Transference theory between quasi-Banach function spaces with applications to the restriction of Fourier multipliers.

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CERTIFICA: Que la present memòria ha estat realitzada sota la seva direcció per Salvador Rodríguez López i que constitueix la tesi d'aquest per a aspirar al grau de Doctor en Matemàtiques.

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(*María Jesús Carro Rossell*)

Per a Itziar

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Contents

Chapter 1

Introduction

In the early 1970's, R. R. Coifman and G. Weiss developed an abstract method of transferring convolution-type operators and their bounds, from general locally compact groups to abstract measure spaces (see [44–47]), extending the techniques introduced by A. Calderón in [30]. To be more specific, let G be a locally compact group that, for simplicity, in this section is assumed to denote $\mathbb{R}$ or $\mathbb{T}$. Let $1 \leq p < \infty$, let B_K be a convolution operator on $L^p(G)$ given by

$$B_K(\phi)(v) = \int_G K(u)\phi(v-u)du,$$

for $\phi \in L^p(G)$ with $K \in L^1(G)$ and let $(\mathcal{M}, \mu)$ be a measure space. The *transferred operator* T_K is defined for $f \in L^p(\mathcal{M})$ by

$$T_K f(x) = \int_G K(u) R_{-u} f(x)\, du, \tag{1.0.1}$$

where R is a *continuous representation* of G on $L^p(\mathcal{M})$. That is,

$$R : G \to \mathfrak{B}(L^p(\mathcal{M}))$$

maps G continuously into the class of bounded linear operators on $L^p(\mathcal{M})$ endowed with the strong operator topology satisfying

$$R_{u+v} = R_u R_v, \quad R_e = Id,$$

for every $u, v \in G$, where e and Id are the identity element of G and $\mathfrak{B}(L^p(\mathcal{M}))$, respectively. The most basic example of representation arises from the action of G into itself by translations, defined by $R_u f(v) = f(v+u)$ which leads to see the transferred operator as a generalization of convolution.

Coifman and Weiss studied the interplay, focusing on the preservation of L^p inequalities, between convolutions and transferred operators. More precisely, assuming that $\sup_{u \in G} \|R_u\|_{\mathfrak{B}(L^p(\mathcal{M}))} =: c < \infty$, their transference main result asserts:

Theorem 1.0.2. [47, Theorem 3.4] *The operator T_K maps $L^p(\mathcal{M})$ into itself and is bounded with operator norm not exceeding $c^2 N(K)$, where $N(K)$ denotes the* *operator norm of B_K on $L^p(G)$.*

Clearly, the norm of T_K is not greater than $c\,\|K\|_{L^1(G)}$. The essential feature of the previous result is that its operator norm is dominated by $c^2 N(K)$ because, in many cases, $N(K)$ is much smaller than $\|K\|_{L^1}$.

Many interesting operators are of the form (1.0.1), and hence transference theory allows to reduce the question of their boundedness to the corresponding problem for a convolution. For instance, the Ergodic Hilbert Transform operator introduced by M. Cotlar in [48] (see also [30]) is essentially defined by

$$\mathcal{H}f(x) = \int_{\mathbb{R}} f(T_{-t}x)\frac{dt}{t},$$

where $\{T_t\}_{t\in\mathbb{R}^n}$ is a one-parameter group of measure-preserving transformations on the measure space $\mathcal{M}$. Then, M. Riesz theorem for the Hilbert Transform, that is precisely the convolution operator of kernel $K(t) = 1/\pi t$, implies the boundedness of $\mathcal{H}$ on $L^p(\mathcal{M})$:

Let us recall that a function $\mathbf{m}$ is a *Fourier multiplier* for $L^p(\mathbb{R})$ if for K satisfying $\widehat{K} = \mathbf{m}$, B_K is a bounded operator on $L^p(\mathbb{R})$, where $\widehat{\ }$ denotes the Fourier transform. A completely analogous definition is given replacing $\mathbb{R}$ by $\mathbb{T}$. Namely, a sequence $\{\mathbf{m}(n)\}_{n\in\mathbb{Z}}$ is a *Fourier multiplier* for $L^p(\mathbb{T})$ if the convolution operator B_K is bounded on $L^p(\mathbb{T})$, where K satisfies that $\widehat{K}(j) = \mathbf{m}(j)$ for $j \in \mathbb{Z}$.

As a first application of transference, Coifman and Weiss recovered the classical theorem of K. De Leeuw [52] on restriction of multipliers, that essentially asserts that if $\mathbf{m}$ is a Fourier multiplier on $L^p(\mathbb{R})$ then, $\mathbf{m}|_{\mathbb{Z}}$, the restriction of $\mathbf{m}$ to $\mathbb{Z}$, is a Fourier multiplier for $L^p(\mathbb{T})$ with norm not exceeding the norm of $\mathbf{m}$ as a Fourier multiplier for $L^p(\mathbb{R})$.

We shall sketch the proof for a particular case to illustrate how transference is well adapted to these type of problems. To this end, let us consider the Hilbert transform, whose associated multiplier is given by

$$\widehat{(1/\pi t)}(x) = \mathbf{m}(x) = -i \,\mathrm{sgn}\, x.$$

If we take $G = \mathbb{R}$, $\mathcal{M} = \mathbb{T}$ and $R : \mathbb{R} \to \mathfrak{B}\left(L^p(\mathbb{T})\right)$ given by $R_t f(x) = f(x+t)$ for a 1-periodic function f, then

$$T_K f(x) = \int_0^1 \left(\sum_{j\in\mathbb{Z}} K(t+j)\right) f(x-t)\,dt = (P_{\mathbb{Z}}K) * f(x), \tag{1.0.3}$$

where $P_{\mathbb{Z}}K$ is the 1-periodization of K. That is,

$$P_{\mathbb{Z}}K(t) = \sum_{j\in\mathbb{Z}} \frac{1}{\pi(t+j)} = \cot \pi t,$$

and by the Poisson Summation Formula, the Fourier coefficients of $P_{\mathbb{Z}}K$ are given by the sequence

$$\mathbf{m}|_{\mathbb{Z}} = \{-i\,\mathrm{sgn}\, j\}_{j\in\mathbb{Z}}.$$

In other words, T_K is the classical *conjugate function operator*

$$\widetilde{f}(x) = \int_0^1 f(x-t) \cot \pi t \, dt,$$

and, in this way, M. Riesz inequality

$$\left\| \widetilde{f} \right\|_{L^p(\mathbb{T})} \leq N_p \left\| f \right\|_{L^p(\mathbb{T})},$$

for $f \in L^p(\mathbb{T})$, follows from Theorem 1.0.2.

In this monograph we handle multipliers for other spaces than L^p, like Lorentz-Zygmund spaces $L^{p,q}(\log L)^a$ and weighted Lebesgue spaces, studying the validity of restriction results of the previous type, as an application of our new transference techniques.

Transference theory has become a powerful and versatile tool in various areas of Analysis like Ergodic Theory, Operator Theory and Harmonic Analysis (see for instance [10, 17, 43, 50, 51])and many authors have contributed to its development, like N. Asmar, E. Berskon and A. Gillespie [1–17]. The theory has been extended to cover weak (p, p) type convolution operators, maximal operators (see [2, 7, 12, 16]) and convolution operators on potential-type spaces as Hardy H^p spaces and Sobolev spaces $W_{p,k}$ (see [40]).

Recent advances on the resolution on Calderón's conjecture on the bilinear Hilbert Transform (see for instance [80]) have motivated the study of transference techniques for multilinear operators on L^p spaces [20, 26] and [64]. It is this framework that the need to study transference techniques for quasi-Banach L^p spaces first appeared, since the bilinear Hilbert Transform is bounded for indices $p < 1$.

However, in a broad sense, in all the previously studied cases there is the restriction that the index p is the same both in the domain and in the range space. But the question whether the transferred operator is bounded for more general classes of spaces other than L^p naturally arises.

Before going on, we shall mention some motivating examples coming from interpolation and extrapolation techniques. If B_K is bounded on $L^p(G)$, by duality it is also bounded on $L^{p'}(G)$. Thus T_K also is bounded on $L^p(\mathcal{M})$ and $L^{p'}(\mathcal{M})$. Hence, both operators are bounded on each intermediate interpolation space between L^p and $L^{p'}$. For instance, by Marcinkiewicz interpolation theorem (see [98, V.3.15] or generalized versions as [53, Theorem 3.5.15]), it follows that both operators are bounded on the intermediate Lorentz-Zygmund space $L^{q,r}(\log L)^a$ and, in particular, in the Lorentz space $L^{q,r}$. *Can we then prove that if B_K is bounded in an intermediate space like $L^{q,r}$, also is T_K?*

On the other hand, if B_K is bounded on $L^p(G)$ for any $1 < p \leq 2$ with norm growing as $(p-1)^{-\alpha}$, for some $\alpha > 0$, the associated transferred operator T_K also satisfies the same bounds on $L^p(\mathcal{M})$, and hence by extrapolation (see

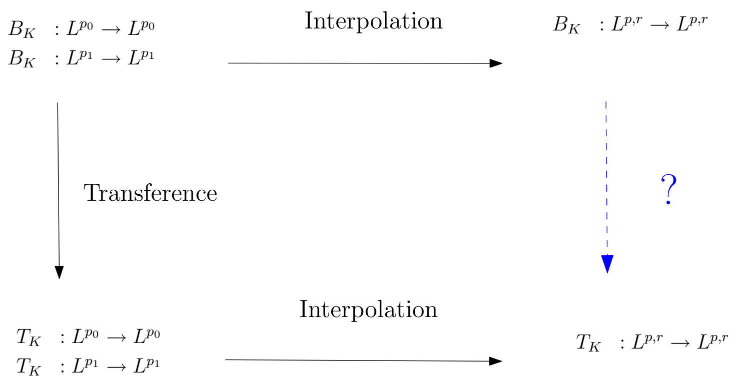

Figure 1.1: Interpolation

[34, Theorem 4.1]), both operators map $L(\log L)^\alpha$ continuously into a weighted Lorentz space $\Gamma^{1,\infty}(w)$. Carleson's operator provides an example of such operators with $\alpha = 1$ and, in this particular case, Yano's extrapolation theorem implies the boundedness for functions in $L(\log L)^2$ into L^1. *Can we directly transfer the endpoints estimates?*

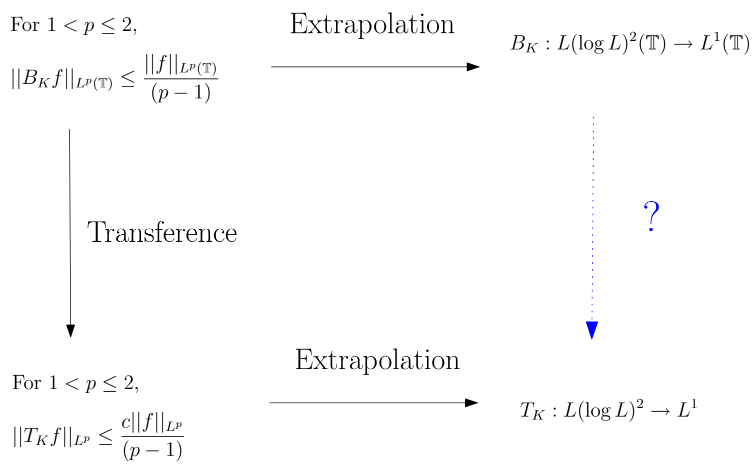

Figure 1.2: Extrapolation

Another motivation arises in the case that B_K is not of strong type but satisfies a stronger estimate than a weak type estimate, like $B_K : L^p \to L^{p,p'}$ for some $1 < p < 2$ (see [56, Theorem 2] for an example). Since $L^{p,p'} \subset L^{p,\infty}$, we can conclude that T_K maps L^p into $L^{p,\infty}$. But in this reasoning we lose a lot of information on B_K. *Can we use the information on the operator to obtain a better estimate on T_K?*

There are also other interesting cases where we simply cannot apply the classical transference results. For instance, if B_K is of restricted weak type, that is $B_K : L^{p,1} \to L^{p,\infty}$, but it is neither of strong type nor of weak type (see for instance [25, Theorem 1]).

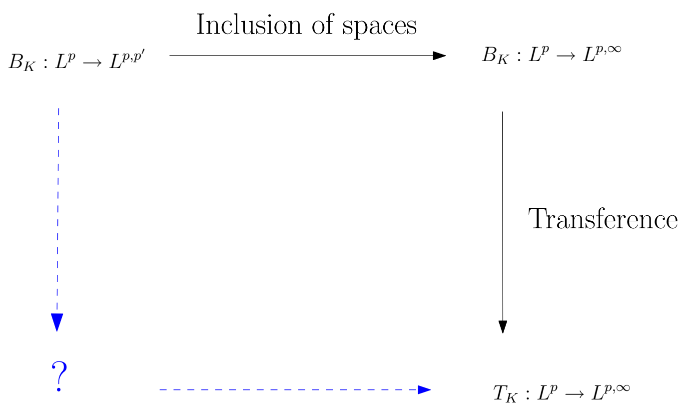

Figure 1.3: Information loss

These considerations lead us to deal with the following general problem: If $B_K : X \to Y$ is bounded, where X, Y are quasi-Banach function spaces defined on G,

What kind of estimate can we obtain for T_K?

The previous situations correspond to the case on which X, Y are rearrangement invariant spaces, and Figures 1.1, 1.2 and 1.3 illustrate them. In this monograph we show that the transference method of Coifman and Weiss can be applied to a more general class of rearrangement invariant spaces other than L^p, including the above mentioned. We deal with these type of problems in Chapter 3 and with analogous questions for bilinear operators in the fist section of Chapter 5.

Let us observe that with the initial hypothesis $B_K : L^p(G) \to L^q(G)$, we cannot expect T_K to map $L^p(\mathcal{M})$ continuously into $L^q(\mathcal{M})$. If such transference result were true, by the same argument given before, the restriction to the integers of any multiplier mapping $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, should be a Fourier multiplier mapping $L^p(\mathbb{T})$ into $L^q(\mathbb{T})$. But this fails to hold for pairs of spaces (L^p, L^q) with $p < q$, as it is shown in [61], leading to a contradiction.

Despite this, we can ask ourself whether it is possible to obtain information on the transferred operator. Let us illustrate this situation with an example: Let $G = \mathbb{T}$, $K \equiv 1$, $\mathcal{M} = \mathbb{R}^2$ with the Lebesgue measure and the representation given by $R_\theta f(z) = f(e^{2\pi i\theta} z)$. Then we can write the radial part of f as

$$T_K f(z) = \int_0^1 f(e^{-2\pi i\theta} z)\, d\theta.$$

Clearly $B_K : L^p(\mathbb{T}) \to L^\infty(\mathbb{T})$, and since $L^\infty(\mathbb{T}) \subset L^p(\mathbb{T})$, by classical transference result, T_K maps $L^p(\mathbb{R}^2)$ into itself. Since, for any radial function $f \in L^p(\mathbb{R}^2)$, $T_K f(z) = f(z)$, T_K does not map $L^p(\mathbb{R}^2)$ into $L^q(\mathbb{R}^2)$ for any $q \in (p, \infty]$. We shall prove that in fact

$$T_K : L^p(\mathbb{R}^2) \longrightarrow Y,$$

is bounded for some space $Y \subset L^p(\mathbb{R}^2)$ and

$$T_K : X \longrightarrow L^\infty(\mathbb{R}^2)$$

is bounded for some space $X \supset L^\infty(\mathbb{R}^2)$. We will discuss this in §3.4 and §4.2.3.

In this monograph we have also considered the same question in the setting of weighted Lebesgue spaces. Some previous works have been done in this context, and particularly in the direction of restricting Fourier multipliers (see for instance [21, 63]). Chapter 4 and last section of Chapter 5 are devoted to this.

Another different situation arises under the initial hypothesis on B_K to satisfy a modular inequality, that is

$$\int_G P\left(B_K\phi(u)\right)\, du \leq C \int_G Q\left(\phi(u)\right)\, du,$$

where P, Q are modular functions (see §5.2 for its definition). Modular estimates for convolution-type operators as the Hilbert Transform have been studied. These estimates do not need to be associated with the boundedness on a quasi-Banach space, and hence, they provide us with an example of operators that cannot be dealt with classical transference, except, of course, for $P(t) = Q(t) = |t|^p$. We handle these type of inequalities in the second section of Chapter 5.

This monograph consists of five chapters, including this introduction, and four appendices.

Chapter 2 contains definitions, notations and preliminary results. It is split in three different parts: quasi-normed spaces, topological groups and Fourier multipliers.

Chapters 3 and 4 contain our main results: Theorems 3.1.4, 3.1.22, 4.1.3 and 4.1.17. We have developed two different techniques. The first one is presented in Chapter 3 and turns out to be very useful to obtain applications on the setting of rearrangement invariant spaces. In particular, we get restriction results for multipliers in general weighted Lorentz spaces $\Lambda^p(w)$ and Orlicz spaces like $L(\log L)^2$. The second technique developed in Chapter 4 applies to the setting of weighted Lebesgue spaces, and becomes particularly useful to obtain restriction results for certain A_p weights.

Chapter 5 deals with four different questions on transference. In the first section, we extend the technique of Chapter 3 to the bilinear setting. The main results of this part are Theorems 5.1.5 and 5.1.9. As in the linear case, we obtain applications on the setting of rearrangement invariant spaces and, in particular, we are able to prove a similar De Leeuw-type result for bilinear multipliers for Lorentz-Zygmund spaces $L^{p,q}(\log L)^\alpha$, extending the results of O. Blasco and F. Villarroya in [27] for Lorentz spaces.

The second section contains results on transference for convolution-type operators satisfying a modular inequality as the above mentioned. The main result of this part is Theorem 5.2.3 which is useful to obtain restriction results on this setting.

In the third part we obtain restriction results for multipliers satisfying a extrapolation-type inequality (Theorem 5.3.6).

In the last section we develop a transference technique for weighted Lebesgue spaces that differs from the one exhibited in Chapter 4. The main result, Theorem 5.4.4, provides a useful tool to obtain restriction results for weak type weighted multipliers, complementing those obtained in [21] for strong type.

In order to make the reading of this monograph easier, some technical details are written in four different appendices.

Chapter 2

Preliminaries

This chapter contains some general preliminary facts that will be used in the forthcoming discussion. It is divided into three thematic parts and its contents are mainly expository.

The first one,contains basic definitions and technical results on quasi-Banach function spaces taking special account to rearrangement invariant ones. In the second part, some definitions and notation on topological group theory are recalled, as well as the definition of the so called *transferred operator*.

The last section is devoted to fix notation, recall definitions and prove some properties on Fourier multipliers on abelian groups. In particular, the problem of approximate Fourier multipliers is considered. This last technical part on approximation can be skipped in a first read, but will play an important role in the development of this dissertation.

2.1 Function spaces

The reader can find more information and technical details on Banach functions spaces, rearrangements and examples in [18, 38, 78] and in [74] on general F-spaces.

A *quasi-norm* $||\cdot||$ defined on a vector space X on a field $\mathbb{K}$ is a map $X \to \mathbb{R}_+$ such that

1. $||x|| > 0$ for $x \neq 0$,
2. $||\alpha x|| = |\alpha|\, ||x||$ for $\alpha \in \mathbb{K}, x \in X$,
3. $||x + y|| \leq C_X(||x|| + ||y||)$ for all $x, y \in X$, where C_X is a constant independent of x, y.

The least constant C_X satisfying the last property is called the modulus of concavity of X. Given $0 < p \leq 1$, we call $||\cdot||$ to be a *p-norm* if we also have $||x + y||^p \leq ||x||^p + ||y||^p$ for all $x, y \in X$. Aoki-Rolewicz theorem (see [74, Theorem 1.3]) states that every quasi normed space has an equivalent p-norm $|||\cdot|||$, where p satisfies $C_X = 2^{\frac{1}{p}-1}$.

If $||\cdot||$ is a quasi-norm (resp. p-norm) on X defining a complete metrizable topology, then X is called a *quasi-Banach space* (resp., p-Banach space). It is said norm and Banach in the case $p = 1$.

Given a pair of quasi-Banach spaces, we shall write $T \in \mathfrak{B}(X, Y)$ or $T \in \mathfrak{B}(X)$ if $X = Y$, to denote a linear and bounded operator mapping X on Y. An operator T mapping a linear space X on a space of functions Y satisfying

$$0 \leq T(x + y) \leq Tx + Ty, \quad T(\lambda x) = |\lambda| \, Tx,$$

for all $x, y \in X$ and all scalar value λ, is called a nonnegative sublinear operator. Observe that such operator satisfies

$$|Tx - Ty| \leq |T(x - y)| = T(x - y).$$

An important example of such operators is given by the maximal operator associated to a family of linear operators $\{T_n\}_{n\geq 1} \subset \mathfrak{B}(X, Y)$, where X, Y are spaces of functions, defined by

$$T^{\sharp} f(x) = \sup_{n\geq 1} |T_n f(x)| \,,$$

for $f \in X$. The following result is well known for Banach spaces and its proof is essentially the same.

Lemma 2.1.1. *Let X be a quasi-normed space and Y be a quasi-Banach space. Let $\widetilde{X} \subset X$ dense in X and let T be a linear (respectively, a nonnegative sublinear) operator defined on $\widetilde{X}$ such that there exists $c > 0$ satisfying, for every $x \in \widetilde{X}$,*

$$\|Tx\|_Y \leq c \, \|x\|_X \,. \tag{2.1.2}$$

Then T admits a unique linear (respectively, a nonnegative sublinear) extension defined on X satisfying

$$\|Tx\|_Y \leq c C_X C_Y \, \|x\|_X \quad \forall x \in X. \tag{2.1.3}$$

Given two positive quantities A, B, if $A \leq cB$ for a positive universal constant c independent of A, B, it is written $A \lesssim B$. If $A \lesssim B$ and $B \lesssim A$, we shall write $A \approx B$.

$(\mathcal{M}, \Sigma, \mu)$, or simply $\mathcal{M}$ if no confusion can arise, denotes a σ-finite measure space. By $L^0(\mathcal{M})$, or simply L^0, it is denoted the space of all complex-valued measurable functions on $\mathcal{M}$, with the topology of local convergence in measure. A quasi-Banach function space (QBFS for short) on $\mathcal{M}$, stands for a complete linear space X continuously embedded in $L^0(\mathcal{M})$, endowed with a (quasi-)norm $\|\cdot\|_X$ with the following properties:

1. $f \in X$ if and only if $|f| \in X$ and $\|f\|_X = \| \, |f| \, \|_X < \infty$;

2. (Lattice property) $g \in X$ and $\|g\|_X \leq \|f\|_X$, whenever $g \in L^0$, $f \in X$, and $|g| \leq |f|$ a.e ;

3. If $0 \leq f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$;

4. $\mu(E) < \infty \Rightarrow ||\chi_E||_X < \infty$.

If moreover $\|\cdot\|_X$ is a norm and it satisfies

5. if $\lambda(E) < \infty$ then $\int_E f d\lambda \leq C_E \|f\|_X$,

it is said that X is a Banach function space (BFS for short). In order to avoid possible misunderstanding, if it is necessary, the underlying measure space is written as a subindex. That is, $X_{\mathcal{M}}$, denotes a QBFS whose functions are defined on $\mathcal{M}$.

For $0 < p < \infty$, write $L^p(\mu)$ for the usual Lebesgue space of p-integrable functions. These are examples of QBFS's that, for $1 \leq p < \infty$ are BFS. A weight on $\mathcal{M}$ is a locally integrable function defined on $\mathcal{M}$ that takes values in $[0, \infty)$. For any measurable set E, write $w(E) = \int_E w \, d\mu$. $L^p(w)$ denotes the Lebesgue space with underlying measure ν such that $d\nu(x) = w(x)d\mu(x)$. In the case that $\mathcal{M} = [0, \infty)$ endowed with the Lebesgue measure, we write

$$W(r) = \int_0^r w(x) \, dx.$$

For any BFS X, the associate space (the Köthe dual) of X is the space X' given by the norm defined by

$$\|g\|_{X'} = \sup \left\{ \left| \int_{\mathcal{M}} fg \, d\mu \right| ; f \in X, \ \|f\|_X \leq 1 \right\}$$

Moreover, $X'' = X$ ([18, Theorem 2.7]) and

$$\|f\|_X = \sup \left\{ \left| \int_{\mathcal{M}} fg \, d\mu \right| ; g \in X', \ \|g\|_{X'} \leq 1 \right\}.$$

Lemma 2.1.4. [18, Lemma I.1.5]*(Fatou's lemma) Let X be a QBFS, and, for $n \in \mathbb{N}$, $f_n \in X$. If $f_n \to f$ a.e. and $\liminf_n \|f_n\|_X < \infty$, then $f \in X$ and*

$$\|f\|_X \leq \liminf_n \|f_n\|_X .$$

Fatou's lemma allow to improve the estimation (2.1.2) for operators defined on QBFS's.

Lemma 2.1.5. *Let X, Y be QBFS. Let $\widetilde{X} \subset X$ be a dense subset of X. Let T be a linear (respectively, a nonnegative sublinear) operator defined on $\widetilde{X}$ such that there exists $c > 0$ satisfying, for every $f \in \widetilde{X}$, $\|Tf\|_Y \leq c \|f\|_X$. Then T admits a unique linear (respectively, a nonnegative sublinear) extension defined on X satisfying*

$$\|Tf\|_Y \leq cC_X \|f\|_X . \tag{2.1.6}$$

for all $f \in X$. Moreover, if $\|\cdot\|_X$ is a p-norm with $0 < p < 1$, C_X can be replaced by 1.

Proof. By Lemma 2.1.1, there exists a unique linear extension of T defined on X satisfying (2.1.3). Fix $f \in X$, and let $(f_n)_n \subset X$ such that $f = X - \lim_n f_n$. By the continuity of T, $Tf = Y - \lim_n Tf_n$. Since Y is continuously embedded in L^0, $Tf = L^0 - \lim_n Tf_n$ and thus there exists a subsequence such that $Tf(x) = \lim_k Tf_{n_k}(x)$ a.e. x. Hence, by Fatou's lemma,

$$\begin{aligned} \|Tf\|_Y &\leq \liminf_k \|Tf_{n_k}\|_Y \leq c \liminf_k \|f_{n_k}\|_X \\ &\leq cC_X \liminf_k \left(\|f_{n_k} - f\|_X + \|f\|_X\right) = cC_X \|f\|_X . \end{aligned} \tag{2.1.7}$$

The last assertion holds since if $\|\cdot\|_X$ is a p-norm, we can use that $\|f_{n_k}\|_X \leq (\|f - f_{n_k}\|_X^p + \|f\|_X^p)^{1/p}$ in the previous argument. □

Definition 2.1.8. [18, Definition 3.1] *Let $(X, ||\cdot||)$, be a QBFS. A function $f \in X$ is said to have absolutely continuous norm if*

$$\lim_{n\to\infty} ||f\chi_{A_n}|| = 0,$$

for every decreasing sequence of measurable sets $(A_n)_n$ with $\chi_{A_n} \to 0$ a.e. If every $f \in X$ has this property, we say that X has an absolutely continuous norm .

A QBFS, or a BFS, X is said to be *rearrangement invariant* (RI for short), if there exists a quasi-norm, respectively a norm, $\|\cdot\|_{X^*}$ defined on the space $L^0\,[0, +\infty)$ endowed with the Lebesgue measure, such that for every measurable function f, $\|f\|_X = \|f^*\|_{X^*}$. Here f^* stands for the non-increasing rearrangement of f, defined, for $t > 0$, by

$$f^*(t) = \inf\{s : \ \mu_f(s) \leq t\},$$

where $\mu_f(s) = \mu\{x : \ |f(x)| > s\}$ is the distribution function of f. Denote the maximal function of f^*

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

A measure space $\mathcal{M}$ is called *resonant* if, for every $f, g \in L^0$,

$$\sup\left|\int_{\mathcal{M}} fh\, d\mu\right| = \int_0^\infty f^* g^*\, ds, \tag{2.1.9}$$

where the supremum is taken over all the functions h such that $h^* = g^*$. And, it is called strongly resonant if the supremum is attained for some function h.

Definition 2.1.10. *Let X be a RIQBFS on a resonant space. For each finite value of t in the range of μ, the fundamental function is defined by*

$$\varphi_X(t) = \|\chi_E\|_X ,$$

for any measurable set $E \subset \mathcal{M}$ such that $\mu(E) = t$. This function is increasing with $\varphi_X(0) = 0$ and quasi-concave, that is, $\varphi_X(t)/t$ is decreasing.

Let X be a RIQBFS, let $D_{\frac{1}{s}} f^*(t) = f^*(\frac{t}{s})$ be the *dilation operator*, and denote by $h_X(s)$ its norm, that is,

$$h_X(s) = \sup_{f\in X\setminus\{0\}} \frac{\left\|D_{\frac{1}{s}} f^*\right\|_{X^*}}{\|f^*\|_{X^*}},\ s>0.$$

h_X is increasing, submultiplicative and, if X is a BFS and X' denotes its Köthe dual space (see [18, Prop. 5.11]), it holds that

$$h_X(t) = t h_{X'}(1/t). \tag{2.1.11}$$

The lower and upper Boyd indices are defined, respectively, by

$$\overline{\alpha}_X = \inf_{t>1} \frac{\log h_X(t)}{\log t}, \quad \underline{\alpha}_X = \sup_{0<t<1} \frac{\log h_X(t)}{\log t}.$$

Proposition 2.1.12. [78, Theorem 1.3]*The Boyd indices of a space X satisfy, $0 \le \underline{\alpha}_X \le \overline{\alpha}_X \le \infty$,*

$$\underline{\alpha}_X = \lim_{t\to 0^+} \frac{\log h_X(t)}{\log t};\ \ \overline{\alpha}_X = \lim_{t\to\infty} \frac{\log h_X(t)}{\log t}.$$

Moreover,

$$\underline{\alpha}_X = \sup\{p:\ \exists c, \forall a<1,\ h_X(a) \le c a^p\}, \tag{2.1.13}$$
$$\overline{\alpha}_X = \inf\{p:\ \exists c, \forall a>1,\ h_X(a) \le c a^p\}. \tag{2.1.14}$$

If in addition X is Banach, $0 \le \underline{\alpha}_X \le \overline{\alpha}_X \le 1$.

Given a BFS X, let us define for $0 < s \le 1$ the *s-convexification* X^s of X as the collection of measurable functions f, for which $\||f|^s\|_X < +\infty$ and set

$$\|f\|_{X^s} = \||f|^s\|_X^{1/s}.$$

It is clear that $\|\cdot\|_{X^s}$ is an s-norm. Moreover, X^s is a QBFS. If X is a RIBFS on a resonant measure space, then X^s is a RIQBFS and

$$\|f\|_{X^s}^s = \sup\left\{\left|\int_{\mathcal{M}} |f|^s g\, d\mu\right|; f\in X,\ \|g\|_{X'} \le 1\right\},$$

from where it follows that

$$h_{X^s}(t) = h_X(t)^{1/s}. \tag{2.1.15}$$

It holds that

$$\underline{\alpha}_{X^s} = \frac{\underline{\alpha}_X}{s}, \quad \overline{\alpha}_{X^s} = \frac{\overline{\alpha}_X}{s}.$$

The classical examples where this situation arises are the L^s spaces, for $0 < s \le 1$, that can be realized as the s-convexification of the L^1 space.

A particular example of RIQBFS is given by the so called weighted Lorentz spaces $\Lambda^p(w,\mathcal{M})$ and $\Lambda^{p,\infty}(w,\mathcal{M})$ whenever w is a weight function in $[0,\infty)$. For convenience of the reader not familiar with these spaces we have collected the definition and some properties in Appendix D. Whenever no confusion can arise we shall write $\Lambda^p(w)$ for $\Lambda^p(w, \mathcal{M})$.

2.2 Topological Groups

The main references for this section are [59, 69, 89, 90] on topological groups, [55, 65, 87] on amenable groups and the monograph [83] on multipliers.

A topological group is a group G endowed with a locally compact topology, with respect to which every point set is closed, and the group operations are continuous. Moreover, it will be assumed that the group is σ-compact, that is, G can be written as a countable union of compact sets. Observe that this assumption on G forces it to have at most a countable number of connex components. Hence topological groups like $(\mathbb{R}, +)$ endowed with the discrete topology will not be considered. From now on, a group will be a topological group as before. Except in concrete cases, the multiplicative notation for the group operation shall be adopted.

Since every group is σ-compact, any Borel measure on G finite on every compact set defines a Radon measure (see [59, Cor. 7.6]). A left (resp. right) Haar measure on G is a nonzero Radon measure λ on G that satisfies $\lambda(xE) = \lambda(E)$ (resp. $\lambda(E) = \lambda(Ex)$), for every Borel set $E \subset G$ and every $x \in G$. It is well known that every locally compact group G possesses a left (resp. right) Haar measure, and that it is unique modulus a multiple constant.

λ_G denotes a left Haar measure on G. Where no confusion can arise it will be written λ for λ_G, $\int f(u)\, du$ for $\int f(u)\, d\lambda(u)$ and du for $d\lambda(u)$. Observe that (G, λ) is σ-finite.

In some applications the groups are required to posses a countable open basis of $\{e_G\}$, where e_G, or e provided that there is no possible confusion with the underlying group, denotes the identity element of the group. This is equivalent to the fact that G is a metrizable space (see [70, (8.3)]).

Lemma 2.2.1. *If G is metrizable, there exists a countable basis of symmetric relatively compact open neighborhoods of e, namely $\{V_n\}_n$, satisfying $\overline{V_{n+1}} \subset V_n$ for all $n \geq 1$, and $\bigcap_{n\geq 1} V_n = \{e\}$.*

Proof. Let $\{W_n\}_n$ be a countable basis of $\{e\}$. Let $V_1 = W_1 \cap W_1^{-1}$ and assume that V_n is defined. Then there exists $W_{k_{n+1}}$ such that $\overline{W_{k_{n+1}}} \subset V_n$. Define $V_{n+1} = W_{k_{n+1}} \cap W_{k_{n+1}}^{-1}$.

By construction $\overline{V_{n+1}} \subset V_n$. It is easy to see that, since $\{W_n\}_n$ is a basis, also is $\{V_n\}_n$. Moreover, $\cap_n V_n = \cap_n W_n = \overline{\{e\}} = \{e\}$. □

Proposition 2.2.2. *If G is metrizable, the following holds:*

1. *The space (G, λ) is completely atomic and each atom has the same measure or, it is non-atomic.*

2. *The space (G, λ) is a resonant measure space.*

3. *The space (G, λ) is strongly resonant if and only if G is compact.*

Proof. Let $\{V_n\}_n$ be the family of sets given by the previous lemma. Since for each n V_n is a non empty relatively compact open subset of G and λ is a Radon measure, $\lambda(V_n) < \infty$, and $\lambda(\{e\}) = \lim_{n\to\infty} \lambda(V_n)$. On the other hand, λ is a non-null σ-finite measure, so there is no atom of infinite measure.

A topological group has the discrete topology if, and only if, $\lambda(\{e\}) > 0$ (see [70, (15.17)]). In this case, λ is a complete atomic measure and, by the left invariance of the measure, each element is an atom and has the same measure as $\lambda(\{e\})$. Assume, that $\lambda(\{e\}) = 0$ and let W be an atom such that $0 < \lambda(W) < \infty$. By inner regularity, there exists a compact $\mathcal{K}$ included in W such that $W \setminus \mathcal{K}$ is a null set. Therefore, we can assume that W is compact. Thus, there exists a natural m_1 such that, $W \subset \bigcup_{i_1=1}^{m_1} x_{i_1} V_1$, where $\{x_{i_1}\}_{i_1=1,\ldots,m_1} \subset W$. Since W is an atom, there exists $j_1 \in \{1,\ldots,m_1\}$ such that

$$W = \left(x_{j_1}\overline{V_1}\right) \cap W \quad \lambda - \text{a.e.}$$

Let $\mathcal{K}_2$ be the set in the right of the last expression. It is an atomic compact set, and then, there exists a natural number m_2 such that

$$\mathcal{K}_2 \subset \cup_{i_2=1}^{m_2} x_{i_1} V_2,$$

where $\{x_{i_2}\}_{i_2=1,\ldots,m_2} \subset \mathcal{K}_2$, and then, there exists $j_2 \in \{1,\ldots,m_2\}$ such that

$$\mathcal{K}_2 = \left(x_{j_1}\overline{V_2}\right) \cap \mathcal{K}_2 \quad \lambda\text{-a.e.},$$

that is, $W = \left(x_{j_1}\overline{V_1}\right) \cap \left(x_{j_2}\overline{V_2}\right) \cap W$ λ-a.e. Repeating the last argument for all $n \geq 1$, we obtain that

$$W = W \cap \bigcap_{i=1}^{n+1} x_{j_i}\overline{V_i} \subset x_{j_{n+1}}\overline{V_{n+1}} \subset x_{j_{n+1}} V_n.$$

Therefore, by the left invariance of the measure, for any $n \geq 1$,

$$\lambda(W) \leq \lambda(x_{j_{n+1}} V_n) = \lambda(V_n).$$

But, taking limit in n, this implies that $0 \leq \lambda(W) \leq \lambda(\{e\}) = 0$ that contradicts the fact that W is an atom. Thus, G is non-atomic. Hence, the first assertion is established.

In [18, Theorem II.2.7] it is shown that, a σ-finite measure is resonant if and only if it is non-atomic or, it is completely atomic and all atoms have the same measure, so this proves the second part.

Finally, in [18, Theorem II.2.6] it is proved that a σ-finite measure is strongly resonant if and only if it is resonant and finite. But, the fact that $\lambda(G) < \infty$ is equivalent to the fact that G is a compact group (see [70, (15.9)]). □

Definition 2.2.3. *A topological group is said to be amenable if for any compact set $\mathcal{K}$ and any $\epsilon > 0$, there exists a open neighborhood V of the identity, with* *compact closure such that*

$$\lambda(V\mathcal{K}) \leq (1+\epsilon)\lambda(V). \tag{2.2.4}$$

The previous condition on the group is called Leptin (or Følner) condition.

Compact, abelian (see for instance [69, (31.36)]) or [90, pp. 52]) and solvable groups are examples of amenable groups. The notion of amenabilty is related to the existence of left invariant means on suitable subspaces of $L^\infty(G)$. Among other equivalent characterizations, Leptin-Følner condition (2.2.4) appears in a natural way on the setting of transference (see [46, Theorem 2.4] or Corollary 3.1.6). For more information on amenability, the reader is referred to the monographs [65, 87] and to [55] for a nice survey on this subject.

Let F denote a Banach space, whose elements are (classes of) measurable functions on the measure space $(\mathcal{M}, \Sigma, \mu)$, continuously embedded in $L^1_{\text{loc}}(\mathcal{M})$. That is, for every set of finite measure $\mathcal{M}_1$, there exists a constant c such that, for all $f \in F$,

$$\int_{\mathcal{M}_1} |f| \, d\mu \leq c \, \|f\|_F \, .$$

Definition 2.2.5. *A homomorphism $u \mapsto R_u$ of G into the group of all topological automorphism of F is called a representation of G on F. That is,*

$$R_{uv} = R_u \circ R_v, \quad R_e = Id_E.$$

Moreover, it is called continuous if the map $(u, f) \mapsto R_u f$ of $G \times F$ into F is continuous.

For any function f on G, we define the left and right translates of f by

$$L_u f(v) = f(u^{-1}v), \quad D_u f(v) = f(vu), .$$

Clearly the maps $u \mapsto L_u$ and $u \mapsto D_u$ are group homomorphisms. Moreover, they induce examples of continuous representations of G on the spaces $F = L^p(G)$ for $1 \leq p < \infty$ and $F = \mathcal{C}_0(G)$, where $\mathcal{C}_0(G)$ denotes the set of all bounded complex-valued continuous functions on G such that for every $\epsilon > 0$, there exists a compact subset $\mathcal{K}$ of G $|f(x)| < \epsilon$ for all $x \notin \mathcal{K}$. It is well known that this space is the Banach closure of the space of continuous and compactly supported functions, that is denoted by $\mathcal{C}_c(G)$ with respect to the uniform norm. Both are vectorial subspaces of the space of continuous and bounded functions, that we denote by $\mathcal{C}_b(G)$.

Definition 2.2.6. *Let $K \in L^1(G)$ with compact support and let R be a continuous representation of G acting on F. The transferred operator is defined to be the continuous linear operator on F determined by*

$$T_K f = \int_G K(u) R_{u^{-1}} f \, du; \quad f \in F.$$

The previous integral is well defined in a vectorial sense (see Appendix A). Let us emphasize that, for any $S \in \mathfrak{B}(F)$,

$$ST_K f = \int_G K(u) S R_{u^{-1}} f \, du,$$

hence, for every $v \in G$, $R_v T_K f = \int_G K(u) R_{vu^{-1}} f \, du$. By technical reasons, we shall assume that in this work all the appearing representations satisfy that, for $f \in F$

$$(u, x) \in G \times \mathcal{M} \mapsto R_u f(x),$$

is jointly measurable. Then, it can be shown that

$$T_K f(x) = \int_G K(u) R_{u^{-1}} f(x) \, du,$$

is well defined μ-a.e. x. Furthermore, for any σ-compact set V, $(\mu \times \lambda)$-a.e. $(x, v) \in \mathcal{M} \times V$,

$$\chi_V(v) R_v T_K f(x) = \chi_V(v) B_K(\chi_{V\mathcal{K}^{-1}} R_{\cdot} f(x))(v), \tag{2.2.7}$$

where $\mathcal{K} \supset \operatorname{supp} K$ and B_K denotes, from now on, the operator given by

$$B_K g(v) = \int_G K(u) g(vu^{-1}) \, du,$$

whenever it is well defined. For a complete account of these technical results see Appendix A.

All the applications we shall present in this work, satisfy the jointly measurability assumption and hence this is not a restrictive assumption for our purpose. Moreover, if F is a BFS, it is possible to avoid it, in the sense that fixed $f \in F$, there exists a jointly measurable function $H_f(u, x)$ such that

$$R_u f \equiv H_f(u, \cdot),$$

and for every $v \in G$,

$$R_v T_K f \equiv \int_G K(u) H_f(vu^{-1}, \cdot) \, du.$$

In this setting, in the foregoing development, $R_u f(x)$ can be interpreted as the corresponding function $H_f(u, x)$ in order to avoid the joint measurability assumption. Details of this last fact are given in Theorem A.3.3.

For a pair of functions f, g on G, the *convolution product* of f and g is defined whenever it makes sense by

$$f * g(v) = \int_G f(u) g(u^{-1} v) \, du.$$

Observe that for the abelian and for the compact groups $B_K g = g * K$.

For any (finite or countably infinite) family of compactly supported functions $\{K_j\}_{j\in J} \subset L^1(G)$, denote by $B^\sharp$ the maximal operator associated to the convolution operators $\{B_{K_j}\}_{j\in J}$, and by $T^\sharp$ the maximal operator associated to the transferred operators $\{T_{K_j}\}_{j\in J}$. That is, for $f \in F$,

$$T^\sharp f = \sup_{j\in J} \left|T_{K_j} f(x)\right| .$$

2.3 Fourier multipliers

Whenever G is a metrizable locally compact abelian group, LCA for short, we shall denote by Γ its dual group (the group of characters of G) that is, the group of continuous homomorphisms of G into $\mathbb{T}$. For $\gamma \in \Gamma$, $f \in L^1(G)$ we define the Fourier transform of f (see [70, (23.9)]) by

$$\widehat{f}(\gamma) = \int_G f(u)\overline{\gamma(u)}\, du,$$

The dual group Γ becomes a locally compact abelian group ([70, (23.15)]) and, hence it has a Haar's measure. Haar's measure on G and Γ can be selected in order that Fourier transform becomes an isometry between $L^2(G)$ and $L^2(\Gamma)$ (Plancherel's Theorem [69, (31.18)]) and, the following inversion formula holds for f such that $\widehat{f} \in L^1(\Gamma)$,

$$f(u) = \int_\Gamma \widehat{f}(\gamma)\gamma(u)\, d\gamma.$$

For $h \in L^1(\Gamma)$, its inverse Fourier transform is given by

$$h^\vee(u) = \int_\Gamma h(\gamma)\gamma(u)\, d\gamma,$$

so inversion formula can be read as $f = (\widehat{f})^\vee$. The requirement on G to be metrizable, is equivalent, by [70, (24.48) and (8.3)], to the fact that Γ is σ-compact.

Given a closed subgroup H, let recall that, (see [90, Theorem 2.1.2], [70, Theorem (23.25)]) the dual group of G/H is isomorphic to the annihilator of H, that it is defined by

$$H^\perp = \{\gamma \in \Gamma :\ \forall u \in H\ \gamma(u) = 1\}.$$

The quotient group G/H is the topological space of (left) cosets of H, with the usual quotient topology, that is itself a topological group. We shall denote by uH the equivalence class of u in G/H. Haar's measure on G/H $\lambda_{G/H}$ is G-invariant and satisfies that for $f \in L^1(G)$ and for $f \geq 0$ measurable (see [69, Theorem (28.54)])

$$\int_G f(u)\, du = \int_{G/H} P_H f\, d\lambda_{G/H}, \tag{2.3.1}$$

where $P_H f(uH) = \int_H f(u\xi)d\lambda_H(\xi)$, is the H-periodization of f. Equation (2.3.1) is called Weil's formula.

By B, C we denote two QBFS defined on G with underlying defining Radon measure. From now, whenever we deal with multipliers we shall assume that $\|\cdot\|_B$ and $\|\cdot\|_C$ are absolutely continuous and that simple and integrable functions are dense in B.

Definition 2.3.2. *Let $\mathbf{m} \in L^\infty(\Gamma)$. $\mathbf{m}$ is a Fourier multiplier for the pair (B, C), and it is denoted by $\mathbf{m} \in M(B, C)$, if the operator defined on functions in $L^2(G)$ by*

$$T_{\mathbf{m}} f(x) = \left(\mathbf{m}\widehat{f}\right)^\vee (x),$$

satisfies that there exists a constant c such that, for every $f \in L^2 \cap B$,

$$\|T_{\mathbf{m}} f\|_C \leq c \|f\|_B . \tag{2.3.3}$$

The least constant satisfying (2.3.3) will be denoted by $\|\mathbf{m}\|_{M(B,C)}$, and will be called the norm of the multiplier.

Observe that, for $f \in L^1(G)$, $\widehat{(D_v f)}(\gamma) = \gamma(v)\widehat{f}(\gamma)$. Hence $D_v(T_{\mathbf{m}} f) = T_{\mathbf{m}}(D_v f)$. That is, multiplier operators are operators that commute with translations.

Definition 2.3.4. *Given $\{\mathbf{m}_j\}_{j\in I} \subset M(B, C)$, where I is a countable set of indices, the associated maximal multiplier operator is defined by*

$$T^\sharp f = \sup_{j\in I} \left|T_{\mathbf{m}_j} f\right| .$$

The family $\{\mathbf{m}_j\}_{j\in I}$ is said to be a maximal Fourier multiplier for the pair (B, C) if there exists a constant c such that for all $f \in L^2 \cap B$

$$\left\|T^\sharp f\right\|_C \leq c \|f\|_B .$$

Let denote by $\left\|\{\mathbf{m}_j\}_j\right\|_{M(B,C)}$, the least constant satisfying the previous inequality.

We write $M(B)$ for $M(B, B)$. Observe that a (maximal) multiplier defines by density (see Lemma 2.1.1) an unique bounded operator $T : B \to C$. In some situations it is useful to consider multipliers defined on more regular classes of functions than on $L^2(G)$. We shall define

$$SL^1(G) = \left\{f \in L^1(G) : \ \widehat{f} \in L^1(\Gamma)\right\},$$

that is a class of functions belonging to $\mathcal{C}_0(G)$, on which the Fourier transform defines a bijection with $SL^1(\Gamma)$. The following proposition ensures that we can indistinctively define the notion of Fourier multiplier on $L^2 \cap B$, $SL^1 \cap B$, $\mathcal{C}_c(G)$ or $\mathcal{C}_c(G) * \mathcal{C}_c(G)$.

Proposition 2.3.5. *Under the above conditions on B, it holds:*

1. $\mathcal{C}_c(G) * \mathcal{C}_c(G) \subset \mathcal{C}_c(G) \cap SL^1(G)$.

2. $\mathcal{C}_c(G)$ *is dense in* B.

3. $\mathcal{C}_c(G) * \mathcal{C}_c(G)$, $L^2(G) \cap B$ *and* $SL^1 \cap B$ *are dense in* B.

Proof. For $f, g \in \mathcal{C}_c(G)$, it is easy to see that $f * g \in \mathcal{C}_c(G)$. Moreover, $\widehat{f * g} = \hat{f}\hat{g} \in L^1(\Gamma)$ because $f, g \in L^2(G)$.

In order to prove the second assertion, it suffices to show that every integrable simple function $f \neq 0$ can be approximated by functions in $\mathcal{C}_c(G)$. Fix $\epsilon > 0$. By [18, Lemma 3.4] (the same proof therein carries over QBFS), there exists $\delta > 0$ such that for any set E with $\mu(E) < \delta$ then $\|\chi_E\|_B < \epsilon/2 \|f\|_\infty$. Since f is supported on a finite measure set, and it is bounded, by Lusin's Theorem (see [59, Theorem 7.10]), there exists $g \in \mathcal{C}_c(G)$ such that $\|g\|_\infty \leq \|f\|_\infty$ and $\mu\{f \neq g\} < \delta$. Hence, $\|g - f\|_B \leq 2\|f\|_\infty \left\|\chi_{\{f \neq g\}}\right\|_B < \epsilon$.

For the last assertion, it is enough to approximate every $f \in \mathcal{C}_c(G)$. Let $\{V_n\}_n$ be a basis of open neighborhoods of e as given in Lemma 2.2.1. Consider an approximation of the identity $\{h_n\}_n$ $h_n \in \mathcal{C}_c(G)$, such that $\int h_n = 1$ and $\operatorname{supp} h_n \subset V_n$. Hence, for any n, $h_n * f - f$ is supported on the relatively compact set $V_1 \operatorname{supp} f$,

$$\|h_n * f - f\|_B \leq \|\chi_{V_1 \operatorname{supp} f}\|_B \sup_{u \in V_n} \|L_u f - f\|_\infty .$$

The result now follows by the uniform continuity of f. □

Observation 2.3.6. *Whenever* $G = \mathbb{R}^d$ *or* $\mathbb{T}$, *in the previous result,* $\mathcal{C}_c(G)$ *can be replaced by* $\mathcal{C}_c^\infty(G)$. *In the case that* $G = \mathbb{T}$, *we can also use trigonometrical polynomials instead.*

Let us observe also that if $\mathbf{m} = \widehat{K}$ for $K \in L^1(G)$ with compact support, $(\mathbf{m}\widehat{f})^\vee = B_K(f)$, for $f \in SL^1(G)$, so by uniqueness, they define the same operator.

2.3.1 Approximation of multipliers

In our forthcoming applications of transference techniques we will need to properly approximate Fourier multipliers by regular ones. In this section, we will revise the notion of normalized multiplier introduced in [46], and we will take care on this approximation procedure.

Lemma 2.3.7. [70, Lemma (18.13)] *There exists a sequence of open sets with compact closure* $\{H_n\}_n$ *such that*

$$\begin{aligned} &H_n \subset H_{n+1} \quad \forall n; \\ &\cup_{n \geq 1} H_n = G; \\ &\lim_n \frac{\lambda(uH_n \cap H_n)}{\lambda(H_n)} = 1 \quad \forall u \in G. \end{aligned} \tag{2.3.8}$$

Lemma 2.3.9. [44, Lemma 3.4] *Let $\{H_n\}_n$ be a family of open sets with compact closure satisfying (2.3.8). Define*

$$\varphi_n(u) := \frac{\left(\chi_{H_n} * \chi_{H_n^{-1}}\right)(u)}{\lambda(H_n)} = \frac{\lambda\left(uH_n \cap H_n\right)}{\lambda(H_n)}.$$

It holds:

1. $\widehat{\varphi_n}(\xi) = \frac{|\widehat{\chi_{H_n}}(\xi)|^2}{\lambda(H_n)} \geq 0$;
2. $\int_\Gamma \widehat{\varphi_n}(\xi)d\xi = 1$;
3. *For every open relatively compact set $\mathcal{K} \subset \Gamma$ such that $e_\Gamma \in \mathcal{K}$,*

$$\lim_n \int_{\xi \notin \mathcal{K}} \widehat{\varphi_n}(\xi)d\xi = 0.$$

Fix a sequence $\{\widehat{\varphi_n}\}_n \subset L^1(\Gamma)$ satisfying $\varphi_n \in \mathcal{C}_c(G)$ and conditions *1.*,*2.*, *3.* of the previous lemma.

Definition 2.3.10. *Given $\mathbf{m} \in L^\infty(\Gamma)$, it is said to be normalized (with respect to $\{\widehat{\varphi}_n\}$) if, for all $\xi \in \Gamma$,*

$$\lim_n \left(\widehat{\varphi_n} * \mathbf{m}\right)(\xi) = \mathbf{m}(\xi).$$

Proposition 2.3.11. *Every $\psi \in \mathcal{C}_b(\Gamma)$ is normalized.*

Proof. Since ψ is uniformly continuous, given $\epsilon > 0$ and $\zeta \in \Gamma$, there exists a relatively compact open neighborhood of e_Γ such that, for any $\xi \in \mathcal{K}$, $|\psi(\xi^{-1}\zeta) - \psi(\zeta)| < \epsilon$. Then

$$|(\widehat{\varphi_n} * \psi - \psi)(\zeta)| = \left| \int_{\mathcal{K} \uplus \mathcal{K}^c} \widehat{\varphi_n}(\xi)\left(\psi(\xi^{-1}\zeta) - \psi(\zeta)\right) d\xi \right| \leq \epsilon + 2\left\|\psi\right\|_\infty \int_{\mathcal{K}^c} \widehat{\varphi_n}(\xi)d\xi,$$

thus taking limit on n and letting ϵ tend to 0, the result follows. □

With minors modifications on the previous proof, it can be proved that a bounded function ψ in $\mathbb{R}^n$ or $\mathbb{T}^n$ is normalized provided that every point is a Lebesgue point.

Definition 2.3.12. *A QBFS C on G is said to be well behaved if there exists a sequence $\{h_n\}_n \subset \mathcal{C}_c(G)$, such that*

1. $\mathfrak{s} := \sup_n ||\widehat{h}_n||_{M(C)} < \infty$,
2. $\sup_n ||\widehat{h}_n||_{L^\infty(\Gamma)} \leq 1$,
3. *for every ξ, $\hat{h}_n(\xi) \to 1$.*

Such a family will be referred as an associated family to C.

Theorem 2.3.13. *Let $\mathbf{m} \in M(B,C) \cap L^\infty(\Gamma)$ be normalized (with respect to $\{\widehat{\varphi_n}\}$). Assume that either B or C is well behaved and for the pair (B,C) there exists $\mathfrak{c} > 0$ such that for every $\varphi \in L^1(\Gamma)$*

$$\|\varphi * \mathbf{m}\|_{M(B,C)} \leq \mathfrak{c}\, \|\varphi\|_1 \|\mathbf{m}\|_{M(B,C)} . \tag{2.3.14}$$

Then there exists a sequence $\{\mathbf{m}_n\} \subset L^\infty(\Gamma)$ satisfying:

1. *For every $\xi \in \Gamma$,*
$$\mathbf{m}(\xi) = \lim_n \mathbf{m}_n(\xi). \tag{2.3.15}$$

2. *$K_n = \mathbf{m}_n{}^\vee \in L^1(G)$ and it is compactly supported.*

3. *$\sup_n \|\mathbf{m}_n\|_{L^\infty(\Gamma)} \leq \|\mathbf{m}\|_{L^\infty(\Gamma)}$.*

4. *$\sup_n \|\mathbf{m}_n\|_{M(B,C)} \leq \mathfrak{c}\mathfrak{s}C_B \|\mathbf{m}\|_{M(B,C)}$, where $\mathfrak{s}$ is given in Definition 2.3.12.*

Proof. We will assume that B is well behaved and that $\{h_n\}_n$ is an associated family to B. The case where C is well behaved is proved in a similar way.

