of period T, parameterized by $\theta = t/T$ as

$$\begin{aligned} \gamma: \quad \mathbb{T} &\to \mathbb{R}^2 \\ \theta &\mapsto \gamma(\theta), \end{aligned} \tag{6.2}$$

so that $\gamma(\theta) = \gamma(\theta + 1)$.

Under these conditions, by the Stable Manifold Theorem (see [Guc75]), there exists a unique scalar function defined in a neighborhood Ω of the limit cycle Γ,

$$\begin{aligned} \Theta: \quad \Omega \subset \mathbb{R}^2 &\to \mathbb{T} \\ \mathbf{x} &\mapsto \Theta(\mathbf{x}) \end{aligned} \tag{6.3}$$

such that

$$\lim_{t \to +\infty} |\phi_t(\mathbf{x}) - \gamma(t/T + \Theta(\mathbf{x}))| = 0. \tag{6.4}$$

if the limit cycle is attracting. If the limit cycle is repelling the same is true with $t \to -\infty$.

The value $\Theta(\mathbf{x})$ is the asymptotic phase of $\mathbf{x}$ and the isochrons are defined as the sets with constant asymptotic phase, that is, the level sets of the function Θ. The same construction can be extended to limit cycles in higher dimensional spaces but, since the applications in this paper will restrict to planar systems, we give the definitions in $\mathbb{R}^2$.

Moreover, we know from [CFdlL05] that there exists an analytic local diffeomorphism

$$\begin{array}{rcl} K : \mathbb{T} \times [\sigma_-, \sigma_+] & \to & \mathbb{R}^2 \\ (\theta, \sigma) & \mapsto & K(\theta, \sigma), \end{array} \tag{6.5}$$

satisfying the invariance equation

$$\left(\frac{1}{T} \partial_\theta + \frac{\lambda \sigma}{T} \partial_\sigma \right) K(\theta, \sigma) = X(K(\theta, \sigma)), \tag{6.6}$$

where T is the period and λ is the characteristic exponent of the periodic orbit.

Remark 6.2.1. In [GH09] we presented a computational method to compute the parameterization K defined in (6.6) numerically.

Given an analytic local diffeomorphism K satisfying (6.6), we know from [GH09, Theorem 3.1] that the isochrons are the orbits of a vector field Y satisfying the Lie symmetry $[Y, X] = \mu Y$ with $\mu = \lambda/T$. That is,

$$Y \circ K(\theta, \sigma) = \partial_\sigma K(\theta, \sigma). \tag{6.7}$$

We can describe (6.6) as saying that if we perform the change of variables given by K, the dynamics of the system (6.1) in the coordinates (θ, σ) consists of a rigid rotation with constant velocity $1/T$ for θ and a contraction (if $\lambda < 0$) with exponential rate λ/T for σ. That is,

$$\begin{array}{rcl} \dot{\theta} & = & 1/T, \\ \dot{\sigma} & = & \lambda \sigma / T, \end{array} \tag{6.8}$$

and $\phi_t(K(\theta, \sigma)) = K(\theta + t/T, \sigma e^{\lambda/T t})$.

Let us assume that a pulse of small modulus ε instantaneously displaces the trajectory in a direction given by the unit vector v. Mathematically, we consider

$$\dot{\mathbf{x}} = X(\mathbf{x}) + \varepsilon v \delta(t - t_s),$$

where $\varepsilon \ll 1$ and $\delta(t)$ is the Dirac delta function. Then, the phase change due to this pulse stimulation at a point $p = K(\theta, \sigma)$ in a neighborhood Ω of the limit cycle Γ is given in first order with respect to ε by $\varepsilon < v, \nabla \Theta(p) >$, where Θ is the phase function defined in (6.3) and $< \cdot, \cdot >$ denotes the dot product. In [GH09], it was shown that

$$\nabla \Theta(K(\theta, \sigma)) = \frac{(\partial_\sigma K)^\perp}{T < (\partial_\sigma K)^\perp, \partial_\theta K >},$$

where $(\partial_\sigma K)^\perp = J(\partial_\sigma K)$ and the matrix J is given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{6.9}$$

We are going to use this notation for the rest of the paper.

Since phase changes studied in neuroscience are due to pulse stimuli in the direction of the voltage, the phase response function (PRF) is defined as the infinitesimal rate of change of the phase in the voltage direction:

$$PRF(p) = \partial_V \Theta(p), \tag{6.10}$$

where ∂_V denotes partial derivative with respect to the variable V.

6.3 The Amplitude Response Function (ARF)

A pulse stimulation displaces the trajectory away from the limit cycle, producing a change both in the phase θ and the transversal variable σ, that we will refer to as the *amplitude* variable. In our notation, the amplitude variable is the time-distance from the limit cycle along the orbits of Y. The phase-reduction approach assumes that the amplitude decreases to zero before the next pulse arrives and, therefore, the amplitude is always zero at the stimulation time. However, if one wants to consider pulses that arrive before the trajectory relaxes back to the limit cycle, one needs to compute also the amplitude displacement in order to predict the coordinates of the point at the next stimulation time.

In this section, we introduce the amplitude function and Amplitude Response Function (ARF); the analogues of the phase function (6.3) and the PRF (6.10) for the variable σ. Finally, we provide a formula to compute them given the diffeomorphism K introduced in (6.5).

6.3.1 Definitions

Given an analytic local diffeomorphism K (6.5) satisfying (6.6), it follows that there exists a unique function Σ, defined in a neighborhood Ω of the limit cycle Γ,

$$\begin{aligned} \Sigma: \quad \Sigma \subset \mathbb{R}^2 &\to \mathbb{R} \\ x &\mapsto \Sigma(x) \end{aligned} \tag{6.11}$$

such that

$$\Sigma(\phi_t(x)) = \Sigma(x) e^{-\lambda t/T},$$

where ϕ_t is the flow associated to the vector field X. The level curves of Σ are closed curves that we will call *amplitude level curves* or, in short, *A-curves.*

Analogous to the phase isochrons, it is easy to see that given an analytic local diffeomorphism K, as in (6.5), satisfying (6.6), the A-curves are the orbits of a vector field Z, satisfying $[X, Z] = [Z, X] = 0$, see Appendix 6.5 for a proof of this result. More specifically,

$$Z(K(\theta, \sigma)) = \partial_\theta K(\theta, \sigma). \tag{6.12}$$

Expressed in the variables (θ, σ) introduced in (6.5), the motion generated by Z is given by $\left\{\dot{\theta} = 1, \dot{\sigma} = 0\right\}$.

In the same way, the gradient $\nabla\Sigma(p)$ of this new function provides the infinitesimal change of the amplitude and we can define the *Amplitude Response Function*(ARF) as

$$ARF(K(\theta, \sigma)) = \partial_V \Sigma(K(\theta, \sigma)),$$

where ∂_V denotes partial derivative with respect to the variable V.

6.3.2 Computation of the PRFs and the ARFs

In this section we provide a formula to compute the functions $\nabla\Theta$ and $\nabla\Sigma$ given the diffeomorphism K introduced in (6.5).

Using the parameterization K introduced in (6.5) and writing:

$$K(x, y) = K(\Theta(x, y), \Sigma(x, y)) = (K_x, K_y),$$

where Θ and Σ are the functions introduced in (6.3) and (6.11), respectively, we have that

$$\begin{pmatrix} \partial_\theta K_x & \partial_\sigma K_x \\ \partial_\theta K_y & \partial_\sigma K_y \end{pmatrix} \begin{pmatrix} \partial_x \Theta & \partial_y \Theta \\ \partial_x \Sigma & \partial_y \Sigma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and therefore $\nabla\Theta = (\partial_x\Theta, \partial_y\Theta)$ and $\nabla\Sigma = (\partial_x\Sigma, \partial_y\Sigma)$ are given by

$$\begin{aligned} \begin{pmatrix} \nabla\Theta \\ \nabla\Sigma \end{pmatrix} &= \begin{pmatrix} \partial_\theta K_x & \partial_\sigma K_x \\ \partial_\theta K_y & \partial_\sigma K_y \end{pmatrix}^{-1} \\ &= \frac{1}{< \partial_\sigma K^\perp, \partial_\theta K >} \begin{pmatrix} \partial_\sigma K_y & -\partial_\sigma K_x \\ -\partial_\theta K_y & \partial_\theta K_x \end{pmatrix} \\ &= \frac{1}{< \partial_\sigma K^\perp, \partial_\theta K >} \begin{pmatrix} \partial_\sigma K^\perp \\ \partial_\theta K^\perp \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \nabla\Theta(K(\theta, \sigma)) &= \frac{\partial_\sigma K^\perp(\theta, \sigma)}{< \partial_\sigma K^\perp(\theta, \sigma), \partial_\theta K(\theta, \sigma) >}, \quad \text{and} \\ \nabla\Sigma(K(\theta, \sigma)) &= \frac{\partial_\theta K^\perp(\theta, \sigma)}{< \partial_\sigma K^\perp(\theta, \sigma), \partial_\theta K(\theta, \sigma) >}. \end{aligned} \tag{6.13}$$

Using the vector field description given in (6.7) and (6.12), we can rewrite the expression above, using that $\partial_\sigma K = Y \circ K$ and $\partial_\theta K = Z \circ K$, as

$$\nabla\Theta(K(\theta,\sigma)) = \left.\frac{Y^\perp}{< Y^\perp, Z >}\right|_{K(\theta,\sigma)}, \quad \text{and} \quad \nabla\Sigma(K(\theta,\sigma)) = \left.\frac{Z^\perp}{< Z^\perp, Y >}\right|_{K(\theta,\sigma)}.$$

By the invariance equation (6.6), we know that $X = \frac{1}{T}Z + \frac{\lambda}{T}\sigma Y$, and therefore

$$\nabla\Theta(K(\theta,\sigma)) = \left.\frac{Y^\perp}{T < Y^\perp, X >}\right|_{K(\theta,\sigma)}, \qquad \text{and} \quad \nabla\Sigma(K(\theta,\sigma)) = \frac{\lambda\sigma}{T}\left.\frac{Z^\perp}{< Z^\perp, X >}\right|_{K(\theta,\sigma)}. \tag{6.14}$$

Remark 6.3.1. Notice that expression for $\nabla\Sigma$ in (6.14) might suggest that it has a singularity at $\sigma = 0$. Nevertheless, the vanishing terms in the numerator and denominator cancel out at $\sigma = 0$, and using that $Z(K(\theta,0)) = \partial_\theta K(\theta,0) = X(K(\theta,0))$, the value at $\sigma = 0$ is given by

$$\nabla\Sigma(\gamma(\theta)) = \frac{X^\perp(\gamma(\theta))}{< X^\perp(\gamma(\theta)), K_1(\theta) >},$$

where $K_1(\theta) = \partial_\sigma K(\theta,0) = Y(K(\theta,0))$.

6.3.3 The adjoint method for the ARF

The most used method to compute the PRC, the so-called *adjoint method*, is based on the fact that the function $\nabla\Theta$ evaluated on the limit cycle Γ is a periodic solution of some adjoint equation (see for instance [Izh07]). In the generalization introduced in [GH09], it was shown that the adjoint method could be extended to compute $\nabla\Theta$ for points in a neighborhood of the limit cycle, for which the periodicity condition is not satisfied. Indeed, $\nabla\Theta$ satisfies the equation

$$\frac{dQ}{dt} = -DX^T(\phi_t(p))Q, \tag{6.15}$$

where DX^T is the transpose of the real matrix DX. In this case, the method just requires an initial condition, so that it can be solved uniquely. The initial condition is provided by formula (6.13).

The same result can be extended to compute the change in the transversal variable σ due to a pulse stimulation. In the following proposition, we provide the differential equation satisfied by $\nabla\Sigma(p)$ where $p = K(\theta,\sigma)$ is a point in a neighborhood Ω of the limit cycle γ evolving under the flow of X.

Proposition 6.3.2. *Let Γ be an isochronous T-periodic orbit of a planar analytic vector field X parameterized by θ according to* (6.3). *Assume that there exists a change of* *coordinates K in a neighborhood Ω satisfying* (6.6)*. Then, the function $\nabla\Sigma$ along the orbits of the vector field X satisfies the adjoint equation*

$$\frac{dQ}{dt} = \left(\frac{\lambda}{T} - DX^T(\phi_t(p))\right) Q, \tag{6.16}$$

where ϕ_t is the flow of the vector field X and λ is the characteristic multiplier of the periodic orbit, with the initial condition

$$Q(0) = \frac{\lambda\Sigma(p)}{T}\frac{Z^{\perp}(p)}{< Z^{\perp}(p), X(p) >,}. \tag{6.17}$$

where $Z^{\perp}(K(\theta,\sigma)) = \partial_\theta K(\theta,\sigma)$.

Proof. We will show that the function $\nabla\Sigma$ evaluated along the orbits $\phi_t(p)$ of X satisfies the adjoint equation (6.16). From expression (6.14) we have that

$$\nabla\Sigma(\phi_t(p)) = \frac{\lambda\Sigma(\phi_t(p))}{T}\frac{Z^{\perp}(\phi_t(p))}{< Z^{\perp}(\phi_t(p)), X(\phi_t(p)) >}. \tag{6.18}$$

We now compute the derivative of $\nabla\Sigma(\phi_t(p))$ with respect to time. In order to simplify notation we set $\mathbf{x} := \phi_t(p)$. We will also use that $Z^{\perp} = JZ$ where J is the matrix (6.9). Using that $\frac{d}{dt}Z(\mathbf{x}) = DZ(\mathbf{x})X(\mathbf{x})$, we have from (6.18)

$$\begin{aligned}\frac{d}{dt}\nabla\Sigma(\mathbf{x}) &= \frac{\lambda\frac{d\Sigma}{dt}(\mathbf{x})J\,Z(\mathbf{x}) + \lambda\Sigma(\mathbf{x})J\,DZ(\mathbf{x})\,X(\mathbf{x})}{T\ < Z^{\perp}(\mathbf{x}), X(\mathbf{x}) >}\\ &\quad -\frac{\lambda\Sigma(\mathbf{x})JZ(\mathbf{x})\,(< JDZ(\mathbf{x})X(\mathbf{x}), X(\mathbf{x}) > + < J\,Z(\mathbf{x}), DX(\mathbf{x})\,X(\mathbf{x}) >)}{T\ < Z^{\perp}(\mathbf{x}), X(\mathbf{x}) >^2}.\end{aligned}$$

Using that $DX\,Z = DZ\,X$, expression (6.18) and dot product properties (namely, $< J\,Z(\mathbf{x}), DX(\mathbf{x})\,X(\mathbf{x}) > = < DX(\mathbf{x})^T\,J\,Z(\mathbf{x}), X(\mathbf{x}) >$), we obtain

$$\begin{aligned}\frac{d}{dt}\nabla\Sigma(\mathbf{x}) &= \frac{\frac{\lambda}{T}\lambda\Sigma(\mathbf{x})J\,Z(\mathbf{x}) + \lambda\Sigma(\mathbf{x})J\,DX(\mathbf{x})\,Z(\mathbf{x})}{T\,g(\mathbf{x})}\\ &\quad -\frac{\nabla\Sigma(\mathbf{x})\left(< JDX(\mathbf{x})Z(\mathbf{x}) + DX(\mathbf{x})^T\,J\,Z(\mathbf{x}), X(\mathbf{x}) >\right)}{g(\mathbf{x})}.\end{aligned}$$

Using that $J\,DX(\mathbf{x}) + DX(\mathbf{x})^T\,J = \text{trace}(DX)(\mathbf{x})\,J$, and denoting $\tau(\mathbf{x}) := \text{trace}(DX)(\mathbf{x})$, we are led to

$$\frac{d}{dt}\nabla\Sigma(\mathbf{x}) = \frac{(\lambda/T - DX(\mathbf{x})^T + \tau(\mathbf{x}))\lambda\Sigma(\mathbf{x})J\,Z(\mathbf{x})}{T\ < Z^{\perp}(\mathbf{x}), X(\mathbf{x}) >} - \frac{\nabla\Sigma(\mathbf{x})\,(< \tau(\mathbf{x})\,J\,Z(\mathbf{x}), X(\mathbf{x}) >)}{< Z^{\perp}(\mathbf{x}), X(\mathbf{x}) >}.$$

Finally, using again (6.18), we have

$$\frac{d}{dt}\nabla\Sigma(\mathbf{x}) = (\lambda/T - DX(\mathbf{x})^T + \tau(\mathbf{x}))\nabla\Sigma(\mathbf{x}) - \nabla\Sigma(\mathbf{x})\,\tau(\mathbf{x}) = (\lambda/T - DX(\mathbf{x})^T)\nabla\Sigma(\mathbf{x}),$$

as we wanted to prove. □

6.4 Periodic pulse-train stimuli

The purpose of this section is to show the convenience of using the response functions beyond the limit cycle to obtain accurate predictions of the ultimate phase advancement. To this end, we force a system with pulse-trains of period $T_s << T_0$, nearby a limit cycle Γ of period T_0 and characteristic exponent λ.

Let us, then, consider a given oscillator, and assume that this oscillator is perturbed with an external instantaneous stimulus of amplitude ε in the voltage direction every T_s time units, that is:

$$\dot{\mathbf{x}} = X(\mathbf{x}) + \varepsilon\, v \sum_{j=0}^{N} \delta(t - jT_s), \tag{6.19}$$

where $v = (1, 0)$ and δ is the Dirac delta function. This system can represent, for example, a neuron which receives an idealized synaptic input from other neurons.

Remark 6.4.1. In the sequel, we will also use $\omega_s = 1/T_s$, the frequency of the stimulus, and $\omega_0 = 1/T_0$, the frequency of the limit cycle Γ. Then, the quotient ω_s/ω_0 indicates how many inputs receives the oscillator in one period. In the context of neuroscience, this could be approximated by the number of connections of a given neuron, assuming that ε is the postsynaptic potential and that the firing rate of the target neuron is close to the mean rate of the population.

