Probing gauge theories: Exact results and holographic computations

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PROBING GAUGE THEORIES: EXACT RESULTS AND HOLOGRAPHIC COMPUTATIONS

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Als meus pares i al meu germà

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"*Exact results for static and radiative fields of a quark in* $\mathcal{N} = 4$ *super Yang-Mills*,"

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A Islandia

De las regiones de la hermosa tierra
Que mi carne y su sombra han fatigado
Éres la más remota y la más íntima,
Última Thule, Islandia de las naves,
Del terco arado y del constante remo,
De las tendidas redes marineras,
De esa curiosa luz de tarde inmóvil
Que efunde el vago cielo desde el alba
Y del viento que busca los perdidos
Velámenes del viking. Tierra sacra
Que fuiste la memoria de Germania
Y rescataste su mitología
De una selva de hierro y de su lobo
Y de la nave que los dioses temen,
Labrada con las uñas de los muertos.
Islandia, te he soñado largamente
Desde aquella mañana en que mi padre
Le dio al niño que he sido, y que no ha muerto
Una versión de la Völsunga Saga
Que ahora está descifrando mi penumbra
Con la ayuda del lento diccionario.
Cuando el cuerpo se cansa de su hombre,
Cuando el fuego declina y ya es ceniza,
Bien está el resignado aprendizaje
De una empresa infinita; yo he elegido
El de tu lengua, ese latín del Norte
Que abarcó las estepas y los mares
De un hemisferio y resonó en Bizancio
Y en las márgenes vírgenes de América.
Sé que no lo sabré, pero me esperan
Los eventuales dones de la busca,
No el fruto sabiamente inalcanzable.
Lo mismo sentirán quienes indagan
Los astros o la serie de los números...
Sólo el amor, el ignorante amor, Islandia.

Jorge Luis Borges

Chapter 1

Introduction

1 Dualities in Physics

One of the most fundamental ingredients in modern theoretical physics, and in string theory in particular, is the notion of *duality*, the exact equivalence between two systems or theories with different descriptions but with the same underlying physics.
The very first discovery of an exact duality in Physics probably dates back to Paul Dirac's crucial observation that Maxwell's equations are invariant under the exchange of the electric and magnetic fields and sources if one is imaginative enough to introduce the concept of magnetic monopole. Even more important, he showed that quantum mechanics does not really preclude the existence of isolated magnetic monopoles but in order to produce a consistent theory at the quantum level we need to require the electric and magnetic charges to satisfy *Dirac's quantization condition* [1] (in natural units)

$$q_e q_m = 2\pi n \quad ; \quad n \in \mathbb{Z} \tag{1.1}$$

Turning the argument around, the existence of a magnetic monopole implies quantization of electric charge. This very first example already shows beautifully the potential predictive power of dualities.
At the same time, equation (1.1) can also be regarded as a prototypical example of a *weak/strong duality* or, in a more modern language, an *S-duality*.
In general, under an S-duality a theory with coupling constant g is mapped to a possibly very different theory with coupling constant $1/g$. It is hard to overestimate the importance of having such a symmetry, since then one might be able to extract information about strongly coupled non-perturbative aspects of one theory by studying the perturbative weak coupling expansion of its S-dual and vice versa.

Entering now the realm of string theory we find, apart from the above described S-duality, new duality transformations such as T-duality and its natural extension, mirror symmetry, which are particular of string theory and, contrary to what happens with S-duality, have no realizations in quantum field theory. Moreover, much more dramatically than what happened in field theory, string dualities play a crucial role in the understanding of the theory.
After the First Superstring Revolution physicists realized that there seemed to be five distinct superstring theories: type I, types IIA and IIB, and heterotic $SO(32)$ and $E_8 \times E_8$. Later, with the discovery of all these string dualities the paradigm shifted and in 1995 the Second Superstring Revolution finished with the certainty that there is indeed a unique string theory and all five string theories, plus the recently discovered M-theory,

are connected through an intricate net of dualities and each one of the previous theories should now better be seen as the appropriate description for a given region of the space of parameters of the Theory.

It came up that one of the crucial ingredients in the development of this final picture was the discovery of D-branes, extended objects which admit a dual interpretation as non-perturbative solitonic solutions of supergravity and as hypersurfaces in flat space where open fundamental strings can end. Precisely due to this dual interpretation, D-branes have also been essential in the construction of dualities between non-gravitational field theories and string theories, which include gravity in a natural manner. These are known as *gauge/gravity* or *gauge/string dualities* and, in turn, can be seen as particular examples of an *open/closed string duality*.

Finally we arrive at which will be the subject of this dissertation, the AdS/CFT correspondence discovered by Juan Maldacena in 1997 [2]. The original conjecture states that maximally supersymmetric $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions, which is a conformal field theory, is exactly equivalent to type IIB superstring theory living on a particular ten-dimensional space, $AdS_5 \times S^5$.
This is one of the most notable and fruitful examples of a gauge/string duality and one of the major breakthroughs in string theory in the last decades. At the same time it is the first explicit realization of the *Holographic Principle*: the idea that string theory, which is a theory of quantum gravity, has a dual description as a quantum field theory living on the boundary of the background space.
At a practical level, the AdS/CFT correspondence represents a powerful tool to explore regions of the moduli space of gauge theories which are not directly accessible by ordinary field theoretical techniques such as the perturbative expansion in small parameters.

The remaining chapters of this introductory section will be devoted to introduce in more detail this correspondence as well as many of the specific ingredients and techniques that I have been using during my PhD.

2 Strings and branes

This thesis is devoted to the study of the AdS/CFT correspondence by means of extended probes. Thus, in this section we provide a very brief review of the building blocks of string theory, namely fundamental strings and D-branes, up to the point of being able to discuss the essential features of type IIB superstring theory necessary for deriving and working with the correspondence.

2.1 Fundamental Strings

Let us start with the simplest object, the bosonic string. Although fundamentally incomplete, it is important as many of its features still play a role in superstring theory and, as a matter of fact, it will prove to be enough for most of the computations involved in this dissertation.
The dynamics of the bosonic string are described by the *Nambu-Goto action*

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det(G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu)} \tag{1.2}$$

which is simply the proper area of the worldsheet. Here, $G_{\mu\nu}$ is the target metric and X^μ describes the embedding of the string. Alternatively, we can introduce an independent metric γ_{ab} on the worldsheet and work with the *Brink-Di Vecchia-Howe action* (often referred to in the literature as the *Polyakov action*)

$$S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \tag{1.3}$$

Both actions are classically equivalent (i.e. when the equations of motions for γ_{ab} are satisfied), but the Polyakov action is more desirable than the Nambu-Goto action since the lack of the square-root allows for quantization more easily and furthermore it exhibits a very important symmetry not present in the first one. Both actions show manifest spacetime Poincaré and worldsheet diffeomorphim invariance but only the second one exhibits worldsheet Weyl invariance.

Quantization of the bosonic string is a fascinating topic, although very technical. Since this thesis is centered mainly in the study of semiclassical strings and branes in supergravity backgrounds, I will prefer not to cover this topic. However, the interested reader is referred to any of the very good books and reviews [3–7].

For future reference, and without entering into too many details, I may outline:

- Quantization of the bosonic string gives a critical dimension of $D = 26$ and a ground state of negative mass-squared, i.e. a *tachyon*.
- There is an ingenious way to get rid of this tachyon by adding fermionic modes on the worldsheet and imposing supersymmetry, hence the name "superstring". Applying now the quantization procedure one finds that the critical dimension of the superstring is $D = 10$ and there is no tachyon in the spectrum. The resulting target space picture also exhibits supersymmetry.
- For the case of the closed superstring, fermionic modes can satisfy either periodic boundary conditions (*Ramond sector* R) or anti-periodic boundary conditions (*Neveu-Schwarz sector* NS). Boundary conditions for right-moving and left-moving modes can be chosen independently, which gives a total of four possiblities: target-space bosons (R,R), (NS,NS) and target-space fermions (R,NS), (NS,R).
- Massless bosonic fields include the graviton $G_{\mu\nu}$, the NSNS Kalb-Ramond 2-form $B_{\mu\nu}$, the dilaton Φ and several RR p-form fields. The exact form of these extra bosonic fields depends on exactly what superstring theory we consider.
- The massless, tree-level approximations of string theories (that is, their low-energy, $g_s \to 0$ limit) become supergravity theories.

As a final remark, fundamental strings can couple to the antisymmetric Kalb-Ramond field $B_{\mu\nu}$ through the term

$$-\int d\tau d\sigma \partial_\tau X^\mu \partial_\sigma X^\nu B_{\mu\nu}, \tag{1.4}$$

but they are neutral with respect to the RR fields.

2.2 Branes in Supergravity and Superstring Theory

As we have seen, superstring theory has two kinds of bosonic gauge fields, from the NSNS and RR sectors of the string Hilbert space, that are quite different in perturbation theory. Some string states carry a worldsheet charge under the NSNS space-time gauge symmetry but, on the other hand, they are all neutral under the RR symmetries. Nevertheless, various dualities interchange NSNS and RR states so string duality requires that states carrying the various RR charges should exist. In a first attempt it was suggested that these objects should be black p-branes, soliton-like classical solutions of supergravity that can be seen as extended versions of charged black holes. In 1995 Polchinski showed that there is a seemingly different class of objects which carry the RR charges, the D(irichlet)-branes [8].

A Dirichlet p-brane (or Dp-brane) is a $p+1$ dimensional hyperplane in a higher D-dimensional space-time where open strings are allowed to end. For the end-points of such strings the $p+1$ longitudinal coordinates satisfy the conventional free (Neumann) boundary conditions, while the $D-p-1$ coordinates transverse to the Dp-brane worldvolume have fixed (Dirichlet) boundary conditions (and hence the name),

$$\begin{aligned} n^a \partial_a X^\mu &= 0\ , \quad \mu = 0, ..., p \\ X^\mu &= 0\ , \quad \mu = p+1, ..., D-1 \end{aligned} \tag{1.5}$$

Polchinski realized that the simplest Dp-brane is a BPS saturated dynamical object which preserves 1/2 of the bulk supersymmetries and carries an elementary unit of charge with respect to the $p+1$ form gauge potential from the RR sector, which is the same kind of charge that carries a black p-brane solution of supergravity. One is then led to think of D-branes as an alternative representation of black p-branes or, better speaking, as objects that give their full string theoretical description. We have, in some sense, two different descriptions of the same object.

But this is not the end of the story. Another fascinating feature of D-branes is that they naturally realize gauge theories on their worldvolumes. The massless spectrum of open strings living in a (single) Dp-brane is that of a maximally supersymmetric $U(1)$ gauge theory in $p+1$ dimensions. The $9-p$ massless scalar fields present in this supermultiplet are the expected Goldstone modes associated with the transverse fluctuations of the Dp-brane, while the photons and fermions may be thought of as providing the unique supersymmetric completion. It can be argued that the low-energy dynamics of a single Dp-brane in a given background is well described by the Dirac-Born-Infeld-Wess-Zumino effective action. In the string frame this action reads as follows

$$S_{Dp} = S_{DBI} + S_{WZ}$$
$$S_{DBI} = -T_{Dp} \int_{\mathcal{M}} d^{p+1}\xi\ e^{-\phi} \sqrt{-|g_{ij} + \mathcal{F}_{ij}|} \qquad S_{WZ} = T_{Dp} \int_{\mathcal{M}} d^{p+1}\xi\ e^{F} \wedge P[C] \tag{1.6}$$

where $g_{ij} = G_{\mu\nu}\partial_i X^\mu \partial_j X^\nu$ is the induced metric on the worldvolume of the brane (i.e. the pullback of the target space metric) and $\mathcal{F}_{ij} = 2\pi\alpha' F_{ij} + B_{ij}$, being F_{ij} the 2-form abelian field-strength inherent in the brane and B_{ij} the pull-back to the worldvolume of the NSNS antisymmetric tensor field. Finally, ϕ is the dilaton field, $C = \bigoplus_n C^{(n)}$ is the collection of all the RR n-form gauge potentials of the target space and $\mathcal{M}$ stands for the D-brane worldvolume.

If we consider now a stack of N coincident D-branes instead of one, we have to associate N degrees of freedom with each of the end-points of the strings in order to specify between which two branes a given string is hanging. This extra labels are the so-called *Chan-Paton indices*. For the case of oriented open strings, the two ends are distinguished, and so it makes sense to associate the fundamental representation N with one end and the antifundamental representation $\bar{N}$ with the other one. In this way one associates N^2 degrees of freedom to each open string that begins and ends on any of the branes so one naturally describes the gauge group $U(N)$. Indeed, at low energies, we find the maximally supersymmetric $U(N)$ gauge theory in this setting.
For unoriented strings, such as type I superstrings or after orientifolding type II, the two ends are indistinguishable and the representations associated with the two ends have to be the same. This forces the symmetry group to be one with a real fundamental representation, specifically an orthogonal or symplectic group.

Both descriptions of a D-brane, as a black brane or as a boundary condition in string perturbation theory, are appropriate at different (complementary) regimes. When there are N D-branes on top of each other, the effective loop expansion parameter for the open strings is $g_s N$ rather than g_s so the D-brane description is good only when $g_s N << 1$. On the other hand, a description in terms of a black brane of charge N under the RR fields is appropriate only when the supergravity approximation is valid, and it can be proved that this happens in the regime $1 << g_s N < N$. As explained in the following subsection, various comparisons of the two descriptions led to the discovery of the AdS/CFT correspondence.

3 The AdS/CFT correspondence

It has been known for a long time that gauge theories and string theories should be related in some way. After all, string theory was born precisely as an attempt to model the strong interaction.
This old idea was first observed by 't Hooft in his seminal paper of 1974 [9], where he showed that the perturbative expansion of any gauge theory with gauge group $U(N)$ can be rewritten in terms of an expansion of double-line Feynman diagrams in a way that is totally reminiscent of the string theory double expansion, with the gauge theory Feynman diagrams seen as string worldsheets. Most notably, if we denote by g_s the coupling constant of the gauge theory, we can write the free energy as

$$F(g_s, N) = \log Z = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} g_s^{2g-2+h} N^h \tag{1.7}$$

The above sum is over double-line diagrams with the topology of an open Riemann surface $\Sigma_{g,h}$ of genus g with h holes, and $F_{g,h}$ can be computed in terms of the Feynman rules associated to the diagram. On the other hand, one could read (1.7) as an open string amplitude in which we sum over all possible topologies of the worldsheet $\Sigma_{g,h}$. The coupling g_s should be interpreted now as the string coupling constant, which weights the contribution of a particular topology by a factor of $g_s^{-\chi}$, with $\chi = 2 - 2g - h$ being the Euler characteristic of the worldsheet. Analogously, the factors N can be seen as Chan-Paton factors associated to the boundary of the open string. From this new point of view the quantities $F_{g,h}$ would be interpreted as open string amplitudes on $\Sigma_{g,h}$.
This can in turn be re-summed by introducing the so-called 't Hooft coupling $\lambda = g_s N$

$$F(g_s, N) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F_{g,h} g_s^{2g-2} \lambda^h, \tag{1.8}$$

in such a way that (1.8) now resembles completely the double expansion of a closed string theory amplitude, g_s playing now the role of a closed string coupling constant.

Obviously, the key question is now the following: Is it possible to make this statement more precise? In other words, given a certain $U(N)$ gauge theory, is it possible to find its particular dual open/closed string theory?
To answer such a question turned out to be an extremely difficult task and, as of today, there are only very few examples where this identification has been carried out in detail.

The first explicit realization of 't Hooft's idea had to wait until 1997, when Maldacena proposed the original conjecture of the AdS/CFT correspondence.

Now, before motivating and discussing the conjecture in more detail, we present next the most relevant features of the main two ingredients of the AdS/CFT correspondence, namely $\mathcal{N}=4$ super Yang-Mills theory and the Anti-de Sitter space (AdS).

3.1 $\mathcal{N}=4$ super Yang-Mills

$SU(N)$ $\mathcal{N}=4$ super Yang-Mills (SYM) theory in four dimensions has only one multiplet, an $\mathcal{N}=4$ gauge multiplet, composed by a gauge field A_μ (with Lorentz index $\mu = 0,\cdots,3$), four Weyl fermions ψ_α^A (with $A=1,\cdots,4$ and spinor index $\alpha=1,2$), and six real scalars Φ^I $(I=1\cdots,6)$. All the fields transform in the adjoint representation of the gauge group $SU(N)$. There is also a global $SU(4)\cong SO(6)$ R-symmetry under which the gauge field is a singlet, while the fermions and scalars transform respectively in the **4** and **6** representations.
Its action reads (in Euclidean signature)

$$S=\frac{1}{g_{YM}^2}\int d^4x\mathrm{Tr}\Big(\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+\frac{g_{YM}^2\theta}{8\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}+D_\mu\Phi^I D^\mu\Phi^I+i\bar{\Psi}\Gamma^\mu D_\mu\Psi-\\ -\frac{1}{2}[\Phi^I,\Phi^J][\Phi^I,\Phi^J]+i\bar{\Psi}\Gamma^I[\Phi^I,\Psi]\Big), \tag{1.9}$$

where g_{YM} is the Yang-Mills coupling constant, we have expressed the four Weyl fermions in terms of a ten-dimensional single Majorana-Weyl spinor Ψ, Γ^μ and Γ^I are ten-dimensional 16×16 Dirac matrices and we have allowed the possibility of a non-vanishing θ angle, which can be relevant for non-trivial instantonic backgrounds.
This action is manifestly scale invariant since g_{YM} is dimensionless and all terms in the Lagrangian have dimension 4. It is actually also conformal invariant, i.e. it is invariant under the whole four-dimensional conformal group $SO(4,2)\cong SU(2,2)$ formed by Poincaré transformations P_μ, $M_{\mu\nu}$, dilatations D and special conformal transformations K_μ. Combined with the 16 Poincaré supercharges Q_α^A and $\bar{Q}_{A\dot{\alpha}}$ they form the larger superconformal group $SU(2,2|4)$. This supergroup has, in addition to the 16 Poincaré supercharges, also 16 superconformal charges S_α^A and $\bar{S}_{A\dot{\alpha}}$ stemming from the fact that the Poincaré supersymmetries and the special conformal transformations do not commute. The doubling of the number of supercharges is a very characteristic feature of conformal field theories.

Another crucial particularity of $\mathcal{N} = 4$ SYM is that the superconformal invariance persists also at the quantum level and the theory is then UV complete. As a consequence the coupling constant g_{YM} is actually a non-running parameter with vanishing beta function which can be fixed to the desired value. $\mathcal{N} = 4$ SYM is thus trivial in the sense that it is so constrained by its symmetries that its Lagrangian (when a Lagrangian description is possible) and matter content are completely fixed and the only freedom is the choice of the gauge group and the value of the coupling.

3.2 Anti-de Sitter space

The n-dimensional anti-de Sitter space AdS_n is the maximally symmetric Lorentzian manifold with constant negative scalar curvature ($R < 0$). It is the Lorentzian analogue of n-dimensional hyperbolic space, just as Minkowski space and de Sitter space are the Lorentzian analogues of the Euclidean flat space and sphere, respectively. From the point of view of general relativity, anti-de Sitter space is the maximally symmetric vacuum solution of Einstein's field equations with a negative (attractive) cosmological constant Λ included.

In order to develop a geometric intuition it is useful to imagine anti-de Sitter space as a manifold embedded in a higher dimensional space. In fact, AdS_{d+1} can be represented as a Lorentzian hyperboloid of radius R

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2 \tag{1.10}$$

embedded in flat $(d+2)$-dimensional space $\mathbb{R}^{2,d}$ with metric

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + \sum_{i=1}^{d} dX_i^2 \tag{1.11}$$

By construction, the induced metric on AdS_{d+1} manifestly preserves the symmetry of the ambient flat space (i.e the embedding is isometric), so it has isometry group $SO(d, 2)$. Equation (1.10) can be solved by setting

$$\begin{aligned} X_0 &= R \cosh\rho \cos\tau \quad ; \quad X_{d+1} = R \cosh\rho \sin\tau \\ X_i &= R \sinh\rho\ \Omega_i\ (i = 1, ..., d\ ;\ \sum_i \Omega_i^2 = 1) \end{aligned} \tag{1.12}$$

and substituting this into (1.11), we obtain the following metric of AdS_{d+1}:

$$ds^2 = R^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-1}^2) \tag{1.13}$$

If we take $0 \leq \rho$ and $0 \leq \tau \leq 2\pi$ this solution covers the entire hyperboloid once. Therefore, (τ, ρ, Ω_i) are called *global coordinates.*
In addition to this global parametrization of AdS, there is another set of coordinates $(u, t, \vec{x})$ which will be also very useful in the context of the AdS/CFT correspondence. It is defined by

$$\begin{aligned} X_0 &= \frac{1}{2u}(1 + u^2(R^2 + \vec{x}^2 - t^2)) \quad ; \quad X_{d+1} = Rut \\ X_i &= Rux^i \;\; (i = 1, ..., d-1) \quad ; \quad X_d = \frac{1}{2u}(1 - u^2(R^2 - \vec{x}^2 + t^2)) \end{aligned} \tag{1.14}$$

By taking $0 \leq u, t$ and $\vec{x} \in \mathbb{R}^{d-1}$ these coordinates cover one half of the hyperboloid. Substituting (1.14) into (1.11) we obtain another form of the AdS_{d+1} metric

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right). \tag{1.15}$$

These coordinates are called *Poincaré coordinates.* In doing calculations it is very usual to work with a variation of the AdS metric in Poincaré coordinates, obtained from the one above by setting $u = 1/z$, giving

$$ds^2 = \frac{R^2}{z^2} \left(dz^2 - dt^2 + d\vec{x}^2 \right). \tag{1.16}$$

Using these coordinates the conformal boundary at spatial infinity is placed at $z = 0$.
As a final remark, the Penrose diagram of AdS_5 is best understood from the metric in global coordinates by taking out a Weyl factor of $\cosh^2\rho$ and defining $dx = d\rho/\cosh\rho$. This way one obtains a solid cylinder with boundary given by $S^3 \times \mathbb{R}$, where $\mathbb{R}$ is the global time direction. Poincaré coordinates cover only a triangular slice of the cylinder.

3.3 The statement of the correspondence

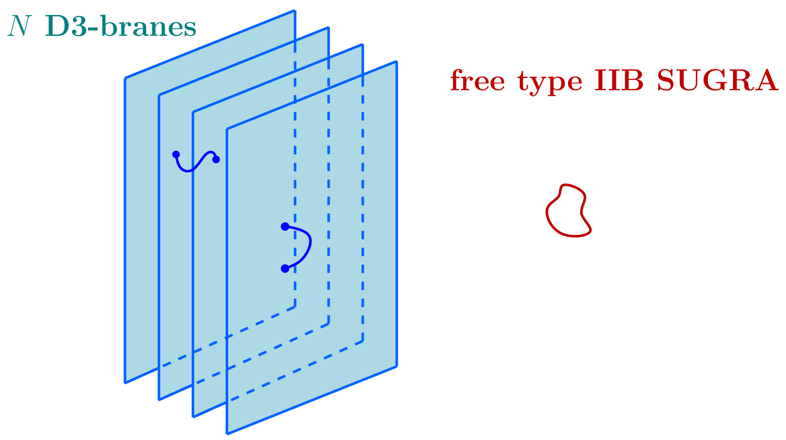
Using the above explained identification between D-branes and black branes, it can be shown that the large N limit of certain CFT's in various dimensions include in their Hilbert space a sector describing supergravity on the product of AdS space-times, spheres and other compact manifolds. For both a pedagogical and an extension reason I will focus my attention to the first and best understood example, the duality between type IIB string theory compactified on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang-Mills theory.

D-brane picture

We start with type IIB string theory in flat, ten dimensional Minkowski space-time. Let us consider N parallel D3 branes sitting together or very close to each other. String theory on this background contains two kinds of perturbative excitations, closed strings (excitations of empty space) and open strings which, as explained in the previous section, end on the D-branes and describe their excitations. If we consider this system at low energies, energies lower than the string scale $1/l_s$, then only the massless string states can be excited and we deal with an effective theory that involves only the massless fields but takes into account the effects of integrating out the massive fields. The low-energy effective Lagrangian of closed string massless states is that of type IIB supergravity while the low-energy effective Lagrangian for the open string massless states is that of four-dimensional $\mathcal{N} = 4$ $U(N)$ SYM. The complete effective action will have the form

$$S = S_{bulk} + S_{brane} + S_{int}$$

where S_{bulk} is the action of $d = 10$ supergravity in the bulk, S_{brane} is the brane action defined on the $(3 + 1)$ dimensional worldvolume of the coincident D3-branes and, finally, S_{int} describes the interactions between the brane modes and the bulk modes.



If we take the low energy limit (sending $l_s \to 0$ while keeping the energy and all the dimensionless parameters fixed) the coupling goes to zero so that the interaction Lagrangian vanishes, leaving just the pure $\mathcal{N} = 4$ $U(N)$ SYM theory in $3 + 1$ dimensions and free supergravity in the bulk. In the low energy limit we have, then, two decoupled systems, free gravity in the bulk and the four dimensional gauge theory.

Black brane picture

Let us consider now the same system from a different point of view. As we saw in the previous section, D-branes give the full string theoretical description of extremal p-branes so they can also be seen as massive charged objects which act as sources for the various supergravity fields. In fact, we can find a D3-brane solution of supergravity for which the metric takes the form

$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + d\Omega_5^2)$$

$$f(r) = 1 + \frac{R^4}{r^4}, \qquad R^4 \equiv 4\pi g_s \alpha'^2 N \tag{1.17}$$

As it happens for the well-known case of the Schwarzschild black hole, since the g_{tt} component of the metric is a non-constant function of r, the energy E_r of an object measured by an observer at a constant position r and the energy E_∞ measured by an observer at the spatial infinity are related by a redshift factor in the following way

$$E_\infty = f^{-1/4}(r) E_r \ . \tag{1.18}$$

Clearly $f(r)$ diverges as r goes to zero, so this means that the same object brought closer and closer to $r = 0$ would appear to have lower and lower energy from an observer at infinity point of view. Now, and as we did for the D-brane picture case, we take the low-energy limit. Since we are dealing now with a curved space-time, we have to choose a specific reference frame, and that would be the one of the observer at infinity. The reason of this choice, beyond the evident simplification, will become much more clear when the conjecture has been stablished.

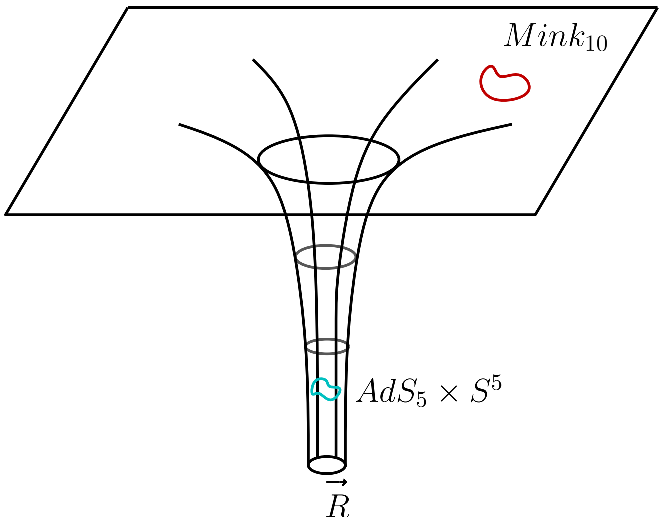
From the point of view of an observer at infinity, there are two kinds of low energy excitations. On the one hand we have massless particles propagating in the bulk with very large wave lengths and, on the other hand, any kind of excitation that we bring close enough to $r = 0$. In this limit, the wavelength of the massless particles becomes much bigger than the typical gravitational size of the stack of branes, which is of order R, so the bulk massless particles decouple from the near horizon region around $r = 0$ (in other words, the low energy absorption cross section is very small). Similarly, the

closer an excitation is to $r = 0$, harder it finds to climb the gravitational potential and escape to the asymptotic region. We see that, like in the D-brane picture case, we have two decoupled pieces. One, as we had in the previous case, is free bulk supergravity and the other is the near horizon region of geometry (1.17).
In the near horizon region $r << R$ we can approximate $f \sim R^4/r^4$ and then the near horizon geometry becomes

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2 \tag{1.19}$$

which is the geometry of $AdS_5 \times S^5$ in Poincaré coordinates.

We see, then, that in both pictures we have two decoupled theories in the low-energy limit. In both cases one of the decoupled systems is supergravity in flat space, so it is natural to identify the second system which appears in both equivalent descriptions. Since $\mathcal{N} = 4$ $d = 3+1$ U(N) SYM is a unitary theory, we conclude that, in the low-energy limit, it includes in its Hilbert space the states of type IIB supergravity on $AdS_5 \times S^5$. It is natural to think that this correspondence goes beyond the supergravity approximation, and thus we are led to the conjecture that $\mathcal{N} = 4$ *U(N) super-Yang-Mills theory in* $3+1$ *dimensions is dual to (is the same as / is equivalent to) type IIB superstring theory on* $AdS_5 \times S^5$.



Having presented the conjecture, let us now be a bit more precise about the validity of various approximations. From the physics of D-branes we know that the Yang-Mills coupling g_{YM} is related to the string coupling g_s through

$$g_{YM}^2 = 4\pi g_s$$

where we have eluded for simplicity the relation between the θ angle and the expectation value of the RR scalar χ. Moreover, we know that we can trust the perturbative analysis in the Yang-Mills theory only when the 't Hooft coupling λ is small, i.e

$$\lambda = g_{YM}^2 N = 4\pi g_s N = \frac{R^4}{l_s^4} << 1 \tag{1.20}$$

It is worthy to notice that we need $g_{YM}^2 N$ to be small and not just g_{YM}^2. On the other hand, the supergravity description is acceptable only when the radius of curvature R of AdS_5 and S^5 becomes large compared to the string length l_s,

$$\frac{R^4}{l_s^4} = 4\pi g_s N = g_{YM}^2 N >> 1 \tag{1.21}$$

We can see that the supergravity regime (1.21) and the perturbative filed theory regime (1.20) are perfectly incompatible, and is in this sense that we call at this correspondence a "duality". The two theories are conjectured to be exactly the same, but when one is weakly coupled the other is strongly coupled and vice versa. This makes the correspondence both hard to prove and useful.
It is important to notice that both in (1.20) and (1.21) we assumed $g_s < 1$ (up to an $SL(2,\mathbb{Z})$ duality $g_s \to 1/g_s$ in case we have $g_s > 1$), so it is always necessary, but not sufficient, to have large N in order to have a weakly coupled supergravity description.

Matching the symmetries

Let us examine now more closely the matching of global symmetries on both sides of the correspondence. In the string theory side we have that the isometry group of $AdS_5 \times S^5$ is $SO(4,2) \times SO(6)$. In the gauge theory side, we know that the $\mathcal{N} = 4$ SYM theory is also invariant under the whole $SO(4,2) \times SO(6)$ Lie group, now understood as the four-dimensional conformal group times the R-symmetry of the theory. We see, then, that the (bosonic) global symmetry groups on both sides of the correspondence agree.
Furthermore, we have some supersymmetries as well. In the gauge theory side the SYM theory is invariant under 16 ordinary supersymmetries as well as under 16 special conformal supersymmetries. In the gravity side, $AdS_5 \times S^5$ is a maximally supersymmetric

solution of type IIB string theory, and so it possesses 32 Killing spinors which generate the fermionic isometries. These can be split into two groups that match those of the dual gauge theory. We therefore conclude that the global symmetries (the whole supergroup) are the same on both sides of the correspondence.

This exact correspondence between global symmetries of both theories gives us, among many other insights, a very nice geometric picture of the renormalization group flow of the dual gauge theory. The identifications stands as follows.
$\mathcal{N} = 4$, $d = 3 + 1$ SYM is a CFT so it is, in particular, invariant under a dilatation transformation

$$x^\mu \to \Lambda x^\mu \tag{1.22}$$

where Λ is a constant. For all we have seen until now, we are led to expect that this transformation is also a symmetry on the gravity side, and indeed this is the case.
We saw that the metric of the AdS_5 part of the near horizon throat geometry, in Poincaré coordinates, reads

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2}dr^2 \tag{1.23}$$

The coordinates x^μ may be thought of as the coordinates along the $d = 3+1$ worldvolume of the stack of D3-branes, while r and the S^5 coordinates span the directions transverse to the branes. This way Poincaré coordinates provide a very simple geometric picture of AdS_5 as a foliation of constant-r slices, each of which is isometric to $d = 4$ Minkowski space. As $r \to \infty$ we approach the conformal boundary of AdS_5 while at $r = 0$ we reach the Poincaré horizon. Moreover, this metric is invariant under (1.22) provided this is accompanied by the rescaling

$$r \to r/\Lambda \tag{1.24}$$

Imposing symmetry correspondence thus links short-distance physics (UV) in the gauge theory with physics near the AdS conformal boundary, whereas long-distance physics (IR) is associated to physics near the horizon of the gravity theory. In other words, r is identified with the renormalization group scale in the gauge theory. Since any UV complete QFT is always defined by an ultraviolet fixed point and a RG flow, it is common to think of the gauge theory as residing at the conformal boundary of AdS.

3.4 A precise prescription

Shortly after the publication of Maldacena's 1997 original paper, in an attempt to make more precise the conjecture Gubser, Klebanov and Polyakov [10], followed by Witten [11], proposed an ansatz whose justification, initially, is just that it combines the ingredients at hand in the most natural way. Gradually, further evidence for this ansatz emerged.
The statement (for the particular case of the original correspondence) is that the partition function of string theory on $AdS_5 \times S^5$ should coincide with the partition function of $\mathcal{N} = 4$, $d = 3 + 1$ SYM theory living "on the boundary" of AdS_5.
Schematically, it reads (working in Euclidean signature for convenience)

$$Z_{bulk} = Z_{gauge} = e^{-W} \tag{1.25}$$

where W is the generating functional for connected Green's functions in the gauge theory. The right-hand side of (1.25) encodes all the physical information in the dual gauge theory, since it allows the calculations of correlation functions of arbitrary gauge-invariant (local) operators. On the other hand, the left-hand side is in general not easy to compute if not directly out of reach, but it simplifies a lot in the classical supergravity approximation (only valid in the large N and large λ limit). At leading order it reduces to

$$Z_{bulk} \sim e^{-S_{sugra}} \tag{1.26}$$

where S_{sugra} is the on-shell supergravity action.

Following from a D-brane physics perspective, the main technical idea behind this bulk-boundary correspondence is that the boundary values of string theory fields act as sources for gauge-invariant operators in the field theory.
Working in Poincaré coordinates, we will write the bulk fields generically as $\phi(\vec{x}, z)$, with value $\phi_0(\vec{x})$ at $z = \epsilon$. Then, in the supergravity approximation, we think of choosing the values ϕ_0 arbitrarily and then extremizing the action $S_{sugra}[\phi]$ in the region $z > \epsilon$ with this boundary condition. In short, we solve the supergravity e.o.m's subject to these Dirichlet boundary conditions on the boundary and then evaluate the action on the solution. So, we can write

$$W[\phi_0] = -\log\langle e^{\int d^4x \phi_0(x)\mathcal{O}(x)}\rangle_{CFT} \simeq S_{sugra}[\phi(\vec{x}, z)|_{z=\epsilon} = \phi_0(\vec{x})] \tag{1.27}$$

where $\mathcal{O}(x)$ denotes the corresponding dual operator. We conclude that the generator of connected Green's functions in the gauge theory is, in the large N, large λ limit, the

on-shell supergravity action. Thus, in this particular limit of the theory, we can obtain correlation functions of the dual operator $\mathcal{O}(x)$ just by taking derivatives of the on-shell supergravity action with respect to ϕ_0 and then setting it to zero at the end.
As a final remark, $\epsilon \ll 1$ serves to avoid divergences of the bulk action integral associated with the infinite volume of AdS which in turn can be seen as a UV cut-off of the dual field theory.
A formula like (1.27) is valid, in general, for any field ϕ in the bulk. Therefore, each field propagating in AdS is in one to one correspondence with an operator in the dual field theory, and we usually talk about this as the AdS/CFT "dictionary". Nevertheless, in general it is not easy to figure out which operator corresponds to which field, but for some very special cases it is easy due to their symmetries.

