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Eigenvalue varieties of abelian trees of groups and link-manifolds

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Abstract

English

The A-polynomial of a knot in $\mathbb{S}^3$ is a two variable polynomial obtained by projecting the $\mathrm{SL}_2\mathbb{C}$-character variety of the knot-group to the character variety of its peripheral subgroup. It distinguishes the unknot and detects some boundary slopes of essential surfaces in knot exteriors.

The notion of A-polynomial has been generalized to 3-manifolds with non-connected toric boundaries; if M is a 3-manifold bounded by n tori, this produces an algebraic subset $\mathfrak{E}(M)$ of $\mathbb{C}^{2n}$ called the *eigenvalue variety* of M. It has dimension at most n and still detects systems of boundary slopes of surfaces in M.

The eigenvalue variety of M always contains a part $\mathfrak{E}^{\mathrm{red}}(M)$ arising from reducible characters and with maximal dimension. If M is hyperbolic, $\mathfrak{E}(M)$ contains *another top-dimensional component*; for which 3-manifolds is this true remains an open question.

In this thesis, this matter is studied for two families of 3-manifolds with toric boundaries and, via two very different technics, we provide a positive answer for both cases.

On the one hand, we study *Brunnian links* in $\mathbb{S}^3$, links in the standard 3-sphere for which any strict sublink is trivial. Using special properties of these links and stability under certain *Dehn fillings* we prove that, if M is the exterior of a Brunnian link different from the trivial link or the Hopf link, then $\mathfrak{E}(M)$ admits a top-dimensional component different from $\mathfrak{E}^{\mathrm{red}}(M)$. This is achieved generalizing the technic applied to knots in $\mathbb{S}^3$, using Kronheimer-Mrowka theorem.

On the other hand, we consider a family of *link-manifolds*, exteriors of links in *integer-homology spheres*. Link-manifolds are equipped with *standard peripheral systems* of *meridians* and *longitudes* and are stable under *splicing*, gluing two link-manifolds along respective boundary components, identifying the meridian of each side to the longitude of the other. This yields a well-defined notion of *torus decomposition* and a link-manifold is called a *graph link-manifold* if there exists such a decomposition for which each piece is *Seifert-fibred*. Discarding trivial cases, we prove that all graph link-manifolds produce another top-dimensional component in their eigenvalue variety.

For this second proof, we propose a further generalization of the eigenvalue variety that also takes into account internal tori and this is introduced in the broader context of *abelian trees of groups*. A tree of group is called *abelian* if all its edge groups are commutative; in that case, we define the *eigenvalue variety of an abelian tree of groups*, an algebraic variety compatible with two natural operations on trees: merging and contraction. This enables to study the eigenvalue variety of a link-manifold through the eigenvalue varieties of its torus splittings. Combining general results on eigenvalue varieties of abelian trees of groups with combinatorial descriptions of graph link-manifolds, we construct top-dimensional components in their eigenvalue varieties.

Français

Le A-polynôme d'un noeud dans $\mathbb{S}^3$ est un polynôme à deux variables obtenu en projetant la variété des $\mathrm{SL}_2\mathbb{C}$-caractères de l'extérieur du noeud sur la variété de caractères du groupe périphérique. Il distingue le noeud trivial et détecte certaines pentes aux bords de surfaces essentielles des extérieurs de noeud.

La notion de A-polynôme a été généralisée aux 3-variétés à bord torique non connexe ; une 3-variété M bordée par n tores produit un sous-espace algebrique $\mathfrak{E}(M)$ de $\mathbb{C}^{2n}$ appelé *variété des valeurs propres* de M. Sa dimension est inférieure ou égale à n et $\mathfrak{E}(M)$ détecte également des systèmes de pentes aux bords de surfaces essentielles dans M.

La variété des valeurs propres de M contient toujours un sous-ensemble $\mathfrak{E}^{\mathrm{red}}(M)$ produit par les caractères réductibles, et de dimension maximale. Si M est hyperbolique, $\mathfrak{E}(M)$ contient *une autre composante de dimension maximale* ; pour quelles autres 3-variétés est-ce le cas reste une question ouverte.

Dans cette thèse, nous étudions cette question pour deux familles de 3-variétés à bords toriques et, via deux techniques distinctes, apportons une réponse positive dans ces deux cas.

Dans un premier temps, nous étudions les *entrelacs Brunniens* dans $\mathbb{S}^3$, entrelacs pour lesquels tout sous-entrelacs strict est trivial. Certaines propriétés de ces entrelacs, et leur stabilité par certains *remplissages de Dehn* nous permettent de prouver que, si M est l'extérieur d'un entrelacs Brunnien non trivial et différent de l'entrelacs de Hopf, $\mathfrak{E}(M)$ contient une composante de dimension maximale différente de $\mathfrak{E}^{\mathrm{red}}(M)$. Ce résultat est obtenu en généralisant la technique préalablement utilisée pour les noeuds dans $\mathbb{S}^3$ grâce au théorème de Kronheimer-Mrowka.

D'autre part, nous considérons une famille de *variétés-entrelacs*, variétés obtenues comme extérieurs d'entrelacs dans des *sphères d'homologie entière*. Les variétés-entrelacs possèdent des *systèmes périphériques standard* de *méridiens* et *longitudes* et sont stables par *splicing*, le recollement de deux variétés-entrelacs le long de tores périphériques en identifiant le méridien de chaque coté avec la longitude opposée. Ceci induit une notion de *décomposition torique* de variété-entrelacs et une telle variété est dite *graphée* si elle admet une décomposition torique où toutes les pièces sont fibrées de Seifert. Nous montrons que, mis-à-part les cas triviaux, toutes les variétés-entrelacs graphées produisent une autre composante de dimension maximale dans leur variétés des valeurs propres.

Pour cette seconde preuve, nous présentons une nouvelle généralisation de la variété des valeurs propres, qui prend également en compte les tores intérieurs, que nous introduisons dans le contexte plus général des *arbres abéliens de groupes*. Un arbre de groupe est appelé *abélien* si tous les groupes d'arête sont commutatifs ; dans ce cas, nous définissions la *variété des valeurs propres d'un arbre abélien de groupe*, une variété algébrique

compatible avec deux opérations naturelles sur les arbres : la fusion et la contraction. Ceci permet d'étudier la variété des valeurs propres d'une variété-entrelacs à travers les variétés des valeurs propres de ses décompositions toriques. En combinant des résultats généraux sur les variétés des valeurs propres d'arbres abéliens de groupe et les descriptions combinatoires des variétés-entrelacs graphées, nous contruisons des composantes de dimension maximale dans leur variétés des valeur propres.

Català

L'A-polinomi d'un nus en $\mathbb{S}^3$ és un poliomi de dues variables obtingut projectant la varietat de $\mathrm{SL}_2\mathbb{C}$-caràcters de l'exterior del nus sobre la varietat de caràcters del grup perifèric. Distingeix el nus trivial i detecta alguns pendents a la vora de superfícies essencials dels exteriors de nus.

El concepte de A-polinomi va ser generalitzat a les 3-varietats amb vores tòriques no connexes; una 3-varietat M amb n tors de vora produeix un sub-espai algebraic $\mathfrak{E}(M)$ de $\mathbb{C}^{2n}$ anomenat *varietat de valors propis* de M. Té dimensió maximal n i $\mathfrak{E}(M)$ també detecta sistemes de pendents a les vores de superfícies essencials en M.

La varietat de valors propis de M sempre conté una part $\mathfrak{E}^{\mathrm{red}}(M)$, de dimensió maximal, produïda pels caràcters reductibles. Si M és hiperbòlica, $\mathfrak{E}(M)$ conté *una altra component de dimensió maximal*; saber quines altres 3-varietats compleixen això encara és una pregunta oberta.

En aquesta tesi, estudiem aquest assumpte per dues famílies de 3-varietats amb vores tòriques i, amb dues tècniques diferents, aportem una resposta positiva en ambdós casos.

Primerament, estudiem els *enllaços Brunnians* en $\mathbb{S}^3$, enllaços per els quals tot sub-enllaç estricte és trivial. Algunes propietats d'aquests enllaços i llur estabilitat sota alguns *ompliments de Dehn* permet mostrar que, si M és l'exterior d'un enllaç Brunnià no trivial i diferent de l'enllaç de Hopf, $\mathfrak{E}(M)$ conté una component de dimensió maximal diferent de $\mathfrak{E}^{\mathrm{red}}(M)$. Aquest resultat s'obté generalitzant la tècnica prèviament utilitzada per els nusos en $\mathbb{S}^3$ fent servir el teorema de Kronheimer-Mrowka.

Per altre banda, considerem una família de *varietats-enllaç*, varietats obtingudes com exteriors d'enllaços en *esferes d'homologia entera*. Les varietats-enllaç tenen *sistemes perifèrics estàndards* de *meridans* i *longituds* i són estables per *splicing*, l'enganxament de dues varietats-enllaç al llarg de tors perifèrics, identificant el meridià de cada costat amb la longitud oposada. El splicing indueix una noció de *descomposició tòrica* per les varietats-enllaç i anomenem *grafejades* les varietats-enllaç que admeten una descomposició tòrica per la qual totes les peces són fibrades de Seifert. Mostrem que, excloent els casos trivials, totes les varietats-enllaç grafejades produeixen una altre component de dimensió maximal en les seves varietats de valors propis.

Per aquesta segona demostració, presentem una nova generalització de la varietat de valors propis, que també té en compte tors interns, i que presentem en el context més general d'*arbres abelians de grups*. Un arbre de grup és *abelià* quan tots els grups de arestes són commutatius; en aquest cas, definim la *varietat de valors propis d'un arbre abelià de grup*, una varietat algebraica compatible amb dues operacions naturales sobre els arbres: la fusió i la contracció. Això permet estudiar la varietat de valors propis d'una varietat-enllaç mitjançant les varietats de valors propis de les seves descomposicions tòriques. Combinant

resultats generals sobre varietats de valors propis d'arbres abelians de grup i les descripcions combinatòries de les varietats-enllaç grafejades, construïm components de dimensió maximal en les seves varietats de valors propis.

Contents

Introduction

Overview

The A-polynomial of a knot in $\mathbb{S}^3$ is a two variable polynomial constructed projecting the $\mathrm{SL}_2\mathbb{C}$-character variety of the knot-group to the character variety of its peripheral subgroup. It was first introduced by Daryl Cooper, Marc Culler, Henri Gillet, Darren Long and Peter Shalen in [CCG$^+$94], where it is also proved that the A-polynomial of any knot contains the A-polynomial of the unknot as a factor. The A-polynomial of a knot is said to be *non-trivial* if it contains other factors and Cooper, Culler, Gillet, Long and Shalen also proved in the same [CCG$^+$94] that hyperbolic knots and non-trivial torus knots always have a non-trivial A-polynomial. This was later proved to be true for all non-trivial knots by Nathan Dunfield and Stavros Garoufalidis in [DG04], and independently by Steve Boyer and Xingru Zhang in [BZ05]; both proofs use a theorem by Peter Kronheimer and Tomasz Mrowka in [KM04] on Dehn-fillings and representations in SU_2.

A remarkable application of the A-polynomial is its hability to detect *Culler-Shalen slopes* on the boundary of the knot exterior. In [CS83] Marc Culler and Peter Shalen had developped a technic to obtain splittings of groups from ideal points of their $\mathrm{SL}_2\mathbb{C}$ character varieties which, when applied to 3-manifolds, also produces essential surfaces. In case of a knot exterior, those surfaces intersect the boundary along a finite set of slopes – Culler-Shalen slopes – and Cooper et al. proved in [CCG$^+$94] that slopes of the Newton polygon of the A-polynomial of the knot are Culler-Shalen slopes of its exterior.

The notion of A-polynomial can be generalized to 3-manifolds with connected toric boundary by specifying a *peripheral system* (a base of $\pi_1\partial M \hookrightarrow \pi_1 M$). Stimulated by the work of Alan Lash[1] in [Las93], it was then extended[2] to manifolds with non-connected boundaries by Stephan Tillmann[3]. In his PhD thesis [Til02] and the subsequent article [Til05], Tillmann presents the *eigenvalue variety* $\mathfrak{E}(M)$ associated to a 3-manifold M with

[1] who I deeply thank for giving me access to his PhD manuscript
[2] I'd like to thank Steve Boyer for this information
[3] who I'd also like to thank for the insight he provided me on this topic when we met

toric boundary. If M has k boundary tori, the associated eigenvalue variety $\mathfrak{E}(M)$ is an algebraic subspace of $\mathbb{C}^{2k}$ essentially corresponding to the peripheral eigenvalues taken by representations (or equivalently, characters) of $\pi_1 M$ in $\mathrm{SL}_2\mathbb{C}$, and a similar construction can also be made using $\mathrm{PSL}_2\mathbb{C}$ characters.

Under these assumptions, Tillmann showed in [Til02] that the possible dimension for components of $\mathfrak{E}(M)$ is at most k; any component of the character variety of M producing a k-dimensional component in the eigenvalue variety will be called here *peripherally maximal*. In the same way as any A-polynomial is divisible by the A-polynomial of the unknot, any eigenvalue variety $\mathfrak{E}(M)$ contains a component $\mathfrak{E}^{\mathrm{red}}(M)$ corresponding to reducible characters. A component in the character variety of M will be called *peripherally abelian* if the corresponding subset in $\mathfrak{E}(M)$ is contained in $\mathfrak{E}^{\mathrm{red}}(M)$; otherwise we call it *peripherally non-abelian*. The component of reducible characters is *peripherally maximal* and we shall search for *peripherally maximal and non-abelian* components. If M is hyperbolic, its character variety contains a distinguished component X_0 called the *geometric component*, containing the character of a discrete faithful representation. Using William Thurston's results of [Thu02], Tillmann proved that the geometric component is peripherally maximal and non-abelian, generalizing the result of [CCG+94]on hyperbolic knots. However, which 3-manifolds produce a peripherally maximal and non-abelian component, or merely whether non-trivial links in $\mathbb{S}^3$ do, remain open questions.

In this thesis, we answer this matter for two specific cases. These two results are mutually independant and obtained using very different technics.

First, we consider *Brunnian links* in $\mathbb{S}^3$, links for which any strict sublink is trivial. Using results of Brian Mangum and Theodore Stanford from [MS01], we push further the technic used for knots by Boyer-Zhang and Dunfield-Garoufalidis and obtain the following Theorem 1 on the $\mathrm{SL}_2\mathbb{C}$-character variety of exteriors of Brunnian links:

Theorem 1. *Let L be a Brunnian link in $\mathbb{S}^3$ and let M denote its exterior, then $X^{\mathrm{SL}_2\mathbb{C}}(M)$ admits a peripherally maximal and non-abelian component if and only if L is neither the trivial link or the Hopf-link.*

Then, we escape the standard 3-sphere to consider links in *integer-homology spheres*; if M is an integer-homology sphere and L is a link in M, the exterior of L in M is called a *link-manifold*[4] and denoted by M_L. Moreover, it is called a *graph* link-manifold if it can be split along essential tori such that each piece of the splitting is Seifert-fibred. Studying thoroughly the $\mathrm{PSL}_2\mathbb{C}$-character varieties of Seifert-fibred manifolds and the merging equations associated to the splitting we prove Theorem 2:

[4]the term link-manifold is sometimes used for any 3-manifolds with toric boundary; here it will only be used for *exteriors of links in integer-homology sphere*.

Theorem 2. *For any non-abelian graph link-manifold M_L with boundary, there exist a peripherally maximal and non-abelian component in $X^{\mathrm{PSL}_2\mathbb{C}}(M_L)$.*

For this second result we need a further generalization of Tillmann's eigenvalue variety which also takes into account internal tori. This new construction is not proper to link-manifolds and is presented in the context of *abelian trees of groups*. We consider *graph of groups* as in Jean-Pierre Serre's [SB77][5] with a slight modification: trees here contain arrows, like edges but connected to a unique vertex with the other end being free; this enables to define tree-*merging* by gluing two trees along chosen arrows on each side and this pairing is compatible with Serre's *contraction* of trees, with the suitable modifications to include arrows.

The $\mathrm{SL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{C}$ character varieties of abelian groups are essentially determined by the eigenvalues of the generators. Considering a tree of groups $(\mathcal{G}, \pi)$, with all the arrow and edge groups abelian – an *abelian tree of group* – we define an algebraic space $E_{\mathcal{G}}(\pi)$ by projecting the character variety of the group π on the different character varieties of the edge and arrow groups and then pulling back at the eigenvalue level in some ${\mathbb{C}^*}^N$. This space $E_{\mathcal{G}}(\pi)$ is the *eigenvalue-variety associated to the abelian tree of group* $\mathcal{G}$; its defining ideal in $\mathbb{C}[Y_1^{\pm 1} \dots Y_N^{\pm 1}]$ will be denoted by $\mathcal{A}_{\mathcal{G}}(\pi)$ and called the *$\mathcal{A}$-ideal associated to* $\mathcal{G}$.

An interesting feature of this new construction is its compatibility with two natural operations on tree of groups mentioned above: *merging* and *contraction*. This compatibility permits to study an eigenvalue-variety through the eigenvalue-varieties associated to contraction or subtrees of the original tree. So far, $\mathcal{G}$ is not necessarily a torus splitting of a link-manifold; however, even in this wider context, we can apply part of Culler-Shalen construction of [CS83] to derive splittings of π from ideal points of $E_{\mathcal{G}}(\pi)$, as well as a criterion to identify wich elements in the edge or arrow groups are in a vertex group of the new Culler-Shalen splitting.

This construction naturally applies to link-manifolds decomposed along tori; the edge groups correspond to splitting tori, the arrow groups are the peripheral subgroups, and the vertex groups are given by the pieces of the toric decomposition. In particular, considering a *trivial decomposition*, the corresponding tree has a single vertex and arrows corresponding to the peripheral tori, and we get back Tillmann's eigenvalue-variety as presented earlier. Even if this whole construction is not fully used for the Brunnian links case, we present a unified definition for all the eigenvalue-varieties that we will consider here. It is only for the case of graph link-manifolds that $E_{\mathcal{G}}$-varieties for non-trivial splittings will really prove themselves useful.

After this brief overview, and before entering the main matter, we follow with a more detailed description of the content of each chapter.

[5]See [Ser03] for an english version.

Trees, characters and links

The first chapter of this thesis is a presentation of all the main concepts and notations we will need in order to properly introduce the $E_{\mathcal{G}}$-variety. As a central piece of the definition, we start with *tree of groups*, setting notations we will use hereafter. As already mentioned, our trees have arrows, and we present the small modifications produced by this extension. We can split a tree of groups along an edge, which becomes two arrows in the two trees obtained after splitting. Symmetrically, given two trees and chosing two arrows with isomorphic groups, we can also merge the trees along the respective arrows to obtain a new, well-defined, tree of groups. In both cases we will write

$$\mathcal{G} = \mathcal{G}^+ \ {}^{a^+}\!\underset{e}{\bowtie}{}^{a^-} \ \mathcal{G}^-$$

when $\mathcal{G}^+$ and $\mathcal{G}^-$ are obtained splitting $\mathcal{G}$ along e, and when $\mathcal{G}$ is obtained by merging $\mathcal{G}^+$ and $\mathcal{G}^-$ along a^+ and a^-. The second natural operation that we will consider on our trees is the *contraction* of a tree, and the *binding*[6] *decomposition* they produce. With the suitable modifications to include arrows, this is quite similar to the contraction of trees as defined by Serre in [SB77].

Let $\mathcal{G}$ be a tree of groups, $\mathcal{E}_0$ a subset of edges of $\mathcal{G}$, and Γ the collection of connected trees obtained by splitting $\mathcal{G}$ along the edges of $\mathcal{E}_0$; we will write

$$\mathcal{G} = (\mathcal{G}_{/\Gamma} \gg= \Gamma)$$

where $\mathcal{G}_{/\Gamma}$ is obtained by contracting each Γ onto a single vertex, keeping all the arrows of $\Gamma \cap \mathcal{G}$ – so the tree $\mathcal{G}$ and its contraction $\mathcal{G}_{/\Gamma}$ have the same arrow-set. The edge set of $\mathcal{G}_{/\Gamma}$ is $\mathcal{E}_{/\Gamma} \cong \mathcal{E}_0$ and the family Γ is indexed by the vertex set $\mathcal{V}_{/\Gamma}$ of $\mathcal{G}_{/\Gamma}$. With a similar symmetry to splitting/merging presented above, $\mathcal{G}$ is also thought as reconstructed from $\mathcal{G}_{/\Gamma}$ by expanding each vertex of $\mathcal{G}_{/\Gamma}$ as the corresponding tree in the collection Γ, and we say that $\mathcal{G}$ is obtained by *binding* the family Γ over the tree $\mathcal{G}_{/\Gamma}$.

We then recall the definition of the *character variety* of a group; using the contravariant properties of the character variety, a splitting of a group over a tree produces various algebraic maps between character varieties. This is all introduced in the second part of Chapter 1, together with important concepts and notations for studying character varieties of groups split over a tree.

Finally, we close our first chapter with a quick review on link-manifolds. With [EN85] as a reference for this matter, we recall how standard peripheral systems are obtained and how two link-manifolds can be *spliced* along respective boundary components. This

[6]the term *binding* as well as the operator ($\gg=$) are borrowed from monad theory and functional programming lexicon. See for example [Mar10] for more details.

produces a well-defined notion of torus splitting, creating splitting trees for the underlying fundamental group. In particular, the splicing of two link-manifolds is compatible with the merging of trees as defined earlier, so a torus splitting of a spliced link-manifold is the merging of the corresesonding trees.

E-varieties

The second chapter presents the construction of the $E_{\mathcal{G}}$-variety associated to an *abelian tree of groups*. The $\mathrm{SL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{C}$ character variety of an abelian group H with h generators has a natural affine structure in $\mathbb{C}^{*h}$ denoted by $E(H)$. Given an abelian tree of groups $(\mathcal{G}, \pi)$, the projection of $X(\pi)$ on all the edge and arrow groups of $\mathcal{G}$ defines the $E_{\mathcal{G}}$-variety of π, $E_{\mathcal{G}}(\pi)$. This algebraic space is defined by an ideal in some $\mathbb{C}[Y_1^{\pm 1}, \ldots, Y_N^{\pm 1}]$, that we call the *$\mathcal{A}$-ideal of the abelian tree* $\mathcal{G}$, and denote by $\mathcal{A}_{\mathcal{G}}(\pi)$. The results concerning the naturality over splitting/merging and contraction/binding naturally follow from the underlying structures. First, we show that, for an abelian tree of groups $(\mathcal{G}, \pi)$, if $\mathcal{G}$ can be split as

$$\mathcal{G} = \mathcal{G}^+ \ {}^{a^+}\!\underset{e}{\bowtie}^{a^-} \ \mathcal{G}^-$$

then there's a natural map

$$E_{\mathcal{G}}(\pi) \to E_{\mathcal{G}^+}(\pi^+) \underset{E(\pi_e)}{\times} E_{\mathcal{G}^-}(\pi^-)$$

and we give a sufficient criterion for belonging to the image. Iterating this result enables to examine the behaviour of the $E_{\mathcal{G}}$-variety under contraction and binding. This is the purpose of Theorem 3:

Theorem 3. *Let $\mathcal{G}$ be an abelian tree of goups. Any binding decomposition $(\mathcal{G}_{/\Gamma} \gg\!= \Gamma)$ of the tree $\mathcal{G}$ produces two regular maps as in the following diagram*

$$\begin{array}{ccc} E_{\mathcal{G}}(\pi) & \overset{i_\Gamma}{\hookrightarrow} & \prod_{v \in \mathcal{V}_{/\Gamma}} E_{\Gamma_v}(\pi_v) \\ {\scriptstyle p}\downarrow & & \\ E_{\mathcal{G}_{/\Gamma}}(\pi) & & \end{array} \tag{1}$$

such that, for any edge $v' \overset{e}{-} v$ of $\mathcal{E}_{/\Gamma}$ in $\mathcal{G}_{/\Gamma}$, if e is sent to ${}^{a'}\!\underset{e}{\bowtie}^{a}$ in $\mathcal{G}$ for some arrows a' and a in $\Gamma_{v'}$ and Γ_v respectively, then

$$(\xi_{v'})_{a'} = (\xi_v)_a \tag{2}$$

Moreover, for any $(\xi_v)_{v\in\mathcal{V}_{/\Gamma}}$ *in* $\prod_{v\in\mathcal{V}_{/\Gamma}} E_{\Gamma_v}(\pi_v)$*, if for every edge* $v' \overset{e}{-} v$ *of* $\mathcal{G}_{/\Gamma}$*, equation* (2) *is satisfied and not all coordinates of* $(\xi_v)_a$ *are* ± 1 *(*1 *in* $\mathrm{PSL}_2\mathbb{C}$*) then* $(\xi_v)_{v\in\mathcal{V}_{/\Gamma}}$ *lies in the image of* i_Γ*.*

This key-result illustrates how we can globally study $E_{\mathcal{G}_{/\Gamma}}(\pi)$ by using a finer decomposition $\mathcal{G}$, while $E_{\mathcal{G}}(\pi)$ itself might be obtained merging the E_{Γ_v}-varieties of the different pieces.

Before focusing on link-manifolds, we present how the $E_{\mathcal{G}}$-variety relates to Culler-Shalen construction, in a quite similar way as the A-polynomial or Tillmann's eigenvalue-variety do. The first part of Culler-Shalen construction produces a splitting tree for a group π derived from an ideal point of the character variety $X(\pi)$. Then, as Tillmann did for his eigenvalue-varieties, we can use the logarithmic-limit set of $E_{\mathcal{G}}(\pi)$ to capture its ideal points and lift them into $X(\pi)$. Moreover, we also get a characterization that determines which edge/arrow elements of the original splitting $\mathcal{G}$ become vertex elements in the new Culler-Shalen splitting.

More precisely, if $E_{\mathcal{G}}(\pi)$ is an algebraic subset of $\mathbb{C}^{*N}$, its logarithmic-limit set $E_{\mathcal{G}}(\pi)_\infty$ is a finite union of rational convex spherical polytopes in $\mathbb{S}^{N-1} \subset \mathbb{R}^N$. For each edge or arrow e of $\mathcal{G}$, we can fix a basis $\mathcal{B}_e$ of the edge or arrow group H_e. Each μ of $\mathcal{B}_e$ determines coordinates m_μ in $E_{\mathcal{G}}(\pi)$, u_μ in $E_{\mathcal{G}}(\pi)_\infty$ and $\mathfrak{m}_\mu$ in $\mathbb{C}[E_{\mathcal{G}}(\pi)]$, enabling the definition of a pairing:

$$\begin{array}{rccc} (\cdot): & H_e \times E_{\mathcal{G}}(\pi)_\infty & \to & \mathbb{R} \\ & ((h_\mu)_\mu, (u_\mu)_{\mu,e}) & \to & \sum_{\mu\in\mathcal{B}_e} h_\mu u_\mu \end{array}$$

Then we obtain the following Theorem 4, a generalization of Tillmann's Lemma 11 of [Til05] to the broader context of abelian trees of groups:

Theorem 4. *For any rational point* ξ_∞ *of* $E_{\mathcal{G}}(\pi)_\infty$*, there exist a splitting of* π *such that, for any edge or arrow* e*, and any* h *in* H_e*,*

$$h \text{ is in a vertex group} \iff h \cdot \xi_\infty = 0$$

When applied to torus splittings of link-manifolds, Culler-Shalen construction will produce essential surfaces, and this new criterion will precisely track down how these surfaces cross or interesect the edge or arrow tori. This is detailed in the rest of this chapter where we refine these general considerations in the specific case of link-manifolds and torus splittings. In that case, the edge and arrow groups H_e are all isomorphic to $\mathbb{Z}^2$ and the spaces $E(H_e)$ will be denoted by $E(T_e)$ where T_e is the corresponding torus.

First, a special attention is dedicated to the *peripheral* E*-variety*, obtained considering a trivial tree with a single vertex and only arrows; as already pointed out, it is equivalent to Tillmann's eigenvalue-variety of [Til05], but the peripheral E-variety of a link-manifold

M_L will be denoted here as $E_\partial(M_L)$. We restate some previously known results, reproving the most relevant ones within our new context. This includes the computation of the component corresponding to reducible characters – given by the first homology, hence the linking numbers – or the study of the peripheral E-varieties of a link-manifold before and after $1/q$-Dehn surgery. As Tillmann showed, there is a dimensional upper bound for the components E of $E_\partial(M_L)$ given by the number of components of the link:

$$\dim E \leq |L|$$

and this dimension is attained for the components of reducible characters and for the *geometric component* of hyperbolic link-manifolds. We re-expose this result here and also give a new interpretation of *strongly geometric cusp isolation* in terms of peripheral eigenvalue-variety. Following Walter Neumann and Alan Reid in [NR93], if $L^+ \sqcup L^-$ is a partition of cusps of a hyperbolic 3-manifold, L^+ is said to be *strongly geometrically isolated* from L^- if, after performing any integral Dehn-fillings replacing the cusps of L^+ by geodesics $(\gamma_{K^+})_{K^+ \subset L^+}$, any deformation on the cusps of L^- leaves the geometry of the γ_{K^+} invariant. We prove that this can be read in the part of the peripheral eigenvalue-variety corresponding to the geometric component X_0 of the character variety:

Theorem 5. *Let M_L be a hyperbolic link-manifold and $L^+ \sqcup L^-$ a partition of L.*
Then L^+ is strongly geometrically isolated from L^- if and only if $E_\partial(X_0)$ splits as a product $E^+ \times E^-$ with E^+ in $\prod_{K \subset L^+} E(T_K)$ and E^- in $\prod_{K \subset L^-} E(T_K)$.

Besides hyperbolic link-manifolds, we would expect the E_∂-variety of generic link-manifolds to admit a component of non-abelian characters whith the maximal dimension, which raises Question 1:

Question 1. *For which link-manifolds M_L does $X(M_L)$ admit peripherally maximal and non-abelian components?*

We will try to address this problem using $E_\mathcal{G}$-varieties associated to torus splittings. In particular, given a torus splitting of M_L over a tree $\mathcal{G}$, a double application of Theorem 3 produces Proposition 3 which relates the $E_\mathcal{G}$-variety of M_L to $E_\partial(M_L)$ and the peripheral E-varieties of the vertex link-manifolds.

The peripheral E-variety corresponds to the trivial splitting of a group, with one vertex and only arrows. In the last section of Chapter 2 we study the $E_\mathcal{G}$-varieties for non-trivial torus splittings of link-manifolds. We present the direct corollary of Theorem 3 for toric splittings of link-manifolds in the form of Corollary 3; finally, we briefly study the case of the $E_{\mathcal{G}_\mathcal{J}}$-variety, associated to the JSJ-dual graph of a link-manifold. The JSJ-decomposition can be thought as a maximal toric decomposition of the underlying link-manifold, and we explain how the $E_{\mathcal{G}_{\mathcal{J}}}$ may also be considered as the maximal $E_{\mathcal{G}}$-variety that may be constructed for M_L.

The next two chapters aim at answering Question 1 for two specific cases; first, in Chapter 3, we study *Brunnian links* in $\mathbb{S}^3$, links for wich all strict sublinks are trivial. Then, Chapter 4 focuses on *graph link-manifolds*, with the property that all the JSJ-pieces are Seifert-fibred. Those chapters both make use of Chapters 1 and 2 but are mutually independant. As a matter of fact, if general results on the $E_{\mathcal{G}}$-varieties are used in Chapter 4, we adopt a much more classical point of view in Chapter 3.

Eigenvalue-variety of Brunnian links

If L is a Brunnian link in $\mathbb{S}^3$, we can perform $1/q$ surgery on any component to produce a new link in $\mathbb{S}^3$. Using finer results of Mangum-Stanford from [MS01], we can precise this stability under Dehn-fillings. Indeed, a corollary of Mangum-Stanford work implies that a non-trivial $1/q$-Dehn-filling on a non-trivial Brunnian link always produces a non-trivial Brunnian link. This enables us to apply Kronheimer-Mrowka Theorem of [KM04] and construct irreducible representations associated to a suitable infinite family of integers, indexed by the components of the link. If L has 3 components or more, all the *linking numbers* are 0 and a special attention has to be drawn on Brunnian links with 2 components and nonzero linking number, for which the computation is a little harder, and the result slightly weaker.

After examining both cases, we finally obtain our first result on the E_∂-variety of Brunnian links in $\mathbb{S}^3$:

Theorem 1. *Let L be a Brunnian link in $\mathbb{S}^3$ and let M denote its exterior, then $X^{\mathrm{SL}_2\mathbb{C}}(M)$ admits a peripherally maximal and non-abelian component if and only if L is neither the trivial link or the Hopf-link.*

This completely answers Question 1 for Brunnian links. However, we do not use here $E_{\mathcal{G}}$-varieties for non-trivial splittings $\mathcal{G}$ but only the peripheral E-variety. Remarking that *splicing* Brunnian links maintains the Brunnian property, we succintly explain how to describe E-varieties of links obtained via *Brunnian trees*. Nonetheless, we do not carry out these considerations any further here, as we will apply the same ideas in the next chapter but in a quite different context.

$E_{\mathcal{G}}$-varieties of graph link-manifolds

In the final chapter, we focus on *graph link-manifolds*, exterior of links obtained by iterated splicing with *Seifert-fibred* link-manifolds. The combinatorial description of Seifert fibrings, such as Allen Hatcher's of [Hat10], produces presentations of the fundamental groups of Seifert-fibred manifolds. We will describe graph link-manifolds using *splice diagrams* as in [EN85]; these are trees with arrows and a new type of edges, ended by nodes (●), representing singular fibres of the fibring. All the edges and arrows are labeled with integers representing orders in the corresponding underlying fibrings. An example of splice diagram is presented in Figure 1.

Splice diagrams enable a quite precise description of the E_{∂}-variety of the link exterior. The labels of the splice diagram determine the linking numbers, which permits to fully describe the component of reducible characters with the standard arguments. In order to address Question 1 for this case, we want to construct components of irreducible characters.

First, we study Seifert-fibred link-manifolds. In this case, the splice diagram of M_L has one vertex, arrows indexed by L, and nodes indexed by the singular fibres C of the fibring. We obtain the following result for $\mathrm{PSL}_2\mathbb{C}$-characters:

Theorem 6. *The group $\pi_1 M_L$ admits irreducible representations in $\mathrm{PSL}_2\mathbb{C}$ if and only if*

$$|L| + |C| \geq 3$$

and, in that case, the peripheral $\mathcal{A}$-ideal corresponding to irreducible characters is

$$\mathcal{A}^{\mathrm{irr}}(M_L) = \langle \mathfrak{m}_K{}^{\alpha_{\widehat{K}}} \mathfrak{l}_K{}^{\alpha_K} - 1,\ K \subset L \rangle \tag{3}$$

where $\alpha_{\widehat{K}}$ and α_K are coefficients computed from the labels of the splice diagram. We also present a similar result for $\mathrm{SL}_2\mathbb{C}$, which gives a full answer to Question 1 for Seifert-fibred manifolds, both for character varieties in $\mathrm{SL}_2\mathbb{C}$ and $\mathrm{PSL}_2\mathbb{C}$.

The rest of this chapter is dedicated to graph manifolds constructed over non-trivial trees. However, the combinatorics involved increases quite rapidly, making it difficult to express precise statements in this introduction. We use Theorem 3 on the splice diagram and different contractions depending on our interests. Without entering into details, this enables us to obtain our final result for graph link-manifolds:

Theorem 2. *For any non-abelian graph link-manifold M_L with boundary, there exist a peripherally maximal and non-abelian component in $X^{\mathrm{PSL}_2\mathbb{C}}(M_L)$.*

Finally, we briefly outline how one could use the very same technics introduced here, to completely describe all the components of all the $E_{\mathcal{G}}$-varieties of a graph link-manifoldM_L;

although it would not determine the character variety, such a description would provide extensive information on all the Culler-Shalen splittings of M_L. The high complexity makes a precise description of all cases hardly manageable; we succintly explain, how, with enough scrutiny, one could study all the possible $E_\Gamma(\pi)$ for subtrees of the splice diagram, and then use Theorem 3 to determine all the possible components of the different $E_{\mathcal{G}}$-varieties.

After this brief presentation, we will now start with Chapter 1 and few recalls on link-manifolds, character varieties, and trees.

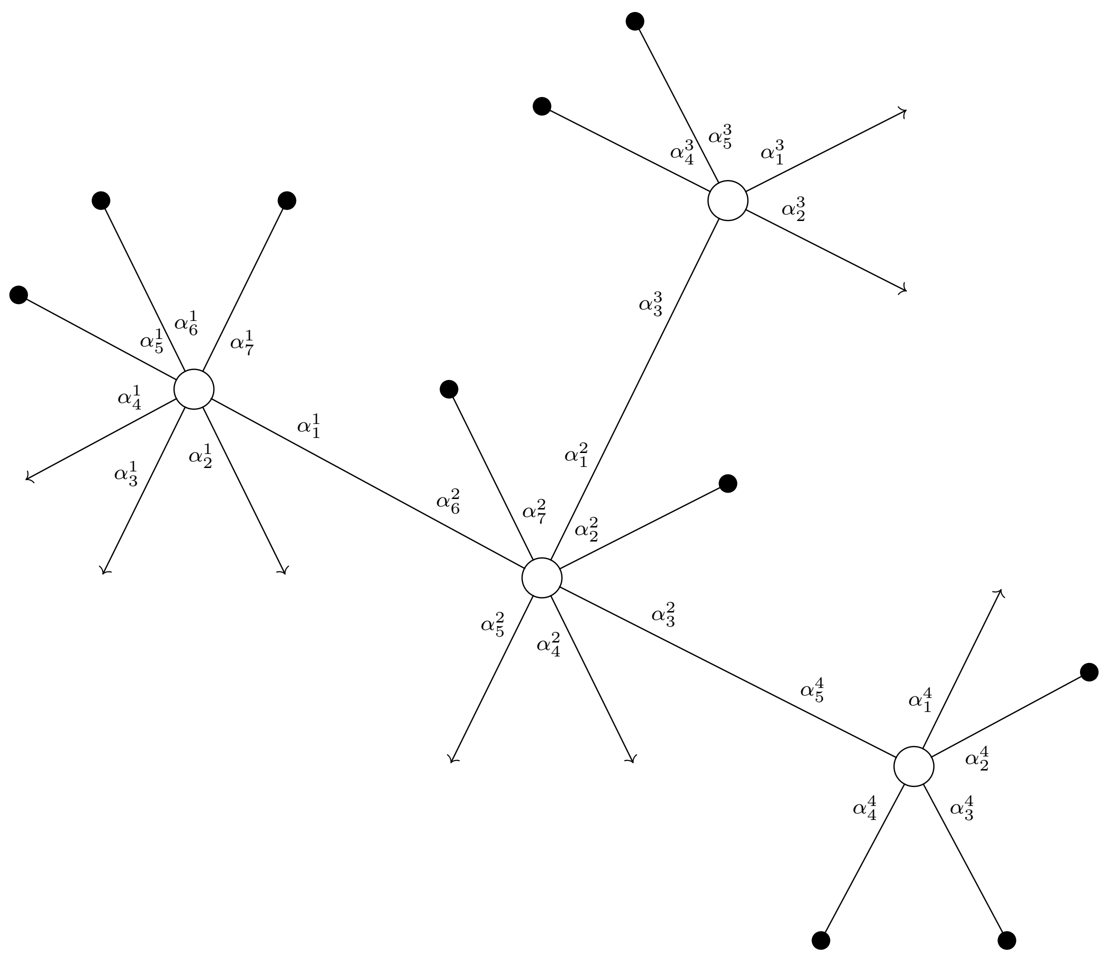


Figure 1 – A generic splice diagram

Chapter 1

Trees, characters and links

This introductive chapter aims at presenting some few classic objects and results that will be used later; first, we start recalling few properties of amalgamated products and splitting trees as in Serre's [Ser03]. The only new feature here is the addition of *arrows*, enabling to split and merge trees .

Next, we briefly present some notions of Algebraic Invariant Theory and we recall the notions of *representation and character variety* of finitely generated groups. Then, we describe how splittings of groups reflect on the character varieties and we give a quick summary of Culler-Shalen theory, using the $\mathrm{SL}_2\mathbb{C}$ character variety of 3-manifolds to produce essential surfaces.

Although the E-varieties that we will define in Chapter 2 could be constructed for generic 3-manifolds, we will restrict to exteriors of links in integer homology spheres. Hence, we follow with some results on *knot and link manifolds*, using [EN85] as a reference for this matter and finally close Section 1.3 with few considerations on *torus splittings* and JSJ *decomposition* of link manifolds.

1.1 Trees, arrows and splittings

In this section we recall the notions of pullback and pushouts in categories. Iterating pushouts yields amalgamating trees as in Serre's [Ser03]. We recall few aspects of this theory here, more precisely trees of groups and splitting trees. Besides few changes in notation, the only difference with [Ser03] is the presence of arrows in the trees, like edges attached to only one vertex.

1.1.1 Pushout, pullback

This section simply recalls the concepts of pullback and pushout in category theory. Even if we will mainly use these to take amalgamated products of groups and fibre products of algebraic varieties, we recall these notions in their broader aspects.

Let $\mathcal{C}$ be a category.

Definition 1.1.1 ((Co)Span)**.** A *(co)span* in $\mathcal{C}$ is a pair of morphisms with same (co)domain. A span will be denoted by

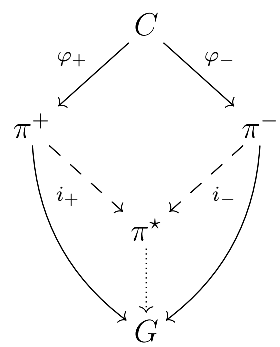
$$\pi^+ \xleftarrow{\varphi_+} C \xrightarrow{\varphi_-} \pi^-$$

and a cospan by

$$Y^+ \xrightarrow{j_+} Z \xleftarrow{j_-} Y^- .$$

The pushout of a span is the co-limit of the corresponding diagram:

Definition 1.1.2 (Pushout of a span)**.** The *pushout* of a span $\pi^+ \xleftarrow{\varphi_+} C \xrightarrow{\varphi_-} \pi^-$ is a cospan $\pi^+ \xrightarrow{i_+} \pi^\star \xleftarrow{i_-} \pi^-$ satisfying the universal property corresponding to following commutative diagram:



so $i_+ \circ \varphi_+ = i_- \circ \varphi_-$ and for any cospan $\pi^+ \xrightarrow{f_+} G \xleftarrow{f_-} \pi^-$, if

$$f_+ \circ \varphi_+ = f_- \circ \varphi_-$$

then there exist a unique morphism

$$f : \pi^\star \to G$$

such that

$$f \circ i_\pm = f_\pm$$

Lemma 1.1.1. *If the pushout* $\pi^+ \xrightarrow{i_+} \pi^\star \xleftarrow{i_-} \pi^-$ *exists, the object* $\pi^\star$ *is unique and will be denoted by*

$$\pi^\star = \pi^+ \ {}_{\varphi_+}\!\underset{C}{\bowtie}{}_{\varphi_-}\ \pi^-$$

Example 1.1.1. *The category* Grp *of groups with group morphisms admits pushouts, the* amalgamated product*. If each* $\pi^\pm$ *has a presentation,* $\pi^\pm = \langle G^\pm \mid R^\pm \rangle$*,* $\pi^\star$ *has a presentation:*

$$\pi^\star = \langle G^+ \sqcup G^- | R^+ \sqcup R^- \sqcup \varphi_+(\delta)\varphi_-(\delta)^{-1},\ \delta \in C \rangle$$

In that case, the pushout will be denoted by $\pi^+ \ {}_{\varphi_+}\!\underset{C}{*}{}_{\varphi_-}\ \pi^-$ *or* $\pi^+ *_C \pi^-$ *if the morphisms can be inferred.*

Example 1.1.2. *The category of commutative rings with ring maps admits pushouts given by the tensor product and the quotient. For a ring span* $A^+ \xleftarrow{\varphi^+} N \xrightarrow{\varphi^-} A^-$ *, we have*

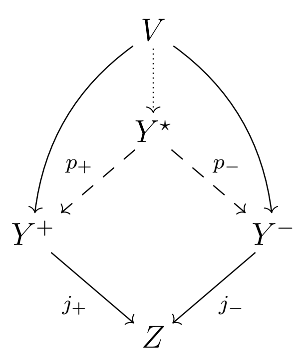
$$\begin{aligned} A^+ \underset{N}{\bowtie} A^- &= A^+ \underset{N}{\otimes} A^- \\ &= A^+ \otimes A^-/K \end{aligned}$$

where K *is the ideal of* $A^+ \otimes A^-/K$ *defined by*

$$K = \langle \varphi^+(x)a^+ \otimes a^- \ - \ a^+ \otimes \varphi^-(x)a^-,\ x \in N,\ a^\pm \in A^\pm \rangle$$

Symmetrically, a cospan can admit a universal span called the *pullback*:

Definition 1.1.3 (Pullback of a cospan)**.** The *pullback* of a cospan $Y^+ \xrightarrow{j_+} Z \xleftarrow{j_-} Y^-$ is a span $Y^+ \xleftarrow{p_+} Y^\star \xrightarrow{p_-} Y^-$ satisfying the universal property described in the following commutative diagram:



so $j_+ \circ p_+ = j_- \circ p_-$ and for any span $Y^+ \xleftarrow{f_+} V \xrightarrow{f_-} Y^-$, if

$$j_+ \circ f_+ = j_- \circ f_-$$

then there exist a unique

$$f : V \to Y^\star$$

such that

$$p_\pm \circ f = f_\pm$$

Lemma 1.1.2. *If the pullback* $Y^+ \xleftarrow{p_+} Y^\star \xrightarrow{p_-} Y^-$ *exists, the object* $Y^\star$ *is unique and will be denoted by*

$$Y^\star = Y^+ \,{}_{j_+}\!\!\underset{Z}{\times}{}_{j_-} Y^-$$

or $Y^+ \underset{Z}{\times} Y^-$ *if the morphisms can be inferred.*

Example 1.1.3. *The category of algebraic affine spaces admits pullbacks where, for a cospan* $Y^+ \xrightarrow{j_+} Z \xleftarrow{j_-} Y^-$ *of regular maps,*

$$Y^+ \underset{Z}{\times} Y^- = \{(y^+, y^-) \in Y^+ \times Y^- \mid j_+(y^+) = j_-(y^-)\}$$

In this case the pullback is also called fibred product.

Iterating these constructions will produce *trees*; we recall few definitions following Serre's [Ser03] with an additional feature, arrows, to facilitate more explicit surgery on trees.

1.1.2 Trees with arrows

As usual, a *tree* is a connected, simply-connected graph. We'll denote by $\mathcal{G}_{(\mathcal{V},\mathcal{E})}$ a tree $\mathcal{G}$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. An edge e of $\mathcal{E}$ between vertices v' and v of $\mathcal{V}$ will be denoted by $v' \overset{e}{-} v$.

Definition 1.1.4 (Tree with arrows)**.** A *tree with arrows* is a tree $\mathcal{G}_{(\mathcal{V},\mathcal{E})}$ and a set $\overrightarrow{\mathcal{E}}$ called *arrows* with an attaching map $\overrightarrow{\mathcal{E}} \to \mathcal{V}$. An arrow a attached to the vertex v will be denoted by $v \xrightarrow{a}$.

We denote by $\overline{\mathcal{E}}$ the union of edges and arrows $\mathcal{E} \sqcup \overrightarrow{\mathcal{E}}$ and $\mathcal{G}_{(\mathcal{V},\overline{\mathcal{E}})}$ the tree $\mathcal{G}$ with the additional arrows.

Remark 1.1.4. A tree in the sense of Serre is a tree with arrows and empty arrow set.

From now on, all trees have a (possibly empty) arrow set.

Arrows on trees enable to split/merge trees by identifying edges with pairs of arrows as in the following definition.

