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Essays on Individual Choice

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ABSTRACT

This thesis can be divided into two (unrelated) parts. The main part (Chapters 1 and 2) focus on addiction models that entail departures from the classical discounting utility model of Individual Intertemporal Choice: Habit-Formation and Self-Control problems. The other part (Chapter 3) studies the famous p-Beauty Contest Game when we restrict the individual's choices to integer numbers.

In the first part, habit formation is the key feature for a product being addictive: a habit is created when past consumption of the product increases current desire for consumption. An addiction can be either beneficial (when past consumption increases current utility, e.g. jogging) or harmful (when past consumption decreases current utility, e.g. drug consumption). In general one could conceive of harmful addictions as habit-forming activities that imply an immediate reward but generate future costs (negative internalities) whereas beneficial addictions imply an immediate cost but generate future rewards (positive internalities). Self-control problems are understood in terms of time inconsistency: they arise when the individual cannot keep up with an intended intertemporal plan of consumption.

In Chapter 1 we analyse a (harmful) addiction model proposed by O'Donoghue and Rabin (O&R) for which they obtain a counterintuitive result: full awareness of self-control problems may exacerbate over-consumption. We show that this result arises from their particular equilibrium selection for the induced intrapersonal game. We provide dominating Markov Perfect equilibria where the paradox vanishes and that seem more "natural" since they capture behaviours often observed in the realm of addiction. We also address the issue of why a person could decide to start consuming and possibly develop an addiction: contrary to O&R, and according to the common intuition, we show that naiveté is at the essence.

In Chapter 2 we obtain an isomorphism between harmful and beneficial addictions in a discrete-time binary choice context (the model of the first chapter being a particular case of this context). The equivalence thus established allows us to study both phenomena (harmful and beneficial addictions) as two sides of the same coin. Besides the theoretical insight it provides, this dualism is also useful: in particular, it permits to readily translate the results obtained in the first chapter to the domain of beneficial addictions. Once the dualism is established, we analyse addictions under both time-consistent and time-inconsistent preferences.

In Chapter 3, we provide a full characterization of the pure-strategy Nash Equilibria for the p-Beauty Contest Game when we restrict individual's choices to integer numbers. Opposed to the case of real number choices, equilibrium uniqueness may be lost depending on the value of p and the number of players: in particular, as p approaches 1 any symmetric profile constitutes a Nash Equilibrium. We also show that any experimental p-Beauty Contest Game can be associated to a game with the integer restriction and thus multiplicity of equilibria becomes an issue. Finally, we show that in these games the iterated deletion of weakly dominated strategies may not lead to a single outcome while the iterated best-reply process always does (though the outcome obtained depends on the initial conditions).

A mis padres y hermanas,

sin cuyo amor y apoyo esto no hubiese sido posible.

Contents

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Chapter 1

Addiction and Self-Control: an Intrapersonal Game*

1.1 Introduction

Mr. XY goes to a party where he is offered a pill - call it Panacea - by Miss XX. He knows for sure that if he takes it he will experience immediate pleasure but he is also aware that some of his neuronal cells will pay for his decision. Since most humans, and particularly XY, use but a small fraction of their brain capacity, he might as well give up those neuronal cells without experiencing a significant loss. However, he is also aware that pill would lead to some more which altogether will produce a certain brain clash. Should I stay or should I go? XY asked to himself. As the indecision bothered him, he evaluated whether the immediate pleasure offset the future brain damage and proceeded accordingly.

We may not know whether XY took Panacea or not, but we certainly know that

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he is a rational forward-looking person: when adopting his decision he knew the future consequences of his choice. Mr. XY was perfectly aware of the two characteristics constituting the crux of an addictive substance, namely, the *habit-forming* property (present pill raising future consumption); and the *negative internalities* induced by consumption (present pill reducing future well-being, via the brain clash).

In their famous work, Becker and Murphy (1988) modeled consumption of a good presenting these two features as a rational process where addiction is understood as the outcome of intended behavior (i.e., intertemporal utility maximization) under perfect foresight. In particular, their Rational Addiction model implies that an addict does not regret his previous decisions and perfectly forecasts his future consumption; two elements that have largely been criticized (surveys of these critics are found in Chaloupka and Warner 1998 and Messinis 1999). On psychological grounds, addiction certainly entails planned behavior but it also involves self-control problems that give rise to regret and misprediction of future conduct. This is clearly illustrated by Heyman (1996):

> Drug consumption is a goal oriented act. The behaviors are learned, not reflexive or innate. It requires planning, effort, and in some cases artfulness to secure drugs in the amount necessary for maintaining an addiction. Yet, according to the diagnostic manuals (e.g., DSM-III-R and ICD-10), the feature that defines addiction is drug use which is 'out of control' or 'compulsive'. By these phrases, the manuals mean that addicts 'take more drug that they initially intended', that drug use persists despite a wide array of ensuing legal, medical, and social problems, and that after periods of abstinence, however long, addicts relapse.

As Gruber and Koszegi (2001) point out, "The term 'rational addiction' obscures the fact that the Becker and Murphy model imposes two assumptions on consumer behavior. The first is that of forward-looking decision-making, which is hard to impugn (...). The second is the assumption that consumers are time consistent. Psychological evidence documents overwhelmingly that consumers are time inconsistent" (page 16).

Recently and in different contexts, many economists have studied self-control problems modeling them in terms of the time inconsistency derived from non-exponential discounting[1]. Supported by empirical evidence showing that subjects exhibit declining discount rates (e.g. Thaler (1991); Loewenstein and Prelec (1992)), most of these studies use hyperbolic discounting (for an excellent review on hyperbolic discounting and time preference see Frederick, Loewenstein and O'Donoghue (2002)).

O'Donoghue and Rabin (2002) (from now on O&R) combine the Becker and Murphy approach with hyperbolic discounting[2] in their modeling of addiction. In their framework, an infinitely lived individual[3] has to decide at each period whether to consume or not a free addictive product; i.e. a product presenting the habit-forming and negative internalities features. As in the Becker-Murphy model, the individual is perfectly aware of these two features. But due to the time inconsistent preferences embodied in hyperbolic discounting the individual may not be able to follow his optimal consumption path thus giving rise to self-control problems. Concerning the awareness of these problems, they distinguish two extreme types of individuals: naifs, who are totally unaware; and sophisticates, who are

[1]This approach to model self-control problems derives from the pioneering work by Strotz (1956) who noted that when using a non-exponentional discount function intertemporal utility gives rise to time inconsistency in the sense that an optimal plan at some particular date may no longer be optimal at further dates. However, self-control problems may also be modeled while maintaining time consistency (i.e. exponential discounting). For instance, Laibson (2001) and Bernheim and Rangel (2003) model self-control problems by introducing cue-conditioned behavior. In their models, environmental cues may trigger a "hot" mode in which the individual consumes the addictive substance disregarding its future consequences (i.e. she "looses control").

[2]The specific discounting functional form they use (which we formally present in Section 2) is not really hyperbolic but it captures the essence of hyperbolic discounting, namely, present-biased preferences. It was first introduced by Phelps and Pollack (1968) and because of its simplicity and tractability, it has been widely used to model self-control problems since the work of Laibson (1994).

[3]They also treat the case of an individual with finite horizon but mainly as a means to understand the infinite horizon case. Indeed, in the context of addiction, an infinite horizon seems a much better approximation of real behavior.

perfectly aware. A naif believes that his future selves will follow his optimal consumption path thus choosing his current action accordingly. But because of his time inconsistent preferences, his future selves will often revise the optimal plan hence yielding a different path from the one intended. As a consequence, a naïf usually falls in over-consumption (note that this result captures Heyman's description "addicts take more drug that they initially intended"). A sophisticate knows that the optimal consumption path he is aiming at may be revised by his future selves and thus may not be followed. Therefore he chooses his current action according to the best path that can be pursued by his future selves. In a sense, a sophisticate is playing an intrapersonal game: he plays against his future selves. The solution concept they propose is that of *perception-perfect strategy equilibrium*[4] (from now on we will refer to it as the ORE). However, the ORE has the shortcoming of producing a counterintuitive result: under some circumstances, sophisticates will consume always (i.e. become addicted) while naifs might not. As they point out, this "contradicts the common intuition that harmful addictions are caused by people naively slipping into an unplanned addiction". Following O&R we will refer to those circumstances as the *inevitability condition* (IC from now on).

In the present chapter we show first that this conterintuitive result is obtained by their particular equilibrium selection (ORE) and that there are more "natural" dominating equilibria where the paradox vanishes; i.e. where sophisticates are less prone to become addicted than naifs. Since in an intrapersonal game the players are just incarnations of the *same* individual, coordination on a dominated equilibrium cannot be supported and

[4]In the induced game with a finite horizon T, there is a unique subgame perfect equilibrium; call it T-equilibrium. In the infinite horizon case, a perception-perfect strategy equilibrium is simply the limit of the sequence of T-equilibria as T becomes long.

therefore we argue that the ORE is not the appropriate selection. Secondly, we address the issue of *developing an addiction,* that is, we analyze the circumstances under which an unnadicted person could decide to start consuming and whether she could become addicted or not. In particular, we show that the ORE solution is of no use for studying this issue since it implies that both naifs and sophisticates will slip into addiction. In contrast, by considering our results, naifs will become addicted while sophisticates will not which is in accordance to the common intuition cited above. Finally, we suggest a very clear-cut way of modeling partial awareness of self-control problems.

The importance of our findings can be motivated in terms of policy implications. Consider for example a public advertising campaign providing information on self-control problems induced by drug consumption. What such a campaign would normally do is a shift from naiveness to sophistication given that people become aware of their time-inconsistency. Under our results such a campaign would be successful in reducing addiction (since sophisticates are less prone to become addicted than naifs) while under the O&R result it would produce the opposite effect. Wide existence of such campaigns favors our results.

The chapter proceeds as follows. In Sections 1.2 and 1.3 we formally present the O&R model and their results, stating clearly under which circumstances the paradox is obtained. In Section 1.4 we study equilibria in cutoff strategies by providing a complete characterization: in particular, we state conditions under which the ORE generates the paradox and yet there is a dominating cutoff equilibrium that solves it. But cutoff equilibria may not exist or may not solve the paradox, therefore, in Section 1.5, we provide non-cutoff dominating equilibria which solve it whenever generated. In Section 1.6 we address the issue of developing an addiction and argue that the ORE solution fails to explain this issue while our results prove to be in accordance to the common intuition. Section 1.7 concludes and suggests a "natural" way of modeling partial awareness of self-control problems, a topic that so far has received very little attention in the literature.

1.2 The O&R Model

An infinitely lived individual decides at each period t, whether to consume (hit) or not (refrain) a free addictive product. Let a_t be the binary variable reflecting the individual's choice at time t: $a_t = 1$ meaning he decides to hit whereas $a_t = 0$ means he decides to refrain.

His period-t instantaneous utility is given by

$$\forall t, u\left(k_t, a_t\right) = \begin{cases} x + f\left(k_t\right) & \text{if} \quad a_t = 1 \\ g\left(k_t\right) & \text{if} \quad a_t = 0 \end{cases} \tag{1.1}$$

where k_t is the individual's level of addiction which captures all the effects of past consumption on current instantaneous utility. The level of addiction is assumed to evolve according to $k_{t+1} = \gamma k_t + a_t$ with $0 < \gamma < 1$. Therefore, there is a maximal addiction level $k^{\max} = \frac{1}{1-\gamma}$. Note that instantaneous utility is stationary in the sense that it depends on the prevailing level of addiction at period t but not on the particular period t. Addiction is modeled by making the following assumptions on f, g and x :

> **Assumption 1:** $f', g' < 0$. This assumption introduces the feature of negative internalities since the more a person has consumed in the past (as captured by his addiction level) the lower his current instantaneous utility. Without loss of generality, it is assumed $f(0) = g(0) = 0$.
>
> **Assumption 2:** $f' - g' > 0$. This assumption introduces the habit-forming feature. To see this, let $h(k) = x + f(k) - g(k)$ be the temptation to hit (i.e.

the marginal instantaneous utility of hitting). Then $h'(k) > 0$ implies that hitting is more desirable the higher the level of addiction; i.e. past consumption of the product (as captured by the addiction level) increases current desire for consumption.

Assumption 3: $f'', g'' \geq 0$. In addition to negative internalities and habit-forming, it is assumed that the more addicted a person becomes the less a given increase in k hurts his instantaneous utility, and therefore less harm hitting induces in future utility.

Assumption 4: $x > 0$. This assumption says that the temptation to hit is positive even for an unaddicted person.

Self-control problems are modeled by assuming present-biased preferences as in the Phelps and Pollak intertemporal utility function given by:

$$U(u_t, u_{t+1}, ...) = u_t + \beta \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u_\tau \quad \text{with } \beta \in (0,1) \tag{1.2}$$

where each u_τ is the period-τ instantaneous utility given by (1.1). The parameter β introduces the present bias.

Before stressing out the implications of (1.2) it is useful to consider the case of a typical intertemporal utility function with exponential discounting, i.e. (1.2) with $\beta = 1$. Following O&R we will refer to a rational forward-looking person having such preferences as a time consistent individual (TC).

1.2.1 Time Consistent Individuals (TC)

A TC's preferences are given by

$$U(u_t, u_{t+1}, ...) = u_t + \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u_\tau \tag{1.3}$$

Definition 1 *A behavior path $A = (a_1, a_2, ...)$ is an infinite sequence of admissible actions; i.e. $\forall i, a_i \in \{0, 1\}$.*

Three particular behavior paths are of special interest and we will label them as follows: the hitting path $H = (1, 1, ...)$; the refraining path $R = (0, 0, ...)$; and the hitting once path $O = (1, 0, 0, ...)$. Let $U_{tc}^{A}(k_t)$ denote the intertemporal utility given in (1.3) associated to following the behavior path A from an initial addiction level k_t (the stationary instantaneous utility function implies that the unique payoff relevant variable at any date t is the prevailing addiction level). Being rational forward-looking amounts to saying that at any given period t a TC solves

$$\max_{A \in \{0,1\}^{\infty}} U_{tc}^{A}(k_t) \tag{1.4}$$

and then chooses the first action corresponding to the solution path . We will refer to such a solution as a desired behavior path (DBP). For a TC there is time consistency: for any starting addiction level k_t a DBP at some date t is still optimal at any further date. Therefore a TC has no self-control problems since future selves have no incentives to deviate from a DBP chosen by a previous self.

O&R show that under stationary instantaneous utility there exists a critical addiction level k^{tc} such that each self solves (1.4) by choosing to hit if and only if $k_t \geq k^{tc}$. As a consequence, a TC's DBP is either hitting always or refraining always. We state this result as a proposition:

Proposition 2 $\exists k^{tc} \in [0, k^{\max}]$ *such that the DBP for a TC with starting addiction level* k *is* H *if* $k \geq k^{tc}$ *and* R *otherwise.*

We turn now to study the consequences of the preferences given by (1.2).

1.2.2 Individuals with Time Inconsistent Preferences (TI): Naifs and Sophisticates

Let $U^{A}(k_t)$ be the intertemporal utility given in (1.2) associated to following the behavior path A from an initial addiction level k_t . A rational forward-looking individual with time inconsistent preferences aims at solving

$$\max_{A\in\{0,1\}^{\infty}} U^{A}(k_t) \tag{1.5}$$

However, in this case there is time-inconsistency: a DBP (a behavior path solving (1.5)) at date t may no longer be optimal at a further date, in the sense that future selves may have incentives to deviate from it thus giving rise to self-control problems. O&R distinguish two types of individuals with preferences induced by (1.2): Naifs, who are totally unaware of their time-inconsistency; and Sophisticates who are fully aware of their time-inconsistency. A Naif believes that he has no self control problems, that is, he believes that any optimal plan he chooses will be followed by his future selves. Thus, at any given period, a naif simply chooses his current action according to the path solving (1.5), but the chosen path may be systematically revised at further periods. A Sophisticate is perfectly aware of his self-control problems, he knows that the path he is aiming at may be revised by his future selves and thus may not be followed. Therefore the best he can do is to maximize (1.5) subject to the condition that the chosen path will be followed by his future selves. A sophisticate is thus playing an intrapersonal game where his opponents are his future selves.

We turn now to study the DBP for a TI. First notice that a TI would like to behave like a TC from next period on . Therefore, given Proposition 2, a TI's DBP (i.e. a path A solving (1.5)) must involve either hitting always or refraining always from next period on.

This leaves us with only four possibilities for the DBP, we discard one of them with the following Lemma.

Lemma 3 $A = (0, 1, 1, ...)$ *cannot be the DBP for a TI.*

Proof. Suppose $\exists k$ such that A is the respective DBP. Then it must be true that $k > \gamma k \geq k^{tc}$ and therefore $U_{tc}^{H}(k) > U_{tc}^{A}(k)$. But then

$$U_{tc}^{H}(k) = u(1,k) + \delta U_{tc}^{H}(\gamma k + 1) > u(0,k) + \delta U_{tc}^{H}(\gamma k) = U_{tc}^{A}(k)$$

which implies

$$u(0,k) - u(1,k) < \delta \left(U_{tc}^{H}(\gamma k + 1) - U_{tc}^{H}(\gamma k) \right) \leq \beta\delta \left(U_{tc}^{H}(\gamma k + 1) - U_{tc}^{H}(\gamma k) \right)$$

where the last inequality follows from $U_{tc}^{H}(\gamma k + 1) - U_{tc}^{H}(\gamma k) \leq 0$. Therefore

$$U^{H}(k) = u(1,k) + \beta\delta U_{tc}^{H}(\gamma k + 1) > u(0,k) + \beta\delta U_{tc}^{H}(\gamma k) = U^{A}(k)$$

A contradiction. ■

Lemma 3 implies the following proposition.

Proposition 4 *For any given starting addiction level k, the DBP of a TI admits only one of the following possibilities (we assume he hits when indifferent): H, R or O.*

Proposition 5 *Let A be any behavior path. Then*

1. $U^{A}(k)$ is decreasing.

2. $\forall k$, $\frac{\partial U^{H}(k)}{\partial k} \geq \frac{\partial U^{A}(k)}{\partial k} \geq \frac{\partial U^{R}(k)}{\partial k}$

Part 1 follows directly from negative internalities while part 2 obtains mainly from the habit-forming assumption (it also requires convexity of f and g or at least them being not too much concave). Note that, in particular, part 2 implies $\frac{\partial U^{H}(k)}{\partial k} \geq \frac{\partial U^{O}(k)}{\partial k} \geq \frac{\partial U^{R}(k)}{\partial k}$. The formal proofs are given in the appendix.

Following O&R, we will define now three important levels of addiction:

- k^{HR}: addiction level such that always hitting is preferred to always refrain if and only if $k \geq k^{HR}$. Formally, let $\widetilde{k}$ be the solution to $U^{H}(k) = U^{R}(k)$, then $k^{HR} = \max\left[0, \widetilde{k}\right]$ follows from Proposition 5, part 2.

- k^{OR}: addiction level such that hitting once is preferred to always refrain if and only if $k \geq k^{OR}$. Formally, let $\widetilde{k}$ be the solution to $U^{O}(k) = U^{R}(k)$, then $k^{OR} = \max\left[0, \widetilde{k}\right]$ follows from Proposition 5, part 2.

- k^{HO}: addiction level such that hitting always is preferred to hitting once if and only if $k \geq k^{HO}$. Formally, let $\widetilde{k}$ be the solution to $U^{H}(k) = U^{O}(k)$, then $k^{HO} = \max\left[0, \widetilde{k}\right]$ follows from Proposition 5, part 2

Remember that the law of motion of k implies a maximum addiction level $k^{\max} = \frac{1}{1-\gamma}$. According to the formal definitions of k^{HR}, k^{OR} and k^{HO} it could be the case that some of them are above $k^{\max}$. We will say that k^{HR}, k^{OR} and k^{HO} exist if all of them are below $k^{\max}$. Because of Proposition 5, part 2, existence of k^{HR}, k^{OR} and k^{HO} is equivalent to requiring $U^{H}(k^{\max}) \geq U^{O}(k^{\max}) \geq U^{R}(k^{\max})$. We will assume throughout that this condition holds.

As O&R point out, in general, k^{HR} and k^{OR} are not rankable so we will usually distinguish two cases as shown in Figure 1.

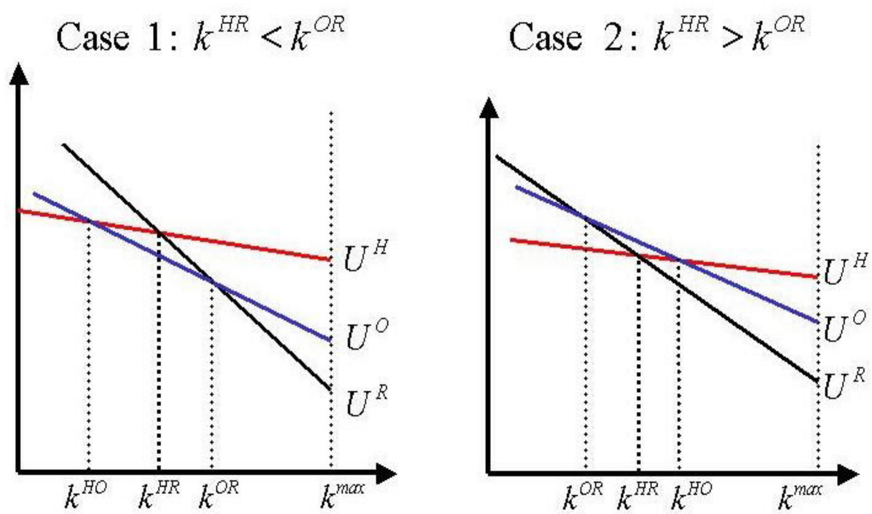


Figure 1: k^{HR} and k^{OR} are not rankable

From Propositions 4 and 5 we obtain

Proposition 6 *a TI's DBP is*

1. R; for any $k < \min\left[k^{HR}, k^{OR}\right]$

2. H; for any $k \geq \max\left[k^{HR}, k^{HO}\right]$

3. O; for any $k \in \left[k^{OR}, k^{HO}\right)$

1.3 The O&R Results

In the previous section we established the DBP for both TC and TI individuals. We are interested now in determining the realized behavior path (RBP), that is, the path actually followed for each type of individuals. This amounts to specifying the actions to be undertaken by an individual in any particular situation. Because of the stationarity of utility functions and the infinite horizon it seems natural to make those actions time-independent: at a particular date, the action of an individual should depend only on the prevailing addiction level since this is the only payoff relevant variable; the calendar time is irrelevant. Since both a TC (correctly) and a Naif (wrongly) believe that they are able to follow their respective DBP, O&R show that they implement cutoff actions. We state their results in the following propositions.

