Programa de Doctorat en Matemàtica Aplicada

Ph.D. Thesis

A Cotangent Bundle Hamiltonian Tube Theorem and its Applications in Reduction Theory

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Abstract

The Marle-Guillemin-Sternberg (MGS) form is an extremely important tool for the theory of Hamiltonian actions on symplectic manifolds. It has been extensively used to prove many local results both in symplectic geometry and in symmetric Hamiltonian systems theory. It provides a model for a tubular neighborhood of a group orbit and puts in normal form the action and the symplectic structure. The main drawback of the MGS form is that it is not an explicit model. Only its existence and main properties can be proved. Moreover, for cotangent bundles, this model does not respect the natural fibration $\tau \colon T^*Q \to Q$.

In the first part of the thesis we build an MGS form specially adapted to the cotangent bundle geometry. This model generalizes previous results obtained by T. Schmah for orbits with fully-isotropic momentum. In addition, our construction is explicit up to the integration of a differential equation on G. This equation can be easily solved for the groups $SO(3)$ or $SL(2)$, thus giving explicit symplectic coordinates for arbitrary canonical actions of these groups on any cotangent bundle.

In the second part of the thesis, we apply this adapted MGS form to describe the structure of the symplectic reduction of a cotangent bundle. We show that, if $\mu \in \mathfrak{g}^*$, the base projection of the μ-momentum leaf $\tau(\mathbf{J}^{-1}(\mu))$ is a Whitney stratified space. Moreover, the set $\mathbf{J}^{-1}(\mu)/G_\mu$ can be decomposed into smooth pieces and each of them fibers over a piece of the stratified space $\tau(\mathbf{J}^{-1}(\mu))/G_\mu$. In the decomposition of $\mathbf{J}^{-1}(\mu)/G_\mu$ there is a maximal piece which is open and dense. Furthermore, this maximal piece is symplectomorphic to a vector subbundle of a certain magnetic cotangent bundle.

Keywords: Cotangent bundles, normal forms, stratified spaces, singular reduction, momentum maps.

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Preamble

In this work we study the symplectic geometry of cotangent-lifted actions induced by a smooth proper action of a Lie group on a smooth manifold. Symplectic manifolds have their origin in the geometric formalization of Hamilton's and Lagrange's equations of classical mechanics. The study of symmetries of these manifolds also has its roots on classical mechanics, where symmetries is the main tool that can be used to simplify the equations of motion.

More precisely, assume that the Lie group G acts on the symplectic manifold (M, ω) leaving the symplectic form ω invariant. Under certain conditions, this implies the existence of an application $\mathbf{J}\colon M \to \mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual of the Lie algebra of the group G. When this happens the action is called a Hamiltonian action. In this case, the fibers of the map $\mathbf{J}$ are invariant under the flow of any Hamiltonian field associated to a G-invariant function. The map $\mathbf{J}$ is called a momentum map, because it becomes the classical notion of angular momentum when $G = SO(3)$ acting in $\mathbb{R}^3$. This result is the well-known Nother's Theorem which implies that the components of the momentum map are preserved under the evolution of a symmetric Hamiltonian system. In fact, the theory of Hamiltonian actions and their momentum maps have deep consequences in fields far away from mechanics, such as the theory of toric manifolds [Can03] or the space of moduli of flat connections [AB83].

It is well known that Darboux's Theorem implies that all symplectic manifolds of the same dimension are locally symplectomorphic. However, the local geometry of symplectic manifolds endowed with Hamiltonian actions is surprisingly rich and constitutes a field originating in the classical papers of Marle [Mar85] and Guillemin and Sternberg [GS84]. These authors obtain a universal model for a tubular neighborhood of the orbit of a point under a Hamiltonian action, which puts in normal form both the symplectic structure and the momentum map.

This model is known as the Hamiltonian tube or Marle-Guillemin-Sternberg form; it is the basis of almost all the local studies concerning Hamiltonian actions of Lie groups on symplectic manifolds. This local normal form has been essential both for the development of singular reduction theory and for the study of qualitative properties of symmetric Hamiltonian systems. Nevertheless, its applications have been limited by the fact that the proof is non-constructive.

In the first part of this thesis we are going to study Hamiltonian tubes when the symplectic manifold is a cotangent bundle. The Marle-Guillemin-Sternberg normal form applied to this case gives, as for every Hamiltonian action, an equivariant local model of (T^*Q, ω_Q) that puts in normal form both the symplectic structure and the momentum map. However, in general, this model does not respect the fibration $T^*Q \to Q$. In the concrete case of cotangent bundles there is a strong motivation coming from geometric mechanics and geometric quantization that makes it desirable to obtain explicit fibrated local models.

The first works studying symplectic normal forms in the specific case of cotangent bundles seem to have been [Sch01; Sch07]. In these references, T. Schmah found a Hamiltonian tube around those points $z \in T^*Q$ whose momentum $\mu = \mathbf{J}(z)$ is totally-isotropic (that is, $G_\mu = G$ with respect to the coadjoint action). One of the main differences between her construction and the classical MGS form for symplectic actions is that the one for cotangent bundles was constructive, unlike the general MGS model. The next step came with [PROSD08]; that work gave a complete description of the symplectic slice of a cotangent bundle, without the assumption $G = G_\mu$. Recently, [SS13] constructed Hamiltonian tubes for free actions of a Lie group G and showed that this construction can be made explicit for $G = SO(3)$.

In this work we obtain a construction of the Hamiltonian tube for a cotangent-lifted action in a cotangent bundle specially adapted to this kind of manifolds. In other words, we give a model space Y that models locally the neighborhood of an orbit of the group in T^*Q. This local model Y has a fibered structure $Y \to W$, and W is a local model for the base Q. Diagramatically we obtain a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\mathcal{T}} & T^*Q \\ \downarrow & & \downarrow \\ W & \longrightarrow & Q. \end{array}$$

We emphasize that the construction of the local model $\mathcal{T}\colon Y \to T^*Q$ is explicit up to the integration of a differential equation on G. The basic geometric ingredient of our model are what we call restricted G-tubes (Definition 5.1.5). These maps are the basic building blocks of $\mathcal{T}$ and are the only non-explicit part of the model. It is worth pointing out that restricted G-tubes depend only on the group G and its algebraic structure, not on the manifold Q.

Restricted G-tubes can be explicitly computed for small dimensional Lie groups; for example, for $SO(3)$ and $SL(2)$. For larger groups the computations get more cumbersome.

In the second part of the dissertation we deal with the symplectic reduction of cotangent bundles. As long as the Hamiltonian action is free, after the work of Marsden and Weinstein in [MW74], it became clear that the elimination of variables in classical mechanics must be understood as the construction of the quotient $\mathbf{J}^{-1}(\mu)/G_\mu$ called the symplectic reduction of M at μ.

This symplectic reduction can be applied to any symplectic manifold with symmetry, but if the symplectic manifold is a cotangent bundle endowed with a cotangent-lifted action, then the reduced space has extra structure (see [Sat77; AM78; MP00]). Intuitively speaking, if T^*Q is a cotangent bundle endowed with a cotangent-lifted action of a Lie group G, the cotangent bundle reduction of T^*Q at μ can be understood as a subbundle of $T^*(Q/G_\mu)$.

Things become much more complicated when we do not assume that the action of G is free. The main reason is because neither $\mathbf{J}^{-1}(\mu)$ nor $\mathbf{J}^{-1}(\mu)/G_\mu$ are even smooth manifolds. We need to enlarge the category of smooth manifolds to allow the singularities that arise from quotienting by a group action. In the early 90's, the work of Sjammar and Lerman [SL91] showed that the reduced space $\mathbf{J}^{-1}(0)/G$ should be understood as a stratified symplectic space, a disjoint union of symplectic manifolds. In fact these pieces, called strata, are determined by the isotropy types of the G-action on M. Later development [BL97; CŚ01; OR04] showed that for a proper Hamiltonian action the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ is a symplectic stratified space.

One expects that, as in the free case, the reduced space will admit additional structure if the symplectic manifold is a cotangent bundle. Up to our knowledge, the first work about singular symplectic reduction in the case of cotangent bundles is [Mon83], in which the author imposes several strong conditions to ensure that all the relevant sets are smooth. Later, [ER90; Sch01] gave a complete description of the zero-momentum reduced space when the action on the base consists only of one orbit-type. The reduction at momentum zero without the single-orbit assumption was studied in [RO04; PROSD07], where it was shown that $\mathbf{J}^{-1}(0)/G$ admits a "coisotropic stratification", a partition of $\mathbf{J}^{-1}(0)/G$ into coisotropic submanifolds, that are well behaved with respect to the base projection. Under the assumption that Q is of single orbit type, [HR06; Hoc08; PRO09] have developed a description of the orbit-reduced space $\mathbf{J}^{-1}(G\cdot\mu)/(G\cdot\mu)$.

In this dissertation we show that, for a general cotangent lifted action, the base projection of the μ-momentum leaf, $\tau(\mathbf{J}^{-1}(\mu))$ and $\tau(\mathbf{J}^{-1}(\mu))/G_\mu$ are not manifolds but stratified spaces. Then we define a partition of $\mathbf{J}^{-1}(\mu)$ and $\mathbf{J}^{-1}(\mu)/G_\mu$ into submanifolds that are well behaved with respect to the fibered structure. Each of the pieces of the partition of $\mathbf{J}^{-1}(\mu)/G_\mu$ is a fiber bundle and is endowed with a constant-rank closed two-form. Moreover on $\mathbf{J}^{-1}(\mu)/G_\mu$ there is an open and dense piece Z endowed with a symplectic form that behaves as in the free cotangent reduction theory; it can be symplectically embedded onto a cotangent bundle.

This thesis is divided into six chapters. The first three chapters summarize the required background and fix the notation used throughout the rest of the dissertation. The last three chapters contain the original contributions.

In Chapter 1 we compile some basic facts about symplectic geometry and regular symplectic reduction. Chapter 2 contains a brief summary of the construction of the standard Marle-Guillemin-Sternberg normal form for a proper Hamiltonian action. Some results in this chapter include short proofs because they play a key role in the results of next chapters. In Chapter 3 we introduce the category of stratified spaces and summarize without proofs the relevant material on singular symplectic reduction and singular cotangent-bundle reduction.

In Chapter 4 we characterize the symplectic slice and the Witt-Artin decomposition of a cotangent-lifted action. Although the description of the symplectic slice for a cotangent bundle has already been described in [RO04; PROSD08], in this chapter we present an alternative approach. As a by-product, we introduce in Proposition 4.2.1 a Lie algebra splitting that will be a key result for all the subsequent development.

Chapter 5 is devoted to the construction of Hamiltonian tubes for cotangent-lifted actions. First we define simple and restricted G-tubes (Definitions 5.1.1 and 5.1.5). Simple G-tubes are, up to technical details, MGS models for the lift of the left action of G on itself to T^*G. Their existence is proved in Proposition 5.1.2. Restricted G-tubes are defined implicitly in terms of a simple G-tube (Proposition 5.1.6) and are the main ingredients that we need later to construct the general Hamiltonian tube.

Using these concepts, we can build a Hamiltonian tube around points on T^*Q with certain maximal isotropy properties (Theorem 5.2.2). Besides, generalizing the ideas of [Sch07] we can write down a Γ map (Proposition 5.2.4). These two maps together give the general Hamiltonian tube in Theorem 5.2.7 for arbitrary points.

In Section 5.3 we present explicit examples of G-tubes for both the groups $SO(3)$ (where we recover the results of [SS13]) and $SL(2,\mathbb{R})$. We finish this chapter writing down the explicit Hamiltonian tube for the natural action of $SO(3)$ on $T^*\mathbb{R}^3$. This example generalizes the final example of [Sch07] to the case $\mu\neq 0$.

In Chapter 6 we study the problem of cotangent bundle reduction in the singular setting. We first use the Hamiltonian tube to construct a fibered analogue of Bates-Lerman lemma (Proposition 6.2.1) that describes the set $\tau^{-1}(U)\cap\mathbf{J}^{-1}(\mu)$ for any small enough open neighborhood $U\subset Q$. Using this fibered description, we introduce in Proposition 6.3.1 local coordinates on Q and T^*Q with good properties with respect to the symplectic structure and the group action.

With these tools, we study in Section 6.4 the single orbit case $Q = Q_{(L)}$ and show that the projection of orbit types of $\mathbf{J}^{-1}(\mu)\subset T^*Q$ are submanifolds of Q. Alternatively, these submanifolds can be written as $\mathbf{L}(H,\mu)\cdot Q_H$ where $\mathbf{L}(H,\mu)$ is a submanifold of G.

Motivated by this fact, we show that, even if $Q\neq Q_{(L)}$, the sets $\mathbf{L}(H,\mu)\cdot Q_H$ are submanifolds of Q and induce a stratification of $Q^\mu=\tau(\mathbf{J}^{-1}(\mu))$ and of Q^μ/G_μ (Propositions

6.5.7 and 6.5.9).

Combining this stratification with the orbit-type stratification of [SL91], we can construct a partition of $\mathbf{J}^{-1}(\mu)/G_\mu$ into smooth pieces with good topological properties (Propositions 6.6.4 and 6.6.5). This partition can be thought as a generalization of the "coisotropic decomposition" of [PROSD07]. After that, in Section 6.8, we study the symplectic properties of each of the pieces and show that each piece is endowed with a closed two-form of constant rank. Moreover, each piece Z is a fiber bundle $Z \to R$ and there is a constant-rank map $f_Z\colon Z/G_\mu \to T^*(R/G_\mu)$ into a magnetic cotangent bundle such that f_Z pulls-back the symplectic form of $T^*(R/G_\mu)$ to the closed two-form of Z/G_μ.

As a corollary of the symplectic properties of the decomposition, we obtain in Corollary 6.8.11 a nice description of the isotropy lattice of $\mathbf{J}^{-1}(\mu)$ that improves the results of [RO06]. Finally, in Section 6.9, we present two detailed examples that illustrate our results.

In Chapter 7 we study the symplectic reduction of $T^*\mathbb{R}^n$ by the action of $O(n)$ and the reduction of $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ by the action of $O(n)$. In both cases the reduced spaces can be explicitly identified with certain coadjoint orbits of the symplectic group. It is specially interesting the symplectic reduction of $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ because it provides a concrete example of the general phenomena described in Chapter 6; some pieces of the decomposition of $\mathbf{J}^{-1}(\mu)/G_\mu$ are symplectic and can be embedded into a cotangent bundle, whereas others are just presymplectic submanifolds that have a constant-rank map into a cotangent bundle.

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Contents

Chapter 1

Background

In this chapter we compile some basic facts about symplectic geometry and regular symplectic reduction. This chapter will also be useful to fix the common notation along different chapters. Proofs can be found in many standard references; for example [AM78].

In Section 1.1 we state the notation that we use for several standard concepts in symplectic geometry. In Section 1.2 we recall basic results about Lie groups and proper actions of Lie groups on manifolds. Section 1.3 contains a brief summary of Hamiltonian group actions and momentum maps. In Section 1.4 we present the well-known symplectic reduction results for the free case, and in Section 1.5 we present the standard theorems regarding cotangent-bundle reduction for free actions.

We would like to remark that, throughout the thesis, by a manifold we mean a smooth, separable, Hausdorff and paracompact manifold of constant dimension.

1.1 Symplectic geometry

A **symplectic manifold** is a pair (M, ω) where M is a manifold of even dimension and $\omega \in \Omega^2(M)$ is a closed non-degenerate two-form. All symplectic manifolds of a given dimension are locally isomorphic in the sense that for any point $z \in M$ there is an open set U containing z and functions $q^1, \dots q^n, p_1, \dots, p_n$ defined on U such that

$$\omega\big|_U = \sum_{i=1}^{n} \mathbf{d}q^i \wedge \mathbf{d}p_i.$$

This is the content of Darboux's Theorem.

By a **presymplectic form** we understand a closed two-form ω_{pre} on M of **constant rank**. We say that a (pre)symplectic manifold (M, ω) is exact if there is $\theta \in \Omega^1(M)$ such that $\omega = -\mathbf{d}\theta$ and we call θ a potential for the (pre)symplectic form ω.

Let V be a vector space and ω a skew-symmetric bilinear form on V. If W is a vector subspace of V, the symplectic orthogonal or the ω-orthogonal is the subspace

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \quad \forall w \in W\}.$$

W is called coisotropic if $W^\omega \subset W$, isotropic if $W^\omega \supset W$, symplectic if $W^\omega \cap W = \emptyset$ and Lagrangian if $W^\omega = W$.

Similarly, if S is a submanifold of the presymplectic manifold (M, ω), we call S coisotropic, isotropic, symplectic or Lagrangian if T_pS is a coisotropic, isotropic, symplectic or Lagrangian subspace of T_pM respectively for all $p \in S$.

A smooth map $f\colon (M_1,\omega_1)\to(M_2,\omega_2)$ between (pre)symplectic manifolds will be called a **(pre)symplectic map** if

$$f^*\omega_2=\omega_1.$$

If $f\colon (M_1,\omega_1)\to(M_2,\omega_2)$ is a symplectic map then by non-degeneracy of the symplectic forms it follows that f is an immersion. A symplectic map which is also a diffeomorphism will be called a symplectomorphism.

A vector field $X\in\mathfrak{X}(M)$ is called a **Hamiltonian vector field** if it satisfies an equation of the form

$$i_X\omega=\mathbf{d}f$$

for some function $f\in C^\infty(M)$.

Consider now $M=T^*Q$ the cotangent bundle of a manifold Q. Let $\tau_Q\colon T^*Q\to Q$ be the natural projection, the formula

$$\theta_Q(p_q)(v_{p_q})=\langle p_q, T_{p_q}\tau_Q(v_{p_q})\rangle$$

where $p_q\in T^*_qQ$ and $v_{p_q}\in T_{p_q}(T^*Q)$ defines a smooth one-form $\theta_Q\in\Omega^1(T^*Q)$ the **Liouville one-form**. The exterior differential of θ_Q defines a symplectic form $\omega_Q=-\mathbf{d}\theta_Q$, the **canonical symplectic form** of the cotangent bundle. From now on, unless otherwise stated, all the cotangent bundles will be endowed with the symplectic canonical form and the symbol τ or τ_Q will always denote the cotangent bundle projection $T^*Q\to Q$.

1.2 Lie groups

Recall that a Lie group G is a smooth manifold with a group structure such that the multiplication map is smooth. We will denote by e the identity element of the group, the left and right translation maps by $g\in G$ are denoted by $L_g, R_g\colon G\to G$. When there is no risk of confusion, the Lie algebra of a Lie group will be represented by the corresponding Gothic letter, that is, the Lie algebra of G will be $\mathfrak{g}$, otherwise we will use $\mathrm{Lie}(G)$.

The adjoint action of $g\in G$ on $\mathfrak{g}$ will be denoted as $\mathrm{Ad}_g\colon\mathfrak{g}\to\mathfrak{g}$ and the linear map $\eta\to[\xi,\eta]$ will be denoted as $\mathrm{ad}_\xi\colon\mathfrak{g}\to\mathfrak{g}$. Similarly, $\mathrm{ad}^*_\xi\colon\mathfrak{g}^*\to\mathfrak{g}^*$ will be the dual of ad_ξ and the coadjoint action of $g\in G$ on $\mathfrak{g}^*$ is given by $\mathrm{Ad}^*_{g^{-1}}\colon\mathfrak{g}^*\to\mathfrak{g}^*$.

Let H be a closed subgroup of a Lie group G. We define the **normalizer** of H in G as

$$N_G(H)=\{g\in G\mid gHg^{-1}=H\}.$$

It can be checked that $N_G(H)$ is a closed subgroup of G, and H is normal as a subgroup of $N_G(H)$. Moreover, if $\xi\in\mathfrak{g}$ satisfies

$$\mathrm{Ad}_h\xi-\xi\in\mathfrak{h}$$

for all $h\in H$ then $\xi\in\mathrm{Lie}(N_G(H))$.

The exponential map

$$\exp\colon\mathfrak{g}\longrightarrow G$$

is a diffeomorphism when restricted to a small enough neighborhood of 0 in $\mathfrak{g}$. The derivative of the exponential mapping can be expressed as a series of commutators (see for example Lemma 4.27 of [KMS93]),

Proposition 1.2.1. *The exponential mapping satisfies*

$$T_\xi\exp=T_eL_{\exp\xi}\circ\left(\frac{\mathrm{Id}-e^{-\mathrm{ad}_\xi}}{\mathrm{ad}_\xi}\right)=T_eL_{\exp\xi}\circ\left(\sum_{n\geq 0}\frac{(-1)^n}{(n+1)!}\mathrm{ad}^n_\xi\right).$$

If the Lie group G is compact there is a volume form Vol_G that is both left and right invariant, the **Haar measure**. This measure will be normalized requiring $\int_G \mathrm{Vol}_G = 1$. The existence of this bi-invariant measure allows the construction invariant objects related to compact groups by simply averaging over it.

1.2.1 Lie group actions

A **left action** of G on a manifold M is a smooth mapping $\mathcal{A}\colon G \times M \to M$ such that

- $\mathcal{A}(e, z) = z$ for all $z \in M$
- $\mathcal{A}(g, \mathcal{A}(h, z)) = \mathcal{A}(gh, z)$ for all $g, h \in G$ and $z \in M$.

we will call the triple $(M, G, \mathcal{A})$ a G-space. Given an element $g \in G$, the translation $\mathcal{A}_g\colon M \to M$ is a diffeomorphism of M. To simplify notation we often use $g \cdot z$ as a shorthand for $\mathcal{A}(g, z) = \mathcal{A}_g(z)$ when the action is clear from the context.

Given an action $\mathcal{A}\colon G \times M \to M$, the infinitesimal generator $\xi_M \in \mathfrak{X}(M)$ associated to $\xi \in \mathfrak{g}$ is the vector field on M defined by

$$\xi_M(x) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}_{\exp(t\xi)}(x) \in T_x M.$$

The fundamental vector fields satisfy $(\mathrm{Ad}_{g^{-1}}\xi)_M = \mathcal{A}_g^*(\xi_M)$ and $[\xi, \eta]_M = -[\xi_M, \eta_M]$. Sometimes we use the notation $\xi \cdot x$ to refer to the fundamental field associated to ξ at the point x.

For $p \in M$, the **isotropy subgroup of** p is $G_p = \{g \in G \mid g \cdot p = p\}$. A map $f\colon M_1 \to M_2$ between two G-spaces is called G**-equivariant** if $f(g \cdot p) = g \cdot f(p)$ for all $p \in M_1$ and $g \in G$. Similarly, a map $f\colon M_1 \to M_2$ is called G**-invariant** if $f(g \cdot p) = f(p)$ for all $p \in M_1$ and $g \in G$.

Note that if $f\colon M_1 \to M_2$ is G-equivariant, then G_x is a subgroup of $G_{f(x)}$ for any $x \in M_1$ if f is a diffeomorphism then $G_x = G_{f(x)}$ for any $x \in M_1$.

It is useful to remark that near a fixed point $p \in M$ of an action of a compact group G, the G-invariant open sets form a basis of neighborhoods of p,

Lemma 1.2.2. *Let G be a compact Lie group acting on the manifold M. If m is a fixed point of the action, any open neighborhood of m contains a G-invariant open neighborhood of m.*

1.2.2 Proper actions

An action of a Lie group on a manifold can be very wild; the quotient topological space M/G may even fail to be Hausdorff. The actions of compact Lie groups are much more well behaved hence it seems reasonable to restrict the study to compact Lie group actions. However, there is a technical condition called properness which allows the study of more general groups, but retaining some nice properties of compact group actions:

Definition 1.2.3. Let G be a Lie group acting on the manifold M via the map $\mathcal{A}\colon G \times M \to M$. We say that the action is **proper** if the map $\Theta\colon G \times M \to M \times M$ defined by $\Theta(g, p) = (p, \mathcal{A}(g, p))$ is proper (i.e. the pre-image of every compact set is compact). This is equivalent to the condition: for any two convergent sequences $\{m_n\}$ and $\{g_n \cdot m_n\}$ in M there is a convergent subsequence $\{g_{n_k}\}$ in G.

This technical condition was introduced by Palais [Pal61] who proved that this hypothesis is enough to ensure that the main properties of compact Lie group actions are available. It can be easily checked that for any Lie group G the left and right actions on itself are proper.

As most of the actions studied in this thesis are proper, we will recall some of their useful properties.

Proposition 1.2.4. *Let $\mathcal{A}\colon G\times M \to M$ be a proper action of a Lie group G on the manifold M, then,*

1. *For any $m \in M$ the isotropy subgroup G_m is compact.*
2. *The quotient space M/G is Hausdorff.*
3. *If the action is free, M/G is a smooth manifold and the canonical projection $\pi\colon M \to M/G$ defines on M the structure of a smooth principal G-bundle.*
4. *If all the isotropy subgroups of the points of M are conjugated to $H \subset G$, then M/G is a smooth manifold and the canonical projection $\pi\colon M \to M/G$ defines on M the structure of a smooth locally trivial fiber bundle with structure group $N(H)/H$ and fiber G/H.*
5. *There exists a G-invariant Riemannian metric on M.*
6. *Let $m \in M$ the orbit $G\cdot m$ is an embedded submanifold of M.*

Assume that G acts properly on M, let H be a closed subgroup of G and let N an embedded submanifold of M such that $h\cdot N \subset N$ for any $h \in H$. Then we can restrict the G-action on M to an H-action on N and this restricted action is again proper.

1.3 Hamiltonian group actions

An action $\mathcal{A}\colon G\times M \to M$, where (M,ω) is a symplectic manifold, is called a **Hamiltonian action** if

- G acts by symplectomorphisms, i.e. $\mathcal{A}_g^*\omega = \omega \quad \forall g \in G$
- There is a map $\mathbf{J}\colon M \to \mathfrak{g}^*$ called a **momentum map** such that:

$$i_{\xi_M}\omega = \mathbf{d}\langle \mathbf{J}(\cdot), \xi\rangle \quad \forall \xi \in \mathfrak{g} \tag{1.1}$$

The tuple $(M,\omega,\mathbf{J},G)$ will be called a **Hamiltonian G-space**. If $\mathbf{J}$ satisfies $\mathbf{J}(g\cdot x) = \mathrm{Ad}^*_{g^{-1}}(\mathbf{J}(x))$, we say that the momentum map is **equivariant**.

Remark 1.3.1. Note that some authors include the equivariance of the momentum map as a required condition for a Hamiltonian action, for example [Can01].

The existence of the momentum map (not necessarily equivariant) is the requirement that the group action not only preserves the symplectic structure but also its **fundamental fields are Hamiltonian**. The existence of the momentum map restricts the dynamics of all Hamiltonian vector fields in the following sense.

Theorem 1.3.2 (Noether's Theorem)**.** *If $H \in C^\infty(M)$ is a G-invariant Hamiltonian on the G-space $(M,\omega,\mathbf{J},G)$, then $\mathbf{J}$ is conserved on the trajectories of the Hamiltonian vector field X_H associated with H.*

From the definition it is easy to check that if $\mathbf{J}_1$ and $\mathbf{J}_2$ are two momentum maps for the same Hamiltonian action then $\mathbf{J}_1 - \mathbf{J}_2$ is a locally constant function.

If $(M, -\mathbf{d}\theta)$ is a symplectic manifold endowed with a G-action such that the symplectic potential $\theta \in \Omega^1(M)$ is G-invariant then the map $\mathbf{J}\colon M \to \mathfrak{g}^*$ defined by:

$$\langle \mathbf{J}(z), \xi \rangle = \langle \theta(z), \xi_M(z) \rangle = (i_{\xi_M}\theta)(z) \tag{1.2}$$

is an equivariant momentum map for the G-action because

$$\mathbf{d}\langle \mathbf{J}(z), \xi \rangle = (\mathbf{d}i_{\xi_M})\theta = (\mathbf{d}i_{\xi_M} - \mathbf{L}_{\xi_M})\theta = -(i_{\xi_M}\mathbf{d})\theta = i_{\xi_M}\omega.$$

where $\mathbf{L}_{\xi_M}$ represents the Lie derivative respect to the vector field ξ_M.

If $(M, \omega, \mathbf{J}, G)$ is a Hamiltonian G-space and H is a closed subgroup of G then $(M, \omega, \mathbf{J}\big|_{\mathfrak{h}}, H)$ is a Hamiltonian H-space where $\mathbf{J}\big|_{\mathfrak{h}}$ is the function $(\mathbf{J}\big|_{\mathfrak{h}})(z) = \mathbf{J}(z)\big|_{\mathfrak{h}}$, that is, the composition of $\mathbf{J}\colon M \to \mathfrak{g}$ with the natural projection $\big|_{\mathfrak{h}}\colon \mathfrak{g}^* \to \mathfrak{h}^*$.

Let $(M, \omega, \mathbf{J}, G)$ be a Hamiltonian G-space and fix $z \in \mathbf{J}^{-1}(\mu)$. If $v \in \operatorname{Ker} T_z\mathbf{J}$, from (1.1), $\omega(\xi_M(z), v) = 0$ for any $\xi \in \mathfrak{g}$, and similarly if $v \in (\mathfrak{g}\cdot z)^\omega$ then $v \in \operatorname{Ker} T_z\mathbf{J}$, that is,

$$\operatorname{Ker} T_z\mathbf{J} = (\mathfrak{g}\cdot z)^\omega \subset T_zM. \tag{1.3}$$

Moreover, if $\mathbf{J}\colon M \to \mathfrak{g}^*$ is an equivariant momentum map and $\mu = \mathbf{J}(z)$, it can be checked that

$$\langle \mu, [\xi, \eta] \rangle = \omega(\xi_M(z), \eta_M(z)) \qquad \forall \xi, \eta \in \mathfrak{g} \tag{1.4}$$

and

$$(\mathfrak{g}\cdot z) \cap (\mathfrak{g}\cdot z)^\omega = \mathfrak{g}_\mu \cdot z \tag{1.5}$$

where $\mathfrak{g}_\mu = \{\xi \in \mathfrak{g} \mid \operatorname{ad}^*_\xi \mu = 0\}$ is the isotropy algebra of $\mu = \mathbf{J}(z) \in \mathfrak{g}^*$ under the coadjoint action.

1.3.1 Coadjoint orbits

Let G be a Lie group and denote $\mathcal{O}_\mu = \{\operatorname{Ad}^*_{g^{-1}}\mu \mid g \in G\}$ the coadjoint orbit through an element $\mu \in \mathfrak{g}^*$. There are two natural symplectic forms on $\mathcal{O}_\mu$, given by

$$\omega^{\pm}_{\mathcal{O}_\mu}(\nu)(\operatorname{ad}^*_{\xi_1}\nu, \operatorname{ad}^*_{\xi_2}\nu)) = \pm\langle \nu, [\xi_1, \xi_2] \rangle, \tag{1.6}$$

for any $\nu \in \mathcal{O}_\mu$.

$\mathcal{O}_\mu$ has a natural G-action given by $g\cdot\nu = \operatorname{Ad}^*_{g^{-1}}\nu$. With respect to this action, $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$ is a G-Hamiltonian space with momentum map $\mathbf{J}(\nu) = \nu \in \mathfrak{g}^*$. Similarly, $(\mathcal{O}_\mu, \omega^+_{\mathcal{O}_\mu})$ is a G-Hamiltonian space with momentum map $\mathbf{J}(\nu) = -\nu \in \mathfrak{g}^*$.

1.3.2 Cotangent lifted actions

If G acts on a manifold Q through $\mathcal{A}\colon G \times Q \to Q$, taking for each fixed g the transpose inverse of the tangent lift, we get: $T^*\mathcal{A}_{g^{-1}}\colon T^*Q \to T^*Q$, which fit together to give a left action of G on T^*Q. This is called the **cotangent lifted action**. It can be checked that this action preserves the symplectic structure. Moreover, as cotangent bundles are exact symplectic manifolds, there exists an equivariant momentum map (see (1.2)) given by

$$\langle \mathbf{J}(p_q), \xi \rangle = \langle p_q, \xi_Q(q) \rangle. \tag{1.7}$$

Hence, if we are given an action of G on Q, we can build the associated **cotangent bundle Hamiltonian G-space** $(T^*Q, \omega, \mathbf{J}, G)$. Moreover, if the action of G on Q is proper, then the cotangent lifted action of G on T^*Q is also proper.

If we assume that G acts linearly on the vector space V, there is also an action of G on V^* given by the inverse transpose of the G action on V. The differential of each of these actions define the fundamental fields $\xi \in \mathfrak{g} \to \xi \cdot a \in V$ and $\xi \in \mathfrak{g} \to \xi \cdot b \in V^*$ for each $a \in V$ and $b \in V^*$. From these observations we can define **the diamond product** of $a \in V$ with $b \in V^*$ as the element $a \diamond b \in \mathfrak{g}^*$ such that

$$\langle a \diamond b, \eta \rangle = \langle b, \eta \cdot a \rangle$$

for all $\eta \in \mathfrak{g}$.

With this notation, if we consider the cotangent lift of the G action on the vector space V to the cotangent bundle $T^*V \cong V \times V^*$ and the resulting G action is Hamiltonian with momentum map

$$\mathbf{J}(a, b) = a \diamond b \in \mathfrak{g}^*.$$

Note that if we consider $G = SO(3)$ acting on $\mathbb{R}^3$, the diamond product becomes the classical cross product under the usual identifications.

If $\mathfrak{h} \subset \mathfrak{g}$ is a subspace then

$$a \diamond_{\mathfrak{h}} b = (a \diamond b)\big|_{\mathfrak{h}} \in \mathfrak{h}^*, \tag{1.8}$$

the restriction of the form $a \diamond b$ to $\mathfrak{h}^*$. If $\mathfrak{h}$ is the Lie algebra of a subgroup $H \subset G$, $a \diamond_{\mathfrak{h}} b$ is the momentum map for the H-action on T^*V induced by restriction of the original G-action, which is in turn the same as the lift of the restricted H-action on V.

1.3.3 Actions on T^*G

From now on we identify TG with $G \times \mathfrak{g}$ and T^*G with $G \times \mathfrak{g}^*$ using left trivializations

$$\begin{aligned} G \times \mathfrak{g} &\longrightarrow TG & \qquad G \times \mathfrak{g}^* &\longrightarrow T^*G \\ (g, \xi) &\longmapsto T_eL_g(\xi) & (g, \nu) &\longmapsto T_e^*L_{g^{-1}}(\nu). \end{aligned} \tag{1.9}$$

Combining them, we can trivialize $T(T^*G) \cong G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^*$.

We need the following well-known properties of the symplectic structure and the cotangent-lifted actions of G on T^*G (see [AM78])

Proposition 1.3.3. *Let G be a Lie group.*

- ***Symplectic structure**: Let $u_i := (\xi_i, \beta_i) \in T_{(g,\nu)}T^*G$ with $i = 1, 2$, the canonical one-form of T^*G is*

$$\theta_G(u_1) = \langle \nu, \xi_1 \rangle \tag{1.10}$$

and the canonical symplectic form $\omega_G = -\mathbf{d}\theta_G$ is

$$\omega_G(u_1, u_2) = \langle \beta_2, \xi_1 \rangle - \langle \beta_1, \xi_2 \rangle + \langle \nu, [\xi_1, \xi_2] \rangle. \tag{1.11}$$

- ***Cotangent-lifted left multiplication**: The G-action given by*

$$h \cdot^L (g, \nu) = (hg, \nu)$$

has as infinitesimal generator $\eta^L_{T^*G}(g, \nu) = (\mathrm{Ad}_{g^{-1}}\eta, 0)$ and is Hamiltonian with momentum map $\mathbf{J}_L(g, \nu) = \mathrm{Ad}^*_{g^{-1}}\nu$.

- ***Cotangent-lifted right multiplication**: The G-action given by*

$$h \cdot^R (g,\nu) = (gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\nu)$$

has as infinitesimal generator $\eta^R_{T^*G}(g,\nu) = (-\eta, -\mathrm{ad}^*_\eta \nu)$ and is Hamiltonian with momentum map $\mathbf{J}_R(g,\nu) = -\nu$.

Note that the actions described by this result are the cotangent lifts of the natural actions of G on itself.

1.4 Regular symplectic reduction

Symplectic reduction is the process that builds a symplectic space out of a given Hamiltonian G-space after the elimination of symmetries and conserved quantities. This strategy can be used to reduce the dimensionality of a given Hamiltonian system. In this section we assume that the action of the group is free, proper, and the momentum map is equivariant.

Theorem 1.4.1 (Regular symplectic point reduction [MW74])**.** *Let (M,ω) be a symplectic manifold and G a Lie group with a free, proper and Hamiltonian action on M. Assume that the momentum map $\mathbf{J}$ is equivariant. Let $\mu \in \mathfrak{g}^*$ and denote by G_μ the isotropy subgroup of μ under the coadjoint action of G on $\mathfrak{g}^*$.*

The space $M_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$ is a smooth manifold with a symplectic form $\omega_\mu \in \Omega^2(M_\mu)$ uniquely characterized by the relation

$$\pi^*_\mu \omega_\mu = i^*_\mu \omega$$

where $i_\mu \colon \mathbf{J}^{-1}(\mu) \to M$ and $\pi_\mu \colon \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ denote the inclusion and the projection, respectively. The pair (M_μ, ω_μ) is the symplectic reduced space at momentum μ.

In fact, the coadjoint orbits introduced in Section 1.3.1 are reduced spaces of T^*G.

Theorem 1.4.2. *Let G be a Lie group and $\mu \in \mathfrak{g}^*$. Using the notation of Proposition 1.3.3, the reduced space of T^*G by the left action at momentum μ, $(\mathbf{J}_L^{-1}(\mu)/G_\mu, \omega_\mu)$ is symplectomorphic to $(\mathcal{O}_\mu, \omega^-_{\mathcal{O}_\mu})$ (see* (1.6)*). Moreover, this symplectomorphism is G-equivariant if $\mathbf{J}_L^{-1}(\mu)/G_\mu$ is endowed with the G-action induced by the G^R-action on T^*G.*

If a group G can be written as a direct product $G = G_1 \times G_2$, then the symplectic reduction by G and the double reduction first by G_1 and then by G_2 yield the same result. This important result is known as the commuting reduction, and although it can be generalized in many ways (see [Mar+07]) we will only use the following version.

Theorem 1.4.3 (Regular commuting reduction [MW74])**.** *Let G and H be two Lie groups acting properly and Hamiltonially on a symplectic manifold (M,ω) with equivariant momentum maps $\mathbf{J}_G$ and $\mathbf{J}_H$, respectively. Assume that both actions are free, commute, $\mathbf{J}_G$ is H-invariant and $\mathbf{J}_H$ is G-invariant. This implies that M is a $G \times H$-Hamiltonian space with $G \times H$-equivariant momentum map $(\mathbf{J}_G, \mathbf{J}_H) \colon M \to \mathfrak{g}^* \times \mathfrak{h}^*$.*

Let $\mu \in \mathfrak{g}^*$ and $\nu \in \mathfrak{h}^*$, G induces a Hamiltonian action on $M_\nu = \mathbf{J}_H^{-1}(\nu)/H_\nu$ with equivariant momentum map $\mathbf{K}_G$ determined by $\mathbf{K}_G \circ \pi_{H_\nu} = \mathbf{J}_G$. If the G action on M_ν is free, then the reduced symplectic space $\mathbf{K}^{-1}(\mu)/G_\mu$ is symplectomorphic to $(\mathbf{J}_G, \mathbf{J}_H)^{-1}(\mu,\nu)/(G_\mu \times H_\nu)$, the symplectic reduced space of M by the product action of $G \times H$.

1.5 Regular cotangent bundle reduction

Let Q be a manifold endowed with a proper G-action. The symplectic reduction of a cotangent bundle T^*Q has more structure than a symplectic manifold. In this section we recall the results that characterize the reduced space as a subset of a certain cotangent bundle. We assume that the action of G on the configuration space Q is free.

The first result of the theory of cotangent bundle reduction, due to [Sat77], deals only with the reduction at zero momentum.

Theorem 1.5.1 (Regular cotangent reduction at zero [Sat77])**.** *Let G act freely and properly by cotangent lifts on T^*Q with momentum map $\mathbf{J}$. Denote $\pi_G\colon Q \to Q/G$, $i\colon \mathbf{J}^{-1}(0) \to T^*Q$ and $\pi_0\colon \mathbf{J}^{-1}(0) \to \mathbf{J}^{-1}(0)/G$ the natural quotient maps and inclusions.*

Consider

$$\varphi\colon \mathbf{J}^{-1}(0) \longrightarrow T^*(Q/G)$$

defined by

$$\langle \varphi(z), T_q\pi_G(v)\rangle = \langle z, v\rangle$$

for every $z \in T_q^*Q$ and $v \in T_qQ$. The map φ is a G-invariant surjective submersion that induces a symplectomorphism

$$\bar{\varphi}\colon \mathbf{J}^{-1}(0)/G \longrightarrow T^*(Q/G)$$

where $\mathbf{J}^{-1}(0)/G$ is endowed with the reduced symplectic form ω_0, that is, the one satisfying $\pi_0^*\omega_0 = i^*\omega_Q$.

The general case $\mu \neq 0$ is more difficult, because to describe the reduced space the symplectic form needs to be deformed. These twisting terms are the geometric analogues of the magnetic or Coriolis terms in classical mechanics.

Theorem 1.5.2 (Embedding cotangent bundle reduction [AM78])**.** *Let G act freely and properly by cotangent lifts on T^*Q with momentum map $\mathbf{J}$. Denote by π_{G_μ} the projection $Q \to Q/G_\mu$.*

There is a G_μ-equivariant map $\alpha_\mu\colon Q \to T^*Q$ such that

$$\mathbf{J}(\alpha_\mu(q)) = \mu \quad \text{and} \quad \tau(\alpha_\mu(q)) = q \qquad \forall q \in Q. \tag{1.12}$$

*A map α_μ satisfying these conditions is known as a **mechanical connection**. Note that α_μ can be understood as one-form on Q. Associated to α_μ there is a two-form $\beta_\mu \in \Omega^1(Q/G_\mu)$ such that $\pi_{G_\mu}^*\beta_\mu = \mathbf{d}\alpha_\mu$.*

The map $\varphi_\mu\colon \mathbf{J}^{-1}(\mu) \to T^*(Q/G_\mu)$ defined by $\langle \varphi_\mu(z), T_q\pi_{G_\mu}(v)\rangle = \langle z - \alpha_\mu(q), v\rangle$ for every $z \in T_q^*Q$ and $v \in T_qQ$ is G_μ-invariant and induces a smooth map

$$\overline{\varphi}_\mu\colon \mathbf{J}^{-1}(\mu)/G_\mu \longrightarrow T^*(Q/G_\mu).$$

*If $\mathbf{J}^{-1}(\mu)/G_\mu$ is endowed with the reduced symplectic structure and $T^*(Q/G_\mu)$ with $\omega_{Q/G_\mu} - \tau_{Q/G_\mu}^*\beta_\mu$, $\overline{\varphi}_\mu$ is a **symplectic embedding** onto a vector subbundle of $T^*(Q/G_\mu)$.*

*Moreover, the map $\overline{\varphi}_\mu$ is **onto if and only if** $\mathfrak{g} = \mathfrak{g}_\mu$.*

The theorem that we have stated corresponds to what is known as the "embedding picture". The "fibrating picture" is an alternative description of the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu = \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ as a bundle over $T^*(Q/G)$ (see [MP00]). We will not enter into further details because this interpretation will not be used in this thesis.

Chapter 2

Local normal forms

In many cases, geometric structures on a manifold have simple local models on which the geometric structure in question has a particularly convenient expression. The importance of normal forms is that usually these easier expressions can simplify many problems. In this chapter we study normal forms of G-spaces and of proper Hamiltonian G-spaces. In most of the results of this chapter we include short proofs because they play a key role in the results of following chapters.

In Section 2.1 we study the normal form for G-spaces. We define twisted products (Proposition 2.1.2) the reference model. Then we prove Palais' theorem (Theorem 2.1.4) and show that by using the Riemannian exponential of an invariant metric we can explicitly construct the local normal form of a G-space.

Our next step is to introduce the normal form for Hamiltonian G-spaces, but before doing so, in Section 2.2 we state the Witt-Artin decomposition of a symplectic vector space. This decomposition can be regarded as a linear analogue of the more general MGS form. In Section 2.3.1 we briefly sketch the construction of a symplectic tube, the analogue of twisted products in the symplectic setting. Later we provide a proof of an equivariant version of Darboux's Theorem 2.3.2. Combining this result with the Witt-Artin decomposition, one can easily obtain Theorem 2.3.4, the MGS form or Hamiltonian tube. Finally, in Proposition 2.3.5 we state without proof an important consequence of the MGS form, a local description of the momentum leaf.

2.1 Proper actions

2.1.1 Twisted products

We first introduce the twisted products, a class of G-spaces that will serve as a local model for any G-space.

If a subgroup H of the Lie group G acts on the manifold A, then the product manifold $G \times A$ can be equipped with a **twisting action** of H defined by

$$h \cdot^T (g, a) = (gh^{-1}, h \cdot a), \quad h \in H$$

and a **left action** of G defined by

$$h \cdot^L (g, a) = (hg, a), \quad h \in G.$$

The left action commutes with the twisting action. This means that both actions can be merged into an action of the direct product $G \times H$ on the manifold $G \times A$.

Note that if we regard $\mathfrak{g}^*$ as a manifold endowed with the G-action $g \cdot \nu = \mathrm{Ad}^*_{g^{-1}}\nu$, then the actions G^L and G^T of the product manifold $G \times \mathfrak{g}^*$ correspond to the trivialized expressions of the cotangent-lifts of the left and right actions of G on itself (see Proposition 1.3.3).

Remark 2.1.1. When it is necessary we will use, as above, a superindex T or L to indicate the twisting or the left action on a product manifold $G \times A$.

Proposition 2.1.2. *Let G be a Lie group and $H \subset G$ a subgroup. Assume that H acts properly on the manifold A.*

Then,

- *the twisting action is free and proper. The quotient space $(G \times A)/H^T$ is a manifold. It will be called the **twisted product** and is represented as $G \times_H A$. Its elements will be denoted as $[g, a]_H \quad g \in G,\ a \in A$.*
- *the twisted product has a proper G action given by $g \cdot [g', a]_H = [gg', a]_H$.*

2.1.2 Palais' tube

Let $p \in M$ and assume that G acts smoothly on M. The isotropy subgroup $G_z = \{g \in G \mid g \cdot z = z\}$ acts naturally on the tangent space T_pM at p. For proper smooth actions this G_p action on T_pM is enough to describe the structure of a whole neighborhood of $G \cdot p$. In fact, only the action of G_p on a subspace $S \subset T_pM$ is important.

Definition 2.1.3. Let M be a smooth manifold with a proper G-action and fix a point $p \in M$. A G_p-invariant complement S of $\mathfrak{g} \cdot p \subset T_pM$ in T_pM will be called a **linear slice** at p.

Note that all linear slices at p are isomorphic as G_p-modules to the quotient $T_pM/\mathfrak{g} \cdot p$ endowed with the natural G_p-action.

The Tube Theorem proved by Koszul in [Kos53] for compact groups and generalized by Palais in [Pal61] shows that in fact every proper G-space is locally a twisted product and this twisted product is determined by a linear slice S.

Theorem 2.1.4 (Tube theorem for G-spaces)**.** *Let M be a manifold and G a Lie group acting properly on M. Fix a point $p \in M$, define $H = G_p$. There exists a G-equivariant diffeomorphism:*

$$\mathbf{t}\colon G \times_H S_r \longrightarrow U \tag{2.1}$$

where U is a G-invariant open neighborhood of $G \cdot p$ and S_r is an open H-invariant neighborhood of 0 in a linear slice S at p.

We sketch the proof of this result given in Theorem 2.3.28 of [OR04].

Proof. Consider a metric g_0 defined on a neighborhood U_0 of p such that the splitting of $T_pM = \mathfrak{g} \cdot q \oplus S$ is orthogonal. Using Lemma 1.2.2, there is a G_p-invariant open set $U_1 \subset U_0$ on U_1. We can define the averaged metric by

$$g(z)(u, v) = \int_H g_0(h \cdot z)(T_z\mathcal{A}_h \cdot u, T_z\mathcal{A}_h \cdot v)dh$$

where the integral is taken with respect to the normalized Haar measure of H. It can be checked that $\mathcal{A}_h^* g = g$ for any $h \in H$.

There is a neighborhood $(T_pM)_0$ of the origin in T_pM such that the restriction of the Riemannian exponential associated to g defines a diffeomorphism

$$\mathrm{Exp}\colon (T_pM)_0 \to U_1 \subset M.$$

Since g is H-invariant, $\mathrm{Exp}(T_z\mathcal{A}_h \cdot a) = h \cdot \mathrm{Exp}(a)$. Define $S_0 = (T_pM)_0 \cap S$ and the map

$$\begin{aligned} \mathbf{t}\colon G \times_H S_0 &\longrightarrow M \\ [g,a]_H &\longmapsto g \cdot \mathrm{Exp}(a) \end{aligned}$$

as Exp is H-equivariant $\mathbf{t}$ is well-defined and G-equivariant.

If $\pi_H\colon G \times S \to G \times_H S$ is the canonical projection, then for any $(\xi, \dot{a}) \in T_eG \times T_0S$

$$T_{(e,0)}(\mathbf{t} \circ \pi_H) \cdot (\xi, \dot{a}) = \xi_M(p) + \dot{a} \in T_pM.$$

Then $T_{(e,0)}(\mathbf{t} \circ \pi_H) \cdot (\xi, \dot{a}) = 0$ implies $\xi \in \mathfrak{h}$ and therefore $T_{(e,0)}\pi_H \cdot (\xi, \dot{a}) = 0$ so $T_{[e,0]_H}\mathbf{t}$ is an isomorphism. Since being a linear isomorphism is an open condition there is an open neighborhood S_1 of the origin in S_0 such that $T_{[e,v]_H}\mathbf{t}$ is an isomorphism for any $v \in S_1$. Again we can assume that S_1 is H-invariant. Finally, by G-equivariance of $\mathbf{t}$, $T_{[g,v]_H}\mathbf{t}$ is an isomorphism for any $g \in G$ and $v \in S_1$; that is, the map

$$\mathbf{t}\colon G \times_H S_1 \longrightarrow M$$

is a local diffeomorphism.

We now show that there must be an H-invariant open neighborhood S_{inj} of the origin of S_1 so that the restricted map $\mathbf{t}\colon G \times_H S_{\mathrm{inj}} \to M$ is injective. If we assume the contrary, this means that there are two sequences $[g_n, a_n]_H$ and $[g_n', a_n']_H$ on $G \times_H S_{\mathrm{inj}}$ such that

$$[g_n, a_n]_H \neq [g_n', a_n']_H \quad \forall n$$

but,

$$\mathbf{t}([g_n, a_n]_H) = \mathbf{t}([g_n', a_n']_H) \quad \forall n$$

and both a_n and a_n' converge to zero. Therefore,

$$\mathbf{t}([e, a_n]_H) = \mathbf{t}([g_n^{-1}g_n', a_n']_H) \quad \forall n \tag{2.2}$$

but as $[e, a_n]_H$ is a convergent sequence, $\mathbf{t}([e, a_n]_H)$ is also convergent and by properness of the action there is a subsequence such that $g_n^{-1}g_n'$ is convergent to $g \in G$. This implies $\mathbf{t}([e,0]_H) = \mathbf{t}([g,0]_H) = g \cdot \mathbf{t}([e,0]_H)$ so $g \in H$. As $\mathbf{t}$ is a local diffeomorphism, there is an open neighborhood U of $[e,0]_H$ such that $\mathbf{t}(x) = \mathbf{t}(y)$ implies $x = y$, but then (2.2) implies $[e, a_{n_k}]_H = [g_{n_k}^{-1}g_{n_k}', a_{n_k}]_H$, which is a contradiction. Therefore, there is an H-invariant open neighborhood S_{inj} of the origin of S_1 so that the restricted map $\mathbf{t}\colon G \times_H S_{\mathrm{inj}} \to M$ is injective.

As $\mathbf{t}\colon G \times_H S_{\mathrm{inj}} \to M$ is a local diffeomorphism onto its image and it is injective, it is a diffeomorphism onto its image. Hence the claim follows if we set $S_r = S_{\mathrm{inj}}$. □

With this semi-local model, in the sense that is global for the G-action, one can prove many properties of G-spaces. It is the main tool used to obtain the smoothness of its isotropy type components, the closeness of group orbits, and many more properties (see [DK00]).

2.2 Witt-Artin decomposition

If M has a proper Hamiltonian action of G, it is not trivial at all how can we adapt the Palais' tube construction in such a way that it becomes a symplectomorphism. The first step towards that goal is to obtain a normal form for the linear space T_pM as a G_p-space with a symplectic form $\omega(p)$.

Definition 2.2.1. Let $(M, \omega, \mathbf{J})$ be a Hamiltonian G-space with equivariant momentum map $\mathbf{J}$. Fix a point $p \in M$, if $\mathbf{J}(p) = \mu$, any G_p-invariant complement N of $\mathfrak{g}_\mu \cdot p$ in $\operatorname{Ker} T_p\mathbf{J} \subset T_pM$, that is,

$$\operatorname{Ker} T_p\mathbf{J} = \mathfrak{g}_\mu \cdot p \oplus N$$

is called a **symplectic slice** at $p \in M$.

The symplectic slice is a symplectic vector subspace of $(T_pM, \omega(p))$. If the action of G on M is free then $T_p\pi_{G_\mu}$ establishes a symplectomorphism of N and $T_{\pi_{G_\mu}(p)}(\mathbf{J}^{-1}(\mu)/G_\mu)$. As it is like a linearization of the reduced space, this subspace plays a key role in the geometry and dynamics of M. For example, the Energy-Momentum method test to ensure non-linear stability of a G invariant Hamiltonian system relies on the evaluation of a certain matrix on a symplectic slice N.

However, even if the action of G is not free, we can always choose a symplectic slice at p. In fact, the symplectic slice is one of the parts of a four-fold linear equivariant splitting of the tangent space known as the Witt-Artin decomposition (see [OR04] and [CB97]).

Proposition 2.2.2. *Let $(M, \omega, \mathbf{J})$ be a Hamiltonian G-space with equivariant momentum map $\mathbf{J}\colon M \to \mathfrak{g}^*$. Fix a point $p \in M$ and denote $\mathbf{J}(p) = \mu$ there is a G_p-invariant splitting*

$$T_zM = \mathfrak{g}_\mu \cdot p \oplus W \oplus \mathfrak{q} \cdot p \oplus N$$

such that

- *$\mathfrak{q}$ is a G_p-invariant complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$. That is, $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$.*
- *N is a symplectic subspace symplectomorphic to a symplectic slice at p.*
- *W is isotropic and G_p-equivariantly isomorphic to $(\mathfrak{g}_\mu/\mathfrak{g}_p)^*$ via the map $f\colon W \to (\mathfrak{g}_\mu/\mathfrak{g}_p)^*$ defined by*
$$\langle f(w), \xi \rangle = \omega(p)(w, \xi_M(p))$$
- *$\mathfrak{g}_\mu \cdot p \oplus W$, $\mathfrak{q} \cdot p$ and N are symplectic subspaces orthogonal with respect to $\omega(z)$.*

This result implies that under this splitting the symplectic form block-diagonalizes as

$$\omega(p) = \begin{array}{c} \begin{array}{cccc} \mathfrak{g}_\mu \cdot p & W & \mathfrak{q} \cdot p & N \end{array} \\ \begin{bmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \end{array}$$

That is, if $u_i = \xi_i \cdot p + w_i + \eta_i \cdot p + v_i \in \mathfrak{g}_\mu \cdot p \oplus W \oplus \mathfrak{q} \cdot p \oplus N$ with $i = 1, 2$ then

$$\begin{aligned} \omega(p)(u_1, u_2) &= \omega(p)(\xi_1, w_2) + \omega(p)(w_1, \xi_2) + \omega(p)(\eta_1 \cdot p, \eta_2 \cdot p) + \omega(p)(v_1, v_2) \\ &= \omega(p)(\xi_1, w_2) + \omega(p)(w_1, \xi_2) + \langle \mu, [\eta_1, \eta_2] \rangle + \omega(p)(v_1, v_2) \\ &= -\langle \xi_1, f(w_2) \rangle + \langle \xi_2, f(w_1) \rangle + \langle \mu, [\eta_1, \eta_2] \rangle + \omega(p)(v_1, v_2) \end{aligned}$$

2.3 MGS form

If G acts Hamiltonially on a symplectic space M, it possible to have a semi-local model like the one for G-spaces. This is the content of the **Marle-Guillemin-Sternberg normal form** proven by Marle Guillemin and Sternberg in [Mar85; GS84] for compact groups and extended to proper actions of arbitrary groups with equivariant momentum map in [BL97]. A similar result can be obtained even if we drop the assumption of equivariance of the momentum map [OR04].

2.3.1 Abstract symplectic tube

Let G be a Lie group, $\mu \in \mathfrak{g}^*$ and $K \subset G_\mu$ a compact subgroup. As K is compact we can choose a K-invariant complement of $\mathfrak{g}_\mu$ in $\mathfrak{g}$. This choice induces a K-equivariant linear inclusion $\iota\colon \mathfrak{g}_\mu^* \to \mathfrak{g}^*$. Consider the product $T_\mu = G \times \mathfrak{g}_\mu^*$ and the embedding

$$\begin{aligned} T_\mu &\longrightarrow T^*G \\ (g,\nu) &\longmapsto T_eL_{g^{-1}}^*(\mu + \iota(\nu)). \end{aligned}$$

With this map we can pull-back the canonical symplectic form of T^*G (see Proposition 1.3.3) obtaining the two-form ω_{T_μ} given by

$$\omega_{T_\mu}(g,\nu)(v_1,v_2) = \langle \dot{\nu}_2, \xi_1 \rangle - \langle \dot{\nu}_1, \xi_2 \rangle + \langle \mu + \iota(\nu), [\xi_1,\xi_2] \rangle$$

where $v_i = (T_eL_g\xi_i, \dot{\nu}_i) \in T_{(g,\nu)}(G \times \mathfrak{g}_\mu^*)$. This form satisfies $\omega_{T_\mu} = -\mathbf{d}\theta_{T_\mu}$ where

$$\theta_{T_\mu}(g,\nu)(v_1) = \langle \mu + \iota(\nu), \xi_1 \rangle. \tag{2.3}$$

It can be checked that, for any $g \in G$ the two-form $\omega_{T_\mu}(g,0)$ is non-degenerate, and therefore there is an open K-invariant neighborhood $(\mathfrak{g}_\mu^*)_r$ of $0 \in \mathfrak{g}_\mu^*$ such that $(G \times (\mathfrak{g}_\mu^*)_r, \omega_{T_\mu})$ is a symplectic space (see Proposition 7.2.2 of [OR04]).

Let (N, ω_N) be a symplectic linear space with a K-Hamiltonian action with momentum map

$$\langle \mathbf{J}_N(v), \xi \rangle := \frac{1}{2}\omega_N(\xi \cdot v, v).$$

The symplectic product $Z := G \times ((\mathfrak{g}_\mu^*)_r \times N)$ with two-form $\omega_{T_\mu} + \omega_N$ is a **symplectic space** and the natural G^L and K^T-actions are free and Hamiltonian with momentum maps:

$$\mathbf{K}_{K^T}(g,\nu,v) = -\nu\big|_{\mathfrak{k}} + \mathbf{J}_N(v), \quad \mathbf{K}_{G^L}(g,\nu,v) = \mathrm{Ad}_{g^{-1}}^*\nu \tag{2.4}$$

By regular symplectic reduction (see Theorem 1.4.1), the quotient $\mathbf{K}_{K^T}^{-1}(0)/K^T$ is a symplectic manifold. Since G^L and K^T actions commute, then the induced G-action on this quotient is also Hamiltonian.

We now build a useful representation of the abstract reduced space $\mathbf{K}_{K^T}^{-1}(0)/K^T$. Choose a K-invariant complement $\mathfrak{m}$ of $\mathfrak{k} = \mathrm{Lie}(K)$ in $\mathfrak{g}_\mu$. There are small enough open neighborhoods $\mathfrak{m}_r^*$ and N_r of the origin in $\mathfrak{m}^*$ and N such that $\nu + \mathbf{J}_N(v) \in (\mathfrak{g}_\mu^*)_r$ for every $\nu \in \mathfrak{m}_r^*$ and $v \in N_r$. In this setting, the map

$$\begin{aligned} L\colon G \times_K ((\mathfrak{m}^*)_r \times N_r) &\longrightarrow \mathbf{K}_{K^T}^{-1}(0)/K^T \\ [g,\nu,v]_K &\longmapsto [g, \nu + \mathbf{J}_N(v), v]_K \end{aligned} \tag{2.5}$$

is a well-defined G-equivariant diffeomorphism onto its image. We can endow the space $Y_r = G \times_K (\mathfrak{m}_r^* \times N_r)$ with the symplectic form

$$\begin{aligned}\Omega_Y(T_{(g,\nu,v)}\pi_K(u_1), T_{(g,\nu,v)}\pi_K(u_2)) = \langle \dot\nu_2 + T_v\mathbf{J}_N(\dot v_2), \xi_1\rangle - \langle \dot\nu_1 + T_v\mathbf{J}_N(\dot v_1), \xi_2\rangle + \\ + \langle \nu + \mathbf{J}_N(v) + \mu, [\xi_1, \xi_2]\rangle + \omega(\dot v_1, \dot v_2)\end{aligned} \tag{2.6}$$

where $u_i = (T_eL_g\xi_i; \dot\nu_i, \dot v_i) \in T_{(g,\nu,v)}(G \times \mathfrak{m}^* \times N)$ and $\pi_K \colon G \times (\mathfrak{m}^* \times N) \to G \times_K (\mathfrak{m}^* \times N)$. Then L becomes a G-equivariant symplectomorphism between (Y_r, Ω_Y) and $\mathbf{K}_{K^T}^{-1}(0)/K^T$ equipped with the reduced symplectic form.

To sum up,

Proposition 2.3.1. *Let G be a Lie group, $\mu \in \mathfrak{g}^*$, $K \subset G_\mu$ a compact subgroup and (N, ω_N) a linear symplectic space with K action preserving the symplectic form ω_N.*

Let $\mathbf{J}_N \colon N \to \mathfrak{k}^*$ be the momentum map for the K-action on N, that is, $\langle \mathbf{J}_N(v), \xi\rangle = \frac{1}{2}\omega_N(\xi \cdot v, v)$. Let $\mathfrak{m}$ be a K-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}_\mu$. There are K-invariant neighborhoods of the origin $\mathfrak{m}_r^* \subset \mathfrak{m}^*$ and $N_r \subset N$ such that the twisted product

$$Y_r := G \times_K (\mathfrak{m}_r^* \times N_r)$$

endowed with the two-form (2.6) is a symplectic space. Moreover, the G-action $g' \cdot [g, \nu, v]_K = [g'g, \nu, v]_K$ is a Hamiltonian action with momentum map

$$\mathbf{J}_Y([g, \nu, v]_K) = \mathrm{Ad}^*_{g^{-1}}(\mu + \nu + \mathbf{J}_N(v)). \tag{2.7}$$

We will say that (Y_r, ω_Y) is a **MGS-model associated** with $(G, \mu, K, (N, \omega_N))$.

2.3.2 Equivariant Darboux

Let G be a Lie group acting properly on a symplectic manifold (M, ω). In this setting, the classical Darboux Theorem can be extended to an equivariant version, as the following result shows.

Theorem 2.3.2 ([BL97], Theorem 6)**.** *Let M be a manifold and ω_0, ω_1 two symplectic forms on it. Let G be a Lie group acting properly on M and preserving both ω_0 and ω_1. Let $p \in M$ and assume that:*

$$\omega_0(g \cdot p)(v_{g\cdot p}, w_{g \cdot p}) = \omega_1(g \cdot p)(v_{g\cdot p}, w_{g\cdot p})$$

for all $g \in G$ and $v_{g\cdot p}, w_{g\cdot p} \in T_{g\cdot p}M$. Then there exist two open G-invariant neighborhoods U_0 and U_1 of $G \cdot p$ and a G-equivariant diffeomorphism $\Psi \colon U_0 \to U_1$ such that $\Psi|_{G\cdot p} = \mathrm{Id}$ and $\Psi^*\omega_1 = \omega_0$.

Remark 2.3.3. As in the classical Darboux theorem, we can only state the existence of Ψ as the solution of a Moser equation, so it is usually very difficult to construct Ψ explicitly or to have some fine control of its properties even for simple examples.

The proof of this result is just an equivariant refinement of Moser's proof of Darboux Theorem. We sketch the proof given in Theorem 7.3.1 of [OR04].

Proof. Using Theorem 2.1.4 at $p \in M$, there is

$$\mathbf{t} \colon G \times_{G_p} S_r \longrightarrow \mathbf{t}(G \times_{G_p} S_r) \subset M$$

such that $\mathbf{t}([e,0]_{G_p}) = p$ and S_r is a G_p-invariant open subset of a linear slice at p. For any $u = \mathbf{t}([g,v]_{G_p})$ the expression $\phi_t(u) = \mathbf{t}([g,(1-t)v]_{G_p})$ defines a diffeomorphism $\phi_t\colon U \to \phi_t(U)$ for any t. Then,

$$\begin{aligned}\omega_0 - \omega_1 &= \phi_1^*(\omega_1 - \omega_0) - (\omega_1 - \omega_0) \\ &= \int_0^1 \frac{d}{dt}\phi_t^*(\omega_1 - \omega_0)dt \\ &= \int_0^1 \phi_t^*(\mathbf{L}_{Y_t}(\omega_1 - \omega_0))dt \\ &= \int_0^1 \phi_t^*(\mathbf{d}i_{Y_t}(\omega_1 - \omega_0))dt \\ &= \mathbf{d}\int_0^1 \phi_t^*(i_{Y_t}(\omega_1 - \omega_0))dt\end{aligned}$$

where Y_t is the vector field defined by $Y_t(z) = \frac{d}{dt}(\phi_t(z))$. Let $\alpha = \int_0^1 \phi_t^*(i_{Y_t}(\omega_1 - \omega_0))dt \in \Omega^1(U)$; note that this one-form is G-invariant, $\omega_0 - \omega_1 = \mathbf{d}\alpha$ and $\alpha(g\cdot p) = 0$.

Consider the family of two-forms $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$; since $\omega_t(g\cdot p) = \omega_0(g\cdot p) = \omega_1(g\cdot p)$, $\omega_t(g\cdot p)$ is non-degenerate for any $g \in G$ and $t \in [0,1]$. As non-degeneracy is an open condition, for any $t_0 \in [0,1]$, there is $\epsilon > 0$ and $V_{t_0} \subset U$ G-invariant such that $\omega_t(z)$ is non-degenerate if $z \in V_{t_0}$. As $[0,1]$ is compact we can cover it with a finite number of open sets, and thus there is V G-invariant open set such that $\omega_t(z)$ is non degenerate if $t \in [0,1]$ and $z \in V$. Therefore, ω_t is a family of symplectic forms on V.

Now we apply Moser's trick to the family ω_t; the Moser equation

$$i_{X_t}\omega_t = \alpha$$

defines a time-dependent vector field X_t on the open set $G \times V$. If Ψ_t is the local flow of the vector field X_t then,

$$\begin{aligned}\frac{d}{dt}\Psi_t^*\omega_t &= \Psi_t^*(\mathbf{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t) \\ &= \Psi_t^*(i_{X_t}\mathbf{d}\omega_t + \mathbf{d}i_{X_t}\omega_t + \omega_1 - \omega_0) \\ &= \Psi_t^*(\mathbf{d}i_{X_t}\omega_t + \omega_1 - \omega_0) \\ &= \Psi_t^*(\mathbf{d}\alpha + \omega_1 - \omega_0) = 0.\end{aligned}$$

As $\Psi_0 = \mathrm{Id}$, this implies $\Psi_t^*\omega_t = \omega_0$. As X_t is G-invariant, Ψ_t is a G-equivariant diffeomorphism, and as $X_t(g\cdot p) = 0$ there is a G-invariant neighborhood $W \subset V$ such that Ψ_1 is well defined on it. Therefore $\Psi_1\colon W \to \Psi_1(W)$ and $\Psi_1^*\omega_1 = \omega_0$, as we wanted to show. □

2.3.3 Hamiltonian Tube

From the Witt-Artin decomposition and the G-relative Darboux theorem, Marle Guillemin and Sternberg in [Mar85; GS84] built the **Marle-Guillemin-Sternberg normal form**, which is the normal form for each proper Hamiltonian action of a Lie group G on a symplectic manifold M.

Theorem 2.3.4 (Hamiltonian Tube Theorem)**.** *Let (M,ω) be a symplectic manifold and let G be a Lie group acting properly and Hamiltonially on M with equivariant momentum map $\mathbf{J}\colon M \to \mathfrak{g}^*$. Let $z \in M$, denote $\mu = \mathbf{J}(z)$, choose a G_z-invariant splitting $\mathfrak{g}_\mu = \mathfrak{g}_z \oplus \mathfrak{m}$ on $\mathfrak{g}_\mu$ and a G_z-invariant splitting* $\operatorname{Ker} T_z\mathbf{J} = \mathfrak{g}_\mu \cdot z \oplus N$.

There exists $Y_r = G \times_{G_z} (\mathfrak{m}_r^* \times N_r)$ *a MGS model associated to* $(G, \mu, G_z, (N, \omega(z)\big|_N))$ *and a map*

$$\mathcal{T} \colon Y_r \longrightarrow M$$

such that

- $\mathcal{T} \colon Y_r \longrightarrow \mathcal{T}(Y_r) \subset M$ *is a* G*-equivariant diffeomorphism onto the open set* $\mathcal{T}(Y_r) \subset M$ *and* $\mathcal{T}([e, 0, 0]_{G_z}) = z$.
- $\mathcal{T}^*\omega = \Omega_Y$.

A map $\mathcal{T} \colon Y_r \longrightarrow M$ *will be called a* ***Hamiltonian tube around*** z *if it satisfies the conditions above.*

A detailed proof can be found in Theorem 7.4.1 of [OR04]; here we only briefly sketch the main points.

Proof. The tangent space T_zM can be decomposed using the Witt-Artin decomposition (Proposition 2.2.2) as

$$T_zM = \mathfrak{g}_\mu \cdot z \oplus W \oplus \mathfrak{q} \cdot p \oplus N.$$

Note that $W \oplus N \subset T_zM$ is a linear slice at $z \in M$. Using this linear slice we can build a Palais' tube

$$\mathbf{t} \colon G \times_{G_z} (W \oplus N)_r \to U \subset M.$$

Let $Y_r = G \times_{G_z} (\mathfrak{m}_r^* \times N_r)$ be MGS-model associated with $(G, \mu, G_z, (N, \omega(z)\big|_N))$, consider the map

$$\begin{aligned} \Psi \colon G \times_{G_z} (\mathfrak{m}_r^* \times N_r) &\longrightarrow M \\ [g, \nu, v]_{G_z} &\longmapsto \mathbf{t}([g, f^{-1}(\nu) + v]_{G_z}). \end{aligned}$$

Via Ψ on $Y_r = G \times_{G_z} (\mathfrak{m}_r^* \times N_r)$, we can consider the two-form $\Psi^*\omega$ and it can be checked that $\Psi^*\omega$ and Ω_Y are G-invariant forms such that $(\Psi^*\omega)(g \cdot z) = \Omega_Y(g \cdot z)$ for any $g \in G$. Using Theorem 2.3.2, there is a diffeomorphism $\Theta \colon U_0 \to U_1$ such that $\Theta^*\Psi^*\omega = \Omega_Y$, then the map $\Psi \circ \Theta$ is a Hamiltonian tube at z.