Consider
$$\widehat{K_n}(\xi) = \mathbf{m}_n(\xi) = (\widehat{\varphi_n} * \mathbf{m})(\xi)\widehat{h_n}(\xi).$$

Since $\sup_n \|\widehat{h_n}\|_{L^\infty(\Gamma)} \leq 1$, it is clear that $K_n \in L^2(G)$, $\lim_n \mathbf{m}_n(\xi) = \mathbf{m}(\xi)$ for every $\xi \in \Gamma$, and
$$\|\mathbf{m}_n\|_{L^\infty(\Gamma)} \leq \|\mathbf{m}\|_{L^\infty(\Gamma)} .$$

Define, for all N, $K_{n,N} = (\varphi_n(\mathbf{m}\chi_{H_N})^\vee) * h_n$, where $H_N \uparrow \Gamma$ and H_N is compact, and observe that $\operatorname{supp} K_{n,N}$ is contained in the compact set $A_n = \operatorname{supp}\varphi_n \operatorname{supp} h_n$. Moreover, since for all $\xi \in \Gamma$, $\lim_{N\to\infty} \mathbf{m}(\xi)\chi_{H_N^c}(\xi) = 0$, $|\mathbf{m}\chi_{H_N^c}| \leq \|\mathbf{m}\|_{L^\infty(\Gamma)}$, and $\left|\widehat{\varphi_n} * (\mathbf{m}\chi_{H_N^c})\right| \leq \|\mathbf{m}\|_{L^\infty(\Gamma)}$, it follows that

$$\lim_{N\to\infty} \|K_n - K_{n,N}\|_{L^2(G)} = \lim_{N\to\infty} \left\| \left(\widehat{\varphi_n} * (\mathbf{m}\chi_{H_N^c})\right) \widehat{h_n} \right\|_{L^2(\Gamma)} = 0.$$

Then K_n is supported in A_n. Hence $K_n \in L^1(G)$.

It holds that, for every $f \in L^2 \cap B$,

$$K_n * f = \left((\widehat{\varphi_n} * \mathbf{m})\widehat{h_n f} \right)^\vee ,$$

and then, since B is well behaved,

$$\begin{aligned} \|K_n * f\|_C &\leq \mathfrak{c}\, \|\widehat{\varphi_n}\|_1 \|\mathbf{m}\|_{M(B,C)} \|h_n * f\|_B \\ &\leq \mathfrak{c} \left\{ \|\mathbf{m}\|_{M(B,C)} \sup_n \left\|\widehat{h_n}\right\|_{M(B)} \right\} \|f\|_B . \end{aligned}$$

By the Dominated Convergence Theorem and Fatou's lemma, it follows that for $f \in SL^1 \cap B$,

$$\left\|(\mathbf{m}\widehat{f})^\vee\right\|_C \leq \liminf_n \|K_n * f\|_C \leq \mathfrak{c}\mathfrak{s}\, \|\mathbf{m}\|_{M(B,C)} \|f\|_B .$$

The result follows by density and Lemma 2.1.5. □

Doing slight modifications on the previous proof we can state the following maximal counterpart.

Theorem 2.3.16. *Let $\{\mathbf{m}_j\}_j \subset M(B,C) \cap L^\infty(G)$ be normalized (with respect to $\{\widehat{\varphi}_n\}$). Assume that either B or C is well behaved and for the pair (B,C) there exists $\mathfrak{c} > 0$ such that for every $\varphi \in L^1(\Gamma)$,*

$$\|\{\varphi * \mathbf{m}_j\}_j\|_{M(B,C)} \leq \mathfrak{c}\, \|\varphi\|_1 \, \|\{\mathbf{m}_j\}_j\|_{M(B,C)} \, .$$

Then there exists a sequence $\{\mathbf{m}_{n,j}\} \subset L^\infty(\Gamma)$ satisfying:

1. *For every j and all $\xi \in \Gamma$,*
$$\mathbf{m}_j(\xi) = \lim_n \mathbf{m}_{n,j}(\xi);$$
2. *$K_{n,j} = \mathbf{m}_{n,j}{}^\vee \in L^1(G)$ and it is compactly supported.*
3. *$\sup_n \left\|\{\mathbf{m}_{n,j}\}_j\right\|_{L^\infty(\Gamma)} \leq \left\|\{\mathbf{m}_j\}_j\right\|_{L^\infty(\Gamma)}$.*
4. *$\sup_n \|\{\mathbf{m}_{n,j}\}_j\|_{M(B,C)} \leq \mathfrak{c}\mathfrak{s} C_B \, \|\{\mathbf{m}_{n,j}\}_j\|_{M(B,C)}$.*

The remain part of the section is devoted to study situations where we can apply Theorems 2.3.13 and 2.3.16. In Table 2.3.16.1 we present examples of pairs of spaces where these theorems hold that we will use later.

Proposition 2.3.17. *Assume that C is a BFS and B is a QBFS. Suppose that $\mathbf{m} \in M(B,C) \cap L^\infty(\Gamma)$ and $\varphi \in L^1(\Gamma)$. Then the convolution $\varphi * \mathbf{m} \in M(B,C)$ and*

$$||\varphi * \mathbf{m}||_{M(B,C)} \leq C_B ||\varphi||_1 ||\mathbf{m}||_{M(B,C)}.$$

Proof. Observe that given $f \in SL^1 \cap B$

$$\int_\Gamma (\varphi * \mathbf{m})(\xi) \widehat{f}(\xi)\xi(u)\, d\xi = \int_\Gamma \varphi(\eta)\eta(u) \int_\Gamma \mathbf{m}(\xi)\widehat{f\eta}(\xi)\xi(u)\, d\xi d\eta. \qquad (2.3.18)$$

So taking norm in C, since $||\eta f||_B = ||f||_B$, by Minkowski's integral inequality,

$$||(\varphi * \mathbf{m})^\vee * f||_C \leq ||\varphi||_1 ||\mathbf{m}||_{M(B,C)} ||f||_B,$$

from where the result follows by density. □

Proposition 2.3.19. *Let C be a BFS and B be a QBFS with an underlying Radon measure. Suppose that $\{\mathbf{m}_j\}_j \subset M(B,C) \cap L^\infty(\Gamma)$ and $\varphi \in L^1(\Gamma)$. Then $\{\varphi * \mathbf{m}_j\}_j \subset M(B,C)$ and*

$$\left\|\{\varphi * \mathbf{m}_j\}_j\right\|_{M(B,C)} \leq C_B ||\varphi||_1 \, \|\{\mathbf{m}_j\}_j\|_{M(B,C)} \, .$$

	B	C
I	RIBFS	RIBFS
II	RIQBFS	RIBFS well behaved
III	$L^1(G)$	$L^{1,s}(G)$ $1 < s \leq \infty$
IV	u or v is Beurling or A_p weight	
	$L^p(u)$, $1 \leq p < \infty$	$L^p(v)$ $1 \leq p < \infty$
V	$L^p(u)$, $1 \leq p < \infty$ u Beurling or A_p weight	$L^{p,\infty}(v)$ $1 \leq p < \infty$

Table 2.3.16.1: Approximation of multipliers

Proof. Observe that given $f \in SL^1 \cap B$, by (2.3.18) for every j,

$$\sup_j \left|T_{\varphi * \mathbf{m}_j} f(x)\right| \leq \int_\Gamma |\varphi(\eta)| \sup_j \left| \int_\Gamma \mathbf{m}_j(\xi) \widehat{f\eta}(\xi)\xi(u)\, d\xi \right| \, d\eta.$$

By Minkowski's integral inequality, taking norm in C and using that $||\eta f||_B = ||f||_B$ and density, the result follows. □

In the case that C is not a Banach space, Minkowski's integral inequality does not hold. Despite this lack of convexity in the space, in some cases it is possible to ensure (2.3.14) to hold. This situation appears for (maximal) multipliers that continuously map $L^1(G)$ into $L^{1,\infty}(G)$ (see [1] for the case $G = \mathbb{R}^d$ and [14, 88] for arbitrary G).

Proposition 2.3.20. *Let u, v be weights on G and $1 < q \leq \infty$. For every $\varphi \in L^1(\Gamma)$ and all $\mathbf{m} \in M(L^1(u), L^{1,q}(v)) \cap L^\infty(\Gamma)$, there exists $c_q > 0$ such that*

$$||\varphi * \mathbf{m}||_{M(L^1(u),L^{1,q}(v))} \leq c_q ||\varphi||_1 ||\mathbf{m}||_{M(L^1(u),L^{1,q}(v))}.$$

Proposition 2.3.21. *Let u, v be weights on G, $1 < q \leq \infty$. For every $\varphi \in L^1(\Gamma)$ and every $\{\mathbf{m}_j\}_j \subset L^\infty(\Gamma) \cap M(L^1(u), L^{1,q}(v))$ there exists $c_q > 0$ such that*

$$\|\{\varphi * \mathbf{m}_j\}_j\|_{M(L^1(u),L^{1,q}(v))} \leq c_q \|\varphi\|_{L^1} \|\{\mathbf{m}_j\}_j\|_{M(L^1(u),L^{1,q}(v))}.$$

See Appendix B for a proof. The technique used involves a linearization procedure and a Marcinkiewicz-Zygmund's type inequality (see Theorems B.1.5

and B.1.8) that allows to recover the case $u = v = 1$ and $q = \infty$ with a better constant than that obtained in [14, 88].

Well behaved spaces

Proposition 2.3.22. *If C is a RIBFS with Haar's measure as its underlying measure such that integrable simple functions are dense, then C is well behaved. Moreover, there exists an associated family to C $\{h_j\}_j$ such that*

$$\mathfrak{s} = \sup_j ||\widehat{h}_j||_{M(C)} \leq 1.$$

Proof. Let $\{V_n\}$ be a family of open relatively compact neighborhoods of e like in Lemma 2.2.1. For all n, let $h_n \in \mathcal{C}_c(G)$ such that $h_n(v) = h_n(v^{-1})$, $h_n \geq 0$, $\operatorname{supp} h_n \subset V_n$ and $\int h_n = 1$. Hence, $\|\widehat{h_n}\|_{L^\infty} \leq 1$. For every $f \in SL^1 \cap C$, by Minkowski's inequality, and since translation is an isometry on C, $\|h_n * f\|_C \leq \|f\|_C$. Then, by density $\|\widehat{h_n}\|_{M(C)} \leq 1$.

Finally, since every $\xi \in \Gamma$ is a continuous function on G and $\xi(e) = 1$, for every $\epsilon > 0$ there exists n_0 such that for all $u \in V_{n_0}$, $|1 - \xi(u)| < \epsilon$. Hence, for every $n \leq n_0$,

$$\left|1 - \widehat{h_n}(\xi)\right| \leq \int h_n(u)\,|1 - \xi(u)|\ du \leq \epsilon.$$

□

Definition 2.3.23. [29, 89] *A Beurling weight on G is a measurable locally bounded function satisfying, $w > 0$ a.e. and, for each $u, v \in G$, $w(uv) \leq w(u)w(v)$.*

Proposition 2.3.24. *Let w be a Beurling weight. Then, for $1 \leq p < \infty$, $L^p(w)$ is well behaved. Moreover, if $\mathfrak{l} = \limsup_{u\to e} w(u)$, fixed $\epsilon > 0$, the associated family to $L^p(w)$ can be taken such that*

$$\mathfrak{s} = \sup_n ||\widehat{h}_n||_{M(L^p(w))} \leq \mathfrak{l}^{1/p} + \epsilon.$$

Proof. Consider the family $\{h_n\}_n$ and the open relatively compact sets $\{V_n\}$ given in the proof of the previous proposition. Observe that, for $f \in SL^1 \cap L^p(w)$,

$$\|h_n * f\|_{L^p(w)} \leq \int_G h_n(u)\,\|L_u f\|_{L^p(w)}\ du \leq \left(\int_G h_n(u) w(u)^{1/p}\, du\right) \|f\|_{L^p(w)}\,.$$

Since w is locally bounded, $\int h_n = 1$, $\operatorname{supp} h_n \subset V_n \subset V_1$,

$$\int_G h_n(u) w(u)^{1/p}\, du \leq \sup_{u\in V_1} w(u)^{1/p} < \infty.$$

Then by density, $\|h_n\|_{M(L^p(w))} \leq \sup_{u\in V_1} w(u)^{1/p}$, uniformly on n.

Let us observe that in fact we can prove that, for all n_0 and every $n \geq n_0$,

$$\sup_{n\leq n_0} \left\|\widehat{h_n}\right\|_{M(L^p(w))} \leq \sup_{u\in V_{n_0}} w(u)^{1/p},$$

from where the last assertion follows as $\mathfrak{l} = \inf_n \sup_{u\in V_n} w(u)$. □

Definition 2.3.25. *Given a weight in $\mathbb{R}^d$, we say that w belongs to the Muckenhoupt class $A_p(\mathbb{R}^d)$, and we write $w \in A_p(\mathbb{R}^d)$ if,*

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)^{1/p}\left(\frac{1}{|Q|}\int_Q w(x)^{1/1-p}\,dx\right)^{1-1/p} < \infty,$$

if $1 < p < \infty$, and

$$[w]_{A_1} = \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left\|w^{-1}\chi_Q\right\|_\infty < +\infty.$$

if $p = 1$, where the supremum is considered over the family of cubes Q with sides parallel to the coordinate axes.

We will see that in the case that $G = \mathbb{R}^d$ and $w \in A_p(\mathbb{R}^d)$, $L^p(w)$ is well behaved. To this end we need the following result.

Theorem 2.3.26. [79, Theorem 2] *Let $1 < p < \infty$, $\varphi \in \mathcal{C}_c^d(\mathbb{R}^d \setminus \{0\})$ satisfying that*

$$c_{\alpha,\varphi} = \sup_{\substack{|\alpha|\le d \\ \alpha=(\alpha_1,\dots,\alpha_d)}} \sup_{r>0}\left(r^{2|\alpha|-d}\int_{r<|x|<2r}\left|\frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1}\dots\partial x_d^{\alpha_d}}(x)\right|^2 dx\right)^{1/2} < \infty, \quad (2.3.27)$$

and $w \in A_p(\mathbb{R}^d)$. Then there exists a constant c depending only on $c_{\alpha,\phi}$ such that,

$$||B_{\varphi^\vee} f||_{L^p(w)} \le c||f||_{L^p(w)}, \quad (2.3.28)$$

In particular, for every $s > 0$, $\overline{D}_{\frac{1}{s}}\varphi(x) = \varphi\left(\frac{x}{s}\right)$ satisfies (2.3.27) with the same constant than φ.

Proposition 2.3.29. *Let $1 \le p < \infty$ and $w \in A_p(\mathbb{R}^d)$. The space $L^p(w)$ is well behaved. Moreover the associated family to $L^p(w)$ $\{h_n\}_n$ can be taken such that h_n is radial and belongs to $\mathcal{C}_c^\infty(\mathbb{R}^d)$.*

Proof. It suffices to consider $h \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $h \ge 0$, radial, supported in $(-1,1)^d$, such that $\int h = 1$ and define $h_n(x) = n^d h(nx)$. With the same argument as in Proposition 2.3.22, it is proved that $\widehat{h_n} \to 1$ and $\left\|\widehat{h_n}\right\|_\infty \le 1$. It remains to find a constant c such that $\left\|\widehat{h_n}\right\|_{M(L^p(w))} \le c$ uniformly on n.

Let consider first the case $p > 1$. Let $\varphi = \widehat{h} \in \mathcal{S}(\mathbb{R}^d)$. Hence

$$\max_{|\alpha|\le d}\sup_{x\in\mathbb{R}^n}\left(|x|^{|\alpha|}\left|\frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1}\dots\partial x_d^{\alpha_d}}(x)\right|\right) = c_{\alpha,\varphi} < +\infty,$$

from where it is easy to see that φ satisfies (2.3.27). Then, by Theorem 2.3.26 it follows that for all n, $\varphi(x/n) = \widehat{h_n}(x) \in M(L^p(w))$ with norm uniformly bounded on n.

Let finally prove the case $p = 1$. Fix $\alpha \in \mathbb{N}$, $\alpha > d$. By Minkowski's integral inequality

$$\|h_n * f\|_{L^1(w)} \leq \int |f(y)| \int h_n(x-y)w(x)\ dxdy.$$

Thus, fixed $y \in \mathbb{R}^d$ and $n > 0$, the inner integral can be split in

$$\int_{|x-y|<n^{-1}} + \sum_{j\geq 0} \int_{2^j n^{-1}<|x-y|\leq 2^{j+1}n^{-1}} h\left(n(x-y)\right) n^d w(x)\ dx.$$

The first term can be bounded by

$$\|h\|_\infty n^d \int_{|x-y|<n^{-1}} w(x)\ dx \leq \|h\|_\infty [w]_{A_1} 2^d w(y).$$

On the other hand, if $p_{0,\alpha}(\varphi) = \sup_{x\in\mathbb{R}^n} |\varphi(x)|\,|x|^\alpha$, each term on the sum can be bounded by

$$\begin{aligned} p_{0,\alpha}(h)n^{d-\alpha} &\int_{2^j n^{-1}<|x-y|\leq 2^{j+1}n^{-1}} |x-y|^{-\alpha} w(x)\ dx \\ &\leq p_{0,\alpha}(h)2^{-j\alpha}n^d \int_{|x-y|\leq 2^{j+1}n^{-1}} w(x)\ dx \\ &\leq p_{0,\alpha}(h)4^d 2^{j(d-\alpha)}[w]_{A_1} w(y). \end{aligned}$$

Then the sum is bounded by $\frac{p_{0,\alpha}(h)4^d}{1-2^{d-\alpha}}[w]_{A_1}w(y)$. From the previous bounds, it follows that

$$\|h_n * f\|_{L^1(w)} \leq c_{\alpha,d,h,w} \|f\|_{L^1(w)},$$

where $c_{\alpha,d,h,w} = [w]_{A_1}2^d\left(\|h\|_\infty + p_{0,\alpha}(h)\frac{2^d}{1-2^{d-\alpha}}\right)$. □

Chapter 3

Amalgam approach

3.1 Transference Wiener amalgams

Wiener amalgams were introduced by H. Feichtinger in the 80's and there have been widely studied thereafter being very useful in time-frequency analysis and sampling theory (see for instance [57, 58, 68] and the references therein). It is not our intention to describe these spaces in whole generality, so we will avoid details and technical hypotheses restricting ourselves to a particular case of them. So, let B and C be BFS's of measurable functions defined on G, and assume that left translation is a continuous isometry on the space B (for example, if $B = L^p(G)$).

Definition 3.1.1 (see [68] and references therein)**.** *Given a relatively compact non empty open set $V \subset G$, the Wiener amalgam space $W(B, C)$ is defined by*

$$W(B,C) := \left\{ f \in L^1_{\text{loc}}(G) : \ ||f||_{W(B,C)} < \infty \right\},$$

where $\|f\|_{W(B,C)} = \left\| \left\| L_x\left(\chi_V\right)(v) f(v) \right\|_B \right\|_C$, and the inner norm is taken with respect to the variable v and the outer with respect to the variable x.

Since, we have assumed that left translations are isometries on B, it holds that, given $f \in W(B, C)$,

$$\left\| \left\| L_x\left(\chi_V\right)(v) f(v) \right\|_B \right\|_C = \left\| \left\| \chi_V(y) f(xv) \right\|_B \right\|_C = \left\| \left\| \chi_V(v) D_v f(x) \right\|_B \right\|_C,$$

that is,

$$W(B,C) = \left\{ f \in L^1_{\text{loc}}(G) : \ \left\| \left\| \chi_V(v) D_v f(x) \right\|_B \right\|_C < \infty \right\},$$

where D denotes the right translation on the group, that is also a representation of the group on $L^1_{\text{loc}}(G)$. This trivial observation, jointly with our aim of extend transference theorems, gave us the idea for giving the definition of transference Wiener amalgams, that as we will see, naturally appears in our framework. Essentially, the key consists in replacing the translation acting on L^1_{loc} by a general representation R acting on a Banach space.

Throughout this chapter F denotes a Banach space whose elements are (classes of) measurable functions defined on $\mathcal{M}$, which is continuously embedded in $L^1_{\text{loc}}(\mathcal{M})$, and R is a continuous representation of G on F. We will denote by B, C QBFS's on G and by E a QBFS on $\mathcal{M}$.

Definition 3.1.2. *Let V be a non empty open set. Assume that the function*

$$x \mapsto \|\chi_V(v)R_v f(x)\|_B\,, \tag{3.1.3}$$

is μ-measurable. The transference Wiener amalgam $W(B,E,V)$, TWA for short, is defined by

$$W(B,E,V) := \left\{ f \in F :\ \|f\|_{W(B,E,V)} = \|\|\chi_V(v)R_v f(x)\|_B\|_E < \infty \right\}.$$

The previous definition depends on F and on the representation R, but we omit this on the notation by simplicity. In the case that the group is compact, we will uniquely consider the space $W(B,E,G)$.

Observe that if $\mathcal{M} = G$, the representation is given by right translations, V is a locally compact open set, B and C are BFS and B is such that left translations is an isometry, then

$$W(B,E,V) = F \cap W(B,E),$$

where $W(B,E)$ is the usual Wiener amalgam. The difference relies in that we only allow F to be a Banach space, instead of a general Frèchet space, like L^1_{loc}.

Measurability condition (3.1.3), is imposed in order that the defining expression of TWA makes sense, but in the applications we are going to present it is satisfied. Hence we implicitly assume it to hold in the appearing amalgams.

3.1.1 General Transference Results

Theorem 3.1.4. *Let $K \in L^1(G)$ with compact support such that $B_K : B \to C$ is bounded with norm less than or equal to $N_{B,C}(K)$. Let $\mathcal{K}$ be a compact set containing* supp K. *Given a non empty open set $V \subset G$, it holds that*

$$\|T_K f\|_{W(C,E,V)} \le N_{B,C}(K)\, \|f\|_{W(B,E,V\mathcal{K}^{-1})}\,.$$

Proof. Fix a non empty open set V. Let $f \in F$. Observe that in a vectorial sense, for every $v \in G$,

$$R_v T_K f = \int_G K(u) R_{vu^{-1}} f\, du.$$

Fix a compact set $W \subset G$. Let $c_W = \sup_{v\in W} \|R_v\|_{\mathfrak{B}(F)}$ and similarly let $c_K = \sup_{u\in(\operatorname{supp} K)^{-1}} \|R_u\|_{\mathfrak{B}(F)}$ that are finite by the uniform boundedness principle. Since $(u,x) \mapsto R_u f(x)$ is jointly measurable in $G \times \mathcal{M}$, it follows that also is $R_{vu^{-1}} f(x)$ in $G\times G\times \mathcal{M}$. On the other hand, by Tonelli's Theorem, the mapping $(v,x) \mapsto \int_G |R_{vu^{-1}} f(x)|\ |K(u)|\ du$, is measurable, and by Minkowski's integral inequality, for any $v \in W$,

$$\left\| \int_G |R_{vu^{-1}} f(x)|\ |K(u)| du \right\|_F \le c_W c_K \|K\|_{L^1(G)} \|f\|_F < +\infty.$$

Hence, fixed a set of finite measure $\mathcal{M}_1 \subset \mathcal{M}$,

$$\begin{aligned}
&\int_{G\times W\times \mathcal{M}_1} |R_{vu^{-1}} f(x)|\; |K(u)|\, d(\lambda\times\lambda\times\mu)(u,v,x) \\
&= \int_W \int_{\mathcal{M}_1} \int_G |R_{vu^{-1}} f(x)|\; |K(u)|\, du\, d\mu(x)\, dv \\
&\le c_{\mathcal{M}_1} \int_W \left\| \int_G |R_{vu^{-1}} f(x)|\; |K(u)| du \right\|_E dv < +\infty.
\end{aligned}$$

Then, by Fubini's Theorem and the σ-finiteness of $G\times\mathcal{M}$, it follows that $(v,x)\mapsto \int_G R_{vu^{-1}} f(x)K(u)\,du$, is $(\lambda\times\mu)$-measurable and locally integrable. Since for every set of finite measure $\mathcal{M}_1$, $\chi_{\mathcal{M}_1}\in E^*$, by Fubini's theorem, for λ-a.e. $v\in G$,

$$\begin{aligned}
\int_{\mathcal{M}_1}\int_G R_{vu^{-1}} f(x)K(u)\,du d\mu(x) &= \int_G <\chi_{\mathcal{M}_1}, R_{vu^{-1}} f(\cdot)> K(u)\,du \\
&= <\chi_{\mathcal{M}_1}, R_v T_K f> .
\end{aligned}$$

Thus, $R_v T_K f(x) = \int_G R_{vu^{-1}} f(x)K(u)\,du$, μ-a.e. $x\in\mathcal{M}$. By the joint measurability it follows that the equality holds $(\lambda\times\mu)$-a.e. $(v,x)\in G\times\mathcal{M}$

Let V be a non empty open set and let $\mathcal{K}$ be a compact set containing supp K. Then, for every $v\in V$ and $x\in\mathcal{M}$,

$$K(u)R_{vu^{-1}} f(x) = K(u)\chi_{V\mathcal{K}^{-1}}(vu^{-1})R_{vu^{-1}} f(x),$$

hence $(\lambda\times\mu)$-a.e. $(v,x)\in G\times\mathcal{M}$,

$$\begin{aligned}
\chi_V(v)R_v T_K f(x) &= \chi_V(v)\int K(u)R_{vu^{-1}} f(x)\; du \\
&= \chi_V(v)\int K(u)\chi_{V\mathcal{K}^{-1}}(vu^{-1})R_{vu^{-1}} f(x)\; du \\
&= \chi_V(v)B_K\left(\chi_{V\mathcal{K}^{-1}}R_{\cdot} f(x)\right)(v).
\end{aligned} \tag{3.1.5}$$

Thus, enlarging the domain, by the lattice property of C and the boundedness assumption, μ-a.e. x

$$\|\chi_V(v)R_v T_K f(x)\|_C \le \|B_K\left(\chi_{V\mathcal{K}^{-1}}R_{\cdot} f(x)\right)\|_C \le N_{B,C}(K)\,\|\chi_{V\mathcal{K}^{-1}}R_{\cdot} f(x)\|_B .$$

Therefore, by the lattice property of E and the definition of TWA, it follows that

$$\|T_K f\|_{W(C,E,V)} \le N_{B,C}(K)\,\|f\|_{W(B,E,V\mathcal{K}^{-1})} .$$

□

The usefulness of the classical Transference Theorem ([46, Theorem 2.4]) is that the obtained bound for the transferred operator does not depend either on the L^1 norm or on the support of the kernel K. The bound obtained in the previous theorem depends on the support of K, but in comparison with the classical results, neither the amenability condition on the group nor the uniformly boundedness of the representation is assumed. In the case of G to be compact, this dependency disappears so V is taken to be G.

Before going on, we will show how this theorem can be applied to recover the classical result.

Corollary 3.1.6. [46, Theorem 2.4] *Let G be an amenable group, and let R be a continuous representation of G acting on $L^p(\mathcal{M})$, with $1 \le p < \infty$, satisfying*

$$c = \sup_{u\in G} \|R_u\|_{\mathfrak{B}(L^p(\mathcal{M}))} < \infty.$$

If $K \in L^1(G)$ with compact support is such that B_K maps boundedly $L^p(G)$ into itself with norm $N_p(K)$, then for $f \in L^p(\mathcal{M})$,

$$\|T_K f\|_{L^p(\mathcal{M})} \le c^2 N_p(K) \|f\|_{L^p(\mathcal{M})}.$$

Proof. Let $E = F = L^p(\mathcal{M})$, $B = C = L^p(G)$, $\mathcal{K} = \operatorname{supp} K$ and let V be an open relatively compact set. For every $f \in F$ condition by Fubini's Theorem,

$$\begin{aligned} \|f\|_{W(L^p(\mathcal{M}),L^p(G),V)} &= \left\{ \int_{\mathcal{M}} \int_V |R_v f(x)|^p \, dv d\mu(x) \right\}^{1/p} \\ &= \left\{ \int_V \|R_v f\|^p_{L^p(\mathcal{M})} \, dv \right\}^{1/p} \end{aligned}$$

Since, for any $v \in G$, $\frac{\|f\|_{L^p(\mathcal{M})}}{c} \le \|R_v f\|_{L^p(\mathcal{M})} \le c \|f\|_{L^p(\mathcal{M})}$, it follows that

$$\|f\|_{L^p(\mathcal{M})} \frac{\lambda(V)^{1/p}}{c} \le \|f\|_{W(L^p(\mathcal{M}),L^p(G),V)} \le \|f\|_{L^p(\mathcal{M})} c\lambda(V)^{1/p}. \tag{3.1.7}$$

So applying Theorem 3.1.4 and using the previous inequalities,

$$\frac{\lambda(V)^{1/p}}{c} \|T_K f\|_{L^p(\mathcal{M})} \le N_p(K) c \lambda \left(V\mathcal{K}^{-1}\right)^{1/p} \|f\|_{L^p(\mathcal{M})}.$$

Then

$$\|T_K f\|_{L^p(\mathcal{M})} \le c^2 \left(\frac{\lambda\left(V\mathcal{K}^{-1}\right)}{\lambda(V)} \right)^{1/p} N_p(K) \|f\|_{L^p(\mathcal{M})},$$

and the statement follows by amenability Følner condition (2.2.4). □

Observe that the key for removing the dependency on the support of the kernel has been to identify the TWA appearing in the proof. That is, (3.1.7) can be read as $L^p(\mathcal{M}) = W\left(L^p(G), L^p(\mathcal{M}), V\right)$, with equivalent norms. In the general case, the main idea consists in finding spaces X and Y such that $X \subset W(B, E, V\mathcal{K}^{-1})$, $W(C, E, V) \subset Y$, assuming some control on the norm of the embeddings. This idea seems to be difficult to apply in whole generality, but it can be done in some particular situations.

In the study of transference of weak type inequalities, stronger conditions than uniformly bounded on the representation naturally appear. Initially, Coifman and Weiss assumed the representation to be given by a family of measure preserving transformations of $\mathcal{M}$ (see [46]). In [5, 12] it was introduced the distributionally bounded representations as a generalization of this type of representations.

Definition 3.1.8. *A representation R of G on $L^1 \cap L^\infty$ is called distributionally bounded if, for some $c > 0$, $\mu_{R_u f}(t) \leq c\mu_f(t)$ for all $f \in L^1 \cap L^\infty$, $u \in G$, $t > 0$.*

Let us observe that if R is a distributionally bounded representation, for any $f \in L^1 \cap L^\infty$, $s > 0$ and $u \in G$

$$\mu_f(s) = \mu_{R_{u^{-1}} R_u f}(s) \leq c\mu_{R_u f}(s),$$

and hence $\mu_f \approx \mu_{R_u f}$.

The following result can be found in [12, Theorem 2.7] for abelian groups. But the proof therein carries over non abelian case.

Lemma 3.1.9. *Let R be a distributionally bounded representation of G and let $1 \leq p < \infty$. Then there exists an unique representation of G on L^p,$R^{(p)}$, such that, for all $u \in G$ $R_u^{(p)}|_{L^1 \cap L^\infty} = R_u$, $\sup_{u \in G} \|R_u^{(p)}\|_{\mathfrak{B}(L^p)} \leq c^{1/p}$ and for all $f \in L^p$, $u \in G$ and $t > 0$,*

$$\mu_{R_u^{(p)} f}(t) \leq c\mu_f(t). \tag{3.1.10}$$

Definition 3.1.11. *We say that a distributionally bounded representation R is continuous, if its extension to L^1 defines a strongly continuous representation.*

In the particular case that $1 \leq p = r < \infty$, $s = \infty$, the following result recovers [46, Theorem 2.6] whenever R is given by a family of measure preserving transformations, and [12, Theorem 4.1] in the single kernel situation, whenever R is given by a continuous distributionally bounded representation of an abelian group G, at least in the case that $\mathcal{M}$ is a σ-finite measure space. Even in the case that the representation is distributionally bounded, the following result is new in the given range of indices.

Corollary 3.1.12. *Let G be an amenable group and let R be a continuous distributionally bounded representation of G. If $K \in L^1(G)$ with compact support such that for $0 < p < \infty$, $0 < r \leq p \leq s \leq \infty$ and for every $f \in L^{p,r}(G)$,*

$$\|B_K f\|_{L^{p,s}(G)} \leq N(K) \|f\|_{L^{p,r}(G)},$$

then, for $f \in L^{p,r}(\mathcal{M})$,

$$\|T_K f\|_{L^{p,s}(\mathcal{M})} \leq c^{2/p} N(K) \|f\|_{L^{p,r}(\mathcal{M})},$$

where c is given in (3.1.10).

Proof. Observe that, by density, it suffices to prove the inequality for $f \in L^1 \cap L^{p,r}(\mathcal{M})$. Let $F = L^1(\mathcal{M})$, $E = L^p(\mathcal{M})$, $B = L^{p,r}(G)$, $C = L^{p,s}(G)$, $\mathcal{K} = \operatorname{supp} K$. Fixed $\epsilon > 0$, let V be an open relatively compact set such that $\lambda(V\mathcal{K}^{-1}) \leq$

$(1+\epsilon)\lambda(V)$. Observe that, for $f \in L^1(\mathcal{M})$, by (3.1.10),

$$\begin{aligned}
&\|f\|_{W(L^{p,r}(G),L^p(\mathcal{M}),V\mathcal{K}^{-1})}\\
&= \left\{ \int_{\mathcal{M}} \left[\int_0^\infty p \left(\lambda_{R_\cdot^{(1)} f(x)\chi_{V\mathcal{K}^{-1}}}(t) \right)^{\frac{r}{p}} t^{r-1} dt \right]^{p/r} d\mu(x) \right\}^{\frac{r}{p}\frac{1}{r}}\\
&\le \left\{ \int_0^\infty \left[\int_{\mathcal{M}} \lambda_{R_\cdot^{(1)} f(x)\chi_{V\mathcal{K}}}(t)\, d\mu(x) \right]^{r/p} p t^{r-1} dt \right\}^{1/r} \qquad (3.1.13)\\
&\le \left\{ \int_0^\infty \left[\int_{V\mathcal{K}^{-1}} \mu_{R_u^{(1)} f}(t)\, du \right]^{r/p} p t^{r-1} dt \right\}^{1/r}\\
&\le c^{1/p} \lambda(V\mathcal{K}^{-1})^{1/p} \|f\|_{L^{p,r}}.
\end{aligned}$$

In other words, $L^1 \cap L^{p,r} \hookrightarrow W(L^{p,r}(G), L^p(\mathcal{M}), V\mathcal{K})$. On the other hand, for $0 < t < \infty$, since $\mu_f(t) \le c\frac{1}{\lambda(V)} \int_V \mu_{R_u^{(1)} f}(t)\ du$,

$$\begin{aligned}
\frac{\lambda(V)^{1/p}}{c^{1/p}} \|f\|_{L^{p,s}} &\le \left\{ \int_0^\infty \left[\int_V \mu_{R_u^{(1)} f}(t)\, du \right]^{s/p} p t^{s-1} dt \right\}^{1/s}\\
&= \left\{ \int_0^\infty \left[\int_{\mathcal{M}} \lambda_{R_\cdot^{(1)} f(x)\chi_V}(t)\, d\mu(x) \right]^{s/p} p t^{s-1} dt \right\}^{\frac{p}{s}\frac{1}{p}}\\
&\le \left\{ \int_{\mathcal{M}} \left[\int_0^\infty \left(\lambda_{R_\cdot^{(1)} f(x)\chi_V}(t) \right)^{s/p} p t^{s-1}\, dt \right]^{p/s} d\mu(x) \right\}^{1/p}\\
&= \|f\|_{W(L^{p,s}(G),L^p(\mathcal{M}),V)}
\end{aligned} \qquad (3.1.14)$$

with the suitable modifications for $s = \infty$. Thus applying Theorem 3.1.4 and using the previous inequalities,

$$\frac{\lambda(V)^{1/p}}{c^{1/p}} \|T_K f\|_{L^{p,s}(\mathcal{M})} \le N(K) c^{1/p} \lambda\left(V\mathcal{K}^{-1}\right)^{1/p} \|f\|_{L^{p,r}(\mathcal{M})}.$$

Then, for $f \in L^1 \cap L^{p,r}(\mathcal{M})$,

$$\|T_K f\|_{L^{p,s}(\mathcal{M})} \le c^{2/p} (1+\epsilon)^{1/p} N(K) \|f\|_{L^{p,r}(\mathcal{M})}.$$

Hence, letting ϵ tends to 0, the result follows. □

Imposing some extra assumptions we can obtain the result for spaces more general than Lorentz spaces $L^{p,q}$.

Definition 3.1.15. *Let h be a positive function h defined on $(0, \infty)$ that is equivalent to an increasing function. We call h to be quasi-concave (respectively quasi-convex) if $\frac{h(t)}{t}$ is equivalent to a decreasing (respectively increasing) function. Let us observe that if h is quasi-concave, then h is equivalent to a concave function* *(see [19]). On the other hand, if h is quasi-convex and h satisfies Δ_2 condition, that is $h(2t) \lesssim h(t)$, it is equivalent to a convex function (see [31, Lemma 2.2]).*

For example, the functions $h_1(t) = t(1+\log^+ \frac{1}{t})^{-1}$ and $h_2(t) = t(1+\log^+ \frac{1}{t})$ are respectively quasi-concave and quasi-convex functions. More generally, a function $h(t) = t\gamma(t)$ where $\gamma(t)$ is a slowly varying function (see Appendix D for its definition) is quasi-concave (respectively quasi-convex), provided γ is decreasing (respectively increasing).

Corollary 3.1.16. *Let G be an amenable group and let R be a continuous distributionally bounded representation of G. Assume that $\mu(\mathcal{M}) = 1$. Let $K \in L^1(G)$ with compact support such that for $0 < p \leq q < \infty$,*

$$\|B_K f\|_{\Lambda^q(w,G)} \leq N(K)\, \|f\|_{\Lambda^p(u,G)}\,.$$

Assume that there exists $r \in [p,q]$ such that $U^{r/p}$ is quasi-concave and $W^{r/q}$ is quasi-convex, where $U(t) = \int_0^t u$ and $W(t) = \int_0^t w$ and assume also that $W \in \Delta_2$. Then, for $f \in \Lambda^p_{\mathcal{M}}(u)$,

$$\|T_K f\|_{\Lambda^q(w,\mathcal{M})} \lesssim N(K)\, \|f\|_{\Lambda^p_{\mathcal{M}}(u,\mathcal{M})}\,,$$

Proof. Let $F = L^1(\mathcal{M})$, $E = L^r(\mathcal{M})$, $B = \Lambda^p(u,G)$, $C = \Lambda^q(w,G)$. Observe that, since $U^{r/p}$ is quasi-concave, it holds that for every $s > 0$

$$\frac{U^{r/p}(2s)}{2s} \lesssim \frac{U^{r/p}(s)}{s},$$

and hence U satisfies Δ_2 condition. Since $U, W \in \Delta_2$, B, C are QBFS (see [38, Thm. 2.2.13 and Thm 2.3.1]).

We can assume that G is not compact. Let $\mathcal{K}$ be a compact set containing $\{e\} \cup \operatorname{supp} K$ such that $\lambda(\mathcal{K}^{-1}) > 1$. Then for every relatively compact open neighborhood V of e, $\frac{\lambda(V\mathcal{K}^{-1})}{\lambda(V)} \geq \max\left(1, \frac{\lambda(\mathcal{K}^{-1})}{\lambda(V)}\right)$. Since G is amenable

$$1 = \inf_V \frac{\lambda(V\mathcal{K}^{-1})}{\lambda(V)} = \inf_{\lambda(V)\geq\lambda(\mathcal{K}^{-1})} \frac{\lambda(V\mathcal{K}^{-1})}{\lambda(V)}.$$

In other words, fixed $\epsilon > 0$ there exists a relatively compact open set V such that $\lambda(V\mathcal{K}^{-1}) \leq (1+\epsilon)\lambda(V)$ and $\lambda(V) > 1$. Since R is distributionally bounded, there exists a constant $c \geq 1$ such that, for any $f \in L^1(\mathcal{M})$ and $u \in G$, $\mu_{R_u f}(t) \leq c\mu_f(t)$.

Observe that, for $f \in L^1(\mathcal{M})$ by Minkowski's integral inequality,

$$\begin{aligned}
&\|f\|_{W(\Lambda^p(u,G),L^r(\mathcal{M}),V\mathcal{K}^{-1})} = \\
&= \left\{ \int_{\mathcal{M}} \left[\int_0^\infty pt^{p-1} U\left(\lambda_{R.f(x)\chi_V}(t)\right) dt \right]^{r/p} d\mu(x) \right\}^{1/r} \\
&\leq \left\{ \int_0^\infty pt^{p-1} \left[\int_{\mathcal{M}} U\left(\lambda_{R.f(x)\chi_{V\mathcal{K}^{-1}}}(t)\right)^{r/p} \, d\mu(x) \right]^{p/r} dt \right\}^{1/p}.
\end{aligned}$$

But, by Jensen's inequality

$$\left[\int_{\mathcal{M}} U\left(\lambda_{R.f(x)\chi_{V\mathcal{K}^{-1}}}(t)\right)^{r/p}\, d\mu(x)\right]^{p/r} \lesssim U\left(\int_{\mathcal{M}} \lambda_{R.f(x)\chi_V}(t)\, d\mu(x)\right) \leq U\left(c\lambda(V\mathcal{K}^{-1})\mu_f(t)\right).$$

And hence, by Proposition D.1.4,

$$\begin{aligned} \|f\|_{W(\Lambda^p(u,G),L^r(\mathcal{M}),V\mathcal{K}^{-1})} &\lesssim \left\{\int_0^\infty pt^{p-1}U\left(c\lambda(V\mathcal{K}^{-1})\mu_f(t)\right)\, dt\right\}^{1/p} \\ &\lesssim h_{\Lambda^p(u)}(c\lambda(V\mathcal{K}^{-1}))\left\{\int_0^\infty U\left(\mu_f(t)\right)pt^{p-1}dt\right\}^{1/p} \\ &= h_{\Lambda^p(u)}(c\lambda(V\mathcal{K}^{-1}))\,\|f\|_{\Lambda^p(u,\mathcal{M})}. \end{aligned} \tag{3.1.17}$$

In a similar way it is shown that $\|f\|_{W(\Lambda^q(w),L^r(\mathcal{M}),V)} \gtrsim \frac{1}{h_{\Lambda^q(w)}\left(\frac{c}{\lambda(V)}\right)}\|f\|_{\Lambda^q(w,\mathcal{M})}$. So applying Theorem 3.1.4 and using the previous inequalities it follows that for $f \in L^1 \cap \Lambda^p(w,\mathcal{M})$,

$$\|T_K f\|_{\Lambda^q(w,\mathcal{M})} \lesssim \left\{h_{\Lambda^p(u)}\left(c\lambda(V\mathcal{K}^{-1})\right)h_{\Lambda^q(w)}\left(\frac{c}{\lambda(V)}\right)\right\}N(K)\,\|f\|_{\Lambda^p(u,\mathcal{M})}.$$

Since the dilation norm is submultiplicative, the term in curly brackets is less than or equal to

$$h_{\Lambda^p(u)}(c)h_{\Lambda^q(w)}(c)\left\{h_{\Lambda^p(u)}\left(\lambda(V\mathcal{K}^{-1})\right)h_{\Lambda^q(w)}\left(\frac{1}{\lambda(V)}\right)\right\}.$$

Since $U^{r/p}$ is quasi-concave and $W^{r/p}$ quasi-convex, using Proposition D.1.4, it follows that for $t \geq 1$, $h_{\Lambda^p(u)}(t) \leq t^{1/r}$ and, for $t \leq 1$ $h_{\Lambda^q(w)}(t) \leq t^{1/r}$. Hence, the term inside curly brackets in the last expression is bounded by

$$\left(\frac{\lambda(V\mathcal{K}^{-1})}{\lambda(V)}\right)^{1/r} \leq (1+\epsilon)^{1/r}.$$

Then, letting $\epsilon \to 0$, for $f \in L^1(\mathcal{M}) \cap \Lambda^p(u,\mathcal{M})$,

$$\|T_K f\|_{\Lambda^q(w,\mathcal{M})} \lesssim N(K)\,\|f\|_{\Lambda^p(u,\mathcal{M})}.$$

Since integrable simple functions are dense in $\Lambda^p(u,\mathcal{M})$ (see [38, Thm. 2.3.4]) the result follows by density. □

Examples of weights u, w satisfying the hypotheses of the previous result are given by $u(t) = t^{\frac{p}{r}-1}\gamma(t)$ and $w(t) = t^{\frac{q}{r}-1}\beta(t)$, where $\gamma(t) = (1+\log^+\frac{1}{t})^{Ap}$, $\beta(t) = (1+\log^+\frac{1}{t})^{Bq}$ and $B \leq 0 \leq A$ (more generally, the same hold if γ, β are slowly-varying functions decreasing and increasing, respectively). In this case, the involved spaces are $\Lambda^p(u) = L^{r,p}(\log L)^A$ and $\Lambda^q(w) = L^{r,q}(\log L)^B$, that are Lorentz-Zygmund spaces. Multiplier operators between Lorentz-Zygmund spaces are considered for instance in [67]. Observe that the case $A = B = 0$ is recovered as a particular situation of Corollary 3.1.12.

3.1.2 General maximal Transference results

Definition 3.1.18. *A linear mapping T on F is called separation-preserving (respectively, positivity-preserving) provided that whenever $f \in F$, $g \in F$, and $fg = 0$ μ-a.e. on $\mathcal{M}$ (respectively, $f \in F$ and $f \geq 0$ μ-a.e. on $\mathcal{M}$), we have $(Tf)(Tg) = 0$ μ- a.e. (respectively, $Tf \geq 0$ μ-a.e.).*

Definition 3.1.19. *The representation R is said to be separation preserving (respectively, positivity-preserving), provided that R_u is a separation-preserving (respectively, positivity-preserving) operator for each $u \in G$.*

The study of separation-preserving operators on L^p spaces goes back to Banach in the characterization of the linear, norm-preserving operators on L^p spaces. They are also called Lamperti operators (see [75, 82]). Separation and positivity preserving properties permit to transfer bounds for maximal convolution operators (see [7, 46]).

The following results are proved in [75, Proposition 3.1, Theorem 3.1] for the case $F = L^p(\mathcal{M})$. But the proof therein automatically carries over arbitrary BFS F provided integrable simple functions are dense.

Lemma 3.1.20. *Let T be an invertible linear map on F. If T and T^{-1} are positivity-preserving, then T is separation-preserving.*

Lemma 3.1.21. *Assume that F is a BFS on which integrable simple functions are dense. Let T be a linear continuous operator on F. Then T is separation-preserving if and only if there exists a positivity-preserving operator P on F, called the linear modulus of T, such that*

$$|Tf| = P\,|f| = |Pf|\,, \quad f \in F.$$

Observe that, if R is a positivity-preserving representation, R is separation-preserving and, for every $f \in F$ and all $u \in G$, $|R_u f| = R_u\,|f|$. Observe also that if T is a positivity-preserving operator and f, g are positive functions, then $T(\max(f,g)) \leq \max(Tf, Tg)$.

Theorem 3.1.22. *Let $\{K_j\}_{j=1,\ldots,N} \subset L^1(G)$ whose support is contained in a compact set $\mathcal{K}$, such that $B^\sharp : B \to C$ is bounded with norm $N(\{K_j\})$. Assume that R is a separation-preserving continuous representation of G on F, satisfying that, for all $u \in G$ there exists a positivity-preserving mapping P_u such that for every $f \in F$, $P_u\,|f| = |R_u f|$. Then, fixed a non empty open set $V \subset G$,*

$$\left\|T^\sharp f\right\|_{W(C,E,V)} \leq N\left(\{K_j\}\right) \|f\|_{W(B,E,V\mathcal{K}^{-1})}\,.$$

Proof. Fixed $u \in G$, $P_u\left|T_{K_{j_0}} f\right| \leq \sup_{1\leq j\leq N} P_u\left|T_{K_j} f\right|$ because P_u is positivity-preserving. Hence

$$\left|R_u T^\sharp f\right| = P_u \sup_{1\leq j\leq N}\left|T_{K_j} f\right| \leq \sup_{1\leq j\leq N} P_u\left|T_{K_j} f\right| = \sup_{1\leq j\leq N}\left|R_u T_{K_j} f\right|.$$

By (3.1.5), it follows that for $(\lambda \times \mu)$-a.e. $(v,x) \in G \times \mathcal{M}$,

$$\chi_V(v) R_v T^\sharp f(x) \leq \sup_{1\leq j\leq N} \left| B_{K_j} \left(\chi_{V\mathcal{K}^{-1}} R_. f(x) \right)(v) \right| .$$

Thus, by the lattice property of C and the boundedness assumption, μ-a.e. $x \in \mathcal{M}$,

$$\begin{aligned} \left\| \chi_V(v) R_v T^\sharp f(x) \right\|_C &\leq \left\| B^\sharp \left(\chi_{V\mathcal{K}^{-1}} R_. f(x) \right) \right\|_C \\ &\leq N\left(\{K_j\}\right) \left\| \chi_{V\mathcal{K}^{-1}}(v) R_v f(x) \right\|_B . \end{aligned}$$

The result follows by the lattice property of E and the definition of TWA. □

As it is the case in Corollary 3.1.6, if $1 \leq p < \infty$, $E = F = L^p(\mathcal{M})$, $B = C = L^p(G)$, this result recovers [7, Theorem (2.3)] in the situation that $\mathcal{M}$ is σ-finite and $\{K_j\}$ is a finite family. As in the single kernel case, the problem consists in properly identifying the amalgams.

The following result is proved in [12, Theorem 2.19] with the hypothesis on G to be abelian, but the proof carries over the non abelian case.

Proposition 3.1.23. *Let $1 \leq p < \infty$. If R is a continuous distributionally bounded representation of G, it is separation-preserving and there exists a continuous distributionally bounded representation ρ of G, that defines a positivity-preserving representation of G on L^p and such that for $f \in L^p(\mathcal{M})$ $\rho_u(|f|) = |R_u f| = |\rho_u f|$.*

If $1 \leq p = r < \infty$ and $s = +\infty$ and G is abelian, the following theorem recovers [12, Theorem 4.1], at least when $\mathcal{M}$ is σ-finite.

Corollary 3.1.24. *Let G be an amenable group and let R be a continuous distributionally bounded representation of G. Let $\{K_j\}_{j\in\mathbb{N}} \subset L^1(G)$ with compact support such that for $0 < r \leq p \leq s \leq \infty$, $B^\sharp : L^{p,r}(G) \to L^{p,s}(G)$ is bounded with norm $N(\{K_j\})$. Then, for $f \in L^{p,r}(\mathcal{M})$,*

$$\left\| T^\sharp f \right\|_{L^{p,s}(\mathcal{M})} \leq N\left(\{K_j\}\right) c^{2/p} \left\| f \right\|_{L^{p,r}(\mathcal{M})} ,$$

where c is the constant given in (3.1.10).

Proof. Observe that Fatou's lemma on $L^{p,s}$ allows us, without loss of generality, to assume that we have a finite family of kernels $\{K_j\}_{j=1,\ldots,N}$. Moreover, it suffices to prove the desired inequality for a dense subset of $L^{p,r}$.

Let $F = L^1(\mathcal{M})$, $E = L^p(\mathcal{M})$, $B = L^{p,r}(G)$, $C = L^{p,s}(G)$, let $\mathcal{K}$ be a compact set that contains $\operatorname{supp} K_j$ for $j = 1, \ldots, N$, and let V be an open relatively compact set. By Lemma 3.1.9, R extends to a separation-preserving strongly continuous uniformly bounded representation on $L^1(\mathcal{M})$ and, for all $u \in G$, there exists a positive operator ρ_u such that, for every $f \in L^1(\mathcal{M})$,

$$\rho_u(|f|) = |R_u f| .$$

By Theorem 3.1.22, fixed a relatively compact open set V,

$$\left\|T^{\sharp} f\right\|_{W(L^{p,s}(G),L^{p}(\mathcal{M}),V)} \leq N\left(\{K_j\}\right)\|f\|_{W(L^{p,r}(G),L^{p}(\mathcal{M}),V\mathcal{K}^{-1})}.$$

Now, by (3.1.13) and (3.1.14), we can identify the amalgams and obtain that, for $f \in L^1 \cap L^{p,r}(\mathcal{M})$,

$$\left\|T^{\sharp} f\right\|_{L^{p,s}(\mathcal{M})} \leq c^{2/p} N\left(\{K_j\}\right)\left(\frac{\lambda\left(V\mathcal{K}^{-1}\right)}{\lambda(V)}\right)^{1/p}\|f\|_{L^{p,r}(\mathcal{M})}.$$

By Følner condition (2.2.4), it follows that for $f \in L^1 \cap L^{p,r}(\mathcal{M})$,

$$\left\|T^{\sharp} f\right\|_{L^{p,s}(\mathcal{M})} \leq c^{2/p} N\left(\{K_j\}\right)\|f\|_{L^{p,r}(\mathcal{M})},$$

completing the proof. □

With minor modification on the proof of Corollary 3.1.16, we can prove the following result.

Corollary 3.1.25. *Let G be an amenable group and let R be a continuous distributionally bounded representation of G. Assume that $\mu(\mathcal{M}) = 1$. Let $0 < p \leq q < \infty$. Let $\{K_j\} \subset L^1(G)$ with compact support such that, $B^{\sharp} : \Lambda^p(u, G) \to \Lambda^q(w, G)$ with norm $N(\{K_j\})$. Assume that there exists $r \in [p, q]$ such that $U^{r/p}$ is quasi-concave and $W^{r/q}$ is quasi-convex, where $U(t) = \int_0^t u$ and $W(t) = \int_0^t w$ and $W \in \Delta_2$. Then, for $f \in \Lambda^p(u, \mathcal{M})$,*

$$\left\|T^{\sharp} f\right\|_{\Lambda^q(w,\mathcal{M})} \lesssim N(\{K_j\})\|f\|_{\Lambda^p(u,\mathcal{M})}.$$

3.2 Restriction of multipliers

K. De Leeuw's result in [52] (see also [72]) essentially states that if $\mathbf{m}$ is a continuous and bounded measurable function on $\mathbb{R}$ that is a bounded multiplier for $L^p(\mathbb{R})$, then the restriction of $\mathbf{m}$ to the integers $\mathbb{Z}$ is a multiplier for $L^p(\mathbb{T})$. An analogous result, due to C. E. Kenig and P.A. Tomas (see [76]), holds for maximal multipliers. The analogues results for the restriction of multipliers on $L^p(\mathbb{R}^{d+n})$ to a subspace $\mathbb{R}^d$, also hold (see [52, Prop. 3.2]).

Original De Leeuw's proof is based on the relationships of Fourier multipliers on $L^p(\mathbb{R})$ with $L^p(b\mathbb{R})$ where $b\mathbb{R}$ denotes the Bohr's compactification of $\mathbb{R}$. Following this idea, the result was extended in the context of general locally compact abelian groups in [92]. These type of results for strong and weak type L^p (maximal) multipliers, were obtained using Coifman and Weiss transference techniques (see [44, 46] for the single multiplier case and $1 \leq p < \infty$, [7, Theorem (4.1)] for maximal strong multipliers, [15, 16] for weak (p,p) maximal multipliers, $1 < p < \infty$ and [14] for weak $(1,1)$ maximal multipliers).

We will apply the general transference results developed in the previous section in order to obtain the analogous consequences to De Leeuw's and Kenig-Tomas on restriction of Fourier multipliers, for other spaces different from L^p.

Definition 3.2.1. *If B, C are RIQBFS on $\mathbb{R}^n$, we say that (B, C) is an admissible pair if*

$$\kappa = \liminf_{N\to\infty} h_C\left(\frac{1}{N}\right) h_B(N) < \infty, \tag{3.2.2}$$

Lemma 3.2.3. *If $\kappa \in (0,\infty)$ then $\underline{\alpha}_C = \overline{\alpha}_B$, where $\underline{\alpha}_C$ and $\overline{\alpha}_B$ denote, respectively, the lower and the upper Boyd index of C and B.*

Proof. By the definition of Boyd indices, for every $N \geq 1$,

$$N^{\overline{\alpha}_B - \underline{\alpha}_C} \leq h_C\left(\frac{1}{N}\right) h_B(N),$$

so $\overline{\alpha}_B \leq \underline{\alpha}_C$. On the other hand, if $\overline{\alpha}_B < \underline{\alpha}_C$, there exists p, q such that $\underline{\alpha}_C > q > p > \overline{\alpha}_B$. So by (2.1.13) and (2.1.14), for all $N \geq 1$,

$$h_C\left(\frac{1}{N}\right) h_B(N) \leq C_p C_q N^{p-q},$$

but then $\kappa = 0$. □

Examples of admissible pairs

I) Clearly every pair of spaces $(L^{p,r}, L^{p,s})$ with $0 < p < \infty$, $0 < r, s \leq \infty$ is an admissible pair.

II) More generally, the pairs of spaces $(L^{r,p}(\log L)^\alpha(\mathbb{R}^n), L^{r,q}(\log L)^\beta(\mathbb{R}^n))$ are admissible pairs provided $\beta \leq 0 \leq \alpha$.

III) If $B = \Lambda^p(w)$ or $\Lambda^{p,\infty}(w)$ and $C = \Lambda^q(v)$ or $\Lambda^{q,\infty}(v)$, the pair (B, C) is admissible whenever $v, w \in L^1_{\text{loc}}[0,\infty)$ satisfying

$$\kappa = \liminf_{N\to+\infty} \overline{w}(N)^{1/p} \overline{v}\left(\frac{1}{N}\right)^{1/q} < \infty,$$

where, for $t > 0$, $\overline{u}(t) = \sup_{r>0} \frac{U(rt)}{U(r)}$ for $u = v, w$. This holds because, by Proposition D.1.4, $h_B(t) = \overline{w}(t)$ and $h_C(t) = \overline{v}(t)$.