Proposition 7 *Let $\alpha^{tc}(k)$ be the action taken by a TC when his addiction level is k. Then, $\exists k^{tc} \in [o, k^{\max}]$ such that $\alpha^{tc}(k) = 1 \Longleftrightarrow k \geq k^{tc}$. Therefore, the RBP of a TC is either H or R.*

Proposition 8 *Let $\alpha^{n}(k)$ be the action taken by a Naif when his addiction level is k. Then, $\alpha^{n}(k) = 1 \Longleftrightarrow k \geq \min\left\{k^{OR}, k^{HR}\right\} = k^{n}$. Therefore, the RBP of a Naif is either H or R.*

Proposition 7 comes directly from Proposition 2. Proposition 8 comes from the fact that a naif, believing that he is going to be able to follow his DBP, will decide to hit if and only if his DBP is either H or O. But this happens if and only if $k \geq \min\left\{k^{HR}, k^{OR}\right\} = k^{n}$. O&R also show that $k^{n} \leq k^{tc}$, an intuitive result since a naif discounts the future at a higher rate than a TC and therefore the future harm of hitting is lower for a naif than for a TC.

Let's turn now to the sophisticate case. Because of his awareness of self-control problems, a sophisticate is involved in strategic considerations. The natural solution concept to be called upon for the sophisticate's intrapersonal game is that of Markov Perfect Equilibrium (MPE). Among the multiple MPE for the infinite horizon case, O&R only consider the one corresponding to the limit of the unique finite-horizon MPE as the horizon becomes long. From now on we will refer to this equilibrium as the ORE. The RBP generated by this particular equilibrium selection depends heavily on whether the following condition is satisfied or not.

1.3.1 ORE and the Inevitability Condition (IC)

We say that IC holds if and only if $U^{H}(0) \geq U^{(0,1,1,\ldots)}(0)$. Let $\alpha^{s}(t,k)$ denote the strategy played by a sophisticated self-t in the ORE. Notice that we are allowing for the strategy to depend on the particular period t. This is so because with a finite horizon the strategy usually depends on the prevailing period and nothing ensures us that when taking the limit as the horizon becomes long we obtain a time-independent strategy. O&R completely characterize the ORE when IC holds and partially when it is not satisfied. We state their results in the following proposition.

Proposition 9 *Partial Characterization of the ORE.*

1. If IC holds then $\forall t, k, \alpha^{s}(t,k) = 1$; i.e. the sophisticate's RBP in the ORE is H.

2. If IC is not satisfied then

   (a) If $\gamma k^{OR} + 1 \geq k^{HR}$ then $\alpha^{s}(t,k) = 1$ if and only if $k \geq k^{HR}$

   (b) If $\gamma k^{OR} + 1 < k^{HR}$ then $\alpha^{s}(t,k) = \begin{cases} 0 & \text{if} \quad k < k^{OR} \\ ? & \text{if} \quad k^{OR} \leq k < k^{HR} \\ 1 & \text{if} \quad k \geq k^{HR} \end{cases}$

Two striking features of the ORE are to be mentioned. Concerning part 1 notice that when IC holds a sophisticate is more prone to become addicted than a naif since a sophisticate will always hit while a naif might not (given Proposition 8, a naif will always hit if and only if $k \geq \min\left\{k^{HR}, k^{OR}\right\}$). As O&R point out, this "contradicts the common intuition that harmful addictions are caused by people naively slipping into an unplanned addiction". However, we claim that this counterintuitive result is obtained by the particular equilibrium selection proposed by O&R (i.e. the ORE) and that it vanishes when considering other type of MPE. Moreover, we claim that there are more "natural" MPE where a sophisticate, even under IC, will never be more prone than a naif to develop an addiction. We will address this issue in the following sections.

Concerning part 2, notice that the counterintuitive result vanishes: a sophisticate will never be more prone than a naif to develop an addiction. However, the ORE is left uncharacterized for $k \in \left[k^{OR}, k^{HR}\right)$. This characterization is a very complicated task: as O&R point out, for this case "sophisticates' behavior can be quite complicated...In fact, because of these complications sophisticates need not follow a stationary strategy or a cutoff strategy". However, we will provide conditions under which a cutoff strategy is a MPE. We also construct non-cutoff equilibria that seem very natural (even under IC).

Since IC plays such an important role in the O&R results, we close this section by proving part 1 of Proposition 9 which in fact is an immediate consequence of the following Lemma whose proof is provided in the appendix.

Lemma 10 *IC implies $\forall k, U^H(k) \geq U^{(0,1,1,...,1)}(k)$ for H and $(0,1,1,...,1)$ paths of arbitrary length T .*

To see that Lemma 10 implies $\forall t, k, \alpha^s(t,k) = 1$, consider a finite horizon T. In period T a sophisticated will hit independently of his addiction level because the instantaneous utility from hitting is always bigger than the one from refraining and there are obviously no future costs of hitting. Self-$T-1$, knowing that self-T will hit no matter what he does, only has to choose between paths $(1,1)$ and $(0,1)$. But Lemma 10 says that $(1,1)$ is preferred for any prevailing addiction level at period $T-1$ and therefore self-$T-1$ will hit independently of his addiction level. Proceeding by backward induction we obtain that a sophisticate will hit in every period. Since this holds for an arbitrarily path length T, in the limit we obtain that a sophisticate will always hit; i.e. ORE generates path H.

Since the ORE solution proves to be unsatisfactory, we turn now to study other sort of MPE. Because TCs and naifs follow cutoff actions, we begin by studying MPE in cutoff strategies for sophisticates.

1.4 MPE in Cutoff Strategies (CE)

In this section we will characterize CE, i.e. MPE where all selves play the same cutoff strategy

$$\overline{\alpha}(k) = \begin{cases} 0 & \text{if} \quad k < \overline{k} \\ 1 & \text{if} \quad k \geq \overline{k} \end{cases}$$

Lemma 11 *for $\overline{k} = 0$, $\overline{\alpha}(k)$ is a CE if and only if IC holds.*

Proof. Suppose $\overline{\alpha}(k)$ is a CE and consider an unaddicted self deviating to some strategy prescribing $\alpha(0) = 0$. The path generated by deviating is $(0,1,1,...)$ while the path generated by sticking is H. For the deviation to be non-profitable we need $U^{H}(0) \geq U^{011\cdots}(0)$, i.e. IC must hold. Now suppose IC holds. Then $U^{H}(0) \geq U^{011\cdots}(0)$ which implies $\forall k, U^{H}(k) \geq U^{011\cdots}(k)$ and therefore no deviation from $\overline{\alpha}(k)$ is profitable. ■

Lemma 12 *if* $k^{HR} \leq k^{OR}$*,* $\overline{\alpha}(k)$ *with* $\overline{k} = k^{HR}$ *is a CE and it dominates any other equilibrium.*

Proof. First note that $\forall k < k^{HR}$ the DBP is R whereas $\forall k \geq k^{HR}$ the DBP is H. Therefore, by sticking to strategy $\overline{\alpha}(k)$ each self follows his DBP which proves that it is a dominating equilibrium. ■

Remark 13 *Since in an intrapersonal game the players are different incarnations of the same individual, we believe that equilibrium selection should be resolved, whenever possible, by a Pareto criterion. If there is not a Pareto dominant equilibrium, at least it should be obvious that a dominated equilibrium should not be played. In the case* $k^{HR} \leq k^{OR}$*,* $\overline{\alpha}(k)$ *with* $\overline{k} = k^{HR}$ *pareto-dominates any other equilibrium so it would be quite unnatural to propose any other solution to this game. However, when IC holds, the O&R solution yields the hitting path : They argue that this is sustainable if each self has the pessimistic beliefs that his future selves will hit no matter what his current action is, thus choosing to hit since the path* H *yields a higher utility than* $(0, 1, 1, ...)$*. But why should every self have those pessimistic beliefs when they can coordinate on a dominating equilibrium? We believe this is a major drawback of the ORE.*

In a sense, Lemma 12 says that whenever $k^{HR} \leq k^{OR}$ the are no self-control problems since for any addiction level the DBP for a particular self will be followed by his futures selves. As a consequence, awareness of time-inconsistency is immaterial: the paths followed by a naif and a sophisticate are the same since the solution to (1.5) at any period and for any addiction level is still optimal at further periods.

We will now characterize CE when $k^{HR} > k^{OR}$ (notice that this implies $k^{HR} > 0$). In what follows we will assume that every self is playing the same cutoff strategy $\overline{\alpha}$. We will denote by $V(\overline{\alpha}(k))$ the utility obtained by a self with addiction level k when sticking to $\overline{\alpha}$, whereas $V(\alpha(k))$ will denote his utility when deviating to a particular strategy α while all other selves stick to $\overline{\alpha}$. Throughout we will assume $\overline{k} > 0$ since the case $\overline{k} = 0$ has already been covered in Lemma 11.

Claim 14 *If $\overline{\alpha}$ is a CE then $\overline{k} = k^{HR}$.*

Proof. Suppose $\overline{k} > k^{HR}$ and consider an addiction level $k \in \left(k^{HR}, k^{OR}\right)$. Any strategy α prescribing $\alpha(k) = 1$ is a profitable deviation since $V(\overline{\alpha}(k)) = U^R(k) \leq U^H(k) = V(\alpha(k))$. Now suppose $\overline{k} < k^{HR}$ and consider an addiction level k such that $\gamma k < \overline{k} < k < k^{HR}$ (such a k exists since $\overline{k} > 0$). Any strategy α prescribing $\alpha(k) = 0$ is a profitable deviation since $V(\overline{\alpha}(k)) = U^H(k) \leq U^R(k) = V(\alpha(k))$. ■

Lemma 15 *If $k^{HR} > k^{OR}$, strategy $\overline{\alpha}(k)$ with $\overline{k} = k^{HR}$ is a CE if and only if $\gamma k^{OR} + 1 \geq k^{HR}$.*

Proof. Take any $k \geq k^{HR}$ and consider deviating to a strategy α prescribing $\alpha(k) = 0$. The path generated by this deviation is either R or $(0, 1, 1, ...)$. If R is generated, the deviation is non profitable since $V(\alpha(k)) = U^R(k) \leq U^H(k) = V(\overline{\alpha}(k))$. Now consider $(0, 1, 1, ...)$ being generated. In the appendix we prove that $U^{(0,1,1,...)}(k) \leq U^H(k)$ for any $k \geq k^{OR}$. Since this is the case we have $V(\alpha(k)) = U^{(0,1,1,...)}(k) \leq U^H(k) = V(\overline{\alpha}(k))$ and therefore the deviation is non-profitable.

For any $k < k^{OR}$ there is no profitable deviation since sticking to strategy $\overline{\alpha}$ generates the DBP. So consider any $k \in \left[k^{OR}, k^{HR}\right)$. If $\gamma k^{OR} + 1 \geq k^{HR}$, a deviation to any strategy α prescribing $\alpha(k) = 1$ generates path H thus being non profitable since $V(\alpha(k)) = U^H(k) \leq U^R(k) = V(\overline{\alpha}(k))$. This proves $\gamma k^{OR} + 1 \geq k^{HR}$ implies $\overline{\alpha}$ is a CE.

Now suppose $\gamma k^{OR} + 1 < k^{HR}$. Take any $k \in \left(k^{OR}, k^{HR}\right)$ such that $\gamma k + 1 < k^{HR}$ and consider a deviation to a strategy α prescribing $\alpha(k) = 1$. The deviation is profitable since it generates path O and $V(\alpha(k)) = U^O(k) > U^R(k) = V(\overline{\alpha}(k))$. ■

Remark 16 *Suppose IC does not hold and $\gamma k^{OR} + 1 < k^{HR}$. Under these circumstances O&R claim that for $k \in \left[k^{OR}, k^{HR}\right)$ "sophisticates need not follow a cutoff strategy" . Here we go further since Lemma 11 and Lemma 15 imply that under these circumstances sophisticates **cannot** follow a cutoff strategy because there is no CE. Also notice that if IC holds and $\gamma k^{OR}+1 \geq k^{HR} > k^{OR}$ there are exactly two CE : the one with $\overline{k} = 0$ and the one with $\overline{k} = k^{HR}$. The latter clearly dominates the former so one would expect the different selves to coordinate in the second equilibrium while O&R take as solution to the game the first one.*

We summarize our results concerning CE in the following Proposition.

Proposition 17 *Characterization of CE and comparison with ORE.*

1. If IC holds then

   (a) If $\gamma k^{OR} + 1 \geq k^{HR}$ then $\overline{\alpha}(k)$ with $\overline{k} = k^{HR}$ and ORE are the only CE. $\overline{\alpha}(k)$dominates ORE.

   (b) If $\gamma k^{OR} + 1 < k^{HR}$ then ORE is the unique CE.

2. If IC does not hold then

(a) If $\gamma k^{OR} + 1 \geq k^{HR}$ then $\overline{\alpha}(k)$ with $\overline{k} = k^{HR}$ is the unique CE. $\overline{\alpha}(k)$ and ORE coincide.

(b) If $\gamma k^{OR} + 1 < k^{HR}$ then there exists no CE.

If we are to restrict our attention to CE, the case $\gamma k^{OR} + 1 < k^{HR}$ is a problematic one. If IC holds then we have the ORE solution which proved to be counterintuitive since sophistication could exacerbate over-consumption. If IC doesn't hold then we may have non-existence of CE. This leads us to study non-cutoff equilibria which is done in the following section.

1.5 Non-cutoff Equilibria: some examples.

In what follows, we still restrict ourselves to MPE where the strategies involved are time-independent. We also focus on the case $\gamma k^{OR} + 1 < k^{HR}$ since the opposite has already been covered.

1.5.1 IC holds: "I won't hit because if I do it I will do it forever".

Here we go one step further in resolving IC since we provide a MPE that dominates ORE whenever IC holds.

Lemma 18 *strategy* $\widehat{\alpha}(k) = \begin{cases} 0 & if \quad k = 0 \\ 1 & if \quad k > 0 \end{cases}$ *is a MPE.*

Proof. For $k = 0$, sticking to $\widehat{\alpha}$ generates path R while deviating generates path H. Since $0 \leq k^{OR} < k^{RH}$ the deviation is non profitable. For any $k > 0$ a deviation is neither profitable since it generates path $(0, 1, 1, ...)$ which is clearly dominated by H (Lemma 10). ■

Remark 19 *$\widehat{\alpha}$ dominates the ORE though it does it strictly only for $k = 0$: when sticking to $\widehat{\alpha}$ an unaddicted sophisticate will never develop an addiction, moreover any self is strictly better-off by sticking to $\widehat{\alpha}$ since it generates his DBP. However, $\widehat{\alpha}$ has the shortcoming of being non robust or fragile, since any small deviation by any self develops the addiction. Nevertheless, we claim that this equilibrium deserves attention since it captures strategic decisions often observed in the realm of harmful addictive drugs: Decisions of the sort "I won't do it because if I do it I will do it forever" (think of drugs such as heroin).*

1.5.2 "Take a walk on the wild side"

We already know that a CE fails to exist if IC is not satisfied and is equal to the ORE otherwise (thus being dominated by naive behavior).

Let $\overline{k} = \frac{k^{HR}-1}{\gamma}$, so that hitting with any $k \geq \overline{k}$ drives the addiction level above k^{HR}. Consider each self following strategy

$$\alpha^{wws}(k) = \begin{cases} 0 & \text{if} \quad k < k^{OR} \\ 1 & \text{if} \quad k^{OR} \leq k < \overline{k} \\ 0 & \text{if} \quad \overline{k} \leq k < k^{HR} \\ 1 & \text{if} \quad k \geq k^{HR} \end{cases}$$

which tries to capture the following idea illustrated in Figure 2:

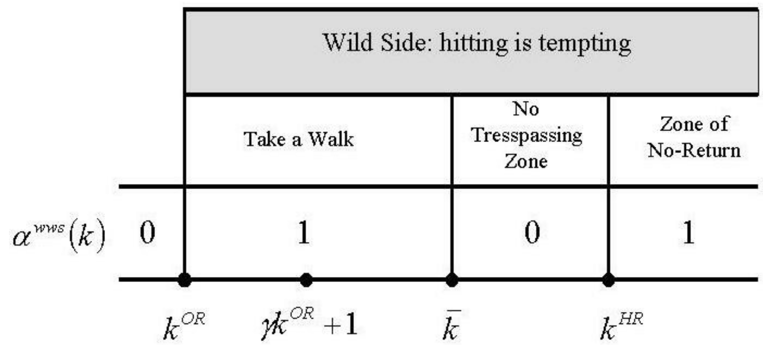


Figure 2: The “Take a Walk on the Wild Side” strategy.

There is a *wild side* ($k \geq k^{OR}$) in which hitting is an effective temptation (for $k < k^{OR}$ hitting is not effectively tempting because always refrain is the DBP) and therefore the individual would like to hit (take a walk). In this *wild side* there is a *zone of no-return* ($k \geq k^{HR}$) since once an individual falls in it he becomes irremediably addicted (he hits forever after). Now notice that hitting leads to the zone of no-return if and only if $k \geq \overline{k}$, therefore $\overline{k}$ marks the point where a *no-trespassing zone* ($\overline{k} < k < k^{HR}$) begins. With strategy α^{wws} the individual hits whenever on the wild side and outside the no-trespassing zone, i.e. he takes a walk but knows when to stop.

Unfortunately, α^{wws} does not always constitute an equilibrium. A counterexample is given in the appendix (Example 1.9.1) . However, we will provide some conditions under which α^{wws} happens to be a MPE.

Lemma 20 *if* $\gamma k^{HR} < k^{OR} < \overline{k} < \gamma\left(\gamma k^{OR}+1\right)+1$ *then* α^{wws} *is a MPE. (This Lemma is illustrated in Figure 3. In the appendix we provide examples satisfying the condition stated with the IC holding, Example 1.9.2, and not holding, Example 1.9.3)*

Proof. For any addiction level k such that $k < k^{OR}$ or $k \geq k^{HR}$, α^{wws} generates the respective DBP so there is no profitable deviation. For any k such that $k^{OR} \leq k < k^{HR}$, by sticking to α^{wws} the DBP is generated (and thus there is no profitable deviation). To see this first notice that the DBP is hitting once. Following α^{wws} the individual hits and then drives the addiction level above $\overline{k}$ but below k^{HR}, therefore his immediate future self will refrain and, since by doing so he drives the addiction level below k^{OR}, all other future selves will refrain as well. For any k such that $\overline{k} \leq k < k^{HR}$, α^{wws} generates path R, while any deviation generates path H. Since R is preferred to H we conclude that there is no profitable deviation.

We tackle now the case $k^{HR} \leq k < k^{HO}$. First notice that necessarily $k^{HO} = \frac{k^{tc}-1}{\gamma}$. Let $k_0 = k^{HO}$, define $k_1 = \frac{k_0-1}{\gamma}$ and suppose the current self has addiction level $k \in [k_1, k_0) \cap \left[k^{HR}, k_0\right)$. Clearly, by sticking to α^{wws} hitting with addiction level k generates path H . A deviation, i.e. refraining with addiction level k, drives next-period's addiction level below $\gamma k_0 = k^{tc} - 1 < k^{tc}$ so from the current self's perspective the best possible behavior path following restraint is R (because he would like to behave like a TC from next period on and a TC would like to refrain always for addiction levels below k^{tc}). That is, the best possible behavior path that could be generated by a deviation is R. Since for addiction level k, H is preferred to R we conclude that there is no profitable deviation. But proceeding by induction, the same logic applies for any $k \in [k_{i+1}, k_i) \cap \left[k^{HR}, k_0\right)$ where $k_{i+1} = \frac{k_i-1}{\gamma}$ $(i = 0, 1, ...)$ so we finally conclude that there is no profitable deviation for any k such that $k^{HR} \leq k \leq k^{HO}$. ■

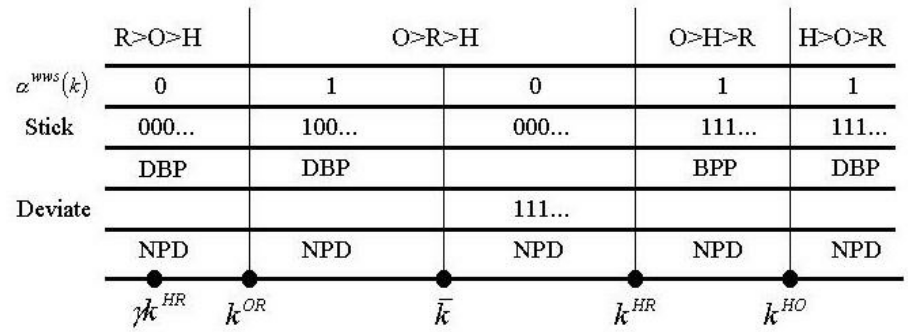


Figure 3: If $\gamma k^{HR} < k^{OR} < \overline{k} < \gamma \left(\gamma k^{OR} + 1\right) + 1$ then α^{wws} is a MPE.