Definition 1.1.5. Let $\mathcal{G}^+_{(\mathcal{V}^+,\overline{\mathcal{E}^+})}$ and $\mathcal{G}^-_{(\mathcal{V}^-,\overline{\mathcal{E}^-})}$ be two trees with arrows and let $v^+ \xrightarrow{a^+}$ and $v^- \xrightarrow{a^-}$ be two arrows of $\overrightarrow{\mathcal{E}}^+$ and $\overrightarrow{\mathcal{E}}^-$ respectively; we denote by

$$\mathcal{G}^+ \;{}^{a^+}\!\underset{e}{\bowtie}^{a^-}\; \mathcal{G}^-$$

the tree $\mathcal{G}$ obtained by merging a^+ and a^- into and edge e defined as follows:

- the vertex set is $\mathcal{V}^+ \sqcup \mathcal{V}^-$
- the edge set is $\mathcal{E}^+ \sqcup \mathcal{E}^- \sqcup \{\, v^+ \overset{e}{-} v^- \,\}$
- the arrow set is $(\overrightarrow{\mathcal{E}}^+ \setminus \{a^+\}) \sqcup (\overrightarrow{\mathcal{E}}^- \setminus \{a^-\})$

We say that $(\mathcal{G}^+, \mathcal{G}^-)$ is obtained by *splitting* $\mathcal{G}$ at e. This pairing is compatible with the *contraction of trees* of Serre, with the suitable modifications to include arrows.

Definition 1.1.6 (Contraction of a tree with arrows)**.** Let $\mathcal{G}_{(\mathcal{V},\overline{\mathcal{E}})}$ be a tree with arrows. Let $\mathcal{E}_0$ be a subset of $\mathcal{E}$ and $\Gamma = (\Gamma_i)_{i\in I}$ the collection of trees obtained by splitting $\mathcal{G}$ on the edges of $\mathcal{E}_0$.

The *contraction* of $\mathcal{G}$ along Γ is the tree $\mathcal{G}_{/\Gamma}$ obtained by retracting each Γ_i in $\mathcal{G}$ into a single vertex. The vertex set $\mathcal{V}_{/\Gamma}$ of $\mathcal{G}_{/\Gamma}$ is in bijection with I so the collection Γ will be denoted by $(\Gamma_v)_{v\in\mathcal{V}_{/\Gamma}}$. Similarly, the edge set $\mathcal{E}_{/\Gamma}$ of $\mathcal{G}_{/\Gamma}$ is in bijection with $\mathcal{E}_0$.

Each edge $v' \overset{e}{-} v$ of $\mathcal{E}_{/\Gamma}$ comes from an edge in $\mathcal{G}$ and corresponds to two arrows in $\Gamma_{v'}$ and Γ_v, attached to the vertices according to their configurations in $\mathcal{G}$. All the other arrows of each Γ_v are attached to the corresponding vertex v of $\mathcal{V}_{/\Gamma}$ in $\mathcal{G}_{/\Gamma}$.

Remark 1.1.5. A tree $\mathcal{G}$ and any of its contraction always have the same arrow-set.

If $\mathcal{G}$ is a tree and $\mathcal{G}_{/\Gamma}$ is a contraction of $\mathcal{G}$, we can think of $\mathcal{G}$ as being obtained from $\mathcal{G}_{/\Gamma}$ by expanding each vertex v of $\mathcal{V}_{/\Gamma}$ into the tree Γ_v. Given a tree $\mathcal{G}'$, one may assign a tree Γ_v to each vertex of v, with a pair of distinguished , to construct a tree $\mathcal{G}$ such that $\mathcal{G}' = \mathcal{G}_{/\Gamma}$.

Definition 1.1.7 (Binding decomposition)**.** We will write

$$\mathcal{G} = (\mathcal{G}' \gg\!= \Gamma)$$

when $\mathcal{G}' = \mathcal{G}_{/\Gamma}$ is the contraction of $\mathcal{G}$ along the collection Γ and this will be called a *binding decomposition* [1] of $\mathcal{G}$.

The contraction as defined in Definition 1.1.6 is compatible with the merging defined in Definition 1.1.5.

Let $\mathcal{G}^+ = (\mathcal{G}^+_{/\Gamma^+} \gg= \Gamma^+)$ and $\mathcal{G}^- = (\mathcal{G}^-_{/\Gamma^-} \gg= \Gamma^-)$ be two trees and let $v^+ \xrightarrow{a^+}$ and $v^- \xrightarrow{a^-}$ be two arrows of $\mathcal{G}^+$ and $\mathcal{G}^-$ respectively. Let $\mathcal{G}$ be the tree $\mathcal{G}^+ \ {}^{a^+}\!\underset{e}{\bowtie}^{a^-} \mathcal{G}^-$ as in Definition 1.1.5.

The vertices v^+ and v^- belong to trees Γ_{u^+} and Γ_{u^-} in the respective collections Γ^+ and Γ^-; the tree $\Gamma_{u^+} \ {}^{a^+}\!\underset{e}{\bowtie}^{a^-} \Gamma_{u^-}$ is a subtree of $\mathcal{G}$ and the family

$$\Gamma = (\Gamma^+ \setminus \Gamma_{u^+}) \sqcup (\Gamma^+ \setminus \Gamma_{u^+}) \sqcup (\Gamma_{u^+} \ {}^{a^+}\!\underset{e}{\bowtie}^{a^-} \Gamma_{u^-})$$

is a partition in subtrees of $\mathcal{G}$.

Lemma 1.1.3. *With these notations, we have:*

$$\mathcal{G}_{/\Gamma} = \mathcal{G}^+_{/\Gamma^+} \ {}^{a^+}\!\underset{e}{\bowtie}^{a^-} \mathcal{G}^-_{/\Gamma^-}$$

We close this section presenting two natural binding decompositions that exist for every trees.

First, any tree $\mathcal{G}$ decomposes as

$$\mathcal{G} = (* \gg= \mathcal{G})$$

where $*$ is the tree $\mathcal{G}_{/\mathcal{G}}$ with a single vertex and all the arrows of $\mathcal{G}$. This decomposition contracts all the tree onto a single vertex.

Definition 1.1.8. The binding decomposition $(* \gg= \mathcal{G})$ of a tree $\mathcal{G}$ is called the *trivial binding decomposition*.

On the other hand, let $\overrightarrow{\mathcal{V}} = \{\overrightarrow{v}, v \in \mathcal{V}\}$ be the collection of vertices of $\mathcal{G}$ with arrows attached for each adjacent edge or arrow in $\mathcal{G}$. We have the binding decompostion

$$\mathcal{G} = (\mathcal{G} \gg= \overrightarrow{\mathcal{V}})$$

which is essentially identical to $\mathcal{G}$.

[1] In an informal type theory (see [Uni13]), if Tree_a denotes the type of trees with vertices of type a, the tree $\mathcal{G}$ has type $\mathrm{Tree}_{\mathcal{V}}$, the tree $\mathcal{G}_{/\Gamma}$ has type $\mathrm{Tree}_{\mathcal{V}_{/\Gamma}}$ and the collection Γ is a map $\mathcal{V}_{/\Gamma} \to \mathrm{Tree}_{\mathcal{V}}$. This is similar to the *binding* operator for a *monad* m in functional programming languages (see [Mar10] for example):

$$(\_ \gg= \_) \ : \ \mathrm{m}\, a \to (a \to \mathrm{m}\, b) \to \mathrm{m}\, b$$

and we chose to use the same name and notation here.

Definition 1.1.9. The binding decomposition $(\mathcal{G} \gg= \overrightarrow{\mathcal{V}})$ of a tree $\mathcal{G}$ is called the *identical binding decomposition*.

1.1.3 Trees of groups

In this section, we present the notion of *tree of groups* as in Serre's [Ser03], slightly modified to include tree with arrows, and using the notations that we have introduced so far.

Any tree $\mathcal{G}_{(\mathcal{V},\overline{\mathcal{E}})}$ defines a category $\mathcal{C}_\mathcal{G}$ with objects $\mathcal{V} \sqcup \mathcal{E} \sqcup \overrightarrow{\mathcal{E}}$ and the following morphisms:

- for each arrow $v \xrightarrow{a}$ a morphism $a \to v$
- for each edge $v' \overset{e}{-} v$ a span $v' \longleftarrow e \longrightarrow v$.

Definition 1.1.10 (Tree of groups)**.** A *tree of groups* is a tree $\mathcal{G}$ and a functor π from $\mathcal{C}_\mathcal{G}$ to the category $(\mathrm{Grp}, \hookrightarrow)$ of groups with monomorphisms.

In other words, the tree $\mathcal{G}_{(\mathcal{V},\overline{\mathcal{E}})}$ is equipped with groups π_v, C_e, and C_a for each vertex v, each edge e and each arrow a, respectively, and with injective morphisms:

- $\varphi_{e_{v'}} : C_e \to \pi_{v'}$ and $\varphi_{e_v} : C_e \to \pi_v$ for each edge $v' \overset{e}{-} v$
- $\varphi_a : C_a \to \pi_v$ for each arrow $v \xrightarrow{a}$.

Let $(\mathcal{G}, \pi)$ be a tree of groups, we want to form the group obtained amalgamating all the groups π_v along the edge groups C_e. As in [Ser03], this is the direct limit in the category $(\mathrm{Grp}, \hookrightarrow)$:

Definition 1.1.11. The *fundamental group of a tree of groups* $(\mathcal{G}, \pi)$ is the group

$$\pi_\mathcal{G} = \varinjlim_{C_e} \pi_v$$

In other words, there exist injective morphisms $i_v : \pi_v \to \pi_\mathcal{G}$ for each vertex v of $\mathcal{G}$ satisfying

$$i_{v'} \circ \varphi_{e_{v'}} = i_v \circ \varphi_{e_v}$$

for any edge $v' \overset{e}{-} v$ and such that, for any group H and any collection of morphisms $f_v : \pi_v \to H$, if $f_{v'} \circ \varphi_{e_{v'}} = f_v \circ \varphi_{e_v}$ for each edge $v' \overset{e}{-} v$ of $\mathcal{G}$, then there exist a unique $f : \pi_\mathcal{G} \to H$ such that $f_v = f \circ i_v$ for each vertex v of $\mathcal{G}$.

Remark 1.1.6. For each arrow a of $\mathcal{G}$, the group C_a injects in $\pi_{\mathcal{G}}$ by composition of φ_a and i_v.

By naturality of the construction, tree of groups covariantly transport the merging operator into amalgamated products:

Lemma 1.1.4. *For any tree of group* $(\mathcal{G}, \pi)$*, if* $\mathcal{G}$ *splits as* $\mathcal{G}^+ \; {}^{a^+}\!\underset{e}{\bowtie}^{a^-} \; \mathcal{G}^-$*, then*

$$\pi_{\mathcal{G}} = \pi_{\mathcal{G}_+} *_{C_e} \pi_{\mathcal{G}_-}$$

Corollary 1.1.5. *Let* $(\mathcal{G}, \pi)$ *be a tree of groups. For any connected subtree* $\mathcal{G}'$ *of* $\mathcal{G}$*, the pair* $(\mathcal{G}', \pi_{|_{\mathcal{G}'}})$ *is a tree of group and the fundamental group of* $\mathcal{G}'$ *injects in the fundamental group of* $\mathcal{G}$*.*

Finally, we present how tree of groups transport binding decompostion.

Lemma 1.1.6. *Let* $(\mathcal{G}, \pi)$ *be a tree of groups with a binding decomposition*

$$\mathcal{G} = (\mathcal{G}_{/\Gamma} \gg= \Gamma)$$

The functor π *induces by restriction a tree of group on each* Γ_v *for* v *in* $\mathcal{V}_{/\Gamma}$*. It also defines a tree of group structure on* $\mathcal{G}_{/\Gamma}$ *defining, for any* v *in* $\mathcal{V}_{/\Gamma}$*,*

$$\pi_v = \pi_{\Gamma_v}$$

By Lemma 1.1.4 and Lemma 1.1.6, the family of edge and arrow groups is preserved by merging and binding. Therefore, any predicate on the edge and arrow groups will be preserved by those operations.

Definition 1.1.12 (Abelian tree of groups)**.** A tree of groups is called *abelian* if all the edge and arrow groups are abelian.

Remark 1.1.7. Similarly, we could define *cyclic tree of groups*, *free tree of groups*, *free abelian tree of groups*, etc. . . when all the edge and arrow groups are cyclic, free, free abelian, etc. . .

1.1.4 Splitting trees

Given a fixed group $\pi^\star$, we consider the differents tree of goups with fundamental group $\pi^\star$.

Definition 1.1.13. Let $\pi^\star$ be a group. A *splitting tree* for $\pi^\star$ is a tree of groups $(\mathcal{G}, \pi)$ whose fundamental group is isomorphic to $\pi^\star$.

As one could expect, most results on trees of groups have their counterpart for splitting trees. Using Lemma 1.1.4 and Lemma 1.1.6, the notion of splitting tree is natural for the inclusion and bindings:

Proposition 1.1.7. *Let* $\pi^+ \xleftarrow{\varphi_+} C \xrightarrow{\varphi_-} \pi^-$ *be a span and and let* $(\mathcal{G}^+, \pi)$ *and* $(\mathcal{G}^-, \pi)$ *be two respective splittings of* π^+ *and* π^-*. If there exist two arrows* $a^\pm$ *of* $\mathcal{G}^\pm$ *such that* $\pi_{a^\pm} = C$ *and* $\varphi_{a^\pm} = \varphi_\pm$ *then*

$$\mathcal{G}^\star = \mathcal{G}^+ \ {}^{a^+}\!\!\bowtie^{a^-} \mathcal{G}^-$$

is a splitting tree for $\pi^\star = \pi^+ \underset{C}{*} \pi^-$.

Corollary 1.1.8. *Let* $\pi^\star$ *be group with a splitting tree* $\mathcal{G}$*. Any subtree of* $\mathcal{G}$ *is a splitting tree of a subgroup of* $\pi^\star$.

Remark 1.1.8. In general, not all subgroups can appear as fundamental groups of subtrees of a given splitting tree.

Proposition 1.1.9. *Let* $\pi^\star$ *be a group with a splitting tree* $\mathcal{G} = (\mathcal{G}_{/\Gamma} \gg\!= \Gamma)$*. Then* $\mathcal{G}_{/\Gamma}$ *is a splitting tree for* $\pi^\star$ *and each* Γ_v *is a splitting tree for the corresponding vertex group.*

Now, following Definition 1.1.12, we use the same naming convention:

Definition 1.1.14. A splitting $(\mathcal{G}, \pi)$ of a group is called an *abelian splitting* if all the edge and arrow groups are *abelian*.

Remark 1.1.9. Similarly, we could define *cyclic splittings*, *free splittings*, *free abelian splittings*, etc… when all the edge and arrow groups are cyclic, free, free abelian, etc…

1.2 Representation and character variety

Before defining character varieties, we need few tools from algebraic geometry.

1.2.1 Algebraic groups & Invariant theory

Let's review some useful results on algebraic groups and invariant theory. More details can be found in [PV94].

Definition 1.2.1. An *algebraic group* is an algebraic variety with a group structure such that the multiplication and inversion are regular functions.

A *map of algebraic groups* is a group homomorphism that is also a regular map.

Definition 1.2.2. Let G be an algebraic group and Z an algebraic set.

An *algebraic action of G over Z* is a morphism from G to the group of birregular self-maps of Z. For an algebraic action σ, we'll denote by σ_g the corresponding birregular self-map.

Example 1.2.1. *Any algebraic group G acts on itself by conjugation via*

$$\kappa_{g_0} : g \to g_0\, g\, g_0^{-1}$$

We will always refer to this action for an algebraic group acting on itself.

Let G be an algebraic group and Z and algebraic set. The quotient of Z by an algebraic action of G may not be an algebraic set. We need the notion of *algebraic quotient*.

An algebraic action σ of G over Z produces, by composition, an action on the ring of regular functions $\mathbb{C}[Z]$ via, for any g in G,

$$\begin{array}{rccc} {\sigma_g}^* : & \mathbb{C}[Z] & \to & \mathbb{C}[Z] \\ & f & \to & f \circ \sigma_{g^{-1}} \end{array}$$

Definition 1.2.3. Let G be an algebraic group acting on an algebraic set Z.

The *ring of G-invariant functions of Z* is the subring of $\mathbb{C}[Z]$, denoted by $\mathbb{C}[Z]^G$, of regular functions invariant under all ${\sigma_g}^*$ for $g \in G$:

$$\mathbb{C}[Z]^G = \{f \in \mathbb{C}[Z] \mid \forall\, g \in G,\ {\sigma_g}^* f = f\}$$

For any algebraic space V, a regular map $f : Z \to V$ is *G-invariant* if $P \circ f$ is G-invariant for any P in $\mathbb{C}[V]$.

Example 1.2.2. *If G is linear (i.e. a subgroup of $GL_n\mathbb{C}$), the trace function* $\mathrm{tr} : G \to \mathbb{C}$ *is G-invariant for the conjugation action.*

Example 1.2.3. *The trace function of square*

$$\begin{array}{rccc} \mathrm{tr}_2 : & \mathrm{PSL}_2\mathbb{C} & \to & \mathbb{C} \\ & A & \to & \mathrm{tr}(A^2) \end{array}$$

is $\mathrm{PSL}_2\mathbb{C}$*-invariant on* $\mathrm{PSL}_2\mathbb{C}$.

Example 1.2.4. *The square of the trace function*

$$\begin{array}{rccc} \mathrm{tr}^2 : & \mathrm{PSL}_2\mathbb{C} & \to & \mathbb{C} \\ & A & \to & (\mathrm{tr}(A))^2 \end{array}$$

is also a $\mathrm{PSL}_2\mathbb{C}$*-invariant on* $\mathrm{PSL}_2\mathbb{C}$.

Remark 1.2.5. The two functions tr_2 and tr^2 differ only by a constant; for any A in $\mathrm{PSL}_2\mathbb{C}$,

$$\mathrm{tr}(A^2) = (tr(A))^2 - 2$$

We want to define the algebraic quotient of Z by G such that its ring of regular functions is $\mathbb{C}[Z]^G$. To be able to do this, we need an additional hypothesis on G.

Definition 1.2.4. A group G is called *reductive* if for any finite dimensional rational representation $\rho : G \to GL(V)$, any G-invariant subspace of V admits a complementary G-invariant subspace in V.

Example 1.2.6. *As explained in [PV94], the Zariski closure of a compact (in the classic topology) subgroup K of $GL_n\mathbb{C}$ is reductive. For example, $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $O_n(\mathbb{C})$, $SO_n(\mathbb{C})$ and $Sp_n(\mathbb{C})$ are reductive.*

Lemma 1.2.1. *[PV94] Let G be an algebraic group acting on an algebraic variety Z. If G is reductive, then the ring of G-invariant $\mathbb{C}[Z]^G$ is finitely generated.*

Moreover, let Y be an other algebraic space with a G-action. For any regular G-invariant map $f : Y \to Z$, if $f^*(\mathbb{C}[Z]) = \mathbb{C}[Y]$, then

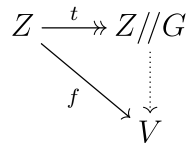
$$f^*(\mathbb{C}[Z]^G) = \mathbb{C}[Y]^G$$

For a reductive algebraic group G acting on Z, we can therefore define the following:

Definition 1.2.5. The algebraic set X such that $\mathbb{C}[X] = \mathbb{C}[Z]^G$ is called the *algebraic quotient* of Z by G and will be denoted

$$X = Z//G$$

There exist a surjective G-invariant map $t : Z \to Z//G$, such that, for any algebraic space V, and for any regular map $f : Z \to V$, f factors by t in the following diagram if and only if f is G-invariant.



1.2.2 Representation & character variety

Definition 1.2.6. Let G be an algebraic group and π a finitely generated group. The *representation variety* of π in G is the set

$$R^G(\pi) = \mathrm{Hom}(\pi, G).$$

Since π is finitely generated, $R^G(\pi)$ inherits the algebraic structure of G. Any finitely generated presentation $\pi \cong \langle \gamma_1, \ldots, \gamma_n | K \rangle$ of π provides an algebraic description of $R^G(\pi)$ as the subset of G^n which satisfy the equations induced by K.

For any morphisms $\delta : G \to G'$ and $\varphi : \pi' \to \pi$, we have the following commutative diagram of regular maps between representation varieties:

$$\begin{array}{ccc} R^G(\pi) & \xrightarrow{\varphi^*} & R^G(\pi') \\ \delta_* \downarrow & & \downarrow \delta_* \\ R^{G'}(\pi) & \xrightarrow[\varphi^*]{} & R^{G'}(\pi') \end{array} \tag{1.1}$$

where $\varphi^* \rho = \rho \circ \varphi$ and $\delta_* \rho = \delta \circ \rho$

In particular, any two finitely generated presentations of the same group π produce birregularly equivalent algebraic structures on $R^G(\pi)$, so the algebraic structure of the representation variety is independent of the particular choice of presentation.

The conjugation action κ of G on itself produces, via κ_{g*} in diagram (1.1), an algebraic action of G on $R^G(\pi)$. For any ρ in $R^G(\pi)$ and g in G, we'll denote by ρ^g the conjugated representation $\kappa_{g^{-1}*}\rho$.

If G is reductive, we can define the character variety of π in G:

Definition 1.2.7. The space $X^G(\pi) = R^G(\pi)//G$ is called the *character variety* of π in G.

We denote by $t : R^G(\pi) \to X^G(\pi)$ the natural projection map.

Remark 1.2.7. Two conjugated representations always have the same image by t but the converse is not true in general.

As explained in Appendix A, any G-invariant regular function τ on G produces, for any γ in π, a regular function τ_γ on $X(\pi)$ by evaluation. In particular, for $G = \mathrm{SL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{C}$, we'll denote by I_γ the functions τ_γ associated to tr in $\mathrm{SL}_2\mathbb{C}$ and $\mathrm{tr}_2 : \pm A \to \mathrm{tr}(A^2)$ in $\mathrm{PSL}_2\mathbb{C}$. Any character χ of $X^{\mathrm{SL}_2\mathbb{C}}(\pi)$ or $X^{\mathrm{PSL}_2\mathbb{C}}(\pi)$ is determined by the function

$$\begin{array}{rccc} \widehat{\chi} : & \pi & \to & \mathbb{C} \\ & \gamma & \to & I_\gamma(\chi) \end{array}$$

and the projection map t can be thought as the map

$$\begin{array}{cccc} t: & R^G(\pi) & \to & X^G(\pi) \\ & \rho & \to & \chi_\rho : \gamma \to \tau(\rho(\gamma)) \end{array}$$

hence the name *character variety*.

The next lemma illustrates the functoriality of the character variety:

Lemma 1.2.2. *For any morphisms $\delta : G \to G'$ and $\varphi : \pi' \to \pi$, diagram* (1.1) *extends to the corresponding character varieties to form the following commutative diagram:*

$$\begin{tikzcd} R^G(\pi) \arrow[rrr] \arrow[dddd] \arrow[dr, "t"] & & & R^G(\pi') \arrow[dddd] \arrow[dl, "t"'] \\ & X^G(\pi) \arrow[r, "\varphi^*"] \arrow[d, "\delta^*"'] & X^G(\pi') \arrow[d, "\delta^*"] & \\ & X^{G'}(\pi) \arrow[r, "\varphi^*"'] & X^{G'}(\pi') & \\ & & & \\ R^{G'}(\pi) \arrow[rrr] \arrow[uur, "t"] & & & R^{G'}(\pi') \arrow[uul, "t"'] \end{tikzcd} \tag{1.2}$$

Example 1.2.8 (Abelianization)**.** *Let's fix an algebraic reductive group G. We'll write R and X for R^G and X^G.*

Any finitely generated group π admits an abelianization π^{ab} given by the exact sequence

$$1 \longrightarrow [\pi, \pi] \longrightarrow \pi \xrightarrow{p} \pi^{\mathrm{ab}} \longrightarrow 1$$

For γ in π, we'll denote by $[\gamma]$ the abelianization of γ, $p(\gamma)$. By Lemma 1.2.2, the morphism $p : \pi \to \pi^{\mathrm{ab}}$ produces the following commutative diagram:

$$\begin{array}{ccc} R(\pi^{\mathrm{ab}}) & \xrightarrow{p^*} & R(\pi) \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ X(\pi^{\mathrm{ab}}) & \xrightarrow[p^*]{} & X(\pi) \end{array} \tag{1.3}$$

Definition 1.2.8. For any finitely generated group π we define

$$R^{\mathrm{ab}}(\pi) = p^*(R(\pi^{\mathrm{ab}}))$$

$$X^{\mathrm{ab}}(\pi) = p^*(R(\pi^{\mathrm{ab}})) = t(R^{\mathrm{ab}}(\pi))$$

Since π is finitely generated, π^{ab} is isomorphic to some $\mathbb{Z}^n \oplus \bigoplus_{i=1}^s \mathbb{Z}_{\alpha_i}$. Therefore, any ρ in $R^{\mathrm{ab}}(\pi)$ is given by $n+s$ elements of G, $(m_j)_{j=1}^n$, and $(c_i)_{i=1}^s$, such that

- $[m_j, m_k] = [m_j, c_i] = [c_i, c_k] = 1$ *for all* i, j, k
- $c_i^{\alpha_i} = 1$ *for all* i

and for any γ in π, if $[\gamma] = (a_1, \dots, a_n, b_1, \dots, b_s)$,

$$\rho(\gamma) = m_1^{a_1} \cdots m_n^{a_n} c_1^{b_1} \cdots c_s^{b_s}$$

In other words, $R^{\mathrm{ab}}(\pi)$ is isomorphic to abelian families of $n+s$ elements of G with the s last ones having torsion α_i for $1 \leq i \leq s$.

If G is linear, a representation ρ produces an action of π over a finite-dimensional $\mathbb{C}$-vector space V. In this context, we have the following definition:

Definition 1.2.9. A representation is *reducible* if there exist a non trivial proper subspace of V, stable under the action of π produced by ρ. Otherwise, ρ is said to be *irreducible*.

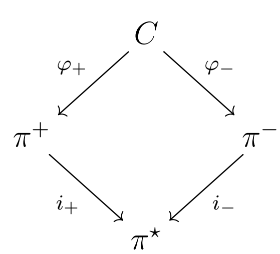
Similarly, a character χ is *irreducible* if there exist ρ irreducible in $t^{-1}\chi$ and χ is *reducible* if all such representations are.

We write $R^{\mathrm{red}}(\pi)$, $R^{\mathrm{irr}}(\pi)$, $X^{\mathrm{red}}(\pi)$ and $X^{\mathrm{irr}}(\pi)$ for the respective subspaces of $R(\pi)$ and $X(\pi)$.

1.2.3 Amalgams, splittings and character varieties

In this section, we study some properties of the character varieties of amalgamated products and trees of groups introduced in Section 1.1.

Let $\pi^\star$ be the amalgamated product of the span $\pi^+ \xleftarrow{\varphi_+} C \xrightarrow{\varphi_-} \pi^-$. The commutative diagram



produces contravariantly a commutative diagram on the representation varieties:

$$\begin{array}{ccccc} & & R(C) & & \\ & \nearrow^{{\varphi_+}^*} & & \nwarrow^{{\varphi_-}^*} & \\ R(\pi^+) & & & & R(\pi^-) \\ & \nwarrow_{{i_+}^*} & & \nearrow_{{i_-}^*} & \\ & & R(\pi^\star) & & \end{array} \tag{1.4}$$

with a natural identification:

$$R(\pi^\star) \;\cong\; \{(\rho_+, \rho_-) \in R(\pi^+) \times R(\pi^-) \mid \rho_+ \circ \varphi_+ = \rho_- \circ \varphi_- \text{ in } R(C)\}$$

In other words, R transforms amalgamated products into fibred products:

$$R(\pi^+ \underset{C}{*} \pi^-) = R(\pi^+) \underset{R(C)}{\times} R(\pi^-) \tag{1.5}$$

In this situation, for any ρ in $R(\pi)$, if ρ_+ and ρ_- denote the corresponding representations in $R(\pi^+)$ and $R(\pi^-)$, we write

$$\rho = \rho_+ \; {}^{\varphi_+}\!\underset{C}{\times}{}^{\varphi_-} \; \rho_-$$

The commutative diagram (1.4) can be pushed down via t to produce the following commutative diagram:

$$\begin{array}{ccccccc} & & & R(C) & & & \\ & \nearrow & & \downarrow & & \nwarrow & \\ & & & X(C) & & & \\ & & \nearrow & & \nwarrow & & \\ R(\pi^+) & \longrightarrow & X(\pi^+) & & X(\pi^-) & \longleftarrow & R(\pi^-) \\ & & \nwarrow & & \nearrow & & \\ & & & X(\pi^\star) & & & \\ & \nwarrow & & \uparrow & & \nearrow & \\ & & & R(\pi^\star) & & & \end{array} \tag{1.6}$$

Where the vertical and horizontal arrows are t and the other ones are pull-backs of the original group morphisms.

However, it is not always true that a pair of characters in $X(\pi^+) \times X(\pi^-)$ with the same image under φ_+ and φ_- produces a character in $X(\pi^\star)$. Indeed, a pair (χ_+, χ_-) of such characters would only ensure the existence of two representations ρ_+ and ρ_- such that $t(\rho_+ \circ \varphi_+) = t(\rho_- \circ \varphi_-)$ in $X(C)$ but, by remark 1.2.7 this is generally not sufficient to find a pair a representations that coincide in $R(C)$. If $t^{-1}({\varphi_+}^*\chi_+)$ (or $t^{-1}({\varphi_-}^*\chi_-)$) is exactly the orbit under conjugation of $\rho_+ \circ \varphi_+$ (resp. $\rho_- \circ \varphi_-$) then we may find g in G so that ${\varphi_+}^*({\rho_+}^g) = {\varphi_-}^*(\rho_-)$. In that case, $\rho'_+ = \rho_+^g$ and ρ_- agree on C and we can form $\rho = \rho'_+ \; {}^{\varphi_+}\!\underset{C}{\bowtie}{}^{\varphi_-} \; \rho_-$. The character $\chi = t(\rho)$ satisfies the expected equations.

$$\begin{aligned} {i_+}^*\chi &= \chi_+ \\ {i_-}^*\chi &= \chi_- \end{aligned}$$

Moreover, let χ be a character of $X(\pi^\star)$, then $({i_+}^*\chi, {i_-}^*\chi)$ does not completely determine χ: let ρ be a representation in $t^{-1}(\chi)$ with image ρ_+ and ρ_- in $R(\pi^+)$ and $R(\pi^-)$ respectively. Then, for any non trivial centralizer g of $\rho(C)$ in G (if it exists),

$$\rho_+ \; {}^{\varphi_+}\!\underset{C}{\times}{}^{\varphi_-} \; (\rho_-)^g$$

defines a new representation ρ_g in $R(\pi^\star)$. Although the new character $\chi_g = t(\rho_g)$ satisfies ${i_\pm}^*\chi_g = {i_\pm}^*\chi$, χ is in general different from χ_g in $X(\pi^\star)$.

It follows that, unlike representation varieties, character varieties **do not** convert amalgamated products into fibred products. However, by commutativity of diagram (1.6), the universal property of the fibred product yields a regular map

$$X(\pi^\star) \to X(\pi^+) \underset{X(C)}{\times} X(\pi^-)$$

Iterating amalgamated products yields trees of groups and splitting trees as in Section 1.1. We introduce here some notations for that case.

Let π be a finitely generated group with a splitting tree $\mathcal{G}$. For each vertex v of $\mathcal{G}$, there exist an injective morphism $i_v : \pi_v \to \pi$. By diagram (1.2), a splitting of π over $\mathcal{G}$ produces, for any vertex v of $\mathcal{G}$, a regular map

$${i_v}^* : X(\pi) \to X(\pi_v)$$

Any property on a character in $X(\pi)$ can then be studied on the different pieces $X(\pi_v)$; for example:

Definition 1.2.10 (Everywhere abelian characters)**.** Let $\mathcal{G}$ be a splitting tree for π. For any χ in $X(\pi)$, we say that χ is *everywhere abelian* if, ${i_v}^*\chi$ is in $X^{\mathrm{ab}}(\pi_v)$ for all the vertices v of $\mathcal{G}$.

If G is linear, we also define the following:

Definition 1.2.11 (Everywhere (ir)reducible characters)**.** Let $\mathcal{G}$ be a splitting tree for π. A character χ in $X(\pi)$ is *everywhere irreducible* (resp. *everywhere reducible*), if, for all vertices v of $\mathcal{G}$, ${i_v}^*\chi$ is in $X^{\mathrm{irr}}(\pi_v)$ (resp. $X^{\mathrm{red}}(\pi_v)$).

Remark 1.2.9. Any reducible character is everywhere reducible and any everywhere irreducible character is irreducible. In particular, any character irreducible on at least one piece is irreducible.

More generally, let $\mathcal{V}$ denote the set of vertices of $\mathcal{G}$, then any irreducible component X of $X(\pi)$ defines a map

$$\eta_X : \mathcal{V} \to \{\mathrm{irr}, \mathrm{red}\}$$

where $\eta_X(v) = \mathrm{irr}$ if ${i_v}^*X$ contains irreducible characters of $X(\pi_v)$ and $\eta_X(v) = \mathrm{red}$ if ${i_v}^*X$ contains only reducible characters.

Definition 1.2.12 (Type of component)**.** For any component X of $X(\pi)$, η_X is called the *type* of X.

Remark 1.2.10. By Definitions 1.2.10 and 1.2.11, a component is *everywhere irreducible* or resp. *everywhere reducible* if it has constant type.

It is then quite natural to ask the following Question:

Question 2. *Given a group π and a splitting tree $\mathcal{G}$ with vertex set $\mathcal{V}$, what maps*

$$\eta : \mathcal{V} \to \{\mathrm{irr}, \mathrm{red}\}$$

can appear as types of components of $X(\pi)$?

Types of components may be studied using binding decompositions (see Definitions 1.1.6 and 1.1.7).

Definition 1.2.13. Given a type η, a binding decomposition $(\mathcal{G}_{/\Gamma} \gg= \Gamma)$ of $\mathcal{G}$ is called *compatible* with η if, for any vertex v of $\mathcal{G}_{/\Gamma}$, η is constant on the tree Γ_v.

Remark 1.2.11. Obviously, the identical decomposition $(\mathcal{G} \gg= \overrightarrow{\mathcal{V}})$ (see Definition 1.1.9) is compatible with any type η.

On the other hand, the trivial decomposition $(* \gg= \mathcal{G})$ of a tree (see Definition 1.1.8) is only compatible with constant types.

Proposition 1. *For any irreducible component X of $X(\pi)$, there exist a binding decomposition*

$$\mathcal{G} = (\mathcal{G}_{/\Gamma} \ggg= \Gamma)$$

which is compatible with η_X and such that, for any binding decomposition $(\mathcal{G}_{/\Gamma'} \ggg= \Gamma')$ of $\mathcal{G}$, if the decomposition is compatible with η_X, then there exist a collection Γ'' of subtrees of $\mathcal{G}' = \mathcal{G}_{/\Gamma'}$ such that

$$\mathcal{G}_{/\Gamma} = \mathcal{G}'_{/\Gamma''}$$

Proof. We can construct $\mathcal{G}_{/\Gamma}$ defining the edge set $\mathcal{E}_{/\Gamma}$ in $\mathcal{G}$.

Let X be an irreducible component of $X(\pi)$; it defines a map $\eta : \mathcal{V} \to \{\mathrm{irr}, \mathrm{red}\}$ as in Definition 1.2.12 corresponding to whether ${i_v}^*X$ contains irreducible characters or not. Let $\mathcal{E}_0$ be the subset of edges of $\mathcal{G}$ between vertices of different types:

$$\mathcal{E}_0 = \{\ v' \overset{e}{-} v\ \in \mathcal{E} \mid \eta(v') = \mathrm{irr},\ \eta(v) = \mathrm{red}\}$$

By Definition 1.1.6, this produces a binding decomposition $\mathcal{G} = (\mathcal{G}_{/\Gamma} \ggg= \Gamma)$ with edge set $\mathcal{E}_{/\Gamma} = \mathcal{E}_0$; by construction, any internal edge of Γ_v connects vertices with the same type on X, so the binding decomposition $\mathcal{G} = (\mathcal{G}_{/\Gamma} \ggg= \Gamma)$ is compatible with η.

Finally, for any binding decomposition $(\mathcal{G}_{/\Gamma'} \ggg= \Gamma')$ of $\mathcal{G}$, compatible with η_X, the tree $\mathcal{G}' = \mathcal{G}_{/\Gamma'}$ must contain at least all the edges of $\mathcal{E}_0$ so we can split $\mathcal{G}'$ along $\mathcal{E}_0$ as before and we obtain

$$\mathcal{G}_{/\Gamma} = \mathcal{G}'_{/\Gamma''}$$

for a collection Γ'' of subtrees of $\mathcal{G}'$. □

1.2.4 Culler-Shalen theory

Culler-Shalen theory produces splittings of groups (and essential surfaces) from the $\mathrm{SL}_2\mathbb{C}$-character variety of a group (the fundamental group of a 3-manifold). All the details can be found in [CS83] and we recall here the two fundamental results that we will use.

In this section $X(\pi)$ is the $\mathrm{SL}_2\mathbb{C}$-character variety and, for any γ in π, I_γ is the regular function of $\mathbb{C}[X(\pi)]$ associated to the evaluation of tr at γ.

By Theorems 2.1.2. and 2.2.1. of [CS83] we have the follwing result:

Theorem 1.2.3. *Let π be a finitely generated group. For any discrete, rank 1 valuation w on $\mathbb{C}[X(\pi)]$, there exist a splitting of π such that, for any γ in π, γ is in a vertex group if and only if*

$$w(I_\gamma) \geq 0$$

This result is achieved using Bass-Serre theory of [SB77]. It is worth noting that, if the valuation is associated to an ideal point of $X(\pi)$, $w(I_\gamma)$ must be negative for some γ in π, so the corresponding splitting is non-trivial.

The splitting of fundamental groups of 3-manifolds can produce essential surfaces. We will not re-expose the construction here, all the details can be found in [CS83] and produce the following proposition:

Proposition 1.2.4 (Proposition 2.3.1 in [CS83])**.** *Let N be a compact, orientable 3-manifold. For any non-trivial splitting of $\pi_1 N$ there exists a non-empty system $\mathcal{S} = S_1 \sqcup \ldots S_m$ of incompressible and non boundary-parallel surfaces in N with the following properties:*

- *for any S in $\mathcal{S}$, $\mathrm{Im}(\pi_1 S \to \pi_1 N)$ is contained in an edge group*
- *for any piece W of $N \setminus \mathcal{S}$, $\mathrm{Im}(\pi_1 W \to \pi_1 N)$ is contained in a vertex group.*

Moreover, if $\mathcal{K}$ is a subcomplex of ∂N such that $\mathrm{Im}(\pi_1 K \to \pi_1 N)$ is contained in a vertex group for each component K of $\mathcal{K}$, we may take $\mathcal{S}$ disjoint from $\mathcal{K}$.

Combining Theorem 1.2.3 and Proposition 1.2.4 enables the detection of essential and non boundary-parallel surfaces from discrete rank 1 valuations on the character variety of exteriors of 3-manifolds.

1.3 Links in integer homology spheres

Let's now review some few facts about exteriors of knots and links in **integer homology 3-spheres**.

1.3.1 Knot-manifolds

Definition 1.3.1 (Knot-manifold)**.** Let M denote an integer-homology sphere. A knot K in M is an oriented embedded circle. The *exterior* of K in M is the complement of the interior of a tubular neighbourhood of K:

$$M_K = M \setminus \overset{\circ}{N}(K).$$

The manifold M_K is called a *knot-manifold*.

Let M_K be a knot-manifold. The boundary of M_K is a torus T which splits M into two integer-homology solid tori M_K and $N(K)$ and there exist oriented simple closed curves μ, λ on T such that

- μ generates $H_1(M_K, \mathbb{Z})$ and is nullhomologous in $N(K)$
- λ is nullhomologous in M_K and homologous to K in $H_1(N(K), \mathbb{Z})$
- $< \mu, \lambda >= 1$, where $< \, , \, >$ denotes the algebraic intersection on T.

Definition 1.3.2 (Standard peripheral system)**.** The pair μ, λ is called a *standard peripheral system* for T. The simple closed curve μ (resp. λ) is called a *meridian* (resp. *longitude*) of T. A meridian-longitude pair gives a basis for homology of the boundary:

$$H_1(T, \mathbb{Z}) = \mathbb{Z}\mu \oplus \mathbb{Z}\lambda.$$

Let M_K and $M'_{K'}$ be two knot-manifolds with respective boundary T and T' and standard peripheral systems (μ, λ) and (μ', λ'). Let $M^\star$ denote the closed 3-manifold obtained by gluing M_K and $M'_{K'}$ along their boundaries, via the orientation-reversing homeomorphism identifying μ with λ' and λ with μ'.

Definition 1.3.3 (Splicing along knots)**.** The manifold $M^\star$ is an integer-homology sphere called the *splice* of M_K and $M'_{K'}$. The original tori T and T' are identified with a single torus S in $M^\star$ and we write:

$$M^\star = M' \; {}^{K'}\!\underset{S}{\bowtie}{}^{K} \; M.$$

On the other hand, let $M^\star$ be an integer-homology sphere and S an embedded torus in $M^\star$. Because $M^\star$ is an integer-homology sphere, S is separating and splits $M^\star$ into two integer-homology solid tori W and W'. There exist simple closed curves λ and λ' on S such that λ (resp. λ') is nullhomologous in W (resp. W') and $< \lambda, \lambda' >= 1$.

Let M denote the integer-homology sphere obtained by gluing a solid torus $\mathbb{S}^1 \times D^2$ on W, gluing $\{1\} \times \partial D$ along λ'. Let K denote the core $\mathbb{S}^1 \times \{0\}$ in M, K is a knot in M and W identifies with M_K.

Symmetrically, W' can be identified with a knot manifold $M'_{K'}$. The pair (λ', λ) is a standard peripheral system for M_K and (λ, λ') a standard peripheral system for $M'_{K'}$.

Definition 1.3.4 (Desplicing along a torus)**.** With these notations, $M^\star$ is exactly the spliced integer-homology sphere $M' \; {}^{K'}\!\underset{S}{\bowtie}{}^{K} \; M$.

The pair $(M_K, M'_{K'})$ is well-defined up to changing the both orientations of K and K'. It is called the *desplicing* of $M^\star$ along S.

The inclusion of S as T and T' in M_K and $M'_{K'}$, respectively, produces the following group morphisms:

$$\begin{array}{rcl} \varphi : \mathbb{Z}^2 & \to & \pi_1 M_K \\ \lambda & \to & \lambda_K \\ \lambda' & \to & \mu_K \end{array}$$

$$\begin{aligned} \varphi' : \mathbb{Z}^2 &\to \pi_1 M'_{K'} \\ \lambda &\to \mu_{K'} \\ \lambda' &\to \lambda_{K'} \end{aligned}$$

and applying Seifert-van Kampen Theorem we obtain:

Proposition 1.3.1 (Fundamental group of spliced integer-homology sphere)**.** *Let $M^\star$ be a spliced integer-homology sphere $M' \; {}^{K'}\!\underset{S}{\bowtie}^{K} \; M$. The fundamental group of $M^\star$ is the amalgamated product of the span*

$$\pi_1 M'_{K'} \xleftarrow{\varphi'} \mathbb{Z}^2 \xrightarrow{\varphi} \pi_1 M_K \ .$$

Let M_K be a knot-manifold and (p, q) be a pair of coprime integers.

Definition 1.3.5. The *Dehn-filling* of M_K with slope p/q is the manifold obtained by gluing a solid torus $\mathbb{S}^1 \times D^2$ on ∂M, identifying $\{1\} \times \partial D^2$ with the slope $\mu^p \lambda^q$. It will be denoted by $M_K(p/q)$.

The fundamental group of $M_K(p/q)$ is the quotient of $\pi_1 M_K$ by the normal closure $\langle\langle \mu^p \lambda^q \rangle\rangle$. Taking the abelianization we have:

Lemma 1.3.2. *The manifold $M_K(p/q)$ is an integer-homology sphere if and only if $p = \pm 1$.*

1.3.2 Link-manifold

Link-manifolds are obtained removing several disjoint knots in an integer-homology sphere.

Definition 1.3.6 (Link-manifold)**.** Let M denote an integer homology sphere. A link L in M is a disjoint union of knots $L = K_1 \sqcup \ldots \sqcup K_{|L|}$. We denote by M_L the exterior of L in M:

$$M_L = M \setminus \bigsqcup_{K \subset L} \overset{\circ}{N}(K).$$

The manifold M_L is called a *link-manifold*.

Remark 1.3.1. Knot and link manifolds are not necessarily *irreducible*.

The boundary of a link-manifold M_L is a disjoint union of tori T_K for $K \subset L$. Each component K of L determines a standard meridian-longitude system (μ_K, λ_K) for T_K.

For any components K, K' of L, there exist a unique integer α such that

$$\lambda_K = \alpha \mu_{K'} \text{ in } H_1(M_{K'}, \mathbb{Z}).$$

Definition 1.3.7. The integer α is called the *linking number* of K and K' in M and is denoted by $lk(K, K')$.

Remark 1.3.2. The linking number is symmetric: for any components K, K' of L,

$$lk(K, K') = lk(K', K).$$

The homology of M_L is given by

$$H_1(M_L, \mathbb{Z}) = \bigoplus_{K \subset L} \mathbb{Z}\mu_K$$

and, for any component K of L, the longitude of T_K is characterized by the following equation in $H_1(M_L, \mathbb{Z})$:

$$\lambda_K = \sum_{K' \subset L \setminus \{K\}} lk(K, K')\mu_{K'}. \tag{1.7}$$

Let M_L and ${M'}_{L'}$ be two link-manifolds. Let K and K' be components of L and L', so $L = K \sqcup L_0$ and $L' = K' \sqcup L'_0$. The splicing of M_K and ${M'}_{K'}$ produces the integer-homology sphere$M^\star = M' \ {}^{K'}\!\underset{S}{\bowtie}^{K} \ M$ and the union of components of L_0 and L'_0 identify with a link $L^\star$ in $M^\star$.

Definition 1.3.8 (Splicing link-manifolds)**.** With these notations, ${M^\star}_{L^\star}$ is a link-manifold, called the *splice* of M_L and ${M'}_{L'}$ along K and K'. As before, the tori $T_{K'}$ and T_K identify with a single torus S in ${M^\star}_{L^\star}$ and we write:

$${M^\star}_{L^\star} = {M'}_{L'_0} \ {}^{K'}\!\underset{S}{\bowtie}^{K} \ M_{L_0}$$

For any component J of L_0, a standard peripheral system for T_J in M_L is a standard peripheral system for T_J in ${M^\star}_{L^\star}$. The linking numbers in M, M' and $M^\star$ satisfy the following proposition:

Proposition 1.3.3. *Let M_L and ${M'}_{L'}$ be two link-manifolds and ${M^\star}_{L^\star} = M'_{L'_0} \ {}^{K'}\!\underset{S}{\bowtie}^{K} \ M_{L_0}$ for some components K and K' of L and L' respectively. Let lk, lk' and $lk^\star$ denote the respective linking numbers in M, M' and $M^\star$. For any components J and J' of $L^\star$ we have:*

- $lk^\star(J, J') = lk(J, J')$ *if* $J, J' \in L$,
- $lk^\star(J, J') = lk'(J, J')$ *if* $J, J' \in L'$,
- $lk^\star(J, J') = lk(J, K)lk'(K', J')$ *if* $J \in L$ *and* $J' \in L'$.

Proof. For any component J of $L^* \cap L$, we have in $H_1(M_L, \mathbb{Z})$:

$$\lambda_J = \sum_{K_0 \subset L \setminus \{J\}} lk(J, K_0)\mu_{K_0}$$

The gluing identifies μ_K with $\lambda_{K'}$ and in $H_1(M'_{L'}, \mathbb{Z})$:

$$\lambda_{K'} = \sum_{J' \subset L' \setminus \{K'\}} lk(K', J')\mu_{J'}.$$

so, in $H_1(M^\star{}_{L^\star}, \mathbb{Z})$:

$$\lambda_J = \sum_{K_0 \subset L \setminus \{J, K\}} lk(J, K_0)\mu_{K_0} + lk(J, K) \sum_{J' \subset L' \setminus \{K'\}} lk(K', J')\mu_{J'}.$$

Therefore, using equation 1.7, we recover $lk^\star$ with the formulae of Proposition 1.3.3. □

Let $M^\star{}_{L^\star}$ be a link-manifold; let S be an embedded torus in $M^\star{}_{L^\star}$ and let $(M_K, M'_{K'})$ be the desplicing of $M^\star$ along S. The torus S separates $L^\star$ into a disjoint union of sublinks $L_0 \subset M_K$, $L'_0 \subset M'_{K'}$ and this produces two links $L = K \sqcup L_0$ and $L' = K' \sqcup L_0$ in M and M', respectively, so that $M^\star{}_{L^\star}$ is the splicing of M_L and $M'_{L'}$ along K and K'.