□

One of the most important consequences of the MGS model is that it provides a local description of the set of points with momentum μ, which is very useful in the theory of singular reduction. This is the content of the following result of [OR04] based on [BL97].

Proposition 2.3.5 ([OR04], Proposition 8.1.2)**.** *Let* (M, ω) *be a symplectic manifold supporting a Hamiltonian* G*-action with momentum map* $\mathbf{J}$*. Let* $m \in M$, $\mu = \mathbf{J}(m)$ *and* $\mathcal{T} \colon G \times_{G_m} (\mathfrak{m}_r^* \times N_r) \to M$ *a Hamiltonian tube around* m*. There is an open* G_μ*-invariant neighborhood* U_M *of* $G_\mu \cdot m$ *such that*

$$U_M \cap \mathbf{J}^{-1}(\mu) = \mathcal{T}(Z)$$

where

$$Z = \{[g, \nu, v]_{G_m} \in \mathcal{T}^{-1}(U_M) \mid g \in G_\mu,\ \nu = 0,\ \mathbf{J}_N(v) = 0\}.$$

Chapter 3

Stratifications and singular reduction

In the general case of a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω), neither the quotient M/G nor the quotients $\mathbf{J}^{-1}(\mu)/G_\mu$ are smooth manifolds. Nevertheless, they are always topological spaces. In fact, they have much more structure than simple topological spaces. They have the important property that, although they are not manifolds, they admit a partition onto locally closed subsets where each of them has the structure of a smooth manifold.

Following this idea, in Section 3.1 we introduce the category of stratified spaces and its properties. In Section 3.2 we describe how a G-space M can be decomposed into different submanifolds and how this decomposition endows the quotient M/G with a smooth stratified structure. Using all these tools, in Theorem 3.3.1 we can state the singular analogue of Marsden-Weinstein reduction procedure. Finally, in Section 3.4, we briefly review recent developments regarding singular cotangent-bundle reduction.

3.1 Stratified spaces

In the literature, there are several and often non-equivalent ways of defining stratified spaces and related concepts. In this work we will follow the conventions of [Pfl01].

3.1.1 Decompositions

Definition 3.1.1. Let X be a paracompact Hausdorff space with countable topology and $\mathcal{Z}$ a set of sets of X. The pair $(X, \mathcal{Z})$ is a **decomposed space** if the following conditions are satisfied:

1. The pieces $S \in \mathcal{Z}$ cover X and are disjoint.
2. Every piece $S \in \mathcal{Z}$ is a locally closed subset of X and it has a manifold structure compatible with the induced topology.
3. The collection $\mathcal{Z}$ is locally finite.
4. If $R \cap \overline{S} \neq \emptyset$ for a pair of pieces $R, S \in \mathcal{Z}$ then $R \subset \overline{S}$. This requirement is usually called the **frontier condition**.

Recall that a locally closed subset A of a topological space is a subset such that each of its points has an open neighborhood U such that $U \cap A$ is closed in U.

The elements of the set of subsets $\mathcal{Z}$ are usually called **pieces**. The boundary of a piece $R \in \mathcal{Z}$ is the set $\partial R = \overline{R} \setminus R$. Note that if $S \in \mathcal{Z}$, $R \neq S$ and $R \cap \overline{S} \neq \emptyset$ then the frontier condition implies that $R \subset \partial S$. In this case we say that R is **incident** to S and we write $R \prec S$.

When a pair $(X, \mathcal{Z})$ satisfies all the properties in Definition 3.1.1 except the frontier condition, we will say that $(X, \mathcal{Z})$ is a **generalized decomposition** or a **generalized decomposed space**. This concept appears in [TT13] under the name of a prestratification.

A continuous mapping $f\colon P \to Q$ between (generalized) decomposed spaces $(P, \mathcal{Z})$ and $(Q, \mathcal{Y})$ is a **morphism of decomposed spaces** if for every piece $S \in \mathcal{Z}$ there is a piece $T \in \mathcal{Y}$ such that $f(S) \subset T$ and the restriction $f\big|_S\colon S \to T$ is smooth. We say that $(X, \mathcal{Z}_1)$ is a coarser decomposition than $(X, \mathcal{Z}_2)$ if the identity mapping $(X, \mathcal{Z}_2) \to (X, \mathcal{Z}_1)$ is a morphism of decomposed spaces.

If $(X_1, \mathcal{Z}_1)$ and $(X_2, \mathcal{Z}_2)$ are stratified spaces, then the cartesian product $X_1 \times X_2$ is a decomposed space with pieces the product of pieces of X_1 and X_2.

Note that a smooth manifold M is a decomposed space if we consider the single piece decomposition $\mathcal{Z} = \{M\}$.

In general, a subspace of a decomposed space is not a decomposed space. Let Y be a subset of $(X, \mathcal{Z})$ even if, for any piece $S \in \mathcal{Z}$ the intersection $X \cap S$ is a manifold, the collection of sets $\{X \cap S\}_{S \in \mathcal{Z}}$ could not satisfy the frontier condition.

Remark 3.1.2. Consider, for example, $X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$ stratified by $S_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ and $S_2 = \{(x, y) \mid x > 0\}$. Let $Y = \{(x, 0)\} \cup \{(0, y)\}$; both $\{S_1 \cap N\}$ and $\{S_2 \cap N\}$ are manifolds, but the partition $\{S_1 \cap N, S_2 \cap N\}$ does not satisfy the frontier condition. This example is an adaptation of an example presented in [MTP03].

3.1.2 Stratifications

Let X be a topological space; the set germ of a subset $A \subset X$ is the equivalence class of subsets $[A]_x$ defined by $[A]_x = [B]_x$ if A and B are subsets of X and there is an open neighborhood U of x such that $A \cap U = B \cap U$.

Definition 3.1.3. A map $\mathcal{S}$ from X to set germs of subsets of X is a **stratification** if for any $x \in X$ there is a neighborhood U containing x and a decomposition $\mathcal{Z}$ of U such that, for any $y \in U$, $\mathcal{S}_y = [Z]_y$ where $Z \in \mathcal{Z}$ is the unique piece of $\mathcal{Z}$ containing y.

The pair $(X, \mathcal{S})$ is called a **stratified space**.

Note that any decomposition $\mathcal{Z}$ induces a stratification by associating to each of its points the set germ of the piece on which it is sitting. In some sense a stratification should be understood as a way of identifying **equivalent decompositions** of a set. For example, the set $\mathbb{R}$ can be decomposed as $\{(-\infty, 0), \{0\}, (0, \infty)\}$ or as $\{\mathbb{R} \setminus \{0\}, \{0\}\}$, and both decompositions although different induce the same stratification.

A continuous map $f\colon X \to Y$ between stratified spaces $(X, \mathcal{S}_1)$ and $(Y, \mathcal{S}_2)$ is a **stratified map** if for every $z \in X$ there are neighborhoods V of $f(z)$ and $U \subset f^{-1}(V)$ of z and decompositions $\mathcal{Z}_1$ of U and $\mathcal{Z}_2$ of V inducing $\mathcal{S}_1\big|_U$ and $\mathcal{S}_2\big|_V$ such that the restricted map $f\big|_U\colon U \to V$ is a decomposed map.

In fact, every stratification is induced by a canonical decomposition associated to it, as the following proposition shows.

Proposition 3.1.4 ([Pfl01], Proposition 1.2.7)**.** *Let $(X, \mathcal{S})$ be a stratified space; there is a decomposition $\mathcal{Z}_{\mathcal{S}}$ with the following maximal property: for every open subset $U \subset X$ and every decomposition $\mathcal{D}$ of U inducing $\mathcal{S}$ on U the restriction of $\mathcal{Z}_{\mathcal{S}}\big|_U$ is coarser than $\mathcal{D}$.*

*This unique decomposition will be called the **canonical decomposition** and its pieces are called the **strata** of* $(X, \mathcal{S})$.

3.1.3 Local triviality

Among the class of stratified spaces, those that around a given strata are the product of a strata and a stratified space seem to be the simplest ones; this idea is the one behind the local triviality of a stratified space.

Definition 3.1.5. A stratified space $(X, \mathcal{S})$ is called **topologically locally trivial** if for every $x \in X$ there is a neighborhood U, a stratified space $(F, \mathcal{F})$, a distinguished point $o \in F$ and an isomorphism of stratified spaces

$$h\colon U \longrightarrow (S \cap U) \times F \tag{3.1}$$

such that $h^{-1}(y, o) = y$ for all $y \in S \cap U$ where S is the stratum of X containing x and $\mathcal{F}(o)$ is the germ set of $\{o\}$. In other words, the stratum of $(F, \mathcal{F})$ containing o is $\{o\}$. We call the stratified set $(F, \mathcal{F})$ the typical fiber over x.

3.1.4 Smooth structure

A decomposed or stratified subspace is the union of submanifolds, but the stratified spaces we are going to work with have even more structure, a set of smooth functions on the whole space X. This smooth structure is generated by local charts as in usual manifolds.

Definition 3.1.6. Let $(X, \mathcal{S})$ be a stratified space and $\mathcal{S}$ the family of its strata. A **singular chart** is a homeomorphism $\mathbf{x}\colon U \to \mathbf{x}(U) \subset \mathbb{R}^n$ such that for each stratum $S \in \mathcal{S}$ the image $\mathbf{x}(S)$ is a submanifold of $\mathbb{R}^n$ and the restriction $\mathbf{x}\big|_{U\cap S}\colon U \cap S \to \mathbf{x}(U \cap S)$ is a diffeomorphism. Two charts $\mathbf{x}_1\colon U_1 \to \mathbf{x}_1(U_1) \subset \mathbb{R}^{n_1}$ and $\mathbf{x}_2\colon U_2 \to \mathbf{x}_2(U_2) \subset \mathbb{R}^{n_2}$ are called compatible if there is a diffeomorphism $H\colon O_1 \to O_2$ where O_1, O_2 are open sets of $\mathbb{R}^m$ such that

$$(i^m_{n_1} \circ \mathbf{x}_1)(y) = (H \circ i^m_{n_2} \circ \mathbf{x}_2)(y) \quad \forall y \in U_1 \cap U_2$$

where i^m_n is the canonical embedding of $\mathbb{R}^n$ into $\mathbb{R}^m \cong \mathbb{R}^n \times \mathbb{R}^{m-n}$. As in standard differential geometry, this allows us to define a singular atlas on X as set of compatible charts, and a maximal atlas will be called a **smooth structure** on X.

A continuous function $f\colon X \to \mathbb{R}$ is called **smooth** if for each singular chart $\mathbf{x}\colon U \to \mathbf{x}(U) \subset \mathbb{R}^n$ there is a smooth function $g\colon \mathbb{R}^n \to \mathbb{R}$ such that $g \circ \mathbf{x} = f\big|_U$; we will denote by $C^\infty(X)$ the set (and sheaf) of smooth functions. We say that a map $f\colon X \to Y$ between two stratified spaces with smooth structures is smooth, continuous and $f^*g \in C^\infty(X)$ for all $g \in C^\infty(Y)$.

Remark 3.1.7. The smooth structure we have defined corresponds to the concept of **weakly smooth structures** of [Pfl01] and to the concept of **stratified subcartesian spaces** of [Sni13]. In the notation of [Pfl01], a singular chart $\mathbf{x}\colon U \to \mathbf{x}(U) \subset \mathbb{R}^n$ satisfies the additional condition that $\mathbf{x}(U)$ is a locally closed subset of $\mathbb{R}^n$. We are not going to impose this condition because we will encounter some spaces that do not have smooth structure in the sense of [Pfl01], but does have one in the sense of the previous definition.

3.1.5 Whitney condition

Even in the class of stratified spaces with smooth structure there are spaces that look very pathological. The Whitney condition is a property concerning how the different strata fit together and is closely related to the local triviality of the stratification.

Definition 3.1.8. Consider a smooth stratified space X and two embedded submanifolds R, S. Let $\mathbf{x}\colon U \to \mathbb{R}^n$ be a chart of X around $x \in R$; we say that (R, S) satisfies **Whitney condition** (or Whitney condition (B)) at $x \in R$ if:

For any sequences $\{x_n\} \subset R \cap U$ and $\{y_n\} \subset S \cap U$ such that:

- $x_n \neq y_n$ and $x_n \to x$ and $y_n \to x$.
- The sequence of lines $\overline{\mathbf{x}(x_n)\mathbf{x}(y_n)} \subset \mathbb{R}^n$ converges in the projective space to a line ℓ
- The sequence of tangent spaces $\{T_{\mathbf{x}(y_n)}(\mathbf{x}(S))\}$ converges in the Grassmann bundle of dim S-dimensional subspaces of $T\mathbb{R}^n$ to $\Sigma \subset T_{\mathbf{x}(x)}\mathbb{R}^n$

then

$$\ell \subset \Sigma.$$

If the Whitney condition is satisfied for one singular chart $\mathbf{x}$, they are satisfied for all singular charts (see [Pfl01] Lemma 1.4.4). We say that the pair (R, S) satisfies the Whitney condition if they for any $x \in R$, (R, S) satisfy Whitney condition at x. Similarly, a stratified space $(X, \mathcal{S})$ with smooth structure satisfies Whitney conditions if for each pair (R, S) of strata the pair (R, S) satisfies Whitney condition.

It can be checked that the preimage of a Whitney pair (R, S) under a submersion is again a Whitney pair.

Lemma 3.1.9 ([Gib+76], Lemma 1.4)**.** *Let N, M be two smooth manifolds. If R, S are two submanifolds of M, $f\colon N \to M$ is a submersion and (R, S) satisfies the Whitney condition, then $(f^{-1}(R), f^{-1}(S))$ satisfies the Whitney condition.*

Under mild topological assumptions, the Whitney condition even implies a frontier condition,

Theorem 3.1.10 ([Gib+76], Theorem 5.6)**.** *Let $(X, \mathcal{S})$ be a smooth stratified space and $\mathcal{M}$ a set of embedded submanifolds of X. Assume that each pair (R, S) in $\mathcal{M}$ satisfies the Whitney condition.*

If $Y = \bigcup_{M \in \mathcal{M}} M$ is a locally compact subspace then the set of connected components of manifolds in $\mathcal{M}$ forms a decomposition of Y.

See [Pfl01] Examples 1.4.8 and 1.4.9 for examples of simple stratified sets that do not satisfy Whitney condition.

The importance of the Whitney condition is that it provides an easily computable property ensuring that a given stratification is topologically locally trivial

Theorem 3.1.11 ([Gib+76], Theorem 5.2)**.** *Let $(X, \mathcal{S})$ be smooth stratified space satisfying the Whitney condition; if X is a locally compact topological space the stratification $\mathcal{S}$ is topologically locally trivial.*

Many interesting classes of sets can be Whitney stratified; for example, real and complex algebraic varieties [Whi65] and semianalytic sets of analytic manifolds [Loj65] admit Whitney stratifications. However, in this thesis we will only use the more restrictive case of semialgebraic sets.

A **semialgebraic subset** of $\mathbb{R}^n$ (Definition 2.1.1 of [Cos00]) is a subset of $\mathbb{R}^n$ determined by a Boolean combination of polynomial equations and inequalities with real coefficients.

One important result in semialgebraic geometry is **Tarski-Seidenberg's Theorem** (Theorem 2.3 in [Cos00]), which states that if $p\colon \mathbb{R}^n \to \mathbb{R}^m$ is a polynomial mapping and X is a semialgebraic subset of $\mathbb{R}^n$ then $p(\mathbb{R}^n)$ is a semialgebraic subset of $\mathbb{R}^m$.

For our purposes we will not need the fact that each semialgebraic set has a Whitney stratification; we only need a weaker form that states that the union of a semialgebraic submanifold X and a single point of its closure $\overline{X}$ form a Whitney stratified set.

Proposition 3.1.12 ([Loj65], Proposition 19.3)**.** *Let X be a semialgebraic subset $\mathbb{R}^n$ and let $a \in (\overline{X} \setminus X)$. If X is an embedded manifold then $\{X, \{a\}\}$ is a Whitney stratification of the subset $X \cup \{a\} \subset \mathbb{R}^n$ with the induced smooth structure.*

Moreover, $\{\mathbb{R}^k \times X, \mathbb{R}^k \times \{a\}\}$ is a Whitney stratification of the subset $\mathbb{R}^k \times (X \cup \{a\}) \subset \mathbb{R}^{n+k}$.

3.2 Orbit types and quotient structures

The proper action of a Lie group G on a manifold M induces a stratification of M satisfying Whitney conditions and can be used to endow the quotient M/G with the structure of a Whitney stratified space. In this section we introduce the partitions induced by the G-action and their properties.

3.2.1 Orbit types

The **conjugacy class** of a subgroup $H \subset G$ is the set of subgroups of G that are conjugated to H, that is,

$$(H) = \{H' \subset G \mid H' = gHg^{-1},\ g \in G\}$$

in the set of conjugacy classes of a group there is a partial ordering defined as

$$(H) \leq (K) \text{ if } \exists g \in G \text{ such that } gHg^{-1} \subset K.$$

In later chapters we will need to refer to conjugacy classes; conjugacy classes where the conjugation is only allowed on a subgroup and related concepts. In order to be consistent, we introduce here the notation of Definition 2.4.2 in [OR04]:

Definition 3.2.1. Let G be a Lie group acting on a manifold M. Let $H,\ J,\ K$ be closed subgroups of G such that $H \subset K \subset G$. We define

$$\begin{aligned}
(H) &= \{L \subset G \mid L = gHg^{-1},\ g \in G\} \\
(H)^J &= \{L \subset G \mid L = gHg^{-1},\ g \in J\} \\
(H)^J_K &= \{L \subset G \mid L = gHg^{-1} \subset K,\ g \in J\} \\
(H)_K &= \{L \subset G \mid L = gHg^{-1} \subset K,\ g \in G\}.
\end{aligned}$$

Using these sets of subgroups we can define the following subsets of M

$$M_{(H)} = \{z \in M \mid G_z \in (H)\}$$
$$M_H = \{z \in M \mid G_z = H\}$$
$$M^H = \{z \in M \mid H \subset G_z\}.$$

Similarly, $M_{(H)^J}, M_{(H)^J_K}$ and $M_{(H)_K}$ are the subsets of M determined by the condition $G_z \in (H)^J$, $G_z \in (H)^J_K$ or $G_z \in (H)_K$, respectively.

Analogously, if X is a subset of M, $X_{(H)}, X_H, X^H, \ldots$ represent the sets $M_{(H)} \cap X, M_H \cap X, M^H \cap X, \ldots$

The set $M_{(H)}$ is called the **(H)-orbit type** set because for any $x \in M_{(H)}$ the orbit $G \cdot x$ is G-equivariantly diffeomorphic to G/H; all orbits are of the same type. The set M_H is called the **H-isotropy type** set because all its points have isotropy H. Finally, M^H is called the set of **H-fixed points**.

Proposition 3.2.2 ([OR04], Proposition 2.4.4)**.** *Let G be a Lie group acting properly on a manifold M, then:*

- *M^H is closed in M,*
- *$M_{(H)} = G \cdot M_H$ and $M_{(H)^K} = K \cdot M_H$.*
- *$M_H = M^H \cap M_{(H)}$ is closed in $M_{(H)}$.*

If the underlying manifold is a twisted product, many of these sets have a simple description.

Proposition 3.2.3 ([OR04], Proposition 2.4.6)**.** *Let A be an H-manifold, and $G \times_H A$ be the twisted product with $H \subset G$ a compact subgroup. Then, relative to the left-action of G on $G \times_H A$ we have:*

- *The isotropy group of $[g, a]_H \in G \times_H A$ is $G_{[g,a]_H} = gH_a g^{-1}$.*
- *$(G \times_H A)_{(H)} = G \times_H A^H$.*
- *$(G \times_H A)_H = N_G(H) \times_H A^H$.*
- *If K is an isotropy group for the G action on $G \times_H A$ then: $(G \times_H A)_{(K)} = G \times_H A_{(K)^G_H} = G \times_H A_{(K)_H}$.*
- *There is only a finite number of different G-orbit types on $G \times_H A$.*

One important corollary of this result is that, as Palais tubes are local models for proper G-spaces, if a Lie group G acts properly on a manifold M, then given any point x there is an open neighborhood V of x such that for each $y \in V$,

$$(G_y) \leq (G_x);$$

that is, for neighboring points the **isotropy can only be smaller**. Similarly, using Palais tubes,

Proposition 3.2.4 ([OR04], Proposition 2.4.7)**.** *Let G be a Lie group acting properly on M; let H be an isotropy subgroup of this action and K a closed subgroup such that $H \subset K \subset G$.*

- *$M_{(H)}$, $M_{(H)^K}$, M^H and M_H are disjoint unions of embedded submanifolds of M.*
- *M_H is open in M^H.*

One problem of the orbit type subsets $M_{(H)}$ is that they can be the disjoint union of manifolds of different dimensions. To deal with this class of sets, it is convenient to introduce the following definition.

Definition 3.2.5. A topological space S is a **Σ-manifold** if it is the topological sum of countably many connected smooth and separable manifolds.

In general, a Σ-manifold is not a manifold unless all its connected components have the same dimension. A map $f\colon S_1 \to S_2$ between two Σ-manifolds is smooth if the restriction of f to each connected component of S_1 is smooth as a map between smooth connected manifolds. Most of the concepts in differential geometry have an analogue in the category of Σ-manifolds; for example, vector fields, bundles, diffeomorphisms, embeddings, ...

The Σ-submanifolds $M_{(H)} \subset M$ can be further partitioned into smaller manifolds so that all the points are locally equivalent with respect to the G-action.

Proposition 3.2.6 ([DK00] Theorem 2.6.7)**.** *Let M be a smooth manifold endowed with a proper action of G. Two points $x, y \in M$ are said to be of the same local orbit type if there is a G-equivariant diffeomorphism $f\colon U_x \to U_y$, where U_x, U_y are G-invariant neighborhoods of x and y, respectively. Define*

$$\begin{aligned} M^{\mathrm{loc}}_{G\cdot x} &= \{y \in M \mid x \text{ and } y \text{ have the same local orbit type}\} \\ M^{\mathrm{loc}}_{x} &= M^{\mathrm{loc}}_{G\cdot x} \cap M^{G_x} \end{aligned}$$

- *$M^{\mathrm{loc}}_{G\cdot x}$ is an open and closed submanifold of $M_{(G_x)}$, that is, a union of connected components of $M_{(G_x)}$.*
- *M^{loc}_{x} is open and closed in M^{G_x}, $N_G(G_x)$-invariant and satisfies $M^{\mathrm{loc}}_{G\cdot x} = G \cdot (M^{\mathrm{loc}}_{x})$.*

In fact, using Palais' tube, two points lie on the same local type manifold only if the action of G_x on $T_xM/(\mathfrak{g}\cdot x)$ is isomorphic to the action of G_y on $T_yM/(\mathfrak{g}\cdot y)$.

Proposition 3.2.7 ([DK00] Theorem 2.7.4)**.** *Let G be a Lie group acting properly on M. The orbit type sets $M_{(H)}$ form a partition of the manifold M. This partition induces a stratification, called the **orbit-type stratification** of M. This stratification satisfies the Whitney condition.*

Note that as the local-orbit type sets are unions of connected components of orbit-type sets, then both local orbit-types and orbit-types induce the same stratification.

Remark 3.2.8. Although the orbit type sets $M_{(H)}$ are disjoint, cover M and are locally finite (see [Pfl01] Lemma 4.3.6), they do not form a decomposition because the frontier condition can be violated. However, the connected components of the orbit type sets do form a decomposition of M.

3.2.2 Linear representations of compact groups

We need some results that characterize the stratification and smooth structure of the quotient of a vector space by a compact linear group. The proof of each of the parts of the Theorem can be found in [Pfl01] and [Bie75].

Theorem 3.2.9. *Let H be a compact group acting linearly on the vector space V. Denote by $\pi_H\colon V \to V/H$ the quotient map and by $C^\infty(V)^H$ the set of smooth H-invariant functions on V.*

- *The sets $V_{(K)}/H = \pi_H(V_{(K)})$ induce a stratification $\mathcal{S}$ of the topological space V/H. More precisely, the set*
$$\mathcal{Z}_{V/H} = \left\{\pi_H(Z) \mid Z \text{ is a connected component of } V_{(K)} \text{ for some } K \subset H\right\}$$
is a decomposition of V/H that induces the stratification $\mathcal{S}$.
- *The stratification $\mathcal{S}$ is minimal among all the stratifications of the topological space V/H.*
- *There are H-invariant polynomials $p_1, \ldots, p_k\colon V \to \mathbb{R}$ such that the Hilbert map*
$$\begin{aligned} \mathrm{Hilb}\colon V &\longrightarrow \mathbb{R}^k \\ v &\longmapsto (p_1(v), \ldots, p_k(v)). \end{aligned}$$
induces a homeomorphism $\overline{\mathrm{Hilb}}\colon V/H \longrightarrow \mathrm{Hilb}(V) \subset \mathbb{R}^k$ between V/H and the semialgebraic subset $\mathrm{Hilb}(V)$.
- $\overline{\mathrm{Hilb}}\colon V/H \longrightarrow \mathrm{Hilb}(V) \subset \mathbb{R}^k$ *is a singular chart of the stratified space $(V/H, \mathcal{S})$.*
- *The set of smooth functions on V/H induced by $\overline{\mathrm{Hilb}}$ satisfies*
$$C^\infty(V/H) = \{f\colon V/H \to \mathbb{R} \mid \exists g \in C^\infty(V)^H \text{ and } f \circ \pi_H = g\}.$$
- *For any K isotropy subgroup of V, $V_{(K)}$ is a semialgebraic subset of V and $\overline{\mathrm{Hilb}}(V_{(K)}/H)$ is a semialgebraic subset of $\mathbb{R}^k$.*

3.2.3 Quotient stratifications

Using a Palais' tube, we can essentially reduce the problem of the structure of M/G for a proper action to a quotient of a linear space by a compact group. In this sense, from the previous theorem:

Theorem 3.2.10 ([Pfl01], Theorem 4.4.6)**.** *Let G be a Lie group acting properly on M. Denote by $\pi_G\colon M \to M/G$ the quotient map, the sets $M_{(H)}/G = \pi_G(M_{(H)})$ induce a Whitney stratification of M/G. This stratification is minimal among all the stratifications of M/G. M/G has a smooth structure and the set of smooth functions satisfies*
$$C^\infty(M/G) = \{f\colon M/G \to \mathbb{R} \mid \exists g \in C^\infty(M)^G \text{ and } f \circ \pi_G = g\}.$$

Assume that M/G is connected, then there is one strata $Z \subset M/G$ such that it is maximal, in the sense that $\overline{Z} = M/G$, and not only that, but this stratum is connected. This is the content of the principal orbit type theorem.

Theorem 3.2.11 ([DK00], Theorem 2.8.5)**.** *Let G be a Lie group acting properly on M and assume that M/G is connected. There is a subgroup H such that $M_{(H)}$ is open and dense in M and $M_{(H)}/G$ is open, dense and connected. (H) is called the* ***principal orbit type****.*

Remark 3.2.12. The assumption that M/G is connected does not imply any restriction of generality, since an arbitrary G-manifold can be decomposed as $\pi_G^{-1}(Z)$, where Z runs through the connected components of M/G. In this case, the theorem ensures that on each Z of these components there is an open, dense and connected principal orbit type set.

3.3 Singular symplectic reduction

In the general case of a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) neither the quotient (M, ω) nor the quotients $\mathbf{J}^{-1}(\mu)/G_\mu$ are smooth manifolds. [SL91] showed that reduction at momentum value $\mu = 0$ of a compact group gives a stratified space in which all the strata are symplectic manifolds. Later, [BL97] extended this result showing that for proper actions the set $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ has a symplectic stratification if $\mathcal{O}_\mu \subset \mathfrak{g}^*$ is locally closed. [OR04] studied the point reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ and proved that they also have a symplectic stratification.

Theorem 3.3.1 ([OR04], Theorem 8.3.2)**.** *Let (M, ω) be a symplectic manifold with a proper Hamiltonian action with equivariant momentum map $\mathbf{J}\colon M \to \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ and G_μ the isotropy subgroup of μ.*

1. *Consider $\mathbf{J}^{-1}(\mu)$ as a topological subspace of M. The connected components of $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$ are embedded submanifolds that induce a Whitney stratification of $\mathbf{J}^{-1}(\mu)$.*

2. *The connected components of $M_\mu^{(K)} := (G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu$ are smooth symplectic manifolds and they form a Whitney stratification of the quotient $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$.*

3. *(Theorem 5.9 of [SL91]) If W is a connected component of $\mathbf{J}^{-1}(\mu)/G_\mu$, there is an isotropy subgroup K such that $M_\mu^{(K)} \cap W$ is an open, dense and connected subset of W.*

In the singular setting we also have an analogue of Theorem 1.4.3. The first version for compact group actions at zero momentum appeared in [SL91]. The general case for proper actions is discussed in [Mar+07], but we will only need the following version.

Theorem 3.3.2. *Let G and H be two Lie groups acting proper and Hamiltonially on a symplectic manifold (M, ω) with equivariant momentum maps $\mathbf{J}_G$ and $\mathbf{J}_H$, respectively. Assume that both actions commute, $\mathbf{J}_G$ is H-invariant and $\mathbf{J}_H$ is G-invariant. This implies that M is a proper $G \times H$-Hamiltonian space with $G \times H$-equivariant momentum map $(\mathbf{J}_G, \mathbf{J}_H)\colon M \to \mathfrak{g}^* \times \mathfrak{h}^*$.*

Let $\mu \in \mathfrak{g}^*$ and $\nu \in \mathfrak{h}^*$ assume that G_μ, H_ν are compact subgroups and that the coadjoint orbits $\mathcal{O}_\mu$ and $\mathcal{O}_\nu$ are embedded submanifolds. G induces a proper Hamiltonian action on the singular quotient $M_\nu = \mathbf{J}_H^{-1}(\nu)/H_\nu$ with equivariant momentum map $\mathbf{K}_G$ determined by $\mathbf{K}_G \circ \pi_{H_\nu} = \mathbf{J}_G$.

The reduced (stratified) symplectic space $\mathbf{K}_G^{-1}(\mu)/G_\mu$ is isomorphic as a stratified symplectic space to $(\mathbf{J}_G, \mathbf{J}_H)^{-1}(\mu, \nu)/(G_\mu \times H_\nu)$, the symplectic reduced space of M by the product action of $G \times H$.

3.4 Singular cotangent bundle reduction

As in the free case, one expects that the reduced space will admit additional structure if the symplectic manifold is a cotangent bundle. Up to our knowledge, the first work on singular symplectic reduction in the case of cotangent bundles is [Mon83], where the author imposes several strong conditions to ensure that all the relevant sets are smooth.

Later, [ER90] gave an analogue of Theorem 1.5.1 when the action on the base consists of only one orbit type. They showed that $\mathbf{J}^{-1}(0)$ has only one orbit type and its quotient is symplectomorphic to a cotangent-bundle. More precisely;

Theorem 3.4.1 ([ER90]). *Let G be a Lie group acting properly on a manifold Q and on T^*Q by cotangent lifts. If all the points of Q have the same isotropy type, that is, $Q = Q_{(H)}$ for a subgroup H of G, then,*

- $\{G_z \mid z \in \mathbf{J}^{-1}(0)\} = \{G_q \mid q \in Q\} = (H)$.
- $\mathbf{J}^{-1}(0)/G$ *is a symplectic manifold symplectomorphic to $T^*(M/G)$ endowed with the canonical symplectic form.*

Under the assumption that Q is of single orbit type [HR06; Hoc08; PRO09] have developed a description of the orbit-reduced space $\mathbf{J}^{-1}(G \cdot \mu)/(G \cdot \mu)$.

Even for $\mu = 0$ the reduction of a cotangent bundle without the single-orbit presents some difficulties. The cotangent-bundle projection $\tau \colon T^*Q \to Q$ induces the continuous surjective map $\tau^0 \colon \mathbf{J}^{-1}(0)/G \to Q/G$; using the results of [RO06], it can be checked that

$$\tau^0((\mathbf{J}^{-1}(0))_{(H)}/G) = \overline{Q_{(H)}/G} \subset Q/G.$$

Therefore, if $\mathbf{J}^{-1}(0)/G$ is endowed with the stratification given by Theorem 3.3.1 and Q/G is stratified according to Theorem 3.2.10, then τ^0 is not a stratified map because it does not map strata onto strata. To solve this problem we need to define a finer stratification on $\mathbf{J}^{-1}(0)/G$.

Theorem 3.4.2 ([PROSD07]). *Let G be a Lie group acting properly on the manifold Q and on T^*Q by cotangent lifts. Let $\mathbf{J} \colon T^*Q \to \mathfrak{g}^*$ be the associated momentum map. Given two compact subgroups $H, K \subset G$, the set*

$$s_{H \to K} = \mathbf{J}^{-1}(0) \cap \tau^{-1}(Q_{(H)}) \cap (T^*Q)_{(K)} \tag{3.2}$$

is a Σ-submanifold of $\mathbf{J}^{-1}(0)$. Similarly, its G-quotient

$$\mathcal{S}_{H \to K} = s_{H \to K}/G \subset \mathbf{J}^{-1}(0)/G$$

is also a Σ-manifold.

The collection of Σ-submanifolds $\{s_{H \to K} \subset \mathbf{J}^{-1}(0) \mid H, K \subset G\}$ induces a stratification of $\mathbf{J}^{-1}(0)$ and the collection $\{\mathcal{S}_{H \to K} \subset \mathbf{J}^{-1}(0)/G \mid H, K \subset G\}$ induces a stratification of $\mathbf{J}^{-1}(0)/G$. Additionally,

1. *If Q/G is endowed with the orbit type stratification, then the map*

$$\tau^0 \colon \mathbf{J}^{-1}(0)/G \to Q/G$$

is a stratified fibration, and $\tau^0(\mathcal{S}_{H \to K}) = Q_{(H)}/G$, for any pair $K \subset H \subset G$.

2. *$\mathcal{S}_{H\to K}$ is a coisotropic submanifold of the symplectic strata $(\mathbf{J}^{-1}(0)_{(K)}/G, \omega_0^{(H)})$ of Theorem 3.3.1.*

3. *$\mathcal{S}_{H\to H}$ is an open and dense symplectic submanifold of $(\mathbf{J}^{-1}(0)_{(H)}/G, \omega_0^{(H)})$ and it is symplectomorphic to $T^*(Q_{(H)}/G)$ endowed with the canonical symplectic form.*

This result can be generalized to μ totally isotropic ($G = G_\mu$) (see [RO04]).

Chapter 4

Witt-Artin decomposition for cotangent-lifted actions

In this chapter we characterize the symplectic slice and the Witt-Artin decomposition in the case of a cotangent-lifted action. The computation of the symplectic slice for non-free cotangent-lifted actions was studied in [Sch01; Sch07]; using commuting reduction, T. Schmah described the symplectic slice for several cases. Later, [RO04; PROSD08] gave a full explicit description of the symplectic slice for any cotangent-lifted action.

In this chapter we will give alternative proofs of the results of [RO04; PROSD08] and extend them to the construction of a full Witt-Artin decomposition. Most of the results in this chapter can be regarded as a linearization of the Hamiltonian cotangent tube described in the next chapter.

In Section 4.1 we define a four-fold splitting of the tangent space that will be used throughout this chapter. In Section 4.3 we introduce a splitting of the Lie algebra which will be crucial for all the thesis (Proposition 4.2.1). With these tools, in Proposition 4.3.1 we describe the symplectic slice for a cotangent-lifted action. This description can be extended (Proposition 4.4.1) to a full Witt-Artin decomposition of the tangent space. In Section 4.6 we show that, using the appropriate Lie algebra splitting, based on the ideas of [Sch01; Sch07], we can give an alternative description of the symplectic slice. Finally, in Section 4.7 we study the Witt-Artin decomposition in the case of a homogeneous space. This example will be the linear analogue of the restricted G-tubes defined in Definition 5.1.5.

Throughout this chapter, Q is a smooth manifold acted properly by the Lie group G; T^*Q is a symplectic manifold equipped with the canonical symplectic form ω; $\tau\colon T^*Q \to Q$ is the natural projection, and T^*Q is endowed with the cotangent-lifted action with momentum map $\mathbf{J}\colon T^*Q \to \mathfrak{g}^*$ given by (1.7). We will fix a point $z \in T^*Q$ and denote $q = \tau(z)$ and $H = G_q$.

4.1 Initial trivialization

The space of vertical vectors at $z \in T^*Q$, that is, the kernel of $T_z\tau$, is a G_z-invariant subspace of $T_z(T^*Q)$. In fact it is a G_z-invariant Lagrangian subspace of $T_z(T^*Q)$. The following result from symplectic linear algebra ensures that there are G_z-invariant Lagrangian complements to $\operatorname{Ker} T_z(T^*Q)$.

Lemma 4.1.1 (Lemma 7.1.2 [OR04]). *Let (E, Ω) be a symplectic vector space with an action of a compact Lie group G. Then any G-invariant Lagrangian subspace of (E, Ω) has a G-invariant Lagrangian complement.*

A choice of a Lagrangian splitting $T_z(T^*Q) = \operatorname{Ker} T_z\tau \oplus \operatorname{Hor}_z$ is the starting point of a trivialization of $T_z(T^*Q)$ that will be very useful to compute the symplectic slice and the Witt-Artin decomposition.

Proposition 4.1.2. *Let G be a Lie group acting by cotangent-lifts on T^*Q. Fix a point $z \in T^*Q$ and denote $q = \tau(z) \in Q$ and $H = G_q$.*

Fix a Lagrangian complement Hor_z to $\operatorname{Ker} T_z\tau$, a linear slice $S \subset T_qQ$ and an H-invariant complement $\mathfrak{r}$ to $\mathfrak{h}$ in $\mathfrak{g}$. Let $\mu = \mathbf{J}(z)$ and $\alpha = z\big|_S \in S^*$

There is a linear isomorphism

$$\mathbf{I}\colon \mathfrak{r} \oplus S \oplus \mathfrak{r}^* \oplus S^* \to T_z(T^*Q)$$

that is G_z-equivariant and symplectic, $\mathbf{I}(\mathfrak{r} \oplus S) = \operatorname{Hor}_z$ and $\mathbf{I}(\mathfrak{r}^* \oplus S^*) = \operatorname{Ker} T_z\tau$.

Moreover, there are G_z-equivariant linear maps $F_1\colon \mathfrak{r} \to \mathfrak{r}^*$ and $F_2\colon \mathfrak{r} \to S^*$ such that

$$\mathbf{I}(\xi, 0, F_1(\xi), F_2(\xi)) = \xi \cdot z \text{ if } \xi \in \mathfrak{r}$$

and

$$\mathbf{I}(0, 0, -\mathrm{ad}^*_\xi \mu, \xi \cdot \alpha) = \xi \cdot z \text{ if } \xi \in \mathfrak{h}$$

Proof. The linear slice S and $\mathfrak{r}$ can be used to define an H-equivariant isomorphism

$$\begin{aligned} f\colon \mathfrak{r} \oplus S &\longrightarrow T_qQ \\ (\xi, a) &\longmapsto \xi \cdot q + a \end{aligned}$$

the dual f^* of this map can be used to identify T_q^*Q with $\mathfrak{r}^* \oplus S^*$. Under this identification, the point $z \in T_q^*Q$ satisfies $f^*(z) = (\mu, \alpha) \in \mathfrak{r}^* \oplus S^*$.

Since Hor_z is complementary to $\operatorname{Ker} T_z\tau$, the restriction

$$T_z\tau\big|_{\operatorname{Hor}_z} : \operatorname{Hor}_z \to T_qQ$$

is G_z-equivariant linear isomorphism. As $T^*Q \to Q$ is a vector bundle, the vertical lift

$$\begin{aligned} \mathrm{VertLift}_z\colon T_q^*Q &\longrightarrow T_z(T^*Q) \\ p_q &\longmapsto \frac{d}{dt}(z + tp_q)\Big|_{t=0} \end{aligned}$$

is a G_z-equivariant linear injective map. Combining both maps,

$$\begin{aligned} \mathbf{I}\colon \mathfrak{r} \oplus S \oplus \mathfrak{r}^* \oplus S^* &\longrightarrow T_z(T^*Q) \\ (\xi, a, \nu, b) &\longmapsto (T_z\tau\big|_{\operatorname{Hor}_z})^{-1}(f(\xi, a)) + \mathrm{VertLift}_z((f^{-1})^*(\nu, b)) \end{aligned} \tag{4.1}$$

is a G_z-equivariant linear isomorphism.

Since Hor_z is Lagrangian,

$$\omega(z)(\mathbf{I}(\xi_1, a_1, 0, 0), \mathbf{I}(\xi_2, a_2, 0, 0)) = 0$$

and, as $\operatorname{Ker} T_z\tau$ is a Lagrangian subspace,

$$\omega(z)(\mathbf{I}(0, 0, \nu_1, b_1), \mathbf{I}(0, 0, \nu_2, b_2)) = 0.$$

Moreover, as the canonical symplectic form of a cotangent bundle satisfies

$$\omega(z)(\mathrm{VertLift}_z(p_q), v_z) = \langle p_q, T_z\tau(v_z)\rangle$$

then, if $v_i = (\xi_i, a_i, \nu_i, b_i) \in \mathfrak{r} \oplus S \oplus \mathfrak{r}^* \oplus S^*$ with $i = 1, 2$

$$(\mathbf{I}^*\omega)(v_1, v_2) = \langle \nu_2, \xi_1\rangle - \langle \nu_1, \xi_2\rangle + \langle b_2, a_1\rangle - \langle b_1, a_2\rangle.$$

That is, $\mathbf{I}$ is a G_z-equivariant linear symplectomorphism.

It remains to study the preimage of $\mathfrak{g} \cdot z$ under $\mathbf{I}$; let $\xi \in \mathfrak{r}$, as τ is G-equivariant, $(T_z\tau)(\xi_{T^*Q}(z)) = \xi_Q(q)$, but $\xi_Q(q) = f(\xi, 0)$. Hence there are two linear maps $F_1 \colon \mathfrak{r} \to \mathfrak{r}^*$ and $F_2 \colon \mathfrak{r} \to S^*$ such that

$$\xi_{T^*Q}(z) = \xi \cdot z = \mathbf{I}(\xi, 0, F_1(\xi), F_2(\xi)).$$

Since $\mathbf{I}$ is G_z-equivariant, if $g \in G_z$, $F_1(\mathrm{Ad}_g\xi) = \mathrm{Ad}^*_{g^{-1}}(F_1(\xi))$ and $F_2(\mathrm{Ad}_g\xi) = g \cdot F_2(\xi)$.

If $\xi \in \mathfrak{h}$, then $\tau(\exp(t\xi) \cdot z) = q$. Therefore $\exp(t\xi) \cdot z \in T^*_qQ$, using H-equivariance of f^*,

$$f^*(\exp(t\xi) \cdot z) = (\mathrm{Ad}^*_{\exp(-t\xi)}\mu, \exp(t\xi) \cdot \alpha)$$

but then taking the derivative at $t = 0$ of this expression

$$\xi \cdot z = \mathrm{VertLift}_z((f^{-1})^*(-\mathrm{ad}^*_\xi\mu, \xi \cdot \alpha)),$$

that is,

$$\xi \cdot z = \mathbf{I}(0, 0, -\mathrm{ad}^*_\xi\mu, \xi \cdot \alpha).$$

□

Remark 4.1.3. The isomorphism $\mathbf{I}$ is a generalization of the identification $\mathfrak{r} \oplus S \oplus \mathfrak{r}^* \oplus S^* \cong T_z(T^*Q)$ that appears in [RO04; PROSD08]. The main difference is that they do not consider a Lagrangian splitting of $T_z(T^*Q)$; they start from an invariant Riemannian metric on Q and choose Hor_z as the orthogonal to $\mathrm{Ker}\, T_z\tau$ with respect to the induced Sasaki metric on T^*Q.

The equivariance of the momentum map forces F_1 to satisfy certain relations, due to (1.4)

$$\omega(\xi_{T^*Q}(z), \eta_{T^*Q}(z)) = \langle \mu, [\xi, \eta]\rangle,$$

but using the $\mathbf{I}$ isomorphism,

$$\begin{aligned}\omega(\xi_{T^*Q}(z), \eta_{T^*Q}(z)) &= (\mathbf{I}^*\omega)((\xi, 0, F_1(\xi), F_2(\xi)), (\eta, 0, F_1(\eta), F_2(\eta))\\ &= \langle F_1(\eta), \xi\rangle - \langle F_1(\xi), \eta\rangle = \langle F_1(\eta) - F_1^*(\eta), \xi\rangle.\end{aligned}$$

Hence,

$$\langle \mu, [\xi, \eta]\rangle = \langle F_1(\eta) - F_1^*(\eta), \xi\rangle \qquad \forall \xi, \eta \in \mathfrak{r} \tag{4.2}$$

where $F_1^* \colon \mathfrak{r} \to \mathfrak{r}^*$ is the dual of the map $F_1 \colon \mathfrak{r} \to \mathfrak{r}^*$.

4.2 Lie algebra splitting

The following result shows that if we are given a tuple (G, H, μ) where G is a Lie group, H a compact subgroup and $\mu \in \mathfrak{g}^*$ satisfying the compatibility condition $\mu \in [\mathfrak{h}, \mathfrak{h}]^\circ$, then we can build a splitting (4.3) adapted to the triple (G, H, μ). This construction will be the starting point of local models for cotangent-lifted actions.

Proposition 4.2.1. *Let (G, H, μ) where G is a Lie group, H a compact subgroup and $\mu \in \mathfrak{g}^*$ with $[\mathfrak{h}, \mathfrak{h}] \in \operatorname{Ker} \mu$.*

There is an H_μ-invariant splitting

$$\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n} \tag{4.3}$$

such that:

1. $\mathfrak{h} = \mathfrak{h}_\mu \oplus \mathfrak{l}$ *and* $\mathfrak{g}_\mu = \mathfrak{h}_\mu \oplus \mathfrak{p}$

2. $\langle \mu, [\xi_1, \xi_2] \rangle$ *with* $\xi_1, \xi_2 \in (\mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n})$ *defines a non-degenerate H_μ-invariant 2-form on the vector space $\mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ that block-diagonalizes as*

$$\begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix}.$$

Proof. Let Ω^μ be the skew-symmetric bilinear form on $\mathfrak{g}$ given by $\Omega^\mu(\xi_1, \xi_2) = -\langle \mu, [\xi_1, \xi_2] \rangle$. The splitting is a kind of generalization of the Witt-Artin decomposition (Proposition 2.2.2) for the degenerate form Ω^μ, and therefore this proof is based on Theorem 7.1.1 of [OR04].

As H is compact, we can endow $\mathfrak{g}$ with a Ad_H-invariant metric. We will first note that Ω^μ restricted to $\mathfrak{g}_\mu^\perp$ is non-degenerate because if $\xi \in \operatorname{Ker} \Omega^\mu \Big|_{\mathfrak{g}_\mu^\perp}$ then $\langle \mu, [\xi, \eta] \rangle = 0 \quad \forall \eta \in \mathfrak{g}_\mu^\perp$, but if now $\eta \in \mathfrak{g}_\mu$, then $0 = \langle \mathrm{ad}^*_\eta \mu, \xi \rangle = -\langle \mu, [\xi, \eta] \rangle = \langle \mathrm{ad}^*_\xi \mu, \eta \rangle$ for any $\eta \in \mathfrak{g}$. However, this implies that $\mathrm{ad}^*_\xi \mu = 0$ and as $\xi \in \mathfrak{g}_\mu^\perp$, then $\xi = 0$. Denote by $\omega = \Omega^\mu \Big|_{\mathfrak{g}_\mu^\perp}$ the restriction. The form ω is symplectic on $\mathfrak{g}_\mu^\perp$.

Define now $\mathfrak{l} := \mathfrak{h} \cap \mathfrak{g}_\mu^\perp$ and

$$\mathfrak{o} = \{\lambda \in \mathfrak{g}_\mu^\perp \cap \mathfrak{h}^\perp \subset \mathfrak{g} \mid \langle \mathrm{ad}^*_\lambda \mu, \eta \rangle = 0 \quad \forall \eta \in \mathfrak{h}\}.$$

If $\xi \in \mathfrak{g}_\mu^\perp$ is ω-orthogonal to $\mathfrak{l}$, then it must lie in $\mathfrak{o} \oplus \mathfrak{l}$ because ξ can be decomposed as $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathfrak{h} \cap \mathfrak{g}_\mu^\perp = \mathfrak{l}$ and $\xi_2 \in \mathfrak{h}^\perp \cap \mathfrak{g}_\mu^\perp$. But then as $\langle \mu, [\xi_2, \eta] \rangle = \langle \mu, [\xi, \eta] \rangle = 0$ for any $\eta \in \mathfrak{h}$, then $\xi_2 \in \mathfrak{o}$; that is, $\mathfrak{l}^\omega \subset \mathfrak{o} \oplus \mathfrak{l}$. Conversely, if $\xi \in \mathfrak{o}$ then by definition of $\mathfrak{o}$ $\xi \in \mathfrak{l}^\omega$, and if $\xi \in \mathfrak{h}$ for any $\eta \in \mathfrak{l}$ we have $\langle \mu, [\xi, \eta] \rangle = 0$ because $\mathfrak{l} \in \mathfrak{h}$ and $\mu \in [\mathfrak{h}, \mathfrak{h}]^\circ$ so $\xi \in \mathfrak{l}^\omega$, and therefore $\mathfrak{l}^\omega = \mathfrak{o} \oplus \mathfrak{l}$.

Let $\xi \in \mathfrak{o} \cap \mathfrak{o}^\omega$. Noting that $\xi \in \mathfrak{l}^\omega$ we have $\xi \in \mathfrak{o}^\omega \cap \mathfrak{l}^\omega = (\mathfrak{o} \oplus \mathfrak{l})^\omega = (\mathfrak{l}^\omega)^\omega = \mathfrak{l}$, but as $\mathfrak{o} \cap \mathfrak{l} = 0$ this implies that $\xi = 0$. Hence, the restriction $\omega \Big|_{\mathfrak{o}}$ is non-degenerate.

To build the space $\mathfrak{n}$ we will need a preliminary standard result in linear algebra.

Lemma 4.2.2. *Let $A, B, C \subset E$ be three linear subspaces of a linear space E such that $A \subset B$ and $A \cap C = 0$. Then*

$$B \cap (C \oplus A) = (B \cap C) \oplus A$$

Note that $\mathfrak{l} \subset \mathfrak{o}^\omega$, and as $\langle \mu, [\xi, \eta] \rangle = 0$ for any $\xi, \eta \in \mathfrak{l}$, then $\mathfrak{l}$ is an isotropic subset of the symplectic subspace $\mathfrak{o}^\omega$, but in fact

$$\mathfrak{l}^\omega \cap \mathfrak{o}^\omega = \mathfrak{o}^\omega \cap (\mathfrak{o} \oplus \mathfrak{l}) = (\mathfrak{o}^\omega \cap \mathfrak{o}) \oplus \mathfrak{l} = \mathfrak{l}$$

where we applied the previous lemma with $A = \mathfrak{l}$, $B = \mathfrak{o}^\omega$ and $C = \mathfrak{o}$. This implies that $\mathfrak{l}$ is a Lagrangian subspace of $\mathfrak{o}^\omega$ and it is clearly H_μ-invariant. By Lemma 4.1.1 there must exist an H_μ-invariant complement $\mathfrak{n} \subset \mathfrak{g}_\mu^\perp$ of $\mathfrak{l}$; that is, using $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ and Ω^μ block diagonalizes as

$$-\langle \mu, [\cdot,\cdot] \rangle = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Omega\Big|_{\mathfrak{o}} & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{bmatrix}. \tag{4.4}$$

□

Note that two different splittings (4.3) have to be isomorphic as H_μ-modules because as vector spaces with an H_μ-action both $\mathfrak{l}$ and $\mathfrak{n}$ are isomorphic to the quotient $\mathfrak{h}/\mathfrak{h}_\mu$, $\mathfrak{o}$ is isomorphic to $\mathfrak{h}^{\Omega_\mu}/(\operatorname{Ker}\Omega_\mu + \mathfrak{h})$ and $\mathfrak{p}$ is isomorphic to $\mathfrak{g}_\mu/\mathfrak{h}_\mu$.

The non-degeneracy of $\langle \mu, [\cdot,\cdot] \rangle$ on $\mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ implies that

$$\begin{aligned} \sigma \colon \mathfrak{n} &\longrightarrow \mathfrak{l}^* \\ \xi &\longmapsto -\langle \mu, [\xi, \cdot] \rangle \end{aligned} \tag{4.5}$$

is a linear H_μ-equivariant isomorphism. This isomorphism will be used several times throughout the thesis, both for the linear study of this chapter and for the construction of Hamiltonian tubes in later chapters.

Remark 4.2.3. Endow the coadjoint orbit $\mathcal{O}_\mu$ with the symplectic form $\omega^-_{\mathcal{O}_\mu}$ (see (1.6)). The H-action $h \cdot \nu = \operatorname{Ad}^*_{h^{-1}}\nu$ is Hamiltonian with momentum map $\mathbf{J}_H \colon \mathcal{O}_\mu \to \mathfrak{h}^*$, $\mathbf{J}_H(\nu) = \nu\big|_{\mathfrak{h}}$. As $\mu \in [\mathfrak{h},\mathfrak{h}]^\circ$, $\mathfrak{h}_{\mathbf{J}_H(\mu)} = \mathfrak{h}$ and

$$T_\mu \mathcal{O}_\mu = \underbrace{\mathfrak{l} \cdot \mu}_{\mathfrak{h}\cdot\mu} \oplus \underbrace{\mathfrak{n} \cdot \mu}_{W} \oplus 0 \oplus \underbrace{\mathfrak{o} \cdot \mu}_{N}$$

is a Witt-Artin decomposition (see Proposition 2.2.2) of $T_\mu \mathcal{O}_\mu$ due to the decomposition (4.4).

In other words, $\mathfrak{o}$ is isomorphic to the symplectic slice at μ in $\mathcal{O}_\mu$ and $\mathfrak{l}$ and $\mathfrak{n}$ are the remaining parts of a Witt-Artin decomposition. In [PROSD08] the subspace $\mathfrak{o}$ was introduced in this way, as a symplectic slice at $\mu \in \mathcal{O}_\mu$ for the H-action.

4.3 Symplectic slice

Recall that we have fixed $z \in T^*Q$ and we have denoted $q = \tau(z)$ and $H = G_q$. Using the expression of a cotangent-lifted momentum map (1.7), we can see that $\mu \in \mathfrak{h}^\circ$ because $\langle \mu, \xi \rangle = \langle \mathbf{J}(z), \xi \rangle = \langle z, \xi_Q(q) \rangle = 0$ if $\xi \in \mathfrak{h}$. This implies that (G, H, μ) satisfies the hypothesis of Proposition 4.2.1, so there is an H_μ-invariant splitting

$$\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$$

and we will represent by

$$\mathfrak{g}^* = \mathfrak{h}_\mu^* \oplus \mathfrak{p}^* \oplus \mathfrak{o}^* \oplus \mathfrak{l}^* \oplus \mathfrak{n}^*$$

the induced dual splitting. Choose $\mathfrak{r} = \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{n}$ as a complement to $\mathfrak{h} = \mathfrak{h}_\mu \oplus \mathfrak{l}$ in $\mathfrak{g}$; choose an H-invariant Lagrangian splitting $\operatorname{Hor}_z \oplus \operatorname{Ker} T_z\mathbf{J} = T_z(T^*Q)$, and let $\mathbf{I}$, F_1 and F_2 be the maps given by Proposition 4.1.2.

Recall that the symplectic slice N (Definition 2.2.1) is a G_z-invariant complement to $\mathfrak{g}_\mu \cdot z$ in $\operatorname{Ker} T_z\mathbf{J}$. We first construct a vector subspace V such that

$$V \oplus \mathfrak{g}_\mu \cdot z = (\mathfrak{r} \cdot z)^\omega,$$

then, by (1.5), $\mathfrak{g}_\mu \cdot z \subset (\mathfrak{g}\cdot z)^\omega \subset (\mathfrak{h}\cdot z)^\omega$ and using (1.3)

$$(V \oplus \mathfrak{g}_\mu \cdot z) \cap (\mathfrak{h}\cdot z)^\omega = (V \cap (\mathfrak{h}\cdot z)^\omega) \oplus \mathfrak{g}_\mu \cdot z = (\mathfrak{r}\cdot z)^\omega \cap (\mathfrak{h}\cdot z)^\omega = (\mathfrak{g}\cdot z)^\omega = \operatorname{Ker} T_z\mathbf{J}.$$

Therefore, $N = V \cap (\mathfrak{h}\cdot z)^\omega$ will be a symplectic slice at z.

Let $v = (\eta, a, \nu, b) \in \mathfrak{r} \times S \times \mathfrak{r}^* \times S^*$ and $\xi \in \mathfrak{r}$. Then, using Proposition 4.1.2,

$$\begin{aligned}\omega(\mathbf{I}(v), \xi\cdot z) &= \langle F_1(\xi), \eta\rangle - \langle \nu, \xi\rangle + \langle F_2(\xi), a\rangle - \langle b, 0\rangle \\ &= \langle \xi, F_1^*(\eta)\rangle - \langle \nu, \xi\rangle + \langle \xi, F_2^*(a)\rangle \\ &= \langle F_1^*(\eta) + F_2^*(a) - \nu, \xi\rangle\end{aligned}$$

it follows that,

$$(\mathfrak{r}\cdot z)^\omega = \{\mathbf{I}(\eta, a, \nu, b) \mid \nu = F_1^*(\eta) + F_2^*(a), \quad \eta \in \mathfrak{r}, \quad a \in S, \quad b \in S^*\}.$$

Let $B = (\mathfrak{h}_\mu \cdot \alpha)^\circ \subset S$ and choose a G_z-invariant splitting $S = B \oplus C$. The induced dual splitting is $S^* = B^* \oplus (\mathfrak{h}_\mu \cdot \alpha)$; define

$$V = \{\mathbf{I}(\eta, a, \nu, b) \mid \nu = F_1^*(\eta) + F_2^*(a), \quad b = F_2(\eta) + b', \quad \eta \in \mathfrak{o}\oplus\mathfrak{n}, \quad a \in S, \quad b' \in B^*\}. \tag{4.6}$$

Clearly, $V \subset (\mathfrak{r}\cdot z)^\omega$; moreover, if $\xi \in \mathfrak{p}$, using (4.2),

$$\mathbf{I}^{-1}(\xi\cdot z) = (\xi, 0, F_1(\xi), F_2(\xi)) = (\xi, 0, F_1^*(\xi), F_2(\xi)).$$

Therefore, $(\mathfrak{p}\cdot z)\cap V = 0$. If $\xi \in \mathfrak{h}_\mu$,

$$\mathbf{I}^{-1}(\xi\cdot z) = (0, 0, -\mathrm{ad}^*_\xi \mu, \xi\cdot\alpha) = (0,0,0,\xi\cdot\alpha),$$

that is, $(\mathfrak{h}_\mu \cdot z) \cap V = 0$ and

$$V \oplus \mathfrak{g}_\mu \cdot z = (\mathfrak{r}\cdot z)^\omega,$$

as we claimed.

Consider now $v = (\eta, a, \nu, b)$ and $\xi \in \mathfrak{h}$; then, using Proposition 4.1.2 and the diamond notation (see (1.8))

$$\begin{aligned}\omega(\mathbf{I}(v), \xi\cdot z) &= \langle -\mathrm{ad}^*_\xi\mu, \eta\rangle + \langle \xi\cdot\alpha, a\rangle \\ &= \langle \mu, -[\xi, \eta]\rangle + \langle -a \diamond_{\mathfrak{h}} \alpha, \xi\rangle \\ &= \langle \mathrm{ad}^*_\eta \mu - a\diamond_{\mathfrak{h}} \alpha, \xi\rangle.\end{aligned}$$

Moreover, if $\xi \in \mathfrak{h}_\mu$, then $\omega(\mathbf{I}(v), \xi\cdot z) = \langle -a\diamond_{\mathfrak{h}}\alpha, \xi\rangle$. Hence, $v \in (\mathfrak{h}_\mu\cdot z)^\omega$ implies that $a \in (\mathfrak{h}_\mu \cdot \alpha)^\circ = B \subset S$, that is

$$(\mathfrak{h}_\mu \cdot z)^\omega = \{(\eta, a, \nu, b) \mid \eta \in \mathfrak{r}, \quad \nu \in \mathfrak{r}^*, \quad a \in B \subset S, \quad b \in S^*\}. \tag{4.7}$$

Consider now $\xi \in \mathfrak{l}$, and decompose $\eta \in \mathfrak{r}$ as $\eta_{\mathfrak{p}} + \eta_{\mathfrak{o}} + \eta_{\mathfrak{n}} \in \mathfrak{p}\oplus\mathfrak{o}\oplus\mathfrak{n}$, then by Proposition 4.2.1, $\omega(\mathbf{I}(v), \xi\cdot z) = \langle \mathrm{ad}^*_\eta\mu - a\diamond_{\mathfrak{h}}\alpha, \xi\rangle = \langle \mathrm{ad}^*_{\eta_{\mathfrak{n}}}\mu - a\diamond_{\mathfrak{l}}\alpha, \xi\rangle$. Using the isomorphism σ of (4.5), this implies

$$(\mathfrak{l}\cdot z)^\omega = \{(\eta, a, \nu, b) \mid \eta = \eta_{\mathfrak{p}} + \eta_{\mathfrak{o}} + \sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha), \quad \eta_{\mathfrak{p}} \in \mathfrak{p}, \quad \eta_{\mathfrak{o}} \in \mathfrak{o}, \quad a \in S, \quad b \in S^*\} \tag{4.8}$$

Therefore, from (4.7) and (4.8),

$$(\mathfrak{h}\cdot z)^\omega = \{(\eta, a, \nu, b) \mid \eta = \eta_{\mathfrak{p}} + \eta_{\mathfrak{o}} + \sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha), \quad \eta_{\mathfrak{p}} \in \mathfrak{p}, \quad \eta_{\mathfrak{o}} \in \mathfrak{o}, \quad a \in B, \quad b \in S^*\},$$

combining this description with (4.6),

$$\begin{aligned}V \cap (\mathfrak{h}\cdot z)^\omega = \{(\eta, a, \nu, b + F_2(\eta)) \mid \eta = \eta_{\mathfrak{o}} + \sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha),& \\ \nu = F_1^*(\eta) + F_2^*(a), \quad \eta_{\mathfrak{o}} \in \mathfrak{o}, \quad a \in B, \quad b \in B^*\}&.\end{aligned}$$

Note that as a vector space $V \cap (\mathfrak{h}\cdot z)^\omega$ is isomorphic to $\mathfrak{o} \times B \times B^*$.

Proposition 4.3.1. *In the present context, consider the linear map*

$$\begin{aligned}\psi_N\colon \mathfrak{o}\times B\times B^* &\longrightarrow T_z(T^*Q) \\ (\lambda,a,b) &\longmapsto \mathbf{I}(\lambda+\sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha),a,F_1^*(\eta)+F_2^*(a),F_2(\eta)+b)\end{aligned} \tag{4.9}$$

where $\eta=\lambda+\sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha)$*. The image of* ψ_N*,* $\operatorname{Im}\psi_N\subset T_z(T^*Q)$*, is a symplectic slice at* z*. Moreover,* ψ_N *is a* G_z*-equivariant symplectomorphism with* $\operatorname{Im}\psi_N$ *if on* $\mathfrak{o}\times B\times B^*$ *we consider the* G_z*-invariant non-degenerate two-form*

$$\omega_N((\lambda_1,a_1,b_1),(\lambda_2,a_2,b_2))=-\langle\mu,[\lambda_1,\lambda_2]\rangle+\langle b_2,a_1\rangle-\langle b_1,a_2\rangle, \tag{4.10}$$

the associated momentum map for the G_z*-action on* $\mathfrak{o}\times B\times B^*$ *is*

$$\mathbf{J}_N(\lambda,a,b)=\frac{1}{2}\lambda\diamond_{\mathfrak{g}_z}\operatorname{ad}^*_\lambda\mu+a\diamond_{\mathfrak{g}_z}b. \tag{4.11}$$

Proof. We only need to check G_z-equivariance and compute $\psi_N^*\omega$, because $\operatorname{Im}\psi_N=V\cap(\mathfrak{h}\cdot z)^\omega$.

If $g\in G_z$, then $\operatorname{Ad}_g\lambda+\sigma^{-1}((g\cdot a)\diamond_{\mathfrak{l}}\alpha)=\operatorname{Ad}_g(\lambda+\sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha))$ and

$$\begin{aligned}\psi_N(\operatorname{Ad}_g\lambda,g\cdot a,g\cdot b)&=\mathbf{I}(\operatorname{Ad}_g(\lambda+\sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha)),g\cdot a,F_1^*(\operatorname{Ad}_g\eta)+F_2^*(g\cdot a),F_2(\operatorname{Ad}_g\eta)+g\cdot b)\\&=\mathbf{I}(\operatorname{Ad}_g(\lambda+\sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha)),g\cdot a,\operatorname{Ad}^*_{g^{-1}}(F_1^*(\eta)+F_2^*(a)),g\cdot(F_2(\eta)+b))\\&=g\cdot\mathbf{I}(\lambda+\sigma^{-1}(a\diamond_{\mathfrak{l}}\alpha),a,F_1^*(\eta)+F_2^*(a),F_2(\eta)+b)\\&=g\cdot\psi_N(\lambda,a,b),\end{aligned}$$

that is, ψ_N is a G_z-equivariant linear map.

If $v_i=\psi_N(\lambda_i,a_i,b_i)$ and $\eta_i=\lambda_i+\sigma^{-1}(a_i\diamond_{\mathfrak{l}}\alpha)$ for $i=1,2$, then

$$\begin{aligned}\omega(v_1,v_2)&=\omega(\mathbf{I}(\eta_1,a_1,F_1^*(\eta_1)+F_2^*(a_1),F_2(\eta_1)+b_1),\\&\qquad\quad\mathbf{I}(\eta_2,a_2,F_1^*(\eta_2)+F_2^*(a_2),F_2(\eta_2)+b_2))\\&=\langle F_1^*(\eta_2)+F_2^*(a_2),\eta_1\rangle-\langle F_1^*(\eta_1)+F_2^*(a_1),\eta_2\rangle\\&\quad+\langle F_2(\eta_2)+b_2,a_1\rangle-\langle F_2(\eta_1)+b_1,a_2\rangle\\&=\langle F_1^*(\eta_2),\eta_1\rangle-\langle F_1^*(\eta_1),\eta_2\rangle+\langle b_2,a_1\rangle-\langle b_1,a_2\rangle\\&=\langle F_1(\eta_1)-F_1^*(\eta_1),\eta_2\rangle+\langle b_2,a_1\rangle-\langle b_1,a_2\rangle\\&=-\langle\mu,[\eta_1,\eta_2]\rangle+\langle b_2,a_1\rangle-\langle b_1,a_2\rangle\\&=-\langle\mu,[\lambda_1,\lambda_2]\rangle+\langle b_2,a_1\rangle-\langle b_1,a_2\rangle\end{aligned}$$

where we use (4.2) and the fact that $\mathfrak{n}$ and $\mathfrak{o}$ are $\langle\mu,[\cdot,\cdot]\rangle$-orthogonal.

Finally, as the momentum map $\mathbf{J}_N\colon N\to\mathfrak{g}_z^*$ of a linear action on a symplectic vector space satisfies

$$\begin{aligned}\langle\mathbf{J}_N(\lambda,a,b),\xi\rangle&=\frac{1}{2}\omega_N(\xi\cdot(\lambda,a,b),(\lambda,a,b))\\&=\frac{1}{2}\omega_N(\operatorname{ad}_\xi\lambda\mu,\xi\cdot a,\xi\cdot b),(\lambda,a,b))\\&=-\frac{1}{2}\langle\mu,[\operatorname{ad}_\xi\lambda,\lambda]\rangle+\frac{1}{2}\langle b,\xi\cdot a\rangle-\frac{1}{2}\langle\xi\cdot a,b\rangle\\&=\frac{1}{2}\langle\operatorname{ad}^*_\lambda\mu,\operatorname{ad}_\xi\lambda\rangle+\langle b,\xi\cdot a\rangle\\&=\left\langle\frac{1}{2}\lambda\diamond_{\mathfrak{g}_z}\operatorname{ad}^*_\lambda\mu+b\diamond_{\mathfrak{g}_z}a,\xi\right\rangle.\end{aligned}$$

Hence, $\mathbf{J}_N$ is given by (4.11). □

Remark 4.3.2. When we choose a G-invariant metric on Q and Hor_z to be the orthogonal complement of $\operatorname{Ker} T_z\tau$ in $T_z(T^*Q)$ with respect to the associated Sasaki metric the subspace $\operatorname{Im}\psi_N$ coincides with the symplectic slice given by Theorem 6.1 of [PROSD08].

4.4 Witt-Artin decomposition

Once we have described the symplectic slice in Proposition 4.3.1 using similar techniques we can construct the full Witt-Artin decomposition of Proposition 2.2.2.

Note that since $\tau\colon T^*Q \to Q$ is equivariant, $G_z \subset G_q = H$. Similarly, since $\mathbf{J}\colon T^*Q \to \mathfrak{g}^*$ is equivariant $G_z \subset G_\mu$. Hence, $G_z \subset G_\mu \cap H = H_\mu$. At a linear level, we fix a G_z-invariant splitting $\mathfrak{h}_\mu = \mathfrak{g}_z \oplus \mathfrak{s}$.

Recall that $B = (\mathfrak{h}_\mu \cdot \alpha)^\circ \subset S$ and we have chosen a G_z-invariant splitting $S = B \oplus C$. Since $\zeta \mapsto \zeta \cdot \alpha$ is a G_z-equivariant isomorphism between $\mathfrak{s}$ and $\mathfrak{h}_\mu \cdot \alpha \subset S^*$, there is a G_z-equivariant linear isomorphism $\mathfrak{s} \times B^* \cong S^*$. Therefore, there is a G_z-equivariant map $\widetilde{\Gamma}\colon \mathfrak{s}^* \to C \subset S$.

Proposition 4.4.1. *In the present context, consider the G_z-equivariant linear map*

$$\begin{aligned} \psi_W\colon \mathfrak{s}^* \times \mathfrak{p}^* &\longrightarrow T_z(T^*Q) \\ (\zeta,\rho) &\longmapsto \mathbf{I}(\gamma, \widetilde{\Gamma}(\zeta), \rho + F_2^*(\widetilde{\Gamma}(\zeta)) + F_1^*(\gamma), F_2(\gamma)) \end{aligned} \tag{4.12}$$

where $\gamma = \sigma^{-1}(\widetilde{\Gamma}(\zeta) \diamond_{\mathfrak{l}} \alpha)$ (see (4.5)).

The splitting

$$T_z(T^*Q) = (\mathfrak{p} \oplus \mathfrak{s}) \cdot z \oplus \operatorname{Im}\psi_W \oplus (\mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}) \cdot z \oplus \operatorname{Im}\psi_N \tag{4.13}$$

is a ***Witt-Artin decomposition*** *in the sense of Proposition 2.2.2.*

Proof. By the results of Proposition 4.3.1, we only have to check that $\operatorname{Im}\psi_W$ is isotropic and that it is symplectically orthogonal to $((\mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}) \cdot z) \oplus \operatorname{Im}\psi_N$, but

- if $v_i = \psi_W(\zeta_i, \rho_i)$ for $i = 1, 2$,

$$\begin{aligned} \omega(v_1, v_2) &= \langle \rho_2 + F_2^*(\widetilde{\Gamma}(\zeta_2)) + F_1^*(\gamma_2), \gamma_1\rangle - \langle \rho_1 + F_2^*(\widetilde{\Gamma}(\zeta_1)) + F_1^*(\gamma_1), \gamma_2\rangle \\ &\quad + \langle F_2(\gamma_2), \widetilde{\Gamma}(\zeta_1)\rangle - \langle F_2(\gamma_1), \widetilde{\Gamma}(\zeta_2)\rangle \\ &= \langle \rho_2, \gamma_1\rangle - \langle \rho_1, \gamma_2\rangle + \langle F_1^*(\gamma_2) - F_1(\gamma_2), \gamma_1\rangle \\ &= \langle \mu, [\gamma_2, \gamma_1]\rangle = 0. \end{aligned}$$

  because of (4.2) and $\langle \mu, [\gamma_1, \gamma_2]\rangle = 0$, because γ_1, γ_2 lie on $\mathfrak{l}$ and $\mathfrak{l}$ is $\langle \mu, [\cdot,\cdot]\rangle$-isotropic.