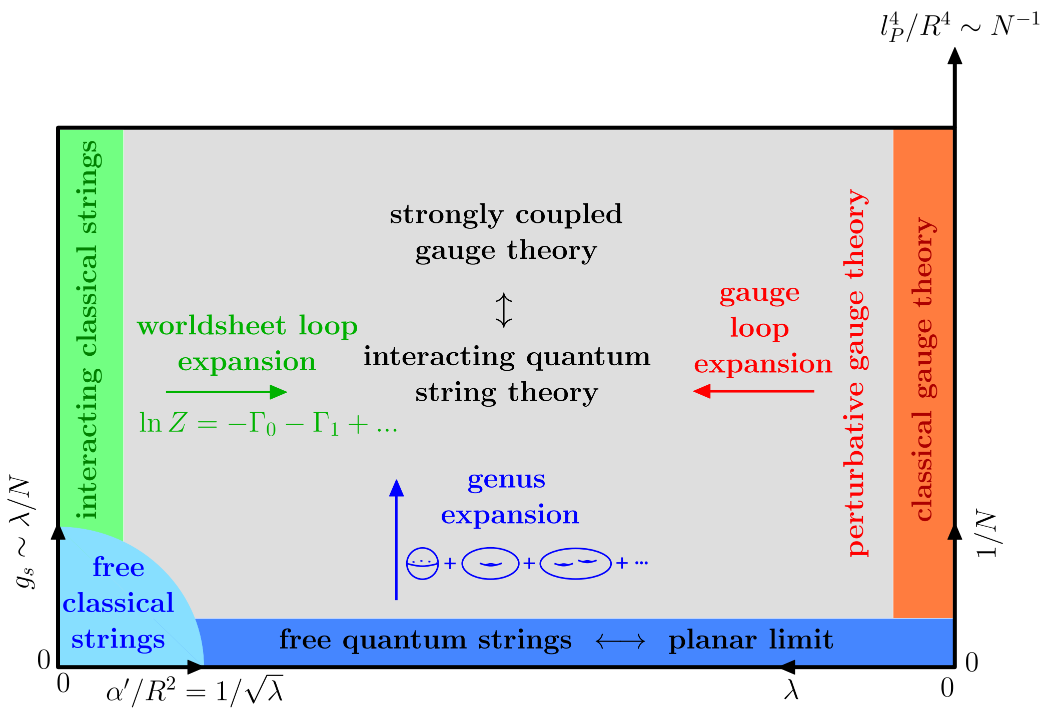
3.5 Beyond the supergravity approximation

As we have just seen, the AdS/CFT correspondence offers us a very useful tool for analyzing the strongly coupled regime of gauge theories and indeed many tests have been done comparing correlation functions in the CFT and propagators and scattering processes in $AdS_5 \times S^5$ in the supergravity approximation. Nevertheless, we may not forget that the original derivation of the correspondence was build completely out of string theory and (the strongest version of) the duality states that the full string theory on AdS is exactly dual to $\mathcal{N} = 4$ super-Yang-Mills theory on its boundary, for all values of λ and N.
This way, the AdS/CFT correspondence provides us with a full non-perturbative definition of string theory on an AdS-like background, including all the stringy objects and effects thats you may think of. A very convincing reason of why the CFT can't be equivalent to "just supergravity" is that the pure supergravity is inconsistent as a quantum theory (it is non-renormalizable) while the CFTs we are dealing with are well-behaved and consistent. Nevertheless, the conjecture cannot be rigorously proved as of today and we would like to have precise checks in order to support the strongest version of the duality. At the same time, it is certainly necessary to understand how holography works beyond the well-stablished supergravity approximation and how can we translate complicated string theory computations in simpler conformal field theory ones.
This will be one of the main topics of the present dissertation and thus it will be important to present it in some more detail.

String theory has fundamentally only two relevant parameters, the string length $l_s = \sqrt{\alpha'}$, which controls fluctuations given a fixed worldsheet topology and the string coupling constant g_s, which weights the probability of emission and absorption of strings, that is, the change in worldsheet topology. As we discussed, these have to be related with the two unique parameters of $\mathcal{N} = 4$ SYM, the coupling constant g_{YM} and the number of colors N, the precise relationship being

$$g_s = g_{YM}^2 \quad ; \quad \lambda = \frac{L^4}{l_s^2} = g_{YM}^2 N. \tag{1.28}$$

In the 't Hooft limit $N \to \infty$, $g_{YM} \to 0$ with λ kept fixed you see that when λ is smaller than one, then the Yang-Mills theory is weakly coupled and the perturbative gauge-theory diagrams are guaranteed to approximate physics well. On the contrary, when λ is (much) greater than one, the AdS radius R is (much) greater than the string length which means that one may approximate the physics by string theory on a "mildly curved" background. In this limit, when the curvature radius is (much) longer than the string length, it is always possible to approximate low-energy physics of string theory by supergravity. In string theory, the supergravity approximation means to neglect the α' stringy corrections while in the gauge-theoretical language, it is equivalent to focus on the planar limit for large λ and neglect subleading $1/N$ nonplanar contributions.



Ideally, in both regimes we could now incorporate corrections to the leading order:

$$\begin{aligned} 1/N \text{ corrections (fixed } \lambda) \quad &\leftrightarrow \quad g_s \text{ corrections (higher genus topologies)} \\ 1/\sqrt{\lambda} \text{ corrections} \quad &\leftrightarrow \quad \alpha' \text{ corrections (worldsheet fluctuations)} \end{aligned}$$

Realistically, computing corrections directly in the bulk is a very difficult task. On the string side, testing AdS/CFT beyond the planar limit involves calculating higher genus string amplitudes which, although presumably a well-defined problem, is currently out of reach. Furthermore, we certainly do not know how to reliably compute quantum g_s corrections in backgrounds spacetimes with RR fluxes. Another approach comes from the realization that higher curvature (or more broadly higher derivative) interactions are expected to arise on general grounds, as quantum or stringy corrections to the classical action. Hence a more refined description beyond the leading order will be given by an effective action supplemented with such higher derivative corrections.

In the present work we will circumvent such difficulties by addressing the problem in a completely different manner. On one hand, the first line of research will be the use of certain D-brane probes with electric fluxes as a way to resum an infinite series of string worldsheet topologies. On the other hand we will use the so-called supersymmetric localization technique in order to get exact results in the dual field theory, that is, exact analytic functions of λ and N. Finally, combining these results and making use of the holographic dictionary, we will infer new predictions for string theory.

4 Supersymmetric localization

Exact results in quantum field theory are certainly rare and very particular. In most of the cases, they rely on large amounts of symmetry and on sophisticated and powerful mathematical theorems.
The most notable examples of exact results accessible in supersymmetric gauge theories are maybe the topological theory constructed by twisting $\mathcal{N}=2$ super Yang-Mills, where the path integral of the twisted theory localizes to the (zero-dimensional) moduli space of instantons and can be used to compute the Donaldson-Witten invariants of four-manifolds [12, 13], the Seiberg-Witten exact low-energy effective action [14, 15] and Nekrasov's instanton partition function [16].
In this section we will introduce briefly the basic features of another technique, the so-called supersymmetric localization technique of Pestun [17].

The fundamental ingredient is to start with a fermionic (Grassmann-odd) symmetry $\mathcal{Q}$ of a theory described by the action $S[\Phi]$, depending on a set of fields Φ

$$\delta S = \mathcal{Q}S[\Phi] = 0 \tag{1.29}$$

Consider now defoming the partition function corresponding to the previous action perturbed by a δ-exact term as follows

$$Z(t) = \int \mathcal{D}\Phi e^{-S-t\delta V}, \tag{1.30}$$

where V is a fermionic Grassman-valued functional of the fields, invariant under the bosonic (Grassmann-even) symmetry $\delta^2 = \mathcal{Q}^2 = \mathcal{L}_B$, and where t is a free real parameter. It is worth noticing that $\mathcal{L}_B$ is made of other possible bosonic symmetries of S and, since we are dealing with Lorentz and gauge invariant theories, it has to be made out of combinations of gauge and Lorentz transformations.

With this conditions satisfied, it is immediate to see that the modified partition function $Z(t)$ is independent of t, since

$$\frac{dZ}{dt} = -\int \mathcal{D}\Phi \delta V e^{-S-t\delta V} = -\delta\Big(\int \mathcal{D}\Phi V e^{-S-t\delta V}\Big) = 0. \tag{1.31}$$

In the second equality we have integrated by parts and the missing term vanishes precisely because of the premise $\delta S = \delta^2 V = 0$. We have also supposed that the δ-symmetry leaves the path integral measure invariant, that is, we presuppose that the theory doesn't

suffer from anomalies. In the last equality we have used the fact that δ is a symmetry of the path integral. However, this last result may not hold if the boundary term does not decay sufficiently fast in field configurations, but this does not happen in general and certainly will not be the case in our computations.
Most notably, the same derivation also applies for the expectation value of any operator preserving this very fermionic symmetry, that is, any $\mathcal{O}$ such that $\delta\mathcal{O} = 0$. The argument is completely analogous:

$$\frac{d}{dt}\langle\mathcal{O}\rangle_t = \frac{d}{dt}\int \mathcal{D}\Phi\mathcal{O}e^{-S-t\delta V} = -\delta\Big(\int \mathcal{D}\Phi\mathcal{O}Ve^{-S-t\delta V}\Big) = 0. \tag{1.32}$$

If the modified partition function or the vev of the operator $\mathcal{O}$ do not depend on the parameter t, we can compute them for several values of t and all of them coincide with the original $t = 0$ integrals. Typically, one chooses V such that δV has a positive definite bosonic part, $(\delta V)_B > 0$. Therefore, when we take the $t \to \infty$ limit, the partition functions and the vev of the operator localizes to the submanifold of field configurations $\{\Phi_c\}$ that satisfy

$$(\delta V)_B = 0 \ . \tag{1.33}$$

It turns out that, in most of the cases cases, this localized set of field configurations $\{\Phi_c\}$ is independent of the space-time coordinates, leading to a zero-dimensional matrix model integral. In particular, one computes the path integral by a saddle point approximation which, in the strict $t \to \infty$ limit, happens to be one-loop exact. The final expression reads

$$Z = Z(0) = \int \mathcal{D}\Phi_c Z_{\text{1-loop}} Z_{\text{inst}} e^{-S[\Phi_c]}, \tag{1.34}$$

where $Z_{\text{1-loop}}$ is the one-loop determinant of all field fluctuations due to the saddle-point and the factor Z_{inst} is Nekrasov's partition function of point instantons.

As we see, the fact that S has to be invariant under a fermionic (Grassman-odd) symmetry, makes supersymmetric quantum field theories the ideal context in which to apply such technique.
In section 5.3 we will apply these results in order to compute the exact vev of the $\frac{1}{2}$-BPS circular Wilson loop in $\mathcal{N} = 4$ super Yang-Mills theory.

5 Wilson loops

This thesis is devoted mainly to the study of supersymmetric Wilson loops in $\mathcal{N} = 4$ SYM and their relation with many relevant observables of the quantum field theory like the total radiated power, the vev of the Lagrangian density or the momentum diffusion coefficient. Such operators are interesting per se, but in addition they exhibit many interesting features. First, they can be computed both at weak coupling by standard perturbative techniques as well as at strong coupling by means of the AdS/CFT correspondence. Secondly, for very specific contours and very ideal and symmetric theories, such operators can be evaluated exactly using the supersymmetric localization technique. This way, they can be used as remarkable precision tests for the conjectured holographic duality.

5.1 Wilson loops in $\mathcal{N} = 4$ SYM

Wilson loops are among the most interesting operators in any gauge theory. They are non-local gauge invariant operators (and so they are observables) which essentially are phase factors associated with the trajectory of a charged point particle along a closed path. Thus, from a physical point of view, they codify the response of the gauge field to the insertion of an external point-like source passing around a closed contour. Mathematically, they correspond to the holonomies of the gauge connection and they play the role of parallel transporters for charged particles moving in a gauge field background.
They were proposed originally by Kenneth Wilson in his seminal paper [18] as order parameters in the lattice formulation of quantum chromodynamics, and hence the name. In pure gauge theory, Wilson loops form a complete set of observables. That is, in principle you can generate all the other (local) observables by applying algebraic operations and taking certain limits. This comes from the mathematical fact that a (gauge) connection is completely determined up to a gauge transformation by its holonomies. And this is not just an statement of the classical theory, the same claim is true in the quantum theory via the path integral formalism: anything you can write down in terms of the gauge fields, you can also write down in terms of Wilson loops [19–23].

In order to be definite, consider the simplest example. Let's take the propagator of a scalar field in flat space. Using Schwinger's proper time formalism we can write it as a sum over histories

$$G(x,y) = \langle x|\frac{i}{p^2-m^2}|y\rangle = \int_0^\infty dT \int_{\substack{X(0)=x \\ X(T)=y}} [\mathcal{D}X] \exp\left[-im\int_0^T dt\sqrt{\dot{X}^2}\right] \quad (1.35)$$

Consider now a charged scalar particle with charge q minimally coupled to a background $U(1)$ gauge field A_μ. As usual, all we have to do is just to replace the derivatives with covariant ones $\partial_\mu \to D_\mu = \partial_\mu - iqA_\mu$ or, equivalently, modify the conjugate momenta as $p_\mu \to p_\mu - qA_\mu$. The propagator now reads

$$G(x,y) = \int_0^\infty dT \int_{\substack{X(0)=x \\ X(T)=y}} [\mathcal{D}X] \exp\left[-im\int_0^T dt\sqrt{\dot{X}^2}\right] \exp\left[-iq\int_0^T A_\mu \dot{x}^\mu dt\right] \quad (1.36)$$

This extra (abelian) phase factor is precisely the *Wilson line* $U(y,x)$. Under an (Abelian) gauge transformation, we have

$$\begin{aligned} \psi(x) &\xrightarrow{g.t.} e^{i\alpha(x)}\psi(x) \\ A_\mu(x) &\xrightarrow{g.t.} A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x) \\ U(y,x) &\xrightarrow{g.t.} e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)} \sim \text{“}\psi(y)\psi^\dagger(x)\text{”}. \end{aligned} \quad (1.37)$$

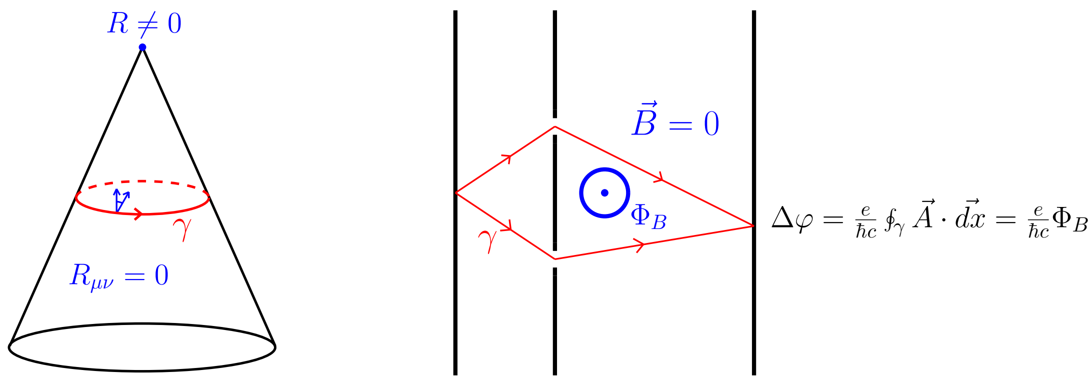
It is then clear that we can construct two manifestly gauge invariant operators out of these fields. The first would be $\psi^\dagger(y)U(y,x)\psi(x)$, which can be seen as the abelian analogous of the familiar two quarks linked by a confining string. Otherwise, if we consider closed trajectories γ, we get our beloved *Wilson loop*

$$W(\gamma) = e^{iq\oint_\gamma A_\mu \dot{x}^\mu dt}. \quad (1.38)$$

The clearest example for presenting the physics behind Wilson loops is probably the classic Aharonov-Bohm effect [24]: Imagine performing a double-slit interference experiment with electrons, and supoose we place a solenoid carrying magnetic flux in between the two slits. The solenoid is perfectly shielded, so that no electron can penetrate inside and detect the magnetic field directly. Yet, we find that the electrons know that the field is there. As the magnetic flux in the solenoid changes, the interference fringes shift. From the amount of the shift, we can infer that there is a field-dependent contribution to the relative phase of the electron paths that pass through the top and bottom slits given by

$$e^{ie\Phi_B/\hbar c} = \exp\left[\frac{ie}{\hbar c}\oint_\gamma \vec{A}\cdot d\vec{x}\right]. \quad (1.39)$$

This is due to the well know fact that in quantum mechanics the wave-function phase itself is unobservable but, in contrast to the classical theory, the phase differences are indeed observables.
A very direct geometric analogy of this effect is to consider parallel transporting vectors around the tip of a cone. Although it certainly doesn't look flat, a cone is flat everywhere just except by the tip. Any closed path not containing the tip will return the vector to its original position, while if we consider a closed path around the tip, the vector will return to the same point with a deficit angle, directly related with the opening angle of the cone. In this second case we will talk about the holonomy of the Levi-Civita connection capturing trhe curvature of the manifold.



Of course, we can generalize the previous results to the case of generic non-Abelian gauge theories. Now the gauge fields A_μ, transforming in the adjoint, are matrix valuated vectors and we have to introduce a notion of path ordering in order to be definite. Usually we will understand

$$e^{iq\epsilon A(x_1)\dot{x}_1} e^{iq\epsilon A(x_2)\dot{x}_2} \dots e^{iq\epsilon A(x_N)\dot{x}_N} \underset{\epsilon\to 0}{=} \mathcal{P} e^{iq\int A_\mu \dot{x}^\mu dt} , \tag{1.40}$$

where the path ordering acting on the exponential have to be understood through its Taylor expansion,

$$\mathcal{P}\exp\left(\int_0^T M(t)dt\right) = \mathbb{1} + \int_0^T M(t)dt + \int_0^T dt_1 \int_{t_1}^T dt_2 M(t_1)M(t_2) + \dots \tag{1.41}$$

Although the matrices $A_\mu(x)$ do not commute, the path-ordered exponential is now defined unambiguously. At the end of the day the Wilson loop for a generic non-Abelian gauge theory will read

$$W(\gamma) \propto \mathrm{Tr}\,\mathcal{P}\exp\left(ig\oint_\gamma A_\mu dx^\mu\right) , \tag{1.42}$$

where we take the trace in order to deal with a gauge invariant quantity. Finally, we could also consider probes transforming in different representations of the gauge group as well as supersymmetric theories, where the vector multiplet includes, apart from the gauge field, scalars and fermions (transforming in the adjoint) that will also interact with our probes.

In order to be explicit, if we concentrate on the particular case of $\mathcal{N} = 4$ $SU(N)$ SYM, which is the theory we will focus on later on, the most general Wilson loop we could consider is naturally defined (in Euclidean signature and supressing all fermions fields for the moment) [25–27]

$$W_{\mathcal{R}}(\gamma; \lambda, N) = \frac{1}{\dim \mathcal{R}} \mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp\left(\oint \left[i A_\mu \dot{x}^\mu + \dot{y}^I \Phi^I \right] d\tau \right) \tag{1.43}$$

where Φ^I are the six scalar fields that belong to the $\mathcal{N} = 4$ gauge multiplet and we chose the standard normalization. The irreducible representations $\mathcal{R}$ of $SU(N)$ can be expressed in terms of Young tableaux. We will start next subsection with the simplest $\mathcal{R} = \square$ case, the fundamental representation, but later on we will consider more general cases, which will be in fact the main focus of the section.

Finally, let us now study the invariances of the $\mathcal{N} = 4$ SYM loop (1.43) under the Poincaré and conformal supersymmetry transformations of the gauge and scalar fields

$$\begin{aligned} \delta_Q A_\mu &= \bar{\Psi} \Gamma_\mu \epsilon_0 \quad ; \qquad \delta_Q \Phi^I = \bar{\Psi} \Gamma^I \epsilon_0 \\ \delta_S A_\mu &= \bar{\Psi} \Gamma_\mu x^\nu \Gamma_\nu \epsilon_1 \quad ; \qquad \delta_S \Phi^I = \bar{\Psi} \Gamma^I x^\nu \Gamma_\nu \epsilon_1 . \end{aligned} \tag{1.44}$$

As before, Ψ is a ten-dimensional spinor and the transformation parameters $\epsilon_{0,1}$ are two ten-dimensional 16-components Majorana-Weyl fermions of opposite chirality. Focussing for the moment on the Poincaré supercharges one finds that $\delta_Q W(\mathcal{C}) = 0$ implies [27],

$$\left(i\Gamma^\mu \dot{x}_\mu + \Gamma^I \dot{y}^I \right) \epsilon_0 = 0 \tag{1.45}$$

This equation has eight independent solutions if $\left(i\Gamma^\mu \dot{x}_\mu + \Gamma^I \dot{y}^I \right)$ squares to zero. Recalling that the matrices Γ^μ amd Γ^I anticommute, this happens only if

$$\dot{x}^2 - \dot{y}^2 = 0. \tag{1.46}$$

This equation is solved for $\dot{y}^I(\tau) = |\dot{x}| \Theta^I(\tau)$, where $(\Theta^I)^2 = 1$, that is, it is a unit vector on $\mathbb{R}^6$, constrained to move over the S^5. With these specifications the Wilson loop (1.43) becomes now

$$W_{\mathcal{R}}(\gamma;\lambda,N)=\frac{1}{\dim\mathcal{R}}\mathrm{Tr}_{\mathcal{R}}\mathcal{P}\exp\left(\oint_{\gamma}\left[iA_{\mu}(\tau)\dot{x}^{\mu}+|\dot{x}|\Phi^{I}\Theta^{I}(\tau)\right]d\tau\right). \tag{1.47}$$

Notice that the previous remarks lead only to *locally supersymmetric* Wilson loops. If one considers the supersymmetry variations of the loop, then at every point along the loop one finds a different condition for preserved supersymmetry. Only if all these conditions commute will the loop be *globally supersymmetric*.
Considering the simplest case and taking Θ^I to be a constant vector, the two only maximally (globally) supersymmetric trajectories are the straight line and the circle.

5.2 Wilson loops in the $AdS_5 \times S^5$

Now, equipped with all the insights from the previous sections, we are ready to present the dual bulk description of supersymmetric Wilson loops as sources of fundamental strings and D-branes of type IIB string theory in $AdS_5 \times S^5$.

Fundamental representation

It was proposed originally in [25, 26] that an $\mathcal{N}=4$ SYM Wilson loop transforming in the fundamental representation of the gauge group should be associated to a fundamental string extending on the bulk of AdS and reaching the boundary along the very same contour of the dual loop. In the large N, large λ limit the expectation value of the loop operator is then given by the regularized area of the minimal surface swept by the string. We will now try to motivate this proposal and discuss it in detail.
First of all, we saw that in $\mathcal{N}=4$ super Yang-Mills there are no dynamical "quarks", there are only "gluons" transforming in the adjoint, together with all their superpartners. Nevertheless, a fundamental particle can be engineered in the following way. Consider a stack of $N+1$ D3-branes and then move one of them far away from the remaining N. There will be a very long string stretching in-between, which can be interpreted as a very heavy "W-boson". The endpoints of this string are the external non-dynamical sources that we call quarks. After taking the decoupling limit characteristic of the string derivation of the AdS/CFT correspondence, the string extends in the bulk of AdS_5 and lands on the loop on the boundary. At the gauge theory level this consists in giving a large expectation value to one scalar and in breaking the gauge group $U(N+1)\to U(N)\times U(1)$. The resulting off-diagonal bosons are in the fundamental of $U(N)$. The amplitude for

one of these W-bosons with mass m to go around a loop C of length $l(C)$ is given (in the limit of very large m) by

$$\mathcal{A} \sim e^{-ml(C)} \langle W(C) \rangle. \tag{1.48}$$

According to the proposal in [25, 26], this amplitude should be equal to the worldsheet area of the string associated to the contour C (with appropriate boundary conditions)

$$\mathcal{A} = \int [dX^\mu][dY^I][dh_{ab}] \exp(-S_{NG}[X, Y, h]) \quad | \quad X^\mu|_C = x^\mu \quad ; \quad Y^I|_C = y^I. \tag{1.49}$$

Considering now the particular limit of large λ, large N the integral can be evaluated on the saddle point, where string fluctuations are suppressed and the action is the (minimal) area of the classical surface

$$\lim_{\lambda \to \infty} \mathcal{A} = e^{-\sqrt{\lambda} Area(C)} \tag{1.50}$$

Then, our prescription for the regularized vev of the Wilson loop is

$$\langle W(C) \rangle \simeq \exp\Big[-(\sqrt{\lambda} Area(C) - ml(C)) \Big] \tag{1.51}$$

Higher representations

Let's move on to presenting the bulk description of higher rank Wilson loops, i.e. loops in representations other than the fundamental.

A first naive guess for the bulk description of a higher rank loop, many coincident loops, or a multiply wrapped loop would be to consider a set of coincident fundamental strings all landing along the loop on the boundary [28]. Although natural and well motivated, this direct approach presents serious technical difficulties, starting from the very fact that we don't know the appropriate (effective) action that describes a stack of coinciding strings. Furthermore, some attempts to bypass this fundamental obstacle lead to string worldsheets developing conical singularities and branch cuts, whose locations need to be integrated over.

A more effective way to describe such loops was first proposed in [29] (see also [30, 31]), where it was suggested that a multiply wrapped loop might be associated to a D3-brane extending in the bulk and pinching off at the boundary landing on the loop. This proposal is based on the merging of two older ideas: the first one is to generalize to the

$AdS_5 \times S^5$ background the BIon picture, first put forward by Callan and Maldacena in ten-dimensional flat space [32], that k fundamental string ending on a Dp-brane in flat space can be described in terms of a curved Dp-brane with a localized spike carrying k unit of electric flux, and the second one is the idea, known as Emparan-Myers polarization effect [33–36], that coincident strings can polarize into a single D-brane.

This brane picture has the advantage of automatically encoding the interactions between the coincident strings and yields all non-planar contributions to the expectation value of the higher rank Wilson loop [29].
If we indicate with k the rank of the loop, the number of coincident loops or the number of windings, the D-brane must have k units of fundamental string charge dissolved in its worldvolume. Furthermore, it must preserve the $SO(1,2) \times SO(3) \times SO(5)$ global symmetry group of the gauge theory operator.
It turns out that there are two kinds of branes with these characteristics: an electrically charged D3-brane with $AdS_2 \times S^2$ worldvolume and charge k, and a D5-brane with $AdS_2 \times S^4$ worldvolume and, again, charge k. Both the D3 and D5-brane are 1/2 BPS and both have an AdS_2 factor which can be associated to the fundamental string worldsheet. The difference between them is that the D3 is completely embedded in AdS_5, whereas the S^4 factor of the D5 is inside S^5.
All these realizations ended with a very precise entrence of the holographic dictionary [29, 30, 37] connecting the D3-branes to Wilson loops in the rank k symmetric representation and the D5-branes to the rank k antisymmetric one.
This brane picture is very reminiscent of the *giant/dual giant graviton picture* for chiral primary operators [38–40]. A giant graviton is a D3-brane wrapping an S^3 inside S^5, whereas a dual giant graviton is a D5 wrapping an S^3 inside AdS_5. Both describe excitations with large angular momentum $J \sim N$. The role of J is played in this context by the charge k.
The probe brane approximation discussed so far is valid when $k \sim N$ and breaks down when $k \gg N$. In that regime the brane backreacts and the $AdS_5 \times 5^5$ geometry is deformed into the supergravity solutions called *bubbling geometries* and studied in [41, 42].
In the comming sections we will make explicit use of the solutions for the D3 and D5-branes associated to higher rank loops. We therefore review them in detail in the following.

D5-brane $\leftrightarrow$ *k*-antisymmetric representation

As it was shown in [30, 37], Wilson loops in the k-th antisymmetric representation are dual to D5-branes embedded in the dual geometry carrying k units of worldvolume electric flux. For the particular case of a $\frac{1}{2}$-BPS circular Wilson loop transforming in the antisymmetric representation, the precise embedding was originally found by Yamaguchi in [30]. Here we will prefer to follow the more general derivation of [43], where the embedding for a probe D5 ending on an arbitrarily shaped contour at the boundary was found.

Consider a general type IIB background of the form $M \times S^5$, where the manifold M can have any asymptotically (locally) AdS_5 metric. We can write the metric as

$$ds^2 = R^2\Big(ds_M^2 + d\theta^2 + \sin^2\theta d\Omega_{S^4}^2\Big) \tag{1.52}$$

with polar angle $\theta \in [0, \pi]$.
Recall that the effective action for a D5-brane embedded in $AdS_5 \times S^5$ reads (in Euclidean signature)

$$S_{D5} = T_{D5}\int d\tau d^5\sigma\sqrt{\det(g + 2\pi\alpha' F)} - ig_s T_{D5}\int 2\pi\alpha' F \wedge *C_4, \tag{1.53}$$

with brane tension $T_{D5} = N\sqrt{\lambda}/8\pi^4 R^6$ and where the relevant part of the RR four form potential is

$$C_4 = \frac{R^4}{g_s}\Big[\frac{3(\theta - \pi)}{2} - \sin^3\theta\cos\theta - \frac{3}{2}\cos\theta\sin\theta\Big]\text{vol}S^4 \tag{1.54}$$

θ being the polar angle in the five sphere, as it appears in (1.52).
As we just discussed, the D5-brane embedding we are looking for will be of the form $\Sigma \times S^4 \hookrightarrow M \times S^4$, the S^4 set to be at an angle θ in the S^5. Thus, this embedding preserves an $SO(5)$ subgroup of the R-symmetry, as did the original Σ worldsheet. Furthermore, in order to have a dissolved string charge k over the brane, we need to turn on an electric worldvolume gauge field (constrained by the symmetries of the system)

$$F_{\tau\sigma} = iF\frac{\sqrt{\lambda}}{2\pi}. \tag{1.55}$$

With this ansatz the equations of motion for θ and F allow for constant solutions $\theta = \theta_0$, provided theta θ_0 satisfies

$$\pi\left(\frac{k}{N} - 1\right) = \sin\theta_0 \cos\theta_0 - \theta_0. \tag{1.56}$$

For these solutions,

$$F = -\cos\theta_0 \sqrt{\det g_\Sigma}. \tag{1.57}$$

Using now the constancy of θ and the fact that F is proportional to the volume form on Σ, it is straightforward to show that the equations of motions reduce to the ones following from the Nambu-Goto action. Therefore, we have found that any embedding Σ into M of a fundamental string defines an embedding $\Sigma \times S^4$ of a D5 brane for every $k \in \mathbb{Z}$.
Finally, after adding the appropriate boundary conditions, the total renormalized action is of course proportional to the on-shell Nambu-Goto action and reads

$$S_{D5} = \frac{2N}{3\pi} \sin^3\theta_0 S_{F1}. \tag{1.58}$$

As a final example, we can compute holographically the vacuum expectation value of the $\frac{1}{2}$-BPS circular Wilson loop transforming in the k-antisymmetric representation of $SU(N)$ in the large N, large λ [30]:

$$\langle W_{A_k} \rangle = e^{-S_{D5}} = \exp\left(\frac{2N}{3\pi}\sqrt{\lambda}\sin^3\theta_0\right) \tag{1.59}$$

D3-brane $\leftrightarrow$ *k*-symmetric representation

Very different from the D5-brane case, for many years we only knew the precise embedding for a D3-brane ending on a straight line or a circle. Since in this dissertation we will concentrate only in these two possibilities, we discuss them in detail. At the very end we will comment on new results for general embeddings for probe D3-branes ending on arbitrary trajectories at the conformal boundary.

We start with the D3-brane originally found in [29] and consider a circular Wilson loop of radius a placed on the boundary of AdS_5 . It will be convenient to write the metric of AdS_5 in polar coordinates as

$$ds^2_{AdS_5} = \frac{L^2}{z^2}(dz^2 + dr_1^2 + r_1^2 d\psi^2 + dr_2^2 + r_2^2 d\phi^2) \quad ; \quad r_1^2 = x_0^2 + x_1^2 \ , \ r_2^2 = x_2^2 + x_3^2 \tag{1.60}$$

The position of the loop is defined by $r_1 = a$ and $z = r_2 = 0$. As explained above, we look for a D3-brane which pinches off on this circle as $z \to 0$ and preserves a $SO(1,2) \times SO(3) \times SO(5)$ isometry.
Again, the bulk action includes a DBI part and a Wess-Zumino term, which captures the coupling of the background Ramond-Ramond field to the brane

$$S_{D3} = T_{D3} \int \sqrt{|g + 2\pi\alpha' F|} - T_{D3} \int P[C_{(4)}], \tag{1.61}$$

where $T_{D3} = \frac{N}{2\pi^2}$ is the tension of the brane, g is the induced metric, F the electromagnetic field strength, and $P[C_{(4)}]$ is the pull-back of the RR 4-form

$$C_{(4)} = \frac{r_1 r_2}{z} dr_1 \wedge d_\psi \wedge dr_2 \wedge d\phi \tag{1.62}$$

to the D3-brane worldvolume. It turns out to be more convenient to use a new set of coordinates obtained by transforming z, r_1, r_2 into

$$z = \frac{a \sin\eta}{\cosh\rho - \sinh\rho\cos\theta} \ , \quad r_1 = \frac{a\cos\eta}{\cosh\rho - \sinh\rho\cos\theta} \ , \quad r_2 = \frac{a\sinh\rho\sin\theta}{\cosh\rho - \sinh\rho\cos\theta} \tag{1.63}$$

In these coordinate system the metric on AdS_5 reads

$$ds^2_{AdS_5} = \frac{1}{\sin^2\eta}\Big(d\eta^2 + \cos^2\eta d\psi^2 + d\rho^2 + \sinh^2\rho(d\theta^2 + \sin^2\theta d\phi^2)\Big) \tag{1.64}$$

where $\rho \in [0, \infty)$, $\theta \in [0, \pi]$, and $\eta \in [0, \pi/2]$. The Wilson loop is located at $\eta = \rho = 0$. One can pick a static gauge in which the worldvolume coordinates of the brane are identified with ψ, ρ, θ, ϕ and the brane sits at a fixed point of the S^5. The remaining coordinate can be seen as a scalar field, $\eta = \eta(\rho)$. Because of the symmetries of the problem, the electromagnetic field has only one non-vanishing component, $F_{\psi\rho}(\rho)$. In these coordinates the DBI action reads

$$S_{DBI} = 2N \int d\rho d\theta \frac{\sin\theta\sinh^2\rho}{\sin^4\eta}\sqrt{\cos^2\eta(1+\eta'^2) + (2\pi\alpha')^2\sin^4\eta F^2_{\psi\rho}} \tag{1.65}$$

while the Wess-Zumino term is

$$S_{WS} = -2N \int d\rho d\theta \frac{\cos\eta \sin\theta \sinh^2\rho}{\sin^4\eta} \Big(\cos\eta + \eta' \sin\eta \frac{\sinh\rho - \cosh\rho\cos\theta}{\cosh\rho - \sinh\rho\cos\theta} \Big) \tag{1.66}$$

The solution to the equations of motion reads [80]

$$\sin\eta = \frac{1}{\kappa}\sinh\rho \ , \quad F_{\psi\rho} = \frac{ik\lambda}{8\pi N \sinh^2\rho} \ , \quad \kappa = \frac{k\sqrt{\lambda}}{4N} \tag{1.67}$$

From this solution one can see that k is not constrained for the D3-brane. In fact k determines a particular position in an $AdS_2 \times S^2$ foliation of AdS_5, which is a non-compact space. This has to be contrasted with the D5-brane case where we clearly find the bound $k \leq N$.

The bulk action has to be complemented with boundary terms for the worldvolume scalar η and for the electric field $F_{\psi\rho}$. These terms do not change the solution but alter the final value of the on-shell action which finally reads

$$S_{D3} = S_{DBI} + S_{WS} + S_{_{boundary}} = -2N \Big(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa \Big) \tag{1.68}$$

The expectation value of the $\frac{1}{2}$-BPS circular Wilson loop transforming in the k-symmetric representation is then

$$\langle W_{S_k}(C) \rangle = \exp\Big[2N \Big(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa \Big) \Big] \tag{1.69}$$

For small κ this expression coincides with the result of k coincident non-interacting fundamental strings

$$\langle W_{S_k}(C) \rangle \sim e^{k\sqrt{\lambda}}. \tag{1.70}$$

As a final remark, I would like to stress that it was not until very recently that the authors of [44] constructed the D3-brane embedding dual to a point-like particle of $\mathcal{N} = 4$ SYM transforming in the k-symmetric representation of $SU(N)$ and undergoing *arbitrary* motion. Their method consists on a generalization of the ansatz of Mikhailov for the fundamental string [45] and proceeds analogously by shooting light rays inwards in the bulk from the AdS conformal boundary.

At the same time this very method can be applied to the case of a D5-brane embedding, finding exact agreement with the previously dervied embeddings.

