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Correlated stochastic dynamics in financial markets

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La física dels mercats financers

La física és una branca de la ciència dedicada a estudiar els fenomens i els cossos de la natura, per tal de trobar tant la formulació matemàtica de les lleis que regeixen com els constituents fonamentals dels fenomens i cossos de la natura. D'aquesta definició se'n desprèn que la física abarca un camp tan ampli que qualsevol altra disciplina científica podria arribar-se a denominar física. No obstant, el saber científic, a mesura que ha concretat més i més, s'ha vist obligat a disgregar-se en cada cop més nombroses i desconnectades disciplines. I la física no ha estat menys. Arran d'això, la figura del físic (o del científic) com a posseïdor d'una noció global del món ha passat forçosament de moda doncs, amb el seu esperit d'universalitzar les lleis de la natura, ha tendit a abstreure's més i més de la realitat més tangible.

Malgrat tot, cada dia són més els físics que fan recerca des de la pluridisciplinarietat. Aquesta nova tendència es justifica a ella mateixa entenent que l'avenç desaforat de cadascuna de les disciplines científiques ha provocat una absoluta incomprensió d'unes amb les altres i que, en conseqüència, cadascuna hagi avançat pel seu compte. Degut a aquest fet, la ciència ha deixat parts de la "natura" sense estudiar i d'altres han estat tractades des d'una única perspectiva. Els "físics pluridisciplinars" consideren que caldria aproximar-se a moltes d'aquestes qüestions simultàniament des de diverses disciplines. Els temes d'interès pluridisciplinar acostumen a ser més prosaics i menys fonamentals, de contacte directe amb el ciutadà.

La present memòria de tesi s'inscriu dins aquesta tendència d'estendre el camp d'estudi de la física a nous objectius, com és en aquest cas el món de la dinàmica financera. En els darrers trenta anys, el món borsari ha desenvolupat un creixement vertiginós i la societat té un cada cop major interés envers els mercats financers. La borsa és un ens generador d'un volum enorme de dades que necessiten d'un estudi molt acurat. La computació dels mercats de valors ha permès tenir un registre molt ampli d'informació que arriba a guardar els valors negociats fins a variacions registre a registre, el que s'anomena "*tick a tick*"[1]. A partir de l'estudi empíric de la borsa, s'ha observat que aquesta està subjecte a una molt variada i complexa fenomenologia que exigeix una descriptiva matemàtica d'alt nivell. Així doncs, els mercats contenen tots els ingredients per ser un camp especialment llaminer als ulls

[1] Les dades resultants s'anomenen "dades d'alta freqüència" ja que la distància entre *ticks* arriba a ser de l'ordre dels pocs segons.

de la física i, recentment, nombrosos físics s'han sentit atrets per les finances i han intentat dir-hi la seva.

No obstant, la contribució dels físics al món de les finances ve de lluny. Per exemple, el físic matemàtic ja als anys seixanta va estudiar la distribució dels preus i Mandelbrot va observar que les cues de la distribució es poden descriure amb una llei de potències (Mandelbrot (1963)). Les cues, en qualsevol cas, no poden ser explicades pel procés brownià geomètric acceptat fins aleshores. Mandelbrot proposà les distribucions Lévy doncs aquestes distribucions tenen les cues amb el decaïment desitjat. Ara bé, no ha estat fins a principis dels noranta que la física ha entés com a disciplina pròpia el món de les finances i l'economia en general. Des d'aleshores, la natura que observa la física també inclou als mercats financers.

Un dels primers articles publicats en una revista de física va ser el de Mantegna l'any 1991. En aquest, s'estudia la distribució empírica de l'índex de la borsa de Milano i es quantifiquen els coeficients de la distribució Lévy consistent amb les dades. Aquest article és un treball preliminar al publicat a Nature el 1995 que, certament, ha tingut molta més repercussió dins la comunitat de la física. En aquest article, Mantegna i Stanley analitzen les propietats estadístiques de l'índex borsari Standard & Poor's 500 de New York. Per fer-ho, es van servir de dades d'alta freqüència i del major conjunt de dades històriques de què s'havia disposat fins aleshores. L'article aprofundeix en la tesi de que la dinàmica dels preus podria ser descrita per un procés Lévy i que la sèrie temporal de l'índex mostra un comportament auto-similar, és a dir, que el perfil de les distribucions de canvi de preus minut a minut, diaris, setmanals i mensuals és semblant. En aquesta mateixa línia existeix una extensió interessant de Bouchaud i Sornette (1994). El treball estudia un mètode de valoració d'opcions alternatiu al que típicament es fa servir en finances. La intenció de Bouchaud i Sornette era de formalitzar un mètode que accepti altres dinàmiques com ara la dels processos Lévy.

Paral·lelament, altres físics van començar a estudiar els mercats basant-se en simulacions de la dinàmica del preu de l'acció un cop fixades les pautes de comportament dels inversors. Dins d'aquest àmbit, es poden destacar els treballs de Bak *et al.* (1997), Challet i Zhang (1997) o Johnson *et al.* (1998). Cadascú a la seva manera tracta de mecanitzar la generació dels preus especulatius des d'allò que se'n diu la microestructura dels mercats. És a dir, cerquen les interaccions dels agents que operen a borsa que generin la complexitat observada en l'evolució dels preus.

Totes aquestes primerenques aproximacions a les finances s'encabeixen dins d'una sola disciplina que Stanley batejà amb el nom d'"econofísica". Va ser tota una declaració d'intencions per tal d'incitar els físics a estudiar els mercats financers i encabir la borsa dins la física. L'article de Mantegna i Stanley a Nature l'any 1995 es considera el tret de sortida de la publicació d'estudis sobre finances en revistes especialitzades de física. Des d'aleshores, existeixen unes quantes publicacions de física que accepten articles d'econofísica, la quantitat d'articles publicats per físics dins d'aquesta disciplina s'ha disparat i ja s'estan cel·lebrant nombrosos congressos i *workshops* sobre les aplicacions de la física a les finances[2]. Alguns com Bouchaud (França), Wilmott (UK) i Farmer (US) s'han atrevit, amb més o meys fortuna, a crear empreses usant els seus propis coneixements per especular a borsa. I d'altres han publicat tot el coneixement adquirit en llibres o han reescrit conceptes de la matemàtica financera de manera més agradable als ulls d'un físic (*e.g.* Wilmott (1998), Mantegna i Stanley (2000) o Bouchaud i Potters (2000)).

Després de la primera embranzida, les contribucions publicades pels físics s'han perllongat en vàries direccions. Bona part de la feina s'ha fet en l'anàlisi de dades. Els físics han rendibilitzat la gran quantitat de dades que els hi ha arribat i han extret conclusions importants. Per exemple, el grup dirigit per Eugene Stanley ha continuat estudiant les propietats estadístiques dels mercats com ho havia anat fent fins aleshores[3]. Stanley i altres han quantitzat l'exponent polinòmic del decaïment a les cues de la distibució de preus i comprovat l'esperit universal del comportament auto-similar a diferents mercats del món. Amb tot, aquests estudis conclouen que les distribucions Lévy tenen unes cues massa altes en comparació a les del mercat[4]. L'estudi empíric ha anat també per altres camins. S'ha estudiat les correlacions entre mercats (Plerou *et al.* (1999b), Laloux *et al.* (1999)) que són de gran utilitat per a la gestió de carteres i l'estadística de la volatilitat (Liu *et al.* (1999), Lillo i Mantegna (2000)) que és tant o més important que el propi preu de l'acció.

Tots aquests resultats s'han anat compil·lant en un llista de requeriments que ha de complir tot model proposat (Cont (2001)). Després de conèixer la naturalesa dels mercats financers, els físics s'han esforçat a model·litzar la seva dinàmica. Les aproximacions són fetes des de dos punts de vista.

Un dels dos punts de vista estudia la generació dels preus a partir del comportament dels agents que operen a borsa. Com ja hem vist, aquests models s'han anat presentant des del 1997 basant-se, sovint, amb models similars als que s'apliquen a altres sistemes físics. Podem mencionar-ne uns quants. Bak *et al.* (1999) és un dels treballs més curiosos ja que proposa un model per descriure la generació de la dinàmica del preu del diner. D'altres ja estan més centrats en els mercats especulatius. S'interessen per les causes a escala microscòpica que generen certes situacions crítiques com ara la coincidència entre un alt volum de negociació amb períodes de gran fluctuació de la volatilitat (Iori (1999), Capocci i Zhang (2001) o Lux i Marchesi (1999)).

El segon punt de vista tracta amb models que proposen directament una dinàmica dels preus, sense preocupar-se per la pauta de comportament microscòpic que les genera. Dins aquest conjunt de publicacions cal destacar les del nostre propi grup del Department de Física Fonamental de la Universitat de Barcelona. Masoliver *et*

[2]Les conferències *Applications of Physics to Financial Analysis* (APFA) són les més importants i ja han arribat a la seva quarta edició.

[3]Veure, per exemple, Plerou *et al.* (1999a).

[4]A més, en contra de les Lévy, també cal afegir que no tenen tots els moments definits cosa que resulta poc realista.

al. (2000) i Masoliver *et al.* (2001) han proposat una dinàmica estocàstica[5] tal que les variacions de preu vinguin donades per un conjunt de processos a salts. Allí, observen propietats d'escalament i descriuen les cues de manera més refinada a la dels processos Lévy. En aquest sentit, hi ha molts pocs treballs de físics que s'hagin dedicat a estudiar models adequats per a la dinámica. Sornette *et al.* (1996) busca precursors de cracks de la borsa valent-se de l'anàlisi utilitzat per a la previsió de terratrèmols i Sornette (1998) intenta model·litzar certs aspectes de l'estadística dels mercats a partir d'aquesta aproximació.

Cal admetre, però, que l'entrada dels físics al món de les finances també s'ha caracteritzat per una certa ignorància envers el treball desenvolupat fins aleshores per part dels matemàtics especialistes en finances. Aquesta actitud ha desacreditat moltes de les pretensions de multidisciplinarietat i de diàleg amb què els físics han justificat la seva entrada al món de la borsa. Tot i haver existit un boom en la contractació de físics per exercicir de brokers al principi dels noranta, la física és ara escoltada amb una major prevenció fins i tot des dels parquets borsaris.

Malgrat tot, alguns físics insisteixen amb la idea pluridisciplinar i encara existeix un esperit que busca concil·liar els professionals del sector, matemàtics financers i físics. La recent creació de revistes que serveixin de plataforma per a poder-se escoltar i fer-se entendre mutúament vol afavorir aquesta situació. Aquestes revistes són l'*International Journal of Theoretical and Applied Finance* i el *Quantitative Finance*. Alguns físics, fins i tot, ja han publicat o estan en fase de fer-ho en revistes de matemàtica financera (*e.g.* Cont i Bouchaud (2000), Canning *et al.* (1998), Bouchaud *et al.* (1996) o Iori (2001)). D'altres simplement han deixat de ser físics per posar-se a treballar en serveis d'estudis de cases de borsa o bé ja pertanyen i són acceptats com a matemàtics financers.

Paral·lelament a aquests precedents, aquest treball compil·la bona part de la feina que he realitzat des de 1996 fins ara. L'estudi dels mercats borsaris s'ha fet valent-se de mètodes estocàstics. El grup dins del qual s'ha realitzat aquesta tasca s'ha dedicat típicament a estudiar les dinàmiques estocàstiques que intervenen en fenòmens com ara els làsers, el transport de la llum dins medis desordenats i la cinètica dels processos químics entre d'altres. Esperonats per la repercussió del treball de Mantegna i Stanley, ens hem decidit a introduir-nos en l'estudi de les finances i aplicar-hi el nostre coneixement en processos estocàstics. La borsa té l'avantatge que les dades que genera poden ser fàcilment confrontades amb qualsevol model·lització estocàstica mitjançant un senzill ordinador. Aquesta aplicació més directa dels resultats matemàtics obtinguts és molt atractiva i permet estar molt a prop d'una realitat, la borsa.

Com ja hem dit, aquesta aproximació cap a les finances és una via poc explorada pels físics. No obstant, els mètodes estocàstics pròpiament dits han estat implementats des de 1900 pels matemàtics creant una nova disciplina anomenada matemàtica

[5] És a dir, assumint que els mercats tenen una naturalesa aleatòria.

financera. Per aquest raó, ens ha calgut fer un intens estudi dels treballs més notables en matemàtica finacera i, tot seguit, ens hem reexplicat la seva progressió d'una manera més comprensible per a un físic. Cal resaltar un treball que ha estat fruit d'aquesta revisió. A Masoliver *et al.* (2001a), hi hem estudiat les diferències existents pel fet d'analitzar les dades prenent les variacions de preu i les de la rendibilitat.

El problema més complet i emblemàtic és el de la valoració d'opcions i aquest és un dels dos temes cabdals dels capítols que vénen. I si la dinàmica financera estocàstica ja ha estat poc estudiada pels físics encara ho ha estat menys la valoració d'opcions[6]. La teoria bàsica de Black i Scholes (1973) ha estat revisada i ens hem fixat en algunes qüestions que potser no havien preocupat massa als matemàtics però que creiem que són d'importància, com a mínim des del punt de vista d'un físic. Aquesta feina ha estat presentada en format d'article de recerca en una revista de física (Perelló *et al.* (2000)). La generalització de la teoria Black-Scholes també és present en aquesta memòria. Cal destacar-ne la representació, inèdita a la literatura, del preu de l'opció en termes de la transformada de Fourier de la densitat de probabilitat del procés i basant-se amb teoria de martingala (Perelló and Masoliver (2001e)).

Les correlacions en el mercat és la segona quëstió important. Els físics hem fet un gran esforç per a quantificar-les (Cont (2001)). El coneixement de les correlacions és cabdal a nivell pràctic doncs permet fer predicció, encara que sigui estadística, sobre la futura evolució del mercat. Concretament, presentem dos models difussius bidimensionals i n'estudiem les seves pecularietats com a model de mercat i com a extensió al mètode Black-Scholes de valoració d'opcions (Masoliver i Perelló (2001b), Masoliver i Perelló (2001c), Perelló i Masoliver (2001d)). De fet, d'aquí prové el títol de la tesi: "Dinàmica estocàstica correlacionada en mercats financers".

En resum, aquest treball representa, per una banda, un compendi de tot allò que hem anat aprenent sobre les finances i que ja coneixen els matemàtics financers. I, per una altra banda, el treball també presenta les nostres primeres aportacions a aquest camp d'estudi. Crec que tant se val d'on vinguin les contribucions perquè allò que realment interessa és entendre millor el comportament dels mercats financers. Tant la matemàtica financera com la física tenen molt per explicar-se i els treballs matemàtics millorarien amb un major esperit de físic, essent aleshores menys abstractes. Mentrestant, la física li convindria aprendre tot allò que els matemàtics financers ja coneixen i tracten amb una formulació molt robusta.

La present memòria està organitzada de la següent manera. Després d'aquesta introducció com a justificació del treball, el primer capítol defineix el que és una opció i n'explica la seva utilitat. El segon capítol dóna les condicions que han de complir el preu per a l'opció per a no permetre arbitratge i narra els progressos fets des de 1900 per tal de trobar un preu just per a l'opció. El tercer explica el famós mètode Black-

[6]Existeix alguna notable excepció (Bouchaud i Sornette (1994), Bouchaud *et al.* (1996)), Matacz (2000)).

Scholes per a valoració d'opcions mentres que el quart exposa les generalitzacions realitzades assumint encara la teoria Black-Scholes. El cinquè capítol presenta un nou model que estén Black-Scholes relaxant la hipòtesi de mercat eficient. El sisè capítol tracta sobre un model de volatilitat estocàstica i n'estudia totes les seves propietats estadístiques. La memòria acaba amb unes conclusions sobre el treball realitzat i anticipant la nostra possible futura recerca sobre els mercats financers.

Bibliografia

Bak, P., M. Paczuski, M. Shubik, 1997, Price variations in a stock market with many agents, *Physica A* **246**, 430-453.

Bak, P., S. Norrelykke, M. Shubik, 1999, Dynamics of money, *Physical Review E* **60**, 2528-2532.

Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* **81**, 637-659.

Bouchaud, J.-P., D. Sornette, 1994, The Black-Scholes option pricing problem in mathematical finance: generalization and extension for a large class of stochastic processes, *Journal de Physique I* **4**, 863-881.

Bouchaud, J.-P., G. Iori, D. Sornette, 1996, Real-world options, *Risk* **9**, 61-65.

Bouchaud, J.-P., R. Cont, 1998, A Langevin approach to stock market fluctuations and crashes, *European Journal of Physics B* **6**, 543-550.

Bouchaud, J. P., and M. Potters, 2000, *Theory of financial risks* (Cambridge University Press, Cambridge, UK).

Canning, D., L. A. N. Amaral, Y. Lee, M. Meyer, H. E. Stanley, 1998, Scaling the volatility of GDP growth rates, *Economic Letters* **60**, 335-341.

Capocci, A. and Y.-C. Zhang, 2001, Market ecology of active and passive investors, *Physica A* **298**, 488-498.

Challet, D. and Y.-C. Zhang, 1997, Emergence of cooperation and organization in an evolutionary Game, *Physica A* **246**, 407-415.

Cont, R., 2001, Empirical properties of asset returns: stylized facts and statistical issues, *Quantitative Finance* **1**, 223-236.

Cont, R., J.-P. Bouchaud, 2000, Herd behavior and aggregate fluctuations in financial markets, *Journal of Macroeconomic Dynamics* **4**, 170.

Iori, G., 1999, Avalanche dynamics and trading friction effects on stock market Returns, *International Journal of Modern Physics* **10**, 1149-1162.

Iori, G., 2001, A microsimulation of traders activity in the stock market: the role of heterogeneity, agents' interactions and trade frictions, *Journal of Economic Behaviour and Organization* in press.

Johnson, N. F., *et al.*, 1998, Volatility and agent adaptability in a self organizing market, *Physica A* **256**, 230-238.

Laloux, L., P. Cizeau, J.-P. Bouchaud and M. Potters, 1999, Noise dressing of financial correlation matrices, *Physical Review Letters* **83**, 1467-1470.

Lillo, F., R. N. Mantegna, 2000, Variety and volatility in financial markets, *Physical Review E* **62**, 6126-6134.

Liu, Y., P. Gopikrishnan, P. Cizeau, M. Meyer, C.-K. Peng and H. E. Stanley, 1999, Statistical properties of the volatility of price fluctuations, *Physical Review E* **60**, 1390-1400.

Lux, T., M. Marchesi, 1999, Scaling and criticality in a stochastic multi-agent model of a financial market, *Nature* **397**, 498-500.

Mandelbrot, B. B., 1963, The variation of certain speculative prices, *Journal of Business* **36**, 394-419.

Mantegna, R. N., 1991, Lévy walks and enhanced diffusion in Milan stock exchange, *Physica A* **179**, 232-242.

Mantegna, R. N. and H. E. Stanley, 1995, Scaling behavior in the dynamics of an economic index, *Nature* **376**, 46-49.

——, 2000, *An introduction to econophysics: Correlations and complexity in finance* (Cambridge University Press, Cambridge).

Masoliver, J., M. Montero and J. M. Porrà, 2000, A dynamical model describing stock market price distributions, *Physica A* **283**, 559-567.

Masoliver, J., M. Montero and A. McKane, 2001, Integrated random processes exhibiting long tails, finite moments, and power-law spectra, *Physical Review E* **64**, 011110 (11 pages).

Masoliver, J., M. Montero, J. Perelló, 2001a, Return or stock price differences, submitted for publication.

Masoliver, J., J. Perelló, 2001b, Option pricing and perfect hedging on correlated stocks, submitted for publication.

Masoliver, J., J. Perelló, 2001c, A correlated stochastic volatility model measuring leverage and other stylized facts, submitted for publication.

Matacz, A., 2000, Financial modeling and option theory with the truncated Lévy process, *International Journal of Theoretical and Applied Finance* **3**, 143-160.

Perelló, J., J. M. Porrà, M. Montero, J. Masoliver, 2000, Black-Scholes option pricing within Itô and Stratonovich conventions, *Physica A* **278**, 260-274.

Perelló, J., and J. Masoliver, 2001d, Correlated stochastic volatility models and the leverage effect. Poster session: 4th "Applications of Physics to Financial Analysis (APFA)" conferences, London, 5-7 December 2001.

Perelló, J., and J. Masoliver, 2001e, The effect of colored noise and heavy tails on financial options, to appear in *Physica A*. Invited talk in: "Horizons in complex systems", Messina, 5-8 December 2001.

Plerou, V., P. Gopikrishnan, L. N. Amaral, M. Meyer, E. Stanley, 1999a, Scaling of the distribution of price fluctuations of individual companies, *Physical Review E* **60**, 6519-6528.

Plerou, V., P. Gopikrishnan, B. Rosenow, L. A. N. Amaral and H. E. Stanley, 1999b, Universal and nonuniversal properties of cross correlations in financial time series, *Physical Review Letters* **83**, 1471-1474.

Sornette, D., A. Johansen, J.-P. Bouchaud, 1996, Stock market crashes, precursors and replicas, *Journal de Physique I* **6**, 167-175.

Sornette, D., 1998, Linear stochastic dynamics with nonlinear fractal properties, *Physica A* **250**, 295-314.

Wilmott, P., 1998, *Derivatives* (Wiley, New York).

Agraiments

La feina aquí compil·lada no hauria estat possible sense el Dr. Jaume Masoliver, el meu director de tesi. A ell li he d'agrair la seva tutel·la i la seva gran implicació en aquest treball de recerca. Tots dos hem hagut d'aprendre des de zero totes les qüestions sobre el món de la borsa que aquí s'expliquen. El meu pare ha estat qui em va convèncer de que els mercats financers no són una cosa tan aliena a la física com pugui semblar d'entrada. Sense la seva insistència no m'hauria decidit a fer el pas i li estic molt agraït per aquest fet.

Tots els articles que aquí es presenten no són només exclusivament meus i del meu director de tesi. L'apèndix A del capítol 2 forma part d'un article realitzat amb Miquel Montero. L'article que configura el capítol 3 ha estat realitzat en col·laboració amb Miquel Montero i Josep Maria Porrà. Les discussions i comentaris de Josep Maria Porrà, Jaume Puig, Miquel Montero i Alan McKane han ajudat a realitzar el capítol 5. Estem particularment agraïts al Dr. Santiago Carrillo i al Dr. George H. Weiss, doncs han millorat amb les seves diverses suggerències aquest capítol 5. També he d'agrair a la Dra. Giulia Iori i al Dr. Jean-Philippe Bouchaud la seva hospitalitat durant les meves estades a Essex i Paris.

A la gent del departament també els hi de donar gràcies. Sobretot, a aquells que m'han ajudat desinteressadament sempre que els hi ho he demanat. Primordialment, al Miquel. Al posar-se a estudiar els mercats financers i integrar-se al nostre grup de recerca, hem passat hores discutint i, certament, m'ha ajudat a aclarir-me en alguns dels problemes que aquí es tracten. Després d'ell també hi he d'incloure tots els altres estudiants de doctorat amb qui he compartit aquests anys de tesi. Especialment, a aquells amb qui acabo de manera "gairebé" sincronitzada: en David i en Xavi.

Ja en l'àmbit estrictament personal, vull donar gràcies a tots aquells que m'han hagut d'aguantar fora de les hores de tesi. Han estat ells qui han vist la pitjor cara d'aquesta tesi. La meva família, especialment a la meva mare, ha estat sempre al meu costat. També, tota la meva "nova" família, la de la Núria, han mostrat un constant interès sobre això tan estrany que és fer una tesi doctoral. Evidentment, la Núria és la qui realment sap el que m'ha costat portar a terme aquesta "bogeria" i li agraeixo el suport incondicional i la seva paciència. Finalment, als amics de tota la vida, els jacksons, els hi vull demanar disculpes per no haver-me vist tant com jo mateix hauria desitjat.

Contents

Chapter 1

Options

A *derivative*, or a *derivative security*, is a financial instrument whose value depends on the values of other more basic underlying variables such as stocks, financial indices, etc.. In recent years, derivatives have become increasingly important in the world of finance. Derivatives are now traded actively on many exchange markets. Huge volumes of derivatives are also regularly negotiated outside exchange by banks, financial institutions, fund managers and corporations in what is termed the *over-the-counter* (*OTC*) market.

Traditionally, the variables underlying derivatives have been stocks, stock indices, foreign currencies, debt instruments, commodities, and future contracts. However, other underlying variables are becoming increasingly common. For example, the payoff from *credit derivatives* depends on the credit worthiness of one or more companies; *weather derivatives* have payoffs dependent on the average temperature at a particular location; *insurance derivatives* have payoffs dependent on the dollar amount of insurance claims of a specified type during a specified period; *electricity derivatives* have payoffs dependent on the spot price of electricity; and so on. Although their youth, these derivatives are also having great success and they are today the derivatives more in need of theoretical research. The studies on these derivatives are very sophisticated since they require the knowledge of an underlying variables from outside the financial markets. Weather derivatives are a good example of these new derivatives whose underlying variables have no relation with any economic index. Indeed, weather derivatives represents one of the many challenging problems that physicists can find in financial markets. And, in our opinion, physicists have a sufficient background to face to the problem and extract significant results.

Options are one of the most important from a large class of existing derivatives. As their name indicate, options are financial contracts that give to its owner the opportunity but not the obligation of performing a financial transaction during a future time, and according to their underlying variables. Very often, the variables underlying options are the prices of traded assets. Thus, for instance, *stock op-*

tions are derivatives whose value is dependent on the price of a stock, and similar definitions apply for index options, currency options, etc..

Options, as any other derivative, have their origin in the need of protecting investors from the market randomness. They had existed since a long time ago but it was only in 1973 that were first traded on an exchange, when the Chicago Board Options Exchange (CBOE) created the first standardized listed options. Initially, there were only calls on 16 stocks. And puts were not introduced until 1977. In the US, options are traded on CBOE, on the American Stock Exchange, and on the Philadelphia Stock Exchange. Worldwide, there are over 50 exchanges on which options and many other derivatives are traded. In Spain, derivatives gradually started been negotiated in the Mercado Español de Futuros Financieros (MEFF) from 1990.

This chapter introduces the topic that is central to this work, the subject of options. We particularize to the case of the European stock options but this does not avoid extending to any other underlying the results given herein. Chapter contents are non technical and divided in three sections. Section 1.1 gives a description of European option contracts. Section 1.2 comments on the most usual way of classifying options, and an explanation of the trading practice in real markets with those options is left to the Section 1.3. Chapter contains the main financial concepts which must be handled in the forthcoming chapters. Documentation for writing this introduction to the simplest derivatives is taken from the books of Hull (2000), Wilmott (1999), and Fernández (1996).

1.1 The European options

The *European options* are the most simple of a large variety of options since they only depend on the stock price at a one future date. These options are traded all over the world, and the European proper adjective only pretends locate their origin. The most common European options are the *European call* and *European put* contracts. Due to their extreme success in financial markets, they are often called *plain vanilla options*[1].

This section gives the definition of those two European options when the underlying is a stock share. Clearly, the same definitions apply when the underlying is any other *security*[2].

An *European call stock option* is a derivative giving to its owner the right but not the obligation to buy a share at a fixed date T for a certain price K. The

[1] This picturesque denomination comes from US where ice creams are very popular and, indeed, vanilla ice creams are most consumed.

[2] The name of security refers to any piece of paper that proves the ownership of stocks, bonds and other investments.

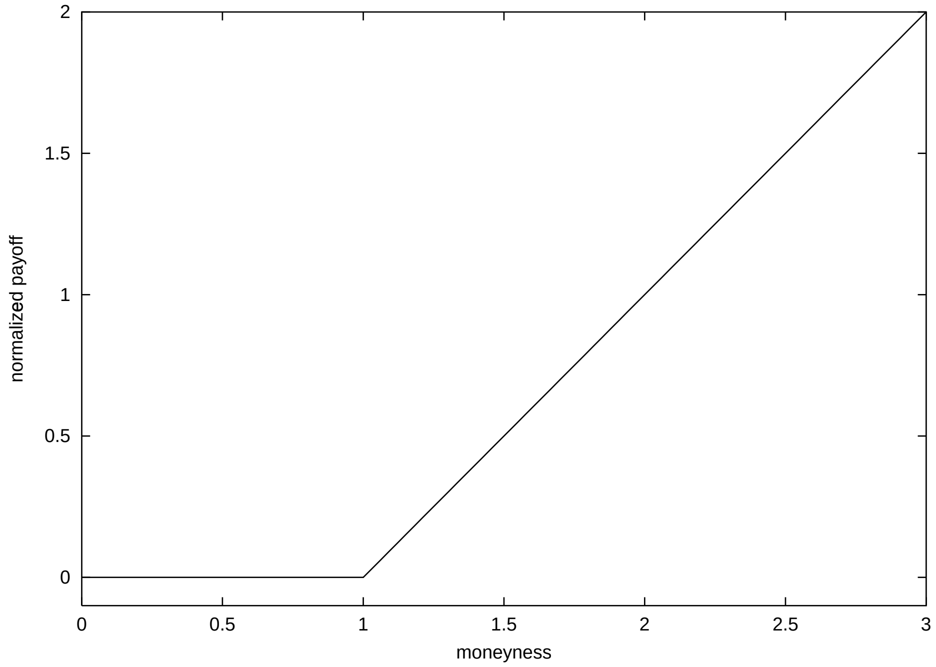

Figure 1.1: The normalized call payoff in terms of the moneyness

We plot the normalized payoff $C(S,T)/K$ as a function of the moneyness S/K. We observe that holding a call at maturity T may lead to a huge gain and avoids losses.

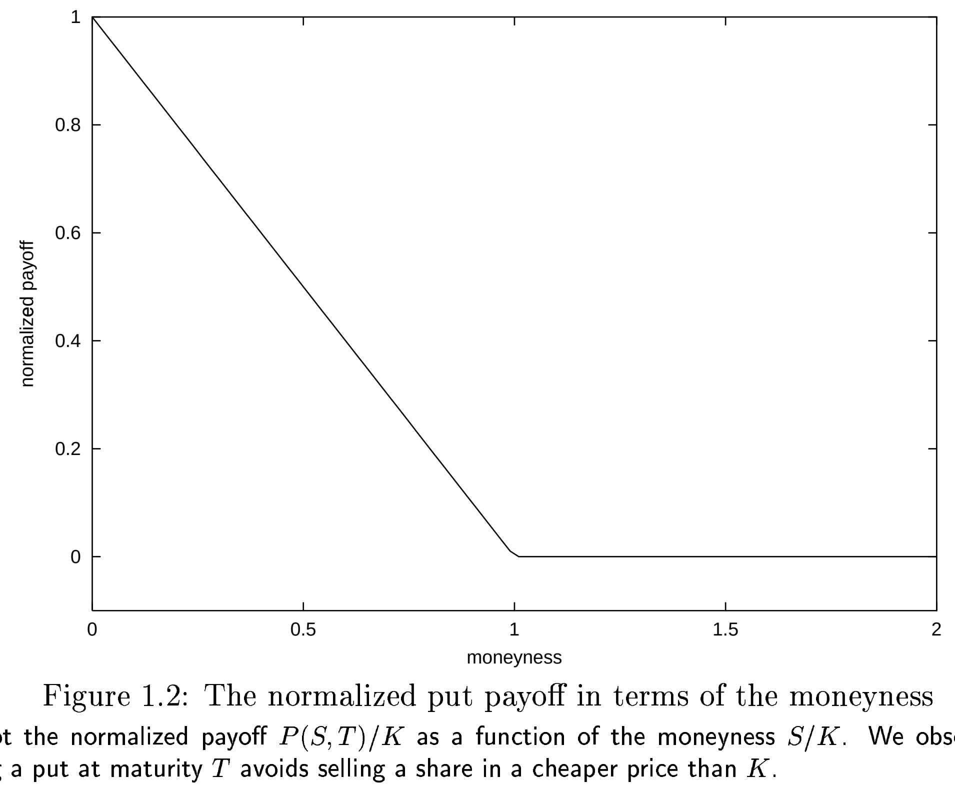

Figure 1.2: The normalized put payoff in terms of the moneyness

We plot the normalized payoff $P(S,T)/K$ as a function of the moneyness S/K. We observe that holding a put at maturity T avoids selling a share in a cheaper price than K.

preestablished price, K, is known as the *exercise price* or *strike price*[3]; the future date, T, is known as *expiration time* or *maturity time*[4].

The call option contract is mainly specified by the gain due to holding this option at the maturity time T. This gain called *payoff* is given by

$$(S-K)^+ \equiv \max[S-K,0], \tag{1.1}$$

where $S = S(T)$ is the share price at time T, and K is the strike price. In other words, the payoff reads

$$(S-K)^+ = \begin{cases} S-K & \text{if} \quad S \geq K, \\ 0 & \text{if} \quad S < K; \end{cases}$$

whose value is always greater or equal than zero.

We can define a kind of normalized payoff if we divide the payoff by the strike price. That is:

$$(S/K-1)^+ = \max[S/K-1,0]. \tag{1.2}$$

Observe that the normalized payoff is only function of the quotient between the stock and the strike price S/K called *moneyness.* Note also that moneyness represents the derivative and underlying prices in a dimensionless manner. Moneyness is a positive quantity since both share and exercise prices are positive. In other words, the moneyness is enclosed between the interval

$$0 \leq S/K < \infty.$$

In Fig. 1.1 we plot the normalized call profit of Eq. (1.2). We there see how the payoff raises linearly without limit as the moneyness grows, and how it avoids negative gain (*i.e.*, losses) when the moneyness becomes smaller than 1.

An *European stock put option* is a derivative giving to its owner the right but not the obligation to sell a share at a future maturity date T, and for a strike price K. The put option contract at maturity T has the following payoff

$$(K-S)^+ \equiv \max[K-S,0], \tag{1.3}$$

where again $S = S(T)$ is the share price at time T, and K is the strike price. Similarly to the call, we can also represent the put payoff as

$$(K-S)^+ = \begin{cases} K-S & \text{if} \quad S \leq K, \\ 0 & \text{if} \quad S > K. \end{cases}$$

[3] Other usual synonyms for giving a name to the variable K are *exercising price* or *striking price*. We will use all of them without distinction.

[4] Those are the most common synonyms for giving a name to the time T. As to the case for K, we will use them without distinction.

Observe that, since the stock and strike price are positive, the put value at time T is enclosed between K and 0.

Also analogously to the call case, we can define a normalized payoff by dividing the payoff by the strike price. For the European put, the normalized payoff reads

$$(1 - S/K)^{+} = \max[1 - S/K, 0], \tag{1.4}$$

where, as before, the normalized put profit is dimensionless and only function of the moneyness. In Fig. 1.2 we plot the normalized put profit of Eq. (1.4). We there show that the put decreases linearly with moneyness from 0 until 1 where put becomes worthless. The normalized payoff is thus enclosed between the values 0 and 1.

1.2 Options in terms of their moneyness

At any given time and for a given underlying security, the markets offer to traders different option contracts depending on their moneyness at time t when options contracts are negotiated. This classification is useful since, as we will see, the option price is basically a function of two variables: moneyness and time to maturity $T - t$. We can thus divide options into the following three categories:

> An *out the money (OTM) option* is either (a) a call option where asset price is less than the exercising price or (b) a put option where the asset price is greater than the strike price.
>
> An *at the money (ATM) option* is an option (call or put) which exercising price equals to the price of the underlying.
>
> An *in the money (ITM) option* is either (a) a call option where asset price is greater than the strike price or (b) a put option where the asset price is less than the strike price.

However, according to the definition given above, the ATM options are just ideal contracts. It is almost impossible to find in real markets an option contract whose moneyness is exactly 1. Hence, the ATM equality definition is extensible to a finite interval. We choose here the moneyness interval for an ATM option to be $0.97 < S/K < 1.03$. In Table 1.1, we summarize those definitions, and show the differences between ITM and OTM put and call options.

1.3 Trading European options

There are many reasons why investors may find options useful. Here we give a broad overview on this interest. However, two facts should be emphasized concerning to

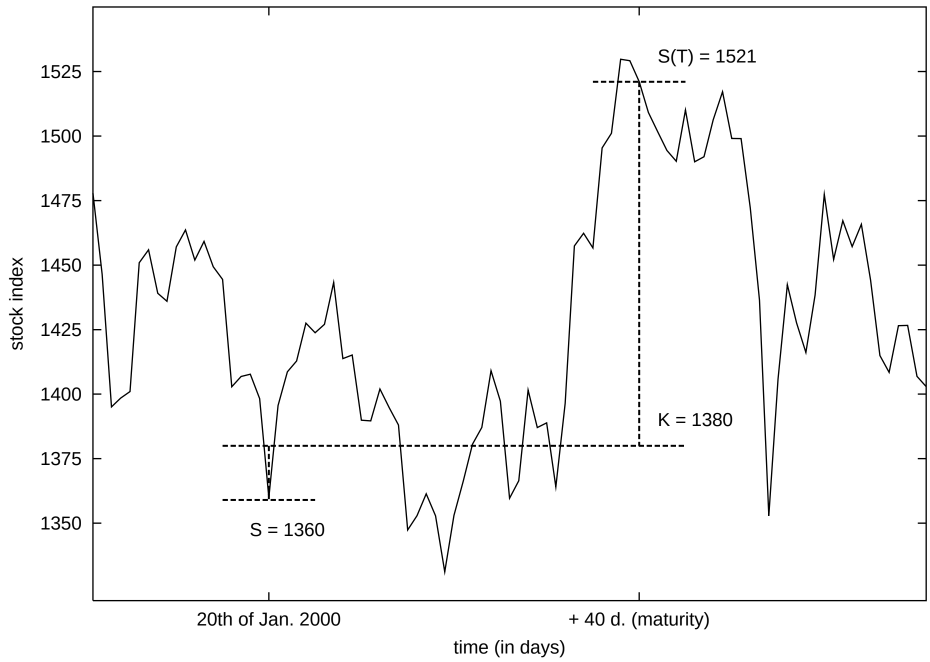

Figure 1.3: Stock index evolution holding a call

We follow the evolution of the S&P 500 stock cash index during 40 days. We buy a call the 20th October of 2000 whose maturity is at 40 days later and its strike price is $K = 1380$. In this case, exercising the call at maturity represents a positive profit quantity $S - K = 1521 - 1380 = 141$ points.

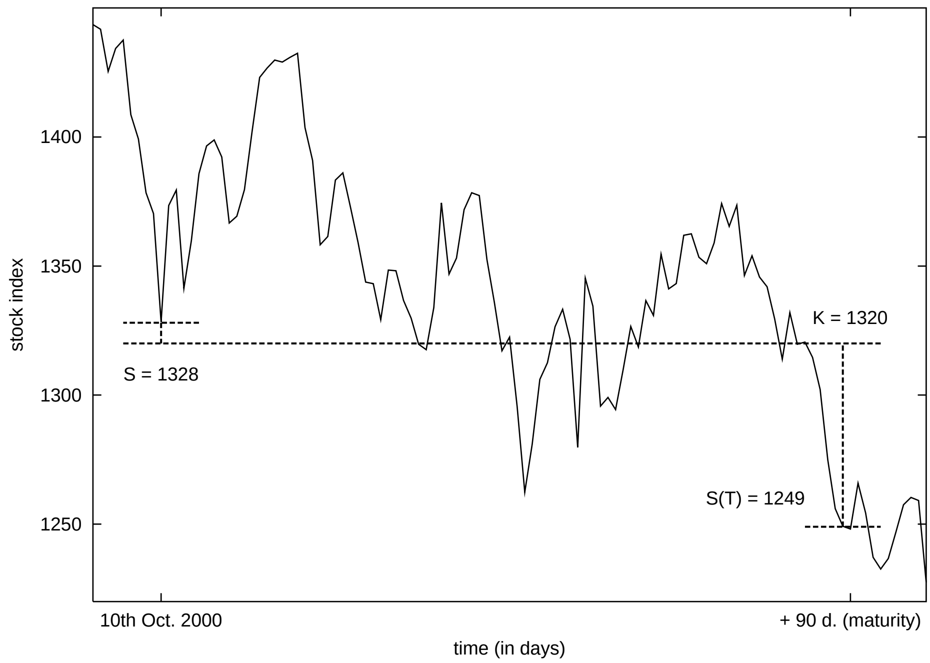

Figure 1.4: Stock index evolution holding a put

We follow the evolution of the S&P 500 stock cash index during 90 days. On the 10th October of 2000, we buy a put whose maturity is 90 days later and its strike price is $K = 1320$. At maturity, we can exercise the put and have a profit $K - S = 1320 - 1249 = 71$.

Table 1.1: Options in terms of their moneyness

Table classifies call and put options according to the value of the moneyness. Observe that options with moneyness greater than 1.03 are ITM calls or OTM puts. Conversely, options with moneyness smaller than 0.97 are referred to OTM calls or to ITM puts.

	$S/K \leq 0.97$	$0.97 < S/K < 1.03$	$1.03 \leq S/K$
Call	OTM	ATM	ITM
Put	ITM	ATM	OTM

the contract specifications before proceeding further. First, European options can be exercised only on the expiration date itself. If the investor prefers a contract specifying that the derivative may be exercised at any time before maturity, a different option type must be hold: the *American options*. Second, an option gives to the holder the right to perform a financial transaction but he/she holder does not have to exercise this right. This feature distinguishes option contracts from *forward* and *future* contracts where there exists the obligation to buy or sell the underlying assets.

As we have said, a call gives the opportunity to buy the stock for a price K at time T and, conversely, a put gives the chance to sell the stock for K at T. Observe that these contracts offer two opposite choices, and traders obviously would be interested on taking advantage of puts or calls in accordance to their own trading strategies. Roughly speaking, we say that there are two groups of traders interested in buying a call or a put.

There exists a groups of traders called *hedgers* whose purpose is to buy a security against the random fluctuations of underlying stocks. Hedgers buy today a call in case that they plan to buy a stock share at maturity. By holding a call, hedgers have the guaranty of buying the underlying at a non higher price than the strike price K. Hedgers are in this way protected against sudden increases of the underlying share. However, hedgers prefer buying today a put option (instead of a call) if they expect to sell (instead of buying) a share at maturity. Then, hedgers will not sell the share in cheaper price than K, and they are therefore preserved to an undesirable decay of the stock share price.

The second group of traders, called *speculators*, is much more aggressive. Whereas hedgers want to avoid an exposure to adverse movements in the underlying, speculators wish to take a position in the market, either they are betting that price will go up or down. When the speculators have the certainty of an stock rise, they decide to buy a call since they will buy an undervalued share. But if the feeling is that the stock will decrease, they buy a put since to sell an overvalued share afterwards.

In Fig. 1.3, we plot the random evolution of the S&P 500 stock cash index from the day of buying a call until maturity. The plot shows a time interval where index tends to go up. For this situation, trader gets some profit due to holding the call since exercising the call means having a 141 points payoff. In this case, hedgers who are holding calls are safe against the large index rise, and speculators earn large amounts of money if they predict the positive trend. In accordance to Tab. 1.1, we see that call bought is an in the money (ATM) option with moneyness 0.99. In Fig. 1.4, we plot the random evolution of the S&P 500 stock cash index from the day of buying the put until the maturity. The plot depicts days when index goes basically down. In this case, trader holding a put gains money at time T since exercising the put means earning 71 points. Hence, hedgers holding puts are safe against these large index decrease, and speculators earn large amounts if they forecast this negative trend. As Tab. 1.1 indicates, the put bought is also an ATM option since its moneyness 1.00.

Clearly, these investor profiles are the extreme ways of describing the trader behavior. In real life, all traders behave either like hedgers or like speculators. Depending on many different factors, traders may be interested in not to lose money and they reduce or eliminate underlying risk (hedgers) buying options. Alternatively, traders may bet that there will be sudden stock price change and they assume the risk involved and expect a big profit holding an option (speculators).

Summary

Derivatives are financial products designed to protect investors from the market randomness. In this chapter we give main definitions of the concepts concerning the options and, more generally, the derivatives. We focus our attention on the European options. We describe their contract specifications, the way they are usually classified and their utility in worldwide financial markets in terms of the traders demand.

References

Fernández, P., 1996, *Opciones, Futuros e Intrumentos Derivados* (Deusto, Bilbo).

Hull, J. C., 2000, *Options, Futures, & Other Derivatives* (Prentice-Hall, Upper Saddle River, NJ).

Wilmott, P., 1999, *Derivatives* (Wiley, New York).

Chapter 2

Market equilibrium and incomplete option pricing methods

An European option, as any other derivative, needs a price. The only information given by the contract specifications is its payoff that fixes the option price at maturity. However, the price to be paid for the option when it is bought remains unknown. It seems reasonable to believe that the price for the option depends on the underlying dynamics, from the day option is bought until the maturity date. Good knowledge of the underlying is thus indispensable, and investigations in this way look for an adequate *market model*, or *asset model*, consistent with real market behavior.

But not only the underlying behavior is important since there even exists a more fundamental problem. Thus, answers to the questions concerning on how an option price is given and why is the resulting price the *fair price* are also crucial. All along the 20th century, there have been a strong effort to obtain a fair option price to the option. The so-called *option pricing methods* pretend to give a fair price for the option which must be neutral either to the seller and to the buyer of the option. Obviously, the pricing problem can be extended to a more general framework of obtaining the fair price of any *asset* (*i.e.*, any generic financial contract that is susceptible to be sold or bought).

Option pricing and asset modeling involve many disciplines. Tools used basically come from probability theory, stochastic methods, data analysis and, more generally, mathematics and lately even from physics. But hypothesis and assumptions on financial markets are based on economic theory. This mixture resulted in an specific field called *mathematical finance* or *econometrics* which has become very important because of the necessity of having an accurate description of the financial markets nature. The achievement by Fischer Black and Myron Scholes (1973) of a consistent theory and its easy application to the real markets greatly enhanced research activity in this field. But prior to the "path-breaking (Smith (1976))" article of Black and Scholes, there had been other works which pursued not only a good market model but also a fair option price along with its reasonable justification. These preliminary studies constitute the core of this chapter and the first step which will drive us in a future chapter to the Black-Scholes option pricing method.

The test for any option pricing theory is based on the absence of *arbitrage* or the *market equilibrium* demands. Absence of arbitrage and market equilibrium are here taken as synonyms and result to the same economic theoretical restrictions. Arbitrage absence denies the existence of trading strategies which represents a sure profit for the investor. In other words, arbitrage avoids the possibility of money profit for free, *i.e.*, without taking any risk.

This chapter is divided in two sections. Section 2.1 is devoted to European options theoretical restrictions, and the review of several attempts for deriving a fair option price are left to Section 2.2. We there summarize main option pricing works and make them more legible with an unified and coherent explanation but also with criticisms based on the restrictions of Section 2.1. An special attention has been given to the mathematical aspects of those works and, specifically, to the stochastic methods needed for building up any of those option pricing methods. In Appendix A, we show the existent differences between performing data analysis with stock and log-price fluctuations. This Appendix is a brief summary of the research paper Masoliver *et al.* (2001a)

2.1 Properties of European option prices

Robert Merton (1973b) established an exhaustive collection of equilibrium restrictions[1] on call and put option prices without making any assumptions on the underlying process. In this section we will review some of these properties. We base this exposition on Merton (1973b), Fernández (1996), and Bergman *et al.* (1996).

Harry Markowitz (1952) asserted that risk coming from stocks fluctuations can be diversified away, at least minimized, by holding a collection of different assets, that is, a *portfolio*. Thus, the trading strategies are basically aimed to correctly manage an *optimal portfolio*, *i.e.*, a portfolio which has a maximum profit with a minimum risk. Within this framework, Merton (1973b) took a portfolio with *net zero investment*, in other words, a portfolio which does not need an initial positive (or negative) amount of money to be carried on. This portfolio buys an asset with the money obtained by selling another different asset of the portfolio. It is usual to add to the collection of assets an asset called *bond* which is a risk-free asset with deterministic dynamics

$$B(t_2) = B(t_1)e^{r(t_2-t_1)} \qquad (t_2 \geq t_1), \tag{2.1}$$

where $B(t)$ is the bond price and r is the *risk-free interest rate.*

[1]From now on, we will use without distinction the terms "equilibrium restrictions on option prices" and "properties of the option".

The properties derived by Merton (1973b) depend on the so-called *dominant arguments*. Merton says that "portfolio A is dominant over portfolio B if over some given time interval the profit due to holding A is not less than B profit for all states of the world, and the A gain is strictly greater than the B gain for at least one state of the world". In other words, portfolio A is dominant over portfolio B if A has a higher profit than B in one future situation and, at least, the same profit for the rest of possible situations. Thus, the investor that holds portfolio A can have a higher profit than investor holding B, and he also does not take any risk of losing more than the investor holding B. In equilibrium no dominant or dominated portfolio, or even no dominant or dominated individual security can exist. If a dominant security exists, everyone would prefer to hold this security. The price will be bid up until the dominance disappears. The results derived through the dominance arguments are completely general. If the implications based on an specific dynamics for the underlying violate these restrictions, economic fundamentalists would reply that the proposed model must be deficient in some way.

As it has already mentioned, the results of these dominance arguments will be used as a general consistency criteria against which subsequent models, or even option pricing methods, may be conveniently measured. We therefore study the case for the European call option and, afterwards, we will thus see how to extend the restrictions to the put case.

2.1.1 Call price properties

We here enumerate the most important restrictions on the European call option price. Restrictions are given by several inequalities between two call prices evaluated in different conditions or between the call, underlying and strike prices. Hence, we prove a call price restriction by showing that if restriction did not exist there would be hidden arbitrage opportunities with a certain trading strategy. In some cases, we specify an strategy which would allow for profits without taking risk. And, in other cases, we compare two portfolios and demonstrate that if inequality is not true one portfolio is dominant over the other.

Portfolio strategies described assume no *transaction costs*[2], *market liquidity*[3], and constant risk-free interest ratio. In addition, all portfolios presented contain no other assets than calls, underlying assets or bonds.

(i) Call prices are non-negative. A call option exercise is voluntary and this opportunity must be charged on the option price since otherwise option not only would give the right to do an advantageous financial transaction but also would

[2] The costs associated with the process of buying and selling.

[3] The degree to which an asset (or, more generally, the market) can be quickly and cheaply turned into money.

Table 2.1: Call price must be cheaper than underlying price

We here show that $C(S,t) > S(t)$ would allow for arbitrage opportunities. The strategy consisting on selling a call and buying a share and bonds with quantity $S(t) - C(S,t)$ would represent a sure profit at maturity. We here define the time t' as time to maturity $T - t$.

present time t		maturity time T $S(T) \leq K$	$S(T) > K$
Portfolio strategy:			
Buy a share	$-S(t)$	$S(T)$	$S(T)$
Sell a call	$C(S,t)$	0	$K - S(T)$
Buy bonds	$S(t) - C(S,t)$	$e^{rt'}[C(S,t) - S(t)]$	$e^{rt'}[C(S,t) - S(t)]$
Portfolio	$\Pi = 0$	$e^{rt'}[C(S,t) - S(t)] + S(T)$	$e^{rt'}[C(S,t) - S(t)] + K$
	If $C(S,t) > S(t)$:	$\Pi > 0$	$\Pi > 0$

represent a sure profit even if it is exercised or not[4]. Thus,

$$0 \leq C(S,t) \qquad \text{for } t \leq T. \tag{2.2}$$

This question is deeper discussed and rigorously proved in restriction *(iii)*.

(ii) Buying a call is never more expensive than its corresponding underlying. If this is not the case, value of the portfolio buying today a share with the money of selling a call and bonds would imply a sure profit at T. If option expires without been exercised, portfolio still contains the share which never can have negative prices. But if option is exercised, then portfolio manager is obligated to give the share to the call's holder but still keeps a positive amount of money. In Table 2.1 we show that this strategy contains arbitrage opportunities in the case that call is more expensive than underlying share. Hence, for avoiding arbitrage opportunities, call must obey

$$C(S,t) \leq S(t), \qquad t \leq T. \tag{2.3}$$

(iii) At any date before maturity, the call must be sold at least for the difference between the stock price and its discounted exercise price. By discounted exercise

[4] At the expiration date, *i.e.*, $t = T$, the call will be priced as the maximum of either the difference between the stock price and the exercise price, $S - K$, or zero. See Sec. 1.1 for further details concerning the contract specifications and definitions of the different parameters here presented.

Table 2.2: Call is more expensive than the discounted payoff

We here show that $0 < C(S,t) < S - Ke^{-r(T-t)}$ would allow for arbitrage opportunities. The strategy consisting on buying a call but selling a share and bonds (with quantity $S(t) - C(S,t)$) would represent a sure profit at maturity. We also express time evolution in terms of the time to maturity $t' = T - t$.

	present time t	maturity time T: $S(T) \leq K$	maturity time T: $S(T) > K$
Portfolio strategy:			
Sell a share	$S(t)$	-	-
Buy a call	$-C(S,t)$	0	$S(T) - K$
Buy bonds	$C(S,t) - S(t)$	$e^{rt'}[S(t) - C(S,t)]$	$e^{rt'}[S(t) - C(S,t)]$
Portfolio worth	$\Pi = 0$	$e^{rt'}[S(t) - C(S,t)]$	$e^{rt'}[S(t) - C(S,t)] + S(T) - K$
Terminal values if $0 \leq C(S,t) < S(t) - Ke^{-rt'}$:		$\Pi > 0$	$\Pi > 0$

price, we understand $Ke^{-r(T-t)}$. This restriction is only partially asserted in *a.* since now call has a more precise lower boundary. In this way,

$$\left(S(t) - Ke^{-r(T-t)}\right)^+ \leq C(S,t) \qquad (t \leq T). \tag{2.4}$$

If this would not be the case, a portfolio with net zero investment that buys today a call and bonds but sells a share will provide a sure profit. In Table 2.2 we show the time evolution of this portfolio and prove that restriction is necessary for avoiding dominant positions.

And if we take into account the restrictions (2.3) and (2.4), we will then confine the European call price into a lower and upper bound. That is:

$$\left(S(t) - Ke^{-r(T-t)}\right)^+ \leq C(S,t) \leq S(t), \qquad \text{for } t \leq T. \tag{2.5}$$

There is also a case of academic interest where upper and lower bounds collapse becoming an equality. For $T \to \infty$ and finite t, the option is called *perpetual option* whose price, for the call, is

$$C_p(S,t) = S(t), \tag{2.6}$$

and thus equal to the underlying asset price.

(iv) If two options differ only on exercise price then the option with lower exercise must be sold for a price which is not less than the option with the higher exercise. This can be demonstrated by constructing two portfolios, A and B, where portfolio

Table 2.3: Call price dominance argument for the exercise price

Demonstration that a call with exercise K will have a higher price than or an equal price to option with higher exercise $K \leq K'$. We show terminal values of portfolios A and B for different situations at the maturity T.

Portfolio worth	present time t	maturity time T $S(T) \leq K$	$K < S(T) \leq K'$	$K' < S(T)$
Π_A	$C(S,t)$	0	$S(T) - K$	$S(T) - K$
Π_B	$C'(S,t)$	0	0	$S(T) - K'$
	Terminal values:	$\Pi_A = \Pi_B$	$\Pi_B \leq \Pi_A$	$\Pi_B \leq \Pi_A$

A contains a call $C(S,t)$ with exercise price K, and portfolio B contains one call $C'(S,t)$ with exercise price $K \leq K'$. We observe in Table 2.3 that, for all possible terminal stock prices, value for the portfolio A is always greater than or equal to portfolio B terminal value. Thus, for avoiding dominance, current price of A must be greater than or equal to the current price of B. In case that current price of B were greater than that of A, the portfolio A would have arbitrage opportunities and B would be a dominated portfolio. Therefore,

$$C'(S,t) \leq C(S,t), \tag{2.7}$$

where call with prime represents a call whose exercising price $K \leq K'$, and also recall that time t is always shorter than T.

(v) If two options differ only on the maturity time then the option with the longer maturity must be sold for a price not lower than the option with the shorter maturity. If this is not the case, strategy consisting in buying today the longest maturity option and selling shortest maturity call gives arbitrage opportunities[5]. In Table 2.4 we specify this strategy when options have maturities $T \leq T'$, and show that arbitrage opportunities exist at time T. Thus, equilibrium of the markets assumption asks for

$$C(S,t) \leq C'(S,t) \qquad (t \leq T \leq T'). \tag{2.8}$$

[5]There are some subtleties concerning this strategy. Econometricians assert that this portfolio cannot fully prove that there exists arbitrage in this strategy since it is necessary that call with maturity T' is "correctly priced by the market at time T". For this reason, they say that this strategy only contains "quasi-arbitrage opportunities" and propose a more accurate proof by indirect arguments which needs the American option price restrictions (see for more details *e.g.* Smith (1976)).

Table 2.4: Call price arbitrage argument for the maturity date

Strategy consists on buying a call with maturity T' and selling a call with maturity $T \leq T'$ having both options the same striking price. Our strategy finishes at time T where, in case that $C'(S,t)$ is correctly priced by the market (*i.e.*, $C'(S,T)$ obeys restriction (2.5)), portfolio will have a non negative value. Absence of arbitrage requirement asserts that portfolio at current time t must follow the same inequality as the one of time T which corresponds to Eq.(2.8).

	present time t	maturity time T: $S(T) \leq K$	maturity time T: $K < S(T)$
Portfolio strategy:			
Buy a call (maturity T')	$-C'(S,t)$	$C'(S,T)$	$C'(S,T)$
Sell a call (maturity T)	$C(S,t)$	0	$K - S(T)$
Portfolio worth Π	$C(S,t) - C'(S,t)$	$C'(S,T)$	$C'(S,T) - [K - S(T)]$
	Terminal values:	$\Pi \geq 0$	$\Pi \geq 0$

Finally, we take into account Eqs. (2.5) and (2.6), and a trivial consequence of Eq. (2.8) is that option price is also enclosed between two critical call prices

$$(S(t) - K)^+ = C'(S,t) \leq C(S,t) \leq C_p(S,t) = S(t) \qquad (t \leq T), \qquad (2.9)$$

where $C'(S,t)$ contract expires at the current date $t = T'$, and $C_p(S,t)$ is the perpetual which never expires (*i.e.*, $T' \to \infty$). Observe that inequality is identical to that of Eq. (2.5) but now bounds are expressed in terms of the contract maturity date.

2.1.2 Put price properties

The European put option price has analogous properties to that of the call. The put price restrictions, $P(S,t)$, can be derived easily using similar dominant arguments[6]. The so-called *put-call parity* is the most transcendental equation because relates the call price $C(S,t)$, the put price $P(S,t)$, and the underlying $S(t)$. The equation reads

$$P(S,t) + S(t) = C(S,t) + Ke^{-r(T-t)}. \qquad (2.10)$$

[6]In general, the portfolio strategies for the put case would then consist on doing the inverse operations to the ones of the call. If we bought a certain asset in the call portfolios case, we now must sell this asset. And if we sold the asset we now have to buy it.

Table 2.5: Put-call parity proof by dominance argument

We here construct two portfolios, A and B. These portfolios have the same terminal values, independently on the price of the underlying $S(T)$. Hence, for avoiding dominance, the portfolios must have the same price at time t. Observe that this implies that Eq. (2.10) must hold.

	present time t	maturity date T	
		$S(T) \leq K$	$K < S(T)$
Strategies:			
Portfolio A			
Buy a call	$-C(S,t)$	0	$S(T) - K$
Buy bonds	$-Ke^{-r(T-t)}$	K	K
	$\Pi_A = -[C(S,t) + Ke^{-r(T-t)}]$	K	$S(T)$
Portfolio B			
Buy a put	$-P(S,t)$	$K - S(T)$	0
Buy a share	$-S(t)$	$S(T)$	$S(T)$
	$\Pi_B = -[P(S,t) + S(t)]$	K	$S(T)$
	Terminal values:	$\Pi_A = \Pi_B$	$\Pi_A = \Pi_B$

As is represented in Table 2.5, the put-call parity (2.10) is proved by dominance arguments. We there show the evolution of a portfolio A that contains bonds and a call, and of a portfolio B that contains a put and a share. The two portfolios have the same value at maturity, independently on the underlying dynamics. In consequence, the dominance argument dictates that both portfolios must be equivalent at time t. And we see in Table 2.5 that this finally implies the validity of Eq. (2.10).

The put-call parity helps us to translate call properties to the put price with very simple manipulations. For instance, the boundaries for the call price given by Eq. (2.5) can be represented in terms of the put price with the help of the put-call parity (2.10). In terms of the put, call boundaries (2.5) now read

$$\left(S(t) - Ke^{-r(T-t)}\right)^+ \leq P(S,t) + S(t) - Ke^{-r(T-t)} \leq S(t),$$

where t is the current time always enclosed between $(0, T)$. We are mainly interested in giving the upper and lower bounds but to the put price. After trivial calculus,

Table 2.6: Put price dominance argument for the exercise price

Demonstration that a put with exercise K will have a lower price than or an equal price to option with higher exercise $K \leq K'$. We show terminal values of portfolios A and B for different situations at the maturity T.

Portfolio worth	present time t	maturity time T $S(T) \leq K$	$K < S(T) \leq K'$	$K' < S(T)$
Π_A	$P(S,t)$	$K - S(T)$	0	0
Π_B	$P'(S,t)$	$K' - S(T)$	$K' - S(T)$	0
	Terminal values:	$\Pi_A \leq \Pi_B$	$\Pi_A \leq \Pi_B$	$\Pi_A = \Pi_B$

put boundaries result to be

$$\left(Ke^{-r(T-t)} - S(t)\right)^+ \leq P(S,t) \leq Ke^{-r(T-t)} \qquad (t \leq T). \tag{2.11}$$

In the call option case, there was a very particular case where the upper and lower bounds collapsed. This happens for the perpetual option limit, *i.e.*, when its maturity tends to infinity . From Eq. (2.11), we can get the perpetual put by performing the limit $T \to \infty$ and thus obtain that

$$P_p(S,t) = 0. \tag{2.12}$$

In contrast with the perpetual call (2.6), the perpetual put is worthless.