In order to know the evolution of this perturbed oscillator after each time period T_s, it is enough to know how the variables θ and σ change. We recall that the variation of the variable θ produced by an external stimulus is given, in first order of the stimulus strength ε, by the PRF. Similarly, the variation of the variable σ is given in first order by the ARF. Hence, we can consider the following map, which approximates the position of the oscillator at the moment of the next kick:

$$\begin{aligned} \theta_{n+1} &= \theta_n + \varepsilon\, PRF(\theta_n, \sigma_n) + \frac{T_s}{T_0} \qquad (\text{ mod } 1), \\ \sigma_{n+1} &= (\sigma_n + \varepsilon\, ARF(\theta_n, \sigma_n))\, e^{\lambda T_s/T_0}. \end{aligned} \tag{6.20}$$

Moreover, we can compare it with the map obtained by considering the classical PRC (see, for instance, [Izh07, Ch. 10]), which is:

$$\theta_{n+1} = \theta_n + \varepsilon\, PRC(\theta_n) + \frac{T_s}{T} \qquad (\text{ mod } 1). \tag{6.21}$$

In the latter case we are assuming that the perturbation happens always on the limit cycle, and therefore $\sigma_n = 0$ for all n. The possibility that this might not be a realistic assumption (for example, if the stimulus period T_s is too small, the limit cycle is weakly hyperbolic or the strength of the stimulus ε is too large) has been already pointed out in the literature, see for instance [RF06] or [Izh07, Ch. 10]. However, other factors could play a role, for example the geometry of the isochrons (curvature, transversality to the limit cycle...). Our aim is to consider some examples and see in which cases the 1D map (6.21) gives a correct prediction or, on the contrary, one requires the 2D map (6.20) to correctly assess the true dynamics of the phase variable.

To quantify the long-term agreement or disagreement between the 1D and the 2D predictions, we compute an approximation of rotation numbers after N iterations of both maps respectively. More precisely, given an initial condition (θ_0, σ_0), we are entitled to compute

$$\overline{\rho} = \lim_{N\to+\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\tilde{\theta}_j - \tilde{\theta}_{j-1}\right), \tag{6.22}$$

where $\tilde{\theta}$ denotes the lift of θ to $\mathbb{R}$. Then, for the 2-dimensional map (6.20) and N large enough, the rotation number can be approximated by

$$\rho_{2D} := \frac{T_s}{T_0} + \frac{1}{N}\varepsilon \sum_{j=0}^{N-1} PRF(\theta_j, \sigma_j), \tag{6.23}$$

and by

$$\rho_{1D} := \frac{T_s}{T_0} + \frac{1}{N}\varepsilon \sum_{j=0}^{N-1} PRC(\theta_j), \tag{6.24}$$

for the 1-dimensional map (6.21).

These approximate rotation numbers will be our main indicator to compare the dynamics predicted by the 1D map with that of the 2D map. In order to dissect the causes that create the eventual differences between the two maps and highlight the shortcomings of the phase-reduction approach, we have first considered a "canonical" example in which the isochrons can be analytically computed. Next, we consider a conductance-based model, in which the isochrons have to be computed analytically and we obtain similar comparative results.

6.4.1 Examples

A simple canonical model

We use the simplest model having a limit cycle adding two parameters to play with the isochron-limit cycle angle and the hyperbolicity of the limit cycle. We consider, then, the following vector field X in polar coordinates:

$$\begin{aligned} \dot{r} &= \alpha r(1-r^2), \\ \dot{\varphi} &= 1 + \alpha a r^2, \end{aligned} \tag{6.25}$$

for $a, \alpha \in \mathbb{R}$, which has the following expression in cartesian coordinates:

$$\begin{aligned} \dot{x} &= \alpha x(1-(x^2+y^2)) - y(1+\alpha a(x^2+y^2)), \\ \dot{y} &= \alpha y(1-(x^2+y^2)) + x(1+\alpha a(x^2+y^2)). \end{aligned} \tag{6.26}$$

The limit cycle corresponds to $r=1$ and the dynamics on it is given by $\dot{\varphi} = 1+\alpha a$; therefore, $\varphi(t) = \varphi_0 + (1+\alpha a)t \bmod 2\pi$. The period of the limit cycle Γ is $T_0 = 2\pi/(1+\alpha a)$. A parameterization of the limit cycle in terms of the phase $\theta = t/T_0$, for $\theta \in [0,1)$ is $\gamma(\theta) = (\cos(2\pi\theta), \sin(2\pi\theta))$.

Now, consider the vector field Y, given in the different coordinate systems by

$$\begin{aligned} \dot{r} &= \alpha r^3, \\ \dot{\varphi} &= -\alpha a r^2; \end{aligned} \qquad \text{or} \qquad \begin{aligned} \dot{x} &= \alpha(x^2+y^2)(x+ay), \\ \dot{y} &= \alpha(x^2+y^2)(y-ax). \end{aligned}$$

It is easy to check that Y satisfies $[X,Y] = -2Y$. Then, using (6.7) we have that $\mu = -2$ and

$$K(\theta,\sigma) = \left(\sqrt{\frac{1}{1-2\alpha\sigma}} \cos(2\pi\theta + \frac{1}{2} a \ln(1-2\alpha\sigma)), \sqrt{\frac{1}{1-2\alpha\sigma}} \sin(2\pi\theta + \frac{1}{2} a \ln(1-2\alpha\sigma)) \right), \tag{6.27}$$

with $\theta \in [0,1)$ and $\sigma < 1/(2\alpha)$.

Notice that the function K can be easily inverted using that $r^2 = x^2+y^2 = (1-2\alpha\sigma)^{-1}$ and $\arctan\left(\frac{y}{x}\right) = 2\pi\theta + \frac{1}{2} a \ln(1-2\alpha\sigma)$. Then, $K^{-1}(x,y) = (\Theta(x,y), \Sigma(x,y))$, where

$$\Theta(x,y) = \frac{1}{2\pi}\left(\arctan\left(\frac{y}{x}\right) - \frac{1}{2} a \ln\left(\frac{1}{r^2}\right)\right), \Sigma(x,y) = \frac{1}{2\alpha}\left(1 - \frac{1}{r^2}\right).$$

Thus, the dynamics in (θ,σ) is given by

$$\begin{aligned} \dot{\theta} &= 1/T_0, \\ \dot{\sigma} &= -2\alpha\sigma. \end{aligned} \tag{6.28}$$

The vector field Z, defined in (6.12), has the following expression in Cartesian coordinates and polar coordinates, respectively:

$$\begin{aligned} \dot{x} &= -2\pi y, \\ \dot{y} &= 2\pi x; \end{aligned} \qquad \text{or} \qquad \begin{aligned} \dot{r} &= 0, \\ \dot{\varphi} &= 2\pi. \end{aligned}$$

Therefore, we have that $\nabla\Theta(p) = \frac{1}{2\pi r^2}(-y+ax, x+ay)$, and, by the parameterization γ of the limit cycle, $\nabla\Theta(\gamma(\theta)) = \frac{1}{2\pi}(-\sin(2\pi\theta) + a\cos(2\pi\theta), \cos(2\pi\theta) + a\sin(2\pi\theta))$. Similarly, $\nabla\Sigma(p) = \left(\frac{x}{\alpha r^4}, \frac{y}{\alpha r^4}\right)$, and $\nabla\Sigma(\gamma(\theta)) = (\cos(2\pi\theta), \sin(2\pi\theta))$.

From the last equations, we can then obtain:

$$PRF(K(\theta,\sigma)) = -\frac{\sqrt{1-2\alpha\sigma}}{2\pi}\left(\sin(2\pi\theta + \frac{1}{2}a\ln(1-2\alpha\sigma)) - a\cos(2\pi\theta + \frac{1}{2}a\ln(1-2\alpha\sigma))\right) \tag{6.29}$$

and:

$$ARF(K(\theta,\sigma)) = \frac{(1-2\alpha\sigma)^{3/2}}{\alpha}\cos(2\pi\theta + \frac{1}{2}a\ln(1-2\alpha\sigma)). \tag{6.30}$$

Let us stress out the role of the parameters α and a. On one hand, the parameter α determines the hyperbolicity of the limit cycle, since its characteristic exponent is:

$$\lambda = -2\alpha T_0.$$

Hence the larger is α, the stronger will be the contraction to the limit cycle. On the other hand, the parameter a determines the transversality of the isochrons to the limit cycle. Indeed, denoting β the angle between the isochron $\{p \in \mathbb{R}^2 : \Theta(p) = \theta\}$ and the limit cycle at the point $\gamma(\theta)$, we have that:

$$\cos\beta = \frac{\gamma'(\theta)\cdot\nabla\Theta^{\perp}(\gamma(\theta))}{\|\gamma'(\theta)\|\|\nabla\Theta^{\perp}(\gamma(\theta))\|}.$$

Computing explicitly the right-hand side of the equality, it is straightforward to check that:

$$\cos\beta = \frac{2\pi a}{\sqrt{1+a^2}}.$$

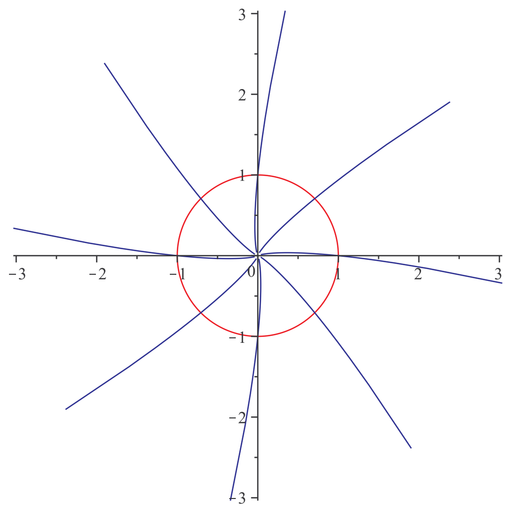
In particular, note that β is independent of the variable θ. Moreover, for $a = 0$ the isochrons will be orthogonal to the limit cycle, and they will become tangent to it as a goes to infinity (see Figure 6.1).

Numerical simulations

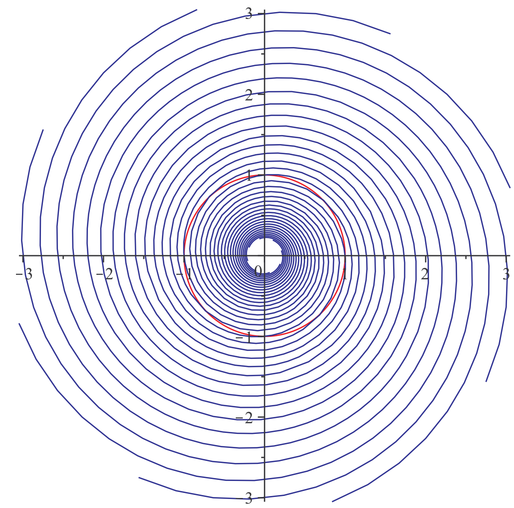
In this section we use the analytic expressions obtained in the previous subsection for the PRF, ARF and PRC to compute and compare the maps defined in (6.20) and (6.21). Moreover, as we also have an explicit formula for the parameterization K and its inverse K^{-1}, we can integrate numerically system (6.26), perturb it periodically, and obtain a sequence (x_n, y_n). Then, we can compute analytically

$$(\theta_n, \sigma_n) = K^{-1}(x_n, y_n), \tag{6.31}$$

and compare it with the iterations obtained using the maps (6.20) and (6.21). In the following, we will call the approximation of the rotation number obtained by this method simply ρ, to distinguish it from ρ_{2D} and ρ_{1D} defined previously in (6.23) and (6.24) respectively. Next lemma gives a description of the dynamics expected in the 1-dimensional map.



(a) $a = 0.1$



(b) $a = 10$

Figure 6.1: The limit cycle (red) of system (6.26) and some isochrons (blue) for different values of the parameter a. In both cases, $\alpha = 10$.

Lemma 6.4.2. *Let $k \in \mathbb{Z}$, and denote*

$$C_k = \frac{2\pi}{\varepsilon}\left(\frac{T_s}{T_0} - k\right).$$

Then, the fixed points of the 1-dimensional map (6.21) *can be computed analytically and:*

- *If $1 + a^2 - C_k^2 < 0$, the map* (6.21) *has no fixed points.*
- *If $1 + a^2 - C_k^2 \geq 0$ and*

$$\left|\frac{-aC_k + \sqrt{1 + a^2 - C_k^2}}{1 + a^2}\right| \leq 1,$$

the map (6.21) *has the fixed point*

$$\theta_+^* = \frac{1}{2\pi}\arccos\left(\frac{-aC_k + \sqrt{1 + a^2 - C_k^2}}{1 + a^2}\right).$$

Moreover, if $a \leq C_k$ and

$$\left|\frac{-aC_k - \sqrt{1 + a^2 - C_k^2}}{1 + a^2}\right| \leq 1,$$

there exists also another fixed point

$$\theta_-^* = \frac{1}{2\pi} \arccos\left(\frac{-aC_k - \sqrt{1+a^2-C_k^2}}{1+a^2}\right).$$

Proof. The fixed points θ^* of map (6.21) must satisfy

$$\varepsilon PRC(\theta^*) + \frac{T_s}{T_0} = k$$

for some $k \in \mathbb{Z}$, or equivalently

$$PRC(\theta^*) + \frac{C_k}{2\pi} = 0.$$

Substituting $PRC(\theta^*)$ for expression (6.29) with $\sigma = 0$ and rearranging terms we have

$$\sin(2\pi\theta^*) = a\cos(2\pi\theta^*) + C_k. \tag{6.32}$$

Taking squares in both sides of the equality and using trigonometric properties, we obtain that

$$(1+a^2)\cos^2(2\pi\theta^*) + 2aC_k\cos(2\pi\theta^*) + C_k^2 - 1 = 0,$$

which is an equation of degree 2 in $\cos(2\pi\theta^*)$. Solving it, after some simplifications we obtain

$$\cos(2\pi\theta^*) = \frac{-aC_k \pm \sqrt{1+a^2-C_k}}{1+a^2}. \tag{6.33}$$

It is clear that in order that equation (6.33) has real solutions, the right hand side must have modulus at most 1 and that $1 + a^2 - C_k \geq 0$. In this case the solutions of (6.33) are

$$\theta_\pm^* = \frac{1}{2\pi} \arccos\left(\frac{-aC_k \pm \sqrt{1+a^2-C_k}}{1+a^2}\right).$$

However, as we have taken squares in equation (6.32), we still have to check whether θ_+^* and θ_-^* are solutions of (6.32). It is easy to check that θ_+^* always solves (6.32), while θ_- is a solution just when $a \leq C$. □

Remark 6.4.3. A natural question is whether the 2-dimensional map (6.20) and the sequence (6.31) have the same qualitative behavior. As an example, let us take $\varepsilon = 0.03$, $\alpha = 0.1$ and $\alpha = 10$. In this case there exists just the fixed point θ_+^* for the 1-dimensional map (6.21). So let us take the initial condition $(\theta_0, \sigma_0) = (\theta_+^*, 0)$ and compute its iterates by the three different maps (6.20), (6.21) and (6.31). In Figure 6.2 we plot the sequences $K(\theta_n, \sigma_n)$ (for clarity, we have just plotted those with $n > 200$). As one can see, map (6.21) fails to predict correctly the qualitative behavior of the solution, since (6.31) seems to be attracted to a quasi-periodic orbit and not a fixed point. On the contrary, map (6.20) predicts correctly this qualitative behavior.

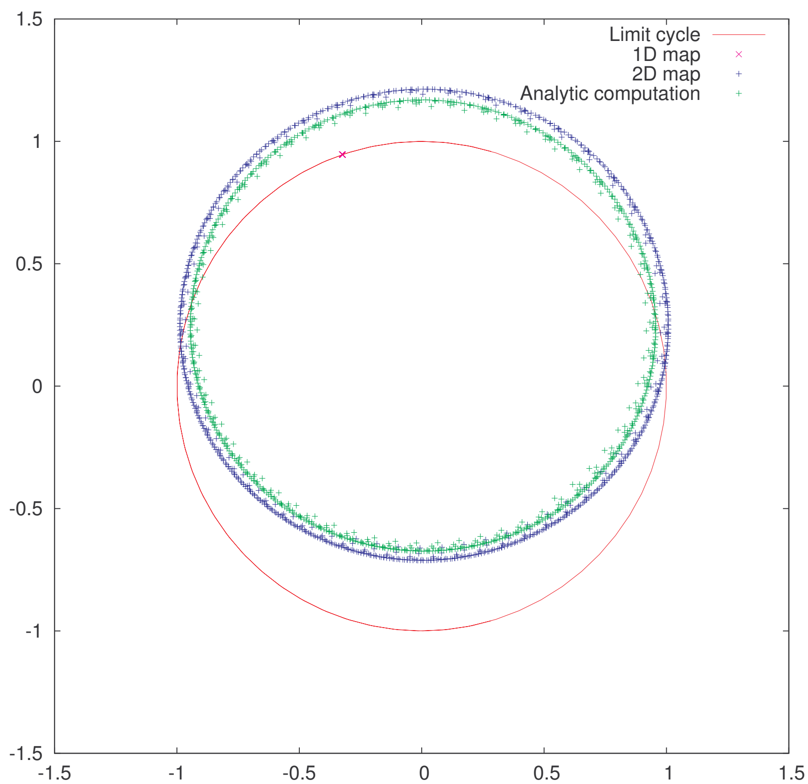


Figure 6.2: Sequences $K(\theta_n, \sigma_n)$, for $n > 200$, computed using maps (6.20), (6.21) and (6.31) respectively.

From now on we will take the initial condition to be $(\theta_0, \sigma_0) = (0.8, 0)$, that is, $(x_0, y_0) \approx (0.30901, -0.95106)$. In order to see the effect that both the hyperbolicity and the transversality of the isochrons to the limit cycle may have, we will plot the different approximations of the rotation numbers ρ, ρ_{2D} and ρ_{1D} for different values of the parameters a and α.

First of all we will take $\alpha = 0.1$ and $a = 10$. This corresponds to consider a weakly hyperbolic limit cycle with isochrons that are almost tangent to it. In Figure 6.3 we show the rotation numbers obtained for different amplitudes and for two different stimulus periods T_s. In this case, in order to make the rotation number ρ_{1D} stabilize, we have taken $N = 1000$.

Observe also the matching with the result in Lemma 6.4.2, which predicts the appearance of the fixed point of the 1D map when $1 + a^2 - C^2 = 0$, that is, when $C^2 = 101$ or, equivalently after substituting $T_s = T_0/m$, $\varepsilon = 2\pi/(\sqrt{101}\, m)$. This gives $\varepsilon \approx 0.0125$ for $m = 50$ (panel (a) in Figure 6.3) and $\varepsilon \approx 0.0312$ for $m = 20$ (panel (b) in Figure 6.3); both values coincide with the downstroke of the corresponding values of ρ_{1D}.