- if $\xi \in \mathfrak{l}$

$$\begin{aligned} \omega(\xi \cdot z, \psi_W(\zeta, \rho)) &= \langle \mathrm{ad}_\xi^* \mu, \gamma\rangle - \langle \xi \cdot \alpha, \widetilde{\Gamma}(\zeta)\rangle \\ &= \langle \mu, [\xi, \gamma]\rangle + \langle \widetilde{\Gamma}(\zeta) \diamond_{\mathfrak{l}} \alpha, \xi\rangle \\ &= \langle -\sigma(\gamma) + \widetilde{\Gamma}(\zeta) \diamond_{\mathfrak{l}} \alpha, \xi\rangle = 0 \end{aligned}$$

  because of (4.5) and $\gamma = \sigma^{-1}(\widetilde{\Gamma}(\zeta) \diamond_{\mathfrak{l}} \alpha)$. If $\xi \in \mathfrak{o} \oplus \mathfrak{n}$,

$$\begin{aligned} \omega(\xi \cdot z, \psi_W(\zeta, \rho)) &= \omega(\xi \cdot z, \psi_W(\zeta, 0)) + \omega(\xi \cdot z, \psi_W(0, \rho)) \\ &= 0 + \langle \rho, \xi\rangle = 0 \end{aligned}$$

  because $\rho \in \mathfrak{p}^*$ and in particular $\rho \in (\mathfrak{o} \oplus \mathfrak{n})^\circ$.

- if $v = \psi_N(\lambda, a, b)$,

$$\begin{aligned}\omega(v, \psi_W(\zeta,\rho)) &= \langle \rho + F_2^*(\widetilde{\Gamma}(\zeta)) + F_1^*(\gamma), \eta\rangle - \langle F_1^*(\eta) + F_2^*(a), \gamma\rangle \\ &\qquad + \langle F_2(\gamma), a\rangle - \langle F_2(\eta) + b, \widetilde{\Gamma}(\zeta)\rangle \\ &= \langle \rho, \eta\rangle + \langle F_1^*(\gamma) - F_1(\gamma), \eta\rangle + \langle b, \widetilde{\Gamma}(\zeta)\rangle \\ &= \langle \mu, [\gamma, \eta]\rangle = 0\end{aligned}$$

  using again (4.2) and Proposition 4.2.1.

Therefore (4.13) is a Witt-Artin decomposition of $T_z(T^*Q)$. □

4.5 Adapted horizontal spaces

The isomorphism of Proposition 4.1.2 depends on the choice of a horizontal Lagrangian subspace at z. We will show that there are horizontal subspaces for which the symplectic slice and the Witt-Artin decomposition have simpler expressions.

Consider the G_z-equivariant endomorphism

$$\begin{aligned}\Sigma\colon (\mathfrak{r} \oplus S) \oplus (\mathfrak{r}^* \oplus S^*) &\longrightarrow (\mathfrak{r} \oplus S) \oplus (\mathfrak{r}^* \oplus S^*) \\ (\xi, a, \nu, b) &\longmapsto (\xi, a, \nu + \frac{1}{2}F_1(\xi) + \frac{1}{2}F_1^*(\xi) + F_2^*(a), b + F_2(\xi));\end{aligned}$$

then $\Sigma^*(\mathbf{I}^*\omega) = \mathbf{I}^*\omega$, because if $v_1 = (\xi_1, a_1, \nu_1, b_1)$ and $v_2 = (\xi_2, a_2, \nu_2, b_2)$

$$\begin{aligned}(\mathbf{I}^*\omega)(\Sigma(v_1), \Sigma(v_2)) &= \langle \nu_2 + \frac{1}{2}F_1(\xi_2) + \frac{1}{2}F_1^*(\xi_2) + F_2^*(a_2), \xi_1\rangle \\ &\qquad - \langle \nu_1 + \frac{1}{2}F_1(\xi_1) + \frac{1}{2}F_1^*(\xi_1) + F_2^*(a_1), \xi_2\rangle \\ &\qquad + \langle b_2 + F_2(\xi_2), a_1\rangle - \langle b_1 + F_2(\xi_1), a_2\rangle \\ &= \langle \nu_2, \xi_1\rangle - \langle \nu_1, \xi_2\rangle + \langle b_2, a_1\rangle - \langle b_1, a_2\rangle \\ &\qquad + \langle \frac{1}{2}F_1(\xi_2) + \frac{1}{2}F_1^*(\xi_2), \xi_1\rangle - \langle \frac{1}{2}F_1(\xi_1) + \frac{1}{2}F_1^*(\xi_1), \xi_2\rangle \\ &= \langle \nu_2, \xi_1\rangle - \langle \nu_1, \xi_2\rangle + \langle b_2, a_1\rangle - \langle b_1, a_2\rangle = (\mathbf{I}^*\omega)(v_1, v_2).\end{aligned}$$

The subspace $\widetilde{\mathrm{Hor}}_z = \mathbf{I}(\Sigma(\mathfrak{r} \oplus S))$ is a G_z-invariant Lagrangian subspace of $T_z(T^*Q)$, because

$$\omega(\mathbf{I}(\Sigma(\xi_1, a_1, 0, 0)), \mathbf{I}(\Sigma(\xi_2, a_2, 0, 0)) = \omega(\mathbf{I}(\xi_1, a_1, 0, 0), \mathbf{I}(\xi_2, a_2, 0, 0) = 0$$

and as $\mathbf{I}(\Sigma(\mathfrak{r}^* \oplus S^*)) = \operatorname{Ker} T_z\tau$, the splitting

$$T_z(T^*Q) = \widetilde{\mathrm{Hor}}_z \oplus \operatorname{Ker} T_z\tau$$

is a G_z-invariant Lagrangian splitting. Applying Proposition 4.1.2 to this splitting, we get a map $\widetilde{\mathbf{I}}\colon (\mathfrak{r}^* \oplus S^*) \oplus (\mathfrak{r} \oplus S) \to T_z(T^*Q)$, but by (4.1)

$$\begin{aligned}\widetilde{\mathbf{I}}(\xi, a, \nu, b) &= (T_z\tau\big|_{\widetilde{\mathrm{Hor}}_z})^{-1}(f(\xi, a)) + \mathrm{VertLift}_x((f^{-1})^*(\nu, b)) \\ &= \mathbf{I}(\Sigma(\xi, a, 0, 0)) + \mathbf{I}(0, 0, \nu, b) \\ &= \mathbf{I}(\Sigma(\xi, a, 0, 0)) + \mathbf{I}(\Sigma(0, 0, \nu, b)) \\ &= (\mathbf{I} \circ \Sigma)(\xi, a, \nu, b).\end{aligned}$$

Since $\widetilde{\mathbf{I}} = \mathbf{I} \circ \Sigma$, this implies that the fundamental fields have a simpler expression under $\widetilde{\mathbf{I}}$, because if $\xi \in \mathfrak{r}$

$$\widetilde{\mathbf{I}}^{-1}(\xi \cdot z) = \Sigma^{-1}(\xi, 0, F_1(\xi), F_2(\xi)) = (\xi, 0, \frac{1}{2}F_1(\xi) - \frac{1}{2}F_1^*(\xi), 0) = (\xi, 0, \frac{1}{2}\mathrm{ad}^*_\xi \mu, 0).$$

This computation implies that we can choose the horizontal space adapted for our decomposition more precisely,

Proposition 4.5.1. *Let G be a Lie group acting by cotangent lifts on T^*Q. Fix a point $z \in T^*Q$ and denote $q = \tau(z) \in Q$ and $H = G_q$.*

Fix a linear slice $S \subset T_qQ$ and an H-invariant complement $\mathfrak{r}$ to $\mathfrak{h}$ in $\mathfrak{g}$. Let $\mu = \mathbf{J}(z)$ and $\alpha = z\big|_S \in S^*$.

There is a Lagrangian complement Hor_z to $\mathrm{Ker}\, T_z\tau$ and a linear isomorphism:

$$\mathbf{I} \colon \mathfrak{r} \oplus S \oplus \mathfrak{r}^* \oplus S^* \to T_z(T^*Q)$$

G_z-equivariant and symplectic such that $\mathbf{I}(\mathfrak{r} \oplus S) = \mathrm{Hor}_z$, $\mathbf{I}(\mathfrak{r}^* \oplus S^*) = \mathrm{Ker}\, T_z\tau$ and

$$\mathbf{I}(\xi, 0, -\frac{1}{2}\mathrm{ad}^*_\xi \mu, 0) = \xi \cdot z \text{ if } \xi \in \mathfrak{r}$$
$$\mathbf{I}(0, 0, -\mathrm{ad}^*_\xi \mu, \xi \cdot \alpha) = \xi \cdot z \text{ if } \xi \in \mathfrak{h}.$$

Note that using the adapted splitting given by Proposition 4.5.1 as $F_1(\xi) = -\frac{1}{2}\mathrm{ad}^*_\xi \mu$ and $F_1^*(\xi) = \frac{1}{2}\mathrm{ad}^*_\xi \mu$, the expressions of both ψ_N and ψ_W ((4.9) and (4.12)) are simplified to

$$\begin{aligned}
\psi_N(\lambda, a, b) &= \mathbf{I}(\lambda + \sigma^{-1}(a \diamond_{\mathfrak{l}} \alpha), a, \frac{1}{2}\mathrm{ad}^*_\eta \mu, b) \\
&= \mathbf{I}(\lambda + \sigma^{-1}(a \diamond_{\mathfrak{l}} \alpha), a, \frac{1}{2}\mathrm{ad}^*_\lambda \mu + \frac{1}{2}(a \diamond_{\mathfrak{l}} \alpha), b) \\
\psi_W(\zeta, \rho) &= \mathbf{I}(\gamma, \widetilde{\Gamma}(\zeta), \rho + \frac{1}{2}\mathrm{ad}^*_\gamma \mu, 0) \\
&= \mathbf{I}(\gamma, \widetilde{\Gamma}(\zeta), \rho + \frac{1}{2}\widetilde{\Gamma}(\zeta) \diamond_{\mathfrak{l}} \alpha, 0).
\end{aligned}$$

4.6 Alternative approach: Commuting reduction

In this section we will prove Proposition 4.3.1 again using a different technique. The advantage is that the proof is clearer and more direct, but the disadvantage is that we are going to obtain an abstract model for the symplectic slice; we are not going to realize it as a subspace of $T_z(T^*Q)$. The approach we use is based on commuting reduction and is a generalization of the study of symplectic slices done in [Sch01; Sch07]. In those works, T. Schmah computed the symplectic slice either when $H \subset G_\mu$ and when $\alpha = 0$. Now, using (4.3), we check that her approach can also give the symplectic slice in the general case.

The starting point of this proof is the following tangent-level commuting reduction theorem

Theorem 4.6.1 ([Sch07], Theorem 10)**.** *Let G and H be free symplectic, commuting actions on a symplectic manifold (M, ω) with equivariant momentum maps $\mathbf{J}_G$ and $\mathbf{J}_H$, respectively. The product action $G \times H$ has momentum map $\mathbf{J}_{G\times H}(x) = (\mathbf{J}_G(x), \mathbf{J}_H(x)) \in \mathfrak{g}^* \times \mathfrak{h}^*$. Let $x \in M$ and $(\mu, \nu) = \mathbf{J}_{G\times H}(x)$. Let $\pi_{G_\mu} \colon \mathbf{J}_G^{-1}(\mu) \to \mathbf{J}_G^{-1}(\mu)/G_\mu$ be the projection and denote $[x] = \pi_{G_\mu}(x)$.*

There is an induced quotient action of H on $\mathbf{J}_G^{-1}(\mu)/G_\mu$, symplectic and with equivariant momentum map $\overline{\mathbf{J}_H}$ that satisfies $\overline{\mathbf{J}_H} \circ \pi_{G_\mu} = \mathbf{J}_H$.

The map $(g, k) \mapsto k$ is a Lie group isomorphism from $(G \times H)_x$ to $H_{[x]}$ and we will refer to both groups by K. Then,

$$\begin{aligned}(\operatorname{Ker} T_x \mathbf{J}_{G\times H})/(\mathfrak{g}_\mu \cdot x + \mathfrak{h}_\nu \cdot x) &\longrightarrow (\operatorname{Ker} T_{[x]} \overline{\mathbf{J}_H})/(\mathfrak{h}_\nu \cdot [x]) \\ v + \mathfrak{g}_\mu \cdot x + \mathfrak{h}_\nu \cdot x &\longmapsto T_x \pi_{G_\mu}(v) + (\mathfrak{h}_\nu \cdot [x])\end{aligned}$$

is a K-equivariant linear symplectomorphism.

In particular, this result implies that the symplectic slices are isomorphic if one first reduces by G and then by H, or if one first reduces by H and then by G.

Using Theorem 2.1.4, the linear splitting $T_q Q = \mathfrak{g} \cdot q \oplus S$ can be extended to a Palais' tube

$$\mathbf{t} \colon G \times_H S \to U \subset Q$$

that maps $[e, 0]_H$ to q. The cotangent lift of this diffeomorphism gives

$$T^* \mathbf{t}^{-1} \colon T^*(G \times_H S) \to T^* U \subset T^* Q$$

As in Section 2.1.1, the product $G \times S$ can be endowed with a left G-action and a twisting H-action, both of them free. These actions can be cotangent-lifted to $T^*(G \times S)$ and then, using Theorem 1.5.1, the space $T^*(G \times_H S)$ is the H-reduced space of $T^*(G \times S)$ by the H^T-action at zero-momentum.

Using left-trivializations (1.9) and similarly to the computations in (2.4), the H^T-momentum of a point $(g, \nu, a, b) \in G \times \mathfrak{g}^* \times S \times S^* \cong T^*(G \times S)$ is $-\nu\big|_{\mathfrak{h}} + a \diamond_{\mathfrak{h}} b$.

By the definition of the cotangent bundle reduction map $\varphi \colon \mathbf{J}_{H^T}^{-1}(0) \to T^*(G \times_H S)$ (see Theorem 1.5.1), $\varphi(e, \mu, 0, \alpha) = (T^* \mathbf{t})(z)$ if $\mu = \mathbf{J}(z)$ and $\alpha = z\big|_S$.

Therefore, by Theorem 4.6.1, the symplectic slice for the G-action at z must be symplectomorphic to the symplectic slice for the H action at $\pi_{G_\mu}(e, \mu, 0, \alpha)$ of the G^L-reduced space $\mathbf{J}_{G^L}^{-1}(\mu)/G^L_\mu$. One can check using Theorem 1.4.2 that the reduction at momentum μ of $T^*(G \times S)$ is the space

$$(\mathcal{O}_\mu \times T^* S, \omega_\mu)$$

where $\mathcal{O}_\mu$ is endowed with the symplectic form $\omega^-_{\mathcal{O}_\mu}$, T^*S is endowed with canonical symplectic form and the induced H action is

$$h \cdot (\nu, a, b) = (\mathrm{Ad}^*_{h^{-1}} \nu, h \cdot a, h \cdot b)$$

with equivariant momentum map

$$\overline{\mathbf{J}_H}(\nu, a, b) = -\nu\big|_{\mathfrak{h}} + a \diamond_{\mathfrak{h}} b \in \mathfrak{h}^*.$$

Denote $x = \pi_{G_\mu}(e, \mu, 0, \alpha) \in \mathbf{J}_{G^L}^{-1}(\mu)/G_\mu$. To build the symplectic slice we need to compute $\operatorname{Ker} T_x \overline{\mathbf{J}_H}$. As $T_x M = T_\mu \mathcal{O}_\mu \oplus S \oplus S^*$, using the isomorphism $\mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n} \to T_\mu \mathcal{O}_\mu$, $\xi \mapsto \mathrm{ad}^*_\xi \mu$ the linearized momentum map $T_x \mathbf{J}_H$ is

$$T_x \overline{\mathbf{J}_H}(\mathrm{ad}^*_{\lambda + \xi_\mathfrak{l} + \xi_\mathfrak{n}} \mu, \dot a, \dot b) = -(\mathrm{ad}^*_{\lambda + \xi_\mathfrak{l} + \xi_\mathfrak{n}} \mu)\big|_{\mathfrak{h}} + \dot a \diamond_{\mathfrak{h}} \alpha = -(\mathrm{ad}^*_{\xi_\mathfrak{n}} \mu) + \dot a \diamond_{\mathfrak{h}} \alpha$$

but then using (4.5) and $B = (\mathfrak{h}_\mu \cdot \alpha)^\circ \subset S$

$$\operatorname{Ker} T_x \overline{\mathbf{J}_H} = \{(\mathrm{ad}^*_\xi \mu, \dot a, \dot b) \mid \xi = \sigma^{-1}(\dot a \diamond_\mathfrak{l} \alpha) + \xi_\mathfrak{o} + \xi_\mathfrak{l}, \quad \xi_\mathfrak{o} \in \mathfrak{o}, \quad \xi_\mathfrak{l} \in \mathfrak{l}, \quad \dot a \in B, \quad \dot b \in S^*\}.$$

If $\xi \in \mathfrak{l}$ then

$$\xi \cdot x = (-\mathrm{ad}^*_\xi \mu, 0, \xi \cdot \alpha),$$

if $\xi \in \mathfrak{h}_\mu$ then

$$\xi \cdot x = (0, 0, \xi \cdot \alpha).$$

These expressions imply that

$$\{(\mathrm{ad}^*_\xi \mu, \dot{a}, \dot{b}) \mid \xi = \sigma^{-1}(\dot{a} \diamond_{\mathfrak{l}} \alpha) + \xi_{\mathfrak{o}}, \quad \xi_{\mathfrak{o}} \in \mathfrak{o}, \quad \dot{a} \in B, \quad \dot{b} \in B\}$$

is a symplectic slice at x. And the map

$$\begin{aligned} \overline{\psi_N} \colon \mathfrak{o} \times B \times B^* &\longrightarrow T_x(\mathcal{O}_\mu \times T^*S) \\ (\lambda, a, b) &\longmapsto (\mathrm{ad}^*_{\lambda + \sigma^{-1}(a \diamond_{\mathfrak{l}} \alpha)} \mu, a, b) \end{aligned} \tag{4.14}$$

is a symplectic map if $\mathfrak{o} \times B \times B^*$ has the symplectic structure (4.10), because

$$\begin{aligned} (\overline{\psi_N}^* \omega_\mu)((\lambda_1, a_1, b_1), (\lambda_2, a_2, b_2)) &= \omega^-_{\mathcal{O}_\mu}(\mathrm{ad}^*_{\eta_1}\mu, \mathrm{ad}^*_{\eta_2}\mu) + \langle b_2, a_1 \rangle - \langle b_1, a_2 \rangle \\ &= -\langle \mu, [\eta_1, \eta_2] \rangle + \langle b_2, a_1 \rangle - \langle b_1, a_2 \rangle \\ &= -\langle \mu, [\lambda_1, \lambda_2] \rangle + \langle b_2, a_1 \rangle - \langle b_1, a_2 \rangle \end{aligned}$$

where $\eta_i = \lambda_i + \sigma^{-1}(a \diamond_{\mathfrak{l}} \alpha)$ and $\langle \mu, [\eta_1, \eta_2] \rangle = \langle \mu, [\lambda_1, \lambda_2] \rangle$ due to the block-decomposition (4.4).

Remark 4.6.2. Note that if $H \subset G_\mu$, then the $\mathfrak{l}$ term in the decomposition (4.3) vanishes and the map σ is the zero map. Similarly, if $\alpha = 0$, the term $\sigma^{-1}(a \diamond_{\mathfrak{l}} \alpha)$ vanishes and in both cases (4.14) is simplified to $\overline{\psi_N}(\lambda, a, b) = (\mathrm{ad}^*_\lambda \mu, a, b)$. These are the two cases for which the symplectic slice was computed in [Sch01; Sch07].

4.7 Example: $T^*(G/H)$

Let G be a Lie group and H a compact subgroup, the quotient space G/H is a smooth manifold and $\pi_H \colon G \to G/H$ is a submersion. The left action of G on itself induces a G-action on G/H. We endow $T^*(G/H)$ with the canonical symplectic structure $\omega_{G/H}$ and the cotangent-lifted action of G on G/H. As an example, we compute the Witt-Artin decomposition at a point $z \in T^*_{\pi_H(e)}(G/H)$.

As we observed on the last section, $T^*(G/H)$ is the symplectic reduced space of (T^*G, ω_G) by the cotangent lift of the H^T-action $h \cdot g = gh^{-1}$ on G (see Proposition 1.3.3). This action is Hamiltonian, and using the left trivialization (1.9) the momentum map for the H^T-action is

$$\mathbf{J}_{H^T}(g, \nu) = -\nu\big|_{\mathfrak{h}}.$$

By Theorem 1.5.1, there is a submersion

$$\varphi \colon \mathbf{J}^{-1}_{H^T}(0) \to T^*(G/H)$$

that induces the symplectomorphism $\overline{\varphi} \colon \mathbf{J}^{-1}_{H^T}(0)/H^T \to T^*(G/H)$. Therefore, there is $\mu \in \mathfrak{g}^*$ such that $\varphi(e, \mu) = z$ and μ lies in $\mathfrak{h}^\circ$ because $\mathbf{J}_{H^T}(e, \mu) = 0$.

As $\mu \in \mathfrak{h}^\circ \subset [\mathfrak{h}, \mathfrak{h}]^\circ$, Proposition 4.2.1 gives an H_μ-invariant splitting $\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ and the induced dual splitting $\mathfrak{g}^* = \mathfrak{h}^*_\mu \oplus \mathfrak{p}^* \oplus \mathfrak{o}^* \oplus \mathfrak{l}^* \oplus \mathfrak{n}^*$. Using the left trivialization

$$\mathbf{J}^{-1}_{H^T}(0) = \{(g, \nu) \mid g \in G, \quad \nu \in \mathfrak{p}^* \oplus \mathfrak{o}^* \oplus \mathfrak{n}^*\} \subset G \times \mathfrak{g}^* \cong T^*G$$

and therefore,

$$\overbrace{(\mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{n})}^{\mathfrak{r}} \oplus \overbrace{(\mathfrak{p}^* \oplus \mathfrak{o}^* \oplus \mathfrak{n}^*)}^{\mathfrak{r}^*} \longrightarrow T_z(T^*(G/H))$$
$$(\xi, \nu) \longmapsto (T_{(e,\mu)}\varphi)(\xi, \nu)$$

is a linear isomorphism. Using this trivialization of $T_z(T^*(G/H))$, the vertical subspace is

$$\operatorname{Ker} T_z\tau = \{(T_{(e,\mu)}\varphi)(0, \nu) \mid \nu \in \mathfrak{p}^* \oplus \mathfrak{o}^* \oplus \mathfrak{n}^*\}$$

the most reasonable candidate for a horizontal subspace is

$$\{(T_{(e,\mu)}\varphi)(\xi, 0) \mid \xi \in \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{n}\} \subset T_z(T^*Q).$$

However, it is not a Lagrangian subspace, because (using (1.11))

$$\omega_{G/H}(z)((T_{(e,\mu)}\varphi)(\xi_1, 0), (T_{(e,\mu)}\varphi)(\xi_2, 0)) = \omega_G(e, \mu)((\xi_1, 0)(\xi_2, 0)) = -\langle \mu, [\xi_1, \xi_2]\rangle$$

Nevertheless, the subspace

$$\operatorname{Hor}_z = \{(T_{(e,\mu)}\varphi)(\xi, \frac{1}{2}\operatorname{ad}^*_\xi \mu) \mid \xi \in \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{n}\} \subset T_z(T^*Q)$$

is complementary to $\operatorname{Ker} T_z\tau$ and Lagrangian, because if $v_i = (\xi_i, \frac{1}{2}\operatorname{ad}^*_{\xi_i}\mu)$

$$\omega_{G/H}(z)\left((T_{(e,\mu)}\varphi)(v_1), (T_{(e,\mu)}\varphi)(v_2)\right) = \omega_G(e, \mu)(v_1, v_2) =$$
$$= \frac{1}{2}\langle \operatorname{ad}^*_{\xi_2}\mu, \xi_1\rangle - \frac{1}{2}\langle \operatorname{ad}^*_{\xi_1}\mu, \xi_2\rangle + \langle \mu, [\xi_1, \xi_2]\rangle = 0.$$

Let $\xi \in \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{n}$; the fundamental field at z (see Proposition 1.3.3) is

$$\xi \cdot z = (T_{(e,\mu)}\varphi)(\xi, 0) = (T_{(e,\mu)}\varphi)(\xi, \frac{1}{2}\operatorname{ad}^*_\xi \mu) - (T_{(e,\mu)}\varphi)(0, \frac{1}{2}\operatorname{ad}^*_\xi \mu).$$

Therefore, if we apply Proposition 4.1.2, the associated F_1 map is simply $F_1(\xi) = -\frac{1}{2}\operatorname{ad}^*_\xi\mu$, which means that this horizontal-vertical Lagrangian splitting is adapted in the sense of Proposition 4.5.1. This means that,

$$\xi \cdot z = \mathbf{I}(\xi, -\frac{1}{2}\operatorname{ad}^*_\xi \mu) = (T_{(e,\mu)}\varphi)(\xi, 0) \qquad \text{if} \quad \xi \in \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{n}$$
$$\xi \cdot z = \mathbf{I}(0, -\operatorname{ad}^*_\xi \mu) = (T_{(e,\mu)}\varphi)(0, -\operatorname{ad}^*_\xi \mu) = (T_{(e,\mu)}\varphi)(\xi, 0) \qquad \text{if} \quad \xi \in \mathfrak{h}.$$

In this case (4.9) and (4.12) become

$$\begin{aligned} \psi_N \colon \mathfrak{o} &\to T_z(T^*(G/H)) \\ \lambda &\mapsto \mathbf{I}(\lambda, \frac{1}{2}\operatorname{ad}^*_\lambda \mu) = (T_{(e,\mu)}\varphi)(\lambda, \operatorname{ad}^*_\lambda \mu) \end{aligned} \qquad \begin{aligned} \psi_W \colon \mathfrak{p}^* &\to T_z(T^*(G/H)) \\ \rho &\mapsto \mathbf{I}(0, \rho) = (T_{(e,\mu)}\varphi)(0, \rho). \end{aligned}$$

and

$$T_z(T^*(G/H)) = \underbrace{\mathfrak{g}_\mu \cdot z}_{\mathfrak{p}\cdot z} \oplus \operatorname{Im}\psi_W \oplus ((\mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}) \cdot z) \oplus \operatorname{Im}\psi_N$$

is a Witt-Artin decomposition of $T_z(T^*(G/H))$.

Chapter 5

Hamiltonian Tubes for Cotangent-Lifted Actions

In this chapter we obtain a construction of the Hamiltonian tube for a canonical cotangent-lifted action on a cotangent bundle that puts both the fibration and the symplectic in a normal form (Theorem 5.2.7). This construction is explicit up to the integration of a differential equation on G. Moreover, we show that for groups with easy algebraic structure the Hamiltonian tube can be obtained explicitly.

In Section 5.1 we introduce simple and restricted G-tubes (Definitions 5.1.1 and 5.1.5). Simple G-tubes are, up to technical details, MGS models for the lift of the left action of G on itself to T^*G. Their existence is proved in Proposition 5.1.2. Restricted G-tubes are defined implicitly in terms of a simple G-tube (Proposition 5.1.6) and are the technical tool that we need later to construct the general Hamiltonian tube.

In Section 5.2 we construct the general Hamiltonian tube for a cotangent-lifted action in such a way that it is explicit up to a restricted G-tube. First, we construct a Hamiltonian tube around points in T^*Q with certain maximal isotropy properties (Theorem 5.2.2). Then, an adaptation of the ideas of [Sch07] can be used to construct a Γ map (Proposition 5.2.4). The composition of these two maps gives the general Hamiltonian tube of Theorem 5.2.7.

Finally, in Section 5.3 we present explicit examples of G-tubes for both the groups $SO(3)$ (where we recover the recent results of [SS13]) and $SL(2,\mathbb{R})$. In Subsection 5.3.4 we present an explicit Hamiltonian tube for the natural action of $SO(3)$ on $T^*\mathbb{R}^3$ which generalizes the final example of [Sch07] to the case $\mu \neq 0$.

5.1 G-tubes

In this section we define both simple and restricted G-tubes. These maps will be the building blocks needed to find an explicit Hamiltonian tube for cotangent-lifted actions.

Recall that, as in (1.9), we use the trivializations $TG \cong G \times \mathfrak{g}$, $T^*G \cong G \times \mathfrak{g}^*$ and $T(T^*G) \cong G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^*$.

5.1.1 Simple G-tubes

Definition 5.1.1. Let H be a compact subgroup of G and $\mu \in \mathfrak{g}^*$. Given a splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$ invariant under the H_μ-action, a **simple G-tube** is a map

$$\Theta\colon G \times U_\Theta \subset G \times (\mathfrak{g}_\mu^* \times \mathfrak{q}) \longrightarrow G \times \mathfrak{g}^* \cong T^*G$$

such that:

1. U_Θ is a connected H_μ-invariant neighborhood of 0 in $\mathfrak{g}_\mu^* \times \mathfrak{q}$.
2. Θ is a G^L-equivariant diffeomorphism onto $\Theta(G \times U_\Theta)$ satisfying $\Theta(e, 0, 0) = (e, \mu)$.
3. Let $u_i := (T_e L_g \xi_i, \dot{\nu}_i, \dot{\lambda}_i) \in T_{(g,\nu,\lambda)}(G \times \mathfrak{g}_\mu^* \times \mathfrak{q})$ with $i = 1, 2$, then
$$(\Theta^* \omega_{T^*G})(u_1, u_2) = \langle \dot{\nu}_2, \xi_1 \rangle - \langle \dot{\nu}_1, \xi_2 \rangle + \langle \nu + \mu, [\xi_1, \xi_2] \rangle - \langle \mu, [\dot{\lambda}_1, \dot{\lambda}_2] \rangle. \tag{5.1}$$
4. Θ is H_μ^T-equivariant.
5. Let $u = (\xi, \dot{\nu}, \dot{\lambda}) \in T_{(e,0,0)}(G \times \mathfrak{g}_\mu^* \times \mathfrak{q})$, then
$$(T_{(e,0,0)}\Theta)(u) = (\xi + \dot{\lambda}; \dot{\nu} + \mathrm{ad}^*_{\dot{\lambda}} \mu) \in \mathfrak{g} \times \mathfrak{g}^* \cong T_{(e,0)}(T^*G). \tag{5.2}$$

If $\mathfrak{q}$ is defined as above; note that the symplectic slice for the cotangent-lifted left multiplication of G on T^*G at $(e, \mu) \in T^*G$ is precisely $\mathfrak{q}$. Indeed, as $T_{(e,\mu)}\mathbf{J}_L(e, \mu) \cdot (\xi, \dot{\nu}) = -\mathrm{ad}^*_\xi \mu + \dot{\nu}$, then a complement to $\mathfrak{g}_\mu \cdot (e, \mu)$ can be chosen to be the space $\{(\xi, \mathrm{ad}^*_\xi \mu) \mid \xi \in \mathfrak{q}\}$, and using (1.11), this linear space is symplectomorphic to $(\mathfrak{q}, \Omega^\mu\big|_{\mathfrak{q}})$.

According to Theorem 2.3.4, the MGS model at $(e, \mu) \in T^*G$ for the free cotangent-lifted left multiplication of G on T^*G will be of the form $G \times \mathfrak{g}_\mu^* \times \mathfrak{q}$, and in this case the symplectic form (2.6) is precisely the one given by (5.1). In other words, a simple G-tube is a Hamiltonian tube for T^*G at (e, μ) (properties 1–3), but we further require H_μ^T-equivariance and a prescribed property on its linearization (properties 4–5).

The next result ensures the existence of simple G-tubes. The idea is that an adaptation of the proof of Theorem 2.3.2 will be enough; the only difference is that we are going to apply it to an explicit, well-behaved family of symplectic potentials.

Proposition 5.1.2 (Existence of simple G-tubes)**.** *Given an H_μ-invariant splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$ there exists an H_μ-invariant open neighborhood U_Θ of $0 \in \mathfrak{g}_\mu^* \times \mathfrak{q}$ and a simple G-tube*
$$\Theta \colon G \times U_\Theta \subset G \times \mathfrak{g}_\mu^* \times \mathfrak{q} \to G \times \mathfrak{g}^*.$$

Proof. As a first approximation, we consider the map
$$\begin{aligned} F \colon G \times \mathfrak{g}_\mu^* \times \mathfrak{q} &\longrightarrow G \times \mathfrak{g}^* \\ (g, \nu, \lambda) &\longmapsto (g \exp(\lambda), \mathrm{Ad}^*_{\exp(\lambda)}(\nu + \mu)). \end{aligned} \tag{5.3}$$

The map F is G^L-equivariant and also H_μ^T-equivariant, because
$$F(g'g, \nu, \lambda) = (g'g \exp(\lambda), \mathrm{Ad}_{\exp(\lambda)}(\nu + \mu)),$$
and
$$\begin{aligned} F(gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\nu, \mathrm{Ad}_h \lambda) &= (gh^{-1} \exp(\mathrm{Ad}_h \lambda), \mathrm{Ad}^*_{\exp(\mathrm{Ad}_h \lambda)}(\mathrm{Ad}^*_{h^{-1}}\nu + \mu) \\ &= (g \exp(\lambda) h^{-1}, \mathrm{Ad}^*_{h^{-1}} \mathrm{Ad}^*_{\exp(\lambda)}(\nu + \mu)). \end{aligned}$$

Consider now the one-form on $G \times (\mathfrak{g}_\mu^* \times \mathfrak{q})$ given by
$$\theta_Y(g, \nu, \lambda)(\xi, \dot{\nu}, \dot{\lambda}) = \langle \nu + \mu, \xi \rangle + \frac{1}{2} \langle \mu, \mathrm{ad}_\lambda \dot{\lambda} \rangle + \langle \mu, \dot{\lambda} \rangle.$$

It is clearly G^L-invariant and H_μ^T-invariant, because

$$\theta_Y(gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\nu, \mathrm{Ad}_h\lambda)(\mathrm{Ad}_h\xi, \mathrm{Ad}^*_{h^{-1}}\dot\nu, \mathrm{Ad}_h\dot\lambda) = \\ = \langle \mathrm{Ad}^*_{h^{-1}}(\nu) + \mu, \mathrm{Ad}_h\xi\rangle + \frac{1}{2}\langle \mu, [\mathrm{Ad}_h\lambda, \mathrm{Ad}_h\dot\lambda]\rangle + \langle \mu, \mathrm{Ad}_h\dot\lambda\rangle = \theta_Y(g,\nu,\lambda)(\xi,\dot\nu,\dot\lambda).$$

Let $u_i := (\xi_i, \dot\nu_i, \dot\lambda_i) \in T_{(g,\nu,\lambda)}(G \times \mathfrak{g}_\mu^* \times \mathfrak{q})$ with $i = 1, 2$. Note that $(-\mathbf{d}\theta_Y)(u_1, u_2)$ is the right-hand side of equation (5.1). Consider now the family of $G^L \times H_\mu^T$-invariant one-forms

$$\theta_t = tF^*\theta_{T^*G} + (1-t)\theta_Y$$

and define $\omega_t := -\mathbf{d}\theta_t$. Using (1.11) and

$$(T_{(e,0,0)}F)(\xi, \dot\nu, \dot\lambda) = (\xi + \dot\lambda, \dot\nu + \mathrm{ad}^*_{\dot\lambda}\mu)$$

it can be checked that

$$(-\mathbf{d}\theta_t)(g,0,0)(\xi_1,\dot\nu_1,\dot\lambda_1)(\xi_2,\dot\nu_2,\dot\lambda_2) = \langle\dot\nu_2,\xi_1\rangle - \langle\dot\nu_1,\xi_2\rangle + \langle\mu,[\xi_1,\xi_2]\rangle - \langle\mu,[\dot\lambda_1,\dot\lambda_2]\rangle,$$

but this two-form is non-degenerate because it corresponds precisely to Ω_Y of Theorem 2.3.4. This implies that Moser's equation $i_{X_t}\omega_t = \frac{\partial\theta_t}{\partial t}$ defines a time-dependent vector field X_t on an open set $G \times V \subset G \times \mathfrak{g}_\mu^* \times \mathfrak{q}$. If Ψ_t is the local flow of X_t then $\Psi_t^*\omega_t = \omega_0$ (see Theorem 2.3.2 for technical details). As θ_t and $-\mathbf{d}\theta_t$ are $G^L \times H_\mu^T$ invariant differential forms, then the vector field X_t is $G^L \times H_\mu^T$ invariant, and therefore the local flow Ψ_t is $G^L \times H_\mu^T$-equivariant for any t.

Note that $\theta_Y(g,0,0) = \langle\mu,\xi\rangle + \langle\mu,\dot\lambda\rangle$ and $F^*\theta_{T^*G}(g,0,0) = \langle\mu,\xi\rangle + \langle\mu,\dot\lambda\rangle$. This implies that $\frac{\partial\theta_t}{\partial t}\big|_{(g,0,0)} = 0$ and $X_t(g,0,0) = 0$ so $\Psi_t(g,0,0) = (g,0,0)$ for any $t \in \mathbb{R}$, and then there is an H_μ-invariant open set $U_\Theta \subset V$ such that Ψ_1 is a diffeomorphism with domain $G \times U_\Theta$.

The simple G-tube will then be the composition $\Theta = F \circ \Psi_1 \colon G \times U_\Theta \longrightarrow T^*G$. It is $G^L \times H_\mu^T$-equivariant and it satisfies $\omega_Y = \omega_0 = \Psi_1^*\omega_1 = \Psi_1^*F^*\omega_{T^*G} = \Theta^*\omega_{T^*G}$ and $\Theta(e,0) = (e,\mu)$. Let Ψ_t be the local flow of X_t and η_t be any time-dependent tensor field then

$$\frac{d}{dt}\Psi_t^*\eta_t = \Psi_t^*\left(\mathcal{L}_{X_t}\eta_t + \frac{d}{dt}\eta_t\right). \tag{5.4}$$

This expression can be used to compute $T_{(e,0,0)}\Theta$. To this end, let Y be any time-independent vector field on $G \times \mathfrak{g}_\mu^* \times \mathfrak{q}$ not vanishing at $(e,0,0)$. As X_t vanishes at $(e,0,0)$ then $\mathcal{L}_{X_t}Y\Big|_{(e,0,0)} = 0$. Setting $\eta_t = Y$ in (5.4), it gives $\frac{d}{dt}\Psi_t^*Y = 0$, but this implies $T_{(e,0,0)}\Psi_1 = \mathrm{Id}$, and therefore

$$T_{(e,0,0)}\Theta(\xi,\dot\nu,\dot\lambda) = (\xi + \dot\lambda, \dot\nu + \mathrm{ad}^*_{\dot\lambda}\mu).$$

In other words, Θ satisfies all the five required conditions for a simple G-tube. □

Remark 5.1.3. Note that if $\mathfrak{g} = \mathfrak{g}_\mu$, which is the hypothesis used in [Sch07], then $\mathfrak{q} = 0$ and the shifting map

$$\begin{aligned} G \times \mathfrak{g}^* &\longrightarrow G \times \mathfrak{g}^* \\ (g,\nu) &\longmapsto (g, \nu + \mu) \end{aligned}$$

is a simple G-tube.

The main shortcoming with the previous existence result is that, as happens with Theorem 2.3.4, it does not produce an explicit map and relies on the integration of a time-dependent field. However, we will see in Section 5.3 that in some particular cases we can explicitly describe these objects. Nevertheless, using momentum maps, we can still find a simpler expression for the simple G-tube Θ. Decompose Θ as

$$\Theta(g,\nu,\lambda) = (A(g,\nu,\lambda), B(g,\nu,\lambda)) \in G \times \mathfrak{g}^*.$$

The property of G^L-equivariance implies that $A(g,\nu,\lambda) = gA(e,\nu,\lambda)$. As $\Theta(e,0,0) = (e,\mu)$, then $A(e,0,0) = e$ and $B(e,0,0) = \mu$.

Using Section 2.3.1, we have that the product $G \times \mathfrak{g}_\mu^* \times \mathfrak{q}$ is equipped with G^L and H_μ^T Hamiltonian actions with momentum maps $\mathbf{K}_{G^L}$ and $\mathbf{K}_{H_\mu^T}$, respectively (see (2.4)). We also have G^L and H_μ^T Hamiltonian actions on $G \times \mathfrak{g}^*$ and their momentum maps are $\mathbf{J}_{G^L}$ and $\mathbf{J}_{H_\mu^T}$ (see Proposition 1.3.3). As the difference between two momentum maps is a locally constant function and both $\mathbf{J}_{G^L}$ and $\mathbf{K}_{G^L}$ are equivariant, then $\mathbf{J}_{G^L} \circ \Theta = \mathbf{K}_{G^L}$, that is

$$\mathrm{Ad}^*_{A(g,\nu,\lambda)^{-1}} B(g,\nu,\lambda) = \mathrm{Ad}^*_{g^{-1}}(\nu+\mu).$$

Hence, $B(g,\nu,\lambda) = \mathrm{Ad}^*_{A(g,\nu,\lambda)}\mathrm{Ad}^*_{g^{-1}}(\nu+\mu) = \mathrm{Ad}^*_{g^{-1}A(g,\nu,\lambda)}(\nu+\mu) = \mathrm{Ad}^*_{A(e,\nu,\lambda)}(\nu+\mu)$. If we denote $E(\nu,\lambda) = A(e,\nu,\lambda)$, then we can write

$$\begin{aligned}\Theta\colon G \times \mathfrak{g}_\mu \times \mathfrak{q} &\longrightarrow T^*G\\ (g,\nu,\lambda) &\longmapsto (gE(\nu,\lambda), \mathrm{Ad}^*_{E(\nu,\lambda)}(\nu+\mu)).\end{aligned} \tag{5.5}$$

Therefore, a simple G-tube is determined by a function $E\colon U_\Theta \subset \mathfrak{g}_\mu^* \times \mathfrak{q} \to G$.

By rewriting Definition 5.1.1 in terms of the function E, one can obtain necessary and sufficient conditions for E; that is, if E is a function defined on a connected open neighborhood $U_\Phi \subset \mathfrak{g}_\mu^* \times \mathfrak{q}$ of $(0,0)$ with values on G that satisfies

- $E(0,0) = e$.
- If $u_i = (\dot\nu_i, \dot\lambda_i) \in T_{(\nu,\lambda)}(\mathfrak{g}_\mu^* \times \mathfrak{q})$ for $i = 1,2$ then
$$-\langle \mu, [\lambda_1,\lambda_2]\rangle = \langle \dot\nu_2, A_1\rangle - \langle \dot\nu_1, A_2\rangle + \langle \nu+\mu, -[A_1,A_2]\rangle \tag{5.6}$$
where $A_i = T_e R^{-1}_{E(\nu,\lambda)} T_{(\nu,\lambda)}E \cdot (\dot\nu_i, \dot\lambda_i)$ with $i = 1,2$.
- For any $h \in H_\mu$ and $(\nu,\lambda) \in \mathfrak{g}_\mu^* \times \mathfrak{q}$, $E(\mathrm{Ad}^*_{h^{-1}}\nu, \mathrm{Ad}_h\lambda) = hE(\nu,\lambda)h^{-1}$.
- Let $(\dot\nu, \dot\lambda) \in T_{(0,0)}(\mathfrak{g}_\mu^* \times \mathfrak{q})$ then
$$T_{(0,0)}E \cdot (\dot\nu,\dot\lambda) = \dot\lambda \in \mathfrak{g} \cong T_eG.$$

Then (5.5) defines a map that satisfies all the properties of Definition 5.1.1. All the conditions apart from the second one are straightforward consequences of Definition 5.1.1. Equation (5.6) is just the condition (5.1) in terms of the function E. In Section 5.3 we will show the equivalence of (5.1) and (5.6) in detail.

Remark 5.1.4. As Θ is H_μ^T-equivariant, the momentum preservation argument that we used to define E gives

$$\mathbf{J}_{H_\mu^T}(\Theta(g,\nu,\lambda)) = \mathbf{K}_{H_\mu^T}(g,\nu,\lambda) = -\nu\big|_{\mathfrak{h}_\mu} + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu. \tag{5.7}$$

Thus, we have the condition

$$(\mathrm{Ad}^*_{E(\nu,\lambda)}(\nu+\mu))\big|_{\mathfrak{h}_\mu} = \nu\big|_{\mathfrak{h}_\mu} - \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda\mu.$$

This property will be useful later during the proof of Proposition 6.2.1.

5.1.2 Restricted G-tubes

If G acts freely on Q, we will see in Section 5.2.4 that the simple G-tube is enough to construct explicitly the Hamiltonian tube for T^*Q, but for non-free actions we will need to adapt a simple G-tube to the corresponding isotropy subgroup, the result being the restricted G-tube.

Definition 5.1.5. Given an adapted splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ as in Proposition 4.2.1, a **restricted G-tube** is a map

$$\Phi\colon G \times U_\Phi \subset G \times \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^* \longrightarrow T^*G$$

such that:

1. U_Φ is a connected H_μ-invariant neighborhood of 0 in $\mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^*$.
2. Φ is a $G^L \times H_\mu^T$-equivariant diffeomorphism between $G \times U_\Phi$ and $\Phi(G \times U_\Phi)$ such that $\Phi(e, 0, 0; 0) = (e, \mu)$.
3. Let $u_i := (T_e L_g \xi_i, \dot\nu_i, \dot\lambda_i, \dot\varepsilon_i) \in T_{(g,\nu,\lambda,\varepsilon)}(G \times \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^*)$ with $i = 1, 2$, then $\Phi^*\omega_{T^*G}$ is
$$\omega_{\mathrm{restr}}(u_1, u_2) = \langle \dot\nu_2, \xi_1\rangle - \langle \dot\nu_1, \xi_2\rangle + \langle \nu + \mu, [\xi_1, \xi_2]\rangle - \langle \mu, [\dot\lambda_1, \dot\lambda_2]\rangle. \tag{5.8}$$
4. $\mathbf{J}_R(\Phi(g, \nu, \lambda, \varepsilon))\Big|_{\mathfrak{l}} = -\varepsilon$ for any $(g, \nu, \lambda, \varepsilon)$ where $\mathbf{J}_R$ is the momentum map for the G^R-action on T^*G (see Proposition 1.3.3).
5. Let $u := (\xi, \dot\nu, \dot\lambda, \dot\varepsilon) \in T_{(e,0,0,0)}(G \times \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^*)$ then
$$(T_{(e,0,0,0)}\Phi)(u) = (\xi + \dot\lambda - \sigma^{-1}(\dot\varepsilon), \dot\nu + \mathrm{ad}^*_{\dot\lambda}\mu + \dot\varepsilon) \in \mathfrak{g} \times \mathfrak{g}^* \cong T_{(e,\mu)}T^*G \tag{5.9}$$
where $\sigma\colon \mathfrak{n} \to \mathfrak{l}^*$ is the H_μ-equivariant linear isomorphism $\zeta \mapsto -\langle \mu, [\zeta, \cdot]\rangle$ (see (4.5)).

If we are given a simple G-tube Θ, then we can build a restricted G-tube Φ solving a non-linear equation. In fact, the restricted G-tube will be of the form $\Phi(g, \nu, \lambda, \varepsilon) = \Theta(g, \nu, \lambda + \zeta(\nu, \lambda, \varepsilon))$ for some map $\zeta\colon \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^* \to \mathfrak{n}$. This is the main idea behind the following result.

Proposition 5.1.6 (Existence of restricted G-tubes)**.** *Given an adapted splitting*

$$\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$$

as in Proposition 4.2.1, there is an H_μ-invariant open neighborhood U_Φ of $0 \in \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^*$ and a restricted G-tube

$$\Phi\colon G \times U_\Phi \longrightarrow T^*G.$$

Proof. Define $\mathfrak{q} = \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$. Using Proposition 5.1.2, there exists a simple G-tube Θ defined on the symplectic space $Y := G \times U_\Theta \subset G \times (\mathfrak{g}_\mu^* \times \mathfrak{q})$ with symplectic form ω_Y (5.1). As $U_\Theta \subset \mathfrak{g}_\mu^* \times \mathfrak{q}$ is a neighborhood of 0, there are H_μ-invariant neighborhoods of the origin $(\mathfrak{g}_\mu^*)_r \subset \mathfrak{g}_\mu^*$, $\mathfrak{o}_r \subset \mathfrak{o}$ and $\mathfrak{n}_r \subset \mathfrak{n}$ such that $(\mathfrak{g}_\mu^*)_r \times (\mathfrak{o}_r + \mathfrak{n}_r) \subset U_\Theta$. Consider now the map

$$\begin{aligned} \iota_W\colon W = G \times ((\mathfrak{g}_\mu^*)_r \times \mathfrak{o}_r \times \mathfrak{n}_r) &\longrightarrow Y = G \times U_\Theta \subset G \times (\mathfrak{g}_\mu^* \times \mathfrak{q}) \\ (g, \nu, \lambda, \zeta) &\longmapsto (g, \nu, \lambda + \zeta) \end{aligned} \tag{5.10}$$

This map is a $G^L \times H_\mu^T$-equivariant embedding. By the properties of the adapted splitting (see Proposition 4.2.1), $\Omega^\mu(\lambda,\zeta)=0$ if $\lambda\in\mathfrak{o}$ and $\zeta\in\mathfrak{n}$. Therefore,

$$(\iota_W^*\omega_Y)(u_1,u_2)=\langle\dot\nu_2,\xi_1\rangle-\langle\dot\nu_1,\xi_2\rangle+\langle\nu+\mu,[\xi_1,\xi_2]\rangle-\langle\mu,[\dot\lambda_1,\dot\lambda_2]\rangle$$

where $u_i:=(\xi_i,\dot\nu_i,\dot\lambda_i,\dot\zeta_i)\in T_{(g,\nu,\lambda,\zeta)}(G\times\mathfrak{g}_\mu^*\times\mathfrak{o}\times\mathfrak{n})$ with $i=1,2$. In order to obtain the restricted G-tube, we need to impose the relationship between ε and $\mathbf{J}_R$. To do so, define the map

$$\begin{aligned}\psi\colon W&\longrightarrow G\times\mathfrak{g}_\mu^*\times\mathfrak{o}\times\mathfrak{l}^*\\(g,\nu,\lambda,\zeta)&\longmapsto(g,\nu,\lambda;-\mathbf{J}_R(\Theta(g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}}).\end{aligned}$$

Note that this map is $G^L\times H_\mu^T$-equivariant because

$$\begin{aligned}(g'g,\nu,\lambda;-\mathbf{J}_R(\Theta(g'g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}})&=(g'g,\nu,\lambda;-\mathbf{J}_R(g'\Theta(g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}})\\&=(g'g,\nu,\lambda;-\mathbf{J}_R(\Theta(g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}})\end{aligned}$$

and

$$\begin{aligned}\psi(h\cdot^T(g,\nu,\lambda,\zeta))&=\left(gh^{-1},\mathrm{Ad}^*_{h^{-1}}\nu,\mathrm{Ad}_h\lambda;-\mathbf{J}_R(\Theta(h\cdot^T(g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}}\right)\\&=\left(gh^{-1},\mathrm{Ad}^*_{h^{-1}}\nu,\mathrm{Ad}_h\lambda;-\mathbf{J}_R(h\cdot^T\Theta((g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}}\right)\\&=\left(gh^{-1},\mathrm{Ad}^*_{h^{-1}}\nu,\mathrm{Ad}_h\lambda;-\mathrm{Ad}^*_{h^{-1}}\mathbf{J}_R(\Theta((g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}}\right)\\&=h\cdot^T\left(g,\nu,\lambda;-\mathbf{J}_R(\Theta(g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}}\right).\end{aligned}$$

Moreover, if we endow $G\times\mathfrak{g}_\mu^*\times\mathfrak{o}\times\mathfrak{l}^*$ with the two-form (5.8), then $\psi^*\omega_{\mathrm{restr}}=\iota_W^*\omega_Y$. We will now check that Ψ is invertible. Let $v:=(\xi,\dot\nu,\dot\lambda,\dot\zeta)\in T_{(e,0,0,0)}(G\times\mathfrak{g}_\mu^*\times\mathfrak{o}\times\mathfrak{n})$, then

$$\begin{aligned}(T_{(e,0,0,0)}\psi)(v)&=\left(\xi,\dot\nu,\dot\lambda;-T_{(e,0,0)}\left(\mathbf{J}_R\big|_{\mathfrak{l}}\circ\Theta\right)\cdot(\xi,\dot\nu,\dot\lambda+\dot\zeta)\right)\\&=\left(\xi,\dot\nu,\dot\lambda;-T_{(e,0)}\left(\mathbf{J}_R\big|_{\mathfrak{l}}\right)\cdot(\xi+\dot\lambda+\dot\zeta,\dot\nu+\mathrm{ad}^*_{\dot\lambda+\dot\zeta}\mu)\right)\\&=\left(\xi,\dot\nu,\dot\lambda;(\dot\nu+\mathrm{ad}^*_{\dot\lambda+\dot\zeta}\mu)\big|_{\mathfrak{l}}\right)\\&=\left(\xi,\dot\nu,\dot\lambda;(\mathrm{ad}^*_{\dot\zeta}\mu)\big|_{\mathfrak{l}}\right)\end{aligned}\tag{5.11}$$

where we have used the expression for $T_{(e,0,0)}\Theta$ given in Definition 5.1.1, and that $\mathrm{ad}^*_{\dot\lambda}\mu\big|_{\mathfrak{l}}=0$ since $\mathfrak{o}$ and $\mathfrak{l}$ are Ω^μ-orthogonal (see Proposition 4.2.1).

As the map $\sigma\colon\mathfrak{n}\to\mathfrak{l}^*$ given by $\sigma(\zeta)=-\mathrm{ad}^*_\zeta\mu\Big|_{\mathfrak{l}}$ is a linear H_μ-equivariant isomorphism (see (4.5)), $T_{(e,0,0,0)}\psi$ is invertible. By the Inverse Function Theorem, there is a neighborhood of $(e,0,0,0)\in G\times\mathfrak{g}_\mu^*\times\mathfrak{o}\times\mathfrak{l}^*$ on which ψ^{-1} is well defined. Due to $G^L\times H_\mu^T$ equivariance of Ψ, this neighborhood must be of the form $G\times U_\Phi$ with $U_\Phi\subset\mathfrak{g}_\mu^*\times\mathfrak{o}\times\mathfrak{l}^*$ an H_μ-invariant neighborhood of zero.

Note that the composition $\Theta\circ\iota_W\circ\psi^{-1}$ is a restricted G-tube because it satisfies

$$(\Theta\circ\iota_W\circ\psi^{-1})^*\omega_{T^*G}=(\iota_W\circ\psi^{-1})^*\omega_Y=\omega_{\mathrm{restr}}.$$

It is $G^L\times H_\mu^T$-equivariant (because it is the composition of $G^L\times H_\mu^T$-equivariant maps), the origin $(e,0,0,0)$ is mapped to $(e,\mu)\in T^*G$, and it is a diffeomorphism onto its image (because it is a composition of diffeomorphisms onto its images). Finally, if $(g,\nu,\lambda,\varepsilon)=\psi(g,\nu,\lambda,\zeta)$

then $\mathbf{J}_R(\Theta(g,\nu,\lambda+\zeta))\big|_{\mathfrak{l}} = -\varepsilon$, that is $(\mathbf{J}_R\big|_{\mathfrak{l}} \circ \Theta \circ \iota_W \circ \psi^{-1})(g,\nu,\lambda,\varepsilon) = -\varepsilon$, which is the condition needed for a restricted G-tube.

Let $u := (\xi, \dot\nu, \dot\lambda, \dot\varepsilon) \in T_{(e,0,0,0)}(G \times \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^*)$, using (5.11),

$$(T_{(e,0,0,0)}\psi^{-1})(u) = (\xi, \dot\nu, \dot\lambda, -\sigma^{-1}(\dot\varepsilon)),$$

as $\Phi = \Theta \circ \iota_W \circ \psi^{-1}$ using (5.10) and (5.2),

$$\begin{aligned}(T_{(e,0,0,0)}\Phi)(u) &= \left(T_{(e,0,0)}\Theta \circ T_{(e,0,0,0)}\iota_W \circ T_{(e,0,0,0)}\psi^{-1}\right)(u)\\ &= \left(T_{(e,0,0)}\Theta \circ T_{(e,0,0,0)}\iota_W\right)(\xi, \dot\nu, \dot\lambda, -\sigma^{-1}(\dot\varepsilon))\\ &= (T_{(e,0,0)}\Theta)(\xi, \dot\nu, \dot\lambda - \sigma^{-1}(\dot\varepsilon))\\ &= (\xi + \dot\lambda - \sigma^{-1}(\dot\varepsilon), \dot\nu + \mathrm{ad}^*_{\dot\lambda}\mu + \dot\varepsilon)\end{aligned}$$

that is, (5.9) is satisfied.

To sum up, the composition $\Phi = \Theta \circ \iota_W \circ \psi^{-1} \colon G \times U_\Phi \to T^*G$ is a restricted G-tube. This map can also be written as

$$\begin{aligned}\Phi \colon G \times U_\Phi \subset G \times \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^* &\longrightarrow T^*G\\ (g,\nu,\lambda;\varepsilon) &\longmapsto \Theta(g,\nu,\lambda+\zeta(\nu,\lambda;\varepsilon))\end{aligned} \tag{5.12}$$

where $\zeta \colon U_\Phi \subset \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^* \to \mathfrak{n}$ is determined by the equation $\mathbf{J}_R\big|_{\mathfrak{l}}(\Phi(g,\nu,\lambda+\zeta,\varepsilon)) = -\varepsilon$. □

5.2 Cotangent bundle Hamiltonian tubes

Let G be a Lie group acting properly on Q, and fix $z \in T^*Q$. In this section we construct a Hamiltonian tube for the cotangent-lifted action of G on T^*Q around z that will be explicit except for the computation of a restricted G-tube. This Hamiltonian tube will be a generalization of the construction in [Sch07] under the hypothesis $G_\mu = G$.

5.2.1 Cotangent-lifted twisted product

We first reduce the problem on T^*Q to a problem on $T^*(G \times_H S)$. This first simplification is already discussed in [Sch07] and is based on regular cotangent reduction at zero momentum (Theorem 1.5.1).

Proposition 5.2.1. *Let Q be a manifold with a proper G action, consider T^*Q with the cotangent lifted action. Fix a point $z = (q,p) \in T^*Q$ and a G_q-invariant metric on Q. Define $S := (\mathfrak{g} \cdot q)^\perp \subset T_qQ$, $\mu := \mathbf{J}_{T^*Q}(z)$ and $\alpha := z\big|_S \in S^*$.*

Then

$$G_z = G_\mu \cap (G_q)_\alpha.$$

If $S_r \subset S$ is a small enough G_q-invariant neighborhood of 0 the map

$$\begin{aligned}\mathbf{t} \colon G \times_{G_q} S_r &\longrightarrow U \subset T^*Q\\ [g,a]_{G_q} &\longmapsto g \cdot \mathrm{Exp}_q(a)\end{aligned}$$

is a G-equivariant diffeomorphism onto the G-invariant open set U where Exp_q represents the Riemannian exponential map at q. The cotangent lift of $\mathbf{t}$ induces a G-equivariant symplectomorphism:

$$T^*\mathbf{t}^{-1} \colon T^*(G \times_{G_q} S_r) \longrightarrow T^*U \subset T^*Q$$

and $T^*\mathbf{t}^{-1}(\varphi(e,\mu,0,\alpha)) = z$ if φ is the G_q-cotangent reduction map (see Theorem 1.5.1).

Proof. Let $q = \tau(z) \in Q$ where $\tau \colon T^*Q \to Q$ is the projection and denote $H = G_q$.

Using Theorem 2.1.4, there is an H-invariant neighborhood $S_r \subset S$ and a G-equivariant diffeomorphism $\mathbf{t} \colon G \times_H S_r \longrightarrow U \subset T^*Q$ of the given form such that $\mathbf{t}([e,0]_H) = q$. As $\mathbf{t}$ is a diffeomorphism, the cotangent lift $T^*\mathbf{t}^{-1} \colon T^*(G \times_H S_r) \longrightarrow \tau^{-1}(U) \subset Q$ is a G-equivariant symplectomorphism onto $T^*U = \tau^{-1}(U) \subset T^*Q$.

The symplectic space $T^*(G \times S_r)$ that can be identified with $G \times \mathfrak{g}^* \times S_r \times S^*$ using the left-trivialization of G and the linear structure of S. In Section 2.1.1 we introduced the G^L and H^T actions on the space $G \times S_r$. These actions can be lifted to Hamiltonian actions on $T^*(G \times S_r)$. More explicitly, using Proposition 1.3.3 and the diamond notation, we have

- cotangent-lifted G^L-action: $g' \cdot^L (g, \nu, a, b) = (g'g, \nu, a, b)$ with momentum map
$$\mathbf{J}_{G^L}(g, \nu, a, b) = \mathrm{Ad}^*_{g^{-1}}\nu.$$
- cotangent-lifted H^T-action: $h \cdot^T (g, \nu, a, b) = (gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\nu, h \cdot a, h \cdot b)$ with momentum map
$$\mathbf{J}_{H^T}(g, \nu, a, b) = -\nu\big|_{\mathfrak{h}} + a \diamond b.$$

Then Theorem 1.5.1 applied to $G \times S_r$ with the H^T-action gives the diagram

$$\begin{array}{ccc} \mathbf{J}^{-1}_{H^T}(0) & \hookrightarrow & T^*(G \times S_r) \\ \downarrow \pi_{H^T} & \searrow \varphi & \\ \mathbf{J}^{-1}_{H^T}(0)/H^T & \xrightarrow{\bar{\varphi}} & T^*(G \times_H S_r) \end{array} \tag{5.13}$$

and the quotient $\mathbf{J}^{-1}_{H^T}(0)/H^T$ supports a Hamiltonian G-action with momentum map

$$\begin{aligned} \mathbf{J}_{\mathrm{red}} \colon \mathbf{J}^{-1}_{H^T}(0)/H^T &\longrightarrow \mathfrak{g}^* \\ \pi_{H^T}(g, \nu, a, b) &\longmapsto \mathrm{Ad}^*_{g^{-1}}\nu. \end{aligned}$$

If we denote $\alpha = z\big|_S$ and $\mu = \mathbf{J}_{T^*Q}(z)$, then

$$T^*\mathbf{t}^{-1}(\varphi(e, \mu, 0, \alpha)) = z,$$

because $\tau(\varphi(e, \mu, 0, \alpha)) = [e, 0]_H = \mathbf{t}^{-1}(q)$ and as any $v \in T_qQ$ can be decomposed as $v = \xi \cdot q + \dot{a}$ with $\xi \in \mathfrak{g}$ and $\dot{a} \in S$ it follows that

$$\begin{aligned} \langle T^*_q\mathbf{t}^{-1}(\varphi(e, \mu, 0, \alpha)), v\rangle &= \langle \varphi(e, \mu, 0, \alpha), T_q\mathbf{t}^{-1} \cdot v\rangle = \langle \varphi(e, \mu, 0, \alpha), (\xi, \dot{a})\rangle = \\ &= \langle (\mu, \alpha), (\xi, \dot{a})\rangle = \langle \mu, \xi\rangle + \langle \alpha, \dot{a}\rangle = \\ &= \langle \mathbf{J}_{T^*Q}(z), \xi\rangle + \langle z, \dot{a}\rangle = \langle z, \xi \cdot q\rangle + \langle z, \dot{a}\rangle = \langle z, v\rangle. \end{aligned}$$

Moreover,

$$\begin{aligned} G_{\varphi(e,\mu,0,\alpha)} &= G_{\pi_{H^T}(e,\mu,0,\alpha)} \\ &= \{g \in G \mid g \cdot \pi_{H^T}(e, \mu, 0, \alpha) = \pi_{H^T}(e, \mu, 0, \alpha)\} \\ &= \{g \in G \mid g \in H, \quad \mathrm{Ad}^*_{g^{-1}}\mu = \mu, \quad g \cdot \alpha = \alpha\} \\ &= H_\mu \cap H_\alpha \end{aligned} \tag{5.14}$$

□

Therefore, from now on we will assume without loss of generality $Q = G \times_H S_r$ and $z = \varphi([e, \mu, 0, \alpha]_H)$ with $\mu \in \mathfrak{g}^*$ and $\alpha \in S^*$. Note that this simplification is explicit up to the exponential of a metric.

In this setting, using the adapted splitting of Proposition 4.2.1, Proposition 4.3.1 and Theorem 2.3.4, the Hamiltonian tube at $z = \varphi(e, \mu, 0, \alpha)$ must to be of the form

$$\mathcal{T}\colon G \times_{G_z} \big(\underbrace{(\mathfrak{s}^* \oplus \mathfrak{p}^*)}_{\mathfrak{m}^*} \times \underbrace{\mathfrak{o} \times B \times B^*}_{N} \big) \longrightarrow T^*(G \times_H S). \tag{5.15}$$

where $\mathfrak{s}$ is a G_z-invariant complement of $\mathfrak{g}_z$ in $\mathfrak{h}_\mu$ and $B = (\mathfrak{h}_\mu \cdot \alpha)^\circ \subset S$.

The first difficulty that we find is that the MGS model is a G_z-quotient, but the target space is an H-quotient. For this reason, instead of constructing the tube directly we are going to split it as the composition of two maps: one that goes from an H_μ-quotient to an H-quotient, and another that goes from a G_z-quotient to an H_μ-quotient. We will explain this construction in the following sections.

5.2.2 The $\alpha = 0$ case

In this section we construct a Hamiltonian around a point of the form $z_0 = \varphi(e, \mu, 0, 0) \in T^*(G \times_H S)$, which is explicit up to a restricted G-tube. Using (5.14), the isotropy of z_0 is

$$G_{z_0} = H_\mu \cap H = H_\mu$$

and by Proposition 4.3.1 and the adapted splitting of Proposition 4.2.1, the symplectic slice at z_0 is

$$N_0 = \mathfrak{o} \times S \times S^*$$

with symplectic form (4.10). Then the map (5.15) reduces in this case to

$$\mathcal{T}_0\colon G \times_{H_\mu} \big(\underbrace{\mathfrak{p}^*}_{\mathfrak{m}^*} \times \underbrace{\mathfrak{o} \times S \times S^*}_{N_0} \big) \longrightarrow T^*(G \times_H S)$$

where $G \times_{H_\mu} (\mathfrak{p}^* \times \mathfrak{o} \times S \times S^*)$ is equipped with the symplectic form (2.6).

As we will later use $\mathcal{T}_0$ to construct a Hamiltonian tube around a general point $z \in T^*Q$, we need to ensure that the domain of $\mathcal{T}_0$ is large enough. More precisely, we will show that the domain of $\mathcal{T}_0$ contains all the points of the form $[e, 0, 0, 0, b]_{H_\mu}$.

Theorem 5.2.2. *Consider the point $z_0 = \varphi(e, \mu, 0, 0) \in T^*(G \times_H S)$. Let $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ be an adapted splitting in the sense of Proposition 4.2.1 and let $\Phi\colon G \times U_\Phi \longrightarrow T^*G$ be an associated restricted G-tube.*

In this setting, there are H_μ-invariant open neighborhoods of zero: $\mathfrak{p}^*_r \subset \mathfrak{p}^*$, $\mathfrak{o}_r \subset \mathfrak{o}$ and an H-invariant open neighborhood of zero $\mathfrak{h}^*_r \subset \mathfrak{h}^*$ such that the map

$$\begin{aligned} \mathcal{T}_0\colon G \times_{H_\mu} (\mathfrak{p}^*_r \times \mathfrak{o}_r \times (T^*S)_r) &\longrightarrow T^*(G \times_H S) \\ [g, \nu, \lambda; a, b]_{H_\mu} &\longmapsto \varphi(\Phi(g, \tilde{\nu}, \lambda; a \diamond_{\mathfrak{l}} b); a, b) \end{aligned} \tag{5.16}$$

is a Hamiltonian tube around the point z_0, where

$$\tilde{\nu} = \nu + \underbrace{\frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b}_{\mathbf{J}_{N_0}(\lambda, a, b)}$$

and $(T^*S)_r := \{(a, b) \in T^*S \mid a \diamond_{\mathfrak{h}} b \in \mathfrak{h}^*_r\}$.

Proof. If we assume the existence of $(\mathfrak{p}^*)_r, \mathfrak{o}_r$ and $\mathfrak{h}_r$ such that the map $\mathcal{T}_0$ is well defined then it follows from the properties of Φ that

$$\mathcal{T}_0([e,0,0;0,0]_{H_\mu}) = \varphi(\Phi(e,0,0;0);0,0) = \varphi(e,\mu;0,0)$$

and by the G-equivariance of Φ it is also clear that

$$\begin{aligned}\mathcal{T}_0(g'\cdot[g,\nu,\lambda;a,b]_{H_\mu}) &= \mathcal{T}_0([g'g,\nu,\lambda;a,b]_{H_\mu}) = \varphi(\Phi(g'g,\tilde{\nu},\lambda;a\diamond_{\mathfrak{l}} b);a,b) = \\ &= \varphi(g'\cdot\Phi(g,\tilde{\nu},\lambda;a\diamond_{\mathfrak{l}} b);a,b) = g'\cdot\varphi(\Phi(g,\tilde{\nu},\lambda;a\diamond_{\mathfrak{l}} b);a,b) = \\ &= g'\cdot\mathcal{T}_0([g,\nu,\lambda;a,b]_{H_\mu}).\end{aligned}$$

We will divide the rest of the proof in three steps. In the first one we prove that there is a set $G\times_{H_\mu}(\mathfrak{p}^*_{\mathrm{dom}}\times\mathfrak{o}_{\mathrm{dom}}\times(T^*S)_{\mathrm{dom}})$ such that the map $\mathcal{T}_0$ is well defined; it pulls-back the natural symplectic form of $T^*(G\times_H S)$ to the MGS form $G\times_{H_\mu}(\mathfrak{p}^*_{\mathrm{dom}}\times\mathfrak{o}_{\mathrm{dom}}\times(T^*S)_{\mathrm{dom}})$ and it is a local diffeomorphism. In the second one we will show that it is injective in a certain subset, and in the third we will prove that it is a diffeomorphism onto its image.

1- $\mathcal{T}_0$ is a local symplectomorphism:

Let $N_0=\mathfrak{o}\times S\times S^*$ be the symplectic slice at $z_0=\varphi(e,\mu,0,0)$. As in Section 2.3.1, there must be an H_μ-invariant neighborhood $(\mathfrak{g}^*_\mu)_r$ such that the product $Z:=G\times(\mathfrak{g}^*_\mu)_r\times(\mathfrak{o}\times S\times S^*)$ with $\omega_Z:=\omega_{T_\mu}+\Omega_{N_0}$ is a symplectic manifold with G^L and H^T_μ Hamiltonian actions with momentum maps $\mathbf{K}_{G^L}$ and $\mathbf{K}_{H^T_\mu}$ (see (2.4)).