Let us study the relation between two options which only differs on their exercises. By dominance arguments, we then conclude that

$$P(S,t) \leq P'(S,t) \qquad (K \leq K'). \tag{2.13}$$

Thus, an European put option is cheaper than another European put with a higher striking price. And this is just the opposite behavior than the one described for the call by Eq. (2.7). In Table 2.6, we demonstrate the inequality (2.13).

The bounds of Eq. (2.11) allows us to determine the behavior of the put price in terms of its maturity date. The difference between two puts that only differs on their maturity date satisfies the following inequality

$$\left(Ke^{-r(T-t)} - S\right)^+ - Ke^{-r(T'-t)} \leq P(S,t) - P'(S,t),$$

where we have taken into account inequality (2.11). Observe that inequality depends on the moneyness S/K. Hence, assuming that $T \leq T'$, the lower bound let us prove us that

$$P'(S,t) \leq P(S,t) \qquad \text{for } S/K \leq e^{-r(T-t)} - e^{-r(T'-t)}. \tag{2.14}$$

Table 2.7: Summary of the European call and put option price properties

We enumerate some properties of the option prices given by Eqs. (2.2)–(2.9) for the call, and given by the Eqs. (2.11)–(2.15) for the put. Observe that put prices have not an specific inequality depending on the their maturity date. See Eqs. (2.14) and (2.15) for more details. We recall that $t' = T - t$.

	Call option	Put option
Current price	$C(S,t)$	$P(S,t)$
Payoff	$(S(t)-K)^+$	$(K-S(t))^+$
Price Bounds		
$S(t)-Ke^{-rt'}>0$	$S(t)-Ke^{-rt'} \leq C(S,t) \leq S(t)$	$0 \leq P(S,t) \leq Ke^{-rt'}$
$S(t)-Ke^{-rt'} \leq 0$	$0 \leq C(S,t) \leq S(t)$	$Ke^{-rt'} - S(t) \leq P(S,t) \leq Ke^{-rt'}$
Exercise		
$K' \geq K$	$C'(S,t) \leq C(S,t)$	$P'(S,t) \geq P(S,t)$
Maturity		
$T' \geq T$	$C'(S,t) \geq C(S,t)$	?
Perpetual option	$S(t)$	0

A similar analysis but with the difference $P - P'$ instead of $P' - P$, let us also show that

$$P'(S,t) \geq P(S,t) \qquad \text{for } S/K \geq e^{-r(T-t)} - e^{-r(T'-t)}. \tag{2.15}$$

Recall that in both inequalities (2.14) and (2.15) $T \leq T'$. Roughly speaking, the first inequality says that longer maturity date implies higher price for the put only if moneyness is cheaper enough. Meanwhile, the second one dictates that the put price lowers when the moneyness is high enough.

During this section, we have been giving the main properties of the put and call prices. In Table 2.7, we express their payoff and summarize all the restrictions related to the exercise or maturity date option variables. We recall that these inequalities are not only independent on the market model but also independent on the method employed to obtain a price for the option. The restrictions serve us as a test for the adequacy of the models and option pricing methods proposed. An adequate model or method is the one that accomplishes the market equilibrium theoretical demands. Forthcoming section will comment several historical market models and option pricing methods, and restrictions enunciated in Table 2.7 will thus let us criticize them in the name of the market equilibrium requirements.

2.2 Incomplete call pricing methods

Before the Black and Scholes (1973) option pricing method was developed, there had been several works proposing a fair price for the call. These studies, which we will now outline in a chronological order, trace an historical trajectory through the main features involving the option pricing problem. They are representative of the main difficulties for deriving an adequate market model and an option price consistent with market equilibrium restrictions summarized in Table 2.7.

Louis Bachelier is considered the father of mathematical finance because, in 1900, he presented his PhD thesis containing, for the first time, an stochastic analysis of the stock and option markets[7]. Despite Bachelier's very early interest in stochastic market modeling, research on this topic is not noticeable again until 1930's. A renewed interest on financial markets appeared in the embryo school of American economists. In an ideal and theoretical framework, they believed that markets are *perfect* in the sense that one cannot forecast future price changes based on past history alone and denied the existence of arbitrage opportunities. In other words, markets are in equilibrium thus implying the restrictions exposed in the last section.

Following this theoretical principles, Kendall (1953) analyzed the American markets and found some disagreements between Bachelier's model and real markets. A decade later, Sprenkle (1964) suggested a new market model with its subsequent option price and Boness (1964) showed the necessity of taking into account the time value of money in the resulting option price. Finally, Samuelson (1967) proposed a general framework in option methodology giving some freedom in the underlying asset dynamics choice. A few years later, Paul Samuelson with his pupil Robert Merton left almost finished a consistent option pricing methodology. However, they were Fischer Black and Myron Scholes (1973) who culminated the efforts, from Bachelier to Samuelson, to obtain a fair option price.

As we have mentioned, the authors of all these works derived a market model and an option pricing methodology trying to avoid any contradiction with the restrictions on the option prices. Nevertheless, these methods are called *incomplete equilibrium methods* since they fail to find an option price consistent with the market equilibrium requirements summarized in Table 2.7. We want to expose as clear as possible these different attempts to solve the pricing problem and make them some allegations sustained on the market equilibrium hypothesis. For doing this, we have particularized to the case of the European call option price.

[7] An interesting work about the Bachelier's life has recently appeared. The authors are Courtault *et al.* (2000) and the article is aimed to celebrate the century of his thesis defense that took place the 29th of March, 1900.

2.2.1 Bachelier (1900): the first option price

The thesis of Bachelier (1900), together with his subsequent works, has deeply influenced the whole development of stochastic calculus and mathematical finance. In spite of his remarkable contributions, he remained in obscurity for decades[8]. Recently, there has appeared a renewed interest on his scientific biography due to the strong success of mathematical finance as a scientific discipline. We follow Courtault *et al.* (2000) for a short summary of the Bachelier's biography. And we follow Bachelier's own work (Bachelier (1900)) to explain, with contemporary notation and terminology, his first derivation of the European call option price.

Louis Bachelier (1870-1946) was born into a respected bourgeois family of Le Havre. His father was a merchant, the vice-consul of Venezuela at Le Havre and an amateur scientist. His mother was a banker's daughter. Unfortunately, just after Louis graduated from the secondary school, his parents died. He therefore had to interrupt his studies to continue his father's business and take care of their brothers.

This event had far-reaching consequences for his academic career, and this explains why Bachelier did not follow any *grande école* with the French scientific elite. Nevertheless, as the head of the family enterprise, he became acquainted with the world of financial markets which may justify the origin of his scientific interest in the financial market nature. Military service soon followed, bringing further delay of his studies. In 1892, Bachelier finally arrived to Paris and continued his university education at the Sorbonne. There is not much information available about these years but it is known that he attended lectures of Joseph Boussinesq and Henri Poincaré, and that his marks of the 1895 register were largely below those of his classmates Langevin and Liénard.

Although his delay, Bachelier's development was fast enough to complete his celebrated thesis. The date March 29, 1900, should be considered as the birth date of mathematical finance. On that day, the French postgraduate student successfully defended his thesis called *Théorie de la spéculation*[9]. The thesis was published in *Annales Scientifiques de l'École Normale Supérieure* and was strongly supported by Henri Poincaré, the Bachelier's supervisor (see Bachelier (1900)). The thesis is an analysis of the stock and option markets and contains several ideas of enormous value in both finance and probability. It was the first attempt to use advanced mathematics in finance and consequently the first attempt to market modeling[10].

The first part of Bachelier's thesis contains a detailed description of products available at that time in French financial markets. After these technical preliminar-

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[8]Indeed, Bachelier was an obscure figure in the mathematics of twentieth century about whom only a few facts could be found in the literature.

[9]In accordance with the tradition, he also defended a second thesis on a subject chosen by the faculty which for his case was on mechanics of fluids. This also helps us to figure out the Bachelier's mathematical background.

[10]Results derived in Bachelier thesis contains several errors. However, they are all rectified in the forthcoming thesis explanation.

ies, Bachelier began with the mathematical modeling of stock price movements and formulated the principle that "the expectation of the speculator is zero"[11]. Bachelier found "the general shape of the probability curve" by two different methods. The first method starts from "the principle of joint probabilities" which is now known as the *Chapman-Kolmogorov equation* and it assumes that underlying process is *Markovian*. The Chapman-Kolmogorov equation reads

$$p_S(S,t|S_0) = \int_{-\infty}^{\infty} p_S(S,t|S',t')p_S(S',t'|S_0)dS' \qquad (t \geq t' \geq 0), \tag{2.16}$$

where $p_S(S,t|S_0)$ is the *probability density function* (*pdf*) of stock price S at time t if at time $t_0 = 0$ the stock price was S_0. Bachelier proposed a Gaussian function for this conditional probability density:

$$p_S(S,t|S_0) = \frac{A}{\sqrt{2\pi}}\, e^{-A^2(S-S_0)^2/2}, \tag{2.17}$$

where $A = A(t)$ may depend on time. Although Bachelier did not study the uniqueness of this solution, he asserted that the probability $p_S(S,t|S_0)$ has to be normalized thus limiting the function $A(t)$ behavior. Implementing the normalization condition into Eq. (2.16) and implicitly demanding time and price *homogeneity* that is $p_S(S,t|S',t') = p_S(S-S',t-t'|0)$, Bachelier found that function $A(t)$ need to be

$$A(t) = \frac{1}{k\sqrt{t}}, \tag{2.18}$$

where k is a constant. We can prove this in a different way using the *characteristic function* of the process defined as the Fourier transform of $p_S(S,t|S_0)$:

$$\varphi_S(\omega,t|S_0) = \int e^{i\omega S} p_S(S,t|S_0)dS. \tag{2.19}$$

We apply this transform to the Chapman-Kolmogorov equation (2.16). The assumption of homogeneity for S and time t allows us to use the convolution theorem with the result

$$\varphi_S(\omega,t|S_0) = \varphi_S(\omega,t-t'|0)\varphi_S(\omega,t'-t_0|S_0). \tag{2.20}$$

On the other hand, the characteristic function of the Gaussian conditional probability (2.17) is

$$\varphi_S(\omega,t|S_0) = e^{i\omega S_0}\, e^{-\omega^2/A^2}. \tag{2.21}$$

Substituting Eq. (2.21) into Eq. (2.20) immediately results in the expression (2.18) for $A(t)$. Finally, the probability density accomplishing all requirements reads

$$p_S(S,t|S_0) = \frac{1}{\sqrt{2\pi k^2 t}}\, e^{-(S-S_0)^2/2k^2 t}, \tag{2.22}$$

[11] In fact, this concept can be associated with the absence of arbitrage or the market equilibrium (see above).

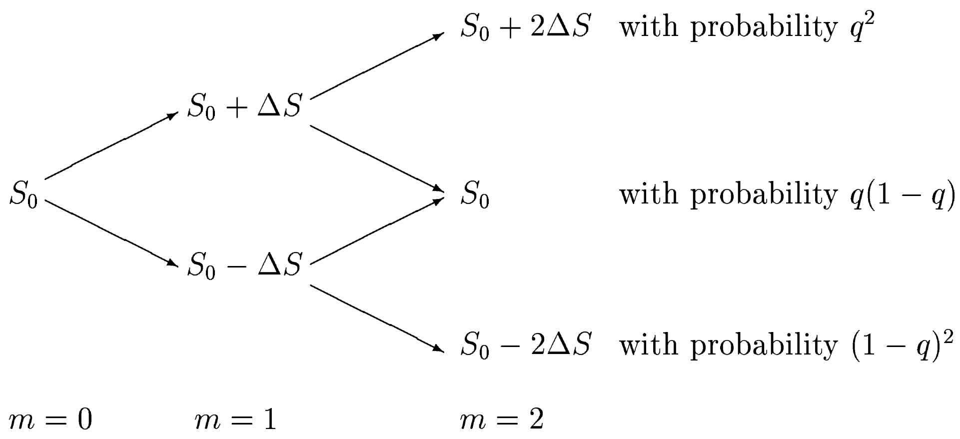

Figure 2.1: Set of the future stock prices assuming a random walk dynamics
We represent the stock share evolution assuming that it follows a binomial random walk process. Process begins at time $t_0 = 0$ and we observe possible prices after two timesteps (*i.e.*, $m = 2$). We show the probability associated with every possible state given by the probability function (2.26).

whose *first moment* and *variance* are

$$E\left[S(t)|S_0\right] = S_0, \tag{2.23}$$

$$\text{Var}[S(t)|S_0] = E\left[S(t)^2|S_0\right] - E\left[S(t)|S_0\right]^2 = k^2 t. \tag{2.24}$$

Bachelier also presented a different approach starting from an specific dynamics of the market. He assumed that the market follows a pure random walk where stock price variations have only two mutually independent events. Stock can go up with probability q or can go down with probability $1-q$. In Fig. 2.1, it is described the random walk in case there are two timesteps within initial and final prices. When $m = 2$, the probability of having α raises of the stock is

$$p(\alpha, 2) = \frac{2}{\alpha!(2-\alpha)!} q^{\alpha}(1-q)^{2-\alpha} \qquad (\alpha = 0, \pm 1, \pm 2).$$

During each timestep stock changes a quantity given by $\pm\Delta S$. Thus, when $m = 2$, the stock price is related to the α value as follows

$$S(2) = 2(\alpha - 1)\Delta S + S_0.$$

We can generalize this relation for an arbitrary m. Note that after m steps the price is given by

$$S(m) = (2\alpha - m)\Delta S + S_0. \tag{2.25}$$

And by induction[12], we can prove that the probability that stock price is $S(m)$ after m timesteps is

$$p(\alpha, m) = \frac{m!}{\alpha!(m-\alpha)!}\; q^{\alpha}(1-q)^{m-\alpha}, \tag{2.26}$$

where we recall that α is the total amount of stock rises needed to achieve the stock price $S(m)$ (see Eq. (2.25)).

Bachelier observed that Eq. (2.26) is a term of the binomial decomposition of the polynomial $[q + (1-q)]^m$. Assuming that stock follows a random walk, the first moment is

$$E\left[\alpha\right] = \sum_{\alpha=0}^{m} \frac{m!}{\alpha!(m-\alpha)!}\; \alpha\; q^{\alpha}(1-q)^{m-\alpha} = q\,\frac{\partial}{\partial\alpha}[q + (1-q)]^m,$$

but since $[q + (1-q)] = 1$ we finally obtain

$$E\left[\alpha\right] = mq. \tag{2.27}$$

The variance can be derived using the same method and results

$$\mathrm{Var}[\alpha] = \left[\sum_{\alpha=0}^{m} \frac{m!}{\alpha!(m-\alpha)!}\; \alpha^2\; q^{\alpha}(1-q)^{m-\alpha}\right] - E\left[\alpha\right]^2 = mq(1-q). \tag{2.28}$$

Bachelier proceeded further and obtained the *continuous-time limit.* He supposed that time intervals $(t - t_0)$ are divided into very small timesteps such that the m number of trials tends to infinity. With the asymptotic Stirling's formula

$$n! \sim e^{-n} n^n \sqrt{2\pi n} \qquad (\text{for } n \to \infty),$$

Bachelier showed in an un rigorous way that the probability density (2.26) becomes (see Gnedenko (1985) for more details)

$$p(\alpha, m) = \frac{1}{\sqrt{2\pi mq(1-q)}}\; e^{-(\alpha - mq)^2/2mq(1-q)}. \tag{2.29}$$

Therefore, Bachelier applied the *central limit theorem* which asserts that a large sum of independent variables tends to a Gaussian probability distribution. Taking into account average (2.27) with the relation (2.25) between α and S, we thus have

$$E\left[S(m)\right] = S_0 + m(2q-1)\Delta S.$$

But if we compare this expression with the first moment obtained in the continuous-time case (2.23), we then conclude that

$$q = 1/2.$$

[12]Random walk theory is very well exposed in Weiss (1994). The reader is encouraged to consult it if more information is wanted.

Otherwise, the variance for the stock in discrete time can be also derived from the α variance (2.28) and the relation between α and the stock (2.25). Hence,

$$\mathrm{Var}[S(m)|S_0] = \mathrm{Var}[\alpha]\Delta S^2 = mq(1-q)\Delta S^2. \tag{2.30}$$

We need to compare this equation with the one derived for the continuous-time case (2.24). In his thesis, Bachelier did not study the many subtleties involving the continuous-time limit. The random walk theory (Weiss (1994)) asks for a certain relation between the timesteps and the stock increments. This relation is

$$\Delta S^2/\Delta t = 4k^2.$$

With this, the variance (2.30) becomes

$$\mathrm{Var}[S(m)|S_0] = q(1-q)4k^2 m\Delta t,$$

but taking into account that $t = m\Delta t$ and $q = 1/2$ we finally obtain

$$\mathrm{Var}[S(m)|S_0] = k^2 t,$$

which equals to the continuous-time variance given by Eq. (2.24). This transformations can be easily implemented in the asymptotic pdf (2.29) and, performing them, we will thus obtain the pdf corresponding to the continuous-time market model (2.22). Therefore, all results are consistent and the two different approaches are equivalent.

These are the two approaches presented by Bachelier to the first mathematical model of the stock movements. We recall that the main hypothesis is that the price evolves as a continuous Markov process homogeneous in time and price. Bachelier observed that the market dynamics derived belongs to the family of distribution functions satisfying the heat equation. We should mention that Einstein defined an identical mathematical model *five years later* for describing the so-called *Brownian motion* (Einstein (1905)). This motion was first observed by Robert Brown in 1827 when he put small pollen grains suspended in water and discovered that they follow very animated and irregular trajectories (see Fig. 2.2). More generally, Brownian motion is followed by any suspension of fine particles, and its posterior mathematical modeling is considered a major scientific discovery of the twentieth century.

However, Bachelier not only defined a market model but he also suggested prices for various options. He asserted that the law governing the relation between the payoff and option price must be based on the assumption that "the mathematical expectation of the buyer of an option is zero". In other words, Bachelier claimed that neither the option seller nor the buyer can adopt dominant positions due to holding the option. In this way, Bachelier proposed that the option price $C(S,t)$ is the mean value of the capital gain at maturity. Therefore,

$$C(S,t) = E\left[(S(T)-K)^+|S(t)=S\right] = \int_K^\infty dS'(S'-K)\, p_S(S',T|S,t). \tag{2.31}$$

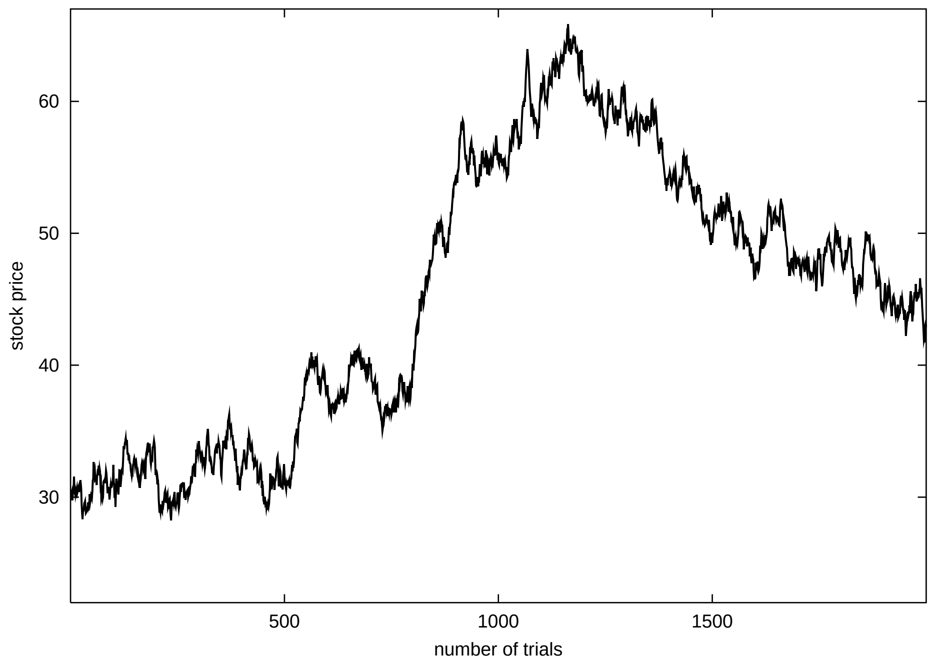

Figure 2.2: A Brownian sample path simulation

We plot a brownian path for the stock with initial price $S_0 = 30$ dollars and constant $k = 1/2$ during 2000 number of trials. Observe that the Brownian motion describes a very animated and irregular trajectory.

Assuming that the pdf (2.22) governs the stock dynamics, this call price reads

$$C(S,t) = \int_K^\infty \frac{dS'}{\sqrt{2\pi k^2(T-t)}}(S'-K)\exp[-(S'-S)^2/2k^2(T-t)].$$

By rewriting the integral on the left hand side of this equation, we finally obtain

$$C(S,t) = (S-K)N(d) + \frac{k\sqrt{T-t}}{\sqrt{2\pi}}e^{-d^2/2}. \tag{2.32}$$

where

$$d = (S-K)/k^2(T-t), \tag{2.33}$$

and

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} dx\ e^{-x^2/2} \tag{2.34}$$

is the *probability integral function.* Observe that the call price only depends on the time to maturity $T-t$, and the difference between the present stock price and the striking price (*i.e.*, $S-K$). These features are consequences of the assumption of homogeneity of time and stock. Another parameter involved in the Bachelier option price is the variance $k^2(T-t)$.

Bachelier performed similar analysis to other more sophisticated options but we are not going to expose them here since they are not of our interest. The thesis report issued by Poincaré, Appell and Boussinesq gave to Bachelier the *honorable* degree. The excellent note is *très honorable* and there are several opinions about the reason why Bachelier did not achieve this excellent mark[13]. It seems that it was the best note he could be awarded for a thesis which was essentially outside mathematics, and that it had a number of arguments far from being rigorous. However, Courtault *et al.* (2000) believe that the report is benevolent and the mild criticisms are positive. The report only expresses the regret that Bachelier did not study in detail the discovered relationship of stochastic processes with equations in partial derivatives. For this reason, Courtault *et al.* (2000) prefer to think that Bachelier did not awarded with the note *très honorable* because of a weaker presentation of his second thesis (see footnote 9).

After the thesis, Bachelier remained quite active in the period from 1900 to 1914. He developed the mathematical theory of diffusion processes. He defined new classes of stochastic processes (now called independent increment processes and Markov processes) and derived the distribution of the Ornstein-Uhlenbeck process. He wrote several books about probability theory and probabilistic modeling of financial markets. A book with greater success was *Le Jeu, la chance, et le hasard* with more than six thousand copies sold. Nevertheless, several events avoided him

[13]See Courtault *et al.* (2000) for further details. Reader can find there the complete report in an appendix.

to be awarded with a permanent position until 1927 when he obtained a permanent professorship in Besançon where he worked until his retirement.

Certain historical research follows the legend that Bachelier results were completely ignored by the scientific community. However, there exist several facts which contradicts this legend. His thesis was published in one of the most prestigious mathematical journals and very quickly appeared their results in other posterior monographs. He successfully published the first book on probability after the famous treatise of Laplace (1814). His ideas influenced the important paper of Kolmogorov (1931) on diffusion processes. In modern textbooks on probability, Brownian motion is traditionally referred to as the Wiener process. However, Feller (1957) in his famous treatise refers to it as the Wiener-Bachelier process. Also Keynes (1921) and Ito and McKean (1965) respectively mentioned the importance of Bachelier works from a mathematical point of view.

2.2.2 The log-Brownian market model

Despite Bachelier's very early interest in stochastic market modeling, the research on financial markets was very slowly developed. While practitioners displayed a continued and strong interest on the financial markets nature, it enjoyed relatively little academic attention until the financial crash of 1929. According to Cootner (1965), this lack of interest is explained by many features: the small importance of organized financial markets; a conviction that stock markets fluctuations were the product of irrational mass gambling; and a short gage among economists of the mathematical and statistical tools necessary for an effective research in this field.

However, research into stock markets nature did attract renewed attention in 1930's by an embryo school of American economists which were highly skilled in mathematics and statistics. One of the important names of this period is considered to be Alfred Cowles. The Cowles Commission organized the first major collection of statistical data on the US stock market. Up to that time, research into financial markets was devoted to predict prices solely based on "outside data" such as industrial production or company earnings. Cowles (1933) approached to financial markets from a new and original point of view since he was interested on the way of forecasting stock market prices from the past history of prices themselves.

At that time, theoretical economists believed that random walk hypothesis for the stock prices was rather reasonable since this market modelisation denied the possibility of predicting the future prices based on past history alone[14]. In this way, Cowles and Jones (1937) empirically tested the validity of the random walk hypothesis. The conclusions were not definitive, although they found some evidences supporting the randomness of the stock price variations. Other evidences indicated that there existed correlations between prices at different times.

[14]Bachelier also followed this way of reasoning but his work did not contain a solid theory to justify this principle.

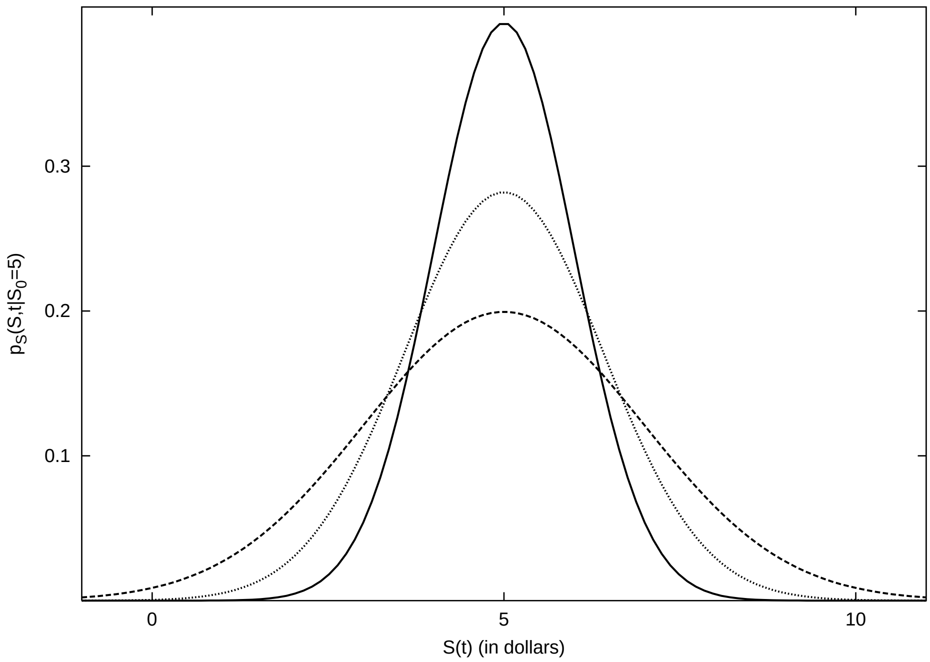

Figure 2.3: The Brownian probability distribution

We plot the pdf given by Bachelier (2.22) for different time intervals, $t - t_0$. We suppose that initial stock price is $S_0 = 5$ dollars and we vary the variance $\text{Var}[S(t)|S_0] = k^2(t - t_0)$dollars^2 by enlarging the time interval. The dashed line corresponds to a variance equals to 4, the dotted line to a variance equals to 2, and solid line to a variance equals to 1. We see that Bachelier model (2.22), *i.e.,* the Brownian dynamics, allows for negatives prices with a non negligible probability that increases as the time period becomes longer.

After the Cowles works in 1930's, there was little or nothing published along these lines until the Kendall's paper. Kendall (1953) performed an empirical study of several American markets time-series, and his investigation was particularly striking since it was not undertaken with the random walk view in mind. Besides a byass problem in the monthly changes averages of cotton prices[15], Kendall (1953) found that an overwhelming proportion of his results supported the independence of price changes assumption.

It remained for Osborne (1959) to try to test the fit of the data to a Brownian market model, and to ask for the form of the pdf for the stock market price changes. The first and more important objection to the Bachelier market model (2.22) is that his model allows for negative prices. Clearly, this is not possible in real markets. Bachelier, and even Kendall and Cowles, had avoided the complications posed by

[15]This empirical feature was already observed by Cowles, and he was unable to give a reasonable explanation for this phenomena. The reader should wait until Osborne's market model for having a reasonable justification (see Eq. (2.39) and Fig. 2.4 below).

the fact that stock prices had a lower bound at zero but unbounded from above because they dealt with short-period (daily, weekly, and monthly) price changes. However, over longer periods, it is obvious that the distribution of prices (2.22) can give a non negligible probability for negative prices. This phenomena is clearly seen in Fig. 2.3 where we plot the Bachelier pdf for several periods of time.

Osborne (1959) found that absolute price changes are correlated over time but that the logarithm of price seemed to have independent increments. Incidentally, we note that the logarithmic transformations are related to the fact that market investors are mostly interested in the proportional changes in stock prices rather than in absolute changes. Reader should go to Appendix A if more details are required. We there perform a brief summary of the Masoliver *et al.* (2001a) paper which studies the differences between taking the stock and return variables. Hence, we define a new stochastic variable called *stock return*

$$R(t) \equiv \ln[S(t)/S_0], \qquad \text{where } S_0 = S(0). \tag{2.35}$$

Observe that the return at time $t = 0$ is always zero, independently on the initial stock price. Osborne proposed that instead of the share price $S(t)$ it was the return the variable whose trajectory describes the Brownian motion given by (2.22). In consequence, the pdf for the return reads

$$p_R(R,t|0) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-R^2/2\sigma^2 t}, \tag{2.36}$$

where σ is a constant called *volatility*, and plays a similar role as the constant k of the Bachelier pdf (2.22). The first moment and variance for the return are respectively

$$E\left[R(t)|0\right] = 0, \qquad \mathrm{Var}[R(t)|0] = \sigma^2 t. \tag{2.37}$$

Osborne proceeded further by studying the corresponding stock price dynamics. He found that prices were described by the so-called *log-Brownian motion* (or *geometric Brownian motion*) after performing the change of variables (2.35). In such a case, the conditional pdf (2.36) in terms of the price changes reads

$$p_S(S,t|S_0) = \frac{1}{S\sqrt{2\pi\sigma^2 t}}\, e^{-[\ln(S/S_0)]^2/2\sigma^2 t}. \tag{2.38}$$

The pdf's (2.36) and (2.38) allow for negative values of the return but deny the possibility of negative stock share prices (see Fig. 2.4). The first moment[16] and variance for the stock are respectively

$$E\left[S(t)|S_0\right] = S_0 e^{\sigma^2 t/2}, \qquad \mathrm{Var}[S(t)|S_0] = S_0^2\left(e^{2\sigma^2 t} - e^{\sigma^2 t}\right). \tag{2.39}$$

[16] The non-zero first moment explains the byass of the cotton prices averages mentioned before. Although the return is zero-mean, stock may have a time dependent first moment. This unexpected exponential term is known as the *spurious drift* component.

Note that, when time is small,

$$E\left[S(t)|S_0\right] = S_0\left[1+\sigma^2 t/2 + \mathrm{O}(\sigma^4 t^2)\right], \quad \mathrm{Var}[S(t)|S_0] = S_0\left[\sigma^2 t + \mathrm{O}(\sigma^4 t^2)\right].$$

And assuming that stock changes are also small

$$\ln(S/S_0) \sim (S-S_0)/S_0,$$

the pdf (2.38) is approximately the Gaussian distribution

$$p_S(S,t|S_0) = \frac{1}{\sqrt{2\pi(S_0\sigma)^2 t}}\exp[-(S-S_0-S_0\sigma^2 t/2)^2/2(S_0\sigma)^2 t],$$

which only differs from that of Bachelier (2.22) due to the spurious drift component $\sigma^2 t/2$.

Indeed, Osborne showed using several tests that empirical results were consistent with the log-Brownian market model assumption. This was the onset of the sharp increase in the interest on this subject by the American academic audience. The University of Chicago proportionated a fertile generation of investigators devoted to analyzing time-series from financial markets. Although Osborne (1959) results were an important factor, the irruption of electronic computing fastened the research focussed on the financial markets nature.

The richest field of application of the random walk theory of stock prices has been in the determination of the value of the derivatives. The option value can be explicitly determined once the stochastic model for the underlying is selected. Indeed, the study of options was the original motivation for the first study of stock prices (Bachelier (1900)). However, the study of options did not end with Bachelier. Even before Bachelier's work was rediscovered by the American economists in the late fifties, Kruizenga (1956) had inaugurated an extensive study of put and call options in his doctoral dissertation at the Massachussets Institute of Technology. Kruizenga (1956) discussed the nature of the option contract, main features of option trading in a systematic way, and relations between the call and the put. The paper dealt for the first time with the idea of "rationality" of the option prices.

The new researchers studying the options started to obtain new results, and an article by Sprenkle (1964) is one the most remarkable works. Recall that the Bachelier model (2.22) allows for negative stock prices. This inadequate feature for a stock market model gives also undesirable properties for an option price. Thus, for instance, we see from Eq. (2.31), we see that the perpetual Bachelier call is

$$C_p(S,t) = \lim_{T\to\infty} C(S,t) = (S-K)N(0) + \lim_{T\to\infty}\frac{k\sqrt{T-t}}{\sqrt{2\pi}}. \tag{2.40}$$

Since $N(0) = 1/2$, the first term is finite but second term grows indefinitely as time to maturity tends to infinity. Hence, the perpetual Bachelier call diverges and

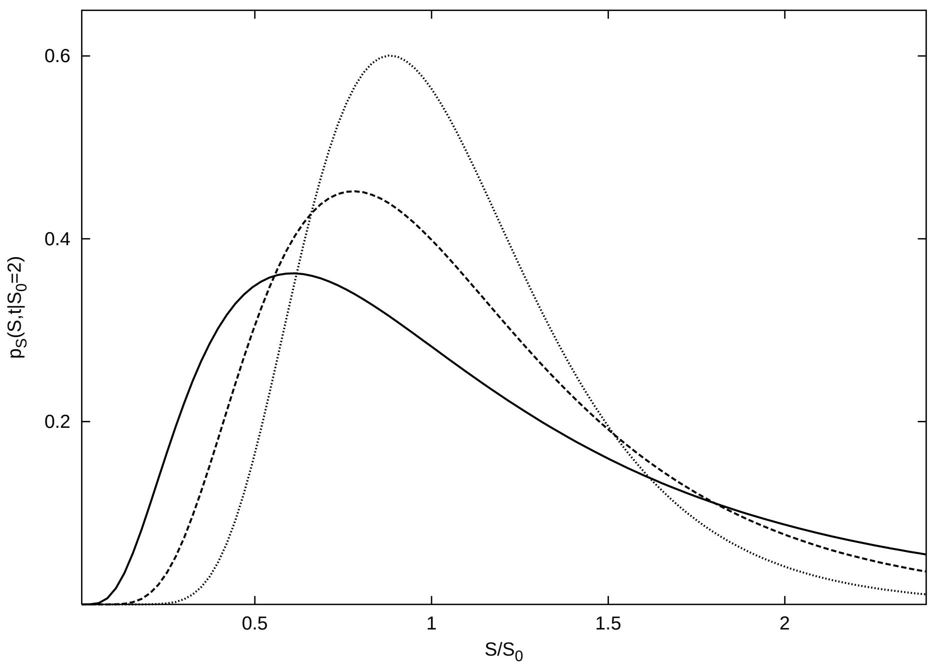

Figure 2.4: The log-Brownian probability distribution

We plot the pdf given by Osborne (2.38) for various time intervals supposing that initial stock prices 2 dollars. Solid line corresponds to $\sigma^2 t = 1/2$, dashed line to $\sigma^2 t = 1/4$, and dotted line to $\sigma^2 t = 1/8$. We have seen that Bachelier pdf of Fig 2.3, *i.e.,* the Brownian dynamics, allows for negatives prices with a non negligible probability. However, the Osborne's pdf (2.38), *i.e.*, the log-Brownian dynamics, denies any chance to negative stock values. Contrary to the Brownian distribution, the log-Brownian pdf is biased.

this disagrees with the market equilibrium result (see Eq. (2.6)). We show there that perpetual call price must be equal to the underlying asset price if arbitrage is avoided.

Sprenkle (1964) tried to rule out this objection by assuming that random walk is followed by the return instead of the stock. In addition, he also included a *drift* μ to the pdf (2.38) proposed by Osborne. The Sprenkle's pdf for the stock thus reads

$$p_S(S,t|S_0) = \frac{1}{S\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{[\ln(S/S_0) - (\mu - \sigma^2/2)t]^2}{2\sigma^2 t}\right\}, \tag{2.41}$$

whose first moment and variance are

$$E\left[S(t)|S_0\right] = S_0 e^{\mu t}, \qquad \mathrm{Var}[R(t)|0] = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right). \tag{2.42}$$

The expected payoff for the call at the expiration date T is

$$E\left[(S(T) - K)^+|S(t) = S\right] = \int_K^\infty dS'(S' - K)p_S(S', T|S, t).$$

And from Eq. (2.41) we get

$$E\left[(S(T) - K)^+\right] = e^{\mu(T-t)} SN(d_1) - KN(d_2), \tag{2.43}$$

where

$$d_1 = \frac{\ln(S/K) + (\mu + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_2 = d_1 - \sigma\sqrt{T - t}, \tag{2.44}$$

and $N(d)$ is the probability integral (2.34).

However, Sprenkle did not directly relate the expected payoff (2.43) to the option price $C(S,t)$. By combining Eq. (2.43) with utility function arguments[17] which escape from our interest here, Sprenkle arrived at the following price

$$C(S,t) = e^{\mu(T-t)} SN(d_1) - (1 - \Re)KN(d_2) \qquad (0 \le \Re \le 1). \tag{2.45}$$

This price has a free parameter $\Re$ which must be chosen in accordance to the "degree of market risk aversion" for each investor. Observe that the price for the call increases as $\Re$ becomes higher, *i.e.*, as the investor's risk aversion grows. Arguments for fixing the value of $\Re$ are roughly exposed by Sprenkle (1964) containing some contradictions in the way of reasoning. The most important objection is that option price is not unique which is not consistent with the "rational" approach for giving a price to the option[18]. Nevertheless, we must keep in mind the necessity of giving a role to the risk taken by the investor when buys a call.

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[17] Roughly speaking, these arguments are referred to the convexity of the stock growth, and that utility function studies try to give a relation between the average of the stock and its inherent risk.

[18] This idea that market has a "rational" behavior was present in the mathematical finance studies from Kruizenga (1956), and they look for a fair and unique price independently on the personal level of risk aversion for each investor.

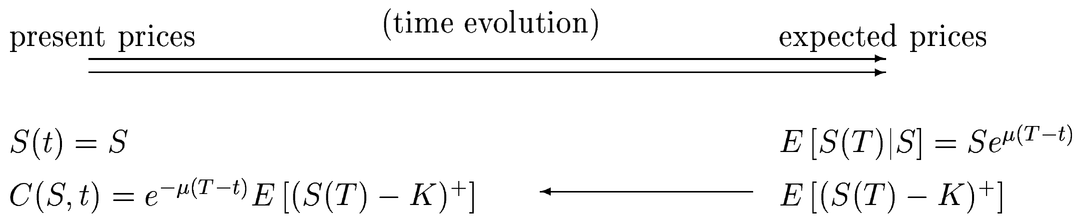


Figure 2.5: The time value of money

According to Boness, the time evolution influences the call price by the exponential growth of the stock average (2.46). Average evaluated at a future date includes implicitly this growth. But the call price is purchased today and not at maturity. It is necessary to go back to time t, and this represents taking out the underlying growth during the period $T - t$.

2.2.3 The time value of money

During 1961-1962, A. James Boness was at the School of Industrial Management of the M.I.T. and studied some features concerning the option pricing. As we have just seen, the Sprenkle's call price (2.45) contains the parameter $\Re$ measuring the "risk aversion" of the investor. In contrast with the Sprenkle's assertion, Boness (1964) assumed that "investors are indifferent to risk". He added another assumption by saying that: "For convenience and in default of better information, all stocks on which options are traded are defined to be of the same risk class". Smith (1976) refers to this assumption saying that the market is competitive in the sense that the equilibrium price of all stocks of the same risk class imply the same expected growing curve. In other words, Boness assumed that all stocks grow in average as

$$E[S(t)|S_0] = S_0 e^{\mu t}, \tag{2.46}$$

and that all obey the same Sprenkle's pdf given by Eq. (2.41) with the same volatility σ.

Furthermore, Sprenkle's call price (2.45) is incomplete since its perpetual call price diverges. Indeed, the limit $T \to \infty$ is not well-behaved due to the exponential of the first term of Eq. (2.45). That is:

$$C_p(S,t) = \lim_{T\to\infty} \left[Se^{\mu(T-t)} - K(1-\Re) \right] \longrightarrow \infty, \tag{2.47}$$

where we have taken into account that $N(\infty) = 1$. Note that if this exponential did not appear, the Sprenkle option would have correctly priced the perpetual call (compare with the equilibrium restriction on the perpetual call given by Eq. (2.6)).

Boness went deeper into this inconsistency, and he distinguished the expected payoff at maturity from the price when the call is bought. Before Boness' work, option price at present was directly related to the expected gain due to holding the call at maturity. Boness sophisticated the relation between expected payoff and call price. He thus assumed that both option and stock have the same expected growth, *i.e.*,

$$\frac{E\left[S(T)|S(t)=S\right]}{S} = \frac{E\left[C(S,T)|S(t)=S\right]}{C(S,t)}. \tag{2.48}$$

And, in consequence, Boness could explicitly express the call price in the form

$$C(S,t) = \frac{E\left[C(S,T)\right]}{E\left[S(T)\right]}\, S.$$

But if we take into account Eq. (2.46) and the payoff for the case of the European call, the Boness' call is

$$C(S,t) = e^{-\mu(T-t)}\; E\left[(S(T)-K)^{+}|S(t)=S\right], \tag{2.49}$$

In this way, the so-called *time value of money* discounts the expected call payoff back to the date t by using the expected rate of return to the stock during the period $T-t$. The diagram of the Fig. 2.5 shows this discounted expected value procedure. We recall that the expected payoff has been already obtained and corresponds to that of Eq. (2.43). Finally, the call reads

$$C(S,t) = SN(d_1) - e^{-\mu(T-t)}KN(d_2), \tag{2.50}$$

where $d_{1,2}$ are those given by Eq. (2.44). For the perpetual call we now have

$$C_p(S,t) = S(t), \tag{2.51}$$

which is consistent with the restriction (2.6). Therefore, the divergences of the perpetual call of Bachelier (2.40) and Sprenkle (2.47) have been removed. Moreover, in Boness' work appeared for the first time the concept of the time value of money applied to options. This important feature allows to move an asset average evaluated at one time and relate it to another different average evaluated at a different time (see Eq. (2.49)).

Paul A. Samuelson closes this option pricing history with "a compact report on desultory researchers stretching over more than a decade (Samuelson (1965))". The professor from the M.I.T. updated previous works (from Bachelier until Boness) and formalized them rigorously with the assistance of McKean[19]. Samuelson generalized the option pricing to any adequate market model. And he postulated that an adequate market model needs to be Markovian thus obeying the Chapman-Kolmogorov equation (2.16)

$$p_S(S,t|S_0) = \int p_S(S,t|S',t')p_S(S',t'|S_0)dS' \qquad (0 \le t' \le t),$$

[19] Harry McKean wrote a mathematical self-containing appendix of the Samuelson (1965) article. He there found some exact explicit solutions for other market models different than the log-Brownian.

and with an average

$$E\left[S(t_2)|S(t_1)=S_1\right]=S_1e^{\mu(t_2-t_1)} \qquad (t_2\geq t_1). \tag{2.52}$$

According to Samuelson (1965), models that fit these requirements are the log-Brownian process, the log-Poisson distribution (or also called jump process or shot noise), and the log-Lévy process. By that time, there appeared several empirical studies that questioned the validity of the log-Brownian model and proposed to take the log-Lévy process instead (see *e.g.* Mandelbrot (1963) and Fama (1963)).

Samuelson derived afterwards several conditions on the absence of arbitrage that are similar, but not identical, to the ones derived in Section 2.1. Based on these results, Samuelson extended the assumption on the average (2.52) to options. He thus assumed that

$$E\left[C(S,T)\right]=e^{\beta(T-t)}C(S,t). \tag{2.53}$$

This axiom is assumed to be valid for any derivative, and a constant and unknown β depending on each particular derivative. In case that the underlying asset follows a log-Brownian motion, Samuelson's call price reads

$$C(S,t)=e^{(\mu-\beta)(T-t)}\ SN(d_1)-e^{-\beta(T-t)}KN(d_2), \tag{2.54}$$

where we have taken into account the expected payoff for the call (2.43) and the call growth axiom (2.53). The postulate (2.53) differs from the Boness assumption (2.48) since now β can be different than μ. Samuelson proceeded by studying the option price depending on the relation between these two parameters. He concluded that

$$0\leq\mu\leq\beta.$$

Samuelson (1965) admitted the possibility of β being greater than μ. He said that the parameter β can be greater than μ if the market perceives the option to be riskier than the underlying security. This obscure condition gives us an idea of the inability for risk managing by the method of Samuelson and even by the other preceding methods. The inequality is not in contradiction with the restrictions on call price assumed by Samuelson. However, the restrictions presented by Samuelson are not completely correct since they do not include risk-free bonds (2.1) in portfolio constructions and, indeed, β being greater than μ is inconsistent with some of the market equilibrium restrictions summarized by Table 2.7. For instance, the perpetual for the Samuelson call (2.53) is

$$\lim_{T\to\infty}C(S,t)=\begin{cases}0 & \text{if } \mu<\beta,\\ S & \text{if } \mu=\beta.\end{cases}$$

In accordance to restriction (2.6), the market equilibrium demand says that $C_p(S,t)\equiv S$. Hence, the Samuelson call obeys the absence of arbitrage demands only when

$\mu = \beta$. Furthermore, the market equilibrium also fixes a bounds between the call price must be enclosed (see Eq. (2.5)). Those are

$$(S - Ke^{-r(T-t)})^+ \leq \text{call price} \leq S.$$

Observe that, assuming $\mu = \beta$, the inequality is hold by the Samuelson's call only if $\beta = r$. Therefore, those two inconsistencies with the market equilibrium restrictions show us that Samuelson method is still an incomplete option pricing method.

Besides these errors, Samuelson was at a single step of imposing that all assets and bonds must have in equilibrium the same growth, *i.e.*, $\mu = \beta = r$ [20]. In that case, bounds given by Eq. (2.5) and all other equilibrium restrictions derived in Section 2.1 would had been accomplished by the Boness (2.50) and Samuelson (2.54) option prices. After the Black and Scholes (1973), Samuelson said that the Black and Scholes article was a "fundamental paper since restores the $\mu = \beta$ case's mathematics primacy" and also asserted that this article "was a valuable breakthrough for science". In reference to his article of (1965) Samuelson exclaimed: "I should have explored this [case] further! (Samuelson (1973), p.16)".

Samuelson (1965) also derived the closed partial differential equation obeyed by the options. Based on a part of the Bachelier (1900) thesis, he found that Eq. (2.54) is the solution to the following partial differential equation

$$\partial_t C = \beta C - \mu S \partial_S C - \frac{1}{2}\sigma^2 S^2 \partial^2_{SS} C, \tag{2.55}$$

with final condition depending on the derivative contract specifications. The final condition for the European call price is

$$C(S,T) = (S-K)^+.$$

The partial differential equation (2.55) is very similar to the *backward Fokker-Planck equation* corresponding to the stock dynamics, and becomes the same equation if the first term, *i.e.*, βC is removed (see *e.g.* Gardiner (1985)).

Summary

We have shown most important properties on the European call and put prices. These properties or restrictions are based on the absence of arbitrage opportunities (or market equilibrium) assumption which denies profits without taking any kind of risk. The derivation of these restrictions has been performed by constructing collections of several assets called portfolios, and proving that option prices must be

[20] This approach has its extension in martingale methods that will assume this equality for deriving the option price (see Section 4.5).

enclosed within several inequalities if arbitrage is to be avoided. The second part of the chapter is devoted to narrate the path described by the mathematical finance discipline, from its origin until the middle 1960's. The option pricing methods herein described are incomplete in the sense that their prices do not hold, in some way, the market equilibrium restrictions. We have updated and presented in a coherent way many works but we have, more extensively, focussed on Bachelier (1900), Sprenkle (1964), Boness (1964), and Samuelson (1965) research papers. All option pricing methods have been given with an historical framework mainly based on the Cootner (1965) monograph.

Appendix A. Data analysis with stock and return differences

This Appendix briefly reports the main ideas enclosed in Masoliver *et al.* (2001). We there study the differences between working with return or with stock price differences. Speculative markets provide us a large time series of stock prices, and from them we can evaluate

$$W(t;\tau) \equiv R(t+\tau) - R(t) \tag{A.1}$$

and

$$Z(t;\tau) \equiv S(t+\tau) - S(t), \tag{A.2}$$

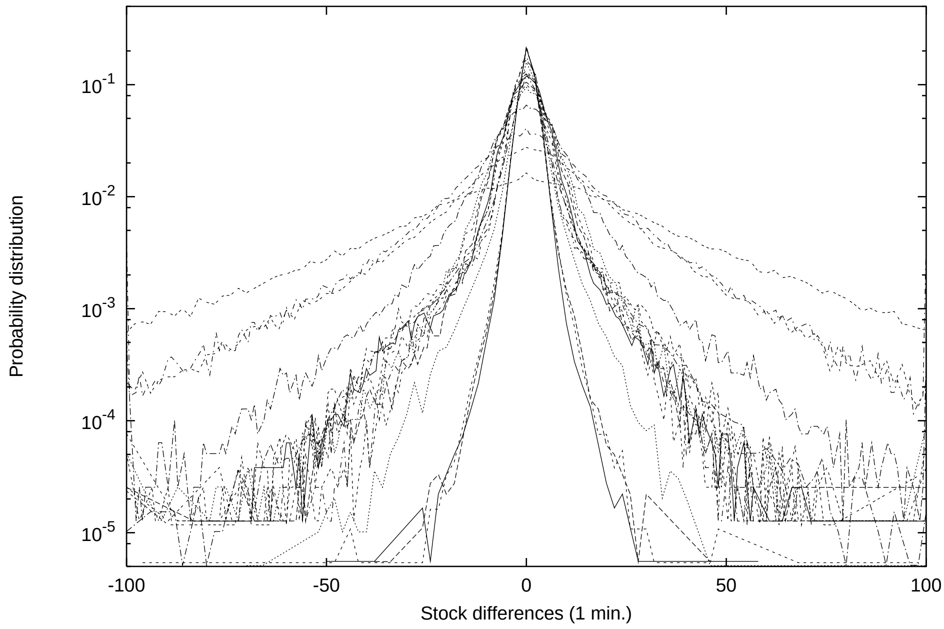
which are, respectively the *return difference* and the *stock price difference*. Let us show some statistical results for $W(t;\tau)$ and $Z(t;\tau)$ taking the *Standard & Poor's cash index* (*S&P 500*) as a data source.

We plot the empirical stock price differences $Z(t;\tau)$ in Figure 2.6. The two graphs give the probability distributions in tick data units when τ equals to 1 minute. The first graph shows one-minute stock differences for seventeen different years, from 1983 to 1999. Meanwhile, the second graph only shows stock differences pdf's for the years: 1983, 1984, 1986, 1990, 1998. In this second plot we can see how the tails become fatter as the time t in years increases exponentially. Data analysis with time series assumes that variable does not change with time t. However, this contradicts the observed probability distribution since, in statistical sense, Figure 2.6 shows that

$$Z(t;\tau) \propto S(\tau).$$

Bachelier (1900) model assumes that, in statistical sense, $Z(t;\tau) = S(\tau) - S(0)$ which does not depend on time t. Clearly, this is not consistent with the empirical pdf's and we therefore conclude that Bachelier's arithmetic Brownian model is not a suitable model for the stock dynamics.

On the other hand, Figure 2.7 shows the pdf's for the return differences $W(t;\tau)$. We there perform the same plots but now plotted for the return differences. In



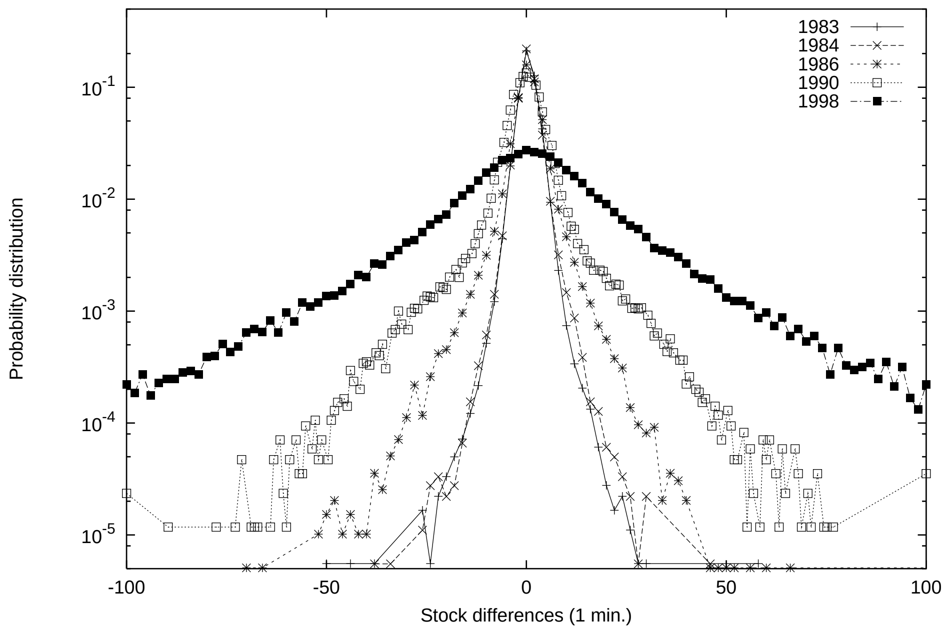


Figure 2.6: Stock difference empirical probability distributions

We show the empirical pdf's for the tick data of the Standard & Poor's 500 stock cash index differences. First graph involves one-minute stock differences for years ranging from 1983 to 1999. Second graph is a detail of the previous graph plotting pdf's of for years exponentially distributed between 1983 and 1998.

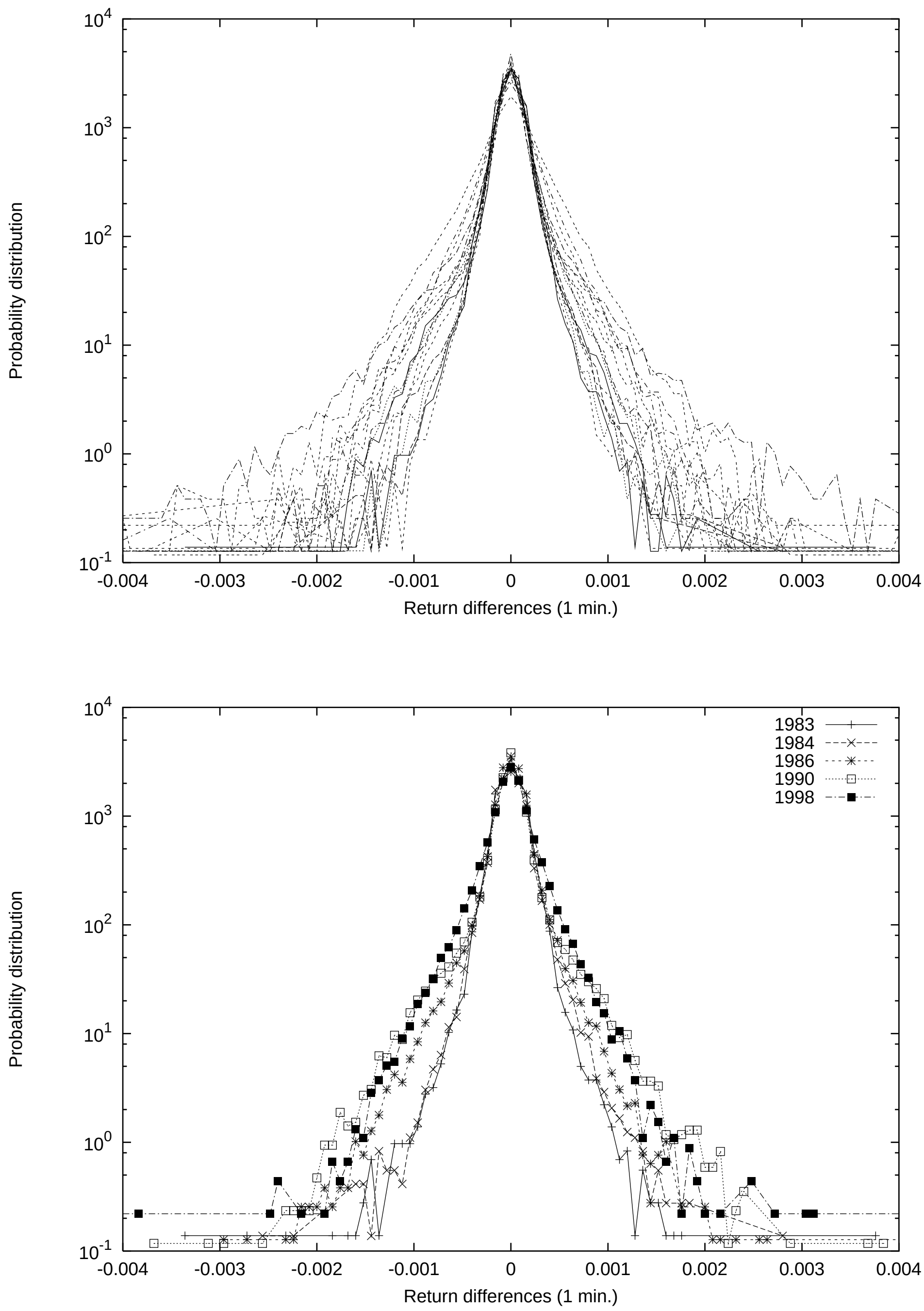


Figure 2.7: Return difference empirical probability distributions

We show the empirical pdf's for the return differences of the Standard & Poor's 500 cash index. First plot involves one-minute returns differences for years ranging from 1983 to 1999. Second graph is a detail of the previous graph plotting pdf's of years exponentially distributed between 1983 and 1998.

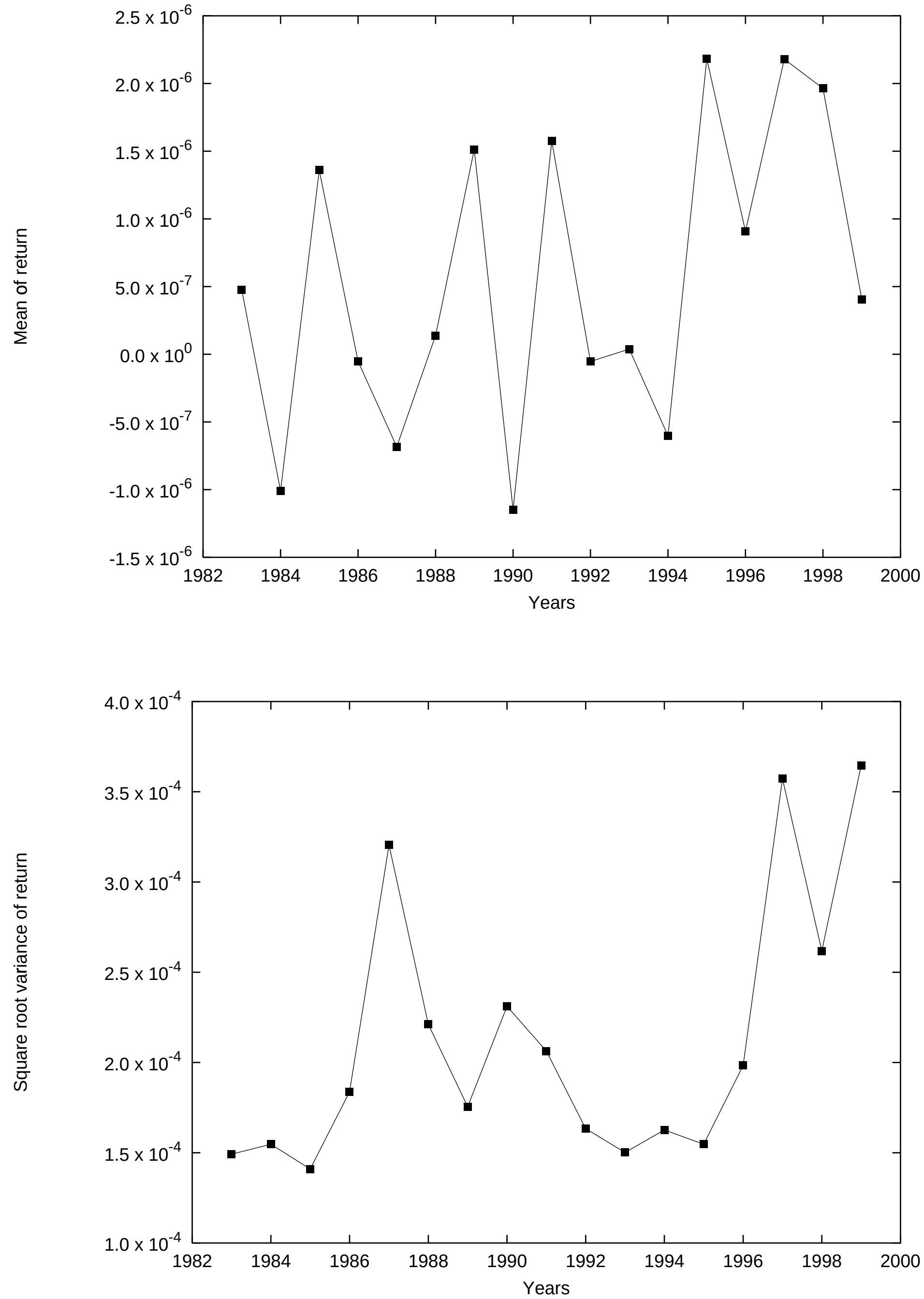


Figure 2.8: First and square root variances estimators of the return

We plot respectively $M_W(t;\tau = 1 \text{ min.}, T = 1 \text{ year})$ and $V_W(t;\tau = 1 \text{ min.}, T = 1 \text{ year})^{1/2}$ as a function of time t in years, from 1983 to 1999. Those functions are defined in Eqs. (A.3) and (A.6).