5.3 Exact results through localization

If we were to compute perturbatively the vev of the Wilson loop (1.43) at small λ we would find that, expanding up to second order, it would read

$$\begin{aligned}\langle W(C)\rangle &= \frac{1}{N}\mathrm{Tr}(\delta_{ab}) - \frac{1}{2}\oint d\tau_1 \oint d\tau_2 \frac{1}{N}\mathrm{Tr}\langle (A_\mu(x_1)\dot{x}_1^\mu - i|\dot{x}_1|\Phi_I(x_1)\Theta^I)(A_\nu(x_2)\dot{x}_2^\nu - i|\dot{x}_2|\Phi_J(x_2)\Theta^J)\rangle + ...\\ &= 1 + \frac{1}{2}\oint d\tau_1 \oint d\tau_2 \frac{1}{N}\mathrm{Tr}\left(\langle \Phi_I^a(x_1)\Phi_J^b(x_2)\rangle |\dot{x}_1||\dot{x}_2|\Theta^I\Theta^J - \langle A_\mu^a(x_1)A_\nu^b(x_2)\rangle \dot{x}_1^\mu \dot{x}_2^\nu\right) T^a T^b + ...\\ &= 1 + \frac{\lambda}{16\pi^2}\oint d\tau_1 \oint d\tau_2 \frac{|\dot{x}_1||\dot{x}_2| - \dot{x}_1\cdot\dot{x}_2}{(x_1-x_2)^2} + \mathcal{O}(\lambda^2)\end{aligned} \tag{1.71}$$

For the particular case of an infinite straight line it is obvious that the combined gauge and scalar propagator vanishes, and one anticipates the expectation value of this operator to be 1. In fact, as we saw, we recover the same result from the dual gravity computation in the bulk, and thus it is expected to be the final answer. But the straight Wilson line is not the only (globally) $\frac{1}{2}$-BPS loop operator. As we briefly commented before, the circular Wilson loop also preserves half of the supercharges (although a different combination than for the case of an infinite straight line) and can be obtained from the Wilson line by performing a special conformal transformation[1].

Thus, one could think that the vev of the circular loop should also be trivially 1, but it turns out that this is not true due to a subtle conformal anomaly coming from mapping the point at infinity (which strictly doesn't belong to $\mathbb{R}^4$) to the origin [48, 49]. In other words, this can be understood considering that the special conformal transformation mapping the line to the circle is not a global symmetry of flat space, but only of S^4.

Taking the explicit parametrization for the circle of unit radius $x^\mu = (\cos\tau, \sin\tau, 0, 0)$, one can immediately see that the relevant propagator in (1.71) reduces to a constant

$$\langle \Big(iA_\mu(x_1)\dot{x}_1^\mu + |\dot{x}_1|\Phi_I(x_1)\Theta^I\Big)\Big(iA_\nu(x_2)\dot{x}_2^\nu + |\dot{x}_2|\Phi_J(x_2)\Theta^J\Big)\rangle = \frac{\lambda}{16\pi^2}\delta^{ab} \tag{1.72}$$

Given the coordinate independence of the propagator it is possible to sum an infinite class of Feynman diagrams without internal vertices, the so-called planar *ladder* or *rainbow* diagrams. It is just a counting problem, doable if one can solve the appropriate recurrence relation. At the end one finds a result valid at leading order in N and for any value of the 't Hooft coupling λ. It is in fact believed that graphs with internal vertices cancel at any order in perturbation theory and one is then left only with free propagators [48]. A formal, albeit incomplete, proof of this conjecture based on the conformal anomaly mentioned above and valid for any values of N (and thus including non-planar corrections) and λ

[1]This is an old result in conformal field theory [46]. The first implementation in a holographic computation appeared in [47]

was presented in [49].
But the very fact that the propagator (1.72) loses coordinate dependence for the $\frac{1}{2}$-BPS circular loop suggests that it might be possible to map the problem of summing the ladder diagrams of the circle to a 0-dimensional matrix model. Furthermore, since interacting graphs are conjectured to cancel in the perturbative expansion we also expect this matrix model to be quadratic [48, 49].
For some years this remained as a promising conjecture, until it was finally proved by Pestun using the supersymmetric localization technique [17].

Applying the localization technique to the particular case of $\mathcal{N} = 4$ super Yang-Mills it turns out that the one-loop determinant is exactly one (that is, the bosonic and fermionic contributions cancel each other), all the instanton corrections vanish and we are led to a Hermitian Gaussian matrix model defined in terms of the partition function [17]

$$Z = \int [dM] \exp\Big(- \frac{2N}{\lambda} \text{Tr} M^2 \Big). \tag{1.73}$$

The $\frac{1}{2}$-BPS circular Wilson loop is a very particular operator that preserves enough supersymmetries off-shell and for which the localization technique can also be applied. As a first particular example, the vacuum expectation value of the loop transforming in the fundamental representation is given by

$$\langle W_\Box(\mathcal{C}) \rangle = \langle \frac{1}{N} \text{Tr} e^M \rangle = \frac{1}{Z} \int [dM] \exp\Big(- \frac{2N}{\lambda} \text{Tr} M^2 \Big) \frac{1}{N} \text{Tr} e^M. \tag{1.74}$$

Being Gaussian, this matrix model integral can be solved exactly by using orthogonal polynomials. The final exact result turns out to be given in terms of a generalized Laguerre polynomial

$$\langle W_\Box(\mathcal{C}) \rangle = \frac{1}{N} L^1_{N-1}(-\frac{\lambda}{4N}) e^{-\frac{\lambda}{8N}}. \tag{1.75}$$

If we compute now the large N, strong coupling limit of this expression, which is the appropriate limit for a comparison with the holographic result, one has

$$\langle W_\Box(\mathcal{C}) \rangle \simeq \sqrt{\frac{2}{\pi}} \frac{e^{\sqrt{\lambda}}}{\lambda^{3/4}}, \tag{1.76}$$

finding perfect agreement with the AdS/CFT results obtained at leading order in $1/N$ and $1/\sqrt{\lambda}$.

References

[1] P.A.M. Dirac, “Quantised Singularities in the Electromagnetic Field,” Proc. Roy. Soc. London **A 133**, 60 (1931).

[2] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)] [hep-th/9711200].

[3] M. Green, J.H. Schwarz and E. Witten, “Sup erstring Theory” (2 volumes), Cambridge University Press (1988).

[4] J. Polchinski, “String Theory” (2 volumes), Cambridge University Press (1998).

[5] K. Becker, M. Becker, J.H. Schwatz, “String Theory and M-Theory: A Modern Intro duction,” Cambridge University Press (2007).

[6] B. Zwiebach, “A First Course in String Theory,” Cambridge University Press (2004/2009).

[7] D. Tong, “String Theory,” lecture notes [hep-th/0908.0333].

[8] J. Polchinski, “Dirichlet Branes and Ramond-Ramond charges,” Phys. Rev. Lett. **75**, 4724 (1995) [hep-th/9510017].

[9] G. ’t Hooft, “A planar diagram theory for strong interactions,” Nuclear Physics, Section B, Volume **72**, Issue 3, p. 461-473 (1974).

[10] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, ”Gauge Theory Correlators from Non-Critical String Theory”, Phys. Lett. B428:105-114,1998 [arXiv:hep-th/9802109v2].

[11] E. Witten, ”Anti-de Sitter space and holography”, Adv. Theor. Math. Phys. **2**, 253 (1998)[arXiv:hep-th/9802150].

[12] E. Witten, “Topological quantum field theory,” Comm. Math. Phys. **117** (1988) 353–386.

[13] E. Witten, "Topological sigma models," Comm. Math. Phys. **118** (1988) 411–449.

[14] N. Seiberg and E. Witten, "Electric-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [hep-th/9407087].

[15] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," Nucl. Phys. B **431**, 484 (1994) [hep-th/9408099].

[16] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," Adv. Theor. Math. Phys. **7**, 831 (2004) [hep-th/0206161].

[17] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," Commun. Math. Phys. **313**, 71 (2012) [arXiv:0712.2824 [hep-th]].

[18] K. Wilson, "Confinement of quark," Physical Review D **10**, 2445 (1974).

[19] S. Mandelstam, "Feynman Rules for Electromagnetic and Yang-Mills Fields from the Gauge-Independent Field-Theoretic Formalism," Phys. Rev. **175**, 1580 (1968).

[20] A. M. Polyakov, "String representations and hidden symmetries for gauge fields," Phys. Lett. B, Volume **82**, Issue 2, p. 247-250 (1979).

[21] Yu. M. Makeenko and A. A. Migdal, "Exact equation for the loop average in multicolor QCD," Phys. Lett. B, Volume **88**, Issues 1–2, 3 p. 135–137 (1979)

[22] A. M. Polyakov, "Gauge Fields as Rings of Glue," Nucl. Phys. B **164**, 171 (1980).

[23] R. Giles, "Reconstruction of gauge potentials from Wilson loops," Phys. Rev. D **24**, 2160 (1981).

[24] Y. Aharonov and D. Bohm, "Significance of electromagnetic potentials in quantum theory," Phys. Rev. **115** p.485–491 (1959).

[25] J. M. Maldacena, "Wilson loops in large N field theories," Phys. Rev. Lett. **80**, 4859 (1998) [hep-th/9803002].

[26] S. J. Rey and J. T. Yee, "Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity," Eur. Phys. J. C **22** (2001) 379 [hep-th/9803001].

[27] N. Drukker, D. J. Gross and H. Ooguri, "Wilson loops and minimal surfaces," Phys. Rev. D **60**, 125006 (1999) [hep-th/9904191].

[28] D. J. Gross and H. Ooguri, “Aspects of large N gauge theory dynamics as seen by string theory,” Phys. Rev. D **58**, 106002 (1998) [arXiv:hep-th/9805129].

[29] N. Drukker and B. Fiol, “All-genus calculation of Wilson loops using D-branes,” JHEP **0502**, 010 (2005) [hep-th/0501109].

[30] S. Yamaguchi, “Wilson loops of anti-symmetric representation and D5-branes,” JHEP 0605, **037** (2006) [arXiv:hep-th/0603208].

[31] D. Rodriguez-Gomez, “Computing Wilson lines with dielectric branes,” Nucl. Phys. B **752**, 316 (2006) [arXiv:hep-th/0604031].

[32] C. G. . Callan and J. M. Maldacena, “Brane dynamics from the Born-Infeld action,” Nucl. Phys. B **513**, 198 (1998) [arXiv:hep-th/9708147].

[33] R. Emparan, “Born-Infeld strings tunneling to D-branes,” Phys. Lett. B **423**, 71 (1998) [arXiv:hep-th/9711106].

[34] R. C. Myers, “Dielectric-branes,” JHEP 9912, **022** (1999) [arXiv:hep-th/9910053].

[35] R. Emparan, “Tubular Branes in Fluxbranes,” Nucl.Phys. B **610** (2001) 169-189, [hep-th/0105062].

[36] I. Bena, “The Polarization of F(1) strings into D2-branes: ’Aut Caesar aut nihil’,” Phys. Rev. D **67**, 026004 (2003) [hep-th/0111156].

[37] J. Gomis and F. Passerini, “Holographic Wilson loops,” JHEP **0608**, 074 (2006) [arXiv:hep-th/0604007]; “Wilson loops as D3-branes,” JHEP **0701**, 097 (2007) [arXiv:hep-th/0612022].

[38] J. McGreevy, L. Susskind and N. Toumbas, “Invasion of the giant gravitons from anti-de Sitter space,” JHEP 0006, **008** (2000) [hep-th/0003075].

[39] M. T. Grisaru, R. C. Myers and O. Tafjord, “SUSY and Goliath,” JHEP 0008, **040** (2000) [hep-th/0008015].

[40] A. Hashimoto, S. Hirano and N. Itzhaki, “Large branes in AdS and their field theory dual,” JHEP 0008, **051** (2000) [hep-th/0008016].

[41] S. Yamaguchi, “Bubbling geometries for half BPS Wilson lines,” Int. J. Mod. Phys. A **22**, 1353 (2007) [arXiv:hep-th/0601089].

[42] O. Lunin, "On gravitational description of Wilson lines," JHEP 0606, **026** (2006) [arXiv:hep-th/0604133].

[43] S. A. Hartnoll, "Two universal results for Wilson loops at strong coupling," Phys. Rev. D **74**, 066006 (2006) [arXiv:hep-th/0606178].

[44] B. Fiol, A. Guijosa and J. F. Pedraza, "Branes from Light: Embeddings and Energetics for Symmetric k-Quarks in $\mathcal{N} = 4$ SYM," arXiv:1410.0692 [hep-th].

[45] A. Mikhailov, "Nonlinear waves in AdS / CFT correspondence," hep-th/0305196.

[46] T. Fulton and F. Rohrlich, "Classical Radiation from a Uniformly Accelerated Charge," Ann. Phys. **9** 499-517 (1960).

[47] D. E. Berenstein, R. Corrado, W. Fischler and J. M. Maldacena, "The Operator product expansion for Wilson loops and surfaces in the large N limit," Phys. Rev. D **59**, 105023 (1999) [hep-th/9809188].

[48] J. K. Erickson, G. W. Semenoff and K. Zarembo, "Wilson loops in N = 4 supersymmetric Yang-Mills theory," Nucl. Phys. B **582**, 155 (2000) [arXiv:hep-th/0003055].

[49] N. Drukker and D. J. Gross, "An Exact Prediction Of N = 4 Susym Theory For String Theory," J. Math. Phys. **42**, 2896 (2001) [arXiv:hep-th/0010274].

Bibliography

[50] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N Field Theories, String Theory and Gravity", Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[51] E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS / CFT correspondence," hep-th/0201253.

[52] David Mateos, "String Theory and Quantum Chromodynamics," Class. Quant. Grav. 24:S713-S740, 2007 [arXiv:hep-th/07091523v1].

[53] Marcos Mariño, "Chern–Simons Theory, Matrix Models, and Topological Strings," Oxford University Press, ISBN-10: 019-856-849-5 (2005).

[54] John McGreevy "8.821 String Theory" (MIT open course).

[55] Juan Maldacena, "TASI 2003 lectures on AdS/CFT," [arXiv:hep-th/0309246v5].

[56] Igor R. Klebanov, “TASI Lectures: Introduction to the AdS/CFT Correspondence,” PUPT-1954 [arXiv:hep-th/0009139v2].

[57] Ingemar Bengtsson, “Anti-de Sitter Space”. Lecture notes.

[58] S. W. Hawking, G. F. R. Ellis, “The large scale structure of space-time,” Cambridge university press (1973).

[59] Aidan J. Keane and Richard K. Barrett, “The conformal group SO(4,2) and Robertson-Walker spacetimes,” Class. Quantum Grav. **17** 201D218 (2000).

[60] P. Di Francesco, P. Mathieu, and D. Sénéchal, “Conformal Field Theory,” Springer-Verlag, New York, 1997. ISBN 0-387-94785-X.

[61] Matthias R. Gaberdiel, “An Introduction to Conformal Field Theory,” Rept. Prog. Phys. 63 (2000) 607-667 [arXiv:hep-th/9910156v2].

Chapter 2

Corrections to all orders in $1/N$ using D-brane probes

This chapter contains the publication:

- B. Fiol and B. Garolera,
  "Energy Loss of an Infinitely Massive Half-Bogomol'nyi-Prasad-Sommerfeld Particle by Radiation to All Orders in $1/N$,"

  Phys. Rev. Lett. **107** (2011) 151601, arXiv:1106.5418 [hep-ph].

Energy Loss of an Infinitely Massive Half-Bogomol'nyi-Prasad-Sommerfeld Particle by Radiation to All Orders in $1/N$

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We use the AdS/CFT correspondence to compute the energy radiated by an infinitely massive half-Bogomol'nyi-Prasad-Sommerfeld particle charged under $\mathcal{N} = 4$ super Yang-Mills theory, transforming in the symmetric or antisymmetric representation of the gauge group, and moving in the vacuum, to all orders in $1/N$ and for large 't Hooft coupling. For the antisymmetric case we consider $D5$-branes reaching the boundary of five-dimensional anti–de Sitter space (AdS_5) at arbitrary timelike trajectories, while for the symmetric case, we consider a $D3$-brane in AdS_5 that reaches the boundary at a hyperbola. We compare our results to the one obtained for the fundamental representation, deduced by considering a string in AdS_5.

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Introduction.—Given a gauge theory, one of the basic questions one can address is the energy loss of a particle charged under such gauge fields, as it follows arbitrary trajectories. For classical electrodynamics this is a settled question, with many practical applications [1]. Much less is known for generic quantum field theories, especially in strongly coupled regimes. This state of affairs has started to improve with the advent of the AdS/CFT correspondence [2], which has allowed us to explore the strongly coupled regime of a variety of field theories. Within this framework, the particular question of the energy radiated by a particle charged under a strongly coupled gauge theory—either moving in a medium or in the vacuum with nonconstant velocity—has received a lot of attention (see [3] for relevant reviews). The motivations are manifold, from the more phenomenological ones, such as modeling the energy loss of quarks in the quark-gluon plasma [4] to the more formal ones, such as the study of the Unruh effect [5]. In most of these studies the heavy particle transforms in the fundamental representation of the gauge group, and the dual computation is in terms of a string moving in an asymptotically AdS space. The main purpose of this note is to extend this prescription to other representations of the gauge group, which will amount to replacing the fundamental string by $D3$ and $D5$-branes (see [6] for a previous appearance of this idea), in complete analogy to the prescription developed for the computation of Wilson loops [7–11].

Besides the intrinsic interest of this generalization, our main motivation in studying it is that, as it happens in the computation of certain Wilson loops, the results for the energy loss obtained with D-branes give an all-orders series in $1/N$. Given the paucity of such results for large N $4d$ gauge theories, this by itself justifies its consideration. Furthermore, these $1/N$ terms might shed some light on some recent results in the study of radiation using the AdS/CFT correspondence. Let us briefly review them.

The case of an infinitely massive particle transforming in the fundamental representation and following an arbitrary timelike trajectory was addressed by Mikhailov [12], who quite remarkably found a string solution in AdS_5 that solves the Nambu-Goto equations of motion and reaches the boundary at any given particle worldline. Working in Poincaré coordinates,

$$ds^2_{AdS_5} = \frac{L^2}{y^2}(dy^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \quad (1)$$

it was furthermore shown that the energy of that string with respect to the Poincaré time is given by

$$E = \frac{\sqrt{\lambda}}{2\pi}\left(\int dt \frac{\vec{a}^2 - |\vec{a}\wedge\vec{v}|^2}{(1-v^2)^3} + \gamma\frac{1}{y}\bigg|_{y=0}\right), \quad (2)$$

where the integral is with respect to the worldline time coordinate, and $\lambda = g^2_{YM}N$ is the 't Hooft coupling. The second (divergent) term corresponds to the (infinite) mass and γ is the Lorentz factor. The first term corresponds to the radiated energy, so in the supergravity regime the total radiated power by a particle in the fundamental representation is

$$P_F = \frac{\sqrt{\lambda}}{2\pi}a^\mu a_\mu, \quad (3)$$

which is essentially Lienard's formula for radiation in classical electrodynamics [1] with the substitution $e^2 \to 3\sqrt{\lambda}/4\pi$. This $\sqrt{\lambda}$ dependence also appears—and has the same origin—in the computation of the vacuum expectation value (VEV) of Wilson loops at strong coupling [8,13].

Having computed the total radiated power, a more refined question is to determine its angular distribution. For a particle moving in the vacuum, this has been done in

[14,15], who found that this angular distribution is essentially like that of classical electrodynamics. This is a somewhat counterintuitive result, as one might have expected that the strong coupling of the gauge fields would tend to broaden the radiating pulses and make radiation more isotropic. In particular, the authors of [15] argue that these results are an artifact of the supergravity approximation, and might go away once stringy effects are taken into account (see [16] for alternative interpretations). Here is where considering particles in other representations might be illuminating, since the $1/N$ expansion of the radiated power we find can be interpreted as capturing string loop corrections [17].

The plan of the present note is as follows: in the next section we introduce $D5$-branes dual to particles in the antisymmetric representation following arbitrary timelike trajectories, and evaluate the corresponding energy loss. We then consider a $D3$-brane dual to a particle in the symmetric representation following hyperbolic motion, and compute its energy loss. We end by discussing the possible connection of this result with the similar one for particles in the fundamental representation, and mentioning possible extensions of this work.

D5-branes and the antisymmetric representation.—Given a string worldsheet that solves the Nambu-Goto action in an arbitrary manifold M, there is a quite general construction due to Hartnoll [18] that provides a solution for the $D5$-brane action in $M \times S^5$, of the form $\Sigma \times S^4$ where $\Sigma \hookrightarrow M$ is the string worldsheet and $S^4 \hookrightarrow S^5$. The evaluation of the respective renormalized actions gives then a universal relation between the VEV of Wilson loops in the antisymmetric and fundamental representations, already observed, in particular, examples [9,19]. More recently, this construction has been used to evaluate the energy loss of a particle in the antisymmetric representation, moving with constant speed in a thermal medium [6]. In this section we combine Mikhailov's string worldsheet solution [12] with Hartnoll's $D5$-brane construction [18] to compute the radiated power for a particle in the antisymmetric representation.

For a given timelike trajectory, we consider a $D5$-brane in $AdS_5 \times S^5$, with worldvolume $\Sigma \times S^4$ where Σ is the corresponding Mikhailov worldsheet [12]. On Σ there is in addition an electric Dirac-Born-Infeld (DBI) field strength with k units of charge [18]. This $D5$-brane is identified as the dual to a particle transforming in the kth antisymmetric representation, and following the given timelike trajectory. As shown in [18] the equations of motion force the angle of S^4 in S^5 to be

$$\sin\theta_0 \cos\theta_0 - \theta_0 = \pi\left(\frac{k}{N} - 1\right). \qquad (4)$$

We now proceed to compute the energy with respect to the Poincaré time coordinate and the radiated power of such particle. The energy density for the $D5$-brane is

$$\mathcal{E}_{D5} = T_{D5} \frac{L^2}{y^2} \frac{|\gamma + F|_s}{\sqrt{-|\gamma + F|}} = T_{D5} \frac{L^2}{y^2} \frac{|\gamma_\Sigma|_s}{\sin\theta_0 \sqrt{-|\gamma_\Sigma|}} \sqrt{|\gamma_{S^4}|},$$

where the subscript s means that the determinant is restricted to the spatial directions of the $D5$-brane or the fundamental string. We have used that in Hartnoll's solution the DBI field strength is purely electric and the DBI determinant is block diagonal. Integrating over the S^4 part of the worldvolume one immediately obtains up to constants the energy density of the fundamental string, so

$$E_{D5} = \frac{2N}{3\pi} \sin^3\theta_0 E_{F1}.$$

This is the same relation as the one found between the renormalized actions of the $D5$-brane and the fundamental string [18], and in [6] for the relation of drag forces in a thermal medium. In the regime of validity of supergravity, the radiated power of a particle in the kth antisymmetric representation is therefore related to the radiated power of a particle in the fundamental representation (3) by

$$P_{A_k} = \frac{2N}{3\pi} \sin^3\theta_0 P_F. \qquad (5)$$

The range of validity of this computation is determined by demanding that backreaction of the $D5$-brane can be neglected and its size is large in string units, yielding $g_s^2 N \sin^3\theta_0 \ll 1$ and $\lambda^{1/4} \sin\theta_0 \gg 1$, respectively. For comparison with the symmetric case it is convenient to write these conditions as $N^2/\lambda^2 \gg N \sin^3\theta_0 \gg N/\lambda^{3/4}$. In particular, this implies that the result cannot be trusted when k/N is very close to 0 or 1.

D3-branes and the symmetric representation.—The computation of Wilson loops of half-BPS particles in the symmetric representation is given by evaluating the renormalized action of $D3$-branes [10], and analogously we propose to compute the radiated power of a half-BPS particle in the symmetric representation by evaluating the energy of a $D3$-brane that reaches the boundary of AdS at the given timelike trajectory. Contrary to what happens for the fundamental or the antisymmetric representations, we currently do not have the generic $D3$-brane solution, so we will focus on a particular trajectory. On the other hand, since these $D3$-branes are fully embedded in AdS_5, we do not use any possible transverse dimensions, so the results should be valid for other $4d$ conformal theories with a gravity dual.

The particular trajectory we will consider is one-dimensional motion with constant proper acceleration, which in an inertial system corresponds to $\gamma^3 a = 1/R$. The trajectory is hyperbolic, $-(x^0)^2 + (x^1)^2 = R^2$. A relevant feature is that a special conformal transformation applied to a straight worldline (static particle) gives the two branches of hyperbolic motion [20]. Besides its prominent role in the study of radiation and the Unruh effect, another reason to choose this trajectory is that the relevant $D3$-brane is the analytic continuation of the one already found in [7].

The radiated energy of a particle in the fundamental representation, Eq. (2) derived in [12], is written in terms of the worldline of the heavy particle. At least in particular cases, it is possible to obtain an alternative derivation that emphasizes the presence of a horizon in the worldsheet metric, which encodes the split between radiative and non-radiative gluonic fields, and therefore signals the existence of energy loss of the dual particle, even in the vacuum [21]. It is convenient to briefly rederive this result for the particular case of hyperbolic motion, since the computation of the energy loss using a $D3$-brane that we will shortly present resembles closely this second derivation. Working in Poincaré coordinates, Mikhailov's string solution for hyperbolic motion can be rewritten as $y^2 = R^2 + (x^0)^2 - (x^1)^2$; the Euclidean continuation of this worldsheet is the one originally used to evaluate the VEV of a circular Wilson loop [22] (see also [23]). This worldsheet is locally AdS_2 and has a horizon at $y = R$, with temperature $T = 1/2\pi R$, which is the Unruh temperature measured by an observer following a $r_1 = R$ trajectory in Rindler space. By integrating the energy density from the horizon to the boundary we obtain

$$E = \frac{\sqrt{\lambda}}{2\pi} \int_0^R \frac{dy}{y^2} \frac{1}{\sqrt{R^2 + (x^0)^2 - y^2}} = \frac{\sqrt{\lambda}}{2\pi}\left(\frac{-x^0}{R^2} + \gamma \frac{1}{y}\bigg|_{y=0}\right). \quad (6)$$

The contribution from the boundary is just the (divergent) second term, corresponding to the mass of the particle. The first term comes from the horizon contribution, and corresponds to the radiated energy.

A. The D3-brane solution.—We are interested in a $D3$-brane that reaches the boundary of AdS_5 at a single branch of the hyperbola $-(x^0)^2 + (x^1)^2 = R^2$. To find it, we change coordinates on the (x^0, x^1) plane of (1) to Rindler coordinates, so the new coordinates cover only a Rindler wedge

$$ds^2 = \frac{L^2}{y^2}(dy^2 + dr_1^2 - r_1^2 d\psi^2 + dr_2^2 + r_2^2 d\phi^2). \quad (7)$$

In these coordinates the relevant $D3$-brane solution found in [7] is given by

$$(r_1^2 + r_2^2 + y^2 - R^2)^2 + 4R^2 r_2^2 = 4\kappa^2 R^2 y^2, \quad (8)$$

where

$$\kappa = \frac{k\sqrt{\lambda}}{4N}.$$

Near the AdS_5 boundary $y = 0$, this solution goes to $r_2 = 0$, $r_1^2 = R^2$, so it reaches a circle in Euclidean signature and the branch of a hyperbola in the Lorentzian one. The $D3$-brane also supports a nontrivial Born-Infeld field-strength on its worldvolume [7]. By a suitable change of coordinates, its worldvolume metric can be written as [7]

$$ds^2 = L^2(1 + \kappa^2)(d\zeta^2 - \sinh^2\zeta d\psi^2) + L^2\kappa^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (9)$$

so it is locally $AdS_2 \times S^2$, with radii $L\sqrt{1 + \kappa^2}$ and $L\kappa$ respectively, and it has a horizon at $\zeta = 0$ [i.e., $r_1 = 0$ in the coordinates of (7)]. The temperature of this horizon can be computed by requiring that the associated Killing vector is properly normalized at infinity; this is easily done in the coordinates of (7) and the resulting temperature is again

$$T = \frac{1}{2\pi R}. \quad (10)$$

B. Evaluation of the energy.—To determine the total radiated power of this solution we will evaluate the energy with respect the Poincaré time coordinate x^0. The energy density is

$$\mathcal{E} = T_{D3}\left(\frac{L^2}{y^2} \frac{|\gamma + F|_s}{\sqrt{-|\gamma + F|}} - \frac{L^4}{y^4}\right). \quad (11)$$

After we substitute the Lorentzian continuation of the solution of [7] in this expression, the energy density is

$$\mathcal{E} = T_{D3}\frac{L^4}{y^4} \times \left(\frac{(1 + \kappa^2)R^2 + (x^0)^2}{\sqrt{\kappa^2(1 + \kappa^2)R^4 - \kappa^2 R^2 r_1^2 - (1 + \kappa^2)R^2 r_2^2}} - 1\right). \quad (12)$$

The energy is the integral of this energy density from the boundary to the worldvolume horizon. A long computation yields

$$E = \frac{2N\kappa}{\pi}\left(-\frac{x^0}{R^2}\sqrt{1 + \kappa^2} + \gamma \frac{1}{y}\bigg|_{y=0}\right). \quad (13)$$

Exactly as it happened for the string, Eq. (6), the boundary contributes only the second term, which is divergent, and is just k times the one for the fundamental string, Eq. (6). The first term is the contribution from the horizon, and from it we can read off the total radiated power

$$P_{S_k} = \frac{2N\kappa}{\pi}\sqrt{1 + \kappa^2}\frac{1}{R^2} = \frac{k\sqrt{\lambda}}{2\pi}\sqrt{1 + \frac{k^2\lambda}{16N^2}}\frac{1}{R^2}.$$

This result was found for a particular timelike trajectory with $a^\mu a_\mu = 1/R^2$. Nevertheless, in classical electrodynamics the radiated power depends on the kinematics only through the square of the 4-acceleration, $a^\mu a_\mu$ and as we have seen, the same is true in theories with gravity duals for particles in the fundamental, Eq. (3), and antisymmetric representations, Eq. (5). It is then natural to conjecture that in the regime of validity of supergravity, the radiated power by a particle in the symmetric representation following arbitrary timelike motion is

$$P_{S_k} = \frac{k\sqrt{\lambda}}{2\pi}\sqrt{1 + \frac{k^2\lambda}{16N^2}a^{\mu}a_{\mu}}. \qquad (14)$$

It would be interesting to check this conjecture by finding $D3$-branes that reach the AdS boundary at arbitrary timelike trajectories and evaluating the corresponding energies.

We now discuss the range of validity of this result, and its possible relevance for the case of a particle in the fundamental representation. By demanding that the radii of the $D3$-brane are much larger than l_s and that its back-reaction can be neglected, one can conclude [7] that this result can be trusted when $N^2/\lambda^2 \gg k \gg N/\lambda^{3/4}$. It is therefore not justified *a priori* to set $k = 1$ in our result, Eq. (14). Nevertheless, the Euclidean continuation of this $D3$-brane was used in [7] to compute the VEV of a circular Wilson loop, which for $k = 1$ is known exactly for all N and λ thanks to a matrix model computation [17,24], and it was found [7] that the $D3$-brane reproduces the correct result in the large N, λ limit with κ fixed, i.e., even for $k = 1$. This better than expected performance (probably due to supersymmetry) of the Euclidean counterpart of this $D3$-brane in a very similar computation suggests the exciting possibility that (14) might capture correctly all the $1/N$ corrections to the radiated power of a particle in the fundamental representation, i.e., for $k = 1$, in the limit of validity of supergravity.

We are investigating whether the angular distributions of the radiated energy obtained with this $D3$-brane and with fundamental strings [14,15] differ qualitatively.

Finally, as already mentioned, the Euclidean version of the $D3$-brane considered here was used in [7] to evaluate the VEV of a circular Wilson loop. That $D3$-brane result is in turn only an approximation to the exact result, available for all N and λ thanks to a matrix model computation [17,24]. It would be extremely interesting to understand whether the radiated power of a particle coupled to a conformal gauge theory can be similarly computed by a matrix model.

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[1] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999), 3rd ed.
[2] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[3] S. S. Gubser, S. S. Pufu, F. D. Rocha, and A. Yarom, arXiv:0902.4041; J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal, and U. A. Wiedemann, arXiv:1101.0618.
[4] C. P. Herzog, A. Karch, P. Kovtun, C. Kozcaz, and L. G. Yaffe, J. High Energy Phys. 07 (2006) 013; S. S. Gubser, Phys. Rev. D **74**, 126005 (2006); J. Casalderrey-Solana and D. Teaney, Phys. Rev. D **74**, 085012 (2006).
[5] A. Paredes, K. Peeters, and M. Zamaklar, J. High Energy Phys. 04 (2009) 015; T. Hirayama, P.-W. Kao, S. Kawamoto, and F.-L. Lin, Nucl. Phys. **B844**, 1 (2011); M. Chernicoff and A. Paredes, J. High Energy Phys. 03 (2011) 063.
[6] M. Chernicoff and A. Guijosa, J. High Energy Phys. 02 (2007) 084.
[7] N. Drukker and B. Fiol, J. High Energy Phys. 02 (2005) 010.
[8] S.-J. Rey and J.-T. Yee, Eur. Phys. J. C **22**, 379 (2001).
[9] S. Yamaguchi, J. High Energy Phys. 05 (2006) 037.
[10] J. Gomis and F. Passerini, J. High Energy Phys. 08 (2006) 074; 01 (2007) 097.
[11] S. A. Hartnoll and S. Prem Kumar, J. High Energy Phys. 08 (2006) 026; K. Okuyama and G. W. Semenoff, J. High Energy Phys. 06 (2006) 057.
[12] A. Mikhailov, arXiv:hep-th/0305196.
[13] J. M. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998).
[14] C. Athanasiou, P. M. Chesler, H. Liu, D. Nickel, and K. Rajagopal, Phys. Rev. D **81**, 126001 (2010).
[15] Y. Hatta, E. Iancu, A. H. Mueller, and D. N. Triantafyllopoulos, J. High Energy Phys. 02 (2011) 065; Nucl. Phys. **B850**, 31 (2011).
[16] V. E. Hubeny, New J. Phys. **13**, 035006 (2011); V. E. Hubeny, arXiv:1011.1270; M. Chernicoff, A. Guijosa, and J. F. Pedraza, arXiv:1106.4059.
[17] N. Drukker and D. J. Gross, J. Math. Phys. (N.Y.) **42**, 2896 (2001).
[18] S. A. Hartnoll, Phys. Rev. D **74**, 066006 (2006).
[19] S. A. Hartnoll and S. Prem Kumar, Phys. Rev. D **74**, 026001 (2006).
[20] T. Fulton and F. Rohrlich, Ann. Phys. (N.Y.) **9**, 499 (1960).
[21] M. Chernicoff and A. Guijosa, J. High Energy Phys. 06 (2008) 005; F. Dominguez, C. Marquet, A. H. Mueller, B. Wu, and B.-W. Xiao, Nucl. Phys. **A811**, 197 (2008).
[22] D. E. Berenstein, R. Corrado, W. Fischler, and J. M. Maldacena, Phys. Rev. D **59**, 105023 (1999).
[23] B.-W. Xiao, Phys. Lett. B **665**, 173 (2008); V. Branding and N. Drukker, Phys. Rev. D **79**, 106006 (2009).
[24] V. Pestun, arXiv:0712.2824.

Chapter 3

Exact results in $\mathcal{N} = 4$ super Yang-Mills

3.1 Energy loss by radiation

This chapter contains the publication:

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Exact results for static and radiative fields of a quark in $\mathcal{N} = 4$ super Yang-Mills

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Abstract: In this work (which supersedes our previous preprint [1]) we determine the expectation value of the $\mathcal{N} = 4$ SU(N) SYM Lagrangian density operator in the presence of an infinitely heavy static particle in the symmetric representation of SU(N), by means of a D3-brane probe computation. The result that we obtain coincides with two previous computations of different observables, up to kinematical factors. We argue that these agreements go beyond the D-brane probe approximation, which leads us to propose an exact formula for the expectation value of various operators. In particular, we provide an expression for the total energy loss by radiation of a heavy particle in the fundamental representation.

Keywords: Supersymmetric gauge theory, AdS-CFT Correspondence

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1 Introduction

Exact results for generic 4d quantum field theories are extremely hard to come by. The situation improves when one considers quantum field theories with additional symmetries, as conformal invariance and/or supersymmetry, since these additional symmetries constrain the parametric dependence of a variety of interesting quantities, that can sometimes be determined exactly.