We now use the restricted G-tube (see Definition 5.1.5) $\Phi\colon G\times U_\Phi\subset G\times\mathfrak{g}^*_\mu\times\mathfrak{o}\times\mathfrak{l}^*\to T^*G$ to relate Z with $T^*(G\times S)$. As Φ is only defined on $G\times U_\Phi$, we will define the open set

$$D:=\{(\nu,\lambda,a,b)\mid(\nu,\lambda,a\diamond_{\mathfrak{l}} b)\in U_\Phi,\quad \nu\in(\mathfrak{g}^*_\mu)_r\}\subset\mathfrak{g}^*_\mu\times\mathfrak{o}\times S\times S^*$$

and the map

$$\begin{aligned} f\colon G\times D &\longrightarrow T^*G\times T^*S \\ (g,\nu,\lambda,a,b) &\longmapsto (\Phi(g,\nu,\lambda,a\diamond_{\mathfrak{l}} b),a,b). \end{aligned} \tag{5.17}$$

The pullback of $\omega_{T^*(G\times S)}$ by f is ω_Z, because

$$\begin{aligned}(f^*\omega_{T^*(G\times S)})(u_1,u_2) &= (\Phi^*\omega_{T^*G})(g,\nu,\lambda,a\diamond_{\mathfrak{l}} b)(v_1,v_2)+\omega_{T^*S}(a,b)(w_1,w_2) = \\ &= \underbrace{\langle\dot{\nu}_2,\xi_1\rangle-\langle\dot{\nu}_1,\xi_2\rangle+\langle\nu+\mu,[\xi_1,\xi_2]\rangle}_{\omega_{T_\mu}} - \underbrace{\langle\mu,[\dot{\lambda}_1,\dot{\lambda}_2]\rangle+\langle\dot{b}_2,\dot{a}_1\rangle-\langle b_1,\dot{b}_1\rangle}_{\Omega_{N_0}}\end{aligned}$$

where $u_i=(\xi_i,\dot{\nu}_i,\dot{\lambda}_i,\dot{a}_i,\dot{b}_i)\in T_{(g,\nu,\lambda,a,b)}(G\times D)$.

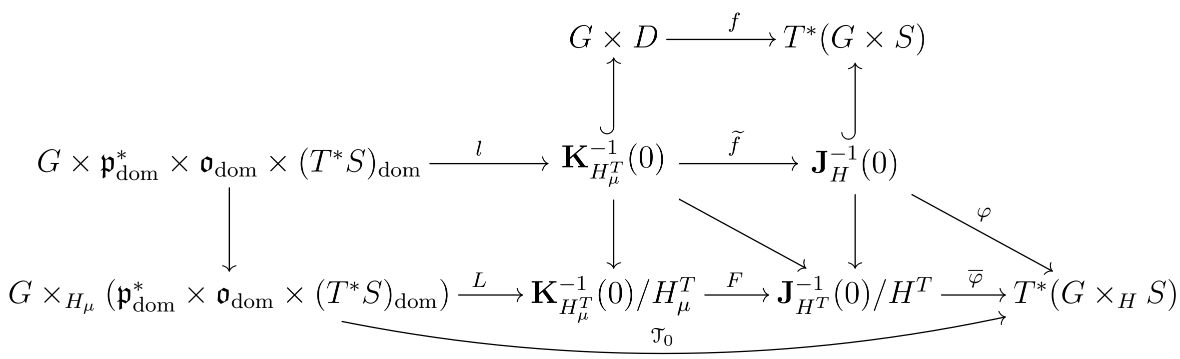
Note that on $G\times D$ there is a $G^L\times H^T_\mu$ action, but on $T^*G\times T^*S\cong G\times\mathfrak{g}^*\times S\times S^*$ there is a $G^L\times H^T$ action. As the map f is $G^L\times H^T_\mu$-equivariant, it preserves the H_μ-momentum, that is, $\mathbf{K}_{H^T_\mu}=\mathbf{J}_{H^T_\mu}\circ f$. In particular $f(\mathbf{K}^{-1}_{H^T_\mu}(0))\subset\mathbf{J}^{-1}_{H^T_\mu}(0)$. However, the $\mathfrak{l}$-momentum property (see Definition 5.1.5) of restricted G-tubes allows us to improve this, since for any $\xi\in\mathfrak{l}$

$$\begin{aligned}\langle\mathbf{J}_{H^T}(f(g,\nu,\lambda;a,b)),\xi\rangle &= \langle\mathbf{J}_R(\Phi(g,\nu,\lambda;a\diamond_{\mathfrak{l}} b))+a\diamond_{\mathfrak{h}} b,\xi\rangle = \\ &= \langle\mathbf{J}_R(\Phi(g,\nu,\lambda;a\diamond_{\mathfrak{l}} b))\big|_{\mathfrak{l}}+a\diamond_{\mathfrak{l}} b,\xi\rangle = \\ &= \langle -a\diamond_{\mathfrak{l}} b+a\diamond_{\mathfrak{l}} b,\xi\rangle = 0.\end{aligned}$$

This means that f can be restricted to a map

$$\widetilde{f}\colon\mathbf{K}^{-1}_{H^T_\mu}(0)\longrightarrow\mathbf{J}^{-1}_{H^T}(0)$$

and this is the key condition that will allow us to relate the H_μ-quotient $G \times_{H_\mu} (\mathfrak{p}^* \times N_0)$ with the H-quotient $\mathbf{J}_{H^T}^{-1}(0)/H^T \cong T^*(G \times_H S)$. To do so, consider the diagram



Composing $\widetilde{f}$ with the projection by H^T in the target we get a smooth map $\mathbf{K}_{H_\mu^T}^{-1}(0) \longrightarrow \mathbf{J}_{H^T}^{-1}(0)/H^T$ which is G^K-equivariant and H_μ^T-invariant, and hence it induces the smooth mapping

$$F\colon \mathbf{K}_{H_\mu^T}^{-1}(0)/H_\mu^T \longrightarrow \mathbf{J}_{H^T}^{-1}(0)/H^T.$$

If $\mathbf{K}_{H_\mu^T}^{-1}(0)/H_\mu^T$ is endowed with the reduced form $(\omega_Z)_{\mathrm{red}}$ and $\mathbf{J}_{H^T}^{-1}(0)/H^T$ with $(\omega_{T^*(G\times S)})_{\mathrm{red}}$ then $F^*(\omega_{T^*(G\times S)})_{\mathrm{red}} = (\omega_Z)_{\mathrm{red}}$, because $f^*\omega_{T^*(G\times S)} = \omega_Z$. In particular, F is an immersion. Also, as the H_μ^T-action on $G \times D$ is free

$$\begin{aligned}
\dim \mathbf{K}_{H_\mu^T}^{-1}(0)/H_\mu^T &= \dim \mathbf{K}_{H_\mu^T}^{-1}(0) - \dim \mathfrak{h}_\mu = \dim(G \times (\mathfrak{g}_\mu^* \times \mathfrak{o})) + 2\dim S - 2\dim \mathfrak{h}_\mu \\
&= \dim \mathfrak{g} + \dim \mathfrak{g}_\mu + \dim \mathfrak{o} - 2\dim \mathfrak{h}_\mu + 2\dim S \\
&= 2\dim \mathfrak{p} + 2\dim \mathfrak{o} + \dim \mathfrak{l} + \dim \mathfrak{n} + 2\dim S \\
&= 2\dim \mathfrak{p} + 2\dim \mathfrak{o} + 2\dim \mathfrak{l} + 2\dim S \\
&= 2(\dim \mathfrak{g} - \dim \mathfrak{h} + \dim S).
\end{aligned}$$

Analogously,

$$\dim \mathbf{J}_H^{-1}(0)/H^T = \dim \mathbf{J}_{H^T}^{-1}(0) - \dim \mathfrak{h} = 2(\dim \mathfrak{g} - \dim \mathfrak{h} + \dim S).$$

This implies that F is a local diffeomorphism because it is an immersion between spaces of the same dimension.

By continuity we can choose H_μ-invariant neighborhoods of the origin $\mathfrak{p}^*_{\mathrm{dom}} \subset \mathfrak{p}^*$, $\mathfrak{o}_{\mathrm{dom}} \subset \mathfrak{o}$ and an H-invariant neighborhood of the origin $\mathfrak{h}^*_{\mathrm{dom}} \subset \mathfrak{h}^*$ such that

$$(\nu + \underbrace{\frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b}_{\mathbf{J}_{N_0}}, \lambda, a, b) \in D$$

for any $\nu \in \mathfrak{p}^*_{\mathrm{dom}}$, $\lambda \in \mathfrak{o}_{\mathrm{dom}}$ and $a, b \in T^*S$ with $a \diamond_{\mathfrak{h}} b \in \mathfrak{h}^*_{\mathrm{dom}}$. The map

$$L\colon G \times_{H_\mu} (\mathfrak{p}^*_{\mathrm{dom}} \times \mathfrak{o}_{\mathrm{dom}} \times (T^*S)_{\mathrm{dom}}) \to \mathbf{K}_{H_\mu^T}^{-1}(0)/H^T$$

given by $[g, \nu, \lambda, a, b]_{H_\mu} \mapsto [g, \nu + \mathbf{J}_{N_0}(\lambda, a, b), \lambda, a, b]_{H_\mu}$ is well defined and, as in (2.5), $L^*(\omega_Z)_{\mathrm{red}} = \Omega_Y$. The conclusion of this first step is that the composition $\mathcal{T}_0 := \bar{\varphi} \circ F \circ L$ is then a local diffeomorphism that pulls-back the canonical form of $T^*(G \times_H S)$ to the MGS form on the set $G \times_{H_\mu} (\mathfrak{p}^*_{\mathrm{dom}} \times \mathfrak{o}_{\mathrm{dom}} \times (T^*S)_{\mathrm{dom}})$.

2- $\mathcal{T}_0$ is locally injective

As $\mathcal{T}_0 \colon G \times_{H_\mu} (\mathfrak{p}^*_{\rm dom} \times \mathfrak{o}_{\rm dom} \times (T^*S)_{\rm dom}) \to T^*(G \times_H S)$ is a local diffeomorphism, there is a neighborhood of $[e,0,0,0,0]_{H_\mu}$ such that $\mathcal{T}_0$ is injective on it.

Using that $\mathcal{T}_0$ is G-equivariant and that the action is proper, this neighborhood can be chosen to be G-invariant (see for example the proof of Theorem 2.1.4); that is, $\mathcal{T}_0$ will be injective when restricted to the set $G \times_{H_\mu} \left(\mathfrak{p}^*_{\rm inj} \times \mathfrak{o}_{\rm inj} \times (T^*S)_{\rm inj}\right)$ where $\mathfrak{p}^*_{\rm inj} \subset \mathfrak{p}^*_{\rm dom}$, $\mathfrak{o}_{\rm inj} \subset \mathfrak{o}_{\rm dom}$ are H_μ-invariant neighborhoods and $(T^*S)_{\rm inj}$ is an H-invariant neighborhood of 0 on $(T^*S)_{\rm dom}$. Note that we cannot ensure that $(T^*S)_{\rm inj}$ will be big enough to contain all the points of the form $(0,b) \in T^*S$. This issue will be addressed in the next step.

3- $\mathcal{T}_0$ is injective

In this step, we efine an open set $(T^*S)_r \subset (T^*S)_{\rm dom}$ such that the restriction

$$\mathcal{T}_0 \colon G \times_{H_\mu} \left(\mathfrak{p}^*_{\rm inj} \times \mathfrak{o}_{\rm inj} \times (T^*S)_r\right) \longrightarrow \mathcal{T}_0\left(G \times_{H_\mu} \left(\mathfrak{p}^*_{\rm inj} \times \mathfrak{o}_{\rm inj} \times (T^*S)_r\right)\right)$$

is a proper map

The key result that we use to prove the properness of $\mathcal{T}_0$ is the following topological result.

Proposition 5.2.3 ([MT03], Lemma 5). *Let H be a Lie group acting on a symplectic vector space (W, ω_W) and denote by $\mathbf{J} \colon W \to \mathfrak{h}^*$ the associated homogeneous momentum map*

$$\langle \mathbf{J}_W(v), \xi \rangle = \frac{1}{2} \omega_W(\xi \cdot v, v).$$

Then $\mathbf{J}$ is H-open relative to its image; that is, if U is an H-invariant open set of W then $\mathbf{J}(U)$ is an H-invariant open set of the topological space $\mathbf{J}(W) \subset \mathfrak{h}^*$.

Let $U_1 \subset S$ and $U_2 \subset S^*$ be H-invariant neighborhoods of the origin such that $\overline{U_1 \times U_2} \subset (T^*S)_{\rm inj}$. Using Proposition 5.2.3, there is $\mathfrak{h}^*_r$ an open neighborhood of $0 \in \mathfrak{h}^*$ such that

$$\mathfrak{h}^*_r \cap (S \diamond_\mathfrak{h} S^*) = U_1 \diamond_\mathfrak{h} U_2 \subset \mathfrak{h}^*.$$

In this setting, define $(T^*S)_r := \{(a,b) \in T^*S \mid a \diamond_\mathfrak{h} b \in \mathfrak{h}^*_r\}$. From the first step of the proof we have the following commutative diagram

$$\begin{array}{ccc}
G \times \mathfrak{p}^*_{\rm dom} \times \mathfrak{o}_{\rm dom} \times (T^*S)_{\rm dom} & \xrightarrow{\;\widetilde{f} \circ l\;} & \mathbf{J}^{-1}_{H^T}(0) \\
\Big\downarrow {\scriptstyle \pi_{H_\mu}} & & \Big\downarrow {\scriptstyle \overline{\varphi} \circ \pi_H} \\
G \times_{H_\mu} (\mathfrak{p}^*_{\rm dom} \times \mathfrak{o}_{\rm dom} \times (T^*S)_{\rm dom}) & \xrightarrow{\;\mathcal{T}_0\;} & T^*(G \times_H S)
\end{array}$$

The problem is that $\widetilde{f} \circ l$ is an injective embedding, but it is not clear if it is proper. We will now show that $\mathcal{T}_0 \circ \pi_{H_\mu}$ is a proper map onto its image when restricted to $G \times \mathfrak{p}^*_{\rm inj} \times \mathfrak{o}_{\rm inj} \times (T^*S)_r$. To this end, let $x_n = (g_n, \nu_n, \lambda_n; a_n, b_n)$ be a sequence in $G \times \mathfrak{p}^*_{\rm inj} \times \mathfrak{o}_{\rm inj} \times (T^*S)_r$ such that

$$\mathcal{T}_0(\pi_{H_\mu}(x_n)) \longrightarrow \mathcal{T}_0(\pi_{H_\mu}(\bar{g}, \bar{\nu}, \bar{\lambda}; \bar{a}, \bar{b}))$$

with $(\bar{g}, \bar{\nu}, \bar{\lambda}; \bar{a}, \bar{b}) \in G \times \mathfrak{p}^*_{\rm inj} \times \mathfrak{o}_{\rm inj} \times (T^*S)_r$. We construct a subsequence $\{x_{\sigma_3(n)}\} \subset \{x_n\}$ which is convergent on $G \times \mathfrak{p}^*_{\rm inj} \times \mathfrak{o}_{\rm inj} \times (T^*S)_r$.

The map $\overline{\varphi} \circ \pi_H \colon \mathbf{J}^{-1}_{H^T}(0) \to T^*(G \times_H S)$ is proper because it is a composition of a homeomorphism and the projection by a compact group. Since $\mathcal{T}_0 \circ \pi_{H_\mu} = \overline{\varphi} \circ \pi_H \circ \widetilde{f} \circ l$, then there is an increasing map $\sigma_1 \colon \mathbb{N} \to \mathbb{N}$ such that the sequence $\{(\widetilde{f} \circ l)(x_{\sigma_1(n)})\}_n$ converges in $\mathbf{J}_{H^T}^{-1}(0) \subset T^*(G \times S)$ (we are just taking a subsequence). But then by uniqueness of the limit there is $h \in H$ such that

$$(\widetilde{f} \circ l)(x_{\sigma_1(n)}) \longrightarrow h \cdot^T ((\widetilde{f} \circ l)(\bar{g}, \bar{\nu}, \bar{\lambda}; \bar{a}, \bar{b})).$$

However, using the expression of f (5.17) this implies that $a_{\sigma_1(n)} \to h \cdot \bar{a}$ and $b_{\sigma_1(n)} \to h \cdot \bar{b}$. By the definition of $(T^*S)_r$ we can choose for each n a pair $(\alpha_n, \beta_n) \in U_1 \times U_2$ satisfying

$$\alpha_n \diamond_{\mathfrak{h}} \beta_n = a_n \diamond_{\mathfrak{h}} b_n.$$

Since $U_1 \times U_2$ is a relatively compact subset of $(T^*S)_{\text{inj}}$, we can find an increasing map $\sigma_2 \colon \mathbb{N} \to \mathbb{N}$ such that $\sigma_2(\mathbb{N}) \subset \sigma_1(\mathbb{N})$ and $(\alpha_{\sigma_2(n)}, \beta_{\sigma_2(n)}) \to (\alpha_\infty, \beta_\infty)$, but then

$$\left\{(\widetilde{f} \circ l)(g_{\sigma_2(n)}, \nu_{\sigma_2(n)}, \lambda_{\sigma_2(n)}, \alpha_{\sigma_2(n)}, \beta_{\sigma_2(n)})\right\}_{\mathfrak{n} \in \mathbb{N}}$$

is a convergent sequence and

$$(\widetilde{f} \circ l)(g_{\sigma_2(n)}, \nu_{\sigma_2(n)}, \lambda_{\sigma_2(n)}, \alpha_{\sigma_2(n)}, \beta_{\sigma_2(n)}) \longrightarrow h \cdot^T \left((\widetilde{f} \circ l)(\bar{g}, \bar{\nu}, \bar{\lambda}, h^{-1}\alpha_\infty, h^{-1}\beta_\infty)\right).$$

As $(g_{\sigma_2(n)}, \nu_{\sigma_2(n)}, \lambda_{\sigma_2(n)}, \alpha_{\sigma_2(n)}, \beta_{\sigma_2(n)})$ lies in $G \times \mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_{\text{inj}}$, using that $\mathcal{T}_0$ restricted to $G \times \mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_{\text{inj}}$ is a diffeomorphism and π_{H_μ} is proper, there is an increasing map $\sigma_3 \colon \mathbb{N} \to \mathbb{N}$ with $\sigma_3(\mathbb{N}) \subset \sigma_2(\mathbb{N})$ such that $(g_{\sigma_3(n)}, \nu_{\sigma_3(n)}, \lambda_{\sigma_3(n)}, \alpha_{\sigma_3(n)}, \beta_{\sigma_3(n)})$ converges in $G \times \mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_{\text{inj}}$. Therefore, $\{x_{\sigma_3(n)}\}$ is a convergent sequence on $G \times \mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r$.

This proves that $\mathcal{T}_0 \circ \pi_{H_\mu} \colon G \times \mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r \to (\mathcal{T}_0 \circ \pi_{H_\mu})(G \times \mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r)$ is proper. But since $\pi_{H_\mu} \colon G \times \mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r \to G \times_{H_\mu} \left(\mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r\right)$ is surjective and continuous this implies that $\mathcal{T}_0 \colon G \times_{H_\mu} \left(\mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r\right) \longrightarrow \mathcal{T}_0\big(G \times_{H_\mu} \left(\mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r\right)\big)$ is a proper map.

As $\mathcal{T}_0$ is a local homeomorphism and a proper map, it follows that it is a covering map. Then if $\varphi(e, \mu, 0, 0)$ has only one preimage, this implies that the covering map is in fact everywhere injective and therefore a global diffeomorphism. But if $\mathcal{T}_0([g, \nu, \lambda, a, b]_{H_\mu}) = \varphi(e, \mu, 0, 0)$ then it is clear from the expression of $\mathcal{T}_0$ (5.16) that $a = b = 0$ and then $[g, \nu, \lambda, 0, 0]_{H_\mu} \in G \times_{H_\mu} \left(\mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_{\text{inj}}\right) \subset G \times_{H_\mu} \left(\mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r\right)$. Therefore, by injectivity we have $[g, \nu, \lambda, 0, 0]_{H_\mu} = [e, 0, 0, 0, 0]_{H_\mu}$. To sum up, the restricted map

$$\begin{aligned} \mathcal{T}_0 \colon G \times_{H_\mu} \left(\mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r\right) &\longrightarrow \mathcal{T}_0(G \times_{H_\mu} \left(\mathfrak{p}^*_{\text{inj}} \times \mathfrak{o}_{\text{inj}} \times (T^*S)_r\right)) \subset T^*(G \times_H S) \\ [g, \nu, \lambda; a, b]_{H_\mu} &\longmapsto \varphi(\Phi(g, \tilde{\nu}, \lambda; a \diamond_{\mathfrak{l}} b); a, b) \end{aligned}$$

where $\tilde{\nu} = \nu + \mathbf{J}_{N_0}(\lambda, a, b) = \nu + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b$, is a bijection.

□

5.2.3 The Γ map

In this section we introduce the Γ map, a technical tool used in [Sch07] to build the Hamiltonian cotangent tube when $G = G_\mu$. Here we will use it as the final step towards generalizing the previous Hamiltonian tube at $\alpha = 0$ to the general case $\alpha \neq 0$.

Let $\varphi(e, \mu, 0, \alpha) \in T^*(G \times_H S)$ and define $K = H_\mu \cap H_\alpha$. Recall that in (5.15) we defined $B := (\mathfrak{h}_\mu \cdot \alpha)^\circ \subset S$ and a K-invariant complement $\mathfrak{s}$ of $\mathfrak{k}$ in $\mathfrak{h}_\mu$. As K is compact, we can choose a K-invariant splitting $S = B \oplus C$ inducing the K-invariant splitting $S^* = B^* \oplus (\mathfrak{h}_\mu \cdot \alpha)$.

However, the previous splitting of S^* is not H_μ-invariant. The following technical result studies how it behaves with respect to the infinitesimal H_μ-action on S. It is a straightforward generalization to the case $\mathfrak{g}_\mu \neq \mathfrak{g}$ of Lemmas 27 and 28 of [Sch07].

Proposition 5.2.4. *In the above situation:*

- *If $a \in B$, $c \in C$ and $b \in B^*$, then*
$$(a+c) \diamond_{\mathfrak{h}_\mu} (\alpha+b) = a \diamond_{\mathfrak{h}_\mu} b + c \diamond_{\mathfrak{s}} (\alpha+b).$$
- *There is a K-invariant neighborhood $(B^*)_r$ of the origin in B^* and a K-equivariant map*
$$\Gamma\colon \mathfrak{s}^* \times (B^*)_r \longrightarrow S$$
defined by
$$\langle \Gamma(\nu; b), \xi \cdot (\alpha+b) + \beta \rangle = -\langle \nu, \xi \rangle \quad \forall \beta \in B^*,\ \forall \xi \in \mathfrak{s}. \tag{5.18}$$
Moreover, Γ satisfies $\Gamma(\nu; b) \diamond_{\mathfrak{s}} (b+\alpha) = \nu$ and $\Gamma(\nu; b) \in C$ for any $\nu \in \mathfrak{s}^*$ and $b \in (B^*)_r$.

Proof. The first part is a generalization of the proof of Lemma 27 of [Sch07],
$$\begin{aligned}(a+c) \diamond_{\mathfrak{h}_\mu} (\alpha+b) &= a \diamond_{\mathfrak{h}_\mu} (\alpha+b) + c \diamond_{\mathfrak{h}_\mu} (\alpha+b) = \\ &= a \diamond_{\mathfrak{h}_\mu} b + c \diamond_{\mathfrak{h}_\mu} (\alpha+b) = \\ &= a \diamond_{\mathfrak{h}_\mu} b + c \diamond_{\mathfrak{s}} (\alpha+b)\end{aligned}$$
where we used $a \diamond_{\mathfrak{h}_\mu} \alpha = 0$ because for any $\xi \in \mathfrak{h}_\mu$: $\langle a, \xi \cdot \alpha \rangle = 0$ as $a \in B = (\mathfrak{h}_\mu \cdot \alpha)^\circ$. By a similar argument $c \diamond_{\mathfrak{k}} (\alpha+b) = 0$ because if $\xi \in \mathfrak{k}$ then $\xi \cdot \alpha = 0$ and $\xi \cdot b \in B^*$ but $c \in C$.

The second part is an adapted version of Lemma 28 of [Sch07]. Consider,
$$\begin{aligned} t\colon H_\mu \times_K B^* &\longrightarrow S^* \\ [h,b]_K &\longmapsto h \cdot (b+\alpha)\end{aligned}$$
then $T_{[e,0]_K}(\xi, \dot{b}) = \xi \cdot a + \dot{b}$ but as $S^* = B^* \oplus (\mathfrak{h}_\mu \cdot \alpha) = B^* \oplus (\mathfrak{s} \cdot \alpha)$, $T_{[e,0]_K} t$ is invertible and, therefore, there is $(B^*)_r$ small enough such that $t\colon H_\mu \times_K (B^*)_r \to t(H_\mu \times_K (B^*)_r) \subset S^*$ is a diffeomorphism.

Then for any $b \in (B^*)_r$
$$\begin{aligned} \mathfrak{s} \times B^* &\longrightarrow T_{\alpha+b} S^* \cong S^* \\ (\xi, \dot{b}) &\longmapsto \xi \cdot (\alpha+b) + \dot{b}\end{aligned}$$
is a linear K-equivariant isomorphism. But then (5.18) defines $\Gamma(\nu, b)$ uniquely for any $\nu \in \mathfrak{s}^*$ and $b \in (B^*)_r$.

As $\langle \Gamma(\nu,b), \beta \rangle = 0$ for any $\beta \in B^*$ it is clear that $\Gamma(\nu,b) \in (B^*)^\circ = C$ and if $\xi \in \mathfrak{s}$
$$\begin{aligned}\langle \Gamma(\nu,b) \diamond_{\mathfrak{s}} (b+\alpha), \xi \rangle &= \langle b+\alpha, \xi \cdot \Gamma(\nu,b) \rangle \\ &= -\langle \xi \cdot (b+\alpha), \Gamma(\nu,b) \rangle \\ &= \langle \nu, \xi \rangle\end{aligned}$$
□

Remark 5.2.5. Note that $\Gamma(\cdot;0) = \widetilde{\Gamma}(\cdot)$, where $\widetilde{\Gamma}\colon \mathfrak{s}^* \to S$ is the linear map appearing in Proposition 4.4.1 that describes the Witt-Artin decomposition of a cotangent bundle.

With the notation that we have already introduced, the symplectic slice at $\varphi(e,\mu,0,0)$ is $N_0 = \mathfrak{o} \times T^*S$, whereas the symplectic slice at $\varphi(e,\mu,0,\alpha)$ is $N_\alpha = \mathfrak{o} \times T^*B$ (see Proposition 4.3.1). The abstract MGS models at $\varphi(e,\mu,0,0)$ and $\varphi(e,\mu,0,\alpha)$ are $G \times_{H_\mu} (\mathfrak{p}^* \times N_0)$ and $G \times_K \left((\mathfrak{s}^* \oplus \mathfrak{p}^*) \times N_\alpha\right)$, respectively. The next result shows that Γ can be used to build a well-behaved map between both spaces.

Theorem 5.2.6. *In the above context there is an open K-invariant neighborhood W of zero in $(\mathfrak{s}^* \oplus \mathfrak{p}^*) \times \mathfrak{o} \times B \times B^*$ such that the G-equivariant map*

$$\begin{aligned} \mathcal{F} \colon G \times_K W &\longrightarrow G \times_{H_\mu} (\mathfrak{p}^* \times \mathfrak{o} \times S \times S^*) \\ [g, \nu_{\mathfrak{s}} + \nu_{\mathfrak{p}}, \lambda, a, b]_K &\longmapsto [g, \nu_{\mathfrak{p}}, \lambda, \widetilde{a}, b + \alpha]_{H_\mu}, \end{aligned}$$

where $\widetilde{a} = a + \Gamma(\nu_{\mathfrak{s}} - a \diamond_{\mathfrak{s}} b - \frac{1}{2}\lambda \diamond_{\mathfrak{s}} \mathrm{ad}^*_\lambda \mu; b)$, is a local symplectomorphism.

Proof. As in the first part of the proof of Theorem 5.2.2, there is a neighborhood $(\mathfrak{g}^*_\mu)_r$ of $0 \in \mathfrak{g}^*$ such that $Z_0 := G \times (\mathfrak{g}^*_\mu)_r \times (\mathfrak{o} \times S \times S^*)$ is a symplectic space with $\omega_{Z_0} := \omega_{T_\mu} + \Omega_{N_0}$. We are in the same setting as in Subsection 2.3.1; therefore, Z_0 supports G^L and H^T_μ Hamiltonian actions with momentum maps that were denoted as $\mathbf{K}_{G^L}$ and $\mathbf{K}_{H^T_\mu}$.

Similarly, $Z_\alpha := G \times (\mathfrak{g}^*_\mu)_r \times (\mathfrak{o} \times B \times (B^*)_r)$ is a symplectic space with symplectic form $\omega_{Z_\alpha} := \omega_{T_\mu} + \Omega_{N_\alpha}$, because $N_\alpha = \mathfrak{o} \times B \times B^*$. Note that Z_α has G^L and K^T-Hamiltonian actions with momentum maps $\mathbf{M}_{G_L}$ and $\mathbf{M}_{K^T}$. Consider now the map

$$\begin{aligned} f \colon Z_\alpha &\longrightarrow Z_0 \\ (g, \nu, \lambda; a, b) &\longmapsto (g, \nu, \lambda; a + \Gamma(\eta; b), b + \alpha) \end{aligned}$$

where $\eta = \nu\big|_{\mathfrak{s}} - a \diamond_{\mathfrak{s}} b - \frac{1}{2}\lambda \diamond_{\mathfrak{s}} \mathrm{ad}^*_\lambda \mu$. As Γ is K-equivariant, then f is $G^L \times K^T$ equivariant. Note that the potential $\theta_{Z_0}(g, \nu, \lambda; a, b)(\xi, \dot{\nu}, \dot{\lambda}; \dot{a}, \dot{b}) = \langle \nu + \mu, \xi \rangle + \frac{1}{2}\langle \mu, [\lambda, \dot{\lambda}] \rangle + \langle b, \dot{a} \rangle - \langle \alpha, \dot{a} \rangle$ generates the symplectic structure ω_{Z_0} (see (2.3)) and

$$\begin{aligned} (f^*\theta_{Z_0})(g, \nu, \lambda, a, b) \cdot v = \theta_{Z_0}(f(g, \nu, \lambda, a, b))(T_{(g,\nu,\lambda,a,b)} f \cdot v) = \\ = \langle \mu + \nu, \xi \rangle + \frac{1}{2}\langle \mu, [\lambda, \dot{\lambda}] \rangle + \langle b, \dot{a} + T_{(\nu,\lambda,a,b)}\Gamma \cdot (\dot{\nu}, \dot{\lambda}, \dot{a}, \dot{b}) \rangle = \\ = \langle \mu + \nu, \xi \rangle + \frac{1}{2}\langle \mu, [\lambda, \dot{\lambda}] \rangle + \langle b, \dot{a} \rangle \end{aligned}$$

where $v = (\xi, \dot{\nu}, \dot{a}, \dot{b}) \in T_{(g,\nu,a,b)} Z_\alpha$. Taking the exterior derivative of this equality, we get $f^*\omega_{Z_0} = \omega_{Z_\alpha}$.

Additionally, the H^T_μ-momentum evaluated at $f(g, \nu, \lambda, a, b)$ is

$$\begin{aligned} \mathbf{K}_{H^T_\mu}(f(g, \nu, \lambda; a, b)) &= -\nu + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + (a + \Gamma(\eta; b)) \diamond_{\mathfrak{h}_\mu} (b + \alpha) = \\ &= -\nu\big|_{\mathfrak{h}_\mu} + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b + \Gamma(\eta; b) \diamond_{\mathfrak{s}} (b + \alpha) = \\ &= -\nu\big|_{\mathfrak{h}_\mu} + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b + \eta = \\ &= -\nu\big|_{\mathfrak{h}_\mu} + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b + \nu\big|_{\mathfrak{s}} - a \diamond_{\mathfrak{s}} b - \frac{1}{2}\lambda \diamond_{\mathfrak{s}} \mathrm{ad}^*_\lambda \mu \\ &= -\nu\big|_{\mathfrak{k}} + \frac{1}{2}\mathrm{ad}^*_\lambda \mu \diamond_{\mathfrak{k}} \lambda + a \diamond_{\mathfrak{k}} b. \end{aligned} \tag{5.19}$$

This means that if $\xi \in \mathfrak{s}$ then $\langle \mathbf{K}_{H^T_\mu}(f(g, \nu, \lambda, a, b)), \xi \rangle = 0$, and as

$$\mathbf{M}_{K^T}(g, \nu, \lambda, a, b) = -\nu\big|_{\mathfrak{k}} + \frac{1}{2}\mathrm{ad}^*_\lambda \mu \diamond_{\mathfrak{k}} \lambda + a \diamond_{\mathfrak{k}} b,$$

f can be restricted to

$$\widetilde{f} \colon \mathbf{M}^{-1}_{K^T}(0) \longrightarrow \mathbf{K}^{-1}_{H^T_\mu}(0).$$

As in Theorem 5.2.2, we can construct from top to bottom all the arrows of the diagram

$$\begin{array}{ccc}
Z_\alpha & \xrightarrow{\quad f \quad} & Z_0 \\
\uparrow & & \uparrow \\
\mathbf{M}_{K^T}^{-1}(0) & \xrightarrow{\quad \widetilde{f} \quad} & \mathbf{K}_{H_\mu}^{-1}(0) \\
\downarrow & \searrow & \downarrow \\
\mathbf{M}_{K^T}^{-1}(0)/K^T & \xrightarrow{\quad F \quad} & \mathbf{K}_{H_\mu}^{-1}(0)/H_\mu^T
\end{array}$$

using the same arguments as in the first part of Theorem 5.2.2. Therefore, as F is an immersion between spaces of the same dimension, it is a local diffeomorphism onto its image.

Adapting the construction of map (2.5) to this setting, define the K-invariant open set

$$W := \{(\nu_{\mathfrak{s}} + \nu_{\mathfrak{p}}, \lambda, a, b) \in (\mathfrak{s}^* \oplus \mathfrak{s}^*) \times \mathfrak{o} \times B \times (B^*)_r \mid \nu_{\mathfrak{s}} + \nu_{\mathfrak{p}} + \underbrace{\frac{1}{2}\mathrm{ad}^*_\lambda \mu \diamond_{\mathfrak{k}} \lambda + a \diamond_{\mathfrak{k}} b}_{\mathbf{J}_{N_\alpha}} \in (\mathfrak{g}^*_\mu)_r\}$$

and the map $L_\alpha \colon G \times_K W \longrightarrow \mathbf{M}_{K^T}^{-1}(0)/K^T$ given by

$$L_\alpha([g, \nu_{\mathfrak{s}} + \nu_{\mathfrak{p}}, \lambda, a, b]_K) = [g, \nu_{\mathfrak{s}} + \nu_{\mathfrak{p}} + \mathbf{J}_{N_\alpha}(\lambda, a, b), \lambda, a, b]_K,$$

then L_α is a G-equivariant symplectomorphism. And similarly,

$$R_0 \colon \mathbf{K}_{H_\mu^T}^{-1}(0)/H_\mu^T \to G \times_{H_\mu} (\mathfrak{p}^* \times \mathfrak{o} \times S \times S^*)$$

defined by $[g, \nu, \lambda, a, b]_{H_\mu} \mapsto [g, \nu\big|_{\mathfrak{p}}, \lambda, a, b]_{H_\mu}$ is a G-equivariant symplectomorphism.

Finally, we can conclude that the composition $\mathcal{F} = R_0 \circ F \circ L_\alpha \colon G \times_K W \to G \times_{H_\mu} (\mathfrak{p}^* \times \mathfrak{o} \times S \times S^*)$ is a G-equivariant local diffeomorphism that pulls-back the MGS symplectic form of the target to the MGS symplectic form of $G \times_K W$.

□

5.2.4 General tube

In this section we deal with the most general situation and will construct a Hamiltonian tube around an arbitrary point $\varphi(e, \mu, 0, \alpha)$. To do so we use Theorem 5.2.2 to obtain a Hamiltonian tube around $\varphi(e, \mu, 0, 0)$, and then we compose it with the map $\mathcal{F}$ of Theorem 5.2.6. The result of this composition will be the desired Hamiltonian tube around $\varphi(e, \mu, 0, \alpha)$.

Theorem 5.2.7. *Consider the point $z \in T^*(G \times_H S)$ defined by $z = \varphi(e, \mu, 0, \alpha)$. Let $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ be an adapted splitting in the sense of Proposition 4.2.1, and let*

$$\begin{aligned}
\mathcal{T}_0 \colon G \times_{H_\mu} (\mathfrak{p}^*_r \times \mathfrak{o}_r \times (T^*S)_r) &\longrightarrow T^*(G \times_H S) \\
[g, \nu, \lambda; a, b]_{H_\mu} &\longmapsto \varphi(\Phi(g, \nu + \mathbf{J}_{N_0}(\lambda, a, b), \lambda; a \diamond_{\mathfrak{l}} b); a, b)
\end{aligned}$$

be a Hamiltonian tube around the point $\varphi(e, \mu, 0, 0)$ given by Theorem 5.2.2, where

$$\mathbf{J}_{N_0}(\lambda, a, b) = \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b$$

and $(T^*S)_r := \{(a, b) \in T^*S \mid a \diamond_{\mathfrak{h}} b \in \mathfrak{h}^*_r\}$.

Define $B = (\mathfrak{h}_\mu \cdot \alpha)^\circ \subset S$ and let the map $\Gamma\colon \mathfrak{s}^* \times B_r^* \to S$ be the one defined in Proposition 5.2.4.

In this setting there are small enough G_z-invariant neighborhoods of zero $\mathfrak{s}_s^* \subset \mathfrak{s}^*$, $\mathfrak{p}_s^* \subset \mathfrak{p}_r^*$, $\mathfrak{o}_s \subset \mathfrak{o}_r$, $B_s \subset B$ and $B_s^* \subset B^*$ such that the map

$$\begin{aligned} \mathcal{T}\colon G \times_{G_z} ((\mathfrak{s}_s^* \oplus \mathfrak{p}_s^*) \times \mathfrak{o}_s \times B_s \times B_s^*) &\longrightarrow T^*(G \times_H S) \\ [g, \nu_\mathfrak{s} + \nu_\mathfrak{p}, \lambda; a, b]_{G_z} &\longmapsto \mathcal{T}_0([g, \nu_\mathfrak{p}, \lambda; \widetilde{a}, b + \alpha]_{H_\mu}) \end{aligned} \tag{5.20}$$

where

$$\tilde{a} = a + \Gamma(\nu_\mathfrak{s} - a \diamond_\mathfrak{s} b - \frac{1}{2}\lambda \diamond_\mathfrak{s} \mathrm{ad}_\lambda^* \mu; b)$$

is a Hamiltonian tube around the point $z = \varphi(e, \mu, 0, \alpha)$.

Equivalently,

$$\mathcal{T}([g, \nu_\mathfrak{s} + \nu_\mathfrak{p}, \lambda; a, b]_{G_z}) = \varphi(\Phi(g, \nu_\mathfrak{s} + \nu_\mathfrak{p} + \mathbf{J}_{N_\alpha}(\lambda, a, b), \lambda; \varepsilon); \tilde{a}, b + \alpha)$$

where

$$\begin{aligned} \tilde{a} &= a + \Gamma(\nu_\mathfrak{s} - a \diamond_\mathfrak{s} b - \frac{1}{2}\lambda \diamond_\mathfrak{s} \mathrm{ad}_\lambda^* \mu; b), \\ \mathbf{J}_{N_\alpha}(\lambda, a, b) &= \frac{1}{2}\lambda \diamond_{\mathfrak{g}_z} \mathrm{ad}_\lambda^* \mu + a \diamond_{\mathfrak{g}_z} b, \\ \varepsilon &= \tilde{a} \diamond_\mathfrak{l} (b + \alpha). \end{aligned}$$

Proof. By Theorem 5.2.6, there is a map $\mathcal{F}\colon G \times_{G_z} W \to G \times_{H_\mu} (\mathfrak{p}^* \times \mathfrak{o} \times T^*S)$ with $W \subset (\mathfrak{p}^* \oplus \mathfrak{s}^*) \times \mathfrak{o} \times B \times B^*$.

Note that $G \times_{H_\mu} (\mathfrak{p}_r^* \times \mathfrak{o}_r \times (T^*S)_r)$ is an open G-invariant subset of $G \times_{H_\mu} (\mathfrak{p}^* \times \mathfrak{o} \times T^*S)$. Since $\mathcal{F}$ is continuous, then the preimage of $G \times_{H_\mu} (\mathfrak{p}_r^* \times \mathfrak{o}_r \times (T^*S)_r)$ by $\mathcal{F}$ is open and contains the point $[e, 0, 0, 0, 0]_{G_z}$ because $\mathcal{F}([e, 0, 0, 0, 0]_{G_z}) = [e, 0, 0, 0, \alpha]_{H_\mu} \subset G \times_{H_\mu} (\mathfrak{p}_r^* \times \mathfrak{o}_r \times (T^*S)_r)$ since $0 \diamond_\mathfrak{h} \alpha = 0 \in \mathfrak{h}_r^*$.

Therefore, we can choose small enough G_z-invariant neighborhoods of zero $\mathfrak{s}_s^* \subset \mathfrak{s}^*$, $\mathfrak{p}_s^* \subset \mathfrak{p}_r^*$, $\mathfrak{o}_s \subset \mathfrak{o}_r$, $B_s \subset B$ and $B_s^* \subset B^*$ such that the composition

$$\mathcal{T} := \mathcal{T}_0 \circ \mathcal{F}\colon G \times_{G_z} ((\mathfrak{s}_s^* \oplus \mathfrak{p}_s^*) \times \mathfrak{o}_s \times B_s \times B_s^*) \to T^*(G \times_H S)$$

is well-defined and injective. Using Theorem 5.2.2 and 5.2.6, we conclude that $\mathcal{T}$ is a Hamiltonian tube around $\varphi(e, \mu, 0, \alpha)$.

More precisely, as $\mathcal{F}([g, \nu_\mathfrak{s} + \nu_\mathfrak{p}, \lambda, a, b]_{G_z}) = [g, \nu_\mathfrak{p}, \lambda, \widetilde{a}, b + \alpha]_{H_\mu}$ with $\widetilde{a} = a + \Gamma(\nu_\mathfrak{s} - a \diamond_\mathfrak{s} b - \frac{1}{2}\lambda \diamond_\mathfrak{s} \mathrm{ad}_\lambda^* \mu; b)$ then $(\mathcal{T}_0 \circ \mathcal{F})([g, \nu_\mathfrak{s} + \nu_\mathfrak{p}, \lambda; a, b]_{G_z}) = \varphi(\Phi(g, \widetilde{\nu}, \lambda; \varepsilon); \widetilde{a}, b + \alpha)$ where $\varepsilon = \widetilde{a} \diamond_\mathfrak{l} (b + \alpha)$, and

$$\widetilde{\nu} = \nu_\mathfrak{p} + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}_\lambda^* \mu + \widetilde{a} \diamond_{\mathfrak{h}_\mu} (b + \alpha).$$

However, using exactly the same computations as in (5.19), we get $\widetilde{a} \diamond_{\mathfrak{h}_\mu} (b + \alpha) = a \diamond_{\mathfrak{g}_z} b + \nu_\mathfrak{s} - \frac{1}{2}\lambda \diamond_\mathfrak{s} \mathrm{ad}_\lambda^* \mu$, that is $\widetilde{\nu} = \nu_\mathfrak{p} + \nu_\mathfrak{s} + \frac{1}{2}\lambda \diamond_{\mathfrak{g}_z} \mathrm{ad}_\lambda^* \mu + a \diamond_{\mathfrak{g}_z} b$.

□

Remark 5.2.8. In general, if we consider a tube around $\varphi(e, \mu, 0, \alpha)$, we cannot expect its image to be global in the B^* direction. This is because all the points in the model space $G \times_{G_z} \left((\mathfrak{s}_r^* \oplus \mathfrak{p}_r^*) \times \mathfrak{o}_r \times B_r \times B_r^*\right)$ must have G-isotropy conjugated to a subgroup of G_z. From this observation we can conclude that, in general, $(\mathfrak{s}_r^* \oplus \mathfrak{p}_r^*) \times \mathfrak{o}_r \times B_r \times B_r^*$ will not be an open neighborhood containing points of the form $(0,0,0,b)$ for arbitrary large $b \in B^*$. Indeed, if that was true, we would have $(0,0,0,\alpha) \in (\mathfrak{s}_r^* \oplus \mathfrak{p}_r^*) \times \mathfrak{o}_r \times B_r \times B_r^*$ which would imply $\mathcal{T}([e,0,0,0,-\alpha]_{G_z}) = \varphi(e,\mu,0,0)$. But this is a point with G-isotropy H_μ and in general, $G_z \subsetneq H_\mu$, thereby producing a contradiction.

Note that if we assume that $\mu \in \mathfrak{g}^*$ satisfies $\mathfrak{g}_\mu = \mathfrak{g}$, then $\mathfrak{o} = 0$ and the Hamiltonian tube $\mathcal{T}$ will be of the form

$$\begin{aligned} \mathcal{T} \colon G \times_{G_z} ((\mathfrak{s}_r^* \oplus \mathfrak{p}_r^*) \times B_r \times B_r^*) &\longrightarrow T^*(G \times_H S) \\ [g, \nu_{\mathfrak{s}} + \nu_{\mathfrak{p}}; a, b]_{G_z} &\longmapsto \varphi(g, \mu + \tilde{\nu}; \tilde{a}, b + \alpha) \end{aligned}$$

where

$$\tilde{\nu} = \nu_{\mathfrak{p}} + \nu_{\mathfrak{s}} + a \diamond_{\mathfrak{g}_z} b, \quad \tilde{a} = a + \Gamma(\nu_{\mathfrak{s}} - a \diamond_{\mathfrak{s}} b; b).$$

This map is the content of Theorem 31 of [Sch07]. In other words, the map $\mathcal{T}$ coincides with the results of [Sch07] when we restrict to their totally isotropic hypothesis $\mathfrak{g} = \mathfrak{g}_\mu$. What happens in this case is that the Hamiltonian tube given by Theorem 5.2.2 becomes the trivial μ-shift (see Remark 5.1.3)

$$\begin{aligned} \mathcal{T}_0 \colon G \times_{H_\mu} (\mathfrak{p}^* \times S \times S^*) &\longrightarrow T^*(G \times_H S) \\ [g, \nu; a, b]_{H_\mu} &\longmapsto \varphi(g, \mu + \nu + a \diamond_{\mathfrak{h}_\mu} b; a, b). \end{aligned}$$

The other extreme case is when Γ becomes trivial. This will happen, for example, if $S = 0$, which is equivalent to assuming that locally $Q = G/H$. Fix a point $z = \varphi(e,\mu) \in T^*(G/H)$, as $G_z = H_\mu$, then $\mathfrak{s} = 0$, and as $S = 0$ then $B = 0$. Therefore, according to (5.20), $\mathcal{T}$ becomes

$$\begin{aligned} \mathcal{T} \colon G \times_{H_\mu} (\mathfrak{p}_r^* \times \mathfrak{o}_r) &\longrightarrow T^*(G/H) \\ [g, \nu_{\mathfrak{p}}, \lambda]_{H_\mu} &\longmapsto \varphi(\Phi(g, \nu_{\mathfrak{p}} + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu, \lambda; 0)). \end{aligned}$$

5.3 Explicit examples

In Proposition 5.1.2 we proved the existence of simple G-tubes using Moser's trick. In this section we write down the actual differential equation that must be solved. We will see that if $\dim \mathfrak{q} = 2$, then the simple G-tube will be a scaling of an exponential map, and we will compute it explicitly for $SO(3)$ and $SL(2,\mathbb{R})$. From the explicit $SO(3)$ restricted tube in Subsection 5.3.4, we will present the Hamiltonian tube for the natural action of $SO(3)$ on $T^*\mathbb{R}^3$ generalizing the final example of [Sch07] to $\mu \neq 0$.

We will compute explicitly the flow that determines a simple G-tube. In order to do this, we are going to use the notation of the proof of Proposition 5.1.2.

Recall that we constructed the simple G-tube as the composition $\Theta = F \circ \Psi_1$, where

$$F(g,\nu,\lambda) = (g\exp(\lambda), \mathrm{Ad}^*_{\exp(\lambda)}(\nu + \mu))$$

(see (5.3)) and Ψ_1 is the time-1 flow of the time dependent vector field X_t which satisfies the Moser equation associated with $\theta_t = tF^*\theta_{T^*G} + (1-t)\theta_Y$, that is,

$$i_{X_t}(-\mathbf{d}\theta_t) = \frac{\partial \theta_t}{\partial t}.$$

This equation for X_t can written explicitly in this case. Using the above expression of F and (1.10), we have

$$\begin{aligned} F^*\theta_{T^*G}(g,\nu,\lambda)(\xi,\dot\nu,\dot\lambda) &= \langle \mathrm{Ad}^*_{\exp(\lambda)}(\nu+\mu), \mathrm{Ad}^{-1}_{\exp(\lambda)}\xi + T_eL^{-1}_{\exp(\lambda)}T_\lambda\exp(\dot\lambda)\rangle \\ &= \langle \nu+\mu,\xi\rangle + \langle \nu+\mu, \mathrm{Ad}_{\exp(\lambda)}T_eL^{-1}_{\exp(\lambda)}T_\lambda\exp(\dot\lambda)\rangle \\ &= \langle \nu+\mu,\xi\rangle + \langle \nu+\mu, T_eR^{-1}_{\exp(\lambda)}T_\lambda\exp(\dot\lambda)\rangle. \end{aligned}$$

Using Proposition 1.2.1, the last term can be expressed as a series of Lie brackets

$$M(\lambda)\cdot\dot\lambda := T_eR^{-1}_{\exp(\lambda)}T_\lambda\exp(\dot\lambda) = \sum_{n\geq 0}\frac{1}{(n+1)!}\mathrm{ad}^n_\lambda\dot\lambda.$$

Furthermore, this series is just the pullback of the right Maurer-Cartan form $\varpi^R(g) = T_eR_g^{-1}$ by the restricted exponential $\exp\big|_{\mathfrak{q}}\colon \mathfrak{q}\to G$. Therefore, using the Maurer-Cartan relation

$$\begin{aligned} (\mathbf{d}M)(X,Y) &= \mathbf{d}(\exp^*\varpi^R)(X,Y) = \exp^*(\mathbf{d}\varpi^R)(X,Y) \\ &= [\exp^*\varpi^R(X), \exp^*\varpi^R(Y)] = [M(X),M(Y)]. \end{aligned} \tag{5.21}$$

Now, since $\theta_t(g,\nu,\lambda)(\xi,\dot\nu,\dot\lambda) = \langle\mu+\nu,\xi\rangle + t\langle\mu+\nu, M(\lambda)\cdot\dot\lambda\rangle + (1-t)\frac{1}{2}\langle\mu,[\lambda,\dot\lambda]\rangle$, (using (5.21)) the exterior derivative $\omega_t = -\mathbf{d}\theta_t$ simplifies to

$$\begin{aligned} \omega_t(g,\nu,\lambda)(\xi_1,\dot\nu_1,\dot\lambda_1)(\xi_2,\dot\nu_2,\dot\lambda_2) &= \langle\dot\nu_2,\xi_1\rangle - \langle\dot\nu_1,\xi_2\rangle + \langle\nu+\mu,[\xi_1,\xi_2]\rangle + \\ &+ t\langle\dot\nu_2, M(\lambda)\dot\lambda_1\rangle - t\langle\dot\nu_1, M(\lambda)\dot\lambda_2\rangle \\ &+ t\langle\nu+\mu, -[M(\lambda)\cdot\dot\lambda_1, M(\lambda)\cdot\dot\lambda_2]\rangle - (1-t)\langle\mu,[\dot\lambda_1,\dot\lambda_2]\rangle. \end{aligned}$$

Also, the expression $\frac{\partial\theta_t}{\partial t} = \theta_1-\theta_0$ can be written as

$$\frac{\partial\theta_t}{\partial t}(g,\nu,\lambda)(\xi,\dot\nu,\dot\lambda) = \langle\nu+\mu, M(\lambda)\cdot\dot\lambda\rangle - \langle\mu, \frac{1}{2}\mathrm{ad}_\lambda\dot\lambda + \dot\lambda\rangle. \tag{5.22}$$

From now on we will assume that $\dim\mathfrak{q} = 2$. Note that the one-form

$$\begin{aligned} \omega_t(g,\nu,\lambda)(0,0,\lambda)(\xi_2,\dot\nu_2,\dot\lambda_2) &= t\langle\dot\nu_2,\lambda\rangle + t\langle\nu+\mu, -[\lambda, M(\lambda)\cdot\dot\lambda_2]\rangle - (1-t)\langle\mu,[\lambda,\dot\lambda_2]\rangle \\ &= t\langle\nu+\mu, -M(\lambda)[\lambda,\dot\lambda_2]\rangle - (1-t)\langle\mu,[\lambda,\dot\lambda_2]\rangle \end{aligned}$$

and (5.22) have the same kernel $\mathfrak{g}\oplus\mathfrak{g}^*_\mu\oplus\mathbb{R}\cdot\lambda \subset T_{(g,\nu,\lambda)}(G\times\mathfrak{g}^*_\mu\times\mathfrak{q})$. Therefore, there is a real-valued function f such that

$$\omega_t(g,\nu,\lambda)(0,0,f(\nu,\lambda,t)\lambda)(\xi_2,\dot\nu_2,\dot\lambda_2) = \frac{\partial\theta_t}{\partial t}(g,\nu,\lambda)(\xi_2,\dot\nu_2,\dot\lambda_2).$$

In other words, $X_t(g,\nu,\lambda) = f(\nu,\lambda,t)\frac{\partial}{\partial\lambda}$ and, in particular, $\Psi_t(g,\nu,\lambda) = (g,\nu,m_t(\nu,\lambda)\lambda)$ for certain scaling factor $m_t\colon\mathfrak{g}^*_\mu\times\mathfrak{q}\to\mathbb{R}$. We will obtain an equation that fully determines m_1 and therefore the map Ψ_1 and the simple G-tube.

Taking the time-derivative of the time-dependent pull-back (see (5.4))

$$\begin{aligned} \frac{\partial}{\partial t}(\Psi_t^*\theta_t) &= \Psi_t^*\left((\mathbf{d}i_{X_t} + i_{X_t}\mathbf{d})\theta_t + \frac{\partial\theta_t}{\partial t}\right) = \Psi_t^*(\mathbf{d}i_{X_t}\theta_t) = \Psi_t^*(\mathbf{d}i_{X_t}(\theta_0 + t(\theta_1-\theta_0)) = \\ &= \Psi_t^*(\mathbf{d}i_{X_t}\theta_0) = \mathbf{d}(\Psi_t^*(i_{X_t}\theta_0)). \end{aligned}$$

Additionally,

$$\Psi_t^*(i_{X_t}\theta_0) = \Psi_t^*(\langle \mu, \lambda \rangle f(\nu, \lambda, t)) = \langle \mu, \lambda \rangle f(\nu, m_t(\nu, \lambda)\lambda, t)) = \frac{\partial}{\partial t}\langle \mu, m_t(\nu, \lambda)\lambda \rangle$$

from where we get

$$\frac{\partial}{\partial t}\left(\Psi_t^*\theta_t - \mathbf{d}\langle \mu, m_t(\nu, \lambda)\lambda \rangle\right) = 0.$$

This equation implies that Ψ_t satisfies the following equation on one-forms

$$\Psi_1^*\theta_1 - \mathbf{d}\langle \mu, m_1(\nu, \lambda)\lambda \rangle = \theta_0 - \mathbf{d}\langle \mu, \lambda \rangle. \tag{5.23}$$

But this equation does not depend on the derivatives of the scaling factor m_1, because

$$\begin{aligned}\Psi_1^*\theta_1(\xi, \dot{\nu}, \dot{\lambda}) &= \langle \mu + \nu, (\mathbf{D}_\nu m_1 \cdot \dot{\nu} + \mathbf{D}_\lambda m_1 \cdot \dot{\lambda})\lambda + M(m_1\lambda) \cdot (m_1\dot{\lambda}) \rangle = \\ &= \langle \mu, \lambda \rangle (\mathbf{D}_\nu m_1 \cdot \dot{\nu} + \mathbf{D}_\lambda m_1 \cdot \dot{\lambda}) + \langle \mu + \nu, M(m_1\lambda) \cdot (m_1\dot{\lambda}) \rangle\end{aligned}$$

and

$$\mathbf{d}\langle \mu, m_1\lambda \rangle(\xi, \dot{\nu}, \dot{\lambda}) = \langle \mu, \lambda \rangle (\mathbf{D}_\nu m_1 \cdot \dot{\nu} + \mathbf{D}_\lambda m_1 \cdot \dot{\lambda}) + \langle \mu, m_1\dot{\lambda} \rangle.$$

Since $\langle \mu, [\lambda, \dot{\lambda}] \rangle$ is a non-vanishing one-form on the two dimensional space $\mathfrak{q}$ with kernel $\mathbb{R} \cdot \lambda$, and $\langle \mu + \nu, M(\lambda) \cdot \dot{\lambda} \rangle - \langle \mu, \dot{\lambda} \rangle$ has also kernel $\mathbb{R} \cdot \lambda$, then there is an analytic function $h(\lambda, \nu)$ such that

$$\langle \mu + \nu, M(\lambda) \cdot \dot{\lambda} \rangle - \langle \mu, \dot{\lambda} \rangle = h(\lambda, \nu) \langle \mu, [\lambda, \dot{\lambda}] \rangle.$$

With this notation, (5.23) becomes the non-linear equation

$$h(m_1(\lambda, \nu)\, \lambda, \nu) \cdot (m_1(\lambda, \nu))^2 = \frac{1}{2}. \tag{5.24}$$

Its solution $m_1(\lambda, \nu)$ is enough to write down explicitly the simple G-tube

$$\Theta(g, \nu, \lambda) = (g \exp(m_1(\lambda, \nu)\lambda), \mathrm{Ad}_{\exp(m_1(\lambda, \nu)\lambda)}(\nu + \mu)).$$

In the following lemmas we will see that under some algebraic assumptions on $\mathfrak{g}$ we can write down m_1 in terms of elementary functions.

Lemma 5.3.1. *Assume that the subspace $\mathfrak{q}$ defined by the splitting $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$ is a 2-dimensional subalgebra. Then the equation (5.24) has the solution $m_1(\nu, \lambda) = \mathcal{E}(-\operatorname{tr}(\mathrm{ad}_\lambda\big|_{\mathfrak{q}}))$ where $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}^+$ the unique analytic function that satisfies*

$$e^{-x\mathcal{E}(x)} = 1 - x\mathcal{E}(x) + \frac{x^2}{2} \tag{5.25}$$

and $\mathcal{E}(0) = 1$.

Proof. As the dimension of $\mathfrak{q}$ is two and ad_ξ is singular, it follows that $\mathrm{ad}_\eta^2\big|_{\mathfrak{q}} - \operatorname{tr}(\mathrm{ad}_\eta\big|_{\mathfrak{q}})\mathrm{ad}_\eta\big|_{\mathfrak{q}} = 0$ for any $\eta \in \mathfrak{q}$. Therefore,

$$\sum_{k\geq 0} \frac{1}{(k+1)!}\mathrm{ad}_\xi^k = \mathrm{Id} + \sum_{k\geq 0} \frac{(\operatorname{tr}(\mathrm{ad}_\xi\big|_{\mathfrak{q}}))^k}{(k+2)!}\mathrm{ad}_\xi = \mathrm{Id} + \frac{e^x - x - 1}{x^2}\mathrm{ad}_\xi$$

where $x = \operatorname{tr}(\mathrm{ad}_\xi\big|_{\mathfrak{q}})$. Then (5.23) becomes

$$\langle \mu + \nu, M(\lambda) \cdot \dot{\lambda} \rangle = \langle \mu, M(\lambda) \cdot \lambda \rangle = \langle \mu, \dot{\lambda} \rangle + \langle \mu, [\lambda, \dot{\lambda}] \rangle \frac{e^x - x - 1}{x^2}.$$

Comparing with (5.24), it follows $h(\nu, \lambda) = \frac{e^{-x}+x-1}{x^2}$ with $x = -\operatorname{tr}(\mathrm{ad}_\xi\big|_{\mathfrak{q}})$. Hence $m_1(\nu, \lambda) = \mathcal{E}(-\operatorname{tr}(\mathrm{ad}_\lambda\big|_{\mathfrak{q}}))$, where $\mathcal{E}$ satisfies (5.25). □

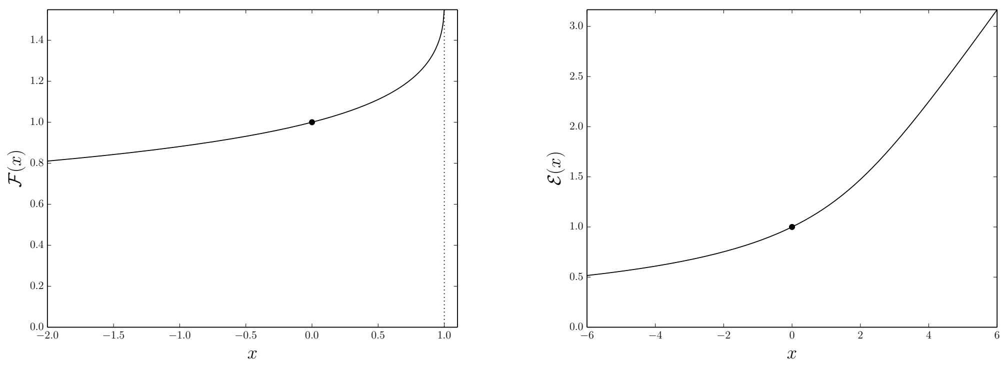


Figure 5.1: Scaling functions $\mathcal{F}(x)$ for $SO(3)$ (left) and $\mathcal{E}(x)$ for $SL(2,\mathbb{R})$ (right).

Remark 5.3.2. The function $\mathcal{E}$ can be written in terms of the **Lambert** W **function** (see [Cor+96])

$$\mathcal{E}(x) = \begin{cases} \frac{x}{2} + \frac{W_0(-\exp(-1-\frac{1}{2}x^2))+1}{x} & \text{if } x > 0 \\ \frac{x}{2} + \frac{W_{-1}(-\exp(-1-\frac{1}{2}x^2))+1}{x} & \text{if } x < 0 \end{cases}$$

where W_0 and W_{-1} are the two main branches of the W function. It can be checked that $\mathcal{E}(x)$ is positive and strictly increasing for all $x \in \mathbb{R}$. Additionally, $\mathcal{E}(x)$ is asymptotic to $\frac{x}{2}$ if $x \to \infty$, and satisfies $\mathcal{E}(0) = 1$ and $\mathcal{E}(x) \to 0$ if $x \to -\infty$ (see Figure 5.1).

Lemma 5.3.3. *Assume that the splitting* $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$ *satisfies*

1. $\mathrm{ad}_\xi^3 + a(\xi)\mathrm{ad}_\xi = 0 \ \forall \xi \in \mathfrak{q}$ *for a certain smooth function* $a\colon \mathfrak{q} \to \mathbb{R}$, *and*

2. $\langle \mu + \nu, \mathrm{ad}_\xi^2 \eta \rangle = 0$ *for any* $\xi, \eta \in \mathfrak{q}$ *and* $\nu \in \mathfrak{q}^\circ$.

In addition, let $b\colon (\mathfrak{g}_\mu^*)_r \to \mathbb{R}$ *be the function satisfying* $\langle \nu + \mu, [\xi, \eta] \rangle = b(\nu)\langle \mu, [\xi, \eta] \rangle$ *for any* $\xi, \eta \in \mathfrak{q}$. *Then equation* (5.24) *has the solution* $m_1(\nu, \lambda) = \mathcal{F}\left(\frac{a(\lambda)}{4b(\nu)}\right)\frac{1}{\sqrt{b(\lambda)}}$, *where* $\mathcal{F}\colon (-\infty, 1) \to \mathbb{R}^+$ *is the analytic function*

$$\mathcal{F}(x) = \begin{cases} \frac{\arcsin(\sqrt{x})}{\sqrt{x}} & \textit{if } x > 0 \\ \frac{\operatorname{arcsinh}(\sqrt{|x|})}{\sqrt{|x|}} & \textit{if } x < 0 \end{cases}$$

Proof. Using the first hypothesis $\displaystyle\sum_{n\geq 0} \frac{1}{(n+1)!}\mathrm{ad}_\xi^n = \mathrm{Id} + A_1(a(\xi))\mathrm{ad}_\xi + A_2(a(\xi))\mathrm{ad}_\xi^2$, where A_1 and A_2 are analytic scalar functions. Then

$$\langle \mu + \nu, M(\lambda)\cdot\dot\lambda \rangle = \langle \mu + \nu, \dot\lambda \rangle + A_1(a(\lambda))\langle \mu + \nu, [\lambda, \dot\lambda] \rangle = \langle \mu, \dot\lambda \rangle + A_1(a(\lambda))b(\nu)\langle \mu, [\lambda, \dot\lambda] \rangle,$$

that is, $h(\lambda, \nu) = A_1(a(\lambda))b(\nu)$. It can be checked that $A_1(x) = \frac{1-\cos(\sqrt{x})}{x}$. If we assume $a(\lambda) > 0$, then using simple formal manipulations (5.24) is equivalent to $m_1\sqrt{a(\lambda)} = \arccos\left(1 - \frac{a(\lambda)}{2b(\nu)}\right)$, and as $2\arcsin x = \arccos(1 - 2x^2)$, then $m_1(\lambda, \nu) = \mathcal{F}\left(\frac{a(\lambda)}{4b(\nu)}\right)\frac{1}{\sqrt{b(\lambda)}}$. If $a(\lambda) \leq 0$, a similar computation gives the same result. $\square$

5.3.1 $SO(3)$ simple tube

Under the hat map, the Lie algebra $\mathfrak{g} = \mathfrak{so}(3)$ can be identified with $\mathbb{R}^3$ equipped with the cross product. The standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3 \cong \mathfrak{g}$ will correspond to the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$, identifying them.

Fix an element $\mu \in \mathfrak{g}^*$. We have two different possibilities:

- $\mu = 0$. In this case, the G-tube is trivial (see Remark 5.1.3).
- $\mu \neq 0$. In this case, $\mathfrak{g}_\mu$ is the subspace generated by μ and we will define $\mathfrak{q}$ as the orthogonal complement to $\mathfrak{g}_\mu$. The subspace $\mathfrak{g}_\mu^*$ being the annihilator of $\mathfrak{q}$ is also identified with the subspace generated by μ.

  The vector identity $\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) = \langle \mathbf{a}, \mathbf{c} \rangle \mathbf{a} - \langle \mathbf{a}, \mathbf{a} \rangle \mathbf{c}$ implies that both conditions of Lemma 5.3.3 hold for $\mathfrak{so}(3)$ with $a(\lambda) = \|\lambda\|^2$. Therefore, the map

$$\begin{aligned} G \times \mathfrak{g}_\mu^* \times \mathfrak{q} &\longrightarrow SO(3) \times \mathbb{R}^3 \cong T^*SO(3) \\ (g, \nu, \lambda) &\longmapsto (gE(\nu, \lambda), E(\nu, \lambda) \cdot (\nu + \mu)) \end{aligned} \tag{5.26}$$

  with $E(\nu, \lambda) = \exp\left(2 \frac{\arcsin\left(\sqrt{\frac{\mu}{\mu+\nu}} \frac{\|\lambda\|}{2} \right)}{\|\lambda\|} \widehat{\lambda} \right)$ is a simple $SO(3)$-tube at $(e, \mu) \in T^*SO(3)$.

Note that this expression is exactly the same as the one obtained in Theorem 3 of [SS13]. In fact, this map was known in celestial mechanics as **regularized Serret-Andoyer-Deprit coordinates** (see [BFG06] and references therein).

5.3.2 $SL(2, \mathbb{R})$ simple tube

On the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, the bilinear form $\langle A, B \rangle = -2\mathrm{Tr}(AB)$ is non-degenerate and we will use it to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. If $\xi, \eta \in \mathfrak{sl}(2, \mathbb{R})$, it can be checked that $\mathrm{ad}_\xi \mathrm{ad}_\xi \eta = \langle \xi, \eta \rangle \xi - \langle \xi, \xi \rangle \eta$, and then for any $\xi \in \mathfrak{g}$ we have $\mathrm{ad}_\xi^3 + \|\xi\|^2 \mathrm{ad}_\xi = 0$.

Fix an element $\mu \in \mathfrak{g}^*$. We now have three different cases:

- $\mu = 0$. In this case, the G-tube is trivial (see Remark 5.1.3)
- $\|\mu\|^2 := \langle \mu, \mu \rangle \neq 0$. Then $\mathfrak{g}_\mu$ is one dimensional and is the space generated by μ. We will define $\mathfrak{q}$ to be the orthogonal space to μ with respect to the pairing. Since the norm of μ is non-zero, $\mathfrak{g} = \mathfrak{g}_\mu \oplus \mathfrak{q}$. As before, $\mathfrak{g}_\mu^* = \mathfrak{q}^\circ = \mathfrak{g}_\mu$, hence we can apply Lemma 5.3.3, obtaining that

$$\begin{aligned} G \times \mathfrak{g}_\mu^* \times \mathfrak{q} &\longrightarrow T^*SL(2, \mathbb{R}) \\ (g, \nu, \lambda) &\longmapsto (gE(\nu, \lambda), \mathrm{Ad}^*_{E(\nu,\lambda)}(\nu + \mu)) \end{aligned}$$

  with

$$E(\nu, \lambda) = \exp\left(\mathcal{F}\left(\frac{\|\lambda\|^2 \mu}{4(\mu + \nu)} \right) \sqrt{\frac{\mu}{\mu + \nu}} \lambda \right)$$

  is a simple $SL(2, \mathbb{R})$-tube at $(e, \mu) \in T^*SL(2, \mathbb{R})$.

- $\|\mu\|^2 = 0$ and $\mu \neq 0$. In this case, using basic linear algebra, it can be shown that there is $k \in SL(2, \mathbb{R})$ such that $\mu = k \begin{bmatrix} 0 & s \\ 0 & 0 \end{bmatrix} k^{-1}$ with $s = 1$ or $s = -1$.

Also in this case, $\mathfrak{g}_\mu$ is the subspace generated by μ, and we will define $\mathfrak{q}$ as the subspace generated by $k\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}k^{-1}$ and $k\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}k^{-1}$. A generic element in $\mathfrak{q}$ will be represented as $k\begin{bmatrix}a & 0\\ b & -a\end{bmatrix}k^{-1}$. It can be checked that $\mathfrak{g}_\mu^* = \mathfrak{q}^\circ$ is the subspace generated by $k\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}k^{-1}$.

A simple computation shows that $\mathfrak{q}$ is a subalgebra of $\mathfrak{g}$, hence we can apply Lemma 5.3.1, obtaining that the map

$$\begin{aligned} G\times\mathfrak{g}_\mu^*\times\mathfrak{q} &\longrightarrow T^*SL(2,\mathbb{R})\\ (g,\nu\mu,\lambda) &\longmapsto \Big(gE(\lambda),\mathrm{Ad}^*_{E(\lambda)}\big((\nu+1)\mu\big)\Big),\end{aligned}$$

where $\lambda = k\begin{bmatrix}a & 0\\ b & -a\end{bmatrix}k^{-1}$, $\nu\in\mathbb{R}$ and $E(\lambda)=\exp\left(\mathcal{E}(2a)\lambda\right)$, is a simple $SL(2,\mathbb{R})$-tube at $(e,\mu)\in T^*SL(2,\mathbb{R})$. Note that for this tube the domain is the whole space $G\times\mathfrak{g}_\mu^*\times\mathfrak{q}$. There are no restrictions on ν or λ but the map is not surjective.