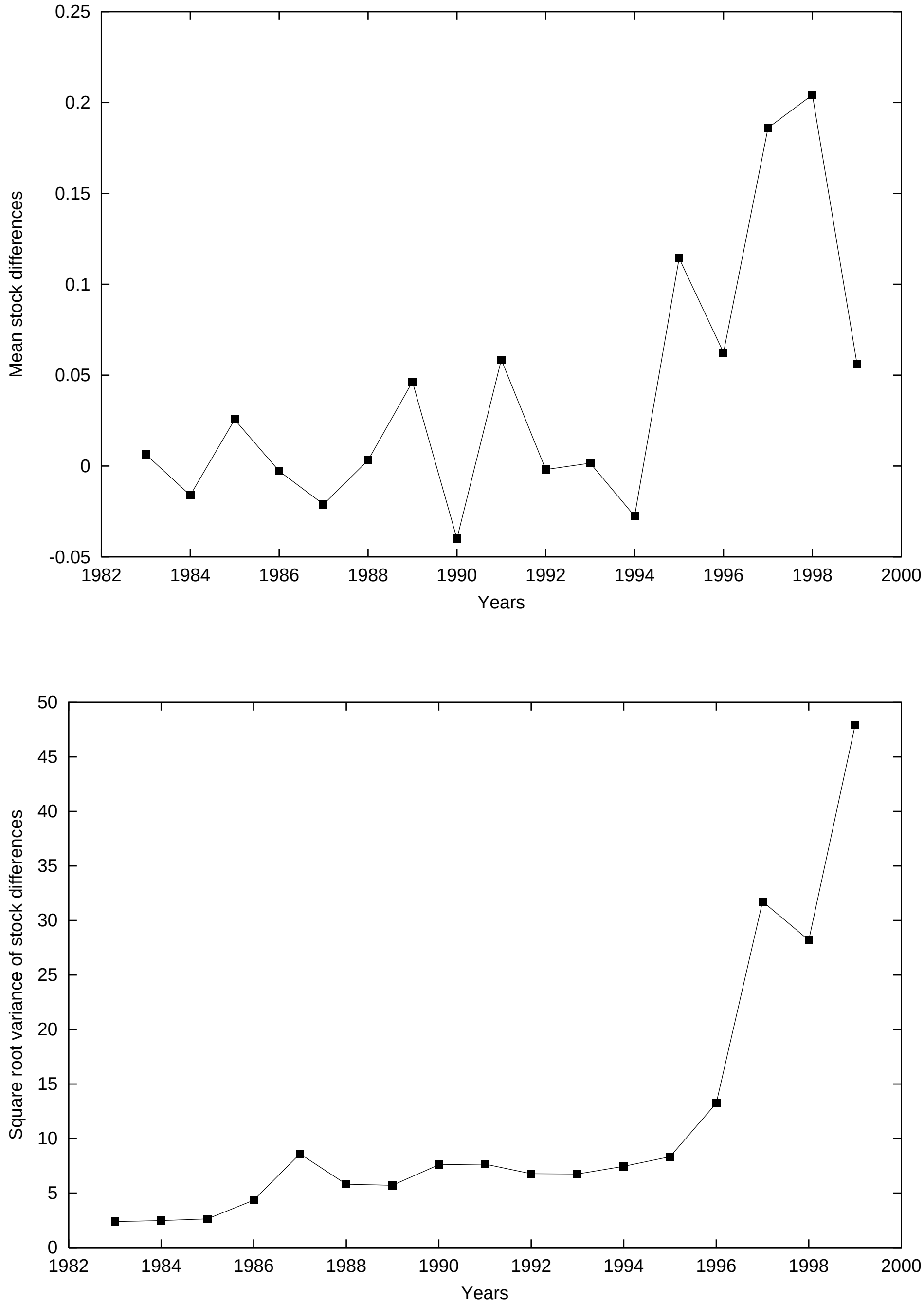


Figure 2.9: First moment and square root variance estimations of the stock

In the first graph, we plot $Z(t; N\tau = 1 \text{ y.})$, for estimating $M_Z(t; \tau = 1 \text{ min.}, T = 1 \text{ year})$, as a function of time t (in years) from 1983 to 1999. In the second graph, we plot $V_Z(t; \tau = 1 \text{ min.}, T = 1 \text{ year})^{1/2}$ as a function of time t in years, from 1983 to 1999. Those are given by Eqs. (A.4) and (A.5) and both cases describe an exponential growth with time t.

contrast with the stock differences, we observe that the shape of the distributions does not change so dramatically over different years (*i.e.*, with time t). Hence, we say that in statistical sense

$$W(t;\tau) = R(\tau).$$

In Masoliver *et al.* (2001), we show that this would be the case of an homogeneous process. Within these class of processes, there exists the geometric Brownian motion which, in terms of the return R, now appears to be a more reasonable candidate to describe stock dynamics.

We can go further in the study and comparison of $W(t;\tau)$ and $Z(t;\tau)$. We define the following two sums for estimating the first moment:

$$M_W(t;\tau,N) \equiv \frac{1}{N}\sum_{n=0}^{N-1} W(t+n\tau,\tau), \tag{A.3}$$

$$M_Z(t;\tau,N) \equiv \frac{1}{N}\sum_{n=0}^{N-1} Z(t+n\tau,\tau), \tag{A.4}$$

In the first plot of Fig. 2.8, we have M_W estimator for the case when $N\tau = 1$ year and $\tau = 1$ minute. We observe that first moment changes from one year to another but with any specific trend. Conversely, the first graph of Fig. 2.9 shows that the average M_Z grows exponentially with time t. This feature is another argument against the stock difference estimator $Z(t;\tau)$.

Moreover, we define the estimator for the variance $Z(t;\tau)$ and $W(t;\tau)$ by

$$V_Z(t;\tau,N) \equiv \frac{1}{N-1}\sum_{n=0}^{N-1}[Z(t+n\tau;\tau) - M_Z(t;\tau,N)]^2. \tag{A.5}$$

$$V_W(t;\tau,N) \equiv \frac{1}{N-1}\sum_{n=0}^{N-1}[W(t+n\tau;\tau) - M_W(t;\tau,N)]^2. \tag{A.6}$$

In the second graph in Fig. 2.8, it is observed that the variance fluctuates without any specific trend. Conversely, second plot of Fig. 2.9 shows that V_W grows exponentially with time t. This exponential trend in the stock differences estimators indicate that process for the stock is multiplicative (or geometric).

This Appendix has intended to stress the importance in the way we manage financial database. From Kendall (1953) research study, it is well-known that stock data follows a multiplicative stochastic process but still nowadays there are research works[21] which still prefer to handle stock differences instead of taking return differences, that is: $S(t+\tau) - S(t)$ instead of $R(t+\tau) - R(t)$. The usual justification for doing this is that when τ is small one can approximate the differences of the stock logarithm with the stock differences. We have showed that, even when $\tau = 1$

[21]See, for instance, Bouchaud (2000). This approximation is been usually taken by physicists who are unaware to the financial studies focussed on these questions decades ago.

minute, it is not true since estimators for the stock differences are biased and not efficient[22].

References

Bachelier, L., 1900, *Théorie de la spéculation.* English version reprinted in P. H. Cootner, 1965, *The random character of stock market prices* (The M.I.T. Press, Cambridge, Massachussets).

Bergman, Y. Z., B. Grundy, and Z. Wiener, 1996, General properties of option prices, *Journal of Finance* **51**, 1573-1610.

Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* **81**, 637-659.

Boness, A. J., 1964, Elements of a theory of stock-option value, *Journal of Political Economy* **72**, 163-175.

Bouchaud, J. P., and M. Potters, 2000, *Theory of financial risks* (Cambridge University Press, Cambridge, UK).

Cramer, H., 1951, *Mathematical methods of statistics* (Princeton University Press, Princeton).

Cootner, P. H., 1965, *The random character of stock market prices* (The M.I.T. Press, Cambridge, Massachussets).

Courtault, J.-M., Y. Kabanov, B. Bru, P. Crépel, I. Lebon, and A. Le Marchand, 2000, Louis Bachelier on the centenary of Théorie de la Spéculation, *Mathematical Finance* **10**, 341-353.

Cowles, A., 1933, Can stock market forecasters forecast?, *Econometrica* **1**, 309-324.

Cowles, A., and H. E. Jones, 1937, Some a posteriori probabilities in stock market action, *Econometrica* **5**, 280-294.

Einstein, A., 1905, Über die von moleKular-Kinetischen Theorie de Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, *Annalen der Physik* **17**, 549-560. English version reprinted in Fürth, R., A. D. Cowper (eds), 1956, *Albert Einstein investigations on the theory of Brownian movement* (Dover, New-York).

Fama, E. F., 1963, Mandelbrot and the Stable Paretian Hypothesis, *Journal of Business* **36**, 420-429. Reprinted in P. H. Cootner, 1965, *The random character of stock market prices* (The M.I.T. Press, Cambridge, Massachussets).

Feller, W., 1957, *An introduction to probability theory and its applications. Vol.2* (Wiley, New-York).

Gardiner, C. W., 1985, *Handbook of stochastic methods* (Springer-Verlag, Heilderberg, Berlin, New-York).

Gnedenko, B. V., 1982, *The theory of probability* (Mir, Moscow).

Itô, K., H. P. McKean, 1965, *Diffusion process and their sample paths* (Springer, Berlin).

Kendall, M. G., 1953, The Analysis of economic time-series, *Journal of the Royal Statistical Society* **96**, 11-25.

Keynes, J. M., 1921, *A treatise on probability* (MacMillan, London).

Kolmogorov, A. N., 1931, Über die analitischen methoden in der wahrscheinlichkeitsrechnung, *Mathematical Annalen* **104**, 415-458.

[22] Again, we refer to Masoliver *et al.* (2001) where we have studied these questions in more detail.

Kruizenga, R. J., 1956, *Introduction to the option contract: A theoretical and market analysis* (Ph. D. dissertation, M.I.T., Cambridge, Massachussets). Partially reprinted in Cootner, P. H., 1965, *The random character of stock market prices* (The M.I.T. Press, Cambridge, Massachussets).

Laplace, P. S., 1814, *Essai philosophique sur les probabilités.* (Gauthier-Villars, Paris). English reprinted version: 1951, *A Philosophical essay of probabilities* (Dover, New-York).

Mandelbrot, B. B., 1963, The variation of certain speculative prices, *Journal of Business* **36**, 394-419. Reprinted in P. H. Cootner, 1965, *The random character of stock market prices* (The M.I.T. Press, Cambridge, Massachussets).

Markowitz, H., 1952, Portfolio selection, *Journal of Finance* **7**, 77-91.

Masoliver, J., M. Montero, J. Perelló, 2001a, Return or stock price differences, submitted for publication.

Merton, R. C., 1973b, Theory of rational option pricing, *Bell Journal of Economics and Management Science* **4**, 141-183.

Osborne, M. F. M., 1959, Brownian motion in the stock market, *Operations Research* **7**, 145-173.

Samuelson, P. A., 1965, Rational theory of warrant pricing, *Industrial Management Review* **3**, 13-39.

Samuelson, P. A., 1973, Mathematics of speculative prices, *SIAM Review* **15**, 1-42.

Smith, C. W., 1976, Option pricing: A review, *Journal of Financial Economics* **3**, 3-51.

Sprenkle, M., 1961, Warrant prices as indicators of expectations and preferences, *Yale Economic Essays* **1**, 178-231.

Weiss, G. H., 1994, *Aspects and applications of the random walk* (Noth-Holland, Amsterdam).

Chapter 3

Black-Scholes option pricing method

As it has been defined in Chapter 1, an European option is a financial instrument giving to its owner the right but not the obligation to buy (call) or to sell (put) a share at a fixed future date, the maturing time T, and at a certain price called exercise or striking price K. In fact, this is the most simple of a large variety of contracts that can be more sophisticated (see for instance Wilmott (1998)). The trading of options and their theoretical study have been known for long although they were relative obscure and unimportant financial instruments until the early seventies. It was then when options experimented an spectacular development. The Chicago Board Options Exchange, created in 1973, was the first attempt to unify options in one market and trade them on only a few stock shares. The market rapidly became a tremendous success and led to a series of innovations in option trading.

The main purpose in option studies is to obtain a fair and presumably riskless price for these instruments. As we have said in Chapter 2, the first solution to the problem was given by Bachelier (1900), and several option prices were proposed without being completely satisfactory (*cf.* Section 2.2). However, in the early seventies it was finally developed a complete option valuation based on equilibrium theoretical hypothesis. The works of Fischer Black and Myron Scholes (1973), and Robert Merton (1973b) were the culmination of this great effort, and left the doors open for extending the option pricing theory in many ways. In addition, the method has been proved to be very useful for investors and has helped to option markets to have the importance that they have nowadays in finance[1].

The option pricing method obtains the so-called Black-Scholes equation which is a partial differential equation of the same kind as the diffusion equation coming from

[1]For the value of their work Myron Scholes and Robert Merton received the Nobel award in 1997. Unfortunately, Fischer Black died in 1995 and could not enjoy with this prestigious award in recognition of his work.

physics. In fact, it was this similarity that led Black and Scholes to obtain their option price formula as the solution of the diffusion equation with the final and boundary conditions stipulated by the option contract terms. Incidentally, these physics studies applied to economy have never been disrupted and there still is a growing effort of the physics community to understand the dynamics of finance from approaches similar to those that tackle complex systems in physics (see for instance Bak *et al.* (1997), Mandelbrot (1997), Lux and Marchesi (1999), Arthur (1999), Bouchaud and Potters (2000), Mantegna and Stanley (2000), Masoliver *et al.* (2000) among many others).

The economic ideas behind the Black-Scholes option pricing theory translated to the stochastic methods concepts are as follows. First, the option price depends on the stock price and this is a random variable evolving with time. Second, the *efficient market* hypothesis (Fama (1965)), *i.e.*, the market incorporates instantaneously any information concerning future market evolution, implies that the random term in the stochastic equation must be *delta-correlated.* That is: speculative prices are driven by *white noise* (Campbell *et al.* (1997)). It is known that any white noise can be written as a combination of the derivative of the Wiener process and white shot noise (Gihman and Skorohod (1972)). In this framework, the Black-Scholes option pricing method was first based on the geometric Brownian option (Osborne (1954), Black and Scholes (1973)), and it was lately extended to include white shot noise[2].

As is well known, any *stochastic differential equation* (*SDE*) driven by a state dependent white noise, such as the geometric Brownian motion, is meaningless unless an interpretation of the multiplicative noise term is given. Two interpretations have been presented: Itô (1951) and Stratonovich (1966). Nevertheless, all derivations of the Black-Scholes equation starting from a SDE are based on the Itô interpretation. A possible reason is that mathematicians prefer this interpretation over the Stratonovich's one, being the latter mostly preferred among physicists. Nonetheless, as we try to point out here, Itô framework is perhaps more convenient for finance being this basically due to the peculiarities of trading (see Section 3.3.1). In any case, Van Kampen (1981) showed that no physical reason can be attached to the interpretation of the SDE modeling price dynamics. However, the same physical process results in two different SDEs depending on the interpretation chosen. In spite of having different differential equations as starting point, we will show that the resulting Black-Scholes equation is the same regardless the interpretation of the multiplicative noise term, and this constitutes the main result of this chapter. In addition, the mathematical exercise that represents this translation into the Stratonovich convention provides a useful review, specially to physicists, of the option pricing theory and the "path-breaking" Black-Scholes method.

There are several monographs aimed to explain the Black-Scholes option pricing theory although their approach to the problem basically comes from economy or

[2]See Chapter 4 for more details on this latter statement.

mathematics disciplines (see *e.g.* Hull (1997), Campbell *et al.* (1997), Baxter and Rennie (1998), Merton (1992), Cox and Rubinstein (1985)). For this reason, this chapter reviews the Black-Scholes theory for the physicist who wants to be introduced to option pricing and subsequent theories. Our intention is to focus on the questions which may be more difficult for a physicist, and show the essential features to the readers which are not familiarized with finance world.

The chapter is divided in 5 sections. The first four sections are basically taken from the article of Perelló *et al.* (2000) and review the Black-Scholes theory from a physicist's point of view[3]. After this introduction, a summary of the differences between Itô and Stratonovich calculus is developed in Section 3.1. The following section is devoted to explain the market model assumed in Black-Scholes option pricing method. Section 3.3 concentrates in the derivation of the Black-Scholes equation using both Itô and Stratonovich calculus. The Section 3.4 presents the European option formula for the call and the put, and we finish this Chapter by studying the Greeks for the European options.

3.1 Itô vs. Stratonovich

It is not our intention to write a formal discussion on the differences between Itô and Stratonovich interpretations of stochastic differential equations since there are many excellent books and reviews on the subject (see for example the book of Gihman and Skorohod (1972), and the review by Lindenberg *et al.* (1983)). During the eighties, there was a quite controversial discussion on Itô and Stratonovich differential calculus conventions, and on which one was more convenient. Studies, mainly done by physicists, finally concluded that both approaches, Itô and Stratonovich, were equivalent although they demanded to be very cautious and consistent at every step of the mathematical operations. Van Kampen (1981) paper is an excellent and synthetic discussion on the main features involving both Itô and Stratonovich calculus conventions. However, we will summarize those elements in these interpretations that change the treatment of the Black-Scholes option pricing method. In all our discussion, we use a notation that it is widely used among physicists.

The interpretation question arises when dealing with a multiplicative stochastic differential equation, also called *multiplicative Langevin equation*,

$$\dot{X} = f(X) + g(X)\xi(t), \tag{3.1}$$

where f and g are given functions, and $\xi(t)$ is *Gaussian white noise*, that is, a Gaussian and stationary random process with zero mean

$$E[\xi(t)] = 0,$$

[3]These sections are an enlarged version of Perelló *et al.* (2000) since they contain more information concerning the Greeks, the effect of dividends, and the put price. We have also added some figures and tables showing the put and call prices and their Greeks.

and correlation

$$E[\xi(t)\xi(t')] = \delta(t - t'),$$

where $\delta(x)$ is the so-called *Dirac delta generalized function* defined as[4]

$$\delta(x) = 0 \quad \text{for } x \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \varphi(x)\delta(x)dx = \varphi(0), \tag{3.2}$$

where $\varphi(x)$ is a regular function fast decreasing to infinity.

Alternatively, Eq. (3.1) can be written in terms of the Wiener process $W(t)$ as

$$dX = f(X)dt + g(X)dW(t), \tag{3.3}$$

where $dW(t) = \xi(t)dt$ and whose first moment and variance are

$$E[dW(t)] = 0 \qquad \text{Var}[dW(t)] = dt. \tag{3.4}$$

When g depends on X, Eqs. (3.1) and (3.3) have no meaning, unless an interpretation of the multiplicative term $g(X)\xi(t)$ is provided. These different interpretations of the multiplicative term must be given because, due to the extreme randomness of white noise, it is not clear which value of X should be used *even during an infinitesimal timestep dt*. According to Itô, this value of X is the one before the beginning of the timestep, *i.e.*, $X = X(t)$, whereas Stratonovich uses the value of X at the middle of the timestep: $X = X(t + dt/2) = X(t) + dX(t)/2$.

Before proceeding further with the consequences of the above discussion, we will first give a precise meaning of the differential of random processes driven by Gaussian white noise and its implications. Obviously, the differential of any function, such as a random process $X(t)$, is defined by

$$dX(t) \equiv X(t + dt) - X(t). \tag{3.5}$$

On the other hand, the differential $dX(t)$ of any random process is considered equal (in the *mean square sense*) to its mean value $E[dX(t)]$ if its variance is, at least, of order dt^2 (Gihman and Skorohod (1972))[5]:

$$\text{Var}[dX(t)] \equiv E\left[(dX(t) - E[dX(t)])^2\right] = \text{O}(dt^2). \tag{3.6}$$

Note that from now on all our results must be interpreted in the mean square sense.

The mean square limit relation can be used to easily show that

$$|dW(t)|^2 = dt. \tag{3.7}$$

[4]See Lighthill (1958) and Vladimirov (1984) for more details on this subject.

[5]In fact, this definition can be extended to a more general framework: A random differential is equal in the mean square sense to another differential if their averages are equal and if, at least, their mean square difference is of order equal or greater than dt^2. We note the averages are performed by taking $X(t) = X$, *i.e.*, the variable value at current time t is known.

Let us prove this. The mean square relation is

$$\mathrm{Var}[dW(t)^2] = E\left[dW(t)^4\right] - E\left[dW(t)^2\right]^2.$$

Recall that the Wiener process is a Gaussian process whose first moment and variance are given by Eq. (3.4) and that those two values fully determine the rest of higher order averages. Hence, $E\left[dW(t)^4\right] = 3E\left[dW(t)^2\right]^2$ and the mean square relation thus reads

$$\mathrm{Var}[dW(t)^2] = 2E\left[dW(t)^2\right]^2 = 2\ dt^2.$$

Note that the average is of order dt^2, and that this allows us to say that Eq. (3.7) is valid in the mean square sense.

Otherwise, we thus have from Eq. (3.3) that

$$|dX|^2 = |g(X)|^2 dt + \mathrm{O}(dt^2), \tag{3.8}$$

and we symbolically write

$$dX(t) = \mathrm{O}\left(dt^{1/2}\right). \tag{3.9}$$

Let us now turn our attention to the differential of the product of two random processes since this differential adopts a different expression depending on the interpretation (Itô or Stratonovich) chosen. In accordance to Eq. (3.5), we define

$$d(XY) \equiv [(X + dX)(Y + dY)] - XY. \tag{3.10}$$

This expression can be rewritten in many different ways. One possibility is

$$d(XY) = \left(X + \frac{dX}{2}\right) dY + \left(Y + \frac{dY}{2}\right) dX,$$

but it is also allowed to write the product as

$$d(XY) = XdY + YdX + dXdY.$$

Therefore, we say that the differential of a product reads in the Stratonovich interpretation when

$$d(XY) \equiv X_S dY + Y_S dX, \tag{3.11}$$

where

$$X_S(t) \equiv X(t + dt/2) = X(t) + dX(t)/2, \tag{3.12}$$

and similarly for $Y_S(t)$. Whereas we say that the differential of a product follows the Itô interpretation when

$$d(XY) \equiv X_I dY + Y_I dX + dXdY, \tag{3.13}$$

where

$$X_I(t) \equiv X(t), \tag{3.14}$$

and $Y_I(t) \equiv Y(t)$. Note that Eq. (3.11) formally agrees with the rules of calculus while Eq. (3.13) does not. Note also that Eqs. (3.11) and (3.13) can easily be generalized to the product of two functions, $U(X)$ and $V(X)$, of the random process $X = X(t)$. Thus,

$$d(UV) = U(X_S)dV(X) + V(X_S)dU(X), \tag{3.15}$$

where X_S is given by Eq. (3.12), and $dV(X) = V(X+dX) - V(X)$ with an analogous expression for $dU(X)$. Within Itô convention we have

$$d(UV) = U(X)dV(X) + V(X)dU(X) + dU(X)dV(X). \tag{3.16}$$

Let us now go back to Eq. (3.1) and see that one important consequence of the above discussion is that the expected value of the multiplicative term, $g(X)\xi(t)$, depends on the interpretation given. In the Itô interpretation, it is clear that $E[g(X)\xi(t)] = 0$ because the value of X (and, hence the value of $g(X)$) anticipates the jump in the noise. In other words, $g(X)$ is independent of $\xi(t)$. On the other hand, it can be proved that within the Stratonovich framework the average of the multiplicative term reads $E[g(X_S)\xi(t)] = g(X)g'(X)/2$ where the prime denotes the derivative[6]. The zero value of the average $E[g(X)\xi(t)]$ makes Itô convention very appealing because the deterministic equation for the mean value of X only depends on the drift term $f(X)$. In this sense, we note that any multiplicative stochastic differential equation has different expressions for the functions $f(X)$ and $g(X)$ depending on the interpretation chosen. In the Stratonovich framework, Eq. (3.3) can be written as

$$dX = f^{(S)}(X_S)dt + g^{(S)}(X_S)dW(t), \tag{3.17}$$

where $X_S = X + dX/2$. In the Itô sense we have

$$dX = f^{(I)}(X_I)dt + g^{(I)}(X_I)dW(t), \tag{3.18}$$

where $X_I = X$. Note that $f^{(S)}$ and $f^{(I)}$ are not only evaluated at different values of X but they are different functions depending on the interpretation given, and the same applies to $g^{(S)}$ and $g^{(I)}$. One can easily show from Eq. (3.12) and Eqs. (3.17)-(3.18) that, after keeping terms up to order dt, the relation between f_S and f_I is[7]

$$f^{(I)}(X) = f^{(S)}(X) - \frac{1}{2}g^{(S)}(X)\frac{\partial g^{(S)}(X)}{\partial X}, \tag{3.19}$$

[6] To prove this, one can use Eq. (3.12) to write $g(X_S) = g(X) + g'(X)dX/2 + \mathrm{O}(dt)$, multiply this equation by $dW(t) = \xi(t)dt$ and then perform the average taking into account Eqs. (3.3) and (3.4) and recalling that $\xi(t)$ is independent of $g(X)$ and $g'(X)$.

[7] The reader can prove this by expanding the diffusive term with Taylor in order to express the SDE in the Itô convention. The expansion, in accordance to the mean square sense equivalence, should discard higher order contributions than dt.

while the multiplicative functions $g^{(S)}$ and $g^{(I)}$ are equal

$$g^{(I)}(X) = g^{(S)}(X).$$

Conversely, it is possible to pass from a Stratonovich SDE to an equivalent Itô SDE (Gardiner (1985)). Note that the difference between both interpretation only affects the drift term given by the function f while the function g remains unaffected. In addition, we see that for an additive SDE, *i.e.*, when g is independent of X, the interpretation question is irrelevant.

Finally, a crucial difference between Itô and Stratonovich interpretations appears when a change of variables is performed on the original equation. Let $h(X,t)$ be an arbitrary function of X and t. In the Itô sense, the differential of $h(X,t)$ reads

$$dh = \frac{\partial h(X,t)}{\partial X} dX + \left[\frac{\partial h(X,t)}{\partial t} + \frac{1}{2} g^2(X,t) \frac{\partial^2 h(X,t)}{\partial X^2} \right] dt. \tag{3.20}$$

In effect, the function $h(X,t)$ is well-behaved and thus its Taylor expansion when dt and dX are small reads

$$dh = \frac{\partial h(X,t)}{\partial t} dt + \frac{\partial h(X,t)}{\partial X} dX + \frac{1}{2} \frac{\partial^2 h(X,t)}{\partial X^2} dX^2 + \mathrm{O}(dt^2, dX^3). \tag{3.21}$$

Recall that Eq. (3.9) gave us the expression for dX^2 and that dh, due to the mean square sense assumption, does not contain terms of higher order than dt. In this case, the reader will finally recover the Eq. (3.20) for the differential of the function $h(X,t)$.

On the other hand, in the Stratonovich sense, we have the usual expression

$$dh = \frac{\partial h(X_S,t)}{\partial X_S} dX + \frac{\partial h(X_S,t)}{\partial t} dt, \tag{3.22}$$

where

$$\frac{\partial h(X_S,t)}{\partial X_S} = \left. \frac{\partial h(X,t)}{\partial X} \right|_{X=X_S},$$

and X_S is given by Eq. (3.12). This can be proved by rewriting the Taylor expansion (3.21) in the form

$$dh = \left[\frac{\partial h(X,t)}{\partial X} + \frac{1}{2} \frac{\partial^2 h(X,t)}{\partial X^2} dX \right] dX + \frac{\partial h(X,t)}{\partial t},$$

and observing that

$$\frac{\partial h(X,t)}{\partial X} + \frac{1}{2} \frac{\partial^2 h(X,t)}{\partial X^2} dX = \frac{\partial h(X + dX/2, t)}{\partial X} + \mathrm{O}(dt).$$

But, since $X + dX/2 = X_S$, this allows us to write the differential of $h(X,t)$ in the form given by Eq. (3.22).

Comparing the change of variables in the Itô sense (3.20) and in the Staratonovich sense (3.22), we observe that using Stratonovich convention, the standard rules of calculus hold but new rules appear when the equation is understood in the Itô sense. From the point of view of this property, the Stratonovich criterion seems to be more convenient. However, the Itô's change of variables (3.20), known as the *Itô lemma*, is the one that it is extensively used in mathematical finance books (Campbell *et al.* (1997), Hull (1997), Baxter and Rennie (1998), Wilmott (1998)).

The information on the properties of the Itô and Stratonovich interpretation of SDE contained in this brief summary is sufficient to follow the derivations of the next sections.

3.2 The log-Brownian market model

Option pricing becomes a problem because market prices or indexes change randomly. Therefore, any possible calculation of an option price is based on a model for the stochastic evolution of the market prices. As we have thoroughly explained in Chapter 2, the first analysis of price changes was given one hundred years ago by Bachelier (1900) who, studying the option pricing problem, proposed a model assuming that price changes behave as an ordinary random walk. Thus, in the continuum limit[8] speculative prices $S(t)$ obey a Langevin equation. Also recall from Chapter 2 that, in order to include the limited liability of the stock prices, *i.e.*, prices cannot be negative, Osborne (1959) proposed the geometric or log-Brownian motion for describing the price changes. Mathematically, the market model assumed by Osborne can be written as a stochastic differential equation of the form

$$dR(t) = \left(\mu - \sigma^2/2\right) dt + \sigma dW(t), \tag{3.23}$$

where $R(t)$ is the so-called return rate after a period t. Therefore, $dR(t)$ is related to the infinitesimal relative change in the stock share price dS/S (see below), $\mu - \sigma^2/2$ is the average rate per unit time, and σ^2 is the volatility per unit time of the rate after a period t, *i.e.*,

$$E\left[dR\right] = (\mu - \sigma^2/2)dt \qquad \text{and} \qquad \mathrm{Var}[dR] = \sigma^2 dt.$$

There is no need to specify an interpretation (Itô's or Stratonovich's) for Eq. (3.23) because σ is a constant independent of $R(t)$ and we are thus dealing with an additive equation. The rate is compounded continuously and, therefore, an initial price S_0 becomes after a period t:

$$S(t) = S_0 \exp[R(t)]. \tag{3.24}$$

[8]In fact, Merton (1992) uses the term "continuos time finance" for the continuum limit assumption on the dynamics of the market.

This equation can be used as a change of variables to derive the SDE for $S(t)$ given that $R(t)$ evolves according to Eq. (3.23). However, as it becomes multiplicative, we have to attach the equation to an interpretation. Indeed, using Stratonovich calculus (see Eq. (3.22)), it follows that $S(t)$ evolves according to the equation

$$dS = \left(\mu - \sigma^2/2\right) S_S dt + \sigma S_S dW(t), \tag{3.25}$$

where $S_S = S + dS/2$. In the Itô sense (see Eq. (3.20)), the equation for $S(t)$ becomes

$$dS = \mu S dt + \sigma S dW(t). \tag{3.26}$$

Therefore, the Langevin equation for $S(t)$ is different depending on the sense it is interpreted. Our main objective here is to show that no matter which equation is used to derive the Black-Scholes equation the final result turns out to be the same.

Before proceeding further, we point out that the average index price after a time t is

$$E\left[S(t)|S_0\right] = S_0 \exp(\mu t), \tag{3.27}$$

regardless the convention being used. In fact, the independence of the averages on the interpretation used holds for moments of any order (Van Kampen (1981)).

3.3 The Black-Scholes equation

There are several different approaches for deriving the Black-Scholes equation starting from the stochastic differential equation point of view. These different derivations only differ in the way the portfolio is defined[9]. In order to get the most general description of the concepts underlying the Black-Scholes theory, our portfolio is similar to the one proposed by Merton (1973b), and it is based on one type of share whose price is the random process $S(t)$. The portfolio is compounded by a certain amount of shares, Δ, a number of calls, Υ, and a quantity of riskless securities (or bonds) Φ. We also assume that short-selling, or borrowing, is allowed. Specifically, we own a certain number of calls worth ΨC money units and we owe $\Delta S + \Phi B$ money units. In this case, the value Π of the portfolio is

$$\Pi = \Upsilon C - \Delta S - \Phi B, \tag{3.28}$$

where S is the share stock price, C is the call price to be determined, and B is the bond price whose evolution is not random and is described according to the value of r, the risk-free interest rate ratio. That is

$$dB = rBdt. \tag{3.29}$$

[9]Black and Scholes (1973), Merton (1973a), and Harrison and Pliska (1981) present different but equivalent approaches.

The so-called *portfolio investor's strategy* decides the quantity to be invested in every asset according to its stock price at time t (Baxter and Rennie (1998)). This is the reason why the asset amounts Δ, Υ, and Φ are functions of stock price and time, although they are nonanticipating functions of the stock price. This somewhat obscure concept is explained in the forthcoming Section 3.3.1. All derivations of Black-Scholes equation assume a *frictionless market*, that is, there are no transaction costs for each operation of buying and selling.

According to Merton (1973a), we assume that, by short-sales, or borrowing, the portfolio (3.28) is constrained to require net zero investment, that is, $\Pi = 0$ for any time t. Then, from Eq. (3.28) we have

$$C = \delta_n S + \phi_n B, \tag{3.30}$$

where, $\delta_n \equiv \Delta/\Upsilon$ and $\phi_n \equiv \Phi/\Upsilon$ are respectively the number of shares per call and the number of bonds per call. As we have mentioned above, δ_n and ϕ_n are nonanticipating functions of the stock price. Note that Eq. (3.30) has an interesting economic meaning, since tells us that having a call option is equivalent to possess a certain number, δ_n and ϕ_n, of shares and bonds thus avoiding any arbitrage opportunity. In fact, Eq. (3.30) is often called the *replicating portfolio* for the call, and represents the starting point of our derivation that we will separate into two subsections according to Itô or Stratonovich interpretations.

3.3.1 Nonanticipating functions and self-financing strategy

Before beginning the Black-Scholes derivation within Itô and Stratonovich conventions, there are two properties concerning the trading mechanism in financial markets that need to be explained. These are the nonanticipating character of the *strategy functions* δ_n and ϕ_n, and the requisite that the only possible strategies are the self-financing ones. Both relations restrict the behavior of the asset quantities to be hold in a net zero investment portfolio. The first refers to the way traders manage the portfolio and deals with the available information that they possess. And the second property restricts the trading operations that investors can perform in accordance to the wealth of their portfolio. Let us explain these trading mechanisms in portfolio management.

The functionals ϕ_n and δ_n representing normalized asset quantities are *nonanticipating functions* with respect to the stock price S. This means that these functionals are in some way independent of $S(t)$ implying a sort of causality in the sense that unknown future stock price cannot affect the present portfolio strategy. The physical meaning of this translated to financial markets is: first buy or sell according to the present stock price $S(t)$ and right after the portfolio worth changes with variation of the prices dS, dB, and dC. In other words, *the investor strategy does not anticipate the stock price change* (Björk (1998)). Therefore, in the Itô sense, the functionals δ_n

and ϕ_n representing the number of assets in the portfolio solely depend on the share price *right before* time t, *i.e.*, they do not depend on $S(t)$ but on $S(t-dt) = S - dS$. That is,

$$\delta_n(S,t) \equiv \delta(S - dS, t), \tag{3.31}$$

and similarly for ϕ_n [10].

The expansion of Eq. (3.31) yields (see Eq. (3.9))

$$\delta_n(S,t) = \delta(S,t) - \frac{\partial \delta(S,t)}{\partial S} dS + \mathrm{O}(dt),$$

but from the Itô lemma (3.20) we see that

$$\frac{\partial \delta(S,t)}{\partial S} dS = d\delta(S,t) + \mathrm{O}(dt),$$

and finally

$$\delta_n(S,t) = \delta(S,t) - d\delta(S,t) + \mathrm{O}(dt).$$

Analogously,

$$\delta(S,t) = \delta_n(S,t) + d\delta_n(S,t) + \mathrm{O}(dt), \tag{3.32}$$

and a similar expression for $\phi(S,t)$.

We assume that traders follow a self-financing strategy (see *e.g.* Harrison and Pliska (1981) and Björk (1998)), that is: variations of wealth are only due to capital gains and not to the withdrawal or infusion of new funds. In other words, we increase, or decrease, the number of shares by selling, or buying, bonds in the same proportion. Observe that $\delta(S, t+dt)$ is the number of shares we have at time $t+dt$, while $\delta(S - dS, t)$ is that number at time t. Therefore,

$$S(t) d\delta(S - dS, t) = [\delta(S, t+dt) - \delta(S - dS, t)] S(t)$$

is the money we need or obtain from buying or selling shares at time t. Analogously, $B(t) d\phi(S - dS, t)$ is the money, needed or obtained at time t, coming from bonds. If we follow a self-financing strategy, both quantities are equal but with different sign, *i.e.*,

$$S(t) d\delta(S - dS, t) = -B(t) d\phi(S - dS, t),$$

or equivalently (see Eq. (3.31))

$$S d\delta_n = -B d\phi_n. \tag{3.33}$$

In forthcoming sections, we will observe that the self-financing strategy assumption restricts the changes in the replicating portfolio for the call given by Eq. (3.30).

[10] The reader should recall that all equalities must be understood in the mean square sense as it is explained in Section 3.1.

3.3.2 Black-Scholes equation derivation (Itô)

We first present the Black-Scholes option pricing method within the Itô rules of calculus. We recall that the starting point is the replicating portfolio given by Eq. (3.30). That is:

$$C(S,t) = S(t)\delta_n(S,t) + B(t)\phi_n(S,t).$$

And we need first to obtain, within the Itô interpretation, the differential of the call price C. Taking into account the Itô product rule Eq. (3.13), we have

$$\begin{aligned} dC = [\delta_n(S,t) + d\delta_n(S,t)]dS + [\phi_n(S,t) + d\phi_n(S,t)]dB \\ +S(t)d\delta_n(S,t) + B(t)d\phi_n(S,t), \end{aligned}$$

which, after using the nonanticipating trading condition (3.32), reads

$$dC = \delta dS + \phi dB + Sd\delta_n + Bd\phi_n + \mathrm{O}(dt^{3/2}), \tag{3.34}$$

where the relationship between δ, ϕ and δ_n, ϕ_n is given by Eq. (3.31). But, if we also take into account the self-financing condition (3.33), the differential for the call thus reads

$$dC = \delta dS + \phi dB. \tag{3.35}$$

Moreover, from Eqs. (3.29)-(3.30) one can easily show that

$$\phi dB = r(C - \delta S)dt + \mathrm{O}(dt^{3/2}),$$

where Eq. (3.9) and Eq. (3.31) have been taken into account. Therefore, the call differential is

$$dC = \delta dS + r(C - \delta S)dt + \mathrm{O}(dt^{3/2}). \tag{3.36}$$

On the other hand, since the call price C is a function of share price S and time t, $C = C(S,t)$, and S obeys the (Itô) SDE (3.26), then dC can be evaluated from the Itô lemma (3.20) with the result

$$dC = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \frac{\partial C}{\partial S} dS. \tag{3.37}$$

Substituting Eq. (3.36) into Eq. (3.37) yields

$$\left(\delta - \frac{\partial C}{\partial S}\right) dS = \left[\frac{\partial C}{\partial t} - r(C - \delta S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right] dt. \tag{3.38}$$

Note that this is an stochastic equation because of its dependence on the Wiener process enclosed in dS. We can thus turn Eq. (3.38) into a deterministic equation that will give the call price functional dependence on share price and time by equating to zero the term multiplying dS. This, in turn, will determine the investor strategy, that is the number of shares per call, the so-called *delta hedging*:

$$\delta = \frac{\partial C(S,t)}{\partial S}. \tag{3.39}$$

The substitution of Eq. (3.39) into Eq. (3.38) results in the Black-Scholes equation:

$$\frac{\partial C}{\partial t} = rC - rS\frac{\partial C}{\partial S} - \frac{1}{2}(\sigma S)^2 \frac{\partial^2 C}{\partial S^2}. \tag{3.40}$$

3.3.3 Black-Scholes equation derivation (Stratonovich)

Let us now derive the Black-Scholes equation, assuming that the underlying asset obeys the Stratonovich SDE (3.25). Firstly, we perform a derivation that uses the Itô interpretation as starting point. We thus begin with Eq. (3.36) that we write in the form

$$dC = \delta(S,t)dS(t) + r\left[C(S,t) - \delta(S,t)S\right]dt + \mathrm{O}(dt^{3/2}). \tag{3.41}$$

Now, we have to express the function δ within Stratonovich interpretation. Note that $S = S_S - dS/2$. Hence $\delta(S,t) = \delta(S_S - dS/2, t)$, whence

$$\delta(S,t) = \delta(S_S,t) - \frac{1}{2}\frac{\partial \delta(S_S,t)}{\partial S_S}dS + \mathrm{O}(dS^2). \tag{3.42}$$

Analogously, $C(S,t) = C(S_S,t) + \mathrm{O}(dS)$. Therefore, from Eqs. (3.41)-(3.42) and taking into account Eq. (3.8) we have

$$dC = \delta(S_S,t)dS + \left[rC(S_S,t) - rS_S\delta(S_S,t) - \frac{1}{2}\sigma^2 S_S^2 \frac{\partial \delta(S_S,t)}{\partial S_S}\right]dt + \mathrm{O}(dt^{3/2}). \tag{3.43}$$

On the other hand, dC will also be given by Eq. (3.22)

$$dC = \frac{\partial C(S_S,t)}{\partial t}dt + \frac{\partial C(S_S,t)}{\partial S_S}dS,$$

From these two equations we get

$$\left[\delta(S_S,t) - \frac{\partial C(S_S,t)}{\partial S_S}\right]dS = \left[\frac{\partial C(S_S,t)}{\partial t} - rC(S_S,t) + rS_S\delta(S_S,t) \right. \\ \left. + \frac{1}{2}\sigma^2 S_S^2 \frac{\partial \delta(S_S,t)}{\partial S_S}\right]dt. \tag{3.44}$$

Again, this equation becomes non stochastic if we set

$$\delta(S_S,t) = \frac{\partial C(S_S,t)}{\partial S_S}. \tag{3.45}$$

In this case, the combination of Eqs. (3.44)-(3.45) agrees with Eq. (3.40). Therefore, the Stratonovich calculus results in the same call price formula and equation than the Itô calculus.

The first part of this Stratonovich derivation, from Eq. (3.41) to Eq. (3.43), is performed by translating the Itô differential into the Stratonovich sense. However, we can derive the differential dC by directly taking the differential of the replicating call (3.30) in the Stratonovich sense (3.15). Thus,

$$dC = S_S(t)d\delta_n + B(t)d\phi_n + \delta_n(S_S,t)dS + \phi_n(S_S,t)dB. \tag{3.46}$$

From Eq. (3.31), we have

$$\delta_n(S_S,t) = \delta(S_S,t) - \frac{\partial\delta(S_S,t)}{\partial S_S}dS + \mathrm{O}(dS^2), \tag{3.47}$$

and analogously for ϕ_n. Substituting Eq. (3.47) into Eq. (3.46), and taking into account Eqs. (3.8)-(3.9), (3.12) and (3.29) we obtain

$$\begin{aligned} dC = {} & [S(t) + dS/2]d\delta_n + B(t)d\phi_n + \delta(S_S,t)dS \\ & + \left[rB(t)\phi(S_S,t) - \sigma^2 S_S^2 \frac{\partial\delta(S_S,t)}{\partial S_S}\right] dt + \mathrm{O}(dt^{3/2}). \end{aligned}$$

But from Eq. (3.31) and the self-financing strategy (3.33), we see that $S(t)d\delta_n + B(t)d\phi_n = 0$. Hence,

$$dC = \frac{1}{2}dSd\delta_n + \delta(S_S,t)dS + \left[rB(t)\phi(S_S,t) - \sigma^2 S_S^2 \frac{\partial\delta(S_S,t)}{\partial S_S}\right] dt + \mathrm{O}(dt^{3/2}).$$

The substitution of the Stratonovich rule (3.22)

$$d\delta_n = \frac{\partial\delta_n(S_S,t)}{\partial S_S}dS + \frac{\partial\delta_n(S_S,t)}{\partial t}dt,$$

yields

$$dC = \delta(S_S,t)dS + \left[rB(t)\phi(S_S,t) - \frac{1}{2}\sigma^2 S_S^2 \frac{\partial\delta(S_S,t)}{\partial S_S}\right] dt + \mathrm{O}(dt^{3/2}), \tag{3.48}$$

where we have taken into account Eq. (3.8) and the fact that $\partial\delta_n/\partial S_S = \partial\delta/\partial S_S + \mathrm{O}(dt^{1/2})$. Thus, Eq. (3.48) agrees with Eq. (3.43) and this shows the consistency of the two calculus conventions. It is only necessary to be cautious at every step of the calculation.

3.3.4 The effect of dividends

An extension of the market model (3.26) is given by including the dividends. The *dividends* are the amount of the company's profits that is distributed to shareholders. Annual dividends are usually handled in several payments made over the year that can be or cannot be preestablished. The easiest way to give reason of them in the market modelisation is to proportionate them continuously with a constant rate d. The dividend delivery makes cheaper in a deterministic manner the stock price since the value of the company is reduced by going part of its resources to the shareholders via dividends. Thus, the stock price SDE reads

$$dS = (\mu - \mathrm{d})Sdt + \sigma SdW(t). \tag{3.49}$$

Let us show that the option can be similarly derived as it was shown in Section 3.3. Once we have been showing the equivalence between Itô and Stratonovich interpretations, it is not necessary to derive the Black-Scholes equation with dividends coming from these two different approaches. Hence, we solely perform the derivation within the Itô sense. We can take the replicating call given by Eq. (3.30) and rewrite its differential (3.34) after the nonanticipating trading condition is been imposed. That is:

$$dC = \delta dS + \phi dB + Sd\delta_n + Bd\phi_n. \tag{3.50}$$

Following the derivation given by Section 3.3.2, we should now impose the self-financing (3.33) condition. Nevertheless, the self-financing condition needs to be defined again because there exists an infusion of new funds due to the dividends distribution. At each timestep, the shareholders receive the extra amount of money:

$$\delta_n S \mathrm{d}dt.$$

Therefore, the variations of wealth for the portfolio now are restricted by a more general equation that reads

$$Sd\delta_n + \delta_n S\mathrm{d}dt = -Bd\phi_n. \tag{3.51}$$

On the left side of the equation we have the wealth due to holding shares and on the right side the wealth invested in bonds. Observe that the share holder has a new source of wealth due to the existence of dividends. From Eq. (3.50) and (3.51), we thus obtain

$$dC = \delta dS + \phi dB - \delta S\mathrm{d}dt,$$

where we take into account the relationship between δ and δ_n given by Eq. (3.31).

From now on, the derivation proceeds in the same way as to the one given by Section 3.3.2 but with always carrying the extra term $\delta S\mathrm{d}dt$. We thus find the same delta hedging as the one given by Eq. (3.39) and derive a more general Black-Scholes equation:

$$\frac{\partial C}{\partial t} = rC - (r - \mathrm{d})S\frac{\partial C}{\partial S} - \frac{1}{2}(\sigma S)^2\frac{\partial^2 C}{\partial S^2}. \tag{3.52}$$

Observe that the dividends per unit time d appear only in one term out of the two terms that contain the risk-free interest ratio r. To take into account the effect of dividends in the option pricing is not as easy as replacing r by $r - \mathrm{d}$. Although it is true that B-S equation does not depend on the drift $\mu - \mathrm{d}$, we must take into account that dividends have modified the self-financing strategy relation and, in consequence, B-S equation must include a new term, namely, $\mathrm{d}S\ \partial C/\partial S$.

3.4 The Black-Scholes formula for the European options

Let us now derive from Eq. (3.40) the well-known Black-Scholes formula[11]. Note that the Black-Scholes equation is a backward parabolic differential equation, we therefore need one "final" condition and, in principle, two boundary conditions in order to solve it (Carslaw and Jaeger (1990)). In fact, Black-Scholes equation is defined on the semi-infinite interval $0 \leq S < \infty$. In this case, since $C(S,t)$ is assumed to be sufficiently well-behaved for all S, we only need to specify one boundary condition at $S = 0$ (see Wilmott *et al.* (1993), Carslaw and Jaeger (1990)), although we specify below the boundary condition at $S = \infty$ as well.

We also note that all financial derivatives have the same boundary conditions but different initial or final condition. Let us first specify the boundary conditions. We see from the multiplicative character of Eq. (3.3) that if at some time the price $S(t)$ drops to zero then it stays there forever. In such a case, it is quite obvious that the call option is worthless:

$$C(0,t) = 0. \tag{3.53}$$

On the other hand, as the share price increases without bound, $S \to \infty$, the difference between share price and option price vanishes, since option is more and more likely to be exercised[12] and the value of the option will agree with the share price, that is:

$$\lim_{S\to\infty} \frac{C(S,t)}{S} = 1. \tag{3.54}$$

In order to obtain the "final" condition for Eq. (3.40), we need to specify the following two parameters: the expiration or maturing time T, and the striking or exercise price K that fixes the price at which the call owner has the right to buy the share at time T. If we want to avoid arbitrage opportunities, it is clear that the value of the option C of a share that at time T is worth S money units must be equal to the payoff for having the option (see Chapter 2). Recall that this payoff is

[11] The same approach will serve for solving the Black-Scholes equation (3.52) where shareholders continuously receive a constant rate dividends.

[12] See Table 2.7 in Chapter 2.

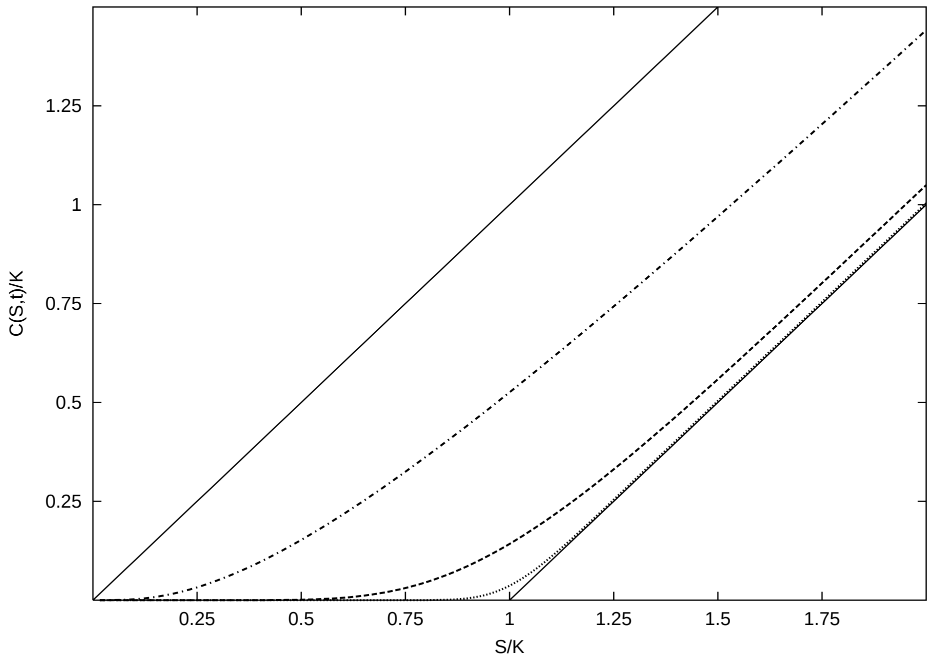


Figure 3.1: The B-S call price in terms of the moneyness

We plot the Black-Scholes normalized call price C/K given by Eq. (3.56) as a function of the moneyness S/K. The two solid lines give the limiting call prices $C_p(S,t) = S$ and $C(S,t=T) = (S-K)^+$ (*i.e.*, when time to maturity is respectively infinity and zero). The intermediate prices correspond to maturities equal to 1 month (dotted line), 1 year (dashed line), and 10 years (dashed and dotted line). For this graph, we take $r = 5\%\ year^{-1}$ and $\sigma = 30\%\ year^{-1/2}$.

either 0 or the difference between share price at time T and option striking price. Hence, the "final" condition for the European call is given by Eq. (1.1) and reads

$$C(S, t = T) = (S - K)^+. \tag{3.55}$$

In Appendix A, we show that the solution to the problem given by Eq. (3.40) and Eqs. (3.53)–(3.55) is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \qquad (0 \le t \le T), \tag{3.56}$$

where $N(z)$ is the probability integral given by Eq. (2.34) and its arguments are

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}. \tag{3.57}$$

It is quite straightforward to derive the put price. With the put-call parity (2.10)

$$P(S,t) = C(S,t) - S + Ke^{-r(T-t)},$$

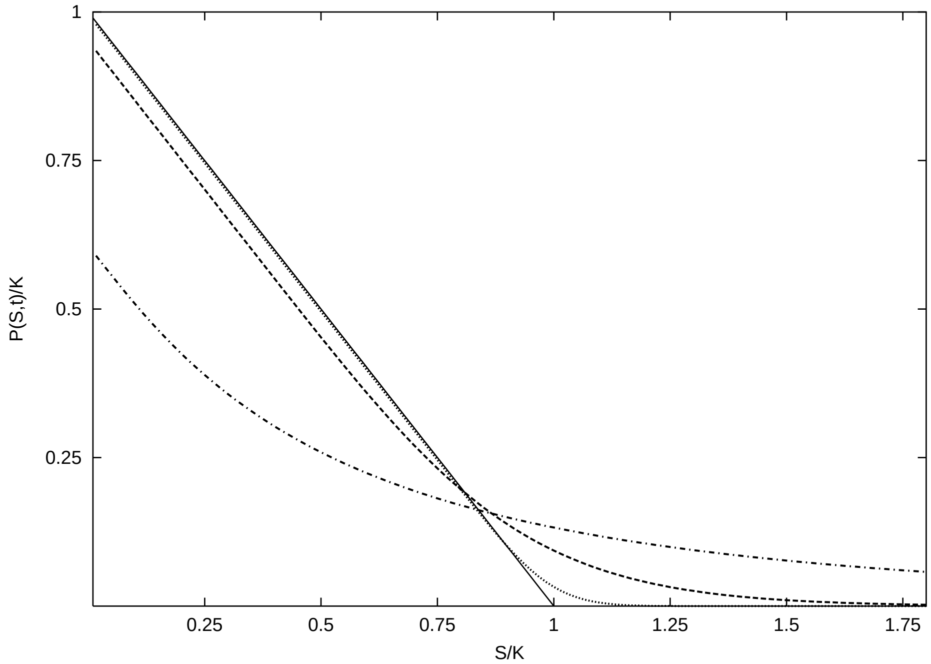


Figure 3.2: The B-S put price in terms of the moneyness

We plot the Black-Scholes normalized put price P/K given by Eq. (3.56) as a function of the moneyness S/K. The prices correspond to maturities equal to 1 month (dotted line), 1 year (dashed line), and 10 years (dashed and dotted line). For this graph, we take $r = 5\%\ year^{-1}$ and $\sigma = 30\%\ year^{-1/2}$. Observe that as time to maturity increases the European put option tends to be worthless as it was predicted in Section 2.1.2.

we give a direct relation between the put price and the call price (3.56). Hence, the Black-Scholes put price formula is

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{3.58}$$

where $d_{1,2}$ are given by Eq. (3.57). Also note that we have used a very characteristic property of the probability integral (2.34) functions: $N(d) + N(-d) = 1$. The put price formula (3.58) can be also obtained by solving the partial differential equation (3.40) but with its own final condition $P(S, t = T) = (K - S)^+$.

For the sake of completeness, we also give the option price solution when underlying share gives dividends. The partial differential (3.52) solution is obtained in the same way as to the case of the Appendix A. The call price formula reads

$$C^{\mathrm{div}}(S,t) = Se^{-\mathrm{d}(T-t)}N(d_1^{\mathrm{div}}) - Ke^{-r(T-t)}N(d_2^{\mathrm{div}}), \tag{3.59}$$

and the put price is

$$P^{\mathrm{div}}(S,t) = Ke^{-r(T-t)}N(-d_2^{\mathrm{div}}) - Se^{-\mathrm{d}(T-t)}N(-d_1^{\mathrm{div}}). \tag{3.60}$$

Table 3.1: The Greeks for the European call and put

The table gives the Greeks for the European call options $C(S,t)$ given by Eqs. (3.63)–(3.67) and show the corresponding Greeks for the put $P(S,t)$. The Greeks for the put can be obtained from the call Greeks and the put-call parity (2.10). We recall that $N'(d) = \exp(-d^2/2)/\sqrt{2\pi}$ and that probability integral function $N(d)$ is given by Eq. (2.34).

Greeks	Call	Put
δ	$N(d_1)$	$N(d_1) - 1$
γ	$\frac{N'(d_1)}{\sigma S\sqrt{T-t}}$	$\frac{N'(d_1)}{\sigma S\sqrt{T-t}}$
θ	$-\frac{\sigma S N'(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$	$-\frac{\sigma S N'(d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$
υ	$S\sqrt{T-t}N'(d_1)$	$S\sqrt{T-t}N'(d_1)$
ϱ	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

In both cases the probability integral arguments are

$$d_1^{\text{div}} = \frac{\ln(S/K) + (r - \mathrm{d} + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2^{\text{div}} = d_1^{\text{div}} - \sigma\sqrt{T-t}. \tag{3.61}$$

3.5 The Greeks

The *Greeks* give the sensitivity of the option price respect to small changes of each variable inside the option price formula. These are: the volatility σ, the risk-free interest rate r, the underlying asset price S, and the time to expiration $T-t$. In real markets, these parameters may fluctuate for many different reasons[13]. Thus, the Greeks are very important in real markets since they allow investors to be hedged against the changes of these variables by giving a first order correction of their consequences in the option price.

Let us define the Greeks and find their expression to the European call price (3.56). In order to derive them it is necessary to take into account a very important identity:

$$SN'(d_1) - Ke^{-r(T-t)}N'(d_2) = 0, \tag{3.62}$$

[13] Although the fluctuations of σ and r parameters are not allowed in the original Black-Scholes theory, there exists ample evidence that this assumption is unrealistic (Hull (1997)).

where prime denotes the partial derivative of $d_{1,2}$. This equality can easily be proved with simple algebraic operations. In effect, the derivative of the probability integral given by Eq. (2.34) is

$$N'(d) = \frac{1}{\sqrt{2\pi}} \exp(-d^2/2).$$

Taking into account that $d_1 = d_2 + \sigma\sqrt{T-t}$ (see Eq. (3.57)), we have

$$\frac{S}{\sqrt{2\pi}} \exp(-d_1^2) = \frac{S}{\sqrt{2\pi}} \exp\Big\{-\big[d_2^2 + \sigma^2(T-t) - 2d_2\sigma\sqrt{T-t}\big]/2\Big\}.$$

And by writing the explicit expression for d_2 (3.57) solely in the third term inside the exponential we see that

$$\frac{S}{\sqrt{2\pi}} \exp(-d_1^2) = \frac{Ke^{-r(T-t)}}{\sqrt{2\pi}} \exp(-d_2^2/2),$$

which proves the identity (3.62).

One of the existent Greeks is already presented: the delta, δ. This Greek is the rate of change of the call with respect to the price of the underlying asset. Recall that the delta gives the hedging necessary to avoid the random fluctuations coming from the stock changes (see Section 3.3). According to the delta hedging definition (3.39) and the call price formula (3.56), the delta reads

$$\delta = \frac{\partial C}{\partial S} = N(d_1), \tag{3.63}$$

where d_1 is given by Eq. (3.57). We show in Fig. 3.3 that δ is enclosed between 0 and 1. In the Black-Scholes theory, δ evaluates the number of shares per call to be hold at every time in a portfolio that follows a riskless strategy.

But if we want to take into account the second order fluctuations of the stock, we must look at the *gamma*. The derivation of the delta (3.63) reads

$$\gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\exp(-d_1^2/2)}{\sigma S\sqrt{2\pi(T-t)}}. \tag{3.64}$$

The gamma gives us an idea of the fluctuations of the delta hedging and allows us to readjust the riskless strategies. Observe that gamma is always positive and this means that the delta hedging must be increased when price S goes up.