One of the most intensively studied theories with such additional symmetries is $\mathcal{N} = 4$ SYM, which is both conformally invariant and maximally supersymmetric. Among its local gauge invariant operators, one encounters the chiral primary operators (CPOs) and their descendants, which fall into short multiplets of the superconformal algebra and enjoy various special properties [2–4]. For the purposes of this work, we will be chiefly interested in the supercurrent multiplet, whose CPO has scaling dimension $\Delta = 2$, since both the Lagrangian density and the stress-energy tensor belong to this multiplet.

Among the non-local gauge invariant operators of $\mathcal{N} = 4$ SYM, locally BPS Wilson loops and Wilson lines have also been intensively studied over the years. They are characterized by a contour in space-time and a representation of the gauge group, and their vacuum expectation value has been computed exactly in a few cases [5–8]. Finally, there have been a number of works devoted to computing correlation functions of Wilson loops with local operators [9–11].

In a superficially different line of research, the behavior of external probes and the response of the fields to such probes, both in vacuum and at finite temperature, have been intensively scrutinized using the AdS/CFT duality [12, 13]. One first goal of the present note is to continue the study of such computations, by presenting the evaluation of the one-point function of the Lagrangian density in the presence of a static heavy probe in the symmetric representation.[1] A second, and in our opinion, farther reaching goal is to argue that some of the questions that appear in the study of external probes can be answered exactly, by relating them to the evaluation of certain correlation functions involving Wilson loops and local operators. To be specific, we propose that from the two-point function of

[1] This computation appeared in our previous preprint [1], which has been superseded by the present work.

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the circular Wilson loop in the fundamental representation and the $\Delta = 2$ CPO, computed exactly in [10] (and normalized by the vev of the circular Wilson loop computed in [5–7]) one can read off the exact one-point functions of this CPO and (more importantly) its descendants, in the presence of a heavy particle following either a static trajectory (straight Wilson line) or a trajectory with constant proper acceleration (hyperbolic Wilson line). Up to kinematic factors, these one-point functions are given by the following function

$$f(\lambda, N) = \frac{\lambda}{64\pi^2 N} \frac{L^2_{N-1}(-\frac{\lambda}{4N}) + L^2_{N-2}(-\frac{\lambda}{4N})}{L^1_{N-1}(-\frac{\lambda}{4N})} \tag{1.1}$$

where L^α_n are generalized Laguerre polynomials.

The structure of this note is as follows. In section 2 we review the study of external probes within the framework of the AdS/CFT correspondence, and the computation of various one-point functions of operators in the presence of such external probes. In section 3 we present[2] the computation of the one-point function of the Lagrangian density for a static heavy particle in the symmetric representation, by means of the study of the perturbation of the dilaton profile caused by a certain D3-brane in the AdS_5 background. We note that the result that we obtain is, up to the respective kinematical factors, exactly the same as in two previous D3-brane probe computations that have appeared in the literature, a first one regarding the two-point function of a circular Wilson loop with a particular chiral primary operator [11],[3] and a second one devoted to the computation for energy loss by radiation of a heavy particle in the symmetric representation [15]. Finally, in section 4 we address why the results of these different computations ought to coincide, and argue that their agreement holds for arbitrary representations of the gauge group and beyond the D-brane probe approximation. This leads us to propose that existing exact results [10] provide an exact formula for all those quantities.

Note Added: Quite recently, a very interesting preprint appeared [14] presenting arguments, complementary to those provided here, for an exact formula for the radiation of a moving quark in $\mathcal{N} = 4$ SYM. While the possible connections between their and our arguments remain to be fully sorted out, happily the proposed formulas for energy loss exactly agree. Indeed, their formula depends on a function $B(\lambda, N)$

$$B(\lambda, N) = \frac{1}{2\pi^2} \lambda \partial_\lambda \log \langle W \rangle$$

where $\langle W \rangle$ is the vev of the circular Wilson loop obtained by a special conformal transformation of the 1/2 BPS straight line, and computed exactly in [5–7]. Considering the explicit form of $\langle W \rangle$ [5–7], it is a simple matter to check that this function B is up to a numerical factor our function f, eq. (1.1). In fact, using that

$$\partial_\lambda L^1_{N-1}\left(-\frac{\lambda}{4N}\right) = \frac{1}{4N} L^2_{N-2}\left(-\frac{\lambda}{4N}\right)$$

and $L^1_{N-1} = L^2_{N-1} - L^2_{N-2}$ one easily sees that $B = 4f$, so the dependence on λ and N is exactly the same.

[2] See footnote 1.

[3] We would like to thank N. Drukker for pointing out this reference to us, and for urging us to compare our results with the ones that appear there.

2 External probes in AdS/CFT

One of the many applications of the AdS/CFT correspondence is the study of the behavior of external probes and the response of the fields to the presence of those probes. The first example of such computations was the evaluation of the static quark-antiquark potential in [16, 17] by means of a particular string configuration reaching the boundary of AdS. Following those seminal works, the key idea of realizing external heavy quarks by strings in the bulk geometry has been generalized in many directions. In particular, as we will briefly review, probes transforming under different representations of the gauge group are holographically realized by considering different types of branes in the supergravity background.

An external probe in the fundamental representation of the gauge group is dual to a string in the bulk. At least for the simplest implementations of this identification (i.e. in the absence of additional scales like finite mass or non-zero temperature), the computed observables reveal a common feature: if we identify at weak coupling λ as the analogous of the charge squared, at strong coupling there is a screening of this charge, in the sense that the results obtained are similar to the ones we would obtain in classical electrodynamics, but with the strong coupling identification

$$e^2_{\square} \sim \sqrt{\lambda} \tag{2.1}$$

This generic behavior stems from the fact that the Nambu-Goto action evaluated for world-sheet metrics embedded in AdS_5 goes like

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-|g|} = -\frac{\sqrt{\lambda}}{2\pi L^2}\int d^2\sigma\sqrt{-|g|}$$

where L is the AdS_5 radius which generically cancels out from this expression when specific world-sheet metrics are plugged-in. Some examples of this are the original quark-antiquark potential [16, 17], the expectation value of gauge invariant operators in the presence of a particle at rest [18, 19] or following arbitrary motion [20–24], and the formula for energy loss by radiation [25].

This leading $\sqrt{\lambda}$ result is expected to receive $1/N$ and $1/\sqrt{\lambda}$ corrections. The computation of $1/\sqrt{\lambda}$ corrections is addressed for instance in [26–29]. As it turns out, a possible venue to compute results that capture $1/N$ corrections is to switch to probes transforming in higher rank representations of the gauge group. It is by now well understood that on the gravity side these probes are realized by D3 and D5 branes. Specifically, the duals of particles in the symmetric or antisymmetric representations of the gauge group are given by D3 and D5 branes respectively, with world-volume fluxes that encode the rank of the representation [30–33]. One of the novel features of this identification is that some computed observables are functions of k/N, where k is the rank of the symmetric/antisymmetric representation. This allows to explore the AdS/CFT correspondence beyond the leading large N, large λ regime.

While the holographic prescription is in principle equally straightforward for the study of probes in the symmetric and the antisymmetric representations, when it comes to actual computations we currently face more difficulties in the symmetric case than in the antisymmetric one. One of the reasons behind this difference comes from the existence of a quite universal result for the embedding of D5 branes in terms of embeddings of fundamental strings, due to Hartnoll [34]: given a string world-sheet that solves the Nambu-Goto action on an arbitrary Ricci flat manifold M, there is a quite general construction that provides a solution for the D5-brane action in $M \times S^5$, of the form $\Sigma \times S^4$ where $\Sigma \hookrightarrow M$ is the string world-sheet and $S^4 \hookrightarrow S^5$. This gives a link between the string used to describe a particle in the fundamental representation and the D5 brane used to represent a probe in the antisymmetric representation. Moreover,

$$\sqrt{\lambda} \to \frac{2N}{3\pi} \sin^3 \theta_k \sqrt{\lambda} \sim e^2_{A_k} \tag{2.2}$$

where θ_k denotes the angle of S^4 inside S^5 and is the solution of [35, 36]

$$\sin \theta_k \cos \theta_k - \theta_k = \pi \left(\frac{k}{N} - 1 \right)$$

This identification is supported by explicit computations of Wilson loops [32, 34] which match matrix model computations [37], energy loss by radiation in vacuum [15] and in a thermal medium [38], or the impurity entropy in supersymmetric versions of the Kondo model [39, 40].

On the other hand, for probes in the symmetric representation we currently don't have a generic construction that links the string that realizes a particle in the fundamental representation with a D3 brane that realizes a probe in the symmetric representation. Furthermore, while the observables analyzed so far in the symmetric representation seem to depend on the combination

$$\kappa = \frac{k\sqrt{\lambda}}{4N} \tag{2.3}$$

introduced in [41], they do not display a common function that replaces the $\sqrt{\lambda}$ dependence of the fundamental representation. For instance, in the computation of the energy loss by radiation in vacuum of a particle moving with constant proper acceleration, it was found in [15] that

$$\sqrt{\lambda} \to 4N\kappa\sqrt{1+\kappa^2} = k\sqrt{\lambda}\sqrt{1+\frac{k^2\lambda}{16N^2}} \overset{?}{\sim} e^2_{S_k} \tag{2.4}$$

while for the vev of a circular Wilson loop, it was found that [41]

$$\sqrt{\lambda} \to 2N\left(\kappa\sqrt{1+\kappa^2} + \sinh^{-1}\kappa\right)$$

While both functions expanded as a power series in κ start with the common term $k\sqrt{\lambda}$ (i.e. k times the result for the fundamental representation) they are clearly different beyond this leading order.

In order to shed some light on the issue of observables for probes in the symmetric representation, in the next section we will compute the expectation value of the Lagrangian density in the presence of an infinitely heavy half-BPS static particle, transforming in the k-symmetric representation of $\mathcal{N} = 4$ SU(N) SYM. This operator is sourced by the asymptotic value of the dilaton [42, 43] (we follow the conventions of [24]) ,

$$\mathcal{O}_{F^2} = \frac{1}{2g_{YM}^2}\text{Tr}\ \left(F^2 + [X_I, X_J][X^I, X^J] + \text{fermions}\ \right)$$

On general grounds, in the presence of a static probe placed at the origin, and transforming in the $\mathcal{R}$ representation of the gauge group, the one-point function will be of the form

$$\langle \mathcal{O}_{F^2}(\vec{x})\rangle_{\mathcal{R}} = \frac{f_{\mathcal{R}}(\lambda, N)}{|\vec{x}|^4}$$

and our objective is to compute the dimensionless function $f_{\mathcal{S}_k}(\lambda, N)$ when the probe transforms in the k-symmetric representation of SU(N). By analogy with the Coulombic case, one might refer to $f_{\mathcal{S}_k}(\lambda, N)$ as the square of the "chromo-electric" charge of the heavy particle. To carry out this computation we will consider a particular half-BPS D3-brane embedded in $AdS_5 \times S^5$ and analyze the linearized perturbation equation for the dilaton, with the D3-brane acting as source. The advantage of considering this operator is that the perturbation equation of the dilaton decouples from the equations for the metric and RR field perturbations, so its study is quite straightforward.

The analogous computation for a particle in the fundamental representation was carried out some time ago, considering in that case the perturbation equation for the dilaton sourced by a fundamental string [18, 19]. For future reference, let's end this section by quoting their final result in our conventions,

$$\langle \mathcal{O}_{F^2}(\vec{x})\rangle_{\square} = \frac{\sqrt{\lambda}}{16\pi^2}\frac{1}{|\vec{x}|^4} \tag{2.5}$$

3 Static fields via a probe computation

In this section we will present the details of the computation of $\langle \mathcal{O}_{F^2}\rangle$ in the presence of a static heavy probe transforming in the symmetric representation. We will first compute the linearized perturbation of the dilaton field caused by the D-brane probe, and from its behavior near the boundary of AdS_5 we will then read off the vev of $\mathcal{O}_{F^2}$. Our computations will closely follow the ones presented in [18, 19] for the case of a probe in the fundamental representation.

We work in Poincaré coordinates and take advantage of the spherical symmetry of the problem

$$ds^2_{AdS_5} = \frac{L^2}{z^2}\left(dz^2 - dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

The D3-brane we will be interested in was discussed in [16, 41]. It reaches the boundary of AdS ($z = 0$ in our coordinates) at a straight line $r = 0$, which is the world-line of the static dual particle placed at the origin. Since we let the D3 brane reach the boundary, the static particle is infinitely heavy. To describe the D3-brane, identify (t, z, θ, φ) as the world-volume coordinates. Then the solution is given by

$$r = \kappa z \qquad F_{tz} = \frac{\sqrt{\lambda}}{2\pi}\frac{1}{z^2}$$

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with κ as defined in eq. (2.3). As shown in detail in [41] this D3-brane is 1/2-BPS.

Our next step is to consider at linear level the backreaction that this D3-brane induces on the $AdS_5 \times S^5$ solution of IIB supergravity. More specifically, since the dilaton is constant in the unperturbed solution, and its stress-energy tensor is quadratic, at the linearized level the equation for the perturbation of the dilaton decouples from the rest of linearized supergravity equations. As in [18, 19], we work in Einstein frame, and take as starting point the action

$$S = -\frac{\Omega_5 L^5}{2\kappa_{10}^2}\int d^5x\sqrt{-|g_E|}\frac{1}{2}g_E^{mn}\partial_m\phi\partial_n\phi - T_{D3}\int d^4\xi\sqrt{-|G_E + e^{-\phi/2}2\pi\alpha' F|}$$

The resulting equation of motion can be written

$$\partial_m\left(\sqrt{-|g_E|}g_E^{mn}\partial_n\phi\right) = J(x)$$

with the source defined by the D3-brane solution

$$J(x) = \frac{T_{D3}\kappa_{10}^2}{\Omega_5 L}\frac{\kappa\sin\theta}{z^2}\delta\left(r - z\kappa\right)$$

To compute $\phi(x)$ we will use its Green function $D(x,x')$

$$\phi(x) = \int d^5x'\ D(x,x')\ J(x')$$

It is convenient to write $D(x,x')$ purely in terms of the invariant distance v defined by

$$\cos v = 1 - \frac{(t-t')^2 - (\vec{x}-\vec{x}')^2 - (z-z')^2}{2zz'} \tag{3.1}$$

The explicit expression for $D(v)$ can be found for instance in [18]

$$D = \frac{-1}{4\pi^2L^3\sin v}\frac{d}{dv}\left(\frac{\cos 2v}{\sin v}\Theta(1-|\cos v|)\right)$$

To carry out the integration, we follow the same steps as [18]. We first define a rescaled dilaton field,

$$\tilde{\phi} \equiv \frac{\Omega_5 L^8}{2\kappa_{10}^2}\phi$$

and use eq. (3.1) to change variables from t' to v to obtain, after an integration by parts

$$\tilde{\phi} = \frac{N\kappa z^2}{16\pi^4}\int_0^\infty dr'\int_0^\pi d\theta'\sin\theta'\int_0^{2\pi}d\varphi'\int_0^\infty\frac{dz'\ \delta(r'-z'\kappa)}{(z^2+z'^2+(\vec{x}-\vec{x}')^2)^{\frac{3}{2}}}\times$$
$$\times\int_0^\pi\frac{dv\ \cos 2v}{\left(1-\frac{2zz'\ \cos v}{z^2+z'^2+(\vec{x}-\vec{x}')^2}\right)^{\frac{3}{2}}}$$

The integral over v is the same one that appeared in the computation of the perturbation caused by a string dual to a static probe [19]. The novel ingredient in the computation comes from the non-trivial angular dependence in the current case. While it might

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be possible to completely carry out this integral, at this point it is pertinent to recall that to compute the expectation value of the dual field theory operator, we only need the leading behavior of the perturbation of the dilaton field near the boundary of AdS_5. Specifically [18],

$$\langle \mathcal{O}_{F^2} \rangle = -\frac{1}{z^3}\partial_z \tilde{\phi}|_{z=0} \tag{3.2}$$

so for our purposes it is enough to expand the integrands in powers of z, and keep only the leading z^4 term. This simplifies the task enormously, and reduces it to computing some straightforward integrals. Skipping some unilluminating steps we arrive at

$$\tilde{\phi} = \frac{N\kappa}{16\pi^2}\frac{z^4}{(z^2+r^2)^2}\frac{1}{(1+\kappa^2)^{3/2}}\frac{1}{\left(1-\frac{\kappa^2}{1+\kappa^2}\frac{r^2}{r^2+z^2}\right)^2} + O(z^5)$$

which upon differentiation, and setting then $z=0$ as required by eq. (3.2), leads to our final result

$$\langle \mathcal{O}_{F^2} \rangle_{S_k} = \frac{N\kappa\sqrt{1+\kappa^2}}{4\pi^2}\frac{1}{|\vec{x}|^4} = \frac{k\sqrt{\lambda}}{16\pi^2}\frac{\sqrt{1+\frac{k^2\lambda}{16N^2}}}{|\vec{x}|^4} \tag{3.3}$$

to be compared with the result for the fundamental representation (2.5).

4 Exact results for static and radiative fields

The coefficient that appears in the one-point function (3.3) computed in the previous section has appeared before in the literature, at least in two different computations. Let us review them in turns. The first place where this coefficient appears is in the computation of the large distance behavior of the correlation function of a circular Wilson loop in the symmetric representation with the $\Delta = 2$ chiral primary operator by means of a D3-brane [11]. A circular Wilson loop of radius R can be expanded in terms of local operators when probed from distances D much larger than its radius, and the coefficients appearing in this OPE can be read off from the large distance behavior of the two-point function of the Wilson loop and the local operators [44]

$$\frac{\langle W(C)\mathcal{O}_n\rangle}{\langle W(C)\rangle} = c_n\frac{R^{\Delta_n}}{D^{2\Delta_n}} + \dots$$

The authors of [11] computed the coefficients c_n in the case that the local operators are chiral primary operators given by symmetric traceless combinations of scalar fields, and for a Wilson loop in the symmetric or antisymmetric representation. For the symmetric representation and in their normalization[4] for $\mathcal{O}_n$, they obtained (eq. (4.20) in [11])

$$c^{GRT}_{S_k,\Delta} = \frac{2^{\Delta/2+1}}{\sqrt{\Delta}}\sinh(\Delta\sinh^{-1}\kappa)$$

[4] [11, 44] consider CPOs whose two-point function is unit-normalized [3].

In the previous section we computed a one-point function for the Lagrangian density, which belongs to the same short multiplet as the chiral primary operator with $\Delta = 2$, so we are interested in comparing our result with the previous formula for $\Delta = 2$

$$c^{GRT}_{S_k,2} = 4\sqrt{2}\kappa\sqrt{1+\kappa^2} \tag{4.1}$$

which displays the same dependence in λ and N as our result (3.3), except for the overall differing normalization conventions for $\mathcal{O}_2$ (as a check, both results reproduce in the corresponding limit the previously known results for the fundamental representation, $k = 1$).

We are now going to argue that this agreement can be understood as following from generic properties of $\mathcal{N} = 4$ SYM, and it is a reflection of exact relations among expectation values of various operators. These relations are valid for any representation of the gauge group, and beyond the regime of validity of the D-brane probe computation. Our argument proceeds in three steps: first, we notice that the one-point function of an operator in the presence of an external probe is equivalent to the two-point function of the operator with the corresponding Wilson loop, normalized by the vev of the Wilson loop.

As the second step in our argument, we claim that the coefficient that appears in the normalized two-point function of a circular Wilson loop with a CPO is the same that the one that appears in the (trivially, since $\langle W\rangle = 1$ for the straight line) normalized two-point function of a straight line Wilson loop with the same CPO. One argument[5] is that when one performs the conformal transformation from the circle to the straight line, the whole anomalous contribution comes from diagrams where some gluon has both ends of its propagator hitting the origin, the space-time point in the contour being sent to infinity by the conformal transformation (see sections 2.2 and 2.3 of the second reference in [5–7] for a detailed discussion). Diagram by diagram, the contributions from these gluons factorize from the rest of the diagram; furthermore the localization results of the third reference in [5–7] rigurously prove that the resulting matrix model is Gaussian, so gluons in interacting vertices don't contribute to the anomaly. Therefore, the full anomalous contribution is insensitive to the rest of the diagram, which implies that both the two-point function and the vev of the Wilson loop pick up the same anomalous contribution, so it will cancel in the ratio.

Finally, since the Lagrangian density and the stress-energy tensor are descendants of $\mathcal{O}_2$, we expect that their normalized two-point functions with the circular Wilson loop are determined by the same coefficient as the normalized two-point function of $\mathcal{O}_2$.

This line of reasoning explains the agreement found between results (3.3) and (4.1), obtained in the D-brane probe approximation for external sources in the symmetric representation, but goes well beyond this particular case. If we are somehow able to exactly compute the normalized two-point function of $\mathcal{O}_2$ and the 1/2 BPS circular Wilson loop in some representation, we claim that this also gives the exact one-point function of the Lagrangian density or the stress-energy tensor in the presence of a heavy particle in that representation. Luckily, some of the relevant computations have already been performed; for instance, in [10], Okuyama and Semenoff computed the exact two-point function of a

[5]due to N. Drukker (private communication).

circular Wilson loop in the fundamental representation and a chiral primary operator, by means of a normal matrix model. In particular, for the unit-normalized chiral primary operator $\mathcal{O}_2$ they obtain (eq. (4.5) in [10])

$$\langle W(C)_{\Box}\mathcal{O}_2\rangle = \frac{\sqrt{2}\lambda}{4N^3}\left[L^2_{N-1}\Big(-\frac{\lambda}{4N}\Big)+L^2_{N-2}\Big(-\frac{\lambda}{4N}\Big)\right]e^{\frac{\lambda}{8N}}$$

so after normalizing by the vev of the circular Wilson loop computed in [5–7] we obtain

$$c_2 = \frac{\langle W(C)_{\Box}\mathcal{O}_2\rangle}{\langle W(C)_{\Box}\rangle} = \frac{\sqrt{2}\lambda}{4N^2}\frac{L^2_{N-1}(-\frac{\lambda}{4N})+L^2_{N-2}(-\frac{\lambda}{4N})}{L^1_{N-1}(-\frac{\lambda}{4N})}$$

The authors of [11] already showed that this matrix model result reduces in the corresponding limit to the one computed with the D3-brane probe approximation, eq. (4.1). Furthermore, in the planar limit it reproduces the result of [9]. Applying the reasoning presented above, we therefore propose that the exact one-point function of $\mathcal{O}_{F^2}$ in the presence of a heavy probe in the fundamental representation is given by

$$\langle \mathcal{O}_{F^2}(\vec{x})\rangle_{\Box} = \frac{\lambda}{64\pi^2 N}\frac{L^2_{N-1}(-\frac{\lambda}{4N})+L^2_{N-2}(-\frac{\lambda}{4N})}{L^1_{N-1}(-\frac{\lambda}{4N})}\frac{1}{|\vec{x}|^4} \tag{4.2}$$

This is one of our main results. It reduces in the large λ, large N limit to the known result (2.5). More than that, as argued in [15, 41] the range of validity of the probe computation is $N^2/\lambda^2 \gg k \gg N/\lambda^{3/4}$, so a priori it does not include setting $k=1$ (i.e. considering the fundamental representation); if we nevertheless go ahead and set $k=1$, it can be checked that (3.3) correctly captures the large λ, large N limit with κ fixed of (4.2). This type of agreement between a matrix model result and a D3-brane computation, beyond the expected regime of validity of the D-brane probe approximation, has been observed before [41].

So far we have been discussing static sources. Let's now turn to particles undergoing accelerated motion, where we will encounter for the second time that the coefficient of the one-point function (3.3) computed in the previous section had appeared before in the literature. In [15], two of the present authors computed the energy loss by radiation of an infinitely heavy particle transforming in the symmetric or antisymmetric representation, in the large λ, N, fixed κ limit. For the symmetric representation, it was possible to carry out the computation only for the particular case of a particle undergoing motion with constant proper acceleration, and the total radiated power obtained was [15]

$$P_{S_k} = \frac{2N\kappa}{\pi}\sqrt{1+\kappa^2}\frac{1}{R^2} \tag{4.3}$$

where R appears in the hyperbolic trajectory $-(x^0)^2+(x^1)^2=R^2$. Note that again, apart from kinematic factors, the coefficient that appears in the total radiated power, eq. (4.3), is the same one as in the one-point function computed in the previous section, eq. (3.3) and in the two-point function of the circular Wilson loop with $\mathcal{O}_2$, eq. (4.1). The agreement between coefficients in eq. (4.3) and in eq. (4.1) is perhaps not too surprising from the field theory point of view, since the radiated power can be read off from the stress-energy tensor, which as already mentioned is in the same short multiplet as $\mathcal{O}_2$, and the D3-brane corresponding to hyperbolic motion [15] is just the continuation to Lorentzian signature of the Euclidean D3-brane used to compute the vev of the circular Wilson loop [41]. Let us nevertheless note that from the point of view of the D3-brane computations, this agreement was not a foregone conclusion, even if the D3-branes used in these three computations are related by a conformal transformation and/or continuation to Lorentzian signature. For instance, the energy loss computation in [15] captured physics of radiative fields, encoded in the bulk by the presence of a horizon in the world-volume of the D3-brane, while in the computation of the previous section, the physics of static fields is captured by the behavior near the AdS boundary, and the D3-brane world-volume has now no horizon.

Furthermore, this agreement between the coefficient of radiated power and the two point function of the circular Wilson loop with $\mathcal{O}_2$ also takes place for the antisymmetric representation, at least at the D-brane probe computation. This can easily be checked by comparing the relevant results obtained respectively in [15] and [11], by means of D5-brane probes.

This second agreement in the coefficients of eq. (3.3) and eq. (4.3), or perhaps more directly between eq. (4.3) and eq. (4.1), leads us again to propose that given a particular representation of the gauge group, the full coefficients are exactly the same. In particular this proposal implies that the exact formula for radiated power of an infinitely heavy particle transforming in the fundamental representation and undergoing hyperbolic motion, valid for arbitrary values of N and λ is

$$P = \frac{\lambda}{8\pi N} \frac{L^2_{N-1}(-\frac{\lambda}{4N}) + L^2_{N-2}(-\frac{\lambda}{4N})}{L^1_{N-1}(-\frac{\lambda}{4N})} \frac{1}{R^2} \tag{4.4}$$

This is our second main result in this section. It reduces in the large λ, large N limit to the one obtained by Mikhailov [25]. Furthermore, if we again set $k = 1$ in the result of the D3-brane probe computation for the k-symmetric representation, eq. (4.3), we obtain an agreement in the large λ, large N, fixed κ limit, even though $k = 1$ is beyond the regime of validity of the computation [15] yielding (4.3).

Let us conclude this paper by commenting on the possibility that the previous formula (4.4) for total radiated power might be valid not only for hyperbolic motion, but for an arbitrary timelike trajectory, with the obvious substitution $1/R^2 \to a^\mu a_\mu$. A first piece of evidence is that Mikhailov's computation [25] of the total radiated power for arbitrary timelike trajectories does indeed give a common $\sqrt{\lambda}$ coefficient, independent of the trajectory. Some further evidence beyond the leading large N, large $\sqrt{\lambda}$ result is given by the fact that for particles in the antisymmetric representation, the D5-brane computation in [15] again gives a Liénard-type formula for the total radiated power. This issue deserves further attention.

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References

[1] B. Fiol, B. Garolera and A. Lewkowycz, *Gluonic fields of a static particle to all orders in* $1/N$, arXiv:1112.2345 [INSPIRE].

[2] F. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, *Annals Phys.* **307** (2003) 41 [hep-th/0209056] [INSPIRE].

[3] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three point functions of chiral operators in* $D = 4$, $N = 4$ *SYM at large-*N, *Adv. Theor. Math. Phys.* **2** (1998) 697 [hep-th/9806074] [INSPIRE].

[4] M. Baggio, J. de Boer and K. Papadodimas, *A non-renormalization theorem for chiral primary* 3*-point functions*, arXiv:1203.1036 [INSPIRE].

[5] J. Erickson, G. Semenoff and K. Zarembo, *Wilson loops in* $N = 4$ *supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 582** (2000) 155 [hep-th/0003055] [INSPIRE].

[6] N. Drukker and D.J. Gross, *An exact prediction of* $N = 4$ *SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[7] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, arXiv:0712.2824 [INSPIRE].

[8] N. Drukker, 1/4 *BPS circular loops, unstable world-sheet instantons and the matrix model*, *JHEP* **09** (2006) 004 [hep-th/0605151] [INSPIRE].

[9] G.W. Semenoff and K. Zarembo, *More exact predictions of SUSYM for string theory*, *Nucl. Phys.* **B 616** (2001) 34 [hep-th/0106015] [INSPIRE].

[10] K. Okuyama and G.W. Semenoff, *Wilson loops in* $N = 4$ *SYM and fermion droplets*, *JHEP* **06** (2006) 057 [hep-th/0604209] [INSPIRE].

[11] S. Giombi, R. Ricci and D. Trancanelli, *Operator product expansion of higher rank Wilson loops from D-branes and matrix models*, *JHEP* **10** (2006) 045 [hep-th/0608077] [INSPIRE].

[12] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-*N *field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [INSPIRE].

[13] E. D'Hoker and D.Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, hep-th/0201253 [INSPIRE].

[14] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in* $N = 4$ *super Yang-Mills*, arXiv:1202.4455 [INSPIRE].

[15] B. Fiol and B. Garolera, *Energy loss of an infinitely massive half-Bogomol'nyi-Prasad-Sommerfeld particle by radiation to all orders in* $1/N$, *Phys. Rev. Lett.* **107** (2011) 151601 [arXiv:1106.5418] [INSPIRE].

[16] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [hep-th/9803001] [INSPIRE].

[17] J.M. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002] [INSPIRE].

[18] U.H. Danielsson, E. Keski-Vakkuri and M. Kruczenski, *Vacua, propagators and holographic probes in AdS/CFT*, *JHEP* **01** (1999) 002 [hep-th/9812007] [INSPIRE].

[19] C.G. Callan Jr. and A. Guijosa, *Undulating strings and gauge theory waves*, *Nucl. Phys.* **B 565** (2000) 157 [hep-th/9906153] [INSPIRE].

[20] C. Athanasiou, P.M. Chesler, H. Liu, D. Nickel and K. Rajagopal, *Synchrotron radiation in strongly coupled conformal field theories*, *Phys. Rev.* **D 81** (2010) 126001 [*Erratum ibid.* **D 84** (2011) 069901] [arXiv:1001.3880] [INSPIRE].

[21] Y. Hatta, E. Iancu, A. Mueller and D. Triantafyllopoulos, *Aspects of the UV/IR correspondence: energy broadening and string fluctuations*, *JHEP* **02** (2011) 065 [arXiv:1011.3763] [INSPIRE].

[22] Y. Hatta, E. Iancu, A. Mueller and D. Triantafyllopoulos, *Radiation by a heavy quark in $N = 4$ SYM at strong coupling*, *Nucl. Phys.* **B 850** (2011) 31 [arXiv:1102.0232] [INSPIRE].

[23] R. Baier, *On radiation by a heavy quark in $N = 4$ SYM*, arXiv:1107.4250 [INSPIRE].

[24] M. Chernicoff, A. Guijosa and J.F. Pedraza, *The gluonic field of a heavy quark in conformal field theories at strong coupling*, *JHEP* **10** (2011) 041 [arXiv:1106.4059] [INSPIRE].

[25] A. Mikhailov, *Nonlinear waves in AdS/CFT correspondence*, hep-th/0305196 [INSPIRE].

[26] S. Förste, D. Ghoshal and S. Theisen, *Stringy corrections to the Wilson loop in $N = 4$ super Yang-Mills theory*, *JHEP* **08** (1999) 013 [hep-th/9903042] [INSPIRE].

[27] N. Drukker, D.J. Gross and A.A. Tseytlin, *Green-Schwarz string in $AdS_5 \times S^5$: semiclassical partition function*, *JHEP* **04** (2000) 021 [hep-th/0001204] [INSPIRE].

[28] V. Forini, *Quark-antiquark potential in AdS at one loop*, *JHEP* **11** (2010) 079 [arXiv:1009.3939] [INSPIRE].

[29] N. Drukker and V. Forini, *Generalized quark-antiquark potential at weak and strong coupling*, *JHEP* **06** (2011) 131 [arXiv:1105.5144] [INSPIRE].

[30] J. Gomis and F. Passerini, *Holographic Wilson loops*, *JHEP* **08** (2006) 074 [hep-th/0604007] [INSPIRE].

[31] J. Gomis and F. Passerini, *Wilson Loops as D3-branes*, *JHEP* **01** (2007) 097 [hep-th/0612022] [INSPIRE].

[32] S. Yamaguchi, *Wilson loops of anti-symmetric representation and D5-branes*, *JHEP* **05** (2006) 037 [hep-th/0603208] [INSPIRE].

[33] S.A. Hartnoll and S.P. Kumar, *Multiply wound Polyakov loops at strong coupling*, *Phys. Rev.* **D 74** (2006) 026001 [hep-th/0603190] [INSPIRE].

[34] S.A. Hartnoll, *Two universal results for Wilson loops at strong coupling*, *Phys. Rev.* **D 74** (2006) 066006 [hep-th/0606178] [INSPIRE].

[35] C.G. Callan Jr., A. Guijosa and K.G. Savvidy, *Baryons and string creation from the five-brane world volume action*, *Nucl. Phys.* **B 547** (1999) 127 [hep-th/9810092] [INSPIRE].

JHEP05(2012)093

[36] J. Camino, A. Paredes and A. Ramallo, *Stable wrapped branes*, *JHEP* **05** (2001) 011 [hep-th/0104082] [INSPIRE].

[37] S.A. Hartnoll and S.P. Kumar, *Higher rank Wilson loops from a matrix model*, *JHEP* **08** (2006) 026 [hep-th/0605027] [INSPIRE].

[38] M. Chernicoff and A. Guijosa, *Energy loss of gluons, baryons and k-quarks in an $N = 4$ SYM plasma*, *JHEP* **02** (2007) 084 [hep-th/0611155] [INSPIRE].

[39] W. Mueck, *The Polyakov loop of anti-symmetric representations as a quantum impurity model*, *Phys. Rev.* **D 83** (2011) 066006 [*Erratum ibid.* **D 84** (2011) 129903] [arXiv:1012.1973] [INSPIRE].

[40] S. Harrison, S. Kachru and G. Torroba, *A maximally supersymmetric Kondo model*, arXiv:1110.5325 [INSPIRE].

[41] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [hep-th/0501109] [INSPIRE].

[42] I.R. Klebanov, *World volume approach to absorption by nondilatonic branes*, *Nucl. Phys.* **B 496** (1997) 231 [hep-th/9702076] [INSPIRE].

[43] I.R. Klebanov, W. Taylor and M. Van Raamsdonk, *Absorption of dilaton partial waves by D3-branes*, *Nucl. Phys.* **B 560** (1999) 207 [hep-th/9905174] [INSPIRE].

[44] D.E. Berenstein, R. Corrado, W. Fischler and J.M. Maldacena, *The operator product expansion for Wilson loops and surfaces in the large-N limit*, *Phys. Rev.* **D 59** (1999) 105023 [hep-th/9809188] [INSPIRE].

3.2 Momentum diffusion coefficient

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Exact momentum fluctuations of an accelerated quark in $\mathcal{N} = 4$ super Yang-Mills

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Abstract: In this work we consider a heavy quark moving with constant proper acceleration in the vacuum of any four dimensional conformal field theory. We argue that the two-point function of its momentum fluctuations is exactly captured by the Bremsstrahlung function that gives the total radiated power. For the particular case of $\mathcal{N} = 4$ SU(N) SYM this function is exactly known, so in this case we obtain an explicit expression for the momentum diffusion coefficient, and check that various limits of this result are reproduced by probe computations in AdS_5. Finally, we evaluate this transport coefficient for a heavy quark accelerated in the vacuum of $\mathcal{N} = 4$ SU(3) SYM, and comment on possible lessons of our results for the study of heavy quarks traversing the super Yang-Mills plasma.

Keywords: Wilson, 't Hooft and Polyakov loops, AdS-CFT Correspondence

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1 Introduction

One of the possible ways to study gauge theories is to introduce heavy external probes, following prescribed trajectories. These external probes can transform under various representations of the gauge group, and can also be coupled to additional fields, besides the gauge potential. A common tool to implement this idea is the use of Wilson loops, where the contour of the loop is given by the world-line of the probe. Wilson loops are among the most interesting operators in a gauge theory, but in general computing their expectation value or their correlation functions with other operators is prohibitively difficult. On the other hand, for gauge theories with additional symmetries (e.g. conformal symmetry and/or supersymmetry) and for particular contours, a variety of techniques allows to prove exact relations among various correlators involving line operators, and sometimes also evaluate exactly these quantities [1–7].