A particular case is given by those weights v, w for which there exist $0 < a, b < \infty$ satisfying $\frac{a}{b} = \frac{p}{q}$ and for $s < t$,

$$\frac{W(s)}{s^a} \gtrsim \frac{W(t)}{t^a} \quad \text{and} \quad \frac{V(s)}{s^b} \lesssim \frac{V(t)}{t^b}. \tag{3.2.4}$$

Examples of such weighs are given by

$$w(t) = t^{\frac{p}{r}-1}\zeta(t), \quad \text{and} \quad v(t) = t^{\frac{q}{r}-1}\gamma(t),$$

where $0 < p, q, r < \infty$, ζ, γ are slowly varying functions on $(0,\infty)$ satisfying that γ is equivalent to a non-decreasing function, and ζ is equivalent to a non-increasing function. These weights satisfy (3.2.4) with $a = p/r$ and $b = q/r$. The associated admissible pairs are the Lorentz-Karamata spaces $(L^{r,p,\zeta}, L^{r,q,\gamma})$ (see Definition D.1.7).

IV) Let Φ be a Young function, that is, $\Phi(t) = \int_0^t \varphi(s)\, ds$, where φ is an increasing left-continuous function with $\varphi(0) = 0$. Assume that Φ satisfies condition Δ_2, $\Phi(\infty) = \infty$ and that, for every t, $\Phi(t) < \infty$. Let $L_\Phi(\mathbb{R}^n)$ be the associated Orlicz space. A slightly modification of the proof of [18, Thm. IV.8.18] allows us to prove that

$$h_{L_\Phi}(t) = \sup_{s>0} \frac{\Phi^{-1}(st)}{\Phi^{-1}(s)} = \sup_{s>s_0} \frac{\Phi^{-1}(\Phi(s)t)}{s},$$

where $\Phi^{-1}(t) = \sup\{s \geq 0 : \ \Phi(s) \leq t\}$, and $s_0 = \sup\{s \geq 0 : \ \Phi(s) = 0\}$.
Hence, if there exists $p \geq 1$ such that, for $t > 1$, $h_{L_\Phi}(t) \lesssim t^{1/p}$, then $(L_\Phi, L^{p,r})$ is an admissible pair. That is the case, for example, of the Young function $\Phi(t) = t^p(\log(1+t))^p$.

V) For all RIQBFS C, (L^∞, C) is an admissible pair since, for $t < 1$, $h_C(t) \leq 1$, and, for $t > 0$, $h_{L^\infty}(t) = 1$.

VI) If (B, C) is an admissible pair of RIBFS, then by (2.1.11) and for $s \in (0, 1]$ by (2.1.15), (C', B') and (B^s, C^s) also are admissible pairs. Moreover, by Proposition D.1.5, if $X \in \{B, M(B), \Lambda(B)\}$, $Y \in \{C, M(C), \Lambda(C)\}$, (X, Y) is an admissible pair provided $\varphi_B(0^+) = \varphi_C(0^+) = 0$.

VII) If (B_i, C_i) for $i = 0, 1$ are admissible pairs of RIQBFS, then, for the range $0 < \theta < 1$, $0 < q, r \leq \infty$, the pair of intermediate spaces (see [18] for details in real interpolation methods) $\left((B_0, B_1)_{\theta,q}, (C_0, C_1)_{\theta,r}\right)$ is also an admissible pair. This is a consequence of the admissibility of (B_i, C_i) for $i = 0, 1$ and the fact that, for all couple of RIQBFS (X_0, X_1), for $s > 0$,

$$h_{(X_0,X_1)_{\theta,q}}(s) \lesssim h_{X_0}(s)^{1-\theta} h_{X_1}(s)^\theta.$$

3.2.1 Restriction to the integers

Definition 3.2.5. *If X is a RIQBFS on $\mathbb{R}$, we define*

$$X_{\mathbb{T}} := \left\{ f \in L^0(\mathbb{T}) : \ \|f\|_{X_{\mathbb{T}}} := \|f^{*_{\mathbb{T}}}\|_{X^*} < \infty \right\},$$

where the $f^{*_{\mathbb{T}}}$ denotes the decreasing rearrangement of the function f with respect to the Lebesgue measure in $\mathbb{T}$ and X^* is a RIQBFS on $\mathbb{R}_+$ such that for $f \in L^0(\mathbb{R})$ $\|g\|_X = \|g^*\|_{X^*}$.

Examples: If $X = L^p(\mathbb{R})$ then $X^* = L^p(0, \infty)$ so

$$f \in L^p(\mathbb{R})_{\mathbb{T}} \Leftrightarrow \int_0^\infty f^{*_{\mathbb{T}}}(s)^p\, ds = \int_0^1 f^{*_{\mathbb{T}}}(s)^p\, ds < +\infty \Leftrightarrow f \in L^p(\mathbb{T}).$$

In a similar way it can be shown that $L^{p,\infty}(\mathbb{R})_{\mathbb{T}} = L^{p,\infty}(\mathbb{T})$, $L^p(\log L)^\alpha(\mathbb{R})_{\mathbb{T}} = L^p(\log L)^\alpha(\mathbb{T})$. More generally, if $\Lambda^p(w, \mathbb{R})_{\mathbb{T}} = \Lambda^p(w, \mathbb{T})$, and $\Lambda^{p,\infty}(w, \mathbb{R})_{\mathbb{T}} = \Lambda^{p,\infty}(w, \mathbb{T})$. On the other hand, if Φ is a Young's function, $L_\Phi(\mathbb{R})_{\mathbb{T}} = L_\Phi(\mathbb{T})$.

Lemma 3.2.6. *$X_{\mathbb{T}}$ is a RIQBFS. If X is RIBFS, also is $X_{\mathbb{T}}$. If $X = Y^p$ for some $0 < p < 1$ and some RIBFS Y, then $X_{\mathbb{T}} = (Y_{\mathbb{T}})^p$.*

Proof. Observe that for $f, g \in L^0(\mathbb{T})$, since for $s > 0$, $(f+g)^*(s) \leq f^*(s/2) + g^*(s/2)$ and $\|\cdot\|_{X^*}$ is a quasi-norm,

$$\|f+g\|_{X_{\mathbb{T}}} \leq C_{X^*} h_X(2) \left(\|f\|_{X_{\mathbb{T}}} + \|g\|_{X_{\mathbb{T}}} \right).$$

The other properties of quasi-norm are easily verified. The completeness proof is (except on some minor modifications) identical to the one in [18, Theorem 1.6].

If X is a RIBFS, then also is X^* and by [18, Theorem 4.9], $X_{\mathbb{T}}$ is a RIBFS. The other assertion is a direct consequence of the previous one. □

By (B, C) we denote an admissible pair of RIQBFS on $\mathbb{R}$ with Lebesgue measure as its underlying measure.

Theorem 3.2.7. *Let $\mathbf{m} \in M(B, C)$ such that $\mathbf{m} = \widehat{K}$, where $K \in L^1(\mathbb{R})$ with compact support. Then $\mathbf{m}|_{\mathbb{Z}} \in M(B_{\mathbb{T}}, C_{\mathbb{T}})$ with norm controlled by $\|\mathbf{m}\|_{M(B,C)}$.*

Proof. $T_{\mathbf{m}}$ coincides with the convolution operator B_K, so we will use Theorem 3.1.4 for proving the result. To this end, let $F = \mathcal{C}(\mathbb{T})$, that is a Banach space of functions that is embedded in $L^1(\mathbb{T})$, and let R be the representation of $\mathbb{R}$ in F given by $R_t f(\theta) = f(\theta + t)$ for $\theta \in \mathbb{T}$ and $t \in \mathbb{R}$. In this case the related transferred operator is given by

$$T_K f(\theta) = \int_{\mathbb{R}} K(t) R_{-t} f(\theta)\, dt = \int_0^1 \left\{ \sum_{j \in \mathbb{Z}} K(\eta + j) \right\} f(\theta - \eta)\, d\eta.$$

And observe that, for every trigonometric polynomial f, $T_K f = T_{\mathbf{m}|_{\mathbb{Z}}} f$.

Fixed $s > 0$, $g \in F$, $\theta \in \mathbb{T}$ and $L \in \mathbb{N}$, since g is 1-periodic it holds that

$$\begin{aligned} \int_{\mathbb{R}} \chi_{\{v \in (-L,L):\, |g(\theta+[v])| > s\}}(u)\, du &= \sum_{j=-L}^{L-1} \int_j^{j+1} \chi_{\{v \in (-L,L):\, |g(\theta+[v])| > s\}}(u)\, du \\ &= 2L \int_0^1 \chi_{\{z \in \mathbb{T}:\, |g(z)| > s\}}(u)\, du, \end{aligned}$$

from where it follows that

$$\left(\chi_{(-L,L)}(v) R_v g(\theta) \right)^* (s) = g^{*_{\mathbb{T}}} \left(\frac{s}{2L} \right), \tag{3.2.8}$$

where the rearrangement is taken in $\mathbb{R}$ with respect to the Lebesgue measure as a function of the variable v in the term on the left, and in the Lebesgue measure of $\mathbb{T}$ on the right. Hence, if X is a RIQBFS,

$$\left\| \chi_{(-L,L)}(v) R_v g(\theta) \right\|_X = \left\| D_{\frac{1}{2L}} g^{*_{\mathbb{T}}} \right\|_{X^*},$$

that is a constant function on θ.

Let $M \in \mathbb{N}$ big enough such that, $\operatorname{supp} K \subset \mathcal{K} = [-M, M]$. Given $N \in \mathbb{N}$, by Theorem 3.1.4 with $V = (-N, N)$ and $E = L^\infty(\mathbb{T})$, it holds that, for $f \in F$,

$$\|T_K f\|_{W(C,L^\infty(\mathbb{T}),(-N,N))} \leq \|\mathbf{m}\|_{M(B,C)} \|f\|_{W(B,L^\infty(\mathbb{T}),(-N-M,N+M))}. \tag{3.2.9}$$

By the previous observation, we can identify these TWA and rewrite the last inequality as $\left\|D_{\frac{1}{2N}}(T_K f)^{*\mathbb{T}}\right\|_{C^*} \leq \|\mathbf{m}\|_{M(B,C)} \left\|D_{\frac{1}{2(N+M)}} f^{*\mathbb{T}}\right\|_{B^*}$. Hence

$$\|T_K f\|_{C_\mathbb{T}} \leq \|\mathbf{m}\|_{M(B,C)} \left\{h_C\left(\frac{1}{2N}\right) h_B\left(2(N+M)\right)\right\} \|f\|_{B_\mathbb{T}}.$$

Since the dilation norm is submultiplicative and increasing, for $N \geq M$,

$$h_C\left(\frac{1}{2N}\right) h_B\left(2(N+M)\right) \lesssim h_C\left(\frac{1}{N}\right) h_B\left(N\right).$$

Therefore, since (B, C) are admissible, by (3.2.2), for every trigonometric polynomial f,

$$\left\|T_{\mathbf{m}|_\mathbb{Z}} f\right\|_{C_\mathbb{T}} \leq c_{B,C}\kappa \|\mathbf{m}\|_{M(B,C)} \|f\|_{B_\mathbb{T}}.$$

So the result concludes by the density of trigonometric polynomial in $B_\mathbb{T}$. □

Theorem 3.2.10. *Let $L \in \mathbb{N}$ and let $\{\mathbf{m}_l\}_{l=1}^L \subset M(B,C) \cap L^\infty$ such that, for all l, $\mathbf{m}_l = \widehat{K_l}$, where $K_l \in L^1(\mathbb{R})$ with compact support. Then $\{\mathbf{m}_l|_\mathbb{Z}\}_{l=1}^L \subset M(B_\mathbb{T}, C_\mathbb{T})$ and*

$$\left\|\{\mathbf{m}_l|_\mathbb{Z}\}_{l=1}^L\right\|_{M(B_\mathbb{T},C_\mathbb{T})} \lesssim \|\{\mathbf{m}_l\}\|_{M(B,C)}.$$

Proof. Since for every $f \in \mathcal{C}_c^\infty(\mathbb{R})$, $T_{\mathbf{m}_l} f = B_{K_l} f$ and $\{\mathbf{m}_l\}_{l=1}^L$ is a maximal Fourier multiplier, the sublinear operator $\sup_{1\leq l\leq L} |T_{\mathbf{m}_l}|$ coincides with the operator given by $\sup_{1\leq l\leq L} |B_{K_l}|$. Consider the same representation of $\mathbb{R}$ in $F = \mathcal{C}(\mathbb{T})$ as that given in the previous theorem. Let $M \in \mathbb{N}$ such that $\mathcal{K} = [-M, M]$ contains the support of K_l for $l = 1, \ldots, L$. Hence, fixed, $N \geq 1$, if we consider $V = (-N, N)$, $E = L^\infty(\mathbb{T})$, applying Theorem 3.1.22, it holds that for $f \in \mathcal{C}(\mathbb{T})$,

$$\left\|\sup_{1\leq l\leq L} |T_{K_l} f|\right\|_{W(C,L^\infty(\mathbb{T}),(-N,N))} \leq \|\{\mathbf{m}_l\}\|_{M(B,C)} \|f\|_{W(B,L^\infty(\mathbb{T}),(-N-M,N+M))}.$$

Now, by the same argument as that exposed in the proof of the previous theorem, the result follows. □

Proposition 3.2.11. *If (B, C) is an admissible pair of RIQBFS such that $\kappa = 0$ where κ is the constant appearing in (3.2.2), it does not exist $K \in L^1$, $K \neq 0$, with compact support such that $\widehat{K} \in M(B,C)$.*

Proof. If $f \in \mathcal{C}_c(\mathbb{R})$, $K * f \in \mathcal{C}_c(\mathbb{R})$. Then, fixed $N \geq 1$, there exist $\{s_j\}_{j=1}^N$, such that $\left\{L_{s_j}(K * f)\right\}_{j=1}^N$ and $\left\{L_{s_j} f\right\}_{j=1}^N$ have disjoint supports. Thus,

$$\left\|\sum_{j=1}^N L_{s_j}(K * f)\right\|_C = \left\|D_{\frac{1}{N}}(K * f)^*\right\|_{C^*},$$

and $\left\| K * \sum_{j=1}^{N} L_{s_j} f \right\|_C \leq N(K) \left\| \sum_{j=1}^{N} L_{s_j} f \right\|_B = \left\| D_{\frac{1}{N}} f^* \right\|_{B^*}$. Hence,

$$\|K * f\|_C \leq h_C\left(\frac{1}{N}\right) h_B(N)\, N(K)\, \|f\|_B .$$

Since $\kappa = 0$, for all $f \in \mathcal{C}_c(\mathbb{R})$, $K * f = 0$, so it follows that $K \equiv 0$. □

In the case that $B = L^p(\mathbb{R})$, $C = L^q(\mathbb{R})$ and $p \geq q$, the constant appearing in (3.2.2) is zero. So, the previous result can be viewed in this case, as a particular case of the well known result in $M(L^p, L^q)$ (see [83]).

Observe that in Theorems 3.2.7 and 3.2.10, the obtained bounds depend only on the bound of the respective multiplier operator. This, jointly with the approximation techniques developed in §2.3.1, will allow us to prove the desired extensions on restriction theorems. To this end, we shall assume also that (B, C) is an admissible pair such that it is of the type I, II or III described in Table 2.3.16.1.

Theorem 3.2.12. *If $\mathbf{m} \in M(B, C) \cap L^\infty(\mathbb{R})$ is normalized, then the restricted function $\mathbf{m}|_{\mathbb{Z}} \in M(B_{\mathbb{T}}, C_{\mathbb{T}})$, with norm controlled by $\|\mathbf{m}\|_{M(B,C)}$.*

Proof. Let $\{K_n\}_{n\geq 1}$ be the functions given in Theorem 2.3.13 for $G = \mathbb{R}$. Now we can use Theorem 3.2.7 for obtaining that $\widehat{K_n}|_{\mathbb{Z}} \in M(B_{\mathbb{T}}, C_{\mathbb{T}})$, and that for every trigonometric polynomial f

$$\left\| T_{\widehat{K_n}|_{\mathbb{Z}}} f \right\|_{C_{\mathbb{T}}} \leq \kappa c_{B,C} \|\mathbf{m}_n\|_{M(B,C)} \|f\|_{B_{\mathbb{T}}} .$$

But, by Theorem 2.3.13, $\|\mathbf{m}_n\|_{M(B,C)} \leq \mathfrak{c} C_B \|\mathbf{m}\|_{M(B,C)}$. On the other hand, since $\lim_n \mathbf{m}_n = \mathbf{m}$ pointwise, for every trigonometric polynomial f,

$$\lim_{n\to\infty} T_{\widehat{K_n}} f(x) = \lim_{n\to\infty} \sum_{k\in\mathbb{Z}} \widehat{K_n}(k) \widehat{f}(k) e^{2\pi i k x} = T_{\mathbf{m}|_{\mathbb{Z}}} f(x).$$

Then the result follows by Fatou's lemma and density of trigonometric polynomials in $B_{\mathbb{T}}$. □

Theorem 3.2.13. *Let $\{\mathbf{m}_l\}_{l\in\mathbb{N}} \subset M(B, C) \cap L^\infty$ normalized. Then $\{\mathbf{m}_l|_{\mathbb{Z}}\}_{l\in\mathbb{N}} \subset M(B_{\mathbb{T}}, C_{\mathbb{T}})$ and*

$$\|\{\mathbf{m}_l|_{\mathbb{Z}}\}_l\|_{M(B_{\mathbb{T}}, C_{\mathbb{T}})} \lesssim \|\{\mathbf{m}_l\}_l\|_{M(B,C)} .$$

Proof. Fix $L \in \mathbb{N}$. Observe that, if we consider the operator T^L defined by $T^L f(x) := \sup_{1\leq l\leq L} |T_{\mathbf{m}_l} f(x)|$, its norm is uniformly bounded by $\|\{\mathbf{m}_l\}_l\|_{M(B,C)}$. Let $\{K_{j,l}\}_{j,l\geq 1}$ be the functions given in Theorem 2.3.16 for $G = \mathbb{R}$. Now we can use Theorem 3.2.10 and Theorem 2.3.16 to obtain that $\widehat{K_{j,l}}|_{\mathbb{Z}} \in M(B_{\mathbb{T}}, C_{\mathbb{T}})$ and that for all trigonometric polynomial f,

$$\left\| \sup_{1\leq l\leq L} \left| T_{\widehat{K_{j,l}}|_{\mathbb{Z}}} f \right| \right\|_{C_{\mathbb{T}}} \leq \mathfrak{c} C_B \kappa c_{B,C} \|\{\mathbf{m}_l\}_l\|_{M(B,C)} \|f\|_{B_{\mathbb{T}}} .$$

On the other hand, since for every l and x, $\lim_j \widehat{K_{j,l}}(x) = \mathbf{m}_l(x)$,

$$\lim_{j\to\infty} \sup_{1\le l\le L} \left| T_{\widehat{K_{j,l}}|_{\mathbb{Z}}} f(\theta) \right| = \sup_{1\le l\le L} \left| T_{\mathbf{m}_l|_{\mathbb{Z}}} f(\theta) \right| =: S^L f(\theta),$$

and, by Fatou's lemma,

$$\left\| S^L f \right\|_{C_{\mathbb{T}}} \le \liminf_{j\to\infty} \left\| \sup_{1\le l\le L} \left| T_{\widehat{K_{j,l}}|_{\mathbb{Z}}} f(\theta) \right| \right\|_{C_{\mathbb{T}}} \lesssim \| \{\mathbf{m}_l\}_l \|_{M(B,C)} \|f\|_{B_{\mathbb{T}}} .$$

But, since $0 \le S^L f \uparrow S^\sharp f := \sup_{l\ge 1} \left| T_{\mathbf{m}_l|_{\mathbb{Z}}} f \right|$ as $L \to \infty$, it follows that

$$\left\| S^\sharp f \right\|_{C_{\mathbb{T}}} \lesssim \| \{\mathbf{m}_l\}_l \|_{M(B,C)} \|f\|_{B_{\mathbb{T}}} .$$

The proof finishes by the density of trigonometric polynomials in B. □

The previous results are directly applied to the examples of admissible pairs given before, IV) and $(L^{r,p,\varsigma}, L^{r,q,\gamma})$ in III), for the range of indices $1 < p < \infty$, $1 \le q, r < \infty$ on which they are BFS (see Proposition D.1.11). In particular, for the Lorentz-Zygmund spaces $(L^{p,q}(\log L)^\alpha(\mathbb{R}), L^{p,r}(\log L)^\beta(\mathbb{R}))$ with $\alpha \ge 0 \ge \beta$ of example II). For the case $\alpha = \beta = 0$, with a convenient renormalization on the spaces, a precise analysis of the constants allow us to derive the next result, counterpart to that proved in [27, Theorem 2.9] in the bilinear setting. Let us observe that for $1 < p < \infty$ and $s < r$, it is known (see [49]) that $M(L^{p,r}(\mathbb{R}), L^{p,s}(\mathbb{R})) = \{0\}$. For the range $s \ge r$ is a consequence of the previous result.

Corollary 3.2.14. *Let $1 < p < \infty$, $1 \le s \le \infty$, $1 \le r < \infty$. Let $\{\mathbf{m}_j\}_{j\in\mathbb{N}} \subset M\left(L^{p,r}(\mathbb{R}), L^{p,s}(\mathbb{R})\right)$ normalized. Then $\{\mathbf{m}_j|_{\mathbb{Z}}\}_{j\in\mathbb{N}} \subset M\left(L^{p,r}(\mathbb{T}), L^{p,s}(\mathbb{T})\right)$, and*

$$\| \{\mathbf{m}_j|_{\mathbb{Z}}\}_j \|_{M(L^{p,r}(\mathbb{T}),L^{p,s}(\mathbb{T}))} \le \| \{\mathbf{m}_j\}_j \|_{M(L^{p,r}(\mathbb{R}),L^{p,s}(\mathbb{R}))} .$$

Even though the argument also works for the case $s < r$, it is known (see [49]) that, $M(L^{p,r}(\mathbb{R}), L^{p,s}(\mathbb{R})) = \{0\}$ for $1 < p < \infty$, and $s < r$. So, for this range of indices the previous result is trivial. The developed procedure allows also to obtain the following outcome that recovers [1, Theorem 1.1] for $s = +\infty$.

Corollary 3.2.15. *Let $1 \le s \le \infty$. Suppose that $\{\mathbf{m}_j\}_{j\in\mathbb{N}} \subset M\left(L^1(\mathbb{R}), L^{1,s}(\mathbb{R})\right)$, and are normalized functions. Then $\{\mathbf{m}_j|_{\mathbb{Z}}\}_{j\in\mathbb{N}} \subset M\left(L^1(\mathbb{T}), L^{1,s}(\mathbb{T})\right)$, and*

$$\| \{\mathbf{m}_j|_{\mathbb{Z}}\} \|_{M(L^1(\mathbb{T}),L^{1,s}(\mathbb{T}))} \le c_{1,s} \| \{\mathbf{m}_j\} \|_{M(L^1(\mathbb{R}),L^{1,s}(\mathbb{R}))} ,$$

where $c_{1,s}$ is the constant appearing in (B.1.7).

3.2.2 Restriction to lower dimension

The method of proof of the results in the previous section works also in the setting of restriction of Fourier multipliers in several variables. In this section, d, d_1, d_2 are natural numbers such that $d \ge 2$, and $d = d_1 + d_2$. Then, for every $x \in \mathbb{R}^d$,

$x = (x_{d_1}, x_{d_2}) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. The proofs are modifications of the ones given in the previous subsection.

Definition 3.2.16. *If X is a RIQBFS on $\mathbb{R}^d$, for $k \in \mathbb{N}$, we define X_k to be*

$$X_k := \left\{ f \in L^0(\mathbb{R}^k) : \ \|f\|_{X_k} := \|f^{*_k}\|_{X^*} < \infty \right\},$$

where f^{*_k} denotes the rearrangement of f with respect to the Lebesgue measure in $\mathbb{R}^k$.

As in the previous subsection (B, C) denotes an admissible pair of RIQBFS defined on $\mathbb{R}^d$ endowed with Lebesgue measure. Let us observe that the analogous result to Lemma 3.2.6 holds.

Theorem 3.2.17. *Let $\mathbf{m} \in M(B, C)$ such that $\mathbf{m} = \widehat{K}$ where $K \in L^1(\mathbb{R}^d)$ with compact support. Then, fixed $\xi \in \mathbb{R}^{d_1}$, $\mathbf{m}(\xi, \cdot) \in M(B_{d_2}, C_{d_2})$ with norm controlled by $\|\mathbf{m}\|_{M(B,C)}$.*

Proof. Observe first that the multiplier operator $T_{\mathbf{m}}$ coincides with the convolution operator B_K. Let $F = \mathcal{C}_0(\mathbb{R}^{d_2})$ that is a Banach space of functions defined on $\mathbb{R}^{d_2}$ that it is continuously embedded in L^1_{loc}. Fixed $\xi \in \mathbb{R}^{d_1}$, let R be the representation of $\mathbb{R}^d$ acting on F given by $R_x f(y) = e^{2\pi \xi x_1} f(y + x_2)$ for $y \in \mathbb{R}^{d_2}$, $x = (x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Hence the associated transferred operator is

$$T_K f(y) = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} K(x_1, x_2) e^{-2\pi i x_1 \xi} dx_1 \right) f(y - x_2)\, dx_2,$$

that corresponds to the operator associated to the Fourier multiplier given by $\mathbf{m}(\xi, \cdot)$. That is, for $f \in \mathcal{C}_c^\infty(\mathbb{R}^{d_2})$, $T_K f = T_{\mathbf{m}(\xi,\cdot)} f$.

Fixed $s > 0$ and $g \in \mathcal{C}_0(\mathbb{R}^{d_2})$, for all $N \geq 1$ and $y \in \mathbb{R}^{d_2}$, if we define $V = (-N, N) \times \mathbb{R}^{d_2}$, it holds that

$$\int_{\mathbb{R}^d} \chi_{\left\{x \in V : \, \left|g(x_{d_2}+y)\right| > s\right\}}(u)\, du \;\; = \;\; (2N)^{d_1} \int_{\mathbb{R}^{d_2}} \chi_{\left\{z \in \mathbb{R}^{d_2} : \, |g(z)| > s\right\}}(u)\, du.$$

Hence $\left(\chi_V(x) R_x g(y)\right)^*(s) = g^{*_{d_2}}\left(\frac{s}{(2N)^{d_1}}\right)$, where the rearrangement is taken in $\mathbb{R}^d$ with respect to the variable $x = (x_{d_1}, x_{d_2})$ in the term on the left, and in $\mathbb{R}^{d_2}$ on the right. Therefore, for every RIQBFS X,

$$\|\chi_V(v) R_v g(y)\|_X = \left\| D_{\frac{1}{(2N)^{d_1}}}\, g^{*_{d_2}} \right\|_{X^*}$$

is constant on y.

Let $M \in \mathbb{N}$ big enough such that, $\operatorname{supp} K \subset \mathcal{K} = [-M, M]^d$. Given $N \in \mathbb{N}$, by Theorem 3.1.4, with $V = (-N, N) \times \mathbb{R}^{d_2}$ and $E = L^\infty(\mathbb{R}^{d_2})$, it holds that, for $f \in F$,

$$\|T_K f\|_{W(C, L^\infty(\mathbb{R}^{d_2}), V)} \leq N_{B,C}(K)\, \|f\|_{W(B, L^\infty(\mathbb{R}^{d_2}), V + (-M,M)^d)}. \tag{3.2.18}$$

By the previous calculations, we can identify these TWA and rewrite (3.2.18) as

$$\left\| D_{\frac{1}{(2N)^{d_1}}} (T_K f)^{*d_2} \right\|_{C^*} \leq N_{B,C}(K) \left\| D_{\frac{1}{(2(N+M))^{d_1}}} f^{*d_2} \right\|_{B^*}.$$

Hence, for $f \in F \cap B_{d_2}$,

$$\|T_K f\|_{C_{d_2}} \leq N_{B,C}(K) \left\{ h_C \left(\frac{1}{(2N)^{d_1}} \right) h_B \left((2(N+M))^{d_1} \right) \right\} \|f\|_{B_{d_2}}.$$

Now, the proof is a straightforward adaptation of the proof of Theorem 3.2.7. □

In order to apply the approximation techniques of §2.3.1, we shall assume that (B, C) is an admissible pair such that it is of the type I, II or III described in table 2.3.16.1.

Theorem 3.2.19. *If $\mathbf{m} \in M(B,C) \cap L^\infty(\mathbb{R}^d)$ is normalized then, fixed $\xi \in \mathbb{R}^{d_1}$, $\mathbf{m}(\xi, \cdot) \in M(B_{d_2}, C_{d_2})$ with norm controlled by $\|\mathbf{m}\|_{M(B,C)}$.*

Proof. Fix $\xi \in \mathbb{R}^{d_1}$. Let $\{K_n\}_{n \geq 1}$ be the functions given in Theorem 2.3.13 for $G = \mathbb{R}^d$. So, applying Theorem 3.2.17 it holds that $\widehat{K_n}(\xi, \cdot) \in M(B_{d_2}, C_{d_2})$, with norm controlled by $\|\mathbf{m}\|_{M(B,C)}$. Since $\|\widehat{K_n}\|_\infty \leq \|\mathbf{m}\|_\infty$ and, for every $y \in \mathbb{R}^{d_2}$, $\lim_n \widehat{K_n}(\xi, y) = \mathbf{m}(\xi, y)$, by the dominated convergence theorem, for any $f \in \mathcal{C}_c^\infty(\mathbb{R}^{d_2})$,

$$\lim_{n\to\infty} T_{\widehat{K_n}} f(y) = \lim_{n\to\infty} \int_{\mathbb{R}^{d_2}} \widehat{K_n}(\xi, u) \widehat{f}(u) e^{2\pi i u y} = T_{\mathbf{m}(\xi,\cdot)} f(y).$$

Then the proof finishes by Fatou's lemma and density. □

Arguing in a similar way, we can obtain the analogous results to Theorems 3.2.10 and 3.2.13.

Theorem 3.2.20. *Let $\{\mathbf{m}_l\}_{l\in\mathbb{N}} \subset M(B,C) \cap L^\infty(\mathbb{R}^d)$ normalized. Then, fixed $\xi \in \mathbb{R}^{d_1}$, $\{\mathbf{m}_l(\xi, \cdot)\}_{l\in N} \subset M(B_{d_2}, C_{d_2})$ and*

$$\|\{\mathbf{m}_l(\xi,\cdot)\}_l\|_{M(B_{d_2}, C_{d_2})} \lesssim \|\{\mathbf{m}_l\}\|_{M(B,C)},$$

independently of ξ.

3.3 Homomorphism Theorem for multipliers

We are going to apply the results of §3.1 to prove a generalization of the Homomorphism Theorem for L^p spaces given in [54, Appendix B], that corresponds to the case $1 \leq r = p = s < \infty$,. A simplified proof of this theorem using transference techniques can be found in [23, Theorem 2.6]. In the same range, a maximal version is proved in [7, Theorem 4.1] and, for $1 \leq r = p < s = \infty$ in [16, Theorem 4.1]. In the range of indices stated below the result is new. We will estate the single kernel version, but using Corollary 3.1.24 it is not difficult to prove its maximal counterpart and obtain a generalization of [7, Theorem 4.1] and [16, Theorem 4.1].

In particular we will obtain a generalization of De Leeuw's restriction result on general LCA groups.

Theorem 3.3.1. *Let G_1, G_2 be LCA groups and let Γ_1, Γ_2 be its respective dual groups. Assume that π is a continuous homomorphism from Γ_1 to Γ_2. Then, if $1 \leq r \leq p < \infty$, $p \leq s \leq +\infty$ and $\mathbf{m} \in L^\infty(\Gamma_2) \cap M(L^{p,r}(G_2), L^{p,s}(G_2))$ normalized then, $\mathbf{m} \circ \pi \in M(L^{p,r}(G_1), L^{p,s}(G_1))$ and*

$$\|\mathbf{m} \circ \pi\|_{M(L^{p,r}(G_1),L^{p,s}(G_1))} \leq C_{p,r,s} \|\mathbf{m}\|_{M(L^{p,r}(G_2),L^{p,s}(G_2))} .$$

Proof. To begin with, let us check that we can apply Theorem 2.3.13 in order to approximate the multiplier $\mathbf{m}$. Let $B = L^{p,r}$ and $C = L^{p,s}$. B is a RIBFS, has absolute continuous norm, integrable simple functions are dense, and thus, by Proposition 2.3.22 it is well behaved. If $p > 1$, C is a RIBFS so we can apply Proposition 2.3.17, and if $p = 1$, we can apply Proposition 2.3.20 to ensure that there exists $\mathfrak{c} > 0$ such that, for every $\phi \in L^1(\Gamma)$, (2.3.14) is satisfied.

Let $\widetilde{\pi} : G_2 \to G_1$ be the adjoint homomorphism of π defined by

$$\gamma_1(\widetilde{\pi}(u_2)) = \pi(\gamma_1)(u_2), \quad \forall u_2 \in G_2 \, \forall \gamma_1 \in \Gamma_1,$$

that, by [70, (24.38)] is a continuous homomorphism.

For $u_2 \in G_2$, let $R_{u_2} f(u_1) = f(\widetilde{\pi}(u_2)u_1)$. By the left invariance of the Haar measure, it is a measure preserving transformation, and thus $\mu_{R_{u_2}f}(s) = \mu_f(s)$. Moreover, by the continuity of the translation, it follows that it defines a continuous distributionally bounded representation of G_2 on functions defined on G_1. Let, for $n \geq 1$, K_n be the compactly supported functions in $L^1(G_2)$ given by Theorem 2.3.13, and T_{K_n} the transferred operator associated to K_n and the previous representation. By Corollary 3.1.12, for $f \in L^{p,r}(G_1)$ it holds that

$$\|T_{K_n} f\|_{L^{p,s}(G_1)} \leq C_{p,r,s} \|\mathbf{m}\|_{M(L^{p,r}(G_2),L^{p,s}(G_2))} \|f\|_{L^{p,r}(G_1)} . \tag{3.3.2}$$

Fixed $f \in SL^1(G_1) \cap L^{p,r}(G_1)$, since by inversion formula,

$$R_{u_2} f(u_1) = \int_{\Gamma_1} \widehat{f}(\gamma_1)\gamma_1(\widetilde{\pi}(u_2))\,\gamma_1(u_1)\, d\gamma_1,$$

we have that

$$\begin{aligned} T_{K_n} f(u_1) &= \int_{G_2} K_n(u_2) f(\widetilde{\pi}(u_2^{-1})u_1)\, du_2 \\ &= \int_{\Gamma_1} \left(\int_{G_2} K_n(u_2)\overline{\pi(\gamma_1)(u_2)}\, du_2 \right) \widehat{f}(\gamma_1)\gamma_1(u_1)\, d\gamma_1 \\ &= \int_{\Gamma_1} \widehat{K_n}\,(\pi(\gamma_1))\, \widehat{f}(\gamma_1)\gamma_1(u_1)\, d\gamma_1. \end{aligned}$$

Hence by the dominated convergence theorem,

$$\lim_n T_{K_n} f(u_1) = \int_{\Gamma_1} \mathbf{m}\left(\pi(\gamma_1)\right) \widehat{f}(\gamma_1)\gamma_1(u_1)\, d\gamma_1 = T_{\mathbf{m}\circ\pi} f(\gamma_1).$$

By Fatou's lemma and (3.3.2),

$$\|T_{\mathbf{m}\circ\pi} f\|_{L^{p,s}(G_1)} \leq C_{p,r,s} \|\mathbf{m}\|_{M(L^{p,r}(G_2),L^{p,s}(G_2))} \|f\|_{L^{p,r}(G_1)},$$

and the result follows by density of $SL^1(G_1) \cap L^{p,r}(G_1)$ in $L^{p,r}(G_1)$. □

Now, if $G_2 = G$ and $G_1 = G/H$ where H is a closed subgroup of H, $\Gamma_1 = H^\perp$, $\Gamma_2 = \Gamma$ and π is the canonical inclusion of $H^\perp$ in Γ, we obtain the following generalization of De Leeuw's result on restriction of Fourier multiplier (see [52]), which is recovered for $1 \leq r = p = s < \infty$.

Corollary 3.3.3. *Let $1 \leq r \leq p < \infty$, $p \leq s \leq +\infty$, G be a LCA group and H be a closed subgroup of G. Let $\mathbf{m} \in M(L^{p,r}(G), L^{p,s}(G)) \cap L^\infty(\Gamma)$ normalized, then $\mathbf{m}|_{H^\perp} \in M(L^{p,r}(G/H), L^{p,s}(G/H))$ and*

$$\|\mathbf{m}|_{H^\perp}\|_{M(L^{p,r}(G/H),L^{p,s}(G/H))} \leq C_{p,r,s} \|\mathbf{m}\|_{M(L^{p,r}(G),L^{p,s}(G))}.$$

Now considering π to be the natural inclusion of Γ_1 into $\Gamma_1 \times \Gamma_2$ (hence $\widetilde{\pi}$ corresponds to the projection of $G_1 \times G_2$ into G_1) it is immediate to obtain the following "extension" result.

Corollary 3.3.4. *Let G_1, G_2 be LCA groups and let Γ_1, Γ_2 be its respective dual groups. Let $G = G_1 \times G_2$. Then, if $1 \leq r \leq p < \infty$, $p \leq s \leq +\infty$ and $\mathbf{m} \in \mathcal{C}_b(\Gamma_1) \cap M(L^{p,r}(G_1), L^{p,s}(G_1))$ then the function defined by $\mathbf{\Psi}(u,v) = \mathbf{m}(u)$, belongs to $M(L^{p,r}(G), L^{p,s}(G))$ and*

$$\|\mathbf{\Psi}\|_{M(L^{p,r}(G),L^{p,s}(G))} \leq C_{p,r,s} \|\mathbf{m}\|_{M(L^{p,r}(G_1),L^{p,s}(G_1))}.$$

If $G_1 = \mathbb{R}$ and $G_2 = \mathbb{Z}$ (then $\Gamma_1 = \mathbb{R}$ and $\Gamma_2 = \mathbb{T}$) and $\pi : \mathbb{R} \to \mathbb{T}$ is the canonical projection, Theorem 3.3.1 implies the following result.

Corollary 3.3.5. *Let $1 \leq r \leq p < \infty$, $p \leq s \leq +\infty$ and let $\mathbf{m} \in \mathcal{C}(\mathbb{T})$ such that $\mathbf{m} \in M(\ell^{p,r}(\mathbb{Z}), \ell^{p,s}(\mathbb{Z}))$. Then, if $\mathbf{\Psi}$ is the 1-periodic extension of $\mathbf{m}$, $\mathbf{\Psi} \in M(L^{p,r}(\mathbb{R}), L^{p,s}(\mathbb{R}))$ and $\|\mathbf{\Psi}\|_{M(L^{p,r}(\mathbb{R}),L^{p,s}(\mathbb{R}))} \leq C_{p,r,s} \|\mathbf{m}\|_{M(\ell^{p,r}(\mathbb{Z}),\ell^{p,s}(\mathbb{Z}))}$.*

3.4 Other applications

In this section, we will give a pair of examples where we can apply Theorem 3.1.4 to obtain transference results and where we can not use the classical results.

For compact groups, Theorem 3.1.4 gives us a complete transfer result in the sense that the obtained bound for the transferred operator does not depend on the L^1 norm of the kernel K. Moreover, if for example, the defining representation is given by a family of measure preserving transformations, the appearing TWA have structure of QBFS (see Proposition C.2.4).

Let fix $G = SO(n)$, $\mathcal{M} = \mathbb{R}^n$, for $n \geq 2$ and let R be the representation given by the expression

$$R_A f(x) = f(Ax), \quad A \in G$$

acting on $L^p(w) = L^p(\mathbb{R}^n, w)$, where w is a radial weight in $\mathbb{R}^n$. Since $\mathcal{C}_c(\mathbb{R}^2)$ is dense in $L^p(w)$ and, for every A, R_A is an isometry on $L^p(w)$, it easily follows that the representation R is continuous. In this case, the transferred operator is defined by the expression

$$T_K f(x) = \int_{SO(n)} K(A) f(Ax)\, dA.$$

Proposition 3.4.1. *Let $1 \leq p \leq q \leq \infty$ and $K \in L^1(G)$ such that*

$$K* : L^p(G) \to L^q(G),$$

with norm $N(K)$. For any radial weight w, it holds that, for any $0 < r < \infty$,

$$T_K : \left(L^r_{rad}\left(L^p_{ang}; u\right) \cap L^p(w), \|\cdot\|_{L^r_{rad}\left(L^p_{ang};u\right)} \right) \to L^r_{rad}\left(L^q_{ang}; u\right),$$

with norm $N(T_K) \lesssim N(K)$, where $u(\rho) = \rho^{n-1} w(\rho)$, $L^r_{rad}\left(L^q_{ang}; u\right)$ is defined by

$$\|f\|^r_{L^r_{rad}\left(L^q_{ang};u\right)} = \int_0^\infty \left[\int_{\Sigma_{n-1}} |f(\rho\theta)|^q \; d\sigma\,(\theta) \right]^{p/q} u(\rho)\, d\rho,$$

and $d\sigma$ denotes the surface measure on Σ_{n-1}. Similarly $L^r_{rad}\left(L^p_{ang}; u\right)$ is defined. In particular

$$T_K : L^p(w) \to L^p_{rad}\left(L^q_{ang}; u\right),$$

Proof. By Theorem 3.1.4 with $V = G$, $F = L^p(w)$, $B = L^p(G)$, $C = L^q(G)$ and $E = L^r(w)$,

$$\|T_K f\|_{W(L^q(G), L^r(w), G)} \leq N(K) \, \|f\|_{W(L^p(G), L^r(w), G)}\, .$$

To end the proof, it suffices to identify the amalgams. Observe that, by the invariance of the Haar's measure on G, fixed $x \in \mathbb{R}^n$, $x \neq 0$,

$$\int_G |f(Ax)|^s \; dA = \int_G |f(|x|\, A\mathbf{e})|^s \; dA = \frac{1}{\omega_{n-1}} \int_{\Sigma_{n-1}} |f(|x|\,\theta)|^s \; d\sigma(\theta)$$

where ω_{n-1} denotes the surface area of Σ_{n-1} and $\mathbf{e} = (0, \ldots, 0, 1)$. Then

$$\begin{aligned}
\omega^{r/s}_{n-1} \|f\|^r_{W(L^s(G), L^r(w), G)} &= \int_{\mathbb{R}^n} \left\{ \int_G |f(Ax)|^s \; dA \right\}^{r/s} w(x) dx \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\Sigma_{n-1}} |f(|x|\,\theta)|^s \right\}^{r/s} w(x) dx \\
&= \omega_{n-1} \int_0^\infty \left\{ \int_{\Sigma_{n-1}} |f(\rho\theta)|^s \right\}^{r/s} \rho^{n-1} w(\rho)\, d\rho.
\end{aligned}$$

Henceforth

$$\|f\|_{W(L^s(G),L^r(w),G)} = \omega_{n-1}^{\frac{1}{r}-\frac{1}{s}} \|f\|_{L^s_{\text{rad}}\left(L^r_{\text{ang}};u\right)},$$

and in particular $\|f\|_{W(L^s(G),L^s(w),G)} = \|f\|_{L^s(w)}$. □

Since $G = SO(n)$, $L^q(G) \subsetneq L^p(G)$ if $p < q$. Therefore, if K is a kernel like in the hypotheses of the previous result, it follows that

$$K* : L^p(G) \to L^p(G), \quad K* : L^q(G) \to L^q(G)$$

Since R_A is an isometry on $L^p(w)$ and $L^q(w)$, we can apply the classical transference theorem [46, Theorem 2.4] to obtain that T_K defines a bounded operator on $L^p(w)$ and on $L^q(w)$.

But this procedure loses information that we have about the operator. On the other hand, our approach uses this information to say something better on the operator. In fact, it gives a parametric family of inequalities. In particular if $r = p$, since $L^p_{\text{rad}}\left(L^q_{\text{ang}}; u\right) \subsetneq L^p(w)$,

$$\|T_K f\|_{L^p(w)} \leq \omega_{n-1}^{1/q-1/p} \|T_K f\|_{L^p_{\text{rad}}\left(L^q_{\text{ang}};u\right)} \leq N(K) \|f\|_{L^p(w)}.$$

Similarly, for $r = q$, since $L^q_{\text{rad}}\left(L^p_{\text{ang}}; u\right) \supsetneq L^q(w)$, for $f \in L^p(w)$,

$$\|T_K f\|_{L^q(w)} \leq N(K)\omega_{n-1}^{1/q-1/p} \|f\|_{L^q_{\text{rad}}\left(L^p_{\text{ang}};u\right)} \leq N(K) \|f\|_{L^q(w)}.$$

In the following example, the classical transference result can not be applied but Theorem 3.1.4 allows us to obtain a transference result.

Proposition 3.4.2. *Let $1 \leq p_0 \leq p_1 < \infty$. Let $K \in \ell^1(\mathbb{Z})$ with compact support and let us assume that the operator*

$$K* : \ell^{p_0}(\mathbb{Z}) \longrightarrow \ell^{p_1}(\mathbb{Z})$$

is bounded. For $0 < r < \infty$, let $X_{p,r}$ be the space of $f \in L^{p_0}(\mathbb{R})$ such that

$$||f||_{X_{p,r}} = \left(\int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} |f(x+n)|^p\right)^{r/p} dx\right)^{\frac{1}{r}} < \infty, \tag{3.4.3}$$

and $Y_{p,r}$ be the space defined by

$$\|f\|_{Y_{p,r}} = \limsup_N \frac{1}{(2N+1)^{1/r}} \left(\int_{\mathbb{R}} \left(\sum_{j=-N}^{N} |f(x+j)|^p\right)^{r/p} dx\right)^{\frac{1}{r}} < +\infty.$$

Then the operator defined on $\mathbb{R}$ by

$$T_K f(x) = \sum_{j\in\mathbb{Z}} K(j) f(x-j)$$

satisfies that $T_K : X_{p_0,r} \longrightarrow Y_{p_1,r}$ for $r \geq p_0$, and $T_K : (L^r \cap L^{p_0}, \|\cdot\|_{L^r}) \to Y_{p_1,r}$ for $r \leq p_0$, are bounded.

In particular, $T_K : L^{p_0}(\mathbb{R}) \to Y_{p_1,p_0}$ *and* $T_K : X_{p_0,p_1} \to L^{p_1}(\mathbb{R})$ *are bounded.*

Proof. Let $G = \mathbb{Z}$ and let us consider representation given by $R_n f(x) = f(x+n)$. Then by Theorem 3.1.4 we have that, if $V = \{-N, \ldots, N\}$ and K is supported in $\mathcal{K} = \{-M, \ldots, M\}$, with $N, M \in \mathbb{N}$,

$$\|T_K f\|_{W(\ell^{p_1}, L^r(\mathbb{R}), V)} \leq N(K) \|f\|_{W(\ell^{p_0}, L^r(\mathbb{R}), V\mathcal{K})}.$$

Observe that

$$\begin{aligned} \|f\|_{W(\ell^{p_0}, L^r(\mathbb{R}), V\mathcal{K})} &= \left(\int_{\mathbb{R}} \left(\sum_{j=-N-M}^{N+M} |f(x+j)|^{p_0} \right)^{r/p_0} dx \right)^{\frac{1}{r}} \\ &= \left(\int_{\mathbb{T}} \left\{ \sum_{n \in \mathbb{Z}} \left(\sum_{j=-N-M}^{N+M} |f_\theta(n-j)|^{p_0} \right)^{r/p_0} \right\} d\theta \right)^{\frac{1}{r}}, \end{aligned}$$

where $f_\theta(n) = f(\theta + n)$. Hence, for $r \geq p_0$, by Minkowski's integral inequality, the term in curly brackets is bounded by

$$(2(M+N)+1) \left(\sum_{j \in \mathbb{Z}} |f_\theta(j)|^{p_0} \right)^{r/p_0},$$

Thus,

$$\|f\|_{W(\ell^{p_0}, L^r(\mathbb{R}), V\mathcal{K})} \leq (2(M+N)+1)^{1/r} \left(\int_{\mathbb{T}} \left(\sum_{j \in \mathbb{Z}} |f(\theta+j)|^{p_0} \right)^{r/p_0} d\theta \right)^{1/r}.$$

Hence,

$$\frac{\left(\int_{\mathbb{R}} \left(\sum_{j=-N}^{N} |T_K f(x+j)|^{p_1} \right)^{r/p_1} dx \right)^{\frac{1}{r}}}{(2N+1)^{1/r}} \leq \left(\frac{2(M+N)+1}{2N+1} \right)^{1/r} N(K) \|f\|_{X_{p_0,r}}$$

from where the result follows by taking limit when $N \to \infty$.

For $r \leq p_0$, $\|f\|_{W(\ell^{p_0}, L^r(\mathbb{R}), V\mathcal{K})} \leq (2(N+M)+1)^{1/r} \|f\|_{L^r}$, from where the result is proved in the same way as before. □

Observe that the spaces $X_{p,r}$, $Y_{p,r}$ are not trivial as for any function f supported in $[0,1)$, $\|f\|_{Y_{p_1,r}} = \|f\|_{L^r} = \|f\|_{X_{p_1,r}}$. Observe also, that for $r \geq p_1$, since $\ell^{p_1} \subset \ell^r$, $\|f\|_{L^r} \leq \|f\|_{Y_{p_1,r}}$ and $X_{p_0,r} \subset L^{p_0} \cap L^r$.

Corollary 3.4.4. *Let* $1 \leq p_0 \leq p_1 < \infty$. *Assume that*

$$\mathbf{m} \in \mathcal{C}(\mathbb{T}) \cap M(\ell^{p_0}(\mathbb{Z}), \ell^{p_1}(\mathbb{Z})).$$

Fixed $r \geq p_1$, *for* $f \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\|T_{\mathbf{m}} f\|_{L^r(\mathbb{R})} \leq \|\mathbf{m}\|_{M(\ell^{p_0}(\mathbb{Z}), \ell^{p_1}(\mathbb{Z}))} \|f\|_{X_{p_0,r}}.$$

Proof. Let $\{K_n\}$ be the kernels given by Theorem 2.3.13 that in this case can be explicitly given by $K_n(j) = \left(1 - \frac{|j|}{n+1}\right)^+$. For $f \in \mathcal{C}_c^\infty(\mathbb{R})$, by the dominated convergence theorem

$$T_{K_n} f(x) = \int_{\mathbb{R}} \widehat{K_n}(\xi) \widehat{f}(\xi) e^{2\pi i \xi x} \, d\xi \to T_{\mathbf{m}} f(x).$$

Then by Fatou's lemma,

$$\|T_{\mathbf{m}} f\|_{L^r} \leq \liminf_n \|T_{K_n} f\|_{L^r} \leq \liminf_n \|T_{K_n} f\|_{Y_{p_1,r}}.$$

But, by the previous result, for each n, $\|T_{K_n} f\|_{Y_{p_1,r}} \leq \|\mathbf{m}\| \, \|f\|_{X_{p_0,r}}$, so the result follows. □

Observe that in the previous examples, the representations are positivity-preserving, so the previous results hold also for maximal operators applying Theorem 3.1.22.

Chapter 4

Duality approach

Weighted L^p inequalities naturally arise whenever one try to study the boundedness of operators in L^p spaces defined with respect to other measures than Haar's measure. We shall devote this chapter to transfer the boundedness of convolution operators

$$B_K : L^{p_0}(w_0) \to L^{p_1}(w_1)$$

for positive locally integrable weights w_0 and w_1 and indices $1 \le p_0, p_1 < \infty$ (see for example Corollary 4.2.8 or Theorem 4.2.9). To this end, we need to introduce the following operators:

$$S_V f(x) = \left(\int_V |R_v f(x)|^{p_0} w_0(v) dv \right)^{1/p_0},$$

and

$$S^*_{V,w_2} f(x) = \left(\int_V |R^*_{v^{-1}} f(x)|^{p'_1} w_2(v)^{p'_1} w_1(v)^{1-p'_1} dv \right)^{1/p'_1},$$

where V is a measurable set in G, R^*_v denotes the adjoint mapping of R_v, and w_2 is a weight on G such that for any set of positive measure E such that $w_1 = 0$ on E, then $w_2 = 0$ in E and w_2/w_1 is considered to be 1 on E.

Observe that both operators may depend on p_0, p_1, w_0 or w_1 but since these parameters will be fixed all over the chapter, we omit these subindexes. We shall also need the following definition, which is the analogue to the amenability condition (2.2.4).

Definition 4.0.1. *Given a weight w on G, a collection $\mathcal{V}$ of measurable sets in G is w-complete if there exists a constant $\mathfrak{i}_{\mathcal{V}} > 0$ such that for every compact subset $\mathcal{K}$ that is symmetric and contains e, there exist $V_0 \in \mathcal{V}$ and $V_1 \in \mathcal{V}$ such that $V_0\mathcal{K} \subset V_1$ and*

$$1 \le \frac{w(V_1)}{w(V_0)} \le \mathfrak{i}_{\mathcal{V}}. \tag{4.0.2}$$

We shall denote by $\mathfrak{i}$ for the infimum of all values $\mathfrak{i}_{\mathcal{V}}$ such that (4.0.2) holds.

Examples of w-complete families:

1. If G is a compact group, then every $\mathcal{V}$ containing G is obviously w-complete, for every weight w. Furthermore, $\mathfrak{i}_{\mathcal{V}}$ can be taken to be 1.

2. If G is amenable, and $\mathcal{V}$ is taken to be the family of relatively compact non empty open sets, $\mathcal{V}$ is 1-complete, because for every $V_0 \in \mathcal{V}$, $V_0\mathcal{K}$ is a relatively compact open set and, by Følner condition (2.2.4), $\inf_{V_0\in\mathcal{V}} \frac{\lambda(V_0\mathcal{K})}{\lambda(V_0)} = 1$. In this case, we can take $\mathfrak{i}_{\mathcal{V}}$ associated to the family $\mathcal{V}$ as close to 1 as we need. That is, $\mathfrak{i} = 1$.

3. Let $G = (\mathbb{R}^n, +)$ and let $\mathcal{V} = \{(-r, r)^n, r > 1\}$. Then $\mathcal{V}$ is w-complete for every weight w such that there exists $\mathfrak{i}_{\mathcal{V}} > 1$ that, for every s,

$$\inf_{r>1} \frac{\int_{(-r-s,r+s)^n} w(x)dx}{\int_{(-r,r)^n} w(x)dx} \leq \mathfrak{i}_{\mathcal{V}}. \tag{4.0.3}$$

If w is a weight with the property that there exists a constant c such that, for every $r > 1$,

$$\int_{(-2r,2r)^n} w(x)dx \leq c \int_{(-r,r)^n} w(x)dx,$$

one can easily see that (4.0.3) holds with $\mathfrak{i}_{\mathcal{V}} = c$. Thus $\mathcal{V}$ is w-complete for every weight with the previous doubling property.

Let $N_{p_0,p_1}(K)$ be the smallest constant c such that, for every $f \in L^{p_0}(w_0)$,

$$\|B_K f\|_{L^{p_1}(w_1)} \leq c\|f\|_{L^{p_0}(w_0)}.$$

In the case that $p_0 = p_1 = p$, we write $N_p(K)$.

As in the preceding chapter, we will apply our results to multiplier restriction problems for weighted Lebesgue spaces. A different approach to these kind of problems, with $p_0 = p_1$ and $w_0 = w_1$ being a periodic weight belonging to Muckenhoupt's class $A_p(\mathbb{R})$, can be found in [21]. Some of the contents of this chapter can be found in [42].

4.1 Main results

The following assumptions will be needed throughout this section unless otherwise stated, and will be called *standard hypotheses* for short: Let X be a class of measurable functions defined on $\mathcal{M}$ and let $||\cdot||_X$ be a non negative functional defined on X and let Y be a BFS over the same measure space. Let R be a strongly continuous representation of G acting on Y' satisfying the condition that, for every $v \in G$ and every $g \in Y$, $R_v^* g \in Y''$.