Definition 1.3.9 (Desplicing of a link-manifold)**.** The pair $(M_L, M'_{L'})$ is called the *desplicing* of $M^\star{}_{L^\star}$ along S.

Let M_L be a link-manifold. Let L' be a sublink of L and, for each component K' of L', let $(p_{K'}, q_{K'})$ be pair of coprime integers. We'll denote the family $(p_{K'}, q_{K'})_{K' \subset L'}$ by $(p_{L'}, q_{L'})$. As in Definition 1.3.5 we can fill M_L along the components K' of L':

Definition 1.3.10 (Dehn-filling)**.** The *Dehn-filling* of M_L, along L', with slopes $p_{L'}/q_{L'}$, is the manifold obtained from M_L by gluing, on each $(T_{K'})_{K' \subset L'}$, a solid torus $\mathbb{S}^1 \times D^2$, identifying $\{1\} \times \partial D^2$ with the slope ${\mu_{K'}}^{p_{K'}} {\lambda_{K'}}^{q_{K'}}$.

It will be denoted by $M_L(L' : p_{L'}/q_{L'})$.

Let M_L be a link-manifold with $L = L_0 \sqcup K$. A Dehn-filling on K with slope $1/0$ is equivalent to removing the component K. Moreover, by Lemma 1.3.2, if $p = 1$, $M_K(1/q)$ is an integer-homology sphere M'. In that case, $M_L(K : 1/q)$ is the exterior of a link L_q in M' whose components naturally identifies with the components of L_0.

Let $(\mu'_J, \lambda'_J)_{J \subset L_q}$ denote the new standard peripheral system of L_q in M' and let $lk_q(J, J')$ denote the linking number in M'.

Proposition 1.3.4. *For J in L_q, (μ_J, λ_J) and (μ'_J, λ'_J) satisfy the following relations in $H_1(T_J)$:*

$$\mu'_J = \mu_J \tag{1.8}$$
$$\lambda'_J = \lambda_J + q\, lk(K, J)^2 \mu_J \tag{1.9}$$

and for any J, J' in L_q,

$$lk_q(J, J') = lk(J, J') - q\, lk(K, J)\, lk(K, J') \tag{1.10}$$

Proof. The homology of M_L is $\bigoplus_{J\subset L} \mu_J\mathbb{Z}$ and the homology of ${M'}_{L_q}$ is isomorphic to the quotient $H_1(M_L)/\langle \mu_K + q\, \lambda_K = 0\rangle$.

By construction meridians of L_0 identifies with meridians of L_q in M'. For any component J of L, $\lambda_J = \sum_{J'\neq J} lk(J, J')\, \mu_{J'}$ so, in $H_1({M'}_{L_q})$,

$$\begin{aligned}
\lambda_J &= \sum_{J'\neq J} lk(J, J')\mu_{J'} \\
&= lk(K, J)\, \mu_K + \sum_{J'\neq J,K} lk(J, J')\, \mu_{J'} \\
&= -q\, lk(K, J)\lambda_K + \sum_{J'\neq J,K} lk(J, J')\mu_{J'} \\
&= -q\, lk(K, J) \sum_{J''\neq K} lk(K, J'')\mu_{J''} + \sum_{J'\neq J,K} lk(J, J')\, \mu_{J'} \\
&= -q\, lk(K, J)^2\mu_J + \sum_{J'\neq K,J} (lk(J, J') - q\, lk(K, J)\, lk(K, J'))\mu_{J'}
\end{aligned}$$

so, for any J in L_q

$$\lambda_J + q\, lk(K, J)^2\mu_J = \sum_{J'\neq K,J} (lk(J, J') - q\, lk(K, J)\, lk(K, J'))\mu_{J'} \tag{1.11}$$

Therefore, in the new peripheral system of L_q in M', $\lambda'_J = \lambda_J + q\, lk(K, J)^2\mu_J$ and for J, J' in L_q the new linking numbers are given by $lk_q(J, J') = lk(J, J') - q\, lk(K, J)\, lk(K, J')$. □

It follows that, if K as linking number zero with all other components, the standard peripheral systems and the linking number pairing remain unchanged.

We define the following:

Definition 1.3.11. A link L in an integer-homology sphere is *homologically trivial* (HT) if the linking number of any two components of L is zero.

By Proposition 1.3.4, the family of HT-links is stable under $1/q$-Dehn-fillings. More precisely:

Proposition 1.3.5. *Let M be an integer-homology sphere and let $L = K \sqcup L_0$ be a link in M; for q in $\mathbb{Z}$, let L_q denote the link obtained by $1/q$-Dehn-filling along K. Then, if L is homologically trivial, so is L_q, and the peripheral system is unchanged.*

Proof. This is a direct consequence of Proposition 1.3.4. □

1.3.3 Torus splittings

Let M_L denote a link-manifold.

For any embedded torus S in M_L, we can desplice M_L along S and form two link-manifolds ${M^+}_{L^+}$ and ${M^-}_{L^-}$. For any family $\mathcal{S}$ of embedded tori in M_L, applying this to some torus S in $\mathcal{S}$ gives a partition of $\mathcal{S}$

$$\mathcal{S} = \{S\} \sqcup \mathcal{S}^+ \sqcup \mathcal{S}^-$$

where $\mathcal{S}^+$ and $\mathcal{S}^-$ are embedded tori in ${M^+}_{L^+}$ and ${M^-}_{L^-}$, respectively.

Applying this process to $\mathcal{S}^+$ in ${M^+}_{L^+}$ and $\mathcal{S}^-$ in ${M^-}_{L^-}$, it follows that any family of embedded tori $\mathcal{S}$ produces a tree decomposition of M_L where

- each vertex is a connected component of $\overline{M_L \setminus N(\mathcal{S})}$,
- each edge is a torus of $\mathcal{S}$.

Definition 1.3.12 (Torus splitting)**.** This process is called a *torus splitting* of M_L. The manifolds associated to the vertices are called the *pieces* of the splitting.

The associated tree is called the *dual graph* of the splitting and a *splitting tree* of M_L.

Remark 1.3.3. The dual graph is a tree because any embedded torus in M_L is separating.

Remark 1.3.4. We may represent components of L by arrows in the dual graph and obtain a splitting trees with arrows as in Section 1.1. Unless stated otherwise, we will always assume **the splitting trees of a link-manifold contains all the arrows of the components of** L.

Let M_L be a link-manifold with a splitting tree Γ. Any edge $v' \overset{e}{-} v$ of Γ, splits Γ into two trees Γ^+ and Γ^- (where the edge e becomes two arroheads in Γ^+ and Γ^-). Then, let $({M^+}_{L^+}, {M^-}_{L^-})$ be the link-manifolds obtained by desplicing M_L along S_e. The following lemma is a direct consequence of the definitions:

Lemma 1.3.6. *The trees Γ^+ and Γ^- are splitting trees for ${M^+}_{L^+}$ and ${M^-}_{L^-}$.*

Proposition 1.3.7. *If all the edge are essential tori in M_L, a splitting tree $\mathcal{G}$ of M_L produces a splitting tree of $\pi_1 M_L$ over $\mathcal{G}$ with vertex groups $\pi_1 {M^v}_{L^v}$ and edge groups $\mathbb{Z}^2$.*

Finally, the JSJ-decomposition of Jaco, Shalen and Johannson produces, for link-manifold s, a toric splitting that may be considered, in a way, maximal.

Definition 1.3.13. A 3-manifold is called *irreducible* if any embedded 2-sphere bounds 3-ball.

Let M_L be an irreducible link-manifold. Let $\mathcal{J}$ denote the family of JSJ-tori of M_L; by definition, $\mathcal{J}$ splits M_L into a family $(N_v)_{v\in\mathcal{V}}$ (the JSJ*-pieces*) such that each N_v is either hyperbolic or Seifert-fibred. Desplicing along each JSJ-torus shows that each N_v is itself some link-manifold ${M^v}_{L^v}$ and M_L is obtained by iterated splicing.

The dual graph of the JSJ decomposition is the tree $\mathcal{G}_\mathcal{J}$ obtained from:

- a vertex $v \in \mathcal{V}$ for each JSJ-piece ${M^v}_{L^v}$.
- an edge $v' \overset{e}{-} v \ \in \mathcal{E}$ for each JSJ-torus S in $\mathcal{J}$ such that, $M_L = {M^+}_{L_+} \, {}^{K^+}\!\underset{S}{\bowtie}^{K^-} \, {M^-}_{L_-}$ with $M^{v'} \subset M^+$ and $M^v \subset M^-$.

The JSJ decomposition induces a splitting of $\pi_1 M_L$ over the tree $\mathcal{G}_\mathcal{J}$ with edge group $\mathbb{Z}^2$ and vertex group $\pi_1 {M^v}_{L^v}$.

Definition 1.3.14 (Graph link-manifold)**.** A link-manifold M_L is called a *graph manifold* if it is irreducible and all its JSJ-pieces are Seifert-fibred.

Chapter 2

E-varieties

The A-polynomial of a knot was first describe by Cooper, Culler, Gillet, Long and Shalen in [CCG$^+$94]. It is a polynomial in 2 variables m and ℓ whose zero-set corresponds to eigenvalues of $\rho(\mu)$ and $\rho(\lambda)$ for ρ in the $\mathrm{SL}_2\mathbb{C}$ representation variety of the knot exterior. It was then naturally generalized to links in $\mathbb{S}^3$ by Tillmann in [Til02, Til05].

In this chapter, we give a more generic construction which generalizes both these objects, the *eigenvalue-varieties* associated to an abelian splitting of a finitely generated group π.

It is also constructed from the $\mathrm{SL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{C}$ character varieties, using the special properties of their abelian subgroups and the algebraic structure of the $\mathbb{C}^*$ character variety. Applying this construction the trivial splitting of the fundamental group of a link-manifold yields precisely Tillmann's eigenvalue-varieties as in [Til02, Til05].

In addition, the generalization presented here is compatible with the natural operations of merging, splitting and binding on splitting trees (see Section 1.1).

The $\mathrm{SL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{C}$ character variety of an abelian group H has a natural affine structure in $\mathbb{C}^{*h}$ denoted by $E(H)$. Given an abelian tree of groups $(\mathcal{G}, \pi)$, the projection of $X(\pi)$ on all the edge and arrow groups of $\mathcal{G}$ defines the $E_{\mathcal{G}}$-variety of π, $E_{\mathcal{G}}(\pi)$. This algebraic space defines the *$\mathcal{A}$-ideal of the abelian tree* $\mathcal{G}$, $\mathcal{A}_{\mathcal{G}}(\pi)$. The main feature of this new construction is the compatibility with the natural operations on trees of groups: merging, splitting, contraction and binding.

We show first that the $E_{\mathcal{G}}$-variety is natural under merging (see Definition 1.1.5) in the following sense:

Lemma 1. *Let $\mathcal{G}$ be a tree of groups. For any splitting $\mathcal{G}^+ \;{}^{a^+}\!\underset{e}{\bowtie}{}^{a^-}\; \mathcal{G}^-$ of the tree $\mathcal{G}$, there exist an injective regular map*

$$E_{\mathcal{G}}(\pi) \hookrightarrow E_{\mathcal{G}^+}(\pi^+) \underset{E(H_e)}{\times} E_{\mathcal{G}^-}(\pi^-) \tag{2.1}$$

Moreover, for any $(\xi^+ \underset{\xi_e}{\times} \xi^-)$ *in* $E_{\mathcal{G}^+}(\pi^+) \underset{E(H_e)}{\times} E_{\mathcal{G}^-}(\pi^-)$*, if not all the coordinates of* ξ_e *are* ± 1 *(or* 1 *if working in* $\mathrm{PSL}_2\mathbb{C}$*), then there exist* ξ *in* $E_{\mathcal{G}}(\pi)$ *with image* $\xi^+ \underset{\xi_e}{\times} \xi^-$*.*

Then, we consider a binding decomposition $(\mathcal{G}_{/\Gamma} \ggg\!= \Gamma)$ of $\mathcal{G}$ as in Definition 1.1.7. The naturality of the construction enables us to prove the following Theorem 3 for binding decompositions:

Theorem 3. *Let* $\mathcal{G}$ *be an abelian tree of goups. Any binding decomposition* $(\mathcal{G}_{/\Gamma} \ggg\!= \Gamma)$ *of the tree* $\mathcal{G}$ *produces two regular maps as in the following diagram*

$$\begin{array}{ccc} E_{\mathcal{G}}(\pi) & \overset{i_\Gamma}{\hookrightarrow} & \prod_{v\in\mathcal{V}_{/\Gamma}} E_{\Gamma_v}(\pi_v) \\ {\scriptstyle p}\downarrow & & \\ E_{\mathcal{G}_{/\Gamma}}(\pi) & & \end{array} \tag{1}$$

such that, for any edge $v' \overset{e}{-} v$ *of* $\mathcal{E}_{/\Gamma}$ *in* $\mathcal{G}_{/\Gamma}$*, if* e *is sent to* ${}^{a'}\!\underset{e}{\bowtie}{}^{a}$ *in* $\mathcal{G}$ *for some arrows* a' *and* a *in* $\Gamma_{v'}$ *and* Γ_v *respectively, then*

$$(\xi_{v'})_{a'} = (\xi_v)_a \tag{2}$$

Moreover, for any $(\xi_v)_{v\in\mathcal{V}_{/\Gamma}}$ *in* $\prod_{v\in\mathcal{V}_{/\Gamma}} E_{\Gamma_v}(\pi_v)$*, if for every edge* $v' \overset{e}{-} v$ *of* $\mathcal{G}_{/\Gamma}$*, equation* (2) *is satisfied and not all coordinates of* $(\xi_v)_a$ *are* ± 1 *(*1 *in* $\mathrm{PSL}_2\mathbb{C}$*) then* $(\xi_v)_{v\in\mathcal{V}_{/\Gamma}}$ *lies in the image of* i_Γ*.*

It follows from Theorem 3 that the eigenvalue-varieties might be a useful contruction in to understand the decomposition of character varieties induced by splitting trees.

Next, we present the logarithmic-limit set of the eigenvalue-varieties, $E_{\mathcal{H}}(\pi)_\infty$ and how it is related with Culler-Shalen theory (presented in Section 1.2.4). The logarithmic limit set is an object from tropical geometry which encodes, in a way, ideal points of algebraic varieties in $\mathbb{C}^{*k}$; the application to $E_{\mathcal{G}}$-varieties and Culler-Shalen's Theorem 1.2.3, produces a relation between Culler-Shalen splittings of groups and the logarithmic-limit set of $E_{\mathcal{G}}(\pi)$. After introducing all the relevant notations we obtain the following:

Theorem 4. *For any rational point* ξ_∞ *of* $E_{\mathcal{G}}(\pi)_\infty$*, there exist a splitting of* π *such that, for any edge or arrow* e*, and any* h *in* H_e*,*

$$h \textit{ is in a vertex group} \iff h \cdot \xi_\infty = 0$$

As we shall see, Theorem 4 applied to the trivial splitting of a link-manifold is equivalent to Tillmann's Lemma 11 of [Til05].

After that, we inspect in more details how this construction applies to torus splittings of link-manifolds. Standard peripheral systems give a canonical description of the tori subgroups (using adjacent pieces for internal tori), wich induces natural coordinates for the eigenvalue-variety; after presenting how Theorems 3 and 4 translate in standard peripheral systems, we will study more in details the case for trivial splittings, to follow with generic ones.

In particular, we will inspect how Theorem 3 applies to different generic or canonical cases.

First, we study the *peripheral eigenvalue-variety*, $E_\partial(M_L)$, associated to the trivial splitting of a link-manifold. In this case, we obtain the eigenvalue-variety of Tillmann, as presented in [Til02, Til05]. We recall some important properties, most of them already present in [Til05], sometimes in a different form.

We start computing the component of reducible characters and obtain Proposition 2:

Proposition 2. *The component of reducible characters in the peripheral eigenvalue-variety of a link-manifold M_L is given by*

$$\mathcal{A}^{\text{red}}(M_L) = \left\langle \mathfrak{l}_K - \prod_{K' \neq K} \mathfrak{m}_{K'}{}^{\pm lk(K,K')},\ K \subset L \right\rangle \tag{2.2}$$

We follow considering the relation with Hatchers *boundary curve space* of [Hat82], using Theorem 4 and Proposition 1.2.4 with a dimensional bound on the logarithmic-limit set. This implies that the peripheral eigenvalue-variety of a link-manifold have dimension lower than the number of components of the link (as in [Til02, Til05]). The components of reducible characters achieve this bound and a natural question is to look out for other components of maximal dimension in the peripheral eigenvalue-variety:

Question 1. *For which link-manifolds M_L does $X(M_L)$ admit peripherally maximal and non-abelian components?*

Later, Chapters 3 and 4 will attempt to give partial answers to this question.

It is now a classic fact that, by Thurston's results of [Thu02], link-manifolds whose interior admit a hyperbolic structure give a positive answer to Question 1. We re-expose this result here, and also show how the eigenvalue-variety can be used to detect more subtle properties, such as strong geometric isolation of hyperbolic cusps as defined in [NR93]; if $L^+ \sqcup L^-$ is a disjoint subset of cusps of a hyperbolic 3-manifold, we say that L^+ is *strongly geometrically isolated* from L^- if, after performing any integral Dehn-fillings allong geodesics $(\gamma_{K^+})_{K^+ \subset L^+}$, any deformation on the cusps of L^- leaves the geometry of the γ_{K^+} invariant. We obtain the following characterization:

Theorem 5. *Let M_L be a hyperbolic link-manifold and $L^+ \sqcup L^-$ a partition of L.*

Then L^+ is strongly geometrically isolated from L^- if and only if $E_\partial(X_0)$ splits as a product $E^+ \times E^-$ with E^+ in $\prod_{K\subset L^+} E(T_K)$ and E^- in $\prod_{K\subset L^-} E(T_K)$.

Then, we study the relations between the peripheral eigenvalue-variety of a link-manifold M_L and the different peripheral eigenvalue-varieties after $1/q$ Dehn-filling on a component K of L (see Definition 1.3.10). In the eigenvalue-variety coordinates, the Dehn-surgery equation $\mu_K {\lambda_K}^q = 1$ becomes a regular function

$$\delta_q = \mathfrak{m}_K {\mathfrak{l}_K}^q - 1 \in \mathbb{C}[E_\partial(M_L)]$$

On the other hand, the boundary of the Dehn-filled manifold is the subset $(T_{K'})_{K'\neq K}$ of the boundary of the original manifold, inducing a projection p between the eigenvalue-varieties. After imposing the equation $\delta_q = 0$ on $E_\partial(M_L)$, the projection by p should correspond to pieces of $E_\partial(M_L(K : 1/q))$. However, the surgery shifts the standard peripheral systems according to Proposition 1.3.4 and this needs to be taken into account to obtain $E_\partial(M_L(K : 1/q))$ in standard peripheral coordinates; all together, this enables us to obtain Theorem 7:

Theorem 7. *With these notations, for any link $L = L_0 \sqcup K$ in an integer-homology sphere M, and for any integer q,*

$$E_\partial(M_L(K : 1/q)) \subset \Phi_q \star V(\delta_q)$$

where $V(\delta_q)$ is the zero set of δ_q in $E_\partial(M_L)$, Φ_q is the projection p composed with the self-map of $E(\mathcal{H}_q)$ given, on each factor $E(T_J)_{J\subset L_q}$, by the 2×2 block:

$$\begin{bmatrix} 1 & 0 \\ q\, lk(J,K)^2 & 1 \end{bmatrix}$$

and $\star$ is the exponential action of $\mathcal{M}_{2,2}(\mathbb{Z})$ on $\mathbb{C}^{*2}$ (see Definition 2.1.3).

Finally, we inspect how Theorem 3 applies to the peripheral eigenvalue-varieties. Given a splitting tree of M_L, the peripheral eigenvalue-variety relates with the different peripheral eigenvalue-varieties of the vertex link-manifolds. Let $E_\mathcal{G}(M_L)$ be the eigenvalue-variety associated to a splitting tree $\mathcal{G}$, and let $\mathcal{V}$ denote the vertex set of $\mathcal{G}$.

Proposition 3. *For any torus splitting tree $\mathcal{G}$ of M_L, there exist two maps in the following diagram*

$$\begin{array}{ccc} E_\mathcal{G}(M_L) & \overset{i^*}{\hookrightarrow} & \prod_{v\in\mathcal{V}} E_\partial({M^v}_{L^v}) \\ {\scriptstyle p}\big\downarrow & & \\ E_\partial(M_L) & & \end{array} \tag{2.3}$$

such that

- *p is the projections induced by the inclusion of ∂M_L as arrows in $\mathcal{G}$*

- *for any point $(\xi^v)_{v\in\mathcal{V}}$ in the image of i, and for any edge $v' \overset{e}{-} v$ of $\mathcal{G}$ connecting some components K' and K of $L^{v'}$ and L^v, respectively, the corresponding coordinates of (ξ^v),*
$$(\xi^{v'})_{K'} = (m_{K'}, \ell_{K'}) \text{ and } (\xi^v)_K = (m_K, \ell_K)$$
satisfy the gluing condition:
$$\begin{array}{rcl} m_K & = & \ell_{K'} \\ \ell_K & = & m_{K'} \end{array} \tag{2.4}$$

Finally, for any (ξ_v) in $\prod_{v\in\mathcal{V}} E_\partial(M^v{}_{L^v})$, such that, for any edge $v' \overset{e}{-} v$,

- *equation* (2.4) *is satisfied,*

- *ℓ_K and $\ell_{K'}$ are not both equal to ± 1 (1 if working in $\mathrm{PSL}_2\mathbb{C}$)*

then there exist ξ in $E_{\mathcal{G}}(M_L)$ with $i^*\xi = (\xi^v)$.

So the peripheral eigenvalue-varieties of the vertex submanifolds can, in a way, be glued together to construct the peripheral eigenvalue-variety.

To conclude this chapter, we apply the same considerations on two more cases; first, we inspect how Theorem 3 applies for generic torus splittings of link-manifolds. An application of Theorem 3 produces Corollary 3, a more generic version of Proposition 3. Then, we present the eigenvalue-variety associated to the JSJ-decomposition of a link-manifold M_L and its dual graph $\mathcal{G}_{\mathcal{J}}$. The JSJ-decomposition can be considered a maximal toric decomposition, and, applying Theorem 3 in that case gives Proposition 4, wich shows that $E_{\mathcal{G}_{\mathcal{J}}}(M_L)$ is a kind of maximal eigenvalue-variety, acting similar to an initial object for the different $E_{\mathcal{G}}$-varieties of the link-manifold M_L.

2.1 $E_{\mathcal{G}}$-varieties, $\mathcal{A}_{\mathcal{G}}$-ideals

Let's start with the $\mathbb{C}^*$ character variety, which will enable us to define our eigenvalue-varieties.

2.1.1 The space $\mathrm{Hom}(\pi, \mathbb{C}^*)$

In this section, we will study the elementary, but fundamental, case of $R^{\mathbb{C}^*}$ and $X^{\mathbb{C}^*}$. Recall that $\mathbb{C}^*$ is algebraic with $\mathbb{C}[\mathbb{C}^*] = \mathbb{C}[Y, Y^{-1}]$. Since $\mathbb{C}^*$ is abelian, the conjugation action is trivial; moreover, for any finitely generated group π, any ρ of $R^{\mathbb{C}^*}(\pi)$

factors through π^{ab}. It follows that the four algebraic spaces of diagram (1.3) are equal to $\mathrm{Hom}(\pi, \mathbb{C}^*)$:

$$R^{\mathbb{C}^*}(\pi) = R^{\mathbb{C}^*}(\pi^{\mathrm{ab}}) = X^{\mathbb{C}^*}(\pi^{\mathrm{ab}}) = X^{\mathbb{C}^*}(\pi) = \mathrm{Hom}(\pi, \mathbb{C}^*)$$

Therefore, it is sufficient to describe $X^{\mathbb{C}^*}(H)$ for finitely generated abelian groups H.

Lemma 2.1.1. *Let $H = \mathbb{Z}^n \oplus \bigoplus_{i=1}^s \mathbb{Z}_{\alpha_i}$ be a finitely generated abelian group, then*

$$\mathrm{Hom}(H, \mathbb{C}^*) \cong \mathbb{C}^{*n} \times \prod_{i=1}^{s} \mathcal{U}_{\alpha_i}$$

where $\mathcal{U}_\alpha$ denotes the set of α^{th} roots of unity.

Proof. This is a simple consequence of the three following elementary facts:

- $\mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*) \cong \mathbb{C}^*$
- $\mathrm{Hom}(\mathbb{Z}_\alpha, \mathbb{C}^*) \cong \mathcal{U}_\alpha$ for any $\alpha \in \mathbb{Z}$
- $\mathrm{Hom}(H^+ \oplus H^-, \mathbb{C}^*) \cong \mathrm{Hom}(H^+, \mathbb{C}^*) \times \mathrm{Hom}(H^-, \mathbb{C}^*)$ for any abelian groups H^+ and H^-.

□

Definition 2.1.1. For $H = \mathbb{Z}^n \oplus \bigoplus_{i=1}^s \mathbb{Z}_{\alpha_i}$, we'll denote by $E(H)$ the set

$$E(H) = \mathbb{C}^{*n} \times \prod_{i=1}^{s} \mathcal{U}_{\alpha_i}$$

From now on, for any finitely generated group π, we identify points ξ of $E(\pi^{\mathrm{ab}})$ with morphisms φ of $\mathrm{Hom}(\pi, \mathbb{C}^*)$.

Let π be a finitely generated group with $\pi^{\mathrm{ab}} = \mathbb{Z}^n \oplus \bigoplus_{i=1}^s \mathbb{Z}_{\alpha_i}$. Under the identification $E(\pi^{\mathrm{ab}}) \cong \mathrm{Hom}(\pi, \mathbb{C}^*)$, we have:

Lemma 2.1.2. *For any φ in $\mathrm{Hom}(\pi, \mathbb{C}^*)$ and any γ in π, if*

$$[\gamma] = (a_1, \ldots, a_n, b_1, \ldots, b_s) \in \pi^{\mathrm{ab}} = \mathbb{Z}^n \oplus \bigoplus_{i=1}^{s} \mathbb{Z}_{\alpha_i}$$

and

$$\varphi \sim \xi = (m_1, \ldots, m_n, y_1, \ldots, y_s) \in E(\pi^{ab}) = \mathbb{C}^{*n} \times \prod_{i=1}^{s} \mathcal{U}_{\alpha_i}$$

then

$$\varphi(\gamma) = m_1^{a_1} \cdots m_n^{a_n} y_1^{b_1} \cdots y_s^{b_s} \in \mathbb{C}$$

Therefore, it makes sense to define the following:

Definition 2.1.2. For ξ in $E(\pi^{\mathrm{ab}})$ and γ in π,

$$\xi^{[\gamma]} = \varphi(\gamma) \text{ for } \varphi \sim \xi$$

Example 2.1.1. *The space $E(\mathbb{Z}^2)$ is $\mathbb{C}^* \times \mathbb{C}^*$. For any $\xi = (m, \ell)$ in $E(\mathbb{Z}^2)$ and $[\gamma] = (p, q) \in \mathbb{Z}^2$,*

$$\xi^{[\gamma]} = m^p \ell^q$$

For any φ in $\mathrm{Hom}(\pi, \mathbb{C}^*)$, $\varphi^{-1} : z \to \varphi(z)^{-1}$ and $\overline{\varphi} : z \to \overline{\varphi(z)}$ define other morphisms from π to $\mathbb{C}^*$. If $\xi \in E(\pi^{ab})$ corresponds to φ we denote by ξ^{-1} (resp. $\overline{\xi}$) for the points of $E(\pi^{ab})$ corresponding to φ^{-1} (resp, $\overline{\varphi}$).

Finally, since $\mathcal{U}_\alpha$ is finite, $E(H)$ is a finite number of copies of $E(H^{\mathrm{free}})$, the torsion-free part of H. Therefore, most algebraic properties won't depend on the torsion part and, from now on, we will often restrict to free abelian groups.

Applying Definition 2.1.2 to a free abelian groups defines a pairing

$$\begin{array}{ccccc} (\star): & & \mathbb{Z}^n \times \mathbb{C}^{*n} & \to & \mathbb{C}^* \\ a & = & (a_1 \ldots, a_n) & & \\ m & = & (m_1, \ldots, m_n) & \to & m^a = {m_1}^{a_1} \ldots {m_n}^{a_n} \end{array}$$

This can be generalized to define the *tropical action* of $\mathcal{M}_{p,n}(\mathbb{Z})$ over $\mathbb{C}^{*n}$. For a matrix A of $\mathcal{M}_{p,n}(\mathbb{Z})$,

$$A = \begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & & \vdots \\ a_{p,1} & \ldots & a_{p,n} \end{bmatrix}$$

and a vector ξ of $\mathbb{C}^{*n}$,

$$\xi = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix}$$

Definition 2.1.3. We define $A \star \xi$ in $\mathbb{C}^{*p}$ as the vector:

$$A \star \xi = \begin{bmatrix} \xi^{A_1} \\ \vdots \\ \xi^{A_p} \end{bmatrix}$$

where $A_1, \ldots, A_p$ denote the lines of the matrix A.

Remark 2.1.2. If we denote by $(\cdot)$ the linear pairing $\mathcal{M}_{p,n}(\mathbb{Z}) \times \mathbb{C}^n \to \mathbb{C}^p$, it forms with the pairing $(\star)$ a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{A\cdot} & \mathbb{C}^p \\ \Big\downarrow{\scriptstyle \exp} & & \Big\downarrow{\scriptstyle \exp} \\ \mathbb{C}^{*n} & \xrightarrow[A\star]{} & \mathbb{C}^{*p} \end{array}$$

where $\exp$ is the componentwise exponential map and A is any matrix of $\mathcal{M}_{p,n}(\mathbb{Z})$.

Finally, we close this section introducing a last operator which will enable us to go from $\mathbb{C}^*$ to $\mathrm{SL}_2\mathbb{C}$ and $\mathrm{PSL}_2\mathbb{C}$. From now on, we work with $G = \mathrm{SL}_2\mathbb{C}$ or $G = \mathrm{PSL}_2\mathbb{C}$ so X and R will denote the corresponding character and representation varieties; the distinction between $\mathrm{SL}_2\mathbb{C}$ and $\mathrm{PSL}_2\mathbb{C}$ will be only done when relevant.

We define the two following morphisms from $\mathbb{C}^*$, to $\mathrm{SL}_2\mathbb{C}$ and $\mathrm{PSL}_2\mathbb{C}$,

$$\begin{array}{cccc} \Delta : & \mathbb{C}^* & \to & \mathrm{SL}_2\mathbb{C} \\ & z & \to & \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \end{array}$$

$$\begin{array}{cccc} \Delta : & \mathbb{C}^* & \to & \mathrm{PSL}_2\mathbb{C} \\ & z & \to & \pm \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \end{array} \quad \text{with } \zeta^2 = z$$

By diagram (1.2), the group morphism $\Delta : \mathbb{C}^* \to G$ induces a regular map

$$\Delta_* : E(\pi^{\mathrm{ab}}) \to X(\pi)$$

By construction, all characters obtained this way are abelian. However, if we consider edges and arrow groups of an abelian splitting of π, we can use this construction to capture more information about $X(\pi)$. This is the purpose of the next section.

2.1.2 Eigenvalue-varieties associated to abelian splittings

Given a group π split over a tree $\mathcal{G}$, we construct an object which witnesses the restriction on the edges and arrows of the character variety. Since the main goal is to study toric splittings of link manifolds, we can consider abelian splittings (where the edge and arrow groups are abelian), enabling the use of Section 2.1.1 to describe the character varieties of edge and arrow groups.

Let π be a finitely generated group with an **abelian** splitting over a tree $\mathcal{G}_{(\mathcal{V},\overline{\mathcal{E}})}$. For e in $\overline{\mathcal{E}}$, we denote the edge or arrow group by H_e instead of C_e as in Section 1.1 to emphasize the fact that they are abelian. We will denote by $\mathcal{H}_{\mathcal{G}}$ the union of edges and arrow groups:

$$\mathcal{H}_{\mathcal{G}} = \bigsqcup_{e\in\overline{\mathcal{E}}} H_e$$

First, we extend the notation of Definition 2.1.1.

Definition 2.1.4. We'll denote by $E(\mathcal{H}_{\mathcal{G}})$ and $X(\mathcal{H}_{\mathcal{G}})$ the spaces

$$E(\mathcal{H}_{\mathcal{G}}) = \prod_{e\in\overline{\mathcal{E}}} E(H_e) \cong E\left(\oplus_{e\in\overline{\mathcal{E}}} H_e\right)$$

$$X(\mathcal{H}_{\mathcal{G}}) = \prod_{e\in\overline{\mathcal{E}}} X(H_e) \ncong X\left(\oplus_{e\in\overline{\mathcal{E}}} H_e\right)$$

and by $d_{\mathcal{G}}$ the regular map, product of Δ_* on each component:

$$\begin{array}{cccc} d_{\mathcal{G}}: & E(\mathcal{H}_{\mathcal{G}}) & \to & X(\mathcal{H}_{\mathcal{G}}) \\ & \xi = (\xi_e)_{e\in\overline{\mathcal{E}}} & \to & (\Delta_*\xi_e)_{e\in\overline{\mathcal{E}}} \end{array}$$

On the other hand, each inclusion of edge or arrow group $i_e : H_e \to \pi$ produces an algebraic map

$$i_e{}^* : X(\pi) \to X(H_e)$$

and we'll denote by $i_{\mathcal{G}}{}^*$ the product map

$$\begin{array}{cccc} i_{\mathcal{G}}{}^*: & X(\pi) & \to & X(\mathcal{H}_{\mathcal{G}}) \\ & \chi & \to & (i_e{}^*\chi)_{e\in\overline{\mathcal{E}}} \end{array}$$

We can represent this in the following diagram:

$$\begin{array}{ccc} & & E(\mathcal{H}_{\mathcal{G}}) \\ & & \downarrow d_{\mathcal{G}} \\ X(\pi) & \xrightarrow[i_{\mathcal{G}}{}^*]{} & X(\mathcal{H}_{\mathcal{G}}) \end{array} \tag{2.5}$$

We define the corresponding $E_{\mathcal{G}}$-variety as the Zariski closure of the pre-image by $d_{\mathcal{G}}$ of the image of $i_{\mathcal{G}}{}^*$:

Definition 2.1.5 ($E_{\mathcal{G}}$-variety)**.** The *eigenvalue-variety (or E-variety) of π associated to the splitting $\mathcal{G}$* is the space $E_{\mathcal{G}}(\pi)$ defined by

$$E_{\mathcal{G}}(\pi) = \overline{d_{\mathcal{G}}{}^{-1}(i_{\mathcal{G}}{}^{*}X(\pi))} \subset E(\mathcal{H}_{\mathcal{G}}).$$

For any subspace X of $X(\pi)$, we define $E_{\mathcal{G}}(X)$ as the subspace $\overline{d_{\mathcal{G}}{}^{-1}(i_{\mathcal{G}}{}^{*}X)}$ of $E_{\mathcal{G}}(\pi)$.

Remark 2.1.3. By definition, for any component $X \subset X(\pi)$, there's a strict closed subset F of $E_{\mathcal{G}}(X)$ such that, for any $\xi = (\xi_1, \ldots, \xi_m)$ in $E(\mathcal{H}_{\mathcal{G}})$,

$$\xi \in E_{\mathcal{G}}(X) \setminus F \iff \exists\, \chi \in X \mid \forall\, e \in \overline{\mathcal{E}},\ i_e{}^{*}\chi = \Delta_{*}\xi_e \text{ in } X(H_e)$$

Definition 2.1.6. The union of the smallest such F for each component X is called the *forbidden set* of the eigenvalue-variety.

Remark 2.1.4. A change of basis for the groups of $\mathcal{H}$ changes the eigenvalue-variety via the corresponding tropical action (see Definition 2.1.3) on each factor $E(H)$.

Recall that, for γ in π, $I_\gamma : X(\pi) \to \mathbb{C}$ denotes evaluation function at γ, associated to tr if $G = \mathrm{SL}_2\mathbb{C}$ or tr_2 for $G = \mathrm{PSL}_2\mathbb{C}$. The following lemma relates $I_\gamma(\chi)$ and ξ for γ in H_j:

Lemma 2.1.3. *If $\chi \in X(\pi)$ and $\xi \in E_{\mathcal{G}}(\pi)$ satisfy*

$$i_{\mathcal{H}}{}^{*}\chi = d_{\mathcal{H}}\xi \text{ in } X(\mathcal{H})$$

then, for any H in $\mathcal{H}$ and any γ in H,

$$I_\gamma(\chi) = \xi^{\gamma} + \xi^{-\gamma}$$

Proof. Since $\mathrm{tr}(\Delta(z)) = z + z^{-1} = \mathrm{tr}_2(\Delta(z))$ in both $\mathrm{SL}_2\mathbb{C}$ and $\mathrm{PSL}_2\mathbb{C}$, the construction of $E_{\mathcal{G}}$ gives the relation of Lemma 2.1.3. □

Example 2.1.5. *For example, if the edge or arrow group H_e is equal to $\mathbb{Z}^2$, we can use Example 2.1.1; as before, $E(H) = \mathbb{C}^{*2}$ and, for any χ in $X(\pi)$, if $i_H{}^{*}\chi = \Delta_{*}(m, \ell)$ in $X(H)$, then, for any $\delta = p\mu + q\lambda$ in H, Lemma 2.1.3 gives:*

$$I_\delta(\chi) = m^p \ell^q + m^{-p}\ell^{-q}$$

On each component $E(H_e)$ of $E(\mathcal{H}_{\mathcal{G}})$, $\mathbb{Z}_2$ acts by inversion $\xi \to \xi^{-1}$. For any ξ in $E(H_e)$, $\Delta_{*}\xi = \Delta_{*}\xi^{-1}$ in $X(H)$. This makes $E_{\mathcal{G}}(X)$ stable under the product action of $\mathbb{Z}_2{}^{|\overline{\mathcal{E}}|}$ on $E(\mathcal{H}_{\mathcal{G}})$:

Proposition 2.1.4. *For any component X of $X(\pi)$, $E_{\mathcal{G}}(X)$ is stable under the componentwise action of $\mathbb{Z}_2{}^{|\overline{\mathcal{E}}|}$ given by $(\varepsilon, \xi) \to \xi^{\varepsilon}$ on each factor $E(H_e)$.*

Finally, as an algebraic subset of $E(\mathcal{H}_{\mathcal{G}})$, each $E_{\mathcal{G}}(X)$ is defined by an ideal of $\mathbb{C}[E(\mathcal{H}_{\mathcal{G}})]$. These ideals can be obtained directly from $\mathbb{C}[X(\pi)]$ reversing diagram (2.5) into maps between the rings of regular functions:

$$\begin{array}{ccc} \mathbb{C}[X(\mathcal{H}_{\mathcal{G}})] & \xrightarrow{i_{\mathcal{G}*}} & \mathbb{C}[X(\pi)] \\ {\scriptstyle d_{\mathcal{G}}{}^*}\big\downarrow & & \\ \mathbb{C}[E(\mathcal{H}_{\mathcal{G}})] & & \end{array} \tag{2.6}$$

and, for any component X of $X(\pi)$, $i_{\mathcal{H}*}$ restricts to

$$i_{\mathcal{G}|_X *} : \mathbb{C}[X(\mathcal{H}_{\mathcal{G}})] \to \mathbb{C}[X]$$

The defining ideals of $E_{\mathcal{G}}(\pi)$ and $E_{\mathcal{G}}(X)$ are given gy Diagram (2.6):

Definition 2.1.7. The $\mathcal{A}_{\mathcal{G}}$*-ideal of* π is the defining ideal of $E_{\mathcal{G}}(\pi)$ given by:

$$\mathcal{A}_{\mathcal{G}}(\pi) = \sqrt{d_{\mathcal{G}}{}^*(\mathrm{Ker}\ i_{\mathcal{G}*})}$$

Similarly, for any component X of $X(\pi)$, $\mathcal{A}_{\mathcal{G}}(X)$ is the defining ideal of $E_{\mathcal{G}}(X)$:

$$\mathcal{A}_{\mathcal{G}}(X) = \sqrt{d_{\mathcal{G}}{}^*(\mathrm{Ker}\ i_{\mathcal{G}|_X *})}$$

2.1.3 Naturality under splitting, merging and contracting

We will now inspect how the E-varieties behave under the cannonical operations on tree of groups: splitting/merging (see Definition 1.1.5) and contracting/binding (see Definition 1.1.6).

Lemma 1. *Let $\mathcal{G}$ be a tree of groups. For any splitting $\mathcal{G}^+ \; {}^{a^+}\!\underset{e}{\bowtie}{}^{a^-} \; \mathcal{G}^-$ of the tree $\mathcal{G}$, there exist an injective regular map*

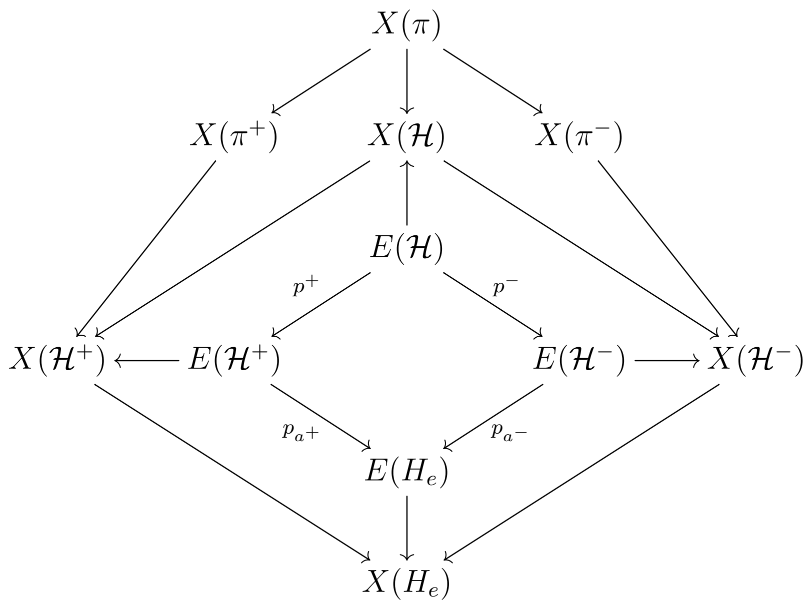
$$E_{\mathcal{G}}(\pi) \hookrightarrow E_{\mathcal{G}^+}(\pi^+) \underset{E(H_e)}{\times} E_{\mathcal{G}^-}(\pi^-) \tag{2.1}$$

Moreover, for any $(\xi^+ \underset{\xi_e}{\times} \xi^-)$ in $E_{\mathcal{G}^+}(\pi^+) \underset{E(H_e)}{\times} E_{\mathcal{G}^-}(\pi^-)$, if not all the coordinates of ξ_e are ± 1 (or 1 if working in $\mathrm{PSL}_2\mathbb{C}$), then there exist ξ in $E_{\mathcal{G}}(\pi)$ with image $\xi^+ \underset{\xi_e}{\times} \xi^-$.

Proof. Let $\mathcal{G}$ be a tree of groups and an edge e with $\mathcal{G} = \mathcal{G}^+ \; {}^{a^+}\!\underset{e}{\bowtie}{}^{a^-} \; \mathcal{G}^-$. Let $\mathcal{H}$, $\mathcal{H}^+$ and $\mathcal{H}^-$ denote the respective collections of edge and arrow groups of $\mathcal{G}$, $\mathcal{G}^+$ and $\mathcal{G}^-$. The family $\mathcal{H}$ splits into a partition

$$\mathcal{H} = (\mathcal{H}^+ \setminus \{a^+\}) \sqcup (\mathcal{H}^- \setminus \{a^-\}) \sqcup \{e\}$$

and this partition induces projections between the X and E varieties of $\mathcal{H}$ to $\mathcal{H}^+$ and $\mathcal{H}^-$, so we get the following commutative diagram:



(2.7)

By Definition 2.1.5 the pair (p^+, p^-) restricts to an injective map

$$E_{\mathcal{G}}(\pi) \to E_{\mathcal{G}^+}(\pi^+) \times E_{\mathcal{G}^-}(\pi^-)$$

such that $p_{a^+} \circ p^+ = p_{a^-} \circ p^-$ so it factors as a map

$$E_{\mathcal{G}}(\pi) \hookrightarrow E_{\mathcal{G}^+}(\pi^+) \underset{E(H_e)}{\times} E_{\mathcal{G}^-}(\pi^-)$$

For the second part of Lemma 1, let $(\xi^+ \underset{\xi_e}{\times} \xi^-)$ be an element of the fibred product $E_{\mathcal{G}^+}(\pi^+) \underset{E(H_e)}{\times} E_{\mathcal{G}^-}(\pi^-)$.

First, let's assume that ξ^+ and ξ^- are outside the forbidden sets of $E_{\mathcal{G}^+}(\pi^+)$ and $E_{\mathcal{G}^-}(\pi^-)$; there exist χ^+ and χ^- in $X(\pi^+)$ and $X(\pi^-)$ such that

$$d_{\mathcal{H}^\pm}\xi^\pm = i_{\mathcal{H}^\pm}{}^*\chi^\pm$$

by the fibre product equation, ${\xi^+}_{a^+} = {\xi^-}_{a^-} = \xi_e \in E(H_e)$ and, if not all the coordinates of ξ_e are ± 1 (or 1 if working in $\mathrm{PSL}_2\mathbb{C}$) the character $\chi_e = d_{H_e}\xi_e \in X(H_e)$ is non central. Therefore, there exist $\rho^\pm$ in $t^{-1}(\chi^\pm)$ such that

$$ {\rho^\pm}_{|_{H_{a^\pm}}} = \Delta_* {\xi^\pm}_{a^\pm} $$

so the pair (ρ^+, ρ^-) factors as $\rho = \rho_+ \underset{H_e}{\times} \rho_-$ in $R(\pi)$. Let χ be the corresponding character $t(\rho)$ in $X(\pi)$, $p^\pm\chi = \chi^\pm$ so there exist ξ such that $d_{\mathcal{H}}\xi = {i_{\mathcal{H}}}^*\chi$, and, by diagram (2.7), ξ has image $\xi^+ \underset{\xi_e}{\times} \xi^-$.