There also exists a Greek giving the rate of change of the call with time. The time t derivative can be easily obtained if identity (3.62) is considered. Thus,

$$\theta = \frac{\partial C}{\partial t} = -\frac{\sigma S\exp(-d_1^2/2)}{2\sqrt{2\pi(T-t)}} - rKe^{-r(T-t)}N(d_2). \tag{3.65}$$

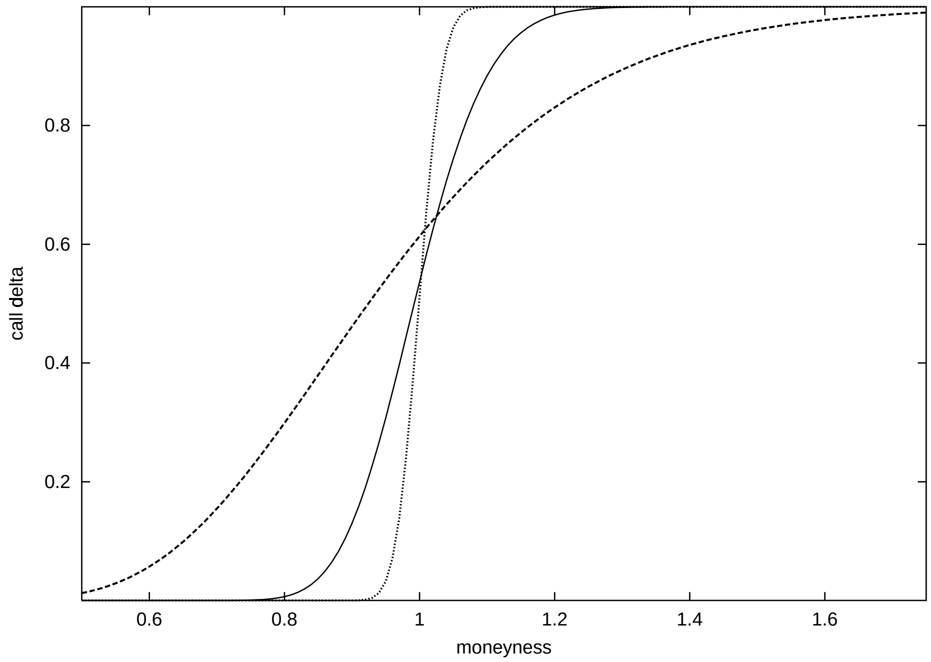


Figure 3.3: The delta for the B-S call in terms of the moneyness

We plot the Black-Scholes delta (3.56) as a function of the moneyness S/K. The solid line corresponds to maturity $1\ month$, the dotted line to $3\ days$, and dashed line to $10\ months$. Observe that function smoothes as time to maturity increases and it becomes stepper as the call approaches to the maturity date. Recall that the delta proportionates the percentage of shares per option that an investor following a risk-free strategy must hold. For this graph, we take $r = 5\%\ year^{-1}$ and $\sigma = 30\%\ year^{-1/2}$.

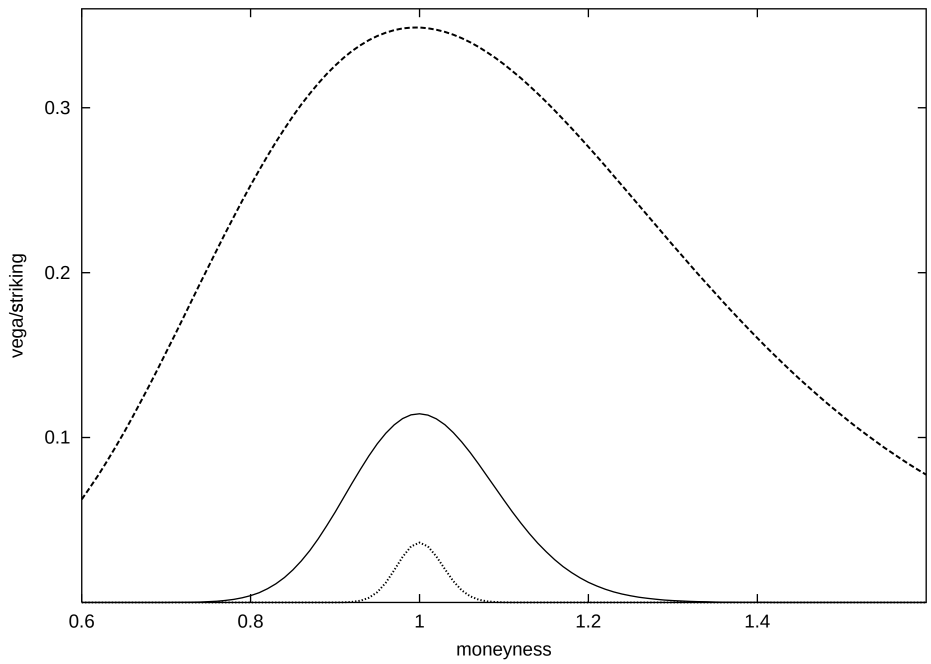


Figure 3.4: The vega for the B-S call in terms of the moneyness

We plot the normalized Black-Scholes vega v/K given by Eq. (3.66) as a function of the moneyness S/K. The solid line corresponds to maturity 1 $month$, the dotted line to 3 $days$, and dashed line to 10 $months$. Observe that function is a log-normal whose maximum value is near to S/K equals to 1. We see that as the time to maturity enlarges vega grows rapidly. Also observe the infinitesimal changes of the call price due to the volatility may lead to call price changes of $0.36 \times K$. This corresponds to a substantial change in the call price thus saying that $C(S,t)$ is very sensible to volatility changes. For this graph, we take $r = 5\%$ $year^{-1}$ and $\sigma = 30\%$ $year^{-1/2}$.

The *theta* gives us the variation of the call with time and informs us about the call price change as time t grows. The theta is negative and this tells us that the call price lowers as the time to maturity $T - t$ decreases.

The *vega* is very important because it gives the option sensitivity to the volatility σ. In practice, the volatility is a very difficult quantity to be evaluated. It is necessary to estimate empirically the variance of the return $R(t)$ then assume that market follows a log-Brownian motion, and indirectly find out that

$$\sigma^2 t = \overline{\mathrm{Var}}[R(t)],$$

where the bar denotes the empirical estimation of the variance. Since the volatility is derived from historical time series, the sigma included in the option price formula is often called the *historical volatility.* The vega[14] is defined as follows

$$\upsilon = \frac{\partial C}{\partial \sigma} = S\sqrt{(T-t)/2\pi}\; e^{-d_1^2/2}. \tag{3.66}$$

This result is derived taking into account the identity (3.62). Observe that the vega is always positive thus indicating that higher volatility implies higher price for the call. Indeed, this is consistent with the idea that a riskier underlying, *i.e.*, a more volatile asset, has a more expensive call option. In Fig. 3.4, we show the high sensitivity to small changes in the volatility. Some studies on the market behavior question the validity of the log-Brownian market model approach since it does not give an adequate description of the volatility (see for instance Masoliver *et al.* (2000), Masoliver and Perelló (2001b), Masoliver and Perelló (2001c)). As we will see in Chapter 5 small corrections on the volatility may significantly modify the price for the option.

The changes of the option price due to the fluctuations of the risk-free interest r are measured by the *rho.* The Black-Scholes theory not only assumes that market follows a log-Brownian process but it also demands the bond evolves with a constant and known interest ratio r (see Eq. (3.29)). Nevertheless, this is not true in real markets and traders want to hedge their changes by measuring their consequences. Hence,

$$\varrho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2), \tag{3.67}$$

where again we have used the identity (3.62) for deriving the Greek. Observe that if r becomes higher then the option becomes more expensive.

Summary

Options are financial instruments designed to protect investors from the stock market randomness. In 1973, Fischer Black, Myron Scholes and Robert Merton proposed

[14] In fact, vega is not a Greek letter.

a very popular option pricing method using stochastic differential equations within the Itô interpretation. We have derived the Black-Scholes equation for the option price using the Stratonovich calculus along with a comprehensive review of the classical option pricing method based on the Itô calculus. We have shown, as could be expected, that the Black-Scholes equation is independent of the interpretation chosen. We have extended the B-S pricing allowing for continuous-time dividends and shown how the self-financing identity must be rewritten if dividends are considered. We have also given the European put and call price formula and derive their Greeks. This Chapter constitutes an enlarged version of the paper Perelló *et al.* (2000).

Appendix A. Solution to the Black-Scholes equation

In this section, we outline the solution to the Black-Scholes equation (3.40) under conditions (3.53)-(3.55). We first transform Eq. (3.40) into a forward parabolic equation with constant coefficients by means of the change of variables

$$z = \ln(S/K), \qquad t' = T - t. \tag{A.1}$$

We have

$$\frac{\partial C}{\partial t'} = -rC(z,t') + \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial C}{\partial z} + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial z^2}, \tag{A.2}$$

$(-\infty < z < \infty,\ 0 < t' < T)$. Moreover, the definition of a new dependent variable:

$$u(z,t') = \exp\left[-\frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)z + \frac{1}{8}\sigma^2\left(1 + \frac{2r}{\sigma^2}\right)(T - t')\right]C(z,t'), \tag{A.3}$$

turns Eq. (A.2) into the ordinary diffusion equation in an infinite medium

$$\frac{\partial u}{\partial t'} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial z^2}, \tag{A.4}$$

with a constant diffusion coefficient given by $\sigma^2/2$, and initial condition:

$$u(z,0) = K\exp\left[-\frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)z + \frac{1}{8}\sigma^2\left(1 + \frac{2r}{\sigma^2}\right)T\right](e^z - 1)^+. \tag{A.5}$$

The solution of problem (A.4)-(A.5) is standard and reads (Carslaw and Jaeger (1990))

$$u(z,t') = \frac{1}{\sqrt{2\pi\sigma^2 t'}}\int_{-\infty}^{\infty} u(y,0)e^{-(z-y)^2/2\sigma^2 t'}dy. \tag{A.6}$$

If we substitute the initial condition (A.5) into the right hand side of this equation and undo the changes of variables (A.1) and (A.3), we finally obtain the Black-Scholes formula Eq. (3.56).

References

Arthur, B., 1999, Complexity and the economy, *Science* **284**, 107-109.

Bachelier, L., 1900, *Théorie de la spéculation.* English version reprinted in P. H. Cootner, 1965, *The random character of stock market prices* (The M.I.T. Press, Cambridge, Massachussets).

Bak, P., M. Paczuski, M. Shubik, 1997, Price variations in a stock market with many agents, *Physica A* **246** 430-453.

Baxter, M., and A. Rennie, 1998, *Financial Calculus* (Cambridge University Press, Cambridge).

Björk, T., 1998, *Arbitrage theory in continuous time* (Oxford University Press, Oxford).

Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* **81**, 637-659.

Bouchaud, J.-P., and M. Potters, 2000, *Theory of financial risks* (Cambridge University Press, Cambridge, UK).

Campbell, J. Y., A. Lo, A.C. MacKinlay, 1997, *The Econometrics of Financial Markets* (Princeton University Press, Princeton).

Carslaw, H. S., and J.C. Jaeger, 1990, *Conduction of Heat in Solids* (Oxford Science Publications, Oxford).

Cox, J. C., and M. Rubinstein, 1985, *Option Markets* (Prentice-Hall, New-Jersey).

Fama, E.,1965, The behavior of stock market prices, *Journal of Business* **38**, 34-105.

Gardiner, C. W., 1985, *Handbook of stochastic methods* (Springer-Verlag, Heilderberg, Berlin, New-York).

Gihman, I. I., and A. V. Skorohod, 1972, *Stochastic Differential Equations* (Springer-Verlag, New York).

Harrison, M, and S. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Their Applications* **11**, 215-260.

Hull, J. C., 1997, *Options, Futures, and other derivatives* (Prentice Hall, New York).

Itô, K., 1951, On Stochastic Differential Equations, *Memmoirs of the American Mathematical Society* **4**, 1-51.

Lighthill, M. J., 1958, *An introduction to Fourier analysis and generalized functions* (Cambridge University Press, Cambridge, UK).

Lindenberg, K. L., V. Seshadri, K. E. Shuler, and B. J. West, 1983, Langevin equations with multiplicative Noise: Theory and applications to physical processes, in A.T. Bharucha-Reid editor: *Probabilistic Analysis and Related Topics* (Academic Press, New York), pp. 81-125.

Lux, T., and M. Marchesi, 1999, Scaling and criticality in a stochastic multi-agent model of a financial market, *Nature* **397**, 498-500.

Mandelbrot, B. B., 1997, *Fractals and scaling in finance* (Springer-Verlag, Berlin).

Mantegna, R. N., H. E. Stanley, 2000, *An introduction to econophysics: Correlations and complexity in finance* (Cambridge University Press, Cambridge).

Masoliver, J., M. Montero and J. M. Porrà, 2000, A dynamical model describing stock market price distributions, *Physica A* **283**, 559-567.

Masoliver, J., and J. Perelló, 2001b, Option pricing and perfect hedging on correlated stocks, submitted for publication.

Masoliver, J., and J. Perelló, 2001c, A correlated stochastic volatility model measuring leverage and other stylized facts, submitted for publication.

Merton, R. C., 1973a, Appendix: Continuous-time speculative processes, *SIAM Review* **15**, 34-38.

——, 1973b, Theory of rational option pricing, *Bell Journal of Economics and Management Science* **4**, 141-183.

——, 1992, *Continous Time Finance* (Blackwell, Cambridge, Massachussets).

Osborne, M. F. M., 1959, Brownian motion in the stock market, *Operations Research* **7**, 145-173.

Perelló, J., J. M. Porrà, M. Montero, J. Masoliver, 2000, Black-Scholes option pricing within Itô and Stratonovich conventions, *Physica A* **278**, 260-274.

Stratonovich, R. L., 1966, A new representation for stochastic integrals and equations, *SIAM Journal on Control* **4**, 362-371.

Van Kampen, N. G., 1981, Itô versus Stratonovich, *Journal of Statistical Physics* **24**,175-187.

Vladimirov, V. S., 1984, *Equations of mathematical physics* (Mir, Moscow).

Wilmott, P., 1998, *Derivatives* (Wiley, New York).

Wilmott, P., J. Dewyne, S. Howison, 1993, *Option Pricing* (Oxford Financial Press, Oxford).

Chapter 4

Generalizations within the Black-Scholes theory

The Black-Scholes theory proportionates a full explanation of the option pricing derivation. It determines a fair price by demanding a riskless behavior of a portfolio containing all assets involved in the option contract. And, finally, not only it gives us the fair price for the option but also dictates us how to manage a portfolio in order to be riskless. Hence, the Black-Scholes pricing method gives a solid justification of the price derived and this was indeed a major contribution to finance.

Few years after Black and Scholes presented their article, Cox and Ross (1976) and Merton (1976) studied which alternative market modelisations might have a fair option price within the B-S pricing formulation. Any alternative models must obey the so-called *ideal market conditions* that not only avoid any arbitrage opportunities but also require the existence of a riskless strategy and its corresponding fair price[1]. Cox and Ross (1976) proposed diffusion models where drift and volatility can deterministically depend on time t and stock price S. In these cases, the B-S equation appears to be the same partial differential equation as the one given by Eq. (3.40) but with constant σ replaced by a function $\sigma(S,t)$. Their main disadvantage is that the B-S partial differential equation is unsolvable for most of the cases.

Moreover, Cox and Ross (1976) and Merton (1976) went further and also included a new class of processes in the market modelisation: the jump process. The jump processes, in contrast with the Brownian processes, describe a discontinuous path where jump events are randomly distributed along its time evolution. This dynamics is due to the arrival of exogenous information to the market that causes an instantaneous stock price change whose amplitude can be or can not be random.

As we have seen in Section 2.2, before the Black and Scholes (1973) article it was not clear how to handle the risk in the pricing derivation and which was the unique and fair option price. Nevertheless, after the B-S solid pricing theory, there appeared some rapid and simple option pricing methods consistent with the absence

[1] The demand of existence of a riskless strategy is also known as the *complete market hypothesis*.

of arbitrage demands. The following sections show how the CAPM theory is able to give the partial differential B-S equation, and how the martingale theory lets us to derive the B-S price formula. The drawback is that these theories are unable to state anything concerning hedging and riskless portfolios.

The Chapter is divided in 6 sections. Section 4.1 present the jump process and Section 4.2 proves the Itô lemma when underlying follows a jump process. Section 4.3 is devoted to explain alternative market models with jumps and derives their option price. The following sections study other pricing theories without assuming any specific market model. The Capital Asset Pricing Model (CAPM) is briefly outlined in Section 4.4, and Section 4.5 concentrates in the derivation of the Black-Scholes option price using the martingale theory. Finally, Section 4.6 presents a new and useful option price formula in terms of the characteristic function of the underlying process. In Appendix A, we test the validity of this representation with the geometric Brownian process, *i.e.*, the market model assumed by the original Black-Scholes theory.

4.1 The jump process

As we have said in the introduction of Chapter 3, the only existing processes obeying the efficient market hypothesis are the ones containing Brownian diffusion or jump processes (see also Gihman and Skorohod (1972)). Hence, a general model fisting the B-S assumptions is also allowed to include the jump process defined as follows. The SDE of a *jump process* $X(t)$ [2] is

$$\frac{dX(t)}{X(t)} = \sum_k (Y_k - 1) d\Theta(t - t_k), \tag{4.1}$$

where $Y_k - 1$ gives the relative change of the stock due to the kth jump which can be random or not, and $d\Theta$ is the differential of the *Heaviside step function*

$$\Theta(t - t_k) = \begin{cases} 1 & \text{if } t \geq t_k, \\ 0 & \text{if } t < t_k. \end{cases} \tag{4.2}$$

Note that although $\Theta(t - t_k)$ is not differentiable as an ordinary function its differential exists in the sense of generalized functions. A simple way of proceeding is the following: since $d\Theta(t - t_k) \equiv \Theta(t + dt - t_k) - \Theta(t - t_k)$ then Eq. (4.2) immediately leads to

$$d\Theta(t - t_k) = \begin{cases} 1 & \text{if } t \leq t_k < t + dt, \\ 0 & \text{if otherwise.} \end{cases} \tag{4.3}$$

[2]We observe that the Itô interpretation given by Section 3.1 evaluates the value of X at the beginning of the timestep. Hence, following Itô the jump occurs at time t while following Stratonovich it occurs at time $t + dt/2$. In contrast with the diffusion models, processes with jumps are typically described in the literature only within the Itô interpretation.

We also observe that, using a notation closer to physics, Eq. (4.1) can be written as

$$\frac{\dot{X}}{X} = \sum_k A_k \delta(t - t_k), \tag{4.4}$$

where $\delta(t) \equiv d\Theta(t)/dt$ is the derivative (in the sense of the generalized functions (Lighthill (1958))) of the step function and, since it results to be the Dirac delta function, it satisfies the properties given by Eq. (3.2). Equation (4.4) provides an interesting interpretation of the jump process. In such a process, the stock dynamics is given by a train of delta-pulses at random times t_k with random amplitudes given by $A_k = Y_k - 1$.

It is further assumed that the occurrence of jumps at random times t_k is a *Poisson process.* Thus the probability $P_n(t)$ of having n jump events during a time interval $(0, t)$ is given by

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \tag{4.5}$$

From this we see that the probability of having one jump during the timestep dt is

$$P_1(dt) = \lambda dt + \mathrm{O}[(\lambda dt)^2], \tag{4.6}$$

while the probability that no jumps has occurred during dt is

$$P_0(dt) = 1 - \lambda dt + \mathrm{O}[(\lambda dt)^2]. \tag{4.7}$$

This is the reason why λ is considered the probability per unit time of having a jump[3]. Usually, λdt is very small therefore it is much less probable to jump than not to jump. Moreover, the probability of having two jump events during dt is

$$P_2(dt) = (\lambda dt)^2/2 + \mathrm{O}[(\lambda dt)^3], \tag{4.8}$$

which is considered to be negligible. From Eq. (4.5) it can be easily proved that the probability of having $n = \{2, 3, ...\}$ jumps during dt is of order dt^n and thus also negligible. See in Fig. 4.1 the resulting path simulation of this dynamics.

Recall that t_k is a random variable representing the occurrence of the kth jump a time t_k. Hence, the average over the stochastic variable $d\Theta(t - t_k)$ given by Eq. (4.3) is

$$\begin{aligned} E\left[d\Theta(t - t_k)\right] &= 1 \times \text{Prob. of having the } k\text{th jump during } \{t, t + dt\} \\ &\quad + 0 \times \text{Prob. of having the } k\text{th jump outside } \{t, t + dt\}. \end{aligned}$$

This probabilities of jump occurrence or absence are respectively given by Eqs. (4.6) and (4.7). We therefore obtain

$$E\left[d\Theta(t - t_k)\right] = \lambda dt. \tag{4.9}$$

[3] It can also be proved that λ^{-1} is the mean time between two consecutive jump events (Rytov *et al.* (1987)).

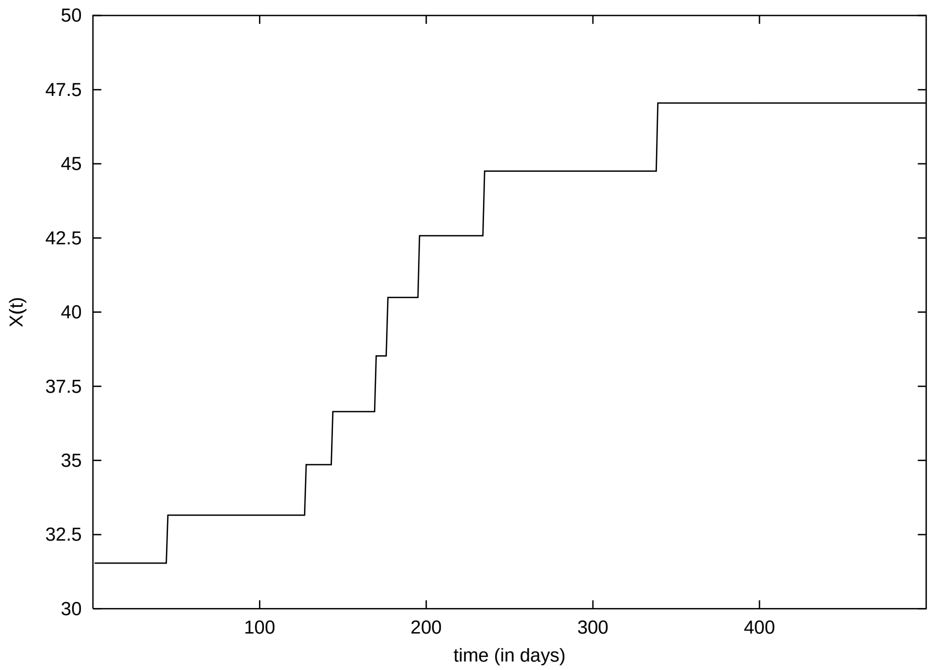


Figure 4.1: Simulation with Poisson jumps

We simulate the dynamics given by Eq. (4.1). We first assume that underlying is solely driven by jumps whose length is deterministic. For this graph, $\lambda = 3\ year^{-1}$ and jumps amplitude $Y = 1.05$.

In the same way, we can average dF's depending on two jump events at times t_k and $t_{k'}$. From Equation (4.8), we see that the probability of having two jumps (or more) during an infinitesimal interval dt is negligible. The only jump occurrence to consider is the one that assumes that $k = k'$ and is λdt (see Eq. (4.6)). Thus,

$$E\left[d\Theta(t-t_k)d\Theta(t-t_{k'})\right] = \begin{cases} \lambda dt & \text{if } t_k = t_{k'}, \\ \mathrm{O}(dt^2) & \text{if } t_k \neq t_{k'}; \end{cases}$$

and

$$E\left[d\Theta(t-t_k)^2 d\Theta(t-t_{k'})^2\right] = \begin{cases} \lambda dt & \text{if } t_k = t_{k'}, \\ \mathrm{O}(dt^2) & \text{if } t_k \neq t_{k'}. \end{cases}$$

And, from them, we see that in the mean square sense[4]

$$d\Theta(t-t_k)d\Theta(t-t_{k'}) = \begin{cases} d\Theta(t-t_k) & \text{if } t_k = t_{k'}, \\ 0 & \text{if } t_k \neq t_{k'}. \end{cases} \tag{4.10}$$

We have given only the essential information of the jump processes in order to derive the Itô lemma for a process with jumps. Two monographs that extensively study this class of processes are Rytov *et al.* (1987), Gihman and Skorohod (1972). We refer the reader to these references for additional information.

4.2 The Itô lemma for the jump process

Before outlining the option pricing method when the underlying is a jump process, let us prove that the Itô lemma now reads

$$df(X,t) = \frac{\partial f(X,t)}{\partial t}dt + \sum_k [f(XY_k,t) - f(X,t)]d\Theta(t-t_k), \tag{4.11}$$

where f is any well-behaved function.

And before proving Eq. (4.11), we first demonstrate that

$$d(X^m) = X^m \sum_k (Y_k^m - 1)d\Theta(t-t_k). \tag{4.12}$$

We proceed by induction. For $m = 1$, this equation is just Eq. (4.1). For $m+1$, we apply the Itô differential product (3.16)

$$d(X\ X^m) = X^m dX + X d(X^m) + dX\ d(X^m),$$

and, taking into account Eq. (4.12), we obtain

$$\begin{aligned} d(X^{m+1}) &= X^{m+1}\Bigg\{\sum_k \left[(Y_k - 1) + (Y_k^m - 1)\right] d\Theta(t-t_k) \\ &\qquad + \sum_{k,k'} (Y_k - 1)(Y_{k'}^m - 1) d\Theta(t-t_k) d\Theta(t-t_{k'})\Bigg\}. \end{aligned}$$

[4]See its definition in Section 3.1.

From Eqs. (4.10), we thus see that the last term, in the mean square sense, is equivalent to

$$\sum_k (Y_k - 1)(Y_k^m - 1) d\Theta(t - t_k)$$

and this implies, after trivial manipulations, the validity of the identity (4.12).

Now we can prove the Itô lemma (4.11). Assuming that $f(X, t)$ is a well-behaved function, the Taylor expansion of the function f is

$$f(X, t) = \sum_j A_j(t) X^j,$$

then

$$df(X, t) = \frac{\partial f(X, t)}{\partial t} dt + \sum_j^{\infty} A_j(t) d(X^j), \tag{4.13}$$

and taking into account Eq. (4.12) we see that

$$df(X, t) = \frac{\partial f(X, t)}{\partial t} dt + \sum_{j,k} A_j(t) X^j (Y_k^j - 1) d\Theta(t - t_k).$$

Since

$$\sum_j A_j(t) X^j (Y_k^j - 1) = f(XY_k, t) - f(X, t),$$

then from Eq. (4.13) we get the Itô lemma (4.11).

4.3 Option pricing for jump processes

We assume that market prices $S(t)$ evolve according to the jump process described above. Therefore, the SDE governing the market dynamics is given by Eq. (4.1) that we now write in the form

$$dS(t) = \sum_k (SY_k - S) d\Theta(t - t_k). \tag{4.14}$$

We will obtain a fair option price when prices evolve following Eq. (4.14). As we have thoroughly explained in Section 3.3, the starting point of B-S option pricing is the differential of the replicating call (*cf.* Eq. (3.36))

$$dC = \delta dS + r(C - \delta S) dt.$$

The use of the Itô lemma, Eq. (4.11), and the substitution of Eq. (4.14) into this equation yield

$$\sum_k \left\{ \delta S(Y_k - 1) - [C(SY_k, t) - C(S, t)] \right\} d\Theta(t - t_k) = \left[\frac{\partial C}{\partial t} - r(C - \delta S) \right] dt. \tag{4.15}$$

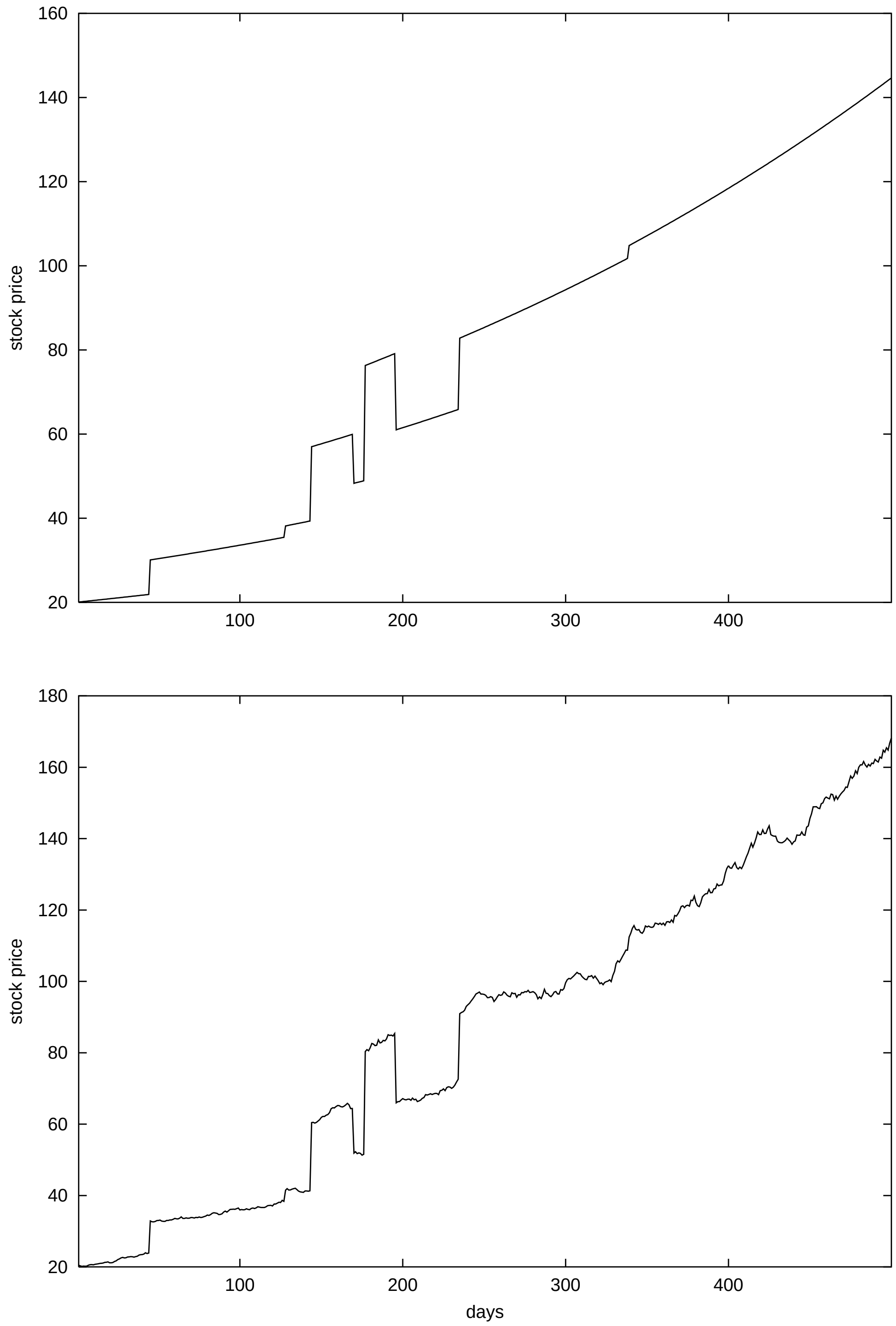


Figure 4.2: Stock path simulations with jumps

We simulate the stock dynamics given by Eqs. (4.14) and (4.20). We first assume that underlying is solely driven by jumps whose length is also stochastic. And we second assume that underlying is a jump-diffusion process as the Merton model given by Eq. (4.20). For these graphs, we take $\mu = 5\%\ year^{-1}$, $\sigma = 10\%\ year^{-1/2}$, $\lambda = 3\ year^{-1}$ and jumps length uniformly distributed between $0.5 \leq Y_k \leq 1.5$.

Note that only in case that Y is not random an independent of k [5], we can turn Eq. (4.15) into a deterministic equation. Indeed,

$$\{\delta S(Y-1) - [C(SY,t) - C(S,t)]\} \sum_k d\Theta(t-t_k) = \left[\frac{\partial C}{\partial t} - r(C - \delta S)\right] dt, \quad (4.16)$$

and, equating to zero the term multiplying the random term $d\Theta(t-t_k)$, the delta hedging is

$$\delta = \frac{C(SY,t) - C(S,t)}{S(Y-1)}. \quad (4.17)$$

The substitution of Eq. (4.17) into Eq. (4.16) results in the Black-Scholes equation when underlying is a jump process:

$$\frac{\partial C(S,t)}{\partial t} = r\left[C(S,t) - \frac{C(SY,t) - C(S,t)}{Y-1}\right]. \quad (4.18)$$

With the final condition $C(S,T) = (S-K)^+$, its solution reads (Cox and Ross (1976))

$$C(S,t) = S\Psi(U, AY) - Ke^{-r(T-t)}\Psi(U,A), \quad (4.19)$$

where

$$\Psi(U,A) \equiv \sum_{k=U}^{\infty} \frac{1}{k!} A^k e^{-A}$$

is the *Laplace function*,

$$U \equiv \left[\frac{\ln(K/S)}{\ln Y}\right],$$

where $[x]$ is the first integer greater or equal than x. Finally,

$$A \equiv \frac{r(T-t)}{Y-1}.$$

Merton (1976) sophisticated the model by adding a diffusion process to the existing jump process. Merton's SDE is

$$dS(t) = S(t)\left[(\mu - \lambda E[Y-1])\, dt + \sigma dW(t) + \sum_k (Y_k - 1) d\Theta(t-t_k)\right], \quad (4.20)$$

where Y_k $(k = 1,2,3,...)$ are independent and identically distributed random variables with $E[Y_k Y_l] = E[Y]^2$ for $k \neq l$, and $E[Y_k^2] = E[Y^2]$. Therefore, the stock price given by Eq. (4.20) satisfies

$$E[dS|S] = \mu S dt \qquad \mathrm{Var}[dS|S] = S^2\left(\sigma^2 + \lambda \mathrm{Var}[Y-1]\right) dt,$$

[5] If Y has a constant and non-random value, the price process experiences jumps of the *same size* but still at random times. This case, which is the simplest case, was first proposed and studied by Cox and Ross (1976). We plot the resulting dynamics in Fig. 4.1.

since we assume that the Wiener and Poisson jump processes are independent.

Let us obtain the call price equation corresponding to Merton's model. The replicating call differential given by Eq. (3.36),

$$dC = dS + r(C - \delta S)dt,$$

and the Itô formulas given by Eqs. (3.37) and (4.11) applied to dC lead to

$$\sum_k \{\delta S(Y-1) - [C(SY_k, t) - C(S,t)]\} \, d\Theta(t-t_k) + \left(\delta - \frac{\partial C}{\partial S}\right) \sigma dW = \left[\frac{\partial C}{\partial t} - r(C - \delta S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right] dt.$$

Observe that there are now two sources of randomness: the jump and the Wiener differentials. This means that one is not able to create a riskless strategy and thus manage a riskless portfolio and, therefore, market is not perfect. At this step Merton (1976) said the "portfolio will be predictable most of the time" but, "on average, once every $1/\lambda$ units of time[6], the portfolio's value will take an unexpected jump (Merton (1976), pp.132-133)". Thus, a "Black-Scholes hedger" will be covered against market risk most of the time but in those rare occasions that a jump occurs hedger can earn or lose an unexpected amount of money. Hence, averaging over the risk coming from jumps and hedging the fluctuations due to dW, we thus derive the B-S delta hedging $\delta = \partial C/\partial S$ and obtain the price equation

$$\frac{\partial C}{\partial t} = rC - (r - \lambda E[Y-1])\, S\frac{\partial C}{\partial S} - \frac{1}{2}(\sigma S)^2 \frac{\partial^2 C}{\partial S^2} - \lambda E[C(SY,t) - C(S,t)], \tag{4.21}$$

where averages are performed over the variable Y. The final condition for an European call is $C(S,T) = (S-K)^+$, and therefore the European call option price is (see the appendix of Merton (1976))

$$C(S,t) = \sum_k \frac{1}{k!} e^{-\lambda(T-t)} [\lambda(T-t)]^k E\left[C_{BS}(SY^k, t)\right] \tag{4.22}$$

where C_{BS} is the original B-S call price given by Eq. (3.56), *i.e.*, assuming that underlying follows a Wiener process whose current stock price is SY^k. In fact, the Merton market model (4.20) is the first model that, starting from the replicating call (3.36), it does not result in a closed equation for the option price.

[6] Recall that λ^{-1} is the average time between two consecutive jumps.

4.4 The Capital Asset Pricing Model (CAPM)

We have used the stochastic differential equation technique in order to derive the option price equation. However, this is only one of the possible routes. Another way, which was also proposed in the original paper of Black and Scholes (1973), uses the *Capital Asset Pricing Model* (*CAPM*) (Sharpe (1964)) where, adducing equilibrium reasons in the asset prices, it is assumed the equality of the so-called *Sharpe ratio* of the stock and the option respectively. The Sharpe ratio of an asset can be defined as its normalized excess of return, therefore CAPM assumption applied to option pricing reads (Merton (1976))

$$\frac{\mu - r}{\sigma} = \frac{\mu_C - r}{\sigma_C}, \tag{4.23}$$

where r is the risk-free interest rate,

$$\mu dt = E\left[\frac{dS}{S}\right], \qquad \sigma^2 dt = \mathrm{Var}\left[\frac{dS}{S}\right];$$

and

$$\mu_C dt = E\left[\frac{dC}{C}\right], \qquad \sigma_C^2 dt = \mathrm{Var}\left[\frac{dC}{C}\right].$$

These averages are performed under the condition that $S(t) = S$ and $C(S,t) = C$ are known quantities.

From this equality it is quite straightforward to derive the Black-Scholes equation (3.40). Indeed, from the Itô differential (3.37), we get

$$E\left[dC\right] = \left[\frac{\partial C}{\partial t} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S}\right] dt,$$

and

$$\mathrm{Var}[dC] = \left(\sigma S\, \frac{\partial C}{\partial S}\right)^2 dt.$$

Note that these averages are performed by taking into account that $E\left[dS\right] = \mu S dt$, and $\mathrm{Var}[dS] = (S\sigma)^2 dt$. The substitution of these equations in the CAPM equality (4.23) leads us to the B-S equation (3.40), which shows the consistency between the two methods. It can be shown that this consistency also appears for to the jump processes given by Eqs. (4.14) and (4.20). Finally, as remarked at the end of Section 3.2, moments are independent of the interpretation chosen, we thus see that either Itô and Stratonovich interpretations lead to the same Black-Scholes equation if we start our derivation from the CAPM since Eq. (4.23) involves moments.

4.5 Martingale theory

As was shown by Harrison and Kreps (1979) and Harrison and Pliska (1981), the B-S option price can also be obtained using martingale methods. This is a shorter, although more abstract way, to derive an expression for the call price. The main advantage is that one only needs to know the probability density function (pdf) governing market evolution. It is not our intention here to write a formal report on the martingale theory and we will just outline the most important features of this pricing methodology.

The equivalent martingale measure theory first discards any underlying process allowing for arbitrage opportunities and then imposes the condition that the stock price $S(t)$ evolves, on average, as a riskless bond (Harrison and Pliska (1981)).

Let $p^*(S,t|S_0,t_0)$ be the *equivalent martingale measure* associated with asset price $S(t)$ under *risk-neutrality* (see below)[7]. Define the following expectation

$$E\left[S(t)|S_0\right]^* = \int_0^\infty S p^*(S,t|S_0,t_0)dS.$$

Then risk-neutrality requires that

$$E\left[S(t)|S_0\right]^* = S_0 e^{r(t-t_0)}, \tag{4.24}$$

where r is the constant risk-free interest rate ratio, in other words, prices must grow on average as a riskless security.

Let us relate the equivalent martingale measure $p^*(S,t|S_0,t_0)$ with $p_S(S,t|S_0,t_0)$ of the price process $S(t)$ which we assume that follows the log-Brownian model:

$$\frac{dS}{S} = \mu dt + \sigma dW.$$

Let $S^*(t)$ be the *risk-neutral stock price* defined by

$$\frac{dS^*}{S^*} = r dt + \sigma dW. \tag{4.25}$$

Note that $S^*(t)$ and $S(t)$ are related by the simple transformation

$$S^*(t) = S(t) e^{-(\mu-r)(t-t_0)}. \tag{4.26}$$

By definition the equivalent martingale measure $p^*(S,t|S_0,t_0)$ is the pdf of process S^*. Thus, the change of variables (4.26) allows us to relate p^* with p_S:

$$p^*(S,t|S_0,t_0) = e^{(\mu-r)(t-t_0)} p_S\left(S e^{(\mu-r)(t-t_0)}, t|S_0,t_0\right), \tag{4.27}$$

[7] It can be proved using the Cameron-Martin-Girsanov theorem (Baxter and Rennie (1998)) that there always exists an equivalent martingale measure associated with any diffusion process such as $S(t)$.

and for the log-Brownian market model we can explicitly write (see Eq. (2.41))

$$p^*(S,t|S_0,t_0) = \frac{1}{S\sqrt{2\pi\sigma^2(t-t_0)}} \exp\left\{-\frac{[\ln(S/S_0)-(r-\sigma^2/2)(t-t_0)]^2}{2\sigma^2(t-t_0)}\right\}, \quad (4.28)$$

A completely analogous analysis can be performed on the return, *i.e.*, $R(t) = \ln[S(t)/S_0]$. We first observe that, in terms of the return, all processes are described by the same conditional pdf, independently of the initial stock price S_0. This is due to the fact that initially the return is always zero. Therefore, from now on we will use the lighter notation $p_R(R,t|t_0)$ instead of $p_R(R,t|0,t_0)$.

Let $R^* = \ln S^*/S_0$ be the *risk-neutral return.* From Eq. (4.26), we have

$$R^*(t) = R(t) - (\mu - r)(t-t_0),$$

and the equivalent relation to Eq. (4.27) is

$$p_R^*(R,t|t_0) = p_R(R+(\mu-r)(t-t_0),t|t_0), \quad (4.29)$$

where $p_R^*(R,t|t_0)$ is the return distribution adjusted for risk-neutrality. Again, for the geometric Brownian motion, we have

$$p_R^*(R,t|t_0) = \frac{1}{\sqrt{2\pi\sigma^2(t-t_0)}} \exp\left\{-\frac{[R-(r-\sigma^2/2)(t-t_0)]^2}{2\sigma^2(t-t_0)}\right\} \quad (4.30)$$

Once we know the risk-neutral density $p^*(S,t|S_0,t_0)$, the price for the European call option is simply the discounted expected gain due to holding the call. That is (Harrison and Pliska (1981)),

$$\begin{aligned} C(S,t) &= e^{-r(T-t)}E\left[(S(T)-K)^+ \mid S(t)=S\right]^* \\ &= e^{-r(T-t)}\int_K^\infty (S'-K)p^*(S',T|S,t)dS', \end{aligned} \quad (4.31)$$

and the final result for the call is obtained by evaluating the expected value with the equivalent martingale measure given by Eq. (4.27) or by Eq. (4.28) for the log-Brownian model. Observe that this price is quite similar to that of Samuelson and Boness (see Section 2.2.3). The main difference being due to the adequate modification of the pdf by "risk-neutralizing" its expression. Note that in terms of the risk-neutral density, $p_R^*(R,t|t_0)$, of the return the call price can be obtained by

$$C(S,t) = e^{-r(T-t)}\int_{\ln(K/S)}^\infty (Se^R - K)p_R^*(R,T|t)dR. \quad (4.32)$$

In the case of the geometric Brownian motion the martingale price agrees exactly with the B-S price Eq. (3.56)[8]. We can thus say that both option pricing

[8] We can perform similar analysis for the processes with jumps described by Eqs. (4.14) and (4.20) and see that martingale pricing method leads to the same call prices as that of Eqs. (4.19) and (4.22).

methods are completely equivalent although martingale theory does not require the construction of a portfolio and ignores any hedging strategy. Again, since averages are independent on the SDE calculus convention chosen, Itô and Stratonovich calculus lead to the same option price formula. As we have mentioned above, the main advantage of martingale methods is that, for a generic underlying price process with a known probability distribution, one only has to determine whether it allows or not for arbitrage opportunities. In case that it does not, one directly obtains the option price by applying the martingale price formula (4.31).

4.6 Martingale option pricing by Fourier analysis

As we have said, the main advantage of martingale pricing is that one can obtain a fair option price when the market obeys a random dynamics different than the geometric Brownian motion. In such a case one only has to know the probability distribution of the underlying, then using Eqs. (4.27) and (4.31) one readily gets a fair price for the option. Nevertheless, knowing an analytical expression of the pdf $p(S,t|S_0,t_0)$ may be, in practice, beyond our reach. There are however many situations where one knows the characteristic function of the market model, although one is not able to invert the Fourier transform and obtain the density $p(S,t|S_0,t_0)$. This is the case of non-Gaussian models (see, for instance, Masoliver *et al.* (2000)) and stochastic volatility models (see Chapter 6) among others. For these cases, we will develop an option pricing based on a combination of martingale methods and harmonic analysis.

We recall that the characteristic function (cf) of a random variable is the Fourier transform of its probability density function. Thus the characteristic function of the return will be given by:

$$\varphi_R(\omega,t|t_0) = E\left[e^{i\omega R} \mid t_0\right] = \int_{-\infty}^{\infty} e^{i\omega R} p_R(R,t|t_0)dR.$$

Note that the knowledge of φ_R allows us to evaluate the first moment of the share price $S(t) = S_0 e^{R(t)}$. In effect,

$$E\left[S(t)|S_0,t_0\right] = S_0 E\left[e^{R(t)} \mid t_0\right] = S_0\varphi_R(-i,t|t_0).$$

On the other hand, the Fourier transform of Eq. (4.29) allows us to write the risk-neutral characteristic function $\varphi^*_R(\omega,t|t_0)$ in terms of the ordinary cf $\varphi_R(\omega,t|t_0)$ as

$$\varphi^*_R(\omega,|t_0) = e^{-i\omega(\mu-r)(t-t_0)}\varphi_R(\omega,|t_0). \tag{4.33}$$

In addition, the risk-neutral market assumption imposes that the stock average must exponentially grow with $r(t-t_0)$ (see Eq. (4.24)). Hence, the equivalent martingale measure must obey

$$\varphi^*_R(-i,t|t_0) = e^{r(t-t_0)}. \tag{4.34}$$

Let us check this for the log-Brownian model. Recall that

$$\varphi_R(-i, t|t_0) = E\left[e^{R(t)}|t_0\right],$$

but for the Brownian motion $dR(t) = (\mu - \sigma^2/2)dt + \sigma dW(t)$, whence

$$E\left[e^{R(t)}|t_0\right] = e^{(\mu-\sigma^2/2)(t-t_0)} E\left[e^{\sigma W(t)}|t_0\right]$$

and from the characteristic function of the Wiener process one finally gets

$$\varphi_R(-i, t|t_0) = e^{\mu(t-t_0)}.$$

Putting $\omega = -i$ in Eq. (4.33) and using this last result yield Eq. (4.34).

After these preliminary settings, we are in the disposition of addressing the option pricing by means of harmonic analysis. The starting point is Eq. (4.32) that we write in the form

$$C(S,t) = e^{-r(T-t)}\left[\int_{-\infty}^{\infty}\left(Se^R - K\right)p_R^*(R,T|t)dR \right.$$
$$\left. - \int_{-\infty}^{\ln(K/S)}\left(Se^R - K\right)p_R^*(R,T|t)dR\right]. \qquad (4.35)$$

The first integral on the right hand side (rhs) of this equation can be solved in closed form with the result

$$\int_{-\infty}^{\infty}\left(Se^R - K\right)p_R^*(R,T|t)dR = SE\left[e^{R(T)} \mid t\right]^* - K = Se^{r(T-t)} - K.$$

As to the second integral on the rhs of Eq. (4.35), we have

$$I \equiv \int_{-\infty}^{\ln(K/S)}\left(Se^R - K\right)p_R^*(R,T|t)dR = K\int_0^{\infty}\left(e^{-z} - 1\right)p_R^*(-z - R_K, T|t)dR,$$

where $R_K \equiv \ln(S/K)$ can be considered as the "*return associated with moneyness*"[9] The inverse Fourier transform of the characteristic function

$$p_R^*(R,T|t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega R}\varphi_R^*(\omega,T|t)d\omega$$

allows us to write this second integral as

$$I = \frac{K}{2\pi}\int_{-\infty}^{\infty}\varphi_R^*(\omega,T|t)e^{i\omega R_K}d\omega\int_0^{\infty} e^{i\omega z}\left(e^{-z} - 1\right)dz,$$

but

$$\int_0^{\infty} e^{i\omega z}\left(e^{-z} - 1\right)dz = \frac{1}{1 - i\omega} - \int_0^{\infty} e^{i\omega z}dz,$$

[9] We recall that the term "moneyness" refers to the ratio S/K.

and

$$\int_0^\infty e^{i\omega z}dz = \lim_{\varepsilon\to 0^+}\int_0^\infty e^{-(\varepsilon - i\omega)z}dz = \lim_{\varepsilon\to 0^+}\frac{i}{\omega+i\varepsilon} = \frac{i}{\omega+i0},$$

where this result has to be understood in the sense of generalized functions as (Vladimirov (1984))

$$\frac{1}{\omega \pm i0} = \mp i\pi\delta(\omega) + \mathcal{P}\left[\frac{1}{\omega}\right],$$

where $\delta(\omega)$ is the Dirac delta function (see Eq. (3.2)), and $\mathcal{P}[1/\omega]$ is the *Cauchy principal value*, *i.e.*,

$$\mathcal{P}[1/\omega] = 1/\omega \qquad \text{for } \omega \neq 0$$

and

$$\mathcal{P}\left[\int_{-\infty}^{\infty}\frac{\phi(\omega)}{\omega}d\omega\right] = \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon}\frac{\phi(\omega)}{\omega}d\omega + \int_{\varepsilon}^{\infty}\frac{\phi(\omega)}{\omega}d\omega\right),$$

where $\phi(\omega)$ is any regular function fast decaying at infinity. Therefore,

$$\int_0^\infty e^{i\omega z}dR = \pi\delta(\omega) + i\mathcal{P}\left[\frac{1}{\omega}\right]. \tag{4.36}$$

Hence

$$I = -\frac{K}{2} + \frac{K}{2\pi}\int_{-\infty}^{\infty}\varphi_R^*(\omega,T|t)e^{i\omega R_K}\frac{d\omega}{1-i\omega} - i\mathcal{P}\left[\int_{-\infty}^{\infty}e^{i\omega R_K}\frac{\varphi_R^*(\omega,T|t)}{\omega}d\omega\right],$$

that is,

$$\begin{aligned} I = -\frac{K}{2} + \frac{K}{2\pi}\int_{-\infty}^{\infty}\varphi_R^*(\omega,T|t)e^{i\omega R_K}\frac{d\omega}{1-i\omega} \\ -i\int_0^\infty\left[\varphi_R^*(\omega,T|t)e^{i\omega R_K} - \varphi_R^*(-\omega,T|t)e^{-i\omega R_K}\right]\frac{d\omega}{\omega}. \end{aligned}$$

Collecting results into Eq. (4.35), we finally get

$$\begin{aligned} C(S,t) = S - \frac{K}{2}e^{-r(T-t)} - \frac{K}{2\pi}e^{-r(T-t)}\Bigg\{\int_{-\infty}^{\infty}\varphi_R^*(\omega,T|t)e^{i\omega R_K}\frac{d\omega}{1-i\omega} \\ -i\int_0^\infty\left[\varphi_R^*(\omega,T|t)e^{i\omega R_K} - \varphi_R^*(-\omega,T|t)e^{-i\omega R_K}\right]\frac{d\omega}{\omega}\Bigg\}. \end{aligned} \tag{4.37}$$

Using the following alternative expression for the Cauchy principal value[10]

$$\mathcal{P}\left[\int_{-\infty}^{\infty}\frac{\phi(\omega)}{\omega}d\omega\right] = \int_{-\infty}^{\infty}\frac{\phi(\omega)-\phi(0)}{\omega}d\omega,$$

[10] See for instance Vladimirov (1984).

we can write Eq. (4.37) in a somewhat simpler form

$$C(S,t) = S - \frac{K}{2}e^{-r(T-t)} - \frac{K}{2\pi}e^{-r(T-t)}\left\{\int_{-\infty}^{\infty} \varphi_R^*(\omega,T|t)e^{i\omega R_K}\frac{d\omega}{1-i\omega} - i\int_0^{\infty}\left[e^{i\omega R_K}\varphi_R^*(\omega,T|t)-1\right]\frac{d\omega}{\omega}\right\}. \quad (4.38)$$

The representations (4.37) and (4.38) are very useful when the pdf is unknown but its characteristic function is known. This would be indeed the case of more sophisticated market models such as those of stochastic volatility (see Chapter 6). A similar result has been presented by Scott (1997) where he used an equivalent form of Eq. (4.37) in order to numerically perform the integration knowing $\varphi_R^*(\omega,T|t)$. Scott (1997) asserts that these Fourier methods allow a fast computing of the option price.

We close this section and chapter presenting an alternative price formula to that of Eq. (4.37) and Eq. (4.38) that only involves real quantities and that can be more convenient in a number of cases. Some definitions are needed. We recall that $R^*(t) = \ln S^*(t)/S_0$ is the risk-neutral return. From Eq. (4.25), we see that $R^*(t)$ obeys the following SDE:

$$dR^* = \mu^* dt + \sigma dW, \quad (4.39)$$

where $\mu^* = r - \sigma^2/2$. Note that $-\sigma^2/2$ is the spurious drift that, due to the Itô lemma, must be added to pass from Eq. (4.25) for the price to Eq. (4.39) for the return. This formalism can be extended to include more general market models. In such cases instead of Eq. (4.25) we will have a more general SDE

$$\frac{dS^*}{S^*} = r dt + \sigma dF(t),$$

where $F(t)$ is a given driving noise with zero mean and unit variance. Equation (4.39) is still correct but changing dW by dF and

$$\mu^* = E\left[dR^*/dt|t_0\right]$$

which includes the risk-free interest rate as well as a term of spurious drift coming from $F(t)$.

Let $X(t)$ be the *zero-mean return* defined by

$$X(t) = R^*(t) - \mu^* t. \quad (4.40)$$

Then

$$\varphi_R^*(\omega,T|t) = e^{i\omega\mu^*(T-t)}\varphi_X(\omega,T|t), \quad (4.41)$$

where $\varphi_X(\omega,T|t)$ is the characteristic function of process $X(t)$[11]. Moreover, since the distribution of $X(t)$, $p_X^*(x,T|t)$, is symmetric around $x=0$, then its Fourier

[11] Since $X(t)$ is driftless, it can be easily proved that it is also a martingale (Baxter and Rennie (1998), p.79).

transform is a real and even function of ω: $\varphi_X(-\omega, T|t) = \varphi_X(\omega, T|t)$ (see, for instance, Lukacs (1970)). These properties allow us to further simplify the integrals appearing in Eqs. (4.37) and (4.38). In effect, in terms of $\varphi_X(\omega, T|t)$ given by Eq. (4.41), the first integral on the rhs of Eq. (4.38) (we call it I_1) reads

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \varphi_X(\omega, T|t) \frac{e^{i\omega\alpha(T-t)}}{1+\omega^2}(1+i\omega)d\omega \\ &= 2\int_0^{\infty} \varphi_X(\omega, T|t) \frac{\cos\omega\alpha(T-t) - \omega\sin\omega\alpha(T-t)}{1+\omega^2}d\omega, \end{aligned}$$

where

$$\alpha(T-t) \equiv R_K + \mu^*(T-t). \tag{4.42}$$

Following an analogous reasoning we obtain, for the second integral on the rhs of Eq. (4.38) the result

$$I_2 = 2i\int_0^{\infty} \varphi_X(\omega, T|t) \frac{\sin\omega\alpha(T-t)}{\omega}d\omega.$$

The substitution of these two integrals into Eq. (4.38) yields our final result

$$\begin{aligned} C(S,t) = S - \frac{K}{2}e^{-r(T-t)} - \frac{K}{\pi}e^{-r(T-t)} \int_0^{\infty} \varphi_X(\omega, T|t) \Big[&\cos\omega\alpha(T-t) \\ &+ \omega\sin\omega\alpha(T-t)\Big] \frac{d\omega}{1+\omega^2}, \end{aligned} \tag{4.43}$$

where α is given by Eq. (4.42). Note that this expression only involves one real integral and it is therefore simpler and faster to compute than Eqs. (4.37) or (4.38). In the Appendix A, we show that for the log-Brownian market model Eq. (4.43) reduces to the Black-Sholes formula.

Summary

After the Black and Scholes (1973) article, there appeared some alternative market models and other equivalent option pricing methods. This Chapter has presented some of the most important market models generalizations that include jumps without contradicting the absence of arbitrage prescriptions. We have shown which conditions must be accomplished by these more general market models if we restrict ourselves within the original B-S theory. Finally, we have outlined the alternative CAPM and martingale pricing theories and give the martingale price formula in terms of the characteristic function for a general probability distribution. To our knowledge, this expression has not been given before in the literature and, as we will see in forthcoming chapters, constitutes a useful expression when we only know the characteristic function.

Appendix A. The Black-Scholes formula by Fourier analysis

Let us check that our price formula (4.43) based on the characteristic function of the driftless process $X(t)$ reduces to the Black-Scholes formula when the market model is taken to be the geometric Brownian motion. Indeed, in such a case the dynamics of process $X(t)$, defined by Eq. (4.40), is given by $dX(t) = \sigma dW(t)$ (see Eq. (4.39). Therefore, the characteristic function of $X(t)$ is[12]

$$\varphi_X(\omega, t) = e^{-\sigma^2\omega^2 t/2}. \tag{A.1}$$

Let $I(t)$ be the integral on the rhs of Eq. (4.43):

$$I(t) \equiv \int_0^\infty \varphi_X(\omega, T|t)\Big[\cos\omega\alpha(t) + \omega\sin\omega\alpha(t)\Big]\frac{d\omega}{1+\omega^2},$$

we write this integral in a slightly different form

$$I(t) = \int_0^\infty \varphi_X(\omega,t)\frac{\cos\omega\alpha(t) - \omega\sin\omega\alpha(t)}{1+\omega^2}d\omega + \int_0^\infty \varphi_X(\omega,t)\frac{\sin\omega\alpha(t)}{\omega}d\omega. \tag{A.2}$$

The substitution of (A.1) into this equation yields

$$\begin{aligned} I(t) = \int_0^\infty e^{-\sigma^2\omega^2 t/2}\cos\omega\alpha(t)\frac{d\omega}{1+\omega^2} + \int_0^\infty e^{-\sigma^2\omega^2 t/2}\omega\sin\omega\alpha(t)\frac{d\omega}{1+\omega^2} \\ - \int_0^\infty e^{-\sigma^2\omega^2 t/2}\frac{\sin\omega\alpha(t)}{\omega}d\omega \equiv I_1(t) + I_2(t) - I_3(t). \end{aligned}$$

But (Gradshteyn and Ryzhik (1994), pp.529-530)

$$I_1(t) = \frac{\pi}{4}e^{\sigma^2 t/2}\left[2\cosh\alpha(t) - e^{-\alpha(t)}\Phi\left(\sqrt{\sigma^2 t/2} - \frac{\alpha(t)}{\sqrt{2\sigma^2 t}}\right) - e^{\alpha(t)}\Phi\left(\sqrt{\sigma^2 t/2} + \frac{\alpha(t)}{\sqrt{2\sigma^2 t}}\right)\right],$$

$$I_2(t) = -\frac{\pi}{4}e^{\sigma^2 t/2}\left[2\sinh\alpha(t) + e^{-\alpha(t)}\Phi\left(\sqrt{\sigma^2 t/2} - \frac{\alpha(t)}{\sqrt{2\sigma^2 t}}\right) - e^{\alpha(t)}\Phi\left(\sqrt{\sigma^2 t/2} + \frac{\alpha(t)}{\sqrt{2\sigma^2 t}}\right)\right],$$

and

$$I_3(t) = \frac{\pi}{2}\Phi\left(\frac{\alpha(t)}{\sqrt{2\sigma^2 t}}\right),$$

where $\Phi(x)$ is the error function defined by

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz.$$

Collecting these integrals into Eq. (A.2), recalling that

$$\alpha(t) = -R_k - (r - \sigma^2/2)t,$$

[12] For simplicity in the notation, in what follows t represents the time to maturity $T - t$.

and using the property $\Phi(-x) = -\Phi(x)$, we get

$$\frac{2}{\pi}I(t) = \frac{S}{K}\left[1 - \Phi\left(\frac{rt + R_K + \sigma^2/2}{\sqrt{2\sigma^2 t}}\right)\right] - \Phi\left(\frac{rt + R_K - \sigma^2/2}{\sqrt{2\sigma^2 t}}\right).$$

Substituting this into Eq. (4.43) we obtain the Black-Sholes formula[13]

$$C(S,t) = \frac{S}{2}\left[1 + \Phi\left(\frac{rt + R_K + \sigma^2/2}{\sqrt{2\sigma^2 t}}\right)\right] - \frac{K}{2}e^{-rt}\left[1 + \Phi\left(\frac{rt + R_K - \sigma^2/2}{\sqrt{2\sigma^2 t}}\right)\right]. \tag{A.3}$$

References

Baxter, M., and A. Rennie, 1998, *Financial Calculus* (Cambridge University Press, Cambridge).

Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* **81**, 637-659.

Cox, J. C., and S. A. Ross, 1976, The valuation of options for alternative stochastic processes, *Journal of Financial Economics* **3**, 145-166.

Gihman, I. I., and A. V. Skorohod, 1972, *Stochastic Differential Equations* (Springer-Verlag, New York).

Gradshteyn, I. S., I. M. Ryzhik, 1994, *Table of integrals, series, and products* (Academic Press, Sand Diego).