In this work we will be concerned with external probes coupled to a four dimensional conformal field theory (CFT), following either a static or a hyperbolic trajectory in vacuum. The probes can transform in an arbitrary representation of the gauge group, and when we consider probes transforming in the fundamental representation, we will often refer to them as quarks. We will extend recent work [8, 9] that provides exact relations among various observables related to these probes. In the particular case of $\mathcal{N} = 4$ U(N) or SU(N) SYM, these exact relations will allow us to compute explicitly the momentum diffusion coefficient of an accelerated quark in vacuum, a transport coefficient than until now was only known in the limit of large N and large 't Hooft coupling $\lambda = g^2_{\text{YM}} N$.

The first observable that appears in our discussion is the energy emitted by a moving quark in accelerated motion, which for small velocities can be written as a Larmor-type formula

$$\Delta E = 2\pi B(\lambda, N) \int dt\, (\dot{v})^2 \ , \tag{1.1}$$

where $B(\lambda, N)$ is a dimensionless function independent of the kinematics that was dubbed the Bremsstrahlung function in [8].

One can also consider inserting operators on the world-line of the probe [10–13]. These operators localized on the world-line are not gauge singlets; nevertheless, their correlation functions evaluated on the world-line are gauge invariant. If the world-line is a straight line, it preserves a $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(3)$ subgroup of the original group [14, 15],[1] and world-line operators can be classified according to representations of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(3)$. Among them, the so called displacement operators $\mathbb{D}_i(t)$ $i = 1, 2, 3$ [8] will play a prominent role in this work. These are operators defined for any line defect in any conformal field theory, that couple to small deviations of the world line, orthogonal to it. They form a $\mathrm{SO}(3)$ triplet and their scaling dimension $\Delta = 2$ is protected for all values of the coupling, so their two-point function evaluated on a static world-line has the form

$$\langle\langle \mathbb{D}_i(t) \mathbb{D}_j(0) \rangle\rangle = \tilde{\gamma}(\lambda, N) \frac{\delta_{ij}}{t^4} \ , \tag{1.2}$$

where again $\tilde{\gamma}(\lambda, N)$ is a dimensionless coefficient and the double ket denotes evaluation on the world-line (see below for a precise definition). Physically, we will interpret correlators of these displacement operators as giving momentum fluctuations of the probe, an interpretation that has appeared before in the literature [16, 17] although not in this language.

A crucial point for what follows is that the two coefficients in (1.1) and (1.2) are exactly related by [8]

$$\tilde{\gamma} = 12B \tag{1.3}$$

for any CFT and any straight line defect operator (Wilson loop, 't Hooft loop,...). This relation is claimed to be exact, valid for any value of the coupling constant, any gauge group and any representation of the gauge group. While it is important to appreciate that this Bremsstrahlung function appears in various observables related to probes coupled to CFTs, it is also important to actually compute it for different line operators in different interacting CFTs. Currently, this has only been done for a probe in the fundamental representation of $\mathcal{N} = 4$ U(N) or SU(N) SYM, for which the Bremsstrahlung function $B(\lambda, N)$ was recently computed in [8, 9] and is given by

$$B_{\mathrm{U}(N)}(\lambda, N) = \frac{\lambda}{16\pi^2 N} \frac{L^2_{N-1}\left(-\frac{\lambda}{4N}\right) + L^2_{N-2}\left(-\frac{\lambda}{4N}\right)}{L^1_{N-1}\left(-\frac{\lambda}{4N}\right)} \ , \tag{1.4}$$

[1] This is the common group preserved by any line defect in any CFT. For particular CFTs with bigger symmetry groups, the preserved group might be much larger.

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where the L_n^α are generalized Laguerre polynomials. It is worth emphasizing that this formula is valid for any value of λ and N, and its derivation ultimately relies on localization techniques. In various limits, it can be checked using the AdS/CFT correspondence [18, 19] or Bethe ansatz techniques [20–23]. To obtain the result for the SU(N) theory, we have to subtract the U(1) contribution [8]

$$B_{\mathrm{SU}(N)} = B_{\mathrm{U}(N)} - \frac{\lambda}{16\pi^2 N^2}\,.$$

The main observation of this paper is that the coefficient $\tilde{\gamma}$ in (1.2) also controls the two-point function of displacement operators when the probe is undergoing motion with constant proper acceleration $a = 1/R$, since this hyperbolic world-line is related to the static one by a special conformal transformation. As it is well-known, a particle moving with constant proper acceleration in vacuum will feel an Unruh temperature $T = a/2\pi$ [24]. The thermal bath felt by the accelerated particle will cause momentum fluctuations, and these can be encoded in a particular transport coefficient, the momentum diffusion coefficient, defined as the zero frequency limit of the two-point function of displacement operators in momentum space,

$$\kappa_{ij} \equiv \lim_{w\to 0}\int_{-\infty}^{\infty} d\tau e^{-iw\tau}\langle\langle \mathbb{D}_i(\tau)\mathbb{D}_j(0)\rangle\rangle\,,$$

where τ is the proper time of the accelerated probe. Since the hyperbolic trajectory still preserves a $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(3)$ subgroup of the original group [14, 15], by isotropy there is only a single transport coefficient as seen by the accelerated observer, $\kappa_{ij} = \kappa\delta_{ij}$, and a straightforward computation yields

$$\kappa = 16\pi^3 B(\lambda, N)T^3\,. \tag{1.5}$$

This is one of the main results of this paper; it relates the momentum diffusion coefficient of an accelerated heavy probe in the vacuum of a 4d CFT with the corresponding Bremsstrahlung function, eq. (1.1), and Unruh temperature. We claim that this result is exact for any 4d CFT, for any value of λ and N and for any representation of the gauge group. In the particular case of a heavy probe in the fundamental representation of $\mathcal{N} = 4$ U(N) or SU(N) SYM, since $B(\lambda, N)$ is given exactly by (1.4), the relation (1.5) provides an explicit expression for the momentum diffusion coefficient. Furthermore, the result thus obtained can be subjected to a non-trivial check, since for $\mathcal{N} = 4$ SYM , κ has been computed in the large λ, planar limit by means of the AdS/CFT correspondence [25, 26]. Reassuringly, in the corresponding limit our result reduces to the previoulsy known one.

Having obtained the exact two-point function of momentum fluctuations of an accelerated heavy quark in the vacuum of $\mathcal{N} = 4$ SU(N) SYM, it's tempting to ask whether we can use it to learn something about momentum fluctuations of a heavy quark in the midst of a finite temperature $\mathcal{N} = 4$ SU(N) SYM plasma. This is a problem that has been extensively scrutinized in the context of the AdS/CFT correspondence [16, 27–29] (see [30–32] for reviews), as a possible model for the momentum fluctuations of a heavy quark traversing the quark-gluon plasma. In general, while an accelerated probe in vacuum registers a non-zero temperature, its detailed response differs from that of a probe in a thermal bath; in particular their respective retarded Green functions and momentum diffusion coefficients are different. Nevertheless, if we take the guess

$$\frac{\kappa^{\text{exact}}_{\text{Unruh}}}{\kappa^{\text{SUGRA}}_{\text{Unruh}}} \approx \frac{\kappa^{\text{exact}}_{\text{thermal}}}{\kappa^{\text{SUGRA}}_{\text{thermal}}}$$

for some range of values of λ as a working hypothesis, we can estimate $\kappa^{\text{exact}}_{\text{thermal}}$ in that range of values of λ, since now the other three quantities in the relation above are known.

The plan of the paper is as follows. In section 2 we recall the definition of displacement operators, and we interpret their correlation functions as characterizing momentum fluctuations of the probe. We then go on to compute their exact two-point function for an accelerated infinitely heavy probe coupled to a CFT, and extract from it the momentum diffusion coefficient. In section 3, we use the AdS/CFT correspondence to verify the relation (1.3) in the particular case of $\mathcal{N}=4$ SU(N) SYM for static heavy probes in the fundamental and the symmetric representation. In section 4 we again use the AdS/CFT correspondence, now to compute the momentum diffusion coefficient of accelerated probes, in the fundamental and in the symmetric representations, and check that the results obtained are compatible with the exact result. Finally, in section 5 we explicitly evaluate the momentum diffusion coefficient for an accelerated quark coupled to $\mathcal{N}=4$ SU(3) SYM. We then evaluate the error introduced when one uses the supergravity expression instead of the exact one, and end by commenting on possible implications for the study of heavy quarks in a thermal bath.

2 Momentum fluctuations and displacement operators

Consider a heavy probe coupled to a four dimensional conformal field theory. This probe transforms in some representation of the gauge group, and perhaps it is also coupled to additional fields, as it is the case for 1/2 BPS probes of $\mathcal{N}=4$ SYM [33, 34]. Since we are considering a heavy probe, we will represent it by the corresponding Wilson line. In this section we will first recall the definition of certain operators inserted along the world-line, the displacement operators, and argue that their physical interpretation is that of momentum fluctuations due to the coupling of the probe to the quantum fields. We will then focus on the two-point function of such displacement operators for static and accelerated world-lines.

To define the displacement operators [8], consider a given Wilson loop, parameterized by t and perform an infinitesimal deformation of the contour $\delta x^\mu(t)$, orthogonal to the contour $\delta\vec{x}(t)\cdot\dot{\vec{x}}(t)=0$. This deformation defines a new contour, and the displacement operator $\mathbb{D}_i(t)$ is defined as the functional derivative of the Wilson loop with respect to this displacement [35]. In particular, the infinitesimal change can be written as

$$\delta W = P\int dt\delta x^j(t)\mathbb{D}_j(t)W\,. \tag{2.1}$$

These operators, and in general other local operators inserted on the world-line, are not gauge invariant. Nevertheless, their n-point functions, evaluated over the world-line are gauge invariant, e.g.

$$\langle\langle \mathbb{D}_i(t_1)\mathbb{D}_j(t_2)\rangle\rangle = \frac{< Tr[P\mathbb{D}_i(t_1)e^{i\int_{t_2}^{t_1} A\cdot dx}\,\mathbb{D}_j(t_2)e^{i\int_{t_1}^{t_2} A\cdot dx}] >}{< Tr[Pe^{i\oint A\cdot dx}] >}\,.$$

What is the physical interpretation of these operators? In general, when we introduce an external heavy probe, its classical trajectory is fixed, giving the contour of the corresponding Wilson line. At the quantum level, this trajectory will suffer fluctuations due to its coupling to quantum fields. By definition, these small deformations in the contour $\delta x^i(t)$ are coupled to the displacement operators, so we identify these operators as forces causing momentum fluctuations. This identification is valid for general quantum field theories (not just conformal ones), and it has appeared before in the literature [16, 17]. For instance, if we consider a charged particle coupled to a U(1) Maxwell field and moving with 4-velocity U^μ, the Lorentz force is $qF_{\mu\nu}U^\nu$ and the displacement operators are

$$\mathbb{D}_\mu = qF_{\mu\nu}U^\nu\,.$$

Since $U^\mu\mathbb{D}_\mu = 0$, we see explicitly that displacement operators are transverse to the world-line. This is easily generalized to additional couplings to scalar fields. For instance, a particularly relevant example for what follows is the 1/2 BPS Wilson loop of $\mathcal{N} = 4$ SYM, $W = \frac{1}{N}\text{tr}\ \mathcal{P}e^{i\int A + \int \vec{n}\cdot\vec{\phi}}$, for which the displacement operator is

$$\mathbb{D}_j = iF_{tj} + \vec{n}\cdot\partial_j\vec{\phi}\,.$$

When the gauge theory under consideration is conformal, there is more we can say about displacement operators. Let's start by considering the world-line corresponding to a static probe, a straight line parameterized by t. In any conformal field theory, any straight line defect (or for that matter, any circular defect in Euclidean signature) preserves a $\text{SL}(2,\mathbb{R})\times\text{SO}(3)$ symmetry group of the original conformal group [14, 15], so operators inserted on the world-line are classified by their $\text{SL}(2,\mathbb{R})\times\text{SO}(3)$ quantum numbers. In particular, displacement operators $\mathbb{D}_i(t)$ form a SO(3) triplet, and since δx^i and t have canonical dimension, we learn from eq. (2.1) that displacement operators have scaling dimension $\Delta = 2$, and this dimension is protected against corrections. This fixes their two-point function evaluated on a straight line to be of the form

$$\langle\langle \mathbb{D}_i(t)\mathbb{D}_j(0)\rangle\rangle = \tilde{\gamma}\frac{\delta_{ij}}{t^4}\,. \tag{2.2}$$

Let's now consider a heavy probe moving with constant proper acceleration $a = 1/R$ in one dimension. It is a textbook result that the resulting trajectory is the branch of a hyperbola in spacetime, which can be written as

$$\tilde{x}^0(\tau) = R\sinh\frac{\tau}{R} \qquad \tilde{x}^1(\tau) = R\left(\cosh\frac{\tau}{R} - 1\right)\,. \tag{2.3}$$

Furthermore, for a conformal field theory, this hyperbolic world-line can be obtained by applying the following special conformal transformation to a static world-line

$$\tilde{x}^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2x\cdot b + b^2 x^2} \qquad \text{with} \qquad b^\mu = (0, \frac{1}{2R}, 0, 0)\,. \tag{2.4}$$

The Euclidean counterpart of this statement is that a special conformal transformation can bring a straight line into a circle. This hyperbolic world-line also preserves a $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(3)$ symmetry group of the original conformal group [14, 15]. The two-point function of displacement operators in terms of these coordinates is

$$\langle\langle \mathbb{D}_i(\tilde{x})\mathbb{D}_j(0)\rangle\rangle = \tilde{\gamma}\frac{\delta_{ij}}{\tilde{x}^{2\Delta}}\,. \tag{2.5}$$

Recalling that $\Delta = 2$ for displacement operators and using (2.3), this two-point function can be immediately written in terms of the proper time of the heavy probe as

$$\langle\langle \mathbb{D}_i(\tau)\mathbb{D}_j(0)\rangle\rangle = \tilde{\gamma}\frac{\delta_{ij}}{16R^4\sinh^4\left(\frac{\tau}{2R}\right)}\,, \tag{2.6}$$

where the coefficient $\tilde{\gamma}$ is the same as for the two-point function evaluated on a straight line, eq. (2.2). This is required so at very short times, when $\tau/2R \ll 1$, we recover the result for the straight line, eq. (2.2).

On the other hand, the two-point functions (2.2) and (2.6) ought to reflect the very different physics felt by a static and an accelerated probe. In particular (2.6) captures the response of the accelerated probe to the non-zero Unruh temperature. To display this, we will now compute the Fourier transform of (2.6) and extract the corresponding transport coefficient. To compute the Fourier transform of (2.6), we notice that it presents poles in the τ complex plane whenever $\tau = 2\pi i n R$, $n \in \mathbb{Z}$. Using the same pole prescription as in [8], we follow [36] and choose the integration contour displayed in figure (1). A straightforward computation then yields

$$G(w)_{ij} = \int_{-\infty}^{\infty} d\tau e^{-iw\tau}\langle\langle \mathbb{D}_i(\tau)\mathbb{D}_j(0)\rangle\rangle = \tilde{\gamma}\delta_{ij}\int_{-\infty}^{\infty} d\tau\frac{e^{-iw\tau}}{16R^4\sinh^4\left(\frac{\tau}{2R}\right)} = \tilde{\gamma}\delta_{ij}\frac{2\pi}{3!}\frac{\frac{w}{R^2}+w^3}{e^{2\pi wR}-1}\,.$$

This Green function displays a thermal behavior with temperature $T^{-1} = 2\pi R$, i.e. the usual Unruh temperature. Note that this temperature depends only on the kinematics, not on dynamical aspects of the theory (e.g. it is coupling independent) [37].

We can take the zero frequency limit of this two-point function to obtain the momentum diffusion coefficients κ_{ij}. In fact, since the hyperbolic trajectory preserves a SO(3) symmetry, there is a single transport coefficient $\kappa_{ij} = \kappa\delta_{ij}$ given by[2]

$$\kappa = \lim_{w\to 0} G(w) = 16\pi^3 B(\lambda, N)T^3\,. \tag{2.7}$$

[2]This transport coefficient is usually obtained from the retarded Green function,

$$\kappa = \lim_{w\to 0} -\frac{2T}{w}\,\mathrm{Im}\,G_R(w)\,.$$

Since for a static particle in a thermal bath, $G(w) = -\coth\frac{w}{2T}\mathrm{Im}\,G_R(w)$, in that case the two expressions are equivalent.

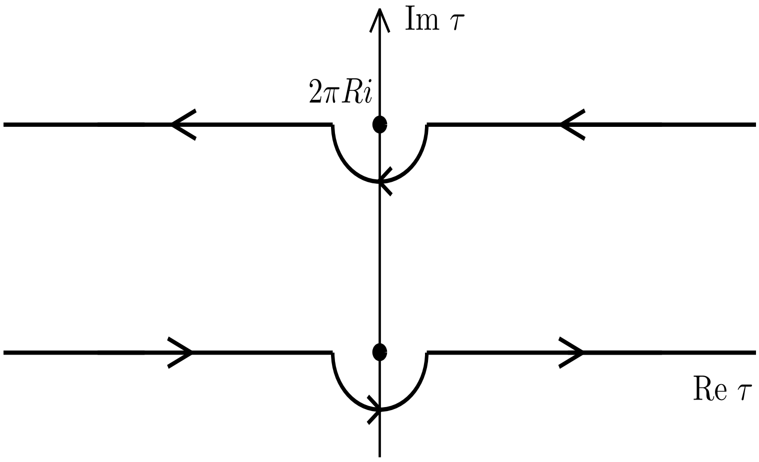

Figure 1. Integration contour for the Fourier transform of the two-point function of displacement operators.

This expression gives the momentum diffusion coefficient of an accelerated heavy probe in terms of the corresponding Bremsstrahlung function, eq. (1.1) and the Unruh temperature T. It is valid for any four-dimensional CFT, and for any heavy probe. Notice that if we Fourier transform the two-point function evaluated on the straight line, eq. (2.2), we obtain that the Green function is proportional to $|w|^3$ [8, 17] so, as expected, the momentum diffusion coefficient defined as in (2.7) vanishes for a particle moving in vacuum with constant speed.

In the particular case of $\mathcal{N} = 4$ SYM, the AdS/CFT correspondence provides the possibility of carrying out a non-trivial check of eq. (2.7). On the one hand, in the planar limit and for large λ, using the asymptotic value of $B \to \frac{\sqrt{\lambda}}{4\pi^2}$, eq. (2.7) reduces to

$$\kappa \to 4\pi\sqrt{\lambda}T^3 \,. \tag{2.8}$$

On the other hand, in this regime, one can use the AdS/CFT correspondence to compute this transport coefficient in an alternative fashion. The heavy probe is dual to a string reaching the boundary of AdS_5 at the hyperbolic world-line. Analysis of the fluctuations of this classical string solution allows to compute the relevant two-point function [25, 26] and from it extract the momentum diffusion coefficient [25, 26] (see also section 4), which precisely reproduces the result above, eq. (2.8). The dependence on T was bound to agree, since is dictated by dimensional analysis, and the $\sqrt{\lambda}$ dependence is ubiquitous in AdS/CFT probe computations using fundamental strings (see e.g. [9] for a discussion of this point), but the agreement on the numerical coefficient 4π in (2.8) is a non-trivial check.

3 Static probes in AdS/CFT

In this section we intend to verify the relation (1.3) for the particular case of $\mathcal{N} = 4$ SYM by means of the AdS/CFT correspondence. To do so, we will compute separately $\tilde{\gamma}$ and B, and check that they are indeed related by $\tilde{\gamma} = 12B$. This relation ought to hold

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for a probe transforming in any representation of the gauge group, and we will perform this check for a heavy probe in the fundamental and in the k-symmetric representations. On the gravity side this corresponds to considering respectively a string and a D3-brane embedded in AdS_5 and reaching the boundary at a straight line. Performing the check for the k-symmetric representation has an interesting plus: while in principle the computation with the D3-brane can't be trusted in the limit when we set $k = 1$ (i.e. we go back to the fundamental representation), by now there are a number of examples [9, 19, 38, 39] where it is known that this procedure nevertheless correctly captures corrections in the large λ, large N limit with fixed $\sqrt{\lambda}/N$. Given that we already know the exact expression of $B(\lambda, N)$ for this probe, eq. (1.4), we will be able to verify that this offers yet another example where a D3-brane probe computation correctly captures all order corrections to the leading large λ large N result.

3.1 Fluctuations of a static string in AdS_5

The fluctuations of a static string in AdS_5 have been computed in many previous works [40–42], so we will be brief. We will work with the Nambu-Goto action in the static gauge, and will be concerned only with the bosonic fluctuations of the transverse coordinates in AdS_5, which we identify as dual to the world-line displacement operators.

Start by writing AdS_5 in Poincaré coordinates

$$ds^2_{\mathrm{AdS}_5} = \frac{L^2}{y^2}\left(dy^2 - dt^2 + d\vec{x}^2\right) . \tag{3.1}$$

The relevant classical solution to the NG action is given by identifying the world-sheet coordinates with (t, y). The induced world-sheet metric is AdS_2 with radius L. We now turn to the quadratic fluctuations around this solution, and focus on the fluctuations of the transverse coordinates in AdS_5, x^i, $i = 1, 2, 3$. To make manifest the geometric content of these fluctuations, it is better to switch to $\phi^i = \frac{L}{y}x^i$. The Lagrangian density for quadratic fluctuations is then

$$\mathcal{L} = \frac{-1}{2\pi\alpha'}\left(-\frac{1}{2}\partial_t\vec{\phi}\,\partial_t\vec{\phi} + \frac{1}{2}\partial_y\vec{\phi}\,\partial_y\vec{\phi} + \frac{1}{y^2}(\vec{\phi})^2\right) , \tag{3.2}$$

so the equation of motion for the fluctuations is

$$-\partial_t^2\phi^i + \partial_y^2\phi^i - \frac{2}{y^2}\phi^i = 0$$

from where we learn that the three transverse fluctuations in AdS_5 are massive $m^2 = 2/L^2$ scalars in the AdS_2 world-sheet [41, 42]. Furthermore, it can also be seen that the five fluctuations of S^5 coordinates are massless [41, 42]. The bosonic symmetries preserved by the classical string solution are $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(3) \times \mathrm{SO}(5)$, which is the bosonic part of the supergroup $\mathrm{OSp}(4^*|4)$ [43, 44]. Therefore, fluctuations should fall into representations of this supergroup, and indeed it is shown in [45] that together with the fermionic excitations they form an ultra-short multiplet.

These bosonic fluctuations are massive and massless scalars in the AdS_2 world-sheet, and according to the AdS/dCFT correspondence, "holography acts twice" [46] and they source dual operators in the boundary of AdS_2, which is just the heavy quark world-line. The conformal dimensions Δ of these operators are determined by the usual relation

$$2\Delta = d + \sqrt{d^2 + 4(mL)^2}\,.$$

In our case $d = 1$, so the three fluctuations ϕ^i in AdS_2 with $m^2 = 2/L^2$ are dual to a SO(3) triplet of operators with $\Delta = 2$: these are the displacement operators $\mathbb{D}_i(t)$. Furthermore, the operators dual to the five massless S^5 fluctuations have $\Delta = 1$ and are in the same supermultiplet as the displacement operators [8]. We will not consider this second set of operators in the rest of the paper.

Our next objective is to compute the two-point function of displacement operators (1.2) in the regime of validity of SUGRA, i.e. the leading large $\sqrt{\lambda}$ large N behavior of $\tilde{\gamma}(\lambda, N)$. This was essentially done in [17], with the minor difference that there the fluctuating fields were x^i rather than ϕ^i. After introducing the Fourier transform $x^i_F(w, y)$, the author of [17] solved the corresponding equation and imposing purely outgoing boundary conditions, obtained the following Green function [17]

$$G(w) = \frac{L^2}{2\pi\alpha'}|w|^3 \Rightarrow G(t) = \frac{3\sqrt{\lambda}}{\pi^2}\frac{1}{t^4}$$

from where we finally deduce

$$\tilde{\gamma} = \frac{3\sqrt{\lambda}}{\pi^2}\,. \tag{3.3}$$

To complete the check, we need the coefficient of energy loss by radiation, defined in eq. (1.1). The computation of B for a heavy probe in this regime was first carried out by Mikhailov in a beautiful paper [18], obtaining $B = \frac{\sqrt{\lambda}}{4\pi^2}$. Putting together these two results, we have verified $\tilde{\gamma} = 12B$ to this order.

3.2 Fluctuations of a static D3-brane in AdS_5

We will now check relation (1.3) for a heavy probe in the k-symmetric representation. To do so, we will consider a D3-brane dual to a static probe in AdS_5, with k units of electric flux that encode the representation of the heavy probe. The relevant static D3-brane solution was found in [33, 38], but for our purposes it will be convenient to present it in the coordinates introduced in [45, 47]. First, write AdS_5 in the following coordinates

$$ds^2_{\mathrm{AdS}_5} = L^2\left(du^2 + \cosh^2 u\frac{1}{r^2}\left(-dt^2 + dr^2\right) + \sinh^2 u\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right)\,.$$

The D3-brane world-volume coordinates are (t, r, θ, ϕ). The classical solution includes some non-trivial world-volume electric field

$$\sinh u = \nu \qquad F_{tr} = \frac{\sqrt{\lambda}}{2\pi}\frac{\sqrt{1+\nu^2}}{r^2}$$

with[3]

$$\nu = \frac{k\sqrt{\lambda}}{4N}\,. \tag{3.4}$$

[3] This combination was originally dubbed κ in [38] and subsequent works. To avoid any possible confusion with the momentum diffusion coefficient, in this work we change its name to ν.

The induced metric is of the form $\mathrm{AdS}_2 \times S^2$, with radii $L\sqrt{1+\nu^2}$ and $L\nu$ respectively. Consider now fluctuations of u, the S^5 coordinates and the Born-Infeld abelian gauge field A_μ. The main advantage of the coordinates used here is that as shown in detail in [45] these sets of fluctuations decouple in these coordinates, so we can focus exclusively on fluctuations of u. Due to the presence of world-volume fluxes, the Lagrangian density for the fluctuating fields is not controlled by the induced world-volume metric, but by the $\mathrm{AdS}_2 \times S^2$ metric with both radii $L\nu$

$$ds^2_{\mathrm{AdS}_2\times S^2} = \mathcal{G}_{ab} d\xi^a d\xi^b = \frac{L^2\nu^2}{r^2}\left(-dt^2+dr^2\right) + L^2\nu^2\left(d\theta^2+\sin^2\theta d\phi^2\right) .$$

In particular, the Lagrangian density for fluctuations of u is

$$\mathcal{L} = -T_{D3}\frac{\sqrt{1+\nu^2}}{\nu}\sqrt{-|\mathcal{G}|}\left(\frac{1}{2}L^2\mathcal{G}^{ab}\partial_a u \partial_b u\right) .$$

Given that the D3-brane world-volume is of the form $\mathrm{AdS}_2 \times S^2$, the next step is to perform a KK reduction of these fields on the world-volume S^2 to end up with fields living purely on AdS_2. This produces an infinite tower of modes, but the only ones relevant for us are the $l = 1$ triplet, since those are the ones sourcing the displacement operators. This KK reduction is discussed in detail in [45] (see their appendix C), and for the $l = 1$ triplet of modes we are interested in, the resulting fluctuation Lagrangian is $k\sqrt{1+\nu^2}$ times the one computed with the string. Since the computation of the two-point function of displacement operators involves the kinetic term of the fluctuations, the upshot is that the $\tilde{\gamma}$ computed in this regime is $k\sqrt{1+\nu^2}$ times the one computed with the string in the previous subsection, eq. (3.3), so

$$\tilde{\gamma}(\lambda, N) = \frac{3k\sqrt{\lambda}}{\pi^2}\sqrt{1+\frac{k^2\lambda}{16N^2}} .$$

To finish the check, we again need the coefficient B in (1.1) for this case. In [19] the total radiated power of a heavy probe in the k-symmetric representation was computed using the AdS/CFT correspondence by means of a D3-brane, and the result found was

$$B(\lambda, N) = \frac{k\sqrt{\lambda}}{4\pi^2}\sqrt{1+\frac{k^2\lambda}{16N^2}} .$$

Comparing these two results, this proves the $\tilde{\gamma} = 12B$ relation for a static probe in k-symmetric representation, in the regime of validity of the D3-brane probe approximation. Furthermore, if we set $k = 1$ in the previous result we can check [9, 39] that the exact expression for $B(\lambda, N)$ reduces to the one above in the appropriate limit.

4 Accelerated probes in AdS/CFT

In this section we will consider accelerated heavy probes coupled to $\mathcal{N} = 4$ SYM, in the context of the AdS/CFT correspondence. As in the previous section, the probes considered transform in the fundamental and the symmetric representations, so their gravity dual is given respectively by a string and a D3-brane, reaching the boundary at a hyperbola. We will compute the momentum diffusion coefficient in both cases, verifying that they reproduce in appropriate limits our exact result (2.7).

4.1 Fluctuations of the hyperbolic string in AdS_5

The dual of a heavy probe moving with constant proper acceleration is a string reaching the boundary of AdS_5 at a hyperbola, or at a circle in Euclidean signature. This type of world-sheet was first considered in [48], see also [49].[4] The spectrum of fluctuations of this world-sheet was discussed in [42], and it is the same as for the straight line. This world-sheet and its fluctuations were used in [25, 26] to derive the momentum diffusion coefficient of this probe in the supergravity approximation. To do so, [25, 26] made a series of change of coordinates to the gravity background, to work in a frame where the probe is static. We will now show that it is possible to obtain that transport coefficient working with Rindler coordinates. We start by writing the AdS_5 metric in Poincaré patch with Rindler coordinates

$$ds^2_{\mathrm{AdS}[5} = \frac{L^2}{y^2}\left(dy^2 + dr^2 - r^2 d\psi^2 + dx_2^2 + dx_3^2\right) .$$

We identify the world-sheet coordinates with (ψ, y). The classical solution is then given by $r = \sqrt{R^2 - y^2}$ [48]. We consider now fluctuations in the transverse directions x^2, x^3. The Lagrangian density for transverse fluctuations is

$$\mathcal{L}_{fluc} = \frac{1}{2\pi\alpha'}\frac{L^2 R}{y^2}\left(-\frac{1}{2}\frac{1}{R^2 - y^2}\left(\partial_\psi x\right)^2 + \frac{1}{2}\frac{R^2 - y^2}{R^2}\left(\partial_y x\right)^2\right) .$$

As a check, near the boundary ($y \to 0$), defining $\tau = R\psi$ we recover the fluctuation Lagrangian (3.2), except for a global factor of R, since here we are integrating with respect to $\psi = \tau/R$. Defining $z = y/R$, the equation of motion for transverse fluctuations is

$$-\partial_\psi^2 x + (1 - z^2)^2 \partial_z^2 x - 2\frac{1 - z^2}{z}\partial_z x = 0 .$$

We separate variables $x(z, \psi) = e^{-iw\psi} x(z)$ (and keeping in mind that this w is dimensionless, $w_\tau = w/R$), the solutions are

$$x(z) = C_1(1 - iwz)e^{iw\ \mathrm{arctanh}\ z} + C_2(1 + iwz)e^{-iw\ \mathrm{arctanh}\ z} .$$

To compute the retarded Green function, we take the purely outgoing solution ($C_2 = 0$) and following [17] obtain

$$G_R(w) = \frac{-iw}{2\pi\alpha'}\frac{L^2}{R^2} + \mathcal{O}(w^3) ,$$

where as in the static case [17] we dropped a $1/y$ term. This retarded Green function coincides with the one computed by Xiao [25], and from it one arrives at

$$\kappa = 4\pi\sqrt{\lambda}T^3 .$$

[4] Some subtleties associated to this world-sheet solution and its usual interpretation have been recently pointed out in [50], but since they concern the part of the world-sheet below the world-sheet horizon, they don't affect our discussion.

4.2 Fluctuations of a hyperbolic D3-brane in AdS_5

We can also compute the momentum diffusion coefficient for an accelerated probe in the symmetric representation. The relevant D3-brane reaches the boundary at a circle in Euclidean signature, and was first discussed in [38]. As in the previous section, it is convenient to present it in the coordinates introduced in [45, 47], so we start by writing AdS_5 as

$$ds^2 = L^2 \left(du^2 + \cosh^2 u \left(d\zeta^2 - \sinh^2 \zeta d\psi^2\right) + \sinh^2 u \, d\Omega_2^2\right) .$$

The D3-brane has world-volume coordinates $(\zeta, \psi, \theta, \phi)$ and the classical solution is

$$\sinh u = \nu \qquad F_{\zeta\psi} = \frac{\sqrt{\lambda}}{2\pi}\sqrt{1+\nu^2}\sinh\zeta \,,$$

with ν defined in eq. (3.4). Consider now fluctuations for the world-volume fields u, the S^5 fields and the BI gauge field. As it was discussed in detail in [45] these fluctuations decouple, so we can focus on the fluctuations of u. To present the relevant fluctuation Lagrangian, define the metric

$$\mathcal{G}_{ab} d\xi^a d\xi^b = L^2\nu^2(d\zeta^2 - \sinh^2\zeta d\psi^2) + L^2\nu^2 d\Omega_2^2 \,.$$

This is the metric that controls the fluctuations of u (and the gauge field)

$$\mathcal{L}_{fluc} = T_{D3} L^4 \sqrt{1+\nu^2}\nu^3 \sinh\zeta \sin\theta \left(\frac{1}{2} L^2 \mathcal{G}^{ab}\partial_a u \partial_b u\right) .$$

As in the previous section, we now have to KK reduce this world-volume field u on S^2, to obtain an infinite tower of 2d fields on the world-volume AdS_2. Again, the relevant modes are the $l = 1$ triplet, and as it happened for the static probe, the resulting fluctuation Lagrangian is $k\sqrt{1+\nu^2}$ the one we would obtain for the fluctuations of a fundamental string in these coordinates. We then conclude that the resulting momentum diffusion coefficient is again $k\sqrt{1+\nu^2}$ times the one obtained for the fundamental string, so

$$\kappa = 4\pi k\sqrt{\lambda}\sqrt{1+\frac{k^2\lambda}{16N^2}} T^3 \,.$$

5 Lessons for the $\mathcal{N} = 4$ super Yang-Mills plasma?

In section 2 we have found the exact two-point function of momentum fluctuations in vacuum of an accelerated heavy quark coupled to a conformal field theory. As expected, this two-point function presents thermal behavior, and the question arises whether we can use our results to learn something about momentum fluctuations of a heavy probe immersed in a thermal bath of the same conformal theory, now at finite temperature. Besides its intrinsic interest, this question has broader relavance since it is expected that at finite temperature, conformal theories (even superconformal ones) share some properties with the high-temperature deconfined phase of confining gauge theories. More specifically, a particular CFT, $\mathcal{N} = 4$ SYM at $T \neq 0$, has been used by means of the AdS/CFT

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correspondence to model the quark-gluon plasma experimentally observed at RHIC and at the LHC (see [30–32] for reviews). In particular, the momentum fluctuations of a heavy quark (either static or moving at constant velocity) in the quark-gluon plasma have been estimated by considering a dual trailing string in the background of a black Schwarzschild brane in an asympotically AdS_5 background [16, 27–29].

The study of a heavy quark in a strongly coupled conformal plasma by means of the AdS/CFT correspondence is currently limited to the large λ, large N regime where supergravity is reliable (see [51, 52] for computation of the $1/\sqrt{\lambda}$ correction and some $\lambda^{-3/2}$ corrections to the jet quenching parameter in the $\mathcal{N} = 4$ SYM plasma) and it currently seems extremely hard to perform such computations at finite λ and N. For this reason, it would be very interesting if the study of an accelerated quark in the vacuum of a conformal field theory, which as we have seen can be tackled at finite λ and N, can become an indirect route to the study of conformal $T \neq 0$ plasma. However, while a probe accelerated in vacuum and a static probe in a thermal bath experience a non-zero temperature, the details of their response are not identical (see the review [53] for a discussion on this point). We can see this explicitly for the $\mathcal{N} = 4$ SYM plasma, by comparing known expressions of the momentum diffusion coefficients in various regimes. Let's consider first the regime of weak coupling; the momentum diffusion coefficient of a heavy quark in a weakly coupled $\mathcal{N} = 4$ SU(N) SYM plasma has been computed at leading and next-to-leading orders [54, 55]

$$\kappa_{\text{thermal}} = \frac{\lambda^2 T^3}{6\pi}\frac{N^2-1}{N^2}\left(\log\frac{1}{\sqrt{\lambda}} + c_1 + c_2\sqrt{\lambda} + \mathcal{O}(\lambda)\right),$$

with $c_{1,2}$ known coefficients (see the second reference in [54, 55]). This expression differs qualitatively from the weak coupling expansion of our result for the momentum diffusion coefficient for an accelerated quark

$$\kappa_{\text{Unruh}} = \pi\lambda T^3\frac{N^2-1}{N^2}\left(1 - \frac{\lambda}{24} + \mathcal{O}(\lambda^2)\right).$$

Notice that κ_{thermal} starts at λ^2 (versus the leading λ in κ_{Unruh}) and furthermore presents a term logarithmic in λ, absent in κ_{Unruh}. These two features come from the non-trivial coupling dependence of the Debye mass in the thermal bath [56].

