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Hodge numbers of irregular varieties and fibrations

by

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A Cris.

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Contents

Summary

We present here a brief, precise overview of the contents of this Thesis. The main topic of the Thesis is the interplay between numerical invariants of a projective complex variety (or more generally, compact Kähler) and the properties of its fibrations over lower-dimensional varieties (if any). In fact, we have restricted the problem to irregular varieties, that is, varieties admitting non-zero holomorphic differential 1-forms. Because of this restriction, Abelian varieties (in particular, the Picard and Albanese varieties) will appear all along the memory.

Part I: Hodge numbers of irregular varieties

The first part of the Thesis deals with irregular varieties of arbitrary dimension, giving numerical conditions on their Hodge numbers that imply the existence of fibrations, or more generally, obtaining some inequalities between the Hodge numbers of varieties admitting some special subspaces of holomorphic forms.

In order to avoid unnecessary repetitions, if nothing is said explicitly X will denote a smooth irregular variety, either complex projective or compact Kähler, of dimension d and irregularity $q = q(X) = \dim H^0(X, \Omega^1_X)$.

It is known that some class of fibrations of an irregular variety X are closely related to the cohomological support loci $V^i(X)$, which are the closed subsets of $\operatorname{Pic}^0(X)$ defined as

$$V^i(X) = \left\{ L \in \operatorname{Pic}^0(X) \mid h^i(X, \omega_X \otimes L) \neq 0 \right\}.$$

The basic results about these loci can be summarized in the following

Theorem 1.3.2 ([19] Th. 0.1, [41])**.** *Let W be an irreducible component of some $V^i(X)$. Then*

1. *there exist a subtorus $Z \subseteq \operatorname{Pic}^0(X)$ and a torsion point $\beta \in \operatorname{Pic}^0(X)$ such that $W = \beta + Z$, and*
2. *there exists a fibration $f : X \to Y$ onto a normal variety Y of dimension $\dim Y \leq d - i$, such that (any smooth model of) Y is of maximal Albanese dimension and $Z \subseteq f^* \operatorname{Pic}^0(Y)$.*

Partial Euler characteristics

Our first family of results is inspired in the BGG complex introduced by Lazarsfeld and Popa in [27]. By truncating the BGG complex after some steps, and using the same techniques, we study the *partial Euler characteristics* of X, which are defined as follows.

Definition 2.1.1. *For any integer $0 \leq i \leq d$, we define the i-th* partial Euler characteristic *of X as*

$$\chi_i\left(X\right) = h^i\left(X, \omega_X\right) - h^{i+1}\left(X, \omega_X\right) + \cdots + \left(-1\right)^{d-i} h^d\left(X, \omega_X\right).$$

To be precise, we obtain the following general inequalities.

Proposition 2.1.3. *If X is any irregular variety, then*

$$\chi_k\left(X\right) \geq 0 \quad \text{for every } k \geq d - \dim \mathrm{alb}_X\left(X\right).$$

Theorem 2.1.4. *Let k be an integer such that $d - \dim \mathrm{alb}_X\left(X\right) \leq k < d$. If X does not admit any irregular fibration $f : X \to Y$ with $\dim Y < d - k$, or more generally, if $\mathcal{O}_X$ is an isolated point of $V^i\left(X\right)$ for all $i > k$, then*

$$\chi_r\left(X\right) \geq \left(q\left(X\right) - \dim X\right) + r$$

for all $k \leq r < d$.

Proposition 2.1.3 is also a consequence of the generic vanishing results of Green and Lazarsfeld [18] and their relation with the exactness of the derivative complexes studied also in [13], while Theorem 2.1.4 is analogous to the higher-dimensional Castelnuovo-de Franchis inequality proved by Pareschi and Popa in [31], and later by Lazarsfeld and Popa in [27]. In fact, we prove Theorem 2.1.4 following the ideas of the latter.