Harrison, M., and D. Kreps, 1979, Martingales and arbitrage in multi period securities markets, *Journal of Economic Theory* **20**, 381-408.

Harrison, M, and S. Pliska, 1981, Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Their Applications* **11**, 215-260.

Lighthill, M. J., 1958, *An introduction to Fourier analysis and generalized functions* (Cambridge University Press, Cambridge, UK).

Lukacs, E., 1970, *Characteristic functions* (Griffin, London).

Masoliver, J., M. Montero, J. M. Porrà, 2000, A dynamical model describing stock market price distributions, *Physica A* **283**, 559-567.

Merton, 1976, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Econonomics* **3**, 125-144.

Rytov, S. M., Yu. A. Kravtsov, and V. I. Tatarskii, 1987, *Principles of statistical radiophysics 1. Elements of random process theory* (Springer-Verlag, Berlin).

Scott, L. O., 1997, Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Applications of Fourier inversion methods, *Mathematical Finance* **7**, 413-424.

Sharpe, W. F., 1964, Capital asset price: A theory of market equilibrium under conditions of risk, *Journal of Finance* **19**, 425-442.

Vladimirov, V. S., 1984, *Equations of mathematical physics* (Mir, Moscow).

[13] Almost always the Black-Sholes formula is written in terms of the probability integral $N(x)$, we can obtain the usual expression from Eq. (A.3) since the error function and the probability integral are related by $\Phi(x) = -1 + 2N(x\sqrt{2})$.

Chapter 5

Option pricing and perfect hedging on correlated stocks

As we have shown in Chapter 3, Fischer Black and Myron Scholes (1973) and Robert Merton (1973a) obtained a fair option price assuming severe and strict theoretical conditions for the market behavior. The requirements under which these were developed include: (i) Absence of arbitrage opportunities, *i.e.*, identical cash flows have identical values (Sharpe (1964), Cox and Ross (1976)). (ii) Efficient market hypothesis, *i.e.*, the market incorporates instantaneously any information concerning future market evolution (Fama (1965)). (iii) Existence of a unique riskless strategy for a portfolio in a complete market (Markowitz (1952)). Due to the random character of stock market prices, the implementation of these conditions, especially condition (ii), indicates that speculative prices are driven by white (*i.e.*, delta-correlated) random processes. At this point, one has to choose between a Gaussian white process or a white jump process. In this latter case and due to requirement (iii), the jump lengths also have to be known and fixed. There are no other choices for modelling market evolution if the above requirements and ideal conditions are to be obeyed[1].

From these three assumptions, condition (ii) is perhaps the most restrictive and, in fact, disagrees with empirical evidence since real markets are not efficient, at least at short times (Grossman and Stiglitz (1980), Fama (1991)). Indeed, market efficiency is closely related to the assumption of totally uncorrelated price variations (white noise). But white noise is only an idealization since, in practice, no actual random process is completely white. For this reason, white processes are convenient mathematical objects valid only when the observation time is much larger than the auto correlation time of the process[2]. And, analogously, the efficient market hypothesis is again a convenient assumption when the observation time is much

[1]See Sections 3.3 and 4.1 for more information concerning these statements and for a description of the geometric Brownian and the Poisson jump processes.

[2]Throughout this Chapter we will use the terms "correlation" and "auto correlation" without distinction.

larger than time spans in which "inefficiencies" (*i.e.*, correlations, delays, etc.) occur.

Alternative models for describing empirical results of the market evolution have been suggested (Mandelbrot (1963), Fama (1963)). In each of these, an option price can be obtained only by relaxing some or even all of the initial Black-Scholes (B-S) assumptions (Figlewski (1989), Aurell *et al.* (2000)). Our main purpose in this Chapter is to derive a nontrivial option price by relaxing the efficient market hypothesis and allowing for a finite, non-zero, correlation time of the underlying noise process. As a model for the evolution of the market we choose the *Ornstein-Uhlenbeck* (O-U) process (Uhlenbeck and Ornstein (1930)) for three reasons: (a) O-U noise is still a Gaussian random process with an arbitrary correlation time τ and it has the property that when $\tau = 0$ the process becomes Gaussian white noise, as in the original Black-Scholes option case. (b) The O-U process is, by virtue of Doob's theorem, the only Gaussian random process which is simultaneously Markovian and stationary (Doob (1942)). In this sense the O-U process is the simplest generalization of Gaussian white-noise. (c) As we will see later on, the variance of random processes driven by O-U noise seems to agree with the evolution of market variance, at least in some particular but relevant cases.

The Ornstein-Uhlenbeck process is not a newcomer in mathematical finance. For instance, it has already been proposed as a model for stochastic volatility(SV). Our case here is rather different since, contrary to SV models, we only have one source of noise[3]. We therefore suggest the O-U process as the driving noise for the underlying price dynamics when the volatility is still a deterministic quantity (Dumas *et al.* (1998)).

The auto correlation in the underlying driving noise is closely related to the predictability of asset returns, of which there seems to be ample evidence (Breen and Jagannathan (1989), Campbell and Hamao (1992)). Indeed, if for some particular stock the price variations are correlated during some time τ, then the price at time t_2 will be related to the price at a previous time t_1 as long as the time span $t_2 - t_1$ is not too long compared to the correlation time τ. Hence, *correlation* implies partial *predictability*. Other approaches to option pricing with predictable asset returns are based under the assumption the market is still driven by white noise and predictability is induced by the drift (Lo and Wang (1995)). Since the B-S formula is independent of the drift, these approaches apply B-S theory with a conveniently modified volatility. Our approach here is rather different because we assume the asset price variations driven by correlated noise –which implies some degree of predictability.

Summarizing, our purpose is to study option pricing and hedging in a more realistic framework than that of white noise process presented by Black and Scholes. Our model includes colored noise and the dependence of the volatility on time. Both

[3]See next Chapter for more details. Representative contributions on stochastic volatility are Hull and White (1987), Scott (1987), Stein and Stein (1991), Heston (1993), Ghysels *et al.* (1996), Heston and Nandi (2000).

are empirically observed in real markets (Bouchaud and Potters (2000)). Empirical characteristic time scales are at least of the order of minutes and can affect option prices particularly when the exercising date is near and speculative fluctuations are more important. Presumably, this effect is negligible when correlation times are shorter (much shorter than time to expiration). In any case, it is interesting to know how, and by how much, the option price and its properties are modified when correlations in the underlying noise are significant.

The shortest way of getting the call price, and hence quantifying the effect of correlations on prices is by martingale methods. Unfortunately, this procedure does not guarantee that we obtain the fairest price since arbitrage and hedging are not included in this approach[4]. It is therefore our main objective to generalize B-S theory not only to get a new call price but, more importantly, to obtain a hedging strategy that avoids risk and arbitrage opportunities.

From a technical point of view, we apply the B-S option pricing method after projecting the two-dimensional O-U process onto a one-dimensional diffusion process with time varying volatility. As we will show, this projection allows us to maintain the conditions of a perfect hedging and the absence of arbitrage. Moreover, the price obtained using this way completely agrees with the price obtained using two alternative and different methods. One of them is based on martingale theory, and the other one develops a new option pricing with a modified portfolio containing secondary options instead of the underlying stock.

The Chapter is divided into six sections and corresponds to the article Masoliver and Perelló (2001b). In Section 5.1 we present our two-dimensional stochastic model for the underlying asset. In Section 5.2 we find the O-U projection onto the stock price correlated process. Section 5.3 concentrates on the B-S option price derivation with the projected process, and Sections 5.4 and 5.5 show the consistency of this derivation by using two alternative methods for obtaining the option price. The Greeks and the new hedging are presented in Section 5.6, and technical details are left to the appendices.

5.1 The asset model

The standard assumption in option pricing theory is to assume that the underlying price $S(t)$ can be modelled as a one-dimensional diffusion process:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \tag{5.1}$$

where $W(t)$ is the Wiener process with the averages given by Eq. (3.4). In the original B-S theory, both drift μ and volatility σ are constants (see Section 3.2).

[4]See Section 4.5 for a more detailed discussion on this point.

Other models take $\mu = \mu(t, S)$ and $\sigma = \sigma(t, S)$ as functions of time and underlying price (Cox and Ross (1976), Bergman *et al.* (1987)). The parameter σ is assumed to be a random quantity in the SV models (see Chapter 6).

Notice that if the time evolution of the underlying price is governed by Eq. (5.1) then $S(t)$ is an uncorrelated random process in the sense that its zero-mean return rate defined by $Z(t) = d\ln S/dt - \mu$ is driven by white noise, *i.e.*, $E[Z(t_1)Z(t_2)] = \sigma^2\delta(t_1 - t_2)$ where $\delta(t)$ is the Dirac delta function[5]. Hence, the asset model immediately incorporates price return effects and meets the efficient market hypothesis.

As a first step, we assume that the underlying price is not driven by the Wiener process $W(t)$ but by O-U noise $V(t)$. In other words, we say that $S(t)$ obeys a *singular two-dimensional diffusion*

$$\frac{dS(t)}{S(t)} = \mu dt + V(t)dt \tag{5.2}$$

$$dV(t) = -\frac{V(t)}{\tau}\, dt + \frac{\sigma}{\tau}\, dW(t), \tag{5.3}$$

where $\tau \geq 0$ is the correlation time. More precisely, $V(t)$ is O-U noise in the stationary regime, which is a Gaussian colored noise with zero mean and correlation function:

$$E[V(t_1)V(t_2)] = \frac{\sigma^2}{2\tau}e^{-|t_1-t_2|/\tau}. \tag{5.4}$$

We call the process defined by Eqs. (5.2)–(5.3) singular diffusion because, contrary to SV models, the Wiener driving noise $W(t)$ only appears in one of the equations, and this results in a singular diffusion matrix (Gardiner (1985)). Observe that we now deal with auto correlated stock prices since the zero-mean return rate $Z(t)$ is colored noise, *i.e.*, $E[Z(t_1)Z(t_2)] = (\sigma^2/2\tau)\exp[-|t_1 - t_2|/\tau]$. Note that when $\tau = 0$ this correlation goes to $\sigma^2\delta(t_1 - t_2)$ and we thus recover the one-dimensional diffusion discussed above. Therefore the case of positive τ is a measure of the inefficiencies of the market.

There is an alternative, and sometimes more convenient, way of writing the above equations using the asset return $R(t)$ defined by

$$R(t) = \ln[S(t)/S_0],$$

where $S_0 = S(t_0)$ and t_0 is the time at which we start observing the process (5.2)–(5.3). Without loss of generality this time can be set equal to zero (see Appendix A). Instead of Eqs. (5.2)–(5.3), we may have

$$\frac{dR(t)}{dt} = \mu + V(t) \tag{5.5}$$

[5]We recall that $\delta(x)$ is a generalized function with the properties: $\delta(x) = 0$ for $x \neq 0$ and $\int_{-\infty}^{\infty}\delta(x)dx = 1$ (*cf.* Eq. (3.2)).

$$\frac{dV(t)}{dt} = \frac{1}{\tau}\left[-V(t) + \sigma\xi(t)\right], \tag{5.6}$$

where $\xi(t) = dW(t)/dt$ is Gaussian white noise defined as the derivative of the Wiener process (in Section 3.1 we have shown several properties of this process). The combination of relations in Eqs. (5.5) and (5.6) leads to a second-order stochastic differential equation for $R(t)$

$$\tau\frac{d^2R(t)}{dt^2} + \frac{dR(t)}{dt} = \mu + \sigma\xi(t). \tag{5.7}$$

From this equation, we clearly see that when $\tau = 0$ we recover the one-dimensional diffusion case (5.1)[6] . We also observe that the O-U process $V(t)$ is the random part of the *return velocity*, dR/dt, and we will often refer to $V(t)$ as the "velocity" of the return process $R(t)$.

In Appendix A, we give explicit expressions for $V(t)$ and for the return $R(t)$. We prove there that $R(t)$ is a non-stationary process with the conditional mean value

$$m(t, V_0) \equiv E\left[R(t)|V_0\right] = \mu t + \tau\left(1 - e^{-t/\tau}\right)V_0, \tag{5.8}$$

where $V(0) \equiv V_0$ is the initial velocity. The conditional return variance,

$$K_{11}(t) \equiv E\left[(R(t) - m(t, V_0))^2|V_0\right],$$

is given by

$$K_{11}(t) = \sigma^2\left[t - 2\tau\left(1 - e^{-t/\tau}\right) + \frac{\tau}{2}\left(1 - e^{-2t/\tau}\right)\right]. \tag{5.9}$$

We also give in Appendix A explicit expressions for the joint probability density function (pdf) $p(R, V, t)$, the marginal pdf's $p(R, t)$ and $p(V, t)$ of the second-order process $R(t)$, and the marginal pdf $p(S, t|S_0, t_0)$ of the underlying price $S(t)$. We also show that the velocity $V(t)$ is, in the stationary regime, distributed according to the normal density:

$$p_{st}(V) = \frac{1}{\sqrt{\pi\sigma^2/\tau}}\, e^{-\tau V^2/\sigma^2}. \tag{5.10}$$

Suppose now that the initial velocity V_0 is random with mean value $E[V_0]$ and variance $\mathrm{Var}[V_0]$. Thus, the return unconditional mean and variance read

$$\begin{aligned} E\left[R(t)\right] &= \mu t + \tau\left(1 - e^{-t/\tau}\right)E\left[V_0\right], \\ \mathrm{Var}[R(t)] &= K_{11}(t) + \tau\left(1 - e^{-t/\tau}\right)\mathrm{Var}[V_0]. \end{aligned}$$

[6]In the opposite case when $\tau = \infty$, Eq. (5.3) shows that $dV(t) = 0$. Thus $V(t)$ is a constant, which we may equal to zero, and from Eq. (5.2) we have $S(t) = S_0e^{\mu t}$. Therefore, the underlying price evolves as a riskless security. Later on we will recover this deterministic case (see, for instance, Eq. (5.35)).

If, in addition, we assume that the initial velocity V_0 is in the stationary regime then $E[V_0] = 0$ and $\mathrm{Var}[V_0] = \sigma^2/2\tau$. In this case, the return unconditional mean value is

$$m(t) \equiv E[R(t)] = \mu t,$$

and the return unconditional variance

$$\kappa(t) \equiv \mathrm{Var}[R(t)]$$

reads (*cf.* Eq. (5.9))

$$\kappa(t) = \sigma^2 \left[t - \tau\left(1 - e^{-t/\tau}\right)\right]. \tag{5.11}$$

A consequence of Eq. (5.11) is that, when $t \ll \tau$, the variance behaves as

$$\kappa(t) \sim (\sigma^2/2\tau)t^2, \qquad (t \ll \tau). \tag{5.12}$$

Equation (5.11) also shows a crossover to ordinary diffusion (B-S case) when $t \gg \tau$:

$$\kappa(t) \sim \sigma^2 t, \qquad (t \gg \tau). \tag{5.13}$$

In Fig. 5.1, we plot $\kappa(t)$ along with the empirical variance from data of the S&P 500 cash index during the period January 1988-December 1996[7]. The dashed line represents results obtained by assuming normal-diffusion $\kappa(t) \propto t$. Observe that the empirical variance is very well fitted by our theoretical variance $\kappa(t)$ for a correlation time $\tau = 2$ minutes. Furthermore, the result of this correlation affects the empirical volatility for around 100 minutes. These times are probably too small to affect call price to any quantifiable extent. However, the S&P 500 is one of the most liquid, and therefore most efficient, markets. Consequently, the effect of correlations in any other less efficient market might significantly influence option prices and hedging strategies, and this is the main motivation for this work.

5.2 The projected process

One may argue that the O-U process (5.2)–(5.3) is an inadequate asset model since the share price $S(t)$ given by Eq. (5.2) is a continuous random process with bounded variations. As Harrison *et al.* (1984) showed, continuous processes with bounded variations allow arbitrage opportunities and this is an undesirable feature for obtaining a fair price. Thus, for instance, arbitrage would be possible within a portfolio containing bonds and stock whose strategy at time t is buying (or selling) stock shares when $\mu + V(t)$ is greater (or lower) than the risk-free bond rate (Harrison *et al.* (1984)).

[7] Tick by tick data on S&P 500 cash index has been provided by The Futures Industry Institute (Washington, DC).

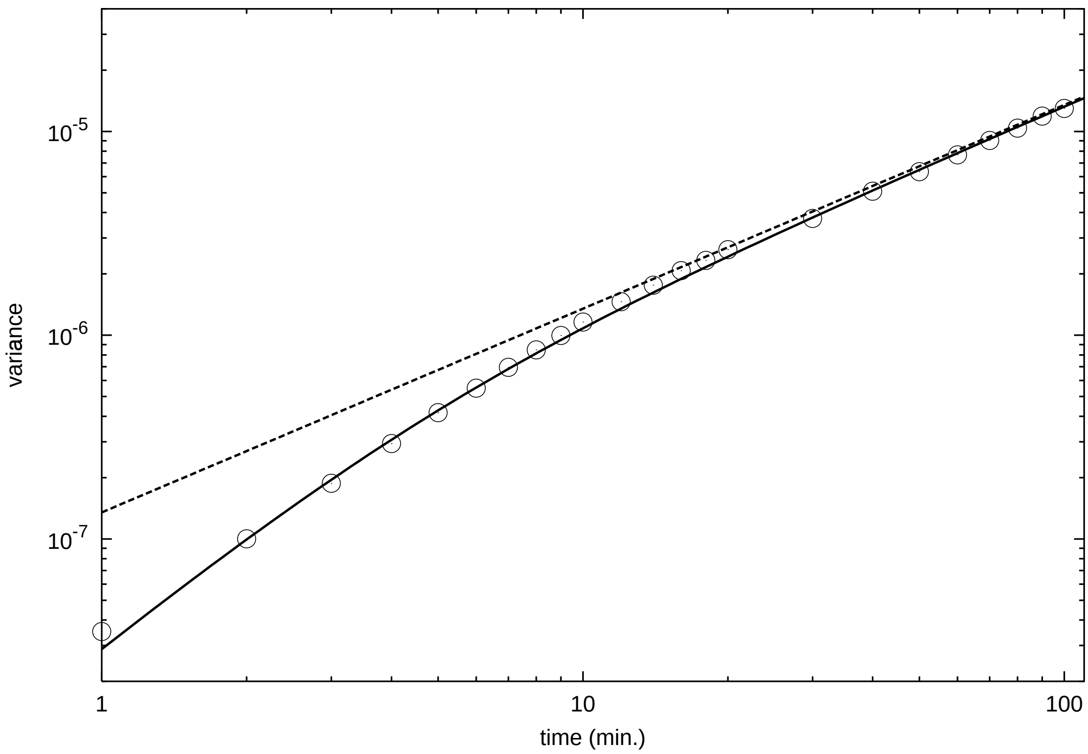


Figure 5.1: The underlying asset variance

The variance of the underlying asset as a function of time (in logarithmic scale). Circles correspond to empirical variance of S&P 500 cash index from 1988 to 1996. Solid line represents the theoretical variance, Eq. (7) with $\tau = 2$ minutes. The dashed line is the B-S variance $\sigma^2 t$. In both cases $\sigma = 3.69 \times 10^{-4} \mathrm{min}^{-1/2}$ which approximately corresponds to an annual volatility $\sigma = 11\%$.

In our case, however, the problem is that in the practice *the return velocity $V(t)$ is non tradable and its evolution is ignored.* In other words, in real markets the observed asset dynamics does not show any trace of the velocity variable[8]. This feature allows us to perform a projection of the two-dimensional diffusion process $[S(t), V(t)]$ onto a *one-dimensional equivalent process* $\bar{S}(t)$ independent of the velocity V. We will show latter in this section that the projected process $\bar{S}(t)$, which is equal to the actual price $S(t)$ in mean square sense, obeys the following one-dimensional SDE

$$\frac{d\bar{S}(t)}{\bar{S}(t)} = [\mu + \dot{\kappa}(T-t)/2]dt + \sqrt{\dot{\kappa}(T-t)}dW(t), \tag{5.14}$$

where $\kappa(t)$ is given by Eq. (5.11), and the dot denotes time derivative. Therefore, the price given by Eq. (5.14) is driven by a noise of unbounded variation, the Wiener process, and the Harrison *et al.* (1984) results do not apply. In consequence, the O-U *projected process* is still a suitable starting point for option pricing since it does not permit arbitrage.

5.2.1 Derivation of the one-dimensional SDE

Note that the dynamics of the return $R(t) = \ln[S(t)/S_0]$ is given by the second-order SDE (5.7) which includes the stochastic evolution of the velocity $V(t)$. Let us now obtain a first-order SDE describing the price dynamics when velocity $V(t)$ has been eliminated.

The starting point of our derivation is the marginal conditional density $p(R, t|R_0, t_0; V_0)$. This density is given by Eq. (A.14) of Appendix A and when $t_0 \neq 0$ it reads

$$p(R, t|R_0, t_0; V_0) = \frac{1}{\sqrt{2\pi K_{11}(t-t_0)}} \exp\left\{-\frac{[R - R_0 - m(t-t_0, V_0)]^2}{2K_{11}(t-t_0)}\right\}, \tag{5.15}$$

where $m(t, V_0)$ and $K_{11}(t)$ are given by Eqs. (5.8) and (5.9). Note that $p(R, t|R_0, t_0; V_0)$ is the solution of the following partial differential equation

$$\frac{\partial p}{\partial t_0} = -\left[\mu + V_0 e^{-(t-t_0)/\tau}\right]\frac{\partial p}{\partial R_0} - \frac{\sigma^2}{2}\left[1 - e^{-(t-t_0)/\tau}\right]^2 \frac{\partial^2 p}{\partial R_0^2}, \tag{5.16}$$

with the final condition $p(R, t|R_0, t; V_0) = \delta(R - R_0)$. Observe that the Eq. (5.16) is a backward Fokker-Planck equation whose drift, $\mu + V_0 \exp[-(t-t_0)/\tau]$, and

[8]Indeed, knowing $V(t)$ would imply knowing the value of the return $R(t)$ at two different times, since

$$V(t) = \lim_{\epsilon \to 0^+} \frac{R(t) - R(t-\epsilon)}{\epsilon} - \mu.$$

Obviously, this operation is not performed by traders who only manage portfolios at time t based on prices at t and not at any earlier time.

diffusion coefficient, $\frac{1}{2}\sigma^2\left\{1-\exp[-(t-t_0)/\tau]\right\}^2$, are both functions of $t-t_0$. As is well-known, there exists a direct relation between the Fokker-Planck equation and the SDE governing the process (Gardiner (1985)). In our case, the corresponding SDE is

$$dR(t_0)=\left[\mu+V_0e^{-(t-t_0)/\tau}\right]dt_0+\sigma\left[1-e^{-(t-t_0)/\tau}\right]dW(t_0), \tag{5.17}$$

and its formal solution is

$$R(t)=R(t_0)+\mu(t-t_0) \quad + \quad V_0\tau\left[1-e^{-(t-t_0)/\tau}\right]+\sigma\int_{t_0}^{t}\left[1-e^{-(t-t_1)/\tau}\right]dW(t_1). \tag{5.18}$$

5.2.2 Equality of processes in mean square sense

To avoid confusion, let $\bar{R}(t)$ be the solution of the first-order SDE (5.17), *i.e.*, $\bar{R}(t)$ is the projected process given by Eq. (5.18). And let $R(t)$ be the solution of the second-order SDE (5.7) where the dynamics of the velocity is still taken into account. Thus, $R(t)$ is explicitly given by Eq. (A.1) of Appendix A.

We will now prove that $\bar{R}(t)$ and $R(t)$ are equal in mean square sense (*cf.* Section 3.1 and specially Eq. (3.6)). That is:

$$E\left[(R(t)-\bar{R}(t))^2\right]=0, \qquad \text{for any time } t. \tag{5.19}$$

In effect, from Eq. (5.18) and assuming, without loss of generality, that $t_0=0$ and $\bar{R}(t_0)=0$ we have

$$\bar{R}(t)=\mu t+V_0\tau(1-e^{-t/\tau})+\int_0^t\left[1-e^{-(t-t_1)/\tau}\right]\xi(t_1)dt_1, \tag{5.20}$$

where $\xi(t_1)=dW(t_1)/dt_1$ is the Gaussian white noise. On the other hand, from Eq. (A.1) we write

$$R(t)=\mu t+V_0\tau(1-e^{-t/\tau})+\frac{\sigma}{\tau}\int_0^t dt'\int_0^{t'}e^{-(t'-t'')/\tau}\xi(t'')dt''. \tag{5.21}$$

Therefore,

$$\begin{aligned}E\left[(R(t)-\bar{R}(t))^2\right]&=2K_{11}(t)\\&-\frac{2\sigma^2}{\tau}\int_0^t dt'\int_0^{t'}dt''e^{-(t'-t'')/\tau}\int_0^t dt_1\left[1-e^{-(t-t_1)/\tau}\right]E\left[\xi(t_1)\xi(t'')\right],\end{aligned}$$

where $K_{11}(t)$ is given by Eq. (5.9). Taking into account that

$$E\left[\xi(t_1)\xi(t'')\right]=\delta(t_1-t''),$$

we have

$$E\left[(R(t)-\bar{R}(t))^2\right] = 2K_{11}(t) - \frac{2\sigma^2}{\tau}\int_0^t dt' \int_0^{t'} dt'' e^{-(t'-t'')/\tau}\left[1-e^{-(t-t'')/\tau}\right].$$

However, (see Eq. (5.9))

$$\frac{\sigma^2}{\tau}\int_0^t dt' \int_0^{t'} dt'' e^{-(t'-t'')/\tau}\left[1-e^{-(t-t'')/\tau}\right] = K_{11}(t).$$

Hence,

$$E\left[(R(t)-\bar{R}(t))^2\right] = 0,$$

and $R(t)$ is equal to $\bar{R}(t)$ in mean square sense.

5.2.3 The projected process when the initial velocity is in the stationary regime

As we have mentioned, we are mainly interested in representing the asset dynamics when the initial velocity V_0 is random and distributed according to the stationary pdf (5.10). We have shown in Section 5.1 that this basically implies the replacement of $K_{11}(t)$ by $\kappa(t)$. In such a case, the SDE for $R(t)$ reads[9]

$$dR(t) = \mu dt + \sqrt{\dot{\kappa}(T-t)}dW(t),$$

where $\kappa(t)$ is given by Eq. (5.11) and the dot denotes time derivative, that is

$$\dot{\kappa}(t) = \sigma^2\left(1-e^{-t/\tau}\right). \tag{5.22}$$

We need the Itô lemma given in Appendix B for deriving the SDE for the stock S. Thus, according to Eq. (B.6), the *effective dynamics* for $S = S_0 e^R$ is

$$\frac{dS(t)}{S(t)} = [\mu + \dot{\kappa}(T-t)/2]dt + \sqrt{\dot{\kappa}(T-t)}dW(t). \tag{5.23}$$

In this way, we have projected the two-dimensional O-U process (S,V) onto a one-dimensional price process which is a Wiener process with time varying drift and volatility. We also note that we need to specify the final condition of the process because the volatility $\sqrt{\dot{\kappa}}$ is a function of the time to maturity $T-t$, and this implies that the projected asset model depends on each particular contract.

[9]Since $R(t)$ and $\bar{R}(t)$ are equal in mean square sense we will drop the bar on $\bar{R}$ as long as there is no confusion. Thus, we will use R for the projected process as well.

5.3 The option price on the projected process

In this section we will present a generalization of the Black-Scholes theory assuming that underlying price is driven by the O-U process. We therefore eliminate the efficient market hypothesis but retain the other two requirements of the original B-S theory: the absence of arbitrage and the existence of a riskless strategy.

We invoke the standard theoretical restrictions –continuos trading without transaction costs and dividends– and apply the original B-S method taking into account that the underlying asset is not driven by white noise but by colored noise modelled as an O-U process.

The starting point of B-S option pricing is a portfolio which contains certain amounts of shares, calls and bonds. In this context, B-S hedging is only able to remove the call risk that comes from stock fluctuations. Therefore, we need to start from the effective one-dimensional market dynamics given by Eq. (5.23) since otherwise we would not be able to remove risk fluctuations arising from $dW(t)$. These fluctuations are only explicitly given in the projected SDE for the stock (see Section 5.4.1 for a deeper discussion on this point).

5.3.1 Black-Scholes option pricing with the equivalent one-dimensional SDE

As we have proved in Section 5.2, there exists an effective one-dimensional diffusion which describes the O-U process (5.2)–(5.3). Assuming that the effective one-dimensional price dynamics is given by Eq. (5.23), it is quite straightforward to derive the European call option price within the original B-S method.

Following the B-S theory presented in Section 3.3, we define a portfolio compounded by a certain amount Δ of shares at price S, a quantity of bonds Φ, and a number Υ of calls with price C, maturity time T and strike price K. We assume that short-selling is allowed and thus the value P of the portfolio is written

$$P = \Upsilon C - \Delta S - \Phi B, \tag{5.24}$$

where the bond price B evolves according to the risk-free interest rate ratio r. That is

$$dB = rBdt. \tag{5.25}$$

Recall that the portfolio is required to obey the net-zero investment hypothesis, which means $P = 0$ for any time t (*cf.* Section 3.3). Hence,

$$C = \delta S + \phi B, \tag{5.26}$$

where $\delta = \Delta/\Upsilon$ and $\phi = \Phi/\Upsilon$ are, respectively, the number of shares per call and the number of bonds per call. Due to the non anticipating character of δ and ϕ we have[10]

$$dC = \delta dS + \phi dB. \tag{5.27}$$

On the other hand, assuming that the market dynamics is described by Eq. (5.23), the differential of the call also reads[11]

$$dC(S,t) = C_t dt + C_S dS + \frac{1}{2}\dot{\kappa}(T-t)S^2 C_{SS} dt, \tag{5.28}$$

where we have used the Itô lemma as expressed by Eq. (B.7) of the Appendix B. From Eqs. (5.27)–(5.28) and (5.26), we get

$$\left[C_t + \frac{1}{2}\dot{\kappa}(T-t)S^2 C_{SS} + r\delta S - rC\right] dt = [\delta - C_S]dS.$$

Now the B-S delta hedging, $\delta = C_S$, removes any random uncertainty in the option price. The partial differential equation for $C(S,t)$ then reads

$$C_t = rC - rSC_S - \frac{1}{2}\dot{\kappa}(T-t)S^2 C_{SS}. \tag{5.29}$$

We note that the delta hedging is able to remove risk because we have projected the two-dimensional SDE (5.2)–(5.3) onto the one-dimensional process. In this way, we directly relate the differential of the stock $dS(t)$ to the random fluctuations of the Wiener process $dW(t)$ (see Eq. (5.23)). Without this projection, the B-S hedging is useless and the random fluctuations persist in the B-S portfolio. We will further discuss this situation in Section 5.4.1.

5.3.2 The price of the European call

For the European call, Eq. (5.29) has to be solved with the following "final condition" at maturity time T

$$C(S,T) = (S(T) - K)^+, \tag{5.30}$$

where $S(T)$ is the underlying price at maturity and K is the strike price. The solution to Eq. (5.29) subject to Eq. (5.30) is a type of solution perfectly known in the literature (*cf.* Appendix A in Chapter 3). Thus, our final price is

$$C_{OU}(S,t) = S\ N(d_1^{OU}) - Ke^{-r(T-t)}\ N(d_2^{OU}), \tag{5.31}$$

where $N(d)$ is the probability integral is given by Eq. (2.34) and

$$d_1^{OU} = \frac{\ln(S/K) + r(T-t) + \kappa(T-t)/2}{\sqrt{\kappa(T-t)}}, \tag{5.32}$$

[10]In Section 3.3.1 we have studied the non anticipating character and the self-financing requirements for these strategy functions (*cf.* Eq. (3.35).

[11]Here we use a different notation to that of the previous chapters. Thus, subscripts indicate partial differentiation, *i.e.*, $C_t = \partial C/\partial t$ and so on.

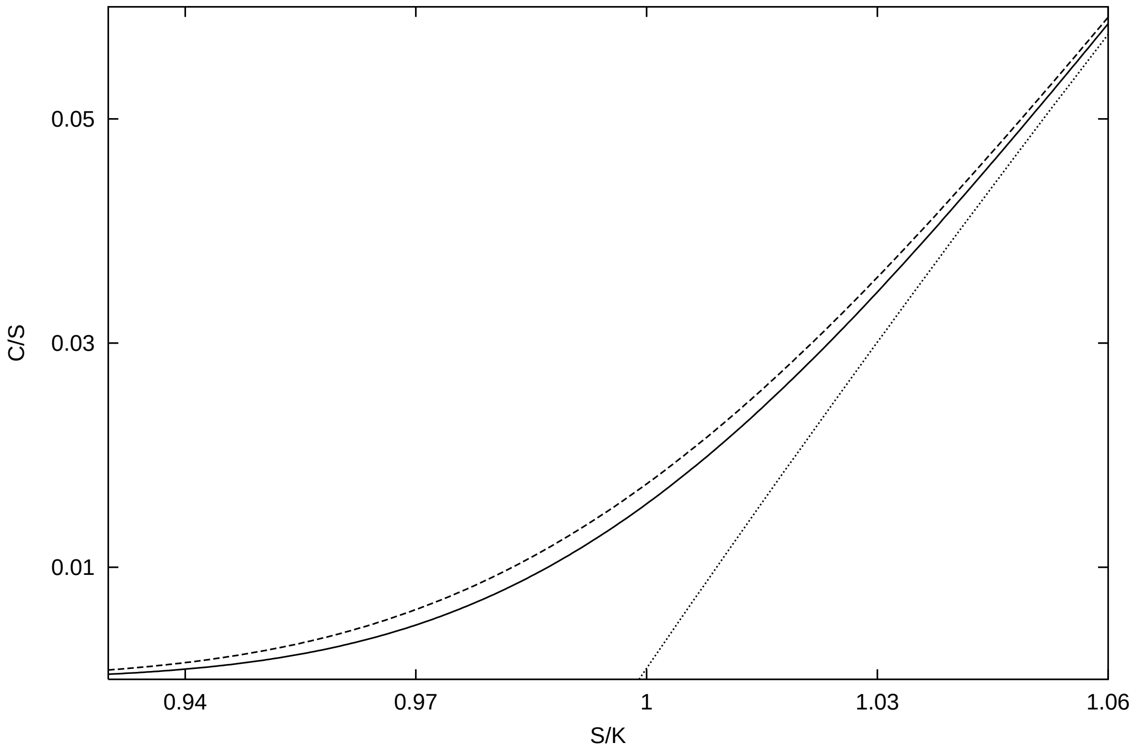


Figure 5.2: The O-U call price in terms of the moneyness

Relative call price C/S as a function of S/K for a given time to expiration $T - t = 5$ days. The solid line represents the O-U call price with $\tau = 1$ day and the dashed line is the B-S price. The dotted line is the deterministic price. In this figure the annual risk-free interest rate $r = 5\%$, and the annual volatility $\sigma = 30\%$.

$$d_2^{OU} = d_1^{OU} - \sqrt{\kappa(T-t)}, \tag{5.33}$$

with $\kappa(t)$ given by Eq. (5.11).

Equation (5.31) constitutes the key result of this Chapter. Note that, when $\tau = 0$, the variance becomes $\kappa(t) = \sigma^2 t$ and the price in Eq. (5.31) reduces to the Black-Scholes price:

$$C_{BS}(S,t) = S\ N(d_1^{BS}) - Ke^{-r(T-t)}\ N(d_2^{BS}), \tag{5.34}$$

where $d_{1,2}^{BS}$ have the form of Eqs. (5.32)–(5.33) with $\kappa(T-t)$ replaced by $\sigma^2(T-t)$. Therefore, the O-U price in Eq. (5.31) has the same functional form as B-S price in Eq. (5.34) when $\sigma^2 t$ is replaced by $\kappa(t)$.

In the opposite case, $\tau = \infty$, where there is no random noise but a deterministic and constant driving force (in our case it is zero), Eq. (5.31) reduces to the deterministic price

$$C_d(S,t) = \left(S - Ke^{-r(T-t)}\right)^+. \tag{5.35}$$

We will now prove that C_{OU} is an intermediate price between B-S price and the deterministic price (see Fig. 5.2)

$$C_d(S,t) \le C_{OU}(S,t) \le C_{BS}(S,t), \tag{5.36}$$

for all S and $0 \le t \le T$. In order to prove this it suffices to show that C_{OU} is a monotone decreasing function of the correlation time τ, since in such a case

$$C_{OU}(\tau = \infty) \le C_{OU}(\tau) \le C_{OU}(\tau = 0).$$

However, $C_{OU}(\tau = \infty) = C_d$ and $C_{OU}(\tau = 0) = C_{BS}$, which leads to Eq. (5.36). Let us thus show that C_{OU} is a decreasing function of τ for $0 \le t \le T$ and all S. Define a function χ as the derivative

$$\chi = \frac{\partial C_{OU}}{\partial \tau}. \tag{5.37}$$

Since the τ dependence in C_{OU} is a consequence of the variance $\kappa(t,\tau)$, we have

$$\chi = \frac{\sigma}{2\kappa(T-t,\tau)} \frac{\partial \kappa(T-t,\tau)}{\partial \tau} \upsilon_{OU},$$

where $\upsilon_{OU} = \partial C_{OU}/\partial \sigma$ (see Section 5.6). But

$$\frac{\partial \kappa(T-t,\tau)}{\partial \tau} = -\sigma^2 \left[1 - (1 + (T-t)/\tau)e^{-(T-t)/\tau}\right] \le 0,$$

for $0 \le t \le T$ which is seen to be non positive. From Eq. (5.67) below we see that $\upsilon_{OU} \ge 0$ for all S and $0 \le t \le T$. Hence, $\chi \le 0$ which proves Eq. (5.36). In Fig. 5.3 we plot the option price C as a function of the correlation time τ and for

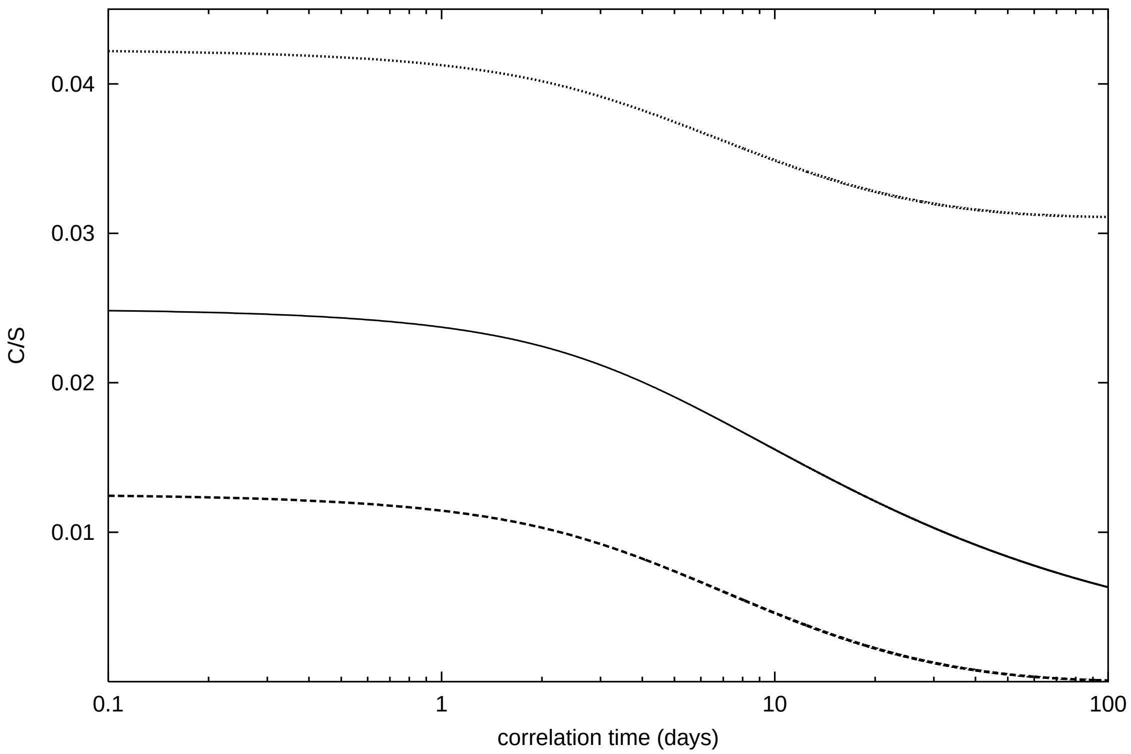


Figure 5.3: The O-U call price as a function of the correlation time

Relative call price C/S as a function of τ for a given time to expiration $T-t=10$ days. The solid line represents the call price with $S/K=1$ (ATM case). The dotted line is the call price when $S/K=1.03$ (ITM case). The dashed line represents an OTM case when $S/K=0.97$. We clearly see that C is a monotone decreasing function of τ having its maximum value when $\tau=0$ (B-S case) and its minimum when $\tau\to\infty$ (deterministic price). The annual risk-free interest rate and the annual volatility are as in Fig. 5.2.

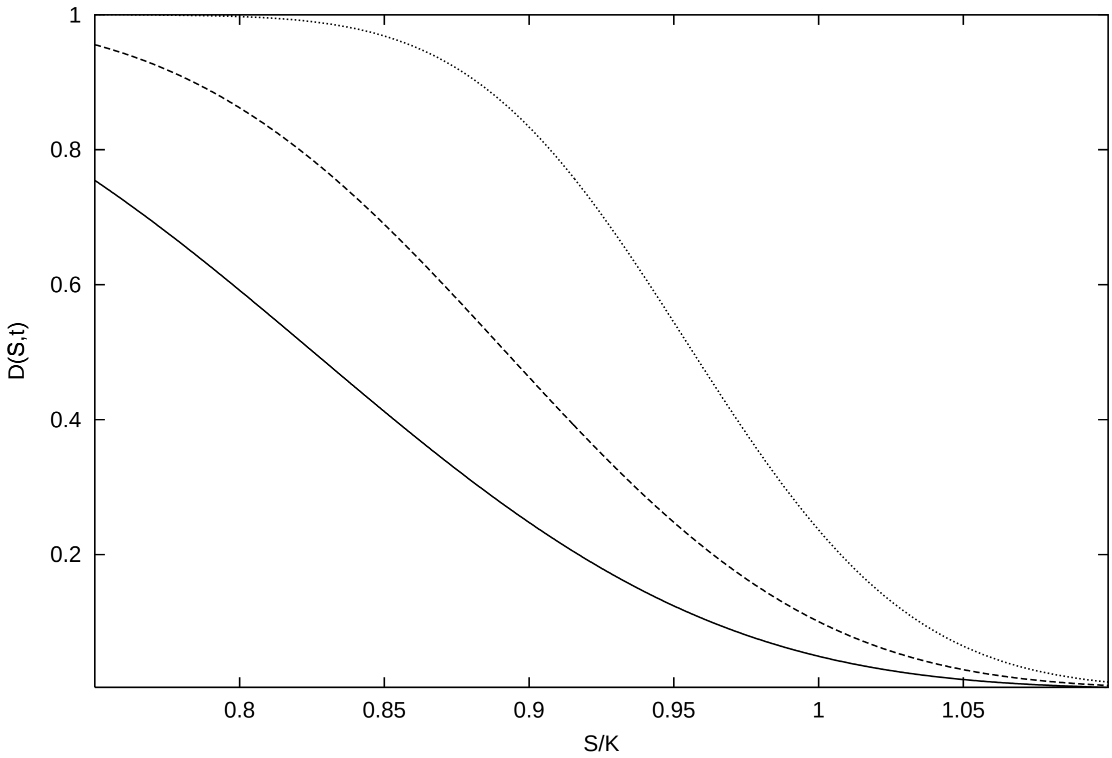


Figure 5.4: The relative call price difference in terms of the moneyness

The function $D(S, t)$ is plotted as a function of S/K for $T - t = 10$ days and $\tau = 1$ day (solid line), $\tau = 2$ days (dashed line) and $\tau = 5$ days (dotted line). Other parameters used to generate the figure are $r = 5\%$ per annum and $\sigma = 30\%$ per annum.

three different values of the moneyness S/K. This figure clearly shows that C is a monotone decreasing function of τ.

Therefore, *the assumption of uncorrelated underlying assets (B-S case) overprices any call option.* This confirms the intuition understanding that correlation implies more predictability and therefore less risk and, finally, a lower price for the option. In fact, we can easily quantify this overprice by evaluating the relative difference

$$D = (C_{BS} - C_{OU})/C_{BS}.$$

Figure 5.4 shows the ratio $D(S,t)$, for a fixed time to expiration, plotted as a function of the moneyness, S/K, and for different values of correlation time τ. We see there that the ratio D is very sensitive to whether the call is in the money (ITM), out of the money (OTM) or at the money (ATM). The biggest difference between prices occurs in the case of OTM options. This is true because when $S/K < 1$, both C_{BS} and C_{OU} are small but $C_{BS} \gg C_{OU}$ (see Fig. 5.2). Depending on the value of correlation time τ this implies that D is approximately equal to 1.

Another interesting point is the behavior of D as a function of the expiration time $T-t$. In this case, D behaves quite differently depending on whether the call is in, out, or at the money. This behavior is evident in Figs. 5.5 and 5.6. Figure 5.5 shows $D(S,t)$ as a function of expiration time $T-t$ for an OTM option ($S/K = 0.95$) and the ATM option ($S/K = 1.00$) and for two different values (1 and 5 days) of the correlation time. Note that B-S notably overprices the option, particularly in the OTM case. In Fig. 5.6 we show plots of $D(S,t)$ as a function of t for an ITM option ($S/K = 1.05$). This exhibits completely different behavior since the B-S overprice is considerably less (no more than 7%). Moreover, contrary to the ATM and OTM cases, the relative difference $D(S,t)$ is a non monotone function of $T-t$, having a maximum value around one or two weeks before maturity. Although perhaps the most striking and interesting feature is *the persistence of the B-S overprice far from maturity* regardless the value of the correlation time. This is clearly shown in Table 5.1 where we quantify the ratio D in percentages for different values of moneyness, time to expiration and correlation time.

5.4 An alternative derivation of the call price

In this section and the next, we present two different and alternative derivations of the final call price C_{OU}. The first of these derivations is based on an extension of the B-S theory but now starting from the two-dimensional diffusion (5.2)–(5.3) and with a different portfolio than the usual one. A second derivation, briefly outlined in the next section, uses the equivalent martingale measure method. Both derivations arrive at the price formula (5.31), thus showing the consistency of the pricing methods.

We will first apply the original B-S method starting from the two-dimensional O-U process (5.2)–(5.3) instead of the equivalent process (5.23). Unfortunately, this

Table 5.1: Relative call price differences in percentages

We present the values of $D \times 100$, where $D = (C_{BS} - C_{OU})/C_{BS}$. $T-t$ is the expiration time in days. Correlation times τ are 1, 2, and 5 days. The rest of columns are divided in three blocks corresponding to a different values of the moneyness S/K. From left to right blocks represent the OTM, ATM, and ITM cases. Notice the importance and the persistence far from maturity of the relative differences in price (r and σ as in Fig. 5.2).

$T-t$	$S/K = 0.95$			$S/K = 1.00$			$S/K = 1.05$		
	τ =1	2	5	τ =1	2	5	τ =1	2	5
1	99.9	100	100	39.3	53.8	69.3	0.1	0.1	0.1
2	87.4	98.7	100	24.6	39.3	58.0	0.6	0.7	0.7
3	62.9	88.3	99.5	17.3	30.5	50.1	1.2	1.7	2.0
4	45.0	73.1	96.5	13.1	24.6	44.1	1.5	2.5	3.4
5	33.5	59.4	90.4	10.5	20.4	39.2	1.6	2.9	4.6
6	26.1	48.6	82.8	8.7	17.3	35.3	1.7	3.1	5.5
7	21.0	40.4	75.1	7.4	14.9	31.9	1.7	3.2	6.1
8	17.4	34.1	67.9	6.4	13.1	29.1	1.6	3.2	6.5
9	14.7	29.2	61.3	5.7	11.6	26.7	1.6	3.2	6.7
10	12.7	25.4	55.6	5.1	10.4	24.5	1.5	3.1	6.8
20	5.0	10.0	25.2	2.5	5.1	13.1	1.1	2.3	5.7
30	2.9	5.9	15.1	1.7	3.4	8.6	0.9	1.8	4.5
40	2.0	4.1	10.5	1.2	2.5	6.4	0.7	1.4	3.7
50	1.6	3.1	7.9	1.0	2.0	5.1	0.6	1.2	3.1
100	0.7	1.4	3.4	0.5	1.0	2.5	0.4	0.7	1.8
150	0.4	0.8	2.2	0.3	0.7	1.6	0.2	0.5	1.3
200	0.3	0.6	1.5	0.2	0.5	1.2	0.2	0.4	1.0
250	0.2	0.5	1.2	0.2	0.4	1.0	0.2	0.3	0.8

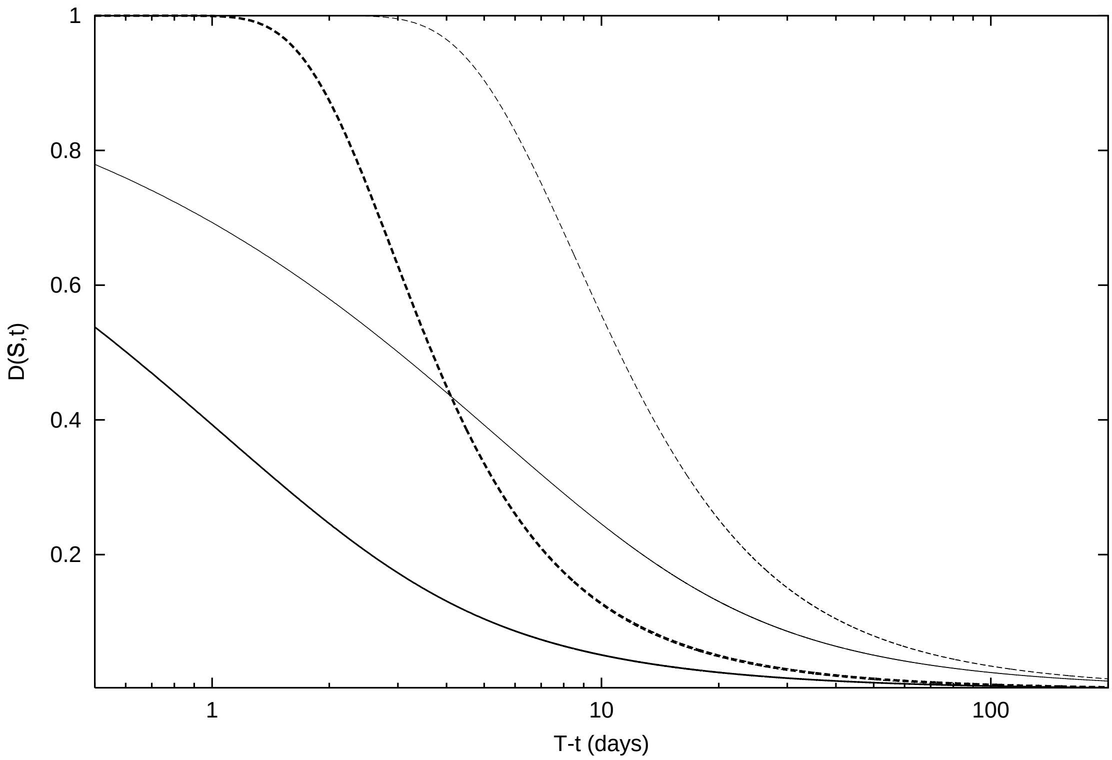


Figure 5.5: The relative call price difference for OTM and ATM options

The function $D(S,t)$ is plotted as a function of $T-t$ (in logarithmic scale) for fixed values of moneyness. The solid lines represent ATM options, the thick line corresponds to $\tau = 1$ day and the thin line corresponds to $\tau = 5$ days. The dashed lines represent an OTM option with $S/K = 0.95$, the thick line corresponds to $\tau = 1$ day and the thin line corresponds to $\tau = 5$ days (r and σ as in Fig. 5.2).

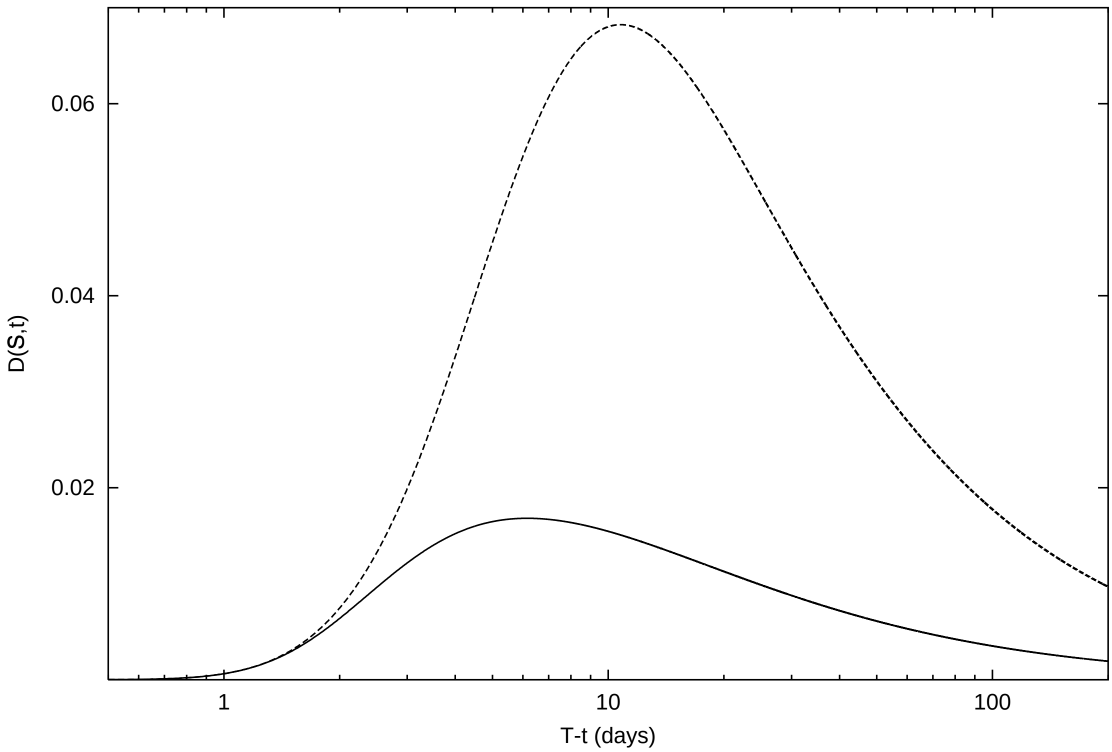


Figure 5.6: The relative call price difference for an ITM option

$D(S,t)$ is plotted as a function of the expiration time $T-t$ (in logarithmic scale) for an ITM option with $S/K = 1.05$. The solid line corresponds to $\tau = 1$ day and the dashed line to $\tau = 5$ days (r and σ as in Fig. 5.2).

procedure yields a trivial expression for the price of the option (see below) and is therefore useless. To avoid this difficulty we will define a different portfolio which is the first step towards the generalization of both B-S equation and formula.

5.4.1 The Black-Scholes method for the two-dimensional O-U process

We assume that market prices are driven by an O-U process as shown in Eqs. (5.2)–(5.3) and that the portfolio is given by Eq. (5.24). That is, $C = \delta S + \phi B$ and

$$dC = \delta dS + \phi dB.$$

Let us now apply the original B-S method starting from the two-dimensional O-U process (5.2)–(5.3) instead of the equivalent process (5.23). Using the Itô lemma for a singular two-dimensional diffusion (see Appendix B),

$$dC(S,V,t) = C_S dS + C_V dV + C_t dt + \frac{\sigma^2}{2\tau^2} C_{VV} dt, \tag{5.38}$$

and taking Eqs. (5.25) and (5.27) into account, we write

$$\left[C_t + \frac{\sigma^2}{2\tau} C_{VV} - r(C - S\delta)\right] dt + (C_S - \delta) dS + C_V dV = 0.$$

Now the assumption of delta hedging $\delta = C_S$, turns this equation into

$$\left[C_t - r(C - SC_S) + \frac{\sigma^2}{2\tau} C_{VV}\right] dt + C_V dV = 0. \tag{5.39}$$

Equation (5.39) is still random due to the term with dV representing velocity fluctuations (see Eq. (5.3)). In consequence, Black-Scholes delta hedging is incomplete since it is not able to remove risk. In this situation, the only way to derive a risk-free partial differential equation for the call price is to assume that the call is independent of velocity. Then, $C_V = 0$ and Eq. (5.39) yields

$$C_t + rSC_S - rC = 0. \tag{5.40}$$

According to the final condition for the European call, $C(S,T) = (S(T) - K)^+$, the call price is $C(S,t) = (S - Ke^{-r(T-t)})^+$. Note that this is a useless expression because it gives a price for the option as if the underlying asset would have evolved deterministically like the risk-free bond without pricing the random evolution of the stock. In fact, there is no hint of randomness, measured by the volatility σ, in Eq. (5.40).

The main reason for the failure of B-S theory is the inappropriateness of B-S hedging for two-dimensional processes such as O-U price process (5.2)–(5.3)[12]. Indeed, delta hedging presumably diversifies away the risk associated with the differential of asset price $dS(t)$ given by Eq. (5.2). Nevertheless, what we have to hedge is the risk associated with $dV(t)$ given by Eq. (5.3), which contains the only source of randomness: the differential of the Wiener process $dW(t)$. All of this clearly shows the uselessness of the B-S delta hedging for the two-dimensional O-U process. Note that we must relate in a direct way the differential $dS(t)$ with the random differential $dW(t)$, otherwise we will not be able to remove risk. This is indeed the case of the projected process (5.23) which leads to the European call price, Eq. (5.31). However, if we do not want to project the process and maintain the two-dimensional formulation (5.2)–(5.3) we have to evaluate the option price from a different portfolio. We will do it next by defining a new portfolio which will allow us to preserve the complete market hypothesis and remove the random component $dW(t)$.

5.4.2 The option pricing method with a modified portfolio

We present a new portfolio in a complete but not efficient market. The market is still assumed to be complete, in other words, there exists a portfolio with assets to eliminate financial risk. However, we relax the efficient market hypothesis by including the correlated O-U process as noise for the underlying price dynamics.

Now, our portfolio is compounded by a number of calls Υ with maturity T and strike K, a quantity of bonds Φ, and another number of "secondary calls" Υ', on the same asset, but with a different strike K' and, eventually, different payoff or maturity time. Note that in the new portfolio there are no shares of the underlying asset. Thus, instead of Eq. (5.24), we have

$$\Pi = \Upsilon C - \Upsilon' C' - \Phi B. \tag{5.41}$$

After assuming the net-zero investment, we obtain

$$C = \phi B + \psi C', \tag{5.42}$$

where $\phi \equiv \Phi/\Upsilon$ is the number of bonds per call, and $\psi \equiv \Upsilon'/\Upsilon$ is the number of secondary calls per call. We proceed as before, thus the non anticipating character of ϕ and ψ allows us to write

$$dC = \phi dB + \psi dC' \tag{5.43}$$

and, after using Itô lemma (5.38) for both dC and dC', some simple manipulations yield

$$\left[\left(C_t + \frac{\sigma^2}{2\tau} C_{VV} - rC + (\mu + V) S C_S\right)\right.$$

[12] A similar situation appears in the stochastic volatility models (Scott (1987)).

$$-\psi\Big(C'_t + \frac{\sigma^2}{2\tau}C'_{VV} - rC' + (\mu + V)SC'_S\Big)\Bigg]dt = (\psi C'_V - C_V)\,dV. \tag{5.44}$$

This equation can be transformed to a deterministic one by equating to zero the term multiplying the random differential $dV(t)$ given by Eq. (5.3). This, in turn, will determine the investor strategy giving the relative number of secondary calls to be held. Thus, instead of B-S delta hedging, we will have the "*psi hedging*":

$$\psi = \frac{C_V}{C'_V}. \tag{5.45}$$

Then

$$\frac{1}{C_V}\Big[C_t + \frac{\sigma^2}{2\tau}C_{VV} - rC + (\mu + V)SC_S\Big]$$
$$= \frac{1}{C'_V}\Big[C'_t + \frac{\sigma^2}{2\tau}C'_{VV} - rC' + (\mu + V)SC'_S\Big]. \tag{5.46}$$

This equation proves, as otherwise expected, that the call has the same partial differential equation independent of its maturity and strike. This has been suggested in a more theoretical setting for any derivative on the same asset (Björk (1998)).

On the other hand, the two options C and C' have different strikes. Then, analogously to the separation of variable method used in mathematics (Mynt-U (1987)) and proceeding in a similar way to that used in the study of SV cases, both sides of Eq. (5.46) are assumed to be equal to an unknown function $\lambda(S, V, t)$ of the independent variables S, V, and t. We thus have

$$C_t + \frac{\sigma^2}{2\tau}C_{VV} + (\mu + V)SC_S - rC = \lambda C_V. \tag{5.47}$$

In the stochastic volatility literature, the arbitrary function $\lambda(S, V, t)$ is known as the "*risk premium*" associated, in our case, with the return velocity (Scott (1987), Heston (1993)). In the Appendix C we show that the risk premium λ is given by

$$\lambda(S, V, t) = \frac{V}{\tau}. \tag{5.48}$$

A substitution of Eq. (5.48) into Eq. (5.47) yields a closed partial differential for the call price $C(S, V, t)$ which is

$$C_t + \frac{\sigma^2}{2\tau}C_{VV} - \frac{V}{\tau}C_V + (\mu + V)SC_S - rC = 0. \tag{5.49}$$

For the European call, Eq. (5.49) has to be solved with the "final condition" (5.30) at maturity time which is $C(S, V, T) = (S(T) - K)^+$. The solution to Eq. (5.49) subject to Eq. (5.30) is given in Appendix D and reads

$$C(S, V, t) = e^{-r(T-t)}\left[Se^{\beta(T-t,V)}N(z_1) - KN(z_2)\right], \tag{5.50}$$

where $z_1 = z_1(S, V, T - t)$, $z_2 = z_2(S, V, T - t)$ are given by Eq. (D.8) of Appendix D, and

$$\beta(t, V) = m(t, V) + K_{11}(t)/2,$$

where $m(t, V)$ and $K_{11}(t)$ are given by Eqs. (5.8) and (5.9).