Remark 21 *If IC does not hold (Example 1.9.3) we know that there is no cutoff equilibrium, α^{wws} proves existence of a non-cutoff MPE. If IC holds (Example 1.9.2), we know that the unique cutoff equilibrium is hitting always, then α^{wws} constitutes an equilibrium that clearly dominates it, moreover, α^{wws} also dominates the fragile equilibrium $\widehat{\alpha}$ given in Lemma 18.*

1.6 Developing an Addiction: $k^{HR} > k^{OR} = 0$

O&R state (page 4) "While Becker and Murphy (1988) argue it can be optimal for a person to *maintain* a severely harmful addiction, their steady-state model provides no formal analysis of why the person would choose to *develop* this harmful addiction in the first place." If we are to study why a person could choose to develop an addiction the pertinent starting addiction level must be $k = 0$; i.e. we must focus on the behavior of an unaddicted person. Suppose $k^{OR} > 0$. If $k^{HR} \geq k^{OR}$ we are in a situation as the one depicted in Figure 4.

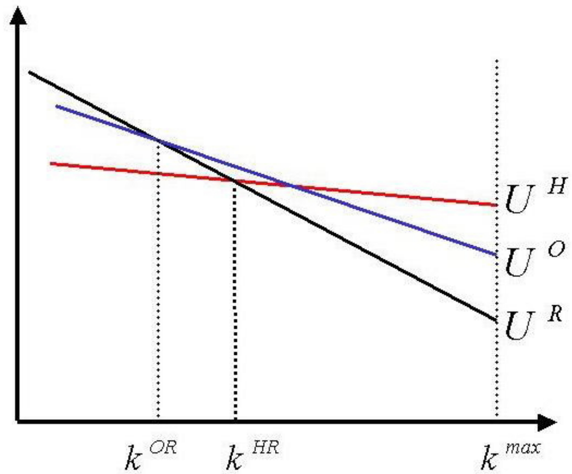


Figure 4: $k^{HR} \geq k^{OR}$

In this case the DBP for an unaddicted person is clearly R, and therefore the addiction will never be developed since the addictive product is not "tempting": no incarnation wants to consume it. If $k^{HR} < k^{OR}$ we should distinguish two cases; $k^{HR} > 0$ and $k^{HR} = 0$; which are illustrated in Figure 5.

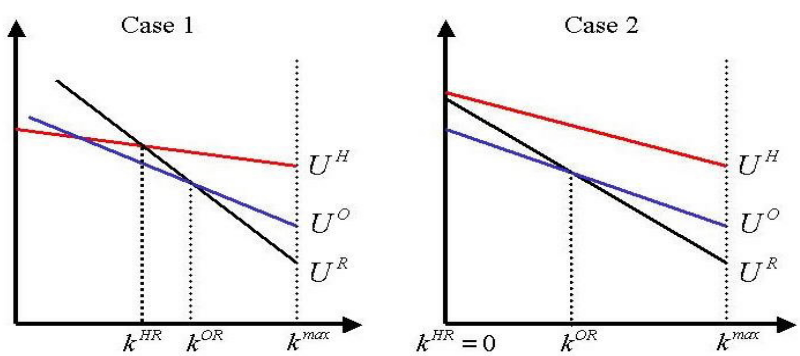


Figure 5: When $k^{HR} < k^{OR}$ we distinguish two cases; $k^{HR} > 0$ and $k^{HR} = 0$

In Case 1 we also obtain the non-tempting condition that ensures that the addiction will never be developed. In Case 2, the DBP for an unaddicted person is H and the addiction is developed (every self will decide to hit). However, in this case there are no self-control problems: each self is following his DBP which amounts to saying that the person is a "happy addict" in the sense that each incarnation behaves precisely as the previous selves desired; each self wants to consume and wants his future selves to consume as well. Therefore, the interesting case (the one presenting self-control problems) for studying the development of an addiction must involve $k^{HR} > k^{OR} = 0$ as depicted in Figure 6. (the case $k^{HR} = k^{OR} = 0$ is similar to Case 2).

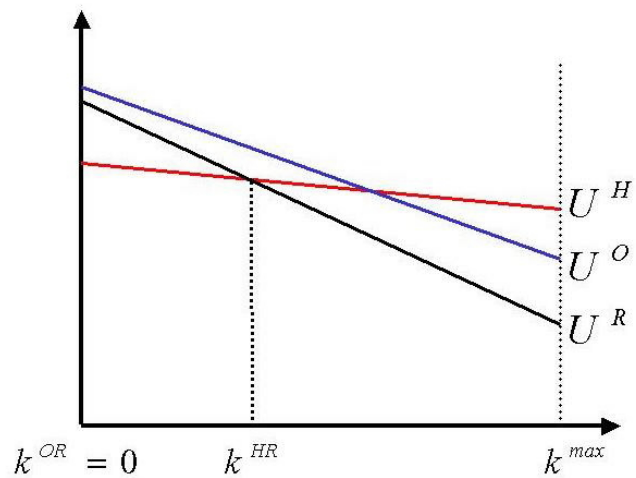


Figure 6: $k^{HR} > k^{OR} = 0$

In this case the unaddicted person would like to hit just once (his DBP is O). If the person is a Naif, he will clearly develop the addiction. We turn now to study a sophisticate's behavior.

First notice that IC holds which implies that in the ORE a sophisticate will also develop an addiction. Are there other equilibria in which a sophisticate does not develop an addiction? The answer is yes since the strategy "I won't hit because if I do it I will do it forever" provided in the previous section is clearly a MPE that induces the refraining path and therefore dominates the ORE. The following Lemma shows an equilibrium that dominates the ORE and generates the hitting once path.

Lemma 22 *Suppose* $k^{HR} \geq 1$ *and consider strategy*

$$\alpha(k) = \begin{cases} 1 & if \quad k = 0 \\ 0 & if \quad k = \gamma^i \text{ for some } i \in \{0, 1, 2, ...\} \\ 1 & if \quad otherwise \end{cases}$$

This strategy is a MPE that generates path O.

Proof. When $k = 0$, sticking to $\alpha(k)$ generates path O which is the DBP, therefore there is no profitable deviation. For any $k = \gamma^i$ with $i \in \{0, 1, 2, ...\}$, sticking to $\alpha(k)$ generates path R. Now fix i and notice that $\gamma(\gamma^i) + 1 > \gamma^j$ for any $j \in \{0, 1, 2, ...\}$, therefore deviating from $\alpha(k)$ generates path H which is non-profitable. For any other k sticking to $\alpha(k)$ generates path H (because for any $j \in \{0, 1, 2, ...\}$, $\gamma k + 1 > \gamma^j$) while deviating generates path $(0, 1, 1, ...)$ (because if $k \neq \gamma^i$ for all $i \in \{0, 1, 2, ...\}$, then $\gamma k \neq \gamma^j$ for all $j \in \{0, 1, 2, ...\}$) which is non-profitable since IC holds. ■

This equilibrium dominates the ORE, but compared to the strategy "I won't hit because if I do it I will do it forever" the initial self (unaddicted person) is strictly better-off while any future self is strictly worse-off.

1.7 Discussion.

The O&R set-up seems appropriate for modeling addiction since it incorporates the two basic features of an addictive substance (habit-formation and negative internalities) and it allows for self-control problems which have been largely documented in the psychological literature. This is an improvement with respect to the Becker-Murphy model of rational addiction in which self-control problems were inexistent. However, their particular equilibrium selection (ORE) for the intrapersonal game induced by sophisticated behavior has the shortcoming of producing a counterintuitive result: awareness of self-control problems may exacerbate over-consumption.

We have shown that this paradox vanishes when considering other sort of equilibria that dominate the ORE and that seem more natural since they capture behaviors often observed in the realm of addiction. Since in an intrapersonal game the players are incarnations of the same individual, coordination on a dominated equilibrium is hard to sustain[5]. This favors our equilibria over the ORE and therefore we readily obtain that naifs are more prone to become addicted than sophisticates. The only cases where the ORE is a dominating equilibrium (and therefore it is the appropriate solution concept to be called upon) is when the desired behavior path is to consume always or when IC does not hold and $\gamma k^{OR} + 1 \geq k^{HR}$.

Another advantage of the O&R set up over the Becker-Murphy model is that it permits to explain why an unaddicted person could decide to consume and develop an addiction. We have seen that for this to be possible consumption should be tempting (in the sense that the desired behavior path cannot be refraining) in which case a naïf will always become addicted. Regarding sophisticate behavior, we proved that when the DBP is either hit always or hit once then the inevitability condition must hold and therefore the ORE implies that a sophisticate will also become addicted. This makes sense only

[5] Carrillo and Mariotti (2000) obtain a similar conclusion. In their model the consumer is uncertain about the degree of addictiveness of the product but may acquire free information which eventually reveals the true degree. In the finite-horizon version each self decides to acquire the information, therefore, when considering the equilibrium of the infinite horizon case that is the limit of the finite case equilibrium, each self decides to get fully informed. But there are other dominating equilibria (refered to as strategic ignorance equilibria) where the selves decide not to get fully informed.

when the DBP is hit always since in this case there are no self-control problems and thus naïf and sophisticate behavior coincide. But when the DBP is hitting once, we provide equilibria where a sophisticate will not become addicted which copes with the general view that addiction is the outcome of naïve behavior.

Naiveness and sophistication are extreme degrees of awareness and one would expect that real-world behaviors lie somewhere in between. We want to conclude by suggesting a way to model partial awareness. Very little has been done in this direction: O'Donoghue and Rabin (2001a) formulate an approach to partial naivete in which a partially naïf agent is simply a sophisticate who overestimates his present-biased parameter β. O'Donoghue and Rabin (2001b) propose an approach to boundedly rational incomplete awareness in which agents "don't do all the rounds of backwards-induction. In other words, instead of starting the backwards-induction logic in the last period, they might start the process, say, three periods hence.". We believe that the first approach is somehow ad-hoc while the second is not applicable to the infinite horizon case since it relies on the backwards-induction logic. We suggest a very natural approach: people are initially naïf and as time elapses they become aware of their self-control problems (i.e. they become sophisticates). This approach is also suggested by Elster (1999): "reversal experiences can give rise to learning. Once the person observes himself reversing his decisions time and again, he will come to know that this is just the way he behaves under these circumstances. In the language of O'Donoghue and Rabin, he is no longer naive, but sophisticated." Obviously there would be persons that become aware more quickly than others; to be more precise, we could define an agent as a t-naif when it takes him t periods to become aware of his time inconsistency. With this formulation, naifs and sophisticates are ∞-naifs and 0-naifs respectively. This would allow to observe behaviors which imply hitting for a finite number of times: an example (Example 1.9.4) is provided in the appendix.

1.8 Appendix - Proofs

1.8.1 Proposition 5

Proof.

1. $U^A(k)$ is decreasing. Let $k_t(A,k)$ be the addiction level prevailing at time t conditional on following path A with starting addiction level k; i.e. $k_1(A,k) = k$; $k_t(A,k) = \gamma^{t-1}k + \sum_{i=1}^{t-1}\gamma^{t-i-1}a_i$ for $t = 2, 3, ...$ Then

$$U^A(k) = \begin{bmatrix} a_1(x + f(k)) + \\ (1-a_1)\, g(k) \end{bmatrix} + \beta \sum_{t=2}^{\infty} \delta^{t-1} \begin{bmatrix} a_t(x + f(k_t(A,k))) + \\ (1-a_t)\, g(k_t(A,k)) \end{bmatrix}$$

   and therefore

$$\frac{\partial U^A(k)}{\partial k} = \begin{bmatrix} a_1 f'(k) + \\ (1-a_1)\, g'(k) \end{bmatrix} + \beta \sum_{t=2}^{\infty} \delta^{t-1} \begin{bmatrix} a_t \gamma^{t-1} f'(k_t(A,k)) + \\ (1-a_t)\, \gamma^{t-1} g'(k_t(A,k)) \end{bmatrix}$$

   which is equal to

$$\begin{bmatrix} a_1 f'(k) + \\ (1-a_1)\, g'(k) \end{bmatrix} + \beta\delta \sum_{t=1}^{\infty} \delta^{t-1}\gamma^t \begin{bmatrix} a_{t+1} f'\left(\gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i\right) + \\ (1-a_{t+1})\, g'\left(\gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i\right) \end{bmatrix} \quad (1.6)$$

   and because of negative internalities (Assumption 1: $f', g' < 0$) we readily obtain $\frac{\partial U^A(k)}{\partial k} \leq 0$.

2. $\forall k$, $\frac{\partial U^H(k)}{\partial k} \geq \frac{\partial U^A(k)}{\partial k} \geq \frac{\partial U^R(k)}{\partial k}$. Since

$$U^H(k) = x\left[\frac{1-\delta+\beta\delta}{1-\delta}\right] + f(k) + \beta\delta\sum_{t=1}^{\infty}\delta^{t-1}f\left(\gamma^t k + \sum_{i=0}^{t-1}\gamma^i\right)$$

we have

$$\frac{\partial U^H(k)}{\partial k} = f'(k) + \beta\delta\sum_{t=1}^{\infty}\delta^{t-1}\gamma^t f'\left(\gamma^t k + \sum_{i=0}^{t-1}\gamma^i\right) \tag{1.7}$$

from (1.6) and (1.7) we can express $\frac{\partial U^H(k)}{\partial k} - \frac{\partial U^A(k)}{\partial k}$ as the sum of the following terms

$$(1-a_1)\left(f'(k) - g'(k)\right) \tag{1.8}$$

$$\beta\delta\sum_{t=1}^{\infty}\delta^{t-1}\gamma^t a_{t+1}\begin{bmatrix} f'\left(\gamma^t k + \sum_{i=0}^{t-1}\gamma^i\right) - \\ f'\left(\gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i\right) \end{bmatrix} \tag{1.9}$$

$$\beta\delta\sum_{t=1}^{\infty}\delta^{t-1}\gamma^t(1-a_{t+1})\begin{bmatrix} f'\left(\gamma^t k + \sum_{i=0}^{t-1}\gamma^i\right) - \\ g'\left(\gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i\right) \end{bmatrix} \tag{1.10}$$

(1.8) is positive because of the habit-forming feature (Assumption 2: $f' - g' > 0$); (1.9) is positive because $\gamma^t k + \sum_{i=0}^{t-1}\gamma^i > \gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i$ and $f'' \geq 0$ (Assumption 3) imply

$$f'\left(\gamma^t k + \sum_{i=0}^{t-1}\gamma^i\right) - f'\left(\gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i\right) \geq 0 \tag{1.11}$$

(1.10) is positive because (1.11) and $f' - g' > 0$ imply

$$\begin{aligned} 0 \quad < \quad & f'\left(\gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i\right) - g'\left(\gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i\right) \leq \\ & f'\left(\gamma^t k + \sum_{i=0}^{t-1}\gamma^i\right) - g'\left(\gamma^t k + \sum_{i=1}^{t}\gamma^{t-i}a_i\right) \end{aligned}$$

therefore we have

$$\forall k, \frac{\partial U^H(k)}{\partial k} - \frac{\partial U^A(k)}{\partial k} \geq 0 \tag{1.12}$$

Since $U^R(k) = g(k) + \beta\delta \sum_{t=1}^{\infty} \delta^{t-1} g(\gamma^t k)$ we have

$$\frac{\partial U^R(k)}{\partial k} = g'(k) + \beta\delta \sum_{t=1}^{\infty} \delta^{t-1} \gamma^t g'(\gamma^t k) \tag{1.13}$$

from (1.6) and (1.13) we can express $\frac{\partial U^A(k)}{\partial k} - \frac{\partial U^R(k)}{\partial k}$ as the sum of the following terms

$$a_1 \left(f'(k) - g'(k) \right) \tag{1.14}$$

$$\beta\delta \sum_{t=1}^{\infty} \delta^{t-1} \gamma^t (1 - a_{t+1}) \left[\begin{array}{c} g'\left(\gamma^t k + \sum_{i=1}^{t} \gamma^{t-i} a_i\right) - \\ g'(\gamma^t k) \end{array} \right] \tag{1.15}$$

$$\beta\delta \sum_{t=1}^{\infty} \delta^{t-1} \gamma^t a_{t+1} \left[\begin{array}{c} f'\left(\gamma^t k + \sum_{i=1}^{t} \gamma^{t-i} a_i\right) \\ -g'(\gamma^t k) \end{array} \right] \tag{1.16}$$

(1.14) is positive by the habit-forming feature; (1.15) is positive because

$$\gamma^t k + \sum_{i=1}^{t} \gamma^{t-i} a_i \geq \gamma^t k \text{ and } g'' \geq 0 \text{ (Assumption3)}$$

imply

$$g'\left(\gamma^t k + \sum_{i=1}^{t} \gamma^{t-i} a_i\right) \geq g'(\gamma^t k) \tag{1.17}$$

and (1.16) is positive because (1.17) and $f' - g' > 0$ imply

$$0 \leq g'\left(\gamma^t k + \sum_{i=1}^{t} \gamma^{t-i} a_i\right) - g'(\gamma^t k) \leq f'\left(\gamma^t k + \sum_{i=1}^{t} \gamma^{t-i} a_i\right) - g'(\gamma^t k) \tag{1.18}$$

therefore we have

$$\forall k, \frac{\partial U^A(k)}{\partial k} - \frac{\partial U^R(k)}{\partial k} \geq 0 \tag{1.19}$$

combining (1.12) and (1.19) completes the proof.

■

1.8.2 Claim used in Lemma 15

Claim 23 $\exists k^{'}$ *such that* $U^{H}(k) \geq U^{(0,1,1,\ldots)}(k) \Leftrightarrow k \geq k^{'}$. *Moreover,* $k^{'} \leq k^{OR}$.

Proof. If IC holds, we obtain trivially $k^{'} = 0 \leq k^{OR}$. So assume that IC doesn't hold, that is, $U^{H}(0) < U^{(0,1,1,\ldots)}(0)$. Define $\Delta(k) = U^{H}(k) - U^{(0,1,1,\ldots)}(k)$. Simple calculations yield

$$\Delta(k) = x + f(k) - g(k) + \beta\delta\sum_{n=1}^{\infty}\delta^{n-1}\left[\begin{array}{c} f\left(\gamma^{n}k + \sum_{i=0}^{n-1}\gamma^{i}\right) - \\ f\left(\gamma^{n}k + \sum_{i=0}^{n-2}\gamma^{i}\right)\end{array}\right]$$

Now notice that $\Delta(k)$ is continuous and increasing in k (by $f^{'} - g^{'} < 0$ and convexity of f). Since $\Delta(0) < 0$ (because IC doesn't hold) and $\Delta(k^{\max}) \geq 0$ such a $k^{'}$ exists, and we must have $\Delta\left(k^{'}\right) = 0$ which can be rewritten as

$$x + f\left(k^{'}\right) - g\left(k^{'}\right) = \beta\delta\sum_{n=1}^{\infty}\delta^{n-1}\left[\begin{array}{c} f\left(\gamma^{n}k^{'} + \sum_{i=0}^{n-2}\gamma^{i}\right) - \\ f\left(\gamma^{n}k^{'} + \sum_{i=0}^{n-1}\gamma^{i}\right)\end{array}\right] \tag{1.20}$$

Suppose now $k^{'} > k^{OR}$. By definition of k^{OR} we must have $U^{O}\left(k^{'}\right) > U^{R}\left(k^{'}\right)$ which is equivalent to

$$x + f\left(k^{'}\right) - g\left(k^{'}\right) > \beta\delta\sum_{n=1}^{\infty}\delta^{n-1}\left[\begin{array}{c} g\left(\gamma^{n}k^{'}\right) - \\ g\left(\gamma^{n}k^{'} + \gamma^{n-1}\right)\end{array}\right] \tag{1.21}$$

Substracting (1.20) from (1.21) we obtain

$$0 > \beta\delta\sum_{n=1}^{\infty}\delta^{n-1}\left[\begin{array}{c} g\left(\gamma^{n}k^{'}\right) - g\left(\gamma^{n}k^{'} + \gamma^{n-1}\right) + \\ f\left(\gamma^{n}k^{'} + \sum_{i=0}^{n-1}\gamma^{i}\right) - f\left(\gamma^{n}k^{'} + \sum_{i=0}^{n-2}\gamma^{i}\right)\end{array}\right] \tag{1.22}$$

by convexity of g we have

$$g\left(\gamma^{n}k^{'}\right) - g\left(\gamma^{n}k^{'} + \gamma^{n-1}\right) \geq g\left(\gamma^{n}k^{'} + \sum_{i=0}^{n-2}\gamma^{i}\right) - g\left(\gamma^{n}k^{'} + \sum_{i=0}^{n-1}\gamma^{i}\right)$$

which together with (1.22) yields

$$0 > \beta\delta \sum_{n=1}^{\infty} \delta^{n-1} \left[\begin{array}{c} g\left(\gamma^n k' + \sum_{i=0}^{n-2} \gamma^i\right) - g\left(\gamma^n k' + \sum_{i=0}^{n-1} \gamma^i\right) + \\ f\left(\gamma^n k' + \sum_{i=0}^{n-1} \gamma^i\right) - f\left(\gamma^n k' + \sum_{i=0}^{n-2} \gamma^i\right) \end{array} \right]$$

which in turn can be rewritten as

$$0 > \beta\delta \sum_{n=1}^{\infty} \delta^{n-1} \left[h\left(\gamma^n k' + \sum_{i=0}^{n-1} \gamma^i\right) - h\left(\gamma^n k' + \sum_{i=0}^{n-2} \gamma^i\right) \right] \tag{1.23}$$

where $h = f - g$. But this is a contradiction because h is increasing and therefore the RHS of (1.23) is positive. This completes the proof. ■

1.9 Appendix - Examples

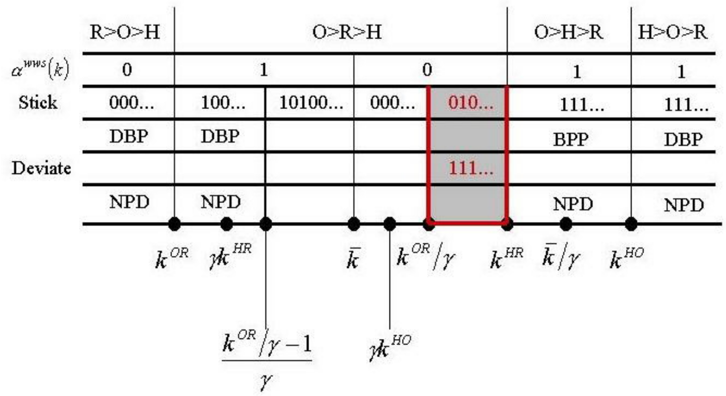
All the numeric examples that follow use the specific instantaneous utility function:

$$\forall t, u(k_t, a_t) = \begin{cases} x - \rho k_t & \text{if} \quad a_t = 1 \\ -(\rho + \sigma) k_t & \text{if} \quad a_t = 0 \end{cases}$$

with x, ρ and σ positive, which clearly satisfies the conditions given in Section 1.2.