Let's move now to the regime where supergravity is reliable, i.e. large λ and large N. As recalled in section 4, an accelerated probe in vacuum is dual to a string reaching the boundary of pure AdS_5 at a hyperbola, while a heavy probe in a thermal bath is represented by a string in the Schwarzschild- Anti de Sitter background, and the respective retarded Green functions are quantitatively different (see [57] for a discussion on this point). In particular, the momentum diffusion coefficient yields [16, 27]

$$\kappa^{\text{SUGRA}}_{\text{thermal}} = \pi\sqrt{\lambda}T^3,$$

which is four times smaller than the supergravity result for the similar transport coefficient for a probe accelerated in vacuum, eq. (2.8),

$$\kappa^{\text{SUGRA}}_{\text{Unruh}} = 4\pi\sqrt{\lambda}T^3.$$

This difference might be surprising at first, since it can be argued that transport coefficients can be read from the world-sheet horizon [58], and the two classical world-sheet metrics (i.e. accelerated string in AdS_5 versus hanging/trailing string in Schwarzschild-AdS_5) while clearly different, have the same near-horizon metric, 1+1 Rindler space. However, the different change of variables used to write these near-horizon metrics imply different normalizations of the corresponding wavefunctions, giving rise to this factor of four discrepancy between the respective transport coefficients.

Keeping this difference in mind, we nevertheless propose to use our exact results to make an educated guess of the impact of using SUGRA instead of the exact result for computing the momentum diffusion coefficient of a static heavy quark, κ_{thermal}, in $\mathcal{N}=4$ SYM at finite temperature. To that end, we start by evaluating the difference between the SUGRA (large λ, large N) and the exact (finite λ, N=3) computations of the coefficient for the accelerated probe in vacuum.

The first ingredient we need in our computation is the Bremsstrahlung function (1.1) for a heavy quark coupled to $\mathcal{N}=4$ SU(3) SYM. For U(N) the Bremsstrahlung function is given in (1.4), and since the SU(N) function is obtained by subtracting the U(1) contribution [8]

$$B_{\mathrm{SU}(N)} = B_{\mathrm{U}(N)} - \frac{\lambda}{16\pi^2 N^2}$$

we obtain

$$B_{\mathrm{SU}(3)} = \frac{1}{4\pi^2}\frac{\lambda}{18}\frac{\lambda^2+144\lambda+3456}{\lambda^2+72\lambda+864}$$

and using the relation derived in this paper, eq. (2.7), we arrive at the following expression for the SU(3) momentum diffusion coefficient, valid for any value of λ,

$$\kappa_{\mathrm{SU}(3)} = 4\pi\frac{\lambda}{18}\frac{\lambda^2+144\lambda+3456}{\lambda^2+72\lambda+864}\,T^3\,. \tag{5.1}$$

Notice that both for small λ and large λ the coefficient grows linearly with λ. This is true for generic fixed N

$$\kappa^{\lambda\ll 1}_{\mathrm{SU}(N)} = \frac{N^2-1}{N^2}\pi\lambda T^3 \qquad \kappa^{\lambda\gg 1}_{\mathrm{SU}(N)} = \frac{N-1}{N^2}\pi\lambda T^3\,.$$

We now consider the quotient of the exact expression for this transport coefficient, eq. (5.1), versus the result obtained in the supergravity limit, eq. (2.8),

$$\text{Unruh} \qquad \frac{\kappa^{\text{EXACT}}}{\kappa^{\text{SUGRA}}} = \frac{\sqrt{\lambda}}{18}\frac{\lambda^2+144\lambda+3456}{\lambda^2+72\lambda+864}\,. \tag{5.2}$$

A first observation is that this ratio is a monotonously increasing function of λ that doesn't go to one as $\lambda\to\infty$. The reason is that the denominator, obtained in the planar limit ($N\to\infty$), grows like $\sqrt{\lambda}$, while the numerator, obtained for $N=3$, grows like λ. This ratio is smaller than one for small λ and becomes larger than one for $\lambda \gtrsim 182.45$. As we discuss below, when modelling the quark-gluon plasma by $\mathcal{N}$=4 SYM the range of values considered for λ is substantially below this point, so another observation is that in that range of values, the supergravity computation gives a value κ^{SUGRA} which is larger than

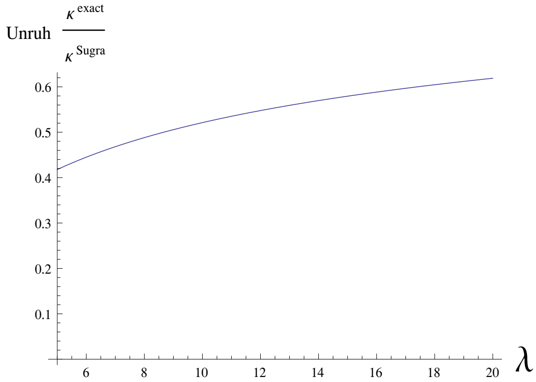

Figure 2. The relation between the exact momentum diffusion coeffient and the supergravity approximation for an accelerated quark in vacuum. The range of λ displayed corresponds to the one considered when modelling the quark-gluon plasma.

κ^{EXACT}. To be more quantitative, we will zoom in the range of values of λ that have been considered when modelling the QCD quark-gluon plasma by $\mathcal{N} = 4$ SYM. Given the differences among these two theories, there are inherent ambiguities in choosing the parameters of $\mathcal{N} = 4$ SYM that might best model the real world QCD plasma. A first choice [59] is to take

$$\text{“obvious” scheme:} \qquad T_{\mathcal{N}=4} = T_{\text{QCD}} \qquad g^2_{\text{YM}} N = 12\pi\alpha_s = 6\pi\,,$$

where in the last equation the value $\alpha_s = 0.5$ was taken. A second choice made in the literature [17, 60] tries to ameliorate the impact of the obvious difference that $\mathcal{N} = 4$ SU(3) SYM has more degrees of freedom than QCD. The main idea is to compare the theories at fixed value of the energy density, rather than temperature. This results in the following identification

$$\text{“alternative” scheme:} \qquad 3^{1/4} T_{\mathcal{N}=4} = T_{\text{QCD}} \qquad g^2_{\text{YM}} N = 5.5\,,$$

where the value $\lambda = 5.5$ is the central value derived from this analysis. This scheme has its own limitations, and the only lesson we want to take from it is that the range of values $\lambda \in [5.5, 6\pi]$ has been considered when modelling the quark-gluon plasma by the $\mathcal{N} = 4$ SYM plasma. Having fixed the range of values for λ we will be zooming in, we can now determine the impact of using supergravity to compute the momentum diffusion coefficient instead of using the exact result: in this range, the ratio (5.2) increases from 0.43 to 0.61, see figure (2). Roughly speaking, in this range of values for the 't Hooft coupling, supergravity gives an answer for this transport coefficient about twice the exact result.

Up until here, we were on firm ground, comparing the result of two computations for the momentum diffusion coefficient of an accelerated quark in the vacuum of $\mathcal{N} = 4$ SYM. Having used the word “exact” or variations exactly thirty-three times so far in this paper, we will end it by indulging in far less precise statements. We have seen that the perturbative expressions (fixed N, $g^2_{\text{YM}} \ll 1$) for the momentum diffusion coefficients in a thermal bath and in the Unruh effect differ qualitatively, while the corresponding expressions in the

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supergravity regime (large λ, large N), share the same parametric behavior, but not the overall numerical coefficient. It is then clear that we are not in a position to estimate κ_{thermal} for $\mathcal{N}=4$ SYM for abitrary λ, N. A more modest goal is to estimate it in the range of values singled out above, that appear when modelling the QCD quark-gluon plasma. If we consider a path in the (λ, N) plane from the range of values considered above to the region of validity of supergravity (i.e. now we don't keep N fixed) the ratio $\kappa_{\text{Unruh}}/\kappa^{\text{SUGRA}}_{\text{Unruh}}$ will uneventfully evolve from the value found above, about 1/2, to 1. In order to proceed, we are going to assume that roughly the same is true for κ_{thermal} so along that path

$$\frac{\kappa^{\text{exact}}_{\text{Unruh}}}{\kappa^{\text{SUGRA}}_{\text{Unruh}}} \approx \frac{\kappa^{\text{exact}}_{\text{thermal}}}{\kappa^{\text{SUGRA}}_{\text{thermal}}} .$$

If this assumption is true, it means that the supergravity computations [16, 27, 28] for κ_{thermal} give an answer $\kappa^{\text{SUGRA}}_{\text{thermal}}$ that is about twice the exact one. While we currently lack solid arguments to substantiate this speculation, let's end by noting that if true, it would in turn imply that the diffusion constant $D = 2T^2/\kappa$ for the $\mathcal{N}=4$ SYM plasma would be about twice the one obtained in supergravity, pushing it in the right direction to match the range of values suggested by RHIC [61].

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References

[1] J. Erickson, G. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B 582** (2000) 155 [hep-th/0003055] [INSPIRE].

[2] N. Drukker and D.J. Gross, *An exact prediction of $N = 4$ SUSYM theory for string theory*, J. Math. Phys. **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[3] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[4] V. Pestun, *Localization of the four-dimensional $N = 4$ SYM to a two-sphere and 1/8 BPS Wilson loops*, JHEP **12** (2012) 067 [arXiv:0906.0638] [INSPIRE].

[5] S. Giombi and V. Pestun, *Correlators of local operators and 1/8 BPS Wilson loops on S^2 from 2D YM and matrix models*, JHEP **10** (2010) 033 [arXiv:0906.1572] [INSPIRE].

[6] J. Gomis, T. Okuda and V. Pestun, *Exact results for 't Hooft loops in gauge theories on S^4*, JHEP **05** (2012) 141 [arXiv:1105.2568] [INSPIRE].

[7] S. Giombi and V. Pestun, *Correlators of Wilson loops and local operators from multi-matrix models and strings in AdS*, *JHEP* **01** (2013) 101 [arXiv:1207.7083] [INSPIRE].

[8] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in $N = 4$ super Yang-Mills*, *JHEP* **06** (2012) 048 [arXiv:1202.4455] [INSPIRE].

[9] B. Fiol, B. Garolera and A. Lewkowycz, *Exact results for static and radiative fields of a quark in $N = 4$ super Yang-Mills*, *JHEP* **05** (2012) 093 [arXiv:1202.5292] [INSPIRE].

[10] A.M. Polyakov and V.S. Rychkov, *Gauge field strings duality and the loop equation*, *Nucl. Phys.* **B 581** (2000) 116 [hep-th/0002106] [INSPIRE].

[11] H. Dorn, *Renormalization of path ordered phase factors and related hadron operators in gauge field theories*, *Fortsch. Phys.* **34** (1986) 11 [INSPIRE].

[12] G. W. Semenoff and D. Young, "Wavy Wilson line and AdS / CFT," Int. J. Mod. Phys. A **20**, 2833 (2005) [hep-th/0405288] G.W. Semenoff and D. Young, *Wavy Wilson line and AdS/CFT*, *Int. J. Mod. Phys.* **A 20** (2005) 2833 [hep-th/0405288] [INSPIRE].

[13] N. Drukker and S. Kawamoto, *Small deformations of supersymmetric Wilson loops and open spin-chains*, *JHEP* **07** (2006) 024 [hep-th/0604124] [INSPIRE].

[14] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev.* **D 74** (2006) 025005 [hep-th/0501015] [INSPIRE].

[15] N. Drukker and S. Kawamoto, *Circular loop operators in conformal field theories*, *Phys. Rev.* **D 74** (2006) 046002 [hep-th/0512150] [INSPIRE].

[16] J. Casalderrey-Solana and D. Teaney, *Heavy quark diffusion in strongly coupled $N = 4$ Yang-Mills*, *Phys. Rev.* **D 74** (2006) 085012 [hep-ph/0605199] [INSPIRE].

[17] S.S. Gubser, *Momentum fluctuations of heavy quarks in the gauge-string duality*, *Nucl. Phys.* **B 790** (2008) 175 [hep-th/0612143] [INSPIRE].

[18] A. Mikhailov, *Nonlinear waves in AdS/CFT correspondence*, hep-th/0305196 [INSPIRE].

[19] B. Fiol and B. Garolera, *Energy loss of an infinitely massive half-Bogomol'nyi-Prasad-Sommerfeld particle by radiation to all orders in $1/N$*, *Phys. Rev. Lett.* **107** (2011) 151601 [arXiv:1106.5418] [INSPIRE].

[20] D. Correa, J. Henn, J. Maldacena and A. Sever, *The cusp anomalous dimension at three loops and beyond*, *JHEP* **05** (2012) 098 [arXiv:1203.1019] [INSPIRE].

[21] D. Correa, J. Maldacena and A. Sever, *The quark anti-quark potential and the cusp anomalous dimension from a TBA equation*, *JHEP* **08** (2012) 134 [arXiv:1203.1913] [INSPIRE].

[22] N. Drukker, *Integrable Wilson loops*, arXiv:1203.1617 [INSPIRE].

[23] N. Gromov and A. Sever, *Analytic solution of Bremsstrahlung TBA*, *JHEP* **11** (2012) 075 [arXiv:1207.5489] [INSPIRE].

[24] W. Unruh, *Notes on black hole evaporation*, *Phys. Rev.* **D 14** (1976) 870 [INSPIRE].

[25] B.-W. Xiao, *On the exact solution of the accelerating string in AdS_5 space*, *Phys. Lett.* **B 665** (2008) 173 [arXiv:0804.1343] [INSPIRE].

[26] E. Caceres, M. Chernicoff, A. Guijosa and J.F. Pedraza, *Quantum fluctuations and the Unruh effect in strongly-coupled conformal field theories*, *JHEP* **06** (2010) 078 [arXiv:1003.5332] [INSPIRE].

[27] C. Herzog, A. Karch, P. Kovtun, C. Kozcaz and L. Yaffe, *Energy loss of a heavy quark moving through $N = 4$ supersymmetric Yang-Mills plasma*, *JHEP* **07** (2006) 013 [hep-th/0605158] [INSPIRE].

[28] S.S. Gubser, *Drag force in AdS/CFT*, *Phys. Rev.* **D 74** (2006) 126005 [hep-th/0605182] [INSPIRE].

[29] J. Casalderrey-Solana and D. Teaney, *Transverse momentum broadening of a fast quark in a $N = 4$ Yang-Mills plasma*, *JHEP* **04** (2007) 039 [hep-th/0701123] [INSPIRE].

[30] S.S. Gubser and A. Karch, *From gauge-string duality to strong interactions: a pedestrian's guide*, *Ann. Rev. Nucl. Part. Sci.* **59** (2009) 145 [arXiv:0901.0935] [INSPIRE].

[31] S.S. Gubser, S.S. Pufu, F.D. Rocha and A. Yarom, *Energy loss in a strongly coupled thermal medium and the gauge-string duality*, arXiv:0902.4041 [INSPIRE].

[32] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U.A. Wiedemann, *Gauge/string duality, hot QCD and heavy ion collisions*, arXiv:1101.0618 [INSPIRE].

[33] S.-J. Rey and J.-T. Yee, *Macroscopic strings as heavy quarks in large-N gauge theory and Anti-de Sitter supergravity*, *Eur. Phys. J.* **C 22** (2001) 379 [hep-th/9803001] [INSPIRE].

[34] J.M. Maldacena, *Wilson loops in large-N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [hep-th/9803002] [INSPIRE].

[35] N. Drukker and V. Forini, *Generalized quark-antiquark potential at weak and strong coupling*, *JHEP* **06** (2011) 131 [arXiv:1105.5144] [INSPIRE].

[36] D. Sciama, P. Candelas and D. Deutsch, *Quantum field theory, horizons and thermodynamics*, *Adv. Phys.* **30** (1981) 327 [INSPIRE].

[37] W.G. Unruh and N. Weiss, *Acceleration radiation in interacting field theories*, *Phys. Rev.* **D 29** (1984) 1656 [INSPIRE].

[38] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [hep-th/0501109] [INSPIRE].

[39] S. Giombi, R. Ricci and D. Trancanelli, *Operator product expansion of higher rank Wilson loops from D-branes and matrix models*, *JHEP* **10** (2006) 045 [hep-th/0608077] [INSPIRE].

[40] C.G. Callan Jr. and A. Guijosa, *Undulating strings and gauge theory waves*, *Nucl. Phys.* **B 565** (2000) 157 [hep-th/9906153] [INSPIRE].

[41] S. Förste, D. Ghoshal and S. Theisen, *Stringy corrections to the Wilson loop in $N = 4$ super Yang-Mills theory*, *JHEP* **08** (1999) 013 [hep-th/9903042] [INSPIRE].

[42] N. Drukker, D.J. Gross and A.A. Tseytlin, *Green-Schwarz string in $AdS_5 \times S^5$: semiclassical partition function*, *JHEP* **04** (2000) 021 [hep-th/0001204] [INSPIRE].

[43] W. Nahm, *Supersymmetries and their representations*, *Nucl. Phys.* **B 135** (1978) 149 [INSPIRE].

[44] M. Günaydin, G. Sierra and P. Townsend, *The unitary supermultiplets of $D = 3$ Anti-de Sitter and $d = 2$ conformal superalgebras*, *Nucl. Phys.* **B 274** (1986) 429 [INSPIRE].

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[45] A. Faraggi and L.A. Pando Zayas, *The spectrum of excitations of holographic Wilson loops*, *JHEP* **05** (2011) 018 [arXiv:1101.5145] [INSPIRE].

[46] A. Karch and L. Randall, *Locally localized gravity*, *JHEP* **05** (2001) 008 [hep-th/0011156] [INSPIRE].

[47] S. Yamaguchi, *Wilson loops of anti-symmetric representation and d5-branes*, *JHEP* **05** (2006) 037 [hep-th/0603208] [INSPIRE].

[48] D.E. Berenstein, R. Corrado, W. Fischler and J.M. Maldacena, *The operator product expansion for Wilson loops and surfaces in the large-N limit*, *Phys. Rev.* **D 59** (1999) 105023 [hep-th/9809188] [INSPIRE].

[49] V. Branding and N. Drukker, *BPS Wilson loops in $N = 4$ SYM: examples on hyperbolic submanifolds of space-time*, *Phys. Rev.* **D 79** (2009) 106006 [arXiv:0902.4586] [INSPIRE].

[50] J.A. Garcia, A. Gijosa and E.J. Pulido, *No line on the horizon: on uniform acceleration and gluonic fields at strong coupling*, *JHEP* **01** (2013) 096 [arXiv:1210.4175] [INSPIRE].

[51] Z.-q. Zhang, D.-f. Hou and H.-c. Ren, *The finite 't Hooft coupling correction on jet quenching parameter in a $\mathcal{N} = 4$ super Yang-Mills plasma*, *JHEP* **01** (2013) 032 [arXiv:1210.5187] [INSPIRE].

[52] N. Armesto, J.D. Edelstein and J. Mas, *Jet quenching at finite 't Hooft coupling and chemical potential from AdS/CFT*, *JHEP* **09** (2006) 039 [hep-ph/0606245] [INSPIRE].

[53] L.C. Crispino, A. Higuchi and G.E. Matsas, *The Unruh effect and its applications*, *Rev. Mod. Phys.* **80** (2008) 787 [arXiv:0710.5373] [INSPIRE].

[54] P. Chesler and A. Vuorinen, *Heavy flavor diffusion in weakly coupled $N = 4$ super Yang-Mills theory*, *JHEP* **11** (2006) 037 [hep-ph/0607148] [INSPIRE].

[55] S. Caron-Huot and G.D. Moore, *Heavy quark diffusion in QCD and $N = 4$ SYM at next-to-leading order*, *JHEP* **02** (2008) 081 [arXiv:0801.2173] [INSPIRE].

[56] M. Le Bellac, *Thermal field theory*, Cambridge University Press, Cambridge U.K. (1996).

[57] T. Hirayama, P.-W. Kao, S. Kawamoto and F.-L. Lin, *Unruh effect and holography*, *Nucl. Phys.* **B 844** (2011) 1 [arXiv:1001.1289] [INSPIRE].

[58] N. Iqbal and H. Liu, *Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm*, *Phys. Rev.* **D 79** (2009) 025023 [arXiv:0809.3808] [INSPIRE].

[59] H. Liu, K. Rajagopal and U.A. Wiedemann, *Calculating the jet quenching parameter from AdS/CFT*, *Phys. Rev. Lett.* **97** (2006) 182301 [hep-ph/0605178] [INSPIRE].

[60] S.S. Gubser, *Comparing the drag force on heavy quarks in $N = 4$ super-Yang-Mills theory and QCD*, *Phys. Rev.* **D 76** (2007) 126003 [hep-th/0611272] [INSPIRE].

[61] A. Francis, O. Kaczmarek, M. Laine and J. Langelage, *Towards a non-perturbative measurement of the heavy quark momentum diffusion coefficient*, *PoS* **LATTICE2011** (2011) 202 [arXiv:1109.3941] [INSPIRE].

Chapter 4

Precision tests through supersymmetric localization

This chapter contains the publication:

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Exact probes of orientifolds

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Abstract: We compute the exact vacuum expectation value of circular Wilson loops for Euclidean $\mathcal{N} = 4$ super Yang-Mills with $G = \mathrm{SO}(N), \mathrm{Sp}(N)$, in the fundamental and spinor representations. These field theories are dual to type IIB string theory compactified on $AdS_5 \times \mathbb{RP}^5$ plus certain choices of discrete torsion, and we use our results to probe this holographic duality. We first revisit the LLM-type geometries having $AdS_5 \times \mathbb{RP}^5$ as ground state. Our results clarify and refine the identification of these LLM-type geometries as bubbling geometries arising from fermions on a half harmonic oscillator. We furthermore identify the presence of discrete torsion with the one-fermion Wigner distribution becoming negative at the origin of phase space. We then turn to the string world-sheet interpretation of our results and argue that for the quantities considered they imply two features: first, the contribution coming from world-sheets with a single crosscap is closely related to the contribution coming from orientable world-sheets, and second, world-sheets with two crosscaps don't contribute to these quantities.

Keywords: Wilson, 't Hooft and Polyakov loops, 1/N Expansion, AdS-CFT Correspondence

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1 Introduction

The AdS/CFT correspondence has drastically changed our view on the interrelations between field theory and quantum gravity. However, at the level of specific results, it seems fair to assess that it has not brought as many new results in quantum gravity as in field theory. Indeed, while it has allowed access to regimes of field theory previously unexplored, the amount of work using field theory results to learn about quantum gravity has been smaller. One of the main reasons of this state of affairs is of course the paucity of known results in the relevant regimes of field theory.

Localization has emerged as a powerful technique to drastically simplify very specific computations in supersymmetric field theories, allowing in some cases to obtain exact results [1–4]. In particular, for 4d $\mathcal{N}=2$ super Yang Mills theories with a Lagrangian description, the evaluation of the vev of certain circular Wilson loops boils down to a matrix model computation [1]. Furthermore, for the particular case of $\mathcal{N}=4$ SYM, the matrix model is Gaussian [1, 5, 6], so all the integrals can be computed exactly. This has been done for $G=\mathrm{U}(N),\mathrm{SU}(N)$ first for a Wilson loop in the fundamental representation, and more recently for other representations [7, 8]. Even though the quantities that can be computed thanks to localization must satisfy a number of conditions that make them non-generic, it seems pertinent to ask whether these exact results in field theories can teach us something about the holographic M/string theory duals, beyond the supergravity regime.

There have been a number of works trying to use the localization of Wilson loops in four dimensional $\mathcal{N}=2$ Yang Mills theories to probe putative string duals [9–11]. This is a potentially very exciting line of research, as it may reveal properties of holographic pairs that have not been fully established to date. In this work we will take a slightly different route, by applying localization to probe a known example of holographic duality. We will consider $\mathcal{N}=4$ SYM with gauge group $G=\mathrm{SO}(N),\mathrm{Sp}(N)$, which is dual to type IIB string theory compactified on $AdS_5\times\mathbb{RP}^5$ with various choices of discrete torsion [12].[1] This duality is closely related to the original proposal for $G=\mathrm{SU}(N)$, but it displays a number of novel features, related to the presence of non-orientable surfaces in the $1/N$ expansion of the field theories, or equivalently to the existence of homologically non-trivial non-orientable subvarieties in the gravity background. Our aim is to explore some of these features at finite g_s and α'/R^2, taking advantage of the possibility of computing exactly the vev of certain Wilson loop operators for these field theories. While our focus is on non-local operators, the physics of local operators of these field theories at finite N has been explored in [16, 17].

Our first task will be to compute the vev of 1/2-BPS circular Wilson loops in various representations, for Euclidean $\mathcal{N}=4$ SYM with gauge groups $G=\mathrm{SO}(N),\mathrm{Sp}(N)$. Even before we start thinking about holography, the evaluation of these vevs has interesting applications within field theory. For instance, for $G=\mathrm{U}(N),\mathrm{SU}(N)$, they immediately allow us to compute the Bremsstrahlung functions for the corresponding heavy probes, using the relation [18]

$$B(\lambda,N)_{\mathcal{R}}=\frac{1}{2\pi^2}\lambda\partial_\lambda\,\log\langle W_{\mathcal{R}}\rangle \tag{1.1}$$

valid for any representation $\mathcal{R}$. These Bremsstrahlung functions in turn completely determine various quantities of physical interest, like the total radiated power [18, 19] and the momentum fluctuations of the corresponding accelerated probe [20]. These vevs also determine the exact change in the entanglement entropy of a spherical region when we add a heavy probe [21].[2] Finally, they can also be used to carry out detailed tests of S-duality in $\mathcal{N}=4$ SYM [7].

The technical computation of these vevs is quite similar to the ones performed for unitary groups, and amounts to introducing a convenient set of orthogonal polynomials to carry out the matrix model integrals. In fact, since for all Lie algebras $\mathfrak{g}$ the matrix model is Gaussian, the relevant orthogonal polynomials are Hermite polynomials, and the computation of vevs ends up amounting to the evaluation of matrix elements for a N-fermion state of the one-dimensional harmonic oscillator,

$$\langle W\rangle=\frac{\langle\Psi_{\mathfrak{g}}|W|\Psi_{\mathfrak{g}}\rangle}{\langle\Psi_{\mathfrak{g}}|\Psi_{\mathfrak{g}}\rangle} \tag{1.2}$$

the only difference being the parity of the one-fermion states involved: for $\mathfrak{su}(\mathfrak{n})$, $|\Psi\rangle$ is built by filling the first N eigenstates of a harmonic oscillator, for $\mathfrak{so}(2\mathfrak{n})$ filling the first

[1] The precise statement is actually more subtle: given a Lie algebra $\mathfrak{g}$, there is a variety of Lie groups G associated to it, and all of them define different gauge theories. These gauge theories have the same correlators of local operators, but differ in the spectrum of non-local operators [13]. In the case of $\mathcal{N}=4$ SYM, theories with the same $\mathfrak{g}$ and different G each have their own holographic dual, differing by a choice of quantization of certain topological term in the type IIB action [14, 15]. We are grateful to Ofer Aharony for clarifying correspondence on this point.

[2] It is worth keeping in mind that for the computation of the entanglement entropy [21], it is convenient to use a normalization of the Wilson loops different from the one used in this work.

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N even states and for $\mathfrak{g} = \mathfrak{so}(2\mathfrak{n}+1), \mathfrak{sp}(\mathfrak{n})$ the first N odd eigenstates [16, 17, 22, 23]. The computations are straightforward, and reveal exact relations among various vevs. For Wilson loops in the respective fundamental representations we find that

$$\langle W(g)\rangle_{\substack{\mathrm{SO}(2N)\\ \mathrm{Sp}(N)}} = \langle W(g)\rangle_{\mathrm{U}(2N)} \mp \frac{1}{2}\int_0^g dg' \, \langle W(g')\rangle_{\mathrm{U}(2N)} \tag{1.3}$$

where $g = \lambda/4N$. This in an exact relation, valid for any value of λ and N.

Once we have obtained these exact field theory results, we shift gears towards string theory. In the past, the exact computation of circular Wilson loops of $\mathcal{N}=4$ SU(N) SYM has been used for precision tests of AdS/CFT [24–26]. Our attitude in the present work will be different, we will take for granted the holographic duality, and we aim to use the exact field theory results to learn about string theory on $AdS_5 \times \mathbb{RP}^5$. Our first observation actually doesn't even rely on the actual computation of the vevs of Wilson loops, it can be made just by noticing that for SU(N), the N-fermion state $|\Psi\rangle$ in (1.2) is the groundstate of the fermionic system dual to the LLM sector [27] of $AdS_5 \times S^5$. We use this observation to revisit the question [28] of what is the analogue of the LLM sector for type IIB on $AdS_5 \times \mathbb{RP}^5$, and argue that it is given by geometries built out of fermions whose wavefunctions have fixed parity, even for SO($2N$) and odd for SO($2N+1$), Sp(N). In this latter case, those are the wavefunctions of the half harmonic oscillator [28]. Still in the LLM sector, we point out that the absence or presence of discrete torsion in the gravity dual correlates with the sign of the one-fermion Wigner quasi-distribution at the origin of phase space.

Another aspect of the holographic duality where we can put our exact results to work is perturbative string theory around $AdS_5 \times \mathbb{RP}^5$. The idea is not new: consider the vev of the circular Wilson loop in the fundamental representation of SU(N), which is known exactly [6]; in principle, string perturbation theory ought to reproduce the $1/N$ expansion of this vev by world-sheet computations at arbitrary genus on $AdS_5 \times S^5$. In practice, these world-sheet computations are currently well out of reach. We would like to claim that some of our results for $G = \mathrm{SO}(N), \mathrm{Sp}(N)$ might have a better chance of being reproduced by direct world-sheet arguments than those of $G = \mathrm{SU}(N)$. To see why, let's recall some generic facts about the large N expansion of gauge theories. In this limit, Feynman diagrams rearrange themselves in a topological expansion of two-dimensional surfaces, weighted by N^χ, where χ is the Euler characteristic of the surface, namely,

$$\chi = -2h + 2 - c - b$$

for a surface with h handles, c crosscaps and b boundaries. For a U(N), SU(N) field theory with all the fields in the adjoint, gauge invariant quantities admit a $1/N^2$ expansion (rather than $1/N$) as befits orientable surfaces. For instance, for the vev of the circular Wilson loop in the fundamental representation of U(N) the relevant world-sheets have a single boundary and an arbitrary number of handles, and in [6] it was explicitly shown that this vev admits a $1/N^2$ expansion. On the other hand, it is well-known that the $1/N$ expansion of field theories with $G = \mathrm{SO}(N), \mathrm{Sp}(N)$ contains both even and odd powers of $1/N$ [29], signaling the presence of non-orientable surfaces.[3] On general grounds, as discussed in detail below,

[3]See [30, 31] for the $1/N$ expansion of 2d Yang-Mills theory with $G = \mathrm{SO}(N), \mathrm{Sp}(N)$.

we can classify the world-sheets as having an arbitrary number of handles, and zero, one or two crosscaps. However, a closer inspection of eq. (1.3) reveals that in a $1/N$ expansion, the first term of the r.h.s. corresponds to orientable world-sheets, while the second one to world-sheets with a single crosscap. We thus learn that, for these quantities, the contribution from world-sheets with a single crosscap is given by an integral of the contribution from orientable world-sheets, while world-sheets with two cross-caps don't contribute. These two features are peculiar to the very specific vevs we have considered. Nevertheless, since they have been derived from exact field theory relations, before actually carrying out the $1/N$ expansion, it is conceivable that they could be deduced in string theory by symmetry arguments, without having to carry out the world-sheet computations.

The structure of the paper is as follows. In section 2 we define the field theory quantities we want to evaluate, and recall that thanks to localization, they boil down to matrix model computations. We then compute the vev of circular Wilson loops for various gauge groups and representations. In section 3 we discuss implications for string theory of the computations presented in the previous section. Some very basic facts about classical simple Lie algebras that we use in the main text are collected in appendix A, while in appendix B we present an alternative derivation of some of the results obtained in section 3.

2 Computations

This section is entirely devoted to the computation of vevs of circular Wilson loops in $\mathcal{N} = 4$ SYM, leaving for the next section the discussion of the implications of the results found here. Technically, the evaluation of these vevs of Wilson loops is possible since they localize to a computation in a Gaussian matrix model [1, 5, 6], with matrices in the Lie algebra $\mathfrak{g}$. To carry out the remaining integrals, we resort to the well-known technique of orthogonal polynomials (see [32, 33] for reviews). Besides the specific results we find, the main point to keep in mind from this section is that for all classical Lie algebras, the orthogonal polynomials are Hermite polynomials, the main difference being the restrictions on their parity. Namely, for the A, B/C and D series, the Hermite polynomials that play a role have unrestricted, odd and even parity, respectively. This observation will become important in the next section.

The field theory quantities we want to compute are vevs of locally BPS Wilson operators. These Wilson loops are determined by a representation $\mathcal{R}$ of the gauge group G and a contour $\mathcal{C}$,

$$W_{\mathcal{R}}[\mathcal{C}] = \frac{1}{\dim\, \mathcal{R}} \mathrm{Tr}_{\mathcal{R}} \mathcal{P}\mathrm{exp}\, \left(i \int_{\mathcal{C}} (A_\mu \dot{x}^\mu + |\dot{x}| \Phi_i \theta^i) ds \right) \tag{2.1}$$

We have fixed the overall normalization of the Wilson loop by the requirement that at weak coupling, $\langle W_{\mathcal{R}} \rangle = 1 + \mathcal{O}(g)$. We will be interested in the case when the signature is Euclidean and the contour is a circle. These Wilson loops are 1/2 BPS and remarkably the problem of the evaluation of their vev localizes to a Gaussian matrix model computation [1, 5, 6],

$$\langle W \rangle_{\mathcal{R}} = \frac{1}{\dim\, \mathcal{R}} \frac{\int_{\mathfrak{g}} dM e^{-\frac{1}{2g}\mathrm{tr}\ M^2}\, \mathrm{Tr}_{\mathcal{R}} e^M}{\int_{\mathfrak{g}} dM e^{-\frac{1}{2g}\mathrm{tr}\ M^2}}$$

where the integrals are over the Lie algebra $\mathfrak{g}$ and $g = \lambda/4N$. These integrals can be reduced to integrals over the Cartan subalgebra $\mathfrak{h}$ (see [7] for details), and one arrives at

$$\langle W \rangle_{\mathcal{R}} = \frac{1}{\dim \mathcal{R}} \frac{\int_{\mathfrak{h}} dX \Delta(X)^2 e^{-\frac{1}{2g}\mathrm{tr}\ X^2}\ \mathrm{Tr}_{\mathcal{R}} e^X}{\int_{\mathfrak{h}} dX \Delta(X)^2 e^{-\frac{1}{2g}\mathrm{tr}\ X^2}} \tag{2.2}$$

where the Jacobian $\Delta(X)^2$ is given by a product over positive roots of the algebra,

$$\Delta(X)^2 = \prod_{\alpha>0} \alpha(X)^2 \tag{2.3}$$

As in [7], it is convenient to write the insertion of the Wilson loop as a sum over the weights of the representation,

$$\mathrm{Tr}_{\mathcal{R}} e^X = \sum_{v\in\Omega(\mathcal{R})} n(v) e^{v(x)} \tag{2.4}$$

where $\Omega(\mathcal{R})$ is the set of weights v of the representation $\mathcal{R}$, and $n(v)$ the multiplicity of the weight. Now that we have introduced the matrix integrals that we want to compute let's very briefly recall the technique we will use to solve them, the method of orthogonal polynomials. Given a potential $W(x)$, we can define a family of orthogonal polynomials $p_n(x)$ satisfying

$$\int_{\infty}^{\infty} dx\ p_m(x) p_n(x) e^{-\frac{1}{g}W(x)} = h_n \delta_{mn}$$

We will choose these polynomials to be monic, namely $p_n(x) = x^n + \mathcal{O}(x^{n-1})$. More precisely, in all the cases in this work, the potential is $W(x) = \frac{1}{2}x^2$, and the orthogonal polynomials are Hermite polynomials,

$$p_n(x) = \left(\frac{g}{2}\right)^{\frac{n}{2}} H_n\left(\frac{x}{\sqrt{2g}}\right) \tag{2.5}$$

so in our conventions

$$h_n = g^n \sqrt{2\pi g}\ n!$$

For future reference, recall that these polynomials have well-defined parity, $p_n(-x) = (-1)^n p_n(x)$. The key point is that in all cases we will encounter in this work, the Jacobian $\Delta(X)^2$ in (2.3) can be substituted by the square of a determinant of orthogonal polynomials. Once we perform this substitution, we expand the determinants using Leibniz formula and carry out the resulting integrals. Note also that the determinant of orthogonal polynomials combined with the Gaussian exponent is (up to a normalization factor) the Slater determinant that gives the wave-function of an N-fermion state,

$$|\Psi_N(x_1, \ldots, x_N)\rangle = C|H_i(x_j) e^{-\frac{1}{4g}x_j^2}|$$

so in all cases the computations we perform can be thought of as normalized matrix elements for certain N-fermion states

$$\langle \mathcal{O} \rangle_{mm} = \frac{\langle \Psi_N | \mathcal{O} | \Psi_N \rangle}{\langle \Psi_N | \Psi_N \rangle} \tag{2.6}$$

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where the specific $|\Psi_N\rangle$ depends on the algebra $\mathfrak{g}$. For $G = \mathrm{SO}(N), \mathrm{Sp}(N)$, these Slater determinants involving one-fermion wavefunctions of definite parity also appear in the study of certain local operators [16, 17].