One can see that the rotation number obtained with the 1-dimensional map diverges from the analytical computation, while the one obtained with the 2-dimensional map does not. This wrong prediction by the 1-dimensional approach is consistent for all intermediate values of T_s (not shown here). We point out that, although the difference between the 1-dimensional approach and the other two seems rather small (it ranges from 10^{-3} to 10^{-2}), we can identify a wrong prediction of the qualitative behavior of the system by the 1-dimensional map. Indeed, in the cases where $\rho_{1D} \approx 0$ but $\rho_{2D}, \rho \neq 0$, the 1-dimensional map (6.21) has a fixed point, while the other two do not (see Remark 6.4.3). For example, in Figure 6.4 we plot in the phase space the first 100 iterates of sequences $K(\theta_n, \sigma_n)$, where (θ_n, σ_n) are obtained, respectively, using the 2-dimensional map (6.20), the 1-dimensional map (6.21) and expression (6.31). While in the 1D map a fixed point is reached, in the 2D and the analytic approaches it seems that dynamics are not so simple. Observe that this different qualitative behavior is obtained despite of the initial condition being on the limit cycle.

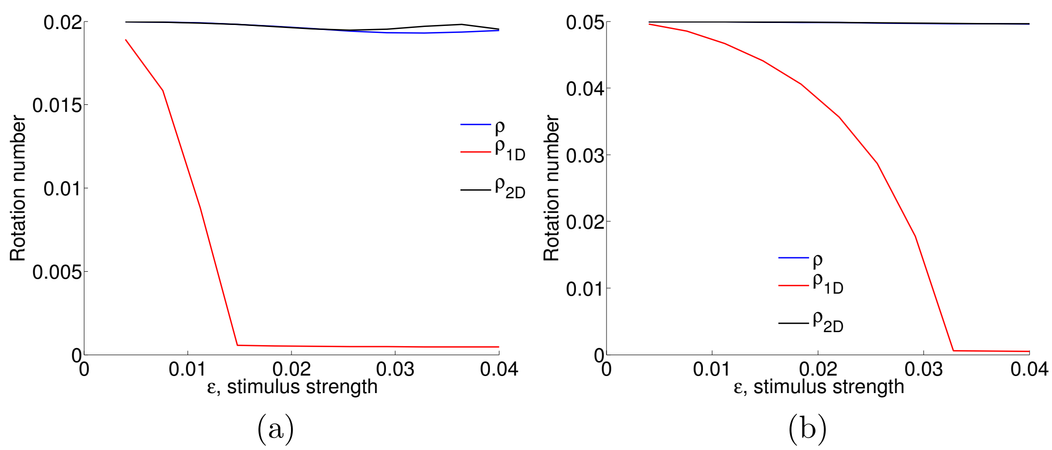


Figure 6.3: Rotation numbers as a function of the stimulus strength for parameter values $\alpha = 0.1$ and $a = 10$. Stimulation periods are (a) $T_s = 0.0628319 \approx T_0/50$, (b) $T_s = 0.1570800 \approx T_0/20$.

Another visualization of the agreement or disagreement between the different approximations of the rotation numbers is provided in Figures 6.5, 6.6 and 6.7. We show the differences between them depending on both ε (that is, the strength of the stimulus) and $\omega_{rel} := \omega_s/\omega_0 = T_0/T_s$ (the ratio between the frequency of the stimulus and the frequency of the limit cycle). In Figure 6.5, we plot the absolute difference between the rotation number obtained with the 2-dimensional approach and the analytic one, namely $|\rho_{2D} - \rho|$, whereas in Figure 6.6, we plot the error when using the phase-reduction hypothesis, namely $|\rho_{1D} - \rho|$. Both errors are compared in Figure 6.7, where the ratio $|\rho_{2D} - \rho|/|\rho_{1D} - \rho|$ is displayed. As expected, one can see in Figures 6.5 and 6.6 that, fixing the stimulus period T_s, the worst approximations of ρ given respectively by ρ_{2D}

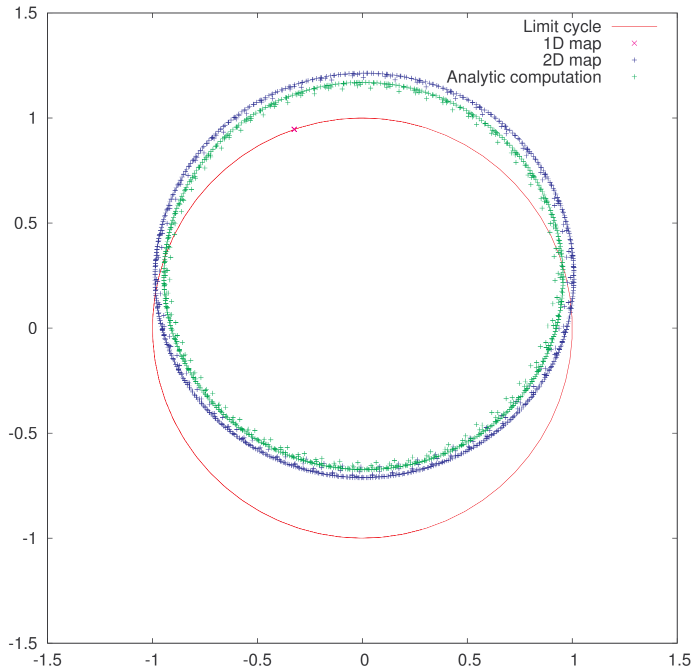


Figure 6.4: First 100 iterates of sequences $K(\theta_n, \sigma_n)$ computed using the three different methods. Parameter values are taken to be $\alpha = 0.1$, $a = 10$, $\varepsilon = 0.022$ and $T_s = 0.0628319 \approx T_0/50$.

and ρ_{1D} are obtained for high values of ε. However, fixing the strength of the stimulus ε, the results for both cases are different: while for the 2-dimensional map the worst results are for high frequency ratios ω_{rel}, in the 1D approach the worst results are obtained, in general, for low ω_{rel}. Finally, as we also expected, in Figure 6.7 we can appreciate that the 2D approach is always better than the 1D. Moreover, the difference between ρ_{2D} and ρ is, in the worst case, two orders of magnitude smaller than the difference between ρ_{1D} and ρ.

As we mentioned before, we use this example to help us understanding the effect of the hyperbolicity of the limit cycle and the transversality of the isochrons to it in the validity of the PRC approach. In Figures 6.8, 6.9 and 6.10, we plot the different approximations of the rotation numbers varying the parameters α ($\alpha = 0$ meaning loss of hyperbolicity) and a ($a = 0$ meaning isochrons normal to the limit cycle). In one hand, when the limit cycle is strongly hyperbolic (for instance, $\alpha = 10$ as in Figures 6.9 and 6.10), all approximations give a very similar result. Hence, in these two cases (even when the isochrons are almost tangent to the limit cycle, which corresponds to Figure 6.9), the use of PRFs and ARFs instead of PRCs seems not necessary. In fact, that is what one can expect intuitively: if the attraction to the limit cycle is very strong, the system relaxes back to the asymptotic

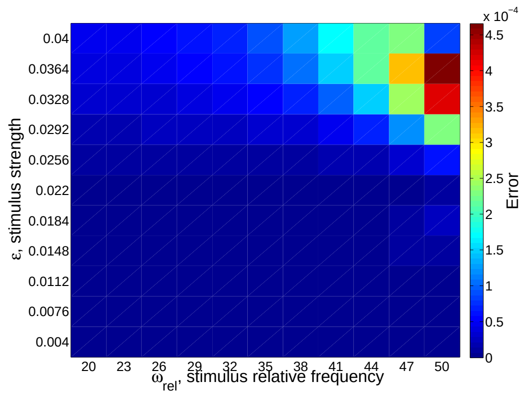


Figure 6.5: Absolute difference between the rotation number obtained with the 2-dimensional approach and the analytic one, that is $|\rho_{2D} - \rho|$, in the two-parametric space $(\omega_{rel}, \varepsilon)$.

state very fast, so that at each kick we can assume that the state variables are on the limit cycle. Of course, this will depend also on the frequency of stimulation ω_s.

On the other hand, in Figure 6.8 one can see that when the contraction to the limit cycle is slow but the isochrons are almost orthogonal to the limit cycle, the 1D approach diverges from the 2D and the analytic ones. However, at least for these ranges of ε and the two different periods of stimulus T_s, the 1D prediction still gives a fairly good approximation. Moreover, unlike the first case (where $\alpha = 0.1$ and $a = 10$), the 1D approach predicts the same qualitative behavior as the other two.

In conclusion, it seems that for the 2D map to represent a qualitative improvement with respect to the 1D it is necessary to have the combination of weak hyperbolicity of the limit cycle and "weak transversality" of isochrons to it. However, the role of hyperbolicity seems to be much more determinant, since in the presence of strong hyperbolicity the use of the 2D approach seems completely unnecessary, but for weak hyperbolicity the differences between the 1D and the 2D maps are present also when the isochrons are orthogonal to the limit cycle.

Remark 6.4.4. Of course, considering a stimulus strength ε large enough, both maps (6.20) and (6.21) will not give correct predictions, since they are based on first-order approximations. In this case, one should consider PRFs of second (or higher) order to obtain a correct result, see for instance [TF10, SD10] for higher-order PRCs. One has to distinguish between these higher order response functions in terms of the stimulus strength from the second-order PRCs above mentioned (see [OC01] for instance) that relate to the second cycle after the stimulus.

In the next example, we apply the same methodology to a more biologically inspired

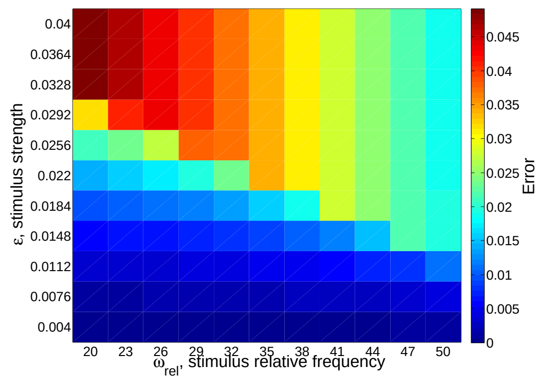


Figure 6.6: Absolute difference between the rotation number obtained with the 1-dimensional approach (phase-reduction hypothesis) and the analytic one, that is $|\rho_{1D} - \rho|$, in the two-parametric space $(\omega_{rel}, \varepsilon)$.

case: a conductance-based model for a point-neuron with one two types of ionic channels.

A conductance-based model

We consider a reduced Hodgkin-Huxley-like system, with sodium and potassium currents, and only one gating variable:

$$\begin{aligned} \dot{V} &= -\tfrac{1}{C_m}(g_{Na} m_\infty(V)(V - V_{Na}) + g_K n(V - V_K) + g_L(V - V_L) - I_{app}), \\ \dot{n} &= n_\infty(V) - n, \end{aligned} \tag{6.34}$$

where V represents the membrane potential, in mV, n is an adimensional gating variable and the open-state probability functions are

$$m_\infty(V) = \frac{1}{1 + \exp(-(V - V_{max,m})/k_m)}, \qquad n_\infty(V) = \frac{1}{1 + \exp(-(V - V_{max,n})/k_n)}.$$

The parameters of the system are $C_m = 1.\mu\, F/cm^2$, $g_{Na} = 20.m\, S/cm^2$, $V_{Na} = 60.mV$, $g_K = 10.m\, S/cm^2$, $V_K = -90.mV$, $g_L = 8.m\, S/cm^2$, $v_L = -80.mV$, $V_{max,m} = -20.mV$, $k_m = 15.$, $V_{max,n} = -25.mV$, $k_n = 5$.

Here, we will take $I_{app} = 190\mu\, A/cm^2$. In this case, the system has a limit cycle with period $T_0 \approx 1.3055442$, and its characteristic exponent is $\lambda \approx -0.6055956$. That is, the limit cycle is weakly hyperbolic, and hence we expect that the 2-dimensional approach will give qualitatively different results with respect to the 1-dimensional approach. Figure 6.11 shows the limit cycle and its isochrons.

Remark 6.4.5. For $I_{app} = 190$, system (6.34) is not a model of a spiking neuron, but one with high voltage oscillations. Thus, this example is not intended to deal with a realistic

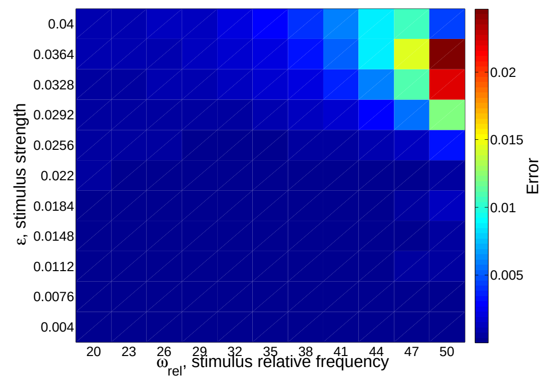


Figure 6.7: .
Ratio of the absolute difference between the 2-dimensional approach and the analytic one (numerator, see Figure 6.5) over the absolute difference between the 1-dimensional approach and the analytic one (denominator, see Figure 6.6), that is, $|\rho_{2D} - \rho|/|\rho_{1D} - \rho|$.

setting of spike synchronization, but to illustrate how to deal with the tools introduced in this paper in the case where one does not have explicitly the parameterization K.

Remark 6.4.6. In order to compute the parameterization K and the PRFs we have used the methods proposed in [GH09]. The same ideas can be applied to compute the ARFs. Roughly speaking, the method consists in two steps. First, to compute the value of a given ARF near the limit cycle, where the numeric approximation of the parameterization K is valid, expression (6.14) is used. Second, to compute the value of some ARF far from the limit cycle, we just integrate the adjoint system (6.16) backwards in time using an initial condition for ARF close to the limit cycle.

Again, we have computed the rotation numbers as defined in (6.23) and (6.24) varying the strength of the stimulus ε with fixed stimulation periods. We have taken $N = 100$ and initial conditions $\theta_0 = 0.089$ and $\sigma_0 = 0$. The results, for two different stimulus periods T_s, are shown in Figure 6.12. Again, note that although the dynamics begins on the limit cycle (since $\sigma_0 = 0$), the behavior of the 1-dimensional approach and the 2-dimensional approach are quite different. Moreover, for $\varepsilon > 0.4$ we have that $\rho_{1D} \approx 0$, while $\rho_{2D} \approx 0.02$. This can be interpreted, similarly to the previous example, as an indicator that the 1D map (6.21) has a fixed point, while the 2D map does not. Furthermore, this indicates that after 100 iterations of the 2D map (6.20), the state variables have turned approximately twice around the fixed point, as one can see from the plots of the sequences $K(\theta_n, \sigma_n)$ computed using both maps (see Figure 6.13).

Remark 6.4.7. Observe that it could happen that the critical point of the continuous system (located inside the limit cycle) was not encircled by the iterates $K(\theta_n, \sigma_n)$. Then,

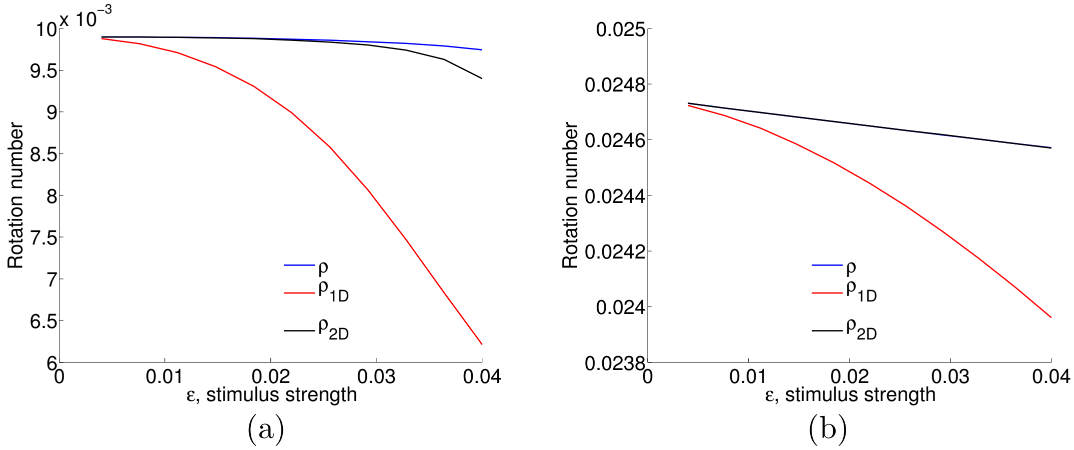


Figure 6.8: Rotation numbers for different stimulus strengths in case of weak hyperbolicity and normal isochrons ($\alpha = 0.1$ and $a = 0$). Stimulation periods are (a) $T_s = 0.0628319 \approx T_0/50$, (b) $T_s = 0.1570800 \approx T_0/20$.

the rotation number defined as (6.22) would not give an intuitive idea of the qualitative behavior of the state variables. This is not the case in our example, though Figure 6.13 shows a nearby situation.

6.5 Discussion

We have introduced general tools (the PRF and the ARF) to study the advancement of both the phase and the amplitude variables for dynamical systems having a limit cycle attractor. These tools allow us to study variations of these variables under general perturbation hypotheses and extend the concept of infinitesimal PRCs which assumes the validity of the phase-reduction and is only true under strong hyperbolicity of the limit cycle or under weak perturbations. In fact, the PRFs and ARFs are first order approximations of the actual variation of the phase and the amplitude, respectively, and so they are supposed to work mainly for weak perturbations; however, being an extension outside the limit cycle makes them more accurate than the PRCs even under strong perturbations. We thus claim that the phase-reduction has to be used with caution since assuming it by default may lead to completely wrong predictions in synchronization problems. We are not reviling phase-reduction but trying to show the limits beyond which an extended scenario is required.