Observation 4.1.1. *The condition that, for every $v \in G$ and every $g \in Y$, $R_v^* g \in Y''$, is automatically satisfied if, for example, either the representation is given by measure preserving transformations or the Köthe dual of Y^* coincides with the topological one.*

Definition 4.1.2. *Let w_1, w_2 be weights in G. We shall write $w_2 \ll w_1$ if they satisfy that, for any set of positive measure E such that $w_1 = 0$ on E, then we have $w_2 = w_1$ in E.*

Theorem 4.1.3. *Let $K \in L^1(G)$ with compact support, let $1 \leq p_0, p_1 < \infty$ and let w_0 and w_1 be two weights in G such that*

$$B_K : L^{p_0}(w_0) \longrightarrow L^{p_1}(w_1)$$

is bounded with constant $N_{p_0,p_1}(K)$. Assume the standard hypotheses and also that the following condition hold: there exists a weight $w_2 \ll w_1$ and a constant $A > 0$ such that, for every $V \in \mathcal{V}$ with $w_2(V) \neq 0$ where $\mathcal{V}$ is a w_2-complete collection of measurable sets in G, we have

$$\frac{1}{w_2(V)} \int_{\mathcal{M}} S_V f(x) S^*_{V,w_2} g(x) d\mu(x) \leq A||f||_X ||g||_Y, \tag{4.1.4}$$

for $f \in X \cap Y'$ and $g \in Y$. Then

$$T_K : (X \cap Y', || \cdot ||_X) \longrightarrow Y'$$

is bounded with norm less than or equal to $\mathfrak{i} A N_{p_0,p_1}(K)$, where $\mathfrak{i}$ is the infimum of the family of $\mathfrak{i}_{\mathcal{V}}$ that satisfy (4.0.2) with $w = w_2$.

Proof. Let $\mathcal{K}$ be the support of K, for which we can assume that $\mathcal{K} = \mathcal{K}^{-1}$ and that $e \in \mathcal{K}$. Let $f \in X \cap Y'$ and let $g \in Y$. Then, for every $v \in G$,

$$L := \left| \int_{\mathcal{M}} T_K f(x) g(x) d\mu(x) \right| = \left| \int_{\mathcal{M}} R_v T_K f(x) R^*_{v^{-1}} g(x) d\mu(x) \right|,$$

and therefore, for every $V \subset G$ measurable set such that $w_2(V) \neq 0$,

$$\begin{aligned} L &= \frac{1}{w_2(V)} \int_V \left| \int_{\mathcal{M}} R_v T_K f(x) R^*_{v^{-1}} g(x) d\mu(x) \right| w_2(v) dv \\ &\leq \frac{1}{w_2(V)} \int_{\mathcal{M}} \left[\int_V \left| R_v T_K f(x) \right| \left| R^*_{v^{-1}} g(x) \right| w_2(v) dv \right] d\mu(x), \end{aligned} \tag{4.1.5}$$

and hence

$$L \leq \frac{1}{w_2(V)} \int_{\mathcal{M}} \left[\int_V |R_v T_K f(x)|^{p_1} w_1(v) dv \right]^{\frac{1}{p_1}} S^*_{V,w_2} g(x) d\mu(x).$$

Now, by the continuity of R_v, it follows, that

$$R_v T_K f(x) = \int_G K(u) R_{vu^{-1}} f(x) du, \tag{4.1.6}$$

and thus, since $\mathcal{K} = \mathcal{K}^{-1}$,

$$\begin{aligned} &\left[\int_V |R_v T_K f(x)|^{p_1} w_1(v) dv \right]^{\frac{1}{p_1}} \\ &\quad = \left[\int_G \left| \int_G K(u) R_{vu^{-1}} f(x) \chi_{V\mathcal{K}}(vu^{-1}) du \right|^{p_1} w_1(v) dv \right]^{\frac{1}{p_1}} \end{aligned}$$

But by the boundedness assumption on B_K, the last term is less that or equal to

$$N_{p_0,p_1}(K)\left[\int_{V\mathcal{K}} |R_v f(x)|^{p_0} w_0(v)dv\right]^{1/p_0} = N_{p_0,p_1}(K)S_{V\mathcal{K}}f(x).$$

Therefore, for every $V \in \mathcal{V}$ and every $g \in Y$,

$$L \leq N_{p_0,p_1}(K)\frac{1}{w_2(V)}\int_{\mathcal{M}} S_{V\mathcal{K}}f(x)S^*_{V,w_2}g(x)d\mu(x). \tag{4.1.7}$$

Choosing $V = V_0 \in \mathcal{V}$ and $V_1 \in \mathcal{V}$ such that $V_0\mathcal{K} \subset V_1$ and $\frac{w_2(V_1)}{w_2(V_0)} \leq \mathfrak{i}_{\mathcal{V}}$, we obtain, using (4.1.4), that

$$\begin{aligned} L &\leq N_{p_0,p_1}(K)\frac{1}{w_2(V_0)}\int_{\mathcal{M}} S_{V_1}f(x)S^*_{V_0,w_2}g(x)d\mu(x) \\ &\leq \mathfrak{i}_{\mathcal{V}}N_{p_0,p_1}(K)\frac{1}{w_2(V_1)}\int_{\mathcal{M}} S_{V_1}f(x)S^*_{V_1,w_2}g(x)d\mu(x) \\ &\leq \mathfrak{i}_{\mathcal{V}}N_{p_0,p_1}(K)A||f||_X||g||_Y, \end{aligned}$$

from which the result follows by taking the infimum on the family of all the $\mathfrak{i}_{\mathcal{V}}$ satisfying (4.0.2) with $w = w_2$. ☐

Corollary 4.1.8. *Let $K \in L^1(G)$ with compact support, let $1 \leq p_0, p_1 < \infty$ and let w_0, w_1 be weights in G such that*

$$B_K : L^{p_0}(w_0) \longrightarrow L^{p_1}(w_1)$$

is bounded with constant $N_{p_0,p_1}(K)$. Assume the standard hypotheses and that the following condition holds: there are a weight $w_2 \ll w_1$, a BFS Z over the measure space $(\mathcal{M}, \mu)$ and a constant $A > 0$ such that the operators

$$S_V : (X \cap Y', ||\cdot||_X) \longrightarrow Z \tag{4.1.9}$$

and

$$S^*_{V,w_2} : Y \longrightarrow Z' \tag{4.1.10}$$

are bounded with constants satisfying

$$||S_V||\,||S^*_{V,w_2}|| \leq Aw_2(V),$$

for every $V \in \mathcal{V}$ such that $w_2(V) \neq 0$, where $\mathcal{V}$ is a w_2-complete collection of measurable sets in G. Then

$$T_K : (X \cap Y', ||\cdot||_X) \longrightarrow Y'$$

is bounded with norm less than or equal to $\mathfrak{i}AN_{p_0,p_1}(K)$, where $\mathfrak{i}$ is the infimum of the family of $\mathfrak{i}_{\mathcal{V}}$ that satisfy (4.0.2) with $w = w_2$.

Proof. It suffices to show that (4.1.4) is satisfied in order to apply Theorem 4.1.3 from where the result then follows. Now, by definition of Z', we have that, for any $f \in X \cap Y'$, $g \in Y$ and $V \in \mathcal{V}$ such that $w_2(V) \neq 0$,

$$\frac{1}{w_2(V)} \int_{\mathcal{M}} S_V f(x) S^*_{V,w_2} g(x) d\mu(x) \leq \frac{1}{w_2(V)} \|S_V f\|_Z \|S^*_{V,w_2} g\|_{Z'}$$
$$\leq \left\{ \frac{1}{w_2(V)} \|S_V\| \|S^*_{V,w_2}\| \right\} ||f||_X ||g||_Y \leq A ||f||_X ||g||_Y,$$

and then (4.1.4) follows. ☐

Observation 4.1.11. *Observe that if X is a normed function space and $||\cdot||_X$ is an associated norm, we can conclude from Theorem 4.1.3 and Corollary 4.1.8 that*

$$T_K : \overline{X \cap Y'}^{||\cdot||_X} \longrightarrow Y',$$

where $\overline{X \cap Y'}^{||\cdot||_X}$ is the Banach completion of $X \cap Y'$ with respect to $||\cdot||_X$.

Corollary 4.1.12. *Let $K \in L^1(G)$ be with compact support, let $1 \leq p_0, p_1 < \infty$ and w_0, w_1 be weights in G. Assume that*

$$B_K : L^{p_0}(w_0) \longrightarrow L^{p_1}(w_1)$$

is bounded with constant $N_{p_0,p_1}(K)$. Let R be a strongly continuous representation of G on $L^{p_1}(\mu)$ such that

$$c = \sup_{u \in G} ||R_u||_{B(L^{p_1}(\mu))} < \infty$$

and let X be the space of $f \in L^{p_1}(\mu)$ such that

$$||f||_X := \sup_{V \in \mathcal{V}} \frac{1}{w_1(V)^{\frac{1}{p_1}}} \|S_V f\|_{p_1} < \infty, \tag{4.1.13}$$

where $\mathcal{V}$ is a w_1-complete family of measurable sets. Then

$$T_K : X \longrightarrow L^{p_1}(\mu)$$

is bounded with norm less than or equal to $\mathfrak{i} c N_{p_0,p_1}(K)$, where $\mathfrak{i}$ is the infimum of the family of $\mathfrak{i}_{\mathcal{V}}$ that satisfy (4.0.2) with $w = w_1$.

Proof. Now, if in Corollary 4.1.8, we take $w_2 = w_1$, $Y = L^{p_1'}(\mu)$ and $Z = L^{p_1}(\mu)$, the hypotheses therein hold immediately, with $A = c$. So by Corollary 4.1.8 and the previous remark, the result follows. ☐

Observation 4.1.14. *Observe that if $||\cdot||_X$ is a norm, then in the preceding result, we can replace X by its Banach completion $\overline{X}^{||\cdot||_X}$. Now, if there exists $V \in \mathcal{V}$ such that $w_0(V) \neq 0$, then $||\cdot||_X$ is a norm. To show this, it suffices to prove that if $||f||_X = 0$, then $f = 0$, because homogeneity and the triangular inequality easily follows by definition of S_V and the analogous properties for $L^{p_1}(\mu)$ and $L^{p_0}(w_0)$. Assume now that $||f||_X = 0$, and hence $||S_V f||_{p_1} = 0$ for every $V \in \mathcal{V}$. Let $V \in \mathcal{V}$ such that $w_0(V) > 0$. Then, there exists a measurable set $W \subset V$ such that for* *all* $v \in W$, $w_0(v) > 0$ *and since* $||S_W f||_{p_1} = ||S_V f||_{p_1} = 0$, *by duality, for any* $g \in L^{p_1'}(\mu)$, *and* $\phi \in L^{p_0'}(w_0)$, *it holds that*

$$0 = \left| \int_{\mathcal{M}} \int_W R_v f(x) g(x) \phi(v)\, w_0(v) dv d\mu(x) \right| = \left| \int_W \langle R_v f, g \rangle\, \phi(v)\, w_0(v) dv \right|.$$

Thus, for any $g \in L^{p_1'}(\mu)$, $||\chi_W \langle R_v f, g \rangle ||_{L^{p_0}(w_0)} = 0$. *Since* $v \mapsto \langle R_v f, g \rangle$ *is continuous and for* $v \in W$ $w_0(v) > 0$, *for all* $v \in W$, $\langle R_v f, g \rangle = 0$. *Then, for all* $v \in W$, $||R_v f||_{L^{p_1}} = 0$ *from where follows that* $f \equiv 0$, *since* $||f||_{L^{p_1}} \leq c||R_v f||_{L^{p_1}} = 0$.

Observation 4.1.15. *In the case that the group* G *is compact, we can take* $\mathcal{V} = \{G\}$ *and, in this case, the operator*

$$S_G f(x) = \left(\int_G |R_v f(x)|^{p_0} w_0(v) dv \right)^{\frac{1}{p_0}} := \overline{R} f(x),$$

and similarly

$$S^*_{G,w_2} f(x) = \left(\int_G |R^*_v f(x)|^{p_1'} w_2(v)^{p_1'} w_1(v)^{1-p_1'} dv \right)^{\frac{1}{p_1'}} := \overline{R}^*_{w_2} f(x).$$

With the above notation the next result is a reworking of Theorem 4.1.3 and Corollaries 4.1.8 and 4.1.12 in the case that G is a compact group.

Corollary 4.1.16. *Assume that* G *is a compact group. Let* $K \in L^1(G)$, *let* $1 \leq p_0, p_1 < \infty$ *and let* w_0, w_1 *be weights in* G *such that*

$$B_K : L^{p_0}(w_0) \longrightarrow L^{p_1}(w_1)$$

is bounded with constant $N_{p_0,p_1}(K)$. *Suppose that the standard hypotheses hold and that there exists a weight* $w_2 \ll w_1$ *such that, at least, one of the following conditions hold:*

a) There exists a constant A *such that, for all* $f \in X \cap Y'$ *and* $g \in Y$,

$$\int_{\mathcal{M}} \overline{R} f(x) \overline{R}^*_{w_2} g(x) d\mu(x) \leq A||f||_X ||g||_Y.$$

b) There exists a BFS Z *such that*

$$\bar{R} : (X \cap Y', || \cdot ||_X) \to Z$$

and

$$\bar{R}^*_{w_2} : Y \to Z'$$

are bounded operators with constant c_1 *and* c_2, *respectively. In this case, let* $A = c_1 c_2$.

c) Suppose that $w_2 = w_1$, $Y' = L^{p_1}(\mu)$, *with* $1 \leq p_1 < \infty$, X *is defined by* *those* $f \in L^{p_1}(\mu)$ *such that*

$$||f||_X = \left\|\overline{R}f\right\|_{p_1} < \infty,$$

and the representation is uniformly bounded on $L^{p_1}(\mathcal{M})$ *by a constant* A.

Then

$$T_K : (X \cap Y', ||\cdot||_X) \longrightarrow Y'$$

is bounded with constant less than or equal to $AN_{p_0,p_1}(K)$,

As is the case with the classical theory (see for instance [7]), if the representation is positivity-preserving, then all the above results can be extended to the case of maximal operators.

Theorem 4.1.17. *Let* $K = \{K_j\}_j \subset L^1(G)$ *be a family of kernels with compact support* $\mathcal{K}_j$, $1 \le p_0, p_1 < \infty$ *and let* w_0 *and* w_1 *be two weights in* G. *Let us assume that the maximal operator* $B_K f = \sup_j |B_{K_j} f|$ *satisfies the condition that*

$$B_K : L^{p_0}(w_0) \longrightarrow L^{p_1}(w_1)$$

is bounded with constant $N_{p_0,p_1}(K)$. *Assume that standard hypotheses hold where* R *is a strongly continuous positive preserving representation of* G. *Suppose also that the following condition holds: there exist a weight* $w_2 \ll w_1$ *on* G *and a constant* $A > 0$ *such that, for every* $V \in \mathcal{V}$ *satisfying* $w_2(V) \neq 0$, *where* $\mathcal{V}$ *is a* w_2*-complete collection of measurable sets in* G, *(4.1.4) holds. Then the operator*

$$T_K^\sharp f(x) = \sup_j \left| \int_G K_j(u) R_{u^{-1}} f(x) du \right|$$

satisfies the condition that $T_K^\sharp : (X \cap Y', ||\cdot||_X) \longrightarrow Y'$ *is bounded with norm less than or equal to* $\mathfrak{i} AN_{p_0,p_1}(K)$, *where* $\mathfrak{i}$ *is the infimum of the family of* $\mathfrak{i}_{\mathcal{V}}$ *that satisfy (4.0.2) with* $w = w_2$.

Proof. Since Fatou's lemma holds in Y', we can assume without loss of generality that the family K is finite. That is $\{K_j\}_{j=1}^n$, for a natural number n.

Let $\mathcal{K}$ be a symmetric compact set containing $\cup_{j=1}^n \mathcal{K}_j$. Let $f \in X \cap Y'$ and let $g \in Y$. Then, for every $v \in G$,

$$L := \left| \int_{\mathcal{M}} T_K^\sharp f(x) g(x) d\mu(x) \right| = \left| \int_{\mathcal{M}} R_v T_K^\sharp f(x) R^*_{v^{-1}} g(x) d\mu(x) \right|,$$

and therefore, for every $V \subset G$ measurable set such that $w_2(V) \neq 0$,

$$\begin{aligned} L &= \frac{1}{w_2(V)} \int_V \left| \int_{\mathcal{M}} R_v T_K^\sharp f(x) R^*_{v^{-1}} g(x) d\mu(x) \right| w_2(v) dv \\ &\le \frac{1}{w_2(V)} \int_{\mathcal{M}} \left[\int_V \left| R_v T_K^\sharp f(x) \right| \left| R^*_{v^{-1}} g(x) \right| w_2(v) dv \right] d\mu(x) \\ &\le \frac{1}{w_2(V)} \int_{\mathcal{M}} \left[\int_V |R_v T_K^\sharp f(x)|^{p_1} w_1(v) dv \right]^{\frac{1}{p_1}} S^*_{V,w_2} g(x) d\mu(x). \end{aligned}$$

Since R is positive preserving, we have

$$R_v\left(\max_{j=1,\dots,n}\left|T_{K_j}f(x)\right|\right)\leq\max_{j=1,\dots,n}\left|R_vT_{K_j}f(x)\right|.$$

Hence, by (4.1.6) and the boundedness hypothesis on B_K,

$$\begin{aligned}&\left[\int_V|R_vT_K^\sharp f(x)|^{p_1}w_1(v)dv\right]^{\frac{1}{p_1}}\\&\leq\left[\int_G\sup_{j=1,\dots,n}\left|\int_G K_j(u)R_{vu^{-1}}f(x)\chi_{V\mathcal{K}}(vu^{-1})du\right|^{p_1}w_1(v)dv\right]^{\frac{1}{p_1}}\\&\leq N_{p_0,p_1}(K)\left[\int_{V\mathcal{K}}|R_vf(x)|^{p_0}w_0(v)dv\right]^{1/p_0}=N_{p_0,p_1}(K)S_{V\mathcal{K}}f(x).\end{aligned}$$

Therefore, for every $V\in\mathcal{V}$ and every $g\in Y$,

$$L\leq N_{p_0,p_1}(K)\frac{1}{w_2(V)}\int_{\mathcal{M}}S_{V\mathcal{K}}f(x)S^*_{V,w_2}g(x)d\mu(x).\tag{4.1.18}$$

Now, the same reasoning as in the proof of Theorem 4.1.3 applies and the result follows.

□

4.2 Examples and Applications

Let us now analyze some examples where the hypotheses of the above theorems and corollaries hold.

4.2.1 The classical case

The classical case (see [46, Theorem 2.4], Corollary 3.1.6) is recovered under the hypotheses of G being amenable, $p_0=p_1=p$, $w_0=w_1=1$, $\mathcal{V}$ is taken to be the family of non-empty relatively compact open sets and $\sup_u\|R_u\|_p=c<\infty$. In this case taking $X=Y'=Z=L^p$ and $w_2=1$ in Corollary 4.1.8, it is enough to see that $S_V:L^p\to L^p$ and $S^*_{V,w_2}:L^{p'}\to L^{p'}$ are bounded uniformly in V, but this follows trivially since

$$\begin{aligned}\|S_Vf\|_p&=\left(\int_{\mathcal{M}}\left[\frac{1}{|V|}\int_V|R_vf(x)|^pdv\right]d\mu(x)\right)^{1/p}\\&=\left(\frac{1}{|V|}\int_V\|R_vf\|_p^pdv\right)^{1/p}\leq c\|f\|_p,\end{aligned}$$

and a similar result is obtained for S^*_{V,w_2}. Therefore, we conclude that T_K is bounded with constant less than or equal to $N_p(K)c^2$.

Observation 4.2.1. *All the results that we shall present from now on, recovers, as a particular case, the classical ones for $p_0=p_1=p$ and $w_0=w_1=1$.*

4.2.2 Restriction of Fourier multipliers

In this section, we will apply the previous transference results to study the restriction of Fourier multiplier to closed subgroups. In fact we will apply them in two particular cases: when $G = \mathbb{R}^d$ and $H = \mathbb{Z}^d$ and when $G = \mathbb{R}^{d_1+d_2}$ and $H = \mathbb{R}^{d_2}$. Moreover, in this section we shall assume that $\mathbf{m}$ is a distribution either on $\mathbb{R}^d$ such that its Fourier transform $K = \mathbf{m}^\vee \in L^1$ and it has compact support. In order to avoid these conditions, we have to consider some kind of "normalized" multipliers. However, in the weighted setting, multipliers will not in general be bounded, and hence we shall give a new definition of normalized function. This technical part will be postponed to the last part, where in particular, we prove that forthcoming Theorem 4.2.9 can be extended to more general multipliers.

It is easy to see that if the kernel K is a positive locally integrable function, then the truncated kernel $K_r(x) = K(x)\chi_{B(0,r)}(x)$ satisfies the same estimate than K and since K_r are in L^1 and has compact support, we can conclude our result for the operators T_{K_r} and deduce the result for T_K by letting r tends to infinity.

Restriction to the integers

Let $G = \mathbb{R}^d$ and let $\mathbf{m} \in \mathcal{S}'(\mathbb{R}^d)$ be a distribution on $\mathbb{R}^d$ such that $K = \mathbf{m}^\vee$ is an integrable function with compact support and the corresponding Fourier multiplier operator

$$\widehat{(B_K f)}(\xi) = \mathbf{m}(\xi)\widehat{f}(\xi),$$

continuously maps $L^{p_0}(w_0)$ into $L^{p_1}(w_1)$, with norm $||\mathbf{m}||_{M(L^{p_0}(w_0),L^{p_1}(w_1))}$. Then, if we take $\mathcal{M} = \mathbb{T}^d$ and R to be the representation acting on periodic functions, given by $R_u f(x) = f(x-u)$, it is easy to see that the transferred operator

$$T_K f(\theta) = \sum_{j\in\mathbb{Z}^d} \mathbf{m}(j)\widehat{f}(j)e^{2\pi i j\theta},$$

coincides with the Fourier multiplier given by $\mathbf{m}|_{\mathbb{Z}^d}$. We can now prove the following extension of De Leeuw restriction result.

Theorem 4.2.2. *Let B_K be as in the preceding paragraph. Assume that the family $\mathcal{V} = \{(-N,N)^d;\ N \geq 1\}$ is w_1-complete. Let U be any periodic function such that for almost every $x \in [0,1)^d$,*

$$U(x) \geq \sup_{N\geq 1} \frac{1}{w_1((-N,N)^d)^{p_0/p_1}} \sum_{j\in[-N,N)^d\cap\mathbb{Z}^d} w_0(x+j).$$

Define X to be the space of $f \in L^{p_1}(\mu)$ such that

$$||f||_X := |||f|^{p_0} * U||_{p_1/p_0}^{1/p_0} < \infty.$$

It holds that,

$$T_K : X \longrightarrow L^{p_1}(\mathbb{T}^d),$$

is bounded with norm no greater than $\mathfrak{i}||\mathbf{m}||_{M(L^{p_0}(w_0),L^{p_1}(w_1))}$ *where* $\mathfrak{i}$ *is the infimum of the family of* $\mathfrak{i}_{\mathcal{V}}$ *that satisfy (4.0.2) with* $w = w_1$.

Proof. Let us take $w_2 = w_1$. Since the family $\mathcal{V}$ is w_1-complete, we can apply Corollary 4.1.12. To this end, we shall study the operator S_V acting on a 1-periodic function f. But, taking $V = (-N,N)^d$, we obtain that

$$\begin{aligned}
\frac{S_V f(x)}{w_1((-N,N)^d)^{\frac{1}{p_1}}} &= \frac{1}{w_1((-N,N)^d)^{\frac{1}{p_1}}}\left(\int_{(-N,N)^d} |f(x-u)|^{p_0} w_0(u)du\right)^{1/p_0} \\
&= \left(\frac{1}{w_1((-N,N)^d)^{p_0/p_1}} \sum_{j\in[-N,N)^d\cap\mathbb{Z}^d} \int_{(j,j+1)^d} |f(x-u)|^{p_0} w_0(u)du\right)^{1/p_0} \\
&= \left(\frac{1}{w_1((-N,N)^d)^{p_0/p_1}} \int_{(0,1)^d} |f(x-u)|^{p_0} \sum_{j\in[-N,N)^d\cap\mathbb{Z}^d} w_0(u+j)du\right)^{1/p_0} \\
&\leq \left(\int_{(0,1)^d} |f(x-u)|^{p_0} U(u)du\right)^{1/p_0} = (|f|^{p_0} * U)(x)^{1/p_0},
\end{aligned}$$

and the result follows. □

Observation 4.2.3. *Observe that if* $w_0 = w_1 = 1$ *and* $p_0 = p_1$, *we can take* $U \approx 1$ *and then* $X = L^{p_0}(\mathbb{T})$.

Restriction to a lower dimension

Fix throughout this section $d = d_1 + d_2$, where $d_1, d_2 \in \mathbb{N}$. As stated at the beginning of this section, $\mathbf{m}$ denotes an element in $\mathcal{S}'(\mathbb{R}^d)$ such that $\mathbf{m} = \widehat{K}$, with $K \in L^1(\mathbb{R}^d)$ with compact support. Fixing $\xi \in \mathbb{R}^{d_1}$, we consider the representation of $\mathbb{R}^d$ on $L^p(\mathbb{R}^{d_2})$ given by

$$R_{(x,y)} f(s) = e^{2\pi i x\xi} f(y+s), \quad (x,y) \in \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}. \tag{4.2.4}$$

Then the associated transferred operator T_K is the operator related to the multiplier $\mathbf{m}(\xi, y)$. Let us consider the space X to be defined by those $f \in L^{p_1}(\mathbb{R}^{d_2})$ for which

$$||f||_X = \sup_{r>1} \frac{\left\|\left(\int_{(-r,r)^d} |f(y+s)|^{p_0} w_0(y,z)dydz\right)^{1/p_0}\right\|_{p_1}}{w_1((-r,r)^d)^{\frac{1}{p_1}}} < \infty. \tag{4.2.5}$$

Theorem 4.2.6. *Suppose that the family of cubes* $\mathcal{V} = \left\{(-r,r)^d : r \geq 1\right\}$ *is* w_1-*complete. Let* $\mathbf{m} \in M\left(L^{p_0}(w_0), L^{p_1}(w_1)\right)$. *Then, fixing* $\xi \in \mathbb{R}^{d_1}$, *the restriction multiplier* $\mathbf{m}(\xi, \cdot)$ *defined on* $\mathbb{R}^{d_2}$ *satisfies the condition that*

$$T_K : X \longrightarrow L^{p_1}(\mathbb{R}^{d_2})$$

with norm less than or equal to $\mathfrak{i}||\mathbf{m}||_{M(L^{p_0}(w_0),L^{p_1}(w_1))}$, *where* $\mathfrak{i}$ *is the infimum of the family of* $\mathfrak{i}_{\mathcal{V}}$ *that satisfy (4.0.2) with* $w = w_1$.

Proof. We have assumed that $\mathbf{m} = \widehat{K}$ with $K \in L^1(\mathbb{R}^d)$ with compact support. The representation given in (4.2.4) is a strongly continuous representation on $L^{p_1}(\mathbb{R}^{d_2})$ and $\|R_{(x,y)}\|_{\mathfrak{B}(L^{p_1})} = 1$, for every $(x, y) \in \mathbb{R}^d$. Since the space defined by (4.2.5) coincides with the defined by (4.1.13), the result follows from Corollary 4.1.12. ☐

Observation 4.2.7. *Since there exists $r \geq 1$ such that $w_0((-r, r)^d) > 0$ (otherwise w_0 is identically 0), the functional given in (4.2.5) is a norm. Thus, in the preceding result, X can be replaced by its Banach completion.*

Corollary 4.2.8. *Let w be a weight in $\mathbb{R}^d$ and suppose that the family of cubes $\mathcal{V} = \left\{(-r, r)^d : \ r \geq 1\right\}$ is w-complete. Let $\mathbf{m} \in M\left(L^p(w)\right)$ and let $\xi \in \mathbb{R}^d$. Then $\mathbf{m}(\xi, \cdot) \in M\left(L^p(\mathbb{R}^{d_2})\right)$ with norm less than or equal to $\mathfrak{i}||\mathbf{m}||_{M(L^p(w))}$, where $\mathfrak{i}$ is the infimum of the family of $\mathfrak{i}_{\mathcal{V}}$ that satisfy (4.0.2).*

Proof. The result easily follow from the previous theorem by considering $p_0 = p_1 = p$ and $w_0 = w_1 = w$ and observing that $X = L^p(\mathbb{R}^{d_2})$ because, by Tonelli's theorem, for any $f \in L^p(\mathbb{R}^{d_2})$, $||f||_X = ||f||_{L^p(\mathbb{R}^{d_2})}$. ☐

To finish with this subsection, we are going to prove a result concerning Muckenhoupt weights, which follows from the proof of our main Theorem 4.1.3. Let us recall (see [60, 84]) that a pair of weights (w_0, w_1) belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$ for $1 < p < \infty$ if

$$[w_0, w_1]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w_0(x)\, dx\right)^{1/p} \left(\frac{1}{|Q|}\int_Q w_1(x)^{1/1-p}\, dx\right)^{1-1/p} < \infty,$$

where the supremum is considered over the family of cubes Q with sides parallel to the coordinate axes.

Theorem 4.2.9. *Let $w_1(x, y) = u_1(x)v_1(y)$ and $w_0(x, y) = u_0(x)v_0(y)$ be weights in $\mathbb{R}^d$ such that $(u_0, u_1) \in A_p(\mathbb{R}^{d_1})$. Assume that $\mathbf{m} \in M(L^p(w_0), L^p(w_1))$. Then, for every $\xi \in \mathbb{R}^{d_1}$, $\mathbf{m}(\xi, \cdot) \in M(L^p(v_0), L^p(v_1))$ and*

$$\|\mathbf{m}(\xi, \cdot)\|_{M(L^p(v_0),L^p(v_1))} \leq [u_0, u_1]_{A_p} \|\mathbf{m}\|_{M(L^p(w_0),L^p(w_1))},$$

uniformly in ξ.

Proof. Let $\xi \in \mathbb{R}^{d_1}$ and R be the representation described in (4.2.4). Let $\mathcal{V} = \left\{(-r, r)^d : \ r \geq 1\right\}$ and let $s > 0$ be sufficiently large that the support of K is contained in $(-s, s)^d$. Then, fixed $l > 0$, for every g supported in $(-l, l)^{d_2}$, by (4.1.5) with $w_2 = 1$,

$$\begin{aligned} L &= \left| \int_{\mathbb{R}^{d_2}} T_K f(x) g(x) dx \right| \\ &\leq \frac{1}{(2r)^d} \int_{\mathbb{R}^{d_2}} \left\{ \int_{(-r,r)^d} \left|R_{(y,z)} T_K f(x)\right| |g(x+z)| \ dydz \right\} dx. \end{aligned}$$

Observe that for $(y, z) \in (-r, r)^d$, since $\operatorname{supp} K \subset (-s, s)^d$,

$$\left|R_{(y,z)} T_K f(x)\right| = \left|B_K(\chi_{(-r-s,r+s)^d} R_{\cdot} f(x))(y, z)\right|.$$

On the other hand, since g is supported in $(-l,l)^{d_2}$ and $z \in (-r,r)^{d_2}$, $x \in (-l-r, r+l)^{d_2}$. Then the right hand term on the last inequality is equal to

$$\frac{1}{(2r)^d}\int_{(-r-l,r+l)^{d_2}}\left\{\int_{(-r,r)^d}|B_K(\chi_W R.f(x))(y,z)|\,|g(x+z)|\ dydz\right\}dx, \quad (4.2.10)$$

where $W=(-r-s,r+s)^d$. Let denote by $N=\|\mathbf{m}\|_{M(L^p(w_0),L^p(w_1))}$.

Fixed $x \in \mathbb{R}^{d_2}$ if we define $F(y,z)=\chi_{(-r-s,r+s)^d+(0,x)}(y,z)e^{2\pi i y\xi}f(z)$,

$$B_K(\chi_{(-r-s,r+s)^d}R.f(x))(y,z)=B_K(F)(y,z+x).$$

Hence, the term in curly brackets can be written as

$$\begin{aligned}\int_{(-r,r)^d}&\left|B_K(\chi_{(-r-s,r+s)^d}R.f(x))(y,z)\right|\,|g(x+z)|\ dydz\\ &=\int_{(-r,r)^d+(0,x)}|B_KF(y,z)|\,|g(z)|\ dydz\end{aligned}$$

Then, by Hölder's, this last term can be bounded by

$$\left[\int|B_KF(y,z)|^p\,w_1(y,z)\ dydz\right]^{1/p}\left[\int_{(-r,r)^d+(0,x)}|g(z)|^{p'}w_1(y,z)^{1-p'}\ dydz\right]^{1/p'}.$$

The first factor is less than or equal to

$$\begin{aligned}N&\left(\int_{(-r-s,r+s)^d+(0,x)}|f(z)|^pw_0(y,z)\ dydz\right)^{1/p}\\ &\le N\left(\int_{\mathbb{R}^{d_2}}|f(z)|^p\int_{(-r-s,r+s)^{d_1}}w_0(y,z)\ dydz\right)^{1/p}\\ &=\|f\|_{L^p(v_0)}\left(\int_{(-r-s,r+s)^{d_1}}u_0(y)\ dy\right)^{1/p}.\end{aligned}$$

Similarly, the second one is not greater than

$$\|g\|_{L^{p'}(v_1^{1-p'})}\left(\int_{(-r,r)^{d_1}}u_1(y)^{1-p'}\ dy\right)^{1/p'}.$$

Hence the term inside curly brackets in (4.2.10) is bounded uniformly on x by

$$\begin{aligned}&N\,\|g\|_{L^p(v_1^{1-p'})}\,\|f\|_{L^p(v_0)}\left(\int_{(-r-s,r+s)^{d_1}}u_0(y)\ dy\right)^{1/p}\left(\int_{(-r,r)^{d_1}}u_1(y)^{1-p'}\ dy\right)^{1/p'}\\ &\le N\,\|g\|_{L^p(v_1^{1-p'})}\,\|f\|_{L^p(v_0)}\,(2(r+s))^{d_1}[u_0,u_1]_{A_p},\end{aligned}$$

where the last inequality holds as $(u_0,u_1)\in A_p(\mathbb{R}^{d_1})$. Using this in (4.2.10), we obtain that

$$L \leq [u_0, u_1]_{A_p} \left(\frac{r+l}{r}\right)^{d_2} \left(\frac{r+s}{r}\right)^{d_1} N \|f\|_{L^p(v_0)} \|g\|_{L^{p'}(v_1^{1-p'})}.$$

Thus, by duality,

$$\left\|T_K f \chi_{(-l,l)^{d_2}}\right\|_{L^p(v_1)} \leq [u_0, u_1]_{A_p} N \left(\frac{r+l}{r}\right)^{d_2} \left(\frac{r+s}{r}\right)^{d_1} \|f\|_{L^p(v_0)}.$$

Taking limit when r tends to infinity, $\left\|T_K f \chi_{(-l,l)^{d_1}}\right\|_{L^p(v_1)} \leq [u_0, u_1]_{A_p} N \|f\|_{L^p(v_0)}$. Therefore, the result follows by taking limit when l tends to infinity by the monotone convergence theorem. □

As an automatic consequence of the above theorem, we obtain the next result.

Corollary 4.2.11. *Assume that* $\mathbf{m} \in M(L^p(w))$ *for* $w(x,y) = u(x)v(y)$ *where* $u \in A_p(\mathbb{R}^{d_1})$ *and* $v \in A_p(\mathbb{R}^{d_2})$*. Then, for every* $\xi \in \mathbb{R}^{d_1}$*,* $\mathbf{m}(\xi, \cdot) \in M(L^p(v))$ *and uniformly in* ξ*,* $\|\mathbf{m}(\xi, \cdot)\|_{M(L^p(v))} \leq C_u \|\mathbf{m}\|_{L^p(w)}$*.*

$\{0\}$-Normalized multipliers

Definition 4.2.12. *Given a measurable function* f *defined on* $\mathbb{R}^n$*, and* $s > 0$*, we denote*

$$D_s f(x) = s^n f(sx), \quad \overline{D}_s f(x) = f(sx).$$

Definition 4.2.13. *A distribution* $\mathbf{m} \in \mathcal{S}'(\mathbb{R}^n)$ *is said to be* $\{0\}$*-normalized if*

i) *for every* $\delta > 0$*,* $\mathbf{m} \in L^\infty(\mathbb{R}^d \setminus B(0,\delta))$*,*

ii) *for every* $\delta > 0$*, there exists* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *satisfying that* $\varphi = 1$ *in* $B(0,\delta)$*,* $\operatorname{supp}\varphi \subset B(0,2\delta)$ *and* $\mathbf{m}(\varphi f) \leq C||f||_\infty$*, for every* $f \in \mathcal{S}(\mathbb{R}^n)$*,*

iii) *there exists* $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ *such that* $||\widehat{\phi}||_1 = 1$ *and such that the sequence defined, for* $j \in \mathbb{N}$*, by* $\widetilde{\mathbf{m}}_j(x) = \mathbf{m} * D_j\widehat{\phi}(x)$*, converges to* $\mathbf{m}(x)$*, for every* $x \neq 0$*.*

Observe that if $\mathbf{m}$ is a bounded function normalized in the sense of Definition 2.3.10 with respect to $\{\phi_j\}_j$, that is $\widetilde{\mathbf{m}}_j(x) = \mathbf{m} * D_j\widehat{\phi}(x)$ converges to $\mathbf{m}(x)$, for every x, it is also $\{0\}$-normalized. Note, however, that these are not the only examples, since every locally integrable function, bounded away from a neighborhood of $\{0\}$ satisfying that every $x \neq 0$ is a Lebesgue point, is a $\{0\}$-normalized function. In particular, for every $0 < \alpha < n$, the Fractional Riesz multiplier $\mathbf{m}(\xi) = |\xi|^{-\alpha}$ is a $\{0\}$-normalized function.

Our purpose is to approximate properly such a type of normalized multipliers in the weighted setting. We shall pay special attention to the weights satisfying an A_p condition.

Before going on, we shall mention some facts on weighted Lebesgue spaces. Let us fix $1 \leq p < \infty$ and a weight w in $\mathbb{R}^n$. Since $w \in L^1_{\text{loc}}$, it defines a Radon measure in $\mathbb{R}^n$ and thus $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is a dense linear subspace of $L^p(w)$. Therefore,

$\mathcal{S}(\mathbb{R}^n) \cap L^p(w)$ is also dense in $L^p(w)$. In particular, if $w \in A_p$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(w)$ as $\mathcal{S}(\mathbb{R}^n) \subset L^p(w)$.

Proposition 4.2.14. *Let $1 \leq p < \infty$ and let $\mathbf{m} \in M(L^p(w_0), L^p(w_1))$ be $\{0\}$-normalized, where either $w_1 \in A_p(\mathbb{R}^n)$ or $w_0 \in A_p(\mathbb{R}^n)$. Then there exist $\{\mathbf{m}_j\}_{j\in\mathbb{N}} \subset M(L^p(w_0), L^p(w_1))$ such that the kernels $\mathbf{m}_j^\vee \in L^1(\mathbb{R}^n)$ are compactly supported,*

$$\mathbf{m}_j(x) \to \mathbf{m}(x), \qquad \forall x \neq 0, \tag{4.2.15}$$

and

$$\sup_j ||\mathbf{m}_j||_{M(L^p(w_0),L^p(w_1))} \lesssim ||\mathbf{m}||_{M(L^p(w_0),L^p(w_1))}. \tag{4.2.16}$$

Furthermore, for every $\delta > 0$,

$$\sup_j \sup_{|x|\geq\delta} |\mathbf{m}_j(x)| < \infty, \tag{4.2.17}$$

and, if $\mathbf{m}$ is a bounded function,

$$\sup_j ||\mathbf{m}_j||_\infty < \infty. \tag{4.2.18}$$

Proof. Assume that $w_1 \in A_p(\mathbb{R}^n)$. The case $w_0 \in A_p(\mathbb{R}^n)$ is proved in a similar way. By Proposition 2.3.29, $L^p(w_1)$ is well behaved and exists $\{h_j\}_{j\in\mathbb{N}} \subset \mathcal{C}_c^\infty(\mathbb{R}^n)$ an associated family to $L^p(w)$ (see Definition 2.3.12). For each $j \in \mathbb{N}$ we define

$$\mathbf{m}_j = \widetilde{\mathbf{m}}_j \widehat{h_j},$$

where $\widetilde{\mathbf{m}}_j$ is given by Definition 4.2.13, and $K_j = \mathbf{m}_j^\vee$. Observe that $\mathbf{m}_j \in \mathcal{S}(\mathbb{R}^n)$, and hence $K_j \in \mathcal{S}(\mathbb{R}^n)$. On the other hand, since

$$K_j(x) = (\overline{D}_{\frac{1}{j}}\phi\, \mathbf{m}^\vee)(h_j(x-\cdot)) = \mathbf{m}^\vee(\overline{D}_{\frac{1}{j}}\phi(\cdot)\, h_j(x-\cdot)),$$

and ϕ, h_j are compactly supported, it follows that K_j has compact support. For $f \in \mathcal{S}(\mathbb{R}^n)$, it is easy to see that

$$\begin{aligned} K_j * f(x) &= \left(\widetilde{\mathbf{m}}_j \widehat{f}\right)^\vee * h_j(x) \\ &= \int D_j\widehat{\phi}(-u) \int h_j(y) e^{-2\pi i(x-y)u} \left[\mathbf{m}^\vee * \left(f e^{2\pi i u\cdot}\right)(x-y)\right] dydu. \end{aligned}$$

Then, since $||\widehat{\phi}||_1 = 1$ and $\mathfrak{s} = \sup_j \left\|\widehat{h_j}\right\|_{M(L^p(w_1))} < \infty$, it follows that

$$||K_j * f||_{L^p(w_1)} \leq \mathfrak{s} ||\mathbf{m}||_{M(L^p(w_0),L^p(w_1))} ||f||_{L^p(w_0)}.$$

By the density of $\mathcal{S}(\mathbb{R}^n) \cap L^{p_0}(w_0)$ in $L^{p_0}(w_0)$ it follows that

$$\sup_j ||\mathbf{m}_j||_{M(L^p(w_0),L^p(w_1))} \lesssim ||\mathbf{m}||_{M(L^p(w_0),L^p(w_1))}.$$

(4.2.15) holds because $\mathbf{m}$ is $\{0\}$-normalized and for every $x \in \mathbb{R}^n$, $\lim_j \widehat{h_j}(x) \to 1$.

Assume first that $\mathbf{m} \notin L^\infty$. Let $\delta > 0$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $\varphi = 1$ in $B(0,\delta)$, supp $\varphi \subset B(0,2\delta)$ and $\mathbf{m}(\varphi f) \leq C||f||_\infty$, for every $f \in \mathcal{S}(\mathbb{R}^n)$. If $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is such that $0 \leq \psi \leq 1$, is equal to 1 outside $B(0,2\delta)$ and equal to 0 in $B(0,\delta)$. For any $x \notin B(0,4\delta)$, we have

$$\left|\mathbf{m}(\varphi(\cdot)D_j\widehat{\phi}(x-\cdot))\right| \leq C\sup_{|y|>\delta}|D_j\widehat{\phi}(y)| \leq C_{\delta,\psi}.$$

On the other hand,

$$\left|\mathbf{m}((1-\varphi)D_j\widehat{\phi}(x-\cdot))\right| \leq C||\mathbf{m}||_{L^\infty(\mathbb{R}^n\setminus B(0,\delta))}||\widehat{\phi}||_1.$$

Thus,

$$\sup_j \sup_{|x|\geq 4\delta} |\widetilde{\mathbf{m}}_j(x)| < \infty.$$

Since $\sup_j ||\widehat{h_j}||_\infty \leq 1$, (4.2.17) follows. Finally, if $\mathbf{m} \in L^\infty$, $\{\mathbf{m}_j\}_j$ are uniformly bounded as $\left|\mathbf{m}(D_j\widehat{\phi}(x-\cdot))\right| \leq ||\mathbf{m}||_{L^\infty}||\widehat{\phi}||_1$. □

Theorem 4.2.19. *Let $1 < p < \infty$, let w_0, w_1 be weights in $\mathbb{R}^{d_1+d_2}$ such that $w_0(x,y) = u(x)v_0(y)$ and $w_1(x,y) = u(x)v_1(y)$ where $u \in A_p(\mathbb{R}^{d_1})$ and either v_0 or $v_1 \in A_p(\mathbb{R}^{d_2})$. If $\mathbf{m} \in M(L^p(w_0), L^p(w_1))$ is $\{0\}$-normalized, for any $\xi \in \mathbb{R}^{d_1}$, the following hold:*

1. *If $\xi \neq 0$ or $\mathbf{m} \in L^\infty$, the restriction multiplier*
$$\mathbf{m}(\xi,\cdot) \in M(L^p(v_0), L^p(v_1)).$$
2. *If $\xi = 0$ and $\mathbf{m} \notin L^\infty$, the restriction multiplier*
$$\mathbf{m}(\xi,\cdot) \in M(X, L^p(v_1))$$
where
$$X = \overline{\cup_{\delta>0}F_\delta}^{L^p(v_0)},$$
and, for each $\delta > 0$,
$$F_\delta = \{f \in \mathcal{S}(\mathbb{R}^{d_2}) \cap L^p(v_0);\ \operatorname{supp}\widehat{f} \cap B(0,\delta) = \emptyset\}.$$

In either case $\|\mathbf{m}(\xi,\cdot)\|_Z \lesssim \|\mathbf{m}\|_{M(L^p(w_0),L^p(w_1))}$, uniformly in ξ, where Z denotes $M(L^p(v_0), L^p(v_1))$ or $M(X, L^p(v_1))$, respectively.

Proof. Let us assume that $v_1 \in A_p(\mathbb{R}^{d_2})$. The other case is proved in a similar way. First observe that, since $u \in A_p(\mathbb{R}^{d_1})$ and $v_1 \in A_p(\mathbb{R}^{d_2})$, $w_1 \in A_p(\mathbb{R}^{d_1+d_2})$. Let us consider the family of multipliers $\{\mathbf{m}_j\}_j$ given by the previous proposition and let T_j be the transferred operator associated to the kernel $K_j = \mathbf{m}_j^\vee$ and to the representation of $\mathbb{R}^{d_1+d_2}$ on $L^p(\mathbb{R}^{d_2})$ given by (4.2.4). That is, for a function $f \in \mathcal{S}(\mathbb{R}^{d_2})$,

$$T_jf(s) = \int_{\mathbb{R}^{d_1+d_2}} K_j(x,y)e^{-2\pi i\xi x}f(y+s)dxdy = \int_{\mathbb{R}^{d_2}} \mathbf{m}_j(\xi,y)\widehat{f}(y)e^{2\pi iys}dy.$$

Let T be the multiplier operator associated to $\mathbf{m}(\xi,\cdot)$. By Theorem 4.2.9 and (4.2.16),

$$T_j : L^p(v_0) \longrightarrow L^p(v_1)$$

are uniformly bounded.

If $\xi \neq 0$, fixed $\delta < |\xi|$, for every $y \in \mathbb{R}^{d_2}$, by (4.2.17),

$$\mathbf{m}_j(\xi, y) \leq \sup_{|(x,z)|\geq\delta} |\mathbf{m}_j(x,z)| < \infty$$

uniformly in j, and by (4.2.15), $\mathbf{m}_j(\xi,y)\to\mathbf{m}(\xi,y)$.

If $\mathbf{m} \in L^\infty$, $\sup_{y\in\mathbb{R}^{d_1}} |\mathbf{m}_j(\xi,y)| \leq ||\mathbf{m}_j||_\infty < \infty$ uniformly in j by (4.2.18). Observe also that by (4.2.15), $\mathbf{m}_j(\xi,y)\to\mathbf{m}(\xi,y)$ a.e. $y \in \mathbb{R}^n$. In fact, if $\xi \neq 0$ the convergence holds for all $y \in \mathbb{R}^{d_2}$ and, for $y \neq 0$ if $\xi = 0$.

In both cases, by the Dominated Convergence Theorem, for any $f \in \mathcal{S}(\mathbb{R}^{d_2})$,

$$T_j f(s) \to Tf(s) := \int_{\mathbb{R}^{d_2}} \mathbf{m}(\xi,y)\widehat{f}(y)e^{2\pi i y s}dy. \tag{4.2.20}$$

By Fatou's lemma and the uniform boundedness of the operators T_j, given $f \in \mathcal{S}(\mathbb{R}^{d_2})$, we have

$$||Tf||_{L^p(v_1)} \leq \liminf_j ||T_j f||_{L^p(v_1)} \lesssim ||\mathbf{m}||_{M(L^p(w_0),L^p(w_1))}||f||_{L^p(v_0)}.$$

So the result follows by the density of $\mathcal{S}(\mathbb{R}^{d_2}) \cap L^p(v_0)$ in $L^p(v_0)$.

Assume now that $\mathbf{m} \notin L^\infty$ and $\xi = 0$. Fixed $\delta > 0$, for any $f \in F_\delta$, since by (4.2.17), $\sup_{|y|>\delta} |\mathbf{m}_j(0,y)| < \infty$, uniformly in j, we can apply the Dominated Convergence Theorem to the functions

$$\widehat{f}(y)\mathbf{m}_j(0,y) = \chi_{\{z:\,|z|\geq\delta\}}(y)\mathbf{m}_j(0,y)\widehat{f}(y),$$

in order to get (4.2.20) for functions in $\cup_{\delta>0}F_\delta$. As we showed above, it holds that, for any $f \in \cup_{\delta>0}F_\delta$,

$$||Tf||_{L^p(v_1)} \lesssim ||\mathbf{m}||_{M(L^p(w_0),L^p(w_1))}||f||_{L^p(v_0)},$$

and then, by the definition of X, the result easily follows. □

Observation 4.2.21. *Observe in the previous result that the local integrability of w_0 implies the local integrability of v_0. Moreover, if $\int v_0(x)^{-1/(p-1)}dx < \infty$, the map $f \mapsto \hat{f}(0)$ is a bounded linear functional on $L^p(v_0)$, and hence X is a proper linear subspace of $L^p(v_0)$.*

Corollary 4.2.22. *Let $1 < p < \infty$ and let w be a weight in $\mathbb{R}^{d_1+d_2}$ such that $w(x,y) = u(x)v(y)$ where $u \in A_p(\mathbb{R}^{d_1})$ and $v \in A_p(\mathbb{R}^{d_2})$. Given $\mathbf{m} \in M(L^p(w)) \cap \mathcal{C}_b(\mathbb{R}^{d_1+d_2})$ for any $\xi \in \mathbb{R}^{d_1}$, $\mathbf{m}(\xi,\cdot) \in M(L^p(v))$, and*

$$\|\mathbf{m}(\xi,\cdot)\|_{M(L^p(v))} \lesssim \|\mathbf{m}\|_{M(L^p(w))},$$

with constant independent of ξ.

4.2.3 An example with G a compact group

Let us now consider $G = \mathbb{T}$, $\mathcal{M} = (\mathbb{R}^2, w(x)dx)$ for some weight $w > 0$. For a radial weight v, we shall consider the continuous representation of G on $L^p(v)$ defined by

$$R_\theta f(x) = f(e^{i\theta}x).$$

For any locally integrable function $u > 0$ defined on $\mathbb{R}^+$, we shall consider the mixed weighted spaces

$$L^{p_1}_{\mathrm{rad}}(L^{p_0}_{\mathbb{T}}; v) = \left\{ f;\ \|f\|_{L^{p_1}_{\mathrm{rad}}(L^{p_0}_{\mathbb{T}};v)} = \left(\int_{\mathbb{R}^+} \left[\int_{-\pi}^{\pi} |f(e^{i\theta}r)|^{p_0} d\theta \right]^{\frac{p_1}{p_0}} v(r)dr \right)^{\frac{1}{p_1}} < \infty \right\}.$$

Since $R^*_\theta g(x) = g(e^{-i\theta}x)\frac{w(e^{-i\theta}x)}{w(x)}$, and, for $w_i = 1$ for $i = 0, 1, 2$, we obtain by Minkowski's inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \overline{R}f(x)\overline{R}^*_{w_2}g(x)w(x)\,dx \\
&\approx \int_{\mathbb{R}^2} \left(\int_{-\pi}^{\pi} |f(e^{i\theta}x)|^{p_0}\,d\theta \right)^{1/p_0} \left(\int_{-\pi}^{\pi} |g(e^{i\theta}x)w(e^{i\theta}x)|^{p_1'}\,d\theta \right)^{1/p_1'} dx \\
&= \int_{\mathbb{R}^+} r \left(\int_{-\pi}^{\pi} |f(e^{i\theta}r)|^{p_0}\,d\theta \right)^{1/p_0} \left(\int_{-\pi}^{\pi} |g(e^{i\theta}r)w(e^{i\theta}r)|^{p_1'}\,d\theta \right)^{1/p_1'} dr \\
&\leq ||f||_{L^{p_1}_{\mathrm{rad}}(L^{p_0}_{\mathbb{T}};rv)} ||g||_{L^{p_1'}(\overline{u})},
\end{aligned}$$

where $\overline{u}(x) = w(x)^{p_1'}v(|x|)^{1-p_1'}$. Therefore, the conclusion of our Corollary 4.1.16 is the following:

Corollary 4.2.23. *If $K \in L^1(\mathbb{T})$ satisfies the condition that $B_K : L^{p_0}(\mathbb{T}) \to L^{p_1}(\mathbb{T})$ is bounded with constant $N_{p_0,p_1}(K)$ and*

$$T_K f(x) = \int_{-\pi}^{\pi} K(e^{i\theta})f(e^{i\theta}x)d\theta,$$

then, for every radial weight $v > 0$, defined on $\mathbb{R}^2$

$$T_K : \left(L^{p_1}_{\mathrm{rad}}(L^{p_0}_{\mathbb{T}}; u) \cap L^{p_1}(v), ||\cdot||_{L^{p_1}_{\mathrm{rad}}(L^{p_0}_{\mathbb{T}};u)} \right) \longrightarrow L^{p_1}(v)$$

is bounded with norm bounded above by a constat multiple of $N_{p_0,p_1}(K)$, where $u(r) = rv(r)$.

4.2.4 An example of a maximal operator

Theorem 4.2.24. *For every $n \in \mathbb{Z}$, let $w_n = (w_n^k)_{k\in\mathbb{Z}}$ be a sequence with finite support and let us assume that the maximal convolution operator*

$$\sup_{n\in\mathbb{Z}} |w_n * \cdot| : \ell^{p_0} \longrightarrow \ell^{p_1}$$

is bounded with $p_0 \le p_1$. *Let* X_{p_0,p_1} *be the space of* $f \in L^{p_1}(\mathbb{R})$ *such that*

$$||f||_{X_{p_0,p_1}} = \left(\int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} |f(x+n)|^{p_0} \right)^{p_1/p_0} dx \right)^{\frac{1}{p_1}} < \infty.$$

Then the operator defined by

$$T^\sharp f(x) = \sup_{n\in\mathbb{Z}} \left| \sum_{k\in\mathbb{Z}} w_n^k f(x+k) \right|$$

satisfies the condition that $T^\sharp : X_{p_0,p_1} \longrightarrow L^{p_1}(\mathbb{R})$ *is bounded.*

Proof. Let $G = \mathbb{Z}$ and let us consider the action on functions on $\mathbb{R}$ given by $R_n f(x) = f(x+n)$. Since R is obviously positive preserving and $||R_n f||_{L^{p_1}} = ||f||_{L^{p_1}}$, we can apply our Theorem 4.1.17, to deduce that

$$T^\sharp : W \longrightarrow L^{p_1}$$

whenever

$$W = \left\{ f \in L^{p_1}(\mathbb{R}^n);\ \sup_N \left\| \left(\frac{1}{N^{p_0/p_1}} \sum_{j=-N}^{N} |f(x+j)|^{p_0} \right)^{1/p_0} \right\|_{p_1} < \infty \right\}.$$

So now we consider the operator $S_N a(\mathbf{m}) = \frac{1}{N^{p_0/p_1}} \sum_{j=-N}^{N} |a_{j+m}|$ and observe that by Young's convolution inequality $S_N : \ell^1 \to \ell^{p_1/p_0}$ is bounded with norm less than or equal to $3^{p_0/p_1}$. Hence

$$\begin{aligned} ||f||_W &= \sup_N \left(\int_{\mathbb{R}} \left(\frac{1}{N^{p_0/p_1}} \sum_{j=-N}^{N} |f(x+j)|^{p_0} \right)^{p_1/p_0} dx \right)^{\frac{1}{p_1}} \\ &\le \sup_N \left(\int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} \left(\frac{1}{N^{p_0/p_1}} \sum_{j=-N}^{N} |f(\theta+n+j)|^{p_0} \right)^{p_1/p_0} d\theta \right)^{\frac{1}{p_1}} \\ &\lesssim \left(\int_{\mathbb{T}} \left(\sum_{n\in\mathbb{Z}} |f(\theta+n)|^{p_0} \right)^{p_1/p_0} d\theta \right)^{\frac{1}{p_1}} = ||f||_X, \end{aligned}$$

and the result follows. □

4.2.5 Radial Kernels

In this section apply the results in §4.1 to the setting of convolution operators with radial kernels. More precisely, we will get results that allow to obtain estimations on a convolution operator on $\mathbb{R}^d$ with radial kernel, from estimations on a certain convolution operator in a lower dimensional space. We will face the problem from two points of view. In the first one we consider the unweighted (L^p, L^q) situation with $p \le q$ and, in the second case, the case $p = q$ with weights.