Since the property on the coordinates of ξ_e is open, we can take the Zariski closure outside the forbidden sets of $E_{\mathcal{G}^+}(\pi^+)$ and $E_{\mathcal{G}^-}(\pi^-)$ to conclude the proof of Lemma 1. □

If $\mathcal{G}$ is a tree of group with a binding decomposition $(\mathcal{G}_{/\Gamma} \ggg\!= \Gamma)$, the edge set $\mathcal{E}_{/\Gamma}$ of $\mathcal{G}_{/\Gamma}$ is a subset of the edge set of $\mathcal{G}$ and Γ is obtained from iterated mergings of the trees Γ_v for v in $\mathcal{V}_{/\Gamma}$. Using the naturality of the construction and Lemma 1, we obtain the following Theorem 3:

Theorem 3. *Let $\mathcal{G}$ be an abelian tree of goups. Any binding decomposition $(\mathcal{G}_{/\Gamma} \ggg\!= \Gamma)$ of the tree $\mathcal{G}$ produces two regular maps as in the following diagram*

$$ \begin{array}{ccc} E_{\mathcal{G}}(\pi) & \overset{i_\Gamma}{\hookrightarrow} & \prod_{v\in\mathcal{V}_{/\Gamma}} E_{\Gamma_v}(\pi_v) \\ {\scriptstyle p}\downarrow & & \\ E_{\mathcal{G}_{/\Gamma}}(\pi) & & \end{array} \tag{1} $$

such that, for any edge $v' \overset{e}{—} v$ of $\mathcal{E}_{/\Gamma}$ in $\mathcal{G}_{/\Gamma}$, if e is sent to ${}^{a'}\!\underset{e}{\bowtie}{}^{a}$ in $\mathcal{G}$ for some arrows a' and a in $\Gamma_{v'}$ and Γ_v respectively, then

$$ (\xi_{v'})_{a'} = (\xi_v)_a \tag{2} $$

Moreover, for any $(\xi_v)_{v\in\mathcal{V}_{/\Gamma}}$ in $\prod_{v\in\mathcal{V}_{/\Gamma}} E_{\Gamma_v}(\pi_v)$, if for every edge $v' \overset{e}{—} v$ of $\mathcal{G}_{/\Gamma}$, equation (2) *is satisfied and not all coordinates of $(\xi_v)_a$ are ± 1 (1 in $\mathrm{PSL}_2\mathbb{C}$) then $(\xi_v)_{v\in\mathcal{V}_{/\Gamma}}$ lies in the image of i_Γ.*

Proof. Let $\mathcal{G}$ be an abelian tree of groups with a binding decomposition $(\mathcal{G}_{/\Gamma} \ggg\!= \Gamma)$. Let $\mathcal{H}$ (resp. $\mathcal{H}_{/\Gamma}$ and $\mathcal{H}_v$ for v in $\mathcal{V}_{/\Gamma}$) denote the family of edge and arrow groups of $\mathcal{G}$, (resp. $\mathcal{G}_{/\Gamma}$, Γ_v for v in $\mathcal{V}_\Gamma$). By definition of the binding decomposition, there exist natural inclusions $\mathcal{H}_{/\Gamma} \subset \mathcal{H}$ and $\bigcup_{v\in\mathcal{V}_{/\Gamma}} \mathcal{H}_v \subset \mathcal{H}$ which induce two algebraic maps

$$p \;:\; E(\mathcal{H}) \to E(\mathcal{H}_{/\Gamma}) \text{ and } i_\Gamma \;:\; E(\mathcal{H}) \to \prod_{v\in\mathcal{V}} E(\mathcal{H}_v)$$

These inclusions also produce maps between the character varieties of the families $\mathcal{H}$, $\mathcal{H}_{/\Gamma}$ and $\mathcal{H}_v$. By Lemma 1.1.6, each v in $\mathcal{V}_{/\Gamma}$ also yields a groups morphism $\pi_v \to \pi$, producing an algebraic map $X(\pi) \to \prod_{v\in\mathcal{V}} X(\pi_v)$.

All together, we obtain the following diagram of algebraic maps :

$$\begin{array}{ccccc} E(\mathcal{H}_{/\Gamma}) & \overset{p}{\twoheadleftarrow} & E(\mathcal{H}) & \overset{i_\Gamma}{\hookrightarrow} & \prod_{v\in\mathcal{V}} E(\mathcal{H}_v) \\ \downarrow & & \downarrow & & \downarrow \\ X(\mathcal{H}_{/\Gamma}) & \twoheadleftarrow & X(\mathcal{H}) & \hookrightarrow & \prod_{v\in\mathcal{V}} X(\mathcal{H}_v) \\ & \nwarrow & \uparrow & & \uparrow \\ & & X(\pi) & \hookrightarrow & \prod_{v\in\mathcal{V}} X(\pi_v) \end{array} \tag{2.8}$$

so the maps p and i_Γ restric to maps

$$p:\; E_{\mathcal{H}}(\pi) \twoheadrightarrow E_{\mathcal{H}_{/\Gamma}}(\pi)$$

and

$$i_\Gamma:\; E_{\mathcal{H}}(\pi) \hookrightarrow \prod_{v\in\mathcal{V}} E_{\mathcal{H}_v}(\pi_v)$$

as expected.

For the rest of Theorem 3, we can inductively apply Lemma 1 on the edges of $\mathcal{G}_{/\Gamma}$. The fibre product equation at each edge gives equation (2) and the reconstruction criterion is obtained by splitting $\mathcal{G}_{/\Gamma}$ along an edge and using Lemma 1 by induction on each part. □

2.1.4 Logarithmic-limit set and Culler-Shalen splittings of groups

Using eigenvalue-varieties in conjunction with Culler-Shalen theory will enable to detect how essential surfaces intersect with toric decomposition of link manifolds. The first part of Culler-Shalen theory produces group splittings from discrete valuations on the character variety. Applying this to an abelian tree of groups $\mathcal{G}$, we can use the E-variety $E_{\mathcal{G}}(\pi)$ to detect when elements of edge and arrow groups of $\mathcal{G}$ become vertex elements in the new Culler-Shalen splitting of π. This generalizes, for $E_{\mathcal{G}}$-variety of abelian trees of groups, the boundary-slopes-detection results known for the A-polynomial as in [CCG+94] and Tillmann's eigenvalue-variety as in [Til02, Til05].

This will be done using the logarithmic-limit set; first let's recall some definitions and few properties. The details can be found in [Til02, Til05], and, more extensively in [Ber71] and [BG86].

Let V be a subvariety of $\mathbb{C}^{*m}$. We denote by $\mathbb{C}[Y^{\pm}]$ the ring $\mathbb{C}[Y_1^{\pm 1}, \dots, Y_m^{\pm 1}]$ of regular functions of $\mathbb{C}^{*m}$. For α in $\mathbb{Z}^m$, we denote by Y^α the monomial of $\mathbb{C}[Y^{\pm}]$,

$$Y^\alpha = Y_1^{\alpha_1} \dots Y_m^{\alpha_m}$$

Any regular function of $\mathbb{C}[Y^{\pm}]$ is written $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha Y^\alpha$ where a_α is 0 except for a finite number of m-tuples, called the *support* of f.

Let $\mathcal{A} \subset \mathbb{C}[Y^{\pm}]$ be the defining ideal of V.

Definition 2.1.8. The logarithmic limit set of V is the subset V_∞ of $\mathbb{S}^{m-1}$ defined by the three following equivalent constructions:

- V_∞ is the limit set in $\mathbb{S}^{m-1}$ of the following subset of the unit ball in $\mathbb{R}^m$:

$$\left\{ \frac{(\log|y_i|, \dots, \log|y_m|)}{\sqrt{1 + \sum_{i=1}^m (\log|y_i|)^2}},\ x \in V \right\}$$

- V_∞ is the set of m-tuples $(-v(Y_1), \dots, -v(Y_m))$ for all real-valued valuations v on $\mathbb{C}[V] \cong \mathbb{C}[Y^{\pm}]/\mathcal{A}$, normalized so that $\sum_{i=1}^m v(Y_i)^2 = 1$
- V_∞ is the intersection of all the spherical duals of Newton polytopes of non-zero elements of $\mathcal{A}$.

Remark 2.1.6. The Newton polytope of a non-zero polynomial is the convex hull of its support and the spherical dual of a bounded convex polytope $\mathcal{P}$ in $\mathbb{R}^m$ is the set of unit vectors v such that $\sup_{p \in \mathcal{P}} v \cdot p$ is achieved for more than one point p in $\mathcal{P}$.

Remark 2.1.7. In particular, if $m = 2$ and V is a curve defined by a polynomial f, the Newton polytope of f is the classical Newton polygon and V_∞ is the set of normal vectors to the edges of the Newton polygon.

Remark 2.1.8. Under the identification $\mathbb{S}^{m-1} \cong (\mathbb{R}^m \setminus \{0\})/\langle v \sim rv,\ r \in \mathbb{R}_+^* \rangle$, the logarithmic limit set also identifies with a positive cone in $\mathbb{R}^m$.

By the remark 2.1.7, the logarithmic limit set of a curve in $\mathbb{C}^{*2}$ is a set of points on $\mathbb{S}^1$. More generally, the results of Bergman, Bieri and Groves in [Ber71] and [BG86] imply the following theorem of [Til02, Til05]:

Theorem 2.1.5 (Bergman, Bieri-Groves). *Let V be an algebraic variety in $\mathbb{C}^{*m}$. Then, its logarithmic limit set is a finite union of rational convex spherical polytopes.*

Moreover, let $\mathrm{dim} V_\infty$ be the maximal (real) dimension of such a polytope, then

$$\mathrm{dim} V_\infty = \mathrm{dim} V - 1$$

The following curve-finding lemma of [Til02, Til05] will enable the use of Culler-Shalen theory with eigenvalue-varieties:

Lemma 2.1.6 (Lemma 6. of [Til05]). *Let V be an algebraic variety in $\mathbb{C}^{*m}$. Then any point of V_∞ with rational coordinates belongs to the logarithmic limit set C_∞ of a curve C in V.*

Now, let $\mathcal{G}$ be an abelian tree of groups and $E_{\mathcal{G}}(\pi)$ the associated E-variety. For any edge or arrow e in $\overline{\mathcal{E}}$, $E(H_e)$ is a subset of some $\mathbb{C}^{*m}$. Since the torsion part of H_e will be sent to 0 in the logarithmic-limit set, we can assume that all the edge and arrow groups are free abelian. Then, $E(\mathcal{H}_{\mathcal{G}}) \cong \mathbb{C}^{*r}$ where r is the sum $\sum r_e$ of all the ranks r_e of H_e for e in $\overline{\mathcal{E}}$, and the logarithmic-limit set of $E_{\mathcal{G}}(\pi)$ is a subset of the unit sphere in $\mathbb{R}^r$.

For each edge or arrow e in H_e, let $\mathcal{B}_e$ be a basis of H_e, so $H = \bigoplus_{\mu \in \mathcal{B}_e} \mu\mathbb{Z}$; we write coordinates of $E(H_e)$ as $(m_\mu)_{\mu \in \mathcal{B}_e}$ and $E_{\mathcal{H}}(\pi)$ have coordinates $((m_\mu)_{\mu \in \mathcal{B}_e})_{e \in \overline{\mathcal{E}}}$. The ideal $\mathcal{A}_{\mathcal{G}}(\pi)$ lies in the ring

$$\mathbb{C}[\mathfrak{m}_{\mathcal{G}}^{\pm}] = \mathbb{C}[((\mathfrak{m}_\mu{}^{\pm 1})_{\mu \in \mathcal{B}_e})_{e \in \overline{\mathcal{E}}}]$$

and the logarithmic limit set $E_{\mathcal{H}}(\pi)_\infty$ has coordinates $((u_\mu)_{\mu \in \mathcal{B}_e})_{e \in \overline{\mathcal{E}}}$.

Remark 2.1.9. The stability of $E_{\mathcal{G}}(\pi)$ under the $\mathbb{Z}_2{}^{|\overline{\mathcal{E}}|}$-action induces a similar stability for $E_{\mathcal{G}}(\pi)_\infty$. The action by inversion becomes an action $(u_\mu)_\mu \to (-u_\mu)_\mu$ on each $\mathbb{R}^{r_e}$.

For each eadge or arrow e, there's a natural bilinear pairing from $H_e \times \mathbb{R}^{r_e}$ to $\mathbb{R}$: for any $h = \sum_\mu h_\mu \mu$ in H, and $u = (u_\mu)_\mu$ in $\mathbb{R}^{r_e}$

$$h \cdot u = \sum_\mu h_\mu u_\mu$$

and this pairing naturally extends to $H_e \times \mathbb{R}^r$ ignoring the other coordinates.

We can now state Theorem 4:

Theorem 4. *For any rational point ξ_∞ of $E_{\mathcal{G}}(\pi)_\infty$, there exist a splitting of π such that, for any edge or arrow e, and any h in H_e,*

$$h \textit{ is in a vertex group} \iff h \cdot \xi_\infty = 0$$

Proof. As stated earlier, this result is quite similar to the boundary-slope detection lemma of Tillmann. However, the proof we present here is slightly different and uses a more direct approach, using Lemma 2.1.3 to relate valuations on $\mathbb{C}[X(\mathcal{H}_{\mathcal{G}})]$ and $\mathbb{C}[E(\mathcal{H}_{\mathcal{G}})]$.

Let $\xi_\infty = ((u_\mu)_{\mu\in\mathcal{B}_e})_{e\in\overline{\mathcal{E}}}$ be a rational point of $E_{\mathcal{H}}(\pi)_\infty$. Since ξ_∞ has rational coordinates, there exist a positive integer r such that $((ru_\mu)_\mu)_e$ are coprime integers. In other words, for each μ, ru_μ is an integer and there exist δ_μ in $\mathbb{Z}$ such that

$$\sum_{e\in\overline{\mathcal{E}}}\sum_{\mu\in\mathcal{B}_e}\delta_\mu u_\mu = 1$$

By the curve-finding Lemma 2.1.6, ξ_∞ is in the logarithmic limit set of a curve C in $E_{\mathcal{H}}(\pi)$. By the second description of the logarithmic limit set, ξ_∞ corresponds to a normalized valuation v on $\mathbb{C}[C]$ via $u_\mu = -v(\mathfrak{m}_\mu)$. Renormalizing v with r gives a valuation rv on $\mathbb{C}[C]$ such that

- each $rv(\mathfrak{m}_\mu)$ is an integer
- $\prod_{e\in\overline{\mathcal{E}}}\prod_{\mu\in\mathcal{B}_e} rv({\mathfrak{m}_\mu}^{-\delta_\mu}) = 1$

so rv is a discrete rank 1 valuation on $\mathbb{C}[C]$.

The curve C in $E_{\mathcal{G}}(\pi)\subset E(\mathcal{H}_{\mathcal{G}})$ lifts to $X(\pi)$ in the following diagram

$$\begin{array}{ccc} & & E(\mathcal{H}_{\mathcal{G}}) \\ & & \downarrow{\scriptstyle d_{\mathcal{G}}} \\ X(\pi) & \xrightarrow[{i_{\mathcal{G}}}^*]{} & X(\mathcal{H}_{\mathcal{G}}) \end{array} \tag{2.5}$$

and there exist a curve D in $X(\pi)$ such that $\overline{d_{\mathcal{G}}(C)} = \overline{{i_{\mathcal{G}}}^*D}$. The ring $\mathbb{C}[D]$ is a finitely generated extension of $\mathbb{C}[C]$ and there exist a positive integer r' and a discrete, rank 1 valuation w on $\mathbb{C}[D]$ such that $w = r'rv$ on $\mathbb{C}[C]$.

Then, Culler-Shalen Theorem 1.2.3 produces a splitting of π with the property that, for any γ in π, γ is contained in a vertex group if and only if $w(I_\gamma)\geq 0$.

For any $\chi\in X(\pi)$ and $\xi\in E_{\mathcal{G}}(\pi)$, if ${i_{\mathcal{G}}}^*\chi = d_{\mathcal{G}}\xi$ then, for any e in $\overline{\mathcal{E}}$ and h in H_e,

$$I_h(\chi) = {\xi_e}^h + {\xi_e}^{-h}$$

It follows that, in the diagram

$$\begin{array}{ccc} \mathbb{C}[X(\mathcal{H}_{\mathcal{G}})] & \xrightarrow{{i_{\mathcal{G}}}_*} & \mathbb{C}[X(\pi)] \\ {\scriptstyle {d_{\mathcal{G}}}^*}\downarrow & & \\ \mathbb{C}[E(\mathcal{H}_{\mathcal{G}})] & & \end{array} \tag{2.6}$$

if $h = \sum_\mu h_\mu \mu \in H$, we have the following identification:

$$I_h = \prod_\mu \mathfrak{m}_\mu{}^{h_\mu} + \prod_\mu \mathfrak{m}_\mu{}^{-h_\mu}$$

and therefore,

$$w(I_h) \geq 0 \iff r'rv(\prod_\mu \mathfrak{m}_\mu{}^{h_\mu}) = 0$$

In other words, h is in a vertex group if and only if

$$\begin{aligned} & r'rv(\prod_\mu \mathfrak{m}_\mu{}^{h_\mu}) = 0 \\ \iff \quad & v(\prod_\mu \mathfrak{m}_\mu{}^{h_\mu}) = 0 \\ \iff \quad & \sum_\mu h_\mu v(\mathfrak{m}_\mu) = 0 \\ \iff \quad & h \cdot \xi_\infty = 0 \end{aligned}$$

which concludes the proof of Theorem 4. □

When applied to a graph with a single vertex and all the arrow groups are $\mathbb{Z}^2$, Theorem 4 is essentially equivalent to Tillmann's Lemma 11 of [Til05]. However, with this extended generalization, the $E_\mathcal{G}$-variety procures more information, relating Culler-Shalen splittings with the orginal structure of tree of groups. The edge elements detected by Theorem 4 identify how the Culler-Shalen splitting traverses the original edge groups in $\mathcal{G}$.

2.1.5 Application to torus splittings

Let M_L be a link-manifold. By Proposition 1.3.7, a torus splitting of M_L produces a splitting of its fundamental group, over the dual tree $\mathcal{G}$, where edge and arrow groups are the fundamental groups of tori. We will apply here the results on abelian trees of groups to such torus splittings of link-manifolds, all of which will be detailed in the next sections of the chapter. In that case, we can also use the second part of Culler-Shalen construction to produce essential surfaces, and, with Theorem 4, detect how such surfaces cross the edge and arrow tori of $\mathcal{G}$.

The link structure of M_L gives a natural basis for $\pi_1 T$ for each edge or arrow torus T; for each arrow torus, it is given by a meridian and a longitude, and fore each edge torus, by longitudes of the two adjacent pieces. We fix a torus splitting $\mathcal{G}$ of M_L and we will denote by $\mathcal{T}$ the family $\mathcal{H}_\mathcal{G}$ of edge and arrow groups of $\mathcal{G}$.

The basis give a canonical description of $E(\mathcal{T})$ as $\mathbb{C}^{*2|\mathcal{T}|}$. We denote the coordinates in $E(\mathcal{T})$ by (m_T, ℓ_T) for arrow tori and $(\ell_T, \ell_{T'})$ for edges $v' \overset{e}{-} v$ corresponding to the splicing ${}^{T'}\!\bowtie^{T}$. The associated elements of $\mathbb{C}[E(\mathcal{T})]$ will be denoted by $(\mathfrak{m}_T, \mathfrak{l}_T)$ and $(\mathfrak{l}_T, \mathfrak{l}_{T'})$ respectively.

Remark 2.1.10. The elements $\mathfrak{m}_T$ and $\mathfrak{l}_T$ are invertible in $\mathbb{C}[E(\mathcal{T})]$.

The general diagram (2.5) takes the following form:

$$\begin{array}{ccc} & & E(\mathcal{T}) \\ & & \downarrow d_{\mathcal{T}} \\ X(M_L) & \xrightarrow[i_{\mathcal{T}}^*]{} & X(\mathcal{T}) \end{array} \tag{2.9}$$

On each component $E(T)$ of $E(\mathcal{T})$, $\mathbb{Z}_2$ acts by inversion $\xi \to \xi^{-1}$, and for any torus T and any ξ in $E(T)$, $\Delta_*\xi = \Delta_*\xi^{-1}$ in $X(T)$. By Proposition 2.1.4, $E_{\mathcal{G}}(X)$ is stable under the product action of ${\mathbb{Z}_2}^{|\mathcal{T}|}$ on $E(\mathcal{T})$:

Corollary 2.1.7. *For any component X of $X(M_L)$, $E_{\mathcal{G}}(X)$ is stable under the following action of ${\mathbb{Z}_2}^{|\mathcal{T}|}$: for any $\varepsilon = (\varepsilon_T)_{T\in\mathcal{T}}$ in ${\mathbb{Z}_2}^{|\mathcal{T}|} \cong \{\pm 1\}^{|\mathcal{T}|}$,*

$$\varepsilon \cdot (\xi_T)_{T\in\mathcal{T}} = ({\xi_T}^{\varepsilon_T})_{T\in\mathcal{T}}$$

Now, the second part of Culler-Shalen construction, Proposition 1.2.4, will produce essential surfaces in M_L. Using Theorem 4, we can detect how they intersect the tori of $\mathcal{T}$.

First, let's recall the construction of the *projective lamination space* $\mathcal{PL}(\mathcal{T})$ of a family of tori $\mathcal{T}$, defined in [Thu02]. We follow the construction made in [Hat82] for the special case of tori.

For any torus T with a given basis, an oriented isotopy class of a closed curve on T determines a pair of integers (p, q); forgetting orientation produces coordinates in $\mathbb{Z}^2/\pm$, which is not $(0, 0)$ if the curve doesn't bound a disc on T.

Let $\mathcal{T} = \{T_1, \dots, T_m\}$ be a collection of tori. A system of closed curves on each T_i, not all bounding discs, produces a system of coordinates in $(\mathbb{Z}^2/\pm)^m \setminus \{0\}$.

If we identify any such system $\mathcal{C}$ with any number of parallel copies of it, we obtain coefficients in the space

$$((\mathbb{Z}^2/\pm)^m \setminus \{0\})/\langle v \sim nv,\ n \in \mathbb{Z}_{>0}\rangle$$

This is the same as $(\mathbb{Q}^2/\pm)^m \setminus \{0\}/\langle v \sim rv,\ r \in \mathbb{Q}_{>0}\rangle$ and taking the completion gives the so-called projective lamination space:

Definition 2.1.9. Let $\mathcal{T} = \{T_1, \ldots, T_m\}$ be a collection of tori. The *projective lamination space* of $\mathcal{T}$ is the $(2m-1)$-dimensional sphere:

$$\mathcal{PL}(\mathcal{T}) = ((\mathbb{R}^2/\pm)^m \setminus \{0\})/\langle v \sim rv\rangle \cong \mathbb{S}^{2m-1}/{\mathbb{Z}_2}^m$$

If $\mathcal{T}$ is a collection of tori in a 3-manifold N, any essential surface S in N defines a curve system $[S]_{\mathcal{T}}$ in $\mathcal{PL}(\mathcal{T})$ whose coordinates are the intersection between S and each torus T of $\mathcal{T}$.

Back to a torus splitting $\mathcal{G}$ of M_L, the logarithmic limit set of $E_{\mathcal{G}}(M_L)$ is a rational polytope in the unit sphere of $\mathbb{R}^{2|\mathcal{T}|}$ with coordinates $(u_T, v_T)_{T\in\mathcal{T}}$. By the symmetry under the ${\mathbb{Z}_2}^{|\mathcal{T}|}$-action of Corollary 2.1.7, $E_{\mathcal{H}}(M_L)_\infty$ is invariant under the ${\mathbb{Z}_2}^{|\mathcal{T}|}$-action of Remark 2.1.9:

$$\varepsilon \cdot (u_T, v_T)_{T\in\mathcal{T}} = (\varepsilon_T u_T, \varepsilon_T v_T)_{T\in\mathcal{T}}$$

Therefore, any ξ_∞ in $E_{\mathcal{G}}(X)_\infty$ defines a class $[\xi_\infty]$ in $\mathbb{S}^{2m-1}/{\mathbb{Z}_2}^m \cong \mathcal{PL}(\mathcal{T})$.

Then, applying Theorem 4 with Culler-Shalen Proposition 1.2.4, we obtain the following result:

Corollary 1. *Let M_L be a link-manifold with a torus splitting over $\mathcal{G}$. For any point ξ_∞ in $E_{\mathcal{G}}(M_L)_\infty$ with rational coordinates, there exist an essential surface in M_L such that*

$$[S]_{\mathcal{T}} = \Phi([\xi_\infty])$$

where Φ is given by the diagonal of m blocks $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Remark 2.1.11. The operator Φ takes the orthogonal on each factor $\mathbb{R}^2$ corresponding to each torus of $\mathcal{T}$ to obtain $[S]_T \cdot (u_T, v_T) = 0$ as in Theorem 4. This is similar to the situation encountered with knots where we have

logarithmic limit set	$\leftrightarrow$	normal to Newton polygon	(see Remark 2.1.7)
boundary slopes	$\leftrightarrow$	edges of Newton polygon	(see [CCG+94])

so, with this vocabulary we would have:

boundary slopes $\leftrightarrow$ normal to logarithmic limit set

Proof. From a rational point ξ_∞ of $E_{\mathcal{G}}(M_L)_\infty$, Theorem 4 gives a splitting of $\pi_1 M_L$ where elements of $\pi_1 T$ that belong to vertex groups are orthogonal to $(\xi_\infty)_T$.

Then, Culler-Shalen Proposition 1.2.4 produces an essential surface S in M_L such that, for any piece W of $M_L \setminus S$, $\mathrm{Im}(\pi_1 W \to \pi_1 M_L)$ is a vertex group.

For any T in $\mathcal{T}$, let $\delta_T = (p_T, q_T) \in \mathcal{PL}(\mathcal{T})$ representing $S \cap T$. There's a parallel of δ_T on T that belongs to a piece of $M_L \setminus S$ so δ_T belongs to a vertex group and, by Theorem 4,

$$\delta_T \cdot (\xi_\infty)_T = 0$$

If $(\xi_\infty)_T = (u_T, v_T)$, $\delta_T \cdot (\xi_\infty)_T = p_T u_T + q_T v_T$ so, up to projectivisation, this is equivalent in $\mathcal{PL}(T)$ to:

$$[S]_T = (-v_T, u_T) = \Phi([(\xi_\infty)_T])$$

This is true on each torus T of $\mathcal{T}$ so, in $\mathcal{PL}(\mathcal{T})$,

$$[S]_{\mathcal{T}} = \Phi([\xi_\infty])$$

□

In the case of a tree with a single vertex, $E_{\mathcal{G}}(M_L)$ is the eigenvalue-variety as defined by Tillmann in [Til02, Til05]. The graph $\mathcal{G}$ has only arrows and all the groups correspond to boundary tori. The correponding E-variety will be called the *peripheral eigenvalue-variety* and will be studied more thoroughly in the next section.

After that, we will study generic torus splitting and how Theorem 3 applies when all edge and arrow groups are tori. We will see that, under contraction and bindings, the peripheral eigenvalue-variety acts as a kind of terminal object, whereas, the JSJ tree produces an algebraic space similar to an initial objects for the different $E_{\mathcal{G}}$-varieties of a link-manifold M_L.

2.2 Peripheral eigenvalue-variety

The application of Definition 2.1.5 to the trivial splitting of a 3-manifold gives the *eigenvalue-variety* defined by Tillman in [Til02, Til05]. All the arrows correspond to boundary tori, and we call it, here, the *peripheral eigenvalue-variety* of the link-manifold M_L.

Definition 2.2.1. Let M_L be a link manifold. The eigenvalue-variety associated to the trivial splitting of M_L will be called the *peripheral eigenvalue-variety* of M_L. It will be denoted by $E_\partial(M_L)$.

Coordinates of $E_\partial(M_L)$ are given by the standard peripheral basis and will be denoted by $(m_K, \ell_K)_{K\subset L}$ in $\mathbb{C}^{*2|L|}$. We'll denote by $\mathbb{C}[\mathfrak{m}, \mathfrak{l}]$ the coordinate ring of $E(\partial M_L)$:

$$\mathbb{C}[\mathfrak{m}, \mathfrak{l}] = \mathbb{C}[(\mathfrak{m}_K, \mathfrak{l}_K)_{K\subset L}]$$

If L has only one component so it is a knot K, $E_\partial(M_K)$ is a curve in $\mathbb{C}^{*2}$, the zero-set of the so-called *A-polynomial* of the knot as defined by Cooper, Culler, Gillet, Long and Shalen in [CCG+94]. For this reason, the ideals $\mathcal{A}_\partial(M_L)$ and $\mathcal{A}_\partial(X)$, defining $E_\partial(M_L)$ and $E_\partial(X)$ for X in $X(M_L)$ (see Definition 2.1.7), will simply be denoted by $\mathcal{A}(M_L)$ and $\mathcal{A}(X)$; they are ideals of in the ring $\mathbb{C}[\mathfrak{m}, \mathfrak{l}]$, called the $\mathcal{A}$-ideals of M_L.

Most of the results of this section already appear, sometimes in a different form, in Tillman's work [Til02, Til05].

2.2.1 Component of reducible characters

Before giving more results on the peripheral eigenvalue-variety we can easily compute the component corresponding to reducible characters.

Let $E_\partial^{\text{red}}(M_L)$ denote the subset $E_\partial(X^{\text{red}}(M_L))$ of $E_\partial(M_L)$ corresponding to reducible characters of M_L. Let $\mathcal{A}^{\text{red}}(M_L)$ be the corresponding defining ideal.

For $G = \mathrm{SL}_2\mathbb{C}$ or $\mathrm{PSL}_2\mathbb{C}$, reducible characters are characters of abelian representations: for any reducible character χ, there exist an upper-triangular representation ρ in $t^{-1}\chi$; then, the diagonal of ρ defines a representation ρ' and there exist φ in $\mathrm{Hom}(\pi_1 M_L, \mathbb{C}^*)$ such that $\rho' = \Delta \circ \varphi$.

Therefore, $E_\partial^{\text{red}}(M_L)$ is isomorphic to $E(H_1(M_L, \mathbb{Z})) \cong \mathbb{C}^{*|L|}$. It is generated by the images of the meridians and the equations for the longitudes are given by the linking numbers of L:

Proposition 2. *The component of reducible characters in the peripheral eigenvalue-variety of a link-manifold M_L is given by*

$$\mathcal{A}^{\text{red}}(M_L) = \left\langle \mathfrak{l}_K - \prod_{K' \neq K} \mathfrak{m}_{K'}{}^{\pm lk(K,K')},\ K \subset L \right\rangle \tag{2.2}$$

Proof. By equation 1.7 each longitude is given in $H_1(M_L, \mathbb{Z}) \cong \bigoplus_{K \subset L} \mathbb{Z}\mu_K$ by

$$\lambda_K = \sum_{K' \neq K} lk(K, K')\mu_{K'}$$

It follows that, for any φ in $\mathrm{Hom}(\pi_1 M_L, \mathbb{C}^*)$,

$$\varphi(\lambda_K) = \prod_{K' \neq K} \varphi(\mu_{K'})^{lk(K,K')}.$$

With the $\mathbb{Z}_2{}^{|L|}$ action of Corollary 2.1.7, this gives the following equations in $\mathbb{C}[E_\partial^{\text{red}}(M_L)]$:

$$\forall\, K \subset L,\ \mathfrak{l}_K = \prod_{K' \neq K} \mathfrak{m}_{K'}{}^{\pm lk(K,K')} \tag{2.10}$$

Conversely, any point $\xi = (m_K, \ell_K)_{K\subset L} \in E(\partial M_L)$ satisfying equation (2.10) may define a morphism φ of $\mathrm{Hom}(\pi_1 M_L, \mathbb{C}^*)$ satisfying, on each component K,

$$\varphi(\mu_K) = m_K$$

and

$$\varphi(\lambda_K) = \ell_K$$

so $\Delta_*\varphi$ is a reducible character of $X(M_L)$ with pre-image ξ. Therefore, the ideal defining $E_\partial^{\mathrm{red}}(M_L)$ is given by:

$$\mathcal{A}^{\mathrm{red}}(M_L) = \left\langle \mathfrak{l}_K - \prod_{K'\neq K} \mathfrak{m}_{K'}{}^{\pm lk(K,K')},\ K \subset L \right\rangle \tag{2.2}$$

□

Remark 2.2.1. Proposition 2 generalizes the fact that the A-polynomial of a knot in an homology sphere always has a factor $\mathfrak{l} - 1$.

Then, a simple computation using the first description of the logarithmic-limit set (see Definition 2.1.8) yields the following equations for $E_\partial^{\mathrm{red}}(M_L)_\infty$:

Corollary 2. *The logarithmic limit set of $E_\partial^{\mathrm{red}}(M_L)$ is the intersection in $\mathbb{R}^{2|L|}$ of $\mathbb{S}^{2|L|-1}$ with the $|L|$-dimensional subspace defined by*

$$\forall\ K \subset L, v_K = \pm \sum_{K'\neq K} lk(K', K) u_K$$

From now on, we will try to focus on components of $E_\partial(M_L)$ different from $E_\partial^{\mathrm{red}}(M_L)$. For a matter of convenience we define the following:

Definition 2.2.2. A component X in $X(M_L)$ is called *peripherally non-abelian* if

$$E_\partial(X) \neq E_\partial^{\mathrm{red}}(M_L)$$

Peripherally non-abelian components contain irreducible characters but the converse is not always true. As the following two examples show, there exist link-manifolds M_L and irreducible characters χ in $X(M_L)$ such that

$$d_\partial{}^{-1}(i_\partial{}^*\chi) \subset E_\partial^{\mathrm{red}}(M_L)$$

Example 2.2.2. *Characters of $X(M_L)$ may be irreducible but still have pre-image in $E_\partial^{\mathrm{red}}(M_L)$. For example, if K is a knot in an integer-homology sphere M and the manifold obtained by Dehn-filling along the longitude, $M_K(K : 0/1)$, admits irreducible characters. By the surgery relation, any irreducible representation of $\pi_1 M_K(K : 0/1)$ trivializes the longitude. The inclusion $M_K \subset M_K(K : 0/1)$ yields an algebraic map*

$$X(M_K(K:0/1)) \to X(M_K)$$

and any irreducible character of $X(M_K(K : 0/1))$ produces an irreducible character χ in $X(M_K)$ with

$$d_\partial{}^{-1}(i_\partial{}^*\chi) \subset E_\partial^{\mathrm{red}}(M_K)$$

Example 2.2.3. *Let U_n be the trivial link with n components in $\mathbb{S}^3$. The fundamental group of ${\mathbb{S}^3}_{U_n}$ is the free group F_n so, if $n \geq 2$, $X({\mathbb{S}^3}_{U_n})$ contains irreducible characters. However, since all the longitudes are nullhomotope in ${\mathbb{S}^3}_{U_n}$, they are trivial under any representation of $\pi_1{\mathbb{S}^3}_{U_n}$. Therefore, although $X({\mathbb{S}^3}_{U_n})$ contains irreducible characters,*

$$E_\partial({\mathbb{S}^3}_{U_n}) = E_\partial^{\mathrm{red}}({\mathbb{S}^3}_{U_n})$$

2.2.2 Dimensional bound, hyperbolic link-manifolds

Let M_L be a link-manifold. Culler-Shalen theory on $X(M_L)$ produces essential incompressible ∂-incompressible surfaces. The points of $\mathcal{PL}(\partial M_L)$ corresponding to intersections $[S]_{\partial M_L}$ of essential incompressible ∂-incompressible surfaces in M_L with its boundary are called *boundary slope* of M_L. The set of boundary slopes will be denoted by $\mathfrak{BC}(M_L)$.

By Corollary 1, as in Tillmann's Lemma 11 of [Til05], rational points of $E_\partial(M_L)$ produce points in $\mathfrak{BC}(M_L)$. On the other hand, by Hatcher's Theorem of [Hat82], the closure of $\mathfrak{BC}(M_L)$ is a polyhedron with dimension at most $|L| - 1$ in $\mathcal{PL}(\partial M_L)$. Put together with Bergman, Bieri-Groves Theorem 2.1.5, we obtain a dimensional bound for the components of $E_\partial(M_L)$ in ${\mathbb{C}^*}^{2|L|}$:

Theorem 2.2.1. *Let M_L be a link-manifold. For any component X in $X(M_L)$,*

$$\dim E_\partial(X) \leq |L|$$

This leads us to define the following:

Definition 2.2.3 (Peripherally maximal components)**.** A component X of the character variety $X(M_L)$ will be called *peripherally maximal* if $\dim E_\partial(X) = |L|$.

Remark 2.2.4. By Proposition 2, the components of $E_\partial(M_L)$ corresponding to reducible characters are peripherally maximal.

Most of the following sections will be dedicated to answer the following question:

Question 1. *For which link-manifolds M_L does $X(M_L)$ admit peripherally maximal and non-abelian components?*

Remark 2.2.5. For knots, Question 1 is equivalent to whether the A-polynomial of a knot admits a component different from $\mathfrak{l} - 1$. This is known to be true for non trivial knots in $\mathbb{S}^3$ and has been proved by Dunfield-Garoufalidis in [DG04] and Boyer-Zhang in [BZ05].

Remark 2.2.6. The exterior of the trivial link ${\mathbb{S}^3}_{U_n}$ (see example 2.2.3) does not admit any peripherally maximal and non-abelian component.

Hyperbolic knots were the first example for which non-triviality of the A-polynomial was established. This was due to Thurston's results on deformation of holonomy for hyperbolic manifolds with cusps (see [Thu02]). As observed by Tillman in [Til02, Til05], this result remains true for hyperbolic link-manifolds.

Theorem 2.2.2. *Let M_L be a link-manifold admitting an hyperbolic structure and let χ_0 be the character in $X(M_L)$ of a lift of the holonomy. Then the component X_0 containing χ_0 (the* geometric component*) is peripherally maximal and non-abelian.*

More precisely, for any family of coprime integers $(p_K, q_K)_{K\subset L}$, the functions

$$({\mathfrak{m}_K}^{p_K} {\mathfrak{l}_K}^{q_K})_{K\subset L}$$

are algebraically free in $\mathbb{C}[E_\partial(X_0)]$.

Proof. The proof is quite similar to the one found in [Til05]. We follow the same idea, using notations of [NZ85] which will use for the next Theorem 5.

Let M_L be an hyperbolic link-manifold. Let $(p_K, q_K)_{K\subset L}$ be a family of coprime integers and $\gamma_K = p_K\mu_K + q_K\lambda_K$ the corresponding system of simple closed curves on ∂M_L. By Thurston's results, there exists a local biholomorphism between a neighbourhood of $(0,\ldots,0)$ in $\mathbb{C}^{|L|}$ and a neighbourhood of χ_0 in X_0. Moreover, let $\underline{\nu} = (\nu_K)_{K\subset L}$ denote Thurston's local parameters and $\chi_{\underline{\nu}}$ the associated character, for each boundary torus T_K,

$$I_{\gamma_K}(\chi_{\underline{\nu}}) = 2\cosh\frac{\nu_K}{2}$$

For each $\chi_{\underline{\nu}}$, let $\xi_{\underline{\nu}} = (m_K, \ell_K)_{K\subset L}$ denote the corresponding point in $E_\partial(X_0)$, the definition of γ_K yields

$${m_K}^{p_K}{\ell_K}^{q_K} = \exp\frac{\nu_K}{2}$$

Since $\underline{\nu}$ are local parameters, this forces the functions $(\mathfrak{m}_K{}^{p_K}\mathfrak{l}_K{}^{q_K})_{K\subset L}$ to be algebraically free in $\mathbb{C}[E_\partial(X_0)]$. Therefore, X_0 is peripherally maximal, and applying this to the longitudinal system $\gamma_K = \lambda_K$ shows that X_0 is also peripherally non-abelian. □

Boundary components of an hyperbolic link-manifold correspond to geometric cusps. Neumann-Reid results on cusp-rigidity in [NR93] include a characterization of cusp-isolation through deformation of the holonomy character. We recall the following definition of [NR93]:

Definition 2.2.4. Let $L^+ \sqcup L^-$ be a disjoint subset of cusps of a hyperbolic 3-manifold. We say that L^+ is *strongly geometrically isolated* from L^- if, after performing any integral Dehn-fillings allong geodesics $(\gamma_{K^+})_{K^+\subset L^+}$, any deformation on the cusps of L^- leaves the geometry of the γ_{K^+} invariant.

Then, using Theorem 4.3 of [NR93], we can give a characterization of strong geometric isolation in terms of the variety $E_\partial(X_0)$.

Theorem 5. *Let M_L be a hyperbolic link-manifold and $L^+ \sqcup L^-$ a partition of L.*
Then L^+ is strongly geometrically isolated from L^- if and only if $E_\partial(X_0)$ splits as a product $E^+ \times E^-$ with E^+ in $\prod_{K\subset L^+} E(T_K)$ and E^- in $\prod_{K\subset L^-} E(T_K)$.

Proof. As stated before, the proof relies on Theorem 4.3 of [NR93] which relates strong geometric isolation with Thurston's deformation parameters of [Thu02]. More details can be found in [NZ85] and we follow up with the same notation.

Let M_L be an hyperbolic link-manifold. As in the proof of Theorem 2.2.2 with $p_K/q_K = 1/0$, let $\underline{\nu} = (\nu_K)_{K\subset L}$ denote local parameters around χ_0 in X_0 corresponding to the meridian system $(\mu_K)_{K\subset L}$.

Thurston constructs holomorphic functions $\tau_K(\underline{\nu})$ for each component K such that

- each $\tau_K(0,\ldots,0)$ is the modulus of T_K in the geometry of M_L
- for any character $\chi_{\underline{\nu}}$ with parameters $\underline{\nu}$,

$$\begin{aligned} I_{\mu_K}(\chi_{\underline{\nu}}) &= 2\cosh\frac{\nu_K}{2} \\ I_{\lambda_K}(\chi_{\underline{\nu}}) &= 2\cosh\frac{\nu_K\tau_K(\underline{\nu})}{2} \end{aligned}$$

Then, as above, the corresponding point $\xi_{\underline{\nu}} = (m_K, \ell_K)_{K\subset L}$ is given by

$$\begin{aligned} m_K &= \exp\frac{\nu_K}{2} \\ \ell_K &= \exp\frac{\nu_K\tau_K(\underline{\nu})}{2} \end{aligned}$$

Let $L^+ \sqcup L^-$ be a partition of L, by Theorem 4.3 of [NR93], L^+ is strongly geometrically isolated from L^- if and only if the functions τ_K for K in L^+ only depend on ν_K for K in L^+. Since strong geometric isolation is symmetric (Theorem 3 of [NR93]), τ_K for K in L^- only depends on ν_K for K in L^-. Therefore, around $\xi_0 = (1, 1, \ldots, 1, 1)$, $E_\partial(X_0)$ splits as a product in $\mathbb{C}^{*2|L^+|} \times \mathbb{C}^{*2|L^-|}$ and, by algebraicity, $E_\partial(X_0)$ is also a product. $\square$

2.2.3 Peripheral eigenvalue-variety and Dehn-fillings

Let L be a link in an integer-homology sphere M, and let K be a component of L, so

$$L = K \sqcup L_0$$

For any integer q, the Dehn-filled manifold $M_L(K : 1/q)$ can be identified with the exterior of a link L_q in the integer-homology sphere $M_K(1/q)$ (see Definitions 1.3.5 and 1.3.10).

Moreover, there is a natural inclusion

$$i_q : M_L \to M_L(K : 1/q)$$

which induces by Diagram (1.2) a regular map

$${i_q}^* : X(M_L(K : 1/q)) \to X(M_L)$$

A component J of L_q can be identified with a component K' of L_0, and the map i_q identifies the boundary of $M_L(K : 1/q)$ with $(T_{K'})_{K' \neq K}$ in ∂M_L. This enables to describe a relation between the respective peripheral eigenvalue-varieties of M_L and $M_L(K : 1/q)$. Let $\mathcal{H}$ denote the boundary of M_L and let $\mathcal{H}_q$ denote the boundary of $M_L(K : 1/q)$, so

$$\mathcal{H} = \mathcal{H}_q \sqcup T_K$$

and there are natural projections p that complete Diagram (2.5) into the following commutative diagram:

$$\begin{array}{ccc} X(M_L(K : 1/q)) & \xrightarrow{{i_q}^*} & X(M_L) \\ \downarrow {i_\partial}^* & & \downarrow {i_\partial}^* \\ X(\mathcal{H}_q) & \xleftarrow{p} & X(\mathcal{H}) \\ \uparrow d_\partial & & \uparrow d_\partial \\ E(\mathcal{H}_q) & \xleftarrow[p]{} & E(\mathcal{H}) \end{array} \tag{2.11}$$

By the surgery relation, for any χ in the image of ${i_q}^*$ and for any ξ in $E(\mathcal{H})$, if

$$d_\partial \xi = {i_\partial}^* \chi \text{ in } X(\mathcal{H})$$

then the coordinates of ξ at the component K satisfy

$$m_K {\ell_K}^q = 1$$

Let δ_q denote the regular function on $E_\partial(M_L)$ given by

$$\delta_q = \mathfrak{m}_K {\mathfrak{l}_K}^q - 1$$

then, in the peripheral eigenvalue-variety of M_L,

$${d_\partial}^{-1}(i_\partial(\mathrm{Im}({i_q}^*))) \subset V(\delta_q)$$

The new standard peripheral system is given by Proposition 1.3.4 and depends, for each component J of L_q, on the linking number $lk(J,K)$ in M. Moreover, by Remark 2.1.4, this is reflected by a tropical action on the eigenvalue-variety. Joining all this together, we obtain the following theorem for peripheral eigenvalue-varieties of Dehn-filled link-manifolds:

Theorem 7. *With these notations, for any link $L = L_0 \sqcup K$ in an integer-homology sphere M, and for any integer q,*

$$E_\partial(M_L(K:1/q)) \subset \Phi_q \star V(\delta_q)$$

where $V(\delta_q)$ is the zero set of δ_q in $E_\partial(M_L)$, Φ_q is the projection p composed with the self-map of $E(\mathcal{H}_q)$ given, on each factor $E(T_J)_{J\subset L_q}$, by the 2×2 block:

$$\begin{bmatrix} 1 & 0 \\ q\ lk(J,K)^2 & 1 \end{bmatrix}$$

and $\star$ is the exponential action of $\mathcal{M}_{2,2}(\mathbb{Z})$ on ${\mathbb{C}^*}^2$ (see Definition 2.1.3).

Proof. Having set the notations, the proof is straightforward. By commutativity of Diagram 2.11,

$$\begin{array}{ccc} X(M_L(K:1/q)) & \xrightarrow{{i_q}^*} & X(M_L) \\ {\scriptstyle {i_\partial}^*}\downarrow & & \downarrow{\scriptstyle {i_\partial}^*} \\ X(\mathcal{H}_q) & \xleftarrow{\ p\ } & X(\mathcal{H}) \\ {\scriptstyle d_\partial}\uparrow & & \uparrow{\scriptstyle d_\partial} \\ E(\mathcal{H}_q) & \xleftarrow[\ p\ ]{} & E(\mathcal{H}) \end{array} \qquad (2.11)$$

the projection $p : E(\mathcal{H}) \to E(\mathcal{H}_q)$ restricts to the respective eigenvalue-varieties:

$$p: \; E_\partial(M_L) \longrightarrow\!\!\!\!\rightarrow E_\partial(M_L(K:1/q))$$

For any representation ρ in $R(M_L(K:1/q))$, $\rho(\mu_K {\lambda_K}^q) = 1$ and we denote by δ_q the regular function of $\mathbb{C}[E_\partial(M_L)]$:

$$\delta_q = \mathfrak{m}_K {\mathfrak{l}_K}^q - 1$$

so, as stated earlier, the surgery equation implies precisely that

$$p^{-1} E_\partial(M_L(K:1/q)) \subset V(\delta_q)$$

Now, for any J in L_q, let K' be the corresponding component of L_0 in M. The coordinates at J in $E_\partial(M_L(K:1/q))$ are (m_J, ℓ_J), given by the standard peripheral system (μ_J, λ_J) of T_J in $M_K(1/q)$. On the other hand, in $E_\partial(M_L)$, the coordinates at K' are $(m_{K'}, \ell_{K'})$ given by $(\mu_{K'}, \lambda_{K'})$ on $T_{K'}$ in M.

By Proposition 1.3.4, the two peripheral systems satisfy:

$$\begin{array}{rcl} \mu_J &=& \mu_{K'} \\ \lambda_J &=& \lambda_{K'} + q\, lk(K', K)^2 \mu_{K'} \end{array}$$

where $lk(K', K)$ is the linking number of the components K and K' in M. Therefore, since K' and J represent the same components of L, $lk(K', K) = lk(J, K)$, and we obtain the equation

$$\begin{array}{rcl} m_J &=& m_{K'} \\ \ell_J &=& {m_{K'}}^{q\, lk(J,K)^2}\, \ell_{K'} \end{array}$$

Let P_J denote the matrix

$$P_J = \begin{bmatrix} 1 & 0 \\ q\, lk(J,K)^2 & 1 \end{bmatrix}$$

we obtain the following equation on each factor $E(T_J)$:

$$\begin{bmatrix} m_J \\ \ell_J \end{bmatrix} = P_J \star \begin{bmatrix} m_{K'} \\ \ell_{K'} \end{bmatrix}$$

Let Φ_q be the selfmap of $E(\mathcal{H}_q)$ equal to $P_J\star$ on each factor $E(T_J)$; Φ_q changes the basis according to the $1/q$-Dehn-filling at T_K and we finally obtain

$$E_\partial(M_L(K:1/q)) \subset \Phi_q \star V(\delta_q)$$

□

Remark 2.2.7. If the link is homologically trivial (all the linking numbers are 0, see Definition 1.3.11), Φ_q is just the projection on the remaining coordinates.