1.9.1 An example where α^{wws} fails to be an equilibrium.

When $k^{OR} < \gamma k^{HR} < \overline{k} < \gamma k^{HO} < k^{HR} < \gamma\left(\gamma k^{OR} + 1\right) + 1$, a necessary condition for α^{wws} to be an equilibrium is $U^{(0,1,0,\ldots)}\left(k^{HR}\right) \geq U^{H}\left(k^{HR}\right)$. To see this, consider the following figure:



In the first row we rank the paths R, H and O. In the second row we put the actions prescribed by the strategy,. The third row establishes the path generated by sticking to the strategy and the fourth establishes whether this path is the desired behavior path (DBP), the best possible path (BPP) or none. The fifth row establishes the path generated by deviating from the strategy proposed in the second row. Finally, the sixth row establishes whether there is no profitable deviation (NPD). Consider for example, an addiction level $k \in \left[\frac{k^{OR}}{\gamma}, k^{HR}\right)$. For α^{wws} to be an equilibrium, the path $(0,1,0,...)$ must be preferred to H. But this occurs only if $U^{(0,1,0,...)}\left(k^{HR}\right) \geq U^{H}\left(k^{HR}\right)$. Parameter values where this doesn't hold (and therefore α^{wws} fails to be an equilibrium), are

$$\beta = 0.78 \quad \delta = 0.9 \quad \rho = 40.5 \quad \sigma = 20 \quad \gamma = 0.8 \quad x = 102$$

1.9.2 An example satisfying $\gamma k^{HR} < k^{OR} < \bar{k} < \gamma\left(\gamma k^{OR} + 1\right) + 1$ and IC.

$$\beta = 0.79 \quad \delta = 0.9 \quad \rho = 40 \quad \sigma = 20 \quad \gamma = 0.8 \quad x = 102$$

1.9.3 An example satisfying $\gamma k^{HR} < k^{OR} < \overline{k} < \gamma\left(\gamma k^{OR} + 1\right) + 1$ but not IC.

$$\beta = 0.9 \quad \delta = 0.9 \quad \rho = 40 \quad \sigma = 20 \quad \gamma = 0.9 \quad x = 101$$

1.9.4 $t-$naiveness: an example where the realized behavior path involves hitting a finite number of times.

Let $k_0 = k^{OR}$ and define $k_i = \gamma k_{i-1} + 1$ for $i = 1, 2, ...$ Let $j = \max\left\{i : k_i \leq k^{HR}\right\}$ so that j is the maximum number of consecutive hits, starting from k^{OR}, that keep the addiction level below k^{HR}. Let $\overline{k}_1 = \frac{k^{HR}-1}{\gamma}$ and define $\overline{k}_i = \frac{\overline{k}_{i-1}-1}{\gamma}$ for $i = 1, 2, ...$ Suppose j is odd and $U^R\left(\overline{k}_2\right) \geq U^{1100...}\left(\overline{k}_2\right)$, then the following strategy is a MPE:

$$\alpha(k) = \begin{cases} 0 & \text{if} \quad k < k^{OR} \\ 1 & \text{if} \quad k^{OR} \leq k < \overline{k}_j \\ 0 & \text{if} \quad \overline{k}_{2n+1} \leq k < \overline{k}_{2n} \text{ for } 2n+1 < j \\ 1 & \text{if} \quad \overline{k}_{2n} \leq k < \overline{k}_{2n-1} \text{ for } 2n < j \end{cases}$$

Starting from k^{OR}, a sophisticate will hit once, a naif will hit forever and:

A 1-naif will hit once

A 2-naif and a 3-naif will hit 3 times

A 4-naif and a 5-naif will hit 5 times

...

A j-1-naif and a j-naif will hit j times

A n-naif for n bigger than j will hit forever.

An example with $j = 5$ is given by

$$\beta = 0.9 \quad \delta = 0.9 \quad \rho = 43 \quad \sigma = 20 \quad \gamma = 0.9 \quad x = 106$$

Bibliography

[1] Becker, G. S. and K. M. Murphy (1988). "A Theory of Rational Addiction." *Journal of Political Economy*, 96, 675-700.

[2] Bernheim, D. and A. Rangel (2002). "Addiction and Cue-conditioned Cognitive Processes" Mimeo.

[3] Carrillo, J. and T. Mariotti (2000). "Strategic Ignorance as a Self-Disciplining Device." *Review of Economic Studies*, 67, 529-544.

[4] Chaloupka, F. J. and K. E. Warner (1998). "The economics of smoking." In *Handbook of Health Economics*.

[5] Elster, J. (1999). Introduction in Jon Elster, ed., *Addiction: Entries and Exits*. New York: Russell Sage Foundation, ix-xx.

[6] Frederick, S., Loewenstein, G. and O'Donoghue, T. (2002). "Time Discounting and Time Preference: A Critical Review." *Journal of Economic Literature*, 40(2), 351-401.

[7] Gruber, J. and B. Koszegi (2001). "Is Addiction 'Rational'? Theory and Evidence." *Quarterly Journal of Economics*, 116, 1261-1303.

[8] Heyman, G. M. (1996). "Resolving the Contradictions of Addiction." *Behavioral and Brain Sciences*, 19, 561-610.

[9] Laibson, D. (1994). "Essays in Hyperbolic Discounting." Economics, MIT.

[10] Laibson, D. (1997). "Golden Eggs and Hyperbolic Discounting." *Quarterly Journal of Economics*, 112, 443-477.

[11] Laibson, D. (2001). "A Cue-Theory of Consumption." *Quarterly Journal of Economics*, 116(1): 81-120.

[12] Loewenstein, G. and D. Prelec (1992). "Anomalies in Intertemporal Choice: Evidence and an Interpretation." *Quarterly Journal of Economics*, 107, 573-597.

[13] Messinis, G. (1999). "Habit Formation and the Theory of Addiction." *Journal of Economic Surveys*, 13, 417-442.

[14] O'Donoghue, T. and M. Rabin (2001a). "Choice and Procrastination." *Quarterly Journal of Economics*, 116(1), 121-160.

[15] O'Donoghue, T. and M. Rabin (2001b). "Self Awareness and Self Control." To appear as a chapter in *Now or Later: Economic and Psychological Perspectives on Intertemporal Choice*, edited by Roy Baumeister, George Loewenstein, and Daniel Read, published by Russell Sage Foundation Press.

[16] O'Donoghue, T. and M. Rabin (2002). "Addiction and Present-Biased Preferences." Mimeo.

[17] Phelps, E. S. and R. A. Pollak (1968). "On Second-best National Saving and Game-equilibrium Growth." *Review of Economic Studies*, 35, 185-199.

[18] Strotz, R. H. (1956). "Myopia and Inconsistency in Dynamic Utility Maximization." *Review of Economic Studies*, 23, 165-180.

[19] Thaler, R. (1991). "Some Empirical Evidence on Dynamic Inconsistency." in *Quasi Rational Economics*. New York: Russell Sage Foundation, 127-133.

Chapter 2

Beneficial and Harmful Addictions: Two sides of the same coin

2.1 Introduction

Since the seminal work by Becker and Murphy (1988) there has been a growing body of literature devoted to the study of addiction. Even though the Becker-Murphy model accounted for both harmful addictions (e.g. drug consumption) and beneficial addictions (e.g. jogging) most of the recent research has focused solely on the former. However we believe that there is a variety of economic contexts where beneficial addictions play an important role. People might get addicted to sports, living standards, work, high levels of human capital, etc... As we pointed out in the previous chapter, the key feature for a product being addictive is that it generates *habit formation*[1]: past consumption of the product increases current desire for consumption. In general one could conceive of *harmful* addic-

[1]Becker and Murphy refer to this feature as adjacent complementarities.

tions as habit-forming activities that imply an immediate reward but generate future costs (negative internalities) whereas *beneficial* addictions imply an immediate cost but generate future rewards (positive internalities). In the present chapter we establish an isomorphism between harmful and beneficial addictions which allows us to study both phenomena as two sides of the same coin: any harmful addiction can be thought of as a beneficial addiction and vice-versa.

The above dualism holds for a specific context to which we refer as a binary activity choice accumulation problem (BACAP). In a BACAP an individual faces at each period the binary choice of undertaking or not an activity; and his (instantaneous) payoff depends on his current choice as well as on the history of past choices (which is captured by a state variable)[2]. To illustrate the dualism consider the following harmful addiction example. Suppose that the activity is "smoking one cigarette today" and that the history of past smoking is captured by the individual's "nicotine level". Negative internalities are captured by assuming that smoking today produces immediate pleasure but generates future costs (because it raises the nicotine level and therefore induces detrimental health effects). Habit formation is captured by assuming that higher nicotine levels induce higher desire for current smoking. Now consider the choice problem for the same individual where the activity is defined as "avoiding to smoke one cigarette today". This problem presents positive internalities: by undertaking the activity, the individual renounces to the immediate pleasure (and thus incurs an immediate cost) but generates future benefits by reducing his addiction level. Moreover, this problem also has the habit-forming feature: the more the individual has avoided smoking, the lower his nicotine level and therefore the higher the

[2] In particular, the model studied in the previous chapter is a BACAP.

desire to keep avoiding smoking because its associated cost is lower. Therefore the original harmful addiction problem may be seen as a beneficial addiction one; in fact we show that both problems are equivalent.

Besides the theoretical insight it provides, this equivalence result is appealing because of its usefulness: it allows to obtain results for both realms of addiction by focusing just on one. For instance, the results obtained in the previous chapter can be readily translated to the domain of beneficial addictions.

Once the dualism is established, we illustrate its attractiveness by analyzing beneficial addictions under two settings: time-consistent preferences (i.e. when the intertemporal utility presents exponential discounting) and time-inconsistent preferences (i.e. intertemporal utility with hyperbolic discounting). Under the first setting we show that the individual's behavior depends crucially on the convexity of his instantaneous utility: when convexity holds the individual either always undertakes the activity or always refrains; when convexity fails we show that there might be other absorbing states. This result is worth mentioning because previous research (on harmful addiction) has assumed convexity and yet we believe that concavity might as well be of interest. Under the second setting, the individual can either be aware (he is *naïf*) or not (he is *sophisticated*) of his time inconsistency. When he is aware he engages in an intrapersonal game. We show that the isomorphism preserves the equilibria of the induced game and then we analyze the implications of naiveté vs. sophistication. In particular, we show that the implications are the same whether we have immediate costs and delayed rewards (beneficial addictions) or immediate rewards and delayed costs (harmful addictions). This contrasts with the implications of naiveté vs.

sophistication in a context where the activity must be performed exactly once in a finite number of periods: as O'Donoghue and Rabin (1999) show, in the doing-it-once context sophistication exacerbates misbehavior under immediate costs while it mitigates it under immediate rewards. Our analysis shows that it would be misleading to extrapolate this conclusion to a full-fledged model of intertemporal decision making, i.e. one where actions are chosen repeatedly.

The chapter proceeds as follows. In Section 2.2 we formally define a BACAP and we establish the isomorphism which allows us to show that a BACAP is equivalent to its dual. In Section 2.3 we consider addictions under time consistency and we show that concavity of the utility function may yield richer patterns of behavior as opposed to the cutoff rule implied by convexity. In Section 2.4 we consider addictions under time inconsistency: we show that the isomorphism is equilibria-preserving and therefore the implications of naiveté vs. sophistication are the same under both realms (beneficial and harmful) of addiction. Section 2.5 concludes.

2.2 The Model

We will restrict ourselves to the following class of problems:

2.2.1 Binary Activity Choice Accumulation Problem (BACAP)

In a BACAP an individual has to choose at each period $t = 1, 2, ...,$ whether to undertake ($a_t = 1$) or not ($a_t = 0$) an activity. The instantaneous per-period payoff u_t depends on the period´s action as well as on the history of past actions which is assumed to be captured by a state variable k that evolves according to $k_{t+1} = \gamma k_t + a_t$, with $0 < \gamma < 1$. Notice that there is a maximum value for the state variable $k^{\max} = \frac{1}{1-\gamma}$. The per-period payoff u_t is given by

$$u_t = u(k_t, a_t) = \begin{cases} \Psi(k_t) & \text{if } a_t = 1 \\ \Gamma(k_t) & \text{if } a_t = 0 \end{cases}$$

where $\Psi : [0, k^{\max}] \to \Re$ and $\Gamma : [0, k^{\max}] \to \Re$. At each period t, the individual aims at maximizing the intertemporal utility given by

$$U_t = U(u_t, u_{t+1}, ...)$$

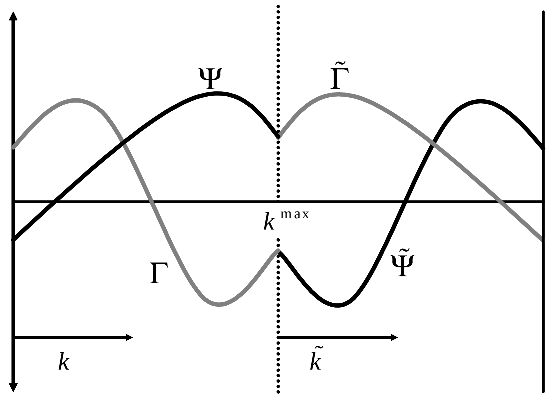
where $U : \Re^{\infty} \to \Re$. If $k_1 \in [0, k^{\max}]$ is the initial condition for the state variable k, a BACAP is completely characterized by $[\Psi, \Gamma, U, \gamma, k_1]$.The main result of the following subsection is that for every BACAP $\mathcal{B} = [\Psi, \Gamma, U, \gamma, k_1]$, there is an equivalent BACAP where the activity may be viewed as the negation of the activity in $\mathcal{B}$.

2.2.2 The DUAL of a BACAP.

Given a BACAP $\mathcal{B} = [\Psi, \Gamma, U, \gamma, k_1]$ define $\widetilde{\Psi} : [0, k^{\max}] \to \Re$ and $\widetilde{\Gamma} : [0, k^{\max}] \to \Re$ as

$$\begin{aligned} \widetilde{\Psi}\left(\widetilde{k}\right) &= \Gamma\left(k^{\max} - \widetilde{k}\right) \\ \widetilde{\Gamma}\left(\widetilde{k}\right) &= \Psi\left(k^{\max} - \widetilde{k}\right) \end{aligned}$$

so that $\widetilde{\Psi}$ and $\widetilde{\Gamma}$ are the reflections of Γ and Ψ with respect to $k^{\max}$ as illustrated in the figure.



Now consider a problem where at each period $t = 1, 2, ...$, an individual faces the choice of undertaking ($\widetilde{a}_t = 1$) or not ($\widetilde{a}_t = 0$) an action and where the instantaneous payoff $\widetilde{u}_t$ is given by

$$\widetilde{u}_t = \widetilde{u}\left(\widetilde{k}_t, \widetilde{a}_t\right) = \begin{cases} \widetilde{\Psi}\left(\widetilde{k}_t\right) & \text{if } \widetilde{a}_t = 1 \\ \widetilde{\Gamma}\left(\widetilde{k}_t\right) & \text{if } \widetilde{a}_t = 0 \end{cases}$$

where $\widetilde{k}$ is assumed to evolve according to $\widetilde{k}_{t+1} = \gamma\widetilde{k}_t + \widetilde{a}_t$. Suppose that at each period t, the individual aims at maximizing the intertemporal utility

$$\widetilde{U}_t = U\left(\widetilde{u}_t, \widetilde{u}_{t+1}, ...\right)$$

Then, for a given initial condition $\widetilde{k}_1$, $\left[\widetilde{\Psi}, \widetilde{\Gamma}, U, \gamma, \widetilde{k}_1\right]$ defines a BACAP.

Definition 24 *Given a BACAP $\mathcal{B} = [\Psi, \Gamma, U, \gamma, k_1]$, its DUAL is the BACAP*

$$D\left(\mathcal{B}\right) = \left[\widetilde{\Psi}, \widetilde{\Gamma}, U, \gamma, \left(k^{\max} - k_1\right)\right]$$

Definition 25 *We say that a BACAP $\mathcal{B}$ is equivalent to a BACAP $\mathcal{B}'$ if*

1. *for any path $A = \{a_1, a_2, ...\} \in \{0, 1\}^\infty$ in $\mathcal{B}$ there is a path $A' = \{a'_1, a'_2, ...\} \in \{0, 1\}^\infty$ in $\mathcal{B}'$ that yields the same payoff; and*

2. *for any path* $A' = \{a'_1, a'_2, ...\} \in \{0,1\}^\infty$ *in* $\mathcal{B}'$ *there is a path* $A = \{a_1, a_2, ...\} \in \{0,1\}^\infty$ *in* $\mathcal{B}$ *that yields the same payoff.*

Proposition 26 *A BACAP* $\mathcal{B}$ *is equivalent to its dual* $D(\mathcal{B})$.

Proof. Let (a_t, k_t) and $\left(\widetilde{a}_t, \widetilde{k}_t\right)$ denote period-t actions and states in $\mathcal{B}$ and $D(\mathcal{B})$ respectively. Consider the isomorphism $\Xi : \{0,1\} \times [0, k^{\max}] \to \{0,1\} \times [0, k^{\max}]$ that maps actions and states in $\mathcal{B}$ to actions and states in $D(\mathcal{B})$ in the following way

$$\Xi(a,k) = (1-a, k^{\max} - k) = \left(\widetilde{a}, \widetilde{k}\right)$$

Then, if $a_t = 1$ we have $\widetilde{a}_t = 0$ and

$$u(a_t, k_t) = \Psi(k_t) = \widetilde{\Gamma}\left(\widetilde{k}_t\right) = \widetilde{u}\left(\widetilde{a}_t, \widetilde{k}_t\right) = \widetilde{u}(\Xi(a_t, k_t))$$

and if $a_t = 0$ we have $\widetilde{a}_t = 1$ and

$$u(a_t, k_t) = \Gamma(k_t) = \widetilde{\Psi}\left(\widetilde{k}_t\right) = \widetilde{u}\left(\widetilde{a}_t, \widetilde{k}_t\right) = \widetilde{u}(\Xi(a_t, k_t))$$

Thus the per-period utilities associated to (a_t, k_t) and $\Xi(a_t, k_t)$ are the same. Moreover, according to the isomorphism the evolution of $\widetilde{k}_t$ is *consistent* with the evolution of k_t:

$$\begin{aligned} \widetilde{k}_{t+1} &= \gamma \widetilde{k}_t + \widetilde{a}_t = \gamma(k^{\max} - k_t) + 1 - a_t \\ &= (\gamma k^{\max} + 1) - (\gamma k_t + a_t) = k^{\max} - k_{t+1} = \Xi(k_{t+1}) \end{aligned}$$

Since $\widetilde{k}_1 = k^{\max} - k_1 = \Xi(k_1)$ so that the initial condition in $D(\mathcal{B})$ is the image of the initial condition of $\mathcal{B}$ under Ξ, we conclude that any path $A \in \{0,1\}^\infty$ in $\mathcal{B}$ is payoff equivalent to path $\widetilde{A} = \Xi(A)$ in $D(\mathcal{B})$. This shows part 1 of Definition 25. To see that part 2 is also satisfied just notice that $D(D(\mathcal{B})) = \mathcal{B}$, i.e. "$\mathcal{B}$ is the dual of its dual". ∎

Definition 27 *We say that a BACAP has Positive Internalities, if* Ψ *and* Γ *are increasing in* k*. That is, the more the activity has been undertaken in the past, the higher the present well-being.*

Definition 28 *We say that a BACAP has Negative Internalities, if* Ψ *and* Γ *are decreasing in* k*. That is, the more the activity has been undertaken in the past, the lower the present well-being.*

Proposition 29 *If a BACAP* $\mathcal{B}$ *has Positive (Negative) Internalities its dual* $D(\mathcal{B})$ *has Negative (Positive) Internalities.*

Proof. *When* Ψ *and* Γ *are increasing (decreasing),* $\widetilde{\Psi}$ *and* $\widetilde{\Gamma}$ *are decreasing (increasing)* ■

Definition 30 *We say that a BACAP has Habit-Formation, if* $\Psi(k) - \Gamma(k)$ *is increasing in* k*. That is, the more the activity has been undertaken in the past (as captured by* k*) the higher the marginal instantaneous utility of undertaking it in the present.*

Proposition 31 *If a BACAP* $\mathcal{B}$ *has Habit-Formation its dual* $D(\mathcal{B})$ *has Habit-Formation*

Proof. *if* $\Psi(k) - \Gamma(k)$ *is increasing then* $\widetilde{\Psi}(k) - \widetilde{\Gamma}(k)$ *is increasing* ■

Definition 32 *We say that a BACAP presents* ***Harmful Addiction*** *if it has Habit Formation and Negative Internalities. We say that a BACAP presents* ***Beneficial Addiction*** *if it has Habit Formation and Positive Internalities.*

Proposition 33 *If a BACAP* $\mathcal{B}$ *presents* ***Harmful (Beneficial) Addiction*** *its dual* $D(\mathcal{B})$ *has* ***Beneficial (Harmful) Addiction.***

Proof. *Direct from Propositions 29 and 31* ■

Let $\alpha : [0, k^{\max}] \to \{0,1\}$ be a rule or strategy in a BACAP $\mathcal{B}$, that is, for each state level k, $\alpha(k)$ prescribes an admissible action. Define $\widetilde{\alpha} : [0, k^{\max}] \to \{0,1\}$ as the rule in $D(\mathcal{B})$ given by

$$\widetilde{\alpha}\left(\widetilde{k}\right) = 1 - \alpha\left(k^{\max} - \widetilde{k}\right) \tag{2.1}$$

Proposition 34 *If α induces path A in a BACAP $\mathcal{B}$ then $\widetilde{\alpha}$ induces path $\Xi(A)$ in $D(\mathcal{B})$.*

Proof. *Suppose that at some period t, state levels are k_t and $\widetilde{k}_t = (k^{\max} - k_t)$ in $\mathcal{B}$ and $D(\mathcal{B})$ respectively. Then α prescribes in $\mathcal{B}$ action $a_t = \alpha(k_t)$ and induces state level $k_{t+1} = \gamma k_t + a_t$; and $\widetilde{\alpha}$ prescribes in $D(\mathcal{B})$ action $\widetilde{a}_t = \widetilde{\alpha}\left(\widetilde{k}_t\right)$ and induces state level $\widetilde{k}_{t+1} = \gamma \widetilde{k}_t + \widetilde{a}_t$.Now notice that*

$$\widetilde{a}_t = \widetilde{\alpha}\left(\widetilde{k}_t\right) = 1 - \alpha\left(k^{\max} - \widetilde{k}_t\right) = 1 - \alpha(k_t) = 1 - a_t = \Xi(a_t)$$

and

$$\widetilde{k}_{t+1} = \gamma \widetilde{k}_t + \widetilde{a}_t = \gamma\left(k^{\max} - k_t\right) + 1 - a_t = \left(\gamma k^{\max} + 1\right) - \left(\gamma k_t + a_t\right) = \left(k^{\max} - k_{t+1}\right)$$

Therefore the proposition follows by induction and the fact that $\widetilde{k}_1 = (k^{\max} - k_1)$. ■

The above proposition shows that given a strategy α in a BACAP $\mathcal{B}$, strategy $\widetilde{\alpha}$ as given by (2.1) has the natural interpretation of being the strategy in $D(\mathcal{B})$ induced by applying the isomorphism Ξ to α. Thus, with a slight abuse of notation we will refer to it as $\Xi(\alpha)$.Notice that as it is shown in the proof of Proposition 26 path A in $\mathcal{B}$ is payoff equivalent to path $\Xi(A)$ in $D(\mathcal{B})$.Therefore, we readily obtain the following proposition.