After checking with some examples that these inequalities do not seem to be sharp, we exploit the functorial behaviour of the BGG complex to get much stronger inequalities for some varieties. Recall (Definition 2.1.12, taken from [35]) that an m-dimensional smooth subvariety $Y \subseteq A$ of an Abelian variety A is *geometrically non-degenerate* if the restriction $H^0\left(A, \Omega_A^m\right) \to H^0\left(Y, \omega_Y\right)$ is injective. We prove some general results about these subvarieties that ultimately give the following

Corollary 2.1.16. *Let X be a smooth n-dimensional irregular variety such that its Albanese image $Y = \mathrm{alb}_X\left(X\right)$ is smooth of dimension $m = \dim Y$. Assume moreover that $\mathcal{O}_X \in V^i\left(X\right)$ is isolated for every $i > n - m$ and that Y is geometrically non-degenerate. Then*

$$\chi_{n-r}\left(X\right) \geq \binom{q\left(X\right) - 1}{r} \qquad \forall\, r = 1, \ldots, m-1.$$

In particular, if X is primitive (hence of maximal Albanese dimension) and its Albanese image is smooth and non-degenerate, then

$$\chi_r(X) \geq \binom{q(X)-1}{\dim(X)-r} \qquad \forall\, r = 1, \ldots, \dim X - 1.$$

Higher-rank derivative complexes

Following the same stream of obtaining new inequalities between the Hodge numbers of X, we introduce the following higher-rank generalization of the derivative complex.

Definition 2.2.1. *Fix integers $r \geq 1$ and $0 \leq j \leq d$, set $n = \min\{r, d\}$, and fix a linear subspace $W \subseteq H^0(X, \Omega_X^1)$. We define $C_{r,W}^j$ as the complex (of vector spaces)*

$$\begin{aligned} 0 \longrightarrow \operatorname{Sym}^r W \otimes H^j(X, \mathcal{O}_X) \longrightarrow \operatorname{Sym}^{r-1} W \otimes H^j\left(X, \Omega_X^1\right) \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Sym}^{r-i} W \otimes H^j\left(X, \Omega_X^i\right) \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Sym}^{r-n} W \otimes H^j(X, \Omega_X^n) \end{aligned}$$

where the maps $\mu_i^j : \operatorname{Sym}^{r-i} W \otimes H^j(X, \Omega_X^i) \to \operatorname{Sym}^{r-i-1} W \otimes H^j\left(X, \Omega_X^{i+1}\right)$ are given by

$$\mu_i^j\left((w_1 \cdots w_{r-i}) \otimes [\alpha]\right) = \sum_{t=1}^{r-i} (w_1 \cdots \widehat{w_t} \cdots w_{r-i}) \otimes [w_t \wedge \alpha].$$

It is worth noting that, although the complexes $C_{r,n,W}^j$ above are generalizations of the derivative complexes, they have not been obtained from a "derivative construction". In fact, they are defined directly as above.

The higher-rank derivative complexes can be seen as the result of applying the j-th cohomology functor to the complex of sheaves

$$\begin{aligned} \mathcal{C}_{r,W} : 0 \longrightarrow \operatorname{Sym}^r W \otimes \mathcal{O}_X \longrightarrow \operatorname{Sym}^{r-1} W \otimes \Omega_X^1 \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Sym}^{r-i} W \otimes \Omega_X^i \longrightarrow \cdots \longrightarrow \operatorname{Sym}^{r-n} W \otimes \Omega_X^n. \end{aligned}$$

Following an approach inspired in Section 3 of [18], we study the exactness of the $C_{r,W}^j$ with the combined use of two spectral sequences, both abutting to the hypercohomology of $\mathcal{C}_{r,W}$. As in [18], one of the spectral sequences degenerates at the second page (see Proposition 2.2.10), but the other one is not so well behaved. In fact, the second spectral sequence depends on the cohomology sheaves $\mathcal{H}^j$ of $\mathcal{C}_{r,W}$, which can be computed in some cases with the help of Eagon-Northcott type complexes. Indeed, $\mathcal{C}_{r,W}$ is dual to such a complex $E_r(\phi_W)$, which is constructed from the dual ϕ_W of the evaluation map

$$ev_W : W \otimes \mathcal{O}_X \longrightarrow \Omega^1_X.$$

Eagon-Northcott complexes have been extensively studied (see for example [14] Appendix A.6, [25] Appendix B, [8] or [1]), and their exactness depends on the degeneracy loci of ϕ. For any positive integer i, let

$$Z_i(W) = \left\{p \in X \mid \operatorname{rk}\left(ev_W(p) : W \longrightarrow \Omega^1_X(p)\right) < i\right\},$$

the locus where the 1-forms in W do not span a subspace of dimension at least i, and make the following

Definition 2.2.12 (Non-degenerate subspace)**.** *We say that a subspace* $W \subseteq H^0(X, \Omega^1_X)$ *is* non-degenerate *if*

$$\operatorname{codim} Z_i(W) \geq d - i + 1 \qquad \forall\, 1 \leq i \leq \min\{\dim W, d\}.$$

It turns out that, if W is non-degenerate, the Eagon-Northcott complex $E_r(\phi)$ is exact and the cohomology sheaves $\mathcal{H}^i$ of $\mathcal{C}_{r,W} = E_r(\phi)^\vee$ are easy to compute (Lemma 2.2.14). Finally, using the two spectral sequences mentioned above, we can prove the following general result.