Having reviewed all the ingredients we now turn to some explicit computations. We use some very basic facts of classical Lie algebras, that we have collected in appendix A.

2.1 $\mathfrak{su}(\mathfrak{n})$

This case is the best studied one, corresponding to the familiar Hermitian matrix model. It is customary to work with $\mathrm{U}(N)$, and we will do so in what follows; the modification needed when dealing with $\mathrm{SU}(N)$ is mentioned below. While none of the results recalled here are new, having them handy will be helpful in what follows. In this case, the Jacobian (2.3) is

$$\prod_{\alpha>0} \alpha(X)^2 = \prod_{1\leq i<j\leq N} |x_i - x_j|^2$$

This Vandermonde determinant can be traded by a determinant of polynomials, which due to the Gaussian potential is convenient to choose to be the first N Hermite polynomials (2.5),

$$\prod_{1\leq i<j\leq N} |x_i - x_j| = |p_{i-1}(x_j)| \tag{2.7}$$

The partition function can be computed using (2.7)

$$\begin{aligned}\mathcal{Z} &= \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N \prod_{1\leq i<j\leq N} |x_i - x_j|^2 \, e^{-\frac{1}{2g}(x_1^2+\dots+x_N^2)} = \\ &= \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N \, |p_{i-1}(x_j)|^2 \, e^{-\frac{1}{2g}(x_1^2+\dots+x_N^2)} = N! \prod_{i=0}^{N-1} h_i\end{aligned} \tag{2.8}$$

In the last step we used the following integral of Hermite polynomials [34], that we will apply repeatedly in this work,

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-(x-y)^2} dx = 2^n \sqrt{\pi} m! \, y^{n-m} L_m^{n-m}(-2y^2) \qquad n \geq m \tag{2.9}$$

where $L_n^\alpha(x)$ are generalized Laguerre polynomials.

Let's recall briefly the computation of Wilson loops. Consider first the Wilson loop in the fundamental representation.[4] The new integral to compute is

$$\begin{aligned}&\int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N \prod_{1\leq i<j\leq N} |x_i - x_j|^2 \; (e^{x_1} + \dots + e^{x_N}) \, e^{-\frac{1}{2g}(x_1^2+\dots+x_N^2)} = \\ &= N \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N |p_{i-1}(x_j)|^2 \; e^{x_1} e^{-\frac{1}{2g}(x_1^2+\dots+x_N^2)}\end{aligned}$$

[4]A Lie algebra of rank r has r fundamental weights, which are the highest weights of the r fundamental representations. In Physics 'fundamental representation' often refers to the representation with highest weight w_1.

where we already used (2.7). Now applying (2.9) and recalling (2.8) we arrive at [6]

$$\langle W(g)\rangle_{\mathrm{U}(N)} = \frac{1}{N}\sum_{k=0}^{N-1} L_k(-g)e^{\frac{g}{2}} = \frac{1}{N}L^1_{N-1}(-g)e^{\frac{g}{2}} \tag{2.10}$$

The remaining U(N) fundamental representations are the k-antisymmetric representation. The exact vevs of the corresponding Wilson loops were computed in [8]. In order to evaluate vevs of Wilson loops for SU(N), we have to modify the insertion to [6, 35]

$$\mathrm{Tr}_{\mathcal{R}} e^X \to e^{-\frac{|\mathcal{R}|}{N}TrX}\ \mathrm{Tr}_{\mathcal{R}} e^X$$

2.2 $\mathfrak{so}(2\mathfrak{n})$

The Jacobian $\Delta(X)^2$ for these algebras is

$$\prod_{\alpha>0}\alpha(X)^2 = \prod_{1\leq i<j\leq N}|x_i^2 - x_j^2|^2$$

The key argument to evaluate all the integrals we will encounter in this case rests on two facts: first, the expression above for $\Delta^2(X)$ is a Vandermonde determinant of $\{x_i^2\}$ and second, even polynomials $p_{2i}(x)$ involve only even powers of x, so it is possible to replace

$$\prod_{1\leq i<j\leq N}|x_i^2 - x_j^2|^2 = |p_{2(i-1)}(x_j)|^2 \tag{2.11}$$

It is worth pointing out that while for $\mathfrak{g}=\mathfrak{su}(\mathfrak{n})$, the Hermite polynomials that appear in eq. (2.7) correspond to the first N eigenstates of the harmonic oscillator, for $\mathfrak{so}(2\mathfrak{n})$ what appears in (2.11) are the first N even eigenstates, so only those will contribute to the computation of the partition function and the vev of Wilson loops. Let's start by evaluating the partition function of the corresponding matrix model,

$$\mathcal{Z} = \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N \prod_{1\leq i<j\leq N}|x_i^2 - x_j^2|^2 e^{-\frac{1}{2g}(x_1^2+\cdots+x_N^2)}$$

Performing the substitution (2.11), we arrive at

$$\mathcal{Z} = N!\prod_{i=0}^{N-1} h_{2i} \tag{2.12}$$

Let's now compute the vev of Wilson loops in various fundamental representations. As a first example, let's choose the representation with highest weight w_1. The $2N$ weights of this representation are e_i and $-e_i$ for $i=1,\dots,N$. After diagonalization, the matrix model that computes the vev of the Wilson loop is

$$\langle W(g)\rangle_{\mathrm{SO}(2N)} = \frac{1}{\mathcal{Z}}\int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N \prod_{1\leq i<j\leq N}|x_i^2 - x_j^2|^2 \frac{e^{x_1}+e^{-x_1}}{2} e^{-\frac{1}{2g}(x_1^2+\cdots+x_N^2)}$$

Performing the substitution (2.11), taking into account (2.12) and using (2.9) we arrive at

$$\langle W(g)\rangle_{\mathrm{SO}(2N)} = \frac{1}{N}\sum_{k=0}^{N-1} L_{2k}(-g)e^{g/2} \tag{2.13}$$

Let's now compute the vev of a Wilson loop in a spinor representation.[5] The spinor representation with highest weight w_{N-1} has weights of the form

$$\frac{1}{2}\left(\pm e_1 \pm e_2 \pm \cdots \pm e_N\right)$$

with an odd number of minus signs, while the representation with highest weight w_N has weights with an even number of minus signs. Let's focus on the representation with highest weight w_N,

$$\langle W\rangle_{w_N} = \frac{1}{\mathcal{Z}}\frac{1}{2^{N-1}}\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_N \prod_{1\le i<j\le N} |x_i^2 - x_j^2|^2 \sum_{\substack{\{s_i=\pm\}\\ \prod_i s_i = 1}} e^{\frac{1}{2}(s_1x_1+\cdots+s_Nx_N)} e^{-\frac{1}{2g}(x_1^2+\cdots+x_N^2)}$$

For each $s_i = -$, we change variables $\tilde{x}_i = -x_i$, and deduce that all 2^{N-1} terms contribute the same to the full integral,

$$\begin{aligned}\langle W\rangle_{w_N} &= \frac{1}{\mathcal{Z}}\int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_N \prod_{1\le i<j\le N} |x_i^2 - x_j^2|^2 e^{\frac{1}{2}(x_1+\cdots+x_N)} e^{-\frac{1}{2g}(x_1^2+\cdots+x_N^2)} = \\ &= \frac{1}{\mathcal{Z}}\int_{0}^{\infty} dx_1 \ldots \int_{0}^{\infty} dx_N \prod_{1\le i<j\le N} |x_i^2 - x_j^2|^2 \prod_{i=1}^{N}\left(e^{\frac{x_i}{2}} + e^{-\frac{x_i}{2}}\right) e^{-\frac{1}{2g}(x_1^2+\cdots+x_N^2)}\end{aligned}$$

Now the remaining integrals can be solved as before. After using the substitution (2.11) the details are quite similar to the computation of the vev of Wilson loops in antisymmetric representations of U(N) [8], so we will skip the details and just present the final result. Define the $N\times N$ matrix D_{ij}, with entries involving generalized Laguerre polynomials $L_n^{\alpha}(x)$,

$$D_{ij} = L_{2i-2}^{2j-2i}(-g/4)e^{g/8}$$

Then, the vev of the Wilson loop in the w_N representation is

$$\langle W\rangle_{w_N} = |D_{ij}|$$

Expanding the determinant, and following identical steps as those presented in [8], we can rewrite this vev as

$$\langle W\rangle_{w_N} = P_N(g)e^{\frac{\lambda}{32}}$$

where $P_N(g)$ is a polynomial in g of degree $N(N-1)/2$ that can be written as a sum involving ordered N-tuples,

$$P_N(g) = \sum_{0\le\tau_1<\tau_2<\ldots\tau_N\le 2N-2}\prod_{m=1}^{N}\frac{\tau_m!}{(2m-2)!}\left|\binom{2i}{\tau_j}\right|^2\left(\frac{g}{4}\right)^{N(N-1)-\sum_{m=1}^{N}\tau_m}$$

[5]In $AdS_5\times\mathbb{RP}^5$, these Wilson loops are dual to a D5-brane wrapping $\mathbb{RP}^4 \subset \mathbb{RP}^5$ [12].

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The other spinor representation, with highest weight w_{N-1}, has weights with an odd numbers of minus signs, but applying the same change of variables $\tilde{x}_i = -x_i$ to all minus signs, we immediately arrive at the same integral as before, so we conclude that both vevs are the same,

$$\langle W \rangle_{w_{N-1}} = \langle W \rangle_{w_N}$$

2.3 $\mathfrak{sp}(\mathfrak{n})$

In this case we have

$$\prod_{\alpha>0} \alpha(X)^2 = \prod_{1\leq i<j\leq N} |x_i^2 - x_j^2|^2 \prod_{i=1}^{N} x_i^2$$

Again, since odd Hermite polynomials involve only odd powers of x, it is possible to substitute the Jacobian by the square of a determinant of orthogonal polynomials

$$\prod_{1\leq i<j\leq N} |x_i^2 - x_j^2|^2 \prod_i x_i^2 = |p_{2i-1}(x_j)|^2 \tag{2.14}$$

where now the polynomials that appear correspond to the first N odd eigenstates of the harmonic oscillator. The partition function can be readily computed

$$\mathcal{Z} = N! \prod_{i=1}^{N} h_{2i-1} \tag{2.15}$$

Let's now turn to the computation of Wilson loops. Let's compute for example the vev of the Wilson loop in the representation with highest weight w_1. The weights are e_i and $-e_i$ for $i = 1, \ldots, N$. After diagonalization, the matrix model that computes the vev of the Wilson loop is

$$\langle W(g) \rangle_{\mathrm{Sp}(N)} = \frac{1}{\mathcal{Z}} \int_{\infty}^{\infty} dx_1 \ldots dx_N \prod_{1\leq i<j\leq N} |x_i^2 - x_j^2|^2 \prod_i x_i^2 \frac{e^{x_1} + e^{-x_1}}{2} e^{-\frac{1}{2g}(x_1^2+\ldots x_N^2)}$$

Using the substitution (2.14), taking into account (2.15) and (2.9), we arrive at

$$\langle W(g) \rangle_{\mathrm{Sp}(N)} = \frac{1}{N} \sum_{k=0}^{N-1} L_{2k+1}(-g) e^{g/2} \tag{2.16}$$

2.4 $\mathfrak{so}(2\mathfrak{n}+1)$

The Jacobian is the same as for $\mathfrak{sp}(\mathfrak{n})$, so it admits the same replacement

$$\prod_{\alpha>0} \alpha(X)^2 = \prod_{1\leq i<j\leq N} |x_i^2 - x_j^2|^2 \prod_{i=1}^{N} x_i^2$$

The partition function is essentially the same as for $\mathfrak{sp}(\mathfrak{n})$, eq. (2.15). Let's compute some vevs of Wilson loops. As a first example, consider the representation with highest weight

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w_1. The weights of this representation are e_i and $-e_i$ for $i = 1, \ldots, N$ plus the zero weight. After diagonalization, the matrix model that computes the vev of the Wilson loop is

$$\langle W(g)\rangle_{\mathrm{SO}(2N+1)} = \frac{1}{2N+1}\frac{1}{\mathcal{Z}}\int_{\infty}^{\infty} dx_1 \ldots dx_N \prod_{1\leq i<j\leq N} |x_i^2 - x_j^2|^2 \prod_i x_i^2$$
$$\left(1 + e^{x_1} + e^{-x_1} + \cdots + e^{x_N} + e^{-x_N}\right) e^{-\frac{1}{2g}(x_1^2+\cdots+x_N^2)}$$

Now, the measure is the same as for $\mathfrak{sp}(\mathfrak{n})$, so the same substitution (2.14) works here, and we arrive at

$$\langle W(g)\rangle_{\mathrm{SO}(2N+1)} = \frac{1}{2N+1}\left(1 + 2\sum_{k=0}^{N-1} L_{2k+1}(-g)e^{g/2}\right)$$

For the spinor representation of $\mathfrak{so}(2\mathfrak{n}+1)$, the computation proceeds along the same lines as for the spinor representations of $\mathfrak{so}(2\mathfrak{n})$. Let's just quote the result; define the $N \times N$ matrix

$$B_{ij} = L_{2i-1}^{2j-2i}(-g/4)e^{g/8}$$

Then

$$\langle W\rangle_{w_N} = |B|$$

3 Implications

In the last section we have computed the exact vev of circular Wilson loops of $\mathcal{N} = 4$ SYM, for various representations of different gauge groups. In what follows, we are going to discuss some features and implications of the results we have obtained. Our main interest is trying to derive lessons for the holographic duals of these gauge theories.

The string dual of $\mathcal{N} = 4$ SYM with gauge group $\mathrm{SU}(N)$ is of course type IIB string theory on $AdS_5 \times S^5$. For $\mathcal{N} = 4$ with gauge groups $\mathrm{SO}(N), \mathrm{Sp}(N)$ one can argue for the string duals as follows [12]. Start by placing N parallel D3-branes at an orientifold three-plane. Taking the near horizon limit, the theory on the world-volume of the D3-branes becomes $\mathcal{N} = 4$ SYM with gauge group $\mathrm{SO}(N), \mathrm{Sp}(N)$ while the supergravity solution becomes $AdS_5 \times \mathbb{RP}^5$ (Recall that $\mathbb{RP}^5$ is $S^5/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting as $x_i \sim -x_i$). This orientifold is common to all the holographic duals for $\mathrm{SO}(2N), \mathrm{SO}(2N+1), \mathrm{Sp}(N)$. The additional ingredients that discriminate among these duals are the possible choices of discrete torsion. Let's recall very briefly the identification of these supergravity duals, referring the interested reader to [12] for the detailed derivation. In the presence of the orientifold, the B-fields B_{NS} and B_{RR} become twisted two-forms. The possible choices of discrete torsion for each of them are classified by $H^3(\mathbb{RP}^5, \tilde{\mathbb{Z}}) = \mathbb{Z}_2$, so calling θ_{NS} and θ_{RR} these two choices, there are all in all four possibilities. Using the transformation properties of $\mathcal{N} = 4$ SYM with different gauge groups under Montonen-Olive duality, it is possible to identify the choices of discrete torsion for the respective gravity duals. The choices $(\theta_{NS}, \theta_{RR}) = (0,0), (0,1/2), (1/2,0), (1/2,1/2)$ correspond to the gauge groups $\mathrm{SO}(2N), \mathrm{SO}(2N+1), \mathrm{Sp}(N), \mathrm{Sp}(N)$ respectively.[6]

[6]These last two $\mathrm{Sp}(N)$ theories differ by their value of the θ angle.

3.1 The LLM sector

The first aspect of the holographic duality that we are going to consider is the analogue of the LLM geometries [27] in $AdS_5 \times \mathbb{RP}^5$. Let's recall briefly that LLM [27] constructed an infinite family of ten dimensional IIB supergravity solutions, corresponding to the backreaction of 1/2 BPS states associated to chiral primary operators built out of a single chiral scalar field. These ten dimensional solutions are completely determined by a single function $u(x_1, x_2)$ of two spacetime coordinates. For regular solutions, this function can take only the values $u(x_1, x_2) = 0, 1$ defining a "black-and-white" pattern on the x_1, x_2 plane.[7] On the field theory side, the dynamics of this sector of operators of $\mathcal{N} = 4$ SU(N) SYM is controlled by the matrix quantum mechanics of N fermions on a harmonic potential [36, 37]. The one-fermion phase space (q, p) gets identified with the (x_1, x_2) plane displaying the "black-and-white" pattern. In particular, the ground state of the system is given by filling the first N states of the harmonic oscillator; in the one-fermion phase space, this corresponds to a circular droplet, which in turn is the pattern giving rise to the $AdS_5 \times S^5$ solution in supergravity. The fermion picture can be inferred directly from the supergravity solutions [38–40].

This is the LLM sector of the duality between type IIB on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SU(N) SYM. What is the similar sector for $\mathcal{N} = 4$ SYM with $G = \text{SO}(N), \text{Sp}(N)$? We are going to propose an answer motivated by the fact that the groundstate of the LLM sector for SU(N) is precisely the N-fermion state $|\Psi_N\rangle$ that appears in the matrix model that computes Wilson loops, eq. (2.6). We then propose that for the other classical Lie algebras, it also holds that the corresponding $|\Psi_\mathfrak{g}\rangle$ in eq. (2.6) is the groundstate of the fermionic system dual to the LLM sector. We can imagine starting with the matrix model for U($2N$), so in the ground state the fermions fill up the first 2N energy levels, and then the orientifold projects out either the even or odd parity eigenstates, depending on the gauge group we consider. The LLM sectors are certainly richer than just the groundstate: they are given by a matrix quantum mechanics that allows for excitations. Our complete proposal is that the full LLM sectors are given by *any* N fermion state built from one-fermion eigenstates of fixed parity: even parity for SO($2N$) and odd parity for SO($2N+1$), Sp(N),

$$\psi(-x) = (-1)^s \psi(x) \tag{3.1}$$

where $s = 0, 1$ depending on the gauge group. This picture is especially easy to visualize for SO($2N+1$), Sp(N) since in these cases we are keeping odd-parity eigenstates, which are the eigenstates of an elementary problem in 1d quantum mechanics: the "half harmonic oscillator" where we place an infinite wall at the origin of a harmonic oscillator potential. This identification between the orientifold in $AdS_5 \times \mathbb{RP}^5$ and the projection from the harmonic oscillator to the half harmonic oscillator was pointed out in [28], where it was suggested to hold for any SO(N), Sp(N) group. According to our argument, this identification holds for SO($2N+1$), Sp(N), but it does not for SO($2N$), since in this case the states preserved by the orientifold action are the even parity ones.

We can formalize this identification as follows. In [28] it was argued that the orientifold projection acts in the (x_1, x_2) plane of LLM geometries as $(x_1, x_2) \sim (-x_1, -x_2)$. Since the

[7] This function $u(x_1, x_2)$ is related to the function $z(x_1, x_2)$ of the original paper [27] by $u = 1/2 - z$.

(x_1, x_2) plane is identified with the one-fermion phase space, this identification amounts to implementing a parity projection in phase space. To do so, one can define [41] the following parity operator in phase space

$$\Pi_{q,p} = \int_{-\infty}^{\infty} ds\, e^{-2ips/\hbar}\, |q-s\rangle \langle q+s| \tag{3.2}$$

and the projectors

$$P^{\pm}_{q,p} = \frac{1}{2}\left(1 \pm \Pi_{q,p}\right)$$

In particular, $\Pi_{(0,0)}$ is the parity operator about the origin of phase space: it changes $\psi(q)$ into $\psi(-q)$ and $\hat{\psi}(p)$ into $\hat{\psi}(-p)$, so the similarity with the orientifold action is apparent. The projectors $P^{\pm}_{0,0}$ project on the space of wavefunctions symmetric or antisymmetric about the origin, and the orientifold projection amounts to keeping one of these subspaces.

Going forward with the argument, we note that $s = 0, 1$ in eq. (3.1), depending on the absence or presence of discrete torsion. We want to provide a new perspective on this discrete torsion, from the phase space point of view. We start by recalling that the function $u(x_1, x_2)$ is identified with the phase space density $u(p, q)$ of one of the fermions in the system of N fermions in a harmonic potential. To go beyond a purely classical description, one can consider a number of phase space quasi-distributions that replace the phase space density, as has been discussed in the LLM context in [42, 43]. One particular such distribution is the Wigner distribution, defined as the Wigner transform of the density matrix,

$$\mathcal{W}(p,q) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} dy\, e^{2ipy/\hbar} \langle q-y|\hat{\rho}|q+y\rangle$$

A salient feature of Wigner quasi-distributions is that they are not positive definite functions over phase space. For instance, if we consider a given eigenstate $|n\rangle$ of the harmonic oscillator, the corresponding Wigner distribution is given again by a Laguerre function [42, 43][8]

$$\mathcal{W}_n(p,q) = \frac{(-1)^n}{\pi\hbar} L_n\left(2\frac{q^2+p^2}{\hbar}\right) e^{-\frac{q^2+p^2}{\hbar}}$$

In particular, for the eigenstate $|n\rangle$, at the origin of phase space we have

$$\mathcal{W}_n(0,0) = (-1)^n \frac{1}{\pi\hbar}$$

so it can have either sign. More generally, the Wigner quasi-distribution is the expectation value of the parity operator defined in (3.2) [41]

$$\mathcal{W}(p,q) = \frac{1}{\pi\hbar} \langle \Pi_{p,q} \rangle$$

and in particular

$$\mathcal{W}(0,0) = \frac{1}{\pi\hbar} \langle \Pi_{0,0} \rangle$$

[8]At this time, we regard the fact that Laguerre functions appear both in the vevs of circular Wilson loops and in Wigner distributions as merely fortuitous. In particular, note that the vevs of Wilson loops have negative argument, while for Wigner distributions the argument is positive.

so it is clear that the sign of $\mathcal{W}(0,0)$ captures the parity properties of the wavefunction with respect to the origin of phase space.[9] For a generic N fermion state with eigenstates $\{j_1,\ldots,j_N\}$, the Wigner function is [42, 43],

$$\mathcal{W}(p,q)=\frac{1}{\pi\hbar}e^{-(q^2+p^2)/\hbar}\sum_{\{j_i\}}(-1)^{j_i}L_{j_i}\left(\frac{2}{\hbar}(q^2+p^2)\right)$$

For $G=\mathrm{SO}(N),\mathrm{Sp}(N)$, the sign $(-1)^{j_i}$ is the same for all states, to it comes out of the sum. In particular, for any N fermion state, at the origin of phase space we get

$$(-1)^s=\text{sign }\mathcal{W}(0,0)$$

3.2 Features of the non-orientable terms

In the previous section we have computed the vevs of circular Wilson loops for various gauge groups and representations. We now want to present some exact relations among these vevs, as well as their large N expansion, which in principle ought to be reproduced by string theory computations on $AdS_5\times\mathbb{RP}^5$. Before we take a detailed look at the results we have obtained, let's recall briefly some general expectations. In the large N expansion, Feynman diagrams rearrange themselves in a topological expansion in terms of two-dimensional surfaces. Each surface is weighted by N^χ, with χ the Euler characteristic of the surface; for a surface with h handles, b boundaries and c crosscaps, the Euler characteristic is

$$\chi=-2h+2-c-b \tag{3.3}$$

As a consequence of the classification theorem for closed surfaces, a general non-orientable surface can be thought of as an orientable surface with a number of crosscaps. Furthermore, according to Dycks' theorem, three crosscaps can be traded for a handle and a single crosscap, so we expect three kinds of contributions, coming from world-sheets with an arbitrary number of handles and with zero (i.e. orientable), one or two crosscaps.

For a $\mathrm{U}(N),\mathrm{SU}(N)$ theory with all fields in the adjoint representation, the large N expansion of any observable is actually a $1/N^2$ expansion (without odd powers of $1/N$) as it befits an expansion in orientable surfaces. For the vev of a circular Wilson loop of $\mathrm{U}(N)$ in the fundamental representation, this $1/N^2$ expansion of the exact result was already carried out in [6].[10] On the other hand, when $G=\mathrm{SO}(N),\mathrm{Sp}(N)$, the adjoint representation can be thought of as the product of two fundamental representations (rather than a fundamental times an antifundamental representation as in $\mathrm{U}(N)$), so propagators can still be represented by a double line notation, but now without any arrows in the lines [29]. As a result, the large N expansion of observables for $\mathrm{SO}(N),\mathrm{Sp}(N)$ theories

[9] Incidentally, negative values of the Wigner function at the origin of phase space have apparently been measured experimentally for single photon fields [44].

[10] The surfaces that appear in the $1/N$ expansion of $\langle W\rangle_{\mathrm{SU}(N)}$ have a single boundary and an arbitrary number of handles, so they all have odd Euler characteristic, eq. (3.3). However, in the normalization for $\langle W\rangle_{\mathrm{SU}(N)}$ followed in [6] and in the present work, there is an additional overall $1/N$, so the expansion ends up being in even powers of N. At any rate, what is relevant is that the expansion parameter is $1/N^2$ and not $1/N$.

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— even when all fields transform in the adjoint representation — involves both even and odd powers of $1/N$, signaling the appearance of non-orientable surfaces [29]. Furthermore, gauge invariant quantities for $\mathrm{Sp}(N)$ are related to those of $\mathrm{SO}(2N)$ by the replacement $N \to -N$ [45, 46]. Finally, we know that $\mathrm{SO}(2N)$ and $\mathrm{Sp}(N)$ theories can be obtained from orientifolding $\mathrm{U}(2N)$. All in all, these general arguments imply that vevs in the respective fundamental representations of various groups ought to be related by[11]

$$\langle W\rangle_{\substack{\mathrm{SO}(2N)\\ \mathrm{Sp}(N)}} = \langle W\rangle_{\mathrm{U}(2N)} \pm \text{unoriented}_{\,c=1} + \text{unoriented}_{\,c=2} \tag{3.4}$$

where *unoriented* refers to terms that in the large N limit arrange themselves into non-orientable surfaces with either one or two crosscaps. In the formula above, we have already imposed the relation $\mathrm{Sp}(N) = \mathrm{SO}(-2N)$, which implies that world-sheets with a single cross-cap contribute the same for *SO* and *Sp* up to a sign, while world-sheets with two cross-caps give the same contribution for the two groups.

We are now going to show that indeed our exact results (2.13) and (2.16) follow the pattern expressed in (3.4). In the process, we will furthermore find a couple of features that do not follow from these general arguments.

To obtain the $1/N$ expansion of $\langle W\rangle_{\mathrm{SO}(2N)}$ and $\langle W\rangle_{\mathrm{Sp}(N)}$, we can analyze them separately, following the steps of [6], as we do in the appendix. However, it is much more efficient to consider their sum and their difference, and expand those. Let's start considering the sum. Recalling eq. (2.10), it is immediate that the results we have found, eqs. (2.13) and (2.16) satisfy

$$\langle W(g)\rangle_{\mathrm{SO}(2N)} + \langle W(g)\rangle_{\mathrm{Sp}(N)} = 2\langle W(g)\rangle_{\mathrm{U}(2N)} \tag{3.5}$$

As for the difference $\langle W(g)\rangle_{\mathrm{Sp}(N)} - \langle W(g)\rangle_{\mathrm{SO}(2N)}$, using properties of the Laguerre polynomials, it is not difficult to prove from the explicit results eqs. (2.13) and (2.16) that the following exact relation holds

$$\frac{\partial}{\partial\lambda}\Big(\langle W(g)\rangle_{\mathrm{Sp}(N)} - \langle W(g)\rangle_{\mathrm{SO}(2N)}\Big) = \frac{1}{4N}\langle W(g)\rangle_{\mathrm{U}(2N)} \tag{3.6}$$

These last two relations, eqs. (3.5) and (3.6), can we rewritten in the following suggestive form

$$\langle W(g)\rangle_{\substack{\mathrm{SO}(2N)\\ \mathrm{Sp}(N)}} = \langle W(g)\rangle_{\mathrm{U}(2N)} \mp \frac{1}{2}\int_0^g dg'\,\langle W(g')\rangle_{\mathrm{U}(2N)} \tag{3.7}$$

Recall that $\langle W(g)\rangle_{\mathrm{U}(2N)}$ has a expansion in $1/N^2$. Furthermore, since $g = \lambda/4N$, the integral brings an extra power of $1/N$. Therefore, equation (3.7) neatly splits the $1/N$ expansions of $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ and $\langle W(g)\rangle_{\mathrm{Sp}(N)}$ into even and odd powers of $1/N$. The $1/N^{2k}$ terms coincide for both vevs and are given $\langle W(g)\rangle_{\mathrm{U}(2N)}$; they correspond to orientable surfaces. Note in particular that since all even powers of $1/N$ come from orientable surfaces, there are no contributions from world-sheets with two crosscaps, as it can be already deduced from eqs. (3.4) and (3.5).

[11] Recall that we are normalizing all vevs such that $\langle W\rangle = 1 + \mathcal{O}(g)$. In other normalizations of the vevs of Wilson loops, this equation might involve a different numerical coefficient in front of $\langle W\rangle_{\mathrm{U}(2N)}$.

Turning now to the $1/N^{2k+1}$ terms in the expansion of $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ and $\langle W(g)\rangle_{\mathrm{Sp}(N)}$, they come from the integral in eq. (3.7), so it is manifest that they differ just by a sign; this, together with the equality of the even terms in the expansions, proves that indeed $\langle W(g)\rangle_{\mathrm{Sp}(N)}$ can be obtained from $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ by the substitution $N \to -N$, as it had to happen according to general arguments [45, 46].

To recapitulate, the $1/N$ expansion of $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ and $\langle W(g)\rangle_{\mathrm{Sp}(N)}$ could in principle involve contributions from three kinds of surfaces, with zero, one or two crosscaps. By a mix of generic arguments and exact field theory computations, we have found that for these quantities, and for any number of handles, contributions from surfaces with one crosscap are given by an integral of the contribution from surfaces without crosscaps, while there is no contribution from surfaces with two crosscaps, eq. (3.7).

The two features that we have just uncovered for the $1/N$ expansion of $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ and $\langle W(g)\rangle_{\mathrm{Sp}(N)}$ bear certain resemblance with properties encountered in other instances of $1/N$ expansion of SO/Sp gauge theories. A first example is the computation of the effective glueball superpotential of $\mathcal{N}=1$ SYM theories with a scalar field in the adjoint, with an arbitrary tree-level polynomial superpotential, $\mathcal{W}(\Phi)$. Dijkgraaf and Vafa [47] pointed out that for $G=\mathrm{U}(N)$, this computation reduces to an evaluation of the planar free energy of a one-matrix model with the matrix model potential given by the tree-level superpotential of the gauge theory. For $\mathcal{N}=1$ SYM with gauge groups $\mathrm{SO}(N), \mathrm{Sp}(N)$ the corresponding matrix models are, like in the present work, valued on the Lie algebras [48–50]. It was found in [48–50] that the effective superpotential of the $\mathcal{N}=1$ SYM gauge theory is fully captured by the contributions from S^2 and $\mathbb{RP}^2$, so there is no contribution from the world-sheet with two crosscaps (Klein bottle); furthermore, the contribution to the free energy coming from $\mathbb{RP}^2$ is given by a derivative of the contribution from S^2,

$$\mathcal{F}_1 = \pm \frac{g_s}{4}\frac{\partial \mathcal{F}_0}{\partial S_0}$$

with S_0 (half) the 't Hooft coupling. Notice however that in this example the properties are only established for world-sheets without any handles or boundaries, while our arguments work for world-sheets with a single boundary and an arbitrary number of handles. A second example comes from the large N expansion of Chern-Simons theory on 3-manifolds. It was observed in [51] that the $1/N$ expansion of the free energy of Chern-Simons on S^3 with gauge groups $\mathrm{SO}(N), \mathrm{Sp}(N)$ involves unoriented world-sheets with one cross-cap, but again world-sheets with two cross-caps are absent in this expansion. Moreover, the large N expansion of Chern-Simons with $G=\mathrm{SO}(N), \mathrm{Sp}(N)$, via its connection with knot theory, displays non-trivial relations for the invariants of $\mathrm{U}(N)$ and $\mathrm{SO}(N), \mathrm{Sp}(N)$ links [52].

While it is interesting that the two features we have uncovered in the $1/N$ expansion of $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ and $\langle W(g)\rangle_{\mathrm{Sp}(N)}$ have superficially similar incarnations in other gauge theories with gauge groups $\mathrm{SO}(N), \mathrm{Sp}(N)$, we don't expect these two features to be generic for other observables of $\mathcal{N}=4$ SYM with $G=\mathrm{SO}(N), \mathrm{Sp}(N)$. For instance, in the case we have studied, the absence of contributions coming from world-sheets with two crosscaps is a consequence of the exact relation (3.5), but this relation appears to be quite specific of vevs of Wilson loops in the respective fundamental representations, and we don't know of similar

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relations for vevs of Wilson loops in other representations. Not surprisingly, in Chern-Simons theory with $G = \mathrm{SO}(N), \mathrm{Sp}(N)$, vevs of Wilson loops in higher representations do get contributions from world-sheets with two crosscaps [53, 54].

Turning now to string theory, reproducing the actual $1/N$ expansion of $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ or $\langle W(g)\rangle_{\mathrm{Sp}(N)}$ from world-sheet computations is as out of reach as for $\langle W(g)\rangle_{\mathrm{U}(2N)}$. On the other hand, granting the AdS/CFT duality for any value of g_s and α'/L^2, our results are also exact results in string theory, even beyond the perturbative regime. It is tantalizing to suspect that the results we have found — e.g. the absence of contributions from world-sheets with two crosscaps and any number of handles — are in the string theory language consequences of some symmetry enjoyed by the particular quantities we are considering. Identifying this symmetry and the stringy argument beyond the relations we have found appears to be a more promising and illuminating task than attempting to reproduce them by carrying out the explicit world-sheet computations.

Everything we have said so far follows from the exact results we have computed, and the exact relations among them. We didn't even have to carry out the explicit $1/N$ expansion of the exact results to arrive at these conclusions. Nevertheless, it is still worth to obtain this $1/N$ expansion explicitly, and this task can be accomplished with very little effort, by combining the exact relation (3.6) with the results in [6]. Drukker and Gross [6] obtained the following $1/N$ expansion of $\langle W\rangle_{\mathrm{U}(N)}$, that we write for $\mathrm{U}(2N)$,

$$\langle W\rangle_{\mathrm{U}(2N)} = \frac{2}{\sqrt{2\lambda}} I_1(\sqrt{2\lambda}) + \sum_{k=1}^{\infty} \frac{1}{N^{2k}} \sum_{i=0}^{k-1} X_k^i \left(\frac{\lambda}{2}\right)^{\frac{3k-i-1}{2}} I_{3k-i-1}(\sqrt{2\lambda})$$

where $I_\alpha(x)$ are modified Bessel functions of the first kind, and X_k^i are coefficients satisfying the recursion relation

$$4(3k-i)X_k^i = X_{k-1}^i + (3k-i-2)X_{k-1}^{i-1} \tag{3.8}$$

with initial values $X_1^0 = 1/12$ and $X_k^k = 0$. A trivial integration then yields

$$\langle W\rangle_{\substack{\mathrm{SO}(2N)\\ \mathrm{Sp}(N)}} = \langle W\rangle_{\mathrm{U}(2N)} \mp \frac{1}{4N}\left[\left(I_0(\sqrt{2\lambda}) - 1\right) + \sum_{k=1}^{\infty} \frac{1}{N^{2k}} \sum_{i=0}^{k-1} X_k^i \left(\frac{\lambda}{2}\right)^{\frac{3k-i}{2}} I_{3k-i}(\sqrt{2\lambda})\right]$$

This result is valid for any λ. We can then use it to obtain a large λ expansion at every order in $1/N$

$$\langle W\rangle_{\mathrm{SO}(2N)} - \langle W\rangle_{\mathrm{Sp}(N)} = \sum_k \frac{1}{(2N)^{2k+1}} \frac{e^{\sqrt{2\lambda}}(2\lambda)^{\frac{6k-1}{4}}}{96^k k! \sqrt{2\pi}} \left(1 - \frac{36k^2 + 144k - 5}{40\sqrt{2\lambda}} + \dots\right)$$

Perhaps the most important feature of this result is that the exponent $(6k-3)/4$ obtained in [6] is now replaced by $(6k-1)/4$.