Proposition 35 *Strategies α and $\Xi(\alpha)$ are payoff equivalent.*

We close this section by giving examples that motivate the study of BACAP´s presenting Addiction.

2.2.3 Some examples presenting Addiction.

Drug consumption

Suppose the activity is "smoking one cigarette" and that k represents the individual's addiction level. Let $f(k_t)$ be the utility associated to addiction level k_t irrespective of what the individual's current choice is. Because of the detrimental effects of past smoking in current health it is natural to assume that f is decreasing in k. Suppose that $x > 0$ reflects the intrinsec pleasure of smoking one cigarette and that $w(k_t)$ is the withdrawal cost (which we assume increasing in k_t because the more the person has smoked in the past the harder for him to refrain current smoking). Then, the instantaneous per-period utility is given by

$$u(k_t, a_t) = \begin{cases} \Psi(k_t) = x + f(k_t) & \text{if } a_t = 1 \\ \Gamma(k_t) = f(k_t) - w(k_t) & \text{if } a_t = 0 \end{cases}$$

an the (harmful) addictive properties of smoking are reflected by negative internalities (Ψ and Γ decreasing) and habit-formation (w increasing)[3].

Jogging

Suppose the activity is "one hour of jogging" so that k represents the fitness level. Let $g(k_t)$ be the utility derived from having fitness level k_t. This is a utility the individual enjoys at time t irrespective of what his current choice is; it includes the health benefits from past exercising and therefore is increasing in k_t. Let $e > 0$ be the effort or cost associated to "one hour of jogging" when the individual has never exercised before (i.e. $k = 0$). It

[3]Notice that this is precisely the kind of model proposed by O'Donoghue and Rabin (2002) which we discussed in the previous chapter.

seems natural to assume that this cost decreases as the fitness level grows. We capture this by letting the cost be $c(k_t) = e - h(k_t)$ with $h(k_t)$ increasing and $h(0) = 0$. Notice that this allows for the jogging activity to even become "pleasurable" (i.e. when for some k, $h(k) > e$). Then, the instantaneous per-period utility is given by

$$u(k_t, a_t) = \begin{cases} \Psi(k_t) = g(k_t) - c(k_t) = -e + h(k_t) + g(k_t) & \text{if } a_t = 1 \\ \Gamma(k_t) = g(k_t) & \text{if } a_t = 0 \end{cases}$$

and the (beneficial) addictive properties of jogging are reflected by positive internalities (Ψ and Γ increasing) and habit-formation (h increasing).

Conspicuous consumption

Suppose that the main source of utility for an individual is his living standard (or status) which we denote by k and that can only be raised by the activity of conspicuous consumption (cf.Veblen). For instance, think of this activity ($a = 1$) as "offering a Great-Gatsby-type party". Let $g(k)$ be the utility associated to status k that the individual enjoys irrespective of his current choice. Offering the party has an intrinsic instantaneous utility but it also produces a positive internality because it raises the host´s status for next period. Let v be the intrinsic utility of the party experienced by the host. We may assume that v is increasing in k (for instance, people attending the party are kinder to the host the higher his status is). If $e > 0$ represents the effort or cost of offering the party, the instantaneous per-period utility is given by

$$u(k_t, a_t) = \begin{cases} \Psi(k_t) = -e + v(k_t) + g(k_t) & \text{if } a_t = 1 \\ \Gamma(k_t) = g(k_t) & \text{if } a_t = 0 \end{cases}$$

and the (beneficial) addictive properties of conspicuous consumption are reflected by positive internalities (Ψ and Γ increasing) and habit-formation (v increasing).

Investment in Human Capital

Suppose that the individual has a unit of free time and a unit of working time each period. The working time unit is exerted in a productive activity whose productivity $g(k)$ depends positively on the accumulated human capital k. The individual may use his free time in either learning ($a = 1$), which might be interpreted as investment in human capital and therefore increases his stock k, or in an alternative activity ($a = 0$), which may be leisure or some other productive activity that does not affect k. When he decides to use his free time in the alternative activity, he obtains an instantaneous utility $e > 0$ which can be thought of as the pleasure derived from leisure or the profits from the alternative productive activity. If he devotes his free time to learning he incurs in a cost $c(k)$ which is decreasing in k and may eventually become negative (i.e. the individual "enjoys" learning). Therefore, the instantaneous per-period utility is given by

$$u(k_t, a_t) = \begin{cases} \Psi(k_t) = g(k_t) - c(k_t) & \text{if } a_t = 1 \\ \Gamma(k_t) = g(k_t) + e & \text{if } a_t = 0 \end{cases}$$

and the (beneficial) addictive properties of investment in human capital are reflected by positive internalities (Ψ and Γ increasing) and habit-formation ($-c$ increasing).

2.3 Beneficial Addictions under Time Consistency

Consider a BACAP $\mathcal{B} = [\Psi, \Gamma, U, \gamma, k_1]$ presenting beneficial addiction and where the functional U takes the specific form

$$U(u_t, u_{t+1,\dots}) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} u_t \text{ with } 0 < \delta < 1 \tag{2.2}$$

i.e. where we have time separability and exponential discounting. Let $A \in \{0,1\}^{\infty}$ denote a behaviour path, that is, an infinite sequence of admissible actions. Keeping in mind the jogging example of the previous section we will use E (from "exercising") to denote the behaviour path where the individual always undertakes the activity, that is, $E = (1, 1, \dots)$, and N (from "never exercising") will denote the behaviour path where the individual never undertakes the activity, that is, $N = (0, 0, \dots)$.

We will see that the optimal decision rule and long-run behavior depend crucially on the convexity of Ψ and Γ.

2.3.1 Ψ and Γ convex

By the previous section we know that a BACAP $\mathcal{B} = [\Psi, \Gamma, U, \gamma, k_1]$ is equivalent to its dual $D(\mathcal{B}) = \left[\widetilde{\Psi}, \widetilde{\Gamma}, U, \gamma, (k^{\max} - k_1)\right]$. By proposition 29 we know that $D(\beta)$ presents harmful addiction. Notice that $\widetilde{\Psi}$ and $\widetilde{\Gamma}$ are also convex. In O'Donoghue and Rabin (2002) it is shown that in a BACAP as $D(\mathcal{B})$ the optimal decision rule of the agent is of the cut-off type. That is, there exists a critical state level $\widetilde{k}^c$ such that the optimal action is $\widetilde{a} = 0$ for levels below $\widetilde{k}^c$ and $\widetilde{a} = 1$ for levels above $\widetilde{k}^c$. Therefore the optimal decision rule takes the form

$$\widetilde{a}_t = \widetilde{\alpha}\left(\widetilde{k}_t\right) = \begin{cases} 0 & \text{if } \widetilde{k}_t < \widetilde{k}^c \\ 1 & \text{if } \widetilde{k}_t \geq \widetilde{k}^c \end{cases}$$

The reason for this is that the time consistency implied by the intertemporal utility (2.2) allows for the use of a value function $V\left(\widetilde{k}_t\right)$ that happens to be convex when $\widetilde{\Psi}$ and $\widetilde{\Gamma}$ are convex. Therefore at any time t the agent chooses $\widetilde{a}_t = 1$ if and only if

$$\widetilde{\Psi}\left(\widetilde{k}_t\right) - \widetilde{\Gamma}\left(\widetilde{k}_t\right) \geq \delta\left[V\left(\gamma\widetilde{k}_t\right) - V\left(\gamma\widetilde{k}_t + 1\right)\right] \tag{2.3}$$

i.e. if and only if the marginal instantaneous benefits $\widetilde{\Psi}\left(\widetilde{k}_t\right) - \widetilde{\Gamma}\left(\widetilde{k}_t\right)$ offset the marginal future costs $\delta\left[V\left(\gamma\widetilde{k}_t\right) - V\left(\gamma\widetilde{k}_t + 1\right)\right]$ of the action. Now, because of the Habit-Forming feature, the LHS of (2.3) is increasing, while the convexity of the value function V implies that the RHS of (2.3) is decreasing and therefore we get the existence of $\widetilde{k}^c$.

Now, because of the equivalence between $\mathcal{B}$ and $D(\mathcal{B})$ we readily obtain

Proposition 36 *In a BACAP $\mathcal{B} = [\Psi, \Gamma, U, \gamma, k_1]$ presenting beneficial addiction with Ψ and Γ convex and intertemporal utility U given by (2.2), the optimal decision rule is of the cutoff type. That is, there exists a critical state level $k^c \in [0, k^{\max}]$ such that*

$$a_t = \alpha(k_t) = \begin{cases} 0 & \textit{if } k_t < k^c \\ 1 & \textit{if } k_t \geq k^c \end{cases}$$

Remark 37 *Notice that when Ψ and Γ are convex, long-run behavior depends totally on the initial condition k_1 : if $k_1 < k^c$ the individual will never undertake the activity (his realized behavior path is N) while when $k_1 \geq k^c$ he will undertake it forever (his realized behavior path is E). Therefore there are only two absorbing states: 0 (when $k_1 < k^c$) and $k^{\max}$ (when $k_1 \geq k^c$). This changes dramatically if we relax the convexity assumption as the following subsection shows.*

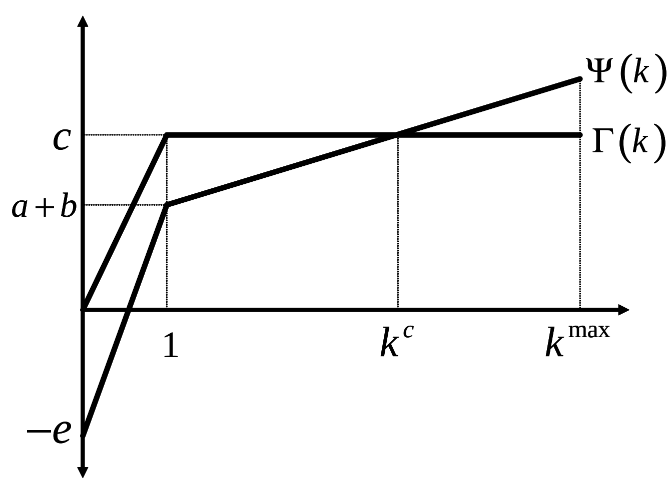
2.3.2 Ψ and Γ concave

Suppose now that Ψ and Γ are concave. Even though we still may make use of a value function by virtue of (2.2), this value function may not be convex nor concave. So the type of analysis used in the previous subsection is of no use. We will show by means of examples that the optimal decision rule may no longer be of the cutoff type, that long-run behavior may be independent of initial conditions and that there might be other absorbing states. For the examples we take Ψ and Γ to be piece-linear with specific functional forms

$$\Psi(k) = \begin{cases} (a+b+e)k - e & \text{if } k < 1 \\ ak + b & \text{if } k \geq 1 \end{cases}$$

$$\Gamma(k) = \begin{cases} ck & \text{if } k < 1 \\ c & \text{if } k \geq 1 \end{cases}$$

where all parameters are positive and $c > a + b$ as illustrated in the figure



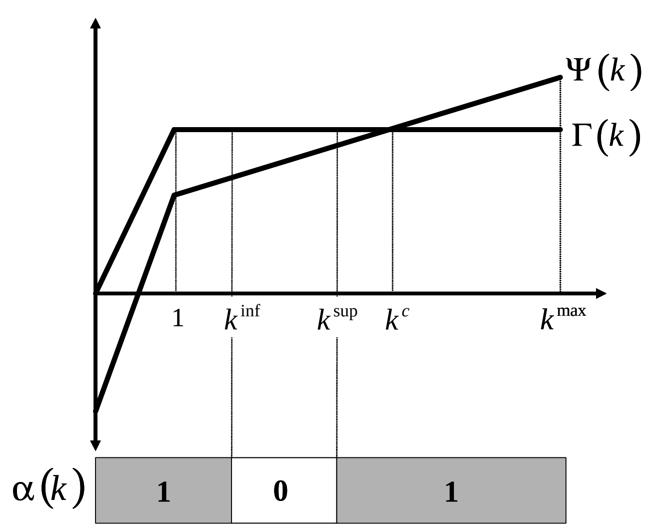
Notice that there is clearly beneficial addiction: Ψ and Γ are increasing (positive internalities) and $\Psi - \Gamma$ is increasing (habit formation). Also notice that the intersection between Ψ

and Γ occurs at $k^c = \frac{c-b}{a}$ and thus for any level $k > k^c$, the individual will always undertake the activity because $\Psi(k) > \Gamma(k)$. Therefore, the state $k^{\max}$ is locally absorbing. For levels below k^c the instantaneous utility of undertaking the activity is lower than the one from refraining but the individual might choose to undertake it because by doing so he raises his state level and therefore increases his future payoffs.

We consider now two examples that differ only in the depreciation rate γ of the state. For both examples the other parameters take the specific values $a = 0.4; b = 9; c = 10; e = 1; \delta = 0.5$. In Example 1, we take $\gamma = \frac{8}{10}$; and in Example 2 we take $\gamma = \frac{65}{100}$.In the appendix we show that for both examples the optimal decision rule takes the non-cutoff form

$$a_t = \alpha(k_t) = \begin{cases} 1 & \text{if } k_t \le k^{\inf} \\ 0 & \text{if } k^{\inf} < k_t < k^{\sup} \\ 1 & \text{if } k_t \ge k^{\sup} \end{cases} \tag{2.4}$$

where $k^{\inf}$ and $k^{\sup}$ differ for both examples but they satisfy $1 < k^{\inf} < k^{\sup} < k^c$ as illustrated in the figure



Claim 38 *In Example 1 ($\gamma = \frac{8}{10}$), the long-run behavior implies undertaking the activity (i.e. $k^{\max}$ is the unique globally absorbing state). To see this it suffices to show that by following the rule $\alpha(k)$ for any initial state level k, we eventually reach a state above $k^{\sup}$ and therefore we keep undertaking the activity from then on. This is shown in the appendix.*

Claim 39 *In Example 2 ($\gamma = \frac{65}{100}$), $k^{\max}$ is not globally absorbing and there are other positive addiction levels that are locally absorbing. To see this consider levels $\overline{k} = \frac{1}{1-\gamma^2}$ and $\underline{k} = \frac{\gamma}{1-\gamma^2}$ and notice that $\gamma\overline{k} = \underline{k}$ and $\gamma\underline{k}+1 = \overline{k}$ (i.e. undertaking the activity when the state level is $\underline{k}$ yields level $\overline{k}$ and refraining when state level is $\overline{k}$ yields level $\underline{k}$).Now, it happens that in this example $\underline{k} < k^{\inf} < \overline{k} < k^{\sup}$ and therefore $\alpha\left(\overline{k}\right) = 0$ and $\alpha\left(\underline{k}\right) = 1$ so that once the individual reaches one of these states he keeps switching from one to another forever. In particular for any initial level $k \in \left(k^{\inf}, k^{\sup}\right)$ the optimal behavior path is 010101.... Hence we have shown that concavity may yield other absorbing states. Also notice that since $k^{\max}$ is locally absorbing the long-run behavior depends heavily on the initial condition.*

As the above examples show allowing for concavity in the functions Ψ and Γ may yield richer patterns of behavior as opposed to Ψ and Γ being convex where we obtain a cutoff decision rule and therefore the behavior observed for a given initial state level is either always undertaking the activity or always refraining. Of special interest is Example 2 because it may yield non-monotonic behavior by alternatively switching from undertaking the activity to refraining.

2.3.3 Assuming Convexity

As we have seen, characterization of behavior is very simple under the convexity (of Ψ and Γ) assumption while it may be a very complicated task when we relax it. Assuming convexity is appealing because of its tractability but it may be unsatisfactory for modeling some addictions. In essence, what the convexity assumption posits is that there are increasing marginal returns in the state variable. However we believe that there are some realms of addiction where the opposite would hold. Consider for instance the jogging example provided in the previous section: perhaps the most reasonable assumption is decreasing marginal returns to the fitness level (i.e. Ψ and Γ concave). The same goes for the investment in human capital example: the standard would be to assume decreasing marginal returns to the accumulated human capital. In their harmful addiction model, O'Donoghue and Rabin posit the convexity assumption arguing that "the more addicted the person becomes, the less a given increase in k hurts his instantaneous utility, and therefore the less harm hitting does to future utility". This is debatable[4]: It could be a reasonable assumption for some drugs, say "soft" drugs, but it may not be too realistic for "hard" drugs such as heroin.

Since the concavity assumption may be appropriate for some addiction contexts we believe that results such as the obtained in the previous two claims deserve attention.

[4] We believe that they are aware of this since they state that "most results hold even if Ψ and Γ are a little concave, and some do not rely at all on them being convex".

2.4 Beneficial Addictions under Time Inconsistency

Suppose now that the functional U takes the specific form[5]

$$U(u_t, u_{t+1}, ...) = u_t + \beta \sum_{\tau=t+1}^{\infty} \delta^{\tau-t} u_\tau \quad \text{with } \beta \in (0,1) \tag{2.5}$$

The present bias reflected in the parameter β induces the time inconsistency: because of the greater taste for immediate gratification an optimal behavior path at some date t may no longer be optimal at a further date. Therefore, an optimal behavior path may not be implementable because future selves may have incentives to deviate (here, of course, we are not allowing for commitment possibilities). When the individual is fully unaware of his time-inconsistency (i.e. he is *naïf*) he chooses his action according to the behavior path that maximizes (2.5) believing (wrongly) that his future selves will stick to it. When the individual is fully aware of his time-inconsistency (i.e. he is *sophisticated*) he engages in an intrapersonal game: he plays against his future selves by maximizing (2.5) *subject to* the intended behavior path will be followed. In the infinite horizon case this intrapersonal game will normally present multiple Markov Perfect equilibria (which is the natural solution concept to be called upon). In what follows it will be useful to distinguish between a *desired behavior path* (DBP), that is, an infinite sequence of actions solving (2.5); and a *realized behavior path* (RBP), that is, a path actually followed by all selves.

The implications of naiveté vs. sophistication have been studied in O'Donoghue and Rabin (2002) and in the previous chapter for a BACAP presenting harmful addiction under time inconsistency as given by (2.5). We may now apply the duality established in Subsection 2.2.2 to translate their results into BACAPs with beneficial addictions. It is

[5] This functional form was first introduced by Phelps and Pollack (1968). Because of its simplicity and tractability, it has been widely used to model self-control problems since the work of Laibson (1994).

important to note that we will consider only the case where Ψ and Γ are convex. We do so because those studies presented this assumption.

2.4.1 Naif behavior

O'Donoghue and Rabin show that in a BACAP $\mathcal{B}$ presenting harmful addiction a naif follows a cutoff rule α^n, that is, there is a critical state level k^n such that

$$\alpha^n(k) = \begin{cases} 0 & \text{if } k < k^n \\ 1 & \text{if } k \geq k^n \end{cases}$$

Moreover, it happens that for levels below k^n the DBP is $R = 00000...$ while for levels above k^n the DBP is either $O = 1000...$ or $H = 1111....$ A naif has self-control problems (or time inconsistency) whenever O is the DBP: Believing that he is able to follow it he will undertake the activity but by doing so he raises his state level for next period and therefore he does not stick to O. In fact, the RBP is either R (for any level below k^n) or H (for any level above k^n).