2.2.4 Torus splittings and peripheral E-variety

Let M_L be a link-manifold with a torus splitting tree $\mathcal{G}$. Let $\mathcal{V}$ denote the set of vertices of $\mathcal{G}$, all the arrows of $\mathcal{G}$ are attached to some vertex v of $\mathcal{V}$ and all the edges $v' \overset{e}{-} v$ correspond to splicing between components of $L^{v'}$ and L^v. This gives a relation between $E_{\mathcal{G}}(M_L)$, $E_\partial(M_L)$ and the different peripheral eigenvalue-varieties of the vertex submanifolds, $E_\partial({M^v}_{L^v})$ for v in $\mathcal{V}$.

Proposition 3. *For any torus splitting tree $\mathcal{G}$ of M_L, there exist two maps in the following diagram*

$$\begin{array}{ccc} E_{\mathcal{G}}(M_L) & \overset{i^*}{\hookrightarrow} & \prod_{v\in\mathcal{V}} E_\partial({M^v}_{L^v}) \\ {\scriptstyle p}\downarrow & & \\ E_\partial(M_L) & & \end{array} \tag{2.3}$$

such that

- *p is the projections induced by the inclusion of ∂M_L as arrows in $\mathcal{G}$*
- *for any point $(\xi^v)_{v\in\mathcal{V}}$ in the image of i, and for any edge $v' \overset{e}{-} v$ of $\mathcal{G}$ connecting some components K' and K of $L^{v'}$ and L^v, respectively, the corresponding coordinates of (ξ^v),*
$$(\xi^{v'})_{K'} = (m_{K'}, \ell_{K'}) \text{ and } (\xi^v)_K = (m_K, \ell_K)$$
satisfy the gluing condition:
$$\begin{array}{rcl} m_K & = & \ell_{K'} \\ \ell_K & = & m_{K'} \end{array} \tag{2.4}$$

Finally, for any (ξ_v) in $\prod_{v\in\mathcal{V}} E_\partial({M^v}_{L^v})$, such that, for any edge $v' \overset{e}{-} v$,

- *equation (2.4) is satisfied,*
- *ℓ_K and $\ell_{K'}$ are not both equal to ± 1 (1 if working in $\mathrm{PSL}_2\mathbb{C}$)*

then there exist ξ in $E_{\mathcal{G}}(M_L)$ with $i^*\xi = (\xi^v)$.

Proof. Let $\mathcal{G}$ be a splitting tree of a link-manifold M_L. We will use Theorem 3 for two specific binding decomposition of $\mathcal{G}$ that will give the result of Proposition 3. Those are the two decompositions of $\mathcal{G}$ given by Definition 1.1.8 and Definition 1.1.9.

We use the notation of Section 1.1. Let $*$ be the tree with one vertex and the same arrows as $\mathcal{G}$ as in Definition 1.1.8, $\mathcal{G}$ trivially decomposes as

$$\mathcal{G} = (* \gg= \mathcal{G})$$

and $E_\partial(M_L)$ is non other than the eigenvalue-variety associated to the trivial splitting $*$ of M_L. Therefore, Theorem 3 gives an epic algebraic map

$$p \ : \ E_{\mathcal{G}}(M_L) \twoheadrightarrow E_\partial(M_L)$$

On the other hand, let $\overrightarrow{\mathcal{V}} = \{\overrightarrow{v}, v \in \mathcal{V}\}$ be the family of vertices of $\mathcal{G}$ with an arrow at each v for each arrow or edge adjacent to v in $\mathcal{G}$. Each $E_\partial({M^v}_{L^v})$ is the eigenvalue-variety associated to the tree $(\overrightarrow{v})_{v\in\mathcal{V}}$ and the identical binding decomposition (Definition 1.1.9)) of $\mathcal{G}$ gives

$$\mathcal{G} = (\mathcal{G} \gg= \overrightarrow{\mathcal{V}})$$

Then, Theorem 3 gives a monic algebraic map

$$i : \ E_{\mathcal{H}}(M_L) \hookrightarrow \prod_{v\in\mathcal{V}} E_\partial({M^v}_{L^v})$$

so we obtain diagram 2.3:

$$\begin{array}{ccc} E_{\mathcal{H}}(M_L) & \overset{i^*}{\hookrightarrow} & \prod_{v\in\mathcal{V}} E_\partial({M^v}_{L^v}) \\ {\scriptstyle p}\downarrow & & \\ E_\partial(M_L) & & \end{array} \tag{2.3}$$

which gives the first part of Proposition 3.

The rest, is, again, a consequence of Theorem 3 in the special case of link-manifolds and torus splittings. □

In other words for any link-manifold M_L, and any splitting tree $\mathcal{G}$, the associated eigenvalue-variety projects onto the peripheral eigenvalue-variety and is contained in the product of the peripheral eigenvalue-varieties of each vertex. This makes $E_\partial(M_L)$ act like a terminal objects for all the $E_{\mathcal{G}}$-varieties of M_L associated to torus splittings $\mathcal{G}$.

2.3 E-varieties associated to non-trivial splittings

Finally, we conclude this chapter with few considerations on eigenvalue-varieties of link-manifolds associated to non-trivial torus splittings.

2.3.1 Generic splittings

Let M_L be a link-manifold with a splitting tree $\mathcal{G}$; as seen in Section 2.2, $E_{\mathcal{G}}(M_L)$ always surjects onto the peripheral eigenvalue-variety. Moreover, as in Corollary 1, coordinates corresponding to internal edges give, in the logarithmic limit set, intersections of essential surfaces with internal tori in M_L.

Besides that, as in Proposition 3, Theorem 3 takes a special form for torus splittings of link-manifolds.

Let $(\mathcal{G}_{/\Gamma} \ggg\!= \Gamma)$ be a binding decomposition of $\mathcal{G}$. For any vertex v of $\mathcal{V}_{/\Gamma}$, Γ_v is a subtree of $\mathcal{G}$, giving a torus splitting of a link-manifold $M^{\Gamma_v}{}_{L^{\Gamma_v}}$ embedded in M_L.

With these notations, Theorem 3 produces the following corollary:

Corollary 3. *For any binding decomposition $(\mathcal{G}_{/\Gamma} \ggg\!= \Gamma)$ of $\mathcal{G}$, there exist two maps:*

$$\begin{array}{ccc} E_{\mathcal{G}}(M_L) & \overset{i_\Gamma}{\hookrightarrow} & \prod_{v\in\mathcal{V}_{/\Gamma}} E_{\Gamma_v}(M^{\Gamma_v}{}_{L^{\Gamma_v}}) \\ {\scriptstyle p}\big\downarrow & & \\ E_{\mathcal{G}_{/\Gamma}}(M_L) & & \end{array} \tag{2.12}$$

such that p is the projection on the corresponding factors of $\overline{\mathcal{E}_{/\Gamma}} \subset \overline{\mathcal{E}}$ and, for any edge $v' \overset{e}{-} v$ of $\mathcal{G}_{/\Gamma}$, if e corresponds to the splicing ${}^{a'}\!\underset{e}{\bowtie}{}^{a}$ for arrows a' and a of $\Gamma_{v'}$ and Γ_v,

$$\begin{array}{rcl} m_{K_a} & = & \ell_{K_{a'}} \\ \ell_{K_a} & = & m_{K_{a'}} \end{array} \tag{2.13}$$

where $K_{a'}$ and K_a are the corresponding components of $L^{\Gamma_{v'}}$ and L^{Γ_v}.

Finally, let $(\xi_v)_{v\in\mathcal{V}_{/\Gamma}}$ be a point of $\prod_{v\in\mathcal{V}_{/\Gamma}} E_{\Gamma_v}(M^{\Gamma_v}{}_{L^{\Gamma_v}})$. If, for any edge $v' \overset{e}{-} v$ of $\mathcal{G}_{/\Gamma}$ with ${}^{a'}\!\underset{e}{\bowtie}{}^{a}$ as above, Equation (2.13) *is satisfied, and not both ℓ_{K_a} and $\ell_{K_{a'}}$ are trivial, then (ξ_v) is in the image of i_Γ.*

Proof. This is just an application of Theorem 3, using the special coordinates given by the meridian-longitude systems of the underlying link-manifolds as in the proof of Proposition 3. □

2.3.2 $JSJ(\partial)$-eigenvalue-variety

Finally, if M_L is a link-manifold, the family of JSJ tori provides another canonical family of embedded tori in M_L.

Let $\mathcal{J}$ denote the family of JSJ tori of M_L, and $\mathcal{G}_{\mathcal{J}}$ the corresponding splitting tree (with all the arrows); we also denote by $\mathcal{G}^0_{\mathcal{J}}$ the graph $\mathcal{G}_{\mathcal{J}}$ without any arrow.

Definition 2.3.1. The eigenvalue-variety $E_{\mathcal{G}^0_{\mathcal{J}}}(M_L)$ is called the JSJ*-eigenvalue-variety* of M_L and denoted by $E_{\mathcal{J}}(M_L)$. It has natural coordinates in $\mathbb{C}^{*2|\mathcal{J}|}$, given by (λ, λ') for each edge $v' \overset{e}{-} v$ in the JSJ-dual graph of M_L.

As the peripheral eigenvalue-variety detects boundary slopes of M_L, the logarithmic limit set of $E_{\mathcal{G}^0_{\mathcal{J}}}(M_L)$ contains information on how Culler-Shalen essential surfaces intersect the JSJ-tori, i.e. how they cross from one JSJ-piece to another.

Now, if we consider the full JSJ-dual graph $\mathcal{G}_{\mathcal{J}}$ the corresponding eigenvalue-variety contains informations from both $E_\partial(M_L)$ and $E_{\mathcal{J}}(M_L)$.

Definition 2.3.2. The eigenvalue-variety $E_{\mathcal{G}_{\mathcal{J}}}(M_L)$ will be denoted by $E_{\mathcal{J}+\partial}(M_L)$ and called the JSJ*-peripheral* (or $JSJ\partial$) *eigenvalue-variety* of M_L. It has natural coordinates in $\mathbb{C}^{*2|\mathcal{J}|+2|L|}$.

By Proposition 3 applied to $\mathcal{G}_{\mathcal{J}}$, there's a monic algebraic map

$$E_{\mathcal{J}+\partial}(M_L) \hookrightarrow \prod_{v\in\mathcal{V}} E_\partial({M^v}_{L^v})$$

where each ${M^v}_{L^v}$ is either hyperbolic or Seifert-fibred.

In Chapter 4, we will study the case where all the pieces are Seifert-fibred (so M_L is a *graph manifold*). Using a combinatorial description of Seifert-fibred link-manifolds, we can describe each $E_\partial({M^v}_{L^v})$ and then, use the gluing criterion of Proposition 3 to describe some components of $E_\partial(M_L)$.

In the opposite case where all the pieces are hyperbolic, any essential torus in the interior of M_L can be isotoped to a torus of $\mathcal{J}$. It follows that, for any splitting tree $\mathcal{G}$ of M_L, there exist a collection Γ of subtrees of $\mathcal{G}_{\mathcal{J}}$ such that $\mathcal{G}_{\mathcal{J}} = (\mathcal{G} \gg= \Gamma)$. Therefore, applying Corollary 3, we get:

Proposition 4. *Let M_L be a link-manifold with all its JSJ pieces hyperbolic. For any splitting tree $\mathcal{G}$ of M_L, the associated eigenvalue-variety $E_{\mathcal{G}}(M_L)$ admits an epic algebraic map*

$$p \ : \ E_{\mathcal{J}+\partial}(M_L) \twoheadrightarrow E_{\mathcal{G}}(M_L)$$

So, in the same way as generic eigenvalue-varieties surject onto the peripheral eigenvalue-variety (see Proposition 3), in that case, the $JSJ\partial$ eigenvalue-variety surjects onto generic eigenvalue-varieties and acts as an initial object for the different $E_{\mathcal{G}}$ varieties of the link-manifold M_L.

Chapter 3

Eigenvalue-variety of Brunnian links

In this chapter, we give an answer to Question 1 for Brunnian links in $\mathbb{S}^3$. The main result here is the following Theorem 1 that will be proved at the end of the chapter:

Theorem 1. *Let L be a Brunnian link in $\mathbb{S}^3$ and let M denote its exterior, then $X^{\mathrm{SL}_2\mathbb{C}}(M)$ admits a peripherally maximal and non-abelian component if and only if L is neither the trivial link or the Hopf-link.*

The proof of this Theorem relies on the same arguments as Boyer-Zhang and Dunfield-Garoufalids proofs of the non-triviality of the A-polynomial of a knot in [BZ05] and [DG04] respectively. With a deep analysis of the peripheral eigenvalue-variety of Brunnian links in $\mathbb{S}^3$ and Dehn-fillings on such links, we show that sufficiently many Dehn-fillings on the link exterior admit irreducible characters, and that these characters span a top-dimensional component of irreducible characters in the peripheral eigenvalue-variety.

First, we start recalling the definition of a Brunnian link in $\mathbb{S}^3$ and review some few properties. We recall some stability properties under Dehn-fillings using [MS01] and similar properties under splicing are deduced using [EN85].

Most Brunnian links are homologically trivial (see Definition 1.3.11) and the proof of Theorem 1 is slightly easier in this case. It is studied first and we define a particular subset X_{KM} of the character variety of the link exterior, using suitable Dehn-fillings and Kronheimer-Mrowka Theorem of [KM04].

We obtain the following result for homologically trivial Brunnian links:

Theorem 8. *Let L be a non trivial HTB-link and M its exterior. The family of longitudinal trace $(I_{\lambda_K})_{K\subset L}$ is algebraically free in $\mathbb{C}[X_{KM}]$.*

In particular, this implies the existence of a peripherally maximal and non-abelian component in X_{KM} for homologically trivial brunnian links.

The case of Brunnian 2-links with nonzero linking number is taken care of with a similar argument; we prove that, besisdes the Hopf-link, all these links also admit peripherally maximal non-abelian components so we finally obtain Theorem 1.

We finally close this chapter with few considerations on non-trivial splittings. Using the stability of Brunnian links in $\mathbb{S}^3$ under *splicing* we can form new links over *Brunnian trees*. We briefly outline how the use of Proposition 3 might enable to describe the E-varieties obtained for those links. The same ideas, will be used in the next chapter to describe E-varieties of *graph link-manifolds*.

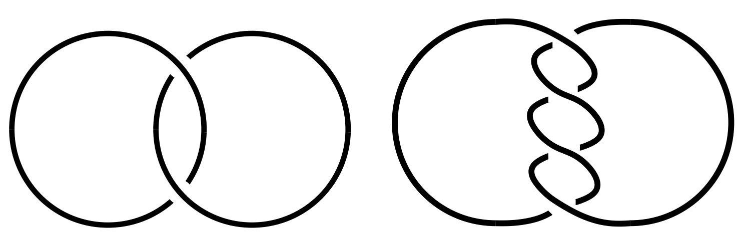
In all this chapter, M denotes the **exterior of a Brunnian link in** $\mathbb{S}^3$.

3.1 Brunnian links in $\mathbb{S}^3$

Let L be a link in $\mathbb{S}^3$ with exterior M. If K is an unknotted component of L, the Dehn-filling of ${\mathbb{S}^3}_K$ with slope $1/q$ is still $\mathbb{S}^3$. Therefore, as in Proposition 1.3.4, if $L = K \sqcup L_0$, $M(K : 1/q)$ identifies with the exterior of a link L_q in $\mathbb{S}^3$.

By Proposition 1.3.5, homologically trivial links are stable under such Dehn-fillings. As the following theorem shows, even if all the components are unknotted, a $1/q$-surgery on a component generally turns the other components into non-trivial knots:

Theorem 3.1.1 (Theorem 3.1 in [Mat92]). *Let $\mathcal{L} = \{K, J\}$ be a link in $\mathbb{S}^3$, K a knot, D an essential disk with $J = \partial D$. There exists an integer n such that the exterior of the knot $J(K; 1/n)$ remains boundary-compressible after $1/n$ Dehn surgery on K if and only if K is the trivial knot in $\mathbb{S}^3$ and $\mathcal{L}$ is one of the two links $\mathcal{L}_1$ or $\mathcal{L}_2$ of figure 3.1. Moreover, for $\mathcal{L}_1$, $J(K : 1/n)$ remains trivial for every n, and for $\mathcal{L}_2$, only the knot $J(K : 1)$ is trivial.*



(a) $\mathcal{L}_1$ (b) $\mathcal{L}_2$

Figure 3.1 – Forbidden links

An other $1/q$-Dehn-filling would then produce a non-trivial integer-homology sphere and escape the scope of links in $\mathbb{S}^3$. In order to remain in the standard 3-sphere we will need some additional property on the link:

Definition 3.1.1. A link L in $\mathbb{S}^3$ is *Brunnian* if any proper sublink of L is trivial.

Remark 3.1.1. Any knot is considered Brunnian; for links with more components we have:

- the components of a Brunnian link with 2 components or more are individually unknotted.
- any Brunnian link with 3 or more components is homologically trivial.

We intend to use Dehn surgeries on exteriors of Brunnian links to produce irreducible characters; the following section recalls some results of Mangum ans Stanford in [MS01] that will enable us to use Kronheimer-Mrowka Theorem in Section 3.2.2.

3.1.1 Dehn-fillings on Brunnian links

If L is the trivial link, any $1/q$-surgery produces the trivial link again. We can use this to show that the family of Brunnian links is stable under $1/q$-Dehn-fillings:

Lemma 3.1.2. *Let L be a Brunnian link with 2 components or more. For any component $K \subset L$ and for any q in $\mathbb{Z}$, the link L_q obtained by $1/q$-Dehn-filling along K is also Brunnian.*

Proof. If L has two components L_q is a knot in $\mathbb{S}^3$ and there's nothing to prove.

Let $L = K \sqcup L_0$ be a link such that L_0 has at least two components; then L is HT so the peripheral systems are unchanged by the surgery. It follows that, for any K' in $L_q \cong L_0$,

$$M(K:1/q)(K':1/0) = M(K':1/0)(K:1/q)$$

Since L is Brunnian, $M(K':1/0)$ is the trivial link and, therefore, so is $M(K':1/0)(K:1/q)$. In other words, removing any component K' of L_q produces the trivial link, so L_q is Brunnian. □

In [MS01], Mangum and Stanford studied when Dehn-fillings on a Brunnian link can produce the trivial link. To keep the peripheral systems unchanged they restrict themselves to links that are both Brunnian and homologically trivial, discarding links with 2 unknotted components and non-zero linking number.

Definition 3.1.2 (HTB links)**.** A link L in $\mathbb{S}^3$ is HTB if it is both homologically trivial and Brunnian.

Then, combining Proposition 1.3.5 and Lemma 3.1.2, we obtain the following stability lemma for HTB-links:

Lemma 3.1.3. *Let L be an HTB-link. For any component K of L and q in $\mathbb{Z}$, the link L_q obtained by $1/q$-Dehn-filling along K is also HTB.*

This stability is used in [MS01] to prove the following theorem about Dehn-fillings on HTB-links:

Theorem 3.1.4 (Theorem 2 in [MS01])**.** *Let L be an n-components HTB-link with exterior M. Suppose that there exist slopes $r_i = p_i/q_i$, with $q_i \neq 0$ for all i, and such that $M(r_1, \ldots, r_n) = \mathbb{S}^3$. Then L is trivial.*

As explained in [MS01], this result implies that non-trivial $1/q$ Dehn-fillings on non-trivial HTB-links can never produce the trivial link:

Corollary 3.1.5. *Let L be a non-trivial HTB-link with 2 components or more. For any component K of L and for any integer $q \neq 0$, the link L_q obtained by $1/q$-Dehn-filling along K is a non-trivial HTB-link.*

Proof. As explained in [MS01], if $M(*, \ldots, *, 1/q)$ is the trivial link, then $M(1, \ldots, 1, 1/q)$ is $\mathbb{S}^3$ so, by Theorem 3.1.4, L must be trivial. □

Remark 3.1.2. For 2-components links, this corollary is a particular case of the more general Theorem 3.1.1 of Mathieu cited earlier.

3.1.2 Splicing of Brunnian links

Let K and K' be two unknots in $\mathbb{S}^3$, then the splicing $\mathbb{S}^3 \,{}^{K}\!\bowtie^{K'} \mathbb{S}^3$ is again $\mathbb{S}^3$. It follows that if $L = L_0 \sqcup K$ and $L' = L'_0 \sqcup K'$ are two links in $\mathbb{S}^3$ with unknotted components K and K', ${\mathbb{S}^3}_{L_0} \,{}^{K}\!\bowtie^{K'} {\mathbb{S}^3}_{L'_0}$ identifies with the exterior of a link $L^\star$ in $\mathbb{S}^3$. This is a special case of *splicing link-manifolds* (see Definition 1.3.8) and the link $L^\star$ will be denoted by $L_0 \,{}^{K}\!\bowtie^{K'} L'_0$.

As explained in [EN85], if L is the trivial link or the Hopf link, the splicing takes special forms:

Lemma 3.1.6. *Let $L = L_0 \sqcup K$ and $L' = L'_0 \sqcup K'$ be two links in $\mathbb{S}^3$.*

- *If L' is the unknot (so $L'_0 = \emptyset$), $L_0 \,{}^{K}\!\bowtie^{K'} L'_0 = L_0$.*
- *If L' is the Hopf link (see Figure 3.2), $L_0 \,{}^{K}\!\bowtie^{K'} L'_0 = L$.*

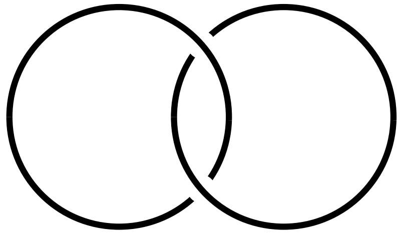


Figure 3.2 – The Hopf link

- *If* L' *is the trivial link (so* L'_0 *is also trivial),* $L_0 \; {}^{K}\!\bowtie^{K'} \; L'_0 = L_0 \sqcup L'_0$.

Proof. We give a rapid sketch of the proof; details can be found in [EN85].

Splicing with the unknot is equivalent to filling the component along the meridian, hence removing the component.

In the Hopf-link, each meridian is a longitude for the other component. Therefore, splicing a component of L with the Hopf-link just leaves L unchanged.

Finally, if L' is the trivial link, ${\mathbb{S}^3}_{L_0} \; {}^{K}\!\bowtie^{K'} \; {\mathbb{S}^3}_{L'_0}$ is the connected sum of ${\mathbb{S}^3}_{L_0} \; {}^{K}\!\bowtie^{K'} \; \mathbb{S}^3$ and ${\mathbb{S}^3}_{L'_0}$ which is simply the exterior of $L_0 \sqcup L'_0$. □

Proposition 3.1.7. *For any Brunnian links* $L = K \sqcup L_0$ *and* $L' = K' \sqcup L'_0$ *in* $\mathbb{S}^3$*, the link* $L_0 \; {}^{K}\!\bowtie^{K'} \; L'_0$ *is also Brunnian.*

Proof. Let $L = K \sqcup L_0$ and $L' = K' \sqcup L'_0$ be two Brunnian links in $\mathbb{S}^3$ and let $L^\star$ be the result of the splicing $L_0 \; {}^{K}\!\bowtie^{K'} \; L'_0$.

Let J be a component of $L^\star$. By Lemma 3.1.6, the link $L^\star \setminus J$ is the result of splicing $L^\star$ with the unknot along J. Any component J of $L^\star$, identifies with a component of L_0 or L'_0; without loss of generality we may assume that J is a component of L_0.

Since the splicing is associative, this is equivalent to splicing L with the unknot along J and, then, splicing the resulting link with L', along K and K'. Because L is Brunnian, the result of the first splicing is the trivial link, and by Lemma 3.1.6, the result of the second splicing is again the trivial link in $\mathbb{S}^3$.

In other words, forgetting any component of $L^\star$ produces the trivial link so any strict sublink of $L^\star$ is trivial and $L^\star$ is Brunnian. □

In the next section, we will prove Theorem 8. The statement of this theorem is quite similar to Theorem 2.2.2 on the character variety of hyperbolic link-manifolds. However, Proposition 3.1.7 shows that not all Brunnian links are hyperbolic and Theorem 8 also applies for manifolds with non-trivial JSJ-decomposition.

3.2 A peripherally maximal component for HTB-links

In this section, L denotes a non-trivial HTB-link and M its exterior.

3.2.1 Reducible characters of HTB-links

By Proposition 2 of Section 3.3.1, the peripheral eigenvalue-variety of the exterior of a HTB-link is particularly simple:

Proposition 3.2.1. *Let L be a* HTB*-link in $\mathbb{S}^3$ and M it exterior; the peripheral eigenvalue-variety of reducible characters is given in $\mathbb{C}[\mathfrak{m}, \mathfrak{l}]$ by:*

$$\mathcal{A}_\partial^{\mathrm{red}}(M) = \langle \mathfrak{l}_K - 1,\ K \subset L \rangle$$

Proof. All the linking numbers are 0. □

In particular, any component in $X(M)$ on which I_{λ_K} is not constant for some longitude λ_K will be peripherally non-abelian.

There's not much more to say about the reducible characters of HTB-links and we will now focus on irreducible characters.

3.2.2 Kronheimer-Mrowka characters

First, let's recall Kronheimer-Mrowka Theorem of [KM04].

Theorem 3.2.2 (Kronheimer-Mrowka Theorem, [KM04])**.** *Let K be a non-trivial knot in S^3, and let Y_r be the 3-manifold obtained by Dehn surgery on K with surgery-coefficient $r \in \mathbb{Q}$. If $|r| \leq 2$, then $\pi_1 Y_r$ is not cyclic. In fact, there is a homomorphism $\rho : \pi_1 Y_r \to \mathrm{SU}_2$ with non-cyclic image.*

Remark 3.2.1. As explained in [DG04], the representation ρ obtained this way is irreducible.

By Section 3.1.1, non-trivial $1/q$-Dehn-fillings on non trivial HTB-links produce non-trivial HTB-links with the peripheral system unchanged. Repeating this process on all but one components produces a non-trivial knot on which we can use Kronheimer-Mrowka Theorem to produce irreducible representations:

Corollary 3.2.3. *Let L be a non trivial* HTB*-link in $\mathbb{S}^3$ and let M denote its exterior. For any family of integers $\underline{q} = (q_K)_{K \subset L}$ in $\mathbb{Z}^{|L|}$, let $M_{\underline{q}}$ denote the homology sphere obtained by $1/q_K$ Dehn-filling on each boundary T_K. Then,*

- *if* $q_K = 0$ *for some* K *in* L, $M_{\underline{q}} = \mathbb{S}^3$
- *otherwise, there exist an irreducible representation*

$$\rho_{\underline{q}} : \pi_1 M_{\underline{q}} \to \mathrm{SU}_2.$$

For any $\underline{q}$ in $(\mathbb{Z} \setminus \{0\})^{|L|}$, Corollary 3.2.3 provides an irreducible representation $\rho_{\underline{q}}$ in $\mathrm{Hom}(\pi_1 M_{\underline{q}}, \mathrm{SU}_2)$ whose restriction to $\pi_1 M$ is an irreducible representation of $R(M)$ (considering $\mathrm{SU}_2 \subset \mathrm{SL}_2\mathbb{C}$). We denote by $\chi_{\underline{q}}$ the corresponding character in $X^{\mathrm{irr}}(M)$.

Definition 3.2.1 (Kronheimer-Mrowka character)**.** Any such $\chi_{\underline{q}}$ is called a *Kronheimer-Mrowka character* of M.

We now consider all the possible Kronheimer-Mrowka characters:

Definition 3.2.2 (Kronheimer-Mrowka components)**.** Let M denote the exterior of a non-trivial HTB-link L in $\mathbb{S}^3$; the *Kronheimer-Mrowka components of* M in $X(M)$, denoted by X_{KM}, is the Zariski closure in $X(M)$ of the set of Kronheimer-Mrowka characters:

$$X_{KM} = \overline{\{\chi_{\underline{q}}, \underline{q} \in (\mathbb{Z} \setminus \{0\})^L\}} \subset X(M).$$

Remark 3.2.2. The space of Kronheimer-Mrowka characters X_{KM} may contain more than one algebraic component.

For any sublink L' of L and for any $\underline{q}$ in $(\mathbb{Z} \setminus \{0\})^{|L'|}$ let $M_{\underline{q}}$ denote the manifold obtained by Dehn-fillings along L' with slopes $1/\underline{q}$. It is the exterior of a non-trivial Brunnian link in $\mathbb{S}^3$ and the natural inclusion

$$M \hookrightarrow M_{\underline{q}}$$

induces an algebraic map on the representation and character varieties with the following commutative diagram:

$$\begin{array}{ccc} R(M_{\underline{q}}) & \longrightarrow & R(M) \\ \downarrow t & & \downarrow t \\ X(M_{\underline{q}}) & \longrightarrow & X(M) \end{array} \tag{3.1}$$

Therefore, the Kronheimer-Mrowka components of M contain all the Kronheimer-Mrowka components $X_{KM}(M_{\underline{q}})$ for any sublink L' of L and coefficients $(q_K)_{K \subset L'}$ in $(\mathbb{Z} \setminus \{0\})^{|L'|}$.

3.2.3 Peripheral maximality

In [BZ05] and [DG04], Boyer-Zhang and Dunfield-Garoufalidis, respectively, use Kronheimer-Mrowka Theorem to prove that the A-polynomial detects the unknot. Following the same ideas, we will use Corollary 3.2.3 to show that for any non trivial HTB-link, the Kronheimer-Mrowka set of characters X_{KM} contains peripherally maximal and non-abelian component. More precisely we will show that there is a component X_0 of X_{KM} such that the family $(I_{\lambda_K})_{K\subset L}$ is algebraically free in $\mathbb{C}[X_0]$. To achieve this, we will need the following lemma which will allow us to separate the different Kronheimer-Mrowka characters.

Lemma 2. *For any Kronheimer-Mrowka character $\chi_{\underline{q}}$ and for any component K of L, $I_{\mu_K}(\chi)$, $I_{\lambda_K{}^{q_K}}(\chi)$, and $I_{\lambda_K}(\chi)$ are not ± 2.*

Proof. Let $\chi_{\underline{q}}$ be a Kronheimer-Mrowka character and ρ an irreducible representation of $\pi_1 M$ to SU_2 with character $\chi_{\underline{q}}$.

If $I_{\lambda_K}(\chi) = \pm 2$, $I_{\lambda_K{}^p}(\chi) = \pm 2$ for any integer p so $I_{\lambda_K{}^{q_K}}(\chi) = \pm 2$. Moreover, the only parabolic elements of SU_2 are $\pm Id$ so, if $I_{\lambda_K^{q_K}}(\chi) = \pm 2$, $\rho(\lambda_K)^{q_K} = \pm Id$ so, by the surgery relation, $\rho(\mu_K) = \pm \mathrm{Id}$ and $I_{\mu_K}(\chi) = \pm 2$.

Therefore, if $I_{\mu_K}(\chi)$ is not equal to ± 2, then neither are $I_{\lambda_K{}^{q_K}}(\chi)$ and $I_{\lambda_K}(\chi)$.

Finally, let's assume that $I_{\mu_K}(\chi) = \pm 2$, so $\rho(\mu_K) = \pm Id$. Then, modulo $\pm Id$, ρ factors through the surgeries $(T_K : 1/0)$ and $(T_J : 1/q_J)$ for $J \neq K$. By Corollary 3.2.3, the latter surgery instructions on M produce $\mathbb{S}^3$ so ρ must be trivial modulo $\pm Id$. Therefore, $\rho(\pi_1 M_{\underline{q}}) \subset \{\pm Id\}$ and ρ cannot be irreducible, a contradiction.

Therefore, $\mathrm{tr}\rho(\mu_K) \neq \pm 2$ for all components K of L. Since ρ factors through the surgery $(T_K : 1/q_K)$, $\rho(\mu_K) = \rho(\lambda_K)^{-q_K}$ so $\mathrm{tr}\rho(\lambda_K^{q_K}) = \mathrm{tr}\rho(\mu_K)$ is also different from ± 2. If $I_{\lambda_K}(\chi) = \pm 2$, then so does $I_{\lambda_K^p}$ for any integer p, so $I_{\lambda_K}(\chi)$ is also different from ± 2. □

This lemma will allow us to prove the following Theorem 8:

Theorem 8. *Let L be a non trivial HTB-link and M its exterior. The family of longitudinal trace $(I_{\lambda_K})_{K\subset L}$ is algebraically free in $\mathbb{C}[X_{KM}]$.*

Proof. We will show that, if $I_{\underline{\lambda}}$ denotes the longitudinal map from $X(M)$ to $\mathbb{C}^{|L|}$:

$$\begin{array}{rccc} I_{\underline{\lambda}}: & X(M) & \to & \mathbb{C}^{|L|} \\ & \chi & \to & (I_{\lambda_k}(\chi))_{k\subset L} \end{array}$$

then $\overline{I_{\underline{\lambda}}(X_{KM})} = \mathbb{C}^{|L|}$ so $(I_{\lambda_K})_{K\subset L}$ is algebraically free in $\mathbb{C}[X_{KM}]$.

We will prove Theorem 8 by induction on the number of components of L. The idea is to construct infinitely many subspaces of X_{KM} which project on different hypersurfaces by $I_{\underline{\lambda}}$.

Base case For links with one component, knots, the proof is essentially equivalent to the ones found in [DG04] and [BZ05] to prove that the A-polynomial distinguishes the unknot. Since the same ideas will be used for the induction step, we re-present this proof here. The fundamental idea in [DG04] and [BZ05] is the same, but the technics to separate the Kronheimer-Mrowka characters differ slightly; we use here the Boyer-Zhang point-of-view which will be more easily adaptable to the induction step.

Let K be a non trivial knot in $\mathbb{S}^3$. Let M denote the exterior of K with boundary T and peripheral system (μ, λ). For any q in $\mathbb{Z}$, let M_q denote the homology sphere obtained by $1/q$ surgery on T. By Kronheimer Mrowka's theorem, for any $q \neq 0$, there exist an irreducible representation

$$\rho_q : \pi_1 M_q \to \mathrm{SU}_2.$$

Let χ_q denote the character of ρ_q in $X(M)$. We will show the following lemma:

Lemma 3.2.4. *I_λ takes infinitely many values on $\{\chi_q, q \in \mathbb{Z} \setminus \{0\}\}$.*

Proof of Lemma 3.2.4. For any $q \neq 0$, let χ_q denote a Kronheimer-Mrowka character of M with surgery instruction $1/q$. By Lemma 2, $I_{\lambda_K}(\chi_q) \neq \pm 2$ so for any irreducible representation ρ_q in $t^{-1}\chi_q$, $\rho_q(\lambda)$ is diagonalizable so, up to conjugation:

$$\rho_q(\lambda) = \begin{bmatrix} \ell_q & 0 \\ 0 & {\ell_q}^{-1} \end{bmatrix}$$

and, by the surgery relation,

$$\rho_q(\mu) = \begin{bmatrix} {\ell_q}^{-q} & 0 \\ 0 & {\ell_q}^{q} \end{bmatrix}$$

For any $q \neq 0$, the set

$$\{p \in \mathbb{Z} \mid I_{\lambda^p}(\chi_q) = \pm 2\} = \{p \in \mathbb{Z} \mid {\ell_q}^p = \pm 1\}.$$

is an ideal $d_q\mathbb{Z}$, $d_q \geq 0$.

For any $q, q' > 0$, if $I_\lambda(\chi_q) = I_\lambda(\chi_{q'})$ then, ρ_q satisfies both surgery relations so

$$\rho_q(\mu\lambda^q) = \mathrm{Id} = \rho_q(\mu\lambda^{q'})$$

and, $\rho_q(\lambda)^{(q'-q)} = Id$. Therefore,

$$I_\lambda(\chi_q) = I_\lambda(\chi_{q'}) \Longrightarrow q' - q \in d_q\mathbb{Z}. \tag{3.2}$$

For any $q \neq 0$, Lemma 2 implies that 1 (and q) is not in $d_q\mathbb{Z}$ so $d_q \neq 1$. Moreover, for any q such that $d_q = 0$, we have:

$$\forall\, q' \neq q,\ I_\lambda(\chi_q) \neq I_\lambda(\chi_{q'}).$$

If the set $\{q \in \mathbb{Z} \mid d_q = 0\}$ is infinite, I_λ takes infinitely many values on $\{\chi_q,\ q \in \mathbb{Z}\}$ and Lemma 3.2.4 is proved.

Otherwise, there exist N such that for any $q \geq N$, $d_q \geq 2$. Let $(q_i)_{i\in\mathbb{N}}$ denote a family of integers such that

- $q_0 \geq N$
- for any j in $\mathbb{N}$, $q_{j+1} \geq q_j$ and $q_{j+1} \in \bigcap_{i=1}^{j} d_{q_j}\mathbb{Z}$.

Then, the following fact concludes the proof of Lemma 3.2.4:

$$\forall\, i < j,\ I_\lambda(\chi_{q_i}) \neq I_\lambda(\chi_{q_j}).$$

Indeed, for any j in $\mathbb{N}$, let's assume that $I_\lambda(\chi_{q_i}) = I_\lambda(\chi_{q_j})$ for some $i < j$. By equation (3.2), this would imply that $q_j - q_i \in d_{q_i}\mathbb{Z}$; by construction, $q_j \in d_{q_i}\mathbb{Z}$ so this would imply $q_i \in d_{q_i}\mathbb{Z}$, a contradiction. □

Since I_λ takes infintely many values on X_{KM}, it contains a curve on which I_λ is non-constant, which concludes the base case for the proof of Theorem 8.

Induction step Now, let L be a non trivial HTB-link, and let's assume that Theorem 8 is true for all non trivial HTB-links with $|L|-1$ components; we denote its exterior by M and ∂M by the collection of tori $(T_J)_{J\subset L}$.

Let K be a component of L so $L = K \sqcup L_0$; for any $q \neq 0$, $M_q = M(T_K : 1/q)$ is the exterior of a non-trivial HTB-link with $|L_0| = |L| - 1$ components, while $M_0 = M(T_K : 1/0)$ is the $(|L|-1)$-trivial link in $\mathbb{S}^3$.

For any $q \neq 0$, we have the following commutative diagram of algebraic maps:

$$\begin{array}{ccc} X(M_q) & \xrightarrow{i_q^*} & X(M) \\ {\scriptstyle I_{\lambda_{L_0}}}\downarrow & & \downarrow{\scriptstyle I_{\underline{\lambda}}} \\ \mathbb{C}^{|L|-1} & \longleftarrow & \mathbb{C}^{|L|} \end{array} \tag{3.3}$$

where $I_{\lambda_{L_0}}$ is the map $(I_{\lambda_J})_{J\neq K}$ so $I_{\underline{\lambda}} = (I_{\lambda_{L_0}}, I_{\lambda_K})$. We can apply the induction hypothesis to M_q and find a component X_q of $X_{KM}(M_q)$ on which $I_{\lambda_{L_0}}$ is an open map. We identify X_q with its image in $X(M)$.

For any q in $\mathbb{Z}\setminus\{0\}$, the family $(I_{\lambda_J})_{J\subset L_0}$ is algebraically free in $\mathbb{C}[X_q]$ so $\overline{I_\lambda(X_q)}$ contains an hypersurface H_q. We will show the the collection $(H_q)_{q\neq 0}$ contains infinitely many different hypersurfaces.

As for the base case, for $q \neq 0$ we define the ideal of $\mathbb{Z}$:

$$\{p \in \mathbb{Z} \mid I_{\lambda_K^p|X_q} \equiv \pm 2\} = d_q\mathbb{Z}.$$

For any Kronheimer-Mrowka character χ of M_q, $I_{\lambda_K}(\chi)$ and $I_{\lambda_K^q}(\chi)$ are different from ± 2 by Lemma 2 so 1 and q are not in $d_q\mathbb{Z}$.

As for the base case, the family $(d_q)_{q\neq 0}$ permits to distinguish the different hypersurfaces H_q:

Lemma 3.2.5. *For any* q, q' *in* $\mathbb{Z} \setminus \{0\}$,

$$H_q \subset H_{q'} \Longrightarrow q - q' \in d_q\mathbb{Z}.$$

Proof of Lemma 3.2.5. For any q in $\mathbb{Z} \setminus \{0\}$, $I_{\mu_K\lambda_K^q|X_q} = 2$.

Assume that $H_q \subset H_{q'}$, then, up to restriction to a Zariski-dense set, for any χ in X_q, $I_{\underline{\lambda}}(\chi) = I_{\underline{\lambda}}(\chi')$ for some χ' in $X_{q'}$. Since I_{λ_K} determines completely χ on T_K, this implies that $I_{\mu_K\lambda_K^q}(\chi) = \pm 2 = I_{\mu_K\lambda_K^{q'}}(\chi)$. Therefore, since the triple

$$I_{\mu_K\lambda_K^q}(\chi), I_{\mu_K\lambda_K^{q'}}(\chi), I_{\lambda_K^{(q-q')}}(\chi)$$

satisfies the relation

$$X^2 + Y^2 + Z^2 - XYZ - 4 = 0$$

we have, then, $I_{\lambda_K^{(q-q')}}(\chi) = \pm 2$.

This is true on a Zariski-dense set of X_q so, by algebraicity, $q - q' \in d_q\mathbb{Z}$. □

The end of the proof is the same as for the base case. We construct a family $(H_q)_{q\in Q\subset\mathbb{Z}\setminus\{0\}}$ of infinitely many distinct hypersurfaces of $\overline{I_{\underline{\lambda}}(X(M))}$ so, by algebraicity, $I_{\underline{\lambda}}(X(M))$ is Zariski-dense in $\mathbb{C}^{|L|}$.

For any $q \neq 0$, if $d_q = 0$ then $H_q \neq H_{q'}$ for all q'. Therefore, if $d_q = 0$ for infinitely many q, $(H_q)_{q\neq 0}$ contains infinitely many different hypersurfaces.

Otherwise, there exist N in $\mathbb{N}$ such that $d_q \geq 2$ for $q \geq N$. Let $(q_i)_{i\in\mathbb{N}}$ be a family integers such that

- $q_0 \geq N$
- for any j in $\mathbb{N}$, $q_{j+1} \geq q_j$ and $q_{j+1} \in \bigcap_{i=1}^{j} d_{q_j}$.

The same argument as in the base case shows that, for $i < j$, $H_{q_i} \neq H_{q_j}$. Therefore, $(H_{q_i})_{i\in\mathbb{N}}$ is a family of infinite many distinct hypersurfaces in $\overline{I_{\underline{\lambda}}(X(M))}$.

Finally, $\overline{I_{\underline{\lambda}}(X(M))}$ contains infinitely many distinct hypersufaces, so, by algebraicity, it must be all $\mathbb{C}^{|L|}$; therefore, the functions $(I_{\lambda_K})_{K\subset L}$ are algebraically free on $\mathbb{C}[X_{KM}]$, which concludes the induction step of Theorem 8.

□

Corollary 4. *The Kronheimer-Mrowka component of a non-trivial* HTB*-link is peripherally maximal and non-abelian.*

Proof. Each $\mathfrak{l}_K$ satisfies $\mathfrak{l}_K + {\mathfrak{l}_K}^{-1} = I_{\lambda_K}$ so, if the functions $(I_{\lambda_K})_{K\subset L}$ are algebraically free on a component X_0 of X_{KM}, the family $(\mathfrak{l}_K)_{K\subset L}$ is algebraically free on $\mathbb{C}[E_\partial(X_0)]$. □

3.3 Brunnian 2-links with nonzero linking number

In this section $L = K \sqcup K'$ is a Brunnian 2-link in $\mathbb{S}^3$ with linking number $\alpha \neq 0$ and M is its exterior.

3.3.1 Reducible characters

As before, the component of reducible characters in the peripheral eigenvalue-variety is given by Proposition 2. In the special case of a link of a 2-link with nonzero linking number α we have:

Proposition 3.3.1. *The peripheral eigenvalue-variety of reducible characters is given in* $\mathbb{C}^{*4}$ *by*

$$\mathcal{A}^{\mathrm{red}}(M)_\partial = \langle \mathfrak{l} - \mathfrak{m}'^{\alpha}, \mathfrak{l}' - \mathfrak{m}^{\alpha} \rangle \times \langle \mathfrak{l}\mathfrak{m}'^{\alpha} - 1, \mathfrak{l}'\mathfrak{m}^{\alpha} - 1 \rangle$$

This will be enough to detect peripherally non-abelian components of the character variety.

3.3.2 Another peripherally maximal component

For two-components Brunnian links with non-zero linking number the peripheral system is changed after $1/q$-Dehn-filling and Corollary 3.2.3 doesn't apply. Moreover, the Hopf link is a Brunnian 2-link but its exterior has abelian fundamental group and therefore admits no irreducible character. However, by Mathieu's Theorem 3.1.1, besides the Hopf link, for $|q| > 1$, a $1/q$-Dehn-filling always produces a non-trivial knot in $\mathbb{S}^3$.

Let $L = K \sqcup K'$ be a Brunnian link with linking number $\alpha \neq 0$, exterior M, and peripheral system $(\mu, \lambda), (\mu', \lambda')$. After $1/q$-Dehn-filling along $T_{K'}$, K becomes a knot K_q and the new longitude for K_q is $\lambda_q = \lambda + q\alpha^2\mu$ (see Proposition 1.3.4). To apply Kronheimer-Mrowka Theorem on K_q, we have to use the new peripheral system. Moreover, Dehn-filling on both components now depends on the order of the surgeries on the boundary tori.

Let M_q denote the exterior of K_q. We cannot exactly apply the ideas of Theorem 8, however, considering the peripheral trace maps in $\mathbb{C}^4$ and $\mathbb{C}^2$ respectively:

$$\begin{array}{rcl} I_{\partial M} : X(M) & \to & \mathbb{C}^4 \\ \chi & \to & (I_\mu(\chi), I_\lambda(\chi), I_{\mu'}(\chi), I_{\lambda'}(\chi)) \\ I_{\partial M_q} : X(M_q) & \to & \mathbb{C}^2 \\ \chi & \to & (I_\mu(\chi), I_\lambda(\chi)) \end{array}$$

we obtain the following commutative diagram

$$\begin{array}{ccc} X(M_q) & \xrightarrow{i_q{}^*} & X(M) \\ {\scriptstyle I_{\partial M_q}}\downarrow & & \downarrow{\scriptstyle I_{\partial M}} \\ \mathbb{C}^2 & \longleftarrow & \mathbb{C}^4 \end{array} \tag{3.4}$$

and we get:

Lemma 3.3.2. *For any $|q| > 1$ there is a curve of irreducible characters C'_q in $X(M_q)$ such that $\overline{I_{\partial M_q}(C'_q)}$ is a curve D'_q in $\mathbb{C}^2$*

Proof. For any q in $\mathbb{Z} \setminus \{-1, 0, 1\}$, M_q is the exterior of a non trivial knot in $\mathbb{S}^3$ and there is a curve C'_q in $X^{\text{irr}}(M_q)$ on which I_{λ_q} is open. For any character χ of C'_q with $I_{\lambda_q}(\chi) \neq \pm 2$, there exist ℓ_q and m in $\mathbb{C}^*$ such that

- $I_{\lambda_q}(\chi) = \ell_q + \ell_q{}^{-1}$
- $I_\mu(\chi) = m + m^{-1}$

and, since $\lambda_q = \lambda + q\alpha^2\mu$, then

$$I_\lambda(\chi) = \ell_q m^{-q\alpha^2} + \ell_q{}^{-1} m^{q\alpha^2}$$

so the image of C'_q by $I_{\partial M_q}$ is also a curve D'_q in $\mathbb{C}^2$. □

The closure of $i_q{}^* C'_q$ contains a curve C_q in $X(M)$ such that $\overline{I_{\partial M}(C_q)}$ is a curve D_q in $\mathbb{C}^4$ whose projection on the first two coordinates is D'_q. Characters of C_q satisfy the surgery relation so $I_{\mu'\lambda'^q|_{C_q}} \equiv 2$ and, as in the HTB case, we can define

$$\{p \in \mathbb{Z} \setminus \{-1, 0, 1\} \mid I_{\lambda'^p|_{C_q}} \equiv \pm 2\} = d_q\mathbb{Z}.$$

Then, as before, $q \notin d_q\mathbb{Z}$ and, for q, q' in $\mathbb{Z}$, $D_q = D_{q'}$, implies that $q - q' \in d_q\mathbb{Z}$. As in the induction step of the proof of Theorem 8, we can construct an infinite family of distinct curves in $\overline{I_{\partial M}(X(M))}$ so there must be a component of irreducible characters whose image by $I_{\partial M}$ has dimension 2 in $\mathbb{C}^4$.