First case

We shall assume that $w_0 = w_1 = w_2 = 1$. Let us define $\mathcal{R}$ to be the set of radial functions $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and such that if

$$\mathbf{m}_\phi(s) = 2\int_0^\infty \phi(t)\cos(2\pi ts)\,dt$$

we have that $||\mathbf{m}_\phi||_1 = 1$ and $\phi(0) = 1$.

Theorem 4.2.25. *Let $K \in L^1_{\text{loc}}(\mathbb{R}^d)$ radial and let us assume that*

$$h = t^{d-1}\chi_{(0,\infty)}K^0 \in \mathcal{S}'(\mathbb{R})$$

where $K^0(t) = K(x)$, whenever $|x| = t$. Suppose that the convolution operator

$$h* : L^p(\mathbb{R}) \longrightarrow L^q(\mathbb{R}),$$

is bounded with norm N for some $p \leq q$. Then the convolution operator

$$K* : X \longrightarrow L^q(\mathbb{R}^d),$$

is bounded, where X is defined as the space of functions $f \in \mathcal{S}(\mathbb{R}^d)$ such that

$$||f||_X = \int_{\Sigma_{d-1}}\left(\int_{H_\theta}\left(\int_{\mathbb{R}}|f(x'+t\theta)|^p dt\right)^{q/p} dx'\right)^{\frac{1}{q}} d\theta < \infty, \tag{4.2.26}$$

H_θ is the orthogonal hyperplane through 0 to the line $[\theta]$, and $d\theta$ is the surface measure on the unit sphere Σ_{d-1}.

Proof. Let $\phi \in \mathcal{R}$ and let us consider $h_r(t) = \phi(\frac{t}{r})h(t)$. Then, since $||\mathbf{m}_\phi||_1 = 1$, we have that, for every $r > 0$, $h_r* : L^p(\mathbb{R}) \longrightarrow L^q(\mathbb{R})$, with norm uniformly bounded. Let $K_r(x) = \phi(\frac{x}{r})K(x)$ and let us observe that

$$(K_r * f)(x) = \int_{\Sigma_{d-1}}\left(\int_0^\infty h_r(t)f(x-t\theta)dt\right)d\theta.$$

Now, for each $\theta \in \Sigma_{d-1}$, the operator in brackets is the transferred operator with convolution kernel h_r under the representation $R_t f(x) = f(x - t\theta)$ and since $h_r \in L^1$ and has compact support, we can apply Corollary 4.1.12. To this end, we need to consider the operators $S^\theta_{L,p/q}$ acting on functions on $\mathbb{R}^d$ and $S_{L,p/q}$ acting on function on $\mathbb{R}$ and defined as follows:

$$\begin{aligned} S^\theta_{L,p/q}f(x) &:= \frac{1}{L^{p/q}}\int_{-L}^{L}|f(x-t\theta)|dt = \frac{1}{L^{p/q}}\int_{-L}^{L}|f(x'-(t-s)\theta)|dt \\ &:= S_{L,p/q}f^\theta_{x'}(s), \end{aligned}$$

where $x = x' + s\theta$ with $x' \in H_\theta$ and $f^\theta_{x'}(s) = f(x' + s\theta)$. Hence, we have that

$$\begin{aligned} ||K_r * f||_q &\leq \int_{\Sigma_{d-1}} \left\| \int_0^\infty h_r(t) f(x - t\theta) dt \right\|_q d\theta \\ &\lesssim \int_{\Sigma_{d-1}} \sup_{L>0} \left(\int_{H_\theta} \int_{\mathbb{R}} S_{L,p/q}[(f^\theta_{x'})^p](s)^{q/p} ds dx' \right)^{\frac{1}{q}} d\theta. \end{aligned}$$

Now, since $S_{L,p/q} : L^1(\mathbb{R}) \longrightarrow L^{q/p}(\mathbb{R})$, uniformly in L, we obtain

$$||K_r * f||_q \leq \int_{\Sigma_{d-1}} \left(\int_{H_\theta} \left(\int_{\mathbb{R}} |f(x' + s\theta)|^p ds \right)^{q/p} dx' \right)^{\frac{1}{q}} d\theta,$$

and hence, taking $f \in \mathcal{S}(\mathbb{R}^d)$ and letting r tends to ∞, we obtain the result, since $K_r \to K$ in $\mathcal{S}'(\mathbb{R}^d)$. □

Observation 4.2.27. *Observe that in the previous theorem, the functional $||\cdot||_X$ is a norm. So the space X given can be replaced by the Banach completion of X. It is easy to see that $||\cdot||_X$ is a seminorm, so in order to see that $||\cdot||_X$ is a norm, it suffices to show that if $||f||_X = 0$ then $f = 0$. However, observe that if $||f||_X = 0$, for any $r > 0$,*

$$\begin{aligned} 0 &= \int_{\Sigma_{d-1}} \left(\int_{H_\theta} \left(\int_{\mathbb{R}} |f(x' + s\theta)|^p \chi_{B(0,r)}(x' + s\theta) ds \right)^{q/p} dx' \right)^{\frac{1}{q}} d\theta \\ &\geq C_r ||f \chi_{B(0,r)}||_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Then, it easily follows that $f = 0$. Observe also that the previous inequality implies that $X \subset L^p_{\text{loc}}(\mathbb{R}^d)$.

In order to give an application of the previous theorem, let us remember the following well known result on Riesz fractional operator.

Theorem 4.2.28 ([96])**.** *Let $d \geq 1$ and $0 < \alpha < d$. Then*

$$I_\alpha : L^p(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d),$$

is bounded, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, $1 < p < q < \infty$, and $I_\alpha f := \frac{1}{|x|^{d-\alpha}} * f$.

Corollary 4.2.29. *Let $0 < \alpha < 1$, $d \in \mathbb{N}$, $d \geq 2$, and $K(x) = \frac{1}{|x|^{d-\alpha}}$. Then, for $1 < p < q < \infty$, and $\frac{1}{q} = \frac{1}{p} - \alpha$,*

$$I_\alpha : \overline{X}^{||\cdot||_X} \longrightarrow L^q(\mathbb{R}^d),$$

is bounded, where $I_\alpha f := K * f$, and X is given by those $f \in \mathcal{S}(\mathbb{R}^d)$ which satisfy (4.2.26).

Proof. Observe that $K \in L^1_{\text{loc}}(\mathbb{R}^d)$ and it is radial function such that $K^0(t) = t^{\alpha-d}$. Therefore, if we define $h(t) = t^{\alpha-1}\chi_{[0,\infty)} \in \mathcal{S}'(\mathbb{R})$, it holds that for $t > 0$, $h(t) = t^{d-1}K^0(t)$. In addition, the convolution operator with kernel h defines a bounded map from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, because $|h * f| \leq \frac{1}{|t|^{1-\alpha}} * |f|$. Thus, the result follows from Theorem 4.2.25 and Remark 4.2.27. □

Second case

Let K be a radial kernel on $\mathbb{R}^d$ with $d \geq 3$, which is continuous and has compact support. In particular, K has the form $K(y) = K^0(|y|)$, where K^0 is a function defined on $(0, \infty)$.

Let $G = \mathrm{SO}(d)$, the group of rotations of $\mathbb{R}^d$. If $\mathbf{1} = (1, 0, \ldots, 0)$ then an element $x' \in \Sigma_{d-1}$ has the form $U\mathbf{1}$ for an appropriate $U \in SO(d)$. Let $\mathbf{e} = (0, \ldots, 0, 1)$. Consider the subgroup H of all $U \in G$ such $U\mathbf{e} = \mathbf{e}$. We can identify Σ_{d-1} with G/H, in the following way: the point $x' \in \Sigma_{d-1}$ corresponds to the coset of all $u \in G$ such that $U\mathbf{e} = x'$.

If f is a right invariant function on G we can associate with it a function f^1 on Σ_{d-1} by the relation $f^1(x) = f(U\mathbf{e})$ whenever $U\mathbf{e} = x \in \Sigma_{d-1}$. Conversely, any function f^1 on Σ_{d-1} determines a right invariant function $f(U) = f^1(Ue)$, $U \in G$. Lebesgue (surface) measure on Σ_{d-1} also corresponds to Haar measure on G in the way that, if f is the right invariant function associated with f^1 on Σ_{d-1}, then $f \in L^1(G)$ if and only if $f^1 \in L^1(\Sigma_{d-1})$. Moreover,

$$\int_{\Sigma_{d-1}} f^1(x')dx' = \omega_{d-1} \int_{SO(d)} f(U)dU \tag{4.2.30}$$

where ω_{d-1} denotes the surface area of Σ_{d-1}. Recall that $\omega_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$. Throughout this section $1 < p < \infty$.

Theorem 4.2.31. *Let K be a function which is continuous, compactly supported and has the form $K(y) = K^0(|y|)$, where K^0 is a function defined on $(0, \infty)$. Let v, w be weights in $\mathbb{R}$ and $\mathbb{R}^{d-1}$, respectively. If $h(y) = |y|\, K^0(|y|)$ satisfies*

$$\int_{\mathbb{R}^{d-1}} \left| \int_{\mathbb{R}^{d-1}} h(y)\phi(z-y)dy \right|^p w(z)\, dz \leq A^p \int_{\mathbb{R}^{d-1}} |\phi(z)|^p\, w(z)\, dz,$$

for all $\phi \in L^p(\mathbb{R}^{d-1}, w)$, then

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} K^0(|y|) f(x-y)dy \right|^p \Upsilon(x)\, dx \leqq (c_d A)^p \int_{\mathrm{R}^d} |f(y)|^p \overline{\Omega}(y)dy$$

for all $f \in L^p(\mathbb{R}^d, \overline{\Omega})$, where $c_d = \frac{\omega_{d-1}}{\omega_{d-2}}$, $\overline{\Omega}(x) = \int_{SO(d)} \Omega(Ux)\, dU$, $\Upsilon(x) = \left(\int_{SO(d)} \Omega^{1-p'}(Ux)\, dU\right)^{1-p}$ and $\Omega(x) = v(x_1)w(\overline{x})$ where $x = (x_1, \overline{x})$.

Proof. Since K is radial and the surface measure of Σ_{d-1} is rotation invariant,

$$\int_{\Sigma_{d-1}} K(ry')f(x-ry')dy' = \int_{\Sigma_{d-1}} K(ry')f(x+ry')dy'.$$

Then

$$\begin{aligned}
(K*f)(x) &= \int_0^\infty r^{d-1}\left\{\int_{\Sigma_{d-1}} K(ry')f(x-ry')dy'\right\}dr \\
&= \frac{1}{2}\int_{\Sigma_{d-1}}\left\{\int_{\mathbb{R}} K(ry')f(x-ry')\,|r|^{d-1}\,dr\right\}dy' \\
&= \frac{\omega_{d-1}}{2}\int_{SO(d)}\left\{\int_{\mathbb{R}} K^0(|r|)f(x-rU\mathbf{1})|r|^{d-1}dr\right\}du.
\end{aligned}$$

Identifying $SO(d-1)$ with the subgroup of $SO(d)$ of all those rotations leaving the vector $\mathbf{e}=(0,\dots,0,1)$ fixed, and $SO(d-2)$ with the subgroup of $SO(d)$ leaving $\mathbf{e}$ and $\mathbf{1}$ fixed. Then $SO(d-1)/SO(d-2)$ can be identified with the set

$$\{0\}\times\Sigma_{d-2}=\{x'\in\Sigma_{d-1}:\ x'\perp\mathbf{1}\}\cong\Sigma_{d-2}.$$

Using the right invariance of Haar measure on G, we see that the last integral equals, for $V\in SO(d-1)$

$$\begin{aligned}
&\int_{SO(d)}\left\{\int_{\mathbb{R}} K^0(|r|)f(x-rUV\mathbf{1})|r|^{d-1}dr\right\}dU \\
&\quad=\int_{SO(d)}\int_{SO(d-1)}\left\{\int_{\mathbb{R}} K^0(|r|)|r|f(x-rUV\mathbf{1})|r|^{d-2}dr\right\}dV\,dU \\
&\quad=\int_{SO(d)}\int_{\Sigma_{d-2}}\left\{\int_{\mathbb{R}} K^0(|r|)|r|f(x-rU\mathbf{y})|r|^{d-2}dr\right\}\frac{dy'}{\omega_{d-2}}dU,
\end{aligned}$$

where $\mathbf{y}=(0,y')\in\Sigma_{d-2}$ and the last equality follows from (4.2.30). Then

$$K*f(x)=\frac{\omega_{d-1}}{\omega_{d-2}}\int_{SO(d)}\left\{\int_{\mathbb{R}^{d-1}}|y|\,K(y)f(x-Uy)\,dy\right\}dU,$$

where we are identifying $y\in\mathbb{R}^{d-1}$ with $(0,y)\in\mathbb{R}^d$.

Let $G=\mathbb{R}^{d-1}$, and fixed $U\in SO(d)$, define

$$(R_y^U f)(x)=f(x+Uy)$$

when f is a function defined on $\mathcal{M}=\mathbb{R}^d$. So the term in curly brackets corresponds to the associated transferred operator

$$T_h^U f(x)=\int_{\mathbb{R}^{d-1}} h(y)R_{-y}^U f(x)\,dy,$$

where $h(y)=|y|\,K(y)$. Observe that

$$(R_y^U f)(Ux)=f(U(x_1,\overline{x}+y))=(R_{\overline{x}+y}^U)f(U(x_1,0)),\quad x=(x_1,\overline{x})\in\mathbb{R}\times\mathbb{R}^{d-1}$$

Then

$$T_h^U f(Ux)=R_{\overline{x}}^U T_h^U(U(x_1,0)).$$

Therefore,

$$\int_{\mathbb{R}^d} T_h^U f(x)g(x)\,dx = \int_{\mathbb{R}^d} T_h^U f(Ux)g(Ux)\,dx$$
$$= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^{d-1}} R_{\overline{x}}^U T_h^U f(U(x_1,0)) R_{\overline{x}}^U g(U(x_1,0))\, d\overline{x} \right\} dx_1.$$

Observe that, fixed $z \in \mathbb{R}^d$,

$$R_{\overline{x}}^U T_h^U(z) = (h * R_{\cdot}^U f(z))(\overline{x}).$$

Then, if h maps $L^p(\mathbb{R}^{d-1}, w)$ into $L^p(\mathbb{R}^{d-1}, w)$ with norm $N(h)$,

$$\left|< T_h^U f, g >\right| \leq N(h) \int_{\mathbb{R}} \left\{ \left(\int_{\mathbb{R}^{d-1}} \left| R_{\overline{x}}^U f(U(x_1,0)) \right|^p w(\overline{x}) d\overline{x} \right)^{1/p} \right.$$
$$\left. \left(\int_{\mathbb{R}^{d-1}} \left| R_{\overline{x}}^U g(U(x_1,0)) \right|^{p'} w(\overline{x})^{1-p'} d\overline{x} \right)^{1/p'} \right\} dx_1$$
$$\leq N(h) \left\{ \int_{\mathbb{R}^d} |f(x)|^p \, \Omega(Ux) dx \right\}^{1/p} \left\{ \int_{\mathbb{R}^d} |g(x)|^{p'} \, \Omega(Ux)^{1-p'} dx \right\}^{1/p'}$$

where $\Omega(x) = v(x_1)w(\overline{x})$. On the other hand, integrating on $SO(d)$, by Hölder's inequality,

$$\left|< K * f, g >\right|$$
$$\leq \frac{\omega_{d-1}}{\omega_{d-2}} N(h) \left\{ \int_{\mathbb{R}^d} |f(x)|^p \, \overline{\Omega}(x) dx \right\}^{1/p} \left\{ \int_{\mathbb{R}^d} |g(x)|^{p'} \, \Upsilon(x)^{1-p'} dx \right\}^{1/p'}.$$

where $\overline{\Omega}(x) = \int_{SO(d)} \Omega(Ux)\,dU$ and $\Upsilon(x) = \left(\int_{SO(d)} \Omega^{1-p'}(Ux)\,dU \right)^{1-p}$. Therefore

$$\|K * f\|_{L^p(\mathbb{R}^d, \Upsilon)} \leq \frac{\omega_{d-1}}{\omega_{d-2}} N(h) \|f\|_{L^p(\mathbb{R}^d, \overline{\Omega})}.$$

□

Let us remark that with minors modification, the previous result holds for a maximal operator associated to a family of convolution operators with radial kernels. Observe also that [46, Theorem 6.3] is recovered as a particular case of the previous theorem with $w = 1$ and $v = 1$. But, now more can be said. In order to give examples we need the following lemma. Let π_1 and π_2 denote the canonical projection of $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ in $\mathbb{R}$ and $\mathbb{R}^{d-1}$ respectively.

Lemma 4.2.32. *If $\Omega(x) = |\pi_1(x)|^\beta \, |\pi_2(x)|^\alpha$ for $\beta > -1$ and $\alpha > 1-d$,*

$$\int_{SO(d)} \Omega(Ux)\,dU = c_{d,\beta,\alpha} \, |x|^{\beta+\alpha}$$

where $c_{d,\beta,\alpha} = \frac{B\left(\frac{d-1+\alpha}{2}, \frac{1+\beta}{2}\right)}{B\left(\frac{d-1}{2}, \frac{1}{2}\right)}$, and B denotes de Beta function.

Proof. Let $\gamma > d-1$. For $x \in \mathbb{R}^d$, denote $\pi_2(x) = \overline{x}$. If $w(\overline{x}) = |\overline{x}|^\gamma$, then by the right invariance of the Haar's measure on $SO(d)$,

$$\int_{SO(d)} w(\pi_2(Ux))\, dU = \int_{SO(d)} w(|x|\, \pi_2(U\mathbf{1}))\, dU = |x|^{\beta+\alpha} c_{d,\beta,\alpha},$$

Observe that $\omega_{d-1} c_{d,\beta,\alpha} = \int_{\Sigma_{d-1}} |\pi_1(x')|^\beta\, |\pi_2(x')|^\alpha \ dx'$. Hence, parameterizing Σ_{d-1},

$$\begin{aligned} \omega_{d-1} c_{d,\gamma} &= \int_0^{2\pi} |\cos\theta_1|^\beta\, |\sin\theta_1|^{d-2+\alpha}\ d\theta_1 \int_{(0,\pi)^{d-2}} \prod_{j=2}^{d-2} |\sin\theta_j|^{d-j-1}\ d\theta_2 \dots d\theta_{d-1} \\ &= \frac{B\left(\frac{d-1+\alpha}{2}, \frac{1+\beta}{2}\right)}{B\left(\frac{d-1}{2}, \frac{1}{2}\right)} \omega_{d-1}. \end{aligned}$$

So the result follows. ∎

By the previous computation, if $w(x) = |\overline{x}|^\alpha$ with $1-d < \alpha < (d-1)(p-1)$, considering $v(x_1) = |x_1|^\beta$ with $-1 < \beta < p-1$,

$$\overline{\Omega}(x) = c_{d,\alpha,\beta}\, |x|^{\alpha+\beta}\,, \text{ and } \Upsilon(x) = c_{d,\alpha(1-p'),\beta(1-p')}^{1-p}\, |x|^{\alpha+\beta}\,,$$

so the following result holds.

Corollary 4.2.33. *Let K be a function which is continuous,compactly supported and $K(y) = K^0(|y|)$, where K^0 is a function defined on $(0,\infty)$. Let $1-d < \alpha < (d-1)(p-1)$. If $h(y) = |y|\, K^0(y)$ satisfies*

$$\|h * \phi\|_{L^p(\mathbb{R}^{d-1},|x|^\alpha)} \le A\, \|\phi\|_{L^p(\mathbb{R}^{d-1},|x|^\alpha)}$$

for all $\phi \in L^p(\mathbb{R}^{d-1}, |x|^\alpha))$, then for $-1 < \beta < p-1$,

$$\|K * f\|_{L^p(|x|^{\alpha+\beta})} \le c_d\, c_{d,\alpha,\beta}^{1/p}\, c_{d,\alpha(1-p'),\beta(1-p')}^{1/p'} A\, \|f\|_{L^p(|x|^{\alpha+\beta})}\,.$$

for all $f \in L^p(\mathbb{R}^d, |x|^{\alpha+\beta})$.

Let, for $n \ge 1$, $\varphi_n = \frac{1}{|B(0,n)|}\chi_{B(0,n)} * \chi_{B(0,n)}$. Observe these functions satisfy Lemma 2.3.9 and that each φ_n is a radial function. The following result generalizes [46, Theorem 6.5], that is recovered for $\alpha = \beta = 0$.

Theorem 4.2.34. *Let $-(d-2) < \alpha < (d-2)(p-1)$, $-1 < \beta < p-1$, $\mathbf{m}(x) = \mathbf{m}^0(|x|)$ be a bounded radial normalized (with respect to $\{\widehat{\varphi_n}\}$) function in $\mathbb{R}^{d-2}$ satisfying that $\mathbf{m} \in M(L^p(\mathbb{R}^{d-2}, |\cdot|^\alpha))$. If we define*

$$M^0(r) = 2\pi \int_0^1 u^{d-3}\mathbf{m}^0(ur)\, du,$$

then $M(y) = M^0(|y|) \in M(L^p(\mathbb{R}^d, |\cdot|^{\alpha+2\beta}))$. Moreover $\|M\|_{M(L^p(\mathbb{R}^d,|\cdot|^{\alpha+2\beta}))} \le c_{d,\alpha,\beta}\, \|\mathbf{m}\|_{M(L^p(\mathbb{R}^{d-2},|\cdot|^\alpha))}$.

Proof. Let us begin observing that $|x|^{\alpha} \in A_p(\mathbb{R}^{d-2})$ and hence, by Proposition 2.3.29, $L^p(|x|^{\alpha})$ is well behaved and the associated family $\{h_n\}_n$ can be taken to satisfy that h_n are radial functions in $\mathcal{C}_c^{\infty}(\mathbb{R}^{d-2})$. On the other hand, since $L^p(w)$ is Banach, we can apply Theorem 2.3.13 to approximate the multiplier. Moreover, following the notation therein, as φ_n, $\mathbf{m}$, and h_n are radial it follows that $\mathbf{m}_n = (\widehat{\varphi_n} * \mathbf{m})\widehat{h_n}$ is also radial.

Observe that if we define $M_n^0(r) = 2\pi \int_0^1 u^{d-3}\mathbf{m}_n^0(ur)\, du$, then for any $r > 0$, $M^0(r) = \lim_n M_n^0(r)$. Moreover $\|M_n^0\|_{\infty} \le c_d \|\mathbf{m}_n\|_{\infty} \le c_d \|\mathbf{m}\|_{\infty}$. And hence, if we proof the result for $\mathbf{m}_n$, as $\|\mathbf{m}_n\|_{M(L^p(\mathbb{R}^{d-2},|\cdot|^{\alpha}))} \lesssim \|\mathbf{m}\|_{M(L^p(\mathbb{R}^{d-2},|\cdot|^{\alpha}))}$, for any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\|T_M f\|_{L^p(\mathbb{R}^d,|\cdot|^{\alpha+2\beta})} \le \liminf_n \|T_{M_n} f\|_{L^p(\mathbb{R}^d,|\cdot|^{\alpha+2\beta})} \lesssim \|f\|_{L^p(\mathbb{R}^d,|\cdot|^{\alpha+2\beta})}.$$

Thus, we can assume without loss of generality, that there exists a radial function h with compact support such that $h \in L^1(\mathbb{R}^{d-2})$ and $\mathbf{m} = \widehat{h}$. Define the radial function K in $L^1(\mathbb{R}^d)$ by $r^2 K^0(r) = h^0(r)$. Observe that

$$M^0(r) = \frac{2\pi}{r^{d-2}} \int_0^r s^{d-3}\mathbf{m}^0(s)\, ds,$$

and hence, $2\pi\mathbf{m}^0(r) = r^{3-d}\left(r^{d-2}M^0(r)\right)'$ is satisfied. Then, by the discussion in [46, p. 35], $M = \widehat{K}$. Thus, the proof finishes iterating Corollary 4.2.33 twice. □

Let us give an example of how the previous theorem can be used in general. Denote by $\mathbf{m}_a(x) = (1-|x|^2)_+^a$, that is the Bochner-Riesz multiplier of order a. If we consider, for $1 < a$,

$$\mathbf{m}(x) = \frac{(d+2a-2)}{2\pi}(1-|x|^2)_+^a - \frac{2a}{2\pi}(1-|x|^2)_+^{a-1},$$

it can be shown that $M(x) = (1-|x|^2)_+^a$. For $d = 4$ and $1 < a$, by [97, §IX.2.2], $\mathbf{m}_a \in M(L^p(\mathbb{R}^2))$ for any $p > 1$, $\mathbf{m}_{a-1} \in M(L^p(\mathbb{R}^2))$ for any p if $a > 3/2$, and for p satisfying

$$\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2a-1}{4}$$

if $a \le 3/2$. By the previous result it follows that for those p on which $\mathbf{m}_a$ and $\mathbf{m}_{a-1}$ are bounded multipliers, for $\beta \in (-2, 2(p-1))$, $M \in M(L^p(|\cdot|^{\beta}, \mathbb{R}^4))$.

Chapter 5

Further results

5.1 Multilineal Transference

In the previous chapters we have developed two ways of transferring the boundedness of a convolution operator whose kernel is an integrable and compactly supported function. In this section we will study how these ideas carry over the multilinear setting restricting our attention, by simplicity, to the bilinear case.

5.1.1 Multilinear transference

We will follow a similar approach to that given in [26] to this problem. Let introduce the notation we will follow in this section.

Let B_1, B_2, B_3 be RIQBFS defined on G. For $K \in L^1(G)$ with compact support, consider the mapping defined by

$$B_K(\phi, \psi)(v) = \int_G K(u)\phi(u^{-1}v)\psi(uv)\, du.$$

Let F^1, F^2, F^3 be Banach spaces of functions such that $F^1F^2 \subset F^3$, that is

$$\|f_1 f_2\|_{F^3} \leq C \|f_1\|_{F^1} \|f_2\|_{F^2}, \tag{5.1.1}$$

and F^3 is continuously embedded in $L^1_{\text{loc}}(\mathcal{M})$. Examples of such spaces are given by $F^1 = F^2 = L^2(\mathcal{M})$, $F^3 = L^1(\mathcal{M})$; $F^1 = F^2 = F^3 = \mathcal{C}_0(G)$; $F^j = L^{p_j}$ with $1 \leq p_j < \infty$ for $j = 1, 2, 3$ satisfying

$$\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}.$$

Let $R^j : G \to \mathfrak{B}(F^j)$ be strongly continuous representations, for $j = 1, 2, 3$, satisfying the property that for $f_j \in F^j$, $j = 1, 2$, and $u, v \in G$,

$$R^3_u \left(R^1_{v^{-1}} f_1 R^2_v f_2\right) = R^1_{uv^{-1}} f_1 R^2_{uv} f_2. \tag{5.1.2}$$

Let us give an example of such representations R^j. Let $\{\tau_u\}_{u \in G}$ be a family of measure preserving transformations that $\tau_v \circ \tau_u = \tau_{uv}$. And let, for $j = 1, 2$, $h^j_t(x)$

be a measurable function such that $\left|h_t^j(x)\right| = 1$ and

$$h_{uv}^j(x) = h_u^j(x)h_v^j(\tau_u x).$$

Consider $h_u^3(x) = h_u^1(x)h_u^2(x)$ and define, for $j = 1,2,3$ the distributionally bounded representation given by

$$R_u^j f(x) = h_u^j(x) f(\tau_u x).$$

It is easy to see that (5.1.2) is satisfied.

Let us define for $f_j \in F^j$, $j = 1,2$, the transferred bilinear operator T_K by

$$T_K(f_1, f_2) = \int_G K(u) R_{u^{-1}}^1 f_1 \; R_u^2 f_2 \;\; du.$$

Observe that, since $u \mapsto R_{u^{-1}}^1 f_1 R_u^2 f_2$ maps G into F^3 continuously, by Proposition A.1.5 the transferred operator is well defined as a vectorial integral. Moreover,

$$R_v^3 T_K(f_1, f_2) = \int_G K(u) R_{vu^{-1}}^1 f_1 R_{vu}^2 f_2 \; du.$$

Let, for $j = 1,2,3$, E_j be QBFS's on $\mathcal{M}$ such that

$$\|f_1 f_2\|_{E_3} \leq c_E \|f_1\|_{E_1} \|f_2\|_{E_2}. \tag{5.1.3}$$

Examples of such 3-tuples of spaces E^j are given by Lorentz-spaces (see [86]),

$$\|f_1 f_2\|_{L^{p_3,s_3}} \leq C_{p_1,p_2,s_1,s_2} \|f_1\|_{L^{p_1,s_1}} \|f_2\|_{L^{p_2,s_2}}, \tag{5.1.4}$$

$0 < p_i \leq \infty$, $0 < s_i \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s_3}$. Observe that if $p_i = s_i$ for $i = 1,2$ and $p_3 \leq s_3$, C_{p_1,p_2,s_1,s_2} is 1.

We shall implicitly assume that for $j = 1,2,3$, for any $f_j \in F^j$ and any open set $V \subset G$, the functions $\|\chi_V(v) R_v^j f(x)\|_{B_j}$ are μ-measurable. Moreover $W(B_j, E_j, V)$ denotes the TWA defined by the representation R^j acting on F^j.

Theorem 5.1.5. *Let $K \in L^1(G)$ with compact support and let $\mathcal{K}$ be a compact set such that $\mathcal{K} \supset \operatorname{supp} K$. Under the above conditions, if $B_K : B_1 \times B_2 \to B_3$ with norm $N(K)$, then, fixed a non empty open set V, for $f_j \in F^j$ with $j = 1,2$,*

$$\|T_K(f_1, f_2)\|_{W(B_3,E_3,V)} \leq c_E N(K) \|f_1\|_{W(B_1,E_1,V\mathcal{K}^{-1})} \|f_2\|_{W(B_2,E_2,V\mathcal{K})}.$$

Proof. Observe that fixed $f_j \in F^j$, $j = 1,2$,

$$(u, v) \mapsto R_{vu^{-1}}^1 f_1 R_{vu}^2 f_2 = H(u, v, \cdot),$$

continuously maps $G \times G$ in F^3, $H(u, v, x)$ is jointly measurable, and for any compact sets $U, V \subset G$,

$$\sup_{v \in V} \sup_{u \in U} \|H(u, v, \cdot)\|_{F^3} \leq \sup_{w \in VU^{-1}} \left\|R_w^1\right\|_{\mathfrak{B}(F^1)} \sup_{w \in VU} \left\|R_w^2\right\|_{\mathfrak{B}(F^2)} \|f_1\|_{F^1} \|f_2\|_{F^2} < +\infty,$$

where the boundedness follows from the uniform boundedness principle. Moreover, by the continuity of R^3 and (5.1.2),

$$R^3_v T_K(f_1, f_2) = \int_G K(u) R^1_{vu^{-1}} f_1 R^2_{vu} f_2 \, du.$$

Hence, similarly as we did in the proof of Theorem 3.1.4, it is shown that for any non-empty open set V, $(\lambda \times \mu)$-a.e. $(v, x) \in V \times \mathcal{M}$,

$$\chi_V(v) R^3_v T_K(f_1, f_2)(x) = \chi_V(v) B_K \left(\chi_{V\mathcal{K}} R^1_\cdot f_1(x), \chi_{V\mathcal{K}^{-1}} R^2_\cdot f_2(x) \right)(v).$$

Thus, by the lattice property of C and the boundedness assumption, μ-a.e. x

$$\begin{aligned} \left\| \chi_V R^3_\cdot T_K(f_1, f_2)(x) \right\|_{B_3} &\le \left\| B_K \left(\chi_{V\mathcal{K}^{-1}} R^1_\cdot f_1(x), \chi_{V\mathcal{K}} R^2_\cdot f_2(x) \right) \right\|_{B_3} \\ &\le N(K) \left\| \chi_{V\mathcal{K}^{-1}} R^1_\cdot f(x) \right\|_{B_1} \left\| \chi_{V\mathcal{K}} R^2_\cdot f(x) \right\|_{B_2}. \end{aligned}$$

Now, by the lattice property of E_3, (5.1.3) and the definition of TWA,

$$\|T_K(f_1, f_2)\|_{W(B_3, E_3, V)} \le c_E N(K) \|f_1\|_{W(B_1, E_1, V\mathcal{K}^{-1})} \|f_2\|_{W(B_2, E_2, V\mathcal{K})},$$

where c_E is the constant on (5.1.3). □

As in the linear case, the problem consists in properly identifies the appearing TWA. As a particular case, we can obtain the following result proved in [26].

Corollary 5.1.6. *Let G be an amenable group and let $1 \le p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $K \in L^1(G)$ with compact support such that $B_K : L^{p_1}(G) \times L^{p_2}(G) \to L^{p_3}(G)$ is bounded with norm less than or equal to $N(K)$.*

Let R^j be continuous representations of G acting on $L^{p_j}(\mathcal{M})$ for $j = 1, 2, 3$, satisfying (5.1.2) and that there exists $c_j > 0$ satisfying,

$$||R^j_v f||_{L^{p_j}} \le c_j ||f||_{L^{p_j}},$$

for any $v \in G$ and any $f \in L^{p_j}(\mathcal{M})$. Then, for $f_j \in L^{p_j}(\mathcal{M})$ ($j = 1, 2$),

$$\|T_K(f_1, f_2)\|_{L^{p_3}(\mathcal{M})} \le c_1 c_2 c_3 N(K) \|f_1\|_{L^{p_1}(\mathcal{M})} \|f_2\|_{L^{p_2}(\mathcal{M})}.$$

Proof. For $j = 1, 2, 3$, let $E_j = F^j = L^{p_j}(\mathcal{M})$, $B_j = L^{p_j}(G)$. By (5.1.4), (5.1.3) is satisfied. Let $\mathcal{K}$ be a symmetric compact set such that $\mathcal{K} \supset \operatorname{supp} K$. Now, for any $\epsilon > 0$, let V be a non-empty open relatively compact set such that $\frac{\lambda(V\mathcal{K})}{\lambda(V)} \le 1 + \epsilon$. As it is shown in (3.1.7), for any $f_1 \in F^1$, $f_2 \in F^2$,

$$\frac{\lambda(V)^{1/p_3}}{c_3} \|T_K(f_1, f_2)\|_{L^{p_3}(\mathcal{M})} \le \|T_K(f_1, f_2)\|_{W(L^{p_3}(G), L^{p_3}(\mathcal{M}), V)},$$

and, for $j = 1, 2$,

$$\|f_j\|_{W(L^{p_j}(G), L^{p_j}(\mathcal{M}), V\mathcal{K})} \le \|f_j\|_{L^{p_j}(\mathcal{M})} c_j \lambda(V\mathcal{K})^{1/p_j}.$$

By Theorem 5.1.5 and the previous inequalities,

$$\begin{aligned}\|T_K(f_1,f_2)\|_{L^{p_3}(\mathcal{M})} &\leq c_1c_2c_3\frac{\lambda(V\mathcal{K})^{\frac{1}{p_1}+\frac{1}{p_2}}}{\lambda(V)^{\frac{1}{p_3}}}N(K)\|f_1\|_{L^{p_1}(\mathcal{M})}\|f_2\|_{L^{p_2}(\mathcal{M})}\\ &\leq c_1c_2c_3(1+\epsilon)^{\frac{1}{p_3}}N(K)\|f_1\|_{L^{p_1}(\mathcal{M})}\|f_2\|_{L^{p_2}(\mathcal{M})}.\end{aligned}$$

from where the result follows. □

Corollary 5.1.7. *Let G be an amenable group. Let $0<s_1\leq p_1<\infty$, $0<s_2\leq p_2<\infty$ and $p_3\leq s_3\leq\infty$ such that $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$. Let $K\in L^1(G)$ with compact support such that $B_K: L^{p_1,s_1}(G)\times L^{p_2,s_2}(G)\longrightarrow L^{p_3,s_3}(G)$ is bounded with norm no greater than $N(K)$.*

Let R^j ($j=1,2,3$) be continuous distributionally bounded representations of G satisfying (5.1.2). Then, for any $f_j\in L^{p_j,s_j}(\mathcal{M})$ ($j=1,2$),

$$\|T_K(f_1,f_2)\|_{L^{p_3,s_3}(\mathcal{M})}\lesssim N(K)\|f_1\|_{L^{p_1,s_1}(\mathcal{M})}\|f_2\|_{L^{p_2,s_2}(\mathcal{M})}.$$

Proof. Let $E_j=L^{p_j}(\mathcal{M})$, $B_j=L^{p_j,s_j}(G)$ for $j=1,2,3$. Let $F^1=F^2=L^2(\mathcal{M})$ and $F^3=L^1(\mathcal{M})$. Let $\mathcal{K}$ be as symmetric compact set $\mathcal{K}\supset\operatorname{supp}K$. By Lemma 3.1.9, for $j=1,2,3$, R^j can be extended to a continuous and uniformly bounded representation of G on F^j and there exists $c_j\geq 1$ such that for any $f\in F^j$, $u\in G$ and $t>0$, $\mu_{R^j_uf}(t)\leq c_j\mu_f(t)$.

For $\epsilon>0$, let V be a open relatively compact set such that $\lambda(V\mathcal{K})\leq(1+\epsilon)\lambda(V)$. As we did in (3.1.13) and (3.1.14), it is shown that for any $f_1,f_2\in L^2(\mathcal{M})$,

$$\frac{\lambda(V)^{1/p_3}}{c_3}\|T_K(f_1,f_2)\|_{L^{p_3,s_3}(\mathcal{M})}\leq\|T_K(f_1,f_2)\|_{W(B_1,E_1,V)},$$

and, for $j=1,2$,

$$\|f_j\|_{W(B_j,E_j,V\mathcal{K})}\leq\lambda(V\mathcal{K})^{1/p_j}\|f_j\|_{L^{p_j,s_j}(\mathcal{M})}.$$

Hence, by Theorem 5.1.5, for $f_j\in L^2\cap L^{p_j,s_j}(\mathcal{M})$ ($j=1,2$),

$$\begin{aligned}&\frac{\lambda(V)^{1/p_3}}{c_3}\|T_K(f_1,f_2)\|_{L^{p_3,s_3}(\mathcal{M})}\leq\\ &\quad\leq N(K)c_1c_2\lambda(V\mathcal{K}^{-1})^{1/p_1}\lambda(V\mathcal{K})^{1/p_2}\|f_1\|_{L^{p_1,s_1}(\mathcal{M})}\|f_2\|_{L^{p_2,s_2}(\mathcal{M})},\end{aligned}$$

from where it follows that

$$\|T_K(f_1,f_2)\|_{L^{p_3,s_3}(\mathcal{M})}\leq c_1c_2c_3N(K)(1+\epsilon)^{1/p_3}\|f_1\|_{L^{p_1,s_1}(\mathcal{M})}\|f_2\|_{L^{p_2,s_2}(\mathcal{M})}.$$

Then the statement follows by letting $\epsilon\to 0$, the density of simple functions on $L^{p_j,s_j}(\mathcal{M})$ and the iterative use of Lemma 2.1.5. □

Corollary 5.1.8. *Let G be an amenable group. Let u_1,u_2,w be weights in $(0,\infty)$ and let $0<p_j\leq r_j\leq q$ for $j=1,2$, such that $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}\geq\frac{1}{q}$, with $W^{r/q}$ quasi-convex, and $U_j^{r_j/p_j}$ quasi-concave for $j=1,2$, where $U_j(t)=\int_0^t u_j$, $W(t)=\int_0^t w_j$*

and $W \in \Delta_2$. Let $K \in L^1(G)$ with compact support such that $B_K : \Lambda^{p_1}(u_1, G) \times \Lambda^{p_2}(u_2, G) \to \Lambda^q(w, G)$ is bounded with norm no greater than $N(K)$.

Assume that $\mu(\mathcal{M}) = 1$. Let R^j for $j = 1, 2, 3$, be continuous distributionally bounded representations of G satisfying (5.1.2). Then, for any $f_j \in \Lambda^{p_j}(u_j, \mathcal{M})$ ($j = 1, 2$),

$$\|T_K(f_1, f_2)\|_{\Lambda^q(w,\mathcal{M})} \lesssim N(K) \|f_1\|_{\Lambda^{p_1}(u_1,\mathcal{M})} \|f_2\|_{\Lambda^{p_2}(u_2,\mathcal{M})}.$$

Proof. Let $F^1 = F^2 = L^2(\mathcal{M})$, $F^3 = L^1(\mathcal{M})$, $E_3 = L^r(\mathcal{M})$, $B_3 = \Lambda^q(w, G)$, $E_j = L^{r_j}(\mathcal{M})$ $B_j = \Lambda^{p_j}(u_j, \mathcal{M})$ for $j = 1, 2$ and let $\mathcal{K} \supset \operatorname{supp} K$ be a symmetric compact set. Under the hypotheses, B_j are QBFS's. Fixed $\epsilon > 0$, let V be a open relatively compact set that $\lambda(V\mathcal{K}) \leq (1 + \epsilon)\lambda(V)$, and that we can assume to be $\lambda(V) \geq 1$ as we did in the proof of Corollary 3.1.16.

By Lemma 3.1.9, for $j = 1, 2, 3$, R^j can be extended to respective continuous and uniformly bounded representation of G on F^j and there exists $c_j \geq 1$ such that, for every $f \in F^j$, $s > 0$ and $v \in G$, $\mu_{R^j_v f}(s) \leq c_j \mu_f(s)$. Hence, applying Theorem 5.1.5, for $f_j \in L^2$ for $j = 1, 2$,

$$\|T_K(f_1, f_2)\|_{W(\Lambda^q(w,G),L^r,V)} \leq N(K) \|f_1\|_{W(\Lambda^{p_1}(w,G),L^{r_1},V\mathcal{K})} \|f_2\|_{W(\Lambda^{p_2}(w,G),L^{r_2},V\mathcal{K})},$$

and using the amalgam identification on Corollary 3.1.16,

$$\|T_K(f_1, f_2)\|_{\Lambda^q(w,\mathcal{M})} \leq AN(K) \|f_1\|_{\Lambda^{p_1}(u_1,\mathcal{M})} \|f_2\|_{\Lambda^{p_2}(u_2,\mathcal{M})},$$

where

$$A = h_{\Lambda^{p_1}(u_1)} \left(c_1 \lambda(V\mathcal{K}^{-1})\right) h_{\Lambda^{p_2}(u_2)} \left(c_2 \lambda(V\mathcal{K})\right) h_{\Lambda^q(w)} \left(\frac{c_3}{\lambda(V)}\right).$$

Similarly as we did in Corollary 3.1.16, since, for $j = 1, 2$, $U_j^{r_j/p_j}$ is quasi-concave and $W^{r/p}$ is quasi-convex, it is shown that

$$A \lesssim \frac{\lambda(V\mathcal{K})^{1/r_1+1/r_2}}{\lambda(V)^{1/r}} \leq (1 + \epsilon)^{1/r}.$$

Then, letting $\epsilon \to 0$, for $f_j \in L^2(\mathcal{M}) \cap \Lambda^{p_j}(u_j, \mathcal{M})$,

$$\|T_K(f_1, f_2)\|_{\Lambda^q(w,\mathcal{M})} \lesssim N(K) \|f_1\|_{\Lambda^{p_1}(u_1,\mathcal{M})} \|f_2\|_{\Lambda^{p_2}(u_2,\mathcal{M})},$$

with constant independent of the support of K. Then the result follows by the density of simple functions on $\Lambda^{p_j}(u_j, \mathcal{M})$ (see [38, Thm. 2.3.4]). □

Let us remark that, with minor modifications, the previous result holds replacing $\Lambda^q(w)$ by $\Lambda^{r,\infty}(w)$, and in particular, for the space $L^{r,\infty}$. Examples of weights u_1, u_2, w satisfying the hypotheses of the previous result are given by $w(t) = t^{\frac{q}{r}-1}\beta(t)$ and $u_j(t) = t^{\frac{p_j}{r_j}-1}\gamma_j(t)$, where $\gamma_j(t) = (1 + \log^+ \frac{1}{t})^{A_j p}$ ($j = 1, 2$), $\beta(t) = (1 + \log^+ \frac{1}{t})^{Bq}$ and $B \leq 0 \leq A_1, A_2$. In this case, the involved spaces are $\Lambda^p(u_j) = L^{r_j,p_j}(\log L)^{A_j}$ ($j = 1, 2$) and $\Lambda^q(w) = L^{r,q}(\log L)^B$, that are Lorentz-Zygmund spaces. For instance, Bilinear Hilbert Transform satisfies bounds in certain Lorentz-Zygmund spaces (see [41]).

5.1.2 Maximal multilinear transference

As in the linear case, we can also establish a maximal counterpart of the previous results. The proofs is an almost immediate adaptation to this bilinear context, so we just outline them.

Theorem 5.1.9. *Let E_i, B_i, F^i be as in Theorem 5.1.5. Let $\{K_j\}_{j=1,\dots,N} \subset L^1(G)$ whose support is contained in a compact set $\mathcal{K}$, such that $B^\sharp : B_1 \times B_2 \to B_3$ is bounded with norm $N(\{K_j\})$, where*

$$B^\sharp(f_1, f_2)(v) = \sup_{1\le j\le N} \left|B_{K_j}(f_1, f_2)(v)\right|.$$

Assume that R^3 is a separation-preserving continuous representation of G on F^3, satisfying the property that, for all $u \in G$ there exists a positivity-preserving mapping P_u^3 such that for every $f \in F^3$, $P_u^3|f| = |R_u^3 f|$. Let R^j be a continuous representation on F^j for $j = 1, 2$, such that (5.1.2) holds. Let us define, for $f_j \in F^j$ $(j = 1, 2)$,

$$T^\sharp(f_1, f_2)(x) = \sup_{1\le j\le N} \left|T_{K_j}(f_1, f_2)(x)\right|.$$

Fixed a non empty open set $V \subset G$, for $f_i \in F^i$ for $i = 1, 2$,

$$\left\|T^\sharp(f_1, f_2)\right\|_{W(B_3,E_3,V)} \le N(\{K_j\}) \left\|f_1\right\|_{W(B_1,E_1,V\mathcal{K}^{-1})} \left\|f_2\right\|_{W(B_2,E_2,V\mathcal{K})}.$$

Proof. As we showed in Theorem 3.1.22, for $f_i \in F^i$, $i = 1, 2$ and $v \in G$,

$$R_v^3 T^\sharp(f_1, f_2) \le \sup_{1\le j\le N} \left|R_v^3 T_{K_j}(f_1, f_2)\right|.$$

Now, $(\mu \times \lambda)$-a.e. $(x, v) \in \mathcal{M} \times V$,

$$\chi_V(v) R_v^3 T^\sharp(f_1, f_2)(x) \le \sup_{1\le j\le N} \left|B_{K_j}\left(\chi_{V\mathcal{K}^{-1}} R^1_{\cdot} f_1(x), \chi_{V\mathcal{K}} R^2_{\cdot} f_2(x)\right)(v)\right|.$$

The proof finishes in the same way as the proof of Theorem 5.1.5. ☐

Corollary 5.1.10. *Let G be an amenable group, and let $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$, where $p_1, p_2, p_3 \ge 1$. For $j = 1, 2, 3$ let R^j be a continuous representation of G on L^{p_j} satisfying (5.1.2), such that there exists $c_j > 0$ satisfying*

$$||R_v^j f||_{L^{p_j}} \le c_j ||f||_{L^{p_j}},$$

for every $v \in G$ and any $f \in L^{p_j}(\mathcal{M})$. Assume that R^3 is a separation-preserving representation.

If $\{K_j\}_j \subset L^1(G)$ with compact support are such that $B^\sharp : L^{p_1}(G) \times L^{p_2}(G) \to L^{p_3}(G)$ is bounded with norm less than or equal to $N(\{K_j\})$, then

$$\left\|T^\sharp(f_1, f_2)\right\|_{L^{p_3}(\mathcal{M})} \le c_1 c_2 c_3 N(\{K_j\}) \left\|f_1\right\|_{L^{p_1}(\mathcal{M})} \left\|f_2\right\|_{L^{p_2}(\mathcal{M})},$$

for $f_j \in L^{p_i}(\mathcal{M})$ *(*$j = 1, 2$*).*

Proof. By Fatou's lemma we can reduce to prove the statement for a finite number of kernels $\{K_j\}_{j=1,\dots,N}$.

Let $E_j = F^j = L^{p_j}(\mathcal{M})$, $B_j = L^{p_j}(G)$, $\mathcal{K} \supset \operatorname{supp} K$ be a symmetric compact set. Let $\epsilon > 0$ and let V be an open relatively compact set such that $\lambda(V\mathcal{K}) \le (1+\epsilon)\lambda(V)$. Since integrable simple functions are dense in F^3, by Lemma 3.1.21, R^3 satisfies the hypotheses of the previous theorem. Hence, jointly with (3.1.7) gives that, for $f_i \in L^{p_i}$,

$$\left\|T^\sharp(f_1, f_2)\right\|_{L^{p_3}(\mathcal{M})} \le c_1 c_2 c_3 N(\{K_j\})(1+\epsilon)^{1/p_3} \left\|f_1\right\|_{L^{p_1}(\mathcal{M})} \left\|f_1\right\|_{L^{p_2}(\mathcal{M})},$$

from where the result follows letting $\epsilon \to 0$. □

Corollary 5.1.11. *Let G be an amenable group and let $0 < s_1 \le p_1 < \infty$ and $0 < s_2 \le p_2 < \infty$, $p_3 \le s_3 \le \infty$ such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $\{K_j\} \subset L^1(G)$ with compact support satisfying that the bilinear mapping $B^\sharp : L^{p_1,s_1}(G) \times L^{p_2,s_2}(G) \to L^{p_3,s_3}(G)$ is bounded with norm no greater than $N(\{K\})$.*

Let R^j for $j = 1, 2, 3$, be continuous distributionally bounded representations of G satisfying (5.1.2). Then for $f_j \in L^{p_j,s_j}(\mathcal{M})$ $(j = 1, 2)$,

$$\left\|T^\sharp(f_1, f_2)\right\|_{L^{p_3,s_3}(\mathcal{M})} \lesssim N(\{K_j\}) \left\|f_1\right\|_{L^{p_1,s_1}(\mathcal{M})} \left\|f_2\right\|_{L^{p_2,s_2}(\mathcal{M})}.$$

Proof. By Fatou's lemma, we can assume that $\{K_j\}$ is a finite family.

Let $F^3 = L^1(\mathcal{M})$, $j = 1, 2$, $E_j = L^{p_j}(\mathcal{M})$, $F^j = L^2(\mathcal{M})$, $B_j = L^{p_j,s_j}(G)$ and $\mathcal{K} \supset \operatorname{supp} K$ a symmetric compact set. Fixed $\epsilon > 0$, let V be a relatively compact open set such that $\lambda(V\mathcal{K}) \le (1+\epsilon)\lambda(V)$.

By Lemma 3.1.9, for $j = 1, 2, 3$, R^j can be extended to respective continuous and uniformly bounded representation of G on F^j. Moreover, by Proposition 3.1.23 R^3 satisfies the hypotheses of Theorem 5.1.9. By this result, and proceeding as in the proof of Corollary 5.1.7 we obtain that for $f_j \in L^2 \cap L^{p_j,s_j}$ for $j = 1, 2$,

$$\left\|T_K(f_1, f_2)\right\|_{L^{p_3,s_3}(\mathcal{M})} \le c_1 c_2 c_3 N(\{K_j\}) \left\|f_1\right\|_{L^{p_1,s_1}(\mathcal{M})} \left\|f_1\right\|_{L^{p_2,s_2}(\mathcal{M})}.$$

Then the statement follows by the density of simple functions on $L^{p_j,s_j}(\mathcal{M})$ and the iterative use of Lemma 2.1.5. □

Corollary 5.1.12. *Let G be an amenable group. Let u_1, u_2, w be weights in $(0, \infty)$ and let $0 < p_j \le r_j \le q$ for $j = 1, 2$, such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \ge \frac{1}{q}$, with $W^{r/q}$ quasi-convex, and $U_j^{r_j/p_j}$ quasi-concave for $j = 1, 2$, where $U_j(t) = \int_0^t u_j$, $W(t) = \int_0^t w_j$ and $W \in \Delta_2$. Let $K \in L^1(G)$ with compact support such that $B_K^\sharp : \Lambda^{p_1}(u_1, G) \times \Lambda^{p_2}(u_2, G) \to \Lambda^q(w, G)$ is bounded with norm no greater than $N(K)$.*

Assume that $\mu(\mathcal{M}) = 1$. Let R^j for $j = 1, 2, 3$, be continuous distributionally bounded representations of G satisfying (5.1.2). Then, for any $f_j \in \Lambda^{p_j}(u_j, \mathcal{M})$ $(j = 1, 2)$,

$$\left\|T^\sharp(f_1, f_2)\right\|_{\Lambda^q(w,\mathcal{M})} \lesssim N(K) \left\|f_1\right\|_{\Lambda^{p_1}(u_1,\mathcal{M})} \left\|f_2\right\|_{\Lambda^{p_2}(u_2,\mathcal{M})}.$$

5.1.3 Application to the restriction of Bilinear multipliers

Let G be LCA group and denote by Γ its dual group. Given $\mathbf{m} \in L^\infty(\Gamma)$, define, for $f, g \in SL^1(G)$, the bilinear form

$$B_{\mathbf{m}}(f,g)(u) = \int_{\Gamma\times\Gamma} \widehat{f}(\xi)\widehat{g}(\eta)\mathbf{m}(\xi\eta^{-1})(\xi\eta)(u)\, d\xi d\eta.$$

We will show how the bilinear transference methods developed in the previous section can be applied to obtain a De Leeuw-type result for these type of bilinear operators, on a range of Lorentz spaces. These type of results have been previously investigated in the case $G=\mathbb{R}$, by O. Blasco an F. Villaroya in [27] following an approach similar to that given by De Leeuw in the linear setting. In [26], bilinear transference techniques are applied to obtain a De Leeuw result, in the case $G=\mathbb{R}$, for Lebesgue spaces.

Restriction of Bilinear multipliers

We will first obtain by transference an bilinear version of Theorem 3.3.1 for operators $B_{\mathbf{m}}$. As in the linear case, this result automatically leads to obtain a De Leeuw's-type result (Corollary 5.1.14) for general LCA groups. Fix a family $\{\widehat{\varphi_n}\}_n \in L^1(\Gamma)$ satisfying $\varphi_n \in \mathcal{C}_c(G)$ and *1*,*2*, *3* of Lemma 2.3.9.

Theorem 5.1.13. *Let G_1, G_2 be LCA groups and let Γ_1, Γ_2 be its respective dual groups. Let π be a continuous homomorphism from Γ_1 to Γ_2. Let $\mathbf{m} \in L^\infty(\Gamma_2)$ be a normalized function with respect to $\{\widehat{\varphi_n}\}_n$. Let $0 < s_1 \le p_1 < \infty$, $0 < s_2 \le p_2 < \infty$, $1 < p_3 \le s_3 \le \infty$ or $1 = p_3 = s_3$ satisfying $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$. Assume that*

$$B_{\mathbf{m}} : L^{p_1,s_1}(G_2) \times L^{p_2,s_2}(G_2) \to L^{p_3,s_3}(G_1)$$

is bounded with norm N. Then

$$B_{\mathbf{m}\circ\pi} : L^{p_1,s_1}(G_1) \times L^{p_2,s_2}(G_1) \to L^{p_3,s_3}(G_1),$$

with norm less than or equal to cN, where c is a constant depending only on p_3 and s_3.