Because of the equivalence between $\mathcal{B}$ and $D(\mathcal{B})$ (in particular because of Propositions 34 and 35) strategy $\widetilde{\alpha}^n = \Xi(\alpha^n)$ describes naif behavior in $D(\mathcal{B})$.Therefore

$$\begin{aligned} \widetilde{\alpha}^n\left(\widetilde{k}\right) &= 1 - \alpha^n\left(k^{\max} - \widetilde{k}\right) \\ &= 1 - \begin{cases} 0 & \text{if } k^{\max} - \widetilde{k} < k^n \\ 1 & \text{if } k^{\max} - \widetilde{k} \geq k^n \end{cases} \\ &= \begin{cases} 1 & \text{if } k^{\max} - k^n < \widetilde{k} \\ 0 & \text{if } k^{\max} - k^n \geq \widetilde{k} \end{cases} \end{aligned}$$

so that we have

$$\widetilde{\alpha}^n\left(\widetilde{k}\right)=\begin{cases}0 & \text{if } \widetilde{k}<\widetilde{k}^n\\ 1 & \text{if } \widetilde{k}\geq\widetilde{k}^n\end{cases}\qquad\text{where } \widetilde{k}^n=k^{\max}-k^n \tag{2.6}$$

and the RBP of a naif in $D(\mathcal{B})$ is either $\Xi(R)=1111...$(for any level above $\widetilde{k}^n$) or $\Xi(H)=$ 000.... (for any level below $\widetilde{k}^n$).

Since any BACAP presenting beneficial addiction can be thought of as the dual of some BACAP presenting harmful addiction, naif behavior for beneficial addictions is completely characterized by (2.6).

2.4.2 Sophisticated behavior

When there is MPE multiplicity (as it is normally the case with an infinite horizon) in the induced intrapersonal game, sophisticated behavior would depend on the particular equilibrium selection. Rather than characterizing sophisticated behavior in a BACAP $\mathcal{B}$ (this would imply characterization of MPE of the induced game and as O'Donoghue and Rabin point out this can be a very complicated task) what we want to point out is that the isomorphism Ξ is equilibrium-preserving:

Proposition 40 *If strategy α constitutes a MPE of a BACAP $\mathcal{B}$, then strategy $\widetilde{\alpha}=\Xi(\alpha)$ constitutes a MPE of $D(\mathcal{B})$.*

Proof. *Because of Propositions 34 and 35, if there were a profitable deviation from $\widetilde{\alpha}$ in $D(\mathcal{B})$ then there would be a profitable deviation from α in $\mathcal{B}$: just apply the isomorphism Ξ to the deviation.* ■

Therefore the dualism also applies for sophisticated behavior: if some strategy α describes sophisticated behavior in a BACAP $\mathcal{B}$, then strategy $\widetilde{\alpha} = \Xi(\alpha)$ describes sophisticated behavior in $D(\mathcal{B})$. We turn now to compare sophisticated versus naif behavior.

2.4.3 Naiveté vs. sophistication

We will use the following notation: For any strategy $\alpha : [0, k^{\max}] \to \{0, 1\}$ and any level $k \in [0, k^{\max}]$, let $V(k; \alpha)$ denote the value (i.e. the maximal intertemporal utility) of following strategy α with starting state level k. For any behavior path $A \in \{0, 1\}^{\infty}$ and any level $k \in [0, k^{\max}]$, let $V(k; A)$ denote the value of following path A with starting state level k.

Suppose that strategy α^s describes sophisticated behavior in a BACAP $\mathcal{B}$ (i.e. α^s is the selected MPE of the induced game). Then, for a given initial state level k a sophisticated would be better off than a naif if and only if

$$V(k; \alpha^s) \geq V(k; \alpha^n)$$

By the previous subsections we know that in $D(\mathcal{B})$ naif and sophisticated behaviors are described by strategies $\widetilde{\alpha}^n = \Xi(\alpha^n)$ and $\widetilde{\alpha}^s = \Xi(\alpha^s)$ respectively. Proposition 35 implies that $V(k; \alpha^s) = V\left(\widetilde{k}; \widetilde{\alpha}^s\right)$ and $V(k; \alpha^n) = V\left(\widetilde{k}; \widetilde{\alpha}^n\right)$ (where $\widetilde{k} = k^{\max} - k$) and therefore we obtain

Proposition 41 *If for some level k a sophisticated is better off than a naif in a BACAP $\mathcal{B}$, for level $\widetilde{k} = k^{\max} - k$ a sophisticated is better off than a naif in $D(\mathcal{B})$.*

Definition 42 *We say that sophistication* ***mitigates (exacerbates) misbehavior*** *in a BACAP $\mathcal{B}$, if*

1. *for all state levels a sophisticated is better off (worse off) than a naif; and*

2. *there are state levels for which a sophisticated is strictly better off (worse off) than a naif.*

The above proposition directly implies

Proposition 43 *If sophistication* ***mitigates (exacerbates) misbehavior*** *in a BACAP* $\mathcal{B}$ *it* ***mitigates (exacerbates) misbehavior*** *in* $D(\mathcal{B})$.

Notice that the above proposition implies that the implications of sophistication vs. naiveté are the same for harmful and beneficial addictions since one may be seen as the dual of the other.

We close this section by discussing a result obtained by O'Donoghue and Rabin (2002), namely, there are BACAPs presenting harmful addiction where sophistication exacerbates misbehavior. It is important to notice that this result depends not only on the characteristics of the BACAP but also on the particular equilibrium selection describing sophisticated behavior. Regarding the characteristics, the BACAP must satisfy a condition which they refer to as the Inevitability Condition (IC). In Chapter 1 we showed that IC is equivalent to

$$V(0;H) \geq V(0;A) \quad \text{for } A = 011111... \tag{2.7}$$

that is, for an unnaddicted individual ($k = 0$) the value of always hitting is higher than the value of refraining today and hitting from tomorrow ever after. We also show that (2.7) implies

$$V(k;H) \geq V(k;A) \quad \text{for } A = 011111... \quad \forall k \tag{2.8}$$

The term Inevitability Condition is due to the following interpretation of (2.8): If sophisticated individuals "perceive that addiction is inevitable in the sense that no matter what they do today their future selves will hit forever after" they might as well start hitting today.

Regarding the equilibrium selection, they propose the equilibrium corresponding to the limit of the unique finite-horizon MPE as the horizon becomes long (we will refer to this equilibrium as the ORE). O&R show that when IC holds the ORE is characterized by strategy

$$\alpha^{ORE}(k) = 1 \quad \forall k$$

And therefore sophistication exacerbates misbehavior: A sophisticated is worse off than a naif for all state levels and strictly worse off for any level below k^n (the threshold level characterizing naif behavior).

Proposition 43 directly implies the following:

Proposition 44 *There are BACAPs presenting beneficial addiction where sophistication exacerbates misbehavior.*

Proof. *Just consider the dual of a BACAP satisfying (2.8) and as the selected equilibrium describing sophisticated behavior strategy* $\Xi\left(\alpha^{ORE}\right)$. ■

We believe that the above result is worth mentioning at least for two reasons. First, it challenges the general view that sophistication mitigates misbehavior when costs are immediate and rewards are delayed (as it is the case of beneficial addictions). O'Donoghue and Rabin (1999) show that this view is correct in a context where the activity has to be undertaken **exactly once** and there is a finite number of periods where the individual can do it. However Proposition 44 shows that extrapolating it to the domain of BACAPs (where the activity may be repeated over time) is misleading.

Second, it relies on the selected equilibrium being the ORE, that is, the limit of the unique finite-horizon MPE as the horizon becomes long. In Chapter 1 we provided other MPE such that, for BACAPs presenting harmful addiction and satisfying IC, sophistication mitigates misbehavior.

2.5 Concluding remarks

We have constructed an isomorphism that establishes a dualism between harmful and beneficial addictions: both phenomena are just two sides of the same coin. The dualism holds for the context of BACAPs, where an individual faces at each period the binary choice of undertaking or not an activity; and his payoff depends on his current choice as well as on the history of past behavior.

From a theoretical perspective the dualism is appealing because it allows to give insights for both realms of addiction by analizing either one. We have shown that in a time consistent setting (i.e. when the intertemporal utility exhibits exponential discounting) whether the instantaneous utility function is convex or concave has very different implications: While under convexity the individual follows a cutoff rule, and therefore for a given initial state either always undertakes the activity or always refrains, assuming concavity may yield richer patterns of behavior. Since for some addiction contexts concavity might be the appropriate assumption, we believe that our results deserve attention and motivate further research.

We have also considered a time inconsistent setting (where the intertemporal utility function exhibits hyperbolic discounting) to study the implications of sophistication (i.e. when the individual is fully aware of his time inconsistent preferences) versus naiveté (i.e. when the individual is unaware of his time inconsistency). The isomorphism allows us to state that whether sophistication hurts or benefits the individual does not depend on whether costs are immediate an rewards delayed (beneficial addicition) or rewards are immediate and costs delayed (harmful addiction). This is worth mentioning because in a doing-it-once context (where the activity must be performed exactly once in a finite number of periods) sophistication benefits the individual in the former while it hurts him in the latter case (as has been shown by O´Donoghue and Rabin (1999)). Therefore, extrapolating this result to the domain of addictions would be misleading. To illustrate this, we make use of the dualism to show that there are BACAPs presenting beneficial addiction where sophistication hurts the individual. Nevertheless, we also point out that to obtain this result (which in fact is a translation of a result obtained by O'Donoghue and Rabin (2002) for harmful addictions) the particular equilibrium selection for the sophisticated's intrapersonal game plays a crucial role. If we allow other dominating MPE (as the ones provided in Sections 1.4 and 1.5 of the first chapter) then sophistication would mitigate misbehavior under both harmful and beneficial addictions.

Finally, we would like to point out that the isomorphism holds for arbitrary instantaneous payoff functions (Ψ and Γ) and arbitrary intertemporal utility function (U); however, the specific evolution of the state variable we have assumed in a BACAP (i.e. $k_{t+1} = \gamma k_t + a_t$) plays an important role in establishing it. Therefore it would be of interest to consider whether the dualism still holds under more general conditions for the evolution of the state variable.

2.6 Appendix

In this appendix we show that for the examples of subsection 3.2 the optimal decision rule takes the form

$$a_t = \alpha(k_t) = \begin{cases} 1 & \text{if } k_t \leq k^{\inf} \\ 0 & \text{if } k^{\inf} < k_t < k^{\sup} \\ 1 & \text{if } k_t \geq k^{\sup} \end{cases}$$

Even though the formal proofs are quite long and tedious the idea is simple: we start by showing that there are levels $k_{\inf 1}$ and $k^{\sup 1}$ with

$$0 < k_{\inf 1} < k^{\sup 1} < k^{\max}$$

such that for any level k in the intervals $L_1 = [0, k_{\inf 1}]$ or $U^1 = \left[k^{\sup 1}, k^{\max}\right]$ the optimal action is to undertake the activity. Then we keep enlarging these lower and upper intervals (with the property that for any k belonging to one of them the optimal action is to undertake the activity) until we get to $L = \left[0, k^{\inf}\right]$ and $U = [k^{\sup}, k^{\max}]$ so that for any $k \in \left(k^{\inf}, k^{\sup}\right)$ the optimal action is to refrain.

In what follows, we will use the following notation: $V(k)$ will denote the value function associated to level k, that is, $V(k)$ is the maximal intertemporal utility the individual can obtain when starting in state k. Notice that because of positive internalities V is increasing in k. $A \in \{0,1\}^{\infty}$ will denote a behavior path, that is, an infinite sequence of admissible actions; we will use E to denote the behavior path where the individual always undertakes the activity, that is, $E = (1, 1, ...)$. By $V(k; A)$ we will denote the intertemporal utility achieved by following path A with initial state level k.

2.6.1 Example 1: $a = 0.4; b = 9; c = 10; e = 1; \delta = 0.5; \gamma = 0.8$

We know that $k^c = \frac{c-b}{a} = 2.5$. For $\gamma = 0.8$ we get $k^{\max} = \frac{1}{1-\gamma} = 5$. We start by finding a level $k_{\inf 1}$ such that if $k < k_{\inf 1}$ then $\alpha(k) = 1$.

Claim 45 *For any level $k < k_{\inf 1} = \frac{c-b-\delta(a+b)}{a+(a-c)\delta\gamma} \approx 1.076$, the optimal rule is $\alpha(k) = 1$.*

Proof. *For some starting level $k < 1$ consider the following cases. Case A: the individual refrains in the first period. By doing so, he obtains instantaneous payoff ck and the maximum intertemporal utility he can achieve is $ck + \delta V(\gamma k)$. Since $V(\gamma k) < c\gamma k + \delta V(\gamma^2 k + \gamma + 1)$ we have*

$$ck + \delta V(\gamma k) < ck + \delta c\gamma k + \delta^2 V\left(\gamma^2 k + \gamma + 1\right) \tag{2.9}$$

Case B: the individual undertakes the activity in the first two periods. In this case the maximum intertemporal utility he can obtain is given by

$$[(a + b + e) k - e] + \delta [a(\gamma k + 1) + b] + \delta^2 V\left(\gamma^2 k + \gamma + 1\right) \tag{2.10}$$

Therefore, whenever (2.10) is greater than RHS of (2.9), i.e. whenever

$$\begin{aligned} [(a + b + e) k - e] + \delta [a(\gamma k + 1) + b] &\geq ck + \delta c\gamma k \\ [(b + e) + (a - c)(1 + \delta\gamma)] k &\geq e - \delta(a + b) \\ k &\leq \frac{e - \delta(a + b)}{(b + e + a - c) + (a - c)\delta\gamma} \end{aligned} \tag{2.11}$$

the optimal rule must be $\alpha(k) = 1$.(Since $[(b + e) + (a - c)(1 + \delta\gamma)]$ is negative, in the last step the inequality reverses). Now consider some level k such that $1 \leq k < \frac{1}{\gamma}$ and the same *cases as before. The maximum intertemporal utility of Case A must satisfy*

$$c + \delta V(\gamma k) < c + \delta c \gamma k + \delta^2 V\left(\gamma^2 k + \gamma + 1\right) \tag{2.12}$$

and the maximum value of Case B is

$$[ak + b] + \delta\left[a(\gamma k + 1) + b\right] + \delta^2 V\left(\gamma^2 k + \gamma + 1\right) \tag{2.13}$$

and therefore whenever (2.13) is greater than RHS of (2.12), i.e. whenever

$$k \leq \frac{c - b - \delta(a + b)}{a + (a - c)\delta\gamma} \tag{2.14}$$

the optimal rule must be $\alpha(k) = 1$. *For the specific parameter values of this example we have* $e = c - b$ *and therefore (2.11) and (2.14) are the same, therefore by making* $k_{\inf 1} = \frac{c-b-\delta(a+b)}{a+(a-c)\delta\gamma}$ *we obtain the claim.* ■

So we have found a lower interval for which the optimal rule is to undertake the activity. We turn now to find an upper interval where this also holds. Obviously for all levels above k^c the optimal rule is $\alpha(k) = 1$ but in the following claim we find a value $k^{\sup 1} < k^c$ such that for all $k \geq k^{\sup 1}$ we have $\alpha(k) = 1$. Before proceeding with the claim note that for any $k \geq 1$ straightforward calculations yield

$$V(k; E) = \left(k + \frac{\delta}{1-\delta}\right)\frac{a}{1-\delta\gamma} + \frac{b}{1-\delta} \tag{2.15}$$

(the reason for the condition $k \geq 1$ is simply that for those levels the instantaneous utility function is linear in k and therefore we can obtain the explicit computation). Therefore, if $k \geq 1$ and the optimal path is $E = (1, 1, ...)$ the value function must satisfy $V(k) = V(k; E)$ and therefore we get

$$\forall k \geq 1, V(k) = V(k; E) \Rightarrow V(k) = \left(k + \frac{\delta}{1-\delta}\right)\frac{a}{1-\delta\gamma} + \frac{b}{1-\delta} \tag{2.16}$$

Claim 46 *Let* $k^{\sup 1}$ *be such that* $V\left(k^{\sup 1}\right)=c+\delta V\left(k^{\sup 1}\right)$, *then* $\forall k\geq k^{\sup 1}$ *the optimal rule is* $\alpha\left(k\right)=1$.

Proof. *Suppose that there exists* $\widetilde{k}$ *such that* $V\left(\widetilde{k}\right)>c+\delta V\left(\widetilde{k}\right)$ *and* $\alpha\left(\widetilde{k}\right)=0$. *By definition of the value function we must have* $V\left(\widetilde{k}\right)=c+\delta V\left(\gamma\widetilde{k}\right)<c+\delta V\left(\widetilde{k}\right)$ *which is a contradiction (the inequality follows by* V *being increasing). Therefore* $\alpha\left(\widetilde{k}\right)=1$ *for all* $\widetilde{k}$ *such that* $V\left(\widetilde{k}\right)>c+\delta V\left(\widetilde{k}\right)$. *But since* V *is increasing by making* $k^{\sup 1}=\inf\left\{k\mid V\left(k\right)>c+\delta V\left(k\right)\right\}$ *we obtain the claim.* ■

Note that since $V\left(k^{\sup 1}\right)=V(k^{\sup 1};E)=\frac{c}{1-\delta}$ we can obtain explicitly $k^{\sup 1}$ from (2.16):

$$k^{\sup 1}=\left(\frac{c-b}{1-\delta}\right)\frac{1-\delta\gamma}{a}-\frac{\delta}{1-\delta} \tag{2.17}$$

Given the parameter values of our example we get

$$k^{\sup 1}=2 \text{ and } V\left(k^{\sup 1}\right)=20$$

Now, since $\gamma^{4}k^{\sup 1}=0.819<k_{\inf 1}$, it is impossible that an optimal path has four consecutive 0's. The following claim shows that no optimal path can have three consecutive 0's.

Claim 47 *There is no level* k *such that its optimal path has three consecutive* 0*'s.*

Proof. *For levels above* $k^{\sup 1}$ *the claim follows from Claim 46. So consider* $k<k^{\sup 1}$.*By time consistency it suffices to show that no path* A *beginning with* 000 *is optimal for level* k. *Let* A *be a path beginning with* 0001 *(if it begins with* 0000 *we already know that it is not optimal). Note that for period five state level is* $\gamma^{4}k+1<\gamma^{4}k^{\sup 1}+1\approx 1.819<k^{\sup 1}$

so that $V\left(\gamma^4 k+1\right) < V\left(k^{\sup 1}\right)$. *We will show that* E *dominates* A.*Suppose first that*

$$k \geq \frac{1}{\gamma^3} \approx 1.953 \tag{2.18}$$

so that in the fourth period (after the three consecutive 0*'s) the state level is above* 1; *then we must have*

$$\begin{aligned} V(k;A) &\leq c\left(1+\delta+\delta^2\right)+\delta^3\left[a\gamma^3 k+b\right]+\delta^4 V\left(\gamma^4 k+1\right) \\ &< c\left(1+\delta+\delta^2\right)+\delta^3\left[a\gamma^3 k+b\right]+\delta^4 V\left(k^{\sup 1}\right) \end{aligned} \tag{2.19}$$

but simple calculations show that (2.19) is lower than $V(k;E)$ *when*

$$k \geq \frac{b\delta^3+c(1+\delta+\delta^2)+\frac{\delta^4 c}{1-\delta}-\frac{b}{1-\delta}-\frac{\delta}{1-\delta}\frac{a}{1-\delta\gamma}}{\frac{a}{1-\delta\gamma}-\delta^3\gamma^3 a} \approx 1.885$$

therefore for $k \geq \frac{1}{\gamma^3}$, E *dominates* A. *Now suppose*

$$k < \frac{1}{\gamma^3} \approx 1.953$$

so that in the fourth period (after the three consecutive 0*'s) the state level is below* 1; *then we must have*

$$\begin{aligned} V(k;A) &\leq c\left(1+\delta+\delta^2\right)+\delta^3\left[(a+b+e)\gamma^3 k-e\right]+\delta^4 V\left(\gamma^4 k+1\right) \\ &< c\left(1+\delta+\delta^2\right)+\delta^3\left[(a+b+e)\gamma^3 k-e\right]+\delta^4 V\left(k^{\sup 1}\right) \end{aligned} \tag{2.20}$$

but (2.20) is lower than $V(k;E)$ *when*

$$k \geq \frac{-e\delta^3+c(1+\delta+\delta^2)+\frac{\delta^4 c}{1-\delta}-\frac{b}{1-\delta}-\frac{\delta}{1-\delta}\frac{a}{1-\delta\gamma}}{\frac{a}{1-\delta\gamma}-\delta^3\gamma^3(a+b+e)} \approx -39.063$$

Therefore for $k < \frac{1}{\gamma^3}$ E *dominates* A. ■

Claim 48 *Let* $A = 00111...$ *If* $k \geq \frac{1}{\gamma^2}$ *then there exists* $k^{001} > \frac{1}{\gamma^2}$ *such that*

$$V(k;E) > V(k;A) \Leftrightarrow k > k^{001}$$

Proof. *Notice that since* $k \geq \frac{1}{\gamma^2}$ *then using (2.15) we have*

$$\begin{aligned} V(k;E) &= \left(k + \frac{\delta}{1-\delta}\right)\frac{a}{1-\delta\gamma} + \frac{b}{1-\delta} \\ V(k;A) &= c(1+\delta) + \delta^2\left[\left(\gamma^2 k + \frac{\delta}{1-\delta}\right)\frac{a}{1-\delta\gamma} + \frac{b}{1-\delta}\right] \end{aligned}$$

Define $\Psi(k) = V(k;E) - V(k;A)$ *and notice that* $\frac{d\Psi(k)}{dk} = a(1+\delta\gamma) > 0.$ *Therefore* k^{001} *is the level solving* $\Psi(k) = 0,$ *that is*

$$k^{001} = \frac{(1+\delta)\left(c - b - \frac{a\delta}{1-\delta\gamma}\right)}{a(1+\delta\gamma)} \approx 1.786$$

■

Claim 49 *Let* $A = 0111...$ *If* $k \geq \frac{1}{\gamma^2}$ *then there exists* $k^{01} > \frac{1}{\gamma^2}$ *such that*

$$V(k;E) > V(k;A) \Leftrightarrow k > k^{01}$$

Proof. *Notice that since* $k \geq \frac{1}{\gamma^2}$ *then using (2.15) we have*

$$\begin{aligned} V(k;E) &= \left(k + \frac{\delta}{1-\delta}\right)\frac{a}{1-\delta\gamma} + \frac{b}{1-\delta} \\ V(k;A) &= c + \delta V(\gamma k;E) = c + \delta\left[\left(\gamma k + \frac{\delta}{1-\delta}\right)\frac{a}{1-\delta\gamma} + \frac{b}{1-\delta}\right] \end{aligned}$$

Define $\Psi(k) = V(k;E) - V(k;A)$ *and notice that* $\frac{d\Psi(k)}{dk} = a > 0.$ *Therefore* k^{01} *is the level solving* $\Psi(k) = 0,$ *that is*

$$k^{01} = \frac{c - b - \frac{\delta a}{1-\delta\gamma}}{a} \approx 1.67$$

■

Let k be such that $\frac{k^{\sup 1}-1}{\gamma^3} \leq k$. Note that $\gamma^2 k+1 > \gamma^3 k+1 \geq k^{\sup 1}$ and therefore we already know the optimal path for levels $\gamma^2 k+1$ and $\gamma^3 k+1$: it is path E.Let A be the optimal path for level k. We will show that $A = E$. By claim 47 we know that A cannot begin with 000. Suppose it begins with 001. Then, by time consistency, $A = 001111$...but by claim 48 E dominates A. Therefore A cannot begin with 00. Suppose it begins with 01. Then, by time consistency, $A = 011111$...but by claim 49 E dominates A. Therefore A cannot begin with 0 which means that A must begin with 1 and by time consistency we obtain $A = E$. Thus we obtain the following claim

Claim 50 *Let* $k^{\sup 2} = \frac{k^{\sup 1}-1}{\gamma^3} \approx 1.953$. *If* $k \geq k^{\sup 2}$ *then the optimal rule is* $\alpha(k) = 1$

We have thus enlarged the upper interval where the optimal rule is $\alpha(k) = 1$ to $\left[k^{\sup 2}, k^{\max}\right]$. The idea is to keep enlarging it in the same way: Consider now a level k such that $\gamma^3 k + 1 \geq k^{\sup 2}$ (i.e. $k \geq \frac{k^{\sup 2}-1}{\gamma^3} \approx 1,862$). We know by the previous claim that the optimal path for levels $\gamma^2 k + 1$ and $\gamma^3 k + 1$ is E. Therefore, repeating the argument used to obtain the previous claim we obtain

Claim 51 *Let* $k^{\sup 3} = \frac{k^{\sup 2}-1}{\gamma^3} \approx 1,862$. *If* $k \geq k^{\sup 3}$ *then the optimal rule is* $\alpha(k) = 1$.