Therefore, $E_\partial(M)$ admits a component of dimension 2, different from $E_\partial^{\text{red}}(M)$, so we have:

Lemma 3. *Except for the Hopf-link, the character variety of a Brunnian* 2*-links with nonzero linking numbers admit a peripherally maximal and non-abelian component.*

In conclusion, we have the following theorem for Brunnian links in $\mathbb{S}^3$:

Theorem 1. *Let* L *be a Brunnian link in* $\mathbb{S}^3$ *and let* M *denote its exterior, then* $X^{\text{SL}_2\mathbb{C}}(M)$ *admits a peripherally maximal and non-abelian component if and only if* L *is neither the trivial link or the Hopf-link.*

Proof. The exteriors of the unknot and the Hopf-link don't admit any irreducible characters so all the characters are peripherally abelian.

The exterior of the n-trivial link with $n \geq 2$ is the free group with n generators. In that case, $\pi_1 M$ admits irreducible characters but, since the longitudes are nullhomotope, any such character is peripherally abelian.

By remark 3.1.1, the other cases correspond to Corollary 4 and Lemma 3. □

We close this chapter with few observations on the links and E-varieties obtained by *splicing* Brunnian links.

3.4 Brunnian trees

By Proposition 3.1.7, if $L^+ = L_0^+ \sqcup K^+$ and $L^- = L_0^- \sqcup K^-$ are two Brunnian links in $\mathbb{S}^3$, the splicing

$$\mathbb{S}^3{}_{L^\star} = \mathbb{S}^3{}_{L_0^+}\ {}^{K^+}\!\!\bowtie^{K^-}\ \mathbb{S}^3{}_{L_0^-}$$

is the exterior of another Brunnian link in $\mathbb{S}^3$. We can iterate this process along trees, as Serre for trees of groups, to obtain links described by *brunnian trees*.

Let $\mathcal{G}$ be a tree with arrows. Let's consider the identical binding decomposition (see Definition 1.1.9) of $\mathcal{G}$,

$$\mathcal{G} = (\mathcal{G} \ggg= \overrightarrow{\mathcal{V}})$$

Definition 3.4.1. A *Brunnian tree* over $\mathcal{G}$ is a collection of a Brunnian link in $\mathbb{S}^3$, L^v, for each vertex v of $\mathcal{G}$, with $|\overrightarrow{\mathcal{E}}(\overrightarrow{v})|$ components; each component of L^v is identified with the an arrow in $\overrightarrow{\mathcal{E}}(\overrightarrow{v})$ for each graph $\overrightarrow{v}$ in the collection $\overrightarrow{\mathcal{V}}$.

By the observations made above, a Brunnian tree $(\mathcal{G}, L)$ defines a Brunnian link $L^{\mathcal{G}}$ in $\mathbb{S}^3$ by splicing according to the edge-data. By Mangum-Stanford results of [MS01], if none of the links L^v is trivial, the link $L^{\mathcal{G}}$ is also non trivial. Since the splicing with the Hopf link leaves the original link unchanged, we also assume none of the L^v is the Hopf link.

This is a special case of toric splitting of link-manifolds; the fundamental group of the exterior of $L^{\mathcal{G}}$ is precisely the one from the tree of groups contructed over $\mathcal{G}$, assigning the fundamental group of the exterior of L^v to each vertex of $\mathcal{G}$. We can then apply Proposition 3 to the tree $\mathcal{G}$, and obtain the diagram:

$$\begin{array}{ccc} E_{\mathcal{G}}({\mathbb{S}^3}_{L^{\mathcal{G}}}) & \stackrel{i^*}{\hookrightarrow} & \prod_{v\in\mathcal{V}} E_\partial({\mathbb{S}^3}_{L^v}) \\ {\scriptstyle p}\big\downarrow & & \\ E_\partial({\mathbb{S}^3}_{L^{\mathcal{G}}}) & & \end{array} \tag{3.5}$$

where, by Theorem 1, $E_\partial({\mathbb{S}^3}_{L^{\mathcal{G}}})$ and each $E_\partial({\mathbb{S}^3}_{L^v})$ contains a component of non-abelian characters with maximal dimension. One could push these observations further and try to apply the merging criterion and study the different components of $E_\partial({\mathbb{S}^3}_{L^{\mathcal{G}}})$ appearing while gluing different components of $\prod_{v\in\mathcal{V}} E_\partial({\mathbb{S}^3}_{L^v})$.

However, we will not go any further in the direction here. In the next chaper, we apply a similar idea in a different case, *graph link-manifolds*. These are also construct over trees as in Definition 3.4.1, but using *Seifert-fibred link-manifolds* instead of Brunnian links in $\mathbb{S}^3$. This provides a combinatorial description of the fundamental groups which will enable us to quite fully describe the $E_{\mathcal{G}}$-varieties in that case.

Chapter 4

$E_{\mathcal{G}}$-varieties of graph link-manifolds

In this chapter, we study a case opposite, in a way, to the Brunnian links that we studied in the previous chapter; here, we apply the theory of $E_{\mathcal{G}}$-varieties to irreducible link exteriors for which all the JSJ pieces admit a Seifert fibration, so-called *graph link-manifolds*,. In that case, the linking numbers are usually non-zero, and grow rapidly with the complexity of the JSJ tree. We describe graph link-manifolds using *splice diagrams* as in Eisenbud-Neumann's [EN85]. The splice diagram is a refinement of the JSJ tree using a description of the Seifert fibration of each piece. Such a generic splice diagram is presented in the following Figure 4.1.

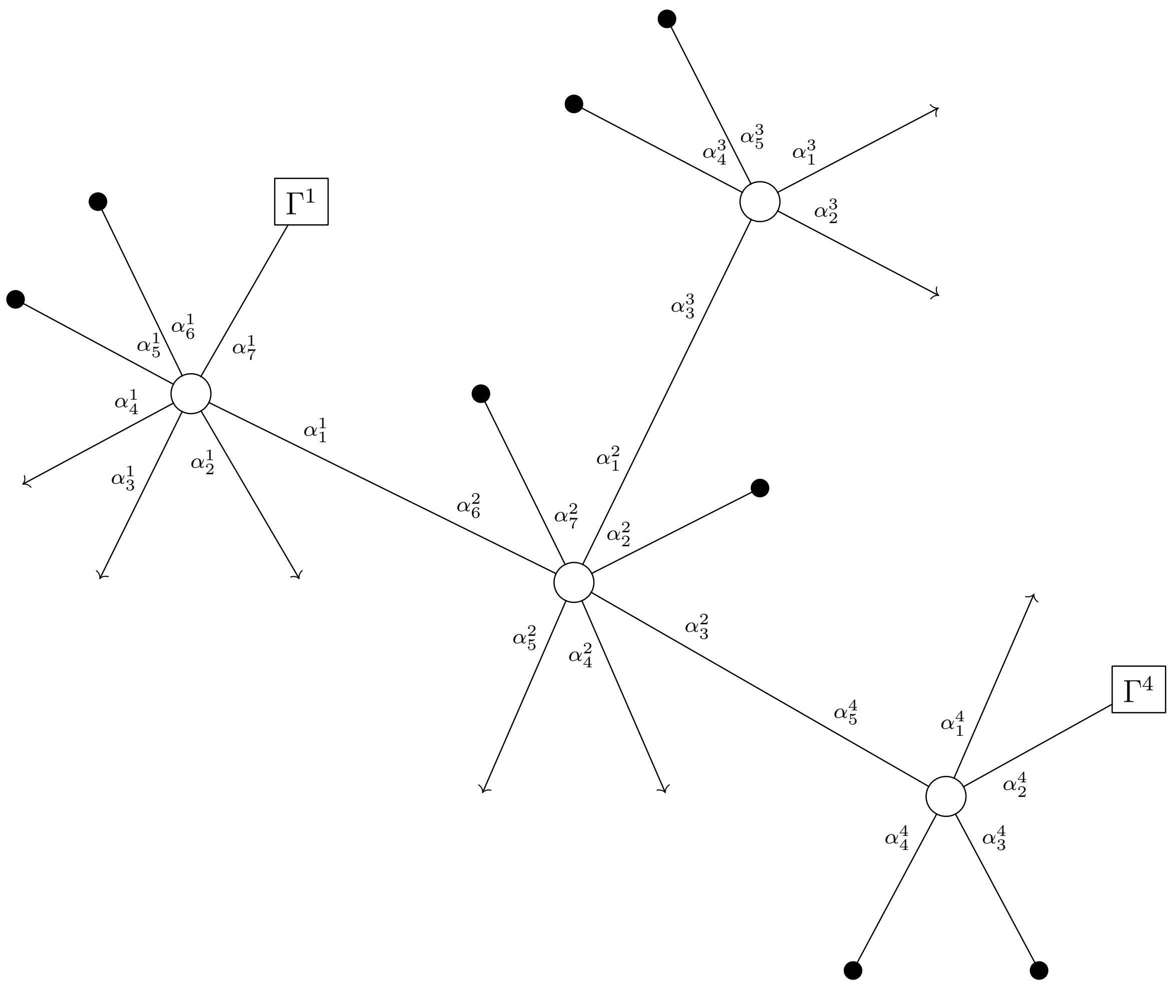


Figure 4.1 – A generic splice diagram

Splice diagrams determine the fundamental group and we can use them to describe eigenvalue-varieties.

We study first the graph links with trivial JSJ decomposition. The exterior of such a link admits a Seifert fibration which extends to the ambient sphere with the components of the link as fibres.

Let M_L be a Seifert-fibred link-manifold, and let C denote the singular fibres of M that are not components of L. We represent M_L by a tree with one central vertex connected to a node for each point of C, and an arrow for each component of L. Each arrow/node is labeled with the order of the corresponding fibre in M, as in Figure 4.2.

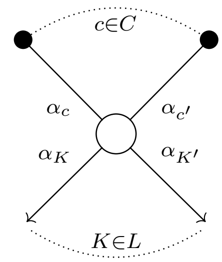


Figure 4.2 – A Seifert splice diagram

We also use the following notation; for any subset D of $L \sqcup C$, α_D is the product of α_d for $d \in D$ and $\alpha_{\widehat{D}}$ is is the product of α_d for $d \notin D$.

The splice diagram enables the description of all the components of the peripheral eigenvalue-variety. We obtain, first, Proposition 5 for the component of reducible characters:

Proposition 5. *The component* $E_\partial^{\mathrm{red}}(M_L)$ *is given by the following ideal of* $\mathbb{C}[\mathfrak{m}, \mathfrak{l}]$*:*

$$\mathcal{A}_\partial^{\mathrm{red}}(M_L) = \left\langle \mathfrak{l}_K - \left(\prod_{K' \subset L \setminus \{K\}} \mathfrak{m}_{K'}{}^{\pm \alpha_{L \setminus \{K, K'\}}} \right)^{\alpha_C}, \ K \subset L \right\rangle \tag{4.1}$$

On the other hand, irreducible representations of Seifert-fibred manifolds are, essentially, representations of the base orbifold. Using this fact, we obtain Theorem 6 for the $\mathrm{PSL}_2\mathbb{C}$ peripheral eigenvalue-variety; let M_L be a Seifert-fibred link-manifold with splice diagram represented in Figure 4.2.

Theorem 6. *The group* $\pi_1 M_L$ *admits irreducible representations in* $\mathrm{PSL}_2\mathbb{C}$ *if and only if*

$$|L| + |C| \geq 3$$

and, in that case, the peripheral $\mathcal{A}$*-ideal corresponding to irreducible characters is*

$$\mathcal{A}^{\mathrm{irr}}(M_L) = \langle \mathfrak{m}_K{}^{\alpha_{\widehat{K}}} \mathfrak{l}_K{}^{\alpha_K} - 1, \ K \subset L \rangle \tag{3}$$

Using a similar approach on $\mathrm{SL}_2\mathbb{C}$, we then obtain the almost identical Theorem 9 for the $\mathrm{SL}_2\mathbb{C}$ peripheral eigenvalue-variety. The main notable difference is that, when the regular Seifert-fibre is trivialized by any irreducible $\mathrm{PSL}_2\mathbb{C}$-representation, it can be $\pm\mathrm{Id}$ in $\mathrm{SL}_2\mathbb{C}$. This enables us to give a complete answer to Question 1 for Seifert-fibred link-manifolds and, besides the obvious counter-example, all Seifert-fibred link manifolds admit a peripherally maximal and non-abelian component in their character variety.

The final step is to study generic graph link-manifolds (with connected splice diagram). The complexity of the combinatorics increases dramatically with the subjacent tree but we can still express some results on the peripheral eigenvalue-varieties of graph link-manifolds. First, with the same formula as Eisenbud-Neumann, we compute the linking numbers and use Proposition 2 to describe the component of reducible characters.

For any two components K and K' in a graph link, let $K{-}K'$ denotes the unique path between the arrows K and K' in the JSJ tree. The linking number $lk(K, K')$ is equal to the product of coefficients adjacent but not on the path $K{-}K'$ in the splice diagram; we denote it by $\alpha_{\widehat{K\text{-}K'}}$ and obtain Proposition 6 for the peripheral $\mathcal{A}$-ideal corresponding to reducible characters:

Proposition 6. *Let M_L be a graph link-manifold. The peripheral eigenvalue-variety corresponding to reducible characters is given by the ideal:*

$$\mathcal{A}^{\mathrm{red}}(M_L) = \left\langle \mathfrak{l}_K - \prod_{K' \subset L \setminus K} \mathfrak{m}_{K'}^{\pm\alpha_{\widehat{K\text{-}K'}}},\ K \subset L \right\rangle \tag{4.2}$$

Now, given a graph link-manifold, one may try to find components of characters with a given type in $\{\mathrm{irr}, \mathrm{red}\}$ (See Definition 1.2.12) on each vertex v of the dual tree (for the JSJ tree, or any other splitting tree). The complexity of the combinatorics involved makes it quite difficult to express precise statements for generic types and splittings. However, using the results on naturality of the $E_{\mathcal{G}}$-varieties under the natural splitting-trees operations (Lemma 1 and Theorem 3 of Chapter 2 and their applications to torus splittings of link-manifolds), we can reduce the complexity and obtain interesting results.

This is the purpose of the last two sections of this chapter, where we study two specific cases; nonetheless, the notations involved remain too heavy to enable stating precise results in this introduction. We may however outline the main ideas presented in these final sections.

First, we study components of *everywhere irreducible* (irreducible on all the JSJ pieces, see Definition 1.2.11). Using Theorem 6 on each piece we obtain, in Theorem 10, a criterion for the existence of everywhere irreducible $\mathrm{PSL}_2\mathbb{C}$-characters and the equations of the resulting peripheral $\mathcal{A}$-ideal. In particular, if they exist, such components are *peripherally maximal and non-abelian*.

In the last section, we study components of characters which are irreducible on one piece and reducible everywhere else. We can combine Proposition 6 and Theorem 6 with the generic splitting-gluing theorems for eigenvalue-varieties and obtain Theorem 11, again a criterion for the existence of such components. Finally, we show that this criterion applies to all graph manifolds with non-abelian fundamental group, which procures an answer to Question 1 for graph link-manifolds:

Theorem 2. *For any non-abelian graph link-manifold M_L with boundary, there exist a peripherally maximal and non-abelian component in $X^{\mathrm{PSL}_2\mathbb{C}}(M_L)$.*

Finally, we close this chapter with a very brief overview on how the very same technics could be used to completely describe the E-varieties of any graph link-manifold.

In this chapter, M denotes an **integer-homology sphere** and M_L the **exterior of a link** L **in** M (see Section 1.3.2).

4.1 Seifert-fibred link-manifolds

First, let's start with the fundamental pieces of graph manifolds, Seifert-fibred link-manifolds.

4.1.1 Seifert fibrations and splice diagrams

Let M_L denote a Seifert-fibred link-manifold. For any component K of L, there may be two cases:

- all the meridians are transverse to the fibre; in that case, the Seifert-fibration of M_L extends to $M_{L\setminus K}$ and K is a fibre.
- a meridian is parallel to the fibre; because M is an integer-homology sphere this can occur for at most one component of L. In that case, M_L has no singular fibres and the longitudes of $L \setminus K$ are all parallel to the fibre, so M_L is a keychain-link in $\mathbb{S}^3$.

Let's assume first that no meridian is parallel to the fibre, so the fibration of M_L extends to M with L as a collection of fibres. These are links in Seifert-fibred integer-homology sphere which were combinatorially described by Seifert in [Sei33] and we recall this description following [EN85]. As we shall see, keychain links naturally appear as degenerated cases of this description.

Let C denote the set of singular fibres of M_L, with orders $(\alpha_c)_{c\in C}$ (so $|\alpha_c| \geq 2$). Let $(\alpha_K)_{K\subset L}$ denote the orders of the fibers K in the induced fibration of M. Assuming that all the meridians are tranverse to the fibre, each and α_K is nonzero; it is equal to ± 1 if K is a regular fibre in the fibration of M.

Let $n = |L|$ denote the number of components of L and $r = |C|$ the number of singular fibres of M_L. Let $\widetilde{L}$ denote the collection $L \sqcup C$ of fibers in M. It is a link in M with $n+r$ components and the link-manifold $M_{\widetilde{L}}$ is a Seifert link-manifold with no singular fibres. Since M is an integer-homology sphere the base orbifold is planar and $M_{\widetilde{L}}$ is the product of a circle and a 2-sphere with $n+r$ discs removed, $\mathcal{S}_{n+r} \times \mathbb{S}^1$.

The boundary of $\mathcal{S}_{n+r} \times \mathbb{S}^1$ may be indexed by the components of $\widetilde{L}$ and we order these components so that $\widetilde{L} = K_1 \sqcup \cdots \sqcup K_{n+r}$ whith $K_i \subset L$ for $1 \leq i \leq n$ and $K_i \subset C$ for $n+1 \leq i \leq n+r$. For each component J of $\widetilde{L}$, let s_J denote the boundary curve of $\mathcal{S}_{n+r}$ dual to the fibre J. The s_J are sections of the fibration of $M_{\widetilde{L}}$ and we denote by t the $\mathbb{S}^1$ fibre. The boundary of $M_{\widetilde{L}}$ consists of tori T_J for $J \subset \widetilde{L}$ with

$$H_1(T_J, \mathbb{Z}) = \mathbb{Z}s_J \oplus \mathbb{Z}t.$$

The homology group $H_1(M_{\widetilde{L}}, \mathbb{Z})$ is the sublattice of $\mathbb{Z}^{n+r+1} \cong \bigoplus_{J \subset \widetilde{L}} \mathbb{Z}s_J \oplus \mathbb{Z}t = \mathbb{Z}s_1 \oplus \cdots \oplus \mathbb{Z}s_{n+r} \oplus \mathbb{Z}t$ such that

$$\sum_{J \subset \widetilde{L}} s_J = 0$$

There exist a family of integers $(\beta_J)_{J \subset \widetilde{L}}$ such that M_L (resp. M) is obtained by α_J/β_J-Dehn-filling – in the basis (s_J, t) – on the components J of C (resp. all components of $\widetilde{L}$).

We'll denote by α (resp. β) the family $(\alpha_J)_{J \subset \widetilde{L}}$ (resp. $(\beta_J)_{J \subset \widetilde{L}}$).

Proposition 4.1.1 (Seifert, See Hatcher [Hat10])**.** *With these notations, the fundamental group of a Seifert-fibred M_L is given by:*

$$\pi_1 M_L = \langle s_1, \ldots, s_n, c_1, \ldots, c_r, t \mid [t, s_i], [t, c_i], c_i^{\alpha_i} t^{\beta_i}, s_1 \cdots s_n c_1 \cdots c_r \rangle \tag{4.3}$$

The homology of M is the kernel in $\mathbb{Z}^{n+r+1}$ of the matrix

$$\mathbf{A}_{\alpha,\beta} = \begin{bmatrix} \alpha_1 & 0 & \ldots & \ldots & 0 & \beta_1 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \alpha_J & \ddots & \vdots & \beta_J \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \ldots & \ldots & 0 & \alpha_{n+r} & \beta_{n+r} \\ 1 & \ldots & 1 & \ldots & 1 & 0 \end{bmatrix} \tag{4.4}$$

For any sublink L' of $\widetilde{L}$ we define:

$$\alpha_{L'} = \prod_{J \subset L'} \alpha_J$$

and

$$\alpha_{\widehat{L'}} = \alpha_{\widetilde{L'}\setminus L'} = \prod_{J \not\subset L'} \alpha_J.$$

Remark 4.1.1. For any sublink L' of $\widetilde{L}$, $\alpha_{L'}\alpha_{\widehat{L'}} = \alpha_{\widetilde{L}}$.

A simple computation shows that $\det \mathbf{A}_{\alpha,\beta} = -\sum_{J\subset\widetilde{L}} \beta_J \alpha_{\widehat{J}}$ and, since M is an integer-homology sphere, this determinant must be ± 1. Up to changing the orientation of t we can assume:

$$\sum_{J\subset\widetilde{L}} \beta_J \alpha_{\widehat{J}} = 1. \tag{4.5}$$

Allowing $\alpha_J = 0$ for some $J \subset L$ in Equation (4.5) forces β_J and all the others $(\alpha_{J'})_{J'\subset\widetilde{L}}$ to be equal to ± 1, which yields a keychain-link in $\mathbb{S}^3$ discussed above. We also allow this degenerated case and, from now on, Seifert-fibred link-manifolds will be described using family of integers $(\alpha_J, \beta_J)_{J\subset\widetilde{L}}$ and satisfying Equation (4.5).

Let's set a little more notation. For any sublink L' of $\widetilde{L}$ we define:

$$\beta_{L'} = \sum_{J\subset L'} \beta_J \alpha_{L'\setminus J}$$

and

$$\beta_{\widehat{L'}} = \beta_{\widetilde{L}\setminus L'} = \sum_{J\not\subset L'} \beta_J \alpha_{\widehat{L'\sqcup J}}$$

Remark 4.1.2. With these notations, for any sublink L' of $\widetilde{L}$, equation (4.5) can be rewritten as:

$$\det \begin{bmatrix} \alpha_{L'} & -\alpha_{\widehat{L'}} \\ \beta_{L'} & \beta_{\widehat{L'}} \end{bmatrix} = \beta_{\widehat{L'}}\alpha_{L'} + \beta_{L'}\alpha_{\widehat{L'}} = 1.$$

Proposition 4.1.2 (Eisenbud-Neumann [EN85]). *A standard peripheral system for M_L is given, for any component J of L, by the following system in $H_1(T_J, \mathbb{Z})$:*

$$\begin{aligned} \mu_J &= \alpha_J s_J + \beta_J t \\ \lambda_J &= -\alpha_{\widehat{J}} s_J + \beta_{\widehat{J}} t \end{aligned}$$

and, for any components J, J' of L, $lk(J, J') = \alpha_{\widehat{J\sqcup J'}}$.

Proof. The integer-homology sphere M is obtained from M_L by Dehn-filling along the slopes $s_J^{\alpha_J} t^{\beta_J}$ for each boundary torus T_J of L. It follows that $\alpha_J s_J + \beta_J t$ is a meridian for J in M_L.

Since $\beta_{\widehat{J}}\alpha_J + \beta_J\alpha_{\widehat{J}} = 1$, the curve $-\alpha_{\widehat{J}}s_J + \beta_{\widehat{J}}t$ has intersection 1 with μ_J. Moreover, in $H_1(M_L, \mathbb{Z})$, $\sum_{J\subset\widetilde{L}} s_J = 0$ so we have the following homological equalities:

$$\begin{aligned} -\alpha_{\widehat{J}}s_J + \beta_{\widehat{J}}t &= \alpha_{\widehat{J}} \sum_{J'\subset\widetilde{L}\setminus J} s_{J'} + \sum_{J'\subset\widetilde{L}\setminus J} \beta_{J'}\alpha_{\widehat{J\sqcup J'}}t \\ &= \sum_{J'\subset\widetilde{L}\setminus J} \alpha_{\widehat{J\sqcup J'}}\alpha_{J'}s_{J'} + \beta_{J'}\alpha_{\widehat{J\sqcup J'}}t \\ &= \sum_{J'\subset\widetilde{L}\setminus J} \alpha_{\widehat{J\sqcup J'}}\left(\alpha_{J'}s_{J'} + \beta_{J'}t\right) \end{aligned}$$

And, since $\alpha_{J'}s_{J'} + \beta_{J'}t$ is 0 for J' in C, and $\mu_{J'}$ for J' in L we have:

$$-\alpha_{\widehat{J}}s_J + \beta_{\widehat{J}}t = \sum_{J'\subset L\setminus J} \alpha_{\widehat{J\sqcup J'}}\mu_{J'} \tag{4.6}$$

Therefore, $-\alpha_{\widehat{J}}s_J + \beta_{\widehat{J}}t$ represents a curve on T_J nullhomologous in M_J so it is the longitude λ_J in $H_1(T_J, \mathbb{Z})$; by Equation (1.7), the last Equation (4.6) shows that, for any pair J, J' of L, $lk(J, J') = \alpha_{\widehat{J\sqcup J'}}$. □

Definition 4.1.1. For any component J of $\widetilde{L}$ we denote by P_J the *peripheral matrix*:

$$P_J = \begin{bmatrix} \alpha_J & -\alpha_{\widehat{J}} \\ \beta_J & \beta_{\widehat{J}} \end{bmatrix}$$

Remark 4.1.3. For each boundary component T_K of M_L, the peripheral matrix P_K gives the coordinates of μ_K and λ_K in the basis (s_K, t) of $\pi_1 T_K$.

We represent M_L with a diagram consisting in:

- a central vertex ◯
- for each c in C, an edge from ◯, labeled by α_c, ending with a node ●
- an arrow labeled by α_K for each component K of L.

Definition 4.1.2 (Splice diagram (1))**.** This presentation is called a *splice diagram* for M_L.

Example 4.1.4. *The splice diagram of a generic Seifert-fibred manifold is represented in Figure 4.3.*

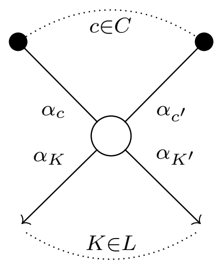


Figure 4.3 – A Seifert splice diagram

Example 4.1.5. *The splice diagram of a p, q-torus knot in $\mathbb{S}^3$ is represented in Figure 4.4.*

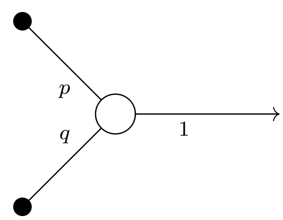
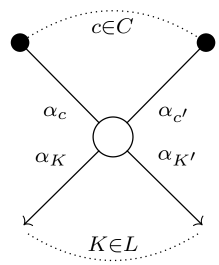


Figure 4.4 – A p, q-torus knot

In the following sections, we compute the components of the peripheral eigenvalue-varieties of Seifert-fibred link-manifolds. The splice diagram of M_L determines the linking numbers so it determines $E_\partial^{\mathrm{red}}(M_L)$; we will see that it also determines components of irreducible characters.

4.1.2 Reducible characters of Seifert-fibred link-manifolds

As usual, the component of reducible characters is given by the linking numbers. In the case of a Seifert-fibred link-manifold M_L, they are given by Proposition 4.1.2; if the splice diagram of M_L is



then each longitude λ_K is given in $H_1(M_L, \mathbb{Z})$ by

$$\lambda_K = \sum_{K' \subset L \setminus \{K\}} \alpha_{\widehat{K \sqcup K'}} \mu_{K'}$$

where $\alpha_{\widehat{K \sqcup K'}} = \prod_{J \subset \widetilde{L} \setminus \{K, K'\}} \alpha_J$. With the notation introduced earlier,

- α_C denotes the product of orders of the singular fibres:

$$\alpha_C = \prod_{c \in C} \alpha_c = \prod_{K \subset \widetilde{L} \setminus L} \alpha_K$$

- $\alpha_{L \setminus \{K, K'\}}$ is the product of the arrowhead coefficients other than K and K':

$$\alpha_{L \setminus \{K, K'\}} = \prod_{J \subset L \setminus \{K, K'\}} \alpha_J$$

Since α_C divides any $\alpha_{\widehat{K \sqcup K'}}$ and does not depend on the particular choice K or K', we can refactor the equations and obtain the following:

Proposition 5. *The component $E_\partial^{\mathrm{red}}(M_L)$ is given by the following ideal of $\mathbb{C}[\mathfrak{m}, \mathfrak{l}]$:*

$$\mathcal{A}_\partial^{\mathrm{red}}(M_L) = \left\langle \mathfrak{l}_K - \left(\prod_{K' \subset L \setminus \{K\}} {\mathfrak{m}_{K'}}^{\pm \alpha_{L \setminus \{K, K'\}}} \right)^{\alpha_C}, \ K \subset L \right\rangle \tag{4.1}$$

We will now focus on irreducible characters of Seifert-fibred link-manifolds. First, we need to make a small detour to study characters of planar orbifolds.

4.1.3 Characters of planar orbifolds

For any positive integer n, let $\mathcal{S}_n$ denote standard 2-sphere with n discs removed:

$$\mathcal{S}_n = \mathbb{S}^2 \setminus D_1, \ldots, D_n.$$

The boundary of $\mathcal{S}_n$ consists in n circles $s_1 \ldots, s_n$ and $\pi_1\mathcal{S}_n$ is isomorphic to the free group with rank $n-1$:

$$\pi_1\mathcal{S}_n \cong \langle s_1, \ldots, s_n \mid \prod_{i=1}^{s} s_i = 1\rangle.$$

Lemma 4.1.3. *There exist an irreducible representation of* $\pi_1\mathcal{S}_n$ *in* $\mathrm{SL}_2\mathbb{C}$ *or* $\mathrm{PSL}_2\mathbb{C}$ *if and only if* $n \geq 3$*.*

In that case, there exist a Zariski-open set U *of* $\mathbb{C}^n$ *such that for any* $u_1, \ldots, u_n$ *in* U*, there exist an irreducible character* χ *in* $X(\mathcal{S}_n)$ *such that for any* $1 \leq i \leq n$*,*

$$I_{s_i}\chi = u_i$$

Moreover, let U_0 *denote the Zariski open set* $\mathbb{C} \setminus \{-2, 2\}$ *(or* $U_0 = \mathbb{C} \setminus \{2\}$ *in* $\mathrm{PSL}_2\mathbb{C}$*), then, for* $n \geq 4$*, we can assume that* U *contains* $(U_0)^n$*.*

Proof. For $n \leq 2$, $\mathcal{S}_n$ has abelian fundamental group so it admits no irreducible representation.

The rest of the proof is by induction on $n \geq 3$.

For $n = 3$, it is known that, for the free group $\mathrm{F}_2 = \langle a, b \mid \rangle$, the map

$$\begin{array}{rcl} X^{\mathrm{SL}_2\mathbb{C}}(\mathrm{F}_2) & \to & \mathbb{C}^3 \\ \chi & \to & (I_a(\chi), I_b(\chi), I_{ab}(\chi)) \end{array}$$

is a birregular map between $\mathbb{C}^3$ and $X^{\mathrm{SL}_2\mathbb{C}}(F_2)$. Moreover, with this coordinates,

$$X^{\mathrm{SL}_2\mathbb{C}}_{\mathrm{red}}(\mathrm{F}_2) = V(f^{\mathrm{red}})$$

where f^{red} is the polynomial of $\mathbb{Q}[x, y, z]$:

$$f^{\mathrm{red}}(x, y, z) = x^2 + y^2 + z^2 - xyz - 4$$

Using the work of Michael Heusener and Joan Porti in [HP04], a similar statement can be made for $X^{\mathrm{PSL}_2\mathbb{C}}(\mathrm{F}_2)$. As Example 4.4 of [HP04] shows, there's a polynomial P in $\mathbb{Q}[U, V, W, Z]$ such that

$$X^{\mathrm{PSL}_2\mathbb{C}}(\mathrm{F}_2) \cong \{(I_a, I_b, I_{ab}, I_{ab^{-1}}) \in \mathbb{C}^4 | P(I_a, I_b, I_{ab}, I_{ab^{-1}}) = 0\}$$

and the reducible part is given by another polynomial $f^{\rm red}$.

In the presentation of $\pi_1\mathcal{S}_3$, $s_1 = a$, $s_2 = b$, and $s_3 = b^{-1}a^{-1}$, so, in either cases, the map $I_s = (I_{s_1}, I_{s_2}, I_{s_3})$ projects $X^{\rm irr}(\mathrm{F}_2)$ on a the Zariski open set of $\mathbb{C}^3$; this complete the proof for $n = 3$.

For any $n \geq 4$, consider a circle s in $\mathcal{S}_n$ splitting $\mathbb{S}^2$ into to discs D^+, D^-, containing respectively $s_1, \dots, s_{n-2}$ and s_{n-1},s_n.

Let B^+ and B^- denote the corresponding pieces of $\mathcal{S}_n$ so

$$\partial B^+ = s_1 \sqcup \ldots \sqcup s_{n-2} \sqcup s_+$$

$$\partial B^- = s_- \sqcup s_{n-1} \sqcup s_n$$

and $\mathcal{S}_n$ can be obtained from B^+ and B^- identifying s_+ with s_-^{-1}. We can apply Lemma 4.1.3 to B^+ and B^- and obtain U^+ and U^- in $\mathbb{C}^{n-1}$ and $\mathbb{C}^3$. For any $(u_1, \dots, u_n, v)$ in $\mathbb{C}^{n+1}$ such that

- $u_1, \dots, u_{n-2}, v \in U^+$
- $v, u_{n-1}, u_n \in U^-$

there exist irreducible characters χ^+ and χ^- in $X(B^+)$ and $X(B^-)$ such that

- for all $1 \leq i \leq b-2$, $I_{s_i}\chi^+ = u_i$
- $I_{s_+}\chi^+ = v$
- $I_{s_-}\chi^- = v$
- for $n-1 \leq i \leq n$, $I_{s_i}\chi^- = u_i$

Without loss of generality, we can assume that $v \neq \pm 2$ (or 2 in $\mathrm{PSL}_2\mathbb{C}$) so there exist irreducible representations $\rho_+ \in t^{-1}\chi^+$ and $\rho_- \in t^{-1}\chi_-$ such that $\rho_+(s_+) = \rho_-(s_-)^{-1}$. This produces an irreducible representation of $\mathcal{S}_n$ with traces u_i on all the s_i.

Moreover, U^- is $\mathbb{C}^3 \setminus V(f^{\rm red})$ and U^+ is either $\mathbb{C}^3 \setminus V(f^{\rm red})$ (if $n = 4$) or contains $(U_0)^{n-1}$ by induction hypothesis. In either case, this can be done for any $u_1, \dots, u_n$ in $(U_0)^n$ and this completes the proof. $\square$

For any positive integers n, r and family α in $(\mathbb{Z})^r$, we can form the orbifold

$$\mathcal{S}_n(\alpha) = \mathcal{S}_n(\alpha_1, \dots, \alpha_r)$$

obtained from $\mathcal{S}_{n+r}$ by gluing discs along $s_i^{\alpha_i}$ for $n+1 \leq i \leq n+r$.

The fundamental group of $\mathcal{S}_n(\alpha)$ is isomorphic to

$$\langle s_1, \ldots, s_{n+r} \mid s_i^{\alpha_i} = 1 \text{ for } n < i \leq n+r, \prod_{i=1}^{n+r} s_i = 1 \rangle$$

For any family α of $\mathbb{Z}^r$ and any positive integer k, we denote by $supp_k(\alpha)$ the subset of indices:

$$supp_k(\alpha) = \{i \in [1 \ldots r] \mid |\alpha_i| \geq k\}.$$

so $i \in supp_k(\alpha)$ if s_i has order at least k in $\pi_1 \mathcal{S}_n(\alpha)$.

Lemma 4.1.4. *The orbifold $\mathcal{S}_n(\alpha)$ admits irreducible representations in $\mathrm{PSL}_2\mathbb{C}$ if and only if $n + supp_2(\alpha) \geq 3$. In that case, there exist a Zariski-open set of $\mathbb{C}^n$ on which, for any $u_1, \ldots, u_n$, there exist an irreducible character χ in $X^{\mathrm{irr}}(\mathcal{S}_n(\alpha))$ with squared trace u_i on s_i.*

Proof. If $n + supp_2(\alpha) \leq 2$, any representation in of $\pi_1 \mathcal{S}_n(\alpha)$ in $\mathrm{PSL}_2\mathbb{C}$ is abelian.

Otherwise, for each $n < i \leq n+r$, let k_i be an integer coprime to $2\alpha_i$ and we set $u_i = 2\cos\left(2\pi \frac{k_i}{\alpha_i}\right)$; it is different from 2 whenever $|\alpha_i|$ is different from 1. When $|\alpha_i| = 1$, the corresponding section becomes trivial and we assume that $|\alpha_i| \geq 2$ for all i, so $supp_2(\alpha) = r$. We can apply Lemma 4.1.3 to $\mathcal{S}_{n+r}$ and there is a Zariski open set in $\mathbb{C}^n$ on which, for any $(u_1, \ldots, u_n)$, there exist χ' in $X^{\mathrm{irr}}(\mathcal{S}_{n+r})$ such that $I_{s_i}\chi' = u_i$ for each $1 \leq i \leq n+r$.

For each $n < i \leq n+r$, let ζ_i denote the root of unity $e^{\pi i \frac{k_i}{\alpha_i}}$ and $z_i = {\zeta_i}^2$. Each $I_{s_i}\chi'$ is equal to $z_i + {z_i}^{-1} \neq 2$, so there exist ρ in $t^{-1}\chi'$ such that $\rho(s_i) = \Delta(z_i) = \pm \begin{bmatrix} \zeta_i & 0 \\ 0 & {\zeta_i}^{-1} \end{bmatrix}$. Therefore $\rho(s_i)^{\alpha_i} = \pm\mathrm{Id}$ and any ρ in $t^{-1}\chi'$ factors by the gluing along $s_i^{\alpha_i}$.

It follows that χ' is in the image of $X(\mathcal{S}_n(\alpha)) \hookrightarrow X(\mathcal{S}_{n+r})$ and there exist χ in $X(\mathcal{S}_n(\alpha))$ with the expected properties. □

For representations in $\mathrm{SL}_2\mathbb{C}$, conic points of order 2 must have image $\pm\mathrm{Id}$ and are also central; we need sufficiently many boundary components or points with order greater than 2 to have irreducible representations. With a similar argument, we have the following result for $\mathrm{SL}_2\mathbb{C}$ characters:

Proposition 4.1.5. *The orbifold $\mathcal{S}_n(\alpha)$ admits irreducible representations in $\mathrm{SL}_2\mathbb{C}$ if and only if $n + supp_3(\alpha) \geq 3$. In that case, there exist a Zariski-open set of $\mathbb{C}^n$ on which, for any $u_1, \ldots, u_n$, there exist an irreducible character χ in $X^{\mathrm{irr}}(\mathcal{S}_n(\alpha))$ with trace u_i on s_i.*

4.1.4 E_∂-variety of Seifert-fibred link-manifolds

The fibre is central in the fundamental group of a Seifert-fibred link-manifold. Therefore, any irreducible representation in $\mathrm{PSL}_2\mathbb{C}$ must trivialize the fibre and factor as a representation of the base orbifold; we can use the results of the previous Section 4.1.3 to describe irreducible characters of Seifert-fibred link-manifold.

Let M_L be a Seifert-fibred link-manifold with splice diagram represented in Figure 4.5.

Theorem 6. *The group $\pi_1 M_L$ admits irreducible representations in $\mathrm{PSL}_2\mathbb{C}$ if and only if*

$$|L| + |C| \geq 3$$

and, in that case, the peripheral $\mathcal{A}$-ideal corresponding to irreducible characters is

$$\mathcal{A}^{\mathrm{irr}}(M_L) = \langle {\mathfrak{m}_K}^{\alpha_{\widehat{K}}} {\mathfrak{l}_K}^{\alpha_K} - 1,\ K \subset L \rangle \tag{3}$$

Proof. First, if $|L| + |C| \leq 2$, the fundamental group of M_L is abelian so $R^{\mathrm{PSL}_2\mathbb{C}}(M_L)$ contains no irreducible representation.

Let's now assume that $|L| + |C| \geq 3$. Let t denote the regular fibre of the Seifert-fibration of M_L; since it is central in $\pi_1 M_L$, any irreducible representation ρ in $R^{\mathrm{PSL}_2\mathbb{C}}(M_L)$ must trivialize t. It follows that any such representation factors as a representation of the base orbifold of M_L, a 2-sphere with $|C|$ conic points of orders α_c and $|L|$ discs removed.

By Proposition 4.1.4, such irreducible representation exist if and only if $|L| + |C| \geq 3$ and, in that case, the traces on the removed discs can be chosen freely in a Zariski open set of $\mathbb{C}^{|L|}$. In other words there's a Zariki open set of $\mathbb{C}^{*|L|}$ on which, for any $(x_K)_{K \subset L}$, there exist an irreducible representation ρ in $R^{\mathrm{PSL}_2\mathbb{C}}(S_{|L|}(\alpha))$ such that, for any K in L,

$$\mathrm{tr}(\rho(s_K)) = x_K + x_K^{-1}.$$

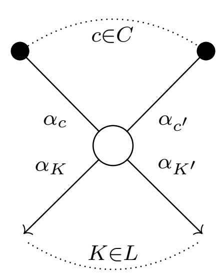


Figure 4.5 – A Seifert-fibred link-manifold

Any such representation pulls back to a representation of $\pi_1 M_L$, trivial on t. For each boundary torus T_K the peripheral matrix gives the following relations between $\rho(s_K)$, $\rho(t)$, $\rho(\mu_K)$ and $\rho(\lambda_K)$:

$$\begin{array}{rcl} \rho(t) & = & \rho(\mu_K)^{\alpha_{\widehat{K}}}\rho(\lambda_K)^{\alpha_K} \\ \rho(s_K) & = & \rho(\mu_K)^{\beta_{\widehat{K}}}\rho(\lambda_K)^{-\beta_K} \end{array} \tag{4.7}$$

and since $\rho(t) = 1$ we obtain the following equations:

$$\begin{array}{rcl} \rho(\mu_K) & = & \rho(s_K)^{\alpha_K} \\ \rho(\lambda_K) & = & \rho(s_K)^{\alpha_{\widehat{K}}} \end{array} \tag{4.8}$$

The previous Equations (4.7) and (4.8) give the following equalities for the eigenvalues of ρ:

$$\begin{array}{rcl} x_K & = & m_K^{\beta_{\widehat{K}}}\ell_K^{-\beta_K} \\ m_K & = & x_K^{\alpha_K} \\ \ell_K & = & x_K^{\alpha_{\widehat{K}}} \end{array}$$

and, we obtain the following equations:

$$m_K{}^{\alpha_{\widehat{K}}}\ell_K{}^{\alpha_K} = 1 \text{ for each component } K$$

Let $\mathcal{A}$ denote the ideal $\langle \mathfrak{m}_K{}^{\alpha_{\widehat{K}}}\mathfrak{l}_K{}^{\alpha_K} - 1,\ K \subset L\rangle$ of $\mathbb{C}[\mathfrak{m},\mathfrak{l}]$. We just showed that $E_\partial^{\text{irr}}(M_L) \subset V(\mathcal{A})$ and, reversing the calculation, there exist a Zariski-dense set of $V(\mathcal{A})$ corresponding to irreducible characters of $X(M_L)$; this makes $\mathcal{A}$ the defining ideal of $E_\partial^{\text{irr}}(M_L)$ so, finally,

$$\mathcal{A}^{\text{irr}}(M_L) = \langle \mathfrak{m}_K{}^{\alpha_{\widehat{K}}}\mathfrak{l}_K{}^{\alpha_K} - 1,\ K \subset L\rangle \tag{3}$$

□

Remark 4.1.6. The result of Theorem 6 generalizes the result of Tillmann in [Til05] stating that the $A^{\text{PSL}_2\mathbb{C}}$-polynomial of a (p,q) torus knots in $\mathbb{S}^3$ is

$$A_{K_{p,q}}(\mathfrak{m},\mathfrak{l}) = \mathfrak{m}^{pq}\mathfrak{l} - 1$$

For representations in $\text{SL}_2\mathbb{C}$, it is quite similar. In that case the centrality of the fibre t implies that $\rho(t) = \pm\text{Id}$ in $\text{SL}_2\mathbb{C}$ and there are two sets of components, depending on whether $\rho(t) = \text{Id}$ or $\rho(t) = -\text{Id}$. We denote by $X^+(M_L), X^-(M_L), E_\partial^+(M_L), E_\partial^-(M_L)$ the corresponding sets in the character and eigenvalue-variety. Naturally, we denote by $\mathcal{A}^\pm(M_L)$ the defining ideals of $E_\partial^\pm(M_L)$.

Theorem 9. *Let M_L be a Seifert-fibred link-manifold with splice diagram represented in Figure 4.5.*

The variety of characters $X^-(M_L)$ is non empty if and only if

$$|L| + |C| \geq 3$$

and, in that case,

$$\mathcal{A}^-(M_L) = \langle {\mathfrak{m}_K}^{\alpha_{\widehat{K}}} {\mathfrak{l}_K}^{\alpha_K} + 1,\ K \subset L \rangle \tag{4.9}$$

On the other hand, the variety of characters $X^+(M_L)$ is non empty if and only if

$$|L| + |C'| \geq 3$$

where C' is the set of singular fibres of order greater than 2. In that case,

$$\mathcal{A}^+(M_L) = \langle {\mathfrak{m}_K}^{\alpha_{\widehat{K}}} {\mathfrak{l}_K}^{\alpha_K} - 1,\ K \subset L \rangle \tag{4.10}$$

Remark 4.1.7. Since the orders are pairwise coprime, there's at most one singular fibre of order 2 in the Seifert fibration of M_L.

Proof. This is essentially the same proof as in Theorem 6.

The only difference is for $X^+(M_L)$; if $\rho(t) = \mathrm{Id}$, conic points of order 2 must have central image in $\mathrm{SL}_2\mathbb{C}$ and, thus, must be put apart to create irreducible representations. □

Remark 4.1.8. As before, the result of Theorem 9 generalizes the result of Tillman in [Til05] stating that the $A^{\mathrm{SL}_2\mathbb{C}}$-polynomial of a (p, q) torus knots in $\mathbb{S}^3$ is

$$A_{K_{p,q}}(\mathfrak{m}, \mathfrak{l}) = \begin{cases} \mathfrak{m}^{pq}\mathfrak{l} + 1 & \text{if } |p| = 2 \text{ or } |q| = 2 \\ (\mathfrak{m}^{pq}\mathfrak{l} + 1)(\mathfrak{m}^{pq}\mathfrak{l} - 1) & \text{otherwise} \end{cases}$$

4.2 Graph link-manifolds

Using splitting-gluing properties of eigenvalue-varieties, we can apply the results of Section 4.1 to describe the peripheral eigenvalue-variety of graph link-manifold (or at least, some components).

4.2.1 Splice diagrams for graph link-manifolds

Let M_L be a graph link-manifold. We follow [EN85], with our notations, to present the splice diagrams describing graph link-manifolds. We merge the description of each piece given by Definition 4.1.2 and the JSJ tree to obtain a combinatorial description of the graph link-manifold.