We have presented a computational analysis to understand the contribution of transient effects in first-order predictions of the phase response, focusing on the importance

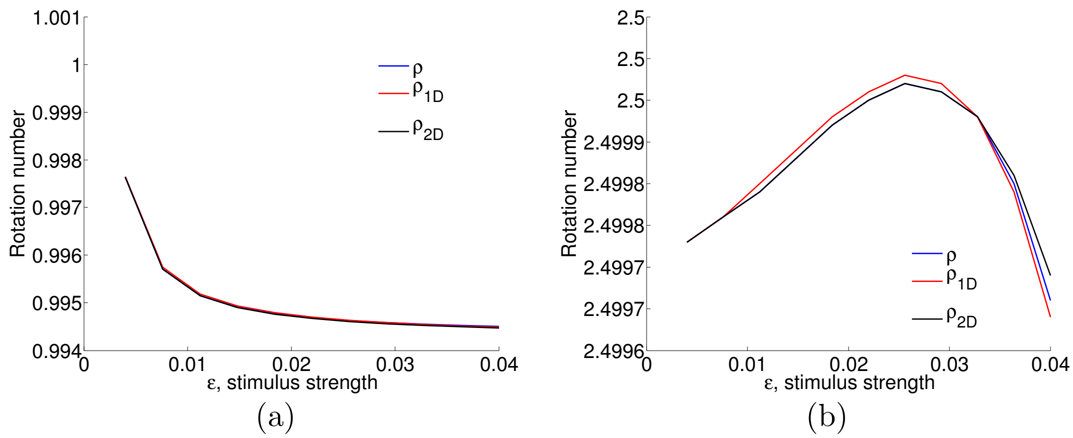


Figure 6.9: Rotation numbers for different stimulus strengths in case of strong hyperbolicity and almost tangent isochrons ($\alpha = 10$ and $a = 10$). Stimulation periods are (a) $T_s = 0.0628319 \approx T_0/50$, (b) $T_s = 0.1570800 \approx T_0/20$.

of the hyperbolicity of the limit cycle, but also on the relative positions of the isochrons with respect to the limit cycle.

In the examples studied, subject to pulse-train stimuli, we have compared the predictions obtained both with the new 2D map defined from the PRF and ARF and the 1D map defined from the classical PRC. Using rotation numbers, we have shown differences up to two orders of magnitude in favor of the 2D predictions, specially when the stimulation frequency is high or the stimulus is too strong. These results confirm previous numerical experiments with specific oscillators, see [RF06]. On the other hand, we have found that both weak hyperbolicity of the limit cycle and "weak transversality" of isochrons to it are important factors, although the role of hyperbolicity seems to be more crucial. In this paper, these achievements have been tested in a canonical model allowing comparisons with the exact solutions and other numerical test have been applied in a conductance-based model. The technique can be applied to other neuron models, and not necessarily for planar systems; n-dimensional systems would only require an additional computational difficulty in computing the associated $n-1$ ARFs.

We would like to emphasize the importance of having good methods to compute isochrons ([GH09, OM09, SG09, MI12]) since they are the cornerstone to study these transient phenomena that we have observed. They can be useful, not only for the problem illustrated here, but for other purposes like testing how far are the experimentally recorded phase variations from the theoretically predicted ones. In fact, they are the key concept to be able to predict the exact phase variation. Indeed, knowing the parameterization K that gives the isochrons, the problem of the phase variation reduces to solve, at each step, $(x, y) = K(\theta, \sigma)$ and $(x', y') = K(\theta', \sigma')$. Here (x, y) denotes the point in

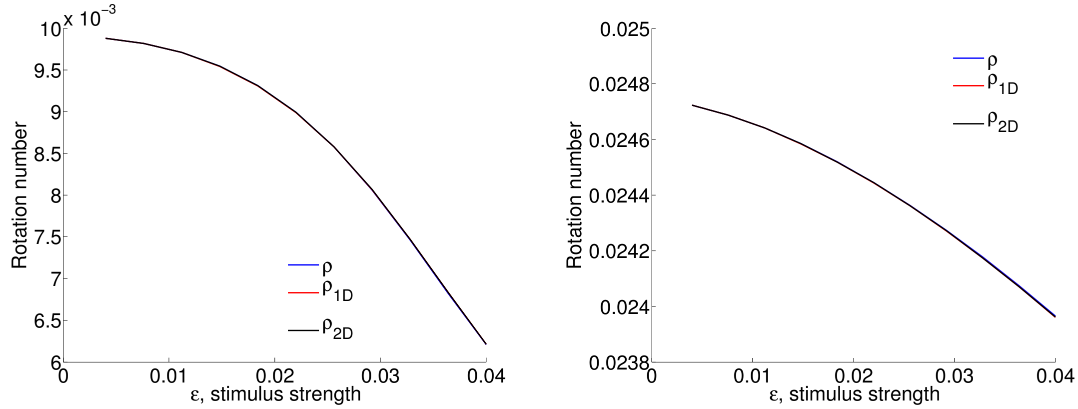


Figure 6.10: Rotation numbers for different stimulus strengths in case of strong hyperbolicity and normal isochrons ($\alpha = 10$ and $a = 0$). Stimulation periods are (a) $T_s = 0.0628319 \approx T_0/50$, (b) $T_s = 0.1570800 \approx T_0/20$.

the phase space where the pulse perturbation, $\varepsilon\, v$, is applied and $(x', y') = (x, y) + \varepsilon\, v$. Note that the PRFs and ARFs can be computed knowing only the first order in K, so that, in principle, they are valid only for weak perturbations. The advantage is that they easier to compute. Other refinements could be obtained by computing second orders of the PRFs and ARFs by using the second-order approximations of the isochrons. Further extensions include also the possibility of computing response curves for long (in time) stimulus rather than pulsatile stimulations.

To conclude, we stress that we have resorted to rotation numbers to give a quantitative illustration of the differences between the 1D and the 2D maps PRF maps under pulse-train stimuli. Rotation numbers turn out to be a good detector of the phase variations with respect to the underlying limit cycle, and are sufficient to make important differences between these maps evident. However, restricting to this descriptive level, we are ignoring other important intrinsic features of these maps: their attractors and the dynamics inside them. Assuming the goal was a fine forecasting of synchronization under pulse-train stimuli, the intrinsic dynamics would not give additional relevant information from a biological point of view.

From a mathematical point of view, however, we cannot just look at the rotation numbers and ignore what do they represent exactly. Indeed, we found interesting to explore the dynamics of these 2D maps that we created by means of the PRFs and ARFs. So, we decided to go forward and use known analytico-numerical methods to compute invariant curves of the PRF-ARF maps and their scenario in the $\varepsilon - \omega_s$ parameter space. This study is done in the next chapter.

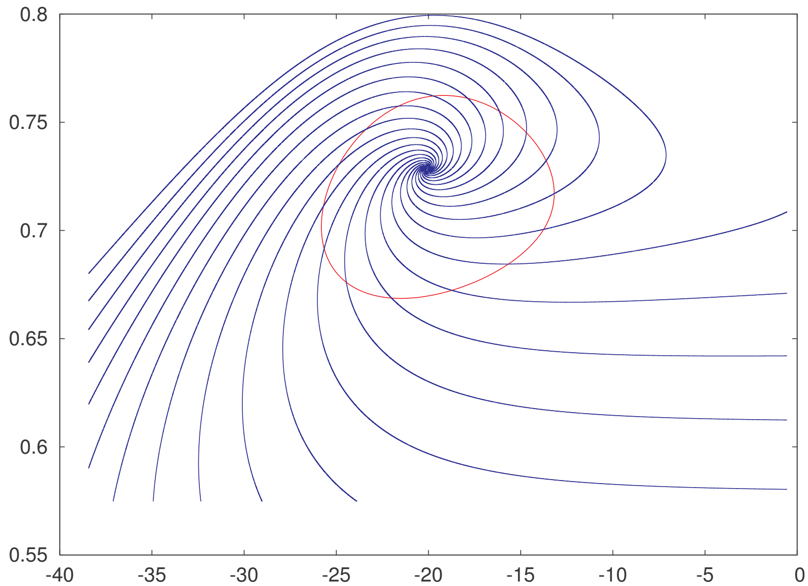


Figure 6.11: The limit cycle (red) and some of its isochrons (blue) for system (6.34) and $I_{app} = 190$.

Appendix A: the vector field for the A-curves

We prove here that given an analytic local diffeomorphism K, as in (6.5), satisfying (6.6), the A-curves are the orbits of a vector field Z, satisfying $[X, Z] = [Z, X] = 0$.

This is equivalent to prove that $DX\, Z = DZ\, X$.

Taking derivatives with respect to θ in equation (6.6), we get

$$\left(\frac{1}{T}\partial_\theta + \frac{\lambda}{T}\sigma\partial_\sigma\right)\partial_\theta K = (DX \circ K)\partial_\theta K,$$

and using (6.12), we get

$$\left(\frac{1}{T}\partial_\theta + \frac{\lambda}{T}\sigma\partial_\sigma\right)(Z \circ K) = (DX \circ K)(Z \circ K).$$

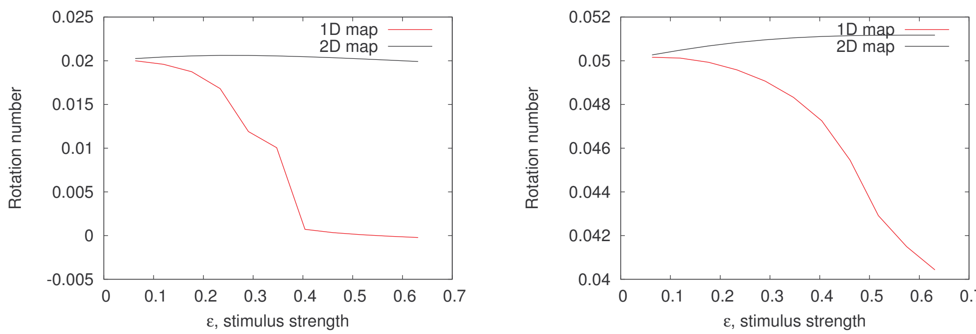
By the chain rule,

$$(DZ \circ K)\left(\frac{1}{T}\partial_\theta + \frac{\lambda}{T}\sigma\partial_\sigma\right)K \quad = \quad (DX \circ K)(Z \circ K),$$

and again, by the invariance equation (6.6), we obtain

$$(DX \circ K)(Z \circ K) = (DZ \circ K)(X \circ K). \tag{6.35}$$

as we wanted to prove.



(a) $T_s = 0.0326060 \approx T_0/50$ (b) $T_s = 0.0815149 \approx T_0/20$

Figure 6.12: Rotation numbers for different stimulus strengths and fixed stimulus periods for system (6.34).

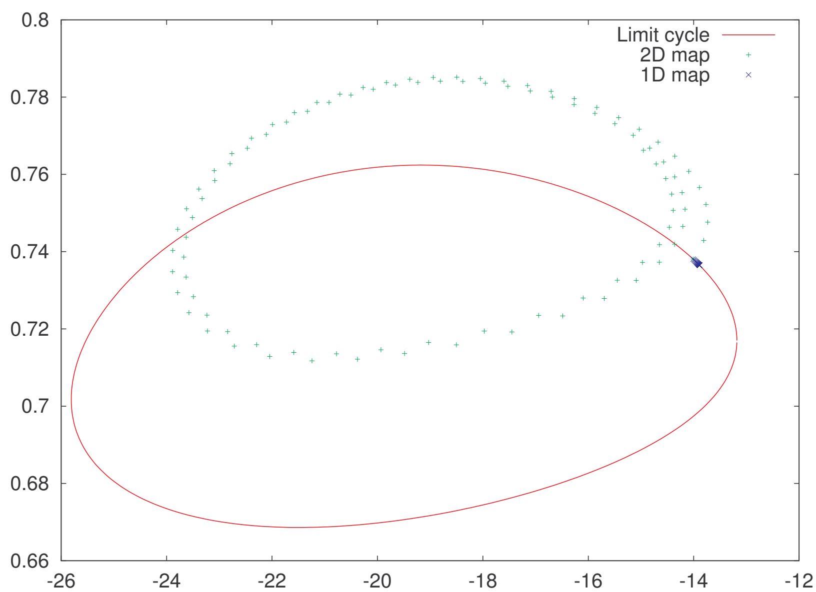


Figure 6.13: Sequences $K(\theta_n, \sigma_n)$ computed using the 2-dimensional map (6.20) and the 1-dimensional map (6.21), respectively, for system (6.34). The strength of the stimulus is $\varepsilon = 0.574604$, while the stimulation period is $T_s = 0.026111 \approx T_0/50$.

Chapter 7

A numerical insight of two-dimensional response maps

7.1 Introduction

In Chapter 6 we have studied the differences in prediction of synchronization or entrainment to an external periodic stimulus, in terms of the hyperbolicity of the limit cycles and the tilt of its isochrons. In that chapter, we have restricted our description to the behavior of the rotation numbers associated to the different ways to forecast the phase, namely: the 1D approach derived from the phase-reduction assumption (see map (6.21)), the 2D approach that we have proposed in order to take into account the transient effects (see (6.20)) and the real phase evolution. We have detected important differences between the two approximate approaches (1D and 2D) but we have not paid attention to the actual attractors underlying those dynamics. The aim of this chapter is to study numerically the attractors of these maps, and more precisely the existence of invariant curves.

As a first exploration, we present simulations of map (6.20) corresponding to the canonical model (6.25) (see Figures 7.1 and 7.2) in which we can see the evolution of the asymptotic attractors when one changes the stimulus amplitude ε or the relative period of the stimulus T_s/T_0, respectively. In the following we shall use the notation $\omega = T_s/T_0$. In both simulations we can observe that:

- the three maps seem to have an invariant curve under weak perturbations (small amplitude or large stimulus period) that breaks down to give rise to a fixed-point attractor when either ε or ω are big enough.

- this breakdown takes place at different perturbation levels according to the map considered: first, the exact map and later on, the 2D and the 1D maps. In particular, the 2D map gives a better approximation compared to the 1D map.

- both the exact map and the 2D map spiral around a focus beyond this bifurcation. Thus, the 2D map is able to predict oscillations of the phase in the transient from the invariant curve to the fixed point attractor that, obviously, the 1D map cannot show up.

- the fixed point of the 2D map gives a better approximation of the fixed point of the exact map. Thus, for big enough perturbations, both the 1D map and the 2D map predict a phase-locking, but the predicted phase of the 1D map is less accurate.

Now we would like to understand, in a more rigorous way, the causes of these differences. In particular, we are interested on:

- checking the existence of the invariant curves of the 2D map, at least for small perturbation values.

- understanding the dynamics inside these invariant curves: the simulations do not show whether the dynamics is quasi-periodic or it presents periodic orbits, for instance.

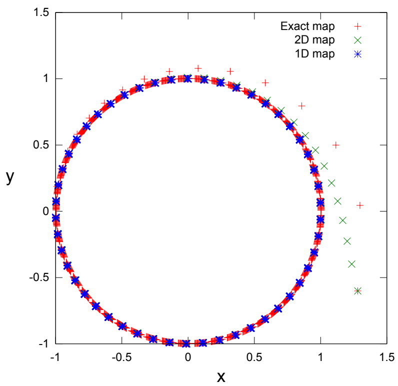
- explaining the underlying bifurcation between the invariant curve attractor regime and the fixed-point attractor regime.

In this chapter we shall implement two different numerical methods that allow us to compute the invariant curves observed in the simulations of the previous chapter using already existing methods adapted to our context. That is, the aim of this chapter is not to develop new numerical tools but to use some methods that shall allow us to understand more deeply the 2D map (6.20) introduced in Chapter 6. We take advantage of some well-known techniques and results in theory of discrete dynamical systems. In Section 7.2, we first test a technique to compute the invariant curves, see Chapter 5 [CH14] of the monograph on computation of invariant manifolds [HCF+14]. It consists of a Newton-like method to solve the invariance equation derived from the parameterization method already introduced in Chapter 6 (see [CFdlL05]). With this approach we are able to compute invariant curves for very small values of ε, but the numerical method diverges before reaching the bifurcation values observed in the simulations given in Figure 7.1. Alternatively, since our 2D map is similar in many aspects to the Arnold family of annulus diffeomorphisms studied in [BST98], we also use the same techniques to characterize the parameter regions having periodic dynamics via Arnold tongues. These techniques are based on Taylor expansions of the invariant curves and their restricted dynamics and will be studied in Section 7.3. This method also presents the shortcoming of diverging before reaching the observed bifurcation, but it allows to compute the Arnold tongues (curves where saddle-nodes of periodic orbits occur) in a simpler way, thus giving a measure of rational dynamics in the parameter space.

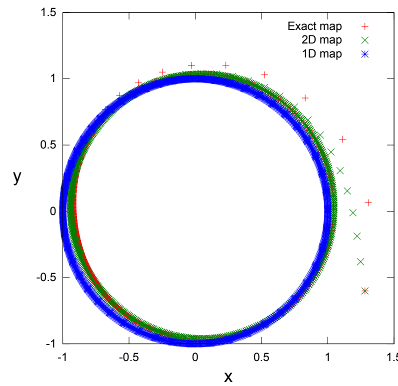
We shall denote by $F_{\varepsilon,\omega} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ the 2D map introduced in (6.20). We recall that it is defined by:

$$\begin{array}{rcl} F_{\varepsilon,\omega} : \mathbb{T} \times \mathbb{R} & \longrightarrow & \mathbb{T} \times \mathbb{R} \\ \begin{pmatrix} \theta \\ \sigma \end{pmatrix} & \longmapsto & \begin{pmatrix} \theta + \omega + \varepsilon\, PRF(\theta, \sigma) & (\bmod 1) \\ (\sigma + \varepsilon\, ARF(\theta, \sigma))\, e^{\lambda\omega} & \end{pmatrix}, \end{array} \tag{7.1}$$

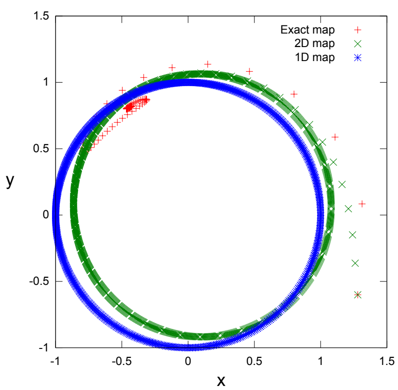
where $\lambda < 0$. We also recall that $\omega = T_s/T_0$ is the ratio between the stimulation period and the period of the underlying limit cycle of the unperturbed system.



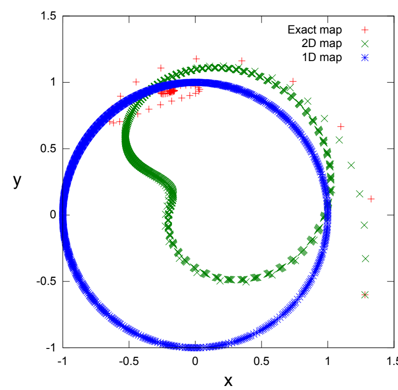
(a) $\varepsilon = 0.001$.



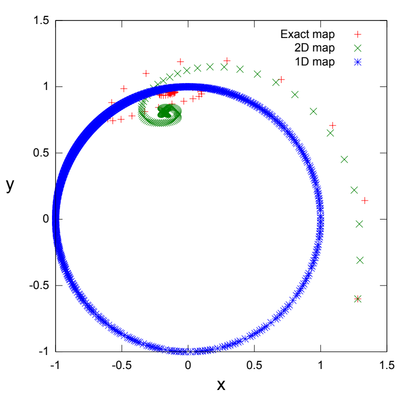
(b) $\varepsilon = 0.017$.