Remark 5.3.4. If $\mu\neq 0$ satisfies $\|\mu\|^2=0$, then we cannot choose a G_μ-invariant splitting $\mathfrak{g}=\mathfrak{g}_\mu\oplus\mathfrak{q}$. In the literature, this situation is known as a non-split momentum and has important consequences for the structure of Hamilton's equation in MGS coordinates [RWL02].

5.3.3 A $SO(3)$ restricted tube

Let H be a compact non-discrete subgroup of $SO(3)$. Note that H must be one-dimensional. We denote by $\xi_{\mathfrak{h}}\in\mathbb{R}^3$ the generator of $\mathfrak{h}$ with unit norm. In this setting the adapted splitting of Proposition 4.2.1 reduces to $\mathfrak{l}=\mathbb{R}\cdot\xi_{\mathfrak{h}}$, $\mathfrak{p}=\mathbb{R}\cdot\mu$ and $\mathfrak{n}=\mathbb{R}\cdot\xi_{\mathfrak{h}}\times\mu$. To obtain the restricted tube we use (5.12), so we need to find $\zeta\in\mathfrak{n}$ satisfying the condition

$$\mathbf{J}_R(\Theta(g,\nu,\zeta)\big|_{\mathfrak{l}} = -\varepsilon \tag{5.27}$$

as a function of ν and ε. Using the notation of (5.5), Θ can be written as

$$\Theta(g,\nu,\lambda) = (gE(\nu,\lambda),\mathrm{Ad}^*_{E(\nu,\lambda)}(\nu+\mu)).$$

Using Proposition 1.3.3, we can rewrite (5.27) as

$$\mathrm{Ad}^*_{E(\nu,\zeta)}(\nu+\mu)\big|_{\mathfrak{l}} = \varepsilon. \tag{5.28}$$

Applying the explicit expression (5.26) for the $SO(3)$ simple tube, we have

$$E(\nu,\zeta) = \exp(\rho(\nu,\zeta)\frac{\xi_{\mathfrak{h}}\times\mu}{\|\xi_{\mathfrak{h}}\times\mu\|}).$$

Then, solving (5.28) is equivalent to finding the real parameter ρ as a function of ν and ε that satisfies

$$\left\langle \exp\Big(-\rho\frac{\xi_{\mathfrak{h}}\times\mu}{\|\xi_{\mathfrak{h}}\times\mu\|}\Big)\cdot(\nu+\mu),\xi_{\mathfrak{h}}\right\rangle = \langle\varepsilon,\xi_{\mathfrak{h}}\rangle.$$

Since $\{\xi_{\mathfrak{h}},\frac{\mu}{\|\mu\|},\frac{\xi_{\mathfrak{h}}\times\mu}{\|\xi_{\mathfrak{h}}\times\mu\|}\}$ is an orthogonal basis, this last equation is equivalent to

$$\langle\sin(\rho)(\nu+\mu),\frac{\mu}{\|\mu\|}\rangle = \langle\varepsilon,\xi_{\mathfrak{h}}\rangle.$$

Therefore, if we denote by r the expression $\arcsin \frac{(\varepsilon\cdot\xi_{\mathfrak{h}})\|\mu\|}{(\nu+\mu)\cdot\mu}$, the equation

$$\Phi(g,\nu;\varepsilon) = \left(g\exp\left(r\frac{\xi_{\mathfrak{h}}\times\mu}{\|\xi_{\mathfrak{h}}\times\mu\|}\right), \exp\left(-r\frac{\xi_{\mathfrak{h}}\times\mu}{\|\xi_{\mathfrak{h}}\times\mu\|}\right)\cdot(\nu+\mu)\right) \in SO(3)\times\mathbb{R}^3 \tag{5.29}$$

defines a restricted $SO(3)$-tube.

5.3.4 Hamiltonian tube for $SO(3)$ acting on $T^*\mathbb{R}^3$

Consider the natural action of $SO(3)$ on $\mathbb{R}^3$ and its cotangent lift to $T^*\mathbb{R}^3$. Fix a point $z=(q,p)\in T^*\mathbb{R}^3$. Note that, under the identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$, the momentum map is $\mu = \mathbf{J}(q,p) = q\times p$. We have four different possibilities

- $\mu = q\times p \neq 0$.
- $q=0$ and $p\neq 0$.
- $q\neq 0$ and $p=\rho q$ for some $\rho\in\mathbb{R}$.
- $q=p=0$.

The last three cases have momentum zero and therefore are covered by the Hamiltonian tube of [Sch07]. However, the first case has non-zero momentum and to compute the Hamiltonian tube we will need all the theory presented in this chapter.

$\mu = q\times p\neq 0$

As $q\neq 0$, the isotropy $H := G_q$ is the group of rotations with axis q. The linear slice $S = (\mathfrak{g}\cdot q)^\perp = \mathbb{R}\cdot p$ is the subspace generated by q, and note that this subspace is fixed by H. As μ and q are perpendicular, the groups H_μ and G_z are trivial. The linear splitting of Proposition 4.2.1 becomes

$$\mathfrak{l} = \mathbb{R}\cdot q, \quad \mathfrak{n} = \mathbb{R}\cdot(\mu\times q), \quad \mathfrak{g}_\mu = \mathbb{R}\cdot\mu.$$

Recall that $\alpha := z\big|_S \in S^*$ (see Proposition 4.3.1); therefore, using standard vector calculus identities

$$\alpha = \frac{p\cdot q}{\|q\|^2}q = p - \frac{\mu\times q}{\|q\|^2}.$$

Theorem 5.2.7 together with the explicit expression for the restricted tube (5.29) give that

$$\begin{aligned} G\times\mathfrak{g}_\mu^*\times T^*S &\longrightarrow T^*(SO(3)\times_H S) \\ (g,\nu,a,b) &\longmapsto \varphi(g,\nu+\mu,a,b+\alpha) \end{aligned} \tag{5.30}$$

is a Hamiltonian tube at $\varphi(e,\mu,0,\alpha)\in T^*(SO(3)\times_H S)$. In this case the parameter $\varepsilon = a\diamond_{\mathfrak{l}} b$ always vanishes because $\xi\cdot a = 0$ for any $\xi\in\mathfrak{h}$ and $a\in S$.

Let $S_r := \{\rho q \mid \rho > -1\}\subset S$. The map $\mathbf{t}\colon G\times_H S_r\to\mathbb{R}^3$ defined by $[g,a]_H\mapsto g\cdot(q+a)$ is a Palais' tube around q. Composing (5.30) with $T^*\mathbf{t}^{-1}$ and after some straightforward manipulations the Hamiltonian tube at (q,p) is

$$\begin{aligned} \mathcal{T}\colon G\times_{\mathrm{Id}}\left(\mathfrak{g}_\mu^*\times S_r\times S^*\right) &\longrightarrow \mathbb{R}^3\times\mathbb{R}^3\cong T^*\mathbb{R}^3 \\ (g,\nu,a,b) &\longmapsto \left(g\cdot(q+a), g\cdot\left((\nu+\mu)\times\frac{q+a}{\|q+a\|^2} + b + \underbrace{p - \frac{\mu\times q}{\|q\|^2}}_{\alpha}\right)\right). \end{aligned}$$

$q = 0, \quad p \neq 0$

In this case $H_\mu = H = G = SO(3)$ is the full rotation group, the linear slice is $S = \mathbb{R}^3$ and $\alpha = p \in \mathbb{R}^3$. The isotropy $K = G_z$ of z is the group of rotations with axis p. Under the identification $\mathfrak{so}(3) \cong \mathbb{R}^3$, the Lie algebra $\mathfrak{g}$ can be split into two terms

$$\mathfrak{k} = \mathbb{R} \cdot p, \quad \mathfrak{s} = \langle p \rangle^\perp \subset \mathbb{R}^3.$$

Note that $B \cong (\mathfrak{h}_\mu \cdot \alpha)^\perp = (\mathfrak{g} \cdot p)^\perp = \mathbb{R} \cdot p$, and if we choose $C = \langle p \rangle^\perp \subset S$ and $(B^*)_r = \{\rho p \mid \rho > -1\}$ the map Γ defined in Proposition 5.2.4 is

$$\Gamma(\nu; b) = -\nu \times \frac{p+b}{\|p+b\|^2}$$

because (5.18) is

$$\begin{aligned} \langle \Gamma(\nu; b), \xi \cdot (\alpha + b) + \beta \rangle &= \left\langle -\nu \times \frac{p+b}{\|p+b\|^2}, \xi \times (p+b) + \beta \right\rangle \\ &= -\left\langle \nu \times \frac{p+b}{\|p+b\|^2}, \xi \times (p+b) \right\rangle. \\ &= -\langle \nu, \xi \rangle. \end{aligned}$$

As H_μ-acts trivially on B, and by Remark 5.1.3, the cotangent Hamiltonian tube of (5.20) becomes

$$\begin{aligned} G \times_K (\mathfrak{s}^* \times B \times (B^*)_r) &\longrightarrow T^*(SO(3) \times_{SO(3)} S) \\ [g, \nu, a, b]_K &\longmapsto \varphi(g, \nu, a + \Gamma(\nu; b), b + \alpha) \end{aligned}$$

after some simplifications

$$\begin{aligned} \mathcal{T}\colon G \times_K (\mathfrak{s}^* \times B \times (B^*)_r) &\longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \cong T^*\mathbb{R}^3 \\ [g, \nu, a, b]_K &\longmapsto \left(g \cdot \left(a - \nu \times \frac{p+b}{\|p+b\|^2} \right), g \cdot (p+b) \right). \end{aligned}$$

$q \neq 0, \quad p = \rho q$

Note that $H = H_\mu = K$, and all equal the group of rotations with axis q. The linear slice is $S = \mathbb{R} \cdot q$, and the adapted splitting of Proposition 4.2.1 has only two terms

$$\mathfrak{p} = \langle q \rangle^\perp, \quad \mathfrak{h}_\mu = \mathbb{R} \cdot q.$$

Let $S_r = \{\lambda q \mid \lambda > -1\}$; the map $\mathbf{t}\colon G \times_H S_r \to \mathbb{R}^3$ defined by $[g, a]_H \mapsto g \cdot (q + a)$ is a Palais' tube around q. In this case, the Γ map vanishes because $\mathfrak{s} = 0$. Therefore, the cotangent Hamiltonian tube (5.20) becomes

$$\begin{aligned} \mathcal{T}\colon G \times_H (\mathfrak{p}^* \times S_r \times S^*) &\longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \cong T^*\mathbb{R}^3 \\ [g, \nu, a, b]_H &\longmapsto \left(g \cdot (q+a), g \cdot \left((\nu + \mu) \times \frac{q+a}{\|q+a\|^2} + b + \underbrace{p}_{\alpha} \right) \right). \end{aligned}$$

$q = p = 0$

In this last case, $H = K = G$ and the linear slice is $S = \mathbb{R}^3$. Therefore, the cotangent Hamiltonian tube (5.20) becomes the trivial map

$$\begin{aligned} \mathcal{T}\colon G \times_G (S \times S^*) &\longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \cong T^*\mathbb{R}^3 \\ [g, a, b]_G &\longmapsto (a, b). \end{aligned}$$

Chapter 6

Cotangent-bundle reduction

In this chapter we study topological and geometric properties of the quotient space $\mathbf{J}^{-1}(\mu)/G_\mu$ of a cotangent-lifted action on T^*Q.

As a first step in this study we check that one of the tubes constructed in the previous chapter contains in its image the set $\tau^{-1}(U) \cap \mathbf{J}^{-1}(\mu)$, where U is an open set of Q. This result can be understood as a fibered Bates-Lerman lemma. Using this fibered description, we introduce in Proposition 6.3.1 a set of coordinates on Q and T^*Q with nice properties with respect to G_μ and $\mathbf{J}^{-1}(\mu)$. This set of adapted coordinates is the key result that allows us to control the local behavior of the projection τ in the MGS model.

In Section 6.4 we study the single orbit case $Q = Q_{(L)}$ and show that the projection of orbit types of $\mathbf{J}^{-1}(\mu) \subset T^*Q$ are submanifolds of Q. Alternatively, we describe those projections as certain manifolds of the form $\mathbf{L}(H, \mu) \cdot Q_H$.

In Section 6.5 we discuss the general case $Q \neq Q_{(L)}$. We check that the sets $\mathbf{L}(H, \mu) \cdot Q_H$ induce a stratification of $Q^\mu = \tau(\mathbf{J}^{-1}(\mu))$ and of Q^μ/G_μ. Using these objects, in Section 6.6, we define a partition of $\mathbf{J}^{-1}(\mu)/G_\mu$ into submanifolds with good properties with respect to the projection $T^*Q \to Q$ and with respect to the induced symplectic structure. This partition is finer than the orbit-type stratification of $\mathbf{J}^{-1}(\mu)/G_\mu$ of Theorem 3.3.1. This partition is a generalization of the one introduced in [PROSD07] for momentum zero. As in that work, we call the elements of that partition seams.

After studying some topological properties of some particular cases in Section 6.7, in Section 6.8 we study the symplectic properties of the seams. We show that each seam is endowed with a closed two-form of constant rank (Proposition 6.8.8) and each seam has a subimmersion to a magnetic cotangent bundle. More importantly, the partition of $\mathbf{J}^{-1}(\mu)$ has a maximal element which is open and dense (Theorem 6.8.15), and in fact this maximal element can be embedded into a magnetic cotangent bundle. As a corollary of this study we have obtained in Corollary 6.8.11 a very clean description of the isotropy lattice of $\mathbf{J}^{-1}(\mu)$ that generalizes the results of [RO06] and can be regarded as the analogue of the first part of Theorem 3.4.1 for $\mu \neq 0$.

Finally, in Section 6.9 we present some examples that show how the results of this chapter can be applied to specific situations.

Throughout this chapter we will use the following notation: Q is a smooth manifold acted properly by the Lie group G; T^*Q is a symplectic manifold with the canonical symplectic form ω; $\tau\colon T^*Q \to Q$ is the natural projection; T^*Q is endowed with the cotangent-lifted action with momentum map $\mathbf{J}\colon T^*Q \to \mathfrak{g}^*$ given by (1.7). We fix $\mu \in \mathfrak{g}^*$ and we denote by Q^μ the set $\tau(\mathbf{J}^{-1}(\mu))$.

6.1 Generalities about Q^μ

One of the crucial differences between regular and singular cotangent-lifted actions is that, in the singular case, the projection of $\mathbf{J}^{-1}(\mu)$ is not the whole base manifold Q. In other words, $Q^\mu = \tau(\mathbf{J}^{-1}(\mu))$ is a proper subset of Q and in many cases this set is not even a manifold.

Nevertheless, the set Q^μ has an simple algebraic characterization.

Lemma 6.1.1. *([Sch01; RO04]) Let $q \in Q$, there exists an element $z \in T_q^*Q$ with $\mathbf{J}(z) = \mu$ if and only if*

$$\mathfrak{g}_q = \mathrm{Lie}(G_q) \subset \mathrm{Ker}\,\mu \subset \mathfrak{g}.$$

Proof. If $z \in T_q^*Q$ is such that $\mathbf{J}(z) = \mu$, then if $\xi \in \mathfrak{g}_q$

$$\langle \mu, \xi \rangle = \langle \mathbf{J}(z), \xi \rangle = \langle z, \xi_Q(q) \rangle = 0.$$

Hence, $\mathfrak{g}_q$ is annihilated by μ.

Fix a splitting $T_qQ = (\mathfrak{g} \cdot q) \oplus S$. Conversely, if μ annihilates $\mathfrak{g}_q$ we can define $z \in T_q^*Q$ by

$$\langle z, v_q \rangle = \langle \mu, \xi \rangle \quad \text{if } v_q = \xi_Q(q) + w \in (\mathfrak{g} \cdot q) \oplus S.$$

This is well defined because if $\xi_Q(q) = \eta_Q(q)$ then $\xi - \eta \in \mathfrak{g}_q$ and we have assumed that μ annihilates $\mathfrak{g}_q$. □

This lemma implies

$$Q^\mu = \bigcup_{\mathrm{Lie}(H) \subset \mathrm{Ker}\,\mu} G_\mu \cdot Q_H$$

where H runs through all the possible isotropy subgroups of Q. Although one may think that the sets $G_\mu \cdot Q_H$ are a good partition of Q^μ, the pieces of this partition are too small. We will see in Section 6.9.2 an example for which the partition $\bigcup_{\mathfrak{h} \subset \mathrm{Ker}\,\mu} G_\mu \cdot Q_H$ is **not locally finite**. This observation suggests that we need to construct bigger sets to decompose (in the sense of Definition 3.1.1) Q^μ.

6.2 A fibered Bates-Lerman Lemma

Recall that, in some cases, we saw that the domain of the cotangent bundle Hamiltonian tube is unbounded in the S^* direction (see Theorem 5.2.2). An important consequence of this fact is that, for cotangent-lifted actions, the open neighborhood U_M in Bates-Lerman Lemma (Proposition 2.3.5) can be global in the vertical direction. In other words, it will be of the form $\tau^{-1}(\mathcal{U}_Q)$ where $\tau\colon T^*Q \to Q$ is the natural projection and $\mathcal{U}_Q$ is a neighborhood in Q. In particular, this implies that the set $\tau^{-1}(\mathcal{U}_Q) \cap \mathbf{J}^{-1}(\mu)$ is fully contained on the image of a certain Hamiltonian tube.

Proposition 6.2.1. *Consider the Hamiltonian tube $\mathcal{T}_0\colon G \times_{H_\mu} (\mathfrak{p}_r^* \times \mathfrak{o}_r \times (T^*S)_r) \to T^*(G \times_H S)$ of Theorem 5.2.2 at the point $\varphi(e, \mu, 0, 0) \in T^*(G \times_H S)$. There is a G_μ-invariant neighborhood $\mathcal{U}_Q$ of $[e, 0]_H \in G \times_H S$ such that*

$$\tau^{-1}(\mathcal{U}_Q) \cap \mathbf{J}^{-1}(\mu) = \mathcal{T}_0(Z) \tag{6.1}$$

where

$$Z = \{[g, \nu, \lambda, a, b]_{H_\mu} \in \mathcal{T}_0^{-1}(\tau^{-1}(\mathcal{U}_Q)) \mid g \in G_\mu,\ \nu = 0,\ \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}_\lambda^* \mu + a \diamond_{\mathfrak{h}_\mu} b = 0\}.$$

Proof. As in the previous chapter, $N_0 = \mathfrak{o} \times T^*S$ will be the symplectic slice at $\varphi(e,\mu,0,0)$ with the symplectic form (4.10). The Hamiltonian tube puts the momentum map $\mathbf{J}$ in the normal form (2.7), that is $\mathbf{J} \circ \mathcal{T}_0 = \mathbf{J}_Y$. We now proceed as in the proof of Proposition 2.3.5 in [BL97] and we factorize $\mathbf{J}_Y = \gamma \circ \beta$, with

$$\begin{aligned} \beta\colon G \times_{H_\mu} (\mathfrak{p}^* \times N_0) &\longrightarrow G \times_{H_\mu} \mathfrak{g}_\mu^*, & \gamma\colon G \times_{H_\mu} \mathfrak{g}_\mu^* &\longrightarrow \mathfrak{g}^* \\ [g,\nu,v]_{H_\mu} &\longmapsto [g,\nu + \mathbf{J}_{N_0}(v)] & [g,\nu]_{H_\mu} &\longmapsto \mathrm{Ad}^*_{g^{-1}}(\mu+\nu). \end{aligned}$$

Using this factorization, it is easy to describe $\mathbf{J}_Y^{-1}(\mu)$. Note that, since the map

$$T_{[e,0]_{H_\mu}}\gamma \cdot (\xi, \dot\nu) = -\mathrm{ad}^*_\xi \mu + \dot\nu$$

is surjective, γ is a submersion near $[e,0]_{H_\mu}$, but by G-equivariance there is a G-invariant open set $U_{\mathrm{subm}} \subset G \times_{H_\mu} \mathfrak{g}^*$ where γ is a submersion. Therefore, $\gamma^{-1}(\mu) \cap U_{\mathrm{subm}}$ is a manifold of dimension $\dim G_\mu - \dim H_\mu$. Hence, $G_\mu \times_{H_\mu} \{0\} \subset \gamma^{-1}(\mu)$, $G_\mu \times_{H_\mu} \{0\}$ must be an open submanifold of $\gamma^{-1}(\mu) \cap U$, that is, there is an open set $U_{\mathrm{BL}} \subset U_{\mathrm{subm}}$ with $G_\mu \times_{H_\mu} \{0\} = \gamma^{-1}(\mu) \cap U_{\mathrm{BL}}$. By equivariance of γ, we can assume that U_{BL} is G_μ-invariant. Applying β^{-1} on the equality $G_\mu \times_{H_\mu} \{0\} = \gamma^{-1}(\mu) \cap U_{\mathrm{BL}}$, we get

$$\{[g,0,v]_{H_\mu} \in G \times_{H_\mu} (\mathfrak{p}^* \times N_0) \mid \mathbf{J}_{N_0}(v) = 0\} = \mathbf{J}_Y^{-1}(\mu) \cap \beta^{-1}(U_{\mathrm{BL}}). \tag{6.2}$$

In this setting, let U_G be a $G^L_\mu \times H^R_\mu$-invariant neighborhood of $e \in G$, $\mathfrak{p}_0^* \subset \mathfrak{p}^*$, $\mathfrak{o}_0 \subset \mathfrak{o}$ H_μ-invariant neighborhoods of zero and $\mathfrak{h}_0^* \subset \mathfrak{h}^*$ an H-invariant neighborhood of zero such that

$$\begin{aligned} \{[g,\nu,\lambda,a,b]_{H_\mu} \mid g \in U_G,\ \nu \in \mathfrak{p}_0^*,\ \lambda \in \mathfrak{o}_0,\ a \diamond_{\mathfrak{h}} b \in \mathfrak{h}_0^*\} \\ \subset \big(G \times_{H_\mu} (\mathfrak{p}_r^* \times \mathfrak{o}_r \times (T^*S)_r)\big) \cap \beta^{-1}(U_{\mathrm{BL}}). \end{aligned}$$

Now let Φ be the restricted tube used in the definition of $\mathcal{T}_0$ (see Theorem 5.2.2) and consider the map

$$\begin{aligned} f\colon U_G \times \mathfrak{p}_0^* \times \mathfrak{o}_0 \times \mathfrak{h}_0^* &\longrightarrow T^*G/H^T \\ (g,\nu,\lambda,\rho) &\longmapsto \pi_{H^T}(\Phi(g, \nu + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + \rho\big|_{\mathfrak{h}_\mu}, \lambda, \rho\big|_{\mathfrak{l}})). \end{aligned}$$

Note the similarity between this expression and (5.16). However, f does not depend on T^*S. We claim that this map is a submersion at $(e,0,0,0)$. Using (5.9) if $v = (\xi, \dot\nu, \dot\lambda, \dot\varepsilon) \in T_{(e,0,0,0)}(G \times \mathfrak{g}_\mu^* \times \mathfrak{o} \times \mathfrak{l}^*)$,

$$T_{(e,0,0,0)}\Phi \cdot v = (\xi + \dot\lambda + \sigma^{-1}(\dot\varepsilon), \dot\nu - \dot\varepsilon + \mathrm{ad}^*_{\dot\lambda}\mu)$$

where $\sigma\colon \mathfrak{n} \to \mathfrak{l}^*$ is the linear isomorphism (4.5). Applying this result to f, we get

$$T_{(e,0,0,0)} f \cdot (\xi, \dot\nu, \dot\lambda, \dot\rho) = T_{(e,\mu)}\pi_{H^T} \cdot (\xi + \dot\lambda + \sigma^{-1}(\rho\big|_{\mathfrak{l}}), \dot\nu + \mathrm{ad}^*_{\dot\lambda}\mu + \dot\rho). \tag{6.3}$$

However, since the splitting of Proposition 4.2.1 induces the dual decomposition $\mathfrak{g}^* = \mathfrak{h}_\mu^* \oplus \mathfrak{p}^* \oplus \mathfrak{o}^* \oplus \mathfrak{l}^* \oplus \mathfrak{n}^*$, each element of $\mathfrak{g}^*$ can be expressed as $\dot\rho\big|_{\mathfrak{h}_\mu} + \dot\nu + \mathrm{ad}^*_{\dot\lambda}\mu + \dot\rho\big|_{\mathfrak{l}} + \mathrm{ad}^*_\eta\mu$ for some $\dot\rho \in \mathfrak{h}^*$, $\dot\nu \in \mathfrak{p}^*$, $\dot\lambda \in \mathfrak{o}^*$ and $\eta \in \mathfrak{h}$. Finally, $\mathrm{Ker}(T_{(e,\mu)}\pi_{H^T}) = \{(\eta, \mathrm{ad}^*_\eta\mu) \mid \eta \in \mathfrak{h}\}$ and (6.3) imply that $T_{(e,0,0,0)}f$ is a surjective linear map.

Since f is G-equivariant, f is a submersion on a neighborhood of $G \cdot (e,0,0,0)$, and since submersions are open maps the image of f contains an open neighborhood of $G \cdot [e,\mu]_H$. Hence, there must exist a neighborhood $U_{\mathfrak{g}^*}$ of $\mu \in \mathfrak{g}^*$ such that

$$\pi_H(G \times_H U_{\mathfrak{g}^*}) \subset f(U_G \times \mathfrak{p}_0^* \times \mathfrak{o}_0 \times \mathfrak{h}_0^*). \tag{6.4}$$

Define $\mathcal{U}_G = \{g \in U_G \mid \mathrm{Ad}^*_g\mu \in U_{\mathfrak{g}^*}\} \cap \{g \in U_G \mid \mathrm{Ad}^*_g\mu\big|_{\mathfrak{h}} \in \mathfrak{h}^*_0\}$, which is an open neighborhood of $e \in G$ and let $\mathcal{U}_Q = \mathcal{U}_G \times_H S$, which is an open neighborhood of $[e, 0]_H \in Q$. We will now check that $\mathcal{U}_Q$ satisfies (6.1).

- $\tau^{-1}(\mathcal{U}_Q) \cap \mathbf{J}^{-1}(\mu) \supset \mathcal{T}_0(Z)$.

  This inclusion is trivial because if $[g, 0, \lambda, a, b]_{H_\mu} \in Z$, then $\mathcal{T}_0([g, 0, \lambda, a, b]_{H_\mu}) \in \tau^{-1}(\mathcal{U}_Q)$ and $(\mathbf{J} \circ \mathcal{T}_0)([g, 0, \lambda, a, b]_{H_\mu}) = \mathbf{J}_Y([g, 0, \lambda, a, b]_{H_\mu}) = \mathrm{Ad}^*_{g^{-1}}(\mu + \mathbf{J}_{N_0}(\lambda, a, b)) = \mathrm{Ad}^*_{g^{-1}}\mu = \mu$.

- $\tau^{-1}(\mathcal{U}_Q) \cap \mathbf{J}^{-1}(\mu) \subset \mathcal{T}_0(Z)$.

  Let $z \in \tau^{-1}(\mathcal{U}_Q) \cap \mathbf{J}^{-1}(\mu)$. Using the cotangent reduction map φ (see (5.13)), there is an element (g, ν, a, b) such that $\varphi(g, \nu, a, b) = z$, but as $\tau(z) \in \mathcal{U}_Q$ then $g \in \mathcal{U}_G$. Since $\varphi(g, \nu, a, b) \in \mathbf{J}^{-1}(\mu)$, using (2.4) we have $\mathrm{Ad}^*_{g^{-1}}\nu = \mu$. Additionally, as $(g, \nu, a, b) \in \mathbf{J}^{-1}_{H^T}(0)$, then $\nu\big|_{\mathfrak{h}} = a \diamond_{\mathfrak{h}} b$. Using $\nu = \mathrm{Ad}^*_g\mu$ this implies the relation $(\mathrm{Ad}^*_g\mu)\big|_{\mathfrak{h}} = a \diamond_{\mathfrak{h}} b$.

  As $g \in \mathcal{U}_G$, using the definition of $\mathcal{U}_G$ we have $(g, \mathrm{Ad}^*_g\mu) \in G \times U_{\mathfrak{g}^*}$. Equation (6.4) implies that there is a point $(g', \nu', \lambda, \rho) \in U_G \times \mathfrak{p}^*_0 \times \mathfrak{o}_0 \times \mathfrak{h}^*_0$ such that $f(g', \nu', \lambda, \rho) = [g, \mathrm{Ad}^*_g\mu]_H$. Therefore, there is $h \in H$ such that

$$(gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\mathrm{Ad}^*_g\mu) = \Phi(g', \nu' + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda\mu + \rho\big|_{\mathfrak{h}_\mu}, \rho\big|_{\mathfrak{l}}). \tag{6.5}$$

  Moreover, using (5.7) and (5.12) it can be checked that the H^T_μ-momentum of a restricted G-tube is $\mathbf{J}_{H^T_\mu}(\Phi(g, \nu, \lambda, \varepsilon)) = -\nu\big|_{\mathfrak{h}_\mu} + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda\mu$. Therefore, taking the H^T_μ-momentum on the previous equation

$$-(\mathrm{Ad}^*_{h^{-1}}\mathrm{Ad}^*_g\mu)\big|_{\mathfrak{h}_\mu} = -\frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda\mu - \rho\big|_{\mathfrak{h}_\mu} + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda\mu = -\rho\big|_{\mathfrak{h}_\mu}.$$

  Now, using item 4. in Definition 5.1.5 we have that H^T-momentum restricted to $\mathfrak{l}^* \subset \mathfrak{g}^*$ in (6.5) becomes the equality

$$-(\mathrm{Ad}^*_{h^{-1}}\mathrm{Ad}^*_g\mu)\big|_{\mathfrak{l}} = -\rho\big|_{\mathfrak{l}}.$$

  In other words, $(\mathrm{Ad}^*_{h^{-1}}\mathrm{Ad}^*_g\mu)\big|_{\mathfrak{h}} = \rho$, but as $\mathrm{Ad}^*_g\mu\big|_{\mathfrak{h}} = \nu\big|_{\mathfrak{h}} = a \diamond_{\mathfrak{h}} b$, it follows that $\rho = \mathrm{Ad}^*_{h^{-1}}(a \diamond_{\mathfrak{h}} b) = (h \cdot a) \diamond_{\mathfrak{h}} (h \cdot b)$, and therefore,

$$\begin{aligned}
\mathcal{T}_0([g', \nu', \lambda, h \cdot a, h \cdot b]_{H_\mu}) &= \varphi(\Phi(g', \nu' + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda\mu + \rho\big|_{\mathfrak{h}_\mu}, \lambda, (h \cdot a) \diamond_{\mathfrak{l}} (h \cdot b); h \cdot a, h \cdot b) \\
&= \varphi(\Phi(g', \nu' + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda\mu + \rho\big|_{\mathfrak{h}_\mu}, \lambda, \rho\big|_{\mathfrak{l}}; h \cdot a, h \cdot b) \\
&= \varphi(gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\mathrm{Ad}^*_g\mu, h \cdot a, h \cdot b) \\
&= \varphi(g, \mathrm{Ad}^*_g\mu, a, b) = \varphi(g, \nu, a, b).
\end{aligned}$$

  Finally, as $g \in \mathcal{U}_G$, $\mathrm{Ad}^*_g\mu\big|_{\mathfrak{h}} \in \mathfrak{h}^*_0$ and $\mathfrak{h}^*_0$ is H-invariant, then $(h \cdot a) \diamond_{\mathfrak{h}} (h \cdot b) = \mathrm{Ad}^*_{h^{-1}}(\mathrm{Ad}^*_g\mu\big|_{\mathfrak{h}}) \in \mathfrak{h}^*_0$. This observation implies that $(g', \nu', \lambda', h{\cdot}a, h{\cdot}b) \in \pi^{-1}_{H_\mu}(\beta^{-1}(U_{\mathrm{BL}}))$. Using the characterization (6.2), $\nu' = 0$, $g \in G_\mu$ and $\mathbf{J}_{N_0}(\lambda', h \cdot a, h \cdot b) = 0$, that is $[g', 0, \lambda', h \cdot a, h \cdot b]_{H_\mu} \in Z$, as we wanted to show.

□

6.3 Induced fibered coordinates

We can combine the results of Theorem 5.2.2 and Proposition 6.2.1 to form a set of fibered coordinates in the sense given by the following result.

Proposition 6.3.1. *Fix $q \in Q^\mu \subset Q$ and define $H = G_q$.*

Let $\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ and $\Phi\colon G \times U_\Phi \longrightarrow T^*G$ be the adapted splitting and restricted tube of (G, H, μ) given by Proposition 4.2.1 and 5.1.6. Let $\mathbf{t}\colon G \times_H S \to Q$ be a Palais' tube around q.

In this setting there are H_μ-invariant neighborhoods of zero $\mathfrak{o}_r \subset \mathfrak{o}$, $\mathfrak{p}^*_r \subset \mathfrak{p}^*$; H-invariant neighborhoods of zero $\mathfrak{h}^*_r \subset \mathfrak{h}$, $S_r \subset S$ and a G_μ-invariant neighborhood $U_Q \ni q$ such that

1. *Denote $\mathfrak{l}^*_r = \mathfrak{h}^*_r \cap \mathfrak{l}^*$, the map*

$$\begin{aligned} \Psi\colon G_\mu \times_{H_\mu} \left(\mathfrak{o}_r \times \mathfrak{l}^*_r \times S_r\right) &\longrightarrow Q \\ [g, \lambda, \varepsilon, a]_{H_\mu} &\longmapsto \tau\Big(T^*\mathbf{t}^{-1}\big(\varphi(\Phi(g, 0, \lambda; \varepsilon), a, 0)\big)\Big) \end{aligned} \tag{6.6}$$

*is a well-defined G_μ-equivariant **diffeomorphism** onto an open neighborhood of $q \in Q$.*

2. *Let $(T^*S)_r = \{(a, b) \in T^*S \mid a \in S_r,\ a \diamond_{\mathfrak{h}} b \in \mathfrak{h}^*_r\}$ the map*

$$\begin{aligned} \mathcal{T}_0\colon G \times_{H_\mu} \left(\mathfrak{p}^*_r \times \mathfrak{o}_r \times (T^*S)_r\right) &\longrightarrow T^*Q \\ [g, \nu, \lambda; a, b]_{H_\mu} &\longmapsto T^*\mathbf{t}^{-1}\big(\varphi(\Phi(g, \tilde{\nu}, \lambda; a \diamond_{\mathfrak{l}} b); a, b)\big) \end{aligned} \tag{6.7}$$

*with $\tilde{\nu} = \nu + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b$ is a **Hamiltonian tube** that satisfies*

$$\tau(\mathcal{T}_0([e, 0, 0, 0, 0]_{H_\mu}) = q.$$

3. *Define*

$$Z = \{[g, \nu, \lambda, a, b]_{H_\mu} \in \mathcal{T}_0^{-1}(\tau^{-1}(U_Q)) \mid g \in G_\mu,\ \nu = 0,\ \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b = 0\},$$

then,

$$\tau^{-1}(U_Q) \cap \mathbf{J}^{-1}(\mu) = \mathcal{T}_0(Z) \tag{6.8}$$

and since τ is surjective, $U_Q \cap Q^\mu = \tau(\mathcal{T}_0(Z))$.

Moreover, if $[g, \nu, \lambda, a, b]_{H_\mu} \in Z$ then $[g, \lambda, a \diamond_{\mathfrak{l}} b]_{H_\mu} \in G_\mu \times_{H_\mu} \left(\mathfrak{o}_r \times \mathfrak{l}^*_r \times S_r\right)$ and

$$\tau(\mathcal{T}_0([g, \nu, \lambda, a, b]_{H_\mu})) = \Psi([g, \lambda, a \diamond_{\mathfrak{l}} b, a]_{H_\mu}) \tag{6.9}$$

Proof. Theorem 5.2.2 at $\varphi(e, \mu, 0, \alpha)$ gives the neighborhoods $\mathfrak{p}^*_s \subset \mathfrak{p}^*$, $\mathfrak{o}_s \subset \mathfrak{o}$, $\mathfrak{h}^*_s \subset \mathfrak{h}^*$ and a map

$$\begin{aligned} \mathcal{T}_0\colon G \times_{H_\mu} \mathfrak{p}^*_s \times \mathfrak{o}_s \times (T^*S)_s &\longrightarrow T^*(G \times_H S) \\ [g, \nu, \lambda; a, b]_{H_\mu} &\longmapsto \varphi(\Phi(g, \tilde{\nu}, \lambda; a \diamond_{\mathfrak{l}} b); a, b) \end{aligned}$$

where $\tilde{\nu} = \nu + \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{h}_\mu} b$ and $(T^*S)_s = \{(a, b) \in S \times S^* \mid a \diamond_{\mathfrak{h}} b \in \mathfrak{h}^*_s\}$.

Besides, the linearization of Ψ at $[e, 0, 0, 0]_{H_\mu}$ can be easily computed because (5.9) gives the linearization of Φ. Therefore,

$$T_{[e,0,0,0]_{H_\mu}} \Psi \cdot T_{(e,0,0,0)} \pi_{H_\mu} \cdot (\dot{\xi}, \dot{\lambda}, \dot{\varepsilon}, \dot{a}) = T^*_{[e,0]_H} \mathbf{t}^{-1} \cdot T_{(e,0)} \pi_H \cdot (\xi + \dot{\lambda} + \sigma^{-1}(\dot{\varepsilon}); \dot{a}) \in T_q Q.$$

Note that $\operatorname{Ker} T_{(e,0)}\pi_H = \{(\xi, 0) \mid \xi \in \mathfrak{h}\}$ but then $T_{[e,0,0,0]_{H_\mu}}\Psi$ is injective, because if $T_{(e,0,0,0)}\pi_{H_\mu} \cdot (\dot{\xi}, \dot{\lambda}, \dot{\varepsilon}, \dot{a}) \in \operatorname{Ker} T_{[e,0,0,0]_{H_\mu}}\Psi$ then $\xi + \dot{\lambda} + \sigma^{-1}(\dot{\varepsilon}) \in \mathfrak{h}$ but as $\xi \in \mathfrak{h}_\mu \oplus \mathfrak{l}$, $\dot{\lambda} \in \mathfrak{o}$ and $\sigma^{-1}(\dot{\varepsilon}) \in \mathfrak{n}$ we must have $\xi \in \mathfrak{h}_\mu$, therefore $T_{(e,0,0,0)}\pi_{H_\mu} \cdot (\dot{\xi}, \dot{\lambda}, \dot{\varepsilon}, \dot{a}) = 0$.

Additionally,

$$\begin{aligned} \dim G_\mu \times_{H_\mu} (\mathfrak{o}_s \times \mathfrak{l}^*_s \times S_s) &= \dim G_\mu + \dim \mathfrak{o} + \dim \mathfrak{l} + \dim S - \dim H_\mu \\ &= \dim \mathfrak{h}_\mu + \dim \mathfrak{p} + \dim \mathfrak{o} + \dim \mathfrak{l} + \dim S - \dim \mathfrak{h}_\mu \\ &= \dim \mathfrak{g} - \dim \mathfrak{h} + \dim S = \dim Q \end{aligned}$$

that is, Ψ is a mapping between spaces of the same dimension, so it is a local diffeomorphism near $[e, 0, 0, 0]_{H_\mu}$.

As Ψ is G_μ-equivariant and the action is proper using the same ideas as in the proof Theorem 2.1.4, we can conclude that there are H_μ-invariant neighborhoods of zero $\mathfrak{o}_r, \mathfrak{l}^*_r, S_r$ small enough so that Ψ restricted to $G_\mu \times_{H_\mu} (\mathfrak{o}_r \times \mathfrak{l}^*_r \times S_r)$ is a diffeomorphism and $\mathfrak{o}_r \subset \mathfrak{o}_s$, $\mathfrak{l}^*_r \subset \mathfrak{h}^*_s \cap \mathfrak{l}^*$. Moreover, we can assume that there is $\mathfrak{h}^*_r \subset \mathfrak{h}^*_s$ an H-invariant neighborhood of zero in $\mathfrak{h}^*$ such that $\mathfrak{h}^*_r \cap \mathfrak{l}^* = \mathfrak{l}^*_r$. Note that we have already checked that the first and the second claim of the Proposition are satisfied.

It remains to check the third claim. Recall that Proposition 6.2.1 gives a neighborhood $\mathcal{U}_Q$ satisfying the relation (6.1); instead we will consider the open neighborhood

$$U_Q = \mathcal{U}_Q \cap (\tau \circ \mathcal{T}_0)\left(G \times_{H_\mu} \left(\mathfrak{p}^*_r \times \mathfrak{o}_r \times ((T^*S)_r \cap (S_r \times S^*))\right)\right).$$

Proposition 6.2.1 gives $\tau^{-1}(U_Q) \cap \mathbf{J}^{-1}(\mu) = \mathcal{T}_0(Z)$. Hence, as τ is surjective, $U_Q \cap Q^\mu = \tau(\mathcal{T}_0(Z))$.

Consider now a point $[g, \nu, \lambda, a, b]_{H_\mu} \in Z$, by definition of Z, $\lambda \in \mathfrak{o}_r$, $a \in S_r$ and $a \diamond_{\mathfrak{h}} b \in \mathfrak{h}_r$, thus, $[g, \lambda, a \diamond_{\mathfrak{l}} b, a]_{H_\mu} \in G_\mu \times_{H_\mu} \left(\mathfrak{o}_r \times \mathfrak{l}^*_r \times S_r\right)$. Moreover, using (6.7), the definition of Z and (6.6)

$$\begin{aligned} \tau(\mathcal{T}_0([g, \nu, \lambda, a, b]_{H_\mu})) = \tau(\varphi(\Phi(g, \widetilde{\nu}, \lambda, a \diamond_{\mathfrak{l}} b), a, b) &= \tau(\varphi(\Phi(g, 0, \lambda, a \diamond_{\mathfrak{l}} b), a, b) \\ = \tau(\varphi(\Phi(g, 0, \lambda, a \diamond_{\mathfrak{l}} b), a, 0) &= \Psi([g, \lambda, a \diamond_{\mathfrak{l}} b, a]_{H_\mu}), \end{aligned}$$

as we wanted to show. □

Fix a point $q \in Q^\mu$; this proposition gives a diagram

$$\begin{array}{ccc} G \times_{H_\mu} (\mathfrak{p}^*_r \times \mathfrak{o}_r \times (T^*S)_r) & \xrightarrow{\mathcal{T}_0} & T^*Q \\ & & \downarrow{\scriptstyle \tau} \\ G_\mu \times_{H_\mu} (\mathfrak{o}_r \times \mathfrak{l}^*_r \times S_r) & \xrightarrow{\Psi} & Q \end{array}$$

where the maps satisfy $q = \tau(\mathcal{T}_0([e, 0, 0, 0, 0]_{H_\mu})) = \Psi([e, 0, 0, 0]_{H_\mu})$. Moreover, if we restrict the previous maps to the appropriate subsets, we have the following commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow[\sim]{\mathcal{T}_0} & \tau^{-1}(U_Q) \cap \mathbf{J}^{-1}(\mu) & \longrightarrow & T^*Q \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ \Psi^{-1}(U_Q) & \xrightarrow[\sim]{\Psi} & U_Q & \longrightarrow & Q \end{array}$$

where

$$\mathrm{pr}([g, 0, \lambda, a, b]_{H_\mu}) = [g, \lambda, a \diamond_{\mathfrak{l}} b, a]_{H_\mu}.$$

The importance of this set of coordinates $(\mathcal{T}_0, \Psi)$ for T^*Q and Q is that, when restricted to $\mathbf{J}^{-1}(\mu)$, they provide a simple expression for the cotangent bundle projection τ.

6.4 Decomposition of Q^μ: single orbit-type

Throughout this section we will assume that the base Q of the cotangent bundle $T^*Q \to Q$ has only one orbit type, that is, $Q = Q_{(L)}$ for some subgroup $L \subset G$. The first important consequence of this assumption is that for any point $q \in Q$ and any linear slice S at q the action of G_q on S is trivial. This follows from the G-equivariant diffeomorphism

$$\mathbf{t}\colon G \times_{G_q} S \to Q$$

because the set of points in $G \times_{G_q} S$ with isotropy in (G_q) is $G \times_{G_q} S^{G_q}$ (Proposition 3.2.3). Hence, $S = S^{G_q}$, and all the points in S are fixed by the whole isotropy subgroup G_q.

Similarly, the condition $Q = Q_{(L)}$ implies that the isotropy type of points of T^*Q with momentum μ depends only on its projection on Q, more precisely,

Proposition 6.4.1. *Assume $Q = Q_{(L)}$. If $z \in \mathbf{J}^{-1}(\mu) \subset T^*Q$, then*

$$G_z = G_{\tau(z)} \cap G_\mu.$$

Proof. Define $q = \tau(z)$ and consider a tube on Q centered at q: $\mathbf{t}\colon G \times_{G_q} S \longrightarrow Q$ with $\mathbf{t}([e,0]_{G_q}) = q$. The cotangent lift $T^*\mathbf{t}^{-1}\colon T^*(G \times_{G_q} S) \to T^*Q$ is also a diffeomorphism. Using the map φ (see Theorem 1.5.1), there exists $\alpha \in S^*$ such that $z = \varphi(e,\mu,0,\alpha)$ and $G_z = G_q \cap G_\mu \cap (G_q)_\alpha$, but as the action of G_q on S is trivial $(G_q)_\alpha = G_q$ and then $G_z = G_q \cap G_\mu$.

□

Recall that in singular cotangent-lifted actions, $Q^\mu = \tau(\mathbf{J}^{-1}(\mu))$ in general is not even a manifold. The following results shows that under single orbit-type assumption the projection Q^μ is a locally closed subset of Q and the projections of the orbit types of $\mathbf{J}^{-1}(\mu)$ are disjoint unions of embedded submanifolds of Q (embedded Σ-submanifolds using the language of Definition 3.2.5).

Proposition 6.4.2. *Assume $Q = Q_{(L)}$, then*

- *Q^μ is a locally closed subset of Q.*
- *Let $z_0 \in T^*Q$, with $\mathbf{J}(z_0) = \mu$, $\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_0}}) \subset Q$ is an embedded Σ-submanifold of Q.*

Proof.
- Let $q \in Q^\mu$ and define $H = G_q$. Proposition 6.3.1 at q gives maps $\mathcal{T}_0$, Ψ and an open set $U_Q \ni q$.

  By the third part of Proposition 6.3.1, $U_Q \cap Q^\mu = \tau(\mathcal{T}_0(Z))$, where

  $$Z = \{[g,\nu,\lambda,a,b]_{H_\mu} \in \mathcal{T}_0^{-1}(\tau^{-1}(U_Q)) \mid g \in G_\mu,\ \nu = 0,\ \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu = 0\},$$

  because as $S = S^H$ then $a \diamond_{\mathfrak{h}} b = 0$ for any a, b. Define $W := \{[g,\lambda,\varepsilon,a]_{H_\mu} \in \Psi^{-1}(U_Q) \mid \frac{1}{2}\lambda \diamond_{\mathfrak{h}_\mu} \mathrm{ad}^*_\lambda \mu = 0,\ \varepsilon = 0\}$ we will check that

  $$\tau(\mathcal{T}_0(Z)) = \Psi(W), \tag{6.10}$$

  - If $[g,\nu,\lambda,a,b]_{H_\mu} \in Z$ then, using (6.9), $\tau(\mathcal{T}_0([g,\nu,\lambda,a,b]_{H_\mu})) = \Psi([g,\lambda,0,a]_{H_\mu}) \subset U_Q$ and it follows $\tau(\mathcal{T}_0(Z)) \subset \Psi(W)$.

- Conversely, if $[g, \lambda, 0, a]_{H_\mu} \in W$ then $[g, 0, \lambda, a, 0]_{H_\mu} \in Z$; therefore $\tau(\mathcal{T}_0(Z)) = \Psi(W)$.

Since W is a locally closed set and Ψ is a diffeomorphism, $\Psi(W) = U_Q \cap Q^\mu$ is a locally closed. As this argument can be applied at each $q \in R$, it implies that R is a locally closed set of Q.

- Let $q \in \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_0}})$, choose $z \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_0}}$ such that $\tau(z) = q$. Define $G_q = H$, using Proposition 6.4.1 $G_z = H_\mu$, which implies that $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_0}} = G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}$. Using Proposition 6.3.1, there are the maps $\mathcal{T}_0$, Ψ and the open set $U_q \ni q$.

  Using again the third part of Proposition 6.3.1, G-equivariance of $\mathcal{T}_0$ and Proposition 3.2.3,
$$\tau^{-1}(U_Q) \cap G_\mu \cdot \mathbf{J}^{-1}(\mu))_{H_\mu} = \mathcal{T}_0(\widetilde{Z})$$
  where $\widetilde{Z} = \{[g, \nu, \lambda, a, b]_{H_\mu} \in \mathcal{T}_0^{-1}(\tau^{-1}(U_Q)) \mid g \in G_\mu,\ \nu = 0,\ \lambda \in \mathfrak{o}^{H_\mu}\}$. Define $\widetilde{W} := \{[g, \lambda, \varepsilon, a]_{H_\mu} \in \Psi^{-1}(U_Q) \mid \lambda \in \mathfrak{o}^{H_\mu},\ \varepsilon = 0\}$, then by similar arguments as in the previous part $\tau(\mathcal{T}_0(\widetilde{Z})) = \Psi(\widetilde{W})$.

  As this argument can be applied at any $q \in \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_0}})$ this implies that $\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_0}})$ is an embedded Σ-submanifold, because locally around each point $\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_0}})$ is a manifold but the different connected components of $\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_0}})$ can have different dimensions.

□

6.4.1 Algebraic characterization

In this section we give an alternative description of the sets $\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)$ as the translation by a certain subset of G of the isotropy type manifolds Q_H of the base.

Definition 6.4.3. If H is a Lie subgroup of the Lie group G and $\mu \in \mathfrak{g}^*$, we define the following subset of G:

$$\mathbf{L}(H, \mu) = \{g \in G \mid \mathrm{Ad}_g^* \mu \in \mathfrak{h}^\circ \text{ and } \exists h \in G_\mu \text{ such that } (gHg^{-1}) \cap G_\mu = hH_\mu h^{-1}\}.$$

This set is in fact a Σ-submanifold of the group G and has some invariance properties.

Proposition 6.4.4. *Let H be a closed subgroup of G and $\mu \in \mathfrak{g}^*$.*

- *The subset $\mathbf{L}(H, \mu) \subset G$ is G_μ-invariant by the left action and $N_G(H)$-invariant under the right action.*
- *$\mathbf{L}(H, \mu)$ is an embedded Σ-submanifold of G (possibly empty).*

Proof.

- Let $g \in \mathbf{L}(G_q, \mu)$ and $g' \in G_\mu$, then, $\mathrm{Ad}_{g'g}^* \mu = \mathrm{Ad}_g^* \mathrm{Ad}_{g'}^* \mu = \mathrm{Ad}_g^* \mu \in \mathfrak{h}^\circ$. Besides, $\big((g'g)H(g'g)^{-1}\big) \cap G_\mu = g'\big((gHg^{-1}) \cap G_\mu\big)g'^{-1}$, but as $g \in \mathbf{L}(G_q, \mu)$ there is $h \in G_\mu$ such that $g'\big((gHg^{-1}) \cap G_\mu\big)g'^{-1} = g'(hH_\mu h^{-1})g'^{-1} = (g'h)H_\mu(g'h)^{-1}$. But since $g'h \in G_\mu$, then, $g'g \in \mathbf{L}(H, \mu)$.

  Consider now $g \in \mathbf{L}(H, \mu)$ and $g' \in N_G(H)$, then $\langle \mathrm{Ad}_{gg'}^* \mu, \xi \rangle = \langle \mathrm{Ad}_{g'}^* \mathrm{Ad}_g^* \mu, \xi \rangle = \langle \mathrm{Ad}_g^* \mu, \mathrm{Ad}_{g'} \xi \rangle$. However, if $\xi \in \mathfrak{h}$ then $\mathrm{Ad}_g \xi \in \mathfrak{h}$ and this shows that $\mathrm{Ad}_{gg'}^* \mu \in \mathfrak{h}^\circ$. Also, $\big((gg')H(gg')^{-1}\big) \cap G_\mu = \big(g(g'Hg'^{-1})g^{-1}\big) \cap G_\mu = (gHg^{-1}) \cap G_\mu = hH_\mu h^{-1}$ for some $h \in G_\mu$, because $g \in \mathbf{L}(H, \mu)$, thus $gg' \in \mathbf{L}(H, \mu)$.

- Consider the manifold G/H endowed with the natural G-action by multiplication on the left and its cotangent bundle $T^*(G/H)$ with the cotangent lifted G-action. Denote by $\mathbf{J}\colon T^*(G/H) \to \mathfrak{g}^*$, $\tau\colon T^*(G/H) \to G/H$ and $\pi_H\colon G \to G/H$ the momentum map and the natural projections. We will show that $\mathbf{L}(H,\mu) = \pi_H^{-1}\left(\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})\right)$.
  - Let $g \in \mathbf{L}(H,\mu)$, then the isotropy subgroup of the point $gH \in G/H$ is gHg^{-1} and by Lemma 6.1.1 $gH \in \tau(\mathbf{J}^{-1}(\mu))$ if and only if $\mathrm{Lie}(gH) \subset \mathrm{Ker}(\mu)$, but $\mathrm{Lie}(gH) = \mathrm{Ad}_g\mathfrak{h}$. However, this is the first condition on the definition of $\mathbf{L}(H,\mu)$, so there is $z \in \mathbf{J}^{-1}(\mu) \subset T^*(G/H)$ projecting on the point $gH \in G/H$, using Proposition 6.4.1, $G_z = G_{\tau(z)} \cap G_\mu = (gHg^{-1}) \cap G_\mu) = hH_\mu h^{-1} \in (H_\mu)^{G_\mu}$, that is, $z \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}$. We can conclude $\mathbf{L}(H,\mu) \subset \pi_H^{-1}\left(\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})\right)$.
  - Conversely, let $z \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}$ and $g \in \pi_H^{-1}(\tau(z))$, that is, $gH = \tau(z)$. By Lemma 6.1.1, $\mathrm{Ker}(\mu) \supset \mathrm{Lie}(gH) = \mathrm{Ad}_g\mathfrak{h}$, and by Proposition 6.4.1, $G_z = (gHg^{-1}) \cap G_\mu \in (H_\mu)^{G_\mu}$. Therefore $g \in \mathbf{L}(H,\mu)$ and we have the equality $\mathbf{L}(H,\mu) = \pi_H^{-1}\left(\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})\right)$.

  Using the second part of Proposition 6.4.2 and that π_H is a submersion, it follows that $\mathbf{L}(H,\mu)$ is an embedded Σ-submanifold.

□

Assume $e \in \mathbf{L}(H,\mu)$ and let $\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ be an adapted Lie-algebra splitting (Proposition 4.2.1). Using the proof of the second part of Proposition 6.4.4 and the proof of Proposition 6.4.2, it follows that the connected component of $\pi_H(\mathbf{L}(H,\mu))$ through H is diffeomorphic to $G_\mu \times_{H_\mu} \mathfrak{o}^{H_\mu}$. Hence, the dimension of the connected of $\mathbf{L}(H,\mu)$ that contains e is equal to

$$\dim \mathfrak{h} + \dim \mathfrak{p} + \dim\left(\mathfrak{o}^{H_\mu}\right).$$

However, the key property of the set $\mathbf{L}(H,\mu)$ is that it characterizes $\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})$ as the set of translates of Q_H through $\mathbf{L}(H,\mu) \subset G$.

Proposition 6.4.5. *Let H be a Lie subgroup of G and $\mu \in \mathfrak{g}^*$. Assume $Q = Q_{(L)}$, then*

$$\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}) = \mathbf{L}(H,\mu) \cdot Q_H$$

Moreover, if $q \in \mathbf{L}(H,\mu) \cdot Q_H$ then $\mathbf{L}(H,\mu) \cdot Q_H = \mathbf{L}(G_q,\mu) \cdot Q_{G_q}$.

Proof. Consider $q' \in \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})$; as $Q = Q_{(L)}$ there must be $g \in G$ such that $G_{q'} = gHg^{-1}$. Hence by Lemma 6.1.1, $\mathrm{Ker}\,\mu \supset \mathrm{Lie}(G_{q'}) = \mathrm{Ad}_g\mathfrak{h}$. Consider now a point $z' \in \tau^{-1}(q') \cap (G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})$, using Proposition 6.4.1, $G_{z'} = (gHg^{-1}) \cap G_\mu$ but then $G_{z'} \in (H_\mu)^{G_\mu}$. We have proved that $g \in \mathbf{L}(H,\mu)$, but then $q' = g \cdot (g^{-1}q') \in \mathbf{L}(H,\mu) \cdot Q_H$.

The converse inclusion is analogous; let $g \cdot q' \in \mathbf{L}(H,\mu) \cdot Q_H$ by the first condition on $\mathbf{L}(H,\mu)$ and Lemma 6.1.1 $g \cdot q' \in \tau(\mathbf{J}^{-1}(\mu))$. Then, by Proposition 6.4.1, for any $z' \in \tau^{-1}(g \cdot q') \cap \mathbf{J}^{-1}(\mu)$ then $G_{z'} = (gHg^{-1}) \cap G_\mu \in (H_\mu)^{G_\mu}$. Therefore we have the equality $\mathbf{L}(H,\mu) \cdot Q_H = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})$.

Consider now $q \in \mathbf{L}(H,\mu) \cdot Q_H = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})$; take $z \in \tau^{-1}(q) \cap \mathbf{J}^{-1}(\mu)$, by Proposition 6.4.1, $G_z = G_q \cap G_\mu \in (H_\mu)^{G_\mu}$, thus, $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu} = G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_q \cap G_\mu}$. Applying the first part to the subgroup G_q,

$$\mathbf{L}(H,\mu) = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}) = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_q \cap G_\mu}) = \mathbf{L}(G_q,\mu)$$

as we wanted to show.

□

6.4.2 Decomposition of Q^μ

We will combine the fact that the connected components of the sets $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$ form a decomposition of $\mathbf{J}^{-1}(\mu)$ (Theorem 3.3.1) with the results of this section to check that the connected components of the sets $\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)$ form a decomposition of Q^μ (Definition 3.1.1).

Proposition 6.4.6. *Assume $Q = Q_{(L)}$, define*

$$\mathcal{Z}_{Q^\mu} = \{G_\mu \cdot Z \mid Z \text{ is a connected component of } \mathbf{L}(H,\mu) \cdot Q_H \text{ where } H \in (L)\}.$$

The pair $(Q^\mu, \mathcal{Z}_{Q^\mu})$ is a decomposed space where $Q^\mu = \tau(\mathbf{J}^{-1}(\mu))$ is endowed with the relative topology as a subset of Q.

Proof. For any subgroup $H \subset G$ we denote $Q^\mu_{[H]} = \mathbf{L}(H,\mu) \cdot Q_H$.

1. **The pieces cover Q^μ and are disjoint**

   Let $q \in Q^\mu$, then $e \in \mathbf{L}(G_q,\mu)$ and $q \in Q_{G_q}$, so $q \in Q^\mu_{[G_q]}$. Conversely, let $q \in Q^\mu_{[H_1]} \cap Q^\mu_{[H_2]}$ using Proposition 6.4.5, $\mathbf{L}(H_1,\mu) \cdot Q_{H_1} = \mathbf{L}(G_q,\mu) \cdot Q_{G_q} = \mathbf{L}(H_2,\mu) \cdot Q_{H_2}$, that is, $Q^\mu_{[H_1]} = Q^\mu_{[H_2]}$.

2. **Each piece is locally closed as a subset of Q^μ and has a manifold structure compatible with the induced topology**

   This follows from the equality $Q^\mu_{[H]} = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})$ of Proposition 6.4.5 and Proposition 6.4.2.

3. **The partition is locally finite**

   Let $z \in \mathbf{J}^{-1}(\mu)$ and W be the connected component of $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}$ which contains z. Consider $w \in \tau^{-1}(\tau(z)) \cap \mathbf{J}^{-1}(\mu)$; the path $\gamma\colon t \mapsto tz + (1-t)w$ connects z and w and satisfies $\mathbf{J}(\gamma(t)) = \mu$ and $\tau(\gamma(t)) = q$. Proposition 6.4.1 implies $G_{\gamma(t)} = G_z$ for all t. Hence $w \in W$, that is

   $$\tau^{-1}(\tau(z)) \cap \mathbf{J}^{-1}(\mu) \subset W \subset G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}. \tag{6.11}$$

   Let $q \in Q^\mu$ and take $z \in \tau^{-1}(q) \cap \mathbf{J}^{-1}(\mu)$. Theorem 3.3.1 states that the connected components of the orbit types in $\mathbf{J}^{-1}(\mu)$ form a decomposition of $\mathbf{J}^{-1}(\mu)$. In particular, there is an open neighborhood $U \subset T^*Q$ of z and finite set of points $\{z_1, \ldots z_N\}$ such that if W_i is the connected component of through z_i of $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_{z_i}}$ then $\mathbf{J}^{-1}(\mu) \cap U \subset \bigcup_{i=1}^N W_i$. But (6.11) implies a stronger condition

   $$\mathbf{J}^{-1}(\mu) \cap \tau^{-1}(\tau(U)) \subset \bigcup_{i=1}^N W_i.$$

   Applying τ to this inclusion,

   $$Q^\mu \cap \tau(U) \subset \bigcup_{i=1}^N \tau(W_i),$$

   but $\tau(W_i)$ is a connected component of the set $Q^\mu_{[G_{\tau(z_i)}]}$. As $q \in \tau(U)$, this implies that the partition of Q^μ given by $\mathcal{Z}_{Q^\mu}$ is locally finite.

4. **The partition satisfies the frontier condition**

   Consider a point $q \in Q^\mu_{[H]} \cap \overline{Q^\mu_{[L]}}$. This means that there is a sequence $\{q_n\}_{n\in\mathbb{N}}$ converging to q with each element in $Q^\mu_{[L]}$.

   Assume $H = G_q$; the use of Proposition 6.3.1 provides us with diffeomorphisms Ψ, $\mathcal{T}_0$ and an open set U_Q containing q satisfying the properties stated in that proposition.

   In particular, there is $N > 0$ big enough such that $q_n \in U_Q$ for any $n > N$. Using the characterization of (6.10), for each $n > N$, there are g_n, λ_n, a_n such that $q_n = \Psi([g_n, \lambda_n, 0, a_n]_{H_\mu})$.

   Consider for any $n > N$, $z_n = \mathcal{T}([g_n, 0, \lambda_n, a_n, 0]_{H_\mu})$. Using (6.8), $z_n \in \mathbf{J}^{-1}(\mu)$ and in fact by Proposition 6.4.5 $z_n \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{L_\mu}$. Note that, since $q_n \to q$, $\{z_n\}_{n\in\mathbb{N}}$ is a sequence that converges to $z_\infty = \mathcal{T}([e, 0, 0, 0, 0]_{H_\mu}) \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}$. Hence

   $$z_\infty \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu} \cap \overline{G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{L_\mu}}.$$

   Let W be the connected component of $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}$ through z_∞. As the connected components of the isotropy types of $\mathbf{J}^{-1}(\mu)$ form a decomposition (Theorem 3.3.1), $W \subset \overline{G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{L_\mu}}$. Applying τ,

   $$\tau(W) \subset \tau(\overline{G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{L_\mu}}) \subset \overline{\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{L_\mu})} = \overline{Q^\mu_{[L]}}.$$

   Since $\tau(W)$ is the connected component of $Q^\mu_{[H]}$ through q, this implies that $(Q^\mu, \mathcal{Z}_{Q^\mu})$ satisfies the frontier condition.

□

6.5 Decomposition of Q^μ: general case

In the general case, when $Q \neq Q_{(L)}$, and motivated by the results of the preceding section, we study the partition of Q^μ into sets of the form $\mathbf{L}(H,\mu) \cdot Q_H$. We check that these sets are Σ-submanifolds of Q^μ inducing a stratification of Q^μ and their G_μ-quotients induce a stratification of Q^μ/G_μ.

From now on, for any compact subgroup H we write $Q^\mu_{[H]}$ to represent the set

$$Q^\mu_{[H]} = \mathbf{L}(H,\mu) \cdot Q_H.$$

Note that if there is no $g \in G$ such that $\mathrm{Lie}(gHg^{-1})$ lies in $\mathrm{Ker}\,\mu$, then $Q^\mu_{[H]} = \emptyset$ because $\mathbf{L}(H,\mu) = \emptyset$, and if H is not an isotropy subgroup of Q then $Q^\mu_{[H]} = \emptyset$. From the single orbit results, we can easily compute under which conditions two subgroups H and L satisfy $Q^\mu_{[H]} = Q^\mu_{[L]}$

Lemma 6.5.1.

- *If $q \in Q^\mu = \tau(\mathbf{J}^{-1}(\mu))$, then $q \in Q^\mu_{[G_q]}$.*
- *If $Q^\mu_{[H]} \cap Q^\mu_{[L]} \neq \emptyset$, then*
  - *$(H) = (L)$, that is, there is $g \in G$ such that $L = gHg^{-1}$.*
  - *$(H_\mu)^{G_\mu} = (L_\mu)^{G_\mu}$, that is, there is $k \in G_\mu$ such that $L_\mu = kH_\mu k^{-1}$.*

– $Q^{\mu}_{[H]} = Q^{\mu}_{[L]}$, *moreover, if* $q \in Q^{\mu}_{[H]} \cap Q^{\mu}_{[L]}$, $Q^{\mu}_{[G_q]} = Q^{\mu}_{[H]} = Q^{\mu}_{[L]}$.

Conversely if H, L *are subgroups such that* $(H) = (L)$ *and* $(H_\mu)^{G_\mu} = (L_\mu)^{G_\mu}$, *then* $Q^{\mu}_{[H]} = Q^{\mu}_{[L]}$.

- *If* $Q^{\mu}_{[H]} \cap \overline{Q^{\mu}_{[L]}} \neq \emptyset$, *then* $(H) \geq (L)$ *and* $(H_\mu)^{G_\mu} \geq (L_\mu)^{G_\mu}$, *that is, there are* $g \in G$ *and* $k \in G_\mu$ *such that* $gHg^{-1} \supset L$ *and* $kH_\mu k^{-1} \supset L_\mu$.

Proof. Let $q \in Q^\mu$ from Lemma 6.1.1 $\mu \in (\mathfrak{g}_q)^\circ$, but then from Definition 6.4.3 $e \in \mathbf{L}(G_q, \mu)$, and clearly $q \in \mathbf{L}(G_q, \mu) \cdot Q_{G_q} = Q^{\mu}_{[G_q]}$.

If $q \in Q^{\mu}_{[H]} \cap Q^{\mu}_{[L]}$, then $q = g_1 \cdot q_1 = g_2 \cdot q_2$ with $q_1 \in Q_H$ and $q_2 \in Q_L$, but then $g_1 H g_1^{-1} = g_2 L g_2^{-1}$ and clearly $(H) = (L)$. As the Σ-submanifold $Q_{(H)} = Q_{(L)}$ is of single-orbit type, we can apply Proposition 6.4.5 and get

$$\mathbf{L}(G_q, \mu) \cdot Q_{G_q} = \mathbf{L}(H, \mu) \cdot Q_H = \mathbf{L}(L, \mu) \cdot Q_L$$

and then $Q^{\mu}_{[G_q]} = Q^{\mu}_{[H]} = Q^{\mu}_{[L]}$, and there must be k_1 and k_2 both in G_μ such that $G_q \cap G_\mu = k_1 H_\mu k_1^{-1} = k_2 L_\mu k_2^{-1}$ implying $(H_\mu)^{G_\mu} = (L_\mu)^{G_\mu}$.

Conversely, if $(H) = (L)$ and $(H_\mu)^{G_\mu} = (L_\mu)^{G_\mu}$, there are $g \in G$ and $k \in G_\mu$ such that $H = gLg^{-1}$ and $H_\mu = kL_\mu k^{-1}$. Then, from Definition 6.4.3

$$\begin{aligned}
\mathbf{L}(L, \mu) &= \{g' \in G \mid \mathrm{Ad}^*_{g'}\mu \in \mathrm{Lie}(L)^\circ, \quad (g'L(g')^{-1}) \cap G_\mu \in (L_\mu)^{G_\mu}\} \\
&= \{g' \in G \mid \mathrm{Ad}^*_{g'}\mu \in (\mathrm{Ad}_{g^{-1}}\mathrm{Lie}(H))^\circ, \quad (g'g^{-1}Lg(g')^{-1}) \cap G_\mu \in (H_\mu)^{G_\mu}\} \\
&= \{g' \in G \mid \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_{g'}\mu \in \mathrm{Lie}(H)^\circ, \quad (g'g^{-1}Lg(g')^{-1}) \cap G_\mu \in (H_\mu)^{G_\mu}\} \\
&= \{g' \in G \mid g'g^{-1} \in \mathbf{L}(H, \mu)\} = \mathbf{L}(H, \mu)g.
\end{aligned}$$

Therefore,

$$\mathbf{L}(L, \mu) \cdot Q_L = \mathbf{L}(H, \mu) \cdot (g \cdot Q_L) = \mathbf{L}(H, \mu) \cdot Q_H$$

as we wanted to show. □

Before studying further properties of the sets $Q^{\mu}_{[H]}$, we need to introduce some notation: let Q be a smooth manifold and $S \subset Q$ a submanifold. $T_S Q$ will be the set $\tau^{-1}_{TQ}(S)$ where $\tau_{TQ} \colon TQ \to Q$ is the canonical projection, that is, $T_S Q$ is a vector bundle over S of rank $\dim Q$. Similarly, $T^*_S Q$ is the set $\tau^{-1}(S)$ where $\tau \colon T^*Q \to Q$, a vector bundle over S of rank $\dim Q$. $T_S S$ is the subset of $T_S Q$ of vectors tangent to S; it can be identified with the tangent bundle TS. The annihilator of $T_S S$ in $T^*_S Q$ is the conormal bundle $N^*S \subset T^*_S Q$.

If Q has a Riemmannian metric, $NS \subset TQ$ will be the normal bundle of S, the vector bundle over S whose fiber at $q \in S$ is given by $N_q S = (T_q S)^\perp$. Similarly, $T^*_S S$ will be the annihilator of NS in $T^*_S Q$. Note that, as vector bundles, we have the decompositions $T_S Q = T_S S \oplus_S NS$ and $T^*_S Q = T^*_S S \oplus_S N^*S$ where $\oplus$ represents the Whitney sum of vector bundles.

Using this notation, we can state the result analogous to Proposition 6.4.5, but without the assumption $Q = Q_{(L)}$.

Proposition 6.5.2. *Let* $q \in Q^\mu$, *and choose a* G*-invariant metric on* Q, *then,*

- *if* $z \in \mathbf{J}^{-1}(\mu)$ *with* $\tau(z) = q$ *and* $z \in T^*_{Q_{(G_q)}} Q_{(G_q)}$, *then the isotropy satisfies*

$$G_z = G_q \cap G_\mu$$

- *define $G_q = H$, then*

$$\mathbf{L}(H,\mu)\cdot Q_H = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu} \cap T^*_{Q_{(H)}} Q_{(H)}). \tag{6.12}$$

Proof. • Define $S = (\mathfrak{g}\cdot q)^\perp$, where the orthogonal is taken with respect to the metric $\langle\langle \cdot,\cdot\rangle\rangle_q$ in T_qQ, denote $H = G_q$. By Theorem 2.1.4, the map: $\mathbf{t}\colon G\times_H S \longrightarrow U \subset Q$, $[g,a]_H \longmapsto g\cdot \mathrm{Exp}(a)$ is a G-equivariant diffeomorphism onto an open neighborhood U of q such that $\mathbf{t}([e,0]_H) = q$. The cotangent lift $T^*\mathbf{t}^{-1}\colon T^*(G\times_H S) \longrightarrow T^*_U Q \subset T^*Q$ is a G-equivariant symplectomorphism.