Proof. Let $\{h_n\}_n \in \mathcal{C}_c^+(G)$ as in the proof of Proposition 2.3.22. Then $\|\widehat{h_n}\|_\infty \le \int h_n = 1$ and, for any $\xi \in \Gamma$, $\widehat{h_n}(\xi) \to 1$. Let us define, as in Theorem 2.3.13, $\widehat{K_n} = (\widehat{\varphi_n} * \mathbf{m})(\xi)\widehat{h_n}(\xi)$. By the discussion therein, $K_n \in L^1(G)$ compactly supported, satisfy $\left\|\widehat{K_n}\right\|_{L^\infty(\Gamma)} \le \|\mathbf{m}\|_{L^\infty(\Gamma)}$ and $\widehat{K_n}(\xi) \to \mathbf{m}(\xi)$ when n tends to infinity. Let us observe that for $f, g \in SL^1(G)$,

$$B_{K_n}(f,g)(v) = \int_G h_n(u) \left\{ \int_{\Gamma\times\Gamma} \widehat{f}(\xi)\widehat{g}(\eta)\eta(uv)\xi(u^{-1}v)(\widehat{\varphi_n} * \mathbf{m})(\xi\eta^{-1}) d\xi d\eta \right\} du.$$

Notice also that $\widehat{f}(\xi)\xi(u^{-1}) = \widehat{L_u f}(\xi)$ and $\widehat{g}(\eta)\eta(u) = \widehat{L_{u^{-1}} g}(\eta)$, where $L_u h(v) =$

$h(u^{-1}v)$. Then the term in curly brackets is equal to

$$\int_\Gamma \widehat{\varphi_n}(\chi)\left(\int_{\Gamma\times\Gamma}\widehat{L_u f}(\xi)\widehat{L_{u^{-1}}g}(\eta)(\xi\eta)(v)\mathbf{m}(\xi\eta^{-1}\chi^{-1})d\xi d\eta\right)\,d\chi.$$

But with the change of variables $\xi\chi^{-1}=\zeta$, the term inside brackets is equal to

$$\chi(v)\int_{\Gamma\times\Gamma}\widehat{(\chi L_u f)}(\zeta)\widehat{L_{u^{-1}}g}(\eta)(\zeta\eta)(v)\mathbf{m}(\zeta\eta^{-1})d\zeta d\eta = \chi(v)B_{\mathbf{m}}\left(\chi L_u f, L_{u^{-1}}g\right).$$

Hence,

$$B_{K_n}(f,g)(v) = \int_G h_n(u)\int_\Gamma \widehat{\varphi_n}(\chi)\chi(v)B_{\mathbf{m}}\left(\chi L_u f, L_{u^{-1}}g\right)(v)\,d\chi\,du.$$

Since on the given range of indices, L^{p_3,s_3} is a RIBFS, by Minkowski's integral inequality

$$\begin{aligned}&\|B_{K_n}(f,g)\|_{L^{p_3,s_3}(G)}\le\\ &\quad\le c_{p_3,s_3}\int_G|h_n(u)|\int_\Gamma|\widehat{\varphi_n}(\chi)|\,\|B_{\mathbf{m}}\left(\chi L_u f, L_{u^{-1}}g\right)\|_{L^{p_3,s_3}(G)}\,d\chi du.\end{aligned}$$

By the density of $SL^1\cap L^{p_i,s_i}(G)$ for $i=1,2$, it follows that B_{K_n} maps $L^{p_1,s_1}(G)\times L^{p_2,s_2}(G)$ into $L^{p_3,s_3}(G)$ with norm uniformly bounded by $c_{p_3,s_3}N$, where c_{p_3,s_3} is a constant, depending on p_3 and s_3.

Similarly as we did in the proof of Theorem 3.3.1, let us consider the continuous distributionally bounded representation of G_2 on functions on G_1 defined by $R_{u_2}f(u_1)=f(\widetilde{\pi}(u_2)u_1)$, where $\widetilde{\pi}: G_2\to G_1$ is the adjoint homomorphism of π. If we consider $R^j=R$ for $j=1,2,3$, (5.1.2) is satisfied. Thus, we can apply Corollary 5.1.7 to the associated transferred bilinear operator T_{K_n} to obtain that it maps $L^{p_1,s_1}(G_1)\times L^{p_2,s_2}(G_1)$ to $L^{p_3,s_3}(G_1)$ with norm uniformly bounded in n by $c_{p_3,s_3}N$.

Fixed $f\in SL^1(G_1)\cap L^{p,r}(G_1)$, by inversion formula,

$$R_{u_2}f(u_1)=\int_{\Gamma_1}\widehat{f}(\gamma_1)\gamma_1\left(\widetilde{\pi}(u_2)\right)\gamma_1(u_1)\,d\gamma_1,$$

and hence, for $f,g\in SL^1(G_1)$,

$$\begin{aligned}T_{K_n}(f,g)(u_1) &= \int_{G_2}K_n(u_2)f(\widetilde{\pi}(u_2^{-1})u_1)g(\widetilde{\pi}(u_2)u_1)\,du_2\\ &= \int_{\Gamma_1}\int_{\Gamma_1}\left(\int_{G_2}K_n(u_2)\overline{\pi(\gamma_1)(u_2)}\pi(\gamma_2)(u_2)\,du_2\right)\widehat{f}(\gamma_1)\widehat{g}(\gamma_2)(\gamma_1\gamma_2)(u_1)\,d\gamma_1 d\gamma_2\\ &= \int_{\Gamma_1}\int_{\Gamma_1}\widehat{K_n}\left(\pi(\gamma_1\gamma_2^{-1})\right)\widehat{f}(\gamma_1)\widehat{g}(\gamma_2)(\gamma_1\gamma_2)(u_1)\,d\gamma_1 d\gamma_2.\end{aligned}$$

Hence by the dominated convergence theorem,

$$\lim_n T_{K_n}(f,g)(u_1) = \int_{\Gamma_1}\int_{\Gamma_1} (\mathbf{m}\circ\pi)\left(\gamma_1\gamma_2^{-1}\right)\widehat{f}(\gamma_1)\widehat{g}(\gamma_2)(\gamma_1\gamma_2)(u_1)\,d\gamma_1 d\gamma_2$$
$$= B_{\mathbf{m}\circ\pi}(f,g)(u_1).$$

By Fatou's lemma and the uniform upper bound for the operators T_{K_n},

$$\|B_{\mathbf{m}\circ\pi}f\|_{L^{p,s}(G_1)} \le c_{p_3,s_3}N\,\|f\|_{L^{p_1,r_1}(G_1)}\,\|g\|_{L^{p_2,r_2}(G_1)},$$

from which the result follows by the density of $SL^1\cap L^{p_i,s_i}(G/H)$ in $L^{p_i,s_i}(G/H)$ for $i=1,2$. ☐

Observe that if $\mathbf{m}=\widehat{K}$, $K\in L^1$ with compact support, the approximation step can be avoided, so the proof works for also for $p_3<1$.

In the following corollaries we shall assume that $0<s_j\le p_j<\infty$ for $j=1,2$, $1<p_3\le s_3\le\infty$ or $1=p_3=s_3$, and $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$.

Corollary 5.1.14. *Let G be a LCA group and let H be a closed subgroup of G. Let $\mathbf{m}\in L^\infty(\Gamma)$ normalized. If $B_{\mathbf{m}}: L^{p_1,r_1}(G)\times L^{p_2,r_2}(G)\to L^{p_3,s_3}(G)$ is bounded then also is $B_{\mathbf{m}|_{H^\perp}}: L^{p_1,r_1}(G/H)\times L^{p_2,r_2}(G/H)\to L^{p_3,s_3}(G/H)$.*

Observe that for $G=\mathbb{R}$ and $H=\mathbb{Z}$, if $p_j=s_j$ $(j=1,2,3)$ the result recovers [26, Theorem 3.2], and for the given range of indices, coincide with the results proved in [27].

Corollary 5.1.15. *Let G_1,G_2 be LCA groups, Γ_1, Γ_2 be its respective dual groups and $G=G_1\times G_2$. Assume that $\mathbf{m}\in L^\infty(\Gamma_1)$ normalized satisfies that $B_{\mathbf{m}}: L^{p_1,r_1}(G_1)\times L^{p_2,r_2}(G_1)\to L^{p_2,s_3}(G_1)$ is bounded. Then the function $\mathbf{\Psi}$ in G defined by $\mathbf{\Psi}(u,v)=\mathbf{m}(u)$ satisfies that $B_{\mathbf{\Psi}}: L^{p_1,r_1}(G)\times L^{p_2,r_2}(G)\to L^{p_2,s_3}(G)$.*

Corollary 5.1.16. *Let $\mathbf{m}\in L^\infty(\mathbb{T})$ normalized such that $B_{\mathbf{m}}:\ell^{p_1,r_1}(\mathbb{Z})\times\ell^{p_2,r_2}(\mathbb{Z})\to\ell^{p_3,s_3}(\mathbb{Z})$ is bounded. Then, if $\mathbf{\Psi}$ is the 1-periodic extension of $\mathbf{m}$, $B_{\mathbf{\Psi}}: L^{p_1,r_1}(\mathbb{R})\times L^{p_2,r_2}(\mathbb{R})\to L^{p_3,s_3}(\mathbb{R})$ is bounded.*

With slight modification in the proof of Theorem 5.1.13, it can be stated the corresponding maximal counterpart.

Theorem 5.1.17. *Let G_1,G_2 be LCA groups and let Γ_1, Γ_2 be their respective dual groups. Let π be a continuous homomorphism from Γ_1 to Γ_2. Let $\{\mathbf{m}_j\}\subset L^\infty(\Gamma_2)$ normalized functions. Let $0<s_1\le p_1<\infty$, $0<s_2\le p_2<\infty$, $1<p_3\le s_3\le\infty$ or $1=p_3=s_3$ satisfying $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$. Assume that*

$$B^\sharp: L^{p_1,s_1}(G_2)\times L^{p_2,s_2}(G_2)\to L^{p3,s_3}(G_2)$$

is bounded, where $B^\sharp(f,g)=\sup_j\left|B_{\mathbf{m}_j}(f,g)(u)\right|$. Then

$$\widetilde{B}^\sharp: L^{p_1,s_1}(G_1)\times L^{p_2,s_2}(G_1)\to L^{p_3,s_3}(G_),$$

is bounded with norm controlled by the norm of $B_{\mathbf{m}}$, where

$$\widetilde{B}^\sharp(f,g)(u)=\sup_j\left|B_{\mathbf{m}_j\circ\pi}(f,g)(u)\right|.$$

Restriction of bilinear multipliers to $\mathbb{Z}$ and $\mathbb{R}^d$

As it is the case with the linear situation, in the case of $G = \mathbb{R}$, as well as $G = \mathbb{R}^{d_1+d_2}$, the relations at measure level between the group and the quotient group allow to obtain a wider class of spaces where the transference techniques can be applied in order to obtain a De Leeuw-type restriction theorem. In this subsection, we will deal with this situation. The proofs are similar to the linear case, using now the corresponding bilinear results. We will prove the case of the restriction to the integers of a single multiplier to show how the arguments can be adapted.

By B_1, B_2, B_3 we denote RIQBFS. As in the linear case we want a family of 'regular functions', that is $\mathcal{C}_c^\infty(\mathbb{R}^d)$ in the case of $G = \mathbb{R}^d$ and trigonometric polynomials in the case of $G = \mathbb{T}$, to be dense in B_i for $i = 1, 2$. So we shall assume that B_i has absolutely continuous norm for $i = 1, 2$. Moreover, since we want to apply the approximation techniques, we shall assume that B_3 is an RIBFS. We will maintain the notations introduced in §3.2.

Definition 5.1.18. *Let B_1, B_2, B_3 be RIQBFS defined on $\mathbb{R}^d$. We say that (B_1, B_2, B_3) is an admissible 3-tuple if*

$$\kappa = \liminf_{N\to\infty} h_{B_3}\left(\frac{1}{N}\right) h_{B_1}(N)\, h_{B_2}(N) < \infty. \tag{5.1.19}$$

We can give examples of admissible 3-tuples similar to the examples we give for admissible pairs. In particular, for $\alpha_1, \alpha_2 \leq 0 \leq \beta$, $\frac{1}{p_1}+\frac{1}{p_2} = \frac{1}{p_3}$, $0 < r_1, r_2, r_3 < \infty$, the 3-tuple $(L^{p_1,r_1}(\log L)^{\alpha_1}(\mathbb{R}^d), L^{p_2,r_2}(\log L)^{\alpha_1}(\mathbb{R}^d), L^{p_3,r_3}(\log L)^{\beta}(\mathbb{R}^d))$ is admissible.

Corollary 5.1.20. *Let B_1, B_2 be RIQBFS's and B_3 a RIBFS defined on $\mathbb{R}$. Assume that (B_1, B_2, B_3) is an admissible 3-tuple. Let $\mathbf{m} \in L^\infty(\mathbb{R})$ be a normalized function such that $B_{\mathbf{m}} : B_1 \times B_2 \to B_3$ is bounded, then*

$$B_{\mathbf{m}|_\mathbb{Z}} : B_{1,\mathbb{T}} \times B_{2,\mathbb{T}} \to B_{3,\mathbb{T}},$$

is also bounded.

Proof. Observe that since $\mathbf{m}$ is normalized, arguing as in the proof of Theorem 5.1.13 for the special case $G_2 = \mathbb{R}$, $G_1 = \mathbb{T}$ and $\pi : \mathbb{R} \to \mathbb{T}$ being the projection, we have a sequence $\{K_n\}_n \subset L^1(\mathbb{R})$ with compact support such that $\widehat{K_n}(x) \to \mathbf{m}(x)$, for any $x \in \mathbb{R}$ and such that the bilinear operators B_{K_n} map $B_1 \times B_2$ into B_3 with norm uniformly bounded in n.

Let R be the representation of $\mathbb{R}$ in $\mathcal{C}(\mathbb{T})$ given by $R_t f(\theta) = f(\theta + t)$, so measurability assumptions are automatically satisfied. Consider $F^i = \mathcal{C}(\mathbb{T})$ and $E_i = L^\infty(\mathbb{T})$ for $i = 1, 2, 3$. Let $V = (-N, N)$ and $\mathcal{K} = [-M, M]$ with $M, N \in \mathbb{N}$ and M being sufficiently large that $\operatorname{supp} K \subset \mathcal{K}$. So we can apply Theorem 5.1.5 and the characterization of the corresponding TWA in the same way as we did in the proof of Theorem 3.2.7 to obtain that, for $f, g \in \mathcal{C}(\mathbb{T})$,

$$\|T_{K_n}(f,g)\|_{B_{3,\mathbb{T}}} \leq N(K)C_{N,M}\, \|f\|_{B_{1,\mathbb{T}}} \|g\|_{B_{2,\mathbb{T}}}\,.$$

where

$$C_{N,M} = h_{B^3}\left(\frac{1}{2N}\right) h_{B^1}\left(2(N+M)\right) h_{B^2}\left(2(N+M)\right).$$

Thus, it follows that

$$\|T_{K_n}(f,g)\|_{B_{3,\mathbb{T}}} \leq N(K)\kappa \|f\|_{B_{1,\mathbb{T}}} \|g\|_{B_{2,\mathbb{T}}}.$$

Now the proof finishes by Fatou's lemma, bilinearity and density of trigonometric polynomials in $B_{i,\mathbb{T}}$ for $i = 1, 2$. □

As an immediate consequence we obtain the following result for bilinear multipliers for Lorentz-Zygmund spaces.

Corollary 5.1.21. *Let $0 < r_1, r_2, r_3 < \infty$, $1 < p_3 \leq p_1, p_2 < \infty$ such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\alpha_3 \leq 0 \leq \alpha_1, \alpha_2$. Let $\mathbf{m} \in L^\infty(\mathbb{R})$ be a normalized function satisfying that*

$$B_{\mathbf{m}} : L^{p_1,s_1}(\log L)^{\alpha_1}(\mathbb{R}) \times L^{p_2,s_2}(\log L)^{\alpha_2}(\mathbb{R}) \to L^{p_3,s_3}(\log L)^{\alpha_3}(\mathbb{R})$$

is bounded. Then

$$B_{\mathbf{m}|_{\mathbb{Z}}} : L^{p_1,s_1}(\log L)^{\alpha_1}(\mathbb{T}) \times L^{p_2,s_2}(\log L)^{\alpha_2}(\mathbb{T}) \to L^{p_3,s_3}(\log L)^{\alpha_3}(\mathbb{T})$$

is bounded with norm controlled by the norm of $B_{\mathbf{m}}$.

In comparison with what we obtained in Corollary 5.1.14 for $G = \mathbb{R}$, $H = \mathbb{Z}$ and $\alpha_i = 0$ for $i = 1, 2, 3$, this last result allows a wider range of indices. This particular situation can be deduced from [27, Theorem 2.9.]. Using Theorem 5.1.9 and Fatou's lemma in a similar way as we did before it can be proved the maximal counterpart.

Corollary 5.1.22. *Let B_1, B_2 be RIQBFS's and B_3 a RIBFS defined on $\mathbb{R}$. Assume that (B_1, B_2, B_3) is an admissible 3-tuple. Let $\{\mathbf{m}_j\} \subset L^\infty(\mathbb{R})$ normalized functions such that $B^\sharp : B_1 \times B_2 \to B_3$, where $B^\sharp(f,g)(u) = \sup_j \left|B_{\mathbf{m}_j}(f,g)(u)\right|$ is bounded, then*

$$\widetilde{B}^\sharp : B_{1,\mathbb{T}} \times B_{2,\mathbb{T}} \to B_{3,\mathbb{T}},$$

is also bounded where $\widetilde{B}^\sharp(f,g)(\theta) = \sup_j \left|B_{\mathbf{m}_j|_{\mathbb{Z}}}(f,g)(\theta)\right|$.

The corresponding theorems in the case $G = \mathbb{R}^{d_1+d_2}$, $H = \mathbb{R}^{d_1}$, are stated below.

Corollary 5.1.23. *Let B_1, B_2 be RIQBFS's and B_3 a RIBFS defined on $\mathbb{R}^{d_1+d_2}$. Assume that (B_1, B_2, B_3) is an admissible 3-tuple. Let $\mathbf{m} \in L^\infty(\mathbb{R}^{d_1+d_2})$ be a normalized function such that $B_{\mathbf{m}} : B_1 \times B_2 \to B_3$, is bounded, then for any $\xi \in \mathbb{R}^{d_1}$ if $\overline{\mathbf{m}} = \mathbf{m}(\xi, \cdot)$,*

$$B_{\overline{\mathbf{m}}} : B_{1,d_2} \times B_{2,d_2} \to B_{3,d_2},$$

is also bounded.

Corollary 5.1.24. *Let B_1, B_2 be RIQBFS's and B_3 a RIBFS defined on $\mathbb{R}^{d_1+d_2}$. Assume that (B_1, B_2, B_3) is an admissible 3-tuple. Let $\{\mathbf{m}_j\} \subset L^\infty(\mathbb{R}^{d_1+d_2})$ normalized functions such that $B^\sharp : B_1 \times B_2 \to B_3$, where*

$$B^\sharp(f,g)(u) = \sup_j \left|B_{\mathbf{m}_j}(f,g)(u)\right|$$

is bounded, then for any $\xi \in \mathbb{R}^{d_1}$

$$\widetilde{B}^\sharp : B_{1,d_2} \times B_{2,d_2} \to B_{3,d_2}$$

is also bounded where $\overline{\mathbf{m}}_j = \mathbf{m}_j(\xi, \cdot)$, $\widetilde{B}^\sharp(f,g)(u) = \sup_j \left|B_{\overline{\mathbf{m}}_j}(f,g)(u)\right|$.

5.2 Transference of modular inequalities

In this section we will show how transference ideas can be carried over the setting of modular inequalities. General terminology is mainly taken from [31, 77, 85].

Let $\mathbf{\Phi}$ be the class of all modular functions . That is the family of all functions P on $\mathbb{R}$, even, nonnegative, increasing on $[0,\infty)$, such that $P(0^+) = 0$ and $P(+\infty) = +\infty$. Observe that if P is convex,

$$P(t) = \int_0^t p(s)\, ds,$$

where $p \in L^1_{\text{loc}}[0,\infty)$ is nonnegative and increasing. A function $\Phi \in \mathbf{\Phi}$ is said to be a Young Function (or N-function) if P is convex and $\lim_{t\to 0^+} P(t)/t = \lim_{t\to +\infty} t/P(t) = 0$. For $P \in \mathbf{\Phi}$, the $P(L)$ class consists in all the measurable functions f such that

$$\int_{\mathcal{M}} P\left(f(x)\right)\, d\mu(x) < +\infty.$$

Whenever P is a Young function, the linear hull L_P of the class $P(L)$, equipped with the Luxemburg norm

$$\|f\|_{L_P,\, d\mu} = \inf\left\{\lambda > 0; \int_{\mathcal{M}} P\left(\frac{f(x)}{\lambda}\right)\, d\mu(x) \le 1\right\},$$

becomes a RIBFS, and it is called to be the Orlicz space L_P.

Given two modular functions P, Q, we say that an operator T satisfies an (P,Q)-modular inequality (in the terminology of [85, p.21], T satisfies condition (e_2)) if there exist $a, M > 0$ such that

$$\int_{\mathcal{M}} P\left(Tf(x)\right)\, d\mu(x) \le M \int_{\mathcal{M}} Q\left(af(x)\right)\, d\mu(x), \tag{5.2.1}$$

for any $f \in L_Q$. If (5.2.1) holds only for f such that $\int_{\mathcal{M}} Q(|f|) \le 1$ it is said that T satisfies condition (e_1).

Clearly, if T satisfies an (P,Q)-modular inequality it also satisfies (e_1) and, if P, Q are Young functions, it turns that property (e_1) implies that the operator

T continuously maps L_Q into L_P. In fact, in this last case, a (P,Q)-modular inequality holds if and only if there exists a constant C such that, for any $\epsilon > 0$,

$$\|Tf\|_{L_P,d\mu_\epsilon} \leq C\,\|f\|_{L_Q,d\mu_\epsilon},$$

where $d\mu_\epsilon = \epsilon d\mu$. That is, an (P,Q)-modular inequality is stronger than an norm inequality (see [28, Proposition 2.5]).

Modular inequalities have been studied for several convolution-type operators as the Hardy-Littlewood maximal operator, Hilbert transform and general Calderón-Zygmund operators (see for instance [28, 33]).

In the sequel we will work with functions $P, Q \in \Phi$ given by

$$P(t) = \int_0^t p(s)\,ds \quad \text{and} \quad Q(t) = \int_0^t q(s)\,ds,$$

for all $t \geq 0$, where p, q are positive continuous functions. The following technical fact is an easy consequence of Tonelli's theorem.

Lemma 5.2.2. *For any $f \in L^1_{\text{loc}}$,*

$$\int_{\mathcal{M}} P\left(f(x)\right)\,d\mu(x) = \int_0^\infty p(t)\mu_f(t)\,dt.$$

Theorem 5.2.3. *Let G be an amenable group, and let $K \in L^1(G)$ compactly supported, such that there exists $C, M > 0$ satisfying*

$$\int_G P\left(\frac{B_K f(v)}{C}\right)\,dv \leq M\int_G Q\left(f(v)\right)\,dv,$$

for any $f \in L^1 \cap L(Q)$. Let R be a continuous distributionally bounded representation of an amenable group G. Then there exist C' and M' such that for all $f \in L^1 \cap Q(L)$,

$$\int_{\mathcal{M}} P\left(\frac{T_K f(x)}{C'}\right)\,d\mu(x) \leq M'\int_{\mathcal{M}} Q\left(f(x)\right)\,d\mu(x).$$

Proof. By Lemma 3.1.9, R extend to an unique representation of G in L^1, $R^{(1)}$ satisfying that, for any $f \in L^1(\mathcal{M})$, $v \in G$ and $t > 0$,

$$\mu_{R_v^{(1)}f}(t) \leq c\mu_f(t).$$

Let $\mathcal{K} \supset \operatorname{supp} K$ be a symmetric compact set and, for any $\epsilon > 0$, let V be an relatively compact open set satisfying that $\lambda(V\mathcal{K}) \leq (1+\epsilon)\lambda(V)$. Observe that, for $f \in L^1(\mathcal{M})$,

$$\begin{aligned} I := \int_{\mathcal{M}} P\left(T_K f(x)\right) d\mu(x) &= \int_0^\infty p(t)\mu_{T_K f}(t)\,dt \\ &\leq \frac{c}{\lambda(V)}\int_V\int_0^\infty p(t)\mu_{R_v^{(1)}T_K f}(t)\,dt\,dv. \end{aligned}$$

But the last term can be written as

$$\begin{aligned}
&\frac{c}{\lambda(V)}\int_{\mathcal{M}}\int_0^\infty p(t)\lambda_{\chi_V R^{(1)}_{\cdot} T_K f(x)}(t)\ dt\ d\mu(x)\\
&\leq \frac{c}{\lambda(V)}\int_{\mathcal{M}}\int_0^\infty p(t)\lambda_{B_K\left(\chi_{V\mathcal{K}^{-1}}R^{(1)}_{\cdot} f(x)\right)}(t)\ dt\ d\mu(x) =\\
&\leq \frac{c}{\lambda(V)}\int_{\mathcal{M}}\left\{\int_G P\left(B_K\left(\chi_{V\mathcal{K}^{-1}}R^{(1)}_{\cdot} f(x)\right)(v)\right)\ dv\right\}\ d\mu(x)
\end{aligned}$$

Observe that $\int_{\mathcal{M}}\int_{V\mathcal{K}^{-1}}|R^{(1)}_v f(x)|\ dvd\mu(x) < +\infty$, so $\chi_{V\mathcal{K}^{-1}}R^{(1)}_v f(x) \in L^1(G)$ μ-a.e. x. Then by the boundedness hypothesis, the term inside curly brackets is bounded by

$$M\int_G Q\left(C\chi_{V\mathcal{K}^{-1}}(v)R^{(1)}_v f(x)\right)\ dv \leq M\int_{V\mathcal{K}^{-1}} Q\left(CR^{(1)}_v f(x)\right)\ dv$$

Then, using the previous lemma and interchanging the order of integration,

$$\begin{aligned}
I &\leq \frac{cM}{\lambda(V)}\int_{\mathcal{M}}\int_{V\mathcal{K}^{-1}} Q\left(CR^{(1)}_v f(x)\right)\ dv = \frac{cM}{\lambda(V)}\int_{V\mathcal{K}^{-1}}\int_{\mathcal{M}} Q\left(CR^{(1)}_v f(x)\right)\ dv\\
&\leq \frac{cM}{\lambda(V)}\int_{V\mathcal{K}^{-1}}\int_0^\infty q(t)\mu_{R^{(1)}_v(Cf)}(t)\ dt\ dv \leq c^2M\frac{\lambda(V\mathcal{K}^{-1})}{\lambda(V)}\int_{\mathcal{M}} Q(Cf(x))\ d\mu(x)\\
&\leq c^2M(1+\epsilon)\int_{\mathcal{M}} Q(Cf(x))\ d\mu(x),
\end{aligned}$$

from which, taking limit when ϵ tends to 0, the result follows. □

Let G be a LCA group. By the density of $SL^1(G)$ in $L^1(G)$ using standard arguments the following holds.

Lemma 5.2.4. *Let $K \in L^1(G)$ compactly supported. Assume that for any $f \in SL^1(G)$,*

$$\int_G P\left(\frac{K * f}{a}\right) \leq M\int_G Q(f).$$

Then the same inequality holds for any $f \in L^1(G)$.

Corollary 5.2.5. *Let G be a LCA group and H a closed subgroup. Let $P, Q \in \mathbf{\Phi}$ quasi-convex. Assume that $\mathbf{m} \in L^\infty(\Gamma)$ is a normalized function such that for $f \in SL^1 \cap L(Q)$,*

$$\int_G P\left((\mathbf{m}\widehat{f})^\vee(u)\right)\ du \leq M\int_G Q\left(Cf\right)\ du.$$

Then, for $f \in SL^1(G/H) \cap L(Q)$,

$$\int_{G/H} P\left((\overline{\mathbf{m}}\widehat{f})^\vee(u)\right)\ du \leq M'\int_{G/H} Q\left(C'f\right)\ du,$$

where $\overline{\mathbf{m}} = \mathbf{m}|_{H^\perp}$.

Proof. Let $\{h_n\}_n \in \mathcal{C}_c^+(G)$ be as in the proof of Proposition 2.3.22, and define the $L^1(G)$ functions with compact support as in Theorem 2.3.13,

$$\widehat{K_n} = (\widehat{\varphi_n} * \mathbf{m})(\xi)\widehat{h_n}(\xi),$$

that satisfy $\left\|\widehat{K_n}\right\|_{L^\infty(\Gamma)} \leq \|\mathbf{m}\|_{L^\infty(\Gamma)}$, $\widehat{K_n}(\xi) \to \mathbf{m}(\xi)$ when n tends to infinity, $\|\varphi_n\|_1 = 1$ and $\|h_n\|_1 = 1$.

Let $f \in SL^1(G)$. Observe that, if we define $g_n = h_n * f$,

$$\begin{aligned} |K_n * f(u)| &= \left| \int_\Gamma (\widehat{\varphi_n} * \mathbf{m})(\xi)\widehat{g_n}(\xi)\xi(u)\, d\xi \right| \\ &= \left| \int_\Gamma \widehat{\varphi_n}(\eta)\overline{\eta(u)} \int_\Gamma \mathbf{m}(\xi)\widehat{g_n\eta}(\xi)\xi(u)\, d\xi d\eta \right| \\ &\leq \int_\Gamma |\widehat{\varphi_n}(\eta)| \left| \int_\Gamma \mathbf{m}(\xi)\widehat{g_n\overline{\eta}}(\xi)\xi(u)\, d\xi \right| d\eta. \end{aligned} \tag{5.2.6}$$

Since P is quasi convex, there exists a convex function $\widetilde{P}$ and a constant c_p such that

$$\widetilde{P}(t) \leq P(t) \leq c_p\widetilde{P}(c_p t),$$

and similarly holds for Q with c_q as associated constant. Then, since $\|\widehat{\varphi_n}\|_1 = 1$, by Jensen's inequality,

$$P\left(K_n * f(u)\right) \leq c_p \int_\Gamma |\widehat{\varphi_n}(\eta)| P\left(c_p \int_\Gamma \mathbf{m}(\xi)\widehat{g_n\overline{\eta}}(\xi)\xi(u)\, d\xi\right) d\eta$$

Then, integrating on u, since $|\eta g_n| = |g_n|$

$$\begin{aligned} \int_G P(K_n * f(u))\, du &\leq c_p \int_\Gamma |\widehat{\varphi_n}(\eta)| \int_G P\left(c_p \int_\Gamma \mathbf{m}(\xi)\widehat{g_n\overline{\eta}}(\xi)\xi(u)\, d\xi\right) d\eta \\ &\leq c_p M \int_G Q\left(c_p C g_n(u)\right)\, du. \end{aligned}$$

By the quasi-convexity of Q,

$$\begin{aligned} \int_G Q(c_p C h_j * f(u))\, du &\leq c_q \int_G |h_j(v)| \int_G Q(f(v^{-1}u))\, dudv \\ &\leq c_q \int_G Q(c_p c_q C f(u))\, du. \end{aligned}$$

Therefore, for $f \in SL^1(G)$,

$$\int_G P\left(K_n * f(u)\right)\, du \leq c_p c_q M \int_G Q\left(c_p c_q a f(u)\right)\, du.$$

By the previous Lemma, the inequality holds for any $f \in L^1(G) \cap L(Q)$.

Consider now the continuous distributionally bounded representation of G on functions defined on G/H

$$R_u f(vH) = f(uvH), \quad u \in G;$$

In this case, the associated transferred operators coincide with the multiplier operator defined by $\widehat{K_n}|_{H^\perp}$. That is, for $f \in SL^1(G/H)$

$$T_{K_n} f(u) = \int_{H^\perp} \widehat{K_n}|_{H^\perp}(\chi)\widehat{f}(\chi)\, \chi(u)\, d\chi.$$

So we can apply Theorem 5.2.3 to obtain that, for $f \in SL^1(G/H) \cap L(Q)$,

$$\int_{G/H} P\left(T_{K_n} f(u)\right)\, d\mu_{G/H}(u) \leq M' \int_{G/H} Q\left(C' f(u)\right)\, d\mu_{G/H}(u),$$

with constants independents on n. Moreover, by dominated convergence theorem,

$$\lim_n T_{K_n} f(u) = \int_{H^\perp} \mathbf{m}|_{H^\perp}(\chi)\widehat{f}(\chi)\, \chi(u)\, d\chi = T_{\mathbf{m}|_{H^\perp}} f(u).$$

Finally, the result follows from Fatou's lemma and the continuity of P. □

In order to illustrate how the previous result can be used, we need the following result that follows by [33, Corollary 4.7].

Proposition 5.2.7. *Let $P, Q \in \mathbf{\Phi}$ such that P satisfies Δ_2 condition, $P \lesssim Q$ and*

$$\int_0^t \frac{P(s)}{s^2}\, ds \lesssim \frac{Q(t)}{t}. \tag{5.2.8}$$

Then it is satisfied the modular inequality for the Hilbert transform

$$\int_{\mathbb{R}} P\left(Hf(x)\right)\, dx \leq M \int_{\mathbb{R}} Q\left(Cf(x)\right)\, dx,$$

for $f \in \mathcal{S}(\mathbb{R}) \cap L(Q)$.

Corollary 5.2.9. *Let P, Q be quasi-convex functions, P satisfies Δ_2 condition such that $P \lesssim Q$ and satisfy (5.2.8). Then, for any trigonometrical polynomial $f \in L(Q)$,*

$$\int_{\mathbb{T}} P(\widetilde{f}(\theta))\, d\theta \leq M \int_{\mathbb{T}} Q(Cf(\theta))\, d\theta,$$

where $\widetilde{f}$ denotes the conjugate function operator.

Proof. Since the modular inequality holds for the Hilbert transform, that is the operator associated to the normalized multiplier $\mathbf{m}(x) = -i\pi \mathrm{sgn} x$. On the other hand, the conjugate function operator is given by the Fourier multiplier $\mathbf{m}|_{\mathbb{Z}}$. Then the result follows from Corollary 5.2.8 witg $G = \mathbb{R}$, $H = \mathbb{Z}$. □

5.3 Transference of extrapolation inequalities

In this section we will briefly discuss how we can deal with some inequalities for convolution operators that arise in the theory of extrapolation, using the transference ideas developed in §3.2.

Let us begin with a pair of examples. Carleson's operator $Sf := \sup_{N\geq 1} |S_N f|$, where $S_N f(x) = \sum_{|n|\leq N} \widehat{f}(n) e^{2\pi i n x}$, naturally arise in the study of convergence of Fourier series. In 1967, R.A. Hunt proved that, for every $1 < p$ and any Borel set E in $\mathbb{T}$,

$$(S\chi_E)^*(s) \leq \frac{|E|^{1/p}}{(p-1)s^{1/p}}. \tag{5.3.1}$$

and then, integrating on $(0, t)$

$$(S\chi_E)^{**}(s) \lesssim \frac{|E|^{1/p}}{(p-1)^2 s^{1/p}}. \tag{5.3.2}$$

This inequality, the sublinear property of the maximal function and Yano's extrapolation Theorem (see [101]) leads to obtain that S maps $L(\log L)^2(\mathbb{T})$ into $L^1(\mathbb{T})$, and hence, a pointwise convergence result of Fourier series for functions in $L(\log L)^2(\mathbb{T})$. Minimizing (5.3.1) in p, it holds that

$$(S\chi_E)^*(t) \lesssim \frac{|E|}{t}\left(1 + \log^+ \frac{t}{|E|}\right).$$

It was proved that, this inequality holds for any $f \in L^1$ such that $\|f\|_{L^\infty} \leq 1$ in the sense that, for these functions it holds that

$$(Sf)^*(t) \lesssim \frac{\|f\|_1}{t}\left(1 + \log^+ \frac{t}{\|f\|_1}\right),$$

which allows to prove the convergence of Fourier series for functions in the Lorentz-Zygmund space $L \log L \log\log\log L(\mathbb{T})$.

In general, if T is an operator such that, for $1 < p \leq 2$, $T : L^p \to L^p$ is bounded with constant $1/(p-1)^\alpha$ for some $\alpha > 0$, then

$$(Tf)^{**}(t) \lesssim \frac{\|f\|_1^{1/p}}{(p-1)^\alpha t^{1/p}},$$

for every $f \in L^1$ such that $\|f\|_\infty \leq 1$, and hence,

$$(Tf)^{**}(t) \lesssim \frac{\|f\|_1}{t}\left(1 + \log^+ \frac{t}{\|f\|_1}\right)^\alpha. \tag{5.3.3}$$

In particular, for $\alpha = 1$, by [34, Theorem 4.1]

$$\sup_{t>0} \frac{t(Tf)^{**}(t)}{1 + \log^+ t} \lesssim \int_0^\infty f^*(s)\left(1 + \log^+ \frac{1}{s}\right) ds.$$

That is, T maps the Lorentz-Zygmund space $L^{1,1,\gamma}$ into $\Gamma^{1,\infty}(w)$ with $w(t) = (1+\log^+ t)^{-1}$ and $\gamma(t) = \left(1+\log^+ \frac{1}{t}\right)$. Let us observe that, by Proposition D.1.4, for $N \geq 1$,

$$h_{\Gamma^{1,\infty}(w)}(1/N) \approx \frac{1}{N} \sup_{r>0} \frac{1+\log^+ r}{1+\log^+ rN} = \frac{1+\log N}{N}$$

and,

$$h_{L^{1,1,\gamma}}(N) = N \sup_{r>0} \frac{1+\log^+ \frac{1}{Nr}}{1+\log^+ \frac{1}{r}} = N.$$

Therefore, the pair $(L^{1,1,\gamma}, \Gamma^{1,\infty}(w))$ is not an admissible pair in the sense of the Definition 3.2.1, and hence, the restriction theorems developed in §3.2 can not be directly applied.

In this section, we are going to see that we can obtain a restriction result if we transfer inequalities of the type (5.3.3). To be more precise, we will study how we can transfer an inequality of the type

$$(B_K f)^{**}(t) \leq cD\left(\frac{\|f\|_{L^1(\mathbb{R})}}{t}\right), \qquad f \in L^1(\mathbb{R}), \quad \|f\|_{L^\infty} \leq 1, \tag{5.3.4}$$

to the periodic case, with the assumption on D to be continuous in $(0,\infty)$. Let us mention that, with minor modifications the proofs carries over the maximal case.

Let us consider the representation of $\mathbb{R}$ in $\mathcal{C}(\mathbb{T})$ defined by $R_t g(\theta) = g(\theta + t)$ introduced in §3.2.1. We have seen in (3.2.8) that, for any $L \in \mathbb{N}$,

$$\left(\chi_{(-L,L)}(v) R_v g(\theta)\right)^*(s) = g^*\left(\frac{s}{2L}\right),$$

where the rearrangement is taken in $\mathbb{R}$ with respect to the variable v in the term on the left, and in $\mathbb{T}$ on the right and hence,

$$\left(\chi_{(-L,L)}(v) R_v g(\theta)\right)^{**}(s) = g^{**}\left(\frac{s}{2L}\right).$$

Lemma 5.3.5. *Let $K \in L^1(\mathbb{R})$ with compact support. Assume that, for every $t > 0$,*

$$(B_K f)^{**}(t) \leq A\, D\left(\frac{\|f\|_{L^1(\mathbb{R})}}{t}\right),$$

for any $f \in L^1$ such that $\|f\|_\infty \leq 1$. Then, for $f \in \mathcal{C}(\mathbb{T})$ with $\|f\|_\infty \leq 1$,

$$(T_K f)^{**}(t) \leq A\, D\left(\frac{\|f\|_{L^1(\mathbb{T})}}{t}\right).$$

Proof. Let M big enough such that $\operatorname{supp} K \subset [-M, M]$. Hence, fixed $L \in \mathbb{N}$, by

(3.2.8), for $t \in (0,1)$ for $\theta \in (0,1)$,

$$\begin{aligned}(T_K f)^{**}(t) &= \left(\chi_{(-L,L)}(v) R_v T_K f(\theta)\right)^{**}(2Lt)\\ &\leq \left(B_K\left(\chi_{(-L-M,L+M)} R_{\cdot} f(\theta)\right)(v)\right)^{**}(2Lt)\\ &\leq A\, D\left(\frac{L+M}{L}\frac{\|f\|_{L^1(\mathbb{T})}}{t}\right)\end{aligned}$$

Thus, by the continuity of D, letting L tend to infinity, the result follows. □

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\widehat{\varphi} \geq 0$, $||\widehat{\varphi}||_1 = 1$. Define $\varphi_j(x) = \varphi(x/j)$. Observe that $\{\widehat{\varphi_j}\}_j$ satisfies *1*,*2* and *3* of Lemma 2.3.9.

Theorem 5.3.6. *Let $\mathbf{m}$ be a L^∞ function normalized with respect to $\{\widehat{\varphi_j}\}_j$. Assume that, for any $t > 0$,*

$$(T_{\mathbf{m}} f)^{**}(t) \leq A\, D\left(\frac{\|f\|_{L^1(\mathbb{R})}}{t}\right),$$

for $f \in L^1$ with $\|f\|_\infty \leq 1$, where D is a continuous function on $(0,\infty)$. Then, if $\overline{\mathbf{m}} = \mathbf{m}|_{\mathbb{Z}}$, for any f such that $\|f\|_\infty \leq 1$, for any $0 < t < 1$,

$$(T_{\overline{\mathbf{m}}} f)^{**}(t) \leq A\, D\left(\frac{\|f\|_{L^1(\mathbb{T})}}{t}\right).$$

Proof. Let $h \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $h \geq 0$, and $\|h\|_1 = 1$. For any $j \geq 1$, consider $h_j(x) = jh(jx)$. A straightforward computation shows that: $\|h_j\|_1 = 1$ for all $j \geq 1$, and $\widehat{h_j}(\xi) \underset{j}{\to} 1$ for every $\xi \in \mathbb{R}$. Define $\widehat{K_j}(\xi) = (\widehat{\varphi_j} * \mathbf{m})(\xi)\widehat{h_j}(\xi)$. Observe that $\widehat{K_j} \in \mathcal{S}(\mathbb{R})$, so $K_j \in \mathcal{S}(\mathbb{R})$. On the other hand, since

$$K_j(x) = (\varphi_j\, \mathbf{m}^\vee)(h_j(x-\cdot)) = \mathbf{m}^\vee(\varphi_j(\cdot)\, h_j(x-\cdot)),$$

and φ_j, h_j are compactly supported, it follows that also is K_j. Thus, $K_j \in \mathcal{C}_c^\infty(\mathbb{R})$. Given $f \in \mathcal{C}_c^\infty(\mathbb{R})$, it holds that

$$K_j * f = T_{\widehat{\varphi_j} * \mathbf{m}}(h_j * f).$$

Observe that given $g \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\|g\|_\infty \leq 1$,

$$\int (\widehat{\varphi_j} * \mathbf{m})(\xi)\widehat{g}(\xi) e^{2\pi i x\xi}\, d\xi = \int \widehat{\varphi_j}(y) e^{2\pi i x y} T_{\mathbf{m}}\left(e^{-2\pi i y\cdot} g\right)(x)\, dy. \tag{5.3.7}$$

Thus, since $||e^{-2\pi i y\cdot} g||_1 = ||g||_1$ and $\|\widehat{\varphi_j}\|_1 = 1$,

$$\begin{aligned}(K_j * g)^{**}(t) &\leq \sup_{|E|=t} \frac{1}{t}\int_E \int \left|\widehat{\varphi_j}(y) T_{\mathbf{m}}\left(e^{-2\pi i y\cdot} g\right)(x)\right|\, dy\, dx.\\ &\leq \int |\widehat{\varphi_j}(y)| \left(T_{\mathbf{m}}\left(e^{-2\pi i y\cdot} g\right)\right)^{**}(t)\, dy \leq AD\left(\frac{\|g\|_1}{t}\right).\end{aligned} \tag{5.3.8}$$

Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ supported in $[-1,1]$, such that $\|\phi\|_1 = 1$, and let $\phi_n(x) = n\phi(nx)$. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, $1 \geq \psi \geq 0$, such that $\psi(x) = 1$ for $|x| \leq 1$ and it is supported in $[-2,2]$. Let $\psi_{(n)}(x) = \psi(x/n)$. Fixed $g \in L^1(\mathbb{R})$ such that $\|g\|_\infty \leq 1$, define $h_n(x) = \psi_{(n)}(x)(\phi_n * g)(x)$. It is easy to see that, $h_n \in \mathcal{C}_c^\infty(\mathbb{R})$, $\|h_n\|_\infty \leq 1$. Moreover,

$$\|h_n - g\|_1 \leq \|\phi_n * g - g\|_1 + \left\|g(1-\psi_{(n)})\right\|_1,$$

so $h_n \to g$ and $K_j * h_n \to K_j * g$ in $L^1(\mathbb{R})$. In particular, there exists a subsequence, that $K_j * h_{n_k}(x) \to K_j * g(x)$ a.e. x. Therefore,

$$\begin{aligned}(K_j * g)^{**}(t) &\leq \liminf_k (K_j * h_{n_k})^{**}(t)\\ &\leq AD\left(\frac{\liminf_n \|h_n\|_1}{t}\right)\\ &= AD\left(\frac{\|g\|_1}{t}\right).\end{aligned}$$

Since $\mathbf{m}$ is normalized, fixed $\xi \in \mathbb{R}$,

$$\lim_{j\to\infty} \widehat{K_j}(\xi) = \lim_{j\to\infty} (\widehat{\phi_j} * \mathbf{m})(\xi) h_j(\xi) = \mathbf{m}(\xi).$$

By the previous lemma $\left(T_{\widehat{K_j}|_\mathbb{Z}} f\right)^{**}(t) \leq AD\left(\frac{\|f\|_1}{t}\right)$, for any $f \in \mathcal{C}(\mathbb{T})$. On the other hand, for all trigonometric polynomial f,

$$\lim_{j\to\infty} T_{\widehat{K_j}} f(x) = \lim_{j\to\infty} \sum_{k\in\mathbb{Z}} \widehat{K_j}(k)\widehat{f}(k)e^{2\pi ikx} = T_{\mathbf{m}|_\mathbb{Z}} f(x).$$

Thus, for every trigonometric polynomial f such that $\|f\|_{L^\infty} \leq 1$,

$$\left(T_{\mathbf{m}|_\mathbb{Z}} f\right)^{**}(t) \leq \liminf_{j\to\infty} \left(T_{\widehat{K_j}|_\mathbb{Z}} f\right)^{**}(t) \leq AD\left(\frac{\|f\|_1}{t}\right). \tag{5.3.9}$$

Since $\mathbf{m}|_Z \in \ell^\infty(\mathbb{Z})$, $T_{\mathbf{m}|_\mathbb{Z}}$ automatically defines a bounded operator on $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. Then, for any f with $\|f\|_\infty \leq 1$, by the density of trigonometrical polynomials in $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, there exists a sequence of trigonometric polynomials $\{p_n\}_n$ with $\|p_n\|_\infty \leq 1$, such that $\|f - p_n\|_1 \to 0$ and $T_{\mathbf{m}|_\mathbb{Z}} p_n \to T_{\mathbf{m}|_\mathbb{Z}} f$ a.e. It follows the (5.3.9) holds for any f with $\|f\|_\infty \leq 1$. □

Whenever T is an operator such that, for $1 < p \leq 2$, $T : L^p \to L^{p,\infty}$ is bounded with constant $1/(p-1)^\alpha$ for some $\alpha > 0$, similarly as in the strong case, it follows that

$$(Tf)^*(t) \lesssim \frac{\|f\|_1}{t}\left(1 + \log^+ \frac{t}{\|f\|_1}\right)^\alpha,$$

for $f \in L^1$ with $\|f\|_\infty \leq 1$. The difficulty of working with the non increasing rearrangement is that it is not sublinear, and let us recall that we have used this property for the maximal function to prove (5.3.8) in the proof of the previous theorem. Then, if we want to prove a result similar to Theorem 5.3.6 with the non increasing rearrangement, we shall deal with this difficulty.

With some extra condition on the function D, the result remains true. We shall assume that there is a constant C such that, for any $t>0$,

$$\frac{1}{t}\int_0^t D(1/s)ds \leq CD(1/t). \tag{5.3.10}$$

This inequality holds, for example for $D(s)=s^{1-a}\gamma(s)$ where $a>0$ and γ is a slowly varying function in $(0,\infty)$ (see Appendix D). In particular, this holds for $D(s)=s^{1-a}\left(1+\log^+ s\right)^\alpha$, with $\alpha\in\mathbb{R}$.

This situation arise, for instance whenever T is a sublinear operator satisfying that, for every $p>2$,

$$T: L^p(\mathbb{R}) \to L^{p,\infty}(\mathbb{R})$$

with constant p. In this case, $D(s)=\sqrt{s}(1+\log^+ s)$ (see for instance [35]).

Theorem 5.3.11. *Let* $\mathbf{m}$ *be a* L^∞ *normalized function. Assume that for any* $t>0$,

$$(T_{\mathbf{m}}f)^*(t) \leq A\,D\left(\frac{\|f\|_{L^1(\mathbb{R})}}{t}\right),$$

for $f\in L^1$ *with* $\|f\|_\infty\leq 1$, *where* D *is a continuous function on* $(0,\infty)$ *satisfying (5.3.10). Then, if* $\overline{\mathbf{m}}=\mathbf{m}|_{\mathbb{Z}}$, *for every trigonometric polynomial* f *such that* $\|f\|_\infty\leq 1$ *and all* $0<t<1$,

$$(T_{\overline{\mathbf{m}}}f)^*(t) \leq A'\,D\left(\frac{3\,\|f\|_{L^1(\mathbb{T})}}{t}\right).$$

Proof. In order to prove the result, it is enough to show that the analogous inequality to (5.3.8) holds for $g\in\mathcal{C}_c^\infty(\mathbb{R})$ such that $\|g\|_\infty\leq 1$. We maintain the notation of the proof in Theorem 5.3.6.

We require to use the approximation and discretization used in the proof of Theorem B.2.2. Following the notation therein, by (B.2.3), for each j, the right hand term of (5.3.7) is not greater than

$$\int\left|\widehat{\varphi_j}(y)T_{\mathbf{m}}\left(e^{-2\pi iy\cdot}g\right)(x)\right|\,dy \leq \liminf_n\left(\liminf_l\sum_{m=1}^{N_l}\lambda_{m,l}^n\left|T_{\mathbf{m}}(e^{-2\pi y_m\cdot}g)\right|(x)\right),$$

where for each l,

$$\sum_{m=1}^{N_l}\lambda_{m,l}^n=\int_{\mathcal{K}_n}\left|\widehat{\phi_l}\right|\leq 1,$$

and $\mathcal{K}_n\uparrow\mathbb{R}$. Thus, by the properties of the non-increasing rearrangement,

$$(K_j*g)^*(t)\leq\liminf_n\left(\liminf_l\left(\sum_{m=1}^{N_l}\lambda_{m,l}^n\left|T_{\mathbf{m}}(e^{-2\pi y_j\cdot}g)\right|(x)\right)^*\right).$$

Now, using [36, Theorem 2.1], fixed n,l, for any sequence of positive numbers

$\{c_m\}$ such that $\sum_{m=1}^{N_l} c_m = 1$,

$$\left(\sum_{m=1}^{N_l} \lambda^n_{m,l} \left|T_{\mathbf{m}}(e^{-2\pi y_j \cdot} g)\right|(x)\right)^* (3t) \leq$$
$$\leq \sum_{m=1}^{N_l} \lambda^n_{m,l} \left(T_{\mathbf{m}}(e^{-2\pi y_j \cdot} g)\right)^* (t) + \sum_{m=1}^{N_l} \lambda^n_{m,l} \frac{1}{t} \int_{c_m t}^{t} \left(T_{\mathbf{m}}(e^{-2\pi y_j \cdot} g)\right)^* (s)\, ds.$$

The first term is bounded above by $AD\left(\frac{\|g\|_1}{t}\right)$. By (5.3.10), each summand on the second one is less than or equal to

$$\frac{1}{t}\int_{c_m t}^{t} D\left(\frac{\|g\|_1}{s}\right) ds \leq ACD\left(\frac{\|g\|_1}{t}\right).$$

Hence,

$$(K_j * g)^*(t) \leq A(1+C) D\left(\frac{3\,\|g\|_1}{t}\right).$$

Now the proof finish with the same argument as used in Theorem 5.3.6. □

5.4 Weak weighted inequalities

In this section we give a weighted transference result for weak type convolution operators. In particular we will obtain results on restriction of fourier multipliers in this setting.

5.4.1 Transference Result

We shall introduce first some notation that we will maintain throughout this section. Let R_t be the continuous representation of G on $L^p(\mu)$ given by

$$R_t f(x) = h_t(x)\Phi_t f(x), \quad h_{ts}(x) = h_t(x)\Phi_t h_s(x).$$

where Φ_t is a σ-endomorphism of $\mathcal{M}$, that we assume preserves measure. That is $\mu(\Phi_t E) = \mu(E)$, and $|h_t(x)| = 1$. Given a weight w on $\mathcal{M}$ we define

$$\mathcal{T} w_x(t) = \Phi_t w(x).$$

It holds that

$$\int_{\mathcal{M}} |R_t f(x)|^p\, \mathcal{T} w_x(t)\, d\mu(x) = \int_{\mathcal{M}} |f(x)|^p\, w(x)\, d\mu(x).$$

The notation is taken from [63] where it is developed a weighted Ergodic theory in the setting of A_p weights.

Let $\mathcal{V}$ be the family of relatively compact open sets if either $G = \mathbb{R}^n$ or $G = \mathbb{Z}$. In the case that $G = \mathbb{T}$, $\mathcal{V} = \mathbb{T}$.

Definition 5.4.1. *A pair of weights* (w_0, w_1) *is admissible if there exists a constant* $c > 0$ *satisfying that*

$$\sup_{\mathcal{K}} \inf_{V\in\mathcal{V}} \frac{w_0\left(V\mathcal{K}\right)}{w_1\left(V\right)} = c_{(w_0,w_1)} < \infty, \tag{5.4.2}$$

where the supremum is taken over the compact sets containing $\{e\}$*.*

Lemma 5.4.3. *Let* w *be a weight such that* (w, w) *is an admissible pair. Then, for any compact set* $\mathcal{K}$ *containing* $\{e\}$, $\inf_{V\in\mathcal{V}} \frac{w(V\mathcal{K})}{w(V)} = 1$*. Hence,* $c_{(w,w)} = 1$*.*

Proof. Let $c_{\mathcal{K}} = \inf_{V\in\mathcal{V}} \left(\frac{w(V\mathcal{K})}{w(V)}\right)$, for any compact set $\mathcal{K}$ such that $e \in \mathcal{K}$. Since, for any $V \in \mathcal{V}$, $w(V) \leq w(V\mathcal{K})$, it follows that $c_{(w,w)} \geq c_{\mathcal{K}} \geq 1$.

Let $\mathcal{K}$ one of such compact sets. Since for any $V \in \mathcal{V}$, $V\mathcal{K} \in \mathcal{V}$ it follows that

$$\frac{w\left(V\mathcal{K}\mathcal{K}\right)}{w\left(V\right)} = \frac{w\left(V\mathcal{K}\mathcal{K}\right)}{w\left(V\mathcal{K}\right)}\frac{w\left(V\mathcal{K}\right)}{w\left(V\right)} \geq c_{\mathcal{K}}^2.$$

On the other hand, since $\mathcal{K}\mathcal{K}$ is also a compact set containing $\{e\}$, it follows that $c_{(w,w)} \geq c_{\mathcal{K}\mathcal{K}} \geq c_{\mathcal{K}}^2$. Then taking the supremum over the compact sets, it follows that $c_{(w,w)} \geq c_{(w,w)}^2$, from where we deduce that $c_{(w,w)} = c_{\mathcal{K}} = 1$. ☐

Examples of admissible pairs of weights:

1. If $w = 1$, it is easy to see that the condition on (w, w) to be admissible is equivalent to the amenability Følner condition on the group (see [46]).

2. A pair of weights (w_0, w_1) belonging to $A_p(\mathbb{R}^n)$ are admissible.

Observe that for any pair of cubes Q_1, Q_2, by Hölder's inequality and A_p condition,

$$\begin{aligned}\frac{w_0(Q_1+Q_2)}{w_1(Q_1)} &\leq \frac{w_0(Q_1+Q_2)}{w_1(Q_1)}\left(\frac{1}{\lambda(Q_1)}w_1^{1-p'}(Q_1)^{1/p'}w_1(Q_1)^{1/p}\right)^p\\ &= \left(\frac{1}{\lambda(Q_1)}w_1^{1-p'}(Q_1)^{1/p'}w_0(Q_1+Q_2)^{1/p}\right)^p\\ &\leq [w_0,w_1]_{A_p}^p\left(\frac{\lambda(Q_1+Q_2)}{\lambda(Q_1)}\right)^p\end{aligned}$$

Let $\widetilde{\mathcal{V}} = \{(-r, r)^n : r > 1\}$. Since for any compact set $\mathcal{K}$ there exists $s > 0$ such that $\mathcal{K} \subset [-s, s]^n$),

$$\begin{aligned}\inf_{V\in\mathcal{V}}\frac{w_0(V\mathcal{K})}{w_1(V)} &\leq \inf_{V\in\widetilde{\mathcal{V}}}\frac{w_0(V\mathcal{K})}{w_1(V)} \leq \inf_{r>1}\frac{w_0((-s-r,s+r)^n)}{w_1((-r,r)^n)}\\ &\leq [w_0,w_1]_{A_p}^p\inf_{r>1}\frac{(s+r)^{pn}}{r^{pn}} = [w_0,w_1]_{A_p}^p.\end{aligned}$$

Hence, taking the supremum over all the compact sets $\mathcal{K}$, it follows that the pair (w_0, w_1) is admissible and $c_{(w_0,w_1)} \leq C$.