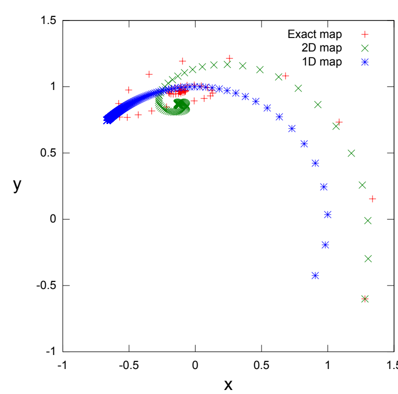
(c) $\varepsilon = 0.033$.



(d) $\varepsilon = 0.063$.

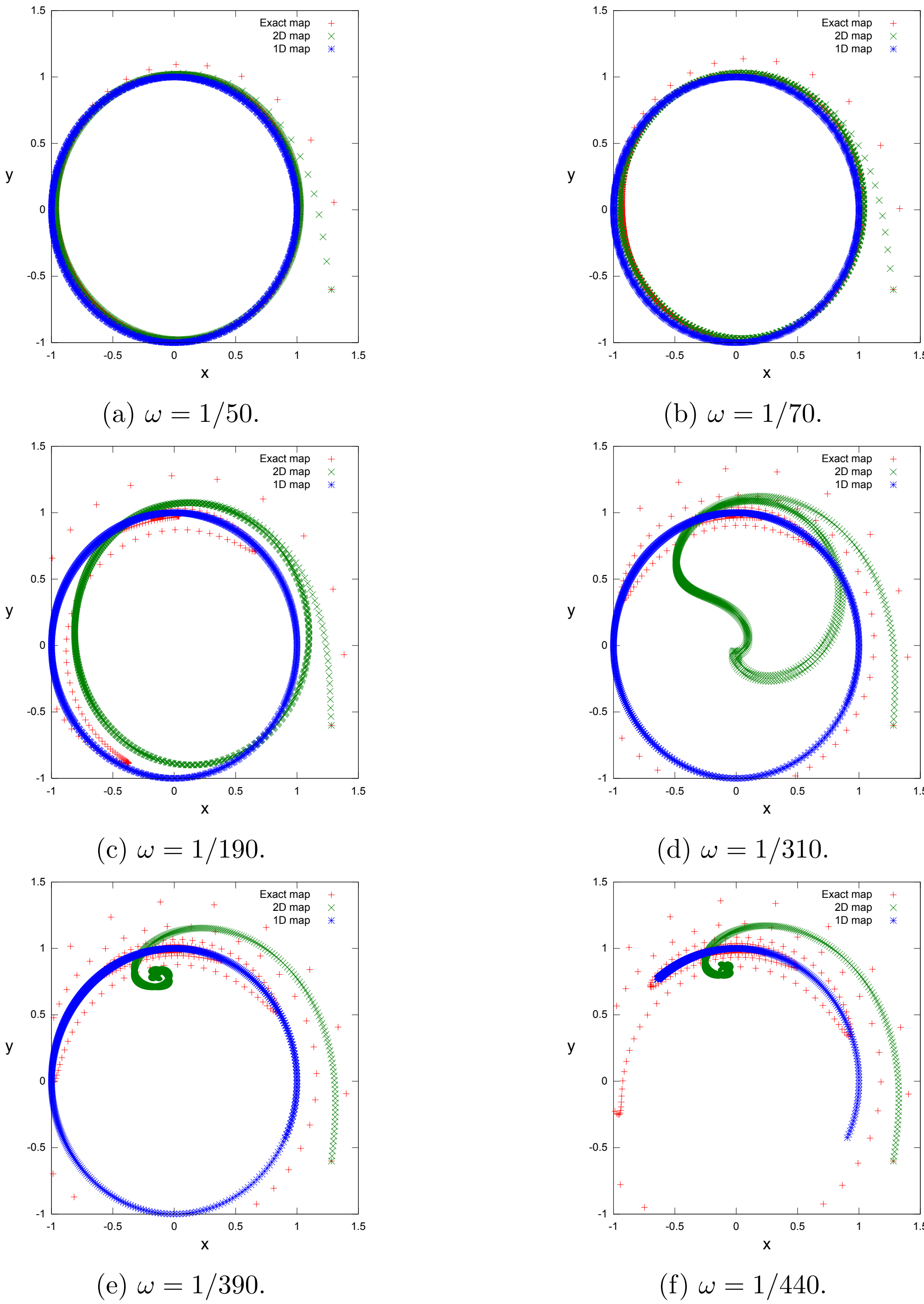


(e) $\varepsilon = 0.079$.



(f) $\varepsilon = 0.089$.

Figure 7.1: Simulations. Evolution of the asymptotic states of the exact, 1D and 2D maps for $\alpha = 5$, $a = 1$, $\omega = 1/50$ and different values of ε.



(a) $\omega = 1/50$.

(b) $\omega = 1/70$.

(c) $\omega = 1/190$.

(d) $\omega = 1/310$.

(e) $\omega = 1/390$.

(f) $\omega = 1/440$.

Figure 7.2: Simulations. Evolution of the asymptotic states of the exact, 1D and 2D maps for $\alpha = 5$, $a = 1$, $\varepsilon = 0.01$ and different values of ω.

We shall just focus on the maps obtained from the canonical model (6.25) with the idea that it can provide an insight of the main differences of the 1D and 2D maps. The PRF and ARF for this model are given in (6.29) and (6.30) respectively. Our purpose studying this particular example is to understand more deeply the 2D map, to establish more precisely the main differences between the 1D and the 2D approaches, and to make better predictions of their long-term behavior. This is a minimal model, so that we expect that this study gives insight of what can happen in more complex and realistic models in neuroscience, for which this numerical study that we shall carry on would be more cumbersome and perhaps less illustrative. Moreover, for the sake of comparison, in the results we will fix the parameters $\alpha = 5$ and $a = 1$. We expect that similar results can be obtained changing these values. However we point out that for $\alpha << 1$ (the more realistic situation, explored in Chapter 6, in which we expected a more dramatic difference with the 1D scenario) the convergence of the methods used below worsens. Finally, since from the simulations in Figures 7.1 and 7.2 we see that the decrease in the stimulus period (that is, ω) and the increase in the amplitude (i.e, ε) give rise to the same bifurcation structures, we are going to take ε as the main bifurcation parameter. Nevertheless, we will examine some properties in terms of the parameter ω.

7.2 Computation of invariant curves using a Newton-like method

In this section we begin the numerical computation of the invariant curves observed in the simulations of the previous chapter. Here we use the Newton-like method proposed in [CH14]. For the sake of self-containedness, we review the main steps of this method adapted to our problem. Since our problem is two-dimensional, there are significant simplifications compared to [CH14], where the method is presented in a setting of arbitrary dimension. However, if this method were to be applied to models of higher dimension, the structure would be basically the same.

Let ε and ω be fixed. For the sake of simplicity, in this section we shall denote simply by F the map $F_{\varepsilon,\omega} : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ defined in (7.1). Our main goal is to find a parameterization of an invariant curve, $\Gamma : \mathbb{T} \to \mathbb{T} \times \mathbb{R}$, of the map F. We note that in the special case $\varepsilon = 0$, the limit cycle of the continuous system (6.25) in (θ, σ) coordinates is an invariant curve of the map (7.1). In this case, one has $\Gamma(\theta) = (\theta, 0)$. For $\varepsilon \neq 0$, an invariant curve can be done by solving an invariance equation of the following form:

$$F(\Gamma(\theta)) = \Gamma(f(\theta)), \tag{7.2}$$

where $\Gamma(\theta)$ (the parameterization of the curve) and $f(\theta)$ (the dynamics inside the curve) are unknowns. To perform the Newton-like method, we also consider the invariant normal (stable) bundle of $\Gamma(\theta)$, denoted by $N(\theta)$, and its linearized dynamics $\Lambda^s(\theta)$. The corresponding invariance equation to $N(\theta)$ and $\Lambda^s(\theta)$ is:

$$DF(\Gamma(\theta))N(\theta) = N(f(\theta))\Lambda^s(\theta). \tag{7.3}$$

In the following we shall also denote $\Lambda(\theta) = \text{diag}(\Lambda^t(\theta), \Lambda^s(\theta))$ the linearized dynamics in both the tangent and normal bundle. Clearly, $\Lambda^t(\theta) = f'(\theta)$.

At the i-th step of the method, we compute successive approximations $\Gamma_i(\theta)$, $f_i(\theta)$, $N_i(\theta)$ and $\Lambda_i(\theta)$ of $\Gamma(\theta)$, $f(\theta)$, $N(\theta)$ and $\Lambda(\theta)$, respectively, in two substeps. In the first substep we compute $\Gamma_i(\theta)$ and $f_i(\theta)$ and in the second substep we compute $N_i(\theta)$ and $\Lambda_i(\theta)$. Let us define $R_i(\theta)$ as the error in the invariance equation of the torus (7.2) at the step i:

$$R_i(\theta) := F(\Gamma_i(\theta)) - \Gamma_i(f_i(\theta)).$$

Let $S_i^s(\theta)$ be the error in the invariance equation of the normal (stable) bundle (7.3) at the step i, that is:

$$S_i^s(\theta) := DF(\Gamma_i(\theta))N_i(\theta) - N_i(f_i(\theta))\Lambda_i^s(\theta).$$

We also define the adapted frame $P_i(\theta) = (D\Gamma_i(\theta), N_i(\theta))$. Let $S_i(\theta)$ be the error of this adapted frame at the step i:

$$S_i(\theta) := DF(\Gamma_i(\theta))P_i(\theta) - P_i(\theta)\Lambda_i(\theta).$$

One has that $S_i(\theta) = (DR_i(\theta), S_i^s(\theta))$. In the following we denote $L_i(\theta) := D\Gamma_i(\theta)$.

In the first substep, we look for $\Gamma_{i+1}(\theta)$ and $f_{i+1}(\theta)$ of the following form:

$$\Gamma_{i+1}(\theta) = \Gamma_i(\theta) + P_i(\theta)\xi_i(\theta), \tag{7.4}$$

$$f_{i+1}(\theta) = f_i(\theta) + \varphi_i(\theta), \tag{7.5}$$

where $\xi_i(\theta)$ and $\varphi_i(\theta)$ are the correction terms. To determine these correction terms, one proceeds as usual in Newton-like methods: first one substitutes expressions (7.4) and (7.5) in the invariance equation (7.2). Then one expands in Taylor series around $\Gamma_i(\theta)$ and $f_i(\theta)$ respectively, up to order two. Finally one imposes that all the terms up to order one in ξ_i and φ_i vanish, obtaining two equations for the unknowns ξ_i and φ_i. Moreover, one can see that in this case we can take $\xi_i(\theta)$ of the form:

$$\xi_i(\theta) = \begin{pmatrix} 0 \\ \xi_i^s(\theta) \end{pmatrix},$$

so that we modify the invariant curve only in the normal (stable) direction. Following this procedure, one finds that $\xi_i^s(\theta)$ is the (unique) solution of

$$\xi_i^s(\theta) = \Lambda_i^s(f_i^-(\theta))\xi_i^s(f_i^-(\theta)) + \tilde{R}_i^s(f_i^-(\theta)), \tag{7.6}$$

and $\varphi_i(\theta)$ as

$$\varphi_i(\theta) = \tilde{R}_i^t(\theta),$$

where $f_i^-(\theta)$ denotes an approximation of $f_i^{-1}(\theta)$, and

$$\tilde{R}_i(\theta) = \begin{pmatrix} \tilde{R}_i^t(\theta) \\ \tilde{R}_i^s(\theta) \end{pmatrix} := P_i^-(f_i(\theta))R_i(\theta),$$

being $P_i^-(\theta)$ an approximation of $P_i^{-1}(\theta)$. Elementary linear algebra shows that $\tilde{R}_i(\theta)$ is simply the error $R_i(\theta)$ in the basis $L_i(f_i(\theta))$, $N_i(f_i(\theta))$.

Remark 7.2.1. We point out that equation (7.6) is a fixed point equation of the form $\xi_i^s = \mathcal{F}(\xi_i^s, \theta)$. Moreover, $\mathcal{F}(\cdot, \theta)$ has Lipschitz constant $\Lambda_i^s(f_i^-(\theta)) < 1$, so that equation (7.6) has a unique fixed point indeed. Moreover, one can solve this equation by iteration: first, one takes $\xi_{i,0}^s(\theta) = \mathcal{F}(0, \theta)$. Then, for $j \geq 1$ one defines $\xi_{i,j}^s(\theta) = \mathcal{F}(\xi_{i,j-1}^s(\theta), \theta)$ and keeps iterating until the error $|\xi_{i,j}^s(\theta) - \mathcal{F}(\xi_{i,j}^s(\theta), \theta)|$ is sufficiently small.

In conclusion, after all these computations, $\Gamma_{i+1}(\theta)$ and $f_{i+1}(\theta)$ are defined as

$$\begin{aligned} \Gamma_{i+1}(\theta) &= \Gamma_i(\theta) + N_i(\theta)\xi_i^s(\theta), \\ f_{i+1}(\theta) &= f_i(\theta) + \tilde{R}_i^t(\theta). \end{aligned}$$

We finish this substep by computing an approximation $f_{i+1}^-(\theta)$ of $f_{i+1}^{-1}(\theta)$, that will be used in the next step of the method. Let

$$e_i(\theta) = f_i^-(f_{i+1}(\theta)) - \theta.$$

Then we define $f_{i+1}^-(\theta)$ as

$$f_{i+1}^-(\theta) = f_i^-(\theta) - e_i(f_i^-(\theta)).$$

This corresponds to one step of Newton's method for the equation

$$f_{i+1}^- \circ f_{i+1}(\theta) - \theta = 0.$$

In the second substep, we shall use $K_{i+1}(\theta)$, $f_{i+1}(\theta)$ and $f_{i+1}^-(\theta)$ for the computation of $N_{i+1}(\theta)$ and $\Lambda_{i+1}^s(\theta)$. Again, we look for $N_{i+1}(\theta)$ and $\Lambda_{i+1}^s(\theta)$ of the following form:

$$N_{i+1}(\theta) = N_i(\theta) + P_i(\theta)Q_i^s(\theta), \tag{7.7}$$
$$\Lambda_{i+1}^s(\theta) = \Lambda_i^s(\theta) + \Delta_i^s(\theta), \tag{7.8}$$

where $Q_i^s(\theta)$ and $\Delta_i(\theta)$ are the correction terms still to be determined. Analogously as in the previous substep, we substitute expressions (7.7) and (7.8) in the invariance equation (7.3), now taking of course $K_{i+1}(\theta)$ and $f_{i+1}(\theta)$. We note that equation (7.3) is linear with respect to $N(\theta)$ and $\Lambda^s(\theta)$, so that we can easily find equations for $Q_i^s(\theta)$ and $\Lambda_i^s(\theta)$ in order that (7.3) vanishes. Similarly as in the previous substep, one can choose $Q_i^s(\theta)$ of the form:

$$Q_i^s(\theta) = \begin{pmatrix} Q_i^{ts}(\theta) \\ 0 \end{pmatrix},$$

that is, we correct the normal bundle in its complementary direction $L_i(\theta)$. Then one obtains that:

$$Q_i^{ts}(\theta) = (Q_i^{ts}(f_{i+1}^{-}(\theta))\Lambda_i^s(\theta) - \tilde{S}_i^{ts}(\theta))(\Lambda_i^t(\theta))^{-1}, \tag{7.9}$$

and:

$$\Delta_i^s(\theta) = \tilde{S}_i^{ss}(\theta),$$

where:

$$\tilde{S}_i^s(\theta) = \begin{pmatrix} \tilde{S}_i^{ts}(\theta) \\ \tilde{S}_i^{ss}(\theta) \end{pmatrix} := P_i^{-}(f_{i+1}(\theta))S_i^s(\theta).$$

We point out that, analogously as $\tilde{P}_i(\theta)$, $\tilde{S}_i^s(\theta)$ is the error of the normal bundle $S_i^s(\theta)$ in the basis $L_i(f_i(\theta))$, $N_i(f_i(\theta))$. Again, equation (7.9) can be solved with the procedure described in Remark 7.2.1. After that, we define $N_{i+1}(\theta)$ and $\Lambda_{i+1}(\theta)$ as:

$$\begin{aligned} N_{i+1}(\theta) &= N_i(\theta) + L_i(\theta)Q_i^{ts}(\theta), \\ \Lambda_{i+1}^s(\theta) &= \Lambda_i^s(\theta) + \tilde{S}_{i+1}^{ss}(\theta), \\ \Lambda_{i+1}^t(\theta) &= f'_{i+1}(\theta). \end{aligned}$$

To finish, we compute the approximation $P_{i+1}^{-}(\theta)$ of $P_{i+1}^{-1}(\theta)$ which shall be used in the next iteration of the method. Let:

$$E_i(\theta) = P_i^{-}(\theta)P_{i+1}(\theta) - \mathrm{Id}.$$

Then we define $P_{i+1}^{-}(\theta)$ as:

$$P_{i+1}^{-}(\theta) = P_i^{-}(\theta) - E_i(\theta)P_i(\theta).$$

Again, this corresponds to one step of Newton's method for the equation:

$$P_{i+1}^{-}(\theta)P_{i+1}(\theta) - \mathrm{Id} = 0.$$

7.2.1 Choosing the initial seeds

In this subsection we indicate how to choose initial seeds for the Newton method, as proposed in [CH14]. We point out that we are in a perturbative setting, that is the map F depends on a parameter ε, so that one can take advantage of it.

Indeed, for an initial value $\varepsilon = \varepsilon_0$, that we assume to be sufficiently small, we can take the initial seeds $\Gamma_0(\theta)$, $f_0(\theta)$, $N_0(\theta)$ and $\Lambda_0(\theta)$ (and also $P_0(\theta)$, $P_0^{-}(\theta)$ and $f_0^{-}(\theta)$) simply as the corresponding objects for $\varepsilon = 0$. In our setting, one has:

$$\begin{aligned}
\Gamma_0(\theta) &= \begin{pmatrix} \theta \\ 0 \end{pmatrix}, \\
f_0(\theta) &= \theta + \omega, \quad f_0^-(\theta) = \theta - \omega, \\
L_0(\theta) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad N_0(\theta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
P_0(\theta) &= P_0^-(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\Lambda_0(\theta) &= \begin{pmatrix} \Lambda_0^t(\theta) & 0 \\ 0 & \Lambda_0^s(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{\lambda\omega} \end{pmatrix}.
\end{aligned}$$

For $\varepsilon > \varepsilon_0$, one can perform a continuation method to find good initial seeds for successive values of ε. In [CH14], the authors propose to perform a continuation method just for the parameterization of the torus, $\Gamma(\theta)$, and its internal dynamics $f(\theta)$, and omit the normal bundle $N(\theta)$ and the linearized dynamics $\Lambda(\theta)$. We now describe this continuation method.