Definition 4.2.1 (Splice diagram (2))**.** A *splice diagram* Γ for M_L is obtained from the JSJ tree $\mathcal{G}_{\mathcal{J}}$ of M_L by adding, on each vertex v, the arrowheads and fibre edges of the splice diagram of the Seifert-fibred manifold ${M^v}_{L^v}$, with the corrsponding labels.

Any edge is called *internal* if it corresponds to a JSJ-torus (so it is neither an arrow or a fibre edge).

Example 4.2.1. *A graph link-manifold with JSJ tree ○——○ is the splice of two Seifert-fibred link-manifolds and is represented by the splice diagram of Figure 4.6.*

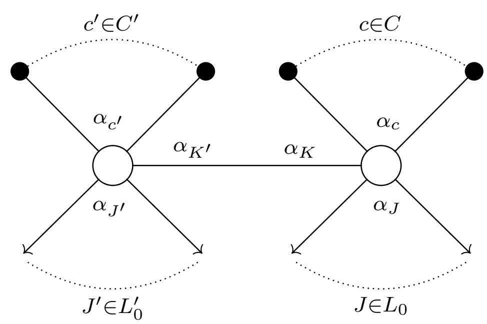


Figure 4.6 – A graph manifold with two pieces

Remark 4.2.2. If the link-manifold M_L splits as a connected sum, the resulting splice diagram is disconnected.

From now on, we will always assume that the splice diagrams are connected.

For any path γ in Γ, any vertex v in Γ, and any component K of $\widetilde{L^v}$ we write:

- $v \in \gamma$ if v is a vertex on the path γ.
- $K \in \gamma$ if the corresponding edge in Γ is on the path γ.

We define the following coefficients:

$$\alpha_\gamma = \prod_{K \in \gamma} \alpha_K$$

$$\alpha_{\widehat{\gamma}} = \prod_{v\in\gamma} \prod_{\substack{J\subset \widetilde{L^v} \\ J\notin\gamma}} \alpha_J^v = \frac{1}{\alpha_\gamma} \prod_{v\in\gamma} \prod_{J\subset \widetilde{L^v}} \alpha_J^v.$$

Remark 4.2.3. For a path γ, $\alpha_{\widehat{\gamma}}$ is the product of the coefficient adjacent to γ but not on it.

Lemma 4.2.1. *The coefficients α_γ and $\alpha_{\widehat{\gamma}}$ are multiplicative:*

Let γ be a path in Γ, let e be an internal edge on γ and let $M^+{}_{L^+}$ and $M^-{}_{L^-}$ be the desplicing of M_L along the JSJ-torus S_e. The path γ splits into two paths γ^+ and γ^- in the splice diagrams of $M^+{}_{L^+}$ and $M^-{}_{L^-}$ respectively and we have

$$\alpha_\gamma = \alpha_{\gamma^+}\alpha_{\gamma^-} \textit{ and } \alpha_{\widehat{\gamma}} = \alpha_{\widehat{\gamma^+}}\alpha_{\widehat{\gamma^-}}$$

Proof. Let γ be a path in Γ. By definition, α_γ is the product of the coefficients of the edges in γ. If splitting Γ along an edge of γ gives two paths γ^+ and γ^-, the edges in γ is the union of edges in γ^+ and edges in γ^- so, taking products,

$$\alpha_\gamma = \alpha_{\gamma^+}\alpha_{\gamma^-}$$

On the other hand, $\alpha_\gamma\alpha_{\widehat{\gamma}} = \prod_{v\in\gamma}\prod_{J\subset\widetilde{L^v}} \alpha_J^v$ so, by the same argument, $\alpha_\gamma\alpha_{\widehat{\gamma}}$ is multiplicative and so is $\alpha_{\widehat{\gamma}}$. □

Example 4.2.4. *Let M_L be the graph link-manifold represented in Figure 4.7. The four vertices represented are numberd from 1 to 4; Γ^1 and Γ^4 are splice diagrams representing other pieces of M_L. Let γ be the path represented by double edges, then, with these notations,*

$$\alpha_\gamma = \alpha_4^1\alpha_1^1\,\alpha_6^2\alpha_1^2\,\alpha_3^3\alpha_2^3$$

and

$$\alpha_{\widehat{\gamma}} = \alpha_2^1\alpha_3^1\alpha_5^1\alpha_6^1\alpha_7^1\,\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2\alpha_7^2\,\alpha_1^3\alpha_4^3\alpha_5^3$$

We can use the previous Lemma 4.2.1 to compute the linking numbers in a graph link-manifold:

Proposition 4.2.2 (Eisenbud-Neumann [EN85])**.** *Let M_L be a graph link-manifold with splice diagram Γ. For any pair of components K, K' of L,*

$$lk(K, K') = \alpha_{\widehat{K\text{-}K'}}$$

where $K{-}K'$ is the unique geodesic path in Γ between the corresponding arrowheads K and K'.

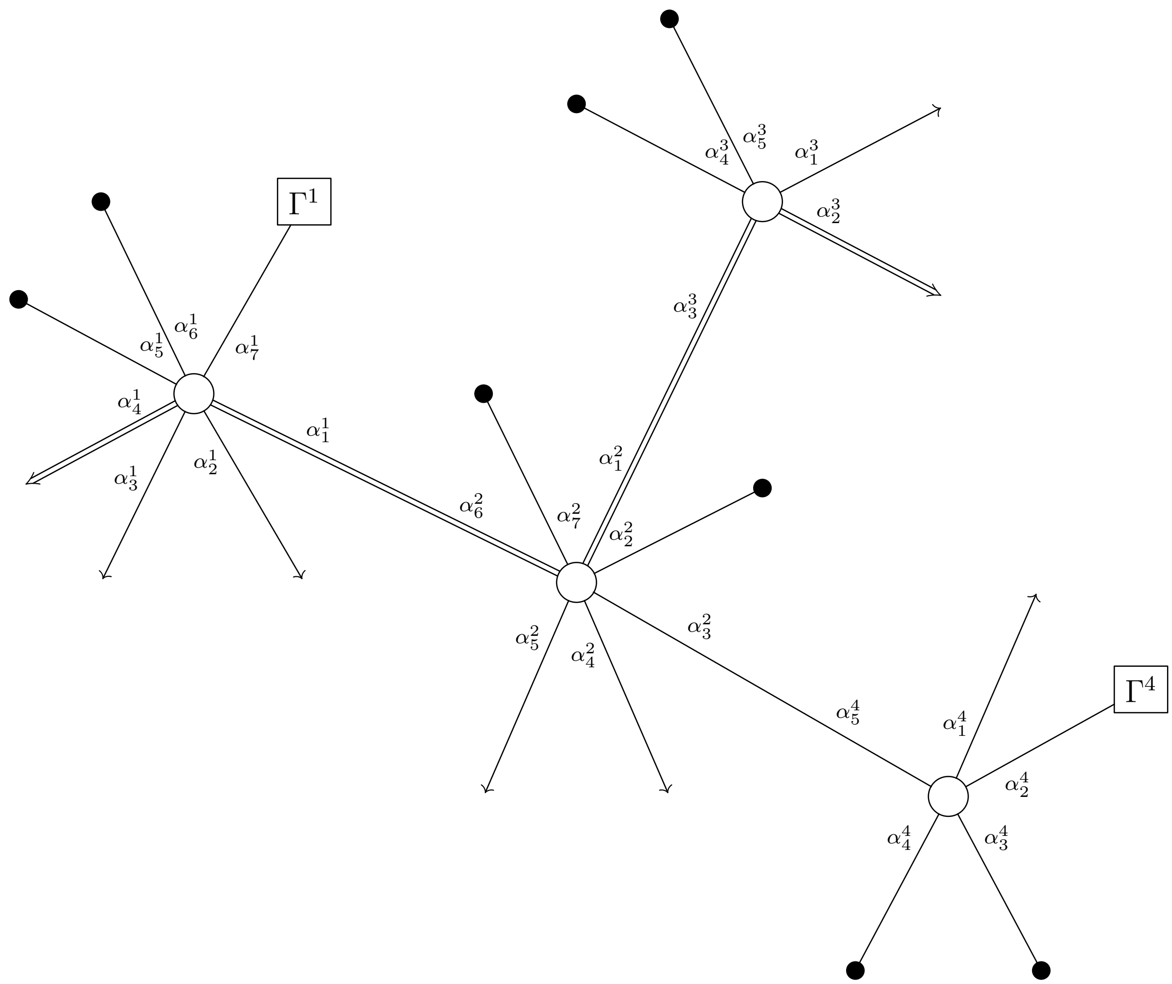


Figure 4.7 – A path in a splice diagram

Proof. Combining propositions 1.3.3 and 4.1.2 with Lemma 4.2.1 we prove this by induction on the length of $\gamma = K$–K'.

If both components are in the same node, $lk(K, K') = \alpha_{\widehat{K \sqcup K'}} = \alpha_{\widehat{\gamma}}$.

Otherwise, γ contains an edge e; desplicing along S_e gives $M_L = {M^+}_{L^+} \; {}^{K^+}\!\!\underset{S_e}{\bowtie}{}^{K^-} \; {M^-}_{L^-}$ and, by Proposition 1.3.3:

$$lk(K, K') = lk^+(K, K^+)lk^-(K^-, K').$$

On the other hand, the path γ splits as $\gamma^+ = K–K^+$ and $\gamma^- = K^-–K'$. We can apply the induction hypothesis to $L^+ \sqcup K^+$ in M^+ and $L^- \sqcup K^-$ in M^-; with Lemma 4.2.1 we obtain:

$$lk(K, K') = \alpha_{\widehat{\gamma^+}} \alpha_{\widehat{\gamma^-}} = \alpha_{\widehat{K–K'}}.$$

$\square$

Let M_L be a graph link-manifold with splice diagram Γ. Let $\mathcal{V}$ and $\mathcal{E}$ denote the vertex and edge sets of Γ. For any vertex v in $\mathcal{V}$, ${M^v}_{L^v}$ is a Seifert-fibred link-manifold and there is a natural partition of L^v:

$$L^v = L^{\partial v} \sqcup L^{\mathcal{E}(v)}$$

where $L^{\partial v} = L^v \cap L$ and $L^{\mathcal{E}(v)}$ are components of L^v spliced along Seifert-fibred neighbours of v in the tree Γ.

Let $v' \overset{e}{—} v$ be an edge in Γ, and let K and K' be the respective components of $L^{\mathcal{E}(v)}$ and $L^{\mathcal{E}(v')}$ corresponding to the splicing at edge e. Around the edge e, Γ can be represented by the diagram in Figure 4.8, where each $\boxed{\Gamma^v_S}$ represents a connected component of the tree $\Gamma \setminus \{v', v\}$.

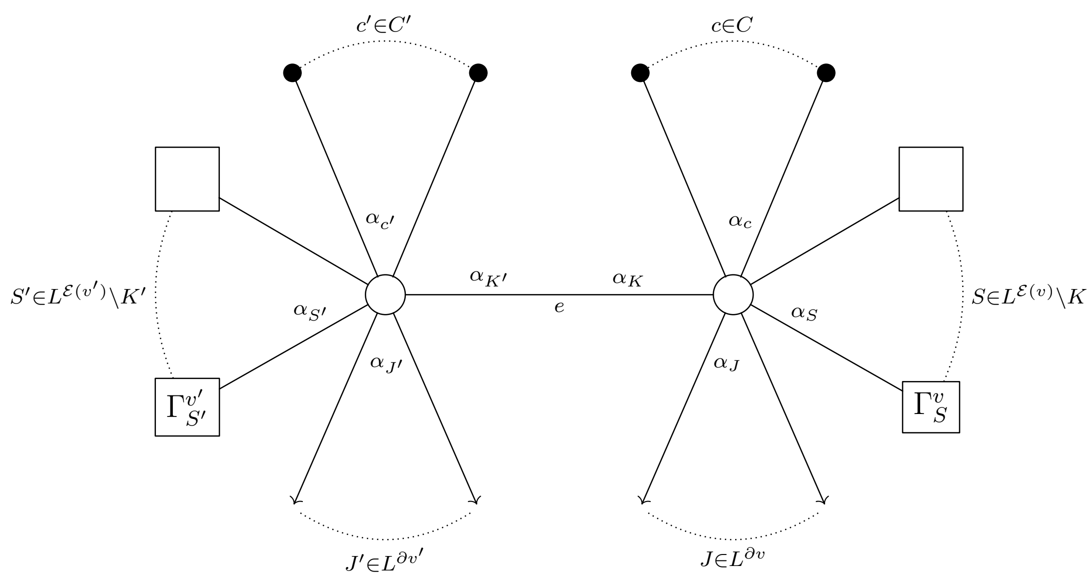


Figure 4.8 – An edge in Γ

Definition 4.2.2 (Determinant of an internal edge [EN85]). With the notations of Figure 4.8, the *determinant* of e is the integer defined by:

$$\det(e) = \alpha_{K'}\alpha_K - \alpha_{\widehat{K'}}\alpha_{\widehat{K}}.$$

Proposition 4.2.3 ([EN85]). *Let t_v and $t_{v'}$ be the regular fibres of the respective Seifert fibrations of ${M^v}_{L^v}$ and ${M^{v'}}_{L^{v'}}$. The algebraic intersection of t_v and $t_{v'}$ on the torus S_e in M_L is given by*

$$< t_{v'}, t_v >= \det(e) \neq 0.$$

Proof. The gluing homeomorphism at S_e identifies $\mu_{K'}$ with λ_K and $\lambda_{K'}$ with μ_K. The peripheral matrices P_K and $P_{K'}$ give, in the basis $(\lambda_{K'}, \lambda_K)$ of S_e:

$$\begin{aligned} t_{v'} &= {\lambda_{K'}}^{\alpha_{K'}}{\lambda_K}^{\alpha_{\widehat{K'}}} \\ t_v &= {\lambda_{K'}}^{\alpha_{\widehat{K}}}{\lambda_K}^{\alpha_K} \end{aligned}$$

and therefore,

$$< t_{v'}, t_v >= \alpha_{K'}\alpha_K - \alpha_{\widehat{K'}}\alpha_{\widehat{K}} = \det(e).$$

If $\det(e)$ vanished for some edge $v' \overset{e}{-} v$ in $\mathcal{E}$, $t_{v'}$ and t_v would be parallel on S_e and the piece ${M^{v'}}_{L^{v'}\backslash K'} \;{}^{K'}\!\underset{S_e}{\bowtie}{}^{K}\; {M^v}_{L^v\backslash K}$ of Γ would be Seifert-fibred, in contradiction with the fact that S_e is a JSJ-torus. □

Definition 4.2.3. For any vertex v in $\mathcal{V}$, and any positive integer d, we'll denote by $\mathcal{E}(v)_d$ the subset of edges adjacent to v with determinant $\pm d$.

4.2.2 Reducible characters of graph link-manifolds

As several times before, the peripheral eigenvalue-variety corresponding to reducible characters is given by linking numbers. By Proposition 4.2.2, these are given by the coefficients on the splice diagram; using the same notation we obtain:

Proposition 6. *Let M_L be a graph link-manifold. The peripheral eigenvalue-variety corresponding to reducible characters is given by the ideal:*

$$\mathcal{A}^{\mathrm{red}}(M_L) = \left\langle \mathfrak{l}_K - \prod_{K' \subset L\backslash K} \mathfrak{m}_{K'}^{\pm\alpha_{\widehat{K\text{-}K'}}},\ K \subset L \right\rangle \tag{4.2}$$

Since reducible characters are everywhere reducible, we could also use Proposition 4.2.2 to compute $\mathcal{A}^{\mathrm{red}}_{\mathcal{G}_{\mathcal{J}}}(M_L)$ and, using Corollary 3, the ideals $\mathcal{A}^{\mathrm{red}}_{\mathcal{G}}(M_L)$ for any tree $\mathcal{G}$ obtained by contraction of the JSJ tree $\mathcal{G}_{\mathcal{J}}$. We shall not make these calculations here, and focus now on irreducible representations.

4.2.3 Everywhere irreducible characters

Theorem 10. *Let M_L be a graph-link manifold with splice diagram Γ; let $\mathcal{V}$ and $\mathcal{E}$ denote the vertex and edge sets of Γ.*

With the notation introduced in Section 4.2.1, there exist an everywhere-irreducible representation in $R^{\mathrm{PSL}_2\mathbb{C}}(M_L)$ if and only if, for any v in $\mathcal{V}$,

$$|L^{\partial v}| + |C^v| + |L^{\mathcal{E}(v)}| - |L^{\mathcal{E}(v)_1}| \geq 3$$

In that case, there exist a component X_0 in $X^{\mathrm{PSL}_2\mathbb{C}}(M_L)$ of everywhere-irreducible characters and for any such component X, $E_\partial(X)$ is the $|L|$-dimensional algebraic manifold V^Γ given in $\mathbb{C}^{2|L|}$ by the following ideal:

$$\left\langle \mathfrak{m}_K{}^{\alpha^v_{\widehat{K}}} \mathfrak{l}_K{}^{\alpha^v_K} - 1,\ v \in \mathcal{V},\ K \subset L^{\partial v} \subset L \right\rangle \tag{4.11}$$

Proof. First, if Γ has a unique vertex, M_L is Seifert-fibred and Theorem 10 is equivalent to Theorem 6.

The next step for the proof of Theorem 10 is to inspect the splicing condition on an internal edge of the splice diagram Γ. As in the proof of Theorem 6, everywhere-irreducible representations will trivialize all the fibres, so the splicing conditions should only involve the sections of the fiberings on each side of the edge.

Any edge $v' \overset{e}{-} v$ of $\mathcal{E}$ splits Γ into two trees Γ^+ and Γ^- containing v' and v respectively. This is represented in the diagram of Figure 4.8, with Γ^+ and v' on the left side of e and Γ^- and v on the right. Let $M^+{}_{L^+}$ and $M^-{}_{L^-}$ be the manifolds obtained by desplicing M_L along S_e; these are graph manifolds over Γ^+ and Γ^- respectively. The link L^+ (resp. L^-) can be written $L_0^+ \sqcup K'$ (resp. $L_0^- \sqcup K$) such that the splicing is done along K' and K in $L^{v'}$ and L^v and we have:

$$M_L = M^+{}_{L_0^+} \ {}^{K'}\!\!\underset{S_e}{\bowtie}{}^{K}\ M^-{}_{L_0^-}$$

We will use the following lemma for the splicing of everywhere-irreducible representations at e:

Lemma 4.2.4. *Two everywhere-irreducible representations, $\rho \in R^{\mathrm{PSL}_2\mathbb{C}}(M^+{}_{L^+})$ and $\rho' \in R^{\mathrm{PSL}_2\mathbb{C}}(M^-{}_{L^-})$ agree with the splicing at S_e if and only if*

$$\begin{array}{rcl} \rho(s_K)^{\alpha_K \alpha_{K'} - \alpha_{\widehat{K}} \alpha_{\widehat{K'}}} & = & 1 \\ \rho'(s_{K'}) & = & \rho(s_K)^{-(\beta_{K'}\alpha_K + \beta_{\widehat{K'}}\alpha_{\widehat{K'}})} \end{array} \tag{4.12}$$

Proof of Lemma 4.2.4. The splicing identifies the longitude of each side with the meridian of the other side so, using the peripheral matrices of T_K and $T_{K'}$ we have in $\pi_1 M_L$:

$$\begin{array}{rcl} {s_K}^{\alpha_K} {t_v}^{\beta_K} & = & {s_{K'}}^{-\alpha_{\widehat{K'}}} {t_{v'}}^{\beta_{\widehat{K'}}} \\ {s_K}^{-\alpha_{\widehat{K}}} {t_v}^{\beta_{\widehat{K}}} & = & {s_{K'}}^{\alpha_{K'}} {t_v}^{\beta_{K'}} \end{array}$$

and since ρ and ρ' trivialize the respective fibres t_v and $t_{v'}$, the images by ρ and ρ' must satisfy.

$$\begin{array}{rcl} \rho(s_K)^{\alpha_K} \rho'(s_{K'})^{\alpha_{\widehat{K'}}} & = & 1 \\ \rho(s_K)^{\alpha_{\widehat{K}}} \rho'(s_{K'})^{\alpha_{K'}} & = & 1 \end{array} \tag{4.13}$$

Finally, since $\alpha_{K'}\beta_{\widehat{K'}} + \beta_{K'}\alpha_{\widehat{K'}} = 1$, the system (4.13) is equivalent to (4.12):

$$\begin{array}{rcl} \rho(s_K)^{\alpha_K \alpha_{K'} - \alpha_{\widehat{K}} \alpha_{\widehat{K'}}} & = & 1 \\ \rho'(s_{K'}) & = & \rho(s_K)^{-(\beta_{K'}\alpha_K + \beta_{\widehat{K'}}\alpha_{\widehat{K'}})} \end{array}$$

A reverse calculation shows that any pair of everywhere-irreducible representations satisfying equations (4.12) will produce an everywhere-irreducible representation for M_L, which concludes the proof of Lemma 4.2.4. □

It follows that, for any everywhere-irreducible representation ρ of $R^{\mathrm{PSL}_2\mathbb{C}}(M_L)$ and for any internal edge e, the restriction $\rho_{|S_e}$ has torsion $\alpha_K\alpha_{K'} - \alpha_{\widehat{K}}\alpha_{\widehat{K'}} = \det(e)$.

Therefore, let e be an internal edge in Γ:

- if $|\det(e)| = 1$, ρ should be trivial on S_e and everywhere-irreducible representations exist if and only if they exist for ${M^+}_{L_0^+}$ and ${M^-}_{L_0^-}$.

- otherwise, for any $z \in \mathbb{C} \setminus \{-1, 0, 1\}$ with $z^{2|\det(e)|} = 1$, any two everywhere-irreducible characters ${M^+}_{L^+}$ and ${M^-}_{L^-}$ with

$$\begin{array}{rcl} I_{s_K}(\chi^+) & = & z + z^{-1} \\ I_{s_{K'}}(\chi^-) & = & w + w^{-1} \quad , \; w = z^{(\beta_{K'}\alpha_K + \beta_{\widehat{K'}}\alpha_{\widehat{K'}})} \end{array} \tag{4.14}$$

  will produce an everywhere-irreducible character of M_L.

Applying this criterion to all the internal edges of Γ, there must exist irreducible representations for each ${M^v}_{L^v \setminus L^{\mathcal{E}(v)_1}}$ which, by Theorem 6 is equivalent to

$$|L^{\partial v}| + |C^v| + |L^{\mathcal{E}(v)}| - |L^{\mathcal{E}(v)_1}| \geq 3.$$

By construction, the traces on the sections of the remaining components can be chosen freely. As in the proof of Theorem 6, the equations of the corresponding component in the eigenvalue-variety are,

$$\forall v \in \mathcal{V},\ \forall K \subset L^{\partial v} \subset L,\ {m_K}^{\alpha^v_{\widehat{K}}} {\ell_K}^{\alpha^v_K} = 1 \tag{4.15}$$

so the corresponding ideal is

$$\left\langle {\mathfrak{m}_K}^{\alpha^v_{\widehat{K}}} {\mathfrak{l}_K}^{\alpha^v_K} - 1,\ v \in \mathcal{V},\ K \subset L^{\partial v} \subset L \right\rangle \tag{4.11}$$

□

If they exist, everywhere-irreducible components are peripherally maximal and non-abelian so we obtain:

Corollary 5. *Let M_L be a graph link-manifold. If, for each vertex v of the dual tree,*

$$|L^{\partial v}| + |C^v| + |L^{\mathcal{E}(v)}| - |L^{\mathcal{E}(v)_1}| \geq 3$$

then $X^{\mathrm{PSL}_2\mathbb{C}}(M_L)$ admits a peripherally maximal and non-abelian component.

The E_∂-variety of everywhere-irreducible characters splits in $\prod_{K \subset L} E(T_K)$ as the product of the curves ${m_K}^{\alpha^v_{\widehat{K}}} {\ell_K}^{\alpha^v_K} = 1$ for each boundary component. Each of these curves has a natural parametrization by x_K, the eigenvalue-variety of the section s_K:

$$\begin{aligned} m_K &= {x_K}^{\alpha_k} \\ \ell_K &= {x_K}^{\alpha_{\widehat{K}}} \end{aligned}$$

Using the same approach, a similar result might be obained for $\mathrm{SL}_2\mathbb{C}$. Everywhere-irreducible representations in $\mathrm{SL}_2\mathbb{C}$ send all the fibres to $\pm\mathrm{Id}$. If the splice diagram has m vertices, there are 2^m possible combinations for the image of the fibres; each combination produces a system similar to (4.12) at each edge e and we can then use Theorem 9 to conclude on the existence of compatible irreducible representations on each piece. We will not go on with these calculation here so this concludes our study of everywhere-irreducible characters.

In the next section, we consider another type of characters, irreducible on one vertex and abelian everywhere else.

4.2.4 A family of peripherally maximal components

We will now consider components of characters that are irreducible on only one vertex, and abelian everywhere else. Let v be a vertex of Γ, represented in Γ by the diagram of Figure 4.9.

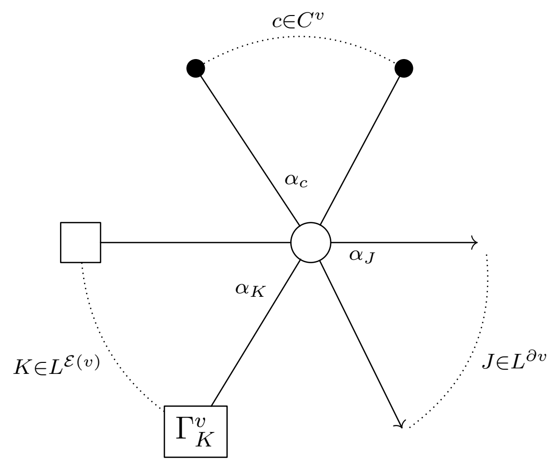


Figure 4.9 – A vertex in Γ

For any K in $L^{\mathcal{E}(v)}$, the tree Γ_K^v defines a graph link-manifold $M'_{L'}$ spliced with $M^v{}_{L^v}$ along K and a component K' of L'.

For any representation ρ of $\pi_1 M_L$, if $\rho_{|_{\pi_1 M'_{L'}}}$ is abelian, it has the form $\Delta \circ \varphi$ for some φ in $H^1(M'_{L'}, \mathbb{C}^*)$. It follows that ρ is determined by the images of the meridians; the images of the longitudes are given by the linking numbers:

$$\forall\, J' \subset L',\ \rho(\lambda_{J'}) = \prod_{J'' \neq J'} \rho(\mu_{J''})^{lk(J', J'')}$$

The following two situations can occur:

- $lk(K', J') = 0$ for all component J' of $L' \setminus \{K'\}$. In that case, $\rho(\lambda_{K'}) = \mathrm{Id}$ for any abelian representation of $\pi_1 M'_{L'}$ so, back in $M^v{}_{L^v}$, ρ must trivialize μ_K. This happens, in particular, if K' is the only component of L' (if Γ_K^v contains no arrowhead in Γ).

- otherwise, for any m', ℓ' in $\mathbb{C}^*$ there exist a morphism φ of $H^1(M'_{L'}, \mathbb{C}^*)$ such that

$$\begin{aligned} \varphi(\mu_{K'}) &= m' \\ \varphi(\lambda_{K'}) &= \ell' \end{aligned}$$

With these notations we consider a new Seifert-fibred link-manifold obtained from v:

Definition 4.2.4. The *isolation* of v in Γ is the Seifert-fibred link manifold $M^v_{\overset{\circ}{L^v}}$ obtained from the diagram of Figure 4.9 by replacing each $\boxed{\Gamma^v_K}$ by

- an node ● if $lk(K', J') = 0$ for all component J' of $L' \setminus \{K'\}$,
- an arrowhead $\rightarrow$ otherwise.

In other words, the diagram of Figure 4.9 becomes Figure 4.10, where the heads ■ for K in $L^{\mathcal{E}(v)}$ are arrow heads $\rightarrow$ or nodes ● depending on the linking number conditions in each Γ^v_K.

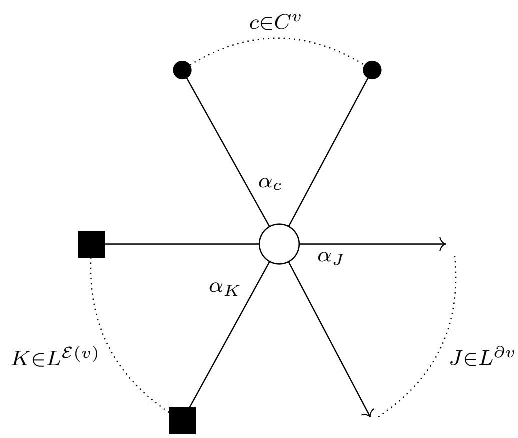


Figure 4.10 – Isolation of a vertex

Remark 4.2.5. Depending on the splice diagram, the isolation of v may become abelian.

Example 4.2.6. *On the cable link represented in Figure 4.11, the isolation of v produces the diagram of Figure 4.12 which reduces to a solid torus.*

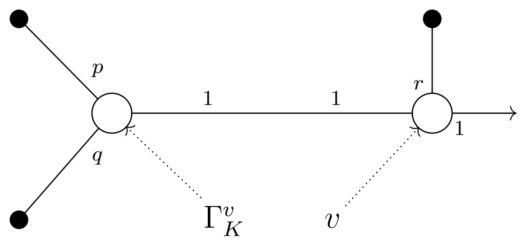


Figure 4.11 – A cable link

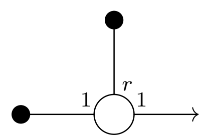


Figure 4.12 – Isolation of v in Figure 4.11

Isolations give a criterion for the existence of representations irreducible on one piece and abelian everywhere else:

Theorem 11. *Let M_L be a graph link-manifold with splice diagram Γ. For any vertex v of Γ, there exist a representation ρ in $R^{\mathrm{PSL}_2\mathbb{C}}(\pi_1 M_L)$ such that*

$$\left\{ \begin{array}{l} \textit{the restriction of } \rho \textit{ to } \pi_1 M^v_{L^v} \textit{ is irreducible} \\ \rho \textit{ is abelian on all the other vertices} \end{array} \right\} \tag{4.16}$$

if and only if the isolation of v in Γ has non-abelian fundamental group.

In that case, there exist a peripherally maximal and non-abelian component X in $X(M_L)$ with the properties (4.16).

Proof. Let M_L be a graph link-manifold with splice diagram Γ. Let v be a vertex of Γ such that the isolation of v, $M^v{}_{\mathring{L}^v}$, has non-abelian fundamental group. By Theorem 6, $R(M^v{}_{\mathring{L}^v})$ contains irreducible representations.

Let ρ^v be such a representation. Let K be a component of $L^{\mathcal{E}(v)}$ and Γ^v_K a tree spliced to v at K as in Figure 4.9. Let $M'_{L'}$ be the link-manifold represented by Γ^v_K. Let's index the components of L' as $K_0, \dots, K_n$, where the splicing is done along K_0 in $M'_{L'}$. We wish to extend ρ^v to $M^v{}_{L^v} \; {}^{K}\!\bowtie^{K_0} \; M'_{L'}$ via an abelian representation on $M'_{L'}$.

First, let's assume that $lk(K_0, K_i) = 0$ for any $1 \leq i \leq n$. Any abelian representation of $M'_{L'}$ trivializes λ_{K_0} and by definition of the isolation, K is filled along μ_K in $M^v_{\overset{\circ}{L^v}}$. Since ρ^v is irreducible, it trivializes the fibre t_v and the following relations hold:

$$\begin{array}{rcl} 1 = \rho^v(\mu_K) & = & \rho^v(s_K)^{\alpha_K} \\ \rho^v(\lambda_K) & = & \rho^v(s_K)^{-\alpha_{\widehat{K}}} \end{array}$$

It follows that $\rho^v(\lambda_K)$ is diagonalizable (of finite order, same as $\rho^v(s_K)$). Let ℓ_K be an eigenvalue of $\rho^v(\lambda_K)$.

For any $m_1, \ldots, m_n$, there exist φ' in $\mathrm{Hom}(\pi_1 M'_{L'}, \mathbb{C}^*)$ such that $\varphi'(\mu_{K_i}) = m_i$ for $1 \leq i \leq n$ and $\varphi'(\mu_{K_0}) = \ell_K$. Then, as usual, φ' defines a diagonal representation ρ' of $M'_{L'}$ and, we can conjugate it so that $\rho'(\mu_{K_0}) = \rho^v(\lambda_K)$. Since $\rho'(\lambda_{K_0}) = 1 = \rho^v(\mu_K)$, ρ' and ρ^v are compatible with the splicing so they define a representation of $R(M^v{}_{L^v} \; {}^{K}\!\bowtie^{K_0} \; M'_{L'})$ satisfying the expected properties.

On the other hand, let's assume that K_0 has non-zero linking number with an other component of L' so K becomes an arrowhead in $M^v_{\overset{\circ}{L^v}}$. Without loss of generality, we can assume that ρ^v is diagonalizable on T_K, and there exist A in $\mathrm{PSL}_2\mathbb{C}$ such that $\rho^v(\mu_K) = A\Delta(m_K)A^{-1}$ and $\rho^v(\lambda_K) = A\Delta(\ell_K)A^{-1}$ with ${m_K}^{\alpha_{\widehat{K}}}{\ell_K}^{\alpha_K} = 1$.

The equations in $\mathbb{C}^{*2n} = \{(m_i, \ell_i),\ 1 \leq i \leq n\}$:

$$\begin{array}{rcl} \forall\, 1 \leq i \leq n,\ \ell_i & = & \ell_K^{lk(K_0,K_i)} \displaystyle\prod_{\substack{j=1 \\ j \neq i}}^{n} m_j^{lk(K_j,K_i)} \\ m_K & = & \displaystyle\prod_{i=1}^{n} m_i^{lk(K_0,K_i)} \end{array}$$

span an $n-1$-dimensional subspace $V_{(m_K,\ell_K)}$ in $\mathbb{C}^{*2n}$. Any ξ' in $V_{(m_K,\ell_K)}$ defines a morphism φ' of $\pi_1 M'_{L'}$ in $\mathbb{C}^*$. This morphism defines a diagonal representation ρ' and $A\rho' A^{-1}$ is compatible with ρ^v for the splicing $M^v{}_{L^v} \; {}^{K}\!\bowtie^{K_0} \; M'_{L'}$ and provides the expected extension of ρ^v. The spaces $(V_{(m_K,\ell_K)})_{\langle {m_K}^{\alpha_{\widehat{K}}} {\ell_K}^{\alpha_K} = 1 \rangle}$ span an n-dimensional space in $\mathbb{C}^{*2n}$ of morphisms compatible with irreducible representations of $\pi_1 M^v_{\overset{\circ}{L^v}}$.

Therefore, provided $M^v_{\overset{\circ}{L^v}}$ is not abelian, we can extend irreducible representations of $\pi_1 M^v_{\overset{\circ}{L^v}}$ to $\pi_1 M_L$ with abelian representations on each $M'_{L'}$. By construction, for each $M'_{L'}$, the extension spans an algebraic space with dimension $|L'| - 1$ in the corresponding part of the eigenvalue-variety. Therefore, the representations obtained this way span an $|L|$-dimensional algebraic manifold in the eigenvalue-variety $E_\partial(M_L)$. □

Remark 4.2.7. For any vertex v, if $|L^{\partial v}| + |C^v| \geq 3$, the isolation of v is never abelian.

Finally, we show that, if M_L is not abelian, the condition of Theorem 11 is always satisfied for at least one vertex of Γ. In fact, it will always be satisfied for *a leaf* of the splice diagram.

Recall that a vertex of a tree is called a *leaf* if it has only one one adjacent vertex in the tree. In that case, the isolation can only take two forms so we can easily apply the criterion of Theorem 11.

Let v be a leaf in Γ as represented in Figure 4.13. The isolation of v depends on the

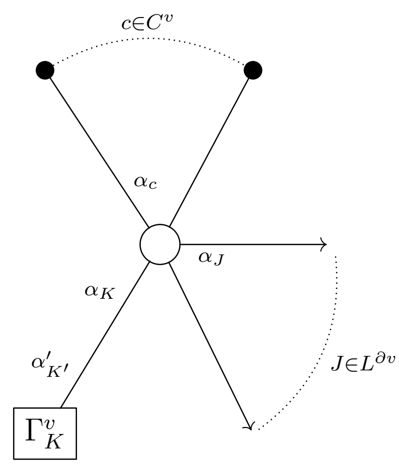


Figure 4.13 – A leaf in Γ

linking number of $\mu_K = \lambda_{K'}$ with eventual boundary components of $\Gamma^v_K = \Gamma \setminus \{v\}$. It is either ${M^v}_{L^v}$ or ${M^v}_{L^v \setminus K}$.

By Proposition 4.2.2, for any component J of L in Γ^v_K, the linking number between K' and J is zero if and only if there's a 0 coefficient adjacent to the path between K' and the arrowhead J in Γ^v_K.

The following lemma shows that, if M_L is a non-abelian graph link-manifold M_L with boundary, there exist a leaf which remains unchanged by isolation.

Lemma 4. *Let Γ be the splice diagram of a non-abelian graph link-manifold M_L with boundary. There exist a path from a leaf v of Γ to a boundary component with no adjacent 0 in $\Gamma \setminus \{v\}$.*

Proof. A vertex has at most one 0 coefficient and it is either on an arrow or an internal edge. We will construct the path starting from an arrow and following any internal edge labelled by 0 until we reach a leaf.

First, let's assume that Γ contains no arrowhead with a 0 coefficient. From an arrowhead K of Γ, we build a path following any possible edge labelled with 0 that we encounter. Since Γ is finite, this path must end to a leaf of Γ and, by construction, can contain no adjacent 0.

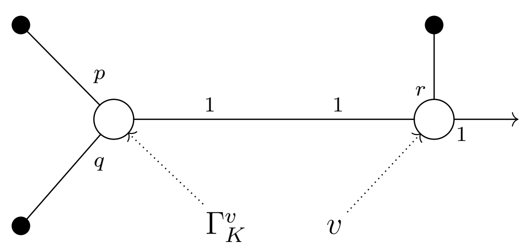
Otherwise, let's start from an arrow labelled with 0. As before, we follow any encountered edge labelled with 0. If, at any point, the path reaches a non-leaf vertex containing an arrow labelled by 0, we start again from this arrow, following any internal edge that was not on the original path; such edge must exist or the vertex would be a leaf. Then, again, this process must reach a leaf and the obtained path contains no adjacent 0. □

So, finally, we obtain the following result for graph link-manifolds:

Theorem 2. *For any non-abelian graph link-manifold M_L with boundary, there exist a peripherally maximal and non-abelian component in $X^{\mathrm{PSL}_2\mathbb{C}}(M_L)$.*

Proof. Let v be leaf obtained from Lemma 4. By construction, ${M^v}_{L^v}$ is unchanged under isolation. Since it is not abelian, Theorem 11 ensures the existence of a component in $E_\partial(M_L)$ with maximal dimension. □

Example 4.2.8. *In the previous example of Figure 4.11,*



we can only isolate the left leaf (corresponding to a p, q-torus knot) to create a curve of irreducible characters on the remaining boundary.

4.3 General components

We close this chapter with few considerations on generic components of the E_∂-variety of a graph link-manifold, or the $E_{\mathcal{G}}$-varieties for contractions $\mathcal{G}$ of the splice diagram.

Applying Proposition 1 to the splice diagram of a graph link-manifold M_L, any component X of $X(M_L)$ determines a binding decomposition ($\mathcal{G} \gg= \Gamma$) of the JSJ tree $\mathcal{G}_{\mathcal{J}}$ where, for each vertex v of the contracted tree $\mathcal{G}$, ${i_v}^*X$ is either everywhere irreducible or everywhere reducible for the splitting Γ_v.

An extensive study of the $2^{|\mathcal{J}|}$ possible binding decompositions of $\mathcal{G}_{\mathcal{J}}$, corresponding to all the subsets of edges of $\mathcal{G}_{\mathcal{J}}$, should enable a complete characterisation of all the possible components in the $E_{JSJ\partial}$-variety. For any subset of $k-1$ edges of $\mathcal{E}$, we obtain a partition of $\mathcal{G}$ into k trees, wich produces 2^k combinations (or 3^k in $\mathrm{SL}_2\mathbb{C}$) of everywhere abelian or everywhere irreducible components on each subtree. These are $2\times 3^{|\mathcal{J}|}$ (or $2\times 4^{|\mathcal{J}|}$ in $\mathrm{SL}_2\mathbb{C}$) possible combinations to inspect.

Then, on each subtree, Proposition 6 and Theorem 10 provide the equations for the possible peripheral eigenvalue-varieties. Examining all the possible combinations, the merging criterion of Theorem 3 should provide a condition of existence for any component with a given type η, as well as the corresponding $\mathcal{A}_{\mathcal{G}_{\mathcal{J}}}$-ideals equations.

Once $E_{JSJ\partial}(M_L)$ is obtained, Theorem 3 would then describe all the possible $E_{\mathcal{G}}$-varieties for contractions $\mathcal{G}$ of $\mathcal{G}_{\mathcal{J}}$.

Appendix A

Regular functions on character varieties

A.1 Regular functions

Let G be a reductive algebraic group and π a finitely generated group.

Any regular function $f \in \mathbb{C}[G]$ produces a family of regular functions $(f_\gamma)_{\gamma\in\pi}$ in $\mathbb{C}[R^G(\pi)]$ via $f_\gamma(\rho) = f(\rho(\gamma))$. As the following Lemma A.1.1 shows, these actually generate the whole ring of regular functions.

For any algebraic space V, for any family $(f_j)_{j\in J}$ of regular functions on V, we denote by $\mathbb{C}[(f_j)_{j\in J}]$ the subring of $\mathbb{C}[V]$ of polynomial combinations of any finite collection $f_{j_1}, \ldots, f_{j_n}$.

Lemma A.1.1.

$$\mathbb{C}[R^G(\pi)] = \mathbb{C}[(f_\gamma)_{\gamma\in\pi,\ f\in\mathbb{C}[G]}]$$

Proof. From the observations made above, $\mathbb{C}[(f_\gamma)_{\gamma\in\pi,\ f\in\mathbb{C}[G]}] \subset \mathbb{C}[R^G(\pi)]$; we show that $\mathbb{C}[(f_\gamma)_{\gamma\in\pi,\ f\in\mathbb{C}[G]}]$ distinguishes points of $R^G(\pi)$ and, therefore, is the whole ring of regular functions.

Let ρ and ρ' be two representations such that

$$\forall\, f \in \mathbb{C}[G],\ \forall\, \gamma \in \pi,\ f_\gamma(\rho) = f_\gamma(\rho')$$

By definition of f_γ,

$$\forall\, \gamma \in \pi,\ \forall\, f \in \mathbb{C}[G],\ f(\rho(\gamma)) = f(\rho'(\gamma))$$

Since, by definition, $\mathbb{C}[G]$ distinguishes points of G, this implies

$$\forall\, \gamma \in \pi,\ \rho(\gamma) = \rho'(\gamma)$$

so $\rho = \rho'$ $\square$

If f is G-invariant, each f_γ is also G-invariant and factors as a function $f_\gamma \in \mathbb{C}[X^G(\pi)]$.

For example, using the characters of examples 1.2.2 to 1.2.4 we can define the following functions:

Definition A.1.1. If G is linear, we denote by $(\tau_\gamma)_{\gamma\in\pi}$ the family of functions in $\mathbb{C}[X^G(\pi)]$ induced by the function tr of $\mathbb{C}[G//G]$. For any γ in π and χ in $X^G(\pi)$,

$$\tau_\gamma(\chi) = \mathrm{tr}(\rho(\gamma)) \text{ for any } \rho \text{ in } t^{-1}\chi$$

Definition A.1.2. For any γ in π,

$$I_\gamma : X^{\mathrm{PSL}_2\mathbb{C}}(\pi) \to \mathbb{C}$$

is the regular map corresponding to the $\mathrm{PSL}_2\mathbb{C}$-invariant function $f = \mathrm{tr}_2$; in other words, $I_\gamma(\chi) = \mathrm{tr}(\rho(\gamma)^2)$ for any ρ in $t^{-1}\chi$.

Definition A.1.3. For any γ in π,

$$J_\gamma : X^{\mathrm{PSL}_2\mathbb{C}}(\pi) \to \mathbb{C}$$

is the regular map corresponding to the $\mathrm{PSL}_2\mathbb{C}$-invariant function $f = \mathrm{tr}^2$; it is characterized by the equation, $J_\gamma(\chi) = (\mathrm{tr}(\rho(\gamma)))^2$ for any ρ in $t^{-1}\chi$.

Remark A.1.1. The two regular functions I_γ and J_γ on $X^{\mathrm{PSL}_2\mathbb{C}}(\pi)$ only differ by a constant. In $\mathbb{C}[X^{\mathrm{PSL}_2\mathbb{C}}(\pi)]$ we have:

$$\forall\, \gamma \in \pi,\ I_\gamma = J_\gamma - 2$$

Remark A.1.2. When working with $G = \mathrm{PSL}_2\mathbb{C}$, the functions tr^2 and J_γ are often used as canonical character (see, for example, [BZ98] or [HP04]). However, we shall prefer the use of tr_2 and I_γ, which reflect more directly the behaviour of tr and τ_γ for $G = \mathrm{SL}_2\mathbb{C}$.

A.2 Generating the ring $\mathbb{C}[X(\pi)]$

The association, for f in $\mathbb{C}[G]^G$, $f \rightsquigarrow (f_\gamma)_{\gamma\in\pi}$, produces two subalgebras of $\mathbb{C}[X^G(\pi)]$:

$$\mathbb{C}[(\tau_\gamma)_{\gamma\in\pi}] \subset \mathbb{C}[(f_\gamma)_{f\in\mathbb{C}[G]^G,\ \gamma\in\pi}] \subset \mathbb{C}[X^G(\pi)] \tag{A.1}$$

For most linear algebraic groups, these tree algebras are equal. Indeed, the following theorem is a consequence of Theorems 3, 5 and 8 of [Sik13].

Theorem A.2.1. *If* G *is special linear, symplectic, orthogonal, or odd special orthogonal,*

$$\mathbb{C}[(\tau_\gamma)_{\gamma\in\pi}] = \mathbb{C}[X(\pi)]$$

If G *is even special orthogonal,*

$$\mathbb{C}[(\tau_\gamma)_{\gamma\in\pi}] \subsetneq \mathbb{C}[(f_\gamma)_{f\in\mathbb{C}[G]^G,\ \gamma\in\pi}] = \mathbb{C}[X^G(\pi)]$$

Proof. See [Sik13]. □

The group $G = \mathrm{PSL}_2\mathbb{C}$ is not a linear group. However, as explained for instance in [HP04], the action of $\mathrm{PSL}_2\mathbb{C}$ on the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ gives an isomorphism

$$\mathrm{Ad} : \mathrm{PSL}_2\mathbb{C} \to \mathrm{SO}_3\mathbb{C}$$

such that $\mathrm{tr}(\mathrm{Ad}(A)) = \mathrm{tr}_2(A) + 1 = \mathrm{tr}^2(A) - 1$; by Theorem A.2.1 we have:

Corollary A.2.2. *For* $G = \mathrm{PSL}_2\mathbb{C}$*,*

$$\mathbb{C}[(I_\gamma)_{\gamma\in\pi}] = \mathbb{C}[(J_\gamma)_{\gamma\in\pi}] = \mathbb{C}[(f_\gamma)_{f\in\mathbb{C}[G]^G,\ \gamma\in\pi}] = \mathbb{C}[X^G(\pi)]$$

Where I_γ *and* J_γ *are the regular functions of definitions A.1.2 and A.1.3, respectively associated to* tr_2 *and* tr^2.

Remark A.2.1. When G is linear, an alternative construction of X^G consists in *defining* $X^G(\pi)$ such that

$$\mathbb{C}[X^G(\pi)] = \mathbb{C}[(\tau_\gamma)_{\gamma\in\pi}]$$

For example, this is the Culler-Shalen construction of $X^{\mathrm{SL}_2\mathbb{C}}$ given in [CS83]. By Theorem A.2.1, this is generally (but not always) equivalent.