3. If a pair of weights (w_0, w_1) satisfies that there exists $\delta > 0$ and a constant C such that, for any pair of centered cubes Q_1, Q_2,

$$\frac{w_0(Q_1 + Q_2)}{w_1(Q_1)} \leq C \left(\frac{\lambda(Q_1 + Q_2)}{\lambda(Q_1)} \right)^{\delta},$$

from the previous discussion, it follows that the pair (w_0, w_1) is admissible.

4. If $(w_0, w_1) \in A_p(\mathbb{Z})$, are admissible. Recall that (w_0, w_1) belongs to the class $A_p(\mathbb{Z})$ for $1 < p < \infty$ (see [71]) if

$$[w_0, w_1]_{A_p} = \sup_{L \leq M,\, L,M \in \mathbb{Z}} \frac{1}{M - L + 1} \left(\sum_{j=L}^{M} w_0(j) \right)^{1/p} \left(\sum_{j=L}^{M} w_1(j)^{1/1-p} \right)^{1-1/p} < \infty.$$

The proof is similar to the case in $\mathbb{R}$.

Theorem 5.4.4. *Let v^0, v^1 be weights in $\mathcal{M}$ and (u_0, u_1) be an admissible pair of weights. Let $0 < q \leq p < \infty$ and define $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, and define*

$$V_i(x,t) \quad = \quad \mathcal{T} v_x^i(t) u_i(t), \quad i = 0, 1.$$

Let $K \in L^1(G)$ such that, for any $V \in \mathcal{V}$, a.e. $x \in \mathcal{M}$, for $f \in L^p(G)$

$$\|B_K f \chi_V\|_{L^q(V_1(x,\cdot))} \leq N(K, x, V) \, \|f\|_{L^p(V_0(x,\cdot))},$$

where $N(K, x, V)$ satisfies that

$$N(K) = \sup_{V \in \mathcal{V}} \frac{\|N(K, \cdot, V)\|_{L^r(\mu)}}{u_1(V)^{1/r}} < \infty.$$

Then, for $f \in L^p(v^0) \cap L^p(\mu)$,

$$\|T_K f\|_{L^q(v^1)} \leq c_{(u_0,u_1)}^{1/p} N(K) \, \|f\|_{L^p(v^0)}.$$

Proof. Denote by $\mathcal{K}$ the support of K. Hence, for every $V \in \mathcal{V}$, fixed $f \in L^p(\mu)$,

$$\begin{aligned}
\|T_K f\|_{L^q(v^1)}^q &= \int_{\mathcal{M}} |R_t T_K f(x)|^q \, \mathcal{T} v_x^1(t) \, d\mu(x) \\
&\leq \frac{1}{u_1(V)} \int_V \int_{\mathcal{M}} |R_t T_K f(x)|^q \, \mathcal{T} v_x^1(t) \, d\mu(x) \, u_1(t) \, d\lambda(t) \\
&\leq \frac{1}{u_1(V)} \int_{\mathcal{M}} \left\{ \int_G \left| B_K \left(R_{(\cdot)} f(x) \chi_{V\mathcal{K}^{-1}} \right)(t) \right|^q V_1(x,t) \, d\lambda(t) \right\} d\mu(x)
\end{aligned}$$

By hypothesis, the term inside curly brackets is bounded by

$$\left\{ \int_{V\mathcal{K}^{-1}} |R_t f(x)|^p \, V_0(x,t) \, d\lambda(t) \right\}^{q/p} (N(K, x, V))^q.$$

Then, integrating on $\mathcal{M}$ and using Hölder's inequality,

$$\begin{aligned}\|T_K f\|^q_{L^q(v^1)} &\leq \frac{1}{u_1(V)} \left\{ \int_{\mathcal{M}} \int_{V\mathcal{K}^{-1}} |R_t f(x)|^p \, V_0(x,t) d\lambda(t) d\mu(x) \right\}^{q/p} \\ &\qquad \left\{ \int_{\mathcal{M}} N(K,x,V)^r d\mu(x) \right\}^{q/r} \\ &\leq \left(\frac{u_0\left(V\mathcal{K}^{-1}\right)}{u_1\left(V\right)} \right)^{q/p} \frac{\|N(K,\cdot,V)\|^q_{L^r(\mu)}}{u_1(V)^{1/r}} \|f\|^q_{L^p(v^0)} \\ &\leq \left(\frac{u_0\left(V\mathcal{K}^{-1}\right)}{u_1\left(V\right)} \right)^{q/p} N(K)^q \|f\|^q_{L^p(v^0)} .\end{aligned}$$

So the result follows considering the infimum in $V \in \mathcal{V}$. □

Since the given representation R_t is positivity-preserving, with minors modifications in the proof, the maximal counterpart of the previous result holds.

Theorem 5.4.5. *Let v^0, v^1 be weights in $\mathcal{M}$ and (u_0, u_1) be an admissible pair of weights. Let $0 < q \leq p < \infty$ and define $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Let, define*

$$V_i(x,t) \quad = \quad \mathcal{T} v^i_x(t) u_i(t), \quad i = 0,1.$$

Let $K = \{K_j\}_j \subset L^1(G)$ such that, for any $V \in \mathcal{V}$, a.e. $x \in \mathcal{M}$, for $f \in L^p(G)$

$$\left\| \sup_j \left| B_{K_j} f \right| \chi_V \right\|_{L^q(V_1(x,\cdot))} \leq N(K,x,V) \|f\|_{L^p(V_0(x,\cdot))} ,$$

where $N(K,x,V)$ satisfies that $N(K) = \sup_{V\in\mathcal{V}} \frac{\|N(K,\cdot,V)\|_{L^r(\mu)}}{u_1(V)^{1/r}} < \infty$. Then, for $f \in L^p(v^0) \cap L^p(\mu)$,

$$\left\| \sup_j \left| T_{K_j} f \right| \right\|_{L^q(v^1)} \leq c^{1/p}_{(u_0,u_1)} N(K) \|f\|_{L^p(v^1)} .$$

5.4.2 Restriction of Fourier multipliers

In this section we will present some applications of Theorem 5.4.4 to the problem of restricting Fourier multipliers in the setting of weighted Lebesgue spaces. We will consider weak multipliers in $\mathbb{R}^n$, for $n \geq 1$ and in the periodic case.

Let $0 < p < \infty$. In $L^{p,\infty}(\mu)$ the quasi-norm $\|f\|_{L^{p,\infty}} = \sup_{t>0} t\mu_f(s)^{1/p}$ satisfies, for every $q < p$, that

$$\|f\|_{L^{p,\infty}(\mu)} \leq \sup \|f\chi_E\|_{L^q(\mu)} \mu(E)^{1/p-1/q} \leq C_{q,p} \|f\|_{L^{p,\infty}(\mu)} , \tag{5.4.6}$$

where $C_{q,p} = \left(\frac{p}{p-q}\right)^{1/q}$ and the supremum is taken over the family of sets of finite measure and $\mu_f(s) = \mu\{x : \ |f(x)| > s\}$. (5.4.6) is called Kolmogorov's condition (see [60, page 485]).

Restriction to lower dimension

Observe that given a weight u defined on $\mathbb{R}^n$, for $d \geq 1$, it induces in a natural way a weight in $\mathbb{R}^{d+n}$. Namely, for $E \subset \mathbb{R}^{d+n}$, $u(E) = \int_E u(y)d(x,y)$.

Throughout this section let $1 \leq p < \infty$, $u \in A_p(\mathbb{R}^d)$, $v \in A_p(\mathbb{R}^n)$ and define $w(x,y) = u(x)v(y)$.

Corollary 5.4.7. *Let $K \in L^1(\mathbb{R}^{d+n})$ with compact support such that the multiplier operator defined by $\widehat{K}$ maps $L^p(w)$ into $L^{p,\infty}(w)$ with norm N. Then, fixed $\xi \in \mathbb{R}^d$, the multiplier given by $\widehat{K}(\xi,\cdot)$ maps $L^p(\mathbb{R}^n, v)$ in $L^{p,\infty}(\mathbb{R}^n, v)$ with norm no greater than $c_{p,w}N$.*

Proof. Fix $\xi \in \mathbb{R}^d$. Consider R to be the continuous representation of $\mathbb{R}^{d+n}$ in $L^p(\mathbb{R}^n)$ given by

$$R_{(x,y)}f(z) = e^{2\pi i x\xi} f(z+y). \tag{5.4.8}$$

In this way, $\mathcal{T}w_z(x,y) = w(z+y)$, and the associated transferred operator T_K coincides with the multiplier operator given by $\widehat{K}(\xi,\cdot)$.

Fixed a set of finite measure $E \subset \mathbb{R}^n$, let consider

$$V_1(z,(x,y)) = \mathcal{T}(\chi_E v)_z(x,y)u(x),$$

and

$$V_0(z,(x,y)) = \mathcal{T}(v)_z(x,y)u(x).$$

Observe that if $A_z = \{(x,y) : \; y+z \in E\}$,

$$\begin{aligned} V_1(z,(x,y)) &= w(x,y+z)\chi_{A_z}(x,y), \\ V_0(z,(x,y)) &= w(x,y+z). \end{aligned}$$

Since convolution commutes with translations, by Kolmogorov's condition it follows that for $q < p$ for every $z \in \mathbb{R}^n$ and every $V \subset \mathbb{R}^{d+n}$,

$$\|B_K g \chi_V\|_{L^q(V_1(z,\cdot))} \leq N(K,z,V) \, \|g\|_{L^p(v(z,\cdot))},$$

where $N(K,z,V) = c_{p,q}N\left(\int_{V\cap A_z} w(x,y+z)dxdy\right)^{1/r}$, and $c_{p,q} = \left(\frac{p}{p-q}\right)^{1/q}$. Observe that,

$$\begin{aligned} \|N(K,\cdot,V)\|^r_{L^r(\mathbb{R}^n)} &= (c_{p,q}N)^r \int_{\mathbb{R}^n}\int_{V\cap A_z} w(x,y+z)dxdydz \\ &= (c_{p,q}N)^r \int_V \int_{\mathbb{R}^n} \chi_E(y+z)v(y+z)\, dz\, u(x)dxdy \\ &= (c_{p,q}N)^r v(E)u(V)\, dxdy. \end{aligned}$$

Hence,

$$\sup_{V\in\mathcal{V}} \frac{\|N(B_K,\cdot,V)\|_{L^r(\mu)}}{u(V)^{1/r}} = c_{p,q}Nv(E)^{1/r}.$$

Since $u \in A_p(\mathbb{R}^d)$ it automatically defines an $A_p(\mathbb{R}^{d+n})$ weight. Hence (u,u) is an admissible pair. So we can apply Theorem 5.4.4 to deduce that, for $f \in$

$L^p(\mathbb{R}^n) \cap L^p(v)$

$$\|T_K f\|_{L^q(\chi_E v)} \leq N c_{p,q} c_u (v(E))^{1/q-1/p} \|f\|_{L^p(v)} .$$

Finally the result follows from Kolmogorov's inequality and by the density of $L^p(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, v)$ in $L^p(\mathbb{R}^n, v)$. □

Corollary 5.4.9. *Let* $\mathbf{m} \in L^\infty(\mathbb{R}^{d+n})$ *be a normalized function such that,* $\mathbf{m} \in M(L^p(w))$. *Fixed* $\xi \in \mathbb{R}^d$, *the multiplier given by* $\mathbf{m}(\xi, \cdot)$ *maps* $L^p(\mathbb{R}^n, v)$ *in* $L^{p,\infty}(\mathbb{R}^n, v)$ *with norm bounded uniformly on* ξ *by* $c_{p,w} \|\mathbf{m}\|_{M(L^p(w), L^{p,\infty}(w))}$.

Proof. First observe that, since $u \in A_p(\mathbb{R}^d)$ and $v \in A_p(\mathbb{R}^n)$, it is easy to see that $w \in A_p(\mathbb{R}^{d+n})$ with A_p constant no greater than the A_p constant of v multiplied by the A_p constant of u. Observe also that (v, v) is admissible.

Since $w \in A_p(\mathbb{R}^{d+n})$, we can apply the approximation techniques of §2.3 (see Table 2.3.16.1). So by Theorem 2.3.13, there exists a sequence $\{\mathbf{m}_j\}_j \subset L^\infty(\mathbb{R}^{d+n})$ such that $\mathbf{m}_j^\vee \in L^1$ with compact support, $\mathbf{m}_j \to \mathbf{m}$ pointwise and with norm less than or equal to $c_{p,w} \|\mathbf{m}\|_{M(L^p(w), L^{p,\infty}(w))}$, where $c_{p,w}$ is a constant depending only on p and the A_p constant of w. Then by Corollary 5.4.7, $\mathbf{m}_j(\xi, \cdot) \in M(L^p(u), L^{p,\infty}(u))$ with norm no greater than $c_{p,w} \|\mathbf{m}\|_{M(L^p(w), L^{p,\infty}(w))}$.

Since for every $f \in \mathcal{S}$, by the dominated convergence theorem, we have

$$\lim_{j\to\infty} (\mathbf{m}_j(\xi, \cdot)\widehat{f})^\vee(x) \to (\mathbf{m}(\xi, \cdot)\widehat{f})^\vee(x),$$

it follows by Fatou's lemma that $\left\|(\mathbf{m}(\xi, \cdot)\widehat{f})^\vee\right\|_{L^{p,\infty}(v)} \lesssim \|\mathbf{m}\| \|f\|_{L^p(v)}$. The result follows by the density of $\mathcal{S}(\mathbb{R}^d)$ in $L^p(v)$. □

Observe that for $p > 1$, since $L^{p,\infty}(w)$ is a Banach space, with slight modifications, the approximation techniques developed in §4.2.2 can be adapted to obtain a restriction result for $\{0\}$-normalized functions analogous to Theorem 4.2.19.

Restriction to the integers

Definition 5.4.10. [24, Definition 4.3] *A weight* w *belongs to the class* W_p *if it is an* $A_p(\mathbb{R})$*-weight essentially constant in the intervals* $[k, k+1)$, *for* $k \in \mathbb{Z}$. *That is, there exists a constant* $\rho \geq 1$ *such that for each* $k \in \mathbb{Z}$

$$\rho^{-1} w(k) \leq w(x) \leq \rho w(k), \quad \text{for all } x \in [k, k+1]. \tag{5.4.11}$$

Observe that if $\{w_k\}_{k\in\mathbb{Z}} \in A_p(\mathbb{Z})$, the continuous function defined by $w(x) = w_k$ for $x \in [k - 1/4, k + 1/4]$ and linear between for $k \in \mathbb{Z}$ (see [71]), are weights in the class W_p. Moreover, by [24, Theorem 4.4] if $w \in W_p$, its restriction to $\mathbb{Z}$ belongs to $A_p(\mathbb{Z})$.

A periodic weight w belonging to $A_p(\mathbb{R})$ is said to be in class $A_p(\mathbb{T})$.

Corollary 5.4.12. *Let* $1 \leq p < \infty$, $u \in A_p(\mathbb{T})$, $v \in W_p$ *and consider* $w = uv$. *Assume that* $K \in L^1(\mathbb{R})$ *with compact support such that,*

$$\widehat{K} \in M\left(L^p(\mathbb{R}, w), L^{p,\infty}(\mathbb{R}, w)\right)$$

with norm N. Hence, the multiplier given by $\widehat{K}|_{\mathbb{Z}}$ maps $L^p(\mathbb{T}, v)$ in $L^{p,\infty}(\mathbb{T}, v)$ with norm less than or equal to $c_{p,w}N$.

Proof. Consider R to be the representation of $\mathbb{R}$ in $L^p(\mathbb{T})$ given by

$$R_x f(\theta) = f(\theta + x). \tag{5.4.13}$$

In this way, $\mathcal{T} w_\theta(x) = w(x + \theta)$ and the associated transferred operator T_K coincides with the operator given by the multiplier $\widehat{K}|_{\mathbb{Z}}$. Fixed a measurable set $E \subset [0, 1)$, define

$$V_1(\theta, x) = \mathcal{T}(\chi_E u)_\theta(x)v(x),$$

and

$$V_0(\theta, x) = \mathcal{T} u_\theta(x)v(x).$$

Observe that if $A_\theta = \{x \in \mathbb{R} : \ x + \theta \in E\}$,

$$\begin{aligned} V_1(\theta, x) &= u(x + \theta)\chi_{A_\theta}(x)v(x), \\ V_0(\theta, x) &= u(x + \theta)v(x). \end{aligned}$$

Since translations and convolution operator commutes, it follows that for every $\theta \in [0, 1)$,

$$\|B_K f\|_{L^{p,\infty}(w(\cdot+\theta))} \leq N \|g\|_{L^p(w(\cdot+\theta))} .$$

Observe also that, since $v \in W_p$, by [24, Theorem 4.4], there exists a constant ζ such that for all $x \in \mathbb{R}$, $\eta \in [0, 1)$, $\frac{1}{\zeta} \leq \frac{v(x)}{v(x+\eta)} \leq \zeta$. Then

$$\begin{aligned} \frac{1}{\zeta} w(x + \theta)\chi_{A_\theta}(x) \leq V_1(\theta, x) \leq \zeta w(x + \theta)\chi_{A_\theta}(x), \\ \frac{1}{\zeta} w(x + \theta) \leq V_0(\theta, x) \leq \zeta w(x + \theta). \end{aligned} \tag{5.4.14}$$

Hence, for $q < p$, if $1/r = 1/q - 1/p$, by Kolmogorov's condition, for any set of finite measure V, fixed $\theta \in [0, 1)$,

$$\begin{aligned} \|B_K g\chi_V\|_{L^q(V_1(\theta,\cdot))} &\leq \zeta^{1/p} \|B_K f\chi_{V\cap A_\theta}\|_{L^q(w(\cdot+\theta))} \\ &\leq \zeta^{1/p+1/q} c_{p,q} N \left(\int_{A_\theta \cap V} w(x + \theta)dx \right)^{1/r} \|g\|_{L^p(V_0(\theta,\cdot))} . \end{aligned}$$

If we define $N(K, z, V) = \zeta^{1/p+1/q} c_{p,q} N \left(\int_{V\cap A_z} w(x + \theta)dx \right)^{1/r}$, it holds that,

$$\begin{aligned} \|N(K, \cdot, V)\|^r_{L^r(\mathbb{T})} &= (\zeta^{1/p+1/q} c_{p,q} N)^r \int_{\mathbb{T}} \int_{V\cap A_\theta} w(x + \theta)dxdz \\ &\leq \zeta(\zeta^{1/p+1/q} c_{p,q} N)^r \int_V \int_{\mathbb{T}} \chi_E(x + \theta)u(x + \theta) \, d\theta v(x)dx \\ &= \zeta(\zeta^{1/p+1/q} c_{p,q} N)^r u(E)v(V). \end{aligned}$$

Hence,

$$\sup_{V\in\mathcal{V}} \frac{\|N(B_K,\cdot,V)\|_{L^r(\mathbb{T})}}{v(V)^{1/r}} \le \zeta^{2/q} c_{p,q} N u(E)^{1/r}.$$

So we can apply Theorem 5.4.4 with the admissible pair (v,v), to deduce that, for $f\in L^p(\mathbb{T})\cap L^p(\mathbb{T},u)$

$$\|T_K f\|_{L^q(\chi_E u)} \le N\zeta^{2/q} c_{p,q} u(E)^{1/q-1/p} \|f\|_{L^p(u)}.$$

Therefore, the result follows by (5.4.6) and by the density of $L^p(\mathbb{T})\cap L^p(\mathbb{T},v_0)$ in $L^p(\mathbb{T},v_0)$. □

Lemma 5.4.15. *If $u\in A_p(\mathbb{T})$ and $v\in W_p$, then $uv\in A_p(\mathbb{R})$.*

Proof. By [24, Theorem 4.4], $\gamma=\sup_{k\in\mathbb{Z}}\frac{v(k)}{v(k+1)}$ and $\Gamma=\sup_{k\in\mathbb{Z}}\frac{v(k+1)}{v(k)}$ are finite.

Let I be an interval on $\mathbb{R}$. Observe that

$$\int_I vu \le \rho\sum_{k\in\mathbb{Z}} v(k)\left(\int_{I\cap[k,k+1)} u\right).$$

So it easily follows

$$\left(\int_I vu\right)^{1/p}\left(\int_I v^{1-p'}u^{1-p'}\right)^{1/p'} \le c_v\rho^{2/p}\|u\|_{L^1(\mathbb{T})}^{1/p}\left\|u^{1-p'}\right\|_{L^1(\mathbb{T})}^{1/p'}(|I|+2).$$

where c_v denotes the $A_p(\mathbb{Z})$ constant of v. On the other hand if $|I|<1/4$, I intersects at most two intervals of the type $[k,k+1)$. If it intersects only one

$$\left(\int_I vu\right)^{1/p}\left(\int_I v^{1-p'}u^{1-p'}\right)^{1/p'} \le \rho^{2/p}\left(\int_I u\right)^{1/p}\left(\int_I u^{1-p'}\right)^{1/p'} \lesssim |I|.$$

If it intersects two different intervals, let them be $[k,k+1)$ and $[k+1,k+2)$, then

$$\begin{aligned}&\left(\int_I vu\right)^{1/p}\left(\int_I v^{1-p'}u^{1-p'}\right)^{1/p'}\\ &\le \rho^{2/p}\left(v(k)\int_{I\cap[k,k+1)} u + v(k+1)\int_{I\cap[k+1,k+2)} u\right)^{1/p}\\ &\qquad\left(v(k)^{1-p'}\int_{I\cap[k,k+1)} u^{1-p'} + v(k+1)^{1-p'}\int_{I\cap[k+1,k+2)} u^{1-p'}\right)^{1/p'}\\ &\le \rho^{2/p}\max(1,\gamma,\Gamma)^{1/p}\left(\int_I u\right)^{1/p}\left(\int_I u^{1-p'}\right)^{1/p'} \lesssim |I|.\end{aligned}$$

In any case, the result follows. □

The following corollary is the weak counterpart of [21, Theorem 1.2], where the strong analogous result is proved with $v=1$.

Corollary 5.4.16. *Let* $1 \le p < \infty$*,* $u \in A_p(\mathbb{T})$*,* $v \in W_p$ *and consider* $w = uv$*. Given* $\mathbf{m} \in M(L^p(w), L^{p,\infty}(w))$ *normalized, it holds that the restriction multiplier* $\mathbf{m}|_{\mathbb{Z}} \in M(L^p(\mathbb{T}, u), L^{p,\infty}(\mathbb{T}, u))$ *and*

$$\|\mathbf{m}|_{\mathbb{Z}}\|_{M(L^p(\mathbb{T},u),L^{p,\infty}(\mathbb{T},u))} \le C_{p,w} \|\mathbf{m}\|_{M(L^p(\mathbb{R},w),L^{p,\infty}(\mathbb{R},w))}.$$

Proof. Since $w \in A_p(\mathbb{R})$, we can apply the approximation techniques of §2.3 (see V in Table 2.3.16.1). So by Theorem 2.3.13, there exists a family $\{\mathbf{m}_n\}_n \subset L^\infty(\mathbb{R})$ such that $\mathbf{m}_n^\vee \in L^1$ with compact support, $\mathbf{m}_n \to \mathbf{m}$ pointwise and with norm no greater than $c_{p,w} \|\mathbf{m}\|_{M(L^p(w),L^{p,\infty}(w))}$, where $c_{p,w}$ is a constant depending only on p and the A_p constant of w.

By Corollary 5.4.12,

$$\|\mathbf{m}_n|_{\mathbb{Z}}\|_{M(L^p(\mathbb{T},u),L^{p,\infty}(\mathbb{T},u))} \le C_{p,w} \|\mathbf{m}\|_{M(L^p(\mathbb{R},w),L^{p,\infty}(\mathbb{R},w))}.$$

Since, for a trigonometric polynomial f,

$$T_n f(s) = \sum_{k\in\mathbb{Z}} \mathbf{m}_n(k)\widehat{f}(k)e^{2\pi iks} \to \sum_{k\in\mathbb{Z}} \mathbf{m}(k)\widehat{f}(k)e^{2\pi iks} = Tf(s),$$

by Fatou's lemma,

$$||Tf||_{L^{p,\infty}(\mathbb{T},u)} \le \liminf_n ||T_n f||_{L^{p,\infty}(\mathbb{T},u)} \le C_{p,w}||\mathbf{m}||_{M(L^p(w),L^{p,\infty}(w))}||f||_{L^p(\mathbb{T},u)}.$$

□

A Remark for strong multipliers

The statement of Theorem 5.4.4 allows not only weak type operators but also of strong type. In particular, we will show how this allows us to obtain a weighted strong type restriction theorem that in particular recovers [21, Theorem 1.2] for $v = 1$.

Corollary 5.4.17. *Let* $1 \le p < \infty$*,* $u \in A_p(\mathbb{T})$ *and* $v \in W_p$*. Define* $w = uv$*. Given* $\mathbf{m} \in M(L^p(w))$ *normalized, it holds that the restriction multiplier* $\mathbf{m}|_{\mathbb{Z}} \in M(L^p(\mathbb{T}, u))$ *and* $\|\mathbf{m}|_{\mathbb{Z}}\|_{M(L^p(\mathbb{T},u))} \le C_{p,w} \|\mathbf{m}\|_{M(L^p(\mathbb{R},w))}$*.*

Proof. Assume first that $\mathbf{m}^\vee = K \in L^1(\mathbb{R})$ with compact support. Denote by $N = \|B_K\|_{L^p(w)\to L^p(w)}$. Let R be the representation of $\mathbb{R}$ in $L^p(\mathbb{T})$ given in (5.4.13). Define

$$V(\theta, x) = \mathcal{T}(v)_\theta(x)u(x).$$

Since translations and convolution commutes, it follows that for every $\theta \in [0, 1)$,

$$\|B_K f\|_{L^p(w(\cdot+\theta))} \le N \|f\|_{L^p(w(\cdot+\theta))}.$$

By (5.4.14) it follows that

$$\|B_K f\|_{L^p(V(\theta,\cdot))} \le \rho^{2/p} N \|f\|_{L^p(V(\theta,\cdot))},$$

where ρ is the constant appearing in (5.4.11) with $w = v$. Then we can apply Theorem 5.4.4 with $p = q$ and the admissible pair (v, v) to deduce that, for $f \in L^p(\mathbb{T}) \cap L^p(u)$,

$$\|T_K f\|_{L^p(\mathbb{T},u)} \leq \rho^{2/p} N \|f\|_{L^p(\mathbb{T},u)} .$$

Observe that the transferred operator coincides with the multiplier operator given by $\mathbf{m}|_{\mathbb{Z}}$. Hence the result follows by the density of $L^p(\mathbb{T}) \cap L^p(\mathbb{T}, u)$ in $L^p(\mathbb{T}, u)$.

In the general case, since $w \in A_p(\mathbb{R})$, we can apply the approximation techniques of §2.3. The proof finishes as the proof of Corollary 5.4.16 □

Appendix A

Representations and the transferred operator

A.1 Representations and the transferred operator

Let E be a Fréchet space, that is a locally convex topological vector space whose topology is described by a family of semi-norms $\{p_n\}_n$ (see [91]).

Definition A.1.1. *A homomorphism $u \mapsto R_u$ of G into the group of all topological automorphism of E is called a representation of G on E. That is,*

$$R_{uv} = R_u \circ R_v, \quad R_e = Id_E.$$

Moreover, it is called continuous if the map $(x, u) \mapsto R_u x$ of $G \times E$ into E is continuous.

Let, from now on R be a representation of G on E.

Proposition A.1.2. *R is continuous if and only if for every $x \in E$, $s \mapsto R_s x$ is a continuous map of G in E.*

Proof. Observe that, as for every $s \in G$, $R_s \in \mathfrak{B}(E)$, the condition that for every $x \in E$, $s \mapsto R_s x$ is a continuous map is equivalent to R being separately continuous. Moreover it suffices to prove that if R is separately continuous, R is continuous.

Fixed $x \in E$, since the map $s \to R_s x$ is continuous it holds that for any compact set $\mathcal{K} \subset G$, $\Gamma(x) = \{R_s x\}_{s \in \mathcal{K}}$ is a compact set of E, and in particular is bounded. Since E is an F-space, Banach-Steinhaus theorem holds (see [91, Thm. 2.6]), hence the family $\{R_s\}_{s \in K}$ is equicontinuous.

Given $(s_0, x_0) \in G \times E$ and given an open neighborhood Ω of $R_{s_0} x_0$, by the continuity of $s \mapsto R_s x_0$, there exists a relatively compact open neighborhood U of s_0, such that for all $s \in U$, $R_s x_0 \in \Omega$. Since the family $\{R_s\}_{s \in \overline{U}}$ is equicontinuous, there exists a neighborhood Γ of x_0 such that, for all $s \in U$ and $x \in \Gamma$, $R_s x \in \Omega$. ☐

Proposition A.1.3. [100, Chapter 4]. *Let $u \mapsto R_u$ be a homomorphism of G into the group of all topological automorphism of a Banach space E. The following conditions are equivalent:*

1. *The map $(x, u) \mapsto R_u x$ is a continuous map of $G \times E$ into E.*
2. *For each fixed $x \in E$, the map $u \mapsto R_u x$ is a continuous map of G into E.*
3. *For each fixed $x \in E$ and $x^* \in E^*$, the map $u \mapsto \langle x^*, R_u x\rangle$ is a continuous map of G into $\mathbb{C}$.*

Local convexity of E allows us to consider vector-valued integrals in a Pettis sense. Let μ be a Radon measure.

Definition A.1.4. *A function $F : X \to E$ is weakly integrable if the scalar functions $\Lambda \circ F$ are integrable with respect to μ, for every $\Lambda \in E^*$. In this case, if there exists a (unique) vector $x \in E$ such that*

$$\Lambda(x) = \int_G \Lambda \circ F \, d\mu$$

for every $\Lambda \in E^*$, then we say that F is integrable in the Pettis sense, x is the integral of F, and we write $\int_{\mathcal{M}} F d\mu = x$. Moreover, it satisfies that for every $T \in \mathfrak{B}(E)$, $Tx = \int_G T \circ F \, d\mu$.

Proposition A.1.5. *If μ is a Radon measure (real or complex) with compact support, and $F : G \to E$ is a continuous map, there exists $x = \int_G F \, d\mu$ and $x \in \|\mu\| A^\sharp$ where $A^\sharp$ is the compact set that denotes the closed balanced convex hull of $A = F(\operatorname{supp}\mu)$, that is $A^\sharp = \overline{\operatorname{co}\left(\cup_{|s|\le 1} sA\right)}^E$.*

Proof. The existence result can be found in [91, Thm. 3.27] for μ being a probability and in this case $x \in A' = \overline{\operatorname{co}A}$. Since any positive Borel measure μ can be expressed as $\|\mu\| \frac{\mu}{\|\mu\|}$, then we can ensure that $x \in \|\mu\| A' \subset \|\mu\| A^\sharp$. If μ is a real-valued Radon measure by the Jordan decomposition theorem $\mu = \mu^+ - \mu^-$, and then $x \in \|\mu\| \left(\frac{\|\mu^+\|}{\|\mu\|} A' + \frac{\|\mu^-\|}{\|\mu\|}(-A') \right) \subset \|\mu\| A^\sharp$. The proof for complex measures runs in the same way.

Since A is compact and E is Frèchet, there exists a balanced convex and relatively compact open set V such that $A \subset V$. Then $A^\sharp \subset \overline{V}$, so it follows that $A^\sharp$ is compact. ∎

Let R be a continuous representation of G on E. Given $\mu \in M_c(G)$, that is μ is a complex measure with compact support, for $f \in E$, the integral

$$T_\mu f = \int_{\operatorname{supp}\mu} R_{u^{-1}} f \, d\mu(u),$$

is well defined as a vector valued integral. Furthermore, if $A = \{R_{u^{-1}} f\}_{u \in \overline{\operatorname{supp}\nu}}$, $T_\mu f \in \|\mu\| A^\sharp$.

Proposition A.1.6. *Following the previous notations, for every $f, g \in E$, every μ,$\nu \in M_c(G)$ and every $\alpha \in \mathbb{C}$,*

1. $T_\mu(f+g) = T_\mu f + T_\mu g$, $T_\mu(\alpha f) = \alpha T_\mu f$;
2. $T_{\mu+\nu} f = T_\mu f + T_\nu f$, $T_{\alpha\mu} f = \alpha T_\mu f$.
3. *For any* $u \in G$, $R_u T_\nu f = \int_G R_{uv^{-1}} f \, d\nu(v)$.
4. $T_{\nu*\mu} = T_\nu \circ T_\mu$.

In particular, T_μ *defines a linear operator on* E.

Proof. We will prove the first assertion. The others are proved in a similar way. For all $\Lambda \in E^*$,

$$\begin{aligned}
\Lambda \circ T_\nu(f+g) &= \int_G \Lambda \circ R_{s^{-1}}(f+g) \, d\nu(s) \\
&= \int_G \left(\Lambda \circ R_{s^{-1}} f + \Lambda \circ R_{s^{-1}} g\right) \, d\nu(s) \\
&= \Lambda \circ T_\nu f + \Lambda \circ T_\nu g = \Lambda \circ \left(T_\nu f + T_\nu g\right),
\end{aligned}$$

from where, by uniqueness, the result follows. ☐

Proposition A.1.7. *Let $\nu \in M_c(G)$. For every $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$ and $n_1, \ldots, n_m \in \mathbb{N}$ such that, for every $f \in E$,*

$$p_n(T_\nu f) \lesssim \max(p_{n_1}(f), \ldots, p_{n_m}(f)). \tag{A.1.8}$$

Then, T_ν is a continuous linear operator defined on E. In the case that E is Banach, this can be read as

$$\|T_\nu f\|_E \leq c_\nu \|f\|_E, \tag{A.1.9}$$

where $c_\nu = \sup_{u \in \operatorname{supp} \nu} \|R_{u^{-1}}\|_{\mathfrak{B}(E)} < \infty$.

Proof. Since, fixed $f \in E$, the mapping $u \mapsto R_{u^{-1}} f$ is continuous and $\overline{V} = \operatorname{supp} \nu$ is compact, $A_f = \{R_{u^{-1}} f\}_{u \in \overline{V}}$ is a compact set of E, hence it is bounded. Then, by the uniform boundedness principle for F-spaces, $\{R_{u^{-1}}\}_{u \in \overline{V}}$ is equicontinuous. Then for every n, it exists $m \in \mathbb{N}$, $c > 0$ and $n_1, \ldots, n_m \in \mathbb{N}$ such that, for every $f \in E$

$$p_n(R_{u^{-1}} f) \leq c \max(p_{n_1}(f), \ldots, p_{n_m}(f)),$$

uniformly in $u \in \overline{V}$. Fixed $n \in \mathbb{N}$ and $f \in E$, since we know that $T_\nu f \in \|\nu\| A^\sharp$, where $A = \overline{\operatorname{co}(A_f)}$, by the continuity and properties of the seminorms, it follows that

$$p_n(T_\nu f) \leq c \|\nu\| \max(p_{n_1}(f), \ldots, p_{n_m}(f)).$$

In the case that E is Banach, we can take $c = \sup_{u \in \overline{V}} \|R_{u^{-1}}\|_{\mathfrak{B}(E)} < \infty$, and hence, using Minkowski's integral inequality,

$$\|T_\nu f\|_E \leq \sup_{u \in \overline{V}} \|R_{u^{-1}}\|_{\mathfrak{B}(E)} \|\nu\| \|f\|_E.$$

□

Observe that the transferred operator of the Definition 2.2.6 is the vector valued integral associated to an absolutely continuous Radon measure with respect to the left Haar's measure on the group with compact support, that is $K = \frac{d\mu}{d\lambda} \in L^1(G)$ with compact support. In particular, it is well defined and satisfies the above mentioned properties.

A.2 Pointwise meaning

We are interested in the case that the space E is a space of functions defined on a measure space $\mathcal{M}$, and we want to give a pointwise meaning of the transferred operator. Observe that if E is a Fréchet space such that, for all $x \in \mathcal{M}$, $\delta_x \in E^*$, by the properties of the vectorial integral, for every $K \in L^1(G)$ with compact support,

$$T_K f(x) = \delta_x \circ T_K f = \int \delta_x \circ R_{u^{-1}} f \; K(u) \; du = \int R_{u^{-1}} f(x) \; K(u), du$$

and, for every $v \in G$, $R_v T_K f(x) = \int R_{vu^{-1}} f(x) \; K(u) du$. This situation arises, for example, if $E = \mathcal{C}_0(\mathcal{M})$ when $\mathcal{M}$ is a locally compact Haussdorff space, or $E = \mathcal{S}(\mathbb{R}^n)$.

But this situation does not hold in general. In this section let E be a Banach space of function defined on $\mathcal{M}$, continuously embedded in $L^1_{\text{loc}}(\mathcal{M})$. That is, it holds that for all set of finite measure $\mathcal{M}_1$, there exists a constant $c_{\mathcal{M}_1}$ such that for all $f \in E$,

$$\int_{\mathcal{M}_1} |f| \leq c_{\mathcal{M}_1} \|f\|_E \, .$$

Observe that this implies that $\chi_{\mathcal{M}_1} \in E^*$.

Proposition A.2.1. *Let H be a jointly measurable function defined in $G \times \mathcal{M}$ such that $u \mapsto H(u,\cdot)$ continuously maps G into E such that for every compact set $U \subset G$*

$$\sup_{u \in U} \|H(u,\cdot)\|_E < +\infty.$$

Then, for $K \in L^1(G)$ with compact support, if in a vectorial sense

$$F = \int_G K(u) H(u,\cdot) \, du,$$

then μ-a.e. $x \in \mathcal{M}$, $F(x) = \int_G K(u)H(u,x) \, du$.

Proof. Let us prove the first assertion. By Proposition A.1.5, F is well defined in a vectorial sense. On the other hand, by Tonelli's Theorem, the mapping $x \mapsto \int_G |H(u,x)| \, |K(u)| \, du$, is measurable, and by Minkowski's integral inequality,

$$\left\| \int_G |H(u,x)| \, |K(u)| du \right\|_E \leq \sup_{u \in \operatorname{supp} K} \|H(u,\cdot)\|_E \, \|K\|_{L^1(G)} < \infty,$$

Thus, fixed a set of finite measure $\mathcal{M}_1$,

$$\begin{aligned}\int_{\mathcal{M}_1\times G} |H(u,x)|\,|K(u)|\,d(\lambda\times\mu)(u,x) &= \int_{\mathcal{M}_1}\int_G |H(u,x)|\,|K(u)|\,dud\mu(x)\\ &\leq c_{\mathcal{M}_1}\left\|\int_G |H(u,x)|\,|K(u)|du\right\|_E < +\infty.\end{aligned}$$

Then, by Fubini's Theorem, $x\mapsto \int_G H(u,x)K(u)\,du$, is $\mu|_{\mathcal{M}_1}$-measurable where $\mu|_{\mathcal{M}_1}$ is the measure μ restricted to $\mathcal{M}_1$. By the σ-finiteness of $\mathcal{M}$, it easily follows that, $x\mapsto\int_G H(u,x)\,K(u)du$, is measurable and locally integrable. Since for every set of finite measure $\mathcal{M}_1$, $\chi_{\mathcal{M}_1}\in E^*$, by Fubini's theorem

$$\int_{\mathcal{M}_1}\int_G H(u,x)K(u)\,dud\mu(x) = \int_G <\chi_{\mathcal{M}_1}, H(u,\cdot)> K(u)\,du =<\chi_{\mathcal{M}_1}, F> .$$

Hence $F(x)=\int_G H(u,x)K(u)\,du$, μ-a.e. $x\in\mathcal{M}$. □

With a similar argument the following statement is proved.

Proposition A.2.2. *Let E be a Banach space of function defined on $\mathcal{M}$, continuously embedded in $L^1_{\rm loc}(\mathcal{M})$. Let H be a jointly measurable function defined in $G\times G\times\mathcal{M}$ such that, for every v, $u\mapsto H(u,v,\cdot)$ continuously maps G in E and that for any compact sets $U,V\subset G$*

$$\sup_{v\in V}\sup_{u\in U}\|H(u,v,\cdot)\|_E < +\infty.$$

Then, for $K\in L^1(G)$ with compact support, if in a vectorial sense $F(v,\cdot) = \int_G K(u)H(u,v,\cdot)\,du$, then $(\lambda\times\mu)$-a.e. $(v,x)\in G\times\mathcal{M}$,

$$F(v,x)=\int_G K(u)H(u,v,x)\,du.$$

Corollary A.2.3. *Let R be a strongly continuous representation of G in E such that, for every $f\in E$, $(x,u)\mapsto R_u f(x)$ is jointly measurable. For every $K\in L^1(G)$ with compact support μ-a.e. x,*

$$T_K f(x) = \int_G K(u)R_{u^{-1}}f(x)\,du.$$

Furthermore, given a non empty σ-compact set V, $(\mu\times\lambda)$-a.e. $(x,u)\in\mathcal{M}\times V$,

$$\chi_V(v)R_vT_Kf(x) = \chi_V(v)B_K(\chi_{V\mathcal{K}^{-1}}R.f(x))(v), \tag{A.2.4}$$

where B_K is the operator given by $B_Kg(v)=\int K(u)g(vu^{-1})\,du$ and $\mathcal{K}=\operatorname{supp}K$.

A.3 Remark on joint measurability

Recall that we have implicitly assumed that our representations satisfy a jointly measurability condition. In [22] it is shown, in the L^p setting how, assuming strongly continuity and uniform boundedness of the representation, joint measurability assumption (as well as σ-finiteness of the measure space $\mathcal{M}$) can be dropped in the sense that there exists a jointly measurable version of the function $(u,x) \mapsto R_u f(x)$. In this section we will discuss a sort of analog result for representations acting on a general BFS. In this section F is a BFS defined on $\mathcal{M}$ and R denotes a strongly continuous representation of G acting on F.

Lemma A.3.1. *Given a compact set $\mathcal{K} \subset G$, there exists a jointly measurable function H such that, a.e. $u \in \mathcal{K}$ $R_u f \equiv H(u,.)$.*

Proof. Let us fix $f \in F$. By the continuity of the representation,

$$J = \{R_s f\}_{s \in \mathcal{K}},$$

is a compact set of F. Thus, since F is an F-space, there exists $\{f_n\}_{n \in I \subset \mathbb{N}} \subset J$ that is dense in J. For all $n \in I$, we define $g_n(u) = R_u f - f_n$, that, is a continuous mapping of G on F. Then the mapping $u \mapsto \|g_n(u)\|_F$, is continuous. Fixed $m \geq 1$, let us consider, for all $n \in I$, the open sets

$$A_n^m = \left\{ u \in G : \ \|g_n(u)\|_F < \frac{1}{m} \right\}.$$

By the density of $\{f_n\}$ in J, for all $u \in \mathcal{K}$, there exists $n_0 \in I$ such that

$$\|R_u f - f_{n_0}\|_F = \|g_{n_0}\|_F < \frac{1}{m}.$$

Thus, $\mathcal{K} \subset \bigcup_{n \in I} A_n^m$. Therefore, there exists $n_m \in I$, such that $\mathcal{K} \subset \bigcup_{n=1}^{n_m} A_n^m$. Hence, we take

$$\begin{aligned} h_1^m(u,x) &= \chi_{A_1^m}(u) f_1(x); \\ h_2^m(u,x) &= \chi_{A_2^m \setminus A_1^m}(u) f_2(x); \\ &\vdots \\ h_{n_m}^m(u,x) &= \chi_{A_{n_m}^m \setminus \bigcup_{i=1}^{n_m - 1} A_i^m}(u) f_{n_m}(x), \end{aligned}$$

and we define

$$h^m(u,x) = \left(\sum_{i=1}^{n_m} h_i^m(u,x) \right) \chi_{\mathcal{K}}(u),$$

that, by construction is jointly measurable. It satisfies that for all $u \in \mathcal{K}$, exists only one index j such that $u \in A_j^m \setminus \bigcup_{i=1}^{j-1} A_i^m$ and then,

$$\|h^m(u,.) - R_u f(.)\|_F = \|f_j - R_u f\|_F = \|g_j\|_F < \frac{1}{m}.$$

Therefore, $\sup_{u \in \mathcal{K}} \|h^m(u,.) - R_u f\|_F \leq \frac{1}{m}$, and

$$\sup_{u \in \mathcal{K}} \|h^m(u,.) - h^n(u,.)\|_F \leq \frac{1}{m} + \frac{1}{n}.$$

Let us fix a finite measure set $E \subset \mathcal{M}$ and $\varepsilon > 0$. Then

$$
\begin{aligned}
&(\lambda \times \mu)\left\{(u,x) \in \mathcal{K} \times E : \ |h^m(u,.) - h^n(u,.)| > \varepsilon\right\} \\
&\quad \le \frac{1}{\varepsilon}\int_{\mathcal{K}}\int_E |h^n(u,x) - h^m(u,x)|\, d\mu(x)du \le \frac{C_E}{\varepsilon}\int_{\mathcal{K}} \|h^n(u,x) - h^m(u,x)\|_F\, du \\
&\quad \le \frac{C_E\,|\mathcal{K}|}{\varepsilon}\left(\frac{1}{m} + \frac{1}{n}\right).
\end{aligned}
$$

In other words, $\{h^n\}_n$ is a Cauchy sequence in the space of measurable functions on $\mathcal{K} \times E$. Then there exists a jointly measurable function H^E such that $\{h^n\}$ converges in measure on $\mathcal{K} \times E$ to H^E, and a subsequence $\{h^{n_j}\}_j$ that converges a.e. $(u,x) \in \mathcal{K} \times E$. In particular, there exists a null set $Z_E \subset \mathcal{K}$, such that for all $u \notin Z_E$, $h^{n_j}(u,x)$ converges to $H^E(u,x)$ a.e. $x \in E$. Since E has finite measure, for all $u \notin Z_E$,

$$h^{n_j}(u,.) \overset{\nu_E}{\to} H^E(u,.).$$

Then, for all $\varepsilon > 0$, and for all $u \notin Z_E$,

$$
\begin{aligned}
\mu\left\{x \in E : \left|R_u f(x) - H^E(u,x)\right| > \varepsilon\right\} &\le \mu\left\{x \in E : |R_u f(x) - h^{n_j}(u,x)| > \varepsilon/2\right\} \\
&\quad + \mu\left\{x \in E : \left|H^E(u,x) - h^{n_j}(u,x)\right| > \varepsilon/2\right\} \\
\le \frac{2C_E}{\varepsilon}\left\|R_u f - h^{n_j}(u,.)\right\|_F + \mu\left\{x \in E : \ \left|H^E(u,x) - h^{n_j}(u,x)\right| > \varepsilon/2\right\},
\end{aligned}
$$

that converges to 0 when n_j tends to infinity. Then, for all $u \notin Z_E$, $R_u f(x) = H^E(u,.)$ μ-a.e $x \in E$. Let $E_n \uparrow \mathcal{M}$ such that, for each n, E_n has finite measure and consider the zero measure subset of G, $Z = \cup_n Z_{E_n}$, and define

$$H(u,x) = \sum_n H^{E_n}(u,x)\chi_{E_n \setminus E_{n-1}}(x)\chi_{Z^c}(u).$$

It holds that for all $u \notin Z, R_u f \equiv H(u,.)$. □

Proposition A.3.2. *For all $f \in F$, there exists a jointly measurable function H_f such that, a.e. $u \in G$, $H_f(u,.) \equiv R_u f(.)$.*

Proof. Since G is σ-compact, there exists an increasing sequence of compact sets $\{\mathcal{K}_n\}_n$ such that $\mathcal{K}_n \uparrow G$. By the previous lemma, for all n, there exists a zero measure set $Z_n \subset \mathcal{K}_n$ and a jointly measurable function H_n satisfying that, for all $u \in \mathcal{K}_n \setminus Z_n$, $R_u f \equiv H_n(u,.)$. Taking $\mathcal{K}_0 = \emptyset$, we define

$$H_f(u,x) = \sum_{m \ge 1} H_m(u,x)\chi_{\mathcal{K}_m \setminus \mathcal{K}_{m-1}}(u).$$

By construction, H_f is jointly measurable and for all $u \notin \cup_{m\ge1} Z_m$, $R_u f \equiv H_f(u,.)$. □

Theorem A.3.3. *Let $K \in L^1(G)$ with compact support. There exists a jointly measurable function H_f such that:*

1. $H_f(u,.) \equiv R_u f(.)$, a.e. $u \in G$;

2. *for all* $v \in G$,

$$R_v T_K f(.) \equiv \int K(u) H_f(vu^{-1}, .)\, du. \tag{A.3.4}$$

Proof. By the previous proposition, there exists a jointly measurable function H_f, and a zero measure set $Z \subset \mathcal{K}$ such that, for all $u \in G \setminus Z$, $R_u f \equiv H_f(u, .)$. By the invariance of the measure, for all $v \in G$, vZ is a zero measure set. Thus, for all $g \in F'$ (the Köthe dual space of F) and all $v \in G$,

$$\begin{aligned}\left\langle R_v \int_G K(u) R_{u^{-1}} f du, g \right\rangle &= \int K(u) \left\langle R_{vu^{-1}} f, g \right\rangle \, du \\ = \int_{G \setminus vZ} K(u) \left\langle R_{vu^{-1}} f, g \right\rangle \, du &= \int_G K(u) \left\langle H_f(vu^{-1}, .), g \right\rangle \, du \\ = \int g(x) \int_{\mathcal{K}} K(u) H_f(vu^{-1}, x) du d\nu(x) &= \left\langle \int_{\mathcal{K}} K(u) H_f(vu^{-1}, .) du, g \right\rangle.\end{aligned}$$

Therefore, it holds that, for all $v \in G$,

$$R_v T_K f(.) \equiv \int_{\mathcal{K}} K(u) H_f(vu^{-1}, .)\, du.$$

□

Theorem A.3.3 gives us a way to assign to each function f a jointly measurable function H_f such that a.e. $u \in G$, $R_u f \equiv H_f(u, \cdot)$. Moreover if G_f is another such function, a.e. $u \in G$, $H_f(u, \cdot) \equiv G_f(u, \cdot)$, and thus, since they are jointly measurable, $H_f = G_f$ $(\lambda \times \mu)$-a.e. $(u, x) \in G \times \mathcal{M}$. Furthermore, if $f, g \in F$ and $f \equiv g$, then $H_f(u, x) = H_g(u, x)$, $(\lambda \times \mu)$-a.e. (u, x). Hence we have well defined a map from F to $L^0(\lambda \times \mu)$ such that, for $f, g \in F$, $\alpha \in \mathbb{C}$,

1. $H_{f+g}(v, x) = H_f(v, x) + H_g(v, x)$, $(\lambda \times \mu)$-a.e. (v, x);
2. $H_{\alpha f}(v, x) = \alpha H_f(v, x)$, $(\lambda \times \mu)$-a.e. (v, x);
3. $H_{T_K f}(v, x) = B_K\left(H_f(\cdot, x)\right)(v)$, $(\lambda \times \mu)$-a.e. (v, x).

Let us prove the first assertion, the others can be proved in a similar way. We know that $R_v(f+g)(\cdot) \equiv H_{f+g}(v, \cdot)$, $R_v f(\cdot) \equiv H_f(v, \cdot)$ and $R_v g(\cdot) \equiv H_g(v, \cdot)$ a.e. $v \in G$. Hence, since for every $v \in G$, $R_v(f+g) \equiv R_v f + R_v g$, it follows that $H_{f+g}(v, \cdot) \equiv H_f(v, \cdot) + H_g(v, \cdot)$ a.e. $v \in G$. But now, by the joint measurability and Tonelli's theorem,

$$\int_{G \times \mathcal{M}} \left| H_{f+g}(v, x) - H_f(v, x) - H_g(v, x) \right| d(\lambda \times \mu)(v, x) = 0,$$

from where the first equality follows.

This procedure allows us to consider the class of functions

$$\widetilde{W}(B, E, V) = \left\{ f \in F : \left\| \left\| \chi_V(v) H_f(v, x) \right\|_B \right\|_E < \infty \right\},$$

where B and E are QBFS defined on G and $\mathcal{M}$, respectively.

This new class can be used to prove results analogous to Theorem 3.1.4 and 3.1.22 without the joint measurability assumption. For instance, similarly as we proved Theorem 3.1.4 the following result is stated.

Theorem A.3.5. *Let $K \in L^1(G)$ with compact support such that $B_K : B \to C$ is bounded with norm less than or equal to $N_{B,C}(K)$. Let $\mathcal{K} = \operatorname{supp} K$. Given a non empty open set $V \subset G$, it holds that*

$$\|T_K f\|_{\widetilde{W}(C,E,V)} \leq N_{B,C}(K)\, \|f\|_{\widetilde{W}(B,E,V\mathcal{K}^{-1})}\,.$$

Then we can recover those results where we have been able to identify the corresponding amalgams without the jointly measurability assumption. For example, if we take $F = E = L^p(\mathcal{M})$, $B = L^p(G)$, for $1 \leq p < \infty$, and R is a continuous representation of G on F, such that $c = \sup_{u\in G} \|R_u\|_{\mathfrak{B}(L^p(\mathcal{M}))} < +\infty$, for any relatively compact open set V,

$$\|f\|_{L^p(\mathcal{M})} \frac{|V|^{1/p}}{c} \leq \|f\|_{\widetilde{W}(L^p(\mathcal{M}),L^p(G),V)} \leq \|f\|_{L^p(\mathcal{M})}\, c|V|^{1/p}. \tag{A.3.6}$$

To see this observe that, fixed $f \in F$, since a.e. $u \in G$, $R_u f \equiv H_f(u,\cdot)$, by Tonelli's Theorem ($H_f$ is jointly measurable),

$$\begin{aligned} \|f\|^p_{\widetilde{W}(B,E,F)} &= \int_{\mathcal{M}}\int_V |H_f(u,x)|^p\, du d\mu(x) = \int_V\int_{\mathcal{M}} |H_f(u,x)|^p\, d\mu(x) du \\ &= \int_V \|R_u f\|^p_{L^p(\mathcal{M})}\, du, \end{aligned}$$

from where (A.3.6) follows and hence, $\widetilde{W}(B,E,F) = L^p(\mathcal{M})$. Hence, by the previous theorem, the same conclusion as Corollary 3.1.6 without the jointly measurability assumption on R.

Appendix B

On $\left(L^1, L^{1,q}\right)$ $1 < q \leq \infty$ multipliers

Throughout this chapter μ,ν denote a pair of Radon measures on G. Here we study the validity of inequalities of type (2.3.14) for multipliers in the class $M(L^1(\mu), L^{1,q}(\nu))$, focusing our attention on the range on indices $1 < q \leq \infty$ where $L^{1,q}$ is not a Banach space, hence Minkowski's inequality fails. This kind of multipliers where studied in [1, 88] for the particular case $q = +\infty$ and λ and μ being the Haar measure.

In the second part of this section, we establish the main results following the spirit of the proof of A. Raposo in [88] for the case $q = +\infty$, where this lack of convexity on the space $L^{1,q}$ is compensated by a discretization technique and a linearization procedure in order to estimate a vectorial inequality. In [1, 88] this last vectorial estimation uses Khintchine's inequality. In the first part, we prove a Marcinkiewicz-Zygmund vectorial type inequality, without using Khintchine's, that allow us to obtain better constants with respect to that obtained in [1, 88].

B.1 Vectorial inequalities

The following result is easy consequence of the homogeneity, so we omit its proof.

Lemma B.1.1. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $S_n = \{y \in \mathbb{K}^n : |y| = 1\}$, v_n be its surface area, $x \in \mathbb{R}^n$ and $0 < p < \infty$. Then,*

$$\|x\|_2 = \frac{1}{\mathfrak{c}_{n,p}} \left(\int_{S_n} |x \cdot y'|^p \, \frac{dy'}{v_n} \right)^{1/p}, \tag{B.1.2}$$

where $\mathbf{1} = (1, 0, \ldots, 0)$ and $\mathfrak{c}_{n,p}^p = \int_{S_n} |\mathbf{1} \cdot y'|^p \, \frac{dy'}{v_n}$.

Let us observe that, if $\mathbb{K} = \mathbb{R}$, $S_n \simeq \Sigma_{n-1}$ and $v_n = \omega_{n-1}$ and, if $\mathbb{K} = \mathbb{C}$, $S_n \simeq \Sigma_{2n-1}$ and $v_n = \omega_{2n-1}$. For our purpose, we need to compute the exact value of $\mathfrak{c}_{n,p}$. Parameterizing S_n and integrating we obtain

$$\mathfrak{c}_{n,p} = \left(\frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\, \Gamma\left(\frac{p+n}{2}\right)} \right)^{1/p},$$

in the real case and

$$\mathfrak{c}_{n,p} = \left(\frac{\Gamma\left(\frac{p}{2}+1\right)\Gamma(n)}{\Gamma\left(\frac{p}{2}+n\right)} \right)^{1/p},$$

in the complex case. For any p, r, we denote by $d_{p,r} := \lim_{n\to\infty} \frac{\mathfrak{c}_{n,p}}{\mathfrak{c}_{n,r}}$. Using Stirling's Formula, it is easy to see that,

$$d_{p,r} = \pi^{\frac{1}{2r}-\frac{1}{2p}} \frac{\Gamma\left(\frac{1+p}{2}\right)^{\frac{1}{p}}}{\Gamma\left(\frac{1+r}{2}\right)^{\frac{1}{r}}}. \tag{B.1.3}$$

if $\mathbb{K} = \mathbb{R}$, and

$$d_{p,r} := \frac{\Gamma\left(\frac{p}{2}+1\right)^{\frac{1}{p}}}{\Gamma\left(\frac{r}{2}+1\right)^{\frac{1}{r}}}. \tag{B.1.4}$$

if $\mathbb{K} = \mathbb{C}$. Observe that $d_{p,p} = 1$.