Consider now a level k such that $\gamma^3 k + 1 \geq k^{\sup 3}$ (i.e. $k \geq \frac{k^{\sup 3}-1}{\gamma^3} \approx 1,683$). We know by the previous claim that the optimal path for levels $\gamma^2 k + 1$ and $\gamma^3 k + 1$ is E. If $k \geq k^{001}$ we can repeat the argument but if $k < k^{001}$ then path $A = 001111$... dominates E. Therefore we have found $k^{\sup}$.

Lemma 52 *Let* $k^{\sup} = k^{001} = \frac{(1+\delta)\left(c-b-\frac{a\delta}{1-\delta\gamma}\right)}{a(1+\delta\gamma)} \approx 1.786$. *If* $k \geq k^{\sup}$ *then* $\alpha(k) = 1$.

Note that by construction $V(k^{\sup}) = V(k^{\sup}; E) = V(k^{\sup}; 00111...)$. Consider now a level k such that $k^{\sup} > k \geq k^{01}$. For such k any path beginning with 1 is dominated by E and any path beginning by 0 is dominated by 001111..which in turn dominates E. Therefore we have obtained

Claim 53 *Let k be such that $k^{\sup} > k \geq k^{01} = \frac{c-b-\frac{\delta a}{1-\delta\gamma}}{a} \approx 1.67$. Then the optimal rule is $\alpha(k) = 0$. Moreover, its optimal path is* 001111...

But now, by time consistency for any k such that $\gamma^2 k^{\sup} > k > \gamma^2 k^{01}$ the optimal path is E. Therefore

Claim 54 *Let k be such that $1,143 \approx \gamma^2 k^{\sup} > k \geq \gamma^2 k^{01} \approx 1,067$, then the optimal rule is $\alpha(k) = 1$.*

This claim together with claim 45 produces

Claim 55 *For every $k \leq \gamma^2 k^{\sup} \approx 1,143$, the optimal rule is $\alpha(k) = 1$.*

Consider now a level k such that $k^{01} > k \geq \frac{1}{\gamma} = 1,25$. Any path beginning with 1 is dominated (by time consistency) by E. But by claim 49 we know that 0111... dominates E therefore the optimal rule must be $\alpha(k) = 0$ which together with claim 53 yields

Claim 56 *For every k such that $k^{\sup} > k \geq \frac{1}{\gamma} = 1,25$ the optimal rule is $\alpha(k) = 0$.*

Consider now k such that $1,228 \approx \frac{k^{\sup}-1}{\gamma^2} < k < \frac{1}{\gamma}$. If its optimal path begins with 1 it must be E by time consistency. If its optimal path begins with 0 then it must begin with 01 by claim 55, but then by time consistency it must be 0111... Since by claim 49 0111... dominates E we conclude

Claim 57 *For every k such that $k^{\sup} > k \geq \frac{k^{\sup}-1}{\gamma^2} \approx 1,228$, the optimal rule is $\alpha(k) = 0$.*

Consider now a level k such that $1,143 \approx \gamma^2 k^{\sup} < k < \frac{k^{\sup}-1}{\gamma^2} \approx 1,228$. If its optimal path begins with 1 it must be E. If its optimal path begins with 0, it must be, by time consistency, $A = 0100111$...But

$$V(k;A) = c + \delta\left[(a+b+e)\gamma k - e\right] + \delta^2 c + \delta^3 c + V(\gamma^2(\gamma^2 k + 1))$$

and we know that $V(\gamma^2(\gamma^2 k+1)) = V\left(\gamma^2(\gamma^2 k + 1); E\right)$. Solving $V(k;E) - V(k;A) \geq 0$ for k we obtain

$$k \leq \frac{c\left(1+\delta^2+\delta^3\right) - \delta e - \left(1-\delta^4\right)\left(\frac{b}{1-\delta} + \frac{\delta a}{(1-\delta)(1-\delta\gamma)}\right) + \frac{a}{1-\delta\gamma}\delta^4\gamma^2}{\frac{a}{1-\delta\gamma}\left(1-\delta^4\gamma^4\right) - \delta\gamma(a+b+e)} \approx 1,203$$

which yields $k^{\inf} = \frac{c\left(1+\delta^2+\delta^3\right) - \delta e - \left(1-\delta^4\right)\left(\frac{b}{1-\delta} + \frac{\delta a}{(1-\delta)(1-\delta\gamma)}\right) + \frac{a}{1-\delta\gamma}\delta^4\gamma^2}{\frac{a}{1-\delta\gamma}\left(1-\delta^4\gamma^4\right) - \delta\gamma(a+b+e)} \approx 1,203$ and therefore the optimal rule is characterized by

$$\alpha(k) = \begin{cases} 1 & \text{if } k \leq k^{\inf} \\ 0 & \text{if } k^{\inf} < k < k^{\sup} \\ 1 & \text{if } k \geq k^{\sup} \end{cases} \tag{2.21}$$

Proposition 58 *The long-run behavior implies undertaking the activity (i.e. initial conditions do not matter for the long-run behavior)*

Proof. Consider k such that $k^{\inf} < k < k^{\sup}$ and notice that the sequence of actions 01 or 001 both lead to a state level above k. Therefore by following the optimal rule $\alpha(k)$ given in (2.21) the individual eventually reaches a state level above $k^{\sup}$. ■

2.6.2 Example 2: $a = 0.4; b = 9; c = 10; e = 1; \delta = 0.5; \gamma = 0.65$

We know that $k^c = \frac{c-b}{a} = 2.5$. For $\gamma = 0.65$ we get $k^{\max} = \frac{1}{1-\gamma} \approx 2.857$. As in Example 1, we start by finding a level $k_{\inf 1}$ such that if $k < k_{\inf 1}$ then $\alpha(k) = 1$.

Claim 59 *For any level $k < k_{\inf 1} = \frac{c-b-\delta(a+b)}{a+(a-c)\delta\gamma} \approx 1.360$ the optimal rule is $\alpha(k) = 1$.*

Proof. *Exactly the same as in claim 45.* ■

Also, by replicating the argument in claim 46, we get

Claim 60 *Let $k^{\sup 1}$ be such that $V\left(k^{\sup 1}\right) = c + \delta V\left(k^{\sup 1}\right)$, then $\forall k \geq k^{\sup 1}$ the optimal rule is $\alpha(k) = 1$.*

Using (2.17) we get

$$k^{\sup 1} = \left(\frac{c-b}{1-\delta}\right)\frac{1-\delta\gamma}{a} - \frac{\delta}{1-\delta} \approx 2.375 \text{ and } V\left(k^{\sup 1}\right) = 20$$

Now, given that $\gamma^2 k^{\sup 1} \approx 1.003 < k_{\inf 1}$, an optimal path cannot have three consecutive 0's. The following claim shows that no optimal path can have two consecutive 0's.

Claim 61 *There is no level k such that its optimal path has two consecutive 0's.*

Proof. Similar to claim 47. ■

Therefore if an optimal path begins with 0 the second action must be a 1. Notice that since $\gamma^2 k^{\sup 1} + 1 < k^{\sup 1}$ then for every $k \in \left[0, k^{\sup 1}\right]$ we have $\gamma^2 k + 1 < k^{\sup 1}$. Let A be a path beginning with 01 and suppose that $k > \frac{1}{\gamma} \approx 1.538$. Since for the third period the state level is below $k^{\sup 1}$, we must have

$$V(k; A) < c + \delta\left[a\gamma k + b\right] + \delta^2 V\left(k^{\sup 1}\right)$$

and thus $V(k;E) - V(k;A) \geq 0$ holds when

$$V(k;E) - \left[c + \delta\left[a\gamma k + b\right] + \delta^2 V\left(k^{\sup 1}\right)\right] \geq 0$$

which in turn is satisfied for

$$k \geq \frac{c + \delta b + \delta^2 V\left(k^{\sup 1}\right) - \frac{\delta}{1-\delta}\frac{a}{1-\delta\gamma} - \frac{b}{1-\delta}}{a\left[\frac{1}{1-\delta\gamma} - \delta\gamma\right]} = k^{\sup 2} \approx 1,962 \tag{2.22}$$

which implies that for any $k \geq k^{\sup 2}$, E dominates A and therefore we have obtained the following claim

Claim 62 *For any $k \geq k^{\sup 2}$ as given by (2.22), the optimal rule is $\alpha(k) = 1$.*

Repeating the above argument with $k^{\sup 2}$ instead of $k^{\sup 1}$ we can keep on enlarging the upper interval for which the optimal rule is to undertake the activity:

Claim 63 *For any $k \geq k^{\sup 3}$ where $k^{\sup 3} = \frac{c+\delta b+\delta^2 V\left(k^{\sup 2}\right) - \frac{\delta}{1-\delta}\frac{a}{1-\delta\gamma} - \frac{b}{1-\delta}}{a\left[\frac{1}{1-\delta\gamma} - \delta\gamma\right]} \approx 1,829$, the optimal rule is $\alpha(k) = 1$.*

Repeating the argument with $k^{\sup 3}$ instead of $k^{\sup 2}$ we get

Claim 64 *For any $k \geq k^{\sup 4}$ where $k^{\sup 4} = \frac{c+\delta b+\delta^2 V\left(k^{\sup 3}\right) - \frac{\delta}{1-\delta}\frac{a}{1-\delta\gamma} - \frac{b}{1-\delta}}{a\left[\frac{1}{1-\delta\gamma} - \delta\gamma\right]} \approx 1,787$, the optimal rule is $\alpha(k) = 1$*

Keeping on with the iterations the process converges to the level where the value of path E is equal to the value of path $A = 010101..$ Straightforward calculations show that for $k > \frac{1}{\gamma} \approx 1,538$

$$V(k;A) = \frac{1}{1-\delta^2}\left[c + b\delta + \frac{a\gamma\delta^3}{1-\delta^2\gamma^2}\right] + a\gamma\delta k\frac{1}{1-\delta^2\gamma^2}$$

But then, using (2.15) and doing some calculations $V(k;E) - V(k;A) \geq 0$ if and only if $k \geq k^{\sup}$ where

$$k^{\sup} = \frac{\left(1-\delta^2\gamma^2\right)}{a}\left[\begin{array}{c} \frac{1}{1-\delta^2}\left(c + b\delta + \frac{a\gamma\delta^3}{1-\delta^2\gamma^2}\right) \\ -\frac{b}{1-\delta} - \frac{a\delta}{(1-\delta)(1-\delta\gamma)} \end{array}\right] \approx 1.765 \tag{2.23}$$

which means that A dominates E for any $k \in \left[\frac{1}{\gamma}, k^{\sup}\right)$. But notice that for any such k undertaking the activity leads to a state level above 2 and therefore to path E, thus we have obtained the following two claims

Claim 65 *For any $k \geq k^{\sup}$, the optimal rule is $\alpha(k) = 1$.*

Claim 66 *For any $k \in \left[\frac{1}{\gamma}, k^{\sup}\right)$, the optimal rule is $\alpha(k) = 0$.*

Moreover, notice that the optimal path for any level $k \in \left[\frac{1}{\gamma}, k^{\sup}\right)$ is in fact $A = 0101010...$ because for any such k, following the rule $\alpha(k) = 0$ leads the state level to the interval $[1, \gamma k^{\sup}) = [1, \gamma k^{\sup}) \subset [1, 1.148)$ so that next period the optimal rule is $\alpha(k) = 1$ by claim 59. But then for the third period the state level falls again in the interval $\left[\frac{1}{\gamma}, k^{\sup}\right)$.

It remains to characterize the optimal rule for levels in the interval $\left(k_{\inf 1}, \frac{1}{\gamma}\right)$. So consider any $k \in \left(k_{\inf 1}, \frac{1}{\gamma}\right)$. If its optimal path begins with a 1 then it must be (by time consistency) path E because $\gamma k_{\inf 1} + 1 > k^{\sup}$. If its optimal path begins with 0, then by claim 61 it must begin with 01. But then for the third period the state level falls in the interval $\left[\frac{1}{\gamma}, k^{\sup}\right)$ because $\gamma^2 k_{\inf 1} + 1 > \frac{1}{\gamma}$ and $\gamma^2 \frac{1}{\gamma} + 1 < k^{\sup}$. Therefore it its optimal path begins with 0 it must be path $A = 0101010...$ So all we have to do is to find for which level below $\frac{1}{\gamma}$ the value of path E is equal to the value of path A. But for any $k \in \left(k_{\inf 1}, \frac{1}{\gamma}\right)$

simple calculations show that

$$V(k;A)=\begin{bmatrix} c+\delta\left[(a+b+e)\gamma k-e\right]+ \\ \delta^2\left[\frac{1}{1-\delta^2}\left[c+b\delta+\frac{a\gamma\delta^3}{1-\delta^2\gamma^2}\right]+a\gamma\delta\left(\gamma^2 k+1\right)\frac{1}{1-\delta^2\gamma^2}\right]\end{bmatrix}$$

and solving for $V(k;A)=V(k;E)$ yields

$$k^{\inf}=\frac{c-\delta e+\frac{\delta^2}{1-\delta^2}\left(c+b\delta+\frac{a\gamma\delta^3}{1-\delta^2\gamma^2}\right)+\frac{a\gamma\delta^3}{1-\delta^2\gamma^2}-\frac{b}{1-\delta}-\frac{a\delta}{(1-\delta)(1-\delta\gamma)}}{\frac{a}{1-\delta\gamma}-\delta\gamma\left(a+b+e\right)-\frac{a\gamma^3\delta^3}{1-\delta^2\gamma^2}}\approx 1.502$$

and therefore the optimal rule is characterized by

$$\alpha(k)=\begin{cases}1 & \text{if } k\le k^{\inf} \\ 0 & \text{if } k^{\inf}<k<k^{\inf} \\ 1 & \text{if } k\ge k^{\sup}\end{cases}$$

Consider levels $\overline{k}=\frac{1}{1-\gamma^2}\approx 1.732$ and $\underline{k}=\frac{\gamma}{1-\gamma^2}\approx 1.126$ and notice that $\gamma\overline{k}=\underline{k}$ and $\gamma\underline{k}+1=\overline{k}$. Since $\underline{k}<k^{\inf}<\overline{k}<k^{\sup}$ we have $\alpha\left(\overline{k}\right)=0$ and $\alpha\left(\underline{k}\right)=1$ so that once the individual reaches one of these states he keeps switching from one to another forever. In particular this happens for any initial level $k\in\left(k^{\inf},k^{\sup}\right)$. Therefore we have the following proposition.

Proposition 67 *For any level* $k\in\left(k^{\inf},k^{\sup}\right)$ *the optimal behavior path is* 010101...

Bibliography

[1] Becker, G. S. and K. M. Murphy (1988). "A Theory of Rational Addiction." *Journal of Political Economy*, 96, 675-700.

[2] Laibson, D. (1994). "Essays in Hyperbolic Discounting." Economics, MIT

[3] O'Donoghue, T. and M. Rabin (1999). "Doing It Now or Later." *American Economic Review*, 89, 103-124.

[4] O'Donoghue, T. and M. Rabin (2002). "Addiction and Present-Biased Preferences." Unpublished paper.

[5] Phelps, E. S. and R. A. Pollak (1968). "On Second-best National Saving and Game-equilibrium Growth." *Review of Economic Studies*, 35, 185-199.

Chapter 3

On p-Beauty Contest Integer Games

3.1 Introduction

The basic p-Beauty Contest Game[1] (p-BCG), consists of a number $N > 1$ of players, a real number $0 < p < 1$, and a closed interval $[l, h]$ with l and h integers. In such a game, N players have to choose simultaneously real numbers from the given interval. The mean of all chosen numbers is calculated and the winner is the person who chose the closest number to p times the mean. The winner receives a fixed prize (in the case of many winners the prize is equally divided among them), while the other players receive nothing.

The game was first introduced by Moulin (1986) as a means to illustrate an equi-

[1] This name was introduced by Duffy and Nagel (1997) and Ho, Camerer, and Weighel (1998) and is due to a famous analogy by Keynes (1936) between stock market investment and "those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole". Other authors such as Nagel (1995) have used the name "guessing game".

librium obtained by iterated deletion of (weakly) dominated strategies. The equilibrium thus obtained was all players playing the lower boundary of the interval, l. Predicting such an equilibrium as an outcome of the game relies in the assumption of rationality of all the players (in the sense that no player is playing a weakly dominated strategy) and that all the players know that the other players are rational and so on ad infinitum. Starting with the work of Nagel (1994) a variety of experiments on the p-BCG have been conducted to study iterated dominance and learning (for a survey see Nagel(1998)). If we apply the process of iterated best-reply to this game (rather than the iterated deletion of weakly dominated strategies), it turns out that all players playing l is also the unique Nash equilibrium. However, if we allow the players to choose only among integer numbers[2] in the given interval, this is no longer true: although every player playing l continues to be a Nash equilibrium, there could be more. The multiplicity of equilibria makes this game appealing to the empirical issue of equilibrium selection.

The purpose of this chapter is first, to characterize the Nash Equilibria of a p-Beauty Contest Integer Game (p-BCIG) and second to give some insights for further experiments. The chapter is organized as follows: In Section 3.2 we formally define a p-Beauty Contest Integer Game and completely characterize its Nash Equilibria in pure strategies for two cases: when the prize a winner earns is fixed and when it is increasing in the winning number. The reason for concentrating in these two cases is simply that those are the cases that have been treated in the experiments on p-BCG conducted so far. In Section 3.3 we show that every experimental p-BCG can be thought of as a p-BCIG. In Section 3.4 we

[2] Osborne and Rubinstein (1994) pose an exercise with this restriction and $p = 2/3$. Some experiments have made the integer restriction explicit (Thaler(1997)) while in other experiments players actually chose only integer numbers though it was not a restriction in the instructions (e.g., Ho et al (1998)). Nagel (1995) and Nagel (1998), mentions the p-BCIG, however their characterization of equilibria is incomplete.

show that the equivalence between the iterated dominance and iterated best-reply holding for the p-BCG may fail for the p-BCIG. Finally, Section 3.5 states the conclusions of our findings.