Using Proposition 3.2.3, the submanifold of (H)-isotropy type is given by

$$\mathbf{t}^{-1}(Q_{(H)}\cap U) = G\times_H S^H.$$

With this characterization it is clear that

$$T_qQ_{(H)} = T_{[e,0]}\mathbf{t}\cdot T_{(e,0)}\pi_H \cdot \{(\xi,v) \mid \xi\in\mathfrak{g},\ v\in S^H\},$$

where $\pi_H\colon G\times S \to G\times_H S$.

The vector subspace S is H-invariant and can be orthogonally split as $S = S^H \oplus W$ where $W := (S^H)^\perp$. But then, if $w\in W$, $\xi\in\mathfrak{g}$ and $v\in S^H$

$$\langle\langle w, \xi_Q(q)+v\rangle\rangle_q = \langle\langle w,\xi_Q(q)\rangle\rangle_q + \langle\langle w,v\rangle\rangle_q = \langle\langle w,v\rangle\rangle_q = 0$$

because $S = (\mathfrak{g}\cdot q)^\perp$ and $W = (S^H)^\perp$. This implies

$$N_qQ_{(H)} \supset T_{[e,0]}\mathbf{t}\cdot T_{(e,0)}\pi_H\cdot\{(0,w)\mid w\in W\}.$$

However,

$$\begin{aligned}\dim(N_qQ_{(H)}) &= \dim(Q) - \dim(T_qQ_{(H)})\\ &= \dim(G\times S) - \dim(H) - (\dim(G)+\dim(S^H)-\dim(H))\\ &= \dim(S)-\dim(S^H) = \dim(W).\end{aligned}$$

Hence,

$$N_qQ_{(H)} = T_{[e,0]}\mathbf{t}\cdot T_{(e,0)}\pi_H\cdot\{(0,w)\mid w\in W\subset T_qQ\}.$$

This implies that

$$(T^*_{Q_{(H)}}Q_{(H)})\cap\tau^{-1}(q) = (N_qQ)^\circ = T^*_{[e,0]_H}\mathbf{t}^{-1}\cdot\{\varphi(e,\nu,0,b)\mid b\in W^\circ\subset S^*,\ \nu\in\mathfrak{g}^*\}$$

If we further impose momentum μ

$$T^*_qQ_{(H)}\cap\mathbf{J}^{-1}(\mu) = T^*_{[e,0]_H}\mathbf{t}^{-1}\cdot\{\varphi(e,\mu,0,b)\mid b\in W^\circ\subset S^*\}.$$

Let $z = T^*_{[e,0]_H}\mathbf{t}^{-1}\cdot\varphi(e,\mu,0,b)\in T^*_qQ_{(H)}\cap\mathbf{J}^{-1}(\mu)$; this point has G-isotropy is $G_z = G_\mu\cap H_b$. Using the metric $W^\circ\subset S^*$ is identified with $S^H\subset S$, and as the metric is H-invariant, W° is a subspace of H-fixed vectors. Hence, $H_b = H$ and $G_z = H\cap G_\mu$, as we wanted to show.

- Let $q \in \mathbf{L}(H,\mu)\cdot Q_H$; as the manifold $Q_{(H)}$ satisfies single-orbit type condition, we can apply Proposition 6.4.5, which in particular shows that there is $z \in T_q^*Q_{(H)} \subset T^*Q_{(H)}$ with $\mathbf{J}_{Q_{(H)}}(z) = \mu$ and $G_z \in (H_\mu)^{G_\mu}$. But as we can G-equivariantly identify $T^*Q_{(H)}$ with $T^*_{Q_{(H)}}Q_{(H)}$, there is $\tilde{z} \in T^*Q$ with $\tau(\tilde{z}) = q$, $\mathbf{J}(\tilde{z}) = \mu$ and $G_{\tilde{z}} = H_\mu$. Hence, we have proved the inclusion $\mathbf{L}(H,\mu)\cdot Q_H \subset \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu} \cap T^*_{Q_{(H)}}Q_{(H)})$.

  The second inclusion follows from the same argument; if $z \in G_\mu\cdot(\mathbf{J}^{-1}(\mu))_{H_\mu} \cap T^*_{Q_{(H)}}Q_{(H)}$ then we can associate $\tilde{z} \in T^*Q_{(H)}$, and using Proposition 6.4.5 there is $q \in \mathbf{L}(H,\mu)\cdot Q_H \subset Q_{(H)}$ such that $\tau_{Q_{(H)}}(\tilde{z}) = q$, but then $\tau(z) = q$. Therefore,

$$\mathbf{L}(H,\mu)\cdot Q_H = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu} \cap T^*_{Q_{(H)}}Q_{(H)}),$$

  as we wanted to show.

□

We will now construct a G-invariant metric in the twisted product $Q = G\times_H S$ such that the cotangent bundles of the isotropy type submanifolds of $T^*(G\times_H S)$ are simpler when viewed as submanifolds of the symplectic reduction of $T^*(G\times S)$.

Lemma 6.5.3. *There is a G-invariant metric on $Q = G\times_H S$ and an H-invariant metric on S such that the following property is satisfied:*

Let $q \in G\times_H S$, if $\varphi(g,\nu,a,b) \in T_q^*(G\times_H S)$ then

$$\varphi(g,\nu,a,b) \in T^*_{Q_{(G_q)}}Q_{(G_q)} \Longleftrightarrow (a,b) \in T^*_{S_{(G_q)_H}}S_{(G_q)_H} \subset T^*S$$

Proof. As H is a compact group, we can assume that $\mathfrak{g}$ has an Ad_H-invariant metric and $\langle\!\langle,\rangle\!\rangle_S$ is an H-invariant metric on S. These two objects give a $G^L\times H^T$-invariant metric on $G\times S$. As this metric is H^T-invariant, the quotient $G\times_H S$ has a quotient Riemmannian metric. The projection $\pi_H\colon G\times S \to G\times_H S$ is a Riemmannian submersion, that is, for any $(g,a)\in G\times S$ the map $T_{(g,a)}\pi_H\big|_{(\mathrm{Ker}\,T_{(g,a)}\pi_H)^\perp} : (\mathrm{Ker}\,T_{(g,a)}\pi_H)^\perp \to T_{[g,a]_H}(G\times_H S)$ is an isometry.

Note that the kernel of the H^T-projection is given by

$$\mathrm{Ker}\,T_{(g,a)}\pi_H = \{(-\xi, \xi\cdot a) \mid \xi \in \mathfrak{h}\}$$

Using Proposition 3.2.3, $(G\times_H S)_{(K)} = G\times_H \left(S_{(H)_K}\right)$. If $[g,a]_H \in (G\times_H S)_{(K)}$, then

$$T_{[g,a]_H}(G\times S)_{(K)} = \{T_{(g,a)}\pi_H(\xi,w) \mid \xi\in\mathfrak{g},\ w \in T_aS_{(K)_H}\}$$

Besides, as $S_{(K)_H}$ is invariant with respect to the H-action, $\xi\cdot a \in T_aS_{(K)_H}$ for any $a\in S_{(K)_H}$ and $\xi\in\mathfrak{h}$. This implies that if $w \in N_aS_{(K)_H}$, then $\langle\!\langle w,\xi\cdot a\rangle\!\rangle_S = 0$ for any $\xi\in\mathfrak{h}$. Therefore, if $w\in N_aS_{(K)_H}$ then the vector $(0,w)\in T_{(g,a)}(G\times S)$ in fact lies in $(\mathrm{Ker}\,T_{(g,a)}\pi_H)^\perp$.

If $w\in N_aS_{(K)_H}$, $\xi\in\mathfrak{g}$ and $v\in T_aS_{(K)_H}$, then

$$\langle\!\langle (0,w),(\xi,v)\rangle\!\rangle_{(g,a)} = \langle\!\langle w,v\rangle\!\rangle_S = 0.$$

Hence, $T_{(g,a)}\pi_H(0,w) \in N_{[g,a]_H}(G\times S)_{(K)}$. Therefore,

$$N_{[g,a]_H}(G\times S)_{(K)} \supset \{T_{(g,a)}\pi_H(0,w) \mid w\in N_aS_{(K)_H}\},$$

but

$$\begin{aligned}\dim(N_{[g,a]_H}(G\times S)_{(K)})&\\ &= \dim(G\times_H S)-\dim(T_{[g,a]_H}(G\times S)_{(K)})\\ &= \dim(G\times S)-\dim(H)-(\dim(G)+\dim(T_aS_{(K)_H})-\dim(H))\\ &= \dim(S)-\dim(T_aS_{(K)_H})=\dim(N_aS_{(K)_H})\end{aligned}$$

and as the dimensions are equal the inclusion above is in fact an equality

$$N_{[g,a]_H}(G\times S)_{(K)}=\{T_{(g,a)}\pi_H(0,w)\mid w\in N_aS_{(K)_H}\}.$$

Consider $\varphi(g,\nu,a,b)\in T^*(G\times_H S)$ and $q=[g,a]_H\in G\times_H S$; clearly,

$$\varphi(g,\nu,a,b)\in T^*_{Q_{(G_q)}}Q_{(G_q)}\iff\langle\varphi(g,\nu,a,b),v\rangle=0\quad\forall v\in N_{[g,a]_H}Q_{(G_q)}.$$

However, using the previous characterization of $N_{[g,a]_H}(G\times S)_{(K)}$, and by definition of φ (see Theorem (1.5.1)),

$$\langle\varphi(g,\nu,a,b),v\rangle=0\quad\forall v\in N_{[g,a]_H}Q_{(G_q)}\iff\langle(\nu,b),(0,w)\rangle=0\qquad\forall w\in N_aS_{(G_q)_H}$$

and this is clearly equivalent to $b\in(N_aS_{(G_q)_H})^\circ\subset S^*$. □

Using the metric given by Lemma 6.5.3 and the induced coordinates of Proposition 6.3.1, we can give a local description of the sets $Q^\mu_{[H]}$.

Proposition 6.5.4. *Let $q\in Q^\mu$ and define $H=G_q$. Using Proposition 6.3.1 at q there is a map Ψ and an open set $U_Q\ni q$ such that*

$$Q^\mu_{[H]}\cap U_Q=\Psi(\{[g,v]_{H_\mu}\in\Psi^{-1}(U_Q)\mid g\in G_\mu,\quad v\in\mathfrak{o}^{H_\mu}\times\{0\}\times S^H\subset\mathfrak{o}\times\mathfrak{l}^*\times S\}).$$

*In particular, for any subgroup H, $Q^\mu_{[H]}$ is an embedded Σ-**submanifold** of Q.*

Consider an H_μ-invariant splitting $\mathfrak{o}=\mathfrak{o}^{H_\mu}\oplus\widetilde{\mathfrak{o}}$ and a H-invariant splitting $S=S^H\oplus\widetilde{S}$. Then

$$\begin{aligned}Q^\mu_{[L]}\cap U_Q=\Psi\big(\{[g,v_1+v_2]_{H_\mu}\in\Psi^{-1}(U_Q)&\\ \mid g\in G_\mu,\quad v_1\in\mathfrak{o}^{H_\mu}\times\{0\}\times S^H,\ v_2\in X_{[L]}\subset\widetilde{\mathfrak{o}}\times\mathfrak{l}^*\times\widetilde{S}\}\big),\end{aligned}$$

*where $X_{[L]}$ is an H_μ-invariant Σ-submanifold of the vector space $\widetilde{\mathfrak{o}}\times\widetilde{S}\times\widetilde{S}^*$. Moreover $X_{[L]}$ is a **semialgebraic set** that satisfies the following conical property: if $(\lambda,a,\varepsilon)\in X_{[L]}$, then $(\rho\lambda,\rho a,\rho^2\varepsilon)\in X_{[L]}$ for any $\rho>0$.*

Proof. Using Theorem 2.1.4, we can assume that Q is $G\times_H S$ around $q=[e,0]_H$. Consider Q with the metric given by Lemma 6.5.3.

For the first part, we consider the set $W=\tau^{-1}(U_Q)\cap G_\mu\cdot(\mathbf{J}^{-1}(\mu))_{H_\mu}\cap T^*_{Q_{(H)}}Q_{(H)}$ using (6.12) $\tau(W)=U_Q\cap\tau(G_\mu\cdot(\mathbf{J}^{-1}(\mu))_{H_\mu}\cap T^*_{Q_{(H)}}Q_{(H)})=U_Q\cap Q^\mu_{[H]}$. But using (6.9) as $\mathfrak{T}_0^{-1}(W)\subset Z$

$$\tau(W)=\Psi(\{[g,\lambda,a,a\diamond_{\mathfrak{l}}b]_{H_\mu}\mid[g,0,\lambda,a,b]_{H_\mu}\in\mathfrak{T}_0^{-1}(W)\})$$

We now check the equality

$$\begin{aligned}\{[g,\lambda,a,a\diamond_{\mathfrak{l}}b]_{H_\mu}\mid[g,0,\lambda,a,b]_{H_\mu}\in\mathfrak{T}_0^{-1}(W)\}=&\\ =\{[g,\lambda,a,0]_{H_\mu}\in\Psi^{-1}(U_Q)\mid\lambda\in\mathfrak{o}^{H_\mu},\ a\in S^H\}\end{aligned}$$

Let $x = [g, 0, \lambda, a, b]_{H_\mu} \in \mathcal{T}_0^{-1}(W) \subset Z$, as $\mathcal{T}_0(x) \in (T^*Q)_{(H_\mu)^{G_\mu}}$ $G_x \in (H_\mu)^{G_\mu}$, but $G_x = gH_{\mu(\lambda,a,b)}g^{-1} = g(H_{\mu_\lambda} \cap H_{\mu_a} \cap H_{\mu_b})g^{-1}$ with $g \in G_\mu$. Therefore λ, a, b are H_μ-fixed. Note that $T^*_{Q_{(H)}}Q_{(H)}$ is the set of points $\varphi(g, \nu, a, b)$ where $a \in S^H$ and $b \in (S^H)^*$. Hence, as $\mathcal{T}_0(x) \in T^*_{Q_{(H)}}Q_{(H)}$, then $a \in S^H$, $b \in (S^H)^*$. For the other inclusion, let $y = [g, \lambda, a, 0]_{H_\mu}$ with $\lambda \in \mathfrak{o}^{H_\mu}$ and $a \in S^H$, then $[g, 0, \lambda, a, 0]_{H_\mu} \in \mathcal{T}_0^{-1}(W)$.

For the second part consider the set $W = \tau^{-1}(U_Q) \cap G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{L_\mu} \cap T^*_{Q_{(L)}}Q_{(L)}$ using (6.12) $\tau(W) = U_Q \cap \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{L_\mu} \cap T^*_{Q_{(L)}}Q_{(L)}) = U_Q \cap Q^\mu_{[L]}$. But using (6.9) as $\mathcal{T}_0^{-1}(W) \subset Z$

$$\tau(W) = \Psi(\{[g, \lambda, a, a \diamond_{\mathfrak{l}} b]_{H_\mu} \mid [g, 0, \lambda, a, b]_{H_\mu} \in \mathcal{T}_0^{-1}(W)\})$$

now we will show

$$\begin{aligned}\{[g, \lambda, a, a \diamond_{\mathfrak{l}} b]_{H_\mu} \mid [g, 0, \lambda, a, b]_{H_\mu} \in \mathcal{T}_0^{-1}(W)\} = \\ = \{[g, v_1 + v_2]_{H_\mu} \in \Psi^{-1}(U_Q) \mid v_1 \in \mathfrak{o}^{H_\mu} \times S^H \times \{0\},\ v_2 \in X \subset \widetilde{\mathfrak{o}} \times \widetilde{S} \times \widetilde{S}^*\}\end{aligned}$$

Consider $(g, \lambda, a, b) \in G_\mu \times \mathfrak{o} \times S \times S^*$ such that $[g, 0, \lambda, a, b]_{H_\mu} \in \mathcal{T}_0^{-1}(W)$. We can decompose the symplectic space $N = \mathfrak{o} \times S \times S^*$ as $(\mathfrak{o}^{H_\mu} \times S^H \times (S^H)^*) \oplus \underbrace{(\widetilde{\mathfrak{o}} \times \widetilde{S} \times \widetilde{S}^*)}_{\widetilde{N}}$. Note that this splitting is H_μ-invariant and each of the two pieces is symplectic. Denote $\mathbf{J}_N : N \to \mathfrak{h}_\mu^*$ and $\mathbf{J}_{\widetilde{N}} : \widetilde{N} \to \mathfrak{h}_\mu^*$ the homogeneous quadratic momentum maps for the H_μ-actions.

Consider the decomposition

$$(\lambda, a, b) = (\lambda_1, a_1, b_1) + (\lambda_2, a_2, b_2) \in (\mathfrak{o}^{H_\mu} \times S^H \times (S^H)^*) \oplus \widetilde{N};$$

then $\mathbf{J}_N(\lambda, a, b) = \mathbf{J}_{\widetilde{N}}(\lambda_2, a_2, b_2)$. Hence $(\lambda, a, b) \in \mathbf{J}^{-1}(0)$ implies $(\lambda_2, a_2, b_2) \in \mathbf{J}_{\widetilde{N}}^{-1}(0)$. As in the previous case using the adapted metric $(a_2, b_2) \in T^*\widetilde{S}_{(L)_H} \subset \widetilde{S} \times \widetilde{S}^*$. Finally, $(\lambda, a, b) \in N_{(L_\mu)^{G_\mu}_{H_\mu}}$, that is $(\lambda_2, a_2, b_2) \in \widetilde{N}_{(L_\mu)^{G_\mu}_{H_\mu}}$. On the other hand, the set

$$Y_{[L]} = (\widetilde{\mathfrak{o}} \times T^*\widetilde{S}_{(L)_H}) \cap \mathbf{J}_{\widetilde{N}}^{-1}(0) \cap \widetilde{N}_{(L_\mu)^{G_\mu}_{H_\mu}} \subset \widetilde{N} \tag{6.13}$$

is a semialgebraic set of $\widetilde{N}$. The image under the algebraic map $(\lambda, a, b) \mapsto (\lambda, a, a \diamond_{\mathfrak{l}} b)$ is semialgebraic. This is the content of Tarski-Seidenberg Theorem. The image of $Y_{[L]}$ under this map will be denoted by $X_{[L]}$.

For the converse, let $[g, v_1 + v_2] \in \Psi^{-1}(U_Q)$. Note that v_2 can be written as $v_2 = (\lambda_2, a_2, a_2 \diamond_{\mathfrak{l}} b_2)$ with $(\lambda_2, a_2, b_2) \in Y_{[L]}$; clearly all the properties are satisfied.

Note that $X_{[L]}$ is a Σ-submanifold, because the first part of the Proposition implies that $Q^\mu_{[L]}$ is a Σ-submanifold of Q.

□

Remark 6.5.5. Using Proposition 6.4.1, the (L)-orbit type subset of $G \times_H \widetilde{S}$ is $G \times_H \widetilde{S}_{(L)_H}$. Similarly, the $(L_\mu)^{G_\mu}$-orbit type of $G_\mu \times_{H_\mu} \widetilde{N}_{(L_\mu)^{G_\mu}_{H_\mu}}$. In other words, $Y_{[L]} \neq \emptyset$ (see (6.13)) if (L) is an orbit type for the G action on $G \times_H \widetilde{S}$ and $(L_\mu)^{G_\mu}$ is an orbit type for the action of G_μ on $G_\mu \times_{H_\mu} \widetilde{N}$. Using again Proposition 6.4.1, on any twisted product there is only a finite number of different orbit types. Hence the local characterization given by Proposition 6.5.4 implies that there is only a finite number of sets $Q^\mu_{[L]}$ with $Q^\mu_{[L]} \cap U_Q \neq \emptyset$, because only a finite number of $Y_{[L]}$ can be non-empty.

6.5.1 Q^μ is a Whitney stratified space

In Proposition 6.4.6 we showed that if $Q = Q_{(L)}$ the projection of $\mathbf{J}^{-1}(\mu)$ has a decomposed structure $\mathcal{Z}_{Q^\mu}$. In the case $Q \neq Q_{(L)}$ we are going to partition the base space Q as $\bigcup_{H\subset G} Q_{(H)}$ and in each $Q_{(H)}$ we are going to use the decomposition given by Proposition 6.4.6. We will show that the union of all these decompositions forms a decomposition of the full set $Q^\mu \subset Q$. To prove this result, we will first need a technical lemma ensuring that a certain union of decompositions is in fact a decomposition.

Lemma 6.5.6. *Let $(M, \mathcal{Z})$ be a generalized decomposed space, N a subset of M endowed with the induced topology, and for each $S \in \mathcal{Z}$ let $(N \cap S, \mathcal{Z}_S)$ be a generalized decomposition of $N \cap S \subset S$. Define $\mathcal{W} = \bigcup_{S\in\mathcal{Z}} \mathcal{Z}_S$, $(N, \mathcal{W})$ is a generalized decomposition of N.*

Proof. We first check that the elements of $\mathcal{W}$ cover N and are disjoint. Let $x \in N \subset M$; as $\mathcal{Z}$ is a generalized decomposition of M, there is $S \in \mathcal{Z}$ with $x \in S \cap N$. Therefore as $\mathcal{Z}_S$ is a generalized decomposition of $N \cap S$, there is $R \in \mathcal{Z}_S \subset \mathcal{W}$ with $x \in R$, that is the elements of $\mathcal{W}$ cover N. Assume $R_1 \cap R_2 \neq \emptyset$; there are $S_1, S_2 \in \mathcal{Z}$ such that $R_1 \cap S_1$ and $R_2 \cap S_2$, but $S_1 \cap S_2 \neq \emptyset$ implies $S_1 = S_2$, and similarly as $\mathcal{Z}_{S_1}$ is a generalized decomposition $R_1 = R_2$.

Let $x \in N \subset M$; as $\mathcal{Z}$ is a generalized decomposition, there is an open set $U \ni x$ such that $U \subset S_{i_1} \cup \cdots \cup S_{i_n}$. For each S_{i_j} there is an open set V_j of M such that $V_j \cap S_{i_j} \subset R_{j_1} \cup \cdots \cup R_{j_{n_j}}$, then the open set $U \cap V_1 \cap \cdots \cap V_N$ containing x intersects finitely many elements of $\mathcal{W}$. Hence $(N, \mathcal{W})$ is a generalized decomposition of N. □

Thanks to this technical result and the local description given by Proposition 6.5.4, we can check that the proposed pieces do form a decomposition of Q^μ.

Proposition 6.5.7. *Consider the set of submanifolds*

$$\mathcal{Z}_{Q^\mu} = \{G_\mu \cdot Z \mid Z \text{ is a connected component of } Q^\mu_{[H]} \text{ for some } H \subset G\}$$

and $Q^\mu = \tau(\mathbf{J}^{-1}(\mu)) \subset Q$ endowed with the relative topology. The pair $(Q^\mu, \mathcal{Z}_{Q^\mu})$ is a ***decomposed space****.*

Proof. The connected components of the orbit type manifolds $Q_{(H)}$ form a decomposition of Q (Proposition 3.2.7). By Proposition 6.4.6 and Proposition 6.5.2, each connected component of $Q_{(H)} \cap Q^\mu$ has a decomposition induced by the sets $Q^\mu_{[H]}$ as H runs through the conjugacy class $(H) \subset G$. Using Lemma 6.5.6, if the frontier condition is satisfied the connected components of the sets $Q^\mu_{[H]}$ form a decomposition of Q^μ.

Consider a point $q \in Q^\mu_{[H]} \cap \overline{Q^\mu_{[L]}}$. This means that there is a sequence q_n converging to q with each element in $Q^\mu_{[L]}$. Using Proposition 6.5.4, there exist a map Ψ and an open set U_Q containing q. Hence there is N big enough such that $q_N \in U_Q \cap Q^\mu_{[L]}$. This implies that there is $[g, \lambda, \varepsilon, a]_{H_\mu}$ such that $\Psi([g, \lambda, \varepsilon, a]_{H_\mu}) = q_N$.

We proved that any $q' \in Q^\mu_{[H]} \cap U_Q$ can be expressed as $q' = \Psi([g', \lambda', 0, a'])$ for $g' \in G_\mu$, $\lambda \in \mathfrak{o}_r^{H_\mu}$ and $a \in S_r^H$, but then $q' \in \overline{Q^\mu_{[L]}}$, because for any $m \in \mathbb{N}$ large enough

$$\Psi\left(\left[g', \frac{1}{m}\lambda + \lambda', \frac{1}{m^2}\varepsilon, \frac{1}{m}a + a'\right]_{H_\mu}\right) \in Q^\mu_{[L]}$$

and

$$\Psi\left(\left[g', \frac{1}{m}\lambda + \lambda', \frac{1}{m^2}\varepsilon, \frac{1}{m}a + a'\right]_{H_\mu}\right) \longrightarrow \Psi([g', \lambda', 0, a']_{H_\mu}) = q'.$$

This implies that $\overline{Q^\mu_{[L]}} \cap Q^\mu_{[H]}$ is open in $Q^\mu_{[H]}$. As $\overline{Q^\mu_{[L]}} \cap Q^\mu_{[H]}$ is also closed in $Q^\mu_{[H]}$, $\overline{Q^\mu_{[L]}} \cap Q^\mu_{[H]}$ contains at least the connected component Z of $Q^\mu_{[H]}$ through q. More precisely, if Z_L is a connected component of $Q^\mu_{[L]}$ such that $q \in \overline{Z_L}$ then $Z \subset \overline{Z_L}$ and using the G_μ-action $G_\mu \cdot Z \subset \overline{G_\mu \cdot Z_L} = G_\mu \cdot \overline{Z_L}$.

Using Lemma 6.5.6, we can conclude that $\mathcal{Z}_{Q^\mu}$ is a decomposition of Q^μ. □

As Q^μ is a subset of the manifold Q, the composition of the inclusion $\iota\colon Q^\mu \to Q$ with charts of Q endows Q^μ with a smooth structure. More precisely, the set of smooth functions on Q^μ is

$$C^\infty(Q^\mu) = \{f\colon Q^\mu \to \mathbb{R} \mid \exists g \in C^\infty(Q) \text{ such that } f = g \circ \iota\}.$$

Due to the local description given by Proposition 6.5.4 and the semialgebraic property of the sets under Ψ, the decomposition will satisfy the Whitney condition.

Proposition 6.5.8. *The decomposition $(Q^\mu, \mathcal{Z}_{Q^\mu})$ of Proposition 6.5.7 satisfies the **Whitney condition** and is **topologically locally trivial** in the sense of Definition 3.1.5.*

Proof. Let $x \in Q^\mu_{[H]} \cap \overline{Q^\mu_{[L]}}$ and apply Proposition 6.5.4 at x. As the map

$$\begin{aligned}(\mathfrak{p} \times \mathfrak{o}^{H_\mu} \times S^H) \times (\widetilde{\mathfrak{o}} \times \mathfrak{l}^* \times \widetilde{S} \times \widetilde{S}^*) &\longrightarrow Q \\ (\xi, \lambda_1, a_1; \lambda_2, \varepsilon, a_2) &\longmapsto \Psi([\exp(\xi), \lambda_1 + \lambda_2, \varepsilon, a_1 + a_2]_{H_\mu})\end{aligned}$$

is a diffeomorphism at $(0,0,0;0,0,0)$, its inverse $\mathbf{x}$ is a well defined diffeomorphism on a neighborhood of $U \subset U_Q$ of x. The restriction

$$\mathbf{x}\colon Q^\mu \cap U \longrightarrow \mathbb{R}^N$$

is a singular chart for Q^μ. We will check that the pair of pieces $Q^\mu_{[H]}$ and $Q^\mu_{[L]}$ satisfy the Whitney condition at $x \in Q^\mu_{[H]}$ with respect to the chart $\mathbf{x}$.

Note that Proposition 6.5.4 gives:

$$\begin{aligned}\mathbf{x}(Q^\mu_{[H]} \cap U) &= \big((\mathfrak{o}^{H_\mu} \times S^H) \times \{0\}\big) \cap \mathbf{x}(U), \\ \mathbf{x}(Q^\mu_{[L]} \cap U) &= \big((\mathfrak{o}^{H_\mu} \times S^H) \times X_{[L]}\big) \cap \mathbf{x}(U).\end{aligned} \tag{6.14}$$

As the set $X_{[L]}$ of (6.14) is semialgebraic and has $\{0\}$ in its closure, from (6.14) and Proposition 3.1.12, we can conclude that $Q^\mu_{[H]}$ and $Q^\mu_{[L]}$ satisfy the Whitney condition at x with respect to the chart $\mathbf{x}$.

Similarly, if $Q^\mu_{[L_i]} \cap U \neq 0$ there is a semialgebraic Σ-manifold $X_{[L_i]}$ such that

$$\mathbf{x}(Q^\mu_{[L_i]} \cap U) = (\mathfrak{p} \times \mathfrak{o}^{H_\mu} \times S^H) \times X_{[L_i]}.$$

But then if we define $X = \{0\} \sqcup \bigsqcup_{L_i} X_{[L_i]}$ with the decomposition induced by connected components of each $X_{[L_i]}$, the map $\mathbf{x}$ restricted to Q^μ becomes the stratified smooth homeomorphism

$$\mathbf{x}\colon Q^\mu \cap U \longrightarrow (\mathfrak{p} \times \mathfrak{o}^{H_\mu} \times S^H) \times X.$$

Therefore, Q^μ is a locally trivial stratified space (see Definition 3.1.5). □

It seems that the frontier condition and local triviality are consequences of Theorems 3.1.10 and 3.1.11 if we check Whitney conditions as we did in the previous result. The problem is that we cannot apply these results because Q^μ does not need to be locally closed. For this reason we had to prove the frontier condition and the local triviality independently.

6.5.2 Q^μ/G_μ is a Whitney stratified space

In Theorem 3.2.10, the orbit-type stratification of a manifold M induced a stratification of the quotient M/G that made the quotient map $M \to M/G$ a smooth map. Using the same idea, we will induce a decomposition on Q^μ/G_μ from the one given in Proposition 6.5.7.

Proposition 6.5.9. *Let $(Q^\mu, \mathcal{Z}_{Q^\mu})$ be the decomposed space of Proposition 6.5.7. The set $\mathcal{Z}_{Q^\mu/G_\mu} = \{Z/G_\mu \mid Z \in \mathcal{Z}_{Q^\mu}\}$ is a decomposition of Q^μ/G_μ.*

Moreover, Q^μ/G_μ has a smooth structure induced by the inclusion $Q^\mu/G_\mu \to Q/G_\mu$ and the set of smooth functions on Q^μ/G_μ is

$$C^\infty(Q^\mu/G_\mu) = \{f\colon Q^\mu/G_\mu \to \mathbb{R} \mid \exists g \in C^\infty(Q)^{G_\mu} \text{ and } f \circ \pi_{G_\mu} = g\}.$$

*With respect to this structure, Q^μ/G_μ is a **Whitney decomposed space and is topologically locally trivial**.*

The G_μ-quotient map

$$\pi_{G_\mu}\colon (Q^\mu, \mathcal{Z}_{Q^\mu}) \to (Q^\mu/G_\mu, \mathcal{Z}_{Q^\mu/G_\mu})$$

*is a **smooth decomposed surjective submersion**.*

Proof. The sets in $\mathcal{Z}_{Q^\mu/G_\mu}$ cover Q^μ and are disjoint because $\mathcal{Z}_{Q^\mu}$ is a set of disjoint G_μ-saturated sets covering Q^μ.

Let $x \in Q^\mu/G_\mu$ and fix $q \in Q^\mu$ projecting on x. Using Proposition 6.3.1 at q, the restriction

$$\Psi\colon \big(G_\mu \times_{H_\mu} (\mathfrak{o}_r \times \mathfrak{l}_r^* \times S_r)\big) \cap \Psi^{-1}(U_Q) \to U_Q \subset Q$$

is a G_μ-invariant diffeomorphism. Define $\widetilde{\mathfrak{o}}$ and $\widetilde{S}$ as in Proposition 6.5.4. Choose $\mathrm{Hilb}\colon \widetilde{\mathfrak{o}} \times \mathfrak{l}^* \times \widetilde{S} \to \mathbb{R}^k$ a Hilbert map for the H_μ action on $\widetilde{\mathfrak{o}} \times \mathfrak{l}^* \times \widetilde{S}$ and define

$$\begin{aligned} \mathbf{y}\colon U_Q/G_\mu &\longrightarrow (\mathfrak{o}^{H_\mu} \times S^H) \times \mathbb{R}^k \\ \pi_{G_\mu}(\Psi([g, v_1 + v_2]_{H_\mu})) &\longmapsto (v_1, \mathrm{Hilb}(v_2)) \end{aligned}$$

where, as in Proposition 6.5.4, $v_1 \in \mathfrak{o}^{H_\mu} \times S^H$ and $v_2 \in \widetilde{o} \times \mathfrak{l}^* \times \widetilde{S}$. Using Theorem 3.2.9, $\mathbf{y}$ is singular chart for Q/G_μ. Then,

$$\mathbf{y}(\pi_{G_\mu}(Q^\mu_{[H]} \cap U_Q)) = \big((\mathfrak{o}^{H_\mu} \times S^H) \times \{0\}\big) \cap \mathbf{y}(U_Q/G_\mu)$$

this implies that $\pi_{G_\mu}(Q^\mu_{[H]} \cap U_Q) = \pi_{G_\mu}(Q^\mu_{[H]}) \cap \pi_{G_\mu}(U_Q)$ is locally closed and has a manifold structure. As this can be applied for any $x \in Q^\mu/G_\mu$, the sets in $\mathcal{Z}_{Q^\mu/G_\mu}$ are locally closed and have a compatible manifold structure.

Moreover, by the second part of Proposition 6.5.4,

$$\mathbf{y}(\pi_{G_\mu}(Q^\mu_{[L]} \cap U_Q)) = \big((\mathfrak{o}^{H_\mu} \times S^H) \times X_{[L]}\big) \cap \mathbf{y}(U_Q/G_\mu),$$

but then, as in Proposition 6.5.7, the set $\mathcal{Z}_{Q^\mu/G_\mu}$ satisfies the frontier condition.

As we noted in Remark 6.5.5, there is only a finite number of sets $Q^\mu_{[L]}$ such that $Q^\mu_{[L]} \cap U_Q \neq \emptyset$. Moreover, for each L, $\Psi^{-1}(Q^\mu_{[L]} \cap U_Q)$ is semialgebraic, and therefore it has a finite number of connected components. This implies that $\mathcal{Z}_{Q^\mu/G_\mu}$ is a locally finite partition and therefore $\mathcal{Z}_{Q^\mu/G_\mu}$ is a decomposition.

As in Proposition 6.5.8, since the sets in the decomposition under $\mathbf{y}$ are semialgebraic, Proposition 3.1.12 ensures that $\mathcal{Z}_{Q^\mu/G_\mu}$ is a Whitney decomposition. Again as in Proposition 6.5.8, the chart $\mathbf{y}$ is a trivializing homeomorphism like (3.1).

Let $q \in Q^\mu$ and $Z \in \mathcal{Z}_{Q^\mu}$ with $q \in Z$. Taking coordinates centered at q, $\Psi^{-1}(Q^\mu_{[H]} \cap U_Q) = G_\mu \times_{H_\mu} (\mathfrak{o}^{H_\mu} \times \{0\} \times S^H)$ and exactly as before $\mathbf{y}(Q^\mu_{[H]}/G_\mu) = (\mathfrak{o}^{H_\mu} \times S^H) \cap \mathbf{y}(U_Q/G_\mu)$. In this setting, the projection $\pi_{G_\mu}\big|_Z$ becomes $\pi_{G_\mu}(\Psi([g, \lambda, 0, a]_{H_\mu}) = \mathbf{y}^{-1}(\lambda + a, 0)$; that is, $\pi_{G_\mu}\big|_Z$ is a submersion at $q \in Z \cap U_Q$. As this can be done at any point of Z, for any $Z \in \mathcal{Z}_{Q^\mu}$ the map $\pi_{G_\mu}\big|_Z : Z \to Z/G_\mu \subset Q^\mu/G_\mu$ is a surjective submersion. Therefore, the quotient map $\pi_{G_\mu} : Q^\mu \to Q^\mu/G_\mu$ is a decomposed map and as it is the restriction of the smooth map $Q \to Q/G_\mu$ it is a smooth decomposed map.

□

Remark 6.5.10. Note that Proposition 6.5.4 is the key result that we used to prove that $(Q^\mu, \mathcal{Z}_{Q^\mu})$ and its quotient $(Q^\mu/G_\mu, \mathcal{Z}_{Q^\mu/G_\mu})$ are locally trivial, Whitney stratified spaces. Although these are the standard notions of regularity in the stratified setting, we would like to remark that $\mathcal{Z}_{Q^\mu}$ and $\mathcal{Z}_{Q^\mu/G_\mu}$ satisfy **stronger conditions**. Proposition 6.5.4 shows that Ψ induces not only a topological trivialization as the one stated in (3.1), but also induces a **smooth isomorphism** of stratified spaces. Exactly as is done in [Jul14] for orbit types, it can be shown that $(Q^\mu, \mathcal{Z}_{Q^\mu})$ satisfies the **strong Verdier condition** or **differentiably regular condition** (see [KTL89; Tro83])

6.6 Seams

Combining the sets $Q^\mu_{[H]}$ with the stratification of $\mathbf{J}^{-1}(\mu)$ given by Theorem 3.3.1, it seems reasonable to study the sets

$$\begin{aligned} s_{H\to K} &= \tau^{-1}(Q^\mu_{[H]}) \cap G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K \subset \mathbf{J}^{-1}(\mu) \\ \mathcal{S}_{H\to K} &= s_{H\to K}/G_\mu \subset \mathbf{J}^{-1}(\mu)/G_\mu. \end{aligned} \tag{6.15}$$

In analogy with the work of [PROSD07] (compare with (3.2)), we will call $s_{H\to K}$ the **pre-seam** $H \to K$ and $\mathcal{S}_{H\to K}$ the **seam** $H \to K$. In this section we will see that these sets are Σ-submanifolds and they induce a (generalized) decomposition of $\mathbf{J}^{-1}(\mu)$. With respect to the appropriate structure, both $\mathbf{J}^{-1}(\mu) \to Q^\mu$ and $\mathbf{J}^{-1}(\mu)/G_\mu \to Q^\mu/G_\mu$ are smooth decomposed surjective submersions.

Remark 6.6.1. In [PROSD07], the reason for calling these sets seams was that they played a "stitching" role; they stitch together different pieces symplectomorphic to cotangent bundles. In our setting, when $\mu \neq 0$, the analogy is not so clear, but we will see in Section 6.8.4 that some pieces $\mathcal{S}_{H\to K}$ are symplectic and all the other pieces $\mathcal{S}_{H'\to K'}$ have a stitching role between those (see (6.28)).

In this setting, the analogue of Lemma 6.5.1 is the following.

Lemma 6.6.2.

- *If* $z \in \mathbf{J}^{-1}(\mu)$ *then* $z \in s_{G_{\tau(z)}\to G_z}$.
- *If* $s_{H_1\to K_1} \cap s_{H_2\to K_2} \neq \emptyset$ *then*
  - $(H_1) = (H_2)$, $(H_1 \cap G_\mu)^{G_\mu} = (H_2 \cap G_\mu)^{G_\mu}$, $(K_1)^{G_\mu} = (K_2)^{G_\mu}$.
  - $s_{H_1\to K_1} = s_{H_2\to K_2}$.

Conversely, if H_1, H_2, K_1, K_2 *are subgroups such that* $(H_1) = (H_2)$, $(H_1 \cap G_\mu)^{G_\mu} = (H_2 \cap G_\mu)^{G_\mu}$, $(K_1)^{G_\mu} = (K_2)^{G_\mu}$ *then* $s_{H_1\to K_1} = s_{H_2\to K_2}$.

- *If* $s_{H_1\to K_1}\cap\overline{s_{H_2\to K_2}}\neq\emptyset$*, then,*
$$(H_1)\geq(H_2),\quad (H_1\cap G_\mu)^{G_\mu}\geq(H_2\cap G_\mu)^{G_\mu},\quad (K_1)\geq(K_2).$$
In other words, there are $g\in G$*,* $k\in G_\mu$*,* $l\in G$ *such that* $gH_1g^{-1}\supset H_2$*,* $k(H_1\cap G_\mu)k^{-1}\supset H_2\cap G_\mu$ *and* $lK_1l^{-1}\supset K_2$*.*

Proof. If $z\in\mathbf{J}^{-1}(\mu)$ by Lemma 6.5.1 $\tau(z)\in Q^\mu_{[G_{\tau(z)}]}$. As it is clear that $z\in G_\mu\cdot(\mathbf{J}^{-1}(\mu))_{G_z}$, then $z\in s_{G_{\tau(z)}\to G_z}$.

If $z\in s_{H_1\to K_1}\cap s_{H_2\to K_2}$ then $\tau(z)\in Q^\mu_{[H_1]}\cap Q^\mu_{[H_2]}$ and using the same lemma $Q^\mu_{[H_1]}=Q^\mu_{[H_2]}$ and $(H_1)=(H_2)$ and $(H_1\cap G_\mu)^{G_\mu}=(H_2\cap G_\mu)^{G_\mu}$. Moreover, as $z\in(G_\mu\cdot(\mathbf{J}^{-1}(\mu))_{K_1})\cap(G_\mu\cdot(\mathbf{J}^{-1}(\mu))_{K_2})$, $(K_1)^{G_\mu}=(K_2)^{G_\mu}$. The converse is analogous. □

Let $z\in s_{H\to L}$; by Proposition 5.2.1 there is a Palais' tube $\mathbf{t}$ and a cotangent reduction map φ such that $z=T^*\mathbf{t}^{-1}(\varphi(e,\mu,0,\alpha))$. Note that for any $n>0$ the point $z_n=T^*\mathbf{t}^{-1}(\varphi(e,\mu,0,\alpha/n)$ is again in $s_{H\to L}$, because $G_z=G_{z_n}$ and $\tau(z)=\tau(z_n)$. But $\{z_n\}_{n\in\mathbb{N}}$ is a sequence converging to $z_\infty=T^*\mathbf{t}^{-1}(\varphi(e,\mu,0,0))$ and $z_\infty\in s_{H\to H_\mu}$, because $G_{z_\infty}=G_{\tau(z)}\cap G_\mu$. Therefore, for any pre-seam $s_{H\to L}$,
$$\emptyset\neq s_{H\to H_\mu}\subset\overline{s_{H\to L}}\implies\emptyset\neq\mathcal{S}_{H\to H_\mu}\subset\overline{\mathcal{S}_{H\to L}};\tag{6.16}$$
that is, the pre-seam $s_{H\to H_\mu}$ is **minimal** among the family $\{s_{H\to L}\}_{L\subset G}$ and similarly for the corresponding seam.

Note that as $\tau\colon T^*Q\to Q$ is a submersion and $Q^\mu_{[H]}$ is a Σ-submanifold, the preimage $\tau^{-1}(Q^\mu_{[H]})$ is a submanifold. Besides, the set $G_\mu\cdot(\mathbf{J}^{-1}(\mu))_K$ is a Σ-manifold because of Theorem 3.3.1. The first problem to solve is if the intersection (6.15) is a Σ-submanifold of T^*Q. Using the cotangent bundle Hamiltonian tube, we can show even more that the intersection is **clean**. More precisely, $s_{H\to K}$ is a Σ-submanifold of T^*Q and, for any $x\in s_{H\to K}$
$$T_x\left(s_{H\to K}\right)=T_x\left(\tau^{-1}(Q^\mu_{[H]}\right)\cap T_x\left(G_\mu\cdot(\mathbf{J}^{-1}(\mu))_K\right).$$

Proposition 6.6.3. *Let* H,K *be closed subgroups of* G*. The intersection of* $\tau^{-1}(Q^\mu_{[H]})$ *and* $G_\mu\cdot(\mathbf{J}^{-1}(\mu))_K$ *is clean.*

Proof. Let $z\in s_{H\to K}$; by Lemma 6.6.2, without loss of generality, we can assume $G_z=K$ and $G_{\tau(z)}=H$. Denote $q=\tau(z)$, Proposition 6.3.1 gives the Hamiltonian tube $\mathcal{T}_0$ and the map Ψ. In this setting, there is $\alpha\in S^*$ such that $z=\mathcal{T}_0([e,0,0,0,\alpha]_{H_\mu})$ and, as $G_z=K$, $H_\mu\cap H_\alpha=K$. Let $B=(\mathfrak{h}_\mu\cdot\alpha)^\circ$, using Theorem 5.2.7 the map
$$\begin{aligned}\mathcal{T}\colon G\times_K((\mathfrak{s}^*_s\oplus\mathfrak{p}^*_s)\times\mathfrak{o}_s\times B_s\times B^*_s)&\longrightarrow T^*Q\\ [g,\nu_{\mathfrak{s}}+\nu_{\mathfrak{p}},\lambda;a,b]_K&\longmapsto\mathcal{T}_0([g,\nu_{\mathfrak{p}},\lambda,\widetilde{a},b+\alpha]_{H_\mu})\end{aligned}\tag{6.17}$$
where
$$\widetilde{a}=a+\Gamma(\nu_{\mathfrak{s}}-a\diamond_{\mathfrak{s}}b-\frac{1}{2}\lambda\diamond_{\mathfrak{s}}\operatorname{ad}^*_\lambda\mu;b)$$
is a Hamiltonian tube centered at z.

Note that, as $B=(\mathfrak{h}_\mu\cdot\alpha)^\circ\subset S$, if $v\in S^H$ for any $\xi\in\mathfrak{h}_\mu\subset\mathfrak{h}$
$$0=\langle\xi\cdot v,\alpha\rangle=-\langle v,\xi\cdot\alpha\rangle,$$
that is, $S^H\subset B$. But then, restricting to K-fixed vectors,
$$S^H\subset B^K.\tag{6.18}$$

In this setting, we can check the following equality
$$\mathcal{T}^{-1}(s_{H\to K}\cap\tau^{-1}(U_Q))=\left(G_\mu\times_K\left(\{0\}\times\mathfrak{o}^{H_\mu}\times S^H\times(B^*)^K\right)\right)\cap\mathcal{T}^{-1}(\tau^{-1}(U_Q)).\tag{6.19}$$

- Let $x = [g, 0, \lambda, a, b]_K \in \left(G_\mu \times_K \left(\{0\} \times \mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K\right)\right) \cap \mathcal{T}^{-1}(\tau^{-1}(U_Q))$, then $\mathbf{J}(\mathcal{T}(x)) = \mu$ because $\mathcal{T}$ is a Hamiltonian tube and $\mathbf{J}_K(\lambda, a, b) = 0$. Regarding the isotropy, we have $g^{-1}G_x g = K_\lambda \cap K_a \cap K_b = K$. Therefore, $\mathcal{T}(x) \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$. Using (6.9),

$$\begin{aligned}\tau(\mathcal{T}(x)) &= \tau(\mathcal{T}_0([g, 0, \lambda, a, b + \alpha]_{H_\mu}))\\ &= \tau(\Phi(g, 0, \lambda, a \diamond_{\mathfrak{l}} (b + \alpha); a, b + \alpha))\\ &= \tau(\Phi(g, 0, \lambda, 0; a, 0)) = \Psi([g, \lambda, 0, a]_{H_\mu}),\end{aligned}$$

  but as $\lambda \in \mathfrak{o}^{H_\mu}$, $a \in S^H$, using Proposition 6.5.4, $\tau(\mathcal{T}(x)) \in Q^\mu_{[H]}$. Then $\mathcal{T}(x) \in s_{H\to K} \cap \tau^{-1}(U_Q)$.

- Conversely, let $x = [g, \nu_{\mathfrak{s}} + \nu_{\mathfrak{p}}, \lambda, a, b]_K$ with $\mathcal{T}(x) \in s_{H\to K} \subset \mathbf{J}^{-1}(\mu)$. Since

$$\mathcal{T}(x) = \mathcal{T}_0([g, \nu_{\mathfrak{p}}, \lambda, \widetilde{a}, b + \alpha]_{H_\mu}$$

  equation (6.8) implies that $\nu_{\mathfrak{p}} = 0$, $g \in G_\mu$ and $\mathbf{J}_{H_\mu}(\lambda, \widetilde{a}, b + \alpha) = 0$. Then, using (6.9),

$$\begin{aligned}\tau(\mathcal{T}(x)) &= \tau(\mathcal{T}_0([g, 0, \lambda, \widetilde{a}, b + \alpha]_{H_\mu}))\\ &= \tau(\Phi(g, 0, \lambda, \widetilde{a} \diamond_{\mathfrak{l}} (b + \alpha); \widetilde{a}, b + \alpha))\\ &= \Psi([g, \lambda, \widetilde{a} \diamond_{\mathfrak{l}} (b + \alpha), \widetilde{a}]_{H_\mu})\end{aligned}$$

  but since this point lies in $Q^\mu_{[H]}$, using the characterization of Proposition 6.5.4, $\widetilde{a} \in S^H$. However, Γ takes values in a complementary of $B \subset S$, but as $S^H \subset B$, this implies that $\widetilde{a} = a$. Hence, $\tau(\mathcal{T}(x)) = \Psi([g, \lambda, 0, a]_{H_\mu})$, and using again Proposition 6.5.4 $\lambda \in \mathfrak{o}^{H_\mu}$. Finally, as $G_x \in (K)^{G_\mu}$, then $b \in B^K$.

The description (6.19) implies that the intersection is a Σ-manifold. We will now check that the intersection is clean. Using Proposition 2.3.5 applied to the Hamiltonian tube $\mathcal{T}$,

$$\begin{aligned}&\mathcal{T}^{-1}(G_\mu \cdot \mathbf{J}^{-1}(\mu)_K) =\\ &\qquad \{T_{(e,0,0;0,0)}\pi_{K^T} \cdot (\xi', 0, \lambda'; a', b') \mid \xi' \in \mathfrak{g}_\mu,\ \lambda' \in \mathfrak{o}^K,\ a' \in B^K,\ b' \in (B^*)^K\}\end{aligned} \tag{6.20}$$

Note that the restriction of τ to $\mathcal{T}(G_\mu \times_K \{0\} \times \mathfrak{o}^{H_\mu} \times S^H \times \{0\})$ is a submersion on the manifold $Q^\mu_{[H]}$ because, using (6.9), if $x = [g, 0, \lambda, a, 0]_K$, then $\tau(\mathcal{T}(x)) = \Psi([g, \lambda, 0, a]_{H_\mu})$. This implies

$$T_z\tau^{-1}(Q^\mu_{[H]}) = T_z\mathcal{T}(G_\mu \times_K \{0\} \times \mathfrak{o}^{H_\mu} \times S^H \times \{0\}) + \operatorname{Ker} T_z\tau.$$

Therefore, we need to compute $\operatorname{Ker} T_z\tau$, and to do so we linearize the tube $\mathcal{T}$ at $[e, 0, 0, 0, 0]_K$. As the map Γ (see (5.18)) is linear with respect to the first variable,

$$T_{(0,0)}\Gamma \cdot (\dot\nu, \dot b) = \Gamma(\dot\nu; 0).$$

Using (5.9) and after some straightforward algebra,

$$T_{(e,0,0,0,0)}(\tau \circ \mathcal{T} \circ \pi_K) \cdot (\xi, \dot\nu_{\mathfrak{s}} + \dot\nu_{\mathfrak{p}}, \dot\lambda, \dot a, \dot b) = (\xi + \dot\lambda + \sigma^{-1}(\widehat{a} \diamond_{\mathfrak{l}} \alpha)) \cdot q + \widehat{a} \in T_qQ$$

where $\widehat{a} = \dot a + \Gamma(\dot\nu_{\mathfrak{s}}; 0)$ and $\sigma \colon \mathfrak{n} \to \mathfrak{l}^*$ is the isomorphism $\eta \mapsto \langle \mu, [\cdot, \eta] \rangle$. Then,

$$\operatorname{Ker} T_{(e,0,0,0,0)}(\tau \circ \mathcal{T}_\alpha \circ \pi_K) = \{(\xi - \dot\lambda, \dot\nu_{\mathfrak{p}}, \dot\lambda; 0, \dot b) \mid \xi \in \mathfrak{h},\ \dot\lambda \in \mathfrak{o},\ \dot\nu_{\mathfrak{p}} \in \mathfrak{p}^*,\ \dot b \in B\}.$$

Hence,

$$\mathfrak{T}^{-1}(\tau^{-1}(Q^{\mu}_{[H]})) = \{T_{(e,0,0,0,0)}\pi_{K^T} \cdot (\xi_1 + \xi_2, 0, \dot{\nu}_{\mathfrak{p}}, \dot{\lambda}_1 + \dot{\lambda}_2; \dot{a}, \dot{b}) \mid \\ \xi_1 \in \mathfrak{g}_\mu,\ \xi_2 = -\dot{\lambda}_2 + \eta,\ \eta \in \mathfrak{h}, \dot{\lambda}_1 \in \mathfrak{o}^{H_\mu},\ \dot{\lambda}_2 \in \mathfrak{o},\ \dot{b} \in B^*,\ \dot{\nu}_{\mathfrak{p}} \in \mathfrak{p}^*,\ \dot{a} \in S^H\} \quad (6.21)$$

After a simple linear algebra computation, comparing the intersection of (6.20) and (6.21) with (6.19) we have

$$T_z(\tau^{-1}(Q^{\mu}_{[H]})) \cap T_z(G_\mu \cdot \mathbf{J}^{-1}(\mu)_K) = T_z(s_{H\to K}).$$

Therefore, the intersection is clean at z. □

Note that during the proof of this result, in (6.19), we have obtained a very useful local description of the pre-seam $s_{H\to K}$ in coordinates induced by a cotangent Hamiltonian tube.

Using that $Q^{\mu}_{[H]}$ form a Whitney stratification of Q^μ, and that $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$ is a Whitney stratification of $\mathbf{J}^{-1}(\mu)$, we can show that the connected components of the pre-seams $s_{H\to K}$ form a generalized decomposition of $\mathbf{J}^{-1}(\mu)$.

Proposition 6.6.4. *The set of submanifolds*

$$\mathcal{Z}_{\mathbf{J}^{-1}(\mu)} = \{G_\mu \cdot Z \mid Z \text{ is a connected component of } s_{H\to K} \text{ for some } H, K \subset G\}$$

forms a ***generalized decomposition*** *of $\mathbf{J}^{-1}(\mu)$. Similarly, the set*

$$\mathcal{Z}_{\mathbf{J}^{-1}(\mu)/G_\mu} = \{Z/G_\mu \mid Z \in \mathcal{Z}_{\mathbf{J}^{-1}(\mu)}\}$$

is a ***generalized decomposition*** *of $\mathbf{J}^{-1}(\mu)/G_\mu$.*

If $\mathbf{J}^{-1}(\mu)/G_\mu$ is endowed with the smooth structure induced by $\mathbf{J}^{-1}(\mu)/G_\mu \to (T^*Q)/G_\mu$, the quotient map

$$\pi_{G_\mu} \colon (\mathbf{J}^{-1}(\mu), \mathcal{Z}_{\mathbf{J}^{-1}(\mu)}) \to (\mathbf{J}^{-1}(\mu)/G_\mu, \mathcal{Z}_{\mathbf{J}^{-1}(\mu)/G_\mu})$$

is a smooth decomposed surjective submersion.

Proof. Using Lemma 6.6.2, the sets in $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)}$ cover $\mathbf{J}^{-1}(\mu)$ and are disjoint. Let Z be a connected component of $s_{H\to K} = \tau^{-1}(Q^{\mu}_{[H]}) \cap G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$. As in Proposition 6.6.3, the local description (6.19) implies that $G_\mu \cdot Z$ is an embedded submanifold of T^*Q; in particular, a locally closed subspace of $\mathbf{J}^{-1}(\mu)$.

Let $z \in \mathbf{J}^{-1}(\mu)$ and define $q = \tau(z)$ using that $\mathcal{Z}_{Q^\mu}$ is a decomposition (see Proposition 6.5.7) there exists a neighborhood U_q of q such that $U_q \cap Q^\mu$ intersects finitely many elements of $\mathcal{Z}_{Q^\mu}$. Similarly, as the partition of T^*Q by G_μ-orbit types is a decomposition, there exists a neighborhood $U_z \subset T^*Q$ such that U_z intersects finitely many G_μ-orbit types. But then $U_z \cap \tau^{-1}(U_q) \cap \mathbf{J}^{-1}(\mu)$ is an open neighborhood of z in $\mathbf{J}^{-1}(\mu)$ intersecting finitely many elements of $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)}$.

Using that $\pi_{G_\mu} \colon T^*Q \to (T^*Q)/G_\mu$ is an open map, the set $\pi_{G_\mu}(U_z \cap \tau^{-1}(U_q))$ intersects finitely many elements of $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)/G_\mu}$. And as the sets $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)}$ cover $\mathbf{J}^{-1}(\mu)$ and are disjoint, $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)/G_\mu}$ is a locally finite partition of the quotient $\mathbf{J}^{-1}(\mu)/G_\mu$. If $Z \in \mathcal{Z}_{\mathbf{J}^{-1}(\mu)}$, Z/G_μ is a manifold, because all the points in Z have the same G_μ-orbit type.

Alternatively, we can check this using the local description (6.19). In these coordinates the inclusion $\mathcal{S}_{H\to K} \subset \mathbf{J}^{-1}(\mu)/G_\mu$ is

$$\mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K \subset (\mathfrak{o} \times B \times B^*)/K.$$

Therefore, $\mathcal{S}_{H\to K}$ is a locally closed set with a submanifold structure. The fact that π_{G_μ} is a smooth decomposed submersion is identical to Proposition 6.5.9.

□

6.6.1 Stratawise projection

We now check that the partition into pre-seams $(\mathbf{J}^{-1}(\mu), \mathcal{Z}_{\mathbf{J}^{-1}(\mu)})$ is well behaved with respect to the base projection $\tau\colon T^*Q \to Q$. More precisely,

Proposition 6.6.5. *Let $Z \in \mathcal{Z}_{\mathbf{J}^{-1}(\mu)}$, then $\tau(Z) \in \mathcal{Z}_{Q^\mu}$ and the restricted map $\tau\big|_Z\colon Z \to \tau(Z) \subset Q^\mu$ is a surjective submersion.*

More globally, the restriction of the cotangent bundle projection

$$\tau^\mu\colon (\mathbf{J}^{-1}(\mu), \mathcal{Z}_{\mathbf{J}^{-1}(\mu)}) \longrightarrow (Q^\mu, \mathcal{Z}_{Q^\mu})$$

is a ***smooth G_μ-equivariant decomposed surjective submersion****.*

Proof. Let Z be a connected component of $s_{H\to K}$ and $z \in Z \subset s_{H\to K}$ with $K = G_z$ and $H = G_{\tau(z)}$. As in (6.19),

$$\mathcal{T}^{-1}(s_{H\to K} \cap \tau^{-1}(U_Q)) = \left(G_\mu \times_K \left(\{0\} \times \mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K\right)\right) \cap \mathcal{T}^{-1}(\tau^{-1}(U_Q)).$$

and

$$\Psi^{-1}(Q^\mu_{[H]} \cap U_Q) = \left(G_\mu \times_{H_\mu} \mathfrak{o}^{H_\mu} \times \{0\} \times S^H\right) \cap \Psi^{-1}(U_Q).$$

Moreover, if $z' \in s_{H\to K} \cap \tau^{-1}(U_Q))$, then by (6.8), $z' = \mathcal{T}([g, 0, \lambda, a, b]_K)$ and using (6.9)

$$\tau(z') = \tau(\mathcal{T}([g, 0, \lambda, a, b]_K) = \tau(\mathcal{T}_0([g, 0, \lambda, a, b + \alpha]_{H_\mu}) = \Psi([g, \lambda, 0, a]_{H_\mu}).$$

Hence, $\tau\big|_Z\colon Z \to Q^\mu_{[H]} \subset Q^\mu$ is a submersion at z and clearly G_μ-equivariant.

We now check that $\tau(Z)$ is the connected component through q of $Q^\mu_{[H]}$ and not just an open subset. There is $\alpha \in S^*$ such that $z = \mathcal{T}_0([e, 0, 0, 0, \alpha]_{H_\mu})$. Let $q' \in \mathbf{L}(H, \mu) \cdot Q^{\mathrm{loc}}_H \subset Q^\mu_{[H]}$ where Q^{loc}_H is a local isotropy type submanifold (Proposition 3.2.6). Denote $H' = G_{q'}$ and let S' be a linear slice at q'. There is $g \in \mathbf{L}(H, \mu)$ such that $H' = gHg^{-1}$ and $k \in G_\mu$ such that $H'_\mu = kH_\mu k^{-1}$. As q and q' have the same local orbit type, there is a linear isomorphism $A\colon S \to S'$ such that

$$A(h \cdot v) = (ghg^{-1}) \cdot A(v) \qquad \forall h \in H$$

and

$$A(h \cdot v) = (khk^{-1}) \cdot A(v) \qquad \forall h \in H_\mu.$$

Consider the Hamiltonian tube $\widetilde{\mathcal{T}}_0$ at q' given by Theorem 5.2.2 associated with the splitting $\mathfrak{g} = \mathrm{Lie}(H' \cap G_\mu) \oplus \widetilde{\mathfrak{p}} \oplus \widetilde{\mathfrak{o}} \oplus \widetilde{\mathfrak{l}} \oplus \widetilde{\mathfrak{n}}$, that is,

$$\widetilde{\mathcal{T}}_0\colon G \times_{H'_\mu} (\widetilde{\mathfrak{p}}^* \times \widetilde{\mathfrak{o}} \times S' \times (S')^*) \longrightarrow T^*Q.$$

Then $z' = \widetilde{\mathcal{T}}([e, 0, 0, 0, A(\alpha)]_{H'_\mu}$ projects onto q' and $G_{z'} = (H'_\mu)_{A(\alpha)} = kLk^{-1} \in (H_\mu)^{G_\mu}$. This implies that $\mathbf{L}(H, \mu) \cdot Q^{\mathrm{loc}}_H \subset \tau(s_{H\to K})$ and as the connected component through q of $Q^\mu_{[H]}$ is contained in $\mathbf{L}(H, \mu) \cdot Q^{\mathrm{loc}}_H$, $\tau(Z)$ is the connected component of $Q^\mu_{[H]}$ through q and $\tau(G_\mu \cdot Z) = G_\mu \cdot \tau(Z) \in \mathcal{Z}_{Q^\mu}$.

Globally, as a map of decomposed sets, $\tau^\mu\colon \mathbf{J}^{-1}(\mu) \to Q^\mu$ is a decomposed map and smooth because it is the restriction of the smooth map $T^*Q \to Q$. ☐

Using this result and Propositions 6.5.9 and 6.6.4, it follows that the maps in the commutative diagram

$$\begin{array}{ccc} \mathbf{J}^{-1}(\mu) & \longrightarrow & \mathbf{J}^{-1}(\mu)/G_\mu \\ \big\downarrow & & \big\downarrow \\ Q^\mu & \longrightarrow & Q^\mu/G_\mu \end{array}$$

are **smooth decomposed surjective submersions**.

6.7 Frontier condition and Whitney condition

The main drawback of Proposition 6.6.4 is that it only states that the (pre-)seams form a generalized decomposition. It does not tell us anything about either the frontier condition or Whitney conditions. In general, we have neither counterexamples nor formal proofs that $(\mathbf{J}^{-1}(\mu), \mathcal{Z}_{\mathbf{J}^{-1}(\mu)})$ or its G_μ-quotient $(\mathbf{J}^{-1}(\mu)/G_\mu, \mathcal{Z}_{\mathbf{J}^{-1}(\mu)/G_\mu})$ satisfy the frontier condition. In this section we study some particular cases in which we can say a little more.

6.7.1 Restriction $G_z = G_q \cap G_\mu$

If instead of considering the whole family of pre-seams $\{s_{H\to K}\}_{H,K\subset G}$, we only consider the subfamily $\{s_{H\to H_\mu}\}_{H\subset G}$ we get a topologically trivial Whitney decomposition. Note that if we endow Q with a G-invariant metric, according to Proposition 6.5.2, these pre-seams cover the subset

$$\bigcup_{H\subset G} (T^*_{Q_{(H)}} Q_{(H)}) \cap \mathbf{J}^{-1}(\mu) \subset \mathbf{J}^{-1}(\mu).$$

As we noted in Remark 6.5.10, Proposition 6.5.4 was the key tool to check that Q^μ was a smooth Whitney stratified space. To prove Whitney conditions in our setting, we will use an analogue of Proposition 6.5.4.

Lemma 6.7.1. *Let $z \in s_{H\to H_\mu}$ with $G_{\tau(z)} = H$, there is a Hamiltonian tube $\mathcal{T}$ centered at z satisfying the following property:*

If $[g, 0, \lambda, a, b]_{H_\mu} \in \mathcal{T}^{-1}(s_{L\to K})$ and $\lambda' \in \mathfrak{o}^{H_\mu}$, $a' \in S^H$, $b' \in B^{H_\mu}$, $g' \in G_\mu$ and $\rho > 0$ satisfy $[g', 0, \rho\lambda + \lambda', \rho a + a', \rho b + b']_{H_\mu} \in G \times_{H_\mu} ((\mathfrak{s}^*_s \oplus \mathfrak{p}^*_s) \times \mathfrak{o}_s \times B_s \times B^*_s)$ then

$$\mathcal{T}([g', 0, \rho\lambda + \lambda', \rho a + a', \rho b + b']_{H_\mu}) \in s_{L\to K}.$$

Proof. We proceed as in Proposition 6.6.3. Denote $q = \tau(z)$, Proposition 6.3.1 gives the Hamiltonian tube $\mathcal{T}_0$ and the map Ψ. In this setting, there is $\alpha \in S^*$ such that $z = \mathcal{T}_0([e, 0, 0, 0, \alpha]_{H_\mu})$ and, as $G_z = H_\mu$, $H_\mu \cdot \alpha = \alpha$. Let $B = (\mathfrak{h}_\mu \cdot \alpha)^\circ = S$, using Theorem 5.2.7, the map (6.17) is the stated Hamiltonian tube. Note that as $K = H_\mu$, the map Γ is always zero.

Let $w = \mathcal{T}([g, 0, \lambda, a, b]_{H_\mu}) \in s_{L\to K}$ and $w' = \mathcal{T}([g', 0, \rho\lambda + \lambda', \rho a + a', \rho b + b']_{H_\mu})$. Then

$$\mathbf{J}(w') = \mu + \mathbf{J}_N(\rho\lambda + \lambda', \rho a + a', \rho b + b') = \mu + \rho^2 \mathbf{J}_N(\lambda, a, b) = \mu$$

because λ', a', b' are all H_μ-fixed but for the same reason

$$(G_w)^{G_\mu} = (G_{w'})^{G_\mu}.$$

Therefore, $w' \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$. But

$$\begin{aligned}
\tau(w') &= \tau(\mathcal{T}_0([g', 0, \rho\lambda + \lambda', \rho a + a', \rho b + b' + \alpha]_{H_\mu}) \\
&= \Psi([g', \rho\lambda + \lambda', (\rho a + a') \diamond_{\mathfrak{l}} (\rho b + b' + \alpha), \rho a + a']_{H_\mu}) \\
&= \Psi([g', \rho\lambda + \lambda', (\rho a) \diamond_{\mathfrak{l}} (\rho b + b' + \alpha), \rho a + a']_{H_\mu}).
\end{aligned}$$

Hence, $(G_{\tau(w')}) = (H_{\rho a + a'}) = (H_{\rho a}) = (H_a) = (G_{\tau(w)})$. Note that $(H_\mu)_{a\diamond_{\mathfrak{l}}(b+\alpha)} \cap (H_\mu)_a = (H_\mu)_{a\diamond_{\mathfrak{l}}(\rho b + b' + \alpha)} \cap (H_\mu)_a$, because if $k \in (H_\mu)_{a\diamond_{\mathfrak{l}}(b+\alpha)} \cap (H_\mu)_a$

$$(a \diamond_{\mathfrak{l}} (b + \alpha)) = k \cdot (a \diamond_{\mathfrak{l}} (b + \alpha)) = a \diamond_{\mathfrak{l}} (k \cdot b + \alpha).$$

But then

$$k \cdot (a \diamond_{\mathfrak{l}} (\rho b + b' + \alpha)) = a \diamond_{\mathfrak{l}} (\rho(k \cdot b) + b' + \alpha) = (a \diamond_{\mathfrak{l}} (\rho b + b' + \alpha)).$$

Hence,

$$\begin{aligned}(g')^{-1}(G_\mu \cap G_{\tau(w')})g' &= (H_\mu)_{\rho\lambda+\lambda'} \cap (H_\mu)_{(\rho a)\diamond_{\mathfrak{l}}(\rho b+b'+\alpha)} \cap (H_\mu)_{\rho a+a'} \\ &= (H_\mu)_\lambda \cap (H_\mu)_{a\diamond_{\mathfrak{l}}(\rho b+b'+\alpha)} \cap (H_\mu)_a \\ &= (H_\mu)_\lambda \cap (H_\mu)_{a\diamond_{\mathfrak{l}}(b+\alpha)} \cap (H_\mu)_a \\ &= g^{-1}(G_\mu \cap G_{\tau(w)})g\end{aligned}$$

and Lemma 6.6.2 implies that $w' \in s_{L\to K}$. □

Using this lemma and the same ideas as in Propositions 6.5.8 and 6.5.9.

Proposition 6.7.2. *The sets of submanifolds*

$$\mathcal{W}_{\mathbf{J}^{-1}(\mu)} = \{G_\mu \cdot Z \mid Z \text{ is a connected component of } s_{H\to H_\mu} \text{ for some } H \subset G\}$$

$$\mathcal{W}_{\mathbf{J}^{-1}(\mu)/G_\mu} = \{Z/G_\mu \mid Z \in \mathcal{W}_{\mathbf{J}^{-1}(\mu)}\}$$

are ***locally trivial Whitney decompositions*** *of* $\mathbf{J}^{-1}(\mu) \cap \bigcup_H s_{H\to H_\mu}$ *and its* G_μ*-quotient respectively.*

Proof. Let $z \in s_{H\to H_\mu} \cap \overline{s_{L\to L_\mu}}$. Analogously as in Proposition 6.5.4, Lemma 6.7.1 at z implies that there is U_z an open G_μ-invariant neighborhood of z and an $\mathbb{R}^+$-invariant semialgebraic subset $X_L \subset \mathbb{R}^k$ with the origin in its closure, such that

$$\begin{aligned} s_{H\to H_\mu} \cap U_z &= \mathcal{T}\left(\left\{[g, v_1]_{H_\mu} \in \mathcal{T}^{-1}(U_z) \mid g \in G_\mu, \quad v_1 \in \mathfrak{o}^{H_\mu} \times S^H \times B^{H_\mu}\right\}\right) \\ s_{L\to L_\mu} \cap U_z &= \mathcal{T}\left(\left\{[g, v_1 + v_2]_{H_\mu} \in \mathcal{T}^{-1}(U_z) \mid g \in G_\mu, \quad v_1 \in \mathfrak{o}^{H_\mu} \times S^H \times B^{H_\mu},\ v_2 \in X_L\right\}\right). \end{aligned}$$

From these characterizations, exactly as in Propositions 6.5.8 and 6.5.9, it follows that $\mathcal{W}_{\mathbf{J}^{-1}(\mu)}$ and $\mathcal{W}_{\mathbf{J}^{-1}(\mu)/G_\mu}$ are locally trivial Whitney decompositions that satisfy the frontier condition.