The option price (5.50) depends on both the price S and the velocity V of the underlying asset at time t, *i.e.*, at the time at which the call is bought. They are therefore the initial variables of the problem. However, while the initial price S is always known, the initial velocity V is unknown. The velocity is thus assumed to be in the stationary regime so that its probability density function is as shown in Eq. (5.10). We therefore average over the unknown initial velocity and define $\overline{C}$ by

$$\overline{C}(S, t) \equiv \int_{-\infty}^{\infty} C(S, V, t) p_{st}(V) dV, \tag{5.51}$$

and from Eqs. (5.10) and (5.50) we have

$$\overline{C}(S, t) = e^{-r(T-t)} \left[S e^{\beta(T-t)} N(\bar{z}_1) - K N(\bar{z}_2) \right], \tag{5.52}$$

where

$$\beta(t) = \mu t + \kappa(t)/2, \tag{5.53}$$

$\kappa(t)$ is the variance defined by Eq. (5.11), and $\bar{z}_{1,2}$ are given by Eq. (D.9) of Appendix D. As mentioned above, Eq. (5.52) cannot be our final price yet because it still depends on the mean return rate μ. This rate could differ depending on whether μ is estimated by the seller or buyer of the option and therefore, in Eq. (5.52), there are hidden arbitrage opportunities[13].

Therefore, we must proceed in a similar way as in the martingale option pricing theory of Eq. (5.62) and define the final call price, $C_{OU}(S, t)$, as price $\overline{C}$ when $\beta(t)$ is replaced by rt. That is:

$$C_{OU}(S, t) \equiv \overline{C}(S, t) \Big|_{\beta(t) \to rt}, \tag{5.54}$$

and this price completely agrees with the one derived in Section 5.2 (see Eq. (5.31)).

5.4.3 The projected process and the modified portfolio

Suppose we start from the modified portfolio (5.42) but assuming that the share price is given by the projected process (5.23) instead of the two-dimensional O-U process (5.2)–(5.3). In this case, one can obtain the same option price as before (*cf.* Eq. (5.31)). However, the hedging strategy will be given by the following function

$$\psi(S, t) = \frac{C_S}{C'_S}. \tag{5.55}$$

[13]See Section 2.1 for the arbitrage restrictions on option prices. These are summarized in Table 2.7.

Let us prove this. We start from Eq. (5.43):

$$dC = \phi dB + \psi dC',$$

Now, instead of Eq. (5.44) we have (see Itô lemma (5.28))

$$\left[\left(C_t + \frac{1}{2}\dot{\kappa}(T-t)C_{SS} - rC\right) - \psi\left(C'_t + \frac{1}{2}\dot{\kappa}(T-t)C'_{SS} - rC'\right)\right]dt$$
$$= (\psi C'_S - C_S)\, dS. \qquad (5.56)$$

And the removal of risk implies Eq. (5.55). The psi hedging given by Eq. (5.55) is equivalent to the psi hedging defined in Eq. (5.45) although now it is represented in terms of the final price $C_{OU}(S,t)$ instead of the intermediate price $C(S,V,t)$. Substituting Eq. (5.55) into Eq. (5.56) and reasoning along the same lines as above (see Eq. (5.47)) we obtain

$$C_t + \frac{1}{2}\dot{\kappa}(T-t)S^2 C_{SS} - rC = \lambda C_S, \qquad (5.57)$$

where $\lambda = \lambda(S,t)$ is the risk premium for the effective process which is now obviously independent of the velocity V. Combining Eqs. (5.23), (5.28) and (5.57), we get

$$dC(S,t) = \left\{rC + \left[\frac{\lambda}{S} + \mu + \frac{1}{2}\dot{\kappa}(T-t)\right] SC_S\right\} dt + \sqrt{\dot{\kappa}(T-t)}SC_S dW(t). \qquad (5.58)$$

Hence, the conditional expected value of dC reads

$$E\left[dC|C\right] = \left\{rC + \left[\frac{\lambda}{S} + \mu + \frac{1}{2}\dot{\kappa}(T-t)\right] SC_S\right\} dt, \qquad (5.59)$$

but the equilibrium of the market implies that $E\left[dC|C\right] = rCdt$. Therefore,

$$\lambda = -S\left[\mu + \frac{1}{2}\dot{\kappa}(T-t)\right], \qquad (5.60)$$

and Eq. (5.57) reads

$$C_t + \frac{1}{2}\dot{\kappa}(T-t)S^2 C_{SS} - rC + \left[\mu + \frac{1}{2}\dot{\kappa}(T-t)\right] SC_S = 0,$$

Finally, the absence of arbitrage opportunities requires the replacement (see Eq. (5.53))

$$\mu + \dot{\kappa}(T-t)/2 \longrightarrow r.$$

Thus, the option price equation is

$$C_t = rC - rSC_S - \frac{1}{2}\dot{\kappa}(T-t)S^2 C_{SS}, \qquad (5.61)$$

which agrees with Eq. (5.29).

Note that both procedures, the original B-S method presented in Section 5.2 and our method, result in the same partial differential equation for the call price. However, each method uses a different hedging strategy because they start from a different portfolio.

5.5 The call price by the equivalent martingale measure method

We will now show that, in the present case, the price obtained by martingale methods completely agrees with our extended B-S price (5.31). As we have seen in Section 4.5, the equivalent martingale measure theory imposes the condition that, in a risk-neutral world, the stock price $S(t)$ evolves, on average, as a riskless bond.

Let $p^*(S,t|S_0,t_0)$ be the equivalent martingale measure associated with asset price $S(t)$ conditioned on $S(t_0) = S_0$. Define the martingale conditional expected value

$$E^*\left[S(t)|S_0\right] = \int_0^\infty Sp^*(S,t|S_0,t_0)dS.$$

Then the risk-neutral assumption requires that

$$E^*\left[S(t)|S_0\right] = S_0 e^{r(t-t_0)},$$

where r is the constant spot interest rate. On the other hand,

$$E\left[S(t)|S_0\right] = \int_0^\infty Sp(S,t|S_0,t_0)dS.$$

Assuming that the initial velocity is in the stationary regime, the marginal density $p(S,t|S_,t_0)$ is given by Eq. (A.16) of Appendix A. Therefore,

$$E\left[S(t)|S_0\right] = S_0 \exp\left[\beta(t-t_0)\right],$$

where $\beta(t) = \mu t + \kappa(t)/2$ with $\kappa(t)$ given by Eq. (5.11). We thus see that the equivalent martingale measure is accomplished by the replacement

$$\beta(t) \longrightarrow rt. \tag{5.62}$$

In consequence,

$$p^*(S,t|S_0,t_0) = \frac{1}{S\sqrt{2\pi\kappa(t-t_0)}} \exp\left\{-\frac{[\ln(S/S_0) - r(t-t_0) + \kappa(t-t_0)/2]^2}{2\kappa(t-t_0)}\right\}, \tag{5.63}$$

which is the so called the risk-neutral pdf for the stock price and it is a consequence of the absence of arbitrage demand. Now, it is possible to express the price for the European call option by defining its value as the discounted expected gain due to holding the call. That is (*cf.* Eq. (4.31))

$$\begin{aligned} C^*(S,t) &= e^{-r(T-t)} E^*[(S(T)-K)^+|S(t)=S] \\ &= e^{-r(T-t)} \int_K^\infty (S'-K)p^*(S',T|S,t)dS', \end{aligned} \tag{5.64}$$

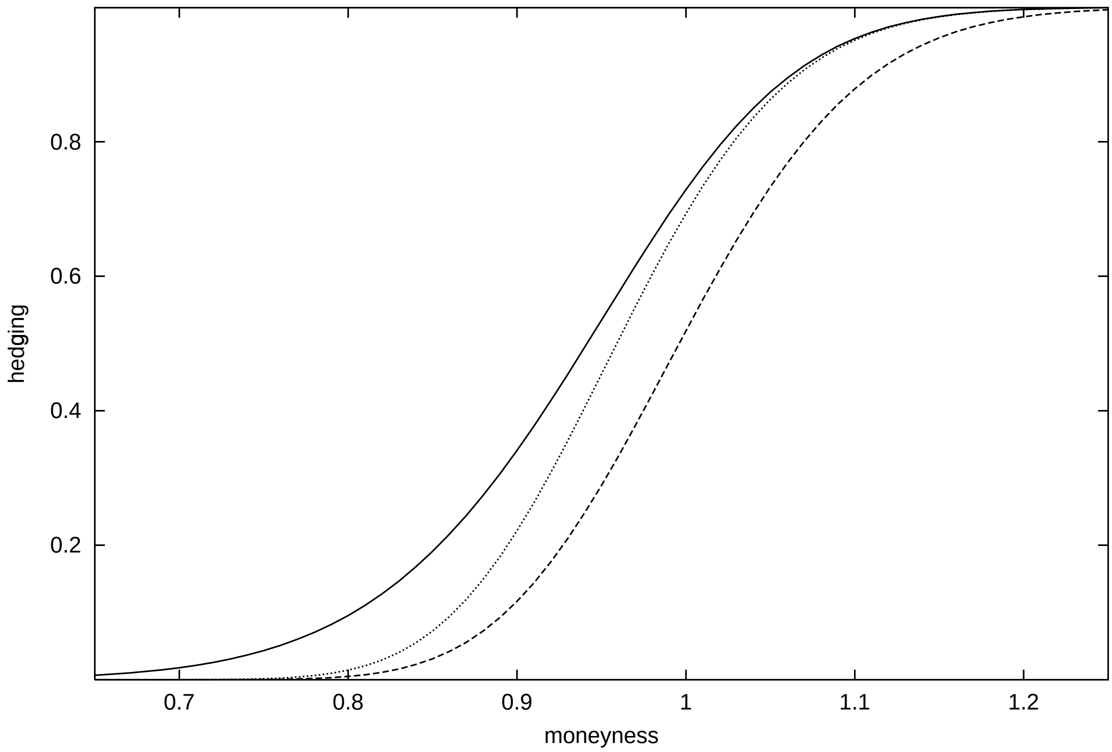


Figure 5.7: Hedging in terms of the moneyness

Psi hedging and delta hedging as a function of the moneyness S/K. The solid line represents psi hedging when $\tau = 1$ day, the time to expiration is $T - t = 20$ days, and the exercising price of the secondary call is $K' = 0.9K$. The dotted line corresponds to the delta hedging still assuming the O-U asset model with the same correlation and expiration time. The dashed line corresponds to B-S delta hedging (r and σ as in Fig. 5.2).

and the final result for the call is obtained by calculating the expected value with the equivalent martingale measure defined in Eq. (5.63). The martingale price agrees exactly with our previous price in Eq. (5.31), $C^*(S,t) = C_{OU}(S,t)$. We can thus say that, in the O-U case, both option pricing methods are completely equivalent although martingale theory does not require the construction of a portfolio and ignores any hedging strategy.

5.6 Greeks and hedging

We briefly derive the Greeks for the O-U case. Since the O-U call price has the same functional form as the B-S price but replaces $\sigma^2(T-t)$ by $\kappa(T-t)$, the O-U Greeks

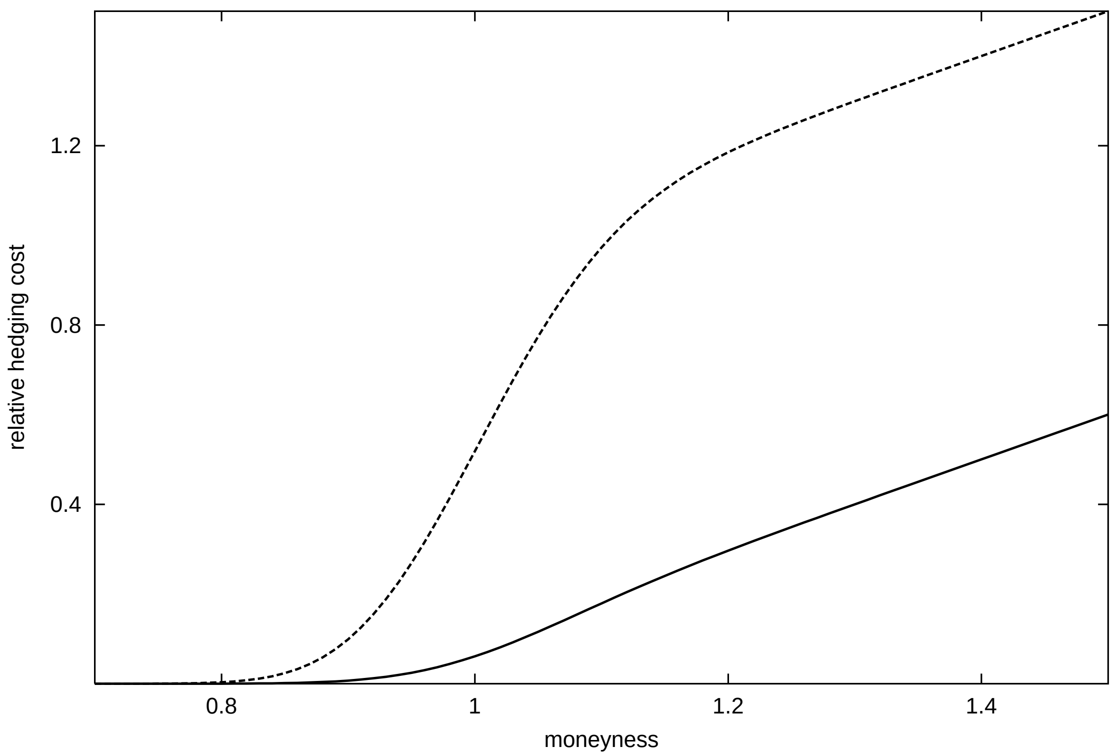


Figure 5.8: Relative hedging cost in terms of the moneyness

We plot the relative hedging costs $\psi C'/K$ and $\delta S/K$ as a function of the moneyness S/K. The solid line represent psi hedging cost when $\tau = 1$ day, the time to expiration is $T - t = 20$ days, and the exercising price of the secondary call is $K' = 0.9K$. The dotted line corresponds to the delta hedging with $\tau = 1$ day and $T - t = 20$ days (r and σ as in Fig. 5.2).

will have the same functional form as B-S Greeks with the same replacement except for vega, $\upsilon = \partial C/\partial\sigma$, and $\theta = \partial C/\partial t$. Thus, for $\delta = \partial C/\partial S$, $\gamma = \partial^2 C/\partial S^2$, and $\varrho = \partial C/\partial r$, we have (cf Section 3.5)

$$\delta_{OU} = N(d_1^{OU}), \qquad \gamma_{OU} = \frac{e^{-(d_1^{OU})^2/2}}{S\sqrt{2\pi\kappa(T-t)}}, \qquad \varrho_{OU} = K(T-t)e^{-r(T-t)}N(d_2^{OU}). \tag{5.65}$$

Since $d_{1,2}^{OU} \geq d_{1,2}^{BS}$ for all S and t and $N(z)$ is a monotone increasing function, we see that $\delta_{OU} \geq \delta_{BS}$ and $\varrho_{OU} \geq \varrho_{BS}$. Hence, the O-U call price is more sensitive to changes in stock price and interest rate than the B-S price.

On the other hand, from Eq. (5.31) and taking into account the identity proved in Section 3.5

$$SN'(d_1) - Ke^{-r(T-t)}N'(d_2) = 0, \tag{5.66}$$

we have

$$\upsilon_{OU} = (S/\sigma)[\kappa(T-t)/2\pi]^{1/2}e^{-(d_1^{OU})^2/2}, \tag{5.67}$$

and

$$\theta_{OU} = -Ke^{-r(T-t)}\left[rN(d_2^{OU}) + \frac{\sigma^2(1-e^{-(T-t)/\tau})}{2\sqrt{2\pi\kappa(T-t)}}\; e^{-(d_2^{OU})^2/2}\right]. \tag{5.68}$$

Since $d_1^{OU} \geq d_1^{BS}$, one can easily see that $\upsilon_{OU} \leq \upsilon_{BS}$ for all values of S/K, $T-t$ and τ. Thus our correlated call price is less sensitive to any change of underlying volatility σ than is the B-S price.

We conclude with the psi hedging. For the two-dimensional O-U case the hedging strategy is given by the function $\psi(S,V,t)$ specifying the number of secondary calls to be hold. However, the hedging given by Eq. (5.45) depends on the velocity V and is not expressed in terms of the final call price $C_{OU} = C(S,t)$. As we have shown in Section 5.4.3, psi hedging in terms of C_{OU} can only be derived from the effective one-dimensional process (5.23). In this case, the removal of the randomness coming from dS implies that hedging is given by Eq. (5.55). Since $C_S = \delta_{OU}$, we see from Eqs. (5.55) and Eq. (5.65) that

$$\psi(S,t) = \frac{N(d_1)}{N(d_1')}. \tag{5.69}$$

Now, we take the secondary option to be an European call with maturity T and exercising price $K' < K$, where T and K refer to the primary option. We plot in Fig. 5.7 the psi hedging as a function of the moneyness. We see there that the ψ hedging is always greater than δ_{OU} and δ_{BS} hedgings. Since $N(d_1') \to 1$ when $K' \to 0$, the psi hedging approaches to the delta hedging δ_{OU} as the moneyness of the secondary call tends to infinity. This is consistent with the fact that secondary calls have the same price as the underlying stock when its exercising price is zero, *i.e.*, $C' \to S$ as $K' \to 0$ (see Eq. (5.31)). Therefore, having secondary calls with exercising price equal to zero is equivalent to own underlying shares and the O-U psi hedging coincides with the O-U delta hedging.

As we have mentioned, psi hedging ψ indicates the number of secondary calls per call to be hold if we follow a risk-free strategy with the modified portfolio (5.41). Therefore, the money invested to carry out this strategy is given by $\psi C'$. That is

$$\psi C' = \frac{N(d_1)}{N(d_1')}\left[SN(d_1') - K'e^{-r(T-t)}N(d_2')\right], \tag{5.70}$$

where we have combined the Eqs. (5.31) and (5.69). On the other hand, delta hedging also indicates the number of shares per call to be hold in a risk-free strategy with the B-S portfolio (5.24). And, analogously, the money necessary to perform this strategy is

$$\delta S = SN(d_1), \tag{5.71}$$

where δ is given by Eq. (5.65). We compare these quantities in order to know which hedging is cheaper for the investor. From Eqs. (5.70) and (5.71), we see

$$\frac{\psi C'}{\delta S} = 1 - \frac{K'e^{-r(T-t)}\ N(d_2')}{S\ N(d_1')},$$

but[14]

$$0 \leq \frac{K'e^{-r(T-t)}\ N(d_2')}{S\ N(d_1')} \leq 1.$$

Therefore, $\psi C' < \delta S$ and psi hedging is always less expensive than delta hedging. Note that when $K' \to 0$ both strategies have the same cost. In Fig. 5.8 we plot, as a function of moneyness, the relative psi hedging cost, $\psi C'/K$, along with the relative delta hedging cost, $\delta S/K$. We see there that psi hedging is considerably less expensive than delta hedging and this difference increases with moneyness. Indeed, for an ATM call ($S/K = 1.00$) and with parameter values as that of Fig. 5.8, delta hedging is approximately 800% more expensive than psi hedging.

Combining Eqs. (5.34), (5.55) and (5.66) one can easily show that when $\tau = 0$ the O-U psi hedging is $\psi_{BS} = N(d_1^{BS})/N(d_1^{BS'})$ [15], where the prime refers to the secondary call. Since $\delta = C_S = N(d_1)$, we have

$$\psi_{BS} = \frac{\delta_{BS}}{\delta'_{BS}}.$$

Finally, for the secondary call, whose exercising price goes to zero, $\delta'_{BS} \to 1$ and, again, B-S psi hedging and B-S delta hedging coincide.

[14] This is straightforward to prove from Eq. (5.70) since $\psi C' \geq 0$.

[15] We use the subscript BS in ψ_{BS} to indicate that this hedging refers to an uncorrelated stock ($\tau = 0$), as in the B-S world

Summary

We have developed a theory for option pricing with perfect hedging in an inefficient market model where the underlying price variations are auto correlated over a time $\tau \geq 0$. This is accomplished by assuming that the underlying noise in the system is derived by an Ornstein-Uhlenbeck, rather than from a Wiener process. After obtaining an effective one-dimensional market model, we have achieved a closed expression for the European call price within the Black-Scholes framework and find that our price is always lower than the Black-Scholes price. We have obtained the same price if we start from a modified portfolio although now we get a different hedging strategy than that of Black-Scholes. We have compared these strategies and study the sensitivity of the call price to several parameters where the correlation effects have been also observed.

Appendix A. Mathematical properties of the model

We present some of the most important properties of the model given by the pair of stochastic equations in Eqs. (5.5) and (5.6). Their formal solutions are

$$V(t) = V_0 e^{-(t-t_0)/\tau} + \frac{\sigma}{\tau} \int_{t_0}^{t} e^{-(t-t')/\tau} dW(t'),$$

and

$$R(t) = \mu(t-t_0) + V_0 \tau (1 - e^{-(t-t_0)/\tau}) + \frac{\sigma}{\tau} \int_{t_0}^{t} dt' \int_{t_0}^{t'} e^{-(t'-t'')/\tau} dW(t''), \qquad \text{(A.1)}$$

where we have assumed that the process begun at time t_0 with initial velocity V_0 and return $R_0 = 0$. The return $R(t)$ has the following conditional mean value

$$E\left[R(t)|V_0\right] = \mu(t-t_0) + \tau(1 - e^{-(t-t_0)/\tau})V_0,$$

and variance

$$\mathrm{Var}[R(t)|V_0] = \sigma^2 \Big[(t-t_0) - 2\tau\left(1 - e^{-(t-t_0)/\tau}\right) + \frac{\tau}{2}\left(1 - e^{-2(t-t_0)/\tau}\right)\Big].$$

Since $(R(t), V(t))$ is a diffusion process in two dimensions, its joint density $p(R, V, t)$ satisfies the following Fokker-Planck equation (Gardiner (1985))

$$p_t = -(\mu + V)p_R + \frac{V}{\tau} p_V + \frac{\sigma^2}{2\tau^2} p_{VV}. \qquad \text{(A.2)}$$

This is to be solved subject to the initial conditions $R(t_0) = 0$ and $V(t_0) = V_0$, that is

$$p(R, V, t_0 | V_0, t_0) = \delta(R)\delta(V - V_0). \qquad \text{(A.3)}$$

A first step towards solving the problem (A.2)–(A.3) is the definition of the joint Fourier transform (*i.e.*, its characteristic function)

$$\varphi(\omega_1, \omega_2, t) = \int_{-\infty}^{\infty} dR e^{i\omega_1 R} \int_{-\infty}^{\infty} dV e^{i\omega_2 V} p(R, V, t).$$

Then problem (A.2)–(A.3) becomes

$$\partial_t \varphi = i\omega_1 \mu \varphi + (\omega_1 - \omega_2/\tau)\omega_1 \partial_{\omega_2} \varphi - (\sigma^2/2\tau^2)\omega_2^2 \varphi, \tag{A.4}$$

$$\varphi(\omega_1, \omega_2, t = 0) = e^{i\omega_2 V_0}. \tag{A.5}$$

We look for a solution of the form

$$\varphi(\omega_1, \omega_2, t) = \exp\left\{ i[\omega_1 m_1(t) + \omega_2 m_2(t)] - [K_{11}(t)\omega_1^2 + K_{12}(t)\omega_1\omega_2 + K_{22}(t)\omega_2^2] \right\}, \tag{A.6}$$

where $m_i(t)$ and $K_{ij}(t)$ are functions to be determined. We substitute Eq. (A.6) into (A.4) and identify term by term. We have

$$\dot{m}_1 = \mu + m_2, \qquad \dot{m}_2 = -m_2/\tau;$$
$$\dot{K}_{22} + (2/\tau)K_{22} = \sigma^2\tau^2, \qquad \dot{K}_{12} + (1/\tau)K_{12} = 2K_{22}(t), \quad \dot{K}_{11} = 2K_{12},$$

with the initial conditions, according to Eqs. (A.5)–(A.6), given by

$$m_2(0) = V_0, \quad m_1(0) = K_{ij}(0) = 0 \quad (i, j = 1, 2).$$

The solution reads

$$m_1(t) = \mu t + V_0 \tau \left(1 - e^{-t/\tau}\right), \quad m_2(t) = V_0 e^{-t/\tau},$$

and $K_{ij}(t)$ are given by

$$K_{11}(t) = \sigma^2 \left[t - 2\tau \left(1 - e^{-t/\tau}\right) + \frac{\tau}{2}\left(1 - e^{-2t/\tau}\right) \right], \tag{A.7}$$

$$K_{12}(t) = \frac{\sigma^2}{2}\left(1 - e^{-t/\tau}\right)^2, \qquad K_{22}(t) = \frac{\sigma^2}{2\tau}\left(1 - e^{-2t/\tau}\right), \tag{A.8}$$

The inverse Fourier transform of Eq. (A.6) yields the Gaussian density

$$p(R, V, t|V_0, t_0) = \frac{1}{2\pi\sqrt{\det[\boldsymbol{K}(t - t_0)]}} \exp\left\{ -\frac{(V - V_0 e^{-(t-t_0)/\tau})^2}{2K_{22}(t - t_0)} - \frac{\left[K_{22}(t - t_0)(R - m(t - t_0, V_0)) - K_{11}(t - t_0)\left(V - V_0 e^{-(t-t_0)/\tau}\right)\right]^2}{2K_{22}(t - t_0)\det[\boldsymbol{K}(t - t_0)]} \right\}, \tag{A.9}$$

where

$$\det[\boldsymbol{K}(t)] \equiv K_{11}(t)K_{22}(t) - K_{12}^2(t). \tag{A.10}$$

and

$$m(t, V_0) = \mu t + V_0\tau\left(1 - e^{-t/\tau}\right). \tag{A.11}$$

Notice that the joint density (A.9) is a function of the time differences $t - t_0$ where t_0 is the initial observation time, so that the two-dimensional diffusion $(S(t), V(t))$ is a time homogeneous process and, without loss of generality, we may assume that $t_0 = 0$.

The marginal pdf of the velocity $V(t)$,

$$p(V, t|V_0) = \int_{-\infty}^{\infty} p(R, V, t|V_0)dR,$$

is

$$p(V, t|V_0) = \frac{1}{\sqrt{2\pi K_{22}(t)}} \exp\left[-\frac{\left(V - V_0 e^{-t/\tau}\right)^2}{2K_{22}(t)}\right]. \tag{A.12}$$

In the stationary regime ($t \to \infty$) we find a normal density independent of the initial velocity:

$$p_{st}(V) = \frac{1}{\sqrt{\pi(\sigma^2/\tau)}} e^{-\tau V^2/\sigma^2}. \tag{A.13}$$

Analogously, the marginal density of the return $R(t)$,

$$p(R, t|V_0) = \int_{-\infty}^{\infty} p(R, V, t|V_0)dV,$$

is

$$p(R, t|V_0) = \frac{1}{\sqrt{2\pi K_{11}(t)}} \exp\left\{-\frac{[R - m(t, V_0)]^2}{2K_{11}(t)}\right\}. \tag{A.14}$$

If we assume that the initial velocity $V_0 = V(0)$ is a random variable distributed according to the pdf in Eq. (A.13). We can therefore average the above densities to obtain a pdf independent of V_0. That is,

$$p(R, V, t) = \int_{-\infty}^{\infty} p(R, V, t|V_0)p_{st}(V_0)dV_0,$$

and similarly for the marginal pdf's $p(R, t)$ and $p(V, t)$. Since we are mainly interested on the marginal distribution of the return we will give its explicit expression. Thus, from Eqs. (A.13) and (A.14) we have

$$p(R, t) = \frac{1}{\sqrt{2\pi\kappa(t)}} \exp\left[-\frac{(R - \mu t)^2}{2\kappa(t)}\right], \tag{A.15}$$

where $\kappa(t)$ is given by Eq. (5.11). Alternatively, the distribution of the underlying price $S = S_0 e^R$ is given by the log-normal density

$$p(S,t|S_0) = \frac{1}{S\sqrt{2\pi\kappa(t)}} \exp\left[-\frac{(\ln S/S_0 - \mu t)^2}{2\kappa(t)}\right]. \tag{A.16}$$

From this we easily see that the conditional probability $p(S',T|S,t)$ when $t \le T$ is

$$p(S',T|S,t) = \frac{1}{S'\sqrt{2\pi\kappa(T-t)}} \exp\left[-\frac{[\ln S'/S - \mu(T-t)]^2}{2\kappa(T-t)}\right]. \tag{A.17}$$

Appendix B. The Itô formula for processes driven by O-U noise

In this Appendix we generalize the Itô formula for processes driven by Ornstein-Uhlenbeck noise. This is applied to the share price $S(t)$ which is governed by the pair of stochastic equations (5.2)–(5.3)

$$dS(t) = S(\mu + V)dt, \qquad dV(t) = -\frac{V)}{\tau}dt + \frac{\sigma}{\tau}dW(t). \tag{B.1}$$

Consider a generic function $f(S,V,t)$ which depends on all of the variables that characterize the underlying asset. The differential of $f(S,V,t)$ is defined by

$$df(S,V,t) \equiv f(S(t+dt),V(t+dt),t+dt) - f(S(t),V(t),t). \tag{B.2}$$

But the Taylor expansion of (B.2) yields

$$df(S,V,t) = f_S dS + f_V dV + f_t dt + \frac{1}{2}f_{SS}dS^2 + \frac{1}{2}f_{VV}dV^2 + f_{SV}dSdV + \cdots, \tag{B.3}$$

where the expansion also involves higher order differentials such as $(dt)^2$, $(dS)^3$, $(dV)^3$, etc. However, the differential of the Wiener process, dW, satisfies the well-known property, in the mean-square sense, $dW(t)^2 = dt$ (*cf.* Section 3.1). And from the pair of equations (B.1) we then see that dS^2 is of order dt^2 while dV^2 is of order dt and $dSdV$ is of order $dt^{3/2}$. Therefore, up to order dt, Eq. (B.3) reads

$$df(S,V,t) = f_S dS + f_V dV + f_t dt + \frac{\sigma^2}{2\tau^2} f_{VV} dt, \tag{B.4}$$

which is the Itô formula for our singular two-dimensional process (5.2)–(5.3).

Suppose now we start from the effective one-dimensional SDE (5.17)

$$dR(t) = \mu dt + \sqrt{\dot{\kappa}(T-t)}dW(t). \tag{B.5}$$

We will prove that the corresponding SDE for the stock price defined as $S = S_0 e^R$ is given by Eq. (5.23). In effect, substituting Eq. (B.5) in the Taylor expansion

$$dS(R) = S_R dR + \frac{1}{2} S_{RR} dR^2 + \cdots,$$

neglecting orders higher than dt and taking into account that $dR^2 = \dot{\kappa}(T-t)dt$ (in mean square sense), we finally obtain

$$\frac{dS(t)}{S(t)} = \left[\mu + \dot{\kappa}(T-t)/2\right] dt + \sqrt{\dot{\kappa}(T-t)}dW(t), \tag{B.6}$$

which is Eq. (5.23).

Moreover, we can also give the differential of a generic function $f(S,t)$ when underlying obeys SDE (B.6). In this case, we have

$$df(S,t) = f_S dS + f_t dt + \frac{1}{2}\dot{\kappa}(T-t)S^2 f_{SS} dt, \tag{B.7}$$

where again we have neglected higher order contributions than dt.

Appendix C. A derivation of the risk premium

We proceed to find a closed expression for the arbitrary function $\lambda(S,V,t)$ that appears in Eq. (5.47). The call price C is a function of S, V, and t. We now consider this function taking into account that $S = S(t)$ and $V = V(t)$ follow Eqs. (5.2) and (5.3), respectively. This therefore allows us to evaluate the random differential dC using the Itô lemma, as a result we find that

$$dC = \left[C_t + (\mu + V)SC_S + \frac{\sigma^2}{2\tau}C_{VV}\right] dt + C_V dV.$$

After using Eqs. (5.47) and (5.3), we have

$$dC = \left[rC + \left(\lambda - \frac{V}{\tau}\right) C_V\right] dt + \frac{\sigma}{\tau} C_V dW. \tag{C.1}$$

The expected value of dC, on the assumption that $C(t) = C$ is known, reads

$$E\left[dC|C\right] = \left[rC + \left(\lambda - \frac{V}{\tau}\right) C_V\right] dt. \tag{C.2}$$

We claim that this average must grow at the same rate as the risk-free bond:

$$E\left[dC|C\right] = rCdt, \tag{C.3}$$

since otherwise the option would not be in equilibrium (Hull (2000)). In some sense, this assumption is similar to that of the equivalent martingale measure demand expecting that markets grow in average as the risk-free bond (*cf.* Section 4.5).

The substitution of Eq. (C.3) into Eq. (C.2) yields the following expression for the risk premium $\lambda(S, V, t)$:

$$\lambda = \frac{V}{\tau}. \tag{C.4}$$

Appendix D. Solution to the problem in Eqs. (5.49)–(5.30)

We will solve Eq. (5.49) subject to the final condition in Eq. (5.30). Define a new independent variable Z

$$S = e^Z, \tag{D.1}$$

where the domain of Z is unrestricted. The problem posed in Eqs. (5.49)–(5.30) now reads

$$C_t = rC - (\mu + V)C_Z + \frac{V}{\tau}C_V - \frac{\sigma^2}{2\tau}C_{VV},$$
$$C(Z, V, T) = \max[e^Z - K, 0].$$

The solution to this problem can be written in the form

$$C(Z, V, t) = \int_{-\infty}^{\infty} dZ' \int_{-\infty}^{\infty} dV' \max[e^{Z'} - K] G(Z, V, t|Z', V', T), \tag{D.2}$$

where $G(Z, V, t|Z', V', T)$ is the Green function for the problem (Mynt-U (1987)), *i.e.*, $G(Z, V, t|Z', V', T)$ is the solution to

$$G_t = rG - (\mu + V)G_Z + \frac{V}{\tau}G_V - \frac{\sigma^2}{2\tau}G_{VV}, \tag{D.3}$$

with the final condition

$$G(Z, V, T|Z', V', T) = \delta(Z - Z')\delta(V - V'), \tag{D.4}$$

where $\delta(X - X')$ is the Dirac delta function. Define $\overline{G} = e^{-rt}G$,then the final-value problem in Eqs. (D.3) and (D.4) reads

$$\overline{G}_t = -(\mu + V)\overline{G}_Z + \frac{V}{\tau}\overline{G}_V - \frac{\sigma^2}{2\tau}\overline{G}_{VV}, \tag{D.5}$$

$$\overline{G}(Z,V,T|Z',V',T) = e^{-rT}\delta(Z-Z')\delta(V-V'). \tag{D.6}$$

Note that Eq. (D.5) is the backward equation corresponding to Eq. (A.2). Therefore, Eq.(A.9) permits us to write the solution to the problem posed in Eqs. (D.5)–(D.6) (Gardiner (1985)). This solution implies that G is

$$G(Z,V,t|Z',V',T) = \frac{e^{-r(T-t)}}{2\pi\sqrt{\det[\boldsymbol{K}(T-t)]}} \exp\left\{-\frac{\left[V' - Ve^{-(T-t)/\tau}\right]^2}{2K_{22}(T-t)} - \frac{\left[K_{22}(T-t)\left(Z'-Z+m(T-t,V)\right) - K_{11}(T-t)\left(V'-Ve^{-(T-t)/\tau}\right)\right]^2}{2K_{22}(T-t)\det[\boldsymbol{K}(T-t)]}\right\}, \tag{D.7}$$

where $\det[\boldsymbol{K}(t)]$, $K_{ij}(t)$, and $m(t,V)$ are defined in Eqs. (A.10)–(A.11).

Substituting Eq. (D.7) into Eq. (D.2) and finally reverting to the original variables we obtain Eq. (5.50) with

$$z_1 = \frac{\ln(S/K) + m(T-t,V) + K_{11}(T-t)}{\sqrt{K_{11}(T-t)}}, \quad z_2 = z_1 - \sqrt{K_{11}(T-t)}. \tag{D.8}$$

Finally it can be shown, after some lengthy but simple manipulations, that the functions $\bar{z}_{1,2} = \bar{z}_{1,2}(S,T-t)$ appearing in the averaged price $\overline{C}(S,t)$, Eq. (5.52), are given by

$$\bar{z}_1 = \frac{\ln(S/K) + \mu(T-t) + \kappa(T-t)}{\sqrt{\kappa(T-t)}}, \qquad \bar{z}_2 = \bar{z}_1 - \sqrt{\kappa(T-t)}, \tag{D.9}$$

where $\kappa(t)$ is given in Eq. (5.11).

References

Aurell, E., R. Baviera, O. Hammarlid, M. Serva, and A. Vulpiani, 2000, A general methodology to price and hedge derivatives in incomplete markets, *International Journal of Theoretical and Applied Finance* **3**, 1-24.

Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* **81**, 637-659.

Björk, T., 1998, *Arbitrage theory in continuous time* (Oxford University Press, Oxford, UK).

Bouchaud, J. P., and M. Potters, 2000, *Theory of financial risks* (Cambridge University Press, Cambridge, UK).

Breen, W., and R. Jagannathan, 1989, Economics significance of predictable variations in stock index returns, *Journal of Finance* **44**, 1177-1189.

Campbell, J., and Y. Hamao, 1992, Predictable stock returns in the United States and Japan: A study of long-term capital market integration, *Journal of Finance* **47**, 43-70.

Cox, J. C., and S. A. Ross, 1976, The valuation of options for alternative stochastic processes, *Journal of Financial Economics* **3**, 145-166.

Doob, J. L., 1942, The Brownian movement and stochastic equations, *Annals of Mathematics* 43, 351-369.

Dumas, B., J. Fleming, and R. Whaley, 1998, Implied volatility smiles: Some empirical tests, *Journal of Finance* **53**, 2059-2106.

Fama, E. F., 1963, Mandelbrot and the stable Paretian hypothesis, *Journal of Business* **36**, 420-429.

——, 1965, The behavior of stock market prices, *Journal of Business* **38**, 34-105.

——, 1991, Efficient capital markets II, *Journal of Finance* **46**, 1575-1617.

Figlewski, S., 1989, Options arbitrage in imperfect markets, *Journal of Finance* **64**, 1289-1311.

Gardiner, C. W., 1985, *Handbook of stochastic methods* (Springer-Verlag, Heilderberg, Berlin, New York).

Ghysels, E., A. C. Harvey, and E. Renault, 1996, Stochastic volatility, in G. S. Mandala and C. R. Rao eds: *Statistical methods in finance* (North-Holland, Amsterdam, New York).

Grossman, S. J., and J. E. Stiglitz, 1980, On the impossibility of informationally efficient markets, *The American Economic Review* **70**, 222-227.

Harrison, J. M., R. Pitbladdo and S. M. Schaefer, 1984, Continuous price processes in frictionless markets have infinite variation, *Journal of Business* **57**, 353-365.

Heston, S., 1993, A closed-form solution for options with stochastic volatility to bond and currency options, *Review of Financial Studies* **6**, 327-343.

Heston, S., and S. Nandi, 2000, A closed-form GARCH option valuation model, *Review of Financial Studies* **13**, 585-625.

Hull, J. C., and A. White, 1987, The pricing of options with stochastic volatilities, *Journal of Finance* **42**, 281-300.

Hull, J. C., 2000, *Options, futures, & other derivatives* (Prentice-Hall, Upper Saddle River, NJ).

Lo, A. W., and J. Wang, 1995, Implementing option pricing models when asset returns are predictable, *Journal of Finance* **50**, 87-129.

Mandelbrot, B., 1963, The variation of certain speculative prices, *Journal of Business* **36**, 394-419.

Markowitz, H., 1952, Portfolio selection, *Journal of Finance* **7**, 77-91.

Masoliver, J., and J. Perelló, 2001b, Option pricing and perfect hedging on correlated stocks, submitted for publication.

Merton, R. C., 1973a, Theory of rational option pricing, *Bell Journal of Economics* **4**, 141-183.

Mynt-U, Tyn, 1987, *Partial differential equations for scientists and engineers* (North-Holland, Amsterdam, New York).

Ross, S., 1976, Arbitrage theory of capital asset pricing, *Journal of Economic Theory* **13**, 341-360.

Sharpe, W. F., 1964, Capital asset prices, *Journal of Finance* **19**, 425-442.

Scott, L.O., 1987, Option pricing when the variance changes randomly: Theory, estimation, and application, *Journal of Financial and Quantitative Analysis* **22**, 419-438.

Stein, E. M., and J. C. Stein, 1991, Stock price distributions with stochastic volatility: An analytic approach, *Review of Financial Studies* **4**, 727-752.

Uhlenbeck, G. E., and L. S. Ornstein, 1930, On the theory of the Brownian motion, *Physical Review* **31**, 823-841.

Chapter 6

A correlated stochastic volatility model

During decades, the diffusion process known as the geometric Brownian motion has been widely accepted as one of the most universal models for speculative markets. It was first proposed by Osborne (1959) when he observed that empirical distributions of prices were biased and in disagreement with the theoretical distribution of the arithmetic Brownian dynamics first proposed by Bachelier (1900).

However, specially after the 1987 crash, the geometric Brownian motion and its subsequent Black-Scholes (B-S) formula were unable to reproduce the option price data of the real markets. Several studies have collected empirical option prices in order to derive their *implied volatility* (*i.e.*, the volatility that underlying should have if B-S price formula hold). These tests conclude that the implied volatility is not constant and it is well-fitted with a U-shaped function of the moneyness and whose minimum is at moneyness near to 1, *i.e.*, when current stock price is equal to the striking price. This effect is known as the *smile effect* and shows the inadequacy of the Black-Scholes model since this assumes a constant volatility[1].

A possible way out to this inconsistency is assuming that volatility is not a constant but an unknown deterministic function of the underlying price. The deterministic volatility still allows to manage the option pricing within the B-S theory although in most of the cases it is not possible to derive an analytic option price. Within this approach, there exists the ARCH-GARCH models (Engle (1982)) and their subsequent extensions (Bollersev *et al.* (1994)). These models do well in describing the implied volatility but their disadvantage is that some of their parameters substantially change with time frequency[2].

The *stochastic volatility* (*SV*) models are another possible choice. These models

[1]The smile effect is fully documented in the literature (see for instance Jackwerth and Rubinstein (1996))

[2]See Engle and Patton (2001) for more details on parameter estimation from data assuming the ARCH-GARCH approach.

assume the original log-Brownian model but, as their name indicates, with volatility being random. Although SV models as known today appeared in the late eighties (see below), there seems to be some precedent like the work of Clark (1973) suggesting to model asset returns with an extra "subordinated stochastic process". This extra process modifies the value of the volatility at randomly distributed times. In Clark's own words: "The different evolution of price series on different days is due to the fact that the information is available to traders at a varying rate. On days when no new information is available, trading is slow, and the price process evolves slowly. On days when new information violates old expectations, trading is brisk, and price process evolves much faster (Clark (1973), p.137)".

At late eighties several different SV models were presented by Scott (1987), Wiggins (1987), and Hull and White (1987). All of them propose a *two-dimensional process* involving two independent variables: the stock and the volatility. These works are basically interested on option pricing theory and ignore the statistical properties of the market model, although they are indeed able to reproduce the smile effect.

After then there have appeared several papers extending and refining the original SV models but again many of them are solely interested on adequately describe empirical option prices. Stein and Stein (1991) is an exception to this tendency because they study the most important statistical properties of a volatility model following an Ornstein-Uhlenbeck (O-U) process.

We believe that the relative small number of works dealing with market dynamics based on SV models is essentially due to two reasons: (i) Their statistical properties are difficult to be analytically derived, and the analysis is even much more involved when there are correlations between volatility and stock. (ii) It is commonly asserted that empirical data available are not enough for obtaining a reliable estimation of all parameters involved in an SV model (see *e.g.* Fouque *et al.* (2000b)).

Our present work wants to modify these statements for the correlated Ornstein-Uhlenbeck stochastic volatility (O-U SV) model because we are able to analytically derive the main statistical properties of it. On the other hand, the leverage correlation recently observed by Bouchaud *et al.* (2001) allows us to estimate all parameters involved not only in our O-U SV process but eventually in any SV process.

Recent research on empirical markets data has provided a set of requirements that a good market model must obey. In this regard Engle and Patton (2001) have listed a number of stylized facts about the volatility. The results we will herein derive conclude that the SV models are good candidates fairly accomplishing these stylized facts. We will prove this and confront the statistical properties of our correlated O-U SV model with the statistical properties of the daily return changes for the Dow-Jones stock index.

The chapter is divided in 5 sections and is a preliminary version to the article Masoliver and Perelló (2001c). After this introduction, we present our stochastic volatility market model in Section 6.1 and study the statistical properties of the volatility in Section 6.2. Section 6.3 is specifically focussed on the leverage effect. Section 6.4 is devoted to show how to estimate from the Dow-Jones index (1900-2000) the parameters of the model. Finally, Section 6.5 concentrates on the derivation of the probability distribution through obtaining the characteristic function. Technical details are left to the Appendices A and B.

6.1 The stochastic volatility market model

The starting point of any stochastic volatility model is the log-Brownian stochastic differential equation:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_1(t), \tag{6.1}$$

where μ is the drift and σ is the volatility. The SV models refine this dynamics taking $\sigma = \sigma(t)$ to be stochastic. There exist a large class of such models but, to our knowledge, the dynamics of σ is not definitively associated to any specific process (see the monograph of Fouque *et al.* (2000b) or the review by Ghysels *et al.* (1996)). We choose the Ornstein-Uhlenbeck (O-U) stochastic volatility model because, as we will see shortly, it is one of the simplest approaches still reproducing the main observed features of markets. We thus assume that the random dynamics of $\sigma(t)$ is given by (Stein and Stein (1991))

$$d\sigma(t) = -\alpha(\sigma - \theta)dt + kdW_2(t). \tag{6.2}$$

Equations (6.1) and (6.2) contain a two-dimensional Wiener process $(W_1(t), W_2(t))$, where $dW_i(t) = \xi_i(t)dt$ $(i = 1, 2)$, and $\xi_i(t)$ is Gaussian white noise processes with zero mean, *i.e.*,

$$E[\xi_i(t)] = 0 \quad \text{and} \quad E[\xi_i(t)\xi_j(t')] = \rho_{ij}\delta(t - t'). \tag{6.3}$$

We note that the cross-correlation is given in terms of the Dirac delta function (*cf.* Eq. (3.2)), *i.e.*, $\delta(x) = 0$ for all $x \neq 0$ and

$$\int_{-\infty}^{a} \phi(z)\delta(x - z)dz = \begin{cases} \phi(x) & -\infty < x < a, \\ 0 & \text{otherwise}; \end{cases} \tag{6.4}$$

where $\phi(x)$ is an arbitrary integrable function. Note that $\rho_{ij} = \rho_{ji}$ and $\rho_{ii} = 1$, hence the components of ρ_{ij} are reduced to

$$\rho_{ij} = \begin{cases} 1 & \text{if } i = j \\ \rho & \text{if } i \neq j, \end{cases} \tag{6.5}$$

where the parameter ρ is given by Eq. (6.3), *i.e.*, $E[\xi_1(t)\xi_2(t')] = \rho\delta(t - t')$ which, in terms of the Wiener process, is equivalent to say that $E[dW_1(t)dW_2(t')] = 0$ when $t \neq t'$, and

$$E[dW_1(t)dW_2(t)] = \rho dt.$$

Recall that $\mathrm{Var}[dW_i(t)] = dt$, therefore

$$\rho = \frac{E\left[dW_1(t)dW_2(t)\right]}{\sqrt{\mathrm{Var}[dW_1(t)]\mathrm{Var}[dW_2(t)]}}$$

is the correlation coefficient and hence[3]

$$-1 \leq \rho \leq 1.$$

The correlation coefficient ρ has no definite sign. However, it is known that a negative ρ is able to provide the skewness observed in financial markets (Fouque *et al.* (2000b)). One of our objectives herein is not only to show that ρ is negative but also to estimate its value from empirical data.

In what follows it turns to be more convenient to work with the zero-mean return defined as

$$dX \equiv \frac{dS}{S} - \mu dt, \tag{6.6}$$

and whose SDE reads

$$dX(t) = \sigma(t)dW_1(t). \tag{6.7}$$

The zero-mean return $X(t)$ has a simpler dynamics than the stock price $S(t)$ because it only contains the random fluctuation σdW_1. Nevertheless, this process still retains the most interesting features of the whole dynamics. In the Appendix A, we give en explicit expression for $X(t)$ and derive some key features.

6.2 The Ornstein-Uhlenbeck volatility process

We will now present the main properties of the O-U volatility. The starting point is the solution of Eq. (6.2):

$$\sigma(t) = \sigma_0 e^{-\alpha(t-t_0)} + \theta(1 - e^{-\alpha(t-t_0)}) + k\int_{t_0}^{t} e^{-\alpha(t-t')}dW_2(t'), \tag{6.8}$$

where we have assumed that the process started at time $t = t_0$, when volatility was σ_0. From now on we will assume that the volatility is in the stationary regime. This means that the market started long time ago, thus $t_0 \to -\infty$, and the stationary volatility reads

$$\sigma(t) = \theta + k\int_{-\infty}^{t} e^{-\alpha(t-t')}dW_2(t'), \tag{6.9}$$

whose average value, variance and correlation are

$$E\left[\sigma\right] = \theta, \qquad \mathrm{Var}[\sigma] \equiv E\left[\sigma^2\right] - E\left[\sigma\right]^2 = k^2/2\alpha, \tag{6.10}$$

[3] This is a direct consequence of the *Cauchy-Schwarz inequality* (see, for instance, Grimmett and Stirzaker (1992)).

and

$$E\left[\sigma(t+\tau)\sigma(t)\right] = \theta^2 + (k^2/2\alpha)e^{-\alpha\tau}. \tag{6.11}$$

Note that these expressions provide a physical interpretation of the parameters of the model, especially θ, the expected volatility, and α, the inverse of the volatility "correlation time" (see below).

Before proceeding further we want to address the question of the sign of $\sigma(t)$. If one adopts the O-U process (6.2) as a model for stochastic volatility one may argue that $\sigma(t)$ has no definite sign which can be seen as an inconvenience for a "good" SV model. Let us see that this is not really the case. First of all, the actual evaluation of volatility is very difficult, not to say impossible, since volatility itself is not observed. In practice, the so-called *instantaneous volatility* is derived from

$$\lim_{\Delta t\to 0}\sqrt{\left[X(t+\Delta t)-X(t)\right]^2/\Delta t}, \tag{6.12}$$

where we have used the zero-mean return defined above. Due to the limit $\Delta t \to 0$, this equation has to be taken as an infinitesimal difference. We thus define

$$\text{instantaneous volatility} \equiv \sqrt{dX(t)^2/dt}. \tag{6.13}$$

¿From Eq. (6.7) and the fact $dW^2 = dt$, we get

$$\text{instantaneous volatility} = \sqrt{\sigma(t)^2} = \left|\sigma(t)\right|. \tag{6.14}$$

Observe that in this definition no sign is attached to the random variable $\sigma(t)$.

6.2.1 The correlated process

As we have mentioned in Section 6.1, our SV model permits correlations between the stock and the volatility. We will now examine the important effects of these correlations. From Eq. (6.9), we see that the correlation between the stationary volatility and the random component of return variations, $dW_1(t)$, is

$$E\left[\sigma(t+\tau)dW_1(t)\right] = k\int_{-\infty}^{t+\tau} e^{-\alpha(t+\tau-t')}E\left[dW_2(t')dW_1(t)\right],$$

which, taking into account Eqs. (6.3)–(6.5), can be written as

$$E\left[\sigma(t+\tau)\frac{dW_1(t)}{dt}\right] = \rho k\int_{-\infty}^{t+\tau} e^{-\alpha(t+\tau-t')}\delta(t-t')dt'. \tag{6.15}$$

Finally (*cf.* Eq. (6.4))

$$E\left[\sigma(t+\tau)dW_1(t)\right] = \begin{cases} \rho k e^{-\alpha\tau}dt & \text{if } \tau > 0, \\ 0 & \text{if } \tau < 0. \end{cases} \tag{6.16}$$

Therefore, for correlated SV processes *future volatility is correlated with past return variations, although past volatility and future return variations are completely uncorrelated.* Moreover, ρ determines the sign of the correlation (6.16) because k is positive. In the next section we will show how this property is able to reproduce the leverage effect observed in real markets.

We finally note that, despite the existence of correlations, $\sigma(t)$ and $dW_1(t)$ are independent random quantities. This is a direct consequence of Itô convention for stochastic integrals, because process $\sigma(t)$ is independent of its driving noise $dW_1(t) = \xi_1(t)dt$. Hence, as $\tau \to 0^-$ we have

$$E\left[\sigma(t)dW_1(t)\right] = E\left[\sigma(t)\right]E\left[dW_1(t)\right] = 0. \tag{6.17}$$

Since $dX(t) = \sigma(t)dW_1(t)$. Eq. (6.17) implies that

$$E\left[dX(t)\right] = 0, \tag{6.18}$$

in accordance with the name "zero-mean return" given to $X(t)$.

6.2.2 Mean reversion

The effect of *mean-reversion* refers to the existence of a normal level of volatility to which volatility will eventually return. This effect can be observed in financial markets (Ghysels *et al.* (1996)). Practitioners believe that the current volatility is high or low compared to a *normal level* of volatility and they assume that in the long run, forecasts of the volatility should all converge to the same normal level. Hence, the average of the instantaneous volatility (6.13) should converge to the normal level as time tends to infinity, this is done, for instance, by Engle and Patton (2001) who require that

$$\lim_{t\to\infty} E\left[\frac{dX(t)^2}{dt}\bigg|\sigma_0\right] \equiv \text{ normal level of volatility.}$$

Note that the limit over time t indicates that process has begun in the infinite past and, therefore, the volatility process is in the stationary state.

Our O-U volatility process appears to be an adequate candidate for describing this effect. Let us show this. From Eq. (6.7) and taking in to account the independence of $\sigma(t)$ and $dW_1(t)$ and that $E\left[dW_1^2\right] = dt$, we get

$$\lim_{t\to\infty} E\left[dX(t)^2|\sigma_0\right] = \lim_{t\to\infty} E\left[\sigma^2(t)\big|\sigma_0\right]dt,$$

but the limit $t \to \infty$ indicates that volatility has reached the stationary state (see Section 6.2). Hence, the second moment of sigma is given by Eq. (6.10), and therefore

$$\text{O-U normal level of volatility} = \theta^2 + \frac{k^2}{2\alpha}. \tag{6.19}$$

Observe that the O-U volatility has a constant and non zero normal level of volatility and this is in accordance with the observed mean-reverting property mentioned above.

Moreover, the average over σ^2 given by Eq. (6.8), when the volatility is not yet in the stationary state, is

$$E\left[\sigma^2(t)|\sigma_0, 0\right] = \left[\sigma_0 e^{-\alpha t} + \theta(1 - e^{-\alpha t})\right]^2 + \frac{k^2}{2\alpha}\left(1 - e^{-2\alpha t}\right).$$

We observe that this average quickly tends to the normal level (6.19) as αt increases. This is the reason why the magnitude of α allows us to classify the SV models into: (i) *fast mean-reverting* processes when $1/\alpha \ll t$, and (ii) *slow mean-reverting* processes when $1/\alpha \gg t$.

6.3 The leverage effect

It is commonly known that positive or negative sudden changes in the return have not the same impact on the volatility. Black (1976) was the first to find empirical evidence on this and observed that the volatility is negatively correlated with return variations. A qualitative explanation of this effect is that a fall in the stock prices implies an increase of the *leverage*[4] of companies, which in turn entails more uncertainty and hence higher volatility. Nevertheless, it has also been argued that the leverage alone is too small to explain empirical asymmetries in prices (Ghysels *et al.* (1996)). Another possible explanation is that news on any increase of volatility reduce the demand of stock shares because of investor's risk aversion. The consequent decline in stock prices is followed by an increment of the volatility as initially forecasted by news, and so on (Ghysels *et al.* (1996)).

Although the mechanism still lacks of a clear explanation, the *leverage effect* denomination indicates this negative correlation. To our knowledge, the leverage effect has only been studied in a qualitative manner[5] until very recently when Bouchaud *et al.* (2001) have performed a complete empirical analysis containing new important information on this issue.

Following Bouchaud *et al.* (2001) we quantify the leverage effect by means of the *leverage correlation function* that we define in the form

$$\mathcal{L}(\tau) \equiv \frac{E\left[dX(t+\tau)^2 dX(t)\right]}{\mathrm{Var}[dX(t)]^2}, \tag{6.20}$$

where $X(t)$ is the zero-mean return defined in Eq. (6.6). Bouchaud *et al.* (2001) have analyzed a large amount of daily relative changes for either market indices and

[4]Leverage is the use of credit to enhance speculative capacity.

[5]See, for instance, Ghysels *et al.* (1996) and Engle and Patton (2001).

stock share prices and find that

$$\mathcal{L}(\tau) = \begin{cases} -Ae^{-b\tau} & \text{if } \tau > 0, \\ 0 & \text{if } \tau < 0, \end{cases} \tag{6.21}$$

$(A > 0)$. Hence, there is a negative correlation with an exponential time decay between future volatility and past returns changes but no correlation is found between past volatility and future price changes. In this way, they provide a sort of causality to the leverage effect which, to our knowledge, has never been previously mentioned in the literature (see for instance Fouque (2000b) or Bekaert and Wu (2000)).

Let us see how our correlated O-U SV model is able to exactly reproduce this result. In effect, in the Appendix A we show that

$$\mathrm{Var}[dX(t)] = \theta^2(1+\nu^2)dt \tag{6.22}$$

where

$$\nu^2 \equiv k^2/(2\alpha\theta^2), \tag{6.23}$$

is the intensity of volatility fluctuations compared to the expected volatility (see Eq. (6.10)). On the other hand we prove in Appendix A that

$$E\left[dX(t+\tau)^2 dX(t)\right] = \begin{cases} 2\rho k\ e^{-\alpha\tau}\theta^2\left[1+\nu^2 e^{-\alpha\tau}\right]dt^2 & \text{if } \tau > 0, \\ 0 & \text{if } \tau < 0. \end{cases} \tag{6.24}$$

Hence the leverage correlation function (6.20) is

$$\mathcal{L}(\tau) = 2\rho\left[\frac{\nu\sqrt{2\alpha}\,(1+\nu^2 e^{-\alpha\tau})}{(1+\nu^2)^2\theta}\right]e^{-\alpha\tau} \qquad \text{for } \tau > 0, \tag{6.25}$$

and

$$\mathcal{L}(\tau) = 0 \qquad \text{for } \tau < 0. \tag{6.26}$$

We observe that sign of the leverage function is solely determined by the sign of ρ. Therefore, the O-U SV model is able to reproduce the empirically observed leverage correlation. In Fig. 6.1 we show the leverage effect for the Dow-Jones index (1900-2000) and plot our O-U SV leverage function given by Eq. (6.25).

6.4 Forecast evaluation

Any acceptable market model is required to "forecast" the dynamics of the market. In other words, the model must be able to reproduce the market behavior and have an easy and systematic methodology for estimating its parameters. In our model these parameters are ρ, k, θ, and α [6]. Fouque *et al.* (2000a) for the same model only

[6] In all SV models there are usually the same number of parameters and they can be estimated in a similar way as follows.