Assume that for a given ε we have good approximations $\Gamma^\varepsilon(\theta)$ and $f^\varepsilon(\theta)$ of $\Gamma(\theta)$, $f(\theta)$ respectively. Then, we define the initial seeds of the Newton method for the parameter $\varepsilon + h$ as:

$$\Gamma_0^{\varepsilon+h}(\theta) = \Gamma^\varepsilon(\theta) + \frac{\partial \Gamma^\varepsilon}{\partial \varepsilon}(\theta) h,$$

$$f_0^{\varepsilon+h}(\theta) = f^\varepsilon(\theta) + \frac{\partial f^\varepsilon}{\partial \varepsilon}(\theta) h.$$

One can obtain the following invariance equation for $\frac{\partial \Gamma^\varepsilon}{\partial \varepsilon}$ and $\frac{\partial f^\varepsilon}{\partial \varepsilon}$ just by differentiating (7.2) with respect to ε:

$$DF(\Gamma^\varepsilon(\theta)) \frac{\partial \Gamma^\varepsilon}{\partial \varepsilon}(\theta) = D\Gamma^\varepsilon(f^\varepsilon(\theta)) \frac{\partial f^\varepsilon}{\partial \varepsilon}(\theta) + \frac{\partial \Gamma^\varepsilon}{\partial \varepsilon}(\theta). \tag{7.10}$$

Now, writing $\frac{\partial \Gamma^\varepsilon}{\partial \varepsilon}$ and $\frac{\partial f^\varepsilon}{\partial \varepsilon}$ in the basis $L^\varepsilon(\theta)$, $N^\varepsilon(\theta)$ we obtain:

$$\frac{\partial \Gamma^\varepsilon}{\partial \varepsilon}(\theta) = P^\varepsilon(\theta) \xi^\varepsilon(\theta),$$

$$\frac{\partial f^\varepsilon}{\partial \varepsilon}(\theta) = \varphi^\varepsilon(\theta),$$

for some unknowns $\xi^\varepsilon(\theta)$ and $\varphi^\varepsilon(\theta)$. Again, $\xi^\varepsilon(\theta)$ can be taken of the form:

$$\xi^\varepsilon(\theta) = \begin{pmatrix} 0 \\ \xi^{s,\varepsilon}(\theta) \end{pmatrix},$$

so that we just correct the torus in the normal direction. Let:

$$R^\varepsilon(\theta) = \frac{\partial F}{\partial \varepsilon}(\Gamma^\varepsilon(\theta)), \qquad \tilde{R}^\varepsilon(\theta) = (P^\varepsilon(f^\varepsilon(\theta)))^{-1} R^\varepsilon(\theta).$$

Then, performing the standard computations of Newton's method, one obtains the following identities for $\xi^{s,\varepsilon}$ and φ^ε:

$$\begin{aligned} \xi^{s,\varepsilon}(\theta) &= \Lambda^{s,\varepsilon}(f_i^{-,\varepsilon}(\theta))\xi^{s,\varepsilon}(f_i^{-,\varepsilon}(\theta)) + \tilde{R}^{s,\varepsilon}(f_i^{-,\varepsilon}(\theta)), \\ \varphi^\varepsilon(\theta) &= \tilde{R}^{t,\varepsilon}(\theta). \end{aligned}$$

As above, the equation for $\xi^{s,\varepsilon}$ has a unique solution that can be found with the method described in Remark 7.2.1. In conclusion, after finding the corrections, we take the initial seeds $\Gamma_0^{\varepsilon+h}(\theta)$ and $f_0^{\varepsilon+h}(\theta)$ as:

$$\Gamma_0^{\varepsilon+h}(\theta) = \Gamma^\varepsilon(\theta) + N^\varepsilon(\theta)\xi^{s,\varepsilon}(\theta)h,$$

$$f_0^{\varepsilon+h}(\theta) = f^\varepsilon(\theta) + \tilde{R}^{t,\varepsilon}(\theta)h,$$

and then we proceed again with the Newton-like method described above.

7.3 Computation of invariant curves using Taylor series

In this section we implement a different method to compute the invariant curves and the dynamics inside them using automatic differentiation tools. This will allow us to compute more easily the so-called Arnold tongues. For the theoretical background of this section we follow the ideas found in [BST98]. The implementation of the numerical methods are based on [Har08] and [Sim90]. See also [JZ05].

Let $p/q \in \mathbb{Q}$. We say that $(\omega, \varepsilon) \in T_{p,q}$ if and only if there exist $(\theta^*, \sigma^*) \in \mathbb{T} \times \mathbb{R}$ such that $F^q_{\varepsilon,\omega}(\theta^*, \sigma^*) = (\theta^* + 2\pi p, \sigma^*)$. We call $T_{p,q}$ the Arnold tongue of rotation number p/q. The boundaries of the Arnold Tongues are saddle-node bifurcation points of the function $F^q_{\varepsilon,\omega}(\theta, \sigma)$, and they can be parameterized in the plane (ω, ε) by some curves $\omega = \omega(\varepsilon)$ such that $\omega(0) = p/q$.

Let us fix $\omega = p/q \in \mathbb{Q}$. To find the invariant curve of the map $F_{\varepsilon,\omega}$ defined in (7.1) and the dynamics inside it, one can proceed as follows. If one looks for invariant curves of the form $\sigma = g(\theta, \varepsilon)$ and denotes the dynamics inside these invariant curves by $h(\theta, \varepsilon)$, these functions are defined implicitly using equations (7.1). Indeed, h and g must satisfy the following invariance equations:

$$h(\theta, \varepsilon) = \theta + \omega + \varepsilon\, PRF(\theta, g(\theta, \varepsilon)). \tag{7.11}$$

and:

$$g(h(\theta,\varepsilon)) = (g(\theta,\varepsilon) + \varepsilon\, ARF(\theta, g(\theta,\varepsilon)))\, e^{\lambda\omega}. \tag{7.12}$$

Moreover, if one expands h and g in orders of ε, that is:

$$h(\theta,\varepsilon) = \sum_{n\geq 0} h_n(\theta)\varepsilon^n, \qquad g(\theta,\varepsilon) = \sum_{n\geq 0} g_n(\theta)\varepsilon^n,$$

one can use equations (7.11) and (7.12) to find numerically the orders $h_n(\theta)$, $g_n(\theta)$, $n = 0,\dots,N$ of these expansions.

Additionally, if besides computing the invariant curves one wants to compute also the boundaries of a given Arnold tongue, one has to introduce a new parameter δ. The boundaries of the Arnold tongue of rotation number $\omega = p/q$ will be given by $\omega = p/q+\delta$. Of course, $\delta = \delta(\varepsilon)$. We observe that for each ε, δ is not unique. More precisely, there exist two different values of valid δ, which will give the two different boundaries of the Arnold tongue.

Then one obtains the following invariance equations:

$$h(\theta,\varepsilon,\delta) = \theta + \omega + \delta + \varepsilon\, PRF(\theta, g(\theta,\varepsilon,\delta)). \tag{7.13}$$

and:

$$g(h(\theta,\varepsilon,\delta),\varepsilon,\delta) = (g(\theta,\varepsilon,\delta) + \varepsilon\, ARF(\theta, g(\theta,\varepsilon,\delta)))\, e^{\lambda(\omega+\delta)}. \tag{7.14}$$

7.3.1 The invariance equations

As we mentioned above, to solve the invariance equations (7.13) and (7.14) we expand h and g in orders of ε and δ:

$$h(\theta,\varepsilon,\delta) = \sum_{j,k\geq 0} h_{jk}(\theta)\varepsilon^j\delta^k, \qquad g(\theta,\varepsilon,\delta) = \sum_{j,k\geq 0} g_{jk}(\theta)\varepsilon^j\delta^k,$$

and then these equations write out as:

$$\sum_{j,k\geq 0} h_{jk}(\theta)\varepsilon^j\delta^k = \theta + \omega + \delta + \varepsilon\, PRF\left(\theta, \sum_{j,k\geq 0} g_{jk}(\theta)\varepsilon^j\delta^k\right). \tag{7.15}$$

and:

$$\begin{aligned}&\sum_{j,k\geq 0} g_{jk}\left(\sum_{m,n\geq 0} h_{mn}(\theta)\varepsilon^m\delta^n\right)\varepsilon^j\delta^k \\ &= \left(\sum_{j,k\geq 0} g_{jk}(\theta)\varepsilon^j\delta^k + \varepsilon\, ARF\left(\theta, \sum_{j,k\geq 0} g_{jk}(\theta)\varepsilon^j\delta^k\right)\right) e^{\lambda(\omega+\delta)}.\end{aligned} \tag{7.16}$$

The idea is to solve equations (7.15) and (7.16) numerically order by order, which can be done using automatic differentiation tools. More precisely, first we solve the term of order zero of both equations analytically, which can be done easily. After that, we also solve analytically the terms of independent of ε. Finally, the higher order terms can be solved numerically once the previous ones are known, so that one just needs to proceed inductively to obtain the subsequent orders. We shall now explain with more detail how this can be done.

As we mentioned above, the first step consists in solving the independent terms of both invariance equations. On the one hand, equating the terms that are independent of ε and δ in both sides of equation (7.15) one obtains readily:

$$h_{00}(\theta) = \theta + \omega.$$

On the other hand, doing the same in equation (7.16) and taking into account that $h_{00}(\theta) = \theta + \omega$, we obtain the following equation:

$$g_{00}(\theta + \omega) = g_{00}(\theta)e^{\lambda\omega}. \tag{7.17}$$

Equation (7.17) can be solved writing both sides in Fourier series and equating the Fourier coefficients. More precisely, if we write:

$$g_{00}(\theta) = \sum_{l\in\mathbb{Z}} g_{00}^l e^{2\pi i l\theta},$$

then equation (7.17) yields the following equation for each $l \in \mathbb{Z}$:

$$g_{00}^l \left(e^{2\pi i l\omega} - e^{\lambda\omega}\right) = 0.$$

Since $\lambda \in \mathbb{R} \setminus \{0\}$, clearly $e^{2\pi i l\omega} - e^{\lambda\omega} \neq 0$ for all $l \in \mathbb{Z}$, so that one obtains straightforwardly:

$$g_{00}^l = 0 \quad \text{for all } l \in \mathbb{Z}, \qquad g_{00}(\theta) = 0.$$

Now we turn to the case h_{0k} and g_{0k} with $k \geq 1$. From equation (7.15) it is straightforward to see that:

$$h_{01}(\theta) \equiv 1,$$

and that for $k \geq 2$:

$$h_{0k}(\theta) \equiv 0.$$

To compute $g_{0k}(\theta)$ we proceed in a different way. First we observe that from the previous computations we know that $h(\theta, 0, \delta) = \theta + \omega + \delta$. Hence, setting $\varepsilon = 0$ in equation (7.14), we obtain:

$$g(\theta + \omega + \delta, 0, \delta) = g(\theta, 0, \delta)e^{\lambda(\omega+\delta)}. \tag{7.18}$$

Writing $g(\theta,\varepsilon,\delta)$ in Fourier series:

$$g(\theta,\varepsilon,\delta)=\sum_{l\in\mathbb{Z}} g^l(\varepsilon,\delta)e^{2\pi i l\theta},$$

equation (7.18) yields for each $l\in\mathbb{Z}$:

$$g^l(0,\delta)\left(e^{2\pi i l(\omega+\delta)}-e^{\lambda(\omega+\delta)}\right)=0.$$

Again, since $\lambda\neq 0$, we have $e^{2\pi i l(\omega+\delta)}-e^{\lambda(\omega+\delta)}\neq 0$, so that $g^l(0,\delta)=0$ for all $l\in\mathbb{Z}$. Consequently, one has that $g(\theta,0,\delta)=0$ for all δ, which implies that for all $k\geq 1$:

$$g_{0k}(\theta)=0.$$

Finally we consider the higher order terms of equations (7.15) and (7.16). In the following, for a series $f(\varepsilon,\delta)=\sum_{m,n\geq 0} f_{mn}\varepsilon^m\delta^n$ we shall denote $[f(\varepsilon,\delta)]_{j,k}:=f_{jk}$. As we mentioned above, for j,k such that $j+k\geq 1$ and $j\geq 1$, one can proceed inductively. Assume one has already computed h_{jk}, g_{jk} with $j+k\leq N$ for some $N\geq 0$, and one wants to compute h_{jk} and g_{jk} with $j+k=N+1$. We note that h_{jk} is simply given by:

$$h_{jk}(\theta)=\left[\theta+\omega+\delta+\varepsilon\, PRF\left(\theta,\sum_{m,n\geq 0} g_{mn}(\theta)\varepsilon^m\delta^n\right)\right]_{j,k}.$$

For $j\geq 1$, $k\geq 0$ we have:

$$h_{jk}(\theta)=\left[PRF\left(\theta,\sum_{m,n\geq 0} g_{mn}(\theta)\varepsilon^m\delta^n\right)\right]_{j-1,k}. \tag{7.19}$$

We note that the right-hand side of (7.19) depends only on g_{mn} with $0\leq m\leq j-1$ and $0\leq n\leq k$, so that $m+n\leq j+k-1=N$, and thus they are already known.

Now assume that one has already computed g_{jk} with $j+k\leq N$ and h_{jk} with $j+k\leq N+1$ and we want to compute g_{jk} with $j+k=N+1$. From (7.16) one can easily see that g_{jk} must satisfy:

$$g_{jk}(\theta+\omega)-g_{jk}(\theta)e^{\lambda\omega}=R_{jk}(\theta), \tag{7.20}$$

where R_{jk} is defined as:

$$\begin{aligned} R_{jk}(\theta) \;=\; & -\left[\sum_{m,n\geq 0} g_{mn}\left(\sum_{r,s\geq 0} h_{rs}(\theta)\varepsilon^r\delta^s\right)\varepsilon^m\delta^n-g_{jk}(\theta+\omega)\varepsilon^j\delta^k\right]_{j,k}\\ & +\left[\left(\sum_{m,n\geq 0} g_{mn}(\theta)\varepsilon^m\delta^n\right)e^{\lambda(\omega+\delta)}-g_{jk}(\theta)\varepsilon^j\delta^k e^{\lambda\omega}\right]_{j,k}\\ & +\left[\varepsilon ARF\left(\theta,\sum_{m,n\geq 0} g_{mn}(\theta)\varepsilon^m\delta^n\right)e^{\lambda(\omega+\delta)}\right]_{j,k}. \end{aligned}$$

It is easy to see that in fact:

$$\begin{aligned} R_{jk}(\theta) &= \left[- \sum_{\substack{0\le m\le j\\ 0\le n\le k\\ m+n\le j+k-1}} g_{mn} \left(\sum_{\substack{0\le r\le j\\ 0\le s\le k\\ r+s\le j+k}} h_{rs}(\theta)\varepsilon^r\delta^s \right) \varepsilon^m\delta^n \right]_{j,k} \\ &\quad + \left[\sum_{\substack{0\le m\le j\\ 0\le n\le k\\ m+n\le j+k-1}} g_{mn}(\theta)\varepsilon^m\delta^n\, e^{\lambda(\omega+\delta)} \right]_{j,k} \\ &\quad + \left[ARF\left(\theta, \sum_{\substack{0\le m\le j-1\\ 0\le n\le k\\ m+n\le j+k-1}} g_{mn}(\theta)\varepsilon^m\delta^n \right) e^{\lambda(\omega+\delta)} \right]_{j-1,k} \end{aligned}$$

Note that R_{jk} depends on g_{mn} with $0 \le m+n \le j+k-1 = N$ and h_{mn} with $0 \le m+n \le j+k = N+1$ and therefore it is known. We point out that, in fact, in our setting R_{jk} does *not* depend on h_{mn} with $m+n = N+1$ since $g_{00}(\theta) = 0$. In order to solve equation (7.20) one can use Fourier series again. Indeed, if we write:

$$g_{jk}(\theta) = \sum_{l\in\mathbb{Z}} g_{jk}^l e^{2\pi i l\theta}, \qquad R_{jk}(\theta) = \sum_{l\in\mathbb{Z}} R_{jk}^l e^{2\pi i l\theta},$$

one can easily see that:

$$g_{jk}^l = \frac{R_{jk}^l}{e^{2\pi i l\omega} - e^{\lambda\omega}}.$$

We note that the denominator is always nonzero since $\lambda \neq 0$.

7.3.2 Implementation of the method

In this section we give some details of how we implemented the method described in Section 7.3.1. The main tool is computing series obtained by operating with two other series (adding, multiplying, etc.) and composing a given series with elementary functions (such as the exponential, sine, cosine, ...). This can be done numerically using automatic differentiation tools, see for instance [Har08] and [JZ05].

However, as one can see in equation (7.16), in our case one must also compute the series of the composition $g(h(\theta, \varepsilon, \delta), \varepsilon, \delta)$. We stress that $g(\theta, \varepsilon, \delta)$ is not known explicitly.

In order to find the series of $g(h(\theta,\varepsilon,\delta),\varepsilon,\delta)$, we proceed as follows. Assume we have computed $g_{jk}(\theta)$, $0 \le j+k \le N$ for some N. Let us define:

$$g_N(\theta,\varepsilon,\delta) = \sum_{0\le j+k\le N} g_{jk}(\theta)\varepsilon^j\delta^k.$$

Since g_N is periodic with respect to θ, we can also write it in its real Fourier series:

$$\begin{aligned} g_N(\theta,\varepsilon,\delta) &= \sum_{l\ge 0} \hat{g}^l_N(\varepsilon,\delta)\cos(2\pi l\theta) + \bar{g}^l_N(\varepsilon,\delta)\sin(2\pi l\theta) \qquad (7.21)\\ &= \sum_{l\ge 0}\sum_{0\le j+k\le N} \left(\hat{g}^l_{j,k}\cos(2\pi l\theta) + \bar{g}^l_{j,k}\sin(2\pi l\theta)\right)\varepsilon^j\delta^k. \end{aligned}$$

To find the Fourier coefficients $\hat{g}^l_{j,k}$ and $\bar{g}^l_{j,k}$ numerically, we compute the values of the function $g_{j,k}(\theta)$ for a discretization $\theta_0,\dots,\theta_n$, and then we use the Fast Fourier Transform (FFT). In the examples below we take $n = 1024$. To compute the FFT we have used the `fftw3` library (see http://www.fftw.org/).