Theorem B.1.5. *Let $0 < p < \infty$ and let $r \leq \min(p,1)$ and let $S \subseteq L^p(\mu)$. Assume that T is a linear operator and that there exists a finite constant $\|T\|$ such that for every $f \in S$, $\|Tf\|_{B^r} \leq \|T\| \, \|f\|_{L^p(\mu)}$, where B^r is the r-convexification of a BFS B. Then*

$$\left\| \left(\sum |Tf_i|^2 \right)^{1/2} \right\|_{B^r} \leq d_{p,r} \|T\| \left\| \left(\sum |f_i|^2 \right)^{1/2} \right\|_{L^p(\mu)},$$

where $(f_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$, and $d_{p,r}$ is the constant appearing on (B.1.3) if T maps real valued functions on real valued functions, and it is the constant appearing on (B.1.4) in the general case.

Proof. Fix $n \geq 1$. Then, by the previous lemma,

$$F_n = \left(\sum_{i=1}^n |Tf_i|^2 \right)^{1/2} = \mathfrak{c}_{n,r}^{-1} \left(\int_{S_n} \left| T\left(\sum_{i=1}^n y_i' f_i \right) \right|^r \frac{dy'}{v_n} \right)^{1/r}$$

Then

$$\begin{aligned} \|F_n\|_{B^r} &= \mathfrak{c}_{n,r}^{-1} \left\| \int_{S_n} \left| T\left(\sum_{i=1}^n y_i' f_i \right) \right|^r \frac{dy'}{v_n} \right\|_B^{1/r} \\ &\leq \mathfrak{c}_{n,r}^{-1} \left\{ \int_{S_n} \left\| T\left(\sum_{i=1}^n y_i' f_i \right) \right\|_{B^r}^r \frac{dy'}{v_n} \right\}^{1/r} \\ &\leq \|T\| \, \mathfrak{c}_{n,r}^{-1} \left\{ \int_{S_n} \left\| \sum_{i=1}^n y_i' f_i \right\|_{L^p}^r \frac{dy'}{v_n} \right\}^{1/r}. \end{aligned}$$

Since $r \leq p$, by the finiteness of the normalized measure on S_n, the last term is bounded by

$$\left\{\int_{S_n} \left\| \sum_{i=1}^n y_i' f_i \right\|_p^p \frac{dy'}{v_n}\right\}^{1/p} = \mathfrak{c}_{n,p} \left(\int \left(\sum_{i=1}^n |f_i|^2 \right)^{p/2} d\mu \right)^{1/p}$$

where the equality follows from the previous lemma and Tonelli's theorem. Then, putting all together, and increasing the last term

$$\|F_n\|_{B^r} \leq \|T\| \frac{\mathfrak{c}_{n,p}}{\mathfrak{c}_{n,r}} \left\| \left(\sum |f_i|^2 \right)^{1/2} \right\|_{L^p(\mu)}.$$

Hence, taking limit on n when $n \to \infty$, by Fatou's lemma,

$$\left\| \left(\sum |Tf_i|^2 \right)^{1/2} \right\|_{B^r} \leq \|T\| \, d_{p,r} \left\| \left(\sum |f_i|^2 \right)^{1/2} \right\|_{L^p(\mu)}.$$

□

Corollary B.1.6. *Let $0 < p < \infty$, $0 < q \leq +\infty$ and $S \subseteq L^p(\mu)$. Assume that T is a linear operator and that there exists a finite constant $\|T\|$ such that for every $f \in S$,*

$$\|Tf\|_{L^{p,q}(\nu)} \leq \|T\| \, \|f\|_{L^p(\mu)},$$

Then

$$\left\| \left(\sum |Tf_i|^2 \right)^{1/2} \right\|_{L^{p,q}(\nu)} \leq c_{p,q} \|T\| \left\| \left(\sum |f_i|^2 \right)^{1/2} \right\|_{L^p(\mu)},$$

where $(f_i)_{i\in\mathbb{N}} \in S^{\mathbb{N}}$, and

$$c_{p,q} = \inf_{0<r<\min(p,q,1)} \left(\frac{p}{p-r} \right)^{1/r} d_{p,r}, \tag{B.1.7}$$

where $d_{p,r}$ denotes the constant appearing on (B.1.3) if T maps real valued functions on real valued functions, and it is the constant appearing on (B.1.4) elsewhere.

Proof. Let $0 < r < \min(p, 1)$, $r \leq q$, and consider $B = L^{\frac{p}{r},\frac{q}{r}}(\nu)$, that is a BFS endowed with a norm $\|\cdot\|_B$ (defined in terms of f^{**}) satisfying (see [98, Thm V.3.21 and V.3.22] or [18, Lemma IV.4.5 and Theorem IV.4.6])

$$\|f\|_{L^{\frac{p}{r},\frac{q}{r}}(\nu)} \leq \|f\|_B \leq \frac{p}{p-r} \|f\|_{L^{\frac{p}{r},\frac{q}{r}}(\nu)}.$$

Then

$$\|f\|_{L^{p,q}(\nu)} = \||f|^r\|_{L^{\frac{p}{r},\frac{q}{r}}(\nu)}^{1/r} \leq \||f|^r\|_B^{1/r} \leq \left(\frac{p}{p-r} \right)^{1/r} \|f\|_{L^{p,q}(\nu)}.$$

Hence $L^{p,q}(\nu) = B^r$ and $\|Tf\|_{B^r} \leq \left(\frac{p}{p-r}\right)^{1/r} \|T\| \, \|f\|_{L^p(\mu)}$. So we can apply the previous theorem to obtain that, for $(f_i) \in S^{\mathbb{N}}$,

$$\left\| \left(\sum |Tf_i|^2 \right)^{1/2} \right\|_{L^{p,q}(\nu)} \leq d_{p,r} \left(\frac{p}{p-r} \right)^{1/r} \|T\| \left\| \left(\sum |f_i|^2 \right)^{1/2} \right\|_{L^p(\mu)} .$$

So the result follows by taking the infimum over r. □

Observe that if in the previous theorem we take $q = +\infty$, we recover the diagonal case of Marcinkiewicz-Zygmund weak type inequality [60, Theorem V.2.9].

Theorem B.1.8. *Let $0 < p < \infty$, $r \leq \min(p, 1)$ and $S \subseteq L^p(\nu)$. Assume that $\{T_j\}$ is a family of linear operators defined on $S \subset L^p(\nu)$, such that there exists a constant $\|T\|$ that for every $f \in S$, $\left\| \sup_j |T_j f| \right\|_{B^r} \leq \|T\| \, \|f\|_{L^p(\mu)}$, where B^r is the r-convexification of a BFS B. Then, for $\{f_i\}_i \in S^{\mathbb{N}}$,*

$$\left\| \sup_j \left(\sum |T_j f_i|^2 \right)^{1/2} \right\|_{B^r} \leq d_{p,r} \|T\| \left\| \left(\sum |f_i|^2 \right)^{1/2} \right\|_{L^p(\mu)} .$$

Proof. Fixed $n \geq 1$, and any set E such that $\nu(E) < \infty$, then for $r < p$

$$\begin{aligned} & \left\| \sup_j \left(\sum_{i=1}^n |T_j f_i|^2 \right)^{1/2} \right\|_{B^r} \\ &= \mathfrak{c}_{n,r}^{-1} \left\| \sup_j \int_{S_n} \left| T_j \left(\sum_{i=1}^n y_i' f_i \right) \right|^r \frac{dy'}{v_n} \right\|_B^{1/r} \\ &= \mathfrak{c}_{n,r}^{-1} \left\| \int_{S_n} \sup_j \left| T_j \left(\sum_{i=1}^n y_i' f_i \right) \right|^r \frac{dy'}{v_n} \right\|_B^{1/r} \\ &\leq \mathfrak{c}_{n,r}^{-1} \left(\int_{S_n} \left\| \sup_j \left| T_j \left(\sum_{i=1}^n y_i' f_i \right) \right| \right\|_{B^r}^r \frac{dy'}{v_n} \right)^{1/r} \\ &\leq \frac{\|T\|}{\mathfrak{c}_{n,r}} \left\{ \int_{S_n} \left\| \sum_{i=1}^n y_i' f_i \right\|_p^r \frac{dy'}{v_n} \right\}^{1/r} . \end{aligned}$$

Now the proof finishes in the same way as Theorem B.1.5 does. □

Slightly modifications of the proof of Corollary B.1.6 allows to prove the following results.

Corollary B.1.9. *Let $0 < p < \infty$, $0 < q \leq +\infty$ and $S \subseteq L^p(\nu)$. Assume that $\{T_j\}$ is a family of linear operators defined on $S \subset L^p(\nu)$, such that there exists a constant $\|T\|$ that for every $f \in S$, $\left\| \sup_j |T_j f| \right\|_{L^{p,q}(\nu)} \leq \|T\| \, \|f\|_{L^p(\mu)}$. Then*

$$\left\| \sup_j \left(\sum |T_j f_i|^2 \right)^{1/2} \right\|_{L^{p,q}(\nu)} \leq c_{p,q} \|T\| \left\| \left(\sum |f_i|^2 \right)^{1/2} \right\|_{L^p(\mu)} ,$$

where $(f_i)_{i \in \mathbb{N}} \in S^{\mathbb{N}}$, $c_{p,q}$ is the constant appearing on (B.1.7).

B.2 Turning back to multipliers

In this section we will prove an inequality of the type (2.3.14) for multipliers in $M(L^1(\mu), L^{1,q}(\nu))$ for $1 < q \leq \infty$. Despite we are more interested in the case $q = +\infty$, we will prove the result for the entire range in order to illustrate how the vectorial results on the previous section can be used.

With minor modifications on the statement, the following key lemma, which will allow us to pass from a continuous context to a discrete setting, is proved in [88, Lemma 2.6].

Lemma B.2.1. *Let μ be a finite measure on Γ supported on a compact set $\mathcal{K}$ and let $f \in SL^1(G)$. Let $\mathbf{m}_j$ be a family of $L^\infty(\Gamma)$ functions. Let $j = 1, \ldots, J$, and $u \in G$, let*

$$F_{j,u}(\gamma) = T_{\mathbf{m}_j}(\overline{\gamma} f)(u).$$

Then, for each $n = 1, 2, \ldots$ there exists a finite family $\{V_i^n\}_{i=1}^{I_n}$ of pairwise disjoint measurable sets in Γ such that

1. *$\mathcal{K} \subset \uplus_{i=1}^{I_n} V_i^n$,*
2. *if $i = 1, \ldots, I_n$ and $\gamma_1, \gamma_2 \in V_i^n$ then*

$$|F_{j,u}(\gamma_1) - F_{j,u}(\gamma_2)| \leq 1/n$$

for $j = 1, \ldots, J$ and $u \in G$.

Theorem B.2.2. *Let $1 < q \leq \infty$. If $\mathbf{m} \in L^\infty(\Gamma) \cap M(L^1(\mu), L^{1,q}(\nu))$, for every finite measure λ on Γ, $\lambda * \mathbf{m} \in M(L^1(\mu), L^{1,q}(\nu))$ and*

$$\|\lambda * \mathbf{m}\|_{M(L^1(\mu), L^{1,q}(\nu))} \leq c_{1,q} \|\lambda\| \|\mathbf{m}\|_{M(L^1(\mu), L^{1,q}(\nu))},$$

where $\|\lambda\|$ is the total variation of λ.

Proof. Assume first that λ is supported on a compact set $\mathcal{K}$. Fix $f \in SL^1(G)$. For every $n \geq 1$, let V_i^n be the sets given by Lemma B.2.1 and for each i pick $\gamma_i^n \in V_i^n$. Then, for every $\gamma \in \mathcal{K}$, and any $n \geq 1$, there exists an unique set $V_{i_{n,\gamma}}^n$ containing γ such that for every $u \in G$,

$$\left| T_{\mathbf{m}}(\overline{\gamma_{i_{n,\gamma}}^n} f)(u) - T_{\mathbf{m}}(\overline{\gamma} f)(u) \right| \leq 1/n.$$

Hence, $\lim_n T_{\mathbf{m}}(\overline{\gamma_{i_{n,\gamma}}^n} f)(u) = T_{\mathbf{m}}(\overline{\gamma} f)(u)$ uniformly in $u \in G$. Therefore, for every $u \in G$ and $\gamma \in \mathcal{K}$

$$\lim_n \sum_{i=1}^{I_n} T_{\mathbf{m}}(\overline{\gamma_i^n} f)(u) \chi_{V_i^n}(\gamma) = T_{\mathbf{m}}(\overline{\gamma} f)(u).$$

Let $\lambda_i^n = \int_\Gamma \chi_{V_i^n}(\gamma)\, d|\lambda|(\gamma)$ and observe that, $\sum_{i=1}^{I_n} \lambda_i^n = \|\lambda\|$. Since

$$|T_{\lambda * \mathbf{m}}(u)| \leq \int_{\mathcal{K}} |T_{\mathbf{m}}(\overline{\gamma} f)(u)|\, d|\lambda|(\gamma),$$

by Fatou's lemma

$$
\begin{aligned}
|T_{\lambda * \mathbf{m}} f(u)| &\leq \liminf_n \sum_{i=1}^{I_n} \lambda_i^n \left| T_{\mathbf{m}}(\overline{\gamma_i^n} f)(u) \right| \\
&\leq \|\lambda\|^{1/2} \liminf_n \left(\sum_{i=1}^{I_n} \left| T_{\mathbf{m}} \left(|\lambda_i^n|^{1/2} f \overline{\gamma_i}(u) \right) \right|^2 \right)^{1/2} . \qquad \text{(B.2.3)}
\end{aligned}
$$

Hence, by Corollary B.1.6, the lattice property of $L^{1,q}(\nu)$ and Fatou's lemma

$$
\begin{aligned}
\|T_{\lambda * \mathbf{m}} f\|_{L^{1,q}(\nu)} &\leq \|\lambda\|^{1/2} c_{1,q} \|\mathbf{m}\| \liminf \left\| \left(\sum_{i=1}^{I_n} \left| |\lambda_i|^{1/2} \overline{\gamma_i^n}(u) f(u) \right|^2 \right)^{1/2} \right\|_{L^1(\mu)} \\
&\leq \|\lambda\| \, c_{1,q} \|\mathbf{m}\| \, \|f\|_{L^1(\mu)} .
\end{aligned}
$$

In the case that λ is not compactly supported, let $\mathcal{K}_n \uparrow \Gamma$ be a sequence of compact sets. Then

$$
|T_{\lambda * \mathbf{m}} f(u)| \leq \int_\Gamma |T_{\mathbf{m}}(f\overline{\gamma})| \, d\,|\lambda| = \lim_n \int_{\mathcal{K}_n} |T_{\mathbf{m}}(f\overline{\gamma})| \, d\,|\lambda| .
$$

Thus, by monotone convergence and the previous result for the measures defined by $\lambda_n(E) = |\lambda| (E \cap \mathcal{K}_n)$,

$$
\|T_{\lambda * \mathbf{m}} f\|_{L^{1,q}(\nu)} \leq \lim_n c_{1,q} \|\lambda_n\| \, \|f\|_{L^1(\mu)} = c_{1,q} \|\lambda\| \, \|f\|_{L^1(\mu)} .
$$

In any case, the result follows by the density of $SL^1 \cap L^1(\mu)$ in $L^1(\mu)$. ☐

Theorem B.2.4. *Let $1 < q \leq \infty$. Let $\{\mathbf{m}_j\}$ be a family of $L^\infty(\Gamma)$ functions such that, for $f \in SL^1(G) \cap L^1(\mu)$,*

$$
\left\| \sup_j \left| T_{\mathbf{m}_j} f \right| \right\|_{L^{1,q}(\nu)} \leq \|\{\mathbf{m}_j\}\|_{M(L^1(\mu), L^{1,q}(\nu))} \|f\|_{L^1(\mu)} .
$$

Then, if λ is a finite measure on Γ, with total variation $\|\lambda\|$, for $f \in S$,

$$
\left\| \sup_j \left| T_{\lambda * \mathbf{m}_j} f \right| \right\|_{L^{1,q}(\nu)} \leq c_{1,q} \|\{\mathbf{m}_j\}\|_{M(L^1(\mu), L^{1,q}(\nu))} \|\lambda\| \, \|f\|_{L^1(\mu)} .
$$

Proof. We can assume without loss of generality that we have a finite family $\{\mathbf{m}_j\}_{j=1,\ldots,J}$, and that λ is supported on a compact set $\mathcal{K}$. Fix $f \in SL^1(G)$. For every $n \geq 1$, let V_i^n be the sets given by Lemma B.2.1 and for each i pick $\gamma_i^n \in V_i^n$. Arguing as in the proof of the previous theorem, it can be shown that, for every $u \in G$, $j \in \{1, \ldots, J\}$, and any $\gamma \in \mathcal{K}$,

$$
\lim_n \sum_{i=1}^{I_n} T_{\mathbf{m}_j}(\overline{\gamma_i^n} f)(u) \chi_{V_i^n}(\gamma) = T_{\mathbf{m}_j}(\overline{\gamma} f)(u).
$$

Hence, for all $u \in G$ and every $j_0 \in \{1, \ldots, J\}$,

$$\left|T_{\lambda * \mathbf{m}_{j_0}} f(u)\right| \leq \|\lambda\|^{1/2} \liminf_n \left\{ \sup_{j=1,\ldots,J} \left(\sum_{i=1}^{I_n} \left| T_{\mathbf{m}_j} \left(|\lambda_i^n|^{1/2} f \overline{\gamma_i}(u) \right) \right|^2 \right)^{1/2} \right\}.$$

Since by Corollary B.1.9, for any $n \geq 1$,

$$\begin{aligned} & \left\| \sup_{j=1,\ldots,J} \left(\sum_{i=1}^{I_n} \left| T_{\mathbf{m}_j} \left(|\lambda_i^n|^{1/2} \overline{\gamma_i^n} f \right) \right|^2 \right)^{1/2} \right\|_{L^{1,q}(\nu)} \\ \leq \; & c_{1,q} \left\| \{\mathbf{m}_j\} \right\|_{M(L^1(\mu), L^{1,q}(\nu))} \left\| \left(\sum_{i=1}^{I_n} \left| |\lambda_i^n|^{1/2} \overline{\gamma_i^n} f \right|^2 \right)^{1/2} \right\|_{L^1(\mu)} \\ = \; & c_{1,q} \|\lambda\|^{1/2} \|f\|_{L^1(\mu)}, \end{aligned}$$

it follows by Fatou's lemma and the lattice properties of $L^{1,q}(\nu)$ that

$$\left\| \sup_{j=1,\ldots,J} \left| T_{\lambda * \mathbf{m}_j} f \right| \right\|_{L^{1,q}(\nu)} \leq \|\lambda\| \, c_{1,q} \left\| \{\mathbf{m}_j\} \right\|_{M(L^1(\mu), L^{1,q}(\nu))} \|f\|_{L^1(\mu)},$$

from where the result follows by the density of $SL^1 \cap L^1(\mu)$ in $L^1(\mu)$. □

In the particular case where μ and ν are absolutely continuous with respect to the Haar's measure, we obtain that propositions 2.3.20 and 2.3.21 hold. In the case that both measures coincide with Haar's, we recover the following known result, proved in [88] and [14].

Corollary B.2.5. *Suppose that* $\{\mathbf{m}_j\}_j \subset L^\infty(\Gamma) \cap M(L^1(G), L^{1,\infty}(G))$ *and* $\phi \in L^1(\Gamma)$. *Then* $\{\phi * \mathbf{m}_j\}_j \subset M(L^1(G), L^{1,\infty}(G))$ *and,*

$$\left\| \{\phi * \mathbf{m}_j\}_j \right\|_{M(L^1(G), L^{1,\infty}(G))} \leq c \|\phi\|_{L^1(\Gamma)} \left\| \{\mathbf{m}_j\}_j \right\|_{M(L^1(G), L^{1,\infty}(G))},$$

where $c > 0$ *is an absolute constant.*

In our procedure, we have obtained that appearing constant c can be taken to be $c_{1,\infty}$ given in (B.1.7). In [14] the obtained constant is $\inf_{0<r<1} \frac{1}{A_r (1-r)^{1/r}}$, and in [88] $\inf_{0<r<1} \left(\frac{e}{e-2}\right)^{1/r} \frac{1}{A_r^2 (1-r)^{1/r}}$, where A_r denotes the best constant on Khintchine's inequality (see [66, 99]). It holds that $c_{1,\infty}$ is smaller that the constants obtained in [14, 88]. To see this, assume first that the multipliers on the previous corollary map real valued function on real valued functions. Then for any $0 < r < 1$, $d_{1,r} = \frac{\pi^{\frac{1}{2r} - \frac{1}{2}}}{\Gamma\left(\frac{1+r}{2}\right)^{\frac{1}{r}}}$. By [99, Remark 2], it follows

$$A_r^{-1} \geq \frac{\pi^{\frac{1}{2r}}}{\sqrt{2}\, \Gamma\left(\frac{r+1}{2}\right)^{1/r}} = \sqrt{\frac{\pi}{2}} d_{1,r} > d_{1,r}.$$

It is known that the best constant on Khintchine's inequality with real coefficients is $2^{1/2-1/r}$ for $0 < r < 1$ (see [66]). Hence $A_r \leq 2^{1/2-1/r}$, and since by [62, (7)] or [81, (1.1)], $\Gamma\left(\frac{r}{2}+1\right) \geq \frac{2^{r/2}}{2}$ it follows that

$$d_{1,r} = \frac{\sqrt{\pi}}{2\Gamma\left(\frac{r}{2}+1\right)^{1/r}} \leq \frac{\sqrt{\pi}}{2} 2^{1/r-1/2} < A_r^{-1}.$$

We finish this chapter with some remarks. We have proved Theorems B.2.2 and B.2.4 for the pair $(L^1, L^{1,q})$ but the proofs carry over the whole range $0 < p \leq q \leq \infty$ for pairs $(L^p, L^{p,q})$. Howvere, observe that for $p > 1$, this result follows by Minkowski's integral inequality . On the other hand, for $p < 1$ and Haar's measure, the analogous result of Theorem B.2.2 is consequence of the fact that any $\mathbf{m} \in M(L^p, L^{p,q})$ satisfies that $\mathbf{m}^\vee = \sum_n a_n \delta_{u_n}$, where $(a_n)_n \in \ell^{p,q}$ and $u_n \in G$, with norm controlled by $\|(a_n)\|_{\ell^{p,q}(\mathbb{N})}$ (see [73, Theorem 10.1]).

With minor modifications on the proofs, the same kind of results hold for other pairs of spaces where the target space is not Banach as $(L^p, \Lambda^q(w))$ or $(L^p, \Lambda^{q,\infty}(w))$.

With some changes on the vectorial results of the chapter, the analogous property of Theorem B.2.2 also holds for multipliers acting on pairs of spaces $(H^1, L^{1,s})$ for $1 < s \leq \infty$, as those described in [94].

Appendix C

Transference Wiener amalgams

C.1 Definition and examples

Let F denote an F-space of measurable functions defined on $\mathcal{M}$, and let R be a representation of G on F which satisfies that for every $f \in F$, the function $(v, x) \mapsto R_v f(x)$ is jointly measurable in $G \times \mathcal{M}$.

Let E and B be QBFS's defined on $\mathcal{M}$ and G respectively and let V be a non empty open set, that in the case that G is compact is considered to be equal to G. Similarly as we did in §3, where we further assumed that F is a Banach space, provided that the function

$$K(f, B, V)(x) = \|\chi_V R_\cdot f(x)\|_B\,,$$

is μ-measurable, we define the transference Wiener amalgam $W(B, E, V)$ to be

$$W(B, E, V) := \Big\{ f \in F :\ \|f\|_{W(B,E,V)} = \|K(f, B, V)\|_E < \infty \Big\}.$$

The definition of the space depends on F and on the representation R, but, by simplicity, we omit this on the notation, so it may be kept in mind.

Give a non-empty open locally compact set V, and $f \in F$ in the following situations the measurability condition on $K(f, B, V)$ is automatically satisfied:

1. If B is a BFS. Since B is Banach, given $x \in \mathcal{M}$,

$$K(f, B, V)(x) = \sup_{\|g\|_{B'} \leq 1} \int_V |R_u f(x) g(u)|\ du,$$

where B' is the Köthe dual space of B. Since $|R_u f(x)|$ is jointly measurable and non-negative, by Luxemburg-Gribanov's Theorem (see [102, Theorem 99.2]), $K(f, B, V)$ is well defined and μ-measurable.

2. If $B = L^p(\mathcal{M})$, with $0 < p < 1$. To see this, it suffices to observe that $|R_u f(x)|^p$ is jointly measurable and that

$$K(f, B, V)(x) = \|(R_\cdot f(x))^p\|_{L^1(G)}^{1/p} = \left(\sup_{\|g\|_{L^\infty} \leq 1} \int_G |R_u f(x)|^p |g(u)|\ du \right)^{1/p}.$$

So result follows from Luxemburg-Gribanov's Theorem.

3. B is the p-convexification of a BFS for some $0 < p \leq 1$. The previous proof carries over this situation.

4. If $B = L^{p,\infty}(\mathcal{M})$, with $0 < p \leq 1$, given $f \in F$, $K(f, B, V)$ is measurable. By joint measurability, for any $t > 0$, the set

$$A_t = \{(v, x) : \ |R_v f(x)| > t\}$$

is measurable in $G \times \mathcal{M}$. Then, by Tonelli's Theorem, the function

$$x \mapsto \lambda_{\chi_V R \cdot f(x)}(t) = \int_V \chi_{A_t}(v)\, dv, \tag{C.1.1}$$

is μ-measurable. Hence,

$$K(f, B, V)(x) = \sup_{t>0} t^p \lambda_{\chi_V R \cdot f(x)}(t) = \sup_{t>0,\, t\in\mathbb{Q}} t^p \lambda_{\chi_V R \cdot f(x)}(t),$$

is also μ-measurable.

5. $B = \Lambda^p(w)$ where $0 < p < \infty$ and w is a weight on $[0, \infty)$. To show this, consider the function given in (C.1.1). For $t, s > 0$, let $A_{t,s} = [0, t] \times \{x \in \mathcal{M} : \ \lambda_{\chi_V R \cdot f(x)}(t) > s\}$ be a measurable set in $[0, \infty) \times \mathcal{M}$. It is not difficult to see that

$$\{(t, x) : \ \lambda_{\chi_V R \cdot f(x)}(t) > s\} = \cup_{t>0,\, t\in\mathbb{Q}} A_{t,s}.$$

Then the function $(t, x) \mapsto \lambda_{\chi_V R \cdot f(x)}(t)$ is jointly measurable. Hence the function $W(\lambda_{\chi_V R \cdot f(x)}(t))$ also is, where $W(s) = \int_0^s w$. Thus, by Tonelli's theorem,

$$x \mapsto \int_0^\infty t^p W(\lambda_{\chi_V R \cdot f(x)}(t))\, dt = K(f, B, V)^p(x).$$

A particular case is given by $B = L^{p,q}$, for $0 < p, q < \infty$.

6. $B = \Lambda^{p,\infty}(w)$ where $0 < p < \infty$ and w is a weight on $[0, \infty)$. It is proved in the same way as in the case $B = L^{p,\infty}$.

Examples of TWA:

1. If we take R to be the trivial representation on F it holds that $\|f\|_{W(B,E,Id,V)} = \|\chi_V\|_B \, \|f\|_E$, so $W(B, E, V) = E \cap F$, provided $\|\chi_V\|_B < +\infty$.

2. Let $F = E = L^p(\mathcal{M})$ for $0 < p < \infty$, and let $B = L^p(G)$. Assume also that $c = \sup_{u\in G} \|R_u\|_{\mathfrak{B}(E)} < \infty$ and let V be a relatively compact open neighborhood of e. It holds that

$$\|f\|^p_{W(L^p(G), L^p(\mathcal{M}), V)} = \int_{\mathcal{M}} \int_V |R_v f(x)|^p \ dv\, d\mu(x) = \int_V \|R_v f\|^p_{L^p(\mathcal{M})} \ dv.$$

Since for all $u \in G$, $R_u R_{u^{-1}} = I$, $1/c \leq \|R_u\|_{\mathfrak{B}(L^p(\mathcal{M}))} \leq c$. Therefore

$$\frac{1}{c}|V|^{1/p}\|f\|_{L^p(\mathcal{M})} \leq \|f\|_{W(L^p(G),L^p(\mathcal{M}),R,V)} \leq c|V|^{1/p}\|f\|_{L^p(\mathcal{M})}.$$

Then $W(L^p(G), L^p(\mathcal{M}), V) = L^p(\mathcal{M})$.

3. If $0 < p < \infty$, $E = F = L^p(G)$, $B = L^{p,\infty}(G)$ and if the representation is given by right translation, it holds that, for all $u \in G$, $f \in L^p(G)$ and $s > 0$,

$$\mu_{R_u f}(s) = \Delta(u)^{-1}\mu_f(s),$$

where Δ is the modular function associated to the left Haar measure on G. Hence

$$\begin{aligned}\|f\|^p_{W(L^{p,\infty}(G),L^p(G),V)} &\geq \left(\int_V \Delta(v)^{-1}dv\right)\sup_{t>0} t^p\mu_f(t)\\ &\geq |V|_r ||f||^p_{p,\infty},\end{aligned}$$

where $|V|_r$ denotes the right Haar's measure of V. Then

$$W(L^{p,\infty}(G), L^p(G), V) \subset L^{p,\infty}(\mathcal{M}).$$

4. Let $G = \left\langle \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix};\ x \neq 0,\ x, y \in \mathbb{R} \right\rangle$ whose left Haar's measure is given by $\frac{dxdy}{x^2}$, where $dxdy$ denotes the Lebesgue's measure in $\mathbb{R}^2$. The representation on $F = L^0(\mathbb{R}^2)$ defined by $R_U f(x) = f(Ux)$ it is not a distributionally bounded representation but, for $0 < r \leq p \leq s \leq \infty$ it follows that, for $V = (1/a, a) \times [-b, b]$,

$$\|f\|_{W(L^{p,s}(G),L^p(\mathbb{R}^2),V)} \geq \left(b\left(a^2 - \frac{1}{a^2}\right)\right)^{1/p}\|f\|_{L^{p,s}(\mathbb{R}^2)},$$

and

$$\|f\|_{W(L^{p,r}(G),L^p(\mathbb{R}^2),V)} \leq \left(b\left(a^2 - \frac{1}{a^2}\right)\right)^{1/p}\|f\|_{L^{p,r}(G)}.$$

5. Let $\mathcal{M} = G$ with $d\mu = wd\lambda$ where w is a Beurling weight (see Definition 2.3.23) and R is the left translation. Then

$$\mu_{R_u f}(s) = \int \chi_{\{z:\ |f(z)|>s\}}(uv)w(v)\,dv = \int \chi_{\{z:\ |f(z)|>s\}}(v)w(u^{-1}v)\,dv.$$

Hence, for $0 < r \leq p \leq s \leq \infty$, it follows that

$$\|f\|_{W(L^{p,s}(G),L^p(w),V)} \geq \left(\int_V \frac{1}{w(u)}\,du\right)^{1/p}\|f\|_{L^{p,s}(w)}$$

and

$$\|f\|_{W(L^{p,r}(G),L^p(w),V)} \leq \left(\int_V w(u^{-1})\,du\right)^{1/p}\|f\|_{L^{p,r}(w)}$$

C.2 Properties of TWA

It is clear that if $\mathcal{M} = G$, $F = L^1_{\text{loc}}(G)$, B is a BFS such that left translation is an isometry, V is a locally compact open set, C is a BFS, and the representation is given by the right translation, it holds that $W(B, C, V)$ coincide with the Wiener amalgam $W(B, C)$. So the natural question that arise is: Which properties of the Wiener amalgams are also satisfied by TWA?

Let us first recall known properties of Wiener amalgams on a locally compact group G.

Proposition C.2.1. [57, Theorem 1] *Let B, C be BFS such that translations act boundedly on them. Then $W(B, C)$ is a Banach space, and the definition of $W(B, C)$ is independent of the choice of V, i.e., different choices of V define the same space with equivalent norms.*

In this section we shall assume that E, B are BFS defined on $\mathcal{M}$ and G, respectively. We also fix $F = L^0(\mathcal{M})$ and assume that the representation is given by

$$R_t f(x) = h_t(x) f(\tau_t x),$$

where $\{\tau_t\}_{t\in G}$ is a family of transformations defined on $\mathcal{M}$, and $\{h_t\}_{t\in G}$ are measurable and positive functions satisfying

$$\tau_t \circ \tau_s = \tau_{st}, \quad h_{st}(x) = h_s(x) h_t(\tau_s x).$$

We will also assume that, there exists a morphism $h : G \to (0, \infty)$ such that $h \in L^1_{\text{loc}}(G)$ and for any $u \in G$,

$$\mu(\tau_u x) = h(u)\mu(x).$$

In the case that $\mathcal{M} = G$ and $\tau_u x = xu$, $h(u) = \Delta(u)$, where Δ is de modular function defined on G, that is continuous and hence it is locally integrable. Observe that, by the properties of τ, h should be a morphism of groups of G on $(0, \infty)$.

We will also assume that, for any $x \in \mathcal{M}$, and any $s \in G$, $0 < h_s(x) < \infty$. Observe that this representation is defined on every measurable function f. Let us consider $F = L^0(\mathcal{M})$ and let B, E be BFS defined on G and $\mathcal{M}$ respectively. Then we denote by $W(B, E, V)$ the associated TWA.

If a measurable function f satisfies that $||f||_{W(B,E,V)} < \infty$, then $f \in L^0(\mathcal{M})$. This holds since, for every set of finite measure $\mathcal{M}_1 \subset \mathcal{M}$,

$$c_{V,\mathcal{M}_1} ||f||_{W(B,E,V)} \geq \int_{V\times\mathcal{M}_1} |R_u f(x)| \, du d\mu(x).$$

Then $R_u f(x)$ is finite a.e. $(u, x) \in V \times \mathcal{M}_1$. By the σ-finiteness of $\mathcal{M}$, it follows that $R_u f(x)$ is finite a.e. $(u, x) \in V \times \mathcal{M}$. However, observe that for all $u \in G$

$$\mu \{x \in \mathcal{M} : |R_u f(x)| = \infty\} = h(u^{-1})\mu \{x \in \mathcal{M} : |f(x)| = \infty\},$$

so

$$\mu\{x \in \mathcal{M} : |f(x)| = \infty\} = \frac{\int_V \mu\{x \in \mathcal{M} : |R_u f(x)| = \infty\}\ du}{\int_V h(u^{-1})\ du} = 0.$$

Proposition C.2.2. *$||\cdot||_{W(B,E,V)}$ is a norm on $W(B,E,V)$.*

Proof. Since E, B are BFS's, triangular inequality and homogeneity easily follows.

Assume that $||f||_{W(B,E,V)} = 0$. Then a.e. $(v,x) \in V \times \mathcal{M}$, $|R_u f(x)| = 0$. Since,

$$\mu\{x \in \mathcal{M} : |R_u f(x)| \neq 0\} = h(u^{-1})\mu\{x \in \mathcal{M} : |f(x)| \neq 0\},$$

we have that $\mu\{x \in \mathcal{M} : |f(x)| \neq 0\} = 0$. □

Lemma C.2.3. *If $f_n \to f$ in $\|\cdot\|_{W(B,E,V)}$, there exists a subsequence that converges pointwise μ-a.e. to f.*

Proof. Given $f \in W(B,E,V)$, for any set of finite measure $\mathcal{M}_1 \subset \mathcal{M}$,

$$C_{V,\mathcal{M}_1}||f||_{W(B,E,V)} \geq \int_{V\times\mathcal{M}_1} |R_u f(x)|\ du d\mu(x).$$

Then, if $f_n \to f$ in $||\cdot||_{W(B,E,V)}$, $R_v f(x) \to R_v f(x)$ in $L^1_{\text{loc}}(V \times \mathcal{M})$, and thus any subsequence f_{n_k} satisfies that a.e. $(v,x) \in V \times \mathcal{M}$, $R_v f_{n_k}(x) \to R_v f(x)$. That is,

$$0 = \int_V \mu\{x \in \mathcal{M} :\ R_v f_{n_k}(x) \not\to R_v f(x)\}\ dv.$$

However, observe that given $v \in V$,

$$\begin{aligned}&\mu\{x \in \mathcal{M} :\ R_v f_{n_k}(x) \not\to R_v f(x)\} = \\ &\quad = h(v^{-1})\mu\{y \in \mathcal{M} :\ f_{n_k}(y)h_v(\tau_{v^{-1}}y) \not\to f(y)h_v(\tau_{v^{-1}}y)\}\end{aligned}$$

but since for all $x \in \mathcal{M}$ and $u \in G$, $0 < h_u(x) < \infty$, the last set coincides with the set $\{y \in \mathcal{M} :\ f_{n_k}(y) \not\to f(y)\}$. So it follows that $f_{n_k} \to f$ μ-a.e. $x \in \mathcal{M}$. □

Proposition C.2.4. *Let $(f_n)_n \in F$. It holds:*

1. *$f \in W(B,E,V)$ if and only if $|f| \in W(B,E,V)$, and*

$$||f||_{W(B,E,V)} = ||\,|f|\,||_{W(B,E,V)}.$$

2. *(Lattice property) If $0 \leq f \leq g$ μ-a.e., $\|f\|_{W(B,E,V)} \leq \|g\|_{W(B,E,V)}$.*

3. *(Fatou property) If $0 \leq f_n \uparrow f$, μ-a.e, then $\|f_n\|_{W(B,E,V)} \uparrow \|f\|_{W(B,E,V)}$*

4. *If $f_n \uparrow f$, μ-a.e, then either $f \notin W(B,E,V)$ and $||f_n||_{W(B,E,V)} \uparrow \infty$, or $f \in X$ and $||f_n||_{W(B,E,V)} \uparrow ||f||_{W(B,E,V)}$.*

5. *(Fatou's Lemma) If* $f_n \to f$, *a.e., and if* $\liminf_{n\to\infty} ||f_n||_{W(B,E,V)} < \infty$, *then* $f \in W(B,E,V)$ *and*

$$||f||_{W(B,E,V)} \leq \liminf_{n\to\infty} ||f_n||_{W(B,E,V)}.$$

6. *If* $\sum_{n\geq 1} ||f_n||_{W(B,E,V)} < \infty$, *there exists a measurable function* f *such that*

$$f = \sum_{n\geq 1} f_n \quad \textit{and} \quad ||f||_{W(B,E,V)} \leq \sum_{n\geq 1} ||f_n||_{W(B,E,V)}.$$

7. $\big(W(B,E,V), ||\cdot||_{W(B,E,V)}\big)$ *is a QBFS and* $||\cdot||_{W(B,E,V)}$ *is a norm.*

Proof. By the assumptions on the representation, $|R_u f| = R_u|f|$, so the first assertion easily follows.

Let f, g such that $f \leq g$ a.e. For any $v \in V$, $R_v f \leq R_v g$ a.e. Then $0 = \mu\{x : R_v f(x) > R_v g(x)\}$. Hence $\{(v,x) \in V \times \mathcal{M} : R_v f(x) > R_v g(x)\}$, is a zero measure set in $V \times \mathcal{M}$. Thus there exists a nul set $Z \subset \mathcal{M}$, such that for any $x \notin Z$, $R_v f(x) \leq R_v g(x)$ a.e. $v \in V$. Thus, for any $x \notin Z$, since B is a BFS,

$$\|\chi_V(v) R_v f(x)\|_B \leq \|\chi_V(v) R_v g(x)\|_B .$$

Consequently, by the lattice property of E, $\|f\|_{W(B,E,V)} \leq \|g\|_{W(B,E,V)}$.

Consider $f_n(x) \uparrow f(x)$ a.e.x. Given $v \in V$, since $h(v^{-1}) > 0$,

$$0 = \mu\{x : \ f_n(\tau_v x) \not\uparrow f(\tau_v x)\}.$$

Then $\{(v,x) \in V \times \mathcal{M} : \ f_n(\tau_v x) \not\uparrow f(\tau_v x)\}$, is a zero measure set in $V \times \mathcal{M}$. Thus there exists a nul set $Z \subset \mathcal{M}$, such that for every $x \notin Z$, a.e.$v \in V$, $f_n(\tau_v x) \uparrow f(\tau_v x)$. Consequently for every $x \notin Z$,

$$\|\chi_V(v) R_v f_n(x)\|_B \uparrow \|\chi_V(v) R_v f(x)\|_B ,$$

and then $\|f_n\|_{W(B,E,V)} \uparrow \|f\|_{W(B,E,V)}$.

Property *4* is a consequence of the definition of $W(B,E,V)$ and the previous property.

For assertion *5*, let $h_n(x) = \inf_{m\geq n} |f_m(x)|$ so that $0 \leq h_n \uparrow |f|$ a.e. By the lattice property and the Fatou property,

$$\begin{aligned} ||f||_{W(B,E,V)} &= \lim_n ||h_n||_{W(B,E,V)} \leq \lim_n \inf_{m\geq n} ||f_n||_{W(B,E,V)} \\ &= \liminf_{n\to\infty} ||f_n||_{W(B,E,V)} < \infty. \end{aligned}$$

Therefore, $f \in W(B,E,V)$ and $||f||_{W(B,E,V)} \leq \liminf_{n\to\infty} ||f_n||_{W(B,E,V)}$.

By Proposition C.2.2, $\|\cdot\|_{W(B,E,V)}$ is a norm, so it suffices to show that the space satisfies *6* in order to prove assertion *7*. Consider $g = \sum_{n\geq 1} |f_n|$, $g_N =$

$\sum_{n=1}^{N} |f_n|$, so $g_N \uparrow g$. Since,

$$||g_N||_{W(B,E,V)} \leq \sum_{n=1}^{N} ||f_n||_{W(B,E,V)} \leq \sum_{n\geq 1} ||f_n||_{W(B,E,V)} < \infty,$$

it follows by the preceding assertions that $g \in W(B,E,V)$. By the previous lemma, the series $\sum_n |f_n(x)|$ converges pointwise μ-a.e. and hence so it does $\sum f_n(x)$. Thus, if for $N \geq 1$,

$$f = \sum_n f_n, \quad f^N = \sum_{n=1}^{N} f_n,$$

$f^N \to f$ μ-a.e. Since $||f^N||_{W(B,E,V)} \leq ||g_N||_{W(B,E,V)} \leq \sum_{n\geq 1} ||f_n||_{W(B,E,V)} < \infty$, by Fatou's lemma, $||f||_{W(B,E,V)} \leq \sum_{n\geq 1} ||f_n||_{W(B,E,V)} < \infty$.

□

Remember that we assumed that, in the case that G is a compact group, the selected set V is the whole group G. But, what happens in the non-compact case? By analogy with the classical amalgam spaces, one would expect that the definition does not depend on the selection. This can be ensured for a particular family of representations.

Proposition C.2.5. *Assume that $h_u(x) \approx 1$, and that R_u induces on E a continuous operator. Given a pair of relatively compact non empty open sets U, V, it holds that*

$$\|\cdot\|_{W(B,E,U)} \approx \|\cdot\|_{W(B,E,V)}.$$

In other word, the space $W(B,E,V)$ is independent of the choice of V.

Proof. By symmetry, it suffices to prove one of the inequalities. By compactness, there exist $n \in \mathbb{N}$, depending on V and U, $s_1, \ldots, s_n \in \overline{V}$, such that $V \subset \cup_{i=1}^{n} s_i U$. Since, for all i,

$$\begin{aligned}\left\|\|\chi_{s_iU}(y)R_y f(x)\|_B\right\|_E &\leq \left\|L_{s_i^{-1}}\right\|_{\mathfrak{B}(B)} \left\|\left\|\chi_U(y)R_{s_i^{-1}y}f(x)\right\|_B\right\|_E \\ &= \left\|L_{s_i^{-1}}\right\|_{\mathfrak{B}(B)} \left\|K(f,B,U)(\tau_{s_i^{-1}}x)\right\|_E \\ &\leq \left\|L_{s_i^{-1}}\right\|_{\mathfrak{B}(B)} \left\|R_{s_i^{-1}}\right\|_{\mathfrak{B}(E)} \|f\|_{W(B,E,U)},\end{aligned}$$

where $\left\|L_{s_i^{-1}}\right\|_{\mathfrak{B}(B)}$ is the norm of the left translation operator acting on B. Then

$$\begin{aligned}\|f\|_{W(B,E,V)} &\leq \left\{\sum_{i=1}^{n} \left\|L_{s_i^{-1}}\right\|_{\mathfrak{B}(B)} \left\|R_{s_i^{-1}}\right\|_{\mathfrak{B}(E)}\right\} \|f\|_{W(B,E,U)} \\ &= c_{(E,B,V,U)} \|f\|_{W(B,E,U)}.\end{aligned}$$

□

Observation C.2.6. *Observe that if* $\mathcal{M} = G$*, and* R *is given by the right translation in the group, the last proposition recover the known property of classical Wiener amalgams.*

Appendix D

Weighted Lorentz spaces

D.1 Weighted Lorentz spaces

Definition D.1.1. *For any $p \in (0,\infty)$ and any weight function w, we consider the weighted Λ and Γ Lorentz spaces defined by*

$$\Lambda^p(w) = \left\{ f \in L^0(\mathcal{M}) : \|f\|_{\Lambda^p(w)} := \left(\int_0^\infty (f^*(t))^p \, w(t)\, dt \right)^{1/p} < \infty \right\};$$

$$\Lambda^{p,\infty}(w) = \left\{ f \in L^0(\mathcal{M}) : \|f\|_{\Lambda^{p,\infty}(w)} := \sup_{0<t<\infty} f^*(t) W(t)^{1/p} < \infty \right\};$$

$$\Gamma^p(w) = \left\{ f \in L^0(\mathcal{M}) : \|f\|_{\Gamma^p(w)} := \left(\int_0^\infty (f^{**}(t))^p \, w(t)\, dt \right)^{1/p} < \infty \right\};$$

$$\Gamma^{p,\infty}(w) = \left\{ f \in L^0(\mathcal{M}) : \|f\|_{\Gamma^{p,\infty}(w)} := \sup_{0<t<\infty} f^{**}(t) W(t)^{1/p} < \infty \right\}.$$

If some confusion can arise, we shall make explicit the underlying defining measure space: $\Lambda^p(w,\mathcal{M}), \Lambda^{p,\infty}(w,\mathcal{M}), \ldots$.

A function W is said to satisfy Δ_2 condition if there exists a constant $C > 0$ such that, for any $r > 0$,

$$W(2r) \leq CW(r).$$

If w is a weight and $W \in \Delta_2$, for $0 < p < \infty$ $\|\cdot\|_{\Lambda^p(w)}$ is a quasi-norm and $\Lambda^p(w)$ is a RIQBFS (see [39, Theorem 2.3.12]).

It is sometimes convenient to express $\|\cdot\|_{\Lambda^p(w)}$ and $\|\cdot\|_{\Lambda^{p,\infty}(w)}$ in terms of the distribution function. It can be shown that

$$\|f\|_{\Lambda^p(w)} = \left(p \int_0^{+\infty} y^{p-1} W(\mu_f(y))\, dy \right)^{1/p}, \tag{D.1.2}$$

$$\|f\|_{\Lambda^{p,\infty}(w)} = \sup_{t>0} t W^{1/p}(\mu_f(t)). \tag{D.1.3}$$

For $p \in (1,\infty)$, $\Lambda^p(w)$ is equivalent to a Banach space if and only if

$$t^p \int_t^\infty s^{-p} w(s) ds \leq CW(t)$$

for some C and all $t > 0$. When w satisfies this inequality for some $p \in (0, \infty)$ we say that $w \in B_p$. Moreover, in this case $\Lambda^p(w)$ and $\Gamma^p(w)$ coincide. On the other hand, $\Lambda^1(w)$ is equivalent to a Banach space if and only if

$$\frac{W(t)}{t} \leq C \frac{W(s)}{s}$$

for $0 < s \leq t$. When this inequality holds, we say that $w \in B_{1,\infty}$. More generally, if for any $p \in (0, \infty)$, for $0 < s \leq t$

$$\frac{W(t)}{t^p} \leq C \frac{W(s)}{s^p}$$

we say that $w \in B_{p,\infty}$. Moreover it is satisfied that, for any $q > p > 0$, (see [95])

$$B_p \subsetneq B_{p,\infty} \subsetneq B_q.$$

The proofs of these facts can be found in [32, 93, 95] respectively.

Proposition D.1.4. *Let w be a weight in $(0, \infty)$. Let $X = \Lambda^p(w), \Lambda^{p,\infty}(w)$ or $\Gamma^p(w)$. In the last case we shall assume that*

$$\int_0^\infty \frac{w(s)}{(1+s)^p}\, ds < \infty, \qquad \int_0^1 \frac{w(s)}{s^p}\, ds = \int_1^\infty w(s)\, ds = \infty.$$

It holds that

$$h_X(t) \approx \sup_{r>0} \frac{\varphi_X(rt)}{\varphi_X(r)},$$

with constants independents of t, where $\varphi_X(t) = \|\chi_E\|_X$, for any set E such that $\mu(E) = t$.

Proof. Observe that, if we define $w_t(s) = \frac{1}{t} w\left(\frac{s}{t}\right)$, it holds that $\|E_t f\|_{\Lambda^p(w)} = \|f\|_{\Lambda^p(w_t)}$. Similar equalities holds for the other spaces. Thus, for any $t > 0$, $h_{\Lambda^p(w)}(1/t)$, is the norm of the embedding $\Lambda^p(w) \hookrightarrow \Lambda^p(w_t)$.

By [37, Thm. 3.1], this is equal to

$$h_{\Lambda^p(w)}\left(\frac{1}{t}\right) = \left(\sup_{r>0} \frac{W(r/t)}{W(r)}\right)^{1/p} = \sup_{r>0} \frac{\varphi_{\Lambda^p(w)}(r/t)}{\varphi_{\Lambda^p(w)}(r)}.$$

The others are proved in a similar way, using the estimations of the norm of the corresponding embedding appearing in [37]. □

Given a RIBFS X, if $w = \frac{d\varphi_X}{dt}$, the Marcinkiewicz and the Lorentz space, that are defined by $M(X) = \Gamma^{1,\infty}(w)$ and $\Lambda(X) = \Lambda^1(w)$ (if $\varphi_X(0^+) = 0$), respectively, have the same fundamental function than X and are the greatest and the smallest RIBFS with fundamental function φ_X (see [18] for more details and proofs), respectively.

Proposition D.1.5. *Let X be a RIBFS such that $\varphi_X(0^+) = 0$. Then*

$$h_{M(X)}(t) = h_{\Lambda(X)}(t) \leq h_X(t).$$

Proof. Since

$$h_X(t) = \sup_f \frac{\|E_{1/t}f\|_X}{\|f\|_X} \geq \sup_r \frac{\varphi_X(tr)}{\varphi_X(r)},$$

and $\frac{\varphi_X(s)}{s} \downarrow$, the statement follows from the previous results. $\square$

Theorem D.1.6. [38, Theorems 2.3.4, 2.3.11 and 2.3.12] *Let $0 < p < +\infty$ and $w \in \Delta_2$, $\Lambda^p(w)$ has absolutely continuous norm and integrable simple functions are dense if $\mu(\mathcal{M}) < +\infty$ or $\mu(\mathcal{M}) = +\infty$ and $\int_0^\infty w = +\infty$.*

Let us introduce the so called Lorentz-Karamata spaces. The definition below slightly varies from that given in [53].

Definition D.1.7. *A measurable function $\gamma : (0,\infty) \to (0,\infty)$ is said to be slowly varying if for any $\varepsilon > 0$, $t \mapsto t^\varepsilon \gamma(t)$ is equivalent to a non-decreasing function and $t \mapsto t^{-\varepsilon}\gamma(t)$ is equivalent to a non-increasing function on $(0,\infty)$.*

It is easy to verify that the following functions $\gamma(t) = b(\max(t, 1/t))$ and $\gamma(t) = b(\max(1, 1/t))$ are slowly varying where:

1. $b(t) = \prod_{i=1}^m l_i^{a_i}(t)$ where $m \in \mathbb{N}$, $a_i \in \mathbb{R}$ for $i = 1, \ldots, m$, and l_i are given on $[1,\infty)$ by $l_0(t) = t$, $l_i(t) = 1 + \log l_{i-1}(t)$, for $i = 1, \ldots, m$.
2. $b(t) = e^{l_m^a(t)}$, where $0 < a < 1$ and $m \in \mathbb{N}$.

Proposition D.1.8. *Let γ be a slowly varying function. Then*

1. *Given any $r \in \mathbb{R}$, the function γ^r is slowly varying. Moreover, $\gamma(1/t)$ is slowly varying.*
2. *If $a > 0$, then for all $t > 0$, $\int_0^t s^{a-1}\gamma(s)\, ds \approx t^a\gamma(t)$.*
3. *If $a > 0$, $\int_0^\infty t^{a-1}\gamma(t)\; dt = \infty$.*
4. *If $a > 0$, and $v(t) = t^{a-1}\gamma(t)$, then $V \in \Delta_2$.*

Proof. The first statement easily follows from the properties of γ. In order to prove the second statement, observe that, for $\varepsilon > 0$, there exist u_ε non-decreasing and v_ε non-increasing such that, for $t > 0$, $t^\varepsilon b(t) \approx u_\varepsilon(t)$ and $t^{-\varepsilon}b(t) \approx v_\varepsilon(t)$. Fix $a, t > 0$. Then

$$\int_0^t s^{a-1}\gamma(s)\, ds = \int_0^t s^a s^{-1}\gamma(s)\, ds \gtrsim t^{-1}\gamma(t) \int_0^t s^a\, ds \approx t^a\gamma(t).$$

On the other hand,

$$\int_0^t s^{a-1}\gamma(s)\, ds = \int_0^t s^{-1+\frac{a}{2}} s^{\frac{a}{2}}\gamma(s)\, ds \lesssim t^{\frac{a}{2}}\gamma(t) \int_0^t s^{-1+\frac{a}{2}}\, ds \approx t^a\gamma(t).$$

Now, since for every $t > 0$, $\int_0^t s^{a-1}\gamma(s)\; ds \approx t^{a/2}\left(t^{a/2}\gamma(t)\right)$, and $t^{a/2}\gamma(t)$ is equivalent to a non decreasing function, $\lim_{t\to\infty}\int_0^t s^{a-1}\gamma(s)\; ds = \infty$. Furthermore, since $V(t) = \int_0^t s^{a-1}\gamma(s)\; ds \approx t^a\gamma(t)$, and $\gamma(2t)(2t)^{-1} \lesssim \gamma(t)(t)^{-1}$, $V(2t) \approx 2^a t^a\gamma(2t) \lesssim 2^{a+1}V(t)$. $\square$

Definition D.1.9. *Let $p, q \in (0, \infty]$, $\in \mathbb{N}$ and let γ be a slowly varying function. The Lorentz-Karamata space $L^{p,q;\gamma}$ is defined to be the weighted Lorentz space $\Lambda^q(w)$ where $w(t) = t^{\frac{q}{p}-1}\gamma(t)$.*

Proposition D.1.10. *Given a slowly varying function γ and $0 < p < \infty$, $0 < q < \infty$, $L^{p,q;\gamma}$ is RIQBFS, $\|\cdot\|_{L^{p,q;\gamma}}$ is absolutely continuous and integrable simple functions are dense.*

Proposition D.1.11. *If $1 < p < \infty$ and $1 \leq q < \infty$, $L^{p,q;\gamma}$ is Banach.*

Proof. Since $L^{p,q;b} = \Lambda^q(w)$ with $w(t) = t^{\frac{q}{p}-1}\gamma(t)$, and for every $\epsilon > 0$,

$$\frac{W(t)}{t^{\frac{q}{p}+\epsilon}} \approx t^{-\epsilon}\gamma(t),$$

that is equivalent to a non-increasing function. Thus $w \in B_{\frac{q}{p}+\epsilon,\infty} \subset B_{\frac{q}{p}+2\epsilon}$. Then $w \in \cup_{r>\frac{q}{p}} B_r$. Hence, for $p > 1$, $w \in B_q$. Therefore, for $1 \leq q < \infty$, $L^{p,q;b}$ is Banach. □

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