3.2 Nash Equilibria of a p-BCIG

For a given $0 < p < 1$, a given interval $[l, h]$ and a set of players $\mathcal{N} = \{1, 2, ..., N\}$ with $N > 1$, define the set of actions (or strategies) for each player $i \in \mathcal{N}$ as

$$A = [l, h] \cap \mathcal{Z}$$

where $\mathcal{Z}$ is the set of integer numbers. Denote by $x_i \in A$ a strategy (or action) for player i and with $S = (x_1, x_2, ..., x_N)$ a strategy profile. Given a strategy profile S define $\mu(S)$ as the mean of S times p :

$$\mu(S) = p\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

The players' payoff functions are given by

$$\forall i \in \mathcal{N}, \;\; \pi_i(S) = \begin{cases} z > 0 & \text{if} \quad \forall j \in \mathcal{N}, \; |x_i - \mu(S)| \leq |x_j - \mu(S)| \\ 0 & \text{otherwise} \end{cases}$$

Definition 68 *Given a strategy profile S, we say that x is a* winning number *if for some $i \in \mathcal{N}$, $x = x_i$ and*

$$\forall j \in \mathcal{N}, \quad |x_i - \mu(S)| \leq |x_j - \mu(S)|$$

Definition 69 *Given a strategy profile S, we say that player i is a* winner *if x_i is a winning number*

Proposition 70 *If for strategy profile S, player i is not a winner then by unilaterally deviating to some strategy profile S' he can become one.*

Proof. *Define* $m = p\left(\frac{1}{n}\sum_{j\neq i} x_j\right)$ *and let player i deviate from S to S' by choosing an integer number $x_i' = x$ such that*

$$\frac{m-\frac{1}{2}}{1-\frac{p}{N}} < x < \frac{m+\frac{1}{2}}{1-\frac{p}{N}} \tag{3.1}$$

Note that such an integer exists since $\frac{m+\frac{1}{2}}{1-\frac{p}{N}} - \frac{m-\frac{1}{2}}{1-\frac{p}{N}} = \frac{1}{1-\frac{p}{N}} > 1$*. But if (3.1) holds then we must have*

$$-\frac{1}{2} < m + \frac{px}{N} - x < \frac{1}{2}$$

which means that x is the closest integer to $\mu(S')$. Therefore player i becomes a winner when deviating to S'. ■

Corollary 71 *In a Nash equilibrium of a p-BCIG, every player must be a winner (thus there can be at most two winning numbers).*

Proposition 72 *In a p-BCIG, if the prize of the game is fixed and equally divided among the winners (FEDAW), then in a Nash equilibrium there is only one winning number.*

Proof. *If $N = 2$, the lowest choice is the winning number. So assume $N > 2$ and suppose strategy profile S is a Nash equilibrium with two winning numbers x and y. Without loss of generality assume $x < y$. Let m be the number of players choosing x (notice that then, by Corollary 71, the number of players choosing y must be $N - m$). Suppose $m > 1$ and let z denote the fixed prize so that every winner is receiving z/N. Let S' be the strategy profile where one of the players choosing x unilaterally deviates by choosing y. Then y would be closer to $\mu(S')$ than x and therefore only the players choosing y will win* *receiving* $z/(N-m+1)$. *Since* $N-m+1<N$ *the player deviating has incentives to do so. Now suppose* $m=1$, *then one of the players choosing* y *can improve by deviating to* x. *Therefore there cannot be a Nash equilibrium with two winning numbers.* ■

Proposition 73 *In a p-BCIG, if the prize of the game is strictly positive, increasing in the winning number and divided by the number of winners (IWND) then, in a Nash equilibrium there is only one winning number.*

Proof. *If* $N=2$, *the lowest choice is the winning number. So assume* $N>2$ *and suppose strategy profile* S *is a Nash equilibrium with two winning numbers* x *and* y. *Without loss of generality assume* $x<y$. *Let* $z(x)$ *and* $z(y)$ *be the prize for players choosing* x *and players choosing* y *respectively. By assumption we must have* $z(x)<z(y)$. *Also, by Corollary 71, the number of winners is* N, *thus, a player choosing* x *is receiving* $z(x)/N$ *and a player choosing* y *is receiving* $z(y)/N$. *Consider a player choosing* x*: if he unilaterally deviates to strategy profile* S' *by choosing* y, *then* y *would be closer to* $\mu(S')$ *than* x. *Therefore the winning number would be* y *and the player deviating would get no less than* $z(y)/N$ *which is greater than* $z(x)/N$. *Since a player choosing* x *has incentives to deviate we conclude that there cannot be a Nash equilibrium with two winning numbers.* ■

Now consider a strategy profile S where each player plays the same integer number x. We have the following results:

Proposition 74 $\forall p\in(0,1)$, *no player has incentives to deviate from* S *by playing an integer number* $y>x$.

Proof. *Trivial.* ■

Proposition 75 *$\forall p \in (0,1)$, if a player has incentives to deviate from S by playing an integer number $y < x$ then he has also incentives to deviate from S by playing $x-1$.*

Proof. *Let $y = x - k$ for some integer $k \geq 1$. Let S' be the strategy profile where one player deviates from S by playing y. Since by assumption the player deviating has incentives to do so it must be true that*

$$p\frac{x(N-1)+x-k}{N} - (x-k) < x - \frac{px(N-1)+x-k}{N} \tag{3.2}$$

where the LHS is the distance to $\mu(S')$ for the player deviating and the RHS is the distance to $\mu(S')$ for the players not deviating. Rearranging (3.2) yields

$$2x(p-1) + k\left(1 - \frac{2p}{N}\right) < 0 \tag{3.3}$$

Since $1 > \frac{2p}{N}$, then if (3.3) holds for some $k \geq 1$ it also holds for $k = 1$. Therefore a player has incentives to deviate by playing $x-1$. ■

Let $F(p,x) = 2x(p-1) + \left(1 - \frac{2p}{N}\right)$.

Proposition 76 *In a p-BCIG, a strategy profile S where every player plays the same integer x is a Nash equilibrium if and only if $F(p,x) \geq 0$ or $x = l$.*

Proof. *Suppose S is a Nash equilibrium. Then a player has no incentives to deviate by playing $x-1$, or, if he has them, he cannot do so (because $x-1 < l$), therefore, either $F(p,x) \geq 0$ or $x = l$. Now suppose $F(p,x) \geq 0$ or $x = l$. If $x = l$ then, by Proposition 74 S is a Nash equilibrium. If $F(p,x) \geq 0$ then a player has no incentives to deviate from S by playing $x-1$ (since $F(p,x)$ is just the LHS of (3.3) with $k = 1$). Therefore, by Proposition 75, a player has no incentives to deviate from S by playing a lower integer than*

x, but by Proposition 74 a player has no incentives to deviate by playing a higher number than x. Therefore S is a Nash equilibrium. ■

Remark 77 *Notice that a player unilaterally deviating from S by playing $x-1$, remains a winner when $F(p,x)=0$ but does not when $F(p,x)>0$.*

Proposition 78 *For $N=2$, all players playing l is the unique Nash equilibrium.*

Proof. *When $N=2$, $F(p,x)=(1-p)(1-2x)>0$ only for the integer $x=0$. Thus both players playing l is the unique Nash equilibrium.* ■

Proposition 78 completely characterizes the Nash Equilibria for the case $N=2$ and Proposition 74 implies that every player playing l is always a Nash Equilibrium. We will now focus on the case where $N>2$ and where the strategy profile is $S=(x,x,...,x)$, i.e. every player is playing the same integer number $x>l$. Let $P(x)$ be such that $F(P(x),x)=0$. Solving for $P(x)$ yields

$$P(x)=\frac{2x-1}{2x-\frac{2}{N}} \tag{3.4}$$

where clearly $0<P(x)<1$ for $N>2$ and $x>l\geq 0$.

Now notice that $\frac{\partial F(p,x)}{\partial p}=2x-\frac{2}{N}>0$ for $N>2$ and $x>l\geq 0$. Since F is strictly increasing in p we have that

$$F(p,x)\geq 0 \quad \text{if} \quad 1>p\geq P(x)$$

$$F(p,x)<0 \quad \text{if} \quad P(x)>p>0$$

Corollary 79 *For every x integer belonging to $(l,h]$, if $p\geq P(x)$ then $S=(x,x,...,x)$ is a Nash equilibrium.*

Proof. *If $p\geq P(x)$ then $F(p,x)\geq 0$ and by Proposition 76 S is a Nash equilibrium.* ■

Now notice that $P(x)$ is strictly increasing in x because for $N > 2$

$$\frac{\partial P(x)}{\partial x} = \frac{2 - \frac{4}{N}}{\left(2x - \frac{2}{N}\right)^2} > 0$$

Corollary 80 *$\forall p \in (0,1)$, if $S = (x, x, ..., x)$ is a NE then for y integer such that $l < y < x$, strategy profile $S' = (y, y, ..., y)$ is also a NE.*

Proof. *Take any y satisfying $l < y < x$. Since S is a NE we must have $p \geq P(x)$. Since $P(x)$ is increasing in x, we have $p \geq P(y)$ and then S' is a NE by Corollary 79.* ■

Notice that for a particular p-BCIG, Corollaries 79 and 80 completely characterize those NE where there is a unique winning number: to see this, we just need to solve for x the inequalities $p \geq \frac{2x-1}{2x-\frac{2}{N}}$ and $x > l$ which yield

$$l < x < \frac{1 - \frac{2p}{N}}{2\,(1-p)} \tag{3.5}$$

Let $B\,(p, N) = \frac{1-\frac{2p}{N}}{2(1-p)}$. Since by Proposition 74 we know that every player playing l is a NE, we have obtained the following

Proposition 81 *In a p-BCIG, a strategy profile S is a NE with only one winning number if and only if*

1. *In S every player plays the same integer x in the interval $[l, h]$; and*

2. *$B\,(p, N) \geq x$ or $x = l$.*

Propositions 72, 73 and 81 imply our first important result which we state as a theorem:

Theorem 82 *In a p-BCIG, if the prize is either FEDAW or IWND then a strategy profile S is a NE if and only if*

1. *In S every player plays the same integer x in the interval $[l, h]$; and*

2. *$B(p, N) \geq x$ or $x = l$.*

Notice from $B(p, N)$ that as p goes to 1 then any integer in the interval is a NE[3]. Notice also that in the case of multiple equilibria, if the prize is IWND, the higher integer x in the interval satisfying $B(p, N) \geq x$ is Pareto dominant.

Remark 83 Mixed Equilibria. *In the p-BCG the unique NE in pure strategies is all players playing the lower bound l and it turns out that this is also the unique mixed equilibrium of the game. Is the set of mixed equilibria equal to the set of pure strategies equilibria in any p-BCIG? The following example shows that the answer is no: consider the FEDAW p-BCIG $(N, p, l, h) = \left(3, \frac{5}{6}, 0, 100\right)$. Since $B\left(\frac{5}{6}, 3\right) = \frac{4}{3}$ we know by Theorem 82 that the pure strategy NE are 0 and 1. It is easy to check that every player playing 0 and 1 each with probability 0.5 is a mixed NE.*

Remark 84 The p-Beauty Contest Decimal Game. *Consider now a p-BCG in the interval $[l, h]$ where players are allowed to choose among decimal numbers up to D decimal positions. Let's call this game a p-BCDG. It is easy to see that this game is equivalent to the p-BCIG in the interval $[l(D), h(D)]$ where $l(D) = l * 10^D$ and $h(D) = h * 10^D$. The equivalence relation is given by $S \Leftrightarrow S^D = S * 10^D$ where S is a strategy in the p-BCDG and S^D is a strategy in the p-BCIG.*

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[3]If the interval is $[0, 100]$, as it has been for most of the experiments on p-BCG, the following are very easy to prove:

1. For $p \geq \frac{3}{4}$, the p-BCIG has multiple equilibria.
2. For $p \leq \frac{1}{2}$, the p-BCIG has a unique NE.
3. For $\frac{1}{2} < p < \frac{3}{4}$, the p-BCIG has multiple equilibria provided the number of players is sufficiently large.

3.3 Experimental Implications

The p-BCG has been widely used to test iterated dominance and learning. In most of the experiments it has been assumed that the game has a unique NE but, in fact, any experimental p-BCG can be thought of as a p-BCIG: the reason for this is that in calculating $\mu(S)$ for any strategy profile S one must use a decimal approximation which implies that we are facing a p-BCDG which in turn (see Remark 84) is equivalent to a p-BCIG. Therefore, all of the results obtained in Section 3.2 also apply to an experimental p-BCG (E-p-BCG) through its equivalent p-BCIG.

It is easy to see that the exact number of equilibria of a E-p-BCG defined in the interval $[l, h]$ is given by

$$E(N, p, l, h) = \max\{\min\{h, \lfloor B(p, N)\rfloor\} - l, 0\} + 1$$

In the following table we show some E-p-BCG for which we have calculated the number of equilibria $E(N, p, l, h)$ in column E.

Authors	N	p	$[l, h]$	Choice	Prize	$B(p, N)$	E
Ho et al. (1998)	7	0.7	$[0, 100]$	reals	\$3.50	1.33	2
	7	0.9			\$3.50	3.71	4
	3	0.7			\$1.50	0.89	1
	3	0.9			\$1.50	2.00	3
Nagel (1995)	12	2/3	$[0, 100]$	reals	\$$x$ winning number	1.33	2
	17	2/3				1.38	2
Bosch and Nagel (1997)	3696	2/3	$[1, 100]$	decim	\$100.000	1.5	1
Thaler (1997)	1460	2/3	$[0, 100]$	integ	2 NY Tickets	1.5	2
Nagel and Selten (1998)	2728	2/3	$[0, 100]$	decim	1000 DM	1.5	2

In order to know the NE of a particular E-p-BCG we just need to know the decimal approximation used in the calculations. As an example, for Nagel (1995) the approximation in the calculations used was of one decimal, this means that the NE for that game were 0 and 0.1[4]

Now, the aim of all these experiments was to find out whether the players tend to equilibrium or not and, if doing so, establishing the way they did. The last three studies were one-shot games. However, in the first three, the game was repeated a number of times to study whether the players "learned" to play an equilibrium or not. Since in all of them a decimal approximation was made, the natural question is: *which are the equilibria players learned to play?*

3.4 Theoretical Predictions for the p-BCIG

Predicting the outcome of a game constitutes one of the main issues of game theory. On simultaneous-move games the concepts of strict dominance and rationalizable strategies (see Fudenberg and Tirole (1991) or Mas-Colell, Whinston and Green (1995)) are useful to restrict the set of possible outcomes relying solely in the assumption of rationality: a rational player should never play a strictly dominated strategy nor a strategy that is never a best-response. Therefore, the iterated deletion of these strategies is justified. It is easy to see that in the p-BCG there are no strictly dominated strategies which implies that every strategy might be a best-response. Therefore, the concepts of strict dominance or rationalizable strategies are of no use for narrowing down the set of possible outcomes of this game. The most used reasoning processes to refine the theoretical predictions of this

[4] For this same example if the aproximation were of D decimal positions the equilibria would be 0 and 10^{-D}.

game have been the iterated deletion of weakly dominated strategies (IDWDS), and the iterated best-reply[5] (IBR). In the first one, it is assumed that players iteratively eliminate weakly dominated strategies, the process ending when no player has a weakly dominated strategy left. In the second one, players act à la Cournot: starting from a hypothetical strategy profile they iteratively best-reply to the previous profile, the process ending when a fixed-point is reached.

It is easy to see that for the p-BCG, both processes lead to the unique prediction of all players playing l (which is actually the unique NE), independently of the order of deletion and the initial strategy profile for the IDWDS and the IBR respectively. However, the situation changes dramatically in the p-BCIG: we will show that under very mild conditions the IDWDS will not lead to a single prediction, while depending on the initial strategy profile, the IBR process will lead to one. Therefore, the equivalence between the IDWDS and the IBR processes that we had in the case of a unique NE fails under multiple equilibria. This is worth noticing since the experimental results show that individuals use rather IBR than IDWDS (see e.g., Nagel (1995), Stahl (1996), Ho et al.(1998)).

Proposition 85 *Consider a p-BCIG where $k \in (l, h]$ is its highest NE. Let*

$$S(t) = (s_1(t), s_2(t), ..., s_N(t))$$

be the strategy profile at iteration t in a IBR process, with $S(0)$ being the initial strategy profile. If $\forall i$, $s_i(0) \geq k$ then the IBR process leads to the unique prediction of all players playing k.

[5] We will focus here only in the simplest IBR procedure which takes into account only the immediately previous period's outcome. This procedure was first introduced by Cournot (1838). For more sophisticated IBR procedures, see Ho, et al. (1998).

Proof. *Since* $s_i(0) \geq k$ *for all* i*, then* $\mu(S(0))$ *is closer to* k *than to* $k-1$*. Therefore, in the next iteration we have* $s_i(1) \geq k$ *for all* i*, and so on:* $\forall t \geq 1$, $s_i(t) \geq k$*. Suppose that at some iteration* t *there are some players choosing higher integers than* k*. Let* $h(t)$ *be the highest of those integers. We claim that* $s_i(t+1) < h(t)$ *for all* i*:* $\mu(S(t)) \leq p * h(t)$ *and* $p * h(t)$ *is closer to* $h(t)-1$ *than to* $h(t)$ *so a best-reply at time* $t+1$ *must be a lower number than* $h(t)$*. Therefore the IBR process leads to every player choosing* k*.* ■

Proposition 86 *Let* k *be an integer such that in the p-BCIG all players playing* k *constitutes a strict*[6] *NE. Then no player can eliminate* k *by IDWDS.*

Proof. *Suppose that at some iteration* t *player* i *is the first to eliminate* k*. Let* $S_i(t)$ *denote the set of all possible strategies for player* i *at iteration* t*. Then it must be true that some strategy* $s' \neq k$ *and* $s' \in S_i(t)$ *weakly dominates* k *for player* i *at iteration* t*. But then, by definition of weak domination we must have*

$$\forall s_{-i} \in S_{-i}(t), \quad U_i\left(s', s_{-i}\right) \geq U_i\left(k, s_{-i}\right)$$

with strict inequality for at least one s_{-i} *(where* s_{-i} *denotes a strategy profile for all players except* i *and* $S_{-i}(t)$ *is the set of possible strategy profiles, at iteration* t*, for all players except* i*). Let* k_{-i} *be the strategy profile where all players but* i *play* k*. Since by assumption at iteration* t *no player except* i *has eliminated* k*, we have* $k_{-i} \in S_{-i}(t)$*. But by strictness of the NE* $U_i(s', k_{-i}) < U_i(k, k_{-i})$ *so that* s' *does not weakly dominate* k *for player* i *at iteration* t*. Therefore no player can eliminate* k *by IDWDS.* ■

Proposition 87 *In a IWND p-BCIG every NE is strict.*

[6] Following Harsanyi (1973), we say that a NE is strict if each player has a unique best-reply to his rival's strategies.

Proof. *Consider any strategy profile S constituting a NE. By Theorem 82 we know that S must be of the form where all players play the same integer k . Since the game is IWND we know that the payoff for player i if not deviating must be $U_i(k, k_{-i}) = z(k)/N > 0$. Clearly, if deviating to $k' > k$ player i obtains 0. If deviating to $k' < k$ player i obtains at most $z(k')/N < z(k)/N$. Therefore the unique best reply to k_{-i} is k.* ■

Proposition 88 *In a FEDAW p-BCIG with $M \geq 2$ equilibria, a NE where all players play the same integer k is strict if and only if $k < B(p, N)$.*

Proof. *Immediate by Remark 77 and the fact that $F(p, k) > 0$.* ■

Notice that by Proposition 88 and Theorem 82 the only way to have a not strict NE where all players play k in a multiple equilibria FEDAW p-BCIG is if $k = B(p, N)$. Therefore an immediate implication of Proposition 88 is that if there are $M \geq 2$ equilibria at least $M - 1$ are strict. Therefore, the Propositions of this section imply our second important result:

Theorem 89 *In a multiple Nash-Equilibria p-BCIG*

1. *The IBR process will lead to the highest NE when starting with a high initial strategy profile.*

2. *The IDWDS process does not lead to a unique prediction if any of the following holds*

   (a) *The game is IWND.*

   (b) *The game is FEDAW and at the highest NE every player plays an integer k satisfying $k < B(p, N)$.*

   (c) *The game is FEDAW and there are at least 3 NE.*

3.5 Conclusions

In this chapter we have completely characterized the NE of a p-BCIG and we have obtained the following interesting results: First, we showed that in the p-BCIG the number of equilibria depends on all the parameters of the game (N, p, l, h) while for the p-BCG the unique NE is all players playing l. Second, we also showed that any experimental p-BCG is in fact a p-BCIG because of the approximation needed to do the calculations. Third, we proved that under very soft conditions the iterated deletion of weakly dominated strategies (IDWDS) does not lead to a unique prediction of the game while the iterated best reply (IBR) might do. This is worth noticing since experimental results show that subjects use IBR rather than IDWDS.

Because of the equilibrium multiplicity that may arise in the p-BCIG it might be interesting to do further experiments with explicit integer restrictions in order to get more insight in the issue of equilibrium selection.

Finally, with the explicit introduction of the p-BCIG and the characterization of its equilibria we have closed the gap between the coordination game (i.e. a p-BCG with $p = 1$) where any number can be an equilibrium and the p-BCG where only one equilibrium exists.

Bibliography

[1] Bosch, A. and R. Nagel (1997, June 16). "El juego de adivinar el número X: una explicación y la proclamación del vencedor" [The game of guessing a number: An explanation and the announcement of the winner]. *Expansión*, 42-43.

[2] Cournot, A. (1838). *Recherches sur les Principes Mathématiques de la Théorie des Richesses.* [English edition: *Researches into the Mathematical Principles of the Theory of Wealth.* New York: Macmillan, 1897.]

[3] Duffy, J. and R. Nagel (1997). "On the Robustness of Behavior in Experimental Beauty-Contest Games.". *Economic Journal*, 107, 1684-1700.

[4] Fudenberg, D. and J. Tirole (1991). *Game Theory.* Cambridge, Mass. MIT Press.

[5] Harsanyi, J. (1973). "Oddness of the number of equilibrium points: A new proof." *International Journal of Game Theory* 2: 235-250.

[6] Ho, T., C. Camerer and K.Weigtel (1998). "Iterated dominance and iterated best-response in experimental 'p-beauty contests'." *American Economic Review*, 88, 947-969.

[7] Mas-Colell, A., M. Whinston and J. Green (1995). *Microeconomic Theory*. New York, NY. Oxford University Press.

[8] Nagel, R. (1995). "Unraveling in guessing games: An experimental study." *American Economic Review*, 85, 1313-1326.

[9] Nagel, R. and R. Selten (1997). "1000 DM zu gewinnen". *Spektrum der Wissenschaft* (German issue of Scientific American).

[10] Nagel, R. (1998). "A Survey on Experimental Beauty Contest Games: Bounded Rationality and Learning." *Games and Human Behavior, Essays in Honor of Amnon Rapoport.* Eds. D. Budescum, I. Erev and R. Zwick. Lawrence Erlbaum Associates Inc. New Yersey. p.105-142

[11] Stahl, D. O. (1996) "Rule Learning in a Guessing Game.", *Games and Economic Behavior*, 1996, 16(2), 303-330.

[12] Osborne, M. and A. Rubinstein (1994). *A Course in Game Theory*. Cambridge, Mass. MIT Press.

[13] Thaler, R. (1997, June 16). "Giving markets a human dimension." *Financial Times*, Mastering Finance section 6, 2-5.