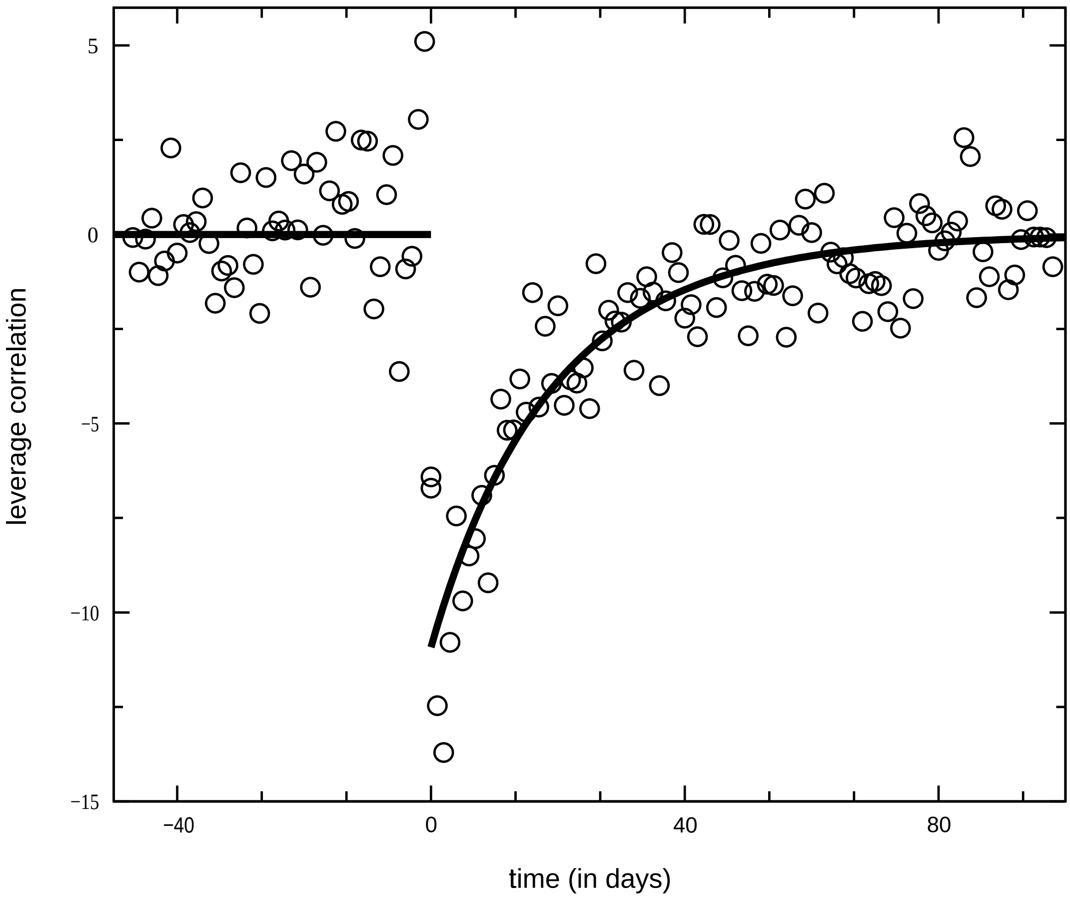


Figure 6.1: The leverage effect in the Dow-Jones index

We plot the leverage function $\mathcal{L}(\tau)$ for the Dow-Jones index from 1900 until 2000. We see that there exists a non-negligible correlation when $\tau > 0$ and negligible when $\tau < 0$. Observe that correlation strongly fluctuates when $-3 < \tau < 2$. We also plot a fit in solid lines with our O-U SV leverage function (6.25). This fit helps us to estimate α and ρ (see Section 6.4 and Table 6.2 for more details).

estimate two of these parameters (k and θ) from the empirical second and fourth moment of daily normalized stock changes. Unfortunately they cannot give a clear estimation of the volatility auto-correlation time $1/\alpha$ and of the magnitude of ρ although it was already known that ρ has to be negative in order to reproduce the desired skewness. Let us now see that with the help of the leverage effect we can completely estimate all parameters of the model.

We will perform the empirical estimation of the Dow-Jones index daily return changes by approximating dX by ΔX, *i.e.*,

$$dX(t) \simeq X(t + \Delta t) - X(t),$$

where $\Delta t = 1$ day. From Eqs. (6.22) and (A.7) of Appendix A, we have

$$\mathrm{Var}[\Delta X] = \theta^2(1+\nu^2)\Delta t, \qquad \mathrm{Var}[\Delta X^2] = 2\theta^4\left[4(1+\nu^2)^2 - 3\right]\Delta t^2,$$

where $\nu^2 = k^2/(2\alpha\theta^2)$. We take the quotient of the variances and obtain

$$\frac{1}{(1+\nu^2)^2} = \frac{4}{3} - \frac{1}{6}\frac{\mathrm{Var}[\Delta X^2]}{\mathrm{Var}[\Delta X]^2}. \tag{6.27}$$

In this way, we are able to estimate the value of ν^2 once we know the empirical values of these variances. The Dow-Jones index proportionates the daily variances of ΔX and ΔX^2 and, subsequently, gives an estimated value for ν^2. Afterwards, θ is estimated with the knowledge of ν^2 and the empirical $\mathrm{Var}[\Delta X]$. In Table 6.1, we briefly report these operations and give the corresponding estimation of ν^2 and θ^2 for the Dow-Jones index time-series from 1900 until 2000.

Since $(1+\nu^2)^2$ is always positive and $(1+\nu^2)^2 \geq 1$, then from Eq. (6.27) we see that

$$0 < \frac{4}{3} - \frac{1}{6}\frac{\mathrm{Var}[\Delta X^2]}{\mathrm{Var}[\Delta X]^2} \leq 1$$

which is equivalent to

$$2 \leq \frac{\mathrm{Var}[\Delta X^2]}{\mathrm{Var}[\Delta X]^2} < 8.$$

Therefore, the kurtosis of our model, $\gamma_2 = \mathrm{Var}[\Delta X^2]/\mathrm{Var}[\Delta X]^2 - 2$, has the bounds

$$0 \leq \gamma_2 < 6, \tag{6.28}$$

which shows that the model is never platykurtic. For the Dow-Jones index $\gamma_2 = 1.72$ and it is therefore consistent with requirement (6.28). However, as Cont (2001) reports, there exists other markets or even intraday tick data possessing a higher kurtosis outside inequality (6.28).

The leverage effect provides the way for estimating the correlation coefficient ρ and the characteristic time $1/\alpha$. Indeed, the best fit of the leverage function (6.25)

Table 6.1: The O-U SV estimation from the return variances

We estimate the parameters of our model from Dow-Jones historical daily returns from 1900 to 2000. We take the variances given by Eq. (6.22) and use the identity (6.27) for deriving the estimated quantities ν^2 and θ.

Estimators	Dow-Jones daily return data	Theoretical values
$\mathrm{Var}[\Delta X(t)]$	1.68×10^{-4}	$\theta^2\ (1+\nu^2)\Delta t$
$\mathrm{Var}\left[\Delta X(t)^2\right]$	10.5×10^{-8}	$2\theta^4\left[4(\nu^2+1)^2-3\right]\Delta t^2$
Parameter estimation	$\nu^2 = 0.18$	
	$\theta = 18.9\%\ \mathrm{year}^{-1/2}$	

Table 6.2: The O-U SV estimation from the leverage

We estimate the parameters ρ, α and k from the fit of the leverage correlation derived from the Dow-Jones stock index data plotted in Fig. 6.1. For doing this, we take the ν^2 estimation given by Table 6.1 and assume that the leverage function is given by Eq. (6.29). Observe that magnitudes $\mathcal{L}(0^+)$ and α estimated from the Dow-Jones index are of the same order as those given by Bouchaud *et al.* (2001) for a combination of several stock indices.

Estimators	Dow-Jones data estimation
$\mathcal{L}(0^+)$	-12.5
α	$0.05\ \mathrm{day}^{-1}$
$1/\alpha$	19.6 days
ρ	-0.58
$k=\sqrt{2\alpha\nu^2\theta^2}$	$1.4 \times 10^{-3}\mathrm{days}^{-1}$

to the Dow-Jones daily data (Fig. 6.1) gives a characteristic time decay $1/\alpha \simeq 20$ days. We thus compare the empirical leverage with our theoretical leverage (6.25) when $\tau \to 0^+$,

$$\mathcal{L}(0^+) = 2\rho \frac{\nu\sqrt{2\alpha}}{(1+\nu^2)\theta}, \tag{6.29}$$

and get an estimation of ρ once we know $\alpha = 0.04$ days^{-1} and $\nu^2 = 0.18$. Finally, we can derive k from the definition given by Eq. (6.23) (see Table 6.1). All these operations are summarized in Table 6.2.

In Fig. 6.2 we simulate the O-U SV resulting process with the parameters estimated above. We follow the random dynamics for $\Delta X(t)$ and compare it with the empirical Dow-Jones time series during approximately one trading year. We there see that our model describes a very similar trajectory than that of the Dow-Jones. This is quite remarkable, because we have simulated last year's trajectory using all past data of the Dow-Jones index with almost equal results to the actual case. This, in turn, shows the stability of parameters. We may thus say that the model is fairly useful for "predicting" the stock dynamics (at least for one year period) using market history.

6.5 The probability distribution

We will now obtain the probability distribution of the model. This problem has been recently addressed by Schöbel and Zhu (1999) who, using the Feynmann-Kac functional (see, for instance, Karlin and Taylor (1981)), end up with an expression for the two-dimensional characteristic function of the joint process $(R(t), \sigma(t))$. Here we take a different path that, besides being simpler, allows us to get an analytical expression of the return characteristic function, which has more practical interest than the joint density. The analysis can be done in terms of the return $R(t) = \ln S(t)/S_0$ but we prefer to deal with the zero-mean return $X(t)$ since, although the calculation is basically identical, the expressions derived are shorter and handier.

6.5.1 The characteristic function

Let $p_2(x, \sigma, t|x_0, \sigma_0, t_0)$ be the joint probability density of the two-dimensional diffusion process $(X(t), \sigma(t))$ described by the pair of SDE's given by Eqs. (6.7) and (6.2). This density obeys the following backward Fokker-Planck equation (Gardiner (1983)):

$$\frac{\partial p_2}{\partial t_0} = \alpha(\sigma_0 - \theta)\frac{\partial p_2}{\partial \sigma_0} - \frac{1}{2}\sigma_0^2 \frac{\partial^2 p_2}{\partial x_0^2} - \rho k \sigma_0 \frac{\partial^2 p_2}{\partial \sigma_0 \partial x_0} - \frac{1}{2}k^2 \frac{\partial^2 p_2}{\partial \sigma_0^2}, \tag{6.30}$$

with final condition

$$p_2(x, \sigma, t|x_0, \sigma_0, t) = \delta(x - x_0)\ \delta(\sigma - \sigma_0). \tag{6.31}$$

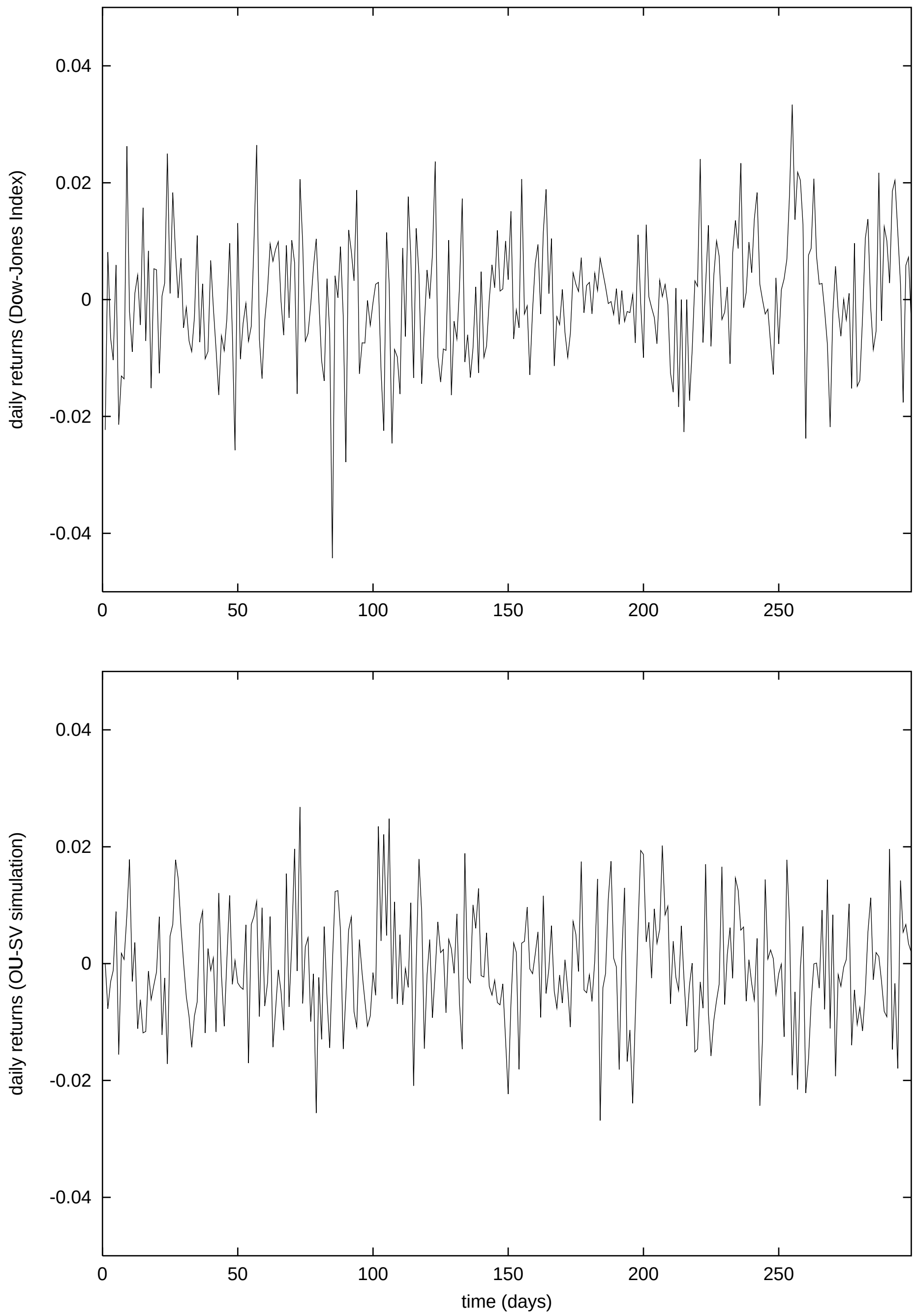


Figure 6.2: Path simulation and Dow-Jones historical time-series

We show a Dow-Jones daily returns sample path and the O-U SV process simulation with the parameters given by Tables 6.1 and 6.2. The dynamics is traced over approximately a trading year (the empirical path approximately corresponds to 1999 trading year).

Before proceeding further we note that, since all coefficients in Eq. (6.30) are independent of x, x_0, t and t_0 and the final condition only depends on the differences $x - x_0$ and $t - t_0$, then $(X(t), \sigma(t))$ is an homogenous process in time and return. Therefore,

$$p_2(x, \sigma, t|x_0, \sigma_0, t_0) = p_2(x - x_0, \sigma, t - t_0|\sigma_0). \tag{6.32}$$

Moreover, one easily sees that the marginal density of the return,

$$p_X(x - x_0, t - t_0|\sigma_0) = \int_{-\infty}^{\infty} p_2(x - x_0, \sigma, t - t_0|\sigma_0)d\sigma,$$

also obeys the same partial differential equation than p_2, Eq. (6.30), which due to homogeneity can be written in the form[7]

$$\frac{\partial p_X}{\partial t} = -\alpha(\sigma_0 - \theta)\frac{\partial p_X}{\partial \sigma_0} + \frac{1}{2}\sigma_0^2\frac{\partial^2 p_X}{\partial x^2} - \rho k \sigma_0 \frac{\partial^2 p_X}{\partial \sigma_0 \partial x} + \frac{1}{2}k^2\frac{\partial^2 p_X}{\partial \sigma_0^2}, \tag{6.33}$$

where $p_X = p_X(x - x_0, t - t_0|\sigma_0)$ and the initial condition is

$$p_X(x - x_0, 0|\sigma_0) = \delta(x - x_0).$$

Partial differential equation (6.33) is the starting point of our analysis. Again, due to homogeneity we can assume, without loss of generality that $x_0 = 0$ and $t_0 = 0$. Observe that Eq. (6.33) is quite involved because of the correlation between the volatility and the return, *i.e.*, because of the crossed derivative term. Therefore, there seems to be a tremendous task if one tries to attack the problem directly from Eq. (6.33). However, Fourier analysis proportionates the necessary tools to obtain an analytic solution of the problem. Solution that is expressed in terms of the return characteristic function (cf) defined by

$$\varphi_X(\omega, t|\sigma_0) = \int_{-\infty}^{\infty} e^{i\omega x} p_X(x, t|\sigma_0)dx.$$

This is done in Appendix B where we prove that

$$\varphi_X(\omega, t|\sigma_0) = \exp[-A(\omega, t)\sigma_0^2 - B(\omega, t)\sigma_0 - C(\omega, t)], \tag{6.34}$$

where

$$A(\omega, t) = \frac{\omega^2}{2}\left(\frac{\sinh \eta t}{\eta \cosh \eta t + \zeta \sinh \eta t}\right), \tag{6.35}$$

$$B(\omega, t) = \frac{\omega^2 \alpha\theta}{\eta}\left(\frac{\cosh \eta t - 1}{\eta \cosh \eta t + \zeta \sinh \eta t}\right), \tag{6.36}$$

$$C(\omega, t) = \left[\frac{(\omega\alpha\theta)^2}{\eta^2} + i\omega\rho k - \alpha\right] t/2 + \frac{1}{2}\ln\left(\cosh \eta t + \frac{\zeta}{\eta}\sinh \eta t\right) - \frac{(\omega\alpha\theta)^2}{2\eta^3}\left[\frac{2\zeta(\cosh \eta t - 1) + \eta \sinh \eta t}{\eta \cosh \eta t + \zeta \sinh \eta t}\right], \tag{6.37}$$

[7]Note that Eq. (6.30) is the backward Fokker-Planck equation while Eq. (6.33) is a forward equation for x and t but not for σ.

and

$$\eta = \sqrt{\alpha^2 - 2i\rho k\alpha\omega + (1-\rho^2)k^2\omega^2}, \qquad \zeta = \alpha - i\omega\rho k. \tag{6.38}$$

Furthermore, we obtain the unconditional characteristic function, $\varphi_X(\omega, t)$, if we average over σ_0 which we assume is in the stationary regime. We thus write

$$\varphi_X(\omega, t) = \int \varphi_X(\omega, t|\sigma_0) p_\sigma(\sigma_0) d\sigma_0 \tag{6.39}$$

with the stationary pdf[8]

$$p_\sigma(\sigma) = \frac{1}{\sqrt{\pi k^2/\alpha}} \exp\left[-\frac{(\sigma-\theta)^2}{k^2/\alpha}\right]. \tag{6.40}$$

Then from Eqs. (6.34) and (6.39)–(6.40), we get

$$\varphi_X(\omega, t) = \frac{1}{\sqrt{1 + k^2 A/\alpha}} \exp\left[-C + \frac{B^2 k^2/\alpha - 4\theta B - 4\theta^2 A}{4(1 + k^2 A/\alpha)}\right]. \tag{6.41}$$

Note that this solution has the right limit when volatility is constant and non-random. Indeed, in such a case $k = 0$ and from Eqs. (6.35)–(6.37) and (6.41) we have

$$\varphi_X(\omega, t) = e^{-\omega^2\theta^2 t/2}, \tag{6.42}$$

which is the cf of the zero-mean return when $X(t)$ follows a one-dimensional diffusive process with constant volatility $\sigma \equiv \theta$ (see Eq. (A.1)). Hence, solution (6.41) appears to be consistent with the geometric Brownian motion model studied in Chapter 4.

6.5.2 Convergence to the Gaussian distribution

Let us see that, as $t \to \infty$, our marginal distribution approaches to the Gaussian density under certain circumstances. In other words, we will prove a Central Limit Theorem for the model. The starting point of this analysis is Eq. (6.41) that as $\alpha t \gg 1$ can be written in the following simpler form

$$\varphi_X(\omega, t) \sim \exp\left[-\left(\frac{\omega^2\alpha^2\theta^2}{\eta^2} + i\omega\rho k - \alpha + \eta\right) t/2\right], \qquad (\alpha t \gg 1). \tag{6.43}$$

This is not a Gaussian distribution yet, since η defined by Eq. (6.38) is an irrational function of ω. We have to assume an extra requirement. Specifically, we suppose that

$$\frac{k}{\alpha} \ll 1, \tag{6.44}$$

[8] In Chapter 5, we show that this is the stationary pdf for an O-U process.

which means that volatility is weakly random. Indeed, k is the strength of the volatility driving noise[9] while α tell us how large is its deterministic drift (see Eq. (6.2)). Therefore the ratio k/α measures in some way the degree of volatility randomness. Taking into account Eq. (6.44) we write (*cf.* Eq. (6.38))

$$\eta = \alpha \left[1 - i\omega\rho\frac{k}{\alpha} + \frac{1}{2}\omega^2\frac{k^2}{\alpha^2} + \mathrm{O}\left(\frac{k^3}{\alpha^3}\right)\right].$$

Substituting this into Eq. (6.43) yields

$$\varphi_X(\omega,t) \sim \exp\left\{-\omega^2\left[1+\nu^2+\mathrm{O}(k/\alpha)\right]\theta^2 t/2\right\}, \qquad (\alpha t \gg 1), \tag{6.45}$$

where $\nu^2 \equiv k^2/2\alpha\theta^2$. The Gaussian density (6.45) proves the Central Limit Theorem in our case.

6.5.3 Cumulants

Cumulants are defined as follows (Kendall (1987))

$$\kappa_n \equiv (-i)^n \left.\frac{\partial^n}{\partial\omega^n}\ln[\varphi_X(\omega,t)]\right|_{\omega=0},$$

and are very useful for deriving statistical properties of the model. For instance, the second cumulant reads

$$\kappa_2 = \theta^2\left(1+\nu^2\right)t, \tag{6.46}$$

which results to be the integrated variance of $dX(t)$ (*cf.* Eq. (A.7)). The third and fourth cumulants are respectively

$$\kappa_3 = 3\rho\frac{\theta^2 k}{\alpha^2}\left\{2\left[\alpha t - \left(1-e^{-\alpha t}\right)\right] + \frac{\nu^2}{2}\left[2\alpha t - \left(1-e^{-2\alpha t}\right)\right]\right\}, \tag{6.47}$$

and

$$\begin{aligned}\kappa_4 \;=\;& k^2\theta^2\frac{3}{2\alpha^3}\Big\{4\Big[2\alpha t + \alpha t\rho^2\left(6+4e^{-\alpha t}\right) - \left(2+12\rho^2\right)\left(1-e^{-\alpha t}\right) + \rho^2\left(1-e^{-2\alpha t}\right)\Big] \\ &+ \nu^2\left[2\alpha t + 8\alpha t\rho^2\left(1+e^{-2\alpha t}\right) - \left(1-e^{-2\alpha t}\right)\left(1+8\rho^2\right)\right]\Big\}.\end{aligned} \tag{6.48}$$

In terms of cumulants kurtosis is given by

$$\gamma_2 \equiv \frac{\kappa_4}{\kappa_2^2}, \tag{6.49}$$

[9] k is often known as the volatility of volatility (vol-vol).

and measures the tails of the distribution compared to the Gaussian distribution. Kurtosis is zero for a Gaussian distribution, negative for a distribution with tails decaying faster than the Gaussian (platykurtic distribution), and positive for a distribution with tails decaying slower (leptokurtic distribution). Typically, markets have positive kurtosis even reaching $\kappa_4 = 50$ in some cases which indicates a very extreme non normality (Cont (2001)).

The *kurtosis* (6.49) of our model can be derived taking into account the cumulants given by Eqs. (6.46) and (6.48). The resulting expression is very similar to that of the fourth cumulant (6.48) but with an extra constant factor. Its asymptotic limits are rather simple to derive and read

$$\gamma_2 \sim \frac{6\nu^2(\nu^2+2)}{(\nu^2+1)^2} \qquad (\alpha t \ll 1), \tag{6.50}$$

and

$$\gamma_2 \sim \frac{6\nu^2[\nu^2(1+4\rho^2)+4(1+\rho^2)]}{(\nu^2+1)^2}\,\frac{1}{\alpha t} \qquad (\alpha t \gg 1). \tag{6.51}$$

¿From Eq. (6.50), we observe that even dealing with an infinitesimal time there will exist a non negligible kurtosis. Conversely, from Eq. (6.51) we see that kurtosis goes to zero as time increases and that the convergence is slow going as $1/t$. In addition, we observe that for short times kurtosis does not contain the correlation ρ^2 but in the long run a non zero ρ^2 magnifies the kurtosis of the distribution (*cf.* Eqs. (6.50)–(6.51)).

Let us now turn our attention to skewness γ_1 defined in terms of third and second cumulant as

$$\gamma_1 \equiv \frac{\kappa_3}{\kappa_2^{3/2}} \tag{6.52}$$

and that quantifies the byass in the return distribution. A negative skewness indicates that returns are more likely to decrease than to increase, while a positive skewness indicates a higher probability of a return raising than a decline. Empirical observations have found that financial markets have an slightly negative skewness (Cont (2001)).

Similarly to the kurtosis derivation, the *skewness* defined by Eq. (6.52) is obtained with Eqs. (6.46)–(6.47) and the resulting expression is very similar to that of Eq. (6.47) but with an extra constant factor. We limit ourselves to the asymptotic cases

$$\gamma_1 \sim 3\rho\,\frac{\nu}{\sqrt{\nu^2+1}}\,\sqrt{2\alpha t} \qquad (\text{for } \alpha t \ll 1), \tag{6.53}$$

and

$$\gamma_1 \sim 6\rho\,\frac{\nu(\nu^2+2)}{(\nu^2+1)^{3/2}}\,\frac{1}{\sqrt{2\alpha t}} \qquad (\text{for } \alpha t \gg 1). \tag{6.54}$$

¿From these equations we see that both at short and long times, skewness vanishes. However, it decreases very slowly, more slowly than kurtosis (compare Eqs. (6.51) and (6.54)). Finally, we note that skewness is proportional to ρ and, in consequence, the sign of ρ not only determines the leverage correlation sign but also the skewness sign. Empirical observations of leverage and skewness indicate that ρ must be negative (Cont (2001)).

6.5.4 Tails

It is well established that distribution of prices have heavy tails. There exists several empirical studies quantifying this fact (see for instance Mantegna and Stanley (1995) or Plerou *et al.* (1999)). Let us now study the existence of fat tails in our SV model. Recall first that for long times, $\alpha t \gg 1$, the probability distribution is practically Gaussian and there is no fat tail to look for. Therefore, we will search for heavy tails at small to moderate times, *i.e.*, when Central Limit Theorem is not applicable.

The tails of the distribution are determined by the shape of the density function $p_X(x,t)$ when x is large. A well known fact from Fourier analysis is that the large x behavior of $p_X(x,t)$ is given by the small ω behavior of its characteristic function $\varphi_X(\omega,t)$ (Weiss (1994)). Therefore, tails are derived from the characteristic function (6.41) by taking the first two orders in the limit when ω is small.

When ω is small but time is not too long (*i.e.*, $\alpha t \sim 1$) the expressions for $A(\omega,t)$, $B(\omega,t)$ and $C(\omega,t)$ are approximately given by (*cf.* Eqs. (6.35)–(6.37))

$$A(\omega,t) \sim \frac{\omega^2}{4\alpha}(1-e^{-2\alpha t}), \qquad B(\omega,t) \sim \frac{\theta\omega^2}{2\alpha}(1-e^{-\alpha t})^2,$$

and

$$C(\omega,t) \sim (\omega^2\theta^2 + i\omega\rho k - \alpha)t/2 - (\theta^2/4\alpha)[2(1-e^{-\alpha t})^2 + 1 - e^{-2\alpha t}]\omega^2 + \frac{1}{2}\ln[1 - i\rho k t\omega + (k^2/4\alpha^2)(2\alpha t - 2\alpha^2 t^2\rho^2 - 1 + e^{-2\alpha t})\omega^2].$$

Thus from Eq. (6.41) we have

$$\varphi_X(\omega,t) \sim \frac{[1+(k^2/4\alpha^2)(1-e^{-2\alpha t})]^{-1/2}\exp(-\omega^2\theta^2 t/2 - i\omega\rho k t/2)}{[1 - i\rho k t\omega + (k^2/4\alpha^2)(2\alpha t - 2\alpha^2 t^2\rho^2 - 1 + e^{-2\alpha t})\omega^2]^{1/2}}.$$

Again, taking into account that ω is small and t is moderate we get

$$\varphi_X(\omega,t) \sim \frac{1}{1 - 2ia(t)\omega + b(t)\omega^2}, \qquad (\omega \to 0), \tag{6.55}$$

where

$$a(t) \equiv \rho k t/4, \qquad \text{and} \qquad b(t) \equiv k^2 t(2-\rho^2\alpha t)/8\alpha. \tag{6.56}$$

The inverse Fourier transform for this asymptotic cf is

$$p_X(x,t) \sim \frac{1}{\sqrt{a(t)^2+b(t)}}\exp\left[-\frac{1}{b(t)}\left(\sqrt{a(t)^2+b(t)}\,|x| - a(t)x\right)\right], \quad (|x|\to\infty).$$

Hence the tails of the zero-mean return have an asymmetric exponential decay given by

$$p_X(x,t) \sim \frac{1}{\sqrt{a(t)^2+b(t)}} \exp\left[-\frac{1}{b(t)}\left(\sqrt{a(t)^2+b(t)}-a(t)\right)x\right] \quad (x \to \infty), \quad (6.57)$$

and

$$p_X(x,t) \sim \frac{1}{\sqrt{a(t)^2+b(t)}} \exp\left[\frac{1}{b(t)}\left(\sqrt{a(t)^2+b(t)}+a(t)\right)x\right], \quad (x \to -\infty). \quad (6.58)$$

Since $a(t) = \rho k t/4$, we see that the sign of ρ will determine which is the fattest tail. When ρ is negative the fattest tail is the one representing losses and when $\rho > 0$ the fattest tail corresponds to profits. If $\rho = 0$ there is no difference between the two tails.

Finally, let us guess how the tails of price (not return) distribution are. We first recall that the asymptotic expressions (6.57)–(6.58) refer to the *marginal* distribution of $X(t)$, that is, regardless the value of volatility at time t and after averaging over the initial volatility. In order to obtain the asymptotic form of price distribution $p_S(S,t)$ out of the asymptotic form of $p_X(x,t)$ we must know what is the relation between $S(t)$ and $X(t)$. For the general case this relation is given by Eq. (A.1) of Appendix A and, since it corresponds to the two-dimensional case when no average and marginal distribution have been performed, Eq. (A.1) involves $X(t)$, $S(t)$ and $\sigma(t)$. We conjecture that if time is not too large $X(t) \sim \ln[S(t)/S_0]$. Therefore, for the price distribution $p_S(S,t)$ we have the following power laws:

$$p_S(S,t) \sim \frac{1}{S^{\nu_-(t)}} \quad (S \to 0), \qquad \text{and} \qquad p_S(S,t) \sim \frac{1}{S^{\nu_+(t)}} \quad (S \to \infty) \qquad (6.59)$$

where

$$\nu_\pm(t) = 1 + \frac{1}{b(t)}\left[\pm\sqrt{a(t)^2+b(t)} - a(t)\right].$$

Summary

The stochastic volatility (SV) models are a possible way out to the observed inconsistencies between the geometric Brownian model and real markets. The SV models, as their name indicates, assume the original log-Brownian model but with the volatility σ being random. We have assumed that the volatility follows one of an Ornstein-Uhlenbeck process that also allows for correlations between the random fluctuations of the price and volatility. With this model we have explained in a quantitative way the leverage effect and other stylized facts, such as mean reversion, leptokurtosis and skewness. We have also estimated all parameters of our model and observed that a simulated path (using the estimated parameters) is very similar to the sample path of the historical evolution of the Dow-Jones daily index (1900-2000). Finally, we have obtained a close analytical expression for the characteristic function and studied the heavy tails of the probability distribution.

Appendix A. The zero-mean return

The zero-mean return has been defined through its differential dX by Eq. (6.6). Let us first prove that $X(t)$ is explicitly given by

$$X(t) = \ln[S(t)/S_0] - \mu t - \frac{1}{2}\int_0^t \sigma^2(t')dt'. \tag{A.1}$$

Indeed, if we apply the Itô lemma to Eq. (A.1) and, as usual, keep orders smaller than $dt^{3/2}$ we have

$$dX(t) = dS/S + \frac{1}{2}(dS/S)^2 - \mu dt - \frac{1}{2}\sigma^2 dt,$$

but $dS/S = \mu dt + \sigma dW_1$ and $(dS/S)^2 = \sigma^2 dt$, then we obtain Eq. (6.7):

$$dX = \sigma dW_1,$$

and this proves the validity of Eq. (A.1).

We will now derive several averages concerning dX. We know from Eq. (6.18) that

$$E\left[dX(t)\right] = 0. \tag{A.2}$$

Again taking into account the independence of $\sigma(t)$ and $dW_1(t)$ we write

$$E\left[dX^2\right] = E\left[\sigma^2\right] E\left[dW_1(t)^2\right], \tag{A.3}$$

but $E\left[dW_1^2\right] = dt$, and using Eq. (6.10) we have

$$E\left[dX^2\right] = \left(\theta^2 + k^2/2\alpha\right) dt. \tag{A.4}$$

As to the fourth moment,

$$E\left[dX^4\right] = E\left[\sigma^4\right] E\left[dW_1^4\right],$$

we note that $E\left[dW_1^4\right] = 3E\left[dW_1^2\right]^2 = 3dt^2$ and we evaluate $E\left[\sigma^4\right]$ using the stationary pdf (6.40). Hence

$$E\left[dX^4\right] = 3\left[3(k^2/2\alpha)^2 + 6(k^2/2\alpha)\theta^2 + \theta^4\right] dt^2. \tag{A.5}$$

The variances of dX^2 and dX are obtained from Eqs. (A.4) and (A.5) and read

$$\begin{aligned} \mathrm{Var}[dX(t)] &= \left(\theta^2 + k^2/2\alpha\right) dt, && \text{(A.6)} \\ \mathrm{Var}[dX(t)^2] &= 2\left[\theta^4 + 4\left(k^2/2\alpha\right)^2 + 8\left(k^2/2\alpha\right)\theta^2\right] dt^2. && \text{(A.7)} \end{aligned}$$

We finally derive the following correlation function:

$$E\left[dX(t+\tau)^2 dX(t)\right] = E\left[\sigma(t)dW_1(t)\sigma(t+\tau)^2 dW_1(t+\tau)^2\right].$$

Note that all variables are Gaussian which allows us to decompose the rhs of this equation into average pairs, taking also into account that $dW_1(t+\tau)^2 = dt$ we can write

$$\begin{aligned} E\left[dX(t+\tau)^2 dX(t)\right] = \Big\{ & 2E\left[\sigma(t)\sigma(t+\tau)\right] E\left[\sigma(t+\tau)dW_1(t)\right] \\ & + E\left[\sigma(t+\tau)^2\right] E\left[\sigma(t)dW_1(t)\right] \Big\} dt \end{aligned}$$

Combining this with Eq. (6.16), we get

$$E\left[dX(t+\tau)^2 dX(t)\right] = \begin{cases} 2\rho k e^{-\alpha\tau} E\left[\sigma(t)\sigma(t+\tau)\right] dt^2 & \text{if } \tau > 0, \\ 0 & \text{if } \tau < 0; \end{cases}$$

and the volatility correlation (6.11) allows us to write

$$E\left[dX(t+\tau)^2 dX(t)\right] = \begin{cases} 2\rho k\ e^{-\alpha\tau}\left[\theta^2 + (k^2/2\alpha)e^{-\alpha\tau}\right] dt^2 & \text{if } \tau > 0, \\ 0 & \text{if } \tau < 0. \end{cases} \qquad \text{(A.8)}$$

Appendix B. The marginal characteristic function

In this Appendix we will obtain the expression given by Eq. (6.34) for the marginal characteristic function $\varphi_X(\omega, t|\sigma_0)$ of the two-dimensional diffusion process $(X(t), \sigma(t))$ whose joint density $p_2(x, \sigma, t|x_0, \sigma_0, t_0)$ is the solution to the final problem posed by Eqs. (6.30)–(6.31).

The marginal characteristic function (cf) of process $X(t)$,

$$\varphi_X(\omega, t|\sigma_0) = \int_{-\infty}^{\infty} e^{i\omega x} p_X(x, t|\sigma_0) dx,$$

allows us to write Eq. (6.33) in the following simpler form:

$$\frac{\partial \varphi_X}{\partial t} = \frac{1}{2}k^2 \frac{\partial^2 \varphi_X}{\partial \sigma_0^2} + \left[i\omega\rho k\sigma_0 - \alpha(\sigma_0 - \theta)\right]\frac{\partial \varphi_X}{\partial \sigma_0} - \frac{1}{2}\sigma_0^2\omega^2\varphi_X. \qquad \text{(B.1)}$$

The initial condition is

$$\varphi_X(\omega, 0|\sigma_0) = 1. \tag{B.2}$$

By direct inspection one can easily see that the solution to problem (B.1)–(B.2) is

$$\varphi_X(\omega, t|\sigma_0) = \exp[-A(\omega,t)\sigma_0^2 - B(\omega,t)\sigma_0 - C(\omega,t)] \tag{B.3}$$

where functions $A(\omega,t), B(\omega,t)$ and $C(\omega,t)$ are the solution of the following set of ordinary differential equations

$$\dot{A} = -2k^2A^2 - 2(\alpha - i\omega\rho k)A + \omega^2/2 \tag{B.4}$$

$$\dot{B} = -\left[2k^2A + (\alpha - i\omega\rho k)\right]B + 2\alpha\theta A \tag{B.5}$$

$$\dot{C} = k^2(A - B^2/2) + \alpha\theta B, \tag{B.6}$$

with initial conditions

$$A(\omega,0) = B(\omega,0) = C(\omega,0) = 0. \tag{B.7}$$

Note that Eq. (B.5) is a linear equation and that the rhs of Eq. (B.6) does not involve $C(t)$. Therefore, their formal solutions are straightforward and read

$$B(t) = 2\alpha\theta\int_0^t A(t')\exp\left[-(\alpha - i\omega\rho k)(t-t') - 2k^2\int_{t'}^t A(t'')dt''\right]dt', \tag{B.8}$$

$$C(t) = k^2\int_0^t \left[A(t') - B^2(t')/2\right]dt' + \alpha\theta\int_0^t B(t')dt'. \tag{B.9}$$

On the other hand Eq. (B.4) is more involved since it is a Ricatti equation. However, the definition of a new dependent variable

$$A = \frac{\dot{y}}{2k^2y} \tag{B.10}$$

turns Eq. (B.4) into the following linear second-order equation with constant coefficients:

$$\ddot{y} + 2(\alpha - i\omega\rho k)\dot{y} - k^2\omega^2 y = 0.$$

The solution to this equation is

$$y(t) = C_1e^{\lambda_+t} + C_2e^{\lambda_-t},$$

where $C_{1,2}$ are arbitrary constants and

$$\lambda_\pm = \alpha - i\omega\rho k \pm \sqrt{(\alpha - i\omega\rho k)^2 + k^2\omega^2}.$$

Substituting this into Eq. (B.10) yields

$$A(t) = \frac{\lambda_+e^{\lambda_+t} + \lambda_-(C_2/C_1)e^{\lambda_-t}}{2k^2[e^{\lambda_+t} + (C_2/C_1)e^{\lambda_-t}]}.$$

Now the initial condition $A(0) = 0$ gives $C_2/C_1 = -\lambda_+/\lambda_-$ and the substitution of $\lambda_\pm$ allows us to write $A(t)$ in the form given by Eq. (6.35). Finally the substitution of Eq. (6.35) into Eqs. (B.8)–(B.9) results in Eqs. (6.36)–(6.37).

References

Bachelier, L., 1900, *Théorie de la spéculation.* English version reprinted in P. H. Cootner, 1965, *The random character of stock market prices* (The M.I.T. Press, Cambridge, Massachussets).

Bekaert, G., G. Wu, 2000, Asymmetric volatility and risk in equity markets, *Review of Financial Studies* **13**, 1-42.

Black, F., 1976, Studies of stock market volatility changes, *Proceedings 1976 of the Meetings of the American Statistical Association, Business and Economic Statistics section*, 177-181.

Bollersev, T., R. Engle, D. Nelson, 1994, ARCH models, in R. Engle and D. McFadden eds: *Handbook of econometrics, vol. IV* (North-Holland, Amsterdam).

Bouchaud, J.-P., A. Matacz, M. Potters, 2001, The leverage effect in financial markets: retarded volatility and market panic, submitted to *Physical Review Letters.*

Clark, P. K., 1973, A subordinated stochastic process model with finite variance for speculative prices, *Econometrica* **41**, 135-156.

Cont, R., 2001, Empirical properties of asset returns: stylized facts and statistical issues, *Quantitative Finance* **1**, 223-236.

Engle, R., 1982, Autorregressive conditional heteroskedasticity with estimates of the variance of UK inflation, *Econometrica* **50**, 987-1002.

Engle, R. and A. Patton, 2001, What good is volatility model?, *Quantitative Finance* **1**, 237-245.

Fouque, J. P., G. Papanicolaou, and K. R. Sircar, 2000a, Mean-reverting stochastic volatility, *International Journal of Theoretical and Applied Finance* **3**, 101-142.

Fouque, J. P., G. Papanicolaou, and R. Sircar, 2000b, *Derivatives in financial markets with stochastic volatility* (Cambridge University Press, Cambridge).

Ghysels, E., A. C. Harvey, E. Renault, 1996, Stochastic volatility, in G. S. Mandala and C. R. Rao eds: *Statistical Methods in Finance. Handbook of Statistics, vol. 14* (North-Holland, Amsterdam, New York).

Grimmet, G. R., D. R. Stirzaker, 1992, *Probability and random processes* (Oxford University Press, Oxford).

Hull, J. C., A. White, 1987, The pricing of options on assets with stochastic volatilities, *The Journal of Finance* **42**, 281-300.

Jackwerth, J., M. Rubinstein, 1996, Recovering probability distributions from contemporaneous security prices, *Journal of Finance* **51**, 1611-1631.

Kendall, M., 1987, *Kendall's advanced theory of statistics. Vol. 1: Distribution theory* (Griffin & Co., London).

Mantegna, R. N., and H. E. Stanley, 1995, Scaling behavior in the dynamics of an economic index, *Nature* **376**, 46-49.

Masoliver, J., and J. Perelló, 2001c, A correlated stochastic volatility model measuring leverage and other stylized facts, submitted for publication.

Osborne, M. F. M., 1959, Brownian motion in the stock market, *Operations Research* **7**, 145-173.

Plerou, V., P. Gopikrishnan, L. N. Amaral, M. Meyer, E. Stanley, 1999, Scaling of the distribution of rice fluctuations of individual companies, *Physical Review E* **60**, 6519-6528.

Schöbel, R., and J. Zhu, 1999, Stochastic volatility with an Ornstein-Uhlenbeck process: An extension, *European Finance Review* **3**, 23-46.

Scott, L., 1987, Option pricing when the variance changes randomly: Theory, estimation and an application, *Journal of Financial and Quantitative Analysis* **22**, 419-438.

Stein, E. M., J. C. Stein, 1991, Stock price distributions with stochastic volatility: An analytic approach, *The Review of Financial Studies* **4**, 727-752.

Weiss, G. H., 1994, *Aspects and applications of the random walk* (Noth-Holland, Amsterdam).

Wiggins, J., 1987, Option values under stochastic volatility, *Journal of Financial Economics* **19**, 351-372.

Chapter 7

Conclusions and perspectives

The year 1995 is considered to be the starting date of physicists bursting research on financial markets. It was then when Mantegna and Stanley[1] published in Nature an statistical analysis of the S&P 500 stock index using large time-series of intra-day data. They concluded that index and price distributions are self-similar and thus follow scaling invariance. Subsequent research by physicists has been mainly focussed on analyzing the statistical properties of prices. Financial markets are very appealing for physicists due to the availability of a huge quantity data which can be easily confronted with proposed models. There is no need of laboratories and a personal computer has all necessary tools to test theories. For this reason, the study of speculative markets has been of great success among physicists and journals of physics has decided to follow a publishing policy accepting papers on this discipline which has been termed "*econophysics*". We can thus say that, nowadays, physics also studies financial markets.

This work represents an introduction to this new topic not only for me but also for the whole research group directed by Prof. Jaume Masoliver. This research group has been typically dedicated to study stochastic dynamics arising in phenomena such as lasers, scattering in disordered media or the kinetics of process arising in chemical physics among many others. In the present case, stochastic methods are applied to financial markets.

Before applying the stochastic methods of physics to finance, we have revisited and studied the work already done on this field by mathematicians. This effort is collected in **Chapter 2** after giving basic definitions in **Chapter 1**. The father of the mathematical finance is considered to be Louis Bachelier since in 1900 he presented his doctoral thesis devoted to study the random behavior of speculative markets. More specifically, the thesis contains a proposed model for the stock, now called the arithmetic Brownian motion, and afterwards gives a price for the

[1]Mantegna, R. N., H. E. Stanley, 1995, Scaling behavior in the dynamics of an economic index, *Nature* **376**, 46-49.

option based on the proposed dynamics. In fact, these two topics still are the two more emblematic problems in mathematical finance. In **Chapter 2** we narrate the development of the mathematical finance discipline during last century and until the path breaking work by Black and Scholes (1973)[2]. We have exposed the main works in a systematic way that allows us to have an idea of main problems involving market modelling and option pricing.

At this point, we have empirically shown that S&P-500 index is, as it is well-known, a geometric instead of an arithmetic process. And we have exemplified the differences between taking stock and return changes and shown its consequences in the data analysis. This constitutes one of our research articles:

- Masoliver, J., M. Montero, J. Perelló, 2001a, Return or stock price differences, submitted for publication.

In **Chapter 3** we have updated the option pricing theory from the physicist's point of view. We have centered our analysis of option pricing to the Black-Scholes equation and formula for the European call, extensions to other kind of options can be straightforward in many cases and are found in several good finance books. We have reviewed Black-Scholes theory using Itô calculus, which is standard to mathematical finance, with a special emphasis in explaining and clarifying the many subtleties of the calculation. Nevertheless, we have not limit ourselves only to review option pricing, but to derive, for the first time to our knowledge, the Black-Scholes equation using the Stratonovich calculus which is standard to physics, thus bridging the gap between mathematical finance and physics.

As we have proved, the Black-Scholes equation obtained using Stratonovich calculus is the same as the one obtained by means of the Itô calculus. In fact, this is the result we expected in advance because Itô and Stratonovich conventions are just different rules of calculus. Moreover, from a practical point of view, both interpretations differ only in the drift term of the Langevin equation and the drift term does not appear in the Black-Scholes equation and formula. But, again, we think that this derivation is still interesting and useful for all the reasons explained above.

We have revisited the Black-Scholes option pricing using the Itô and Stratonovich conventions in the paper:

- Perelló, J., J. M. Porrà, M. Montero, J. Masoliver, 2000, Black-Scholes option pricing within Itô and Stratonovich conventions, *Physica A* **278**, 260-274.

[2]Black, F., and M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* **81**, 637-659.

After Black and Scholes (1973) presented their option pricing theory, there appeared generalizations in two different directions. **Chapter 4** has shown these extensions that still accept to deal with the Black-Scholes ideal conditions. First generalization has asked for the market model that fits the Black-Scholes prescriptions. We have seen that jumps are the only alternative stochastic dynamics to the one initially assumed by Black and Scholes, *i.e.*, the diffusive process. The second generalization wants to simplify the original Black-Scholes derivation. This is done with the Capital Asset Pricing Model (CAPM) theory and with the so-called equivalent martingale measure.

Based on martingale theory, we have presented an *original Fourier analysis* that leads to a useful representation of the European call in terms of the characteristic function of the so-called "zero-mean return". We are now working on applying this representation to underlying models that only information we have is its characteristic function. In this way, we particularize to the Lévy process and to a model already presented by our group[3] and thus observe the consequences in option prices of the existence of *fat tails*. This study is also done assuming a market model with *colored noise* in the way that it is assumed in **Chapters 5** and **6**.

The option price represented in terms of the characteristic function of the return constitutes another research paper:

• Perelló, J., and J. Masoliver, 2001e, The effect of colored noise and heavy tails on financial options, to appear in *Physica A*. Invited talk in: "Horizons in complex systems", Messina, 5-8 December 2001.

The **second part** of this thesis is devoted to relax the Black-Scholes prescriptions on the underlying modelisation. The strong effort on compiling statistical properties of the market has led to a list of *stylized facts* that any proposed market must accomplish. One of the important findings is the existence of time correlations in the prices data series. We have intended to explain these *correlations with two different models*. The first one describes the correlations between log-prices evaluated at distinct times. And the second describes the so-called leverage correlation, *i.e.*, between log-price variations and volatility. Both models are *two-dimensional diffusion processes*. Therefore, extension is done by including a new stochastic differential equation and, in consequence, a new random variable. In **Chapter 5**, this new variable is the *velocity* of the log-price and, in **Chapter 6**, this is the *volatility*.

Chapter 5 has developed option pricing with perfect hedging in an *inefficient*

[3] Masoliver, J., M. Montero and J. M. Porrà, 2000, A dynamical model describing stock market price distributions, *Physica A* **283**, 559-567. Masoliver, J., M. Montero and A. McKane, 2001, Integrated random processes exhibiting long tails, finite moments, and power-law spectra, *Physical Review E* **64**, 011110 (11 pages).

market model. The inefficiency of the market is related to the fact that the underlying price variations are auto correlated over an arbitrary time period τ. In order to take these correlations into account we have modelled the underlying price $S(t)$ as a *singular diffusion process* in two dimensions (with an *Ornstein-Uhlenbeck process*) instead of the standard assumption that $S(t)$ is a one-dimensional diffusion given by the geometric Brownian motion with constant volatility.

The *option pricing* method has been developed by keeping perfect hedging with a riskless strategy which finally results in a closed and exact expression for the European call. Our pricing formula has the same functional form as the B-S price but replaces the variance of the Wiener process by the variance of the O-U process. The O-U variance, $\kappa(t)$, is smaller than the B-S variance, $\sigma^2 t$, which implies that the equivalent volatility in the O-U case is lower than B-S volatility[4]. But less volatility implies a lower option price. We have indeed proved that the B-S call price is always greater than the O-U price. In other words, *the assumption of uncorrelated assets overprices the European call.* This agrees with the fact that correlation, which can be regarded as a form of predictability, implies less risk and therefore a lower price for the option. We have quantified this overprice and showed that B-S formula notably overprices options and, more strikingly, that the overprice *persists for a long time regardless of the strength of correlations.* We have also analyzed the sensitivity of the O-U price to several conditions. Thus we have proved that while C_{OU} is more sensitive to changes in the interest rate and stock price than C_{BS}, it is also less sensitive to any change of the volatility. The practical consequences of this are nontrivial.

The option price and the *hedging strategy* have been obtained using two different approaches. The most straightforward way of getting the call price is by means of a projection onto a one-dimensional process with a time-varying volatility. A second way of obtaining the option price starts with the complete two-dimensional O-U process. This is a longer procedure but opens the door to a new hedging strategy: the psi hedging. We have therefore two ways of achieving the perfect hedging: the usual one consisting in holding underlying assets (delta hedging), and the second one which uses secondary calls instead of assets (psi hedging). We have shown that *this last strategy can be considerable less expensive than the delta hedging* and can avoid a possible lack of liquidity of underlying shares. Finally, the proportion of secondary calls to be held, *i.e.*, the psi hedging, converges towards O-U delta hedging when the exercising price of the secondary call tends to zero.

In practice our method of valuation requires the estimate of one more parameter, the correlation time, than in the B-S Wiener case. Assuming that the underlying asset is driven by O-U noise one can find an estimate for the correlation time τ by evaluating the variance $\kappa(t)$ of the asset return. Once one has an estimate of this

[4] Since the volatility σ is the square root of the variance per unit time, one can define, in the O-U case, an equivalent volatility by $\sigma_{OU} = \sqrt{\dot{\kappa}(t)}$, where the dot denotes time derivative and thus see that $\sigma_{OU}/\sigma = \sqrt{1 - e^{-t/\tau}} \leq 1$.

variance the correlation time.

We finally mention that one interesting extension of the valuation method presented is to the *American options*. Although this case is more involved, one is probably able to obtain, at least an approximate or a numerical result using a combination of first passage times and martingale methods, as recently presented by Bunch and Johnson (2000)[5]. In any case we believe that the effects of auto correlations on the valuation of an American option will be even more critical than for the European call. This case is under present investigation.

This study on option pricing if prices ared driven by an Ornstein-Uhlenbeck correlated process is based on the paper:

- Masoliver, J., J. Perelló, 2001b, Option pricing and perfect hedging on correlated stocks, submitted for publication.

Another possible extension is presented in **Chapter 6**. During decades, the multiplicative diffusion process known as the geometric Brownian motion had been widely accepted as one of the most universal models for speculative markets. However, specially after the 1987 crash, the log-Brownian motion and its subsequent Black-Scholes (B-S) formula were unable to reproduce the observed derivatives prices in real markets. The multiplicative diffusion model assumes that the volatility σ is constant.

The *stochastic volatility (SV)* models are a possible way out to the observed inconsistencies between the geometric Brownian model and real markets. The SV models, as their name indicates, assume the original log-Brownian model but with the volatility σ being random. From 1987 on, there have appeared several works extending and refining stochastic volatility models but most of them are basically designed to reproduce empirical option prices. We have assumed that the volatility follows one of the simplest stochastic volatility models still showing mean reversion, *i.e.*, the *Ornstein-Uhlenbeck* process, but also allowing for *correlations* between the random fluctuations of the *price and volatility* processes.

As we have mentioned, recent efforts in the study of the empirical statistical properties of the speculative markets has led to a list of stylized facts that any proposed model should accomplish. We have shown that our model is able to reproduce these facts. More specifically, our volatility process is *mean reverting* and we have found that it is able to quantitatively reproduce the recently observed *leverage effect*. We have also *estimated all parameters* of our model and observed that a simulated path (using the estimated parameters) is very similar to the sample path of the historical evolution of the Dow-Jones daily index (1900-1999). In this way, we are

[5]Bunch, D. S., and H. Johnson, 2000, The American Put Option and Its Critical Stock Price, *Journal of Finance* **55**, 2333-2356.

able to fairly *reproduce* the stock dynamics. Finally, we have derived the *characteristic function* of the process, obtained its *kurtosis* and *skewness*, and observed the *power-law decay* for the tails of the price distribution. The results herein derived show that the SV models are good candidates for describing not only option prices but market dynamics as well.

Future research based on this approach is as follows. The resulting characteristic function with the parameters already estimated can be useful for deriving option prices and this is one of our possible future works. Analysis can be based on the Fourier represented given in **Chapter 4**. On the hand, observed volatility is not definitively attached to any specific model. There exists a large class of volatility processes and, since we now have a systematic way of estimating parameters of the process, a deeper analysis on which model gives a better explanation of the stylized facts can also be very interesting. Other existent volatility models requires a more involved mathematical analysis and it would be also interesting how far we can go using the methods presented in this Chapter.

The correlated Ornstein-Uhlenbeck stochastic volatility model is been studied also in the paper:

- Masoliver, J., J. Perelló, 2001c, A correlated stochastic volatility model quantifying leverage and other stylized facts, submitted for publication.

and on the poster session:

- Masoliver, J., J. Perelló, 2001d, Correlated stochastic volatility models and the leverage effect, 4th "Applications of Physics to Financial Analysis (APFA)" conferences, London, 5-7 December 2001.

Dinàmica estocàstica correlacionada en mercats financers

1. Les opcions

Els derivats són instruments financers amb un preu que depèn del valor d'altres variables subjacents més bàsiques. Els derivats són negociats activament en nombrosos mercats financers. Tradicionalment, les variables subjacents són accions, índex de mercat o bé altres derivats.

Les opcions són un dels productes més importants dins l'extensa gamma de derivats existents. Com el seu nom indica, les opcions dónen l'oportunitat al seu propietari, però no pas l'obligació, de fer una transacció financera durant un període futur i en funció de certes variables subjacents. Aquest treball es limita a tractar les opcions europees sobre accions. Les opcions europees són les més simples i populars[1].

Una opció de compra europea és un producte financer que dóna dret al seu propietari de poder adquirir una acció a una data futura amb un preu fixat el dia d'avui. La data futura s'anomena és la data d'expiració del contracte a temps T i el preu prefixat és el preu d'exercici de valor K.

En concret, la prima obtinguda pel fet de posseir una opció de compra, anomenada *call* europea, a data d'expiració és

$$[S(T) - K]^+ \equiv \begin{cases} S(T) - K & \text{si } S(T) > K, \\ 0 & \text{si } S(T) \leq K; \end{cases} \tag{R.1}$$

on $S(T)$ és el preu de l'acció a temps T. El propietari d'una *call* s'assegura el dret d'adquirir l'acció a una data fixada amb un preu no superior a l'acordat avui. L'opció permet, així, obtenir fins un guany infinit si la borsa puja i, alhora, restringeix les pèrdues, en cas de descens, fins una quantitat molt minsa. Una quantitat que és, precisament, el preu de l'opció que ha pagar l'inversor pel contracte.

[1]Per aquesta raó, els americans les han batejat amb el nom d'"opcions vainilla".

De la mateixa manera, existeixen les opcions de venda. El guany degut al fet de tenir una opció de venda, anomenada *put* europea, és

$$[K - S(T)]^+ \equiv \begin{cases} K - S(T) & \text{si } K < S(T), \\ 0 & \text{si } S(T) \geq K. \end{cases} \tag{R.2}$$

Així doncs, tenir una *put* representarà tenir el dret de vendre's les accions a una certa data futura i amb un preu fixat avui. Aquesta operativa permetrà al propietari de la *put* d'assegurar-se de poder vendre la seva acció, com a mínim, a un preu K.

Les opcions s'acostumen a classificar en termes de la seva *moneyness* definida com el quocient S/K. Les tres categories són

- Les opcions "fora de diner" [*out the money* (OTM)]. Pertanyen a aquest grup les *calls* amb $S/K < 1$ i les *puts* amb $S/K > 1$.
- Les opcions "a diner" [*at the money* (ATM)]. Són aquelles opcions amb $S/K = 1$.
- Les opcions "dins de diner" [*in the money* (ITM)]. En aquesta classe, s'hi encabeixen les *calls* amb $S/K > 1$ i les *puts* amb $S/K < 1$.

Les opcions estan pensades per evitar el risc derivat de les fluctuacions de les accions al llarg del temps. Així, l'inversor es pot cobrir les espatlles a un baix preu i evitar que qualsevol canvi inesperat i indesitjatble li trastoqui les seves estrategies de compra-venda. No obstant, existeixen inversors amb un ànim més agressiu que poden sentir-se atrets per aquest tipus de productes. El perfil de inversor especulador fa apostes mitjançant les opcions que, com hem dit, són molt barates en comparació amb el subjacent. Mitjançant una opció de compra, l'inversor pot obtenir un guany indefinit en cas de que l'acció augmenti el seu valor. I posseïnt una *put* pot guanyar fins a un quantitat K en cas de que el valor de l'acció es desplomi.

2. Equil·libri de mercat i mètodes incomplets de valoració d'opcions

Un cop sabem què és una opció europea, li hem de donar un preu. La única informació de què diposem és el guany degut al fet de posseir l'opció just quan expira. Sembla raonable pensar que el preu per a l'opció ha ser funció de la dinàmica de l'acció des del moment en què comprem l'opció fins quan expira. Per tant, un bon coneixement del subjacent és indispenable i totes les investigacions sobre el valor de l'opció necessitaran d'un model de mercat adequat que sigui consistent amb els mercats reals.

Però no només el comportament del subjacent és important doncs encara existeix un problema encara més fonamental. Com ve donat el preu d'una opció en termes dels subjacent? Per què el preu resultant és un preu "just" que, a priori, no afavoreix a comprador ni a venedor? Els anomenats mètodes de valoració d'opcions pretenen donar resposta a aquestes preguntes.

La valoració d'opcions i la model·lització del subjacent necessiten el coneixement de vàries disciplines. Les eines necessàries provenen del camp de teoria de probabilitat, mètodes estocàstics i anàlisi de dades. O sigui, de manera més genèrica, de les eines típiques de matemàtics i físics. No obstant, les hipòtesis sobre el comportament mercats són proposades per la teoria econòmica. La barreja de totes aquestes disciplines ha donat com a resultat nous camps d'estudi anomenats "matemàtica financera" i "economètrica" i, darrerament, "econofísica".

2.1 Propietats dels preus de les opcions europees

Robert Merton (1973b) va establir un llista exhaustiva sobre les restriccions que l'assumpció d'equi·libri de mercat imposa al preu de les opcions. Aquí revisarem algunes d'aquestes restriccions sobre el preu de la *call* i *put* europea pel fet de demanar absència d'arbitratge. Enumerem les restriccions més importants que se'n deriven pel preu $C(S,t)$ de la *call* a temps t i quan el subjacent val $S(t)$:

- $[S(t) - Ke^{-r(T-t)}]^+ \leq C(S,t) \leq S(t) \qquad (t \leq T)$.
- La *call* perpètua $C_p(S) \equiv \lim_{(T-t)\to\infty} C(S,t)$ val: $C_p(S,t) = S(t)$.
- $C'(S,t) \leq C(S,t) \qquad (K \leq K')$.
- $C(S,t) \leq C'(S,t) \qquad (t \leq T \leq T')$.

Les restriccions sobre el preu $P(S,t)$ de les opcions de venda, *i.e.*, les *puts*, s'obtenen gràcies a una relació entre ambdues que es fixa adduïnt raons d'equil·libri de mercat. La relació anomenada paritat *put-call* demana que

$$P(S,t) + S(t) = C(S,t) + Ke^{-r(T-t)}. \qquad \text{(R.3)}$$

2.2 Mètodes de valoració incomplets

Abans que Black i Scholes (1973) presentessin el seu mètode de valoració d'opcions, hi han hagut molts treballs intentant de fixar un preu just per a l'opció. Creiem interessant retornar-hi perquè són representatius de les dificultats existents en l'obtenció d'un model de mercat adequat i d'un valor de l'opció consistents amb les restriccions d'equil·libri de mercat.