If $\mathbb{C}[(f_\gamma)_{f\in\mathbb{C}[G]^G,\ \gamma\in\pi}] = \mathbb{C}[X^G(\pi)]$, points of $X^G(\pi)$ are characterized by G-invariants functions G: for any χ, χ' in $X^G(\pi)$,

$$\chi = \chi' \iff \forall f \in C[G]^G,\ \forall\gamma \in \pi, f_\gamma(\chi) = f_\gamma(\chi')$$

Moreover, if there exist θ in $\mathbb{C}[G]^G$ such that $\mathbb{C}[(\theta_\gamma)_{\gamma\in\pi}] = \mathbb{C}[X^G(\pi)]$, then each χ in $X^G(\pi)$ can naturally be identified with the function

$$\begin{array}{rccc} \widehat{\chi}: & \pi & \to & \mathbb{C} \\ & \gamma & \to & \theta_\gamma(\chi) \end{array}$$

hence the name **character** of π.

A.3 Polynomials in the $\mathrm{SL}_2\mathbb{C}$ character varieties

In this section, X denotes, $X^{\mathrm{SL}_2\mathbb{C}}$, the $\mathrm{SL}_2\mathbb{C}$ character variety.

Let F_n be the free group with n generators denoted by $a_1, \dots, a_n$, $R(\mathrm{F}_n)$ is its representation variety $\mathrm{Hom}(\mathrm{F}_n, \mathrm{SL}_2\mathbb{C})$ and $X(\mathrm{F}_n)$ its character variety. For any representation ρ of $R(\mathrm{F}_n)$ and γ in F_n we recall the definition of the trace function:

$$\begin{array}{rcl} \tau_\gamma : R(\mathrm{F}_n) & \to & \mathbb{C} \\ \rho & \to & tr\rho(\gamma). \end{array}$$

The ring of functions of $X(\mathrm{F}_n)$, $\mathbb{C}[X(\mathrm{F}_n)]$ is generated by the functions $\tau_\gamma,\ \gamma \in \mathrm{F}_n$ by Theorem A.2.1; in the case of $\mathrm{SL}_2\mathbb{C}$, the trace relation

$$\forall A, B \in \mathrm{SL}_2\mathbb{C},\ \mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B) = \mathrm{tr}(A)\mathrm{tr}(B) \tag{A.2}$$

enables to specify a finite generating family of $\mathbb{C}[X(\mathrm{F}_n)]$:

Proposition A.3.1. *For any subset $I = \{i_1 < \dots < i_j\}$ of $\mathbb{N}_n$, let τ_I denote the regular function $\tau_I = \tau_{a_{i_1} \dots a_{i_j}}$ of $\mathbb{C}[X(\mathrm{F}_n)]$; then for any γ in F_n, there is a polynomial in $2^n - 1$ variables $P_\gamma \in \mathbb{C}[(Y_I)_{I \subset \mathbb{N}_n}]$ such that*

$$\tau_\gamma = P_\gamma((\tau_I)_{I \subset \mathbb{N}_n}).$$

Proof. The complete proof can be found in [CS83] and is mainly algorithmic, using relation (A.2) do decrease a well-chosen height on the elements of F_n. □

Remark A.3.1. The polynomial P_γ of Proposition A.3.1 is in general not unique.

Example A.3.2. *For exemple*

- $\tau_{a^2} = {\tau_a}^2 - 2$
- $\tau_{a^3} = {\tau_a}^3 - 3\tau_a$
- $\tau_{a^4} = {\tau_a}^4 - 4{\tau_a}^2 - 2$
- $\tau_{[a,b]} = {\tau_a}^2 + {\tau_b}^2 + {\tau_{ab}}^2 - \tau_a\tau_b\tau_{ab} - 2$

There is in general no formula giving the polynomials P_γ for a given word γ. However, in this section, we will proceed to compute the polynomials P_γ for the elements $\gamma = a_1^{\alpha_1} \dots a_n^{\alpha_n}$, for n-uple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$.

For $n \in \mathbb{N}$, let $\mathcal{O}_n$ denote the ring

$$\mathcal{O}_n = \mathbb{C}[(Y_I)_{I \subset \mathbb{N}_n}]$$

with the following convention:

- $Y_\emptyset = 2$
- for singletons of $\mathbb{N}_n$, we will use the notation $Y_k = Y_{\{k\}}$.

On the other and, for $n \in \mathbb{N}$, let $\mathbb{X}_n$ denote the ring

$$\mathbb{X}_n = \mathbb{C}[X(\mathrm{F}_n)]$$

By Theorem A.2.1, $\mathbb{C}[X(\mathrm{F}_n)] = \mathbb{C}[\tau_\gamma, \gamma \in \mathrm{F}_n]$ so we can define a ring-map

$$\begin{array}{rccc} p: & \mathcal{O}_n & \to & \mathbb{X}_n \\ & Y_I & \to & \tau_I \end{array}$$

and, by Proposition A.3.1, p is epic. Given γ in F_n, we want to find P_γ such that $\tau_\gamma = p(P_\gamma)$ in $\mathbb{X}_n$.

Example A.3.3 ($n = 1$)**.** *For* $n = 1$, $\mathrm{F}_1 = \langle a| \ \rangle \cong \mathbb{Z}$*. Applying* (A.2) *with* $B = A^n$ *we obtain*

$$\mathrm{tr}(A^{n+1}) - \mathrm{tr}(A)\mathrm{tr}(A^n) + \mathrm{tr}(A^{n-1}) = 0$$

so, in $\mathcal{O}_1 = \mathbb{C}[Y]$*, we should expect the relation*

$$P_{a^{n+1}} - YP_{a^n} + P_{a^{n-1}} = 0 \tag{A.3}$$

with $P_a = Y$ *and* $P_1 = 2$*, this completely determines* P_γ *for all* γ *in* F_1*.*

We define the two families of polynomials in $\mathbb{C}[Y]$*:*

Definition A.3.1. Let U_n and V_n denote the sequences of polynomials in $\mathbb{C}[Y]$ defined by:

$$\begin{array}{rcl} U_0 &=& 0,\ U_1 = 1 \\ V_0 &=& 1,\ V_1 = 0 \end{array} \tag{A.4}$$

and the recursive relation

$$Q_{n+1} - YQ_n + Q_{n-1} = 0, \text{ for } n \in \mathbb{Z} \tag{A.5}$$

for both $Q = U$ and $Q = V$.

Any family P_n *of polynomials in* $\mathcal{O}_1$ *satisfying* (A.3) *is given by*

$$P_n = P_0V_n + P_1U_n$$

in particular, the polynomials P_{a^n} *are given in* $\mathcal{O}_1$ *by*

Proposition 7. *Let* $P_n = 2V_n + YU_n$ *in* $\mathbb{C}[Y]$*; then, for any* n *in* $\mathbb{Z}$*,* $p(P_n) = \tau_{a^n}$ *in* $\mathbb{X}(\mathrm{F}_1)$*.*

Remark A.3.4. With the convention $Y_\emptyset = 2$, this can also be written

$$P_n = Y_\emptyset V_n + YU_n$$

Let $\mathrm{F}_n = \langle a_1, \dots, a_n | \rangle$ be the free group with rank n. For any α in $\mathbb{Z}^n$, let a^α be the element of F_n:

$$a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$$

We will show that Proposition 7 can be generalized for words a^α in F_n; that is, for any α in $\mathbb{Z}^n$, we give an explicit expression for P_α in $\mathcal{O}_n$ such that

$$p(P_\alpha) = \tau_{a^\alpha}$$

Theorem 12. *For any* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ *and any subset* I *of* $\mathbb{N}_n$*, let* r_α^I *be the polynomial of* $\mathcal{O}_n$*:*

$$r_\alpha^I = \prod_{k \in I} U_{\alpha_k}(Y_k) \prod_{k \notin I} V_{\alpha_k}(Y_k)$$

Then, the polynomial P_α *defined by*

$$P_\alpha = \sum_{I \subset \mathbb{N}} r_\alpha^I \, Y_I$$

satisfies

$$p(P_\alpha) = \tau_{a^\alpha}$$

We will prove that $p(P_\alpha) = \tau_{a^\alpha}$ for all $\alpha \in \mathbb{Z}^{\mathbb{N}_n}$ by induction on

$$||\alpha|| = Max\{|\alpha_i|, i \in \mathbb{N}_n\}.$$

We need to set a few more notation first. For any i in $\mathbb{N}_n$, we denote by ε_i the vector of $\mathbb{Z}^n$:

$$\varepsilon_i = (0, \dots, 0, 1, 0, \dots, 0)$$

where the 1 is at index i.

Lemma 5. *For any* α *in* $\mathbb{Z}^n$ *and any* i *in* $\mathbb{N}$*, Then we have the following identities:*

$$\tau_{a^{\alpha+\varepsilon_i}} - \tau_{a_i}\tau_{a^\alpha} + \tau_{a^{\alpha-\varepsilon_i}} = 0 \textit{ in } \mathbb{X}_n \tag{A.6}$$

$$P_{\alpha+\varepsilon_i} - Y_i P_\alpha + P_{\alpha-\varepsilon_i} = \textit{ in } \mathcal{O}_n \tag{A.7}$$

Proof. For the identity Equation (A.6) we simply use relation (A.2) and the fact that $tr(AB) = tr(BA)$ for any matrices A and B. For any $\alpha_1, \ldots, \alpha_n$ of $\mathbb{Z}^{\mathbb{N}_n}$,

$$\begin{aligned} \tau_{a_1^{\alpha_1}\ldots a_i^{\alpha_i+1}\ldots a_n^{\alpha_n}} &= \tau_{a_{i+1}^{\alpha_{i+1}}\ldots a_n^{\alpha_n} a_1^{\alpha_1}\ldots a_i^{\alpha_i+1}} \\ &= \tau_{a_i}\tau_{a_{i+1}^{\alpha_{i+1}}\ldots a_n^{\alpha_n} a_1^{\alpha_1}\ldots a_i^{\alpha_i}} - \tau_{a_{i+1}^{\alpha_{i+1}}\ldots a_n^{\alpha_n} a_1^{\alpha_1}\ldots a_i^{\alpha_i-1}} \\ &= \tau_{a_i}\tau_{a_1^{\alpha_1}\ldots a_i^{\alpha_i}\ldots a_n^{\alpha_n}} - \tau_{a_1^{\alpha_1}\ldots a_i^{\alpha_i-1}\ldots a_n^{\alpha_n}} \end{aligned}$$

The second equation, on the other hand, is a consequence of the recursive relation (Equation (A.5)) satisfied by U and V. For any subset I of $\mathbb{N}_n$, the definition of r_α^I implies that

$$r^I_{\alpha+\varepsilon_i} - Y_i r^I_\alpha + r^I_{\alpha+2\varepsilon_i} = 0$$

and, taking the sum over the subsets of $\mathbb{N}_n$ this gives

$$P_{\alpha+\varepsilon_i} - Y_i P_\alpha + P_{\alpha+2\varepsilon_i} = 0$$

which completes the proof of Lemma 5 □

These identities will enable the recursion to prove Theorem 12.

Proof of Theorem 12. For α in $\mathbb{Z}^n$, we denote by $||\alpha||$ its l_∞ norm:

$$||\alpha|| = Max\{|\alpha_i|, i \in \mathbb{N}_n\}$$

First, if $||\alpha|| = 0$, all the coordinates α_i are zero. In that case $\tau_{a^\alpha} = \tau_{\mathrm{id}} = 2$ in $\mathbb{X}_n$. On the other hand, $r^I_0 = 0$ for any non-empty subset I of $\mathbb{N}_n$ and $r^\emptyset_0 = 1$. Therefore, $P_0 = 2$ in $\mathcal{O}_n$ and $p(P_0) = \tau_{a^0}$.

Then, assume that $||\alpha|| = 1$, so all the coordinates are in $\{-1, 0, 1\}$. By Lemma 5, we can assume that all the coordinates are in $\{0, 1\}$. Let A denote the subset of indices for which α has nonzero coefficient, with the notations of Proposition A.3.1,

$$\tau_{a^\alpha} = \tau_A \text{ in } \mathbb{X}_n$$

On the other hand, for any $I \subset \mathbb{N}_n$,

$$r^I_\alpha = \prod_{k\in I} U_{\alpha_k}(Y_k) \prod_{k\notin I} V_{\alpha_k}(Y_k)$$

where each α_k is either 0 or 1. With the initial conditions of U and V (Equation (A.4)) we obtain:

$$r^I_\alpha = \begin{cases} 1, & \text{if } I = A \\ 0, & \text{otherwise} \end{cases}$$

Taking the sum of $r_\alpha^I Y_I$ over $I \subset \mathbb{N}_n$ simply leaves $P_\alpha = Y_A$ as expected.

Finally, assume that $||\alpha|| \geq 2$ and that $p(P_\beta) = \tau_{a^\beta}$ for any vector β with $||\beta|| < ||\alpha||$. Let i be an index such that $|\alpha_i| = ||\alpha||$.

- First, let assume that $\alpha_i > 0$ and $|\alpha_j| < |\alpha_i|$ for $j \neq i$. Then $||\alpha - \varepsilon_i|| < ||\alpha||$ and $||\alpha - 2\varepsilon_i|| < ||\alpha||$ so we we have:

$$\begin{aligned} p(P_\alpha) &= p(Y_i P_{\alpha-\varepsilon_i} - P_{\alpha-2\varepsilon_i}) \text{ by Lemma 5} \\ &= p(Y_i)p(P_{\alpha-\varepsilon_i}) - p(P_{\alpha-2\varepsilon_i}) \\ &= \tau_{a_i}\tau_{\alpha-\varepsilon_i} - \tau_{\alpha-2\varepsilon_i} \text{ by induction hypothesis} \\ &= \tau_{a^\alpha} \text{ by Lemma 5} \end{aligned}$$

- Then, if $\alpha_i < 0$ and $|\alpha_j| < |\alpha_i|$ for $j \neq i$, we can use the same argument with $\alpha + \varepsilon_i$ and $\alpha + 2\varepsilon_i$ to conclude.

- Finally, if $||\alpha||$ is attained for more than one coordinates, we can apply the same argument on each of these coordinates until $||\alpha||$ decreases and conclude using the induction hypothesis.

This completes the proof of Theorem 12 so, for any α in $\mathbb{Z}^n$,

$$p(P_\alpha) = \tau_{a^\alpha}.$$

□

Remark A.3.5. All the polynomials P_α have degree 1 in the variables Y_I when I is not a singleton of $\mathbb{N}_n$.

Remark A.3.6. Observing that $r_{\alpha^I} = 0$ if $\alpha_i = 0$ for some $i \in I$, the same formula can be extended to free groups with countable generators and sequences of integeres α with finite support.

Let $\mathrm{F}_\mathbb{N}$ denote the free group with generators $(a_n)_{n\in\mathbb{N}}$. For any α in $\mathbb{Z}^{(\mathbb{N})}$ we define

$$a^\alpha = \prod_{n\in\mathbb{N}} a_n^{\alpha_n}$$

this product is finite since α has finite support and we can also define, for $I \subset \mathbb{N}$,

$$r_\alpha^I = \prod_{n\in I} U_{\alpha_n}(Y_n) \prod_{n\notin I} V_{\alpha_n}(Y_n)$$

where, for the same reason, both products are finite. The observation above implies that $r_\alpha^I = 0$ if I is not contained in the support of α so the sum

$$P_\alpha = \sum_{I \subset \mathbb{N}} r_\alpha^I Y_I$$

is also finite. Then, by Theorem 12, for any α in $\mathbb{Z}^{(\mathbb{N})}$,

$$p(P_\alpha) = \tau_{a^\alpha} \text{ in } \mathbb{C}[X(\mathrm{F}_\mathbb{N})]$$

Theorem 12 can be used to describe maps between character varieties.

Example A.3.7. *Let π be a finited generated group with generators $a_1, \ldots, a_n$ and let F_2 be the free group with two generators x and y. For any n-uple of integers, $\alpha_1, \ldots, \alpha_p, \ldots, \alpha_n$, the map*

$$\begin{array}{rcl} x & \to & a_1^{\alpha_1} \cdots a_p^{\alpha_p} \\ y & \to & a_{p+1}^{\alpha_{p+1}} \cdots a_n^{\alpha_n} \end{array}$$

defines a group morphism $\mathrm{F}_2 \to \pi$ and an ring map

$$\mathbb{C}[X(\mathrm{F}_2)] \to \mathbb{C}[X(\pi)]$$

The character variety of F_2 is $\mathbb{C}^3$ with

$$\mathbb{C}[X(\mathrm{F}_2)] = \mathbb{C}[\tau_x, \tau_y, \tau_{xy}]$$

On the other hand, $\mathbb{C}[X(\pi)]$ is a quotient of $\mathbb{C}[X(\mathrm{F}_n)]$ and $\mathbb{C}[(Y_I)_{I \subset \mathbb{N}_n}]$. With these notations, the ring map is given by the following polynomials:

$$\begin{array}{rcl} \tau_x & \to & P_{\alpha_x}((Y_I)) \\ \tau_y & \to & P_{\alpha_y}((Y_I)) \\ \tau_{xy} & \to & P_\alpha((Y_I)) \end{array}$$

where α_x and α_y are the n-uples

$$\begin{array}{rcl} \alpha_x & = & (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0) \\ \alpha_y & = & (0, \ldots, 0, \alpha_{p+1}, \ldots, \alpha_n) \end{array}$$

Appendix B

Examples of peripheral eigenvalue-varieties

In this chapter we present the results of computation of peripheral $\mathrm{SL}_2\mathbb{C}$-eigenvalue-varieties for few link exteriors in $\mathbb{S}^3$. We took a straightforward approach to compute the equations from the fundamental group, at cost of a high complexity. The computation is done in three steps

1. Compute a presentation of the fundamental group and peripheral systems using Snappy [CDW].

2. Use Culler-Shalen algorithm from Proposition A.3.1 (see [CS83] for the details) to compute some polynomial equations induced by the presentation. This was done in GAP [The12].

3. Eleminate the undesired variables with Macaulay 2 (M2) [GS] to obtain the equations of the E_∂-variety.

These three steps are combined with a small program written in Haskell [Mar10].

Let π be a finitely presented group, $\pi = \langle a_1, \ldots, a_m \mid w \in W \rangle$. The character variety of π is the closed subset of $X(\mathrm{F}_m)$ such that

$$\forall w \in W, \forall 1 \leq i \leq m,\ \tau_w = 2,\ \tau_{a_i w} = \tau_{a_i} \tag{B.1}$$

With the notations of Appendix A.3, we describe $X(\mathrm{F}_m)$ as a subset of $\mathbb{C}^N$ with $N = 2^m - 1$ given by the functions τ_{a_I} for subsets I of $[1 \ldots m]$. Then, using Culler-Shalen algorithm, we compute each polyonmial P_w and $P_{a_i w}$ to obtain the defining equations of the character variety $X(\pi)$.

For each peripheral system μ_T, λ_T, we add variables m_T, ℓ_T and the equations

$$\begin{array}{rcl} m_T^2 - P_{\mu_T}(Y)m_T + 1 & = & 0 \\ \ell_T^2 - P_{\lambda_T}(Y)\ell_T + 1 & = & 0 \\ m_T^2\ell_T^2 - P_{\mu_T\lambda_T}(Y)m_T\ell_T + 1 & = & 0 \end{array} \tag{B.2}$$

where each P_{μ_T}, P_{λ_T}, $P_{\mu_T\lambda_T}$ is also computed using Culler-Shalen algorithm.

Finally, eliminating the variables $Y_1, \dots Y_N$ produces an ideal in $\mathbb{C}[\mathfrak{m}, \mathfrak{l}]$ corresponding to the expected A_∂-ideal.

Unfortunately, the elimination algorithm uses a lot of memory and, although this process should work for any link, the computation terminated only for a handful of examples.

B.1 Code

We reproduce here the most relevant parts of the code; the full source can be found at `http://hub.darcs.net/arbol/E-variety`.

B.1.1 Free groups

If π is a group with n generators, $X(\pi)$ will be computed as a closed subset of $X(\mathrm{F}_n)$; let $\mathcal{O}_n$ denote the ring $\mathbb{C}[(Y_I)_{I\subset\mathbb{N}_n}]$ as in Appendix A.3; The representation variety of the free group F_n is $(\mathrm{SL}_2\mathbb{C})^n$ so

$$R(\mathrm{F}_n) = \{a_i, b_i, c_i, d_i \mid a_i d_i - b_i c_i = 1\}$$

and if M_i denote the matrix $\begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$ for $1 \le i \le n$, $\mathbb{C}[X(\mathrm{F}_n)]$ is given by the kernel of the ring map

$$\begin{array}{rccc} t: & \mathcal{O}_n & \to & \mathbb{C}[R(\mathrm{F}_n)] \\ & Y_I & \to & \mathrm{tr}(\prod_{i\in I} M_i) \end{array}$$

We use GAP to generate the equations and M2 to perform the computation of the kernel. The following GAP code generates the determinant equations for $\mathbb{C}[R(\mathrm{F}_n)]$:

```
Pn:=[1..n];
x:=[1..m];
a:=[1..n];
b:=[1..n];
c:=[1..n];
d:=[1..n];
```

```
for i in Pn do
 a[i]:=Indeterminate(Rationals,i);
 b[i]:=Indeterminate(Rationals,i+n);
 c[i]:=Indeterminate(Rationals,i+2*n);
 d[i]:=Indeterminate(Rationals,i+3*n);
od;

M:=[1..n];
for i in Pn do
  M[i]:=[[a[i],b[i]],[c[i],d[i]]];
od;
D:=[1..n];
for i in D do
  D[i]:=Determinant(M[i])-1;
  od;
```

and the next one generates the list of traces $\mathrm{tr}(\prod_{I\subset\mathbb{N}_n} M_i)$:

```
CPn:=Combinations(Pn);
m:=Length(CPn);
R:=[1..m-1];
for i in R do
  I:=CPn[i+1];
  li:=[1..Length(I)];
  for j in li do
    li[j]:=M[I[j]];
    od;
  R[i]:=Trace(Product(li));
od;
```

Finally, setting `relsDet` and `imTrace` as the list of determinant relations and traces computed above the following piece of M2 code computes the kernel of t in $\mathcal{O}_n$:

```
R = A/ideal(relsDet);
t = map(R,B,imTrace);
K = kernel t;
XFree = B / K;
```

Although this should theoritically work for any n, the computation could not terminate for $n \geq 4$; the character variety $X(\mathrm{F}_3)$ was already known and is given by one polynomial equation in seven variables. With the following convention:

$$\begin{array}{rcl} x_1 & = & Y_{\{1\}} \\ x_2 & = & Y_{\{1,2\}} \\ x_3 & = & Y_{\{1,2,3\}} \\ x_4 & = & Y_{\{1,3\}} \\ x_5 & = & Y_{\{2\}} \\ x_6 & = & Y_{\{2,3\}} \\ x_7 & = & Y_{\{3\}} \end{array} \tag{B.3}$$

we obtain the polynomial:

$x_1x_3x_5x_7 - x_1x_2x_5 - x_3x_4x_5 - x_1x_3x_6 + x_2x_4x_6 - x_2x_3x_7 - x_1x_4x_7 - x_5x_6x_7 +$
$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 - 4$

B.1.2 Polynomial equations

Once $X(\mathrm{F}_n)$ is computed, we need the equations defining $X(\pi)$ induced by the relators of the presentation. To compute these polynomials, we implemented Culler-Shalen algorithm using GAP. Given a word `W` in F_n and if `x` is an array of 2^n variables[1], the following fonction computes a polynomial `Pol` in $\mathcal{O}_n$ such that $p(\texttt{Pol}) = \tau_{\texttt{W}}$ in $\mathbb{C}[\mathrm{F}_n]$. We omit the definition of the `combIndex` function which simply determines the desired index in x for a given subset I of $[1 \ldots n]$.

```
sl2CPol:=function(W,x,n)
  local S,W1,W2,nS,i,Pol,j,W3,W4;
  x[Length(x)]:=2;
  Pol:=0;
  nS:=NumberSyllables(W);
  if
    (ForAll([1..nS],i->
      ExponentSyllable(W,i)=1))
  and
    (ForAll([1..nS-1],i->
      GeneratorSyllable(W,i+1)-GeneratorSyllable(W,i)>0))
  then
    S:=[1..nS];
    for i in [1..nS] do
```

[1] the last entry of `x` will be set to 2, representing $\tau_{a_\emptyset}$

```
      S[i]:=GeneratorSyllable(W,i);
    od;
    Pol:=x[combIndex(S,n)];
  elif
    ForAny([2..nS],i->
      GeneratorSyllable(W,i) - GeneratorSyllable(W,1)=0)
  then
    i:=1;
    repeat i:=i+1;
      until GeneratorSyllable(W,i) - GeneratorSyllable(W,1)=0;
    W1:=SubSyllables(W,1,i-1);W2:=SubSyllables(W,i,nS);
    Pol:=sl2CPol(W1,x,n)*sl2CPol(W2,x,n)-sl2CPol(W1^-1*W2,x,n);
  elif
    ForAll ([1..nS],i->
      ExponentSyllable(W,i)=1)
  then
    i:=0;
    repeat i:=i+1;
      until GeneratorSyllable(W,i+1) - GeneratorSyllable(W,i) < 0;
    j:=0;
    repeat j:=j+1;
      until GeneratorSyllable(W,j) - GeneratorSyllable(W,i+1) > 0;
    W1:=SubSyllables(W,1,j-1);
    W2:=SubSyllables(W,j,i);
    W3:=SubSyllables(W,i+1,i+1);
    W4:=SubSyllables(W,i+2,nS);
    Pol:=sl2CPol(W3*W2,x,n)*sl2CPol(W1*W4,x,n) -
         sl2CPol(W2,x,n)*sl2CPol(W1*W3^-1*W4,x,n) +
         sl2CPol(W1*W3^-1*W2*W4,x,n);
  elif
    not ExponentSyllable(W,1)=1
  then
    W1:=Subword(W,2,Length(W));
    S:=[ ];
    S[1]:=GeneratorSyllable(W,1);
    Pol:=x[combIndex(S,n)]*sl2CPol(W1,x,n) -
          sl2CPol(Subword(W,1,1)^-1*W1,x,n);
  else
```

```
    Pol:=sl2CPol(Subword(W,2,Length(W))*Subword(W,1,1),x,n);
  fi;
  return Pol;
end;
```

Given a link-manifold N with $\pi_1 N = \langle a_1, \ldots, a_n \mid w \in W \rangle$ and a peripheral system $(\mu_T, \lambda_T)_{T \subset \partial N}$ as words in F_n, we use the previous code to output the settings for the algebraic computations performed later in M2. These are four lists of polynomials in $\mathcal{O}_n$, `Relations`, `Meridians`, `Longitudes`, `Products`, such that `Relations` is the list of polynomials $P_w - 2$ and $P_{a_i w} - P_{a_i}$ for each relator w and generator a_i and the three others are the list of peripheral polynomials P_{μ_T}, P_{λ_T} and $P_{\mu_T \lambda_T}$ for each peripheral torus T.

B.1.3 Character variety and peripheral $\mathcal{A}$-ideal

For the final step of the computation, we use M2 to perform the algebraic computations. The ring `B` is the ring $\mathcal{O}_n[\mathfrak{m}_T, \mathfrak{l}_T]$ and, using the settings described in the previous section, the following simple M2 code computes the components of the character variety and list of corresponding A_∂-ideals.

```
X = B / ideal (Relations|FreeIdeal);
dCV = decompose (ideal X);
nCV = #dCV;
dX = for i from 0 to (nCV-1)
  list(B/dCV_i);

periPolyElim = (m,p) -> m^2-p*m+1;

periElimList = for i from 0 to (nP-1)
  list(
    periPolyElim(Evariables_(2*i),Meridians_i),
    periPolyElim(Evariables_(2*i+1),Longitudes_i),
    periPolyElim(Evariables_(2*i)*Evariables_(2*i+1),Products_i)
    );
periElimIdeal = ideal periElimList;

dY = for i from 0 to (nCV-1)
  list(dCV_i + periElimIdeal);
```

```
dEvar = for i from 0 to (nCV-1)
  list(B / eliminate(Xvariables,dY_i));
```

So, after execution, dEvar is the list of ideals $\mathcal{A}_\partial(X)$ for each component X of $X(\pi)$.

In the next section, we present few $\mathcal{A}_\partial$-ideals obtained for different links using the algorithm described above. Links are named following the Thistlethwaite Link Table (see [BNMa]) or their common name if the have one (Hopf link, Whitehead link, etc...).

For each example, we also give the equations of the character variety in the coordinates $x_1, \dots x_{(2^n-1)}$ where n is the number of generators in the presentation we consider. For $n = 2$ the convention is

- $x_1 = Y_{\{1\}}$
- $x_2 = Y_{\{1,2\}}$
- $x_3 = Y_{\{2\}}$

and, for $n = 3$, we follow the notations of Equation (B.3).

B.2 Computed examples

B.2.1 Hopf link

The peripheral eigenvalue-variety of the Hopf link is easily computable by hand since the fundamental group of its exterior is $\mathbb{Z}^2$ where each longitude is a meridian of the other component.

We tested the algorithm with the following presentation and peripheral system (obtained with SnapPy):

```
Generators:
   a,b
Relators:
   abAB
[('a', 'Ab'), ('Ab', 'a')]
```

which gives, as expected, the following equation for the character variety:

$$x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 - 4$$

and produces the following decomposition for the $\mathcal{A}_\partial$-ideal of the Hopf link:

$$\mathcal{A}^{\text{Hopf}} = \langle L_1 - M_2, M_1 - L_2 \rangle \times \langle L_1 M_2 - 1, M_1 L_2 - 1 \rangle$$

which are the components of reducible characters of a 2-components link with linking number 1.

B.2.2 Link l4a1

The link l4a1 is Seifert-fibred so, again, we could compute the $\mathcal{A}_\partial$-ideal using the results Section 4.1.

We use the following presentation and peripheral system:

```
Generators:
   a,b
Relators:
   aBBAbb
[('Ba', 'baBa'), ('A', 'AAbb')]
```

We obtain two components in the character variety; the component of reducible characters is given by the same equation as the one obtained above for the Hopf-link, and the component of irreducible characters is simply given by $\langle x_3 \rangle$.

The $\mathcal{A}_\partial$-ideal is then given by:

$$\mathcal{A}^{\text{l4a1}} = \langle -M_2^2 + L_1, M_1^2 - L_2 \rangle \times \langle L_1 M_2^2 - 1, M_1^2 L_2 - 1 \rangle \times \langle M_2^2 + L_2, M_1^2 + L_1 \rangle$$

The first two components correspond to reducible representations (linking number 2) while the last ideal is produced by the component of irreducible characters, with the fibre sent to $-\mathrm{Id}$.

B.2.3 3-keychain link

Keychain links are the one obtained from the unknot by adding components parallel to the meridian (so the Hopf link is the 2-keychain link). One could use the Seifert fibration to describe the eigenvalue variety but we used it to test our algorithm on some 3-components link. It is the only 3-components links for which the computation terminated.

The presentation and peripheral system given by SnapPy is:

```
Generators:
   a,b,c
Relators:
   aBAb
   bCBc
[('B', 'CA'), ('C', 'B'), ('A', 'B')]
```

We obtain in $\mathbb{C}[x_1, \ldots, x_7]$ three ideals for the character variety. The first component (reducible components) is much more complicated than for 2-generated groups; it is given by twelve equations that we do not reproduce here[2]. On the other hand, the two components of irreducible representations, sending the fibre (here, the first meridian) to $\pm\mathrm{Id}$ are much simpler and given by

$$\langle x_5 - 2, -x_1 + x_2, x_6 - x_7, -x_3 + x_4\rangle \times \langle x_5 + 2, x_1 + x_2, x_6 + x_7, x_3 + x_4\rangle$$

We obtain the four $\mathcal{A}_\partial$-ideals for the reducible characters:

$$\begin{gathered}\langle L_2 - L_3, M_1L_3 - 1, L_1M_2M_3 - 1\rangle \\ \langle L_2 - L_3, -M_2M_3 + L_1, M_1 - L_3\rangle \\ \langle M_1 - L_2, L_2L_3 - 1, L_1M_3 - M_2\rangle \\ \langle M_1 - L_3, L_2L_3 - 1, L_1M_2 - M_3\rangle\end{gathered}$$

and two for the irreducible characters:

$$\begin{gathered}\langle L_3 - 1, L_2 - 1, M_1 - 1\rangle \\ \langle L_3 + 1, L_2 + 1, M_1 + 1\rangle\end{gathered}$$

The previous examples of $\mathcal{A}_\partial$-ideals could have been computed by hand, using the particular simple presentation of their fundamental groups and peripheral systems. As the following examples show, the complexity of the peripheral $\mathcal{A}$-ideals grows quite rapidly.

B.2.4 Whitehead link

The eigenvalue variety had already been computed by Tillmann in [Til02]. We present here the equations obtained using our algorithm.

The presentation and peripheral system obtained with SnapPy is:

[2]they are available at `http://hub.darcs.net/arbol/E-variety/Test/key3`

```
Generators:
   a,b
Relators:
   abAAAbbABaaaBB
[('AAb', 'AAbaBBab'), ('Ba', 'BaBabAAAba')]
```

and the component of irreducible characters is given by

$$-x_1^3x_3^2 + x_1^2x_2x_3 + x_1^3 + 2x_1x_3^2 - x_2x_3 - 2x_1$$

This produces the following $\mathcal{A}_\partial$-ideal:

$$\mathcal{A}^{\mathrm{W}} = \langle L_2 - 1, L_1 - 1\rangle \times \mathcal{A}^{\mathrm{irr}}$$

where $\mathcal{A}^{\mathrm{irr}}$ is given by the four generators:

$M_1^2L_1M_2^2 - M_1^2M_2^2L_2 + M_1^2L_1L_2 - L_1M_2^2L_2 - M_1^2 + M_2^2 - L_1 + L_2$

$M_1^4L_1L_2 + L_1M_2^4L_2 - 2M_1^2M_2^2L_2^2 - L_1M_2^4 - M_1^4L_2 + 2M_1^2M_2^2L_2 + 2M_1^2L_1L_2^2 - 2L_1M_2^2L_2^2 + L_1^2M_2^2 - 2M_1^2L_1L_2 + M_1^2L_2^2 - 2M_1^2L_2 + 2M_2^2L_2 + M_1^2 - M_2^2 - 2L_1L_2 + 2L_2^2 + L_1 - L_2$

$M_1^4M_2^2L_2 - M_1^2M_2^4L_2 - M_1^4L_2 + M_1^2M_2^2L_2 - M_1^2M_2^2 + M_2^4 - L_1M_2^2L_2 + M_1^2L_2^2 - L_1M_2^2 + M_2^2L_2$

$M_1^2M_2^6L_2 - M_1^2M_2^4L_2 - M_2^6 + L_1M_2^4L_2 + L_1M_2^4 - M_1^2M_2^2L_2 - M_2^4L_2 + L_1M_2^2L_2^2 + L_1M_2^2L_2 + M_1^2L_2 - M_2^2L_2 - L_2^2$

B.2.5 Link l6a1

The peripheral eigenvalue variety of this link had, to our knowledge, never been computed. SnapPy provides the following presentation and peripheral system:

```
Generators:
   a,b
Relators:
   aBAABabbbaBAbaabABBBAb
[('bba', 'BAbaabABabba'), ('AB', 'ABABabABBBAbaB')]
```

In this example, we obtain two components in the character variety (besides the component of reducible characters). One is given by

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 - 3$$

and the other, the geometric component, is the zero-set of the polynomial

$$x_1x_2x_3^2 - x_1^2x_3 - x_3^3 - x_1x_2 + x_3$$

This produces four ideals for the peripheral eigenvalue variety; two of reducible characters (linking number 2), and one for each other component. Although the first other component is quite simple, the last one is considerably more complicated.

$$\mathcal{A}_\partial^{\text{l6a1}} = \langle L_1M_2^2 - 1, M_1^2L_2 - 1\rangle \times \langle -M_2^2 + L_1, M_1^2 - L_2\rangle \times \langle L_2 - 1, L_1 - 1\rangle \times \mathcal{A}^0$$

where $\mathcal{A}_0$ is generated by the seven following polynomials:

$M_1^2L_1M_2^2 - M_1^2M_2^2L_2 + M_1^2L_1L_2 - L_1M_2^2L_2 - M_1^2 + M_2^2 - L_1 + L_2$

$-L_1M_2^6L_2^2 - M_1^4M_2^2L_2^3 + 2M_1^2M_2^4L_2^3 + 2L_1M_2^6L_2 + 2M_1^4M_2^2L_2^2 - 4M_1^2M_2^4L_2^2 + M_1^4L_1L_2^3 + 2L_1M_2^4L_2^3 - 3M_1^2M_2^2L_2^4 - L_1M_2^6 - M_1^4M_2^2L_2 + 2M_1^2M_2^4L_2 - 2L_1M_2^4L_2^2 + 2M_1^2M_2^2L_2^3 + 3M_1^2L_1L_2^4 - 3L_1M_2^2L_2^4 + L_1^2M_2^4 - M_1^4L_1L_2 - 2L_1M_2^4L_2 - 2M_1^4L_2^2 + 6M_1^2M_2^2L_2^2 - 2M_2^4L_2^2 - 2M_1^2L_1L_2^3 + 2L_1M_2^2L_2^3 + 2M_1^4L_2 - 6M_1^2M_2^2L_2 + L_1^2M_2^2L_2 + 4M_2^4L_2 - 3M_1^2L_1L_2^2 + 2L_1M_2^2L_2^2 - M_1^2L_2^3 + M_2^2L_2^3 + M_1^2M_2^2 - M_2^4 + 2M_1^2L_1L_2 - 4L_1M_2^2L_2 + 2M_2^2L_2^2 - 3L_1L_2^3 + 3L_2^4 + L_1M_2^2 + M_1^2L_2 - 2M_2^2L_2 + 2L_1L_2^2 - 2L_2^3 + L_1L_2 - L_2^2$

$M_1^6L_1^2L_2 + 2L_1M_2^6L_2^2 - 3M_1^2M_2^4L_2^3 - 2M_1^6L_1L_2 - 4L_1M_2^6L_2 + 2M_1^4L_1^2L_2^2 + 6M_1^2M_2^4L_2^2 + L_1^2M_2^4L_2^2 - 3L_1M_2^4L_2^3 + 2L_1M_2^6 + M_1^6L_2 - 2M_1^4L_1^2L_2 - 3M_1^2M_2^4L_2 - 2L_1^2M_2^4L_2 - 4M_1^4L_1L_2^2 + 2L_1M_2^4L_2^2 + 3M_1^2L_1^2L_2^3 + 6M_1^2M_2^2L_2^3 - 3L_1^2M_2^2L_2^3 - L_1^2M_2^4 + 4M_1^4L_1L_2 + 5L_1M_2^4L_2 + M_1^4L_2^2 - 4M_1^2L_1^2L_2^2 - 12M_1^2M_2^2L_2^2 + 4L_1^2M_2^2L_2^2 + 3M_2^4L_2^2 - 6M_1^2L_1L_2^3 + 6L_1M_2^2L_2^3 - L_1^3M_2^2 + M_1^2L_1^2L_2 + 6M_1^2M_2^2L_2 - 2L_1^2M_2^2L_2 - 6M_2^4L_2 + 5M_1^2L_1L_2^2 - 4L_1M_2^2L_2^2 - 2M_1^2L_2^3 + 3M_2^2L_2^3 - M_1^4 + M_2^4 + 4M_1^2L_1L_2 + 2L_1M_2^2L_2 + 6M_1^2L_2^2 - 3L_1^2L_2^2 - 12M_2^2L_2^2 + 3L_1L_2^3 - 3M_1^2L_1 + L_1M_2^2 - 4M_1^2L_2 + 4L_1^2L_2 + 6M_2^2L_2 + 2L_1L_2^2 - 6L_2^3 - 2L_1^2 - 3L_1L_2 + 5L_2^2$

$M_1^4M_2^4L_2^2 - M_1^2M_2^6L_2^2 - M_1^4M_2^4L_2 + M_1^2M_2^6L_2 - 2M_1^4M_2^2L_2^2 + 2M_1^2M_2^4L_2^2 + 2M_1^4M_2^2L_2 - 3M_1^2M_2^4L_2 + M_2^6L_2 + M_1^2M_2^4 - M_2^6 + L_1M_2^4L_2 + M_1^4L_2^2 - 3M_1^2M_2^2L_2^2 + M_2^4L_2^2 + L_1M_2^4 - M_1^4L_2 + 3M_1^2M_2^2L_2 - 3M_2^4L_2 + L_1M_2^2L_2^2 - M_1^2L_2^3 + L_1M_2^2L_2 + M_1^2L_2^2 - 2M_2^2L_2^2$

$-M_1^6M_2^2L_2^2+M_1^2M_2^6L_2^2+M_1^6M_2^2L_2-M_1^2M_2^6L_2+M_1^6L_1L_2^2+2L_1M_2^6L_2^2-3M_1^2M_2^4L_2^3+$
$M_1^6L_1L_2-4L_1M_2^6L_2-2M_1^4M_2^2L_2^2+5M_1^2M_2^4L_2^2-3L_1M_2^4L_2^3+3M_1^2M_2^2L_2^4+2L_1M_2^6-$
$2M_1^6L_2+3M_1^4M_2^2L_2-2M_1^2M_2^4L_2-M_2^6L_2+3L_1M_2^4L_2^2-3M_1^2M_2^2L_2^3-3M_1^2L_1L_2^4+$
$3L_1M_2^2L_2^4-M_1^4M_2^2-2L_1^2M_2^4+M_2^6+2L_1M_2^4L_2+3M_1^4L_2^2-5M_1^2M_2^2L_2^2+2M_2^4L_2^2+$
$3M_1^2L_1L_2^3-3L_1M_2^2L_2^3-3M_1^4L_2+7M_1^2M_2^2L_2-3L_1^2M_2^2L_2-4M_2^4L_2+3M_1^2L_1L_2^2-$
$2L_1M_2^2L_2^2+2M_1^2L_2^3-2M_1^2M_2^2-L_1^2M_2^2+2M_2^4-3M_1^2L_1L_2+6L_1M_2^2L_2-2M_2^2L_2^2+$
$3L_1L_2^3-3L_2^4-2L_1M_2^2+2M_2^2L_2-3L_1L_2^2+3L_2^3-2M_1^2+2M_2^2-2L_1+2L_2$

$M_1^4M_2^6L_2-M_1^4M_2^4L_2-M_1^2M_2^6L_2-M_1^2M_2^6-M_1^4M_2^2L_2-M_1^2M_2^4L_2-M_1^2M_2^4+M_2^6-$
$L_1M_2^4L_2-M_1^2M_2^2L_2^2-L_1M_2^4+M_1^4L_2-M_1^2M_2^2L_2+M_2^4L_2-L_1M_2^2L_2^2-L_1M_2^2L_2-$
$M_1^2L_2^2-M_1^2L_2+M_2^2L_2+L_2^2$

$M_1^6M_2^4L_2-2M_1^6M_2^2L_2-M_1^4M_2^4+M_1^6L_2-2M_1^4M_2^2L_2-M_1^2M_2^4L_2-M_1^4L_2^2-M_1^2M_2^2L_2^2-$
$L_1M_2^2L_2^2-M_1^2M_2^2+M_2^4-2L_1M_2^2L_2-L_1M_2^2-M_1^2L_2+2M_2^2L_2+L_2^2$

Bibliography

[Ber71] G. Bergman. The logarithmic limit-set of an algebraic variety. *Trans. Amer. Math. Soc*, 157, 1971.

[BG86] Robert Bieri and John R. J. Groves. A rigidity property for the set of all characters induced by valuations. *Trans. Amer. Math. Soc*, 294(2):425–434, 1986.

[BNMa] Dror Bar-Natan, Scott Morrison, and al. The Knot Atlas. `http://katlas.math.toronto.edu/wiki/`.

[BZ98] S. Boyer and X. Zhang. On Culler-Shalen seminorms and Dehn filling. *The Annals of Mathematics*, 148:737–801, 1998.

[BZ05] Steven Boyer and Xingru Zhang. Every nontrivial knot in $\mathbb{S}^3$ has nontrivial A-polynomial. *Proceedings of the American Mathematical Society*, 133(9):2813–2815 (electronic), 2005.

[CCG$^+$94] D. Cooper, M Culler, H. Gillet, D. D. Long, and P B Shalen. Plane curves associated to character varieties of 3-manifolds. *Inventiones Mathematicae*, 118(1):47–84, 1994.

[CDW] M Culler, N M Dunfield, and J R Weeks. SnapPy, a computer program for studying the geometry and topology of 3-manifolds. http://snappy.computop.org.

[CS83] Marc Culler and Peter B. Shalen. Varieties of group representations and splittings of 3-manifolds. *The Annals of Mathematics*, 117(1):109–146, 1983.

[DG04] Nathan M Dunfield and Stavros Garoufalidis. Non-triviality of the A-polynomial for knots in $\mathbb{S}^3$. *Algebraic & Geometric Topology*, 4:1145–1153 (electronic), 2004.

[EN85] David Eisenbud and Walter David Neumann. *Three-dimensional Link Theory and Invariants of Plane Curve Singularities*. Princeton University Press, 1985.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Technical report.

[Hat82] A.E. Hatcher. Boundary curves of incompressible surfaces. *Pacific J. Math.*, 99(2):373–377, 1982.

[Hat10] Allen Hatcher. Notes on Basic 3-Manifold Topology. `http://www.math.cornell.edu/~hatcher/3M/3Mdownloads.html`, pages 1–72, 2010.

[HP04] Michael Heusener and Joan Porti. The variety of characters in $\mathrm{PSL}_2(\mathbb{C})$. *Sociedad Matemática Mexicana. Boletín. Tercera Serie*, 10(Special Issue):221–237, 2004.

[KM04] P B Kronheimer and T S Mrowka. Dehn surgery, the fundamental group and $\mathrm{SU}(2)$. *Mathematical Research Letters*, 11(5-6):741–754, 2004.

[Las93] A E Lash. *Boundary Curve Space of the Whitehead Link Complement*. University of California, Santa Barbara, 1993.

[Mar10] S Marlow. Haskell 2010 language report. *Available online* `http://www.haskell.org/`*(May 2011)*, 2010.

[Mat92] Yves Mathieu. Unknotting, knotting by twists on disks and property P for knots in $\mathbb{S}^3$. In *Knots 90 (Osaka, 1990)*, pages 93–102. de Gruyter, Berlin, 1992.

[MS01] Brian S Mangum and Theodore Stanford. Brunnian links are determined by their complements. *Algebraic & Geometric Topology*, 1(1):143–152, 2001.

[NR93] Walter D Neumann and Alan W. Reid. Rigidity of cusps in deformations of hyperbolic 3-orbifolds. *Mathematische Annalen*, 295(2):223–237, 1993.

[NZ85] Walter D Neumann and Don Zagier. Volumes of hyperbolic three-manifolds. *Topology*, 24(3):307–332, 1985.

[PV94] V L Popov and E B Vinberg. Invariant Theory. In *Algebraic Geometry IV*, pages 123–278. Springer Berlin Heidelberg, Berlin, Heidelberg, 1994.

[SB77] Jean Pierre Serre and Hyman Bass. Arbres, amalgames, SL2. Société mathématique de France, 1977.

[Sei33] H Seifert. Topologie Dreidimensionaler Gefaserter Räume. *Acta Mathematica*, 60(1):147–238, 1933.

[Ser03] Jean Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[Sik13] Adam S Sikora. Generating sets for coordinate rings of character varieties. *Journal of Pure and Applied Algebra*, 217(11):2076–2087, 2013.

[The12] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.4.12. Technical report, 2012.

[Thu02] William P Thurston. *The geometry and topology of 3-manifolds*. `http://library.msri.org/books/gt3m/`, March 2002.

[Til02] Stephan Tillmann. *On caracter varieties*. PhD thesis, 2002.

[Til05] Stephan Tillmann. Boundary slopes and the logarithmic limit set. *Topology*, 44(1):203–216, 2005.

[Uni13] Univalent Foundations Program, The. *Homotopy Type Theory*. `http://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.