□

6.7.2 Decomposition of $\tau^{-1}(Q^\mu_{(H)})$

The subset

$$\tau^{-1}(Q^\mu_{(H)}) \cap \mathbf{J}^{-1}(\mu) = \bigcup_{H'\in(H), K\subset G} s_{H'\to K}$$

is a family of pre-seams that satisfies nicer conditions. As in the previous case, we will use a smooth trivialization lemma to prove it.

Lemma 6.7.3. *Let* $z \in s_{H\to K}$ *with* $G_z = K$, $G_{\tau(z)} = H$, *there is a Hamiltonian tube* $\mathcal{T}$ *centered at* z *satisfying the following property:*

If $H' \in (H)$, $[g, 0, \lambda, a, b]_{H_\mu} \in \mathcal{T}^{-1}(s_{H'\to L})$ *and* $\lambda' \in \mathfrak{o}^{H_\mu}$, $a' \in S^H$, $b' \in B^K$, $g' \in G_\mu$, $\rho > 0$ *satisfy* $[g', 0, \rho\lambda + \lambda', \rho a + a', \rho b + b']_{H_\mu} \in G \times_{H_\mu} ((\mathfrak{s}_s^* \oplus \mathfrak{p}_s^*) \times \mathfrak{o}_s \times B_s \times B_s^*)$ *then*

$$\mathcal{T}([g', 0, \rho\lambda + \lambda', \rho a + a', \rho b + b']_{H_\mu}) \in s_{H'\to L}.$$

Proof. We proceed as in the previous Lemma. Denote $q = \tau(z)$, Proposition 6.3.1 gives the Hamiltonian tube $\mathcal{T}_0$ and the map Ψ. In this setting, there is $\alpha \in S^*$ such that $z = \mathcal{T}_0([e,0,0,0,\alpha]_{H_\mu})$ and, as $G_z = H_\mu$, $H_\mu \cdot \alpha = \alpha$. Let $B = (\mathfrak{h}_\mu \cdot \alpha)^\circ$, using Theorem 5.2.7, the map (6.17) is the stated Hamiltonian tube.

Let $w = \mathcal{T}([g,0,\lambda,a,b]_K) \in s_{H' \to L}$, then

$$\tau(w) = \tau(\mathcal{T}_0([g,0,\lambda,\widetilde{a},b+\alpha]_{H_\mu})) = \Psi([g,\lambda,\widetilde{a} \diamond_{\mathfrak{l}} (b+\alpha),\widetilde{a}]_{H_\mu})$$

as $w \in \tau^{-1}(Q_{(H)})$, $\widetilde{a} = a + \Gamma(\frac{1}{2}\lambda \diamond_{\mathfrak{s}} \mathrm{ad}^*_\lambda \mu + a \diamond_{\mathfrak{s}} b; b) \in S^H$, but as in Proposition 6.6.3 this implies that $a \in S^H$ and $\lambda \diamond_s \mathrm{ad}^*_\lambda \mu = 0$. Therefore,

$$\tau(w) = \Psi([g,\lambda,0,a]_{H_\mu}).$$

Let $w' = \mathcal{T}([g',0,\rho\lambda+\lambda',\rho a+a',\rho b+b']_{H_\mu})$, exactly as in Lemma 6.7.1 $w' \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$. As

$$\Gamma\left(\frac{1}{2}(\rho\lambda+\lambda') \diamond_{\mathfrak{s}} \mathrm{ad}^*_{\rho\lambda+\lambda'}\mu + (\rho a + a') \diamond_{\mathfrak{s}} (\rho b + b'); \rho b + b'\right) =$$
$$= \Gamma\left(\frac{1}{2}(\rho\lambda) \diamond_{\mathfrak{s}} \mathrm{ad}^*_{\rho\lambda}\mu + (\rho a) \diamond_{\mathfrak{s}} (\rho b); \rho b + b'\right) = \Gamma\left(0; \rho b + b'\right) = 0$$

but then

$$\tau(w') = \Psi([g',\rho\lambda+\lambda',0,\rho a + a']_{H_\mu}).$$

Comparing the G and G_μ-isotropies of $\tau(w)$ and $\tau(w')$ by Lemma 6.6.2, we can conclude that $w' \in s_{H' \to L}$. ☐

Using this lemma, we can conclude that $\tau^{-1}(Q_{(H)}) \cap \mathbf{J}^{-1}(\mu)$ is a Whitney stratified space

Proposition 6.7.4. *Let H be an isotropy subgroup of Q^μ; the sets of submanifolds*

$$\mathcal{H}_{\mathbf{J}^{-1}(\mu)} = \{G_\mu \cdot Z \mid Z \text{ is a connected component of } s_{H' \to K} \text{ where } K \subset G, \quad H' \in (H)\}$$

$$\mathcal{H}_{\mathbf{J}^{-1}(\mu)/G_\mu} = \{Z/G_\mu \mid Z \in \mathcal{H}_{\mathbf{J}^{-1}(\mu)}\}$$

are locally trivial Whitney decompositions of $\mathbf{J}^{-1}(\mu) \cap \tau^{-1}(Q_{(H)})$ and its G_μ-quotient respectively.

We omit the proof of this result because it is exactly the same as Proposition 6.7.2.

6.7.3 Frontier conditions if $G_\mu = G$

For $\mu = 0$, the seams (Theorem 3.4.2) were introduced in [RO04; PROSD07] and extended to cosphere bundles in [DRRO07]. As at that time the Hamiltonian tube for cotangent bundles was not available they had to rely on mainly topological considerations to show many of the properties of the seams.

Here, using the local description of the pre-seams given by the cotangent Hamiltonian tube, we present an alternative proof of the frontier condition if μ satisfies $G_\mu = G$. The idea will be to prove that

$$(\pi_G \circ \tau) \colon (\mathbf{J}^{-1}(\mu))_{(K)} \longrightarrow \overline{Q_{(K)}}/G \subset Q/G$$

is an open surjective map. We start with the following elementary lemma in Riemmannian geometry.

Lemma 6.7.5. *Let X be a complete connected Riemmannian manifold and let Y be a closed submanifold. We denote by $NY \subset T_Y X$ the normal bundle of Y. The restriction of the exponential map to the normal bundle NY, $\exp_Y : NY \longrightarrow X$ is surjective.*

Proof. Let $x \in X$. Choose $w \in Y$ and let $m = d(x, w)$ be the Riemmannian distance. The subset $E = \{v \in Y \mid d(x, v) \leq m\}$ is compact, thus there is a point $y \in Y$ minimizing the distance from x to any point of Y. Then the minimizing geodesic arc from x to y has tangent vector at y orthogonal to $T_y Y$ in $T_y X$; that is, there is $v \in N_y Y = (T_y Y)^\perp$ such that $\exp_y(v) = x$. This proves that $\exp_Y$ is surjective. □

This lemma was the main tool in [WZ96] that allowed simpler proofs of certain decompositions of reductive Lie groups. Although it seems completely unrelated to our problem, this lemma is the key ingredient of the following technical result.

Lemma 6.7.6. *Let H be a compact Lie group acting linearly on the vector space S. Endow S with an H-invariant inner product. Let $\alpha \in S^* \cong S$ and define $K = H_\alpha$, $B = (\mathfrak{h} \cdot \alpha)^\perp \subset S$. If $v \in S$ satisfies $(G_v) \geq (K)$, then there is $w \in B^K$ and $h \in H$ such that*

$$h \cdot w = v.$$

Proof. Using the proof of Theorem 2.1.4 at $\alpha \in S$ with the slice B implies that there is an open K-invariant neighborhood B_r of B such that

$$\begin{aligned} \mathbf{t} \colon H \times_K B_r &\longrightarrow U \subset S \\ [g, v]_K &\longmapsto g \cdot (v + \alpha) \end{aligned}$$

is a G-equivariant diffeomorphism. Instead of the restriction to B_r we can consider the G-equivariant map

$$\begin{aligned} F \colon H \times_K B &\longrightarrow S \\ [g, v]_K &\longmapsto g \cdot (v + \alpha). \end{aligned}$$

Note that F is not injective, because $F([h, -\alpha]_K) = 0$ for any $h \in H$. However, the previous Lemma implies that F is surjective, because $H \times_K B$ is the normal bundle of the submanifold $H \cdot \alpha$ and F is just the Riemmannian exponential for the euclidean metric.

Additionally,

$$\begin{aligned} (H \cdot \alpha) \cap S^K &= \{h \cdot \alpha \mid h \in H, \quad k \cdot h \cdot \alpha = h \cdot \alpha \quad \forall k \in K\} \\ &= \{h \cdot \alpha \mid h \in H, \quad (h^{-1} k h) \cdot \alpha = \alpha \quad \forall k \in K\} \\ &= \{h \cdot \alpha \mid h \in H, \quad h^{-1} K h \subset K\} \\ &= \{h \cdot \alpha \mid h \in N_H(K)\} = N_H(K) \cdot \alpha. \end{aligned}$$

By Lemma 3.2.3, using that $\mathbf{t}$ is a tube, $S_K \cap U = \mathbf{t}(N_H(K) \times_K B_r)$. But as S_K is an open set of S^K, $T_\alpha(\mathbf{t}(N_H(K) \times_K B_r)) = S^K$, that is

$$S^K = \mathrm{Lie}(N_H(K)) \cdot \alpha + B^K.$$

Due to the definition of B, $\mathrm{Lie}(N_H(K))$ and B^K are orthogonal subspaces, and this implies that the normal space in S^K at α to $N_H(K) \cdot \alpha$ is B^K.

Using again the Lemma, each element in $v' = S^K$ can be expressed as $v' = g \cdot (\alpha + w')$, where $w' \in B^K$. However, if $v \in S$ is such that $(G_v) \geq (K)$, then there is $f \in G$ such that $fG_vf^{-1} \subset K$, but so $f \cdot v \in S^K$ and then the result follows because

$$v = (f^{-1}g) \cdot (\alpha + w')$$

and as $\alpha \in B^K$, $\alpha + w' \in B^K$. □

Once we have the previous surjectivity lemma, the proof of openness of $\pi_G \circ \tau$ is straightforward.

Proposition 6.7.7. *Assume $G_\mu = G$, the map*

$$\begin{aligned} f\colon (\mathbf{J}^{-1}(\mu))_{(K)} &\longrightarrow \overline{Q_{(K)}}/G \subset Q/G \\ z &\longmapsto \pi_G(\tau(z)) \end{aligned}$$

is continuous, open and surjective.

*This implies that, under the assumption $G = G_\mu$, $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)}$ and $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)/G}$ are **decompositions**, they satisfy the **frontier condition**.*

Proof. We now assume that f is not an open map, therefore there must be $z \in (\mathbf{J}^{-1}(\mu))_{(K)}$, $U_z \subset T^*Q$ an open neighborhood of z such that $f(U_z \cap (\mathbf{J}^{-1}(\mu))_{(K)})$ is not an open neighborhood of $\pi_G(\tau(z)) \in Q/G$.

Let $q = \tau(z)$, $H = G_q$ and S be a linear slice at q with an H-invariant metric. Choose a Palais' tube $\mathbf{t}\colon G \times_H S_r \to Q$ around q and let $\alpha = z\big|_S$.

The fact that $f(U_z \cap (\mathbf{J}^{-1}(\mu))_{(K)})$ is not an open neighborhood of $\pi_G(\tau(z)) \in \overline{Q_{(K)}}/G \subset Q/G$ means that there is a sequence $\pi_G(q_n)$ of points in $\overline{Q_{(K)}}/G$ converging to $\pi_G(q)$ such that $\pi_G(q_n) \notin f(U_z \cap (\mathbf{J}^{-1}(\mu))_{(K)})$ for any n.

Note that using $\mathbf{t}$ locally Q/G is S/H, therefore abusing of the notation $\pi_G(q_n) = \pi_H(v_n)$ where v_n is a sequence in S and for each n $(G_{v_n}) \geq (K)$. Using Lemma 6.7.6 at α, for each v_n there is w_n in B^K such that $\pi_H(w_n) = \pi_H(v_n)$. Moreover, as $\pi_H(v_n) \to \pi_G(q)$, w_n must converge to $0 \in B^K$.

In this case, the Hamiltonian tube of Theorem 5.2.7 at z becomes

$$\begin{aligned} \mathcal{T}\colon G \times_K ((\mathfrak{s}^* \oplus \mathfrak{p}^*) \times B_r \times (B^*)_r) &\longrightarrow T^*Q \\ [g, \nu, a, b]_K &\longmapsto \varphi(g, \nu + \mu; a + \Gamma(a \diamond_{\mathfrak{s}} b; b), b + \alpha)) \end{aligned}$$

but if $a \in B_r^K = B_r \cap B^K$, then $\mathcal{T}([e, 0, a, 0]_K) \in (\mathbf{J}^{-1}(\mu))_{(K)}$. However, for all n large enough $w_n \in B_r^K$, and therefore we can consider the points $z_n = \mathcal{T}([e, 0, w_n, 0]_K) \in (\mathbf{J}^{-1}(\mu))_{(K)}$. As $z_n \to z$, there must be N large enough such that $z_n \in U_z \cap (\mathbf{J}^{-1}(\mu))_{(K)}$. This is a contradiction and therefore f is an open map.

From this point the proof is as in Theorem 7 of [PROSD07], because since f is an open map the incidence relations of $\overline{Q_{(K)}}/G$ can be lifted to $(\mathbf{J}^{-1}(\mu))_{(K)}$, and therefore we can conclude that $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)}$ is a decomposition. Then the fact that $\pi_G\colon T^*Q \to (T^*Q)/G$ is a proper map ensures that $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)/G}$ is a decomposition, because if $\pi_G(s_{H\to K}) \cap \overline{\pi_G(s_{L\to M})} \neq \emptyset$ then

$$\emptyset \neq \pi_G(s_{H\to K}) \cap \overline{\pi_G(s_{L\to M})} = \pi_G(s_{H\to K}) \cap \pi_G(\overline{s_{L\to M}}) = \pi_G(s_{H\to K} \cap \overline{s_{L\to M}})$$

and as $s_{H\to K} \subset \overline{s_{L\to M}}$, $\pi_G(s_{H\to K}) \subset \overline{\pi_G(s_{L\to M})}$. Hence, $\mathcal{Z}_{\mathbf{J}^{-1}(\mu)/G}$ is a decomposition.

□

An important remark already used in the work of [RO04] is that $\emptyset \neq s_{H\to K} \cap \overline{s_{H'\to K'}}$ implies both $\emptyset \neq s_{H\to K} \cap \overline{s_{H'\to K}}$ and $\emptyset \neq s_{H'\to K} \cap \overline{s_{H'\to K}}$, and therefore the inclusions $s_{H\to K} \subset \overline{s_{H'\to K}}$ and $s_{H'\to K} \subset \overline{s_{H'\to K'}}$ give the inclusion $s_{H\to K} \subset \overline{s_{H'\to K'}}$. This allows the study of the frontier condition to be split into the study of pairs $(s_{H\to K}, s_{H'\to K})$ and $(s_{H\to K}, s_{H\to K'})$

For $\mu = 0$, using more elaborate arguments, it can be shown that the pairs $(s_{H\to K}, s_{H'\to K})$ satisfy the Whitney condition. The pairs of the form were studied in the previous section $(s_{H\to K}, s_{H'\to K})$ but the problem is that we do not know how to merge these two results to check the Whitney conditions for the general pair $(s_{H\to K}, s_{H'\to K'})$.

6.8 Symplectic geometry

In this section we will study the symplectic properties of the pre-seams $s_{H\to K} \subset \mathbf{J}^{-1}(\mu)$ and its quotients $\mathcal{S}_{H\to K} \subset \mathbf{J}^{-1}(\mu)/G_\mu$. We show that each seam $\mathcal{S}_{H\to K}$ can be endowed with a presymplectic form. In fact, each space $\mathcal{S}_{H\to K}$ can be subimmersed onto a magnetic cotangent bundle $T^*(Q^\mu_{[H]}/G_\mu)$ and this subimmersion is a presymplectic map. Moreover, we show that, as in the orbit-type decomposition, there is a maximal seam which is open and dense.

6.8.1 Mechanical connection

In the regular cotangent bundle reduction (Theorem 1.5.2), the embedding $\mathbf{J}^{-1}(\mu)/G_\mu \to T^*(Q/G_\mu)$ is given in terms of a mechanical connection (1.12), a G_μ-equivariant section $\alpha_\mu \colon Q \to \mathbf{J}^{-1}(\mu) \subset T^*Q$. We start by showing that in the singular setting we have an analogue of the mechanical connection over a piece $Q^\mu_{[H]}$ of Q^μ.

Lemma 6.8.1. *There is a G_μ-equivariant smooth map*

$$\alpha_\mu \colon Q^\mu_{[H]} \longrightarrow T^*Q$$

such that $\tau(\alpha_\mu(q)) = q$ and $\mathbf{J}(\alpha_\mu(q)) = \mu$ for all $q \in Q^\mu_{[H]}$.

Proof. Let $q \in Q^\mu_{[H]}$ and construct a Palais' tube $\mathbf{t} \colon G \times_H S \to U \subset Q$, where $H = G_q$ and S is a linear slice at q. Define

$$\begin{aligned} \alpha_\mu \colon Q^\mu_{[H]} \cap U &\longrightarrow T^*Q \\ \mathbf{t}([g,s]_H) &\longmapsto T^*\mathbf{t}^{-1}(\varphi(g, \mathrm{Ad}^*_g\mu, s, 0)). \end{aligned} \tag{6.22}$$

We need to check that this correspondence is well defined. As $\mathbf{t}([g,s]_H) \in Q^\mu$, $\mathrm{Ad}_g\mathfrak{h} \in \operatorname{Ker}\mu$, and therefore $(g, \mathrm{Ad}^*_g\mu, s, 0)$ has H^T-momentum 0 and lies on the domain of φ. As φ is H^T equivariant for any $h \in H$,

$$\begin{aligned} \alpha_\mu(\mathbf{t}([gh^{-1}, h\cdot s]_H) &= T^*\mathbf{t}^{-1}(\varphi(gh^{-1}, \mathrm{Ad}^*_{gh^{-1}}\mu, h\cdot s, 0)) \\ &= T^*\mathbf{t}^{-1}(\varphi(gh^{-1}, \mathrm{Ad}^*_{h^{-1}}\mathrm{Ad}^*_g\mu, h\cdot s, 0)) \\ &= T^*\mathbf{t}^{-1}(\varphi(g, \mathrm{Ad}^*_g\mu, s, 0)) = \alpha_\mu(\mathbf{t}([g,s]_H) \end{aligned}$$

Also,

$$\mathbf{J}(\alpha_\mu(\mathbf{t}([g,s]_H)) = \mathbf{J}_{G^L}(g, \mathrm{Ad}^*_g\mu, s, 0) = \mathrm{Ad}^*_{g^{-1}}\mathrm{Ad}^*_g\mu = \mu$$

and clearly $\tau(\alpha_\mu(\mathbf{t}([g,s]_H))) = \mathbf{t}([g,s]_H)$. This implies that we have a mechanical connection defined on $U \cap Q^\mu_{[H]}$.

To obtain a connection over the whole $Q^{\mu}_{[H]}$, we can use partitions of the unity. For any point $q \in Q^{\mu}_{[H]}$ there is an open set U_q containing q that is the image of a Palais' tube $G \times_{G_q} S_q$. As $Q^{\mu}_{[H]} \subset Q$ is paracompact, there is a locally finite collection $\{U_{q_i}\}_{i \in I}$ covering $Q^{\mu}_{[H]}$. Note that each U_{q_i} is G_μ-invariant, and therefore using properness of the action there is a partition of the unity ρ_i subordinate to $\{U_{q_i}\}_{i\in I}$ with G_μ-invariant functions ρ_i. If α^i_μ is the section of $Q^{\mu}_{[H]} \cap U_{q_i}$ defined using equation (6.22), the fiberwise sum

$$\alpha_\mu = \sum_i \rho_i \alpha^i_\mu$$

is a mechanical connection over $Q^{\mu}_{[H]}$. As the momentum is linear on the fibers

$$\mathbf{J}(\alpha_\mu(q)) = \mathbf{J}\left(\sum_i \rho_i(q)\alpha^i_\mu(q)\right) = \sum_i \rho_i(q)\mathbf{J}(\alpha^i_\mu(q)) = \sum_i \rho_i(q)\mu = \mu,$$

similarly we can check that the sum is a section and G_μ-equivariant.

□

Remark 6.8.2. Alternatively, we could have defined the mechanical connection using the singular connection in the sense of [PRO09] or, equivalently, principal connections of a $\mathfrak{g}$-manifold in the sense of [AM95]. Without entering into details, if G acts properly on Q and Q is of single orbit-type, then

$$V = \bigcup_{q\in Q} \mathfrak{g}/\mathfrak{g}_q$$

defines a vector bundle over Q. A **singular connection** for this action is a continuous surjective bundle map

$$\mathcal{A}\colon TQ \longrightarrow V$$

covering the identity, being G-equivariant and satisfying $\mathcal{A}(\xi_Q(z)) = [\xi]_q$ for all $q \in Q$, $\xi \in \mathfrak{g}$. It is possible to show that, if $Q = Q_{(L)}$ for some subgroup $L \subset G$, it always exist a singular connection (see [PRO09; AM95] for further details).

If $\mu \in \mathfrak{g}^*$ and we define $Q^\mu = \{q \in Q \mid \mathfrak{g}_q \subset \operatorname{Ker}\mu\}$ then it can be shown that the formula $A_\mu(q) = \langle \mathcal{A}(v_q), \mu\rangle$ defines a continuous linear map $A_\mu\colon T_{Q^\mu}Q \to \mathbb{R}$ that is equivalent to the mechanical connection α_μ that given in the previous lemma.

6.8.2 Single-orbit type Q

Assume that Q has only one orbit-type, that is, $Q = Q_{(L)}$ for some subgroup $L \subset G$. Note that under this assumption, due to Proposition 6.4.1, the partition of $\mathbf{J}^{-1}(\mu)$ into seams coincides with the decomposition of $\mathbf{J}^{-1}(\mu)$ into orbit types of Theorem 3.3.1.

Using the mechanical connection introduced in Lemma 6.8.1, we can state the following generalization of Theorem 1.5.2.

Proposition 6.8.3. *Assume $Q = Q_{(L)}$. Let $z \in \mathbf{J}^{-1}(\mu)$ and define $H = G_{\tau(z)}$. Given $\alpha_\mu\colon Q^\mu_{[H]} \to T^*Q$ a mechanical connection, there is a map F of fiber bundles*

$$\begin{array}{ccc}
(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z})/G_\mu & \xrightarrow{\;F\;} & T^*(Q^\mu_{[H]}/G_\mu) \\
\downarrow & & \downarrow \\
Q^\mu_{[H]}/G_\mu & \xrightarrow{\;\mathrm{Id}\;} & Q^\mu_{[H]}/G_\mu.
\end{array}$$

Assume that $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z})/G_\mu$ is endowed with the reduced symplectic form and on $T^*(Q^\mu_{[H]}/G_\mu)$ we consider the symplectic form $\omega_{Q^\mu_{[H]}/G_\mu} - \beta_\mu$, where β_μ is the unique two-form that satisfies

$$\pi^*_{G_\mu}\beta_\mu = \mathbf{d}\overline{\alpha_\mu}$$

where $\overline{\alpha_\mu} \in \Omega^1(Q^\mu_{[H]})$ is α_μ regarded as a one-form on $Q^\mu_{[H]}$ and $\pi_{G_\mu} \colon Q^\mu_{[H]} \to Q^\mu_{[H]}/G_\mu$ is the quotient projection.

The image of F is a vector subbundle of $T^*(Q^\mu_{[H]}/G_\mu)$ and F is a ***symplectic embedding****.*

Proof. The idea of the proof is similar to the standard proof of Theorem 1.5.2. More precisely, we consider the shifting map

$$\begin{aligned} \mathrm{shift}_\mu \colon G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z} &\longrightarrow T^*(Q^\mu_{[H]}) \\ z &\longmapsto (z - \alpha_\mu(\tau(z)))\big|_{T_{\tau(z)}Q^\mu_{[H]}} \end{aligned}$$

this map is well defined, because if $z \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}$ then $\tau(z) \in Q^\mu_{[H]}$. Note that it is fibered with respect to the cotangent bundle projections and G_μ-equivariant. $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z} \to Q^\mu_{[H]}$ is an affine subbundle of the vector bundle $T^*_{Q^\mu_{[H]}}Q \to Q^\mu_{[H]}$, and therefore the image of shift_μ is a vector subbundle of $T^*(Q^\mu_{[H]}) \to Q^\mu_{[H]}$.

Since the mechanical connection $\alpha_\mu \colon Q^\mu_{[H]} \to T^*Q$ is a section, its image lies in $T^*_{Q^\mu_{[H]}}Q$, composing with the restriction $T^*_{Q^\mu_{[H]}}Q \to T^*(Q^\mu_{[H]})$, α_μ induces $\overline{\alpha_\mu} \colon Q^\mu_{[H]} \to T^*(Q^\mu_{[H]})$, a G_μ-invariant one-form on $Q^\mu_{[H]}$. Using this definition, we can endow $T^*(Q^\mu_{[H]})$ with the symplectic form

$$\omega_{Q^\mu_{[H]}} - \tau^*(\mathbf{d}\overline{\alpha_\mu}) = -\mathbf{d}\left(\theta_{Q^\mu_{[H]}} + \tau^*\overline{\alpha_\mu}\right).$$

Let $(\dot q, \dot p) = v \in T_z(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z})$ then,

$$\begin{aligned} \left\langle \mathrm{shift}^*_\mu \left(\theta_{Q^\mu_{[H]}} + \tau^*\overline{\alpha_\mu}\right), v\right\rangle &= \langle \theta_{Q^\mu_{[H]}} + \tau^*\overline{\alpha_\mu}, T_z(\mathrm{shift}_\mu)\cdot v\rangle \\ &= \langle z - \alpha_\mu(\tau(z)), \dot q\rangle + \langle \overline{\alpha_\mu}(\tau(z)), \dot q\rangle \\ &= \langle z, \dot q\rangle + \langle -\alpha_\mu(\tau(z)) + \overline{\alpha_\mu}(\tau(z)), \dot q\rangle \\ &= \langle z, \dot q\rangle = \langle \theta_Q, v\rangle. \end{aligned}$$

This computation implies that shift_μ preserves the presymplectic potentials, and therefore it is a presymplectic map.

Similarly, as in Theorem 1.5.1, we can define

$$\varphi_{[H]} \colon (\mathfrak{g}_\mu \cdot Q^\mu_{[H]})^\circ \to T^*(Q^\mu_{[H]}/G_\mu)$$

by the formula $\langle \varphi_{[H]}(z), T_q\pi_{G_\mu}v\rangle = \langle z, v\rangle$ for every $z \in T^*_q(Q^\mu_{[H]})$ and $v \in T_q(Q^\mu_{[H]})$ is a G_μ-invariant surjective submersion that induces the diffeomorphism

$$\overline{\varphi_{[H]}} \colon ((\mathfrak{g}_\mu \cdot Q^\mu_{[H]})^\circ)/G_\mu \to T^*(Q^\mu_{[H]}/G_\mu).$$

As $\overline{\alpha_\mu}$ is G_μ-invariant, it drops to the quotient, that is, there is $\beta_\mu \in \Omega^2(Q^\mu_{[H]}/G_\mu)$ such that $\pi^*_{G_\mu}\beta_\mu = \mathbf{d}\overline{\alpha_\mu}$. Moreover, if $i \colon (Q^\mu_{[H]})^\circ \to T^*(Q^\mu_{[H]})$ is the inclusion, a simple computation shows

$$\varphi^*_{[H]}(\omega_{Q^\mu_{[H]}/G_\mu} - \tau^*\beta_\mu) = i^*(\omega_{Q^\mu_{[H]}} - \tau^*\mathbf{d}\overline{\alpha_\mu}).$$

Consider $\xi \in \mathfrak{g}_\mu$ and $z \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}$, define $q = \tau(z)$ then

$$\begin{aligned}\langle \operatorname{shift}_\mu(z), \xi \cdot q\rangle &= \langle z - \alpha_\mu(\tau(z)), \xi \cdot q\rangle = \langle \mathbf{J}(z - \alpha_\mu(\tau(z)), \xi\rangle \\ &= \langle \mathbf{J}(z) - \mathbf{J}(\alpha_\mu(\tau(z))), \xi\rangle = \langle \mu - \mu, \xi\rangle = 0.\end{aligned}$$

This equality shows that the image of shift_μ is contained in $(\mathfrak{g}_\mu \cdot Q^\mu_{[H]})^\circ \subset T^*(Q^\mu_{[H]})$ and we can then form the composition

$$\varphi_{[H]} \circ \operatorname{shift}_\mu \colon G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z} \longrightarrow T^*(Q^\mu_{[H]}/G_\mu).$$

This composition is presymplectic if we consider on $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}/G_\mu$ the restriction of the symplectic form ω_Q and on $T^*(Q^\mu_{[H]}/G_\mu)$ the symplectic form $\omega_{Q^\mu_{[H]}/G_\mu} - \tau^*\beta_\mu$. Moreover, the image of $\varphi_{[H]} \circ \operatorname{shift}_\mu$ is a vector subbundle of $T^*(Q^\mu_{[H]}/G_\mu)$. Finally, as $\varphi_{[H]} \circ \operatorname{shift}_\mu$ is a G_μ-invariant map, it drops to a smooth map

$$F = \overline{\varphi_{[H]} \circ \operatorname{shift}_\mu} \colon G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}/G_\mu \longrightarrow T^*(Q^\mu_{[H]}/G_\mu).$$

F is a symplectic with respect to the stated structures and therefore it must be an embedding. □

Remark 6.8.4. In regular cotangent bundle reduction (Theorem 1.5.2), the (cohomology class of the) magnetic deformation of the symplectic structure can be related with the curvature of the principal bundle $Q \to Q/G_\mu$. In our setting, using singular connections [PRO09], the (cohomology class of) magnetic term β_μ can be related with the curvature of the bundle $Q^\mu_{[H]} \to Q^\mu_{[H]}/G_\mu$.

Recall that, in regular cotangent reduction, the condition $\mathfrak{g} = \mathfrak{g}_\mu$ is equivalent to the fact that the embedding of Theorem 1.5.2 is in fact a symplectomorphism. In our case we can state a similar condition.

Proposition 6.8.5. *Using the same notation of the previous proposition and assuming that $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}/G_\mu$ is connected, the symplectic embedding $F\colon G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}/G_\mu \to T^*(Q^\mu_{[H]}/G_\mu)$ is* ***bijective if and only if***

$$\mathfrak{o}^{H_\mu} = 0$$

where $\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{l} \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{n}$ is an adapted decomposition of (G, H, μ) in the sense of Proposition 4.2.1.

Proof. We know that the image of F is a vector subbundle of $T^*(Q^\mu_{[H]}/G_\mu)$. Therefore, F is going to be bijective if and only if the dimension of $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}/G_\mu$ is equal to the dimension of $T^*(Q^\mu_{[H]}/G_\mu)$. Let $q \in Q_H \subset Q^\mu_{[H]}$; in the proof of Proposition 6.4.2 we saw that locally $Q^\mu_{[H]}$ is G_μ-diffeomorphic to

$$G_\mu \times_{H_\mu} (\mathfrak{o}^{H_\mu} \times S),$$

where S is a linear slice at q. Similarly, using (6.19), $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}/G_\mu$ is locally diffeomorphic to

$$G_\mu \times_{H_\mu} (\mathfrak{o}^{H_\mu} \times S \times S^*),$$

because as we are in the single orbit case $S = S^H = B$. Then,

$$\dim(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z}/G_\mu) = \dim \mathfrak{o}^{H_\mu} + 2 \dim S,$$

and

$$\dim T^*(Q^\mu_{[H]}/G_\mu) = 2 \dim \mathfrak{o}^{H_\mu} + 2 \dim S.$$

Therefore, F is bijective if and only if $\mathfrak{o}^{H_\mu} = \{0\}$. □

Remark 6.8.6. The work of [Mon83] was, up to our knowledge, the first study of symplectic reduction of cotangent bundles in the singular case. In that work the author considered a Lie group G acting on a manifold Q, the cotangent lifted action on T^*Q and a fixed momentum value $\mu \in \mathfrak{g}^*$. However, the author imposed several important restrictions to ensure that all the relevant sets are submanifolds.

In that setting, he showed that the reduced symplectic space $\mathbf{J}^{-1}(\mu)/G_\mu$ is symplectomorphic to a cotangent bundle if and only if

$$\dim \mathfrak{g} - \dim \mathfrak{g}_\mu = 2\left(\dim \mathfrak{g}_q - \dim\left(\mathfrak{g}_q \cap \mathfrak{g}_\mu\right)\right). \tag{6.23}$$

Using the adapted splitting of Proposition 4.2.1, $\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$, as $\mathfrak{l} \cong \mathfrak{n}^*$ the condition (6.23) is satisfied if and only if $\mathfrak{o} = \{0\}$. In general, the equality $\mathfrak{o}^{H_\mu} = 0$ is weaker than (6.23), but using all the hypothesis of [Mon83] both conditions coincide.

6.8.3 Compatible presymplectic structures

Now we study the general case, when Q has more than one orbit-type.

Since $Q_{(H)}$ is a submanifold of Q, there is a natural inclusion of vector bundles $TQ_{(H)} \to T_{Q_{(H)}}Q$, and dually there is a natural projection of vector bundles

$$p_H \colon T^*_{Q_{(H)}}Q \longrightarrow T^*Q_{(H)}.$$

Note that p_H is G-equivariant. As $Q_{(H)}$ is G-invariant the cotangent-lifted action on $T^*Q_{(H)}$ is a Hamiltonian action with momentum map $\mathbf{J}_{(H)} \colon T^*Q_{(H)} \to \mathfrak{g}^*$. If $\mathbf{J} \colon T^*Q \to \mathfrak{g}^*$ is the momentum map for the G-action and $z \in T^*_{Q_{(H)}}Q$, then

$$\mathbf{J}(z) = \mathbf{J}_{(H)}(p_H(z)), \tag{6.24}$$

because if $\xi \in \mathfrak{g}$, since $\xi \cdot \tau(z) \in T_{\tau(z)}Q_{(H)}$,

$$\langle \mathbf{J}(z), \xi \rangle = \langle z, \xi \cdot \tau(z) \rangle = \langle p_H(z), \xi \cdot \tau(z) \rangle = \langle \mathbf{J}_{(H)}(p_H(z)), \xi \rangle.$$

As $\tau(s_{H\to K}) \subset Q_{(H)}$, we can apply p_H to the whole pre-seam $s_{H\to K}$. If $z \in s_{H\to K}$, clearly $\tau(z) \in Q^\mu_{[H]}$, but then as $p_H(s_{H\to K}) \subset T^*Q_{(H)}$ we can apply Proposition 6.4.1 and Proposition 6.4.5 and then $p_H(s_{H\to K}) = G_\mu \cdot (\mathbf{J}^{-1}_{(H)}(\mu))_{H_\mu}$. In other words, the restriction of p_H to the pre-seam $s_{H\to K}$ induces the G_μ-equivariant projection

$$p_{H\to K} \colon s_{H\to K} \longrightarrow G_\mu \cdot (\mathbf{J}^{-1}_{(H)}(\mu))_{H_\mu}.$$

Using a G-invariant metric and the projection map $p_{H\to K}$, we can characterize a pre-seam $s_{H\to K}$ as the Whitney sum of a set in $T^*Q_{(H)}$ and one piece in the conormal space of $Q_{(H)} \subset Q$. Loosely speaking, only the conormal part is related to the isotropy subgroup K, whereas the cotangent part is simply the momentum $\mathbf{J}^{-1}_{(H)}(\mu)$ of $T^*Q_{(H)}$.

Proposition 6.8.7. *If we fix a G-invariant metric on Q, the bundle $T^*_{Q_{(H)}}Q$ can be written as the Whitney sum of the cotangent and conormal part $T^*_{Q_{(H)}}Q_{(H)} \oplus_{Q_{(H)}} N^*Q_{(H)}$. Let $\mathbf{J}_{(H)} \colon T^*Q_{(H)} \to \mathfrak{g}^*$ be the momentum map for the cotangent lifted action of $Q_{(H)}$. Define*

$$\mathcal{N}_{H\to K} = \{z \in N^*Q_{(H)} \mid (G_\mu)_z \in (K)^{G_\mu}\},$$

then

$$\begin{aligned} s_{H\to K} &= G_\mu \cdot (\mathbf{J}_{(H)}(\mu))_{H_\mu} \oplus_{Q^\mu_{[H]}} \mathcal{N}_{H\to K} \\ &= \{z_1 + z_2 \mid z_1 \in G_\mu \cdot (\mathbf{J}_{(H)}(\mu))_{H_\mu}, \quad z_2 \in \mathcal{N}_{H\to K}, \quad \tau(z_1) = \tau(z_2)\}. \end{aligned} \tag{6.25}$$

Before proving this result, note that $\mathcal{N}_{H\to K}$ is not the same as $G_\mu \cdot (N^*Q_{(H)})_K$, $\mathcal{N}_{H\to K}$ is the subset of elements of $N^*Q_{(H)}$ such that the G_μ-isotropy is G_μ-conjugated to K, whereas $G_\mu \cdot (N^*Q_{(H)})_K$ is the set of elements with G-isotropy G_μ-conjugated to K.

Proof. Let $z \in s_{H\to K}$; using the decomposition $T^*_{Q_{(H)}}Q = T^*_{Q_{(H)}}Q_{(H)} \oplus_{Q_{(H)}} N^*Q_{(H)}$, $z = z_1 + z_2$ and using our previous notation $z_1 = p_{H\to K}(z)$, therefore $z_1 \in G_\mu \cdot (\mathbf{J}_{(H)}(\mu))_{H_\mu}$. Moreover, if $q = \tau(z)$, then

$$(K)^{G_\mu} \ni G_z = G_{z_1} \cap G_{z_2} = G_q \cap G_\mu \cap G_{z_2} = G_\mu \cap G_{z_2}$$

because $G_{z_2} \subset G_q$. Note that this equality is equivalent to $z_2 \in \mathcal{N}_{H\to K}$.

Analogously, if $z = z_1 + z_2 \in G_\mu \cdot (\mathbf{J}_{(H)}(\mu))_{H_\mu} \oplus_{Q^\mu_{[H]}} \mathcal{N}_{H\to K}$, let $q = \tau(z_1) = \tau(z_2)$ by Proposition 6.4.5 $q \in Q^\mu_{[H]}$. Using (6.24) $z \in \mathbf{J}^{-1}(\mu)$ and its G-isotropy is

$$G_z = G_{z_1} \cap G_{z_2} = G_q \cap G_\mu \cap G_{z_2} = G_\mu \cap G_{z_2}$$

and as $z_2 \in \mathcal{N}_{H\to K}$, $z \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$, and therefore $z \in s_{H\to K}$, as we wanted to show. □

Note that, in general, $\mathcal{N}_{H\to K}$ is neither an affine nor vector bundle; it is just a fiber bundle. Therefore by (6.25), unlike in the single orbit case, the pre-seams $s_{H\to K} \to Q^\mu_{[H]}$ are **not affine bundles**.

Using p_H, $p_{H\to K}$ and different canonical projections and inclusions, we have the following commutative diagram:

$$\begin{array}{ccccc}
T^*Q_{(H)} & \xleftarrow{\;p_H\;} & T^*_{Q_{(H)}}Q & \xrightarrow{\;i_H\;} & T^*Q \\
{\scriptstyle j}\uparrow & & {\scriptstyle \phi}\uparrow & & \uparrow{\scriptstyle i_{(K)}} \\
G_\mu \cdot (\mathbf{J}^{-1}_{(H)}(\mu))_{H_\mu} & \xleftarrow{\;p_{H\to K}\;} & s_{H\to K} & \xrightarrow{\;i_{H\to K}\;} & G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K \\
{\scriptstyle \pi^H}\downarrow & & {\scriptstyle \pi^{H\to K}}\downarrow & & \downarrow{\scriptstyle \pi^{(K)}} \\
(G_\mu \cdot (\mathbf{J}^{-1}_{(H)}(\mu))_{H_\mu})/G_\mu & \xleftarrow{\;p^{H\to K}\;} & \mathcal{S}_{H\to K} & \xrightarrow{\;i^{H\to K}\;} & (G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu
\end{array}$$

In view of this diagram, on the seam $\mathcal{S}_{H\to K}$ we can consider the two-forms $\Omega_H, \Omega_K \in \Omega^2(\mathcal{S}_{H\to K})$ defined respectively as the only ones such that

$$(\pi^{H\to K})^*\Omega_H = (p_{H\to K})^* j^* \omega_{Q_{(H)}}$$
$$(\pi^{H\to K})^*\Omega_K = (i_{H\to K})^* (i_{(K)})^* \omega_Q.$$

The following result shows that both forms are equal and are presymplectic in the sense that they are closed forms of constant rank.

Proposition 6.8.8. *In the previous notation, the two-forms Ω_H, Ω_K on the seam $\mathcal{S}_{H\to K}$ induced by the projection onto $(G_\mu \cdot (\mathbf{J}^{-1}_{(H)}(\mu))_{H_\mu})/G_\mu$ and by the inclusion on $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu$ respectively are **equal**. Moreover, $\Omega_H = \Omega_K$ is a **presymplectic form**.*

Proof. Working on local coordinates, it is easy to check the equality

$$p_H^* \omega_{Q_{(H)}} = i_H^* \omega_Q,$$

because if $(U, x_1, \ldots, x_n)$ is a coordinate system on Q adapted to $Q_{(H)}$, then $U \cap Q_{(H)}$ is described as $x_{k+1} = \cdots = x_n = 0$. This coordinates induce fibered coordinates $(T^*U, x_1, \ldots x_n, y_1, \ldots, y_n)$. Then the pullback of Liouville's one-form is

$$p_H^* \theta_{Q_{(H)}} = \sum_{i=1}^{k} y_i \mathbf{d}x_i = \sum_{i=1}^{n} (y_i \mathbf{d}x_i)\big|_{T_{Q_{(H)}}Q} = i_H^* \theta_Q$$

and the exterior derivative of this equality gives $p_H^*\omega_{Q(H)} = i_H^*\omega_Q$.

Using this equality and the commutative diagram

$$\begin{aligned}(\pi^{H\to K})^*\Omega_H &= (p_{H\to K})^* j^*\omega_{Q(H)} = \phi^* p_H^*\omega_{Q(H)} = \phi^* i_H^*\omega_Q \\ &= (i_{H\to K})^* i_{(K)}^*\omega_Q = (\pi^{H\to K})^*\Omega_K,\end{aligned}$$

since $\pi_{H\to K}$ is a submersion this implies $\Omega_H = \Omega_K$.

The rank of Ω_K can be determined using (6.19), because $s_{H\to K}$ in coordinates induced by a Hamiltonian tube is the set

$$G_\mu \times_K (\{0\} \times \mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K).$$

Hence, in these coordinates the inclusion $i^{H\to K}$ becomes

$$\mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K \subset \mathfrak{o}^K \times B^K \times (B^*)^K. \tag{6.26}$$

With respect to the symplectic structure of $\mathfrak{o}^K \times B^K \times (B^*)^K$ given by (4.10), the subset $\mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K$ has constant rank. Therefore $\mathcal{S}_{H\to K}$ has a closed two-form of constant rank.

□

Note that if $\mu = 0$, then $\mathfrak{o} = 0$ and so the inclusion $i^{H\to K}$ of (6.26) becomes

$$S^H \times (B^*)^K \subset B^K \times (B^*)^K,$$

a coisotropic embedding. For this reason, in [PROSD07] the pieces of the secondary stratification were **coisotropic submanifolds**. In the general case $\mu \neq 0$, the seams $\mathcal{S}_{H\to K}$ will only be presymplectic; in fact, the inclusion (6.26) can be factored as

$$\mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K \subset \mathfrak{o}^{H_\mu} \times B^K \times (B^*)^K \subset \mathfrak{o}^K \times B^K \times (B^*)^K$$

so that the first inclusion becomes a coisotropic embedding and the second one a symplectic embedding.

Combining this result with Proposition 6.8.3, we can state the following description of $\mathcal{S}_{H\to K}$.

Theorem 6.8.9. *Let $z \in \mathbf{J}^{-1}(\mu)$ and define $K = G_z$, $H = G_{\tau(z)}$. There is a map $\mathcal{F}^{H\to K}$ of fiber bundles*

$$\begin{array}{ccc} \mathcal{S}_{H\to K} & \xrightarrow{\mathcal{F}^{H\to K}} & T^*(Q^\mu_{[H]}/G_\mu) \\ \downarrow & & \downarrow \\ Q^\mu_{[H]}/G_\mu & \xrightarrow{\mathrm{Id}} & Q^\mu_{[H]}/G_\mu. \end{array}$$

*As a map of smooth manifolds, $\mathcal{F}^{H\to K}$ is a fibration; in particular, it is a **subimmersion, a constant rank map**.*

Moreover, using a mechanical connection, we can construct a closed two-form β_μ on $T^*(Q^\mu_{[H]}/G_\mu)$ such that

$$(\mathcal{F}^{H\to K})^*(\omega_{Q^\mu_{[H]}/G_\mu} - \beta_\mu) = \Omega_H = \Omega_K,$$

that is,

$$\mathcal{F}^{H\to K} : (\mathcal{S}_{H\to K}, \Omega_H) \longrightarrow (T^*(Q^\mu_{[H]}/G_\mu), \omega_{Q^\mu_{[H]}/G_\mu} - \beta_\mu)$$

*is a **pre-symplectic map**. In fact, the image of $\mathcal{F}^{H\to K}$ is a symplectic vector subbundle of $(T^*(Q^\mu_{[H]}/G_\mu), \omega_{Q^\mu_{[H]}/G_\mu} - \beta_\mu)$.*

Proof. Let $z' = p^{H\to K}(z)$. Proposition 6.8.3 at $z' \in \mathbf{J}^{-1}_{(H)}(\mu) \subset T^*Q_{(H)}$ gives a map F and a two-form β_μ on $T^*(Q^\mu_{[H]}/G_\mu)$ such that

$$F\colon G_\mu \cdot (\mathbf{J}^{-1}_{(H)}(\mu))_{H_\mu})/G_\mu \longrightarrow T^*(Q^\mu_{[H]}/G_\mu)$$

is a symplectic embedding onto a vector subbundle of $(T^*(Q^\mu_{[H]}/G_\mu), \omega_{Q^\mu_{[H]}} - \beta_\mu)$. Let $\mathcal{F}^{H\to K} = F \circ p^{H\to K}$, using Propositions 6.8.7 and 6.8.8, it is clear that $\mathcal{F}^{H\to K}$ satisfies all the stated properties. □

As the fiber of $p_{H\to K}$ is $\mathcal{N}_{H\to K}$, this theorem implies that $\mathcal{F}^{H\to K}$ is an embedding if and only if $\mathcal{N}_{H\to K}$ is the zero section. Hence, $\mathcal{S}_{H\to K}$ is a symplectic manifold if and only if $\mathcal{N}_{H\to K}$ is the zero section.

6.8.4 Principal piece

Proposition 6.8.8 shows that each space $\mathcal{S}_{H\to K}$ is endowed with a closed non-degenerate two-form of constant rank. Using the cotangent Hamiltonian tube, we can check that the assumption that $\mathcal{S}_{H\to K}$ is symplectic implies the algebraic condition $K = H \cap G_\mu$.

Proposition 6.8.10. *If a seam $\mathcal{S}_{H'\to K}$ is a symplectic subspace of the symplectic manifold $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu$, then there is a subgroup H such that $s_{H'\to K} = s_{H\to H_\mu}$ and $K = H_\mu$. Moreover, there is $q \in Q^\mu$ such that $H = G_q$.*

Proof. Let $z \in s_{H'\to K}$ with $G_z = K$, an element like this always exists, because if $z' \in s_{H'\to K}$ is any element as $G_{z'} \in (K)^{G_\mu}$ there is $g \in G_\mu$ such that $G_{g\cdot z'} = K$ but by G_μ-invariance $g\cdot z' \in s_{H'\to K}$. Define $q = \tau(z)$, $H = G_{\tau(z)}$, $S = (\mathfrak{g}\cdot q)^\perp \subset T_qQ$ a linear slice at q, $\alpha = z\big|_S \in S^*$ and $B = (\mathfrak{h}_\mu\cdot\alpha)^\circ \subset S$. Let $\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ a splitting adapted to (G, H, μ) of Proposition 4.2.1. As in the proof of Proposition 6.6.3, $s_{H\to K}$ is locally equivalent (see (6.19)) to

$$G_\mu \times_K \left(\mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K\right).$$

Therefore, the seam $\mathcal{S}_{H\to K}$ is locally equivalent to

$$\mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K$$

endowed with the restriction of the symplectic form (4.10). Hence, $\mathcal{S}_{H\to K}$ is symplectic if and only if $\dim S^H = \dim(B^*)^K = \dim B^K$. Note that if we use an H-invariant metric on S to identify S^* and S, then $\alpha \in B^K$ but from the equality $S^H = B^K$ it follows that α is H-fixed and thus as $G_z = K = H_\mu \cap H_\alpha = H_\mu$, this implies that $s_{H'\to K} = s_{H\to H_\mu}$.

□

This property of symplectic seams is very important because it gives a very clean description of the isotropy subgroups that appear in the subset $\mathbf{J}^{-1}(\mu) \subset T^*Q$ for any cotangent lifted action.

Corolally 6.8.11. *If G is a Lie group acting properly on a manifold Q and we consider the cotangent-lifted action on T^*Q with momentum map $\mathbf{J}\colon T^*Q \to \mathfrak{g}^*$, then*

$$\left\{G_z \mid z \in \mathbf{J}^{-1}(\mu)\right\} = \{G_q \cap G_\mu \mid q \in Q, \quad \mathrm{Lie}(G_q) \subset \operatorname{Ker}\mu\}.$$

Proof. This result is almost an immediate consequence of the previous proposition. Let $z \in \mathbf{J}^{-1}(\mu)$ and define $K = G_z$. Since the manifold $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z})/G_\mu$ can be written as a locally finite union of submanifolds $\{\mathcal{S}_{H' \to K}\}$ for different H' there must be a subgroup L such that $\mathcal{S}_{L \to K}$ has the same dimension as $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{G_z})/G_\mu$. Therefore, $\mathcal{S}_{L \to K}$ is an open set of the reduced space and, via the inclusion, the reduced symplectic form induces a symplectic form on $\mathcal{S}_{L \to K}$. Using the previous proposition, there is $q \in Q^\mu$ such that $G_z = G_q \cap G_\mu$.

To prove the other inclusion, let $q \in Q$ with $\mathrm{Lie}(G_q) \subset \operatorname{Ker}\mu$. Using a Palais' tube Q around q is $G \times_{G_q} S$, but then using the cotangent reduction map φ the point $\varphi(e, \mu, 0, 0) \in T_q^*Q$ has momentum μ and isotropy $G_q \cap G_\mu$. □

This result can be understood as a generalization of the first part of Theorem 3.4.1 to $\mathbf{J}^{-1}(\mu)$ and without the single orbit assumption. Note that the description of the isotropy subgroups of $\mathbf{J}^{-1}(\mu)$ is simpler than the one given in [RO06].

In fact, in our setting we can prove a stronger result, in any connected component of an orbit-type stratum of the reduced symplectic space there is a **principal seam** which turns out to be symplectic, open and dense. We call it principal in analogy to Theorem 3.2.11. In fact, the idea of the proof is going to be similar to the proof of Theorem 3.2.11 used in [DK00]: using a transversality argument all the seams of codimension greater or equal than two can be avoided and everything reduces to studying neighborhoods of codimension one seams. The following lemma ensures that around each point of a codimension one seam there is only one seam.

Lemma 6.8.12. *Let $\mathcal{S}_{H \to K}$ be a submanifold of codimension one in the reduced space $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu$, if $x \in \mathcal{S}_{H \to K}$ there is an open neighborhood U of x in $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu$ and a subgroup $H' \subset G$ such that*

$$U = (\mathcal{S}_{H \to K} \cup \mathcal{S}_{H' \to K}) \cap U.$$

Proof. Let $x \in \mathcal{S}_{H \to K}$, as in the proof of 6.8.10, there is $z \in s_{H \to K}$ with $\pi_{G_\mu}(z) = x$, $G_z = K$. Moreover, as $s_{G_{\tau(z)} \to K} = s_{H \to K}$, to avoid unnecessary notation, we can assume $H = G_{\tau(z)}$ and define $q = \tau(z)$.

As usual, define $S = (\mathfrak{g} \cdot q)^\perp \subset T_qQ$ a linear slice at q, $\alpha = z\big|_S \in S^*$, $B = (\mathfrak{h}_\mu \cdot \alpha)^\circ \subset S$ and let $\mathfrak{g} = \mathfrak{h}_\mu \oplus \mathfrak{p} \oplus \mathfrak{o} \oplus \mathfrak{l} \oplus \mathfrak{n}$ the splitting of Proposition 4.2.1 adapted to (G, H, μ). The embedding $\mathcal{S}_{H \to K} \subset (G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu$ is locally equivalent (see (6.26)) to

$$\mathfrak{o}^{H_\mu} \times S^H \times (B^*)^K \subset \mathfrak{o}^K \times B^K \times (B^*)^K.$$

The hypothesis that the codimension of $\mathcal{S}_{H \to K}$ is one is equivalent to

$$\dim \mathfrak{o}^{H_\mu} + \dim S^H + 1 = \dim \mathfrak{o}^K + \dim B^K,$$

but this equation implies that $\dim B^K = 1 + \dim S^H$ because $\dim \mathfrak{o}^{H_\mu} + 1 = \dim \mathfrak{o}^K$ is impossible since both $\mathfrak{o}^{H_\mu}$ and $\mathfrak{o}^K$ are symplectic vector spaces and in particular of even dimension. As $S^H \subset B^K$ (see (6.18)), $\dim B^K = 1 + \dim S^H$ implies that there is v such that $B^K = S^H \oplus \mathbb{R} \cdot v$.

Identifying S and S^* through an H-invariant metric, $\alpha \in B^K$, because $H_\alpha \cap H_\mu = K$ and $\langle \alpha, \xi \cdot \alpha \rangle = 0$ for any $\xi \in \mathfrak{h}_\mu$. At this point we have two different options: either $\alpha \in S^H$ or $\alpha \notin S^H$.

- If $\alpha \in S^H$, then $\xi \cdot \alpha = 0$ for any $\xi \in \mathfrak{h}_\mu$, hence $B = S$, $K = H_\mu$. In this case, if $\mathcal{T}([g, 0, \lambda, a, b]_{H_\mu}) \in G_\mu \cdot \mathbf{J}^{-1}(\mu)_K$, then λ, a, b are H_μ-fixed and, using (6.9),

$$\tau(\mathcal{T}([g, 0, \lambda, a, b]_{H_\mu})) = \Psi([g, \lambda, a, a \diamond_{\mathfrak{l}} (b + \alpha)]_{H_\mu}).$$

As $B^K = S^H \oplus \langle v \rangle$, we can decompose $a = a_H + \rho_1 v$ and $b = b_H + \rho_2 v$ with $\rho_i \in \mathbb{R}$. But then

$$a \diamond_{\mathfrak{l}} (b + \alpha) = (a_H + \rho_1 v) \diamond_{\mathfrak{l}} (b_H + \rho_2 v + \alpha) = \rho_1 \rho_2 (v \diamond_{\mathfrak{l}} v) = 0,$$

because a_H, b_H, α are all H-fixed. Then,

$$\tau(\mathcal{T}([g, 0, \lambda, a_H + \rho_1 v, b]_{H_\mu})) = \Psi([g, \lambda, a_H + \rho_1 v, 0]_{H_\mu}).$$

Note that, if $\rho_1 \neq 0$, $\Psi([g, \lambda, a_H + \rho_1 v, 0]_{H_\mu}) \in Q^\mu_{[H_v]}$ and, if $\rho_1 = 0$, $\Psi([g, \lambda, a_H, 0]_{H_\mu}) \in Q^\mu_{[H]}$. Hence

$$U_z = (s_{H \to K} \cup s_{H_v \to K}) \cap U_z.$$

- If $\alpha \notin S^H$ then we can assume that $v = \alpha$. In this case, if $\mathcal{T}([g, 0, \lambda, a, b]_{H_\mu}) \in G_\mu \cdot \mathbf{J}^{-1}(\mu)_K$ and we decompose $a = a_H + \rho_1 \alpha$ and $b = b_H + \rho_2 \alpha$, using (6.9),

$$\tau(\mathcal{T}([g, 0, \lambda, a_H + \rho_1 \alpha, b_H + \rho_2 \alpha]_{H_\mu})) = \Psi([g, \lambda, \widetilde{a}, \widetilde{a} \diamond_{\mathfrak{l}} (b_H + (1 + \rho_2)\alpha)]_{H_\mu})$$

where

$$\begin{aligned} \widetilde{a} &= a_H + \rho_1 \alpha - \Gamma((a_H + \rho_1 \alpha) \diamond_{\mathfrak{s}} (b_H + \rho_2 \alpha); b) \\ &= a_H + \rho_1 \alpha - \Gamma((\rho_1 \alpha) \diamond_{\mathfrak{s}} (\rho_2 \alpha); b) \\ &= a_H + \rho_1 \alpha. \end{aligned}$$

Therefore,

$$\begin{aligned} &\tau(\mathcal{T}([g, 0, \lambda, a_H + \rho_1 \alpha, b_H + \rho_2 \alpha]_{H_\mu})) \\ &\qquad = \Psi([g, \lambda, a_H + \rho_1 \alpha, (a_H + \rho_1 \alpha) \diamond_{\mathfrak{l}} (b_H + (1 + \rho_2)\alpha)]_{H_\mu}) \\ &\qquad = \Psi([g, \lambda, a_H + \rho_1 \alpha, 0]_{H_\mu}). \end{aligned}$$

As in the previous case, if $\rho_1 \neq 0$, $\Psi([g, \lambda, a_H + \rho_1 \alpha, 0]_{H_\mu}) \in Q^\mu_{[H_\alpha]}$ and, if $\rho_1 = 0$, $\Psi([g, \lambda, a_H, 0]_{H_\mu}) \in Q^\mu_{[H]}$; that is,

$$U_z = (s_{H \to K} \cup s_{H_\alpha \to K}) \cap U_z.$$

□

Theorem 6.8.13. *If W is a connected component of $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu$, there is a subgroup $H \subset G$ such that $(\mathcal{S}_{H \to H_\mu}) \cap W$ is **open and dense** in W.*

Proof. Let x_1, x_2 be points in W with $x_1 \in \mathcal{S}_{H_1 \to K}$, $x_2 \in \mathcal{S}_{H_2 \to K}$ and assume that

$$\dim \mathcal{S}_{H_1 \to K} = \dim \mathcal{S}_{H_2 \to K} = \dim (G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)/G_\mu. \tag{6.27}$$

As W is a connected component of a manifold, it is also path-connected and therefore there is a path $\gamma\colon [0,1] \to W$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$.

For each $t \in [0,1]$ there is an open neighborhood $U_t \ni \gamma(t)$ such that U_t intersects finitely many seams. As $\gamma([0,1])$ is compact, there is a finite collection $\{t_i\}$ such that

$$\gamma([0,1]) \subset \bigcup_i U_{t_i}.$$

This implies that $\gamma([0,1])$ intersects a finite number of seams. Using a transversality principle (see for example [GP74]), there is a C^1 path $\widetilde{\gamma}$ in $\bigcup_i U_{t_i}$ connecting x_1 and x_2 that intersects transversally all the seams. In particular, $\widetilde{\gamma}$ does not intersect any seam of codimension greater or equal than two. Moreover, as the intersection with the codimension one seam must be transversal, there is a finite subset $\{r_i\}_{1\leq i\leq N} \subset (0,1)$ such that the points of $\widetilde{\gamma}([0,1])$ that belong to a codimension one seam is exactly $\{\widetilde{\gamma}(r_i)\}_{1\leq i\leq N}$.

Note that the condition of the dimensions (6.27) implies that x_1 is an interior point of $\mathcal{S}_{H_1\to K}$, so $r_1 > 0$, therefore $\widetilde{\gamma}([0,r_1)) \subset \mathcal{S}_{H_1\to K}$. We can apply the previous lemma at the point $\widetilde{\gamma}(r_1)$ and we have an open neighborhood U_1 of $\widetilde{\gamma}(r_1)$. There is $\varepsilon_1 > 0$ such that $r_1 + \varepsilon_1 < r_2$ and both $\widetilde{\gamma}(r_1 - \varepsilon_1)$ and $\widetilde{\gamma}(r_1 + \varepsilon_1)$ lie in U_1. But using the lemma $\widetilde{\gamma}(r_1 + \varepsilon_1) \in \mathcal{S}_{H_1\to K}$ and $\widetilde{\gamma}([0, r_1 + \varepsilon_1) \setminus \{r_1\}) \subset \mathcal{S}_{H_1\to K}$.

Hence, $\widetilde{\gamma}([0,r_2) \setminus \{r_1\}) \subset \mathcal{S}_{H_1\to K}$. Repeating this argument N times we get

$$\widetilde{\gamma}\left([0,1] \setminus \bigcup_{i=1}^{N} \{r_i\}\right) \subset \mathcal{S}_{H_1\to K}.$$

We have checked that $\mathcal{S}_{H_1\to K} = \mathcal{S}_{H_2\to K}$ and this seam is open and dense in W. □

Remark 6.8.14. Note that in (6.16) we showed that among the family $\{\mathcal{S}_{H\to K}\}_{K\subset G}$ the set $s_{H\to H_\mu}$ was non-empty and **minimal**. This last result states that among the family $\{\mathcal{S}_{L\to K}\}_{L\subset G}$ there is a **maximal** open dense set and it is of the form $\mathcal{S}_{H\to H_\mu}$ for some H.

Note that the maximal piece given by this result is analogous to the maximal piece $C_H = \mathcal{S}_{H\to H}$ given by Theorem 8 of [PROSD07] for $\mu = 0$. However, we would like to remark an important difference with the momentum zero case: in general, given H an isotropy group of Q^μ, the seam $\mathcal{S}_{H\to H_\mu}$ does not need to be symplectic or maximal, whereas in the zero momentum case all the pieces $\mathcal{S}_{H\to H}$ were symplectic and maximal in the appropriate space.

Now we can clarify the assertion made in Remark 6.6.1: assume that H, L are subgroups such that $\mathcal{S}_{H\to H_\mu}$ and $\mathcal{S}_{L\to L_\mu}$ are open and dense in $G_\mu\cdot(\mathbf{J}^{-1}(\mu))_{H_\mu}/G_\mu$ and $G_\mu\cdot(\mathbf{J}^{-1}(\mu))_{L_\mu}/G_\mu$ respectively and

$$\mathcal{S}_{H\to H_\mu} \prec \mathcal{S}_{L\to L_\mu}.$$

Then $\mathcal{S}_{H\to L_\mu} \neq \emptyset$ and

$$\mathcal{S}_{H\to H_\mu} \prec \mathcal{S}_{H\to L_\mu} \prec \mathcal{S}_{L\to L_\mu}. \tag{6.28}$$

In this sense, the piece $\mathcal{S}_{H\to L_\mu}$ "stitches" together the symplectic pieces $\mathcal{S}_{H\to H_\mu}$ and $\mathcal{S}_{L\to L_\mu}$.

Nevertheless, even in the singular case we have that the maximal seam of the whole reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ can be embedded onto a symplectic vector subbundle of a magnetic cotangent bundle.

Theorem 6.8.15. *If W is a connected component of $\mathbf{J}^{-1}(\mu)/G_\mu$, there is H a compact subgroup of G such that $(\mathcal{S}_{H\to H_\mu})\cap W$ is open and dense in W. Moreover, $(\mathcal{S}_{H\to H_\mu})\cap W$ can be* ***embedded onto a vector subbundle of the magnetic cotangent bundle*** *$T^*(Q^\mu_{[H]}/G_\mu)$.*

Proof. Using Theorem 3.3.1, there is $K \subset G$ such that $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K/G_\mu$ is open, dense and connected in W. By Theorem 6.8.13, there is $H \subset G$ such that $(\mathcal{S}_{H\to H_\mu}) \cap W$ is open and dense in $(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K/G_\mu) \cap W$. Assume that H belongs to the principal orbit type for the G action on Q. Then, $s_{H\to H_\mu} \subset T^*_{Q_{(H)}}Q = T^*_{Q_{(H)}}Q_{(H)}$; therefore, we can apply the theory of single-orbit type (Proposition 6.8.3) and we have the desired embedding.

If we assume that H does not belong to the principal orbit type (G_{princ}) for the G-action on Q we arrive at a contradiction: from Lemma 6.1.1 it follows that $\exists q \in Q^\mu \cap Q_{(G_{\text{princ}})}$

if $Q^\mu \neq \emptyset$ but q cannot be in the closure of the projection of a pre-seam $s_{H\to H_\mu}$ because $(H) \lneq (G_{\text{princ}})$ this contradicts the density of $s_{H\to H_\mu}$.

□

Using that for a compact connected Lie group acting by cotangent lifts on a cotangent bundle T^*Q, the level sets of the momentum $\mathbf{J}^{-1}(\mu)$ are connected (see [Kno02]), we can show that on each connected component of $Q^\mu \cap Q_{(L)}$ there is one piece $Q^\mu_{[H]}$ which is open, dense and connected.

Proposition 6.8.16. *Let G be a compact connected Lie group acting properly on a manifold Q. Let L be an isotropy subgroup for the G action on Q and assume that $Q_{(L)}$ is connected. There is $H \in (L)$ such that $Q^\mu_{[H]}/G_\mu$ is* ***open dense and connected*** *on $(Q^\mu \cap Q_{(H)})/G_\mu$.*

Proof. Instead of working with T^*Q, we will consider the symplectic manifold $T^*Q_{(L)}$; we denote $\mathbf{J}\colon T^*Q_{(L)} \to \mathfrak{g}^*$ the associated momentum map. As $T^*Q_{(L)}$ is a connected cotangent bundle acted by a compact connected Lie group, $T^*Q_{(L)}$ is a convex Hamiltonian manifold in the sense of [Kno02]; this implies that $\mathbf{J}^{-1}(\mu)$ is a connected set. Therefore, $G_\mu \cdot (\mathbf{J}^{-1}(\mu))/G_\mu$ is also connected. Using last part of Theorem 3.3.1, there is K such that $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K/G_\mu$ is open dense and connected on $G_\mu \cdot (\mathbf{J}^{-1}(\mu))/G_\mu$. Choose $z \in G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$ and define $H = G_{\tau(z)}$, then by Propositions 6.4.1 and 6.4.5 $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K = G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}$ and

$$Q^\mu_{[H]} = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K).$$

Therefore, $Q^\mu_{[H]}/G_\mu$ is connected.

As $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K$ is open in $\mathbf{J}^{-1}(\mu)$ and $G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K = (\tau^{-1} \circ \tau)(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)$ $Q^\mu_{[H]} = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)$ is open. As τ is continuous

$$Q^\mu = \tau(\mathbf{J}^{-1}(\mu)) = \tau(\overline{G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K}) \subset \overline{\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_K)} \subset \overline{Q^\mu_{[H]}}$$

that is, $\overline{Q^\mu_{[H]}} \cap Q^\mu = Q^\mu$.

□

6.9 Examples

In this section we present three different examples that illustrate most of the concepts introduced in this chapter. In the first we consider a general homogeneous space; after stating some general results we present a result of [Mon83] regarding the symplectic reduction of symmetric spaces and see that in our setting this algebraic condition has deep consequences.

The second example is more explicit; we consider the cotangent bundle of the homogeneous space $Q = SU(3)/H$ and describe all the possible reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$. Although the base has only one orbit type, the stratification of $Q^\mu = \tau(\mathbf{J}^{-1}(\mu))$ is not trivial because in $\mathbf{J}^{-1}(\mu)$ there are several orbit-types.

In the third example we consider an $SU(3)$-action on a twisted product and its cotangent lift. This simple example already shows that although Q^μ is a stratified space, it can have some bad topological properties; for example, Q^μ is not locally compact.

6.9.1 Homogeneous spaces

When the base Q is a homogeneous space, that is, $Q = G/H$, it was already noted in [Mon83] that the restriction of the projection $\tau\colon T^*Q \to Q$ to the momentum leaf $\mathbf{J}^{-1}(\mu)$

$$\tau^\mu\colon \mathbf{J}^{-1}(\mu) \longrightarrow Q^\mu$$

is a bijection. With the tools that we have developed we can go even further.

Proposition 6.9.1. *Let $Q = G/H$ be a homogeneous space where G is a Lie group and H is a compact subgroup. Consider $T^*(G/H)$ with the cotangent lifted action, momentum map $\mathbf{J}\colon T^*(G/H) \to \mathfrak{g}$ and projection $\tau\colon T^*(G/H) \to G/H$. Define $Q^\mu = \tau(\mathbf{J}^{-1}(\mu))$; τ can be restricted to*

$$\tau^\mu\colon \mathbf{J}^{-1}(\mu) \longrightarrow Q^\mu,$$

*where $\mathbf{J}^{-1}(\mu)$ and Q^μ are smooth decomposed spaces. With respect to these structures, τ^μ is a G_μ-**equivariant smooth decomposed isomorphism**.*

Proof. Let $q = gH \in Q^\mu$; this implies that $\operatorname{Lie}(G_q) = \operatorname{Lie}(gHg^{-1}) = \operatorname{Ad}_g \mathfrak{h}$ lies in $\operatorname{Ker}\mu$ and there is $z \in T_q^*Q \cap \mathbf{J}^{-1}(\mu)$. Consider $z' \in T_q^*Q \cap \mathbf{J}^{-1}(\mu)$, then $\mathbf{J}(z - z') = 0$, but then for any $\xi \in \mathfrak{g}$

$$\langle z - z', \xi_Q(q)\rangle = 0$$

as any element in T_qQ is of the form $\xi_Q(q)$. This implies $z = z'$, therefore τ^μ is a bijection. τ^μ is smooth because it is the restriction of a smooth map from $T^*Q \to Q$ and it is decomposed due to Proposition 6.6.5. τ^μ will be a diffeomorphism if for each $q \in Q^\mu$ we can find a local smooth section for τ^μ defined on a neighborhood U_q of q.

Using adapted coordinates at q (Proposition 6.3.1), there exist coordinates Ψ and $\mathfrak{T}$ so that we can define a smooth map

$$\begin{aligned} \sigma\colon U_q &\longrightarrow T^*Q \\ \Psi([g, \lambda, \varepsilon]_{H_\mu}) &\longmapsto \mathfrak{T}([g, 0, \lambda]_{H_\mu}). \end{aligned}$$

But then if $q' = \Psi([g, \lambda, \varepsilon]_{H_\mu}) \in Q^\mu$, $\varepsilon = 0$, $\sigma(q') \in \mathbf{J}^{-1}(\mu)$ and $\tau(\sigma(q')) = q'$. Therefore, σ restricted to Q^μ is a smooth section for τ^μ. □

Also in the same work [Mon83], based on [Mis82], remarks that in the case that $G \to G/H$ is a symmetric space the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ is zero-dimensional. More precisely,

Lemma 6.9.2. *Let G be a Lie group, $\mu \in \mathfrak{g}^*$ and assume that G/H is a symmetric space. If $\mathbf{J}\colon T^*(G/H) \to \mathfrak{g}^*$ is the momentum map for the cotangent-lifted action, then Q^μ is a submanifold and for any $q \in Q^\mu$*

$$T_q(Q^\mu) = T_q(G_\mu \cdot q).$$

Proof. Denote by $\pi_H\colon G \to G/H$ the canonical projection. As G/H is symmetric there is a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

Let $q \in Q^\mu$ and consider a path $\gamma(t)$ with $\gamma(0) = q$ such that $\gamma(t) \in Q^\mu$ for all t. Using a principal connection, γ can be lifted to a path $c(t)$ in G such that $\pi_H(c(t)) = \gamma(t)$. Let $\xi = \frac{d}{dt}\big|_{t=0} c(t)(c(0))^{-1} \in \mathfrak{g}$.