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A Classical simple Lie algebras

In this appendix we collect some very basic facts about classical simple Lie algebras that we use in the main text. A Lie algebra of rank r has r simple roots. For each simple root in the Lie algebra there is a fundamental weight, which is the highest weight of a fundamental representation. A simple Lie algebra has then r fundamental representations. In Physics, the name "fundamental representation" if often reserved for the fundamental representation with highest weight w_1.

$\mathfrak{su}(\mathfrak{n})$. The Lie algebra $\mathfrak{su}(\mathfrak{n})$ has rank $r = n-1$. We introduce the basis $e_i, i = 1, \ldots, n$. The positive roots and the simple roots are

$$\begin{aligned} R_+ &= \{e_i - e_j, i < j\} \\ \Pi &= \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n\} \end{aligned}$$

The $n-1$ fundamental weights of $\mathfrak{su}(\mathfrak{n})$ are

$$w_k = e_1 + e_2 + \cdots + e_k - \frac{k}{n}(e_1 + \cdots + e_n), \qquad k = 1, \ldots, n-1 \tag{A.1}$$

Applying the Weyl dimension formula, the dimensions of the associated fundamental representations are $\binom{n}{k}$, so these are the antisymmetric representations.

$\mathfrak{so}(2\mathfrak{n}+1)$. The Lie algebra $\mathfrak{so}(2\mathfrak{n}+1)$ has rank $r = n$. We introduce the basis $e_i, i = 1, \ldots, n$. The positive roots and the simple roots are

$$\begin{aligned} R_+ &= \{e_i \pm e_j \ (i < j), e_i\} \\ \Pi &= \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n\} \end{aligned}$$

The fundamental weights are

$$\begin{aligned} w_1 &= e_1, \ldots, w_{n-2} = e_1 + \cdots + e_{n-2}, w_{n-1} = e_1 + \cdots + e_{n-1}, \\ w_n &= \frac{1}{2}(e_1 + \cdots + e_n) \end{aligned}$$

The first $n-1$ representations have dimensions $\binom{2n+1}{k}$. The last one is a spinor representation of dimension 2^n.

$\mathfrak{sp}(\mathfrak{n})$. The Lie algebra $\mathfrak{sp}(\mathfrak{n})$ has rank $r = n$. We introduce the basis $e_i, i = 1, \ldots, n$. The positive roots and the simple roots are

$$R_+ = \{e_i \pm e_j\ ,\ i < j\ ; 2e_i\}\Pi \qquad = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = 2e_n\}$$

The corresponding fundamental weights are

$$w_1 = e_1, w_2 = e_1 + e_2, \ldots, w_n = e_1 + \cdots + e_n$$

There are no spinor representations for $\mathfrak{sp}(\mathfrak{n})$.

$\mathfrak{so}(2\mathfrak{n})$. The Lie algebra $\mathfrak{so}(2\mathfrak{n})$ has rank $r = n$. We introduce the basis $e_i, i = 1, \ldots, n$. The positive roots and the simple roots are

$$R_+ = \{e_i \pm e_j \;,\; i < j\}$$
$$\Pi = \{\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_{n-1} + e_n\}$$

The corresponding fundamental weights are

$$w_1 = e_1 \;, w_2 = e_1 + e_2 \;, \ldots, \qquad w_{n-2} = e_1 + \cdots + e_{n-2},$$
$$w_{n-1} = \frac{1}{2}\left(e_1 + \cdots + e_{n-1} - e_n\right), \qquad w_n = \frac{1}{2}\left(e_1 + \cdots + e_{n-1} + e_n\right)$$

The first $n-2$ fundamental representations have dimensions $\binom{2n}{k}$. The last two fundamental weights correspond to spinor representations, both with dimension 2^{n-1}.

B 1/N expansion of $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ and $\langle W(g)\rangle_{\mathrm{Sp}(N)}$

In this appendix we will derive the $1/N$ expansion of $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ and $\langle W(g)\rangle_{\mathrm{Sp}(N)}$ without making use of the exact relations among them found in the main text. We will eventually find out that the expansions involve certain coefficients that satisfy the same recursion relation as the ones that appear in $\langle W(g)\rangle_{\mathrm{U}(2N)}$, eq. (3.8).

To expand $\langle W(g)\rangle_{\mathrm{SO}(2N)}$ given in eq. (2.13) in $1/N$, we will first rewrite

$$\sum_{k=0}^{N-1} L_{2k}(-g) = \sum_{k=0}^{2N-2} d_k \frac{g^k}{k!}$$

with

$$d_k \equiv \sum_{i=0}^{N-1} \binom{2i}{k}$$

These coefficients satisfy the recursion relation

$$d_k + 2d_{k+1} = \binom{2N}{k+2}$$

and with $d_0 = N$ we can now write

$$\langle W(g)\rangle_{\mathrm{SO}(2N)} = \frac{1}{N} \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^n \frac{1}{n!(n+1)!} D(n,N)$$

with

$$D(n,N) \equiv 2\frac{n!(n+1)!}{(2N)^{n+1}} \sum_{k=0}^{n} \frac{d_k}{2^{n-k}(n-k)!k!}$$

$D(n,N)$ is a polynomial in $1/N$ of degree n. Expanding in $1/N$,

$$D(n,N) = 1 - \frac{n+1}{2}\frac{1}{2N} + \frac{(n+1)n(n-1)}{12}\frac{1}{(2N)^2} + \ldots$$

So

$$\langle W(g)\rangle_{\mathrm{SO}(2N)} = \sqrt{\frac{2}{\lambda}} I_1(\sqrt{2\lambda}) - \frac{1}{4N}\left(I_0(\sqrt{2\lambda}) - 1\right) + \dots$$

To expand $\langle W(g)\rangle_{\mathrm{Sp}(N)}$ given in eq. (2.16) in $1/N$, we will first rewrite

$$\sum_{k=0}^{N-1} L_{2k+1}(-g) = \sum_{k=0}^{2N-1} c_k \frac{g^k}{k!}$$

with

$$c_k \equiv \sum_{i=0}^{N-1} \binom{2i+1}{k}$$

These coefficients satisfy the recursion relation

$$c_k + 2c_{k+1} = \binom{2N+1}{k+2}$$

and with $c_0 = N$ we can now write

$$\langle W(g)\rangle_{\mathrm{Sp}(N)} = \sum_{n=0}^{\infty} \left(\frac{\lambda}{2}\right)^n \frac{1}{n!(n+1)!} C(n,N)$$

with

$$C(n,N) \equiv 2\frac{n!(n+1)!}{(2N)^{n+1}} \sum_{k=0}^{n} \frac{c_k}{2^{n-k}(n-k)!k!}$$

$C(n,N)$ is a polynomial in $1/N$ of degree n. Expanding in $1/N$,

$$C(n,N) = 1 + \frac{n+1}{2}\frac{1}{2N} + \frac{(n+1)n(n-1)}{12}\frac{1}{(2N)^2} + \dots$$

So

$$\langle W(g)\rangle_{\mathrm{Sp}(N)} = \sqrt{\frac{2}{\lambda}} I_1(\sqrt{2\lambda}) + \frac{1}{4N}\left(I_0(\sqrt{2\lambda}) - 1\right) + \dots$$

We know from general arguments that the odd powers in $1/N$ of $C(n,N)$ and $D(n,N)$ differ by a sign. Now we want to argue that the even powers are the same, so as polynomials in $1/N$ we have $D(n,-N) = C(n,N)$. Define

$$\Delta(n,N) \equiv C(n,N) - D(n,N) = 2\frac{n!(n+1)!}{(2N)^{(n+1)}} \sum_{k=1}^{n} \frac{d_{k-1}}{2^{n-k}(n-k)!k!}$$

If we prove that $\Delta(n,N)$ is a polynomial in $1/N$ with only odd powers, it will follow that even powers of C and D coincide. The coefficients Δ satisfy the recursion relation

$$\Delta(n+1,N) = \frac{n+2}{n+1}\Delta(n,N) + \frac{(n-1)(n+2)}{16N^2}\Delta(n-1,N)$$

Together with $\Delta(0,N) = 0, \Delta(1,N) = 1/N$ this proves that indeed $\Delta(n,N)$ are odd in $1/N$, and indeed even powers of C and D coincide.

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To carry out the expansion of $\Delta(n,N)$ we follow closely appendix B of [6]. We define

$$\Delta(n,N)=\sum_k \frac{p_k(n)}{(2N)^{2k+1}}$$

where $p_k(n)$ are polynomials in n of degree $3k+1$. We rewrite them as linear combinations of polynomials $(n+1)!/(n-3k+i)!$ with coefficients Y_k^i,

$$p_k(n)=\sum_{i=0}^{k-1}\frac{(n+1)!}{(n-3k+i)!}Y_k^i$$

Using the recursion relation for $\Delta(n,N)$ we derive the relation

$$4(3k-i)Y_k^i=Y_{k-1}^i+(3k-i-2)Y_{k-1}^{i-1}$$

which is the same recursion relation found in [6] for the coefficients X_k^i, eq. (3.8). The initial values can also be seen to coincide, proving that the unoriented term are related to the oriented ones.

References

[1] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[2] A. Kapustin, B. Willett and I. Yaakov, *Exact results for Wilson loops in superconformal Chern-Simons theories with matter*, *JHEP* **03** (2010) 089 [arXiv:0909.4559] [INSPIRE].

[3] N. Drukker, M. Mariño and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511 [arXiv:1007.3837] [INSPIRE].

[4] D.L. Jafferis, *The exact superconformal R-symmetry extremizes Z*, *JHEP* **05** (2012) 159 [arXiv:1012.3210] [INSPIRE].

[5] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N=4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 582** (2000) 155 [hep-th/0003055] [INSPIRE].

[6] N. Drukker and D.J. Gross, *An exact prediction of $N=4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[7] J. Gomis, T. Okuda and D. Trancanelli, *Quantum 't Hooft operators and S-duality in $N=4$ super Yang-Mills*, *Adv. Theor. Math. Phys.* **13** (2009) 1941 [arXiv:0904.4486] [INSPIRE].

[8] B. Fiol and G. Torrents, *Exact results for Wilson loops in arbitrary representations*, *JHEP* **01** (2014) 020 [arXiv:1311.2058] [INSPIRE].

[9] S.-J. Rey and T. Suyama, *Exact results and holography of Wilson loops in $N=2$ superconformal (quiver) gauge theories*, *JHEP* **01** (2011) 136 [arXiv:1001.0016] [INSPIRE].

[10] F. Passerini and K. Zarembo, *Wilson loops in $N=2$ super-Yang-Mills from matrix model*, *JHEP* **09** (2011) 102 [*Erratum ibid.* **10** (2011) 065] [arXiv:1106.5763] [INSPIRE].

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[11] B. Fraser and S.P. Kumar, *Large rank Wilson loops in $N = 2$ superconformal QCD at strong coupling*, *JHEP* **03** (2012) 077 [arXiv:1112.5182] [INSPIRE].

[12] E. Witten, *Baryons and branes in anti-de Sitter space*, *JHEP* **07** (1998) 006 [hep-th/9805112] [INSPIRE].

[13] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, *JHEP* **08** (2013) 115 [arXiv:1305.0318] [INSPIRE].

[14] O. Aharony and E. Witten, *Anti-de Sitter space and the center of the gauge group*, *JHEP* **11** (1998) 018 [hep-th/9807205] [INSPIRE].

[15] E. Witten, *AdS/CFT correspondence and topological field theory*, *JHEP* **12** (1998) 012 [hep-th/9812012] [INSPIRE].

[16] P. Caputa, R. de Mello Koch and P. Diaz, *Operators, correlators and free fermions for* $\mathrm{SO}(N)$ *and* $\mathrm{Sp}(N)$, *JHEP* **06** (2013) 018 [arXiv:1303.7252] [INSPIRE].

[17] P. Caputa, R. de Mello Koch and P. Diaz, *A basis for large operators in $N = 4$ SYM with orthogonal gauge group*, *JHEP* **03** (2013) 041 [arXiv:1301.1560] [INSPIRE].

[18] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in $N = 4$ super Yang-Mills*, *JHEP* **06** (2012) 048 [arXiv:1202.4455] [INSPIRE].

[19] B. Fiol, B. Garolera and A. Lewkowycz, *Exact results for static and radiative fields of a quark in $N = 4$ super Yang-Mills*, *JHEP* **05** (2012) 093 [arXiv:1202.5292] [INSPIRE].

[20] B. Fiol, B. Garolera and G. Torrents, *Exact momentum fluctuations of an accelerated quark in $N = 4$ super Yang-Mills*, *JHEP* **06** (2013) 011 [arXiv:1302.6991] [INSPIRE].

[21] A. Lewkowycz and J. Maldacena, *Exact results for the entanglement entropy and the energy radiated by a quark*, *JHEP* **05** (2014) 025 [arXiv:1312.5682] [INSPIRE].

[22] R.C. Myers and V. Periwal, *The orientability of random surfaces*, *Phys. Rev.* **D 42** (1990) 3600 [INSPIRE].

[23] A. Bilal, *2D gravity from matrix models: an introductory review, and particularities of antisymmetric matrix models*, in *Proceedings, Nonperturbative methods in low dimensional quantum field theories*, Debrecen Hungary (1990), pg. 113 and CERN-TH-5867-90, CERN, Geneva Switzerland (1990) [INSPIRE].

[24] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, *JHEP* **02** (2005) 010 [hep-th/0501109] [INSPIRE].

[25] S. Giombi, R. Ricci and D. Trancanelli, *Operator product expansion of higher rank Wilson loops from D-branes and matrix models*, *JHEP* **10** (2006) 045 [hep-th/0608077] [INSPIRE].

[26] B. Fiol and B. Garolera, *Energy loss of an infinitely massive half-Bogomol'nyi-Prasad-Sommerfeld particle by radiation to all orders in $1/N$*, *Phys. Rev. Lett.* **107** (2011) 151601 [arXiv:1106.5418] [INSPIRE].

[27] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174] [INSPIRE].

[28] S. Mukhi and M. Smedback, *Bubbling orientifolds*, *JHEP* **08** (2005) 005 [hep-th/0506059] [INSPIRE].

[29] G.M. Cicuta, *Topological expansion for* $\mathrm{SO}(N)$ *and* $\mathrm{Sp}(2N)$ *gauge theories*, *Lett. Nuovo Cim.* **35** (1982) 87 [INSPIRE].

[30] S.G. Naculich, H.A. Riggs and H.J. Schnitzer, *Two-dimensional Yang-Mills theories are string theories*, *Mod. Phys. Lett.* **A 8** (1993) 2223 [hep-th/9305097] [INSPIRE].

[31] S. Ramgoolam, *Comment on two-dimensional* $O(N)$ *and* Sp(N) *Yang-Mills theories as string theories*, *Nucl. Phys.* **B 418** (1994) 30 [hep-th/9307085] [INSPIRE].

[32] P. Di Francesco, P.H. Ginsparg and J. Zinn-Justin, 2D *gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [hep-th/9306153] [INSPIRE].

[33] M. Mariño, *Les Houches lectures on matrix models and topological strings*, hep-th/0410165 [INSPIRE].

[34] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*. Academic Press, U.S.A. (2007).

[35] T. Okuda and D. Trancanelli, *Spectral curves, emergent geometry and bubbling solutions for Wilson loops*, *JHEP* **09** (2008) 050 [arXiv:0806.4191] [INSPIRE].

[36] S. Corley, A. Jevicki and S. Ramgoolam, *Exact correlators of giant gravitons from dual* $N = 4$ *SYM theory*, *Adv. Theor. Math. Phys.* **5** (2002) 809 [hep-th/0111222] [INSPIRE].

[37] D. Berenstein, *A toy model for the AdS/CFT correspondence*, *JHEP* **07** (2004) 018 [hep-th/0403110] [INSPIRE].

[38] G. Mandal, *Fermions from half-BPS supergravity*, *JHEP* **08** (2005) 052 [hep-th/0502104] [INSPIRE].

[39] L. Grant, L. Maoz, J. Marsano, K. Papadodimas and V.S. Rychkov, *Minisuperspace quantization of 'bubbling AdS' and free fermion droplets*, *JHEP* **08** (2005) 025 [hep-th/0505079] [INSPIRE].

[40] L. Maoz and V.S. Rychkov, *Geometry quantization from supergravity: the case of 'bubbling AdS'*, *JHEP* **08** (2005) 096 [hep-th/0508059] [INSPIRE].

[41] A. Royer, *Wigner function as the expectation value of a parity operator*, *Phys. Rev.* **A 15** (1977) 449 [INSPIRE].

[42] Y. Takayama and A. Tsuchiya, *Complex matrix model and fermion phase space for bubbling AdS geometries*, *JHEP* **10** (2005) 004 [hep-th/0507070] [INSPIRE].

[43] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, *The library of Babel: on the origin of gravitational thermodynamics*, *JHEP* **12** (2005) 006 [hep-th/0508023] [INSPIRE].

[44] G. Nogues et al., *Measurement of a negative value for the Wigner function of radiation*, *Phys. Rev.* **A 62** (2000) 054101.

[45] R.L. Mkrtchian, *The equivalence of* Sp($2N$) *and* SO($-2N$) *gauge theories*, *Phys. Lett.* **B 105** (1981) 174 [INSPIRE].

[46] P. Cvitanovic and A.D. Kennedy, *Spinors in negative dimensions*, *Phys. Scripta* **26** (1982) 5 [INSPIRE].

[47] R. Dijkgraaf and C. Vafa, *A perturbative window into nonperturbative physics*, hep-th/0208048 [INSPIRE].

[48] H. Ita, H. Nieder and Y. Oz, *Perturbative computation of glueball superpotentials for* SO(N) *and* USp(N), *JHEP* **01** (2003) 018 [hep-th/0211261] [INSPIRE].

[49] S.K. Ashok, R. Corrado, N. Halmagyi, K.D. Kennaway and C. Romelsberger, *Unoriented strings, loop equations and $N = 1$ superpotentials from matrix models*, *Phys. Rev.* **D 67** (2003) 086004 [hep-th/0211291] [INSPIRE].

[50] R.A. Janik and N.A. Obers, SO(N) *superpotential, Seiberg-Witten curves and loop equations*, *Phys. Lett.* **B 553** (2003) 309 [hep-th/0212069] [INSPIRE].

[51] S. Sinha and C. Vafa, SO *and* Sp *Chern-Simons at large-N*, hep-th/0012136 [INSPIRE].

[52] M. Mariño, *String theory and the Kauffman polynomial*, *Commun. Math. Phys.* **298** (2010) 613 [arXiv:0904.1088] [INSPIRE].

[53] V. Bouchard, B. Florea and M. Mariño, *Counting higher genus curves with crosscaps in Calabi-Yau orientifolds*, *JHEP* **12** (2004) 035 [hep-th/0405083] [INSPIRE].

[54] V. Bouchard, B. Florea and M. Mariño, *Topological open string amplitudes on orientifolds*, *JHEP* **02** (2005) 002 [hep-th/0411227] [INSPIRE].

Chapter 5

Summary and conclusions

The holographic duality between gauge theories and string theories has opened a new door to access the strongly coupled regime of quantum field theories and offers, at the same time, a completely new way to understand the elusive nature of quantum gravity and the non-perturbative regime of string theory.
After almost two decades of research, the current status of the correspondence is that of a solid conjecture that has passed a great number of nontrivial tests, to the point that it is generally believed to be true. However, it is fair to say that we still have to face many and sever limitations, among which I may remark:

- Holography is specially well understood and can be made precise only for the specific case of a few ideal and very symmetric theories. Starting from these simpler settings and by breaking manifestly or spontaneously some of their symmetries, it is indeed possible to find the gravity duals of more realistic theories with reduced symmetry. Nevertheless, in general we don't know how to derive precise dualities for less-symmetric theories. We still don't fully understand how holography works.

- When using the correspondence as a tool for analyzing strongly coupled gauge theories, most of the computations are performed at the leading order and using the supergravity approximation. To capture corrections beyond this approximation is in general a very difficult task and the majority of methods and techniques are specific of a particular kind of problem.

- The gauge/gravity correspondence offers us perhaps the best description that we have for a theory of quantum gravity. However, the bulk of the AdS/CFT literature is carried out within the weakly coupled or classical regime of the gravity dual, focusing on the strongly coupled physics of the dual gauge theory. It seems fair

> to assess that it has not brought as many new results in quantum gravity as in quantum field theory.
> Of course, the main reasons for this state of affairs is the paucity of known results in the relevant regimes of field theory. How can we use the duality in order to extract relevant information about the putative quantum gravity theory that leaves on the bulk?

The present thesis includes a collection of four papers published in peer-reviewed scientific journals, all of them in the context of the AdS/CFT correspondence and with a particular focus on studying gauge theories by inserting heavy external probes, following prescribed trajectories and transforming under various representations of the gauge group.
Each of these works reports a little step forward in the development of new strategies for capturing corrections beyond the leading order as well as in using exact results available in quantum field theory in order to derive exact expressions for other relevant observables and new non-trivial string theory predictions.

In chapters 2 and 3 we use the AdS/CFT correspondence in order to compute several observables of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory related with the presence of an infinitely heavy particle transforming in the k-symmetric or the k-antisymmetric representations of the gauge group and following particular trajectories. This is achieved by means of adding certain D-brane probes with electric fluxes turned on and reaching the boundary of AdS on the very trajectories followed by the dual particles. For the antisymmetric case we consider D5-branes reaching the boundary at arbitrary time-like trajectories, while for the symmetric case, we consider a D3-brane fully embedded in AdS_5 that reaches the boundary at either a straight line or a hyperbola. This generalizes previous computations that used fundamental strings, which are claimed to be dual to infinitely heavy point particles transforming in the fundamental.
Besides the intrinsic interest of these generalizations, our main motivation in studying them is that, as it happens in the computation of certain Wilson loops, the results obtained with D3-branes give an all-orders series of corrections in $1/N$ to the leading order result for the fundamental representation obtained by means of fundamental strings. It is important to remark, one more time, that we can not really extrapolate up to $k = 1$, since this is beyond the regime of validity of the supergravity approximation. Therefore, it is not justified a priori to set $k = 1$ in our results. Nevertheless, when compared with the exact results available, we find that the D3-brane computation reproduces the correct result in the large N, λ limit and with $k = 1$.
This better than expected performance suggests the exciting possibility that certain

D3-branes with electric fluxes might capture correctly all the $1/N$ corrections, but it is fair to say that we still lack of a precise string-theoretic argument to prove this.

The remaining part of chapter 3 is devoted to the derivation of exact results for observables related with static and radiative fields. This was achieved by finding exact relations between certain physical observables, the vacuum expectation value of the $\frac{1}{2}$-BPS circular Wilson loop in the fundamental representation and the two-point function of the circular loop and a chiral primary operator, which in turn can be computed exactly by means of the supersymmetric localization technique.
In particular, we provided exact expressions in $\mathcal{N}=4$ super Yang-Mills for the total energy loss by radiation of a heavy particle in the fundamental representation (from now on, a "quark"), the expectation value of the Lagrangian density operator in the presence of a heavy quark and the momentum diffusion coefficient of a heavy quark moving with constant proper acceleration in the vacuum.

Finally, in chapter 4 we compute the exact vacuum expectation value of the $\frac{1}{2}$-BPS circular Wilson loops for Euclidean $\mathcal{N}=4$ super Yang-Mills with gauge group $G = SO(N), Sp(N)$, in the fundamental and spinor representations. These field theories are conjectured to be dual to type IIB string theory compactified on $AdS_5 \times \mathbb{RP}^5$ plus certain choices of discrete torsion, and we use our results to probe this particular holographic duality.
After revisiting the Liu-Lunin-Maldacena-type geometries having $AdS_5 \times \mathbb{RP}^5$ as ground state, we find that our results clarify and refine the identification of these geometries as bubbling geometries arising from fermions on a half harmonic oscillator. We furthermore identify the presence of discrete torsion with the one-fermion Wigner distribution becoming negative at the origin of phase space.
We end with a string world-sheet interpretation of our results. In that case our goal was not that of using the exact results for testing the correspondence, but our attitude was to take for granted the holographic duality and use the exact field theory results to learn about string theory on an $AdS_5 \times \mathbb{RP}^5$ background. The exact relations between the quantities considered imply two main features: first, the contribution coming from world-sheets with a single crosscap is closely related to the contribution coming from orientable world-sheets, and second, world-sheets with two crosscaps don't contribute to these quantities. Finally we end up by carrying the explicit $1/N$ expansion of the exact results and comparing with the known $SU(N)$ case.

Outlook and future directions

The localization technique has emerged as a very powerful and promising technique to drastically simplify very specific computations in supersymmetric gauge theories, allowing in some cases to obtain exact results. It has been stablished that for four-dimensional $\mathcal{N}=2$ super Yang-Mills theories with a Lagrangian description, the evaluation of the partition function and the vev of certain circular Wilson loops boils down to a zero-dimensional matrix model computation. For the particular case of $\mathcal{N}=4$ SYM, the matrix model is Gaussian and all the integrals can be computed exactly, but when we consider less supersymmetric $\mathcal{N}=2$ theories, the one-loop determinant that appears from integrating out field fluctuations becomes a complicated function and an exact evaluation of the integrals is out of reach.
However, there have been a number of works trying to use the localization of the partition function and of certain loop operators in four dimensional $\mathcal{N}=2$ super Yang-Mills theories to probe the putative string duals. As of today, we have analyzed a broad family of four-dimensional $\mathcal{N}=2$ superconformal quiver gauge theories from the matrix model and in the large N limit. In particular, this allowed us to find another evidence for the classification of gauge theories in two big families, with or without a putative classical gravity dual, depending on their matter content.
This is a potentially very exciting line of research, as it may reveal properties of holographic pairs that have not been fully established to date.

Another completely different line of future research will be the extension of our string and D-brane probe computations to more realistic situations, a first approach being that of allowing for finite temperature. Obviously, this breaks completely and explicitly both conformal symmetry and supersymmetry so localization is not applicable.
Studying probes at finite temperature is an old and well established subject, and one of the very first applications of holography for unraveling the mysteries of the strongly coupled regime of QCD. Nevertheless, it has been claimed recently that the standard approach of using fundamental strings and D-branes to probe finite temperature gravity backgrounds can miss important quantitative as well as qualitative information, since these probes are extremal by definition and cannot be in thermal equilibrium with the medium at finite temperature. A promising candidate that fixes this is the so-called *blackfold approach*, which is a technique that consists basically in finding approximate solutions of probe black branes. Being black objects, these will always be in thermal equilibrium with any stationary background.
In this context, my first goal will be to compute the quark-antiquark potential of $\mathcal{N}=4$

SYM at finite temperature and finite chemical potential as a generalization of a previous computation done at zero chemical potential. Although a priori it may look like a straightforward generalization, I think that in this second case one may find richer physics. On the one hand, the relevant phase space is now two-dimensional and was studied in detail in the past. On the other hand, and most importantly, one can approach very low temperatures both in Poincaré and in global coordinates. Maybe this will lead to a separation between classical and quantum temperature fluctuations, as it was observed recently in a slightly different context.

Chapter 6

Resum en Català
Summary in Catalan

La Teoria Quàntica de Camps, la teoria resultant de fer compatibles la Mecànica Quàntica amb els postulats de la Relativitat Especial d'Einstein, és una eina amb una gran diversitat d'aplicacions i que ens permet explicar de manera satisfactòria una gran varietat de fenòmens físics en diferents intervals d'energia. En el camp de la física d'altes energies, la Teoria Quàntica de Camps és la teoria subjacent que fonamenta el Model Estàndard, el qual ofereix una visió unificada i precisa de les interaccions electromagnètica, nuclear feble i forta. En el marc de la física estadística, la Teoria de Camps descriu satisfactòriament les transicions de fase al voltant d'un punt crític així com la física de diversos sistemes de matèria condensada.
Aquestes teories presenten, però, diverses dificultats. La gran majoria de situacions on la Teoria Quàntica de Camps ens és útil tenen en comú el fet de trobar-se en un règim feblement acoblat, el qual ens permet un estudi pertorbatiu de les interaccions. Aquesta descripció pot ser fonamental, com és el cas del Model Estàndard de la Física de Partícules, o bé efectiva, com seria el cas de la teoria de pertorbacions quiral, la teoria BCS per a la superconductivitat o la teoria dels líquids de Fermi.

Al món real trobem també, però, molts sistemes d'interès físic o amb clares aplicacions tecnològiques on no coneixem cap descripció feblement acoblada: QCD a energies baixes (i.e. comparables amb l'energia en repòs del protó), superconductivitat a temperatures altes, sistemes de fermions pesants, etc... Realitzar càlculs en teories com aquestes on el règim d'acoblament és fort resulta molt complicat. Una possibilitat consisteix en discretitzar l'espai-temps substituint-lo per un reticle de punts i portar a terme càlculs numèrics amb ordinadors. Aquesta enfocament pot resultar útil per tal d'avaluar certes

quantitats, però requereix un gran poder computacional i no és fiable per analitzar processos fora de l'equilibri.
Vist l'ampli ventall de sistemes on el paradigma de partícules o quasi-partícules interaccionant feblement no és aplicable, així com les limitacions dels mètodes discrets, és imperatiu trobar noves eines analítiques que vagin més enllà del càlcul pertorbatiu.

Durant les darreres dues dècades ha aparegut un nou paradigma que permet reformular completament certes teories quàntiques de camps i ens aporta una nova eina que ens permet realitzar càlculs analítics en règims fins ara inaccessibles. Aquest nou paradigma sorgeix del descobriment d'una correspondència o dualitat exacta entre dues teories aparentment molt diferents. Per una banda de la dualitat tenim certes teories quàntiques de camps, com per exemple les denominades teories de Yang-Mills, similars a les teories del Model Estàndard. Aquestes descriuen partícules interactuant en un espai pla d-dimensional sense gravetat. A l'altra banda de la dualitat trobem teories que inclouen la gravetat, com ara la Teoria de la Relativitat General d'Einstein o les seves generalitzacions en el marc de la Teoria de Cordes. Aquestes teories de gravetat estan definides sobre espais de dimensió més alta que d, i és per això que aquesta correspondència rep sovint l'adjectiu de "hologràfica". Depenent del context, aquesta rep el nom de dualitat gauge/gravetat, dualitat gauge/corda o AdS/CFT (acrònim anglès per la correspondència particular entre teoria de cordes a espais d'Anti-de Sitter i teories de camps conformes).
Fins ara, una de les correspondències més ben estudiades i que comprenem millor (i sobre la qual es centra la present tesi) és la dualitat entre la teoria quatre-dimensional $\mathcal{N} = 4$ super Yang-Mills amb grup de gauge $SU(N)$ i teoria de cordes tipus IIB en un espai deu-dimensional $AdS_5 \times S^5$.

Aquesta tesi presenta una recopilació de quatre articles publicats en revistes científiques d'alt impacte, tots ells en el camp de la correspondència AdS/CFT i centrats en l'estudi de teories gauge supersimètriques mitjançant la inserció de partícules de prova infinitament massives, seguint trajectòries determinades i transformant sota diverses representacions del grup de gauge. Cadascun d'aquests treballs aporta un pas endavant en el desenvolupament de noves estratègies per calcular correccions més enllà del primer ordre així com en l'ús de resultats exactes accessibles a la Teoria Quàntica de Camps per tal de derivar expressions exactes d'altres observables rellevants de la teoria i realitzar prediccions de Teoria de Cordes.

Als capítols 2 i 3 hem utilitzat la correspondència AdS/CFT per calcular certs observables de la teoria $\mathcal{N} = 4$ super Yang-Mills relacionats amb la presència d'una partícula de prova infinitament massiva, transformant sota les representacions k-simètrica

o k-antisimètrica del grup de gauge i seguint trajectòries concretes. Això es realitza mitjançant la inserció de determinades D-branes de prova amb fluxos elèctrics activats i que arriben al contorn a l'infinit d'AdS precisament sobre les trajectòries descrites per les partícules duals. Pel cas de la representació antisimètrica considerem una D5-brana arribant al contorn en una trajectòria arbitrària tipus temps, mentre que pel cas de la representació simètrica considerem una D3-brana completament immersa dins AdS_5 i que arriba al contorn sobre una línia recta o sobre una hipèrbola. Aquests resultats generalitzen càlculs previs on s'utilitzaven cordes fonamentals.
Tot i l'interès intrínsec d'aquestes generalitzacions, la nostra motivació principal és que, així com també passa amb el càlcul de certs llaços de Wilson, els resultats obtinguts mitjançant D3-branes presenten una sèrie de correccions a tot ordre en $1/N$ que conicdeixen exactament amb les prediccions mitjançant resultats exactes. Aquest resultats, molt millors del que un esperaria donats els rangs de validesa de la tècnica emprada, ens suggereix la interessant possibilitat que certes D3-branes amb fluxos elèctrics activats puguin capturar totes les correccions $1/N$. Cal dir, però, que a dia d'avui encara no hem aconseguit trobar una derivació utilitzant un llenguatge de Teoria de Cordes per tal de demostrar aquest fet.

La part restant del capítol 3 està dedicada a la derivació de resultats exactes per observables a la teoria de camps relacionats amb camps estàtics i radiatius. Això s'aconsegueix trobant relacions exactes entre aquests observables, el valor d'expectació al buit del llaç de Wilson circular $\frac{1}{2}$-BPS transformant sota la representació fonamental i la funció de correlació a dos punts del llaç circular i un operador primari quiral. Per altra banda, aquests dos últims al seu torn poden ésser calculats de manera exacta mitjançant la tècnica de la localització supersimètrica.
D'aquesta manera hem derivat l'expressió exacta a la teoria $\mathcal{N}=4$ super Yang-Mills de l'energia total radiada per una partícula infinitament massiva a la fonamental (d'ara endavant un "quark"), el valor d'expectació de l'operador densitat Lagrangiana en presència d'un quark pesant i el coeficient de difusió de moment d'un quark pesant movent-se amb acceleració pròpia constant al buit.

Per acabar, al capítol 4 calculem exactament el valor d'expectació al buit del llaç de Wilson circular $\frac{1}{2}$-BPS a la versió Euclídea de $\mathcal{N}=4$ super Yang-Mills, amb grups de gauge $G=SO(N), Sp(N)$ i transformant sota les representacions fonamental i espinorial. La conjectura ens diu que aquestes teories de camps son duals a la teoria de corda tipus IIB compactificada sobre $AdS_5 \times \mathbb{RP}^5$, amb l'elecció d'una determinada torsió discreta, així que utilitzem els nostres resultats per fer mesures de prova d'aquesta dualitat.

Després de fer un repàs a les geometries de Liu-Lunin-Maldacena amb $AdS_5 \times \mathbb{RP}^5$ com a estat de mínima energia obtenim, com a primera conclusió, que els nostres resultats clarifiquen i refinen la identificació d'aquestes geometries com a *bubbling geometries* que emergeixen d'un sistema de fermions en un potencial del tipus "mig oscil·lador harmònic".

Acabem finalment amb una interpretació dels nostres resultats encarada a la fulla de temps de la corda dual. En aquest cas el nostre objectiu no és pas el d'utilitzar els resultats exactes per tal d'estudiar la conjectura, sinó que la nostra actitud consisteix en donar per suposada la dualitat hologràfica per així poder utilitzar els resultats exactes obtinguts per obtenir nova informació sobre la teoria de cordes en aquest fons particular. Les relacions exactes entre les diverses quantitats considerades impliquen dos fets principals: en primer lloc, la contribució de les fulles de temps amb un únic *crosscap* està estretament relacionada amb la contribució provinent de les fulles de temps orientables. En segon lloc, fulles de temps amb dos *crosscaps* no contribueixen a aquestes quantitats. En últim lloc acabem realitzant una expansió en $1/N$ explícita dels resultats exactes obtinguts per tal de comparar-los amb els resultats coneguts pel cas de grup de gauge $SU(N)$.