In the numerical implementation, expansions (7.21) must be truncated at a maximum Fourier index l. We choose this maximum Fourier index such that the tails of the Fourier expansion are small relatively to the order. More precisely, we fix two constants E_L and χ and then for each $j,k\ge 0$ we choose $l_{\max} = l_{\max}(j,k)$ such that:

$$g_{j,k}(\theta) = \sum_{l=\lfloor 0.9 l_{\max}\rfloor}^{l_{\max}} \left(|\hat{g}^l_{j,k}| + |\bar{g}^l_{j,k}|\right) < \frac{E_L}{\chi^{j+k}}.$$

We take $\chi < 1$ so that as the order $j+k$ increases a larger error is tolerated, since for small values of ε the contributions due to the terms $g_{j,k}(\theta)$ will be less significant. In the computations shown here we take $E_L = 10^{-10}$ and $\chi = 0.9$. In the following we denote:

$$L = \max_{0\le j+k\le N} l_{\max}(j,k).$$

Following the convention that $\bar{g}^l_{j,k} = \hat{g}^l_{j,k} = 0$ if $l > l_{\max}(j,k)$, equation (7.21) writes out as:

$$\begin{aligned} g_N(\theta,\varepsilon,\delta) &= \sum_{l=0}^{L} \hat{g}^l_N(\varepsilon,\delta)\cos(2\pi l\theta) + \bar{g}^l_N(\varepsilon,\delta)\sin(2\pi l\theta)\\ &= \sum_{l=0}^{L}\sum_{0\le j+k\le N} \left(\hat{g}^l_{j,k}\cos(2\pi l\theta) + \bar{g}^l_{j,k}\sin(2\pi l\theta)\right)\varepsilon^j\delta^k. \end{aligned}$$

Now, after computing the (truncated) Fourier series of g_N, we can write:

$$g_N(h(\theta,\varepsilon,\delta),\varepsilon,\delta) = \sum_{l=0}^{L} \hat{g}^l_N(\varepsilon,\delta)\cos(2\pi l h(\theta,\varepsilon,\delta)) + \bar{g}^l_N(\varepsilon,\delta)\sin(2\pi l h(\theta,\varepsilon,\delta)).$$

We can compute the series of the cosine and sine with methods of automatic differentiation:

$$\begin{aligned} \cos(2\pi l h(\theta,\varepsilon,\delta)) &=: \; c^l(\theta,\varepsilon,\delta) = \sum_{j,k\geq 0} c^l_{j,k}(\theta)\varepsilon^j\delta^k, \\ \sin(2\pi l h(\theta,\varepsilon,\delta)) &=: \; s^l(\theta,\varepsilon,\delta) = \sum_{j,k\geq 0} s^l_{j,k}(\theta)\varepsilon^j\delta^k. \end{aligned} \tag{7.22}$$

Finally, we just need to compute the series of the following products for each l, which can be done again using methods of automatic differentiation:

$$\hat{g}^l_N(\varepsilon,\delta)c^l(\theta,\varepsilon,\delta) = a^l(\theta,\varepsilon,\delta) =: \sum_{j,k\geq 0} a^l_{j,k}(\theta)\varepsilon^j\delta^k,$$

$$\bar{g}^l_N(\varepsilon,\delta)s^l(\theta,\varepsilon,\delta) = b^l(\theta,\varepsilon,\delta) =: \sum_{j,k\geq 0} b^l_{j,k}(\theta)\varepsilon^j\delta^k.$$

The series of $g_N(h(\theta,\varepsilon,\delta),\varepsilon,\delta)$ is then given by:

$$g_N(h(\theta,\varepsilon,\delta),\varepsilon,\delta) = \sum_{l=0}^{L} a^l(\theta,\varepsilon,\delta) + b^l(\theta,\varepsilon,\delta) = \sum_{j,k\geq 0}\left(\sum_{l=0}^{L} a^l_{j,k}(\theta) + b^l_{j,k}(\theta)\right)\varepsilon^j\delta^k.$$

Of course, all the terms in this sum with $j+k>N$ are no taken into account, since they will be modified when computing the series of $g_{N+1}(h(\theta,\varepsilon,\delta),\varepsilon,\delta)$, and so on.

Remark 7.3.1. In the practical implementation, we choose L to be at most 25. The reason is that the error in the the c^l and s^l series (7.22) increases with l. To decrease this error, one needs to compute more orders of these expansions, that is, to increase N. This, in its turn, increases the maximum L needed to control the error of the Fourier expansions (7.21), ending in what seems a vicious circle.

Finally, we point out that one also has to compute the series of $PRF(\theta, g(\theta,\varepsilon,\delta))$ and $ARF(\theta, g(\theta,\varepsilon,\delta))$, see the invariance equations (7.15) and (7.16). In the example below, we know these functions explicitly, so that they can be computed using automatic differentiation tools. However, in realistic models, the PRF and ARF are computed numerically as seen in Chapter 6. In this case, one has these functions expressed as Fourier-Taylor series:

$$PRF(\theta,\sigma) = \sum_{n=0}^{n_{\max}} PRF_n(\theta)\sigma^n, \qquad ARF(\theta,\sigma) = \sum_{n=0}^{n_{\max}} ARF_n(\theta)\sigma^n.$$

One just needs to compute the series of $(g(\theta,\varepsilon,\delta))^n$ for $n=0,\ldots,n_{\max}$, which can be done with standard methods, and then one can easily find the series of $PRF(\theta, g(\theta,\varepsilon,\delta))$ and $ARF(\theta, g(\theta,\varepsilon,\delta))$.

7.4 Computation of Arnold tongues

Once we have computed the series $g(\theta,\varepsilon,\delta)$ (the parameterization of the invariant curve) and $h(\theta,\varepsilon,\delta)$ (the dynamics inside it), we can proceed to look for the Arnold tongue of rotation number p/q. To that aim, we consider the function:

$$\mathcal{F}_{p/q}(\theta,\varepsilon,\delta) = (h^q(\theta,\varepsilon,\delta) - \theta - p, \partial_\theta(h^q(\theta,\varepsilon,\delta)) - 1).$$

Given a fixed ε, we look for $(\theta,\delta) = (\theta(\varepsilon),\delta(\varepsilon))$ such that $\mathcal{F}_{p/q}(\theta(\varepsilon),\varepsilon,\delta(\varepsilon)) = 0$, which ensures that $(\theta(\varepsilon), g(\theta(\varepsilon))$ is a saddle-node bifurcation point of the function $F_{\varepsilon,\frac{p}{q}+\delta(\varepsilon)}$ defined in (7.1). This equation can be solved using Newton's method. We point out that, having computed the coefficients of the series $h(\theta,\varepsilon,\delta)$, the computation of the derivative of h with respect to δ is trivial. To compute its derivative with respect to θ we use the FFT algorithm to compute its Fourier coefficients, and then the derivative is easily obtained. Again, for the FFT algorithm we use a discretization of the function at $n = 1024$ points.

After computing $(\theta(\varepsilon),\delta(\varepsilon))$, we change ε by some small amount $\Delta\varepsilon$ and follow a continuation method to obtain a good initial approximation of $(\theta(\varepsilon+\Delta\varepsilon),\delta(\varepsilon+\Delta\varepsilon))$. This is done again using one step of Newton's method. Then we start again the procedure described above to find it with the desired accuracy.

We increase ε up to some maximum value $\varepsilon_{\max}$ so that the invariance equations (7.13) and (7.14) are satisfied up to some error E_{inv}. That is, we choose $\varepsilon_{\max}$ to be the maximum value of ε such that for all $\varepsilon \le \varepsilon_{\max}$:

$$\sup_{\theta\in[0,1)} |h(\theta,\varepsilon,\delta(\varepsilon)) - \theta + \omega + \delta(\varepsilon) + \varepsilon\, PRF(\theta, g(\theta,\varepsilon,\delta(\varepsilon)))| < E_{\text{inv}}.$$

and:

$$\sup_{\theta\in[0,1)} \left| g(h(\theta,\varepsilon,\delta(\varepsilon)),\varepsilon,\delta(\varepsilon)) - [g(\theta,\varepsilon,\delta(\varepsilon)) + \varepsilon\, ARF(\theta, g(\theta,\varepsilon,\delta(\varepsilon)))]\; e^{\lambda(\omega+\delta(\varepsilon))} \right| < E_{\text{inv}}.$$

In the computations presented here we take $E_{\text{inv}} = 10^{-10}$.

For $\varepsilon > \varepsilon_{\max}$ we can continue the Arnold tongues just by looking for a saddle-node bifurcation point of the 2D-map $F^q_{\varepsilon,p/q+\delta}(\theta,\sigma)$ defined in (7.1). That is, given a certain $\varepsilon > \varepsilon_{\max}$, we look for a point $(\theta(\varepsilon),\sigma(\varepsilon),\delta(\varepsilon))$ such that:

$$F^q_{\varepsilon,p/q+\delta(\varepsilon)}(\theta(\varepsilon),\sigma(\varepsilon)) = 0,$$
$$\det\left(DF^q_{\varepsilon,p/q+\delta(\varepsilon)}(\theta(\varepsilon),\sigma(\varepsilon)) - \text{Id}\right) = 0.$$

We perform a Newton method to obtain such a point $(\theta(\varepsilon),\sigma(\varepsilon),\delta(\varepsilon))$, taking the seed $(\theta(\varepsilon_{\max}),\sigma(\varepsilon_{\max}),\delta(\varepsilon_{\max}))$. We point out that for $\varepsilon > \varepsilon_{\max}$ we cannot ensure that the points $(\theta(\varepsilon),\sigma(\varepsilon))$ lie on an invariant curve anymore.

Remark 7.4.1. This method has a particular drawback for our interests. If one wants to deal with realistic synaptic inputs, one should consider $p/q < 1/20$. To find the series of $h^q(\theta, \varepsilon, \delta)$ with q large, the computation time can be too long, and even the accuracy of the series too bad (that is, $\varepsilon_{\max}$ too small). One possible solution of the latter problem can be to compute a normal form of $h(\theta, \varepsilon, \delta)$ in terms of δ, as is done in [BST98], but then one should expect even longer computation times.

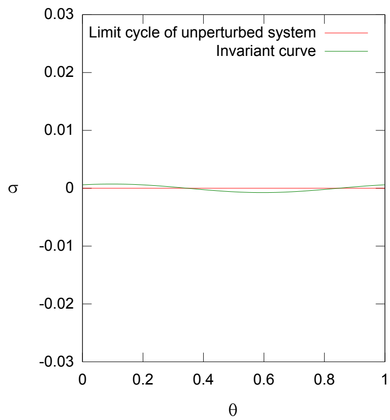
7.5 Results

In this section we show some of the results obtained with the implementation of the methods presented above. The algorithms have been implemented in C language using double precision. All the figures below have been obtained using `gnuplot`. All the examples correspond are done with the canonical example (6.25). We take $\alpha = 5$ and $a = 1$, so that the underlying limit cycle of the continuous system is strongly hyperbolic and the isochrons are slightly tilted.

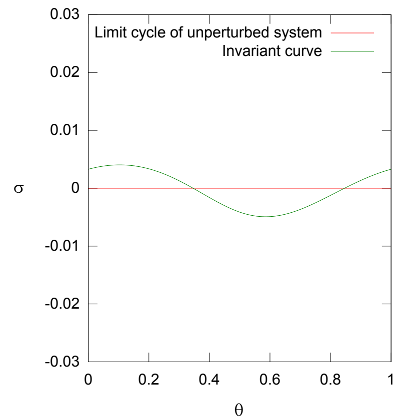
First we show some of the invariant curves of the 2D map (7.1) with $\omega = 1/50$ for different values of ε obtained with the Newton-like method and the Taylor expansion method. On the one hand, we plot the invariant curves in variables (θ, σ) (see Figure 7.3). On the other hand, we plot the same curves in variables $(x, y) = K(\theta, \sigma)$, where K is the function defined in (6.27) (see Figure 7.4). As we mentioned above, in both methods the maximal value of ε that we can reach keeping a low error in the invariance equations is not completely satisfactory, since we are not able to see invariant curves close to the breakdown. However, one can see the evolution of these invariant curves as ε increases. This evolution is much more visible in (θ, σ) variables.

Next, we fix $p/q = 1/3$, and we plot the corresponding Arnold tongue (see Figure 7.5) for $\varepsilon < \varepsilon_{\max}$. We also take some points on the parameter line $\varepsilon = 0.2$ and plot the corresponding invariant curves in Figure 7.6. We can observe a saddle-node bifurcation of periodic orbits: we start having two 1/3-periodic orbits (one attracting and the other repelling) that approach each other until they collide, giving rise to a single 1/3-periodic orbit of saddle-node type. Beyond this parameter value rational dynamics is no longer observed (see Figure 7.6d).

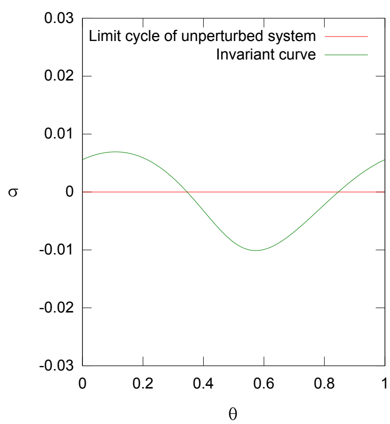
Finally, we show some Arnold tongues of map $F_{\varepsilon,\omega}$. First we plot some tongues for low values of p/q (see Figure 7.7). We also indicate the value of $\varepsilon_{\max}$, that is the value of ε such that $E_{\text{inv}} < 10^{-10}$. In Figure 7.8 we compare some Arnold tongues (for low values of p/q) corresponding to the 2D map $F_{\varepsilon,\omega}$ and the 1-dimensional map (6.21). One can see that, the higher q is, the more the tongues of the 1D and 2D maps differ. We expect that for realistic values of p/q (for instance, $p/q = 1/50$) these differences will be significant. However, working with double precision does not allow us to distinguish between the two boundaries of the Arnold tongues for high values of q (see Figure 7.9). This is due to the fact that the order of contact of the tongues is of ε^q, see [BST98]. In order to be able



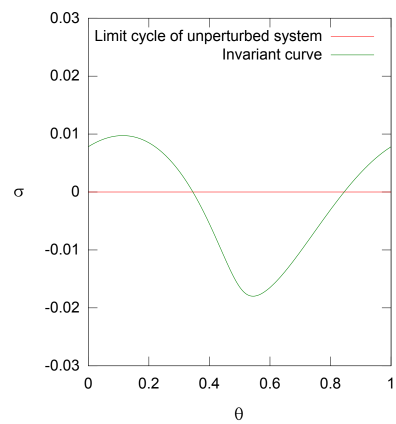
(a) Newton-like method, $\varepsilon = 0.001$.



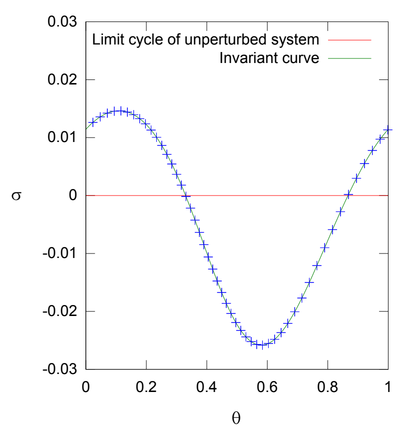
(b) Newton-like method, $\varepsilon = 0.006$.



(c) Newton-like method, $\varepsilon = 0.011$.

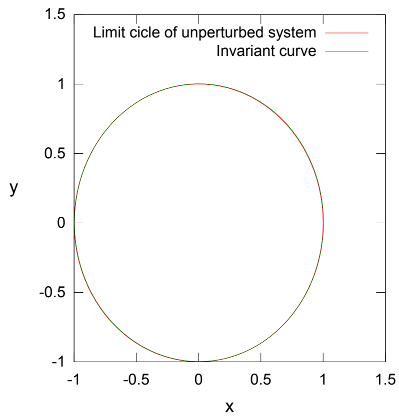


(d) Newton-like method, $\varepsilon = 0.0165$.

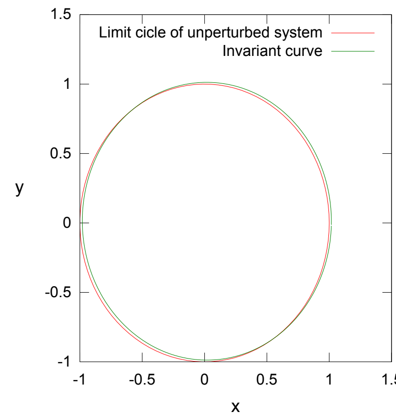


(e) Taylor expansion method. $\varepsilon = 0.021$ and $\delta = 0.0004$ are chosen so that the invariant curve lies on the boundary of the 1/50-Arnold tongue. In blue, the 1/50-periodic orbit.

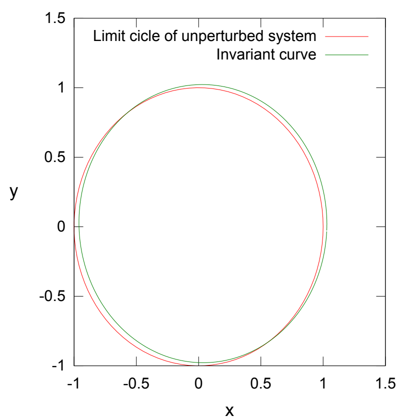
Figure 7.3: Invariant curves and original limit cycle of the map $F_{\varepsilon,\omega}$ with $\omega = 1/50$, $\alpha = 5$, $a = 1$, in variables (θ, σ).



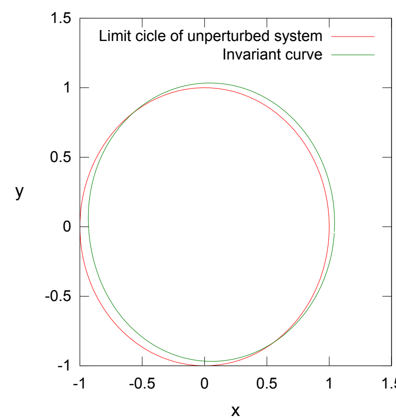
(a) Newton-like method, $\varepsilon = 0.001$.