As $\gamma(t) \in Q^\mu$, then $\operatorname{Ad}_{c(t)}\mathfrak{h} \subset \operatorname{Ker}\mu$ for all t, and taking the derivative at $t = 0$ of this expression

$$\left[\xi, \operatorname{Ad}_{c(0)}\mathfrak{h}\right] \subset \operatorname{Ker}\mu.$$

We can decompose $\xi = \xi_1 + \xi_2$, where $\xi_1 \in \operatorname{Ad}_{c(0)}\mathfrak{h}$ and $\xi_2 \in \operatorname{Ad}_{c(0)}\mathfrak{m}$. Then $\operatorname{Ad}_{c(0)}\mathfrak{h} \subset \operatorname{Ker}\mu$ implies $[\xi_2, \operatorname{Ad}_{c(0)}\mathfrak{h}] \subset \operatorname{Ker}\mu$. As $[\xi_2, \operatorname{Ad}_{c(0)}\mathfrak{m}] \subset \operatorname{Ad}_{c(0)}\mathfrak{m} \subset \operatorname{Ker}\mu$ then,

$$[\xi_2, \mathfrak{g}] = \left[\xi_2, \operatorname{Ad}_{c(0)}\mathfrak{h} + \operatorname{Ad}_{c(0)}\mathfrak{m}\right] \subset \operatorname{Ker}\mu$$

that is, $\xi_2 \in \mathfrak{g}_\mu$. As $\gamma'(0) = T_{c(0)}\pi_H \cdot T_eR_{c(0)} \cdot \xi = T_{c(0)}\pi_H \cdot T_eR_{c(0)} \cdot \xi_2$. This implies that $\gamma'(0) \in T_{c(0)}(G_\mu \cdot c(0))$. Additionally, $G_\mu \cdot q \subset Q^\mu$, and therefore $T_q(Q^\mu) = T_q(G_\mu \cdot q)$. □

Using the results stated in this chapter, this last Proposition has important consequences for non-homogeneous spaces.

Proposition 6.9.3. • *If G is a Lie group, $\mu \in \mathfrak{g}^*$, H is a subgroup such that* $\mathrm{Lie}(H) \subset \mathrm{Ker}(\mu)$ *and G/H is a symmetric space, then the connected components of $\mathbf{L}(H,\mu)$ containing $e \in G$ is the same as the connected component of $G_\mu \cdot H \subset G$. Moreover, the adapted splitting of Proposition 4.2.1 at (G,H,μ) has $\mathfrak{o} = 0$.*

- *Consider G a compact, connected Lie group acting properly on Q and its cotangent lifted action on T^*Q. Let H be an isotropy subgroup on Q. If $G \to G/H$ is a symmetric space and $Q^\mu \cap Q_H \neq \emptyset$, then*

$$Q^\mu_{[H]} = G_\mu \cdot Q_H = Q^\mu \cap Q_{(H)}.$$

Proof. • If we consider the cotangent-lift of the G-action on $Q = G/H$ and we denote by $\pi_H \colon G \to G/H$ the canonical projection, Proposition 6.4.5 gives $\pi_H(\mathbf{L}(H,\mu)) = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu})$. As $H \in \pi_H(\mathbf{L}(H,\mu))$ by G_μ-invariance

$$G_\mu \cdot H \subset \pi_H(\mathbf{L}(H,\mu)) = \tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{H_\mu}) \subset \tau(\mathbf{J}^{-1}(\mu)).$$

Taking the tangent of this inclusion at $H \in G/H$ and using the previous lemma, it follows that the connected component of $\pi_H(\mathbf{L}(H,\mu))$ through $H \in G/H$ coincides with the connected component of $\pi_H(G_\mu) = G_\mu \cdot H$ through $H \in G/H$. Using π_H^{-1}, it follows that the connected component of $\mathbf{L}(H,\mu))$ through $e \in G$ must be equal to the connected component of $G_\mu \cdot H$ through $e \in G$.

Recall that $\mathfrak{o}$ is symplectomorphic to the normal slice at $\mu \subset \mathcal{O}_\mu$ with respect to the induced H-action, but using commuting reduction as G/H is symmetric, $\mathbf{J}^{-1}(\mu)/G_\mu$ is zero dimensional and this implies that $\mathfrak{o} = 0$.

- Without loss of generality, we can assume that $Q_{(H)}$ is connected. Let $\mathbf{J}_{(H)} \colon T^*Q_{(H)} \to \mathfrak{g}^*$ be the momentum map and $\tau_{(H)} \colon T^*Q_{(H)} \to Q_{(H)}$ be the bundle projection. As in the proof of Proposition 6.8.16, $Q^\mu \cap Q_{(H)} = \tau_{(H)}(\mathbf{J}^{-1}_{(H)}(\mu))$ and as G is compact and connected $\mathbf{J}^{-1}_{(H)}(\mu)$ is connected.

  Clearly $G_\mu \cdot Q_H \subset \tau_{(H)}(\mathbf{J}^{-1}_{(H)}(\mu))$ and $G_\mu \cdot Q_H$ is a closed subset of Q.

  Let $q' \in G_\mu \cdot Q_H$; there is $g \in G_\mu$ such that $q' = g \cdot q$ with $q \in Q_H$. Using a Palais' tube at q, Q is the twisted product $G \times_H S$, and $Q_{(H)} \cong G \times_H S^H$. But if $[k,a]_H$ is in Q^μ and is near enough to $[e,0]_H$, we saw in the proof of the last statement that $k = g'h'$, where $g' \in G_\mu$ and $h' \in H$. Therefore, $G_\mu \cdot Q_H$ is open in Q^μ. As $\tau_{(H)}(\mathbf{J}^{-1}_{(H)}(\mu))$ is connected, we must have $G_\mu \cdot Q_H = \tau_{(H)}(\mathbf{J}^{-1}_{(H)}(\mu))$.

□

This Proposition shows that if H is a subgroup such that G/H is symmetric and $\mu \in \mathfrak{g}^*$, then $Q_{(H)} \cap Q^\mu$ has only one orbit type for the G_μ-action.

6.9.2 $Q = SU(3)/H$

In this section we present an example for which $Q = G/H$ is not a symmetric space. Although the base is single-orbit with respect to the G-action, the set Q^μ will have different orbit types with respect to the G_μ-action.

We consider as Q the quotient of the compact group $SU(3)$ by one maximal torus. We fix the following basis for the Lie algebra $\mathfrak{g} = \mathfrak{su}(3)$

$$\xi_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \xi_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \xi_3 = \frac{\sqrt{2}}{2}\begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \xi_4 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$$

$$\xi_5 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \xi_6 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \quad \xi_7 = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \quad \xi_8 = \frac{\sqrt{6}}{6}\begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -2i \end{bmatrix}$$

and the metric on $\mathfrak{g}$

$$\langle\langle \xi, \eta \rangle\rangle = -\mathrm{Trace}(\xi\eta).$$

This metric is G-invariant metric because it is a multiple of the Killing form. The proposed basis is orthonormal with respect to $\langle\langle \cdot, \cdot \rangle\rangle$; from now on we will use this metric to identify $\mathfrak{g}$ and $\mathfrak{g}^*$.

We denote by $H \subset SU(3)$ the maximal torus corresponding to diagonal matrices, infinitesimally it is generated by the abelian subalgebra $\mathfrak{h} = \langle \xi_3, \xi_8 \rangle_{\mathbb{R}} \subset \mathfrak{g}$. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$; we denote by $\alpha_{12} = \mathrm{diag}\{i, -i, 0\} \in \mathfrak{h}$, $\alpha_{23} = \mathrm{diag}\{0, i, -i\} \in \mathfrak{h}$ and $\alpha_{13} = \mathrm{diag}\{i, 0, -i\} \in \mathfrak{h}$ the three simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

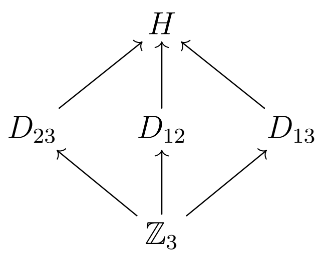
We study the reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ for different values of μ. Note that if $\mu = 0$ then $\mathbf{J}^{-1}(0)$ is the zero section of $T^*(G/H)$, and thus the reduced space $\mathbf{J}^{-1}(0)/G$ is a single point. Therefore, from now on, we assume $\mu \neq 0$.

Note that, using commuting reduction (Theorem 3.3.2), the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ is the G-reduction of $T^*(G/H)$, which can be seen as an H-reduced space. Therefore, as stratified symplectic spaces, $\mathbf{J}^{-1}(\mu)/G_\mu$ must be symplectomorphic to the reduction by the action of H of the coadjoint orbit $\mathcal{O}_\mu \subset \mathfrak{g}^*$.

As a preliminary step, we can describe the different orbit types of 3x3 matrices under the action of H by conjugation: Let $A = (a_{kj})$ be a complex 3x3 matrix and $d = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \in H$, note that $dAd^{-1} = (a_{kj}e^{i(\theta_k - \theta_j)})$. From this expression it is clear that if A satisfies $dAd^{-1} = A$ and A is not diagonal, then either $\theta_1 = \theta_2 = \theta_3$ or A belongs to one of the families

$$\begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{bmatrix}$$

and $\theta_2 = \theta_3$, $\theta_1 = \theta_2$ or $\theta_1 = \theta_3$, respectively. This implies that the lattice of possible isotropy subgroups is



,

where

$$D_{12} = \{\mathrm{diag}(e^{i\theta}, e^{i\theta}, e^{-2i\theta}) \mid \theta \in \mathbb{R}\}$$
$$D_{13} = \{\mathrm{diag}(e^{i\theta}, e^{-2i\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\}$$
$$D_{23} = \{\mathrm{diag}(e^{-2i\theta}, e^{i\theta}, e^{i\theta}) \mid \theta \in \mathbb{R}\}.$$

Note that D_{12}, D_{13}, D_{23} are all isomorphic to S^1.

Let $\mu \in \mathfrak{g}^*$ and denote by $\{i\lambda_1, i\lambda_2, i\lambda_3\}$ its three eigenvalues with $\lambda_i \in \mathbb{R}$ and $\lambda_1+\lambda_2+\lambda_3 = 0$. With this notation, the set of H-fixed points $(\mathcal{O}_\mu)_{(H)}$ contains the diagonal matrix $\mathrm{diag}\{(i\lambda_1, i\lambda_2, i\lambda_3)\}$ and its 6 permutations. The set $(\mathcal{O}_\mu)_{(D_{23})}$ of points with isotropy D_{23} is composed of three different connected components:

$$\begin{bmatrix} i\lambda_1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} i\lambda_2 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \quad \begin{bmatrix} i\lambda_3 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$$

It can be shown that each of these components is diffeomorphic to S^2 because $SU(2)/S^1 \cong S^2$. Similarly, $(\mathcal{O}_\mu)_{(D_{12})}$ and $(\mathcal{O}_\mu)_{(D_{13})}$ are both diffeomorphic to three disjoint S^2.

Let $\Phi\colon \mathcal{O}_\mu \to \mathfrak{h}^*$ be the momentum map for the H-action. Using the Atiyah-Guillemin-Sternberg theorem, the image of Φ is the convex hull of the set of H-fixed points and the set $\Phi^{-1}(0)$ is connected. Using the above description of the H-isotropy, the H-fixed points are exactly the intersection of $\mathcal{O}_\mu$ with $\mathfrak{h}^* \subset \mathfrak{g}^*$ the subspace of diagonal skew-hermitian traceless matrices. In fact, more generally, $\mathcal{O}_\mu \cap \mathfrak{h}^*$ is exactly an orbit of the Weyl group $N_G(H)/H$ (see [BH08] and [Bot79]). Recall that the action of the Weyl group $N_G(H)/H \cong S_3$ on $\mathfrak{h}^*$ is generated by the reflections around the different simple roots.

More graphically, in Figure 6.1 we show $\Phi(\mathcal{O}_\mu)$. The six black dots represent the six diagonal matrices in $\mathcal{O}_\mu$. The three horizontal thick lines are $\Phi((\mathcal{O}_\mu)_{(D_{23})})$; the three lines with slope $\sqrt{3}$ are $\Phi((\mathcal{O}_\mu)_{(D_{12})})$, and the three lines with slope $-\sqrt{3}$ are $\Phi((\mathcal{O}_\mu)_{(D_{13})})$.

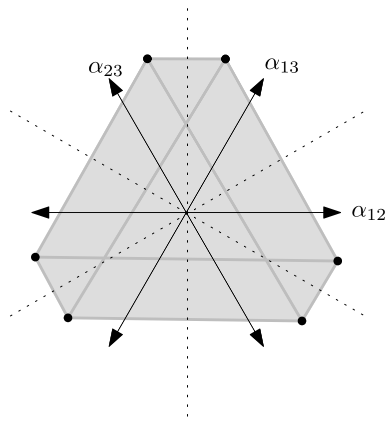


Figure 6.1: The image $\Phi(\mathcal{O}_\mu)$ for μ of generic type.

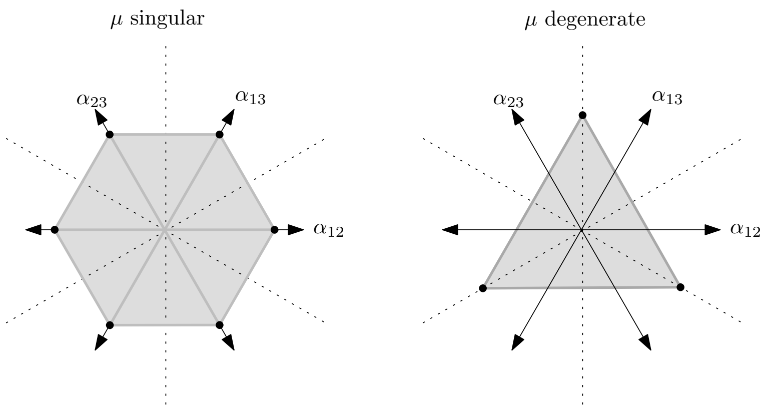
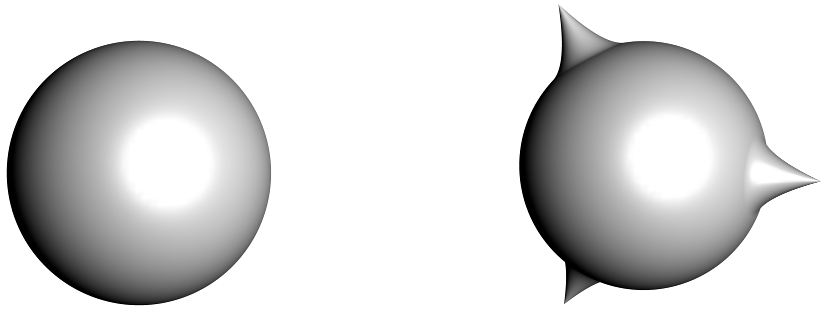


Figure 6.2: The image $\Phi(\mathcal{O}_\mu)$ for μ of singular and degenerate type.



(a) μ generic (b) μ singular

Figure 6.3: The reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ for μ generic or singular.

We can classify $\mu \neq 0$ into three different cases

- **Generic** μ, when $\lambda_1, \lambda_2, \lambda_3$ are all non-zero and different. G_μ is a two dimensional torus, therefore $\mathcal{O}_\mu \cong G/G_\mu$ is a compact connected 6 dimensional manifold.

  The image under Φ of $\mathcal{O}_\mu$ is represented in Figure 6.1. Note that $\Phi^{-1}(0)$ has only one H-isotropy and it is $\mathbb{Z}_3$. As the isotropy subgroup is discrete, Φ is a submersion near $\Phi^{-1}(0)$, therefore $\Phi^{-1}(0)$ is a connected submanifold of dimension 4.

  Hence $\Phi^{-1}(0)/H$ is a compact connected manifold of dimension 2. Using equivariant cohomology or related techniques (see [GM04; Gol99]), it can be shown that in fact $\Phi^{-1}(0)/H$ must be a sphere.

- **Singular** μ, when $\det \mu = 0$. Then μ has eigenvalues $\{i\lambda_1, -i\lambda_1, 0\}$. The image of $\mathcal{O}_\mu$ under Φ is represented in Figure 6.2.

  In this case, $\Phi^{-1}(0)$ contains four different isotropy types $\{D_{12}, D_{23}, D_{13}, \mathbb{Z}_3\}$. Therefore the reduced space $\Phi^{-1}(0)/H$ has four pieces. The open, dense and connected one is $(\Phi^{-1}(0))_{(\mathbb{Z}_3)}/H$ a manifold of dimension 2. $(\Phi^{-1}(0))_{(D_{12})}$ has dimension 1 and is connected, thus $(\Phi^{-1}(0))_{(D_{12})}/D_{12}$ is a single point. The strata corresponding to D_{23} and D_{13} are similar. Therefore, as a stratified space $\Phi^{-1}(0)/H$, contains an open dense stratum of dimension 2 and 3 singular points.

- **Degenerate** μ, when μ has to equal eigenvalues. Then $G_\mu \cong SU(2) \times U(1)$ and it can be shown that $\mathcal{O}_\mu$ is diffeomorphic to $\mathbb{CP}^2$, the complex projective plane (see [BH08]). This case is represented in Figure 6.2.

  $\Phi^{-1}(0)$ contains only points with H-isotropy equal to $\mathbb{Z}_3$, thus $\Phi^{-1}(0)$ is a submanifold of dimension 2 because $\dim \mathcal{O}_\mu = 4$ and Φ is a submersion. As $\Phi^{-1}(0)$ is connected, $\Phi^{-1}(0)/H$ must be a single point.

In the generic case, as the reduced space is a sphere and there is only one stratum, $G_\mu \cdot \mathbf{J}^{-1}(\mu)$ has only one orbit type and is a manifold of dimension 4. Fix μ of generic type and let $z \in \mathbf{J}^{-1}(\mu)$. Define $L = G_{\tau(z)} \in (H)$ then $\mathbf{J}^{-1}(\mu) = G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{\mathbb{Z}_3}$ and $\tau(G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{\mathbb{Z}_3}) = Q^\mu_{[L]} = \mathbf{L}(L, \mu) \cdot Q_L$. Note that $Q_L \cong N_G(H)/H$ a set of 6 different points, as $\mathbf{L}(H, \mu) \cdot Q_H \cong \mathbf{J}^{-1}(\mu)$ and $\mathbf{J}^{-1}(\mu)$ has dimension 4, this implies that $\mathbf{L}(L, \mu)$ has dimension 4. Therefore, in this case it is not possible that $\mathbf{L}(L, \mu) = G_\mu \cdot N_G(H)$, because if so $\mathbf{L}(L, \mu) \cdot Q_H$ would be of dimension 2.

Remark 6.9.4. Note that this case implies that in general the partition

$$Q^\mu = \bigcup_{\mathrm{Lie}(L)\subset \operatorname{Ker}\mu} G_\mu \cdot Q_L$$

cannot be locally finite, because in this example Q^μ is of dimension 4 and each of the sets $G_\mu \cdot Q_L$ is of dimension 2.

As an example of what happens for singular μ, we can choose $\mu = \xi_1$, as $\mathrm{Lie}(H) \in \operatorname{Ker}\mu$ the point $H \in G/H$ lies in Q^μ. In this case, the adapted splitting of Proposition 4.2.1 can be chosen as

$$\mathfrak{h}_\mu = \langle \xi_8 \rangle_\mathbb{R}, \quad \mathfrak{l} = \langle \xi_3 \rangle_\mathbb{R}, \quad \mathfrak{p} = \langle \xi_1 \rangle_\mathbb{R}, \quad \mathfrak{o} = \langle \xi_4, \xi_5, \xi_6, \xi_7 \rangle_\mathbb{R}.$$

If we consider complex coordinates z_1, z_2 in $\mathfrak{o}$ such that $(z_1, z_2) \mapsto \mathrm{Re}(z_1)\xi_4 + \mathrm{Im}(z_1)\xi_5 + \mathrm{Re}(z_2)\xi_6 + \mathrm{Im}(z_2)\xi_7 \in \mathfrak{o}$, then the action of $H_\mu \cong S^1$ on $\mathfrak{o}$ is isomorphic to the S^1-action $e^{i\theta} \cdot (z_1, z_2) = (e^{3i\theta} z_1, e^{-3i\theta} z_2)$. Let $\mathbf{J}_\mathfrak{o} \colon \mathfrak{o} \to \mathbb{R}$ be the momentum map for the H_μ action; in complex coordinates $\mathbf{J}_\mathfrak{o}(z_1, z_2) = |z_1|^3 - |z_2|^3$. $\mathbf{J}_\mathfrak{o}^{-1}(0)$ is topologically a cone of dimension 3, and after a simple calculation $\mathbf{J}_\mathfrak{o}^{-1}(0)/H_\mu$ is isomorphic as a smooth stratified space to the quotient of $\mathbb{C}$ by the natural action of $\mathbb{Z}_3$ by rotations.

Applying Proposition 6.3.1 at $H \in G/H$ and (6.10), locally around $H \in G/H$, $Q^\mu \cong G_\mu \times_{H_\mu} \mathbf{J}_\mathfrak{o}^{-1}(0)$. Recall that for homogeneous spaces (Proposition 6.9.1) $\mathbf{J}^{-1}(\mu)/G_\mu \cong Q^\mu/G_\mu$. This implies that the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ has an orbifold singularity equivalent to the quotient $\mathbb{C}/\mathbb{Z}_3$. Globally, $\mathbf{J}^{-1}(\mu)/G_\mu$ is a sphere with three orbifold singularities (Figure 6.3) and $\mathbf{J}^{-1}(\mu) \subset T^*Q$ is a subset of regular dimension 3 with three singular circles.

6.9.3 Q^μ not locally compact

In this section we present a simple example for which the projection on Q of $\mathbf{J}^{-1}(\mu)$ is not a locally closed subspace of Q; that is, as a topological space Q^μ is not locally compact.

We consider $G = SU(3)$ and $H \cong SU(2)$ the subgroup of $SU(3)$ that has Lie algebra $\langle \xi_1, \xi_2, \xi_3 \rangle_\mathbb{R}$. The vector space $S = \langle \xi_1, \xi_2, \xi_3 \rangle_\mathbb{R}$ has a natural H-action. With respect to this action, we can define the twisted product $Q = G \times_H S$.

Note that the G-action on Q induces the orbit type stratification

$$Q = Q_{(H)} \cup Q_{(S^1)},$$

where $Q_{(S^1)}$ is the principal orbit type and $Q_{(H)} = G \times_H \{0\}$ is the 5-dimensional manifold of isotropy type (H).

Define $\mu = \xi_4$, $q = [e, 0]_H$. As $\mathrm{Lie}(G_q) = H = \langle \xi_1, \xi_2, \xi_3 \rangle_\mathbb{R} \subset \operatorname{Ker}\mu$ then $q \in Q^\mu$ let $z \in \tau^{-1}(q) \cap \mathbf{J}^{-1}(\mu)$. After a simple computation $G_\mu = \exp(\langle \xi_4, \xi_3 - \frac{1}{\sqrt{3}}\xi_8 \rangle_\mathbb{R})$ therefore $H_\mu = \{e\}$ and the adapted splitting is

$$\mathfrak{g} = \underbrace{\langle \xi_1, \xi_2, \xi_3 \rangle_\mathbb{R}}_{\mathfrak{l}} \oplus \underbrace{\left\langle \xi_4, \xi_3 - \frac{1}{\sqrt{3}}\xi_8 \right\rangle_\mathbb{R}}_{\mathfrak{p}} \oplus \mathfrak{n}.$$

Using the map Ψ of Proposition 6.3.1, locally around q, Q is diffeomorphic to

$$G_\mu \times_{\mathrm{Id}} (\mathfrak{l}^* \times S)$$

and using $\mathcal{T}_0$, locally around z, T^*Q is G-symplectomorphic to

$$G \times_{\mathrm{Id}} (S \times S^*).$$

As H_μ is trivial $\mathbf{J}^{-1}_{H_\mu}(0) = S \times S^*$. Then by (6.8) and (6.9),

$$\Psi^{-1}(Q^\mu) = \{(g, a \diamond_{\mathfrak{l}} b, a) \in G_\mu \times \mathfrak{l}^* \times S \mid a, b \in S\}.$$

Note that the $\mathfrak{h}$ action on S is isomorphic to the standard $\mathfrak{so}(3)$ action on $\mathbb{R}^3$, so

$$\{(a \diamond_{\mathfrak{l}} b, a) \mid a, b \in S\} \cong \{(v \times w, v) \mid v, w \in \mathbb{R}^3\} = Y.$$

However, Y is not a locally closed set because $(0,0) \in Y$, $(\rho e_1, 0) \notin Y$ for any $\rho \neq 0$ but $(\rho e_1, e_2\lambda) \in Y$ for any $\lambda \neq 0$; that is, $(\rho e_1, 0) \in \overline{Y}$.

This example shows that, in general, Q^μ is not a locally closed set of Q and similarly for Q^μ/G_μ. Nevertheless, note that Proposition 6.4.2 shows that if Q is of single orbit-type, then Q^μ is locally closed.

Remark 6.9.5. This fact has several non-trivial consequences. First of all it implies that the stratified space Q^μ cannot be a cone space in the usual sense, see [Pfl01], because cone spaces are always locally compact.

Recall that we showed that the partition into pre-seams of $\mathbf{J}^{-1}(0)$ satisfied the frontier condition checking (essentially) the openness of the restriction $\tau^0 \colon \mathbf{J}^{-1}(0) \to Q/G$ (see Proposition 6.7.7). However, using point-set topology it can be shown that the image under an open and continuous map of a locally compact Hausdorff space is again locally compact. This implies that the restriction $\tau^\mu \colon \mathbf{J}^{-1}(\mu) \to Q^\mu/G_\mu$ cannot be open in general. Therefore, if the seams form a decomposition, it cannot be proven using the same ideas as in Proposition 6.7.7.

Chapter 7

Orthogonal actions on $T^*\mathbb{R}^n$ and $T^*(\mathbb{R}^n \times \mathbb{R}^n)$

In this chapter we study the symplectic reduction of the following Hamiltonian spaces

- $O(n)$ acting on $T^*\mathbb{R}^n$ by the cotangent lift of the natural action of $O(n)$ on $\mathbb{R}^n$.
- $O(n)$ acting on $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ by the cotangent lift of the diagonal action of $O(n)$ on $\mathbb{R}^n \times \mathbb{R}^n$.

Using the techniques developed in the last chapter we study the reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ and their partition into seams $\{\mathcal{S}_{H\to K}\}$ for each possible value of μ. The symplectic reduction of $T^*\mathbb{R}^n$ by $O(n)$ was already studied using different techniques in [Mon83]. Although the action is not free, the reduction at $\mu \neq 0$ is quite simple because $\mathbf{J}^{-1}(\mu)$ is a manifold. However, the reduction at $\mu = 0$ requires the introduction of stratifications, but this case could be studied using the techniques of [PROSD07].

The symplectic reduction of $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ for generic μ is again quite simple because $\mathbf{J}^{-1}(\mu)$ and $\tau(\mathbf{J}^{-1}(\mu))$ are submanifolds. Nevertheless, for some special values $\mu \neq 0$ the reduced space is no longer a manifold. In fact, $\mathbf{J}^{-1}(\mu)/G_\mu$ has to be decomposed into four different fibered pieces. This setting exemplifies part of the general behavior described on Chapter 6, while some seams are symplectic and can be embedded into a cotangent bundle, other seams are only presymplectic and the natural map of Theorem 6.8.9 onto a cotangent bundle is just a constant-rank map.

7.1 The orthogonal group $O(n)$

We use the symbol $\mathbf{e}_i \in \mathbb{R}^n$ to represent the vector with a one in the i-th component and zero on the others and we use $\langle v_1, \ldots, v_k\rangle$ to denote the linear span of the set $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$.

The Lie algebra $\mathfrak{o}(n)$ of the orthogonal group $O(n)$ of $\mathbb{R}^n$ is the set of $n\times n$ skew-symmetric matrices, a vector space of dimension $\binom{n}{2}$. We will identify $\mathfrak{o}(n)$ with $\Lambda^2(\mathbb{R}^n)$, the second exterior power of $\mathbb{R}^n$. The metric of $\mathbb{R}^n$ induces a metric on $\Lambda^2(\mathbb{R}^n)$ that can be used to identify $\Lambda^2(\mathbb{R}^n)$ and $\Lambda^2(\mathbb{R}^n)^*$, that is, $\mathfrak{o}(n)$ and $\mathfrak{o}(n)^*$.

After some computations it can be checked that the set

$$\mathfrak{t} = \{\lambda_1\mathbf{e}_1 \wedge \mathbf{e}_2 + \lambda_2\mathbf{e}_3 \wedge \mathbf{e}_4 + \cdots + \lambda_m\mathbf{e}_{2m-1} \wedge \mathbf{e}_{2m} \in \mathfrak{o}(n) \mid \lambda_i \in \mathbb{R}\} \tag{7.1}$$

where $m = \lfloor \frac{n}{2} \rfloor$, is a Cartan subalgebra of $O(n)$ (see [Kna02]) and, as $O(n)$ is compact, for any $\xi \in \mathfrak{o}(n)$ there is $g \in O(n)$ such that $\mathrm{Ad}_g\xi \in \mathfrak{t}$.

Since each element of $\mathfrak{o}(n)$ can be identified with a skew-symmetric endomorphism of $\mathbb{R}^n$ we can assign to each $\xi \in \mathfrak{o}(n)$ its associated rank. Since all the elements in $\mathfrak{t}$ have even rank, the rank of an element $\xi \in \mathfrak{o}(n)$ can only be even. Note that if ξ has rank 2 then it is conjugate to $\lambda \mathbf{e}_1 \wedge \mathbf{e}_2$ for some $\lambda \in \mathbb{R}$. Similarly, if ξ has rank 4, it is conjugate to $\lambda_1 \mathbf{e}_1 \wedge \mathbf{e}_2 + \lambda_2 \mathbf{e}_3 \wedge \mathbf{e}_4$ with $\lambda_i \in \mathbb{R}$. The partition of $\mathfrak{o}(n)$ into subsets of equal rank forms a decomposition (in the sense of Definition 3.1.1) of $\mathfrak{o}(n)$ into semialgebraic sets.

Throughout this chapter if F is a subspace of $\mathbb{R}^n$, $O(F)$ will represent the subgroup of $O(n)$ that fixes $F^\perp$ and by $O(n-r)$ we will mean $O(\langle \mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_r \rangle^\perp)$.

7.2 $O(n)$ action on $T^*\mathbb{R}^n$

Let $Q = \mathbb{R}^n$ endowed with its natural $O(n)$ action and $T^*Q \cong \mathbb{R}^n \times \mathbb{R}^n$ with its cotangent-lift.

Q can be decomposed into two orbit types

$$Q = Q_{O(n)} \sqcup Q_{(O(n-1))}$$

where $Q_{O(n)}$ is just the point $0 \in \mathbb{R}^n$ and $Q_{(O(n-1))}$ is its complement, because if $q \neq 0$ then $G_q = O(\langle q \rangle^\perp)$ is conjugated to $O(n-1)$. If $z = (q,p) \in T^*Q \cong \mathbb{R}^n \times \mathbb{R}^n$ then the conjugacy class of G_z depends only on the dimension of the linear subspace $\langle q, p \rangle$, hence T^*Q decomposes into three orbit-type submanifolds

$$T^*Q = (T^*Q)_{O(n)} \sqcup (T^*Q)_{(O(n-1))} \sqcup (T^*Q)_{(O(n-2))}. \tag{7.2}$$

Using the identification $\mathfrak{o}(n) \cong \mathfrak{o}(n)^*$, the momentum map $\mathbf{J} \colon T^*Q \to \mathfrak{o}(n)^* \cong \mathfrak{o}(n)$ can be written as $\mathbf{J}(q,p) = q \wedge p$. From that expression

$$\mathbf{J}(T^*Q) = \{0\} \sqcup \{\mu \in \mathfrak{o}(n)^* \mid \operatorname{rank}(\mu) = 2\},$$

that is, the image of $\mathbf{J}$ is the subset of $\Lambda^2(\mathbb{R}^n)$ consisting of 0 and the set of decomposable vectors.

We will now study the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ for μ of rank 2 and for $\mu = 0$, but before doing so we introduce a useful map that can be used to the reduced spaces $\mathbf{J}^{-1}(\mu)/G_\mu$ as submanifolds of the set of 2×2 matrices.

Let $Sp(1,\mathbb{R})$ be the group of linear symplectic transformations of $\mathbb{R} \times \mathbb{R}^* = T^*\mathbb{R}$. Note that $Sp(1,\mathbb{R}) \cong SL(2,\mathbb{R})$ and its Lie algebra can be identified with the space of traceless 2×2 matrices. Moreover, using the Killing form we can identify $\mathfrak{sp}(1,\mathbb{R})$ and $\mathfrak{sp}(1,\mathbb{R})^*$. Consider the $O(n)$-invariant smooth map

$$\begin{aligned} \mathbf{K} \colon T^*Q &\longrightarrow \mathfrak{sp}(1,\mathbb{R})^* \\ (q,p) &\longmapsto \begin{bmatrix} q \cdot p & -q \cdot q \\ p \cdot p & -q \cdot p \end{bmatrix}. \end{aligned} \tag{7.3}$$

Fix $\mu \in \mathfrak{o}(n)^*$, then $\mathbf{K}$ induces $\mathbf{K}_\mu \colon \mathbf{J}^{-1}(\mu)/G_\mu \to \mathbf{K}(\mathbf{J}^{-1}(\mu)) \subset \mathfrak{sp}(1,\mathbb{R})^*$. The map $\mathbf{K}_\mu$ satisfies some important properties:

- $\mathbf{K}_\mu$ is an homeomorphism between $\mathbf{J}^{-1}(\mu)/G_\mu$ and the semialgebraic set $\mathbf{K}(\mathbf{J}^{-1}(\mu)) \subset \mathfrak{sp}(1,\mathbb{R})^*$. In fact, $\mathbf{K}_\mu$ is an isomorphism of smooth stratified sets.

- $\mathbf{K}_\mu(\mathbf{J}^{-1}(\mu)/G_\mu)$ is the closure of a single coadjoint orbit in $\mathfrak{sp}(1,\mathbb{R})^*$.

- Let W be a connected component of $G_\mu \cdot \mathbf{J}^{-1}(\mu)_K/G_\mu$ then $\mathbf{K}_\mu(W)$ is a coadjoint orbit and $\mathbf{K}_\mu$ restricts to a symplectomorphism between W and the coadjoint orbit $\mathbf{K}_\mu(W)$.

These properties are the content of Theorem 4.3 in [LMS93]. If we identify $T^*Q = \mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{R}^n \otimes \mathbb{R}^2$ and we endow this space with the $Sp(1,\mathbb{R})$ action $g \cdot (v \otimes w) = v \otimes (gw)$ then $\mathbf{K}$ is the momentum map for this action. The above properties of $\mathbf{K}_\mu$ are due to the fact that the subgroups $(O(n), Sp(1,\mathbb{R}))$ form a dual pair of $Sp(T^*\mathbb{R}^n)$.

7.2.1 μ of rank 2

If μ is of rank 2 then it is conjugated to an element $\lambda \mathbf{e}_1 \wedge \mathbf{e}_2$, $\lambda \in \mathbb{R}$, therefore, without loss of generality we will assume that $\mu = \lambda \mathbf{e}_1 \wedge \mathbf{e}_2$. In this case $G_\mu = SO(2) \times O(n-2)$, the product of orientation-preserving rotations in the $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle$-plane and orthogonal transformations of $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle^\perp \subset \mathbb{R}^n$.

If $(q,p) \in T^*Q$ are such that $q \wedge p = \lambda \mathbf{e}_1 \wedge \mathbf{e}_2$ then $q \neq 0$ and $q \in \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$. Conversely if q satisfies this two conditions we can find p such that $q \wedge p = \mu$, hence

$$Q^\mu = \{q \in Q \mid q \neq 0, q \in \langle \mathbf{e}_1, \mathbf{e}_2 \rangle\}.$$

Note that $Q^\mu \subset Q_{(O(n-1))}$, Q^μ is a connected submanifold of dimension 2 and Q^μ/G_μ is diffeomorphic to $\mathbb{R}$.

Let $q \in Q^\mu$. Then $G_q \cap G_\mu = O(n-2)$. Therefore, using Corollary 6.8.11, all the points in $\mathbf{J}^{-1}(\mu)$ have isotropy $O(n-2)$. Since the set $\mathbf{J}^{-1}(\mu)$ has only one orbit type, by Theorem 3.3.1 $\mathbf{J}^{-1}(\mu)$ is a submanifold of T^*Q with dimension (see (1.3))

$$\dim \mathbf{J}^{-1}(\mu) = \dim T^*Q - \dim \mathfrak{g} \cdot z = 2n - \binom{n}{2} + \binom{n}{4} = 3.$$

Let $z \in \mathbf{J}^{-1}(\mu)$. Since G_μ/G_z has dimension, 1 $\mathbf{J}^{-1}(\mu)/G_\mu$ is a manifold of dimension 2.

Since all the groups G_q with $q \in Q^\mu$ are G_μ-conjugated, by Lemma 6.6.2

$$Q^\mu = Q^\mu_{[O(n-1)]}, \quad \mathbf{J}^{-1}(\mu)/G_\mu = \mathcal{S}_{O(n-1)\to O(n-2)}$$

Note that the map of Proposition 6.8.5 is diffeomorphism of fiber bundles

$$\begin{array}{ccc} \mathcal{S}_{O(n-1)\to O(n-2)} & \xrightarrow{\sim} & T^*(Q^\mu_{[O(n-1)]}/G_\mu) \cong T^*\mathbb{R} \\ \downarrow & & \downarrow \\ Q^\mu_{[O(n-1)]}/G_\mu & \xrightarrow{\sim} & Q^\mu_{[O(n-1)]}/G_\mu \cong \mathbb{R} \end{array}$$

Using $\mathbf{K}_\mu$, since $\mathbf{J}^{-1}(\mu)/G_\mu = \mathcal{S}_{O(n-1)\to O(n-2)}$ we have the isomorphism

$$\mathbf{K}_\mu(\mathcal{S}_{O(n-1)\to O(n-2)}) = Sp(1,\mathbb{R}) \cdot \begin{bmatrix} 0 & -1 \\ a^2 & 0 \end{bmatrix} \subset \mathfrak{sp}(1,\mathbb{R})^*.$$

7.2.2 $\mu = 0$

Note that $\mathbf{J}(q,p) = q \wedge p = 0$ if and only if p and q are parallel. Therefore, in terms of the orbit-type decomposition (7.2) we have the equality

$$\mathbf{J}^{-1}(0) = (T^*Q)_{O(n)} \sqcup (T^*Q)_{O(n-1)}.$$

Since $Q^\mu = Q = Q_{O(n)} \sqcup Q_{(O(n-1))}$, according to Proposition 6.6.4 we have a partition of $\mathbf{J}^{-1}(0)/G$ into three different pieces

$$\mathcal{Z}_{\mathbf{J}^{-1}(0)/G} = \{\mathcal{S}_{O(n-1)\to O(n-1)}, \mathcal{S}_{O(n)\to O(n-1)}, \mathcal{S}_{O(n)\to O(n)}\}.$$

Figure 7.1: In the left plot we have represented the cone $\mathbf{K}(\mathbf{J}^{-1}(0))$. This set can be divided into an open and dense piece, the thick black line $\mathbf{K}_0(\mathcal{S}_{O(n)\to O(n-1)})$, and the vertex of the cone $\mathbf{K}_0(\mathcal{S}_{O(n)\to O(n)})$. In the right plot we have represented $\mathbf{K}(\mathbf{J}^{-1}(\mu))$ for μ of rank 2, which is one sheet of a two-sheeted hyperboloid.

The open and dense piece $\mathcal{S}_{O(n-1)\to O(n-1)}$ is diffeomorphic as a fiber bundle to $T^*\mathbb{R}$. The piece $\mathcal{S}_{O(n)\to O(n-1)}$ is the G-quotient of the pre-seam $\{(0,p) \mid 0 \neq p \in \mathbb{R}^n\}$. Finally $\mathcal{S}_{O(n)\to O(n)}$ is a single point.

Using the map $\mathbf{K}$ we have

$$\begin{aligned}
\mathbf{K}_0(\mathcal{S}_{O(n-1)\to O(n-1)}) &= \left\{ M = g \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} g^{-1} \mid g \in Sp(1,\mathbb{R}), \quad M_{1,2} \neq 0 \right\}, \\
\mathbf{K}_0(\mathcal{S}_{O(n)\to O(n-1)}) &= \left\{ M = g \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} g^{-1} \mid g \in Sp(1,\mathbb{R}), \quad M_{1,2} = 0 \right\}, \\
\mathbf{K}_0(\mathcal{S}_{O(n)\to O(n)}) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \subset \mathfrak{sp}(1,\mathbb{R})^*.
\end{aligned}$$

Since $\mathfrak{sp}(1,\mathbb{R})^*$ is three dimensional, taking coordinates

$$\begin{bmatrix} x & y - z \\ y + z & -x \end{bmatrix} \subset \mathfrak{sp}(1,\mathbb{R})^*$$

we can graphically represent the image of $\mathbf{K}$ for both μ of rank 2 and $\mu = 0$ in Figure 7.1.

7.3 $O(n)$ action on $T^*(\mathbb{R}^n \times \mathbb{R}^n)$

Let $Q = \mathbb{R}^n \times \mathbb{R}^n$ be endowed with the diagonal $O(n)$-action and $T^*Q \cong \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ be endowed with its cotangent-lift.

Let $q = (q_1, q_2) \in Q$. The conjugacy class of the isotropy subgroup G_q depends only on the dimension of $\langle q_1, q_2 \rangle$, therefore Q can be decomposed into three orbit types

$$Q = Q_{O(n)} \sqcup Q_{(O(n-1))} \sqcup Q_{(O(n-2))}. \tag{7.4}$$

Note that $Q_{O(n)}$ is just the point $(0,0)$, $Q_{(O(n-1))}$ is a submanifold of dimension $n+1$ and $Q_{(O(n-2))}$ is an open and dense set. Similarly, the isotropy class of $z = (q_1, q_2, p_1, p_2) \in T^*Q$ depends on the dimension of $\langle q_1, q_2, p_1, p_2 \rangle$ and then T^*Q decomposes into five orbit-type submanifolds

$$T^*Q = (T^*Q)_{O(n)} \sqcup (T^*Q)_{(O(n-1))} \sqcup (T^*Q)_{(O(n-2))} \sqcup (T^*Q)_{(O(n-3))} \sqcup (T^*Q)_{(O(n-4))}.$$

As the momentum map is $\mathbf{J}(q_1, q_2, p_1, p_2) = q_1 \wedge p_1 + q_2 \wedge p_2$ it follows that

$$\mathbf{J}(T^*Q) = \{0\} \sqcup \{\mu \in \mathfrak{o}(n)^* \mid \mathrm{rank}(\mu) = 2\} \sqcup \{\mu \in \mathfrak{o}(n)^* \mid \mathrm{rank}(\mu) = 4\}.$$

Let $Sp(2, \mathbb{R})$ be the group of symplectic linear transformations of $T^*\mathbb{R}^2$. As in (7.3) the map

$$\begin{aligned} \mathbf{K} \colon T^*Q &\longrightarrow \mathfrak{sp}(2, \mathbb{R})^* \\ (q_1, q_2, p_1, p_2) &\longmapsto \begin{bmatrix} q_1 \cdot p_1 & q_1 \cdot p_2 & -q_1 \cdot q_1 & -q_1 \cdot q_2 \\ q_2 \cdot p_1 & q_2 \cdot p_2 & -q_2 \cdot q_1 & -q_2 \cdot q_2 \\ p_1 \cdot p_1 & p_1 \cdot p_2 & -q_1 \cdot p_1 & -q_2 \cdot p_1 \\ p_2 \cdot p_1 & p_2 \cdot p_2 & -q_1 \cdot p_2 & -q_2 \cdot q_2 \end{bmatrix} \end{aligned}$$

induces isomorphisms $\mathbf{K}_\mu$ between $\mathbf{J}^{-1}(\mu)/G_\mu$ and $\mathbf{K}(\mathbf{J}^{-1}(\mu))$. The image of each $\mathbf{K}_\mu$ is the closure of a coadjoint orbit and the appropriate restriction is a symplectomorphism. See [LMS93] for more details.

We will now study the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$ for all possible values of μ depending on its rank.

7.3.1 μ of rank 4

Assume that $\mu \in \mathrm{Im}(\mathbf{J}(T^*Q))$ has rank 4. Without loss of generality we can assume that $\mu = a\mathbf{e}_1 \wedge \mathbf{e}_2 + b\mathbf{e}_3 \wedge \mathbf{e}_4$ with $a, b \neq 0$.

If $\mathbf{J}(q_1, q_2, p_1, p_2) = \mu$, then

$$ab\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 = \mu \wedge \mu = q_1 \wedge p_1 \wedge q_2 \wedge p_2$$

hence $\{q_1, q_2, p_1, p_2\}$ form a basis of the subspace $\langle \mathbf{e}_1, \ldots, \mathbf{e}_4 \rangle$ and satisfy $q_1 \wedge q_2 \wedge \mu = 0$.

Conversely, let $q_1, q_2 \in \langle \mathbf{e}_1, \ldots, \mathbf{e}_4 \rangle$ such that $q_1 \wedge q_2 \neq 0$. After some algebra, there are $x, y \in \langle \mathbf{e}_1, \ldots, \mathbf{e}_4 \rangle$ satisfying

$$q_1 \wedge x + q_2 \wedge y = \mu$$

if and only if $q_1 \wedge q_2 \wedge \mu = 0$. Hence

$$Q^\mu = \{q_1, q_2 \in \langle \mathbf{e}_1, \ldots, \mathbf{e}_4 \rangle \mid q_1 \wedge q_2 \neq 0, \quad q_1 \wedge q_2 \wedge \mu = 0\},$$

and in particular $Q^\mu \subset Q_{(O(n-2))}$. Let $q_i = \sum_{j=1}^4 x_{i,j}\mathbf{e}_j$, then the condition $q_1 \wedge q_2 \wedge \mu = 0$ can be written as

$$b(x_{1,1}x_{2,2} - x_{1,2}x_{2,1}) + a(x_{1,3}x_{2,4} - x_{1,4}x_{2,3}) = 0. \tag{7.5}$$

Therefore, Q^μ is a submanifold of Q of dimension 7.

We now need to split our study into two different cases.

- Assume $a \neq b$. In this case $G_\mu = SO(2) \times SO(2) \times O(n-4)$.

  Let $q = (q_1, q_2) \in Q^\mu$. Then $G_q \cap G_\mu = O(n-4)$ because the condition $q_1 \wedge q_2 \wedge \mu = 0$ forbids the cases $q_1 \wedge q_2 = \lambda \mathbf{e}_1 \wedge \mathbf{e}_2$ and $q_1 \wedge q_2 = \lambda \mathbf{e}_3 \wedge \mathbf{e}_4$. Using Corollary 6.8.11 this implies that $\mathbf{J}^{-1}(\mu)$ has only one G_μ-isotropy type and it is conjugated to $O(n-4)$. Moreover Lemma 6.6.2 implies

  $$Q^\mu = Q^\mu_{[O(\langle \mathbf{e}_1, \mathbf{e}_3 \rangle^\perp)]}, \qquad \mathbf{J}^{-1}(\mu) = s_{O(\langle \mathbf{e}_1, \mathbf{e}_3 \rangle^\perp) \to O(n-4)}.$$

  Let $q \in Q^\mu$, since $G_\mu/(G_q \cap G_\mu)$ has dimension 2, Q^μ/G_μ is a manifold of dimension 5. Since $\dim \mathbf{J}^{-1}(\mu) = 4n - \dim \mathfrak{g} \cdot z = 4n - \binom{n}{2} + \binom{n-4}{2} = 10$, and $\dim G_\mu/O(n-4) = 2$, then $\mathbf{J}^{-1}(\mu)/G_\mu$ is a manifold of dimension 8.

The map of Theorem 6.8.9 becomes a symplectic embedding of the 8-dimensional reduced space $\mathbf{J}^{-1}(\mu)/G_\mu = \mathcal{S}_{O(n-2)\to O(n-4)}$ onto the magnetic cotangent bundle $T^*(Q^\mu/G_\mu)$ of dimension 10.

$$\begin{array}{ccc} \mathcal{S}_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-4)} & \longrightarrow & T^*(Q^\mu_{[O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)]}/G_\mu) \\ \downarrow & & \downarrow \\ Q^\mu_{[O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)]}/G_\mu & \xrightarrow{\text{Id}} & Q^\mu_{[O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)]}/G_\mu \end{array}$$

Alternatively, $\mathbf{J}^{-1}(\mu)/G_\mu = \mathcal{S}_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-4)}$, under the isomorphism $\mathbf{K}_\mu$, can be identified with the 8-dimensional $Sp(2,\mathbb{R})$-coadjoint orbit

$$\mathbf{K}_\mu(\mathcal{S}_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-4)}) = Sp(2,\mathbb{R}) \cdot \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ a^2 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \end{bmatrix} \subset \mathfrak{sp}(2,\mathbb{R})^*.$$

- Assume $a = b$. Consider on $\langle \mathbf{e}_1, \ldots, \mathbf{e}_4\rangle$ the complex structure J defined by

$$J\mathbf{e}_1 = \mathbf{e}_2 \text{ and } J\mathbf{e}_3 = \mathbf{e}_4.$$

Then $G_\mu = U(2) \times O(n-4)$.

Note that $G_q \cap G_\mu = O(n-4)$ unless q_1, Jq_1 and q_2 are not linearly independent. But if $a = b$, (7.5) is equivalent to

$$x_{1,1}x_{2,2} - x_{1,2}x_{2,1} + x_{1,3}x_{2,4} - x_{1,4}x_{2,3} = 0,$$

that is, $q_1 \cdot (Jq_2) = 0$. Therefore if $q_1 \wedge q_2 \neq 0$, q_2 cannot lie in $\langle q_1, Jq_1\rangle$, hence $G_q \cap G_\mu = O(n-4)$.

Again by Corollary 6.8.11 and Lemma 6.6.2

$$Q^\mu = Q^\mu_{[O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)]}, \qquad \mathbf{J}^{-1}(\mu) = s_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-4)}.$$

Let $q \in Q^\mu$. Since G_μ/G_q is conjugated to $U(2)$, it has dimension 4, therefore Q^μ/G_μ has dimension 3. Similarly, as $G_\mu/O(n-4)$ has also dimension 4 and $\mathbf{J}^{-1}(\mu)$ has dimension 10, the reduced space $\mathbf{J}^{-1}(\mu)/G_\mu)$ has dimension 6.

In this case the map of Theorem 6.8.9 becomes a symplectomorphism between the 6 dimensional reduced space $\mathbf{J}^{-1}(\mu)/G_\mu = \mathcal{S}_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-4)}$ and the magnetic cotangent bundle $T^*(Q^\mu/G_\mu)$.

$$\begin{array}{ccc} \mathcal{S}_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-4)} & \xrightarrow{\sim} & T^*(Q^\mu_{[O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)]}/G_\mu) \\ \downarrow & & \downarrow \\ Q^\mu_{[O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)]}/G_\mu & \xrightarrow{\text{Id}} & Q^\mu_{[O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)]}/G_\mu \end{array}$$

Using the isomorphism $\mathbf{K}_\mu$, $\mathcal{S}_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-4)}$ can now be identified with the 6-dimensional $Sp(2,\mathbb{R})$-coadjoint orbit

$$\mathbf{K}_\mu(\mathcal{S}_{O(n-2)\to O(n-4)}) = Sp(2,\mathbb{R}) \cdot \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ a^2 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \end{bmatrix} \subset \mathfrak{sp}(2,\mathbb{R})^*.$$

7.3.2 μ of rank 2

Without loss of generality $\mu = a\mathbf{e}_1 \wedge \mathbf{e}_2$. Therefore $G_\mu = SO(2) \times O(n-2)$.

As for the rank 4 case $\mathbf{J}(q_1, q_2, p_1, p_2) = \mu$ implies $q_1 \wedge q_2 \wedge \mu = 0$, therefore

$$Q^\mu \cap Q_{(O(n-2))} \subset \{(q_1, q_2) \in Q \mid q_1 \wedge q_2 \neq 0, \quad q_1 \wedge q_2 \wedge \mu = 0\}.$$

Conversely, let $(q_1, q_2) \in Q$ such that $q_1 \wedge q_2 \neq 0$ and $q_1 \wedge q_2 \wedge \mu = 0$. This last condition implies that the four vectors $\{q_1, q_2, \mathbf{e}_1, \mathbf{e}_2\}$ span a three dimensional subspace, so we can choose $u \in \mathbb{R}^n$ such that $\{q_1, q_2, u\}$ is a basis of this subspace. Then there are $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$a\mathbf{e}_1 \wedge \mathbf{e}_2 = \lambda_1 q_1 \wedge q_2 + \lambda_2 q_1 \wedge u + \lambda_3 q_2 \wedge u = q_1 \wedge (\lambda_1 q_1 + \lambda_2 u) + q_2 \wedge (\lambda_3 u)$$

hence

$$Q^\mu \cap Q_{(O(n-2))} = \{(q_1, q_2) \in Q \mid q_1 \wedge q_2 \neq 0, \quad q_1 \wedge q_2 \wedge \mu = 0\}.$$

The intersection $Q^\mu \cap Q_{(O(n-1))}$ is simpler because

$$Q^\mu \cap Q_{(O(n-1))} = \{(q_1, q_2) \in Q \mid q_1, q_2 \in \langle \mathbf{e}_1, \mathbf{e}_2 \rangle, \quad (q_1, q_2) \neq (0, 0), \quad q_1 \wedge q_2 = 0\}.$$

Let $q \in Q^\mu \cap Q_{(O(n-1))}$. Then $G_q \cap G_\mu = O(n-2)$. However, if $q_1, q_2 \in Q^\mu \cap Q_{(O(n-2))}$, either $q_1 \wedge q_2 = \mathbf{e}_1 \wedge \mathbf{e}_2$ or $q_1 \wedge q_2 \neq \mathbf{e}_1 \wedge \mathbf{e}_2$. In the former case $G_q \cap G_\mu = O(n-2)$ whereas in the latter $G_q \cap G_\mu \in (O(n-3))^{G_\mu}$.

Using Lemma 6.5.1 we have a partition of Q^μ in three pieces

$$Q^\mu = \underbrace{Q^\mu_{[O(\langle \mathbf{e}_1, \mathbf{e}_3 \rangle^\perp)]} \sqcup Q^\mu_{[O(n-2)]}}_{\subset Q_{(O(n-2))}} \sqcup \underbrace{Q^\mu_{[O(n-1)]}}_{\subset Q_{(O(n-1))}}.$$

Similarly, using Corollary 6.8.11 the set $\mathbf{J}^{-1}(\mu)$ can be decomposed as

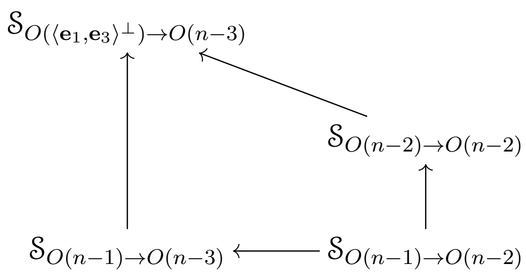
$$\mathbf{J}^{-1}(\mu) = G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{O(n-2)} \sqcup G_\mu \cdot (\mathbf{J}^{-1}(\mu))_{O(n-3)}.$$

With this information Proposition 6.6.4 gives a partition of $\mathbf{J}^{-1}(\mu)/G_\mu$ into six different pieces. However we are now going to check that two of them are empty.

- Let $(q_1, q_2, p_1, p_2) \in s_{O(\langle \mathbf{e}_1, \mathbf{e}_3 \rangle^\perp) \to O(n-2)}$. Then $\langle q_1, q_2, p_1, p_2 \rangle$ has to be two dimensional, since q_1, q_2 are independent we have that $\langle q_1, q_2, p_1, p_2 \rangle = \langle q_1, q_2 \rangle$. But as $\mathbf{J}(z) = \mu$ this implies that $\langle q_1, q_2 \rangle = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$. This is a contradiction with $(q_1, q_2) \in Q^\mu_{[O(\langle \mathbf{e}_1, \mathbf{e}_3 \rangle^\perp)]}$.

- Let $(q_1, q_2, p_1, p_2) \in s_{O(n-2) \to O(n-3)}$. Since $\mu = q_1 \wedge p_1 + q_2 \wedge p_2$ and $\langle q_1, q_2 \rangle = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$, $0 = q_1 \wedge \mu = q_1 \wedge q_2 \wedge p_2$, therefore $p_2 \in \langle q_1, q_2 \rangle$ and analogously $p_1 \in \langle q_1, q_2 \rangle$. Then $(q_1, q_2, p_1, p_2) \notin (T^*Q)_{(O(n-3))}$.

Hence the partition of $\mathbf{J}^{-1}(\mu)/G_\mu$ has four different pieces with the following incidence relations



in this diagram $A \to B$ means that $A \prec B$ and there is no other piece C such that $A \prec C \prec B$. Moreover we have the following equalities

$$
\begin{aligned}
G_\mu \cdot \mathbf{J}^{-1}(\mu)_{O(n-3)} &= s_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-3)} \sqcup s_{O(n-1)\to O(n-3)}\\
G_\mu \cdot \mathbf{J}^{-1}(\mu)_{O(n-2)} &= s_{O(n-2)\to O(n-2)} \sqcup s_{O(n-1)\to O(n-2)}\\
\tau^{-1}(Q^\mu_{[O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)]}) \cap \mathbf{J}^{-1}(\mu) &= s_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-3)}\\
\tau^{-1}(Q^\mu_{[O(n-2)]}) \cap \mathbf{J}^{-1}(\mu) &= s_{O(n-2)\to O(n-2)}\\
\tau^{-1}(Q^\mu_{[O(n-1)]}) \cap \mathbf{J}^{-1}(\mu) &= s_{O(n-1)\to O(n-3)} \sqcup s_{O(n-1)\to O(n-2)}.
\end{aligned}
$$

Using the local description of Proposition 6.3.1 can give more details of each of the pieces.

- $\mathcal{S}_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-3)}$. Using Theorem 6.8.15 this piece is symplectic, open, dense and connected. Fix $z = (\mathbf{e}_1, \mathbf{e}_3, a\mathbf{e}_2, 0) \in s_{O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)\to O(n-3)}$. The dimension of $\mathbf{J}^{-1}(\mu)_{(O(n-3))}$ is equal to $4n - \binom{n}{2} + \binom{n-3}{2} = n+6$ and since G_μ/G_z has dimension $n-2$ this piece is of dimension 8.

  Fix $q = (\mathbf{e}_1, \mathbf{e}_3)$ and let $H = G_q$. We will use Proposition 6.5.4 to model $Q^\mu_{[H]}/G_\mu$ around q. The first step is to compute the Lie algebra splitting (4.3). Since $H = O(\langle \mathbf{e}_1,\mathbf{e}_3\rangle^\perp)$ we have that $H_\mu = O(n-3)$. Then

$$
\begin{aligned}
&\mathfrak{h}_\mu = \langle \mathbf{e}_i \wedge \mathbf{e}_j \mid i \geq 4, \quad j > i\rangle && \mathfrak{p} = \langle \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_3 \wedge \mathbf{e}_4, \ldots, \mathbf{e}_3 \wedge \mathbf{e}_n\rangle\\
&\mathfrak{l} = \langle \mathbf{e}_2 \wedge \mathbf{e}_4, \ldots, \mathbf{e}_2 \wedge \mathbf{e}_n\rangle && \mathfrak{n} = \langle \mathbf{e}_1 \wedge \mathbf{e}_4, \ldots, \mathbf{e}_1 \wedge \mathbf{e}_n\rangle\\
&\mathfrak{o} = \langle \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3\rangle.
\end{aligned}
$$

  Note that $\mathfrak{o}^{H_\mu} = \mathfrak{o} \neq 0$. Therefore Proposition 6.8.5 implies that $\mathcal{S}_{H\to O(n-3)}$ is not diffeomorphic to $T^*(Q^\mu_{[H]}/G_\mu)$.

  The linear slice at q is $S = \langle (\mathbf{e}_1, 0), (0, \mathbf{e}_3), (\mathbf{e}_3, \mathbf{e}_1)\rangle$ and then $S^H = S$. Using Proposition 6.5.4 locally $Q^\mu_{[H]}/G_\mu \cong \mathfrak{o}^{H_\mu} \times S^H$, therefore $Q^\mu_{[H]}/G_\mu$ is a 5 dimensional submanifold of Q.

  The map of Theorem 6.8.9 gives an embedding of the 8 dimensional piece $\mathcal{S}_{H\to O(n-3)}$ onto a symplectic vector subbundle of the 10-dimensional magnetic cotangent bundle $T^*(Q^\mu_{[H]}/G_\mu)$.

- $\mathcal{S}_{O(n-2)\to O(n-2)}$. This piece is open and dense in $\mathbf{J}^{-1}(\mu)_{(O(n-2))}$, therefore it carries a symplectic form. Fix the point $z = (\mathbf{e}_1, \mathbf{e}_2, a\mathbf{e}_2, 0) \in s_{O(n-2)\to O(n-2)}$ that lies on this piece. In this case $H = G_q = O(n-2)$, therefore $H_\mu = H$ and we can choose the (G, H, μ)-adapted splitting

$$
\begin{aligned}
&\mathfrak{h}_\mu = \langle \mathbf{e}_i \wedge \mathbf{e}_j \mid i \geq 3, \quad j > i\rangle && \mathfrak{p} = \langle \mathbf{e}_1 \wedge \mathbf{e}_2\rangle\\
&\mathfrak{l} = 0 && \mathfrak{n} = 0\\
&\mathfrak{o} = \langle \mathbf{e}_1 \wedge \mathbf{e}_3, \ldots, \mathbf{e}_1 \wedge \mathbf{e}_n, \mathbf{e}_2 \wedge \mathbf{e}_3, \ldots, \mathbf{e}_2 \wedge \mathbf{e}_n\rangle
\end{aligned}
$$

  Note that $\mathfrak{o}^{H_\mu} = \mathfrak{o}^H = \{0\}$.

  The linear slice at q is $S = \langle (\mathbf{e}_1, 0), (0, \mathbf{e}_2), (\mathbf{e}_2, \mathbf{e}_1)\rangle$ and clearly $S^H = S$. Using Proposition 6.5.4 $Q^\mu_{[O(n-2)]}/G_\mu$ is a 3 dimensional manifold.

  Moreover, the map of Theorem 6.8.9 gives a symplectomorphism between $\mathcal{S}_{O(n-2)\to O(n-2)}$ and the magnetic cotangent bundle $T^*(Q^\mu_{[O(n-2)]}/G_\mu)$.

- $\mathcal{S}_{O(n-1)\to O(n-3)}$. Fix $z = (\mathbf{e}_1, 0, a\mathbf{e}_2, \mathbf{e}_3) \in s_{O(n-1)\to O(n-1)}$ in this piece. Note that $H = G_q = O(n-1)$ and $H_\mu = O(n-2)$. We can choose the (G, H, μ)-adapted splitting

$$\begin{aligned} \mathfrak{h}_\mu &= \langle \mathbf{e}_i \wedge \mathbf{e}_j \mid i \geq 3, \quad j > i \rangle & \mathfrak{p} &= \langle \mathbf{e}_1 \wedge \mathbf{e}_2 \rangle \\ \mathfrak{l} &= \langle \mathbf{e}_2 \wedge \mathbf{e}_3, \ldots, \mathbf{e}_2 \wedge \mathbf{e}_n \rangle & \mathfrak{n} &= \langle \mathbf{e}_1 \wedge \mathbf{e}_3, \ldots, \mathbf{e}_1 \wedge \mathbf{e}_n \rangle \\ \mathfrak{o} &= 0. \end{aligned} \tag{7.6}$$

  The linear slice at q is $S = \langle (\mathbf{e}_1, 0), (0, \mathbf{e}_1), \ldots, (0, \mathbf{e}_n) \rangle$ and $S^H = \langle (\mathbf{e}_1, 0), (0, \mathbf{e}_1) \rangle$. Using Proposition 5.2.1, $\alpha = z\big|_S = (0, \mathbf{e}_3)$. Then

$$B = (\mathfrak{h}_\mu \cdot \alpha)^\perp \cap S = \langle (\mathbf{e}_1, 0), (0, \mathbf{e}_1), (0, \mathbf{e}_2), (0, \mathbf{e}_3) \rangle$$

  and $B^{O(n-3)} = B$. Hence, using (6.26), $\mathcal{S}_{O(n-1)\to O(n-3)}$ is a coisotropic submanifold of dimension 6 of the symplectic manifold $\mathbf{J}^{-1}(\mu)_{(O(n-3))}$. The seam $\mathcal{S}_{O(n-1)\to O(n-3)}$ fibers over the 2-dimensional manifold $Q^\mu_{[O(n-1)]}/G_\mu$.

  In this case the map $\mathcal{S}_{O(n-1)\to O(n-3)} \to T^*(Q^\mu_{[O(n-1)]}/G_\mu)$ of Theorem 6.8.9 is onto and has two dimensional fibers.

- $\mathcal{S}_{O(n-1)\to O(n-2)}$. Fix $z = (\mathbf{e}_1, 0, a\mathbf{e}_2, 0)$. Since $H = G_q = O(n-1)$ we can the use adapted splitting described in (7.6). Similarly, the linear slice is $S = \langle (\mathbf{e}_1, 0), (0, \mathbf{e}_1), \ldots, (0, \mathbf{e}_n) \rangle$. The difference is that in this case $\alpha = 0$.

  Since $S^{H_\mu} = \langle (\mathbf{e}_1, 0), (0, \mathbf{e}_1), (0, \mathbf{e}_2) \rangle$ then $\mathcal{S}_{O(n-1)\to O(n-2)}$ is a coisotropic submanifold of dimension 5 of the symplectic manifold $\mathbf{J}^{-1}(\mu)_{(O(n-2))}$. The piece $\mathcal{S}_{O(n-1)\to O(n-2)}$ fibers over the 2-dimensional manifold $Q^\mu_{[O(n-1)]}/G_\mu$.

  The map $\mathcal{S}_{O(n-1)\to O(n-2)} \to T^*(Q^\mu_{[O(n-1)]}/G_\mu)$ of Theorem 6.8.9 in onto but has one dimensional fibers.

Using the $\mathbf{K}_\mu$ isomorphism

$$\mathbf{K}_\mu(G_\mu \cdot \mathbf{J}^{-1}(\mu))_{O(n-3)}/G_\mu) = Sp(2, \mathbb{R}) \cdot \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \subset \mathfrak{sp}(2, \mathbb{R})^*$$

which is a coadjoint orbit that can be decomposed as the image of the seams

$$\begin{aligned} \mathbf{K}_\mu(\mathcal{S}_{O(\langle \mathbf{e}_1, \mathbf{e}_3 \rangle^\perp)\to O(n-3)}) &= \{M \in \mathbf{K}_\mu(G_\mu \cdot \mathbf{J}^{-1}(\mu))_{O(n-3)}/G_\mu) \mid \operatorname{rank}(M_{1..2,3..4}) = 2\} \\ \mathbf{K}_\mu(\mathcal{S}_{O(n-1)\to O(n-3)}) &= \{M \in \mathbf{K}_\mu(G_\mu \cdot \mathbf{J}^{-1}(\mu))_{O(n-3)}/G_\mu) \mid \operatorname{rank}(M_{1..2,3..4}) = 1\}. \end{aligned}$$

The closure of this coadjoint orbit contains the set

$$\mathbf{K}_\mu(G_\mu \cdot \mathbf{J}^{-1}(\mu))_{O(n-2)}/G_\mu) = Sp(2, \mathbb{R}) \cdot \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \subset \mathfrak{sp}(2, \mathbb{R})^*$$

that can be decomposed as

$$\begin{aligned} \mathbf{K}_\mu(\mathcal{S}_{O(n-2)\to O(n-2)}) &= \{M \in \mathbf{K}_\mu(G_\mu \cdot \mathbf{J}^{-1}(\mu))_{O(n-2)}/G_\mu) \mid \operatorname{rank}(M_{1..2,3..4}) = 2\} \\ \mathbf{K}_\mu(\mathcal{S}_{O(n-1)\to O(n-2)}) &= \{M \in \mathbf{K}_\mu(G_\mu \cdot \mathbf{J}^{-1}(\mu))_{O(n-2)}/G_\mu) \mid \operatorname{rank}(M_{1..2,3..4}) = 1\}. \end{aligned}$$

7.3.3 $\mu = 0$

Note that in this case since $\mu = 0$ we have $Q^\mu = Q$. Since Q has three orbit types (see (7.4)), Corollary 6.8.11 states that there are only three orbit types in $\mathbf{J}^{-1}(0)$. Hence we have a partition of $\mathbf{J}^{-1}(0)$ into 6 different pieces with the following incidence relations

$$\begin{array}{ccccc}
\mathcal{S}_{O(n-2)\to O(n-2)} & & & & \\
\uparrow & & & & \\
\mathcal{S}_{O(n-1)\to O(n-2)} & \longleftarrow & \mathcal{S}_{O(n-1)\to O(n-1)} & & \\
\uparrow & & \uparrow & & \\
\mathcal{S}_{O(n)\to O(n-2)} & \longleftarrow & \mathcal{S}_{O(n)\to O(n-1)} & \longleftarrow & \mathcal{S}_{O(n)\to O(n)}.
\end{array}$$

Since for $\mu = 0$ the mechanical connection is just the zero section and the condition $\mathfrak{o}^{H_\mu} = 0$ of Proposition 6.8.5 is always satisfied we have the symplectomorphisms $\mathcal{S}_{O(n-2)\to O(n-2)} \cong T^*(Q_{(O(n-2))}/G)$, $\mathcal{S}_{O(n-1)\to O(n-1)} \cong T^*(Q_{(O(n-1))}/G)$ and $\mathcal{S}_{O(n)\to O(n)}$ is a single point. The remaining three seams are coisotropic submanifolds.

Using the $\mathbf{K}_0$ isomorphism we have

$$\mathbf{K}_0(\mathbf{J}^{-1}(0)_{(O(n-2))}/G) = Sp(2,\mathbb{R}) \cdot \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \subset \mathfrak{sp}(2,\mathbb{R})^*$$

Note that this coadjoint orbit can be partitioned into the images by $\mathbf{K}_0$ of $\mathcal{S}_{O(n-2)\to O(n-2)}$, $\mathcal{S}_{O(n-1)\to O(n-2)}$, $\mathcal{S}_{O(n)\to O(n-2)}$ according to the rank (2, 1 or 0) of the upper-right 2×2-matrix.

Analogously,

$$\mathbf{K}_0(\mathbf{J}^{-1}(0)_{(O(n-1))}/G) = Sp(2,\mathbb{R}) \cdot \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \subset \mathfrak{sp}(2,\mathbb{R})^*,$$

and this coadjoint orbit can be partitioned into the images by $\mathbf{K}_0$ of $\mathcal{S}_{O(n-1)\to O(n-1)}$ and $\mathcal{S}_{O(n)\to O(n-1)}$ according to the rank (1 or 0) of the upper-right 2×2-matrix.

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