Louis Bachelier està considerat el pare de la matemàtica financera. L'any 1900 va presentar la seva tesi que contenia, per primer cop, un anàlisi estocàstic de la borsa, dels preus i de la seva opció de compra europea. La primera part de la seva tesi obté la densitat de probabilitat de les variacions dels preus. Bachelier suposa que aquests segueixen un dinàmica tal que només poden moure's una unitat amunt o avall essent aquests dos esdeveniments equiprobables. El límit a temps continu ens permet escriure

$$p_S(S,t|S_0)\frac{1}{\sqrt{2\pi k^2 t}}\ e^{-(S-S_0)^2/2k^2t}, \tag{R.4}$$

on assumim que l'acció val S_0 a temps $t_0 = 0$. De fet, aquesta dinàmica és la mateixa que Einstein, sense conèixer el treball de Bachelier, va emprar per descriure el moviment brownià cinc anys més tard, l'any 1905.

La segona part de la tesi es dedica a donar preu a l'opció. Bachelier va proposar que el preu just fos aquell en què tant comprador com venedor tinguessin un guany zero en promig. En conseqüència,

$$C(S,t) = E\left[(S(T)-K)^+|S(t)=S\right] = \int_K^\infty (S'-K)p_S(S',T|S,t)dS'. \tag{R.5}$$

Si assumim el model de mercat proposat per Bachelier, aleshores l'Eq. (R.5) ens dóna

$$C(S,t) = (S-K)N(d) + \frac{k\sqrt{T-t}}{\sqrt{2\pi}}\ e^{-d^2/2}. \tag{R.6}$$

on

$$d = (S-K)/k^2(T-t) \tag{R.7}$$

i

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}dx \tag{R.8}$$

és la "funció de probabilitat integrada". Observeu, però, que el preu Bachelier per a l'opció (R.6) no obeeix les restriccions d'equi·libri de mercat. Per exemple, es pot comprovar fàcilment que la *call* perpètua de Bachelier no tendeix a $S(t)$ sinò que divergeix.

Tot i el treball de Bachelier, la naturalesa dels mercats no resultava prou atractiva per la comunitat científica. Els brokers tenien un creixent interès sobre els mercats però en cercles acadèmics es continuava considerant que les fluctuacions dels mercats no eren res més que el producte d'una massa irracional que especula. No obstant, a partir dels anys trenta començaren aparèixer un seguit d'estudis fets per investigadors americans amb un alt grau de coneixement de matemàtiques i estadística. Aquest es dedicaren a compil·lar dades i després confrontar-les amb el model de mercat proposat per Bachelier (1900).

Fruit d'aquest esforç, Osborne (1959) i Sprenkle (1964) van concloure que els preus seguien un moviment brownià geomètric. Per tant, les variacions de preu que resultava més adequat tractar-les en termes de la rendibilitat

$$R(t) \equiv \ln[S(t)/S_0] \tag{R.9}$$

donat que la seva densitat de probabilitat era novament una gaussiana com la següent

$$p_R(R,t|0) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-(R-(\mu-\sigma^2/2))^2/2\sigma^2 t}. \qquad \text{(R.10)}$$

En aquesta densitat hi apareix una constant σ que s'anomena "volatilitat". Un cop modificat el model de mercat, Sprenkle (1964) va proposar un nou preu per a l'opció. Es va valer de la fòrmula de Bachelier però modificant el promig amb una constant que dóna raó del grau d'aversió al risc de cada inversor. Sprenkle entenia que el paràmetre d'aversió de risc és, en qualsevol cas, propi de cada inversor. No obstant, això entra en contradicció amb la idea de que existeix un únic preu just per a tot inversor.

La crítica sobre el paràmetre d'aversió de risc no és la única. La *call* perpètua d'Sprenkle també divergeix. Els treballs de Boness (1964) i Samuelson (1965) ataquen la qüestió del valor monetari del temps. Samuelson (1965) imposa que

$$E[S(t_2)|S(t_1) = S_1] = S_1 e^{\mu(t_2-t_1)}, \qquad [C(S,T)|S(t) = S] = C(S,t)e^{\beta(t_2-t_1)}. \quad \text{(R.11)}$$

El preu actual de l'opció $C(S,T)$ no té una relació directa amb el promig del seu valor a temps d'expiració com havia afirmat Bachelier. Suposant que el procés per a l'acció és un brownià geomètric, les Eqs. (R.11) ens permeten escriure

$$C(S,t) = e^{(\mu-\beta)(T-t)} S N(d_1) - e^{-\beta(T-t)} K N(d_2), \qquad \text{(R.12)}$$

amb

$$d_1 = \frac{\ln(S/K) + (\mu + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t},$$

on $N(d)$ és la integral de probabilitat (R.8). Els axiomes de l'equació (R.11) no determinen μ i β i Samuelson només diu que $0 \le \mu \le \beta$. Aquesta desigualtat permet treure la divergència en la *call* perpètua però no dóna el valor desitjat per les restriccions d'equil·libri de mercat. En el proper capítol veurem que si Samuelson (1965) hagués imposat $\mu = \beta = r$ hauria estat ell el primer a presentar el primer preu que acompleix la demanda d'absència d'arbitratge.

3. Mètode Black-Scholes de valoració d'opcions

Els treballs de Black i Scholes (1973) i Merton (1973b) són la culminació del gran esforç fet des del treball de Bachelier (1900) per obtenir un mètode de valoració completament satisfactori. El mètode ha esdevingut de gran utilitat pels inversors i ha ajudat als mercats d'opcions a tenir la importància que actualment tenen.

El mètode obté una equació amb derivades parcials del mateix tipus que l'equació de difussió i rep el nom d'equació Black-Scholes (B-S). La teoria que permet derivar aquesta equació es fixa amb un model de mercat concret. És a dir, dóna una equació diferencial estocàstica tal que descrigui la dinàmica del subjacent. De bon principi, Black i Scholes (1973) proposaren que l'evolució del mercat vingués donada per un procés brownià geomètric.

Ara bé, l'equació diferencial no té sentit fins que no li assignem una interpretació al terme multiplicatiu de soroll. Existeixen dues interpretacions: Itô (1951) i Stratonovich (1966). No obstant, totes les derivacions conegudes de Black-Scholes s'ha fet mitjançant el càlcul d'Itô. Possiblement, això es deu al fet que aquesta és la convenció que típicament utilitzen els matemàtics mentres que la d'Stratonovich és l'emprada normalment pels físics. Potser, la interpretació d'Itô resulta més adequada per descriure la mecànica de les transaccions financeres. En qualsevol cas, però, les dues interpretacions són equivalents (Van Kampen (1981)) i haurien d'arribar als mateixos resultats. En aquest capítol provarem que efectivament les dues convencions deriven la mateixa equació B-S. Aquest exercici matemàtic, a més, ens permetrà estudiar amb cura les subtilitats de càlcul que hi ha en la valoració de Black-Scholes i que poden arribar a ser d'interés per a un físic que es vulgui introduïr en la matèria. Bona part d'aquest capítol està basat en l'article Perelló *et al.* (2000).

3.1 Itô vs. Stratonovich

La diferencial de la variable $X(t)$ es llegeix com

$$dX(t) = X(t + dt) - X(t) \tag{R.13}$$

i direm que la diferential dX és equivalent, en mitjana quadràtica, a la del procés dY sempre i quan

$$\mathrm{Var}[dX(t) - dY(t)] = \mathrm{O}(dt). \tag{R.14}$$

Presentem la següent equació diferencial que determinarà la dinàmica del procés $X(t)$

$$dX(t) = f(X, t)dt + g(X, t)dW(t) \tag{R.15}$$

on $dW(t) = \xi(t)dt$ és el procés de Wiener que és un procés gaussià amb mitjana i variança

$$E\left[dW(t)\right] = 0 \qquad \mathrm{Var}[dW(t)] \equiv E\left[dW(t)^2\right] - E\left[dW(t)\right]^2 = dt. \tag{R.16}$$

Noteu que, en mitjana quadràtica, $dW(t)^2 \to dt$.

El problema de la interpretació apareix quan llegim g i f de l'anterior equació diferencial estocàstica (R.14). Aleshores, la interpretació d'Itô avalua les funcions a l'inici de l'interval infinitessimal de temps dt, *i.e.*, $X = X(t)$. En canvi, Stratonovich dóna la quantitat $X = X(t + dt/2)$, just al mig de l'interval.

En conseqüència, cadascuna de les interpretacions entendrà de manera diferent la diferencial del producte de variables estocàstiques

$$d(XY) \equiv [(X + dX)(Y + dY)] - XY. \tag{R.17}$$

Podem descomposar la darrera equació de dues maneres diferents. Itô llegeix la diferencial

$$d(XY) = XdY + YdX + dXdY, \tag{R.18}$$

però Stratonovich prefereix

$$d(XY) = X_S dY + Y_S dX, \tag{R.19}$$

on $X_S \equiv X + dX/2$.

En general, qualsevol funció $h(X,t)$ ben comportada també s'entendrà de manera diferent si agafem una o altra interpretació. Descartant els termes d'ordre superiors a dt, la sèrie de Taylor de la funció h ens dóna

$$dh = \frac{\partial h(X,t)}{\partial t}dt + \frac{\partial h(X,t)}{\partial X}dX + \frac{1}{2}\frac{\partial^2 h(X,t)}{\partial X^2}dX^2 + \mathrm{O}(dt^2, dX^3), \tag{R.20}$$

que, de fet, correspon a la interpretació d'Itô. Si traduïm la mateixa diferencial en el sentit d'Stratonovich veurem que

$$dh = \frac{\partial h(X_S,t)}{\partial X_S}dX + \frac{\partial h(X_S,t)}{\partial t}dt \tag{R.21}$$

i la diferencial obeeix les regles estàndard de càlcul.

3.2 El model brownià geomètric

Black i Scholes (1973) proposen pel mercat la dinàmica següent:

$$dR(t) = \left(\mu - \sigma^2/2\right) dt + \sigma dW(t). \tag{R.22}$$

La rendibilitat $R(t)$ es defineix com $R \equiv \ln(S/S_0)$ on S és el preu de l'acció. Observeu que

$$E\left[dR\right] = (\mu - \sigma^2/2)dt \qquad \text{i} \qquad \mathrm{Var}[dR] = \sigma^2 dt,$$

L'equació (R.22) té el mateix sentit tant si l'interpretem en Itô com en Stratonovich.

Aplicant els canvis de variables enunciats a la darrera secció (veure Eq. (R.20) i Eq. (R.21)), obtenim (*à* la Stratonovich)

$$dS = \left(\mu - \sigma^2/2\right) S_S dt + \sigma S_S dW(t), \qquad (S_S \equiv S + dS/2) \tag{R.23}$$

i (*à* la Itô)

$$dS = \mu S dt + \sigma S dW(t). \tag{R.24}$$

Així doncs, tot i que el procés és el mateix, aquest ve descrit per dues equacions estocàstiques diferents. El promig de l'acció, en qualsevol cas, és

$$E\left[S(t)|S_0\right] = S_0 \exp(\mu t). \tag{R.25}$$

3.3 L'equació Black-Scholes

Existeixen diverses maneres d'obtenir l'equació Black-Scholes. Nosaltres escollim la derivació de Merton (1973b) que parteix d'una cartera composada per accions de preu S, bons de preu B i opcions de preu C. Així doncs, la cartera serà un certa combinació de cadascun d'aquests valors

$$\Pi = \Upsilon C - \Delta S - \Phi B.$$

Convé recordar que els bons evolucionen sense aleatorietat de manera que $dB = rBdt$. A més, Merton (1973b) demana que la cartera sigui d'inversió nul·la ($\Pi = 0$). Gràcies a aquesta demanda, simples operacions matemàtiques ens permeten escriure

$$C = \delta_n S + \phi_n B, \qquad \text{(R.26)}$$

on $\delta_n \equiv \Delta/\Upsilon$ i $\phi_n \equiv \Phi/\Upsilon$. La carteres que "reprodueixen" el preu de la *call* s'anomena carteres rèplica.

Derivem l'equació amb derivades parcials que assumint la interpretació d'Itô. En aquest cas, la diferencial serà (*cf.* Eq. (R.18))

$$dC = [\delta_n(S,t) + d\delta_n(S,t)]dS + [\phi_n(S,t) + d\phi_n(S,t)]dB \\ +S(t)d\delta_n(S,t) + B(t)d\phi_n(S,t).$$

Certes components d'aquesta diferencial queden cancel·lades degut a dues restriccions degudes a la naturalesa dels mercats. Les funcions δ_n i ϕ_n dónen l'estratègia a seguir. Ara bé, δ_n i ϕ_n són funcions no anticipades en el sentit de que depenen dels preus de l'acció avaluats a un temps immediatament anterior al moment en què decidim l'estratègia. És a dir, $\delta_n(S,t) \equiv \delta(S - dS, t)$ i $\phi_n(S,t) \equiv \phi(S - dS, t)$. Per una altra banda, els inversors fan les operacions segons la quantitat de diners disponibles. O sigui, segueixen estratègies autofinançades que impliquen $Sd\delta_n = -Bd\phi_n$.

Mitjançant aquestes dues restriccions sobre les estratègies de mercat i tenint en compte que $dB = rBdt$, arribem a

$$dC = \delta dS + r(C - \delta S)dt. \qquad \text{(R.27)}$$

Per una altra banda, coneixem la diferencial de $C = C(S,t)$

$$dC = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) dt + \frac{\partial C}{\partial S} dS.$$

Les dues diferencials desemboquen a l'equació

$$\left(\delta - \frac{\partial C}{\partial S}\right) dS = \left[\frac{\partial C}{\partial t} - r(C - \delta S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right] dt. \qquad \text{(R.28)}$$

Però l'equació només té fluctuacions aleatòries degut a dS. Aquestes es poden treure amb l'estratègia

$$\delta = \frac{\partial C}{\partial S}$$

que s'anomena cobertura delta. Finalment obtenim una equació amb derivades parcials on no hi ha cap component aleatòria i que rep el nom d'equació Black-Scholes, *i.e.*,

$$\frac{\partial C}{\partial t} = rC - rS\frac{\partial C}{\partial S} - \frac{1}{2}(\sigma S)^2\frac{\partial^2 C}{\partial S^2}. \tag{R.29}$$

Aquesta equació és una equació diferencial parabòlica amb la condició final especificada pel contracte. La *call* europea té

$$C(S,T) = [S(T) - K]^+ \tag{R.30}$$

i la seva solució és

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \qquad (0 \leq t \leq T) \tag{R.31}$$

amb arguments

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = d_1 - \sigma\sqrt{T-t}. \tag{R.32}$$

En aquesta capítol també derivem l'equació basant-nos amb Stratonovich. De fet, en aquest cas,

$$dC = S_S(t)d\delta_n + B(t)d\phi_n + \delta_n(S_S,t)dS + \phi_n(S_S,t)dB. \tag{R.33}$$

però, assumint que les funcions que determinen l'estratègia siguin no anticipades i que l'estratègia sigui autofinançada,

$$dC = \frac{1}{2}dSd\delta_n + \delta(S_S,t)dS + \left[rB(t)\phi(S_S,t) - \sigma^2 S_S^2\frac{\partial\delta(S_S,t)}{\partial S_S}\right]dt + \mathrm{O}(dt^{3/2}),$$

s'acaba obtenint

$$dC = \delta(S_S,t)dS + \left[rB(t)\phi(S_S,t) - \frac{1}{2}\sigma^2 S_S^2\frac{\partial\delta(S_S,t)}{\partial S_S}\right]dt + \mathrm{O}(dt^{3/2}). \tag{R.34}$$

Per una altra banda,

$$dC = \frac{\partial C(S_S,t)}{\partial t}dt + \frac{\partial C(S_S,t)}{\partial S_S}dS.$$

Si ajuntem ambdues diferencials i intentem substraure les fluctuacions dS de l'equació resultant acabem trobant l'equació Black-Scholes i la cobertura delta.

Tota aquesta anàlisi es pot fer assumint la presència de dividends o bé per a d'altres opcions com ara les *put* europees.

3.4 Les sensibilitats de la *call*

Les "gregues" dónen idea de la sensibilitat de l'opció respecte canvis petits de cadascun dels seus paràmetres. Aquests paràmetres són: r, σ, S i $T-t$. Enumerem les gregues per a la *call* europea:

$$\delta = \frac{\partial C}{\partial S} = N(d_1). \tag{R.35}$$

$$\gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\exp(-d_1^2/2)}{\sigma S\sqrt{2\pi(T-t)}}. \tag{R.36}$$

$$\theta = \frac{\partial C}{\partial t} = -\frac{\sigma S \exp(-d_1^2/2)}{2\sqrt{2\pi(T-t)}} - rKe^{-r(T-t)}N(d_2). \tag{R.37}$$

$$\upsilon = \frac{\partial C}{\partial \sigma} = S\sqrt{(T-t)/2\pi}\; e^{-d_1^2/2}. \tag{R.38}$$

$$\varrho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2). \tag{R.39}$$

La delta mesura les fluctuacions de l'opció respecte l'acció. Fixeu-vos que aquesta quantitat ens especifica la cobertura Black-Scholes. La gamma quantifica els canvis de la pròpia cobertura. La vega és la sensibilitat de l'opció respecte la volatilitat i es pot veure que la *call* és molt sensible a aquest tipus de variacions.

4. Generalitzacions dins la teoria Black-Scholes

Black-Scholes proporciona una completa explicació sobre com es deriva el preu de l'opció. Determina el preu just per a l'opció i, a més, ens dictamina quina és la estratègia de mercat a seguir per tal de que la nostra cartera no tingui cap mena de fluctuació riscosa. Després de l'article de Black i Scholes (1973), aparegueren generalitzacions tant en la model·lització del mercat com en els mètodes valoració de les opcions.

4.1 Processos a salts

Després de Black i Scholes (1973), Cox i Ross (1976) i Merton (1976) estudiaren quins models podrien afitar-se als requeriments de la teoria Black-Scholes. Una possibilitat és la d'admetre que $\sigma = \sigma(S,t)$. Aquesta sofisticació del model de mercat compleix els requeriments per tal d'obtenir l'equació amb derivades parcials de Black-Scholes però la solució de l'equació que ens dóna el preu $C(S,t)$ pot ser molt complicada. Una altra extensió possible és la que assumeix que el mercat segueix un procés a salts.

Cox i Ross (1976) proposen que el preu de l'acció evolucioni com

$$dS(t) = \sum_k (SY_k - S)d\Theta(t - t_k), \tag{R.40}$$

mentres que Merton (1976) composa un procés difussiu amb un de salts

$$dS(t) = S(t)\left[(\mu - \lambda E\left[Y - 1\right])\,dt + \sigma dW(t) + \sum_k (Y_k - 1)d\Theta(t - t_k)\right], \tag{R.41}$$

on Y_k $(k = 1, 2, 3, ...)$ són variables independents and idènticament distribuïdes amb $E\left[Y_k Y_l\right] = E\left[Y\right]^2$ for $k \neq l$, i $E\left[Y_k^2\right] = E\left[Y^2\right]$. Aquest tipus de processos a salts vénen descrits per

$$d\Theta(t - t_k) = \begin{cases} 1 & \text{si } t \leq t_k < t + dt, \\ 0 & \text{en cas contrari.} \end{cases} \tag{R.42}$$

Els esdevinements t_k estan distribuïts segons Poisson amb paràmetre λ.

Aquests dos models arriben a un preu per a l'opció de compra europea. En ambdós cassos necessiten la diferencial d'una funció ben comportada que depengui del procés aleatori S. El lema d'Itô per aquest tipus de processos és

$$df(X,t) = \frac{\partial f(X,t)}{\partial t}dt + \sum_k [f(XY_k, t) - f(X,t)]d\Theta(t - t_k), \tag{R.43}$$

Es pot veure que el model de Cox i Ross dóna el preu de la *call*

$$C(S,t) = S\Psi(U, AY) - Ke^{-r(T-t)}\Psi(U, A), \tag{R.44}$$

on

$$\Psi(U, A) \equiv \sum_{k=U}^{\infty} \frac{1}{k!} A^k e^{-A}$$

és la funció de Laplace,

$$U \equiv \left[\frac{\ln(K/S)}{\ln Y}\right],$$

on $[x]$ és el primer nombre sencer més gran x. Finalment,

$$A \equiv \frac{r(T-t)}{Y-1}.$$

Per poder arribar a aquest preu hem d'assumir que Y és una constant i no és per tant aleatória. La cobertura és diferent a la de Black-Scholes:

$$\delta = \frac{C(SY, t) - C(S, t)}{S(Y - 1)}. \tag{R.45}$$

El model més sofisticat de Merton arriba també a un preu per a la *call*. En aquest cas, però, direm que només és just en mitjana. Merton fa cobertura Black-Scholes i així està a cobert la majoria del temps excepte en aquelles rares ocasions que succeeixi un salt. En aquests cassos, l'inversor pot guanyar o perdre una quantitat inesperada de capital. El preu de Merton és

$$C(S,t) = \sum_k \frac{1}{k!} e^{-\lambda(T-t)} [\lambda(T-t)]^k E\left[C_{BS}(SY^k,t)\right] \tag{R.46}$$

on C_{BS} és el preu B-S de la *call* donat per l'Eq. (R.31), *i.e.*, assumint que el subjacent segueix un procés de Wiener amb preu actual SY^k.

4.2 CAPM i martingales

També després de Black i Scholes (1973), es trobaren altres mètodes de valoració de les opcions. Aquests són més simples però tenen l'incovenient d'ignorar el tipus de cobertura que ha de seguir l'inversor. De tota manera, resulten molt útils i pràctics si la única cosa que es busca és el preu final o l'equació Black-Scholes.

Si el que ens interessa és l'equació amb derivades parcials de Black-Scholes, el mètode anomenat *Capital Asset Pricing Model (CAPM)* (Sharpe (1964)) és el que cal emprar. El quocient Sharpe de cadascun dels valors negociats ha de ser idèntic. És a dir,

$$\frac{\mu - r}{\sigma} = \frac{\mu_C - r}{\sigma_C}, \tag{R.47}$$

on r és tipus d'interès,

$$\mu dt = E\left[\frac{dS}{S}\right], \qquad \sigma^2 dt = \text{Var}\left[\frac{dS}{S}\right];$$

i

$$\mu_C dt = E\left[\frac{dC}{C}\right], \qquad \sigma_C^2 dt = \text{Var}\left[\frac{dC}{C}\right].$$

Aquests promitjos estan fets coneixent $S(t) = S$ i $C(S,t) = C$. Assumim el model brownià geomètric i obtenim

$$E[dC] = \left[\frac{\partial C}{\partial t} + \frac{1}{2}(\sigma S)^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S}\right] dt,$$

i

$$\text{Var}[dC] = \left(\sigma S \frac{\partial C}{\partial S}\right)^2 dt.$$

Observeu que aquests promitjos s'han fet tenint en compte que $E[dS] = \mu S dt$ i $\text{Var}[dS] = (S\sigma)^2 dt$. La substitució d'aquestes equacions a la igualtat CAPM (R.47) ens acaba donant l'equació B-S (R.29) i demostra la consistència dels dos mètodes.

Ara bé, si el que interessa és el preu final de l'opció, cal tractar el problema amb teoria de martingala (Harrison and Kreps (1979), Harrison and Pliska (1981)). La neutralització del risc requereix que

$$E\left[S(t)|S_0\right]^* = S_0 e^{r(t-t_0)}, \tag{R.48}$$

i $S^*(t)$ vindrà definit per

$$\frac{dS^*}{S^*} = rdt + \sigma dW. \tag{R.49}$$

Observeu que aleshores $S^*(t)$ i $S(t)$ estan relacionats

$$S^*(t) = S(t)e^{-(\mu-r)(t-t_0)} \tag{R.50}$$

i

$$p^*(S,t|S_0,t_0) = e^{(\mu-r)(t-t_0)} p_S\left(Se^{(\mu-r)(t-t_0)}, t|S_0,t_0\right), \tag{R.51}$$

Un cop fem el canvi i representem la densitat de probabilitat en termes de la rendibilitat, trobem el preu de l'opció

$$C(S,t) = e^{-r(T-t)} \int_{\ln(K/S)}^{\infty} (Se^R - K) p_R^*(R,T|t) dR. \tag{R.52}$$

A partir d'aquesta equació també podem trobar

$$C(S,t) = S - \frac{K}{2}e^{-r(T-t)} - \frac{K}{\pi}e^{-r(T-t)} \int_0^{\infty} \varphi_X(\omega,T|t) \Big[\cos\omega\alpha(T-t) + \omega\sin\omega\alpha(T-t)\Big] \frac{d\omega}{1+\omega^2}, \tag{R.53}$$

on φ_X és la funció característica

$$\varphi_X(\omega,T|t) \equiv \int_{\infty}^{\infty} e^{i\omega R} p_R^*(R,T|t) dR$$

de la rendibilitat neutra del model de mercat, *i.e.*, havent-li extret el creixement infinitessimal mig. La equació diferencial estocàstica del procés és

$$dX = dF(t) \qquad \text{tal que:} \qquad E\left[dF(t)\right] = 0,$$

on les fluctuacions estocàstiques dF ha d'acomplir la condició d'absència d'arbitratge.

5. Valoració d'opcions i cobertura perfecta sobre accions correlacionades

Un cop hem vist fins on és capaç d'arribar la teoria Black-Scholes, ara ens dedicarem a relaxar alguna de les condicions ideals imposades sobre el comportament del mercat. De les assumpcions imposades, potser la més restrictiva i més en desacord amb les observacions empíriques és la hipòtesi de mercat eficient. Aquesta hipòtesi demana que les variacions de preu estiguin totalment descorrelacionades. Per aquesta demanda, els físics entenem que el soroll que condueix les variacions de preu és blanc. Ara bé, els físics també entenem que els processos blancs són una idealització ja que cap soroll és completament blanc[2]. Només en cas que el temps d'observació sigui molt més llarg que el temps d'autocorrelació del procés podem considerar que el procés és efectivament blanc.

Així doncs, presentem un model de mercat que doni raó de l'existència de correlacions en els preus. Presentem el procés d'Ornstein-Uhlenbeck (O-U) per tres raons: (a) El soroll O-U és encara un procés gaussià que tendeix al model brownià que assumeix Black-Scholes que el seu temps característic $\tau = 0$. (b) És l'únic procés que és simultàniament gaussià, estacionari i markovià. (c) Com veurem, l'evolució de la variança de les rendibilitats està ben descrita per la variança del procés conduït per un procés O-U.

El nostre propòsit és obtenir el preu de l'opció i observar l'efecte de les correlacions en el propi preu de l'opció. Ho hem fet de tres maneres diferents. Amb una cartera com la de Black-Scholes, amb una extensió de la cartera de Black-Scholes contenint una opció secundària i, finalment, via martingala.

5.1 El model de mercat

Recordem que, fins ara, hem assumit que el preu de l'acció $S(t)$ és model·litzat amb un procés difussiu unidimensional

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \tag{R.54}$$

on $W(t)$ és el procés de Wiener amb mitjana zero i

$$E\left[dW(t)dW(t')\right] = \begin{cases} dt & \text{si } t = t', \\ 0 & \text{si } t \neq t'. \end{cases}$$

Ara, el preu no és conduït per un procés Wiener $W(t)$ sinò per un Ornstein-Uhlenbeck $V(t)$. És a dir, $S(t)$ obeeix el procés bidimensional difussiu singular

$$\frac{dS(t)}{S(t)} = \mu dt + V(t)dt \tag{R.55}$$

$$dV(t) = -\frac{V(t)}{\tau}\, dt + \frac{\sigma}{\tau}\, dW(t), \tag{R.56}$$

[2]El soroll amb correlacions en el temps és anomenat soroll de color.

on $\tau \geq 0$ és el temps correlació. $V(t)$ és un soroll Ornstein-Uhlenbeck en l'estat estacionari. Aquest és de color, gaussià, de mitjana zero i funció correlació

$$E\left[V(t_1)V(t_2)\right] = \frac{\sigma^2}{2\tau} e^{-|t_1-t_2|/\tau}. \tag{R.57}$$

Ens resulta convenient expressar el procés en termes de la rendibilitat:

$$R(t) = \ln[S(t)/S_0],$$

I, aleshores, en comptes de l'Eq. (R.55), tindrem

$$\frac{dR(t)}{dt} = \mu + V(t) \tag{R.58}$$

i a la variable $V(t)$ l'anomenarem "velocitat" de la rendibilitat del procés.

Es pot veure que la variança de la rendibilitat (en el règim estacionari)

$$\kappa(t) = \sigma^2 \left[t - \tau\left(1 - e^{-t/\tau}\right)\right], \tag{R.59}$$

i, per $t \ll \tau$,

$$\kappa(t) \sim (\sigma^2/2\tau)t^2, \qquad (t \ll \tau). \tag{R.60}$$

L'equació (R.59) (cas B-S) quan $t \gg \tau$:

$$\kappa(t) \sim \sigma^2 t, \qquad (t \gg \tau). \tag{R.61}$$

La variança empírica del S&P 500 durant el període gener 1988-desembre 1996 es pot explicar mitjançant la variança teòrica $\kappa(t)$ amb temps de correlació $\tau = 2$ minuts. El resultat d'aquesta correlació afecta la volatilitat a 100 minuts.

5.2 El procés projectat

Algú pot dir que el model de mercat presentat és inadequat perquè el preu de l'acció $S(t)$ que obeeix l'Eq. (R.56) és un procés continu de variacions acotades. Harrison *et al.* (1984) van provar que aquest tipus de processos permeten arbitratge i això és una cosa inacceptable per obtenir un preu "just" per a l'opció.

Malgrat això, a la pràctica no coneixem la velocitat de la rendibilitat $V(t)$ doncs aquesta no es negocia als mercats i la seva evolució és ignorada. És a dir, la dinàmica del mercat no dóna cap coneixement de la dinàmica de $V(t)$. Aquest fet ens permet projectar el procés bidimensional (S,V) a un d'unidimensional equivalent $\bar{S}(t)$ (en mitjana quadràtica):

$$\frac{d\bar{S}(t)}{\bar{S}(t)} = [\mu + \dot{\kappa}(T-t)/2]dt + \sqrt{\dot{\kappa}(T-t)}dW(t), \tag{R.62}$$

on $\kappa(t)$ és donada per l'Eq. (R.59), i el punt denota derivada sobre el temps

$$\dot{\kappa}(T-t) \equiv \sigma^2 \left(1 - e^{-t/\tau}\right).$$

En conclusió, el preu donat per l'Eq. (R.62) està, ara sí, conduït per un soroll de variacions no acotades, el procés de Wiener, i els resultats de Harrison *et al.* (1984) no són aplicables. En conseqüència, el procés projectat O-U és encara un punt de partida apropiat per a la valoració d'opcions perquè no permet arbitratge.

5.3 El preu de l'opció amb el procés projectat

La valoració de l'opció amb el procés projectat és la mateixa que l'exposada en el capíyol anterior. Per tant,

$$C_t = rC - rSC_S - \frac{1}{2}\dot{\kappa}(T-t)S^2 C_{SS} \tag{R.63}$$

i la cobertura és també la mateixa: $\delta = C_S$.

La solució per a la *call* europea és:

$$C_{OU}(S,t) = S\ N(d_1^{OU}) - Ke^{-r(T-t)}\ N(d_2^{OU}), \tag{R.64}$$

on $N(d)$ és la funció integral de probabilitat i

$$d_1^{OU} = \frac{\ln(S/K) + r(T-t) + \kappa(T-t)/2}{\sqrt{\kappa(T-t)}}, \tag{R.65}$$

$$d_2^{OU} = d_1^{OU} - \sqrt{\kappa(T-t)}, \tag{R.66}$$

amb $\kappa(t)$ donada per l'Eq. (R.59).

D'aquest resultat en podem extreure alguns resultats. Si fem $\tau = 0$, quan no hi han correlacions, aleshores recuperem el preu B-S

$$C_{BS}(S,t) = S\ N(d_1^{BS}) - Ke^{-r(T-t)}\ N(d_2^{BS}), \tag{R.67}$$

on $d_{1,2}^{BS}$ tenen la forma de les expressions (R.65)-(R.66) però amb $\sigma^2(T-t)$ en comptes de $\kappa(T-t)$. En el cas oposat, $\tau = \infty$, quan la correlació és total i no hi ha soroll aleatori sinò una força determinista i constant, Eq. (R.64) es redueix al preu determinista

$$C_d(S,t) = \left(S - Ke^{-r(T-t)}\right)^+. \tag{R.68}$$

I es pot veure que

$$C_d(S,t) \le C_{OU}(S,t) \le C_{BS}(S,t), \tag{R.69}$$

per tot S i $0 \le t \le T$. Podem concloure que l'assumpció d'absència de correlacions sobrevalora l'opció.

També hem definit la diferència relativa

$$D = (C_{BS} - C_{OU})/C_{BS}$$

i hem estudiat les pecularietats d'aquesta diferència tant en termes de la seva *moneyness* com del temps que manca per a l'expiració del contracte. Els resultats obtinguts conclouen que la sobrevaloració és més important per les opcions fora de diner (OTM) i, en qualsevol cas, els efectes de la correlació són més importants dels que es podria haver pensat d'entrada. La sobrevaloració persisteix a temps d'expiració llargs.

5.4 Derivacions alternatives de la *call*

Una altra manera de trobar el preu de l'opció és modificant i ampliant la cartera. Suposem que la cartera està formada per *calls*, *calls* secundàries i bons:

$$\Pi = \Upsilon C - \Upsilon' C' - \Phi B. \tag{R.70}$$

El mètode de valoració és idèntic al de Black-Scholes però ara considerant que la cobertura està feta amb una altra *call*. Després d'alguns càlculs trobem que

$$\Big[\Big(C_t + \frac{\sigma^2}{2\tau}C_{VV} - rC + (\mu + V)SC_S\Big) \\ -\psi\Big(C'_t + \frac{\sigma^2}{2\tau}C'_{VV} - rC' + (\mu + V)SC'_S\Big)\Big]dt = (\psi C'_V - C_V)\,dV. \tag{R.71}$$

Aquesta equació pot esdevenir determinista si fem zero el terme multiplicant la diferencial estocàstica $dV(t)$ donada per l'Eq. (R.56). I aquesta, al seu torn, fixarà l'estratègia de l'inversor donant, en tot moment, la quantitat relativa de *calls* secundàries que cal mantenir per tenir una cartera sense risc. Llavors, no tenim la cobertura delta de B-S sinò que tenim la cobertura "psi":

$$\psi = \frac{C_V}{C'_V} \tag{R.72}$$

i l'equació esdevé

$$\frac{1}{C_V}\Big[C_t + \frac{\sigma^2}{2\tau}C_{VV} - rC + (\mu + V)SC_S\Big] \\ = \frac{1}{C'_V}\Big[C'_t + \frac{\sigma^2}{2\tau}C'_{VV} - rC' + (\mu + V)SC'_S\Big]. \tag{R.73}$$

Aquest problema ja s'havia tractat en cas de que mercat venia descrit per un model de volatilitat estocàstica (veure proper capítol). Més passos en càlcul que involucren imposicions com l'anomenada prima de risc acaben per obtenir el preu de l'opció (R.64) presentat a la secció anterior.

Per comletitud, també podem avaluar el preu martingala de l'opció quan el subjacent és conduït per un procés d'Ornstein-Uhlenbeck. La mesura equivalent de martingala és

$$p^*(S,t|S_0,t_0) = \frac{1}{S\sqrt{2\pi\kappa(t-t_0)}} \exp\left\{-\frac{[\ln(S/S_0) - r(t-t_0) + \kappa(t-t_0)/2]^2}{2\kappa(t-t_0)}\right\}, \tag{R.74}$$

que també s'anomena probabilitat de risc neutre. Aleshores, el preu martingala serà

$$\begin{aligned} C^*(S,t) &= e^{-r(T-t)}E^*[(S(T)-K)^+|S(t)=S] \\ &= e^{-r(T-t)}\int_K^\infty (S'-K)p^*(S',T|S,t)dS', \end{aligned} \tag{R.75}$$

i el resultat final és consistent amb el preu prèviament donat a l'Eq. (R.64), $C^*(S,t) = C_{OU}(S,t)$.

5.5 Les gregues i la cobertura

Derivem breuement les gregues pel procés conduït per un procés O-U. La *call* O-U té la mateixa relació funcional que el preu B-S però reemplaça $\sigma^2(T-t)$ per $\kappa(T-t)$ i, en conseqüència les gregues O-U tindran la mateix relació funcional que les B-S amb el mateix reemplaçament a excepció de vega, $\upsilon = \partial C/\partial\sigma$, i $\theta = \partial C/\partial t$. Per tant, per $\delta = \partial C/\partial S$, $\gamma = \partial^2 C/\partial S^2$, i $\varrho = \partial C/\partial r$, obtenim

$$\delta_{OU} = N(d_1^{OU}), \qquad \gamma_{OU} = \frac{e^{-(d_1^{OU})^2/2}}{S\sqrt{2\pi\kappa(T-t)}}, \qquad \varrho_{OU} = K(T-t)e^{-r(T-t)}N(d_2^{OU}). \tag{R.76}$$

Donat que $d_{1,2}^{OU} \geq d_{1,2}^{BS}$ per tot S i t, veiem que $\delta_{OU} \geq \delta_{BS}$ i $\varrho_{OU} \geq \varrho_{BS}$. La *call* O-U és més sensible als canvis respecte a S i r.

Per una altra banda, tenim que

$$\upsilon_{OU} = (S/\sigma)[\kappa(T-t)/2\pi]^{1/2}e^{-(d_1^{OU})^2/2}, \tag{R.77}$$

i

$$\theta_{OU} = -Ke^{-r(T-t)}\left[rN(d_2^{OU}) + \frac{\sigma^2(1-e^{-(T-t)/\tau})}{2\sqrt{2\pi\kappa(T-t)}}\, e^{-(d_2^{OU})^2/2}\right]. \tag{R.78}$$

Donat que $d_1^{OU} \geq d_1^{BS}$, es pot veure que $\upsilon_{OU} \leq \upsilon_{BS}$ per tot S/K, $T-t$ i τ. Aleshores, la nostra *call* correlacionada és menys sensible a canvis respecte la volatilitat σ que no pas el preu B-S.

Finalment, estudiarem la cobertura psi

$$\psi(S,t) = \frac{N(d_1)}{N(d'_1)}. \tag{R.79}$$

El cost de mantenir aquesta cobertura serà

$$\psi C' = \frac{N(d_1)}{N(d'_1)} \left[SN(d'_1) - K'e^{-r(T-t)}N(d'_2) \right], \tag{R.80}$$

Anàlogament, el cost de la cobertura delta de Black-Scholes és

$$\delta S = SN(d_1). \tag{R.81}$$

El quocient de les dues cobertures compleix la següent desigualtat

$$0 \leq \frac{K'e^{-r(T-t)}\ N(d'_2)}{S\ N(d'_1)} \leq 1.$$

Per tant, $\psi C' < \delta S$ i la cobertura psi és sempre més barata que la cobertura delta.

6. Un model de volatilitat estocàstica correlacionada

Durant dècades, el procés difussiu conegut com a moviment brownià geomètric ha estat àmpliament acceptat com a un dels models més universals per descriure la dinàmica dels mercats. Malgrat això, especialment després del crac del 1987, el model brownià i la seva subseqüent fórmula Black-Scholes per l'opció no saben reproduir els preus de les opcions negociades als mercats. Existeixen varis estudis que recullen preus empírics de les opcions i en deriven la seva volatilitat implícita (*i.e.*, la volatilitat que hauria de tenir el subjacent si la fórmula B-S fos vàl·lida). Aquests tests conclouen que la volatilitat implícita no és constant. Aqueste efecte és anomenat "efecte somriure", per la forma de u que té la volatilitat implícita en termes de la *moneyness*, i mostra la poca idoneïtat del model assumit per Black-Scholes perquè té volatilitat constant.

Una possible sortida al problema és la que proposen els models de volatilitat estocàstica. Com el seu nom indiquen, aquests models prenen la volatilitat com una varible estocàstica. Des de finals dels vuitanta, s'han presentat varis models difussius bidimensionals que tenen dues variables estocàstiques: el preu $S(t)$ i la volatilitat $\sigma = \sigma(t)$. No obstant, la recerca ha anat gairebé exclusivament destinada a estudiar l'efecte d'assumir la volatilitat estocàstica en la valoració de les opcions.

Existeix alguna excepció, com ara la de Stein i Stein (1991), que estudia les propietats estadístiques d'un model on la volatilitat segueixi un procés d'Ornstein-Uhlenbeck. Creiem que la poca quantitat d'estudis existents que es fixin en la dinàmica d'aquest tipus de models és degut a: (i) Les seves propietats estadístiques són dificils d'obtenir analíticament i encara es complica més el càlcul matemàtic si permetem correlacions entre les fluctuacions de la volatilitat i l'acció. (ii) Generalment es creu que els observables del mercat no són suficients per estimar tots els paràmetres d'aquest tipus de models.

Aquest capítol pretèn modificar aquestes opinions pel cas concret d'una volatilitat que segueixi un procés d'Ornstein-Uhlenbeck (O-U) amb correlacions. Per una banda, som capaços de trobar analíticament les propietats estadístiques del model. I, d'altra banda, la correlació de palanca recentment estudiada per Bouchaud *et al.* (2001) ens ajuda a estimar la resta dels paràmetres que entren en joc.

L'esforç en estudiar empíricament les propietats estadístiques també ha llistat un seguit trets característics que ha de posseir tot "bon" model de mercat proposat. Veurem que el nostre model acompleix la llista de requeriments.

6.1 El model de volatilitat estocàstica

Tot els models de volatilitat estocàstica parteixen de l'equació diferencial

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW_1(t), \tag{R.82}$$

on μ és el ritme de creixement i σ és la volatilitat. Els models SV perfeccionen aquesta dinàmica prenent $\sigma = \sigma(t)$ estocàstica. Existeixen una llarga llista de models d'aquest tipus però encara no hi ha cap dinàmica definitivament associada a σ. Nosaltres triem el procés d'Ornstein-Uhlenbeck (Stein and Stein (1991))

$$d\sigma(t) = -\alpha(\sigma - \theta)dt + kdW_2(t). \tag{R.83}$$

Les equacions (6.1) i (6.2) contenen un procés de Wiener bidimensional (W_1, W_2), on $dW_i(t) = \xi_i(t)dt$ $(i = 1, 2)$, i $\xi_i(t)$ és un soroll gaussià blanc i amb mitjana zero, *i.e.*,

$$E\left[\xi_i(t)\right] = 0 \quad \text{i} \quad E\left[\xi_i(t)\xi_j(t')\right] = \rho_{ij}\delta(t - t'). \tag{R.84}$$

El coeficient de correlació $\rho_{i,j}$ té les propietats $\rho_{ij} = \rho_{ji} = \rho$ amb $-1 \leq \rho \leq 1$ i $\rho_{ii} = 1$. I, en conseqüència,

$$E\left[dW_1(t)dW_2(t)\right] = \rho dt.$$

D'ara en endavant, ens resultarà més cómode analitzar la rendibilitat neutra definida

$$dX \equiv \frac{dS}{S} - \mu dt$$

i aleshores tractar amb

$$dX(t) = \sigma(t)dW_1(t). \tag{R.85}$$

Aquesta variable encara reté les propietats més interessants del procés.

6.2 La volatilitat Ornstein-Uhlenbeck

Estudiem el procés que descriu la dinàmica de la volatilitat. La solució de l'equació diferencial és

$$\sigma(t) = \sigma_0 e^{-\alpha(t-t_0)} + \theta(1 - e^{-\alpha(t-t_0)}) + k \int_{t_0}^{t} e^{-\alpha(t-t')} dW_2(t'), \tag{R.86}$$

entenent que el procés ha començat a temps t_0 just quan la volatilitat és σ_0. Si el procés ha assolit el règim estacionari (*i.e.*, $t_0 \to -\infty$) la volatilitat es llegeix

$$\sigma(t) = \theta + k \int_{-\infty}^{t} e^{-\alpha(t-t')} dW_2(t'), \tag{R.87}$$

amb mitjana, variança i correlació

$$E[\sigma] = \theta, \qquad \mathrm{Var}[\sigma] \equiv E\left[\sigma^2\right] - E[\sigma]^2 = k^2/2\alpha, \tag{R.88}$$

i

$$E[\sigma(t+\tau)\sigma(t)] = \theta^2 + (k^2/2\alpha)e^{-\alpha\tau}. \tag{R.89}$$

Abans de procedir, volem estudiar la qüestió del signe de $\sigma(t)$. Si s'assumeix el procés O-U (R.83) com a model per la volatilitat, es pot argumentar que $\sigma(t)$ pot ser negativa i aquest fet pot ser vist com un inconvenient per poder ser un "bon" model. No obstant, la pròpia volatilitat no es pot observar mai. A la pràctica, l'anomenada volatilitat instantània s'avalua de la manera següent:

$$\lim_{\Delta t \to 0} \sqrt{\left[X(t+\Delta t) - X(t)\right]^2 / \Delta t}, \tag{R.90}$$

Quan fem $\Delta t \to 0$, podem entendre l'equació amb diferències infinitessimals

$$\text{volatilitat instantània } \equiv \sqrt{dX(t)^2/dt}. \tag{R.91}$$

Donat que $dW^2 = dt$,

$$\text{volatilitat instantània } = \sqrt{\sigma(t)^2} = |\sigma(t)|. \tag{R.92}$$

Observeu que en aquesta definició no hi ha assignat el signe que ha de tenir la variable $\sigma(t)$.

Un altre resultat interessant del procés és el comportament tan curiós que té la correlació entre la volatilitat i les fluctuacions provinents de l'acció:

$$E[\sigma(t+\tau)dW_1(t)] = \begin{cases} \rho k e^{-\alpha\tau} dt & \text{if } \tau > 0, \\ 0 & \text{if } \tau < 0. \end{cases} \tag{R.93}$$

D'aquí es dedueix que, per a processos SV correlacionats, la futura volatilitat està correlacionada amb les variacions de rendibilitat passades, però la volatilitat passada no ens diu res sobre les variacions de rendibilitat futures. A més, ρ determina el signe de la correlació (R.93) ja que $k \geq 0$.

El nostre procés descriu reversió en mitjana. Aquest fenòmen empíric es refereix a l'existència d'un nivell normal de volatilitat al qual la volatilitat acaba retornant

$$\lim_{t\to\infty} E\left[\frac{dX(t)^2}{dt}\bigg|\sigma_0\right] \equiv \text{ nivell normal de volatilitat.}$$

El nostre model és adequat donat que

$$\text{nivell normal de volatilitat O-U} = \theta^2 + \frac{k^2}{2\alpha}. \tag{R.94}$$

A més, el promig sobre sigma (6.8) quadrat és

$$E\left[\sigma^2(t)|\sigma_0, 0\right] = \left[\sigma_0 e^{-\alpha t} + \theta^2(1 - e^{-\alpha t})\right]^2 + \frac{k^2}{2\alpha}\left(1 - e^{-2\alpha t}\right).$$

El promig tendeix ràpidament al nivell normal, de manera exponencial. El valor de α determinarà la velocitat exacta.

6.3 L'efecte palanca

És conegut que la volatilitat està negativament correlacionada amb les variacions de la rendibilitat. Aquest efecte ha estat típicament associat a l'efecte palanca[3]. El mecanisme no sembla del tot convincent per justificar aquesta correlació. Tot i amb això, el nom d'efecte palanca ha quedat.

L'efecte no ha estat estudiat de manera quantitativa fins fa ben poc. Definint la correlació de palanca

$$\mathcal{L}(\tau) \equiv \frac{E\left[dX(t+\tau)^2 dX(t)\right]}{\text{Var}[dX(t)]^2}, \tag{R.95}$$

Bouchaud *et al.* (2001) han tractat una gran quantitat de mercats i valors. Han trobat que:

$$\mathcal{L}(\tau) = \begin{cases} -Ae^{-b\tau} & \text{si } \tau > 0, \\ 0 & \text{si } \tau < 0, \end{cases} \tag{R.96}$$

$(A > 0)$.

El nostre model és capaç de descriure aquest efecte. La correlació de palanca és

$$\mathcal{L}(\tau) = 2\rho\left[\frac{\nu\sqrt{2\alpha}\,(1+\nu^2 e^{-\alpha\tau})}{(1+\nu^2)^2\theta}\right] e^{-\alpha\tau} \qquad \text{quan } \tau > 0, \tag{R.97}$$

i

$$\mathcal{L}(\tau) = 0 \qquad \text{quan } \tau < 0. \tag{R.98}$$

Hem definit una nova variable que dóna idea de l'amplitud de les fluctuacions de la volatilitat $\nu^2 \equiv k^2/(2\alpha\theta^2)$. Observem que el signe de la correlació ve donat pel signe de ρ.

[3] "Leverage" en anglés. Es refereix a l'ús de credit per a tenir una major capacitat especulativa.

6.4 Avaluació de la predicció

Qualsevol model de mercat proposat ha de ser capaç de "predir" la dinàmica del mercat. És a dir, ha de poder reproduir la seva evolució. En aquesta secció provem com el nostre model és capaç de fer-ho en cas de que el mercat observat sigui l'índex Dow-Jones de New York.

Les variances

$$\mathrm{Var}[\Delta X] = \theta^2(1+\nu^2)\Delta t \qquad \mathrm{Var}[\Delta X^2] = 2\theta^4\left[4(1+\nu^2)^2 - 3\right]\Delta t^2$$

ens permeten avaluar el valor ν^2 per mitjà del quocient

$$\frac{1}{(1+\nu^2)^2} = \frac{4}{3} - \frac{1}{6}\frac{\mathrm{Var}[\Delta X^2]}{\mathrm{Var}[\Delta X]^2}. \tag{R.99}$$

El valor de theta es pot trobar fàcilment un cop tenim ν^2 i sabem el valor empíric de la variança de ΔX.

No obstant el notre model és capaç de reproduir dinàmiques de mercat tal que la seva curtosi, $\gamma_2 \equiv \mathrm{Var}[\Delta X^2]/\mathrm{Var}[\Delta X]^2 - 2$,

$$0 \leq \gamma_2 < 6, \tag{R.100}$$

Per la rendibilitat diària del Dow-Jones $\gamma_2 = 1.72$ i és dins els marges de consistència requerits.

L'efecte palanca ens subminsitra l'estimació dels altres dos paràmetres: el coeficient de correlació ρ i el temps característic $1/\alpha$. Les dades del Dow-Jones queden afitades amb un temps de decaïment $1/\alpha \simeq 20$ days. A més, el valor empíric de la correlació quan $\tau \to 0^+$

$$\mathcal{L}(0^+) = 2\rho\frac{\nu\sqrt{2\alpha}}{(1+\nu^2)\theta}, \tag{R.101}$$

ens dóna una estimació de ρ que val aproximadament -0.6.

Amb tots aquests paràmetres hem simulat un any d'evolució del mercat i hem vist que és força similar al Dow-Jones. En aquest sentit, el nostre model és predictiu.

6.5 La distribució de probabilitat

No hem estat capaços de trobar la solució de la distribució de probabilitat però sí que hem aconseguit la funció característica de la marginal de la rendibilitat. Aquest fet ja és suficient per trobar les propietats estadístiques més importants.

La funció característica es defineix:

$$\varphi_X(\omega, t|\sigma_0) = \int_{-\infty}^{\infty} e^{i\omega x} p_X(x, t|\sigma_0)dx.$$

Pel nostre cas aquesta val:

$$\varphi_X(\omega,t) = \frac{1}{\sqrt{1+k^2A/\alpha}} \exp\left[-C + \frac{B^2k^2/\alpha - 4\theta B - 4\theta^2 A}{4(1+k^2A/\alpha)}\right]. \tag{R.102}$$

on

$$A(\omega,t) = \frac{\omega^2}{2}\left(\frac{\sinh\eta t}{\eta\cosh\eta t + \zeta\sinh\eta t}\right), \tag{R.103}$$

$$B(\omega,t) = \frac{\omega^2\alpha\theta}{\eta}\left(\frac{\cosh\eta t - 1}{\eta\cosh\eta t + \zeta\sinh\eta t}\right), \tag{R.104}$$

$$C(\omega,t) = \left[\frac{(\omega\alpha\theta)^2}{\eta^2} + i\omega\rho k - \alpha\right] t/2 + \frac{1}{2}\ln\left(\cosh\eta t + \frac{\zeta}{\eta}\sinh\eta t\right) - \frac{(\omega\alpha\theta)^2}{2\eta^3}\left[\frac{2\zeta(\cosh\eta t - 1) + \eta\sinh\eta t}{\eta\cosh\eta t + \zeta\sinh\eta t}\right], \tag{R.105}$$

i

$$\eta = \sqrt{\alpha^2 - 2i\rho k\alpha\omega + (1-\rho^2)k^2\omega^2}, \qquad \zeta = \alpha - i\omega\rho k. \tag{R.106}$$

Es pot veure també que, quan $\alpha t \gg 1$ amb $k/\alpha \ll 1$, la nostra funció característica esdevé

$$\varphi_X(\omega,t) \sim \exp\left\{-\omega^2\left[1+\nu^2+\mathrm{O}(k/\alpha)\right]\theta^2 t/2\right\}, \qquad (\alpha t \gg 1), \tag{R.107}$$

i el teorema del límit central es compleix.

Tenint la funció característica, podem trobar els seus cumulants

$$\kappa_n \equiv (-i)^n \left.\frac{\partial^n}{\partial\omega^n}\ln[\varphi_X(\omega,t)]\right|_{\omega=0}.$$

Aquests, a la seva vegada, ens proporcionen la curtosi i el biaix. Presentem els seus valors assimptòtics. Per una banda, la curtosi

$$\gamma_2 \sim \frac{6\nu^2(\nu^2+2)}{(\nu^2+1)^2} \qquad (\alpha t \ll 1), \tag{R.108}$$

i

$$\gamma_2 \sim \frac{6\nu^2[\nu^2(1+4\rho^2)+4(1+\rho^2)]}{(\nu^2+1)^2}\,\frac{1}{\alpha t} \qquad (\alpha t \gg 1). \tag{R.109}$$

Observeu que malgrat tractar amb temps inifintessimals obtenim encara una curtosi no negligible. La curtosi va a zero a mesura que el temps creix. La seva convergència, però, és lenta. Per una altra banda, el biaix

$$\gamma_1 \sim 3\rho\,\frac{\nu}{\sqrt{\nu^2+1}}\,\sqrt{2\alpha t} \qquad (\alpha t \ll 1), \tag{R.110}$$

i

$$\gamma_1 \sim 6\rho \, \frac{\nu(\nu^2+2)}{(\nu^2+1)^{3/2}} \, \frac{1}{\sqrt{2\alpha t}} \qquad (\alpha t \gg 1). \tag{R.111}$$

En aquest cas, el biaix va a zero per temps petits i per temps grans.

Com ja hem dit, no ens és possible de trobar la distribució de probabilitat al no poder fer la anti-transformada de Fourier de manera analítica. Ara bé, podem donar la probabilitat quan la rendibilitat és gran. Aquesta és

$$p_X(x,t) \sim \frac{1}{\sqrt{a(t)^2+b(t)}} \exp\left[-\frac{1}{b(t)}\left(\sqrt{a(t)^2+b(t)}-a(t)\right)x\right] \quad (x \to \infty) \tag{R.112}$$

i

$$p_X(x,t) \sim \frac{1}{\sqrt{a(t)^2+b(t)}} \exp\left[\frac{1}{b(t)}\left(\sqrt{a(t)^2+b(t)}+a(t)\right)x\right] \quad (x \to -\infty) \tag{R.113}$$

amb

$$a(t) \equiv \rho k t/4, \qquad \text{i} \qquad b(t) \equiv k^2 t(2-\rho^2\alpha t)/8\alpha. \tag{R.114}$$

Donat que $a(t) = \rho k t/4$, el signe de ρ determinarà quina és la cua més alta. Quan rho és negatiu la més alta serà la que representarà les pèrdues i quan $\rho > 0$ la més alta correspondrà a la dels guanys. Si $\rho = 0$, no existeix cap diferència entre les dues cues.

Finalment, la distribució de preus (no pas de rendibilitats) $p_S(S,t)$ tindrà els decaïments polinòmics desitjats:

$$p_S(S,t) \sim \frac{1}{S^{\nu_-(t)}} \quad (S \to 0), \qquad \text{i} \qquad p_S(S,t) \sim \frac{1}{S^{\nu_+(t)}} \quad (S \to \infty) \tag{R.115}$$

on

$$\nu_\pm(t) = 1 + \frac{1}{b(t)}\left[\pm\sqrt{a(t)^2+b(t)} - a(t)\right].$$

7. Conclusions i noves perspectives

Es pot dir que, actualment, la física també estudia els mercats financers. Aquesta tesi representa la introducció a aquest tema no només per a mi sinó per a tot el meu grup de recerca que dirigeix el Prof. Jaume Masoliver. Típicament, el grup de recerca s'havia dedicat a estudiar aquells sistemes dinàmics estocàstics que intervenen en fenomens com el transport de la llum dins medis desordenats o la cinètica de les reaccions químiques entre d'altres. En aquest cas, els mètodes estocàstics són aplicats als mercats financers.

Heus aquí una enumeració dels continguts d'aquest treball:

- Abans d'aportar quelcom d'original en el coneixement dels mercats financers, ens hem dedicat a donar les definicions bàsiques al capítol 1 i després hem revisat la recerca en matemàtica financera al llarg del darrer segle en el capítol 2. A l'apèndix A del capítol 2, resumim un dels nostres articles de recerca. Allí hi exemplifiquem les diferències existents entre prendre diferències de preu i diferències de rendibilitat i ens mostrem les seves conseqüències.

  -Masoliver, J., M. Montero, J. Perelló, 2001a, Return or stock price differences, submitted for publication.

- El mètode valoració d'opcions de Black-Scholes presenta el primer preu just per a l'opció. En el capítol 3 hem revisat aquesta aportació fixant-nos en els aspectes que poden resultar més interessants per a un físic no entés en la matèria. Hem implementat la teoria Black-Scholes amb el càlcul d'Itô i el d'Stratonovich. Itô és la convenció que normalment utilitzen els matemàtics i Stratonovich és el que fan servir els físics. Fins ara, només s'havia derivat l'equació Black-Scholes amb la interpretació d'Itô. De fet, sembla que aquesta és més adequada i que s'adiu més a l'operativa del mercat. En qualsevol cas, hem vist que ambdues interpretacions són equivalents i condueixen a la mateixa solució.

  -Perelló, J., J. M. Porrà, M. Montero, J. Masoliver, 2000, Black-Scholes option pricing within Itô and Stratonovich conventions, *Physica A* **278**, 260-274.

- En el capítol 4 analitzem les generalitzacions al model presentat per Black i Scholes (1973) però que encara s'encabeixen dins la teoria Black-Scholes. Les generalitzacions van per dos camins diferents: incloent noves dinàmiques pel preu i trobant nous mètodes de valoració més simples. Com a aportació novedosa cal resaltar un treball de recerca que es basa amb el preu obtingut via martingala. Representem el preu de l'opció en termes de la funció característica de la rendibilitat neutra.

- El capítol 5 es dedica a relaxar una de les hipòtesis del model de Black-Scholes. Presentem un model de mercat ineficient que ve conduït per un procés Ornstein-Uhlenbeck. Donem preu a l'opció i veiem que les conseqüències de l'existència de correlacions en les variacions de la rendibilitat influeix de manera no trivial al preu de l'opció.

  -Masoliver, J., J. Perelló, 2001b, Option pricing and perfect hedging on correlated stocks, submitted for publication.

- El capítol 6 presenta un model de volatilitat estocàstica on la volatilitat està correlacionada amb la rendibilitat i ve descrita per un procés Ornstein-Uhlenbeck. Amb aquest model expliquem de manera quantitativa l'efecte palanca i altres trets característics com la reversió en mitjana, la leptocurtosi i el biaix negatiu. Gràcies a la correlació de palanca som capaços d'estimar tots els paràmetres del model cosa que no s'havia pogut fer fins ara. També obtenim una expressió analítica per a la funció característica i estudiem les cues de la densitat de probabilitat.

-Masoliver, J., J. Perelló, 2001c, A correlated sotchastic volatility model measuring leverage and other stylized facts, submitted for publication.

-Perelló, J., J. Masoliver, 2001d, Correlated stochastic volatility models and the leverage effect. Poster session: 4th "Applications of Physics to Financial Analysis (APFA)" conferences, London, 5-7 December 2001.

Podem anticipar que la recerca futura es dedicarà a:

- La representació de l'opció en termes de la funció característica de la rendibilitat neutra és útil per derivar el preu de l'opció en cas de que només coneguem la funció característica del subjacent. Podem veure així els efectes de les cues i del soroll de color en l'opció.

  -Perelló, J., and J. Masoliver, 2001e, The effect of colored noise and heavy tails on financial options, to appear in *Physica A*. Invited talk in: "Horizons in complex systems", Messina, 5-8 December 2001

- Hem estudiat els efectes de les correlacions en les opcions europees de compra. Podríem fer el mateix amb les opcions americanes i les *puts* europees.

- Existeixen una gran varietat de models de volatilitat estocàstica. Ara que tenim una manera sistemàtica d'estimar tots els paràmetres del model podem discernir quin és el model més adequat per descriure els mercats en general o bé particularitzar per a cadascun dels diversos mercats existents.

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2. Masoliver, J., M. Montero, J. Perelló, 2001a, Return or stock price differences, submitted for publication.

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5. Perelló, J., and J. Masoliver, 2001d, Correlated stochastic volatility models and the leverage effect. Poster session: 4th "Applications of Physics to Financial Analysis (APFA)" conferences, London, 5-7 December 2001.

6. Perelló, J., and J. Masoliver, 2001e, The effect of colored noise and heavy tails on financial options, to appear in *Physica A*. Invited talk in: "Horizons in complex systems", Messina, 5-8 December 2001.