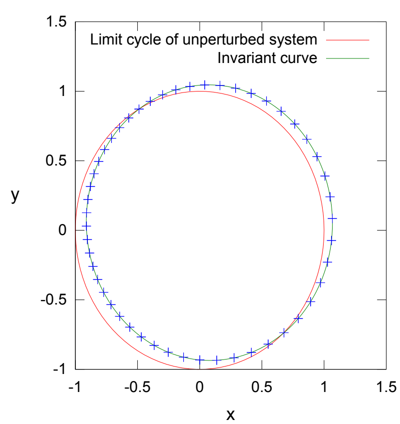
(b) Newton-like method, $\varepsilon = 0.006$.



(c) Newton-like method, $\varepsilon = 0.011$.



(d) Newton-like method, $\varepsilon = 0.0165$.



(e) Taylor expansion method. $\varepsilon = 0.021$ and $\delta = 0.0004$ are chosen so that the invariant curve lies on the boundary of the 1/50-Arnold tongue. In blue, the 1/50-periodic orbit.

Figure 7.4: Invariant curves and original limit cycle of the map $F_{\varepsilon,\omega}$ with $\omega = 1/50$, $\alpha = 5$, $a = 1$, in variables $(x, y) = K(\theta, \sigma)$.

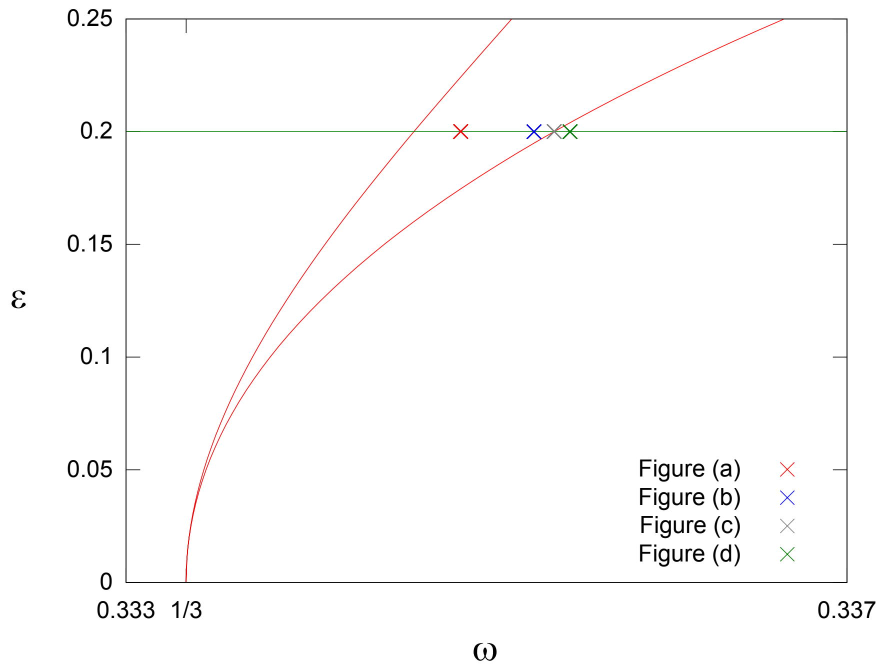


Figure 7.5: 1/3-Arnold tongue for the map $F_{\varepsilon,\omega}$ with $\alpha = 5$, $a = 1$.

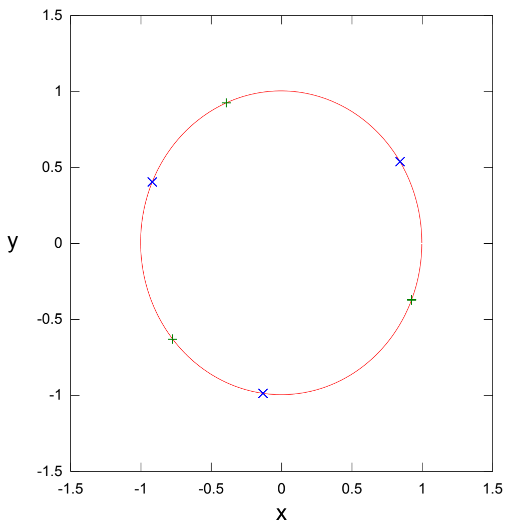
to compute corresponding tongues in these cases, one should work with higher-precision arithmetics or with normal forms, as is done in [BST98].

7.6 Discussion

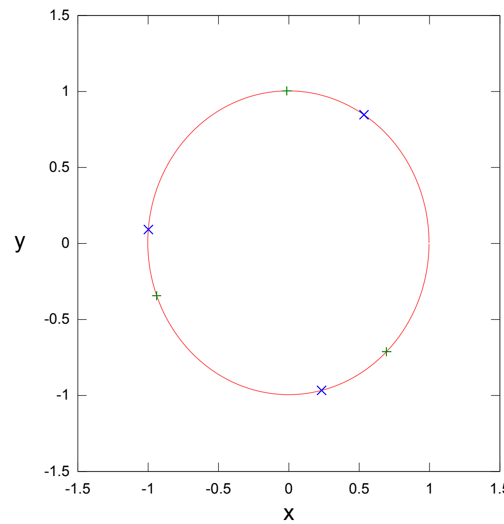
In this chapter we have used two numerical methods to gain more insight into the dynamics of the PRF-ARF map (6.20) defined in Chapter 6. Adapting methods from the literature on 2D maps, we provide two alternatives to compute invariant curves of PRF-ARF maps as well as their intrinsic dynamics. We then apply them to a specific minimal model in which we are able to:

(1) Validate numerically the existence of the invariant curves up to some perturbation level ε_{max} for different stimulation frequencies.

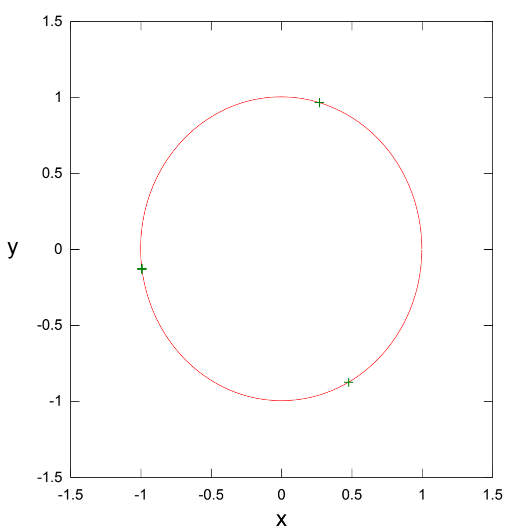
(2) Understand the dynamics on the invariant curves and to distinguish parameter regions with rational dynamics from those with irrational by means of the Arnold tongues. In particular, we have described the saddle-node bifurcation of periodic orbits that the system undergoes when crossing the boundaries of these Arnold tongues.



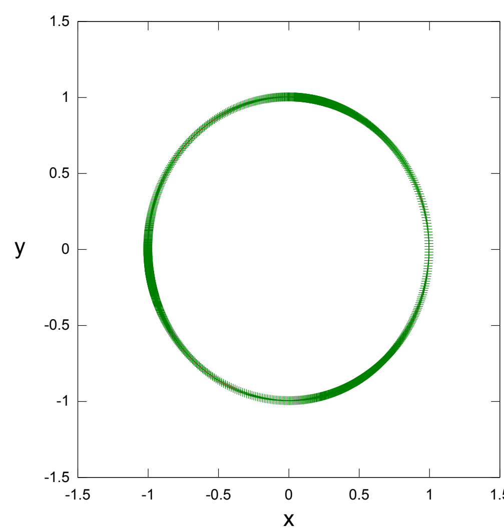
(a) $\omega = 1/3 + 0.0015233263$. One has two 1/3-periodic orbits.



(b) $\omega = 1/3 + 0. + 0.00193$. One has two 1/3-periodic orbits.



(c) $\omega = 1/3 + 0.0020418188$. The two periodic orbits collide, giving rise to a single saddle-node 1/3-periodic orbit.



(d) $\omega = 1/3 + 0.00213$. 1500 iterations of a point inside the invariant curve. Apparent irrational dynamics.

Figure 7.6: Invariant curves of the map $F_{\varepsilon,\omega}$ with $\alpha = 5$, $a = 1$, $\varepsilon = 0.2$ and different values of ω. In variables $(x, y) = K(\theta, \sigma)$.

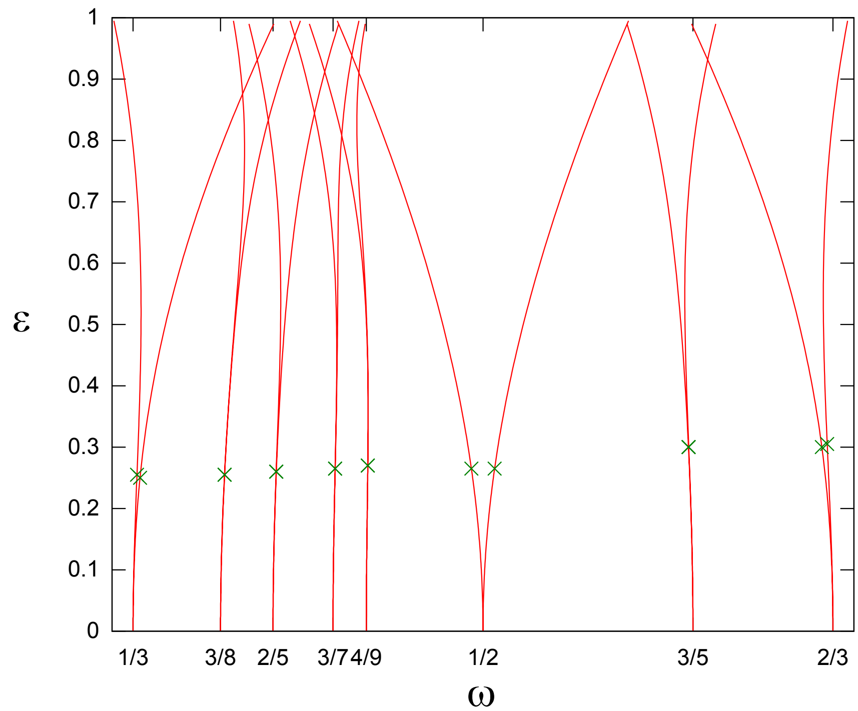
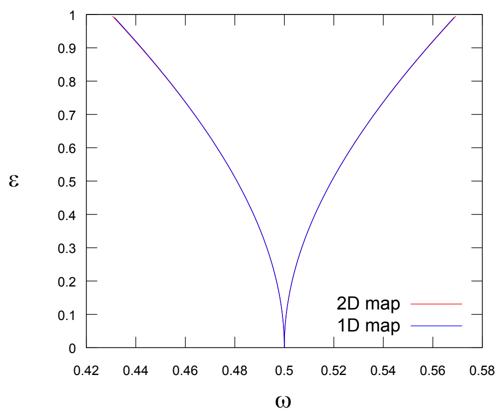


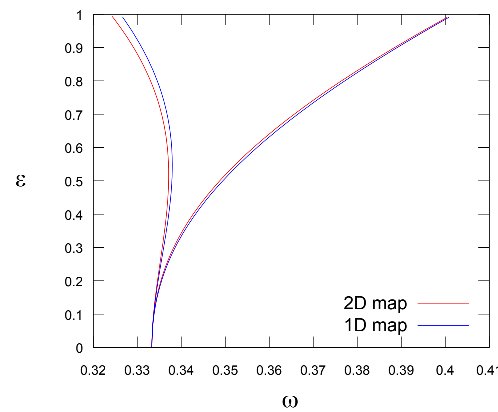
Figure 7.7: Arnold tongues for the map $F_{\varepsilon,\omega}$ with $\alpha = 5$, $a = 1$. Green crosses indicate the value of $\varepsilon_{\max}$ of each tongue (see Section 7.4).

(3) Compare with parallel results obtained with the classical 1D PRC map (6.21). With this study, we aim at providing methodology and giving a proof-of-concept of some issues related to the newly considered PRF-ARF maps (see Chapter 6 and also [WLTC13] for a similar approach), but we acknowledge some shortcomings that are worth to mention in order to go beyond both in the numerical and in the biological aspects in the future.

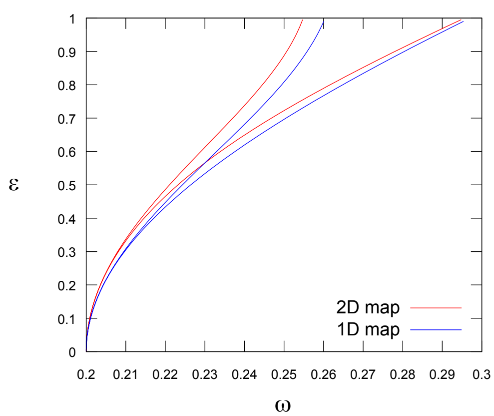
A first challenge would be to improve the implementation of these methods in order to be able to achieve more realistic values of p/q. As we pointed out above, this should be attainable by performing a normal form procedure so that the map $F_{\varepsilon,\omega+\delta}$ is in the simplest form, namely, to be able to write $F_{\varepsilon,\omega+\delta}$ in powers of ε with all coefficients of order n, $0 < n < q$, depending only on δ. Then, the equations to find the boundaries of the Arnold tongues would be also simplified (see [BST98], Proposition 2.9), being able to easily distinguish between the two boundaries even for higher values of q, which are more realistic in the neuroscience paradigm. In addition, one could use higher-precision arithmetics.



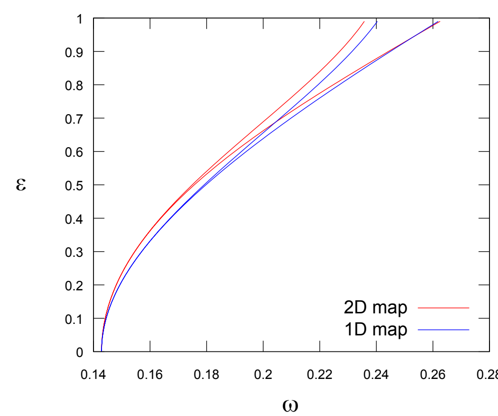
(a) $p/q = 1/2$



(b) $p/q = 1/3$



(c) $p/q = 1/5$



(d) $p/q = 1/7$

Figure 7.8: Comparison between different p/q-Arnold tongues of the 2D and the 1D maps, with $\alpha = 5$, $a = 1$.

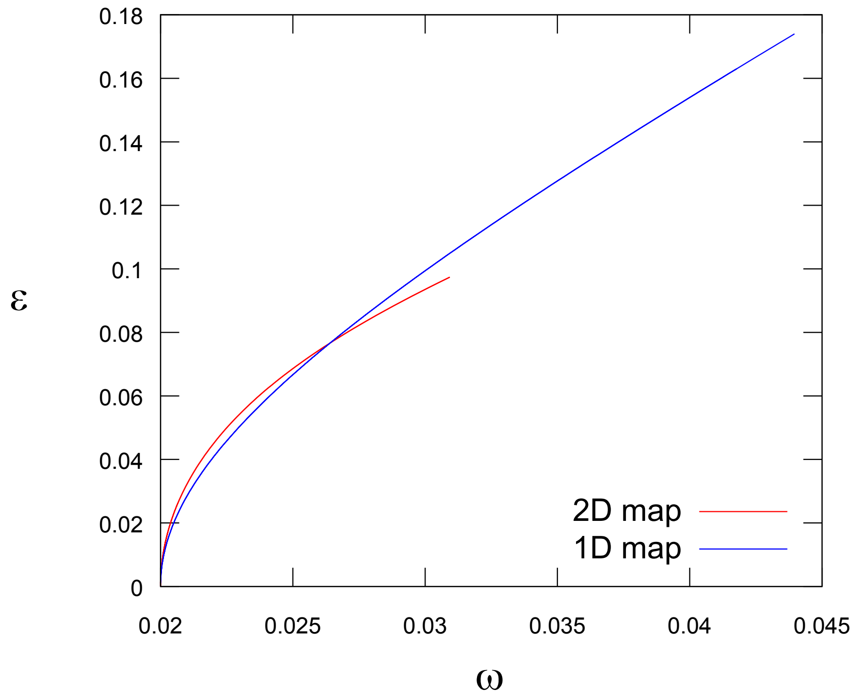


Figure 7.9: Comparison between one of the boundaries of the 1/50-Arnold tongue of the 2D and the 1D maps, with $\alpha = 5$, $a = 1$.

Another goal would be to compute the invariant curves for values of ε that are close to the breakdown. In this direction, similarly as in [BST98], it would be interesting to compute the curves in the (ω, ε) space until which an invariant curve exists, and thus confirm the breakdown phenomenon observed in the simulations of Figures 7.1 and 7.2. We think it is a fairly achievable goal that we will tackle in the next future.

Focusing on the interpretation of these results in the context of neuroscience (see also [Izh07, Section 10.1.9], for a similar discussion with the 1D PRC map), the Arnold tongues inform about the strength and periodicity of periodic pulse stimuli in order to achieve or not an entrainment of the cell to the stimulus. The differences between the 1D map and the 2D map predictions shown in Figure 7.9 are not striking for small ε and "large" ω, but they show the trend of increasing dissimilarity as ε increases and $\omega = T_s/T_0$ decreases (that is, when the stimulation period T_s decreases). In particular, for realistic ε and T_s, one expects stronger differences between the two predictions, meaning that an external control exerted on a neuron model might not have the synchronization properties forecast by the 1D map. Our results show differences between the intervals predicted by the 1D map and the 2D map (supposedly closer to the actual one), corresponding to the interior of the respective ω Arnold tongues, and warn about the validity of this control using only

1D maps.

It is worth to note that this discussion is not only valid in the context of neuroscience. In fact, this was only our leit-motiv and we have brought the problem to a more mathematical (and so, universal) framework. Not surprisingly, this methodology can be applied to any model in which we have an oscillator, namely a limit cycle. As far as we know, only PRCs have been systematically used in other fields like electrical circuits, see [SD10], or cellular oscillators, see [JBB84] and [JK12], which gives promising avenues for future work.

Finally, we point out that this is just a first exploration of these numerical methods in this context. As future work, one could also try to implement these methods for more realistic models, and play also with the parameters (pushing them to limiting values, for instance) and see how these play a role in determining the asymptotic states. One could also try to study other type of stimulus (for instance non-periodic) or other protocols of stimulation (two different periods of stimulation, pulse train, etc.). Doing so, one would obtain another map different from (6.20), but the same questions could be posed and it seems that the same methodology should work.

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