Theorem 2.2.15. *If W is non-degenerate, then the complex*

$$\begin{aligned} C^j_{r,W} : 0 \longrightarrow \operatorname{Sym}^r W \otimes H^j(X, \mathcal{O}_X) \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Sym}^{r-i} W \otimes H^j\left(X, \Omega^i_X\right) \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Sym}^{r-n} W \otimes H^j(X, \Omega^n_X) \end{aligned}$$

is exact at least in the first $d - \dim W - j + 1$ steps.

Subvarieties of Abelian varieties

In the case that X is a subvariety of an Abelian variety A such that $H^0(X, \Omega^1_X) = H^0(A, \Omega^1_A)$, there are non-degenerate subspaces of any dimension $1 \leq k \leq q(X)$ (Proposition 2.2.20). Hence Theorem 2.2.15 gives in particular the following

Corollary 2.2.22. *If X is a subvariety of an Abelian variety A such that $H^0(X, \Omega^1_X) = H^0(A, \Omega^1_A)$, and $p, j \geq 0$ satisfy $\max\{p, j\} \leq d + 1 - (p + j)$, then*

$$h^{p,j}(X) \geq \binom{d + 1 - (p + j)}{p}\binom{d + 1 - (p + j)}{j}.$$

Improvements for $h^{2,0}(X)$

All the previous results are not directly related with the fibrations of X. In fact, they depend on the existence of non-degenerate subspaces of 1-forms, with the further restriction that they must have dimension $k \leq d$. Fortunately, the complex $C^0_{2,W}$ is easy to study more or less by hand, and much better results can be obtained.

Consider the wedge product map $\psi_2 : \bigwedge^2 H^0(X, \Omega^1_X) \to H^0(X, \Omega^2_X)$. An element $v \in \bigwedge^2 H^0(X, \Omega^1_X)$ has rank $2k$ if it can be expressed as

$$v = v_1 \wedge v_2 + \cdots + v_{2k-1} \wedge v_{2k}$$

for some linearly independent $v_1, \ldots, v_{2k} \in H^0(X, \Omega^1_X)$, and there is no such a expression with fewer terms (Definition 2.3.1).

Consider also the Grassmannian variety $\mathbb{G}_m = Gr(m, H^0(X, \Omega^1_X))$ of m-dimensional subspaces of $H^0(X, \Omega^1_X)$. It is possible to glue all the $C^0_{2,W}$ into the following complex of vector bundles on $\mathbb{G}_m$,

$$C^0_2 : 0 \longrightarrow \operatorname{Sym}^2 S \longrightarrow S \otimes H^0\left(X, \Omega^1_X\right) \longrightarrow \mathcal{O}_{\mathbb{G}} \otimes H^0\left(X, \Omega^2_X\right),$$

where S is the tautological subbundle of $\mathbb{G}_m$. In fact, this can be done for any $C^j_{r,W}$, obtaining higher-rank analogues of the BGG complex.

With these notations, the main result concerning C^0_2 is the following

Theorem 2.3.3. *Fix a positive integer $k \leq \frac{q}{2}$. If every non-zero element in $\ker \psi_2$ has rank bigger than $2k$, then the complex C^0_2 on $\mathbb{G}_{2k}$ is generically exact.*

Counting dimensions we obtain an inequality for $h^{2,0}(X)$.

Corollary 2.3.4. *If there is no non-zero element of rank $2k \leq q$ in $\ker \psi_2$, then*

$$h^{2,0}(X) \geq 2rq - \binom{2r+1}{2}$$

for all $1 \leq r \leq k$.

If X is not fibred, then all non-zero elements in $\ker \psi_2$ have rank at least $2d$ (Lemma 2.3.7). Hence, taking the maximum of the right-hand sides of the above inequalities for $1 \leq r < d$, we obtain the final

Theorem 2.3.9. *Let X be an irregular variety without fibrations over smaller-dimensional irregular varieties. Then it holds*

$$h^{2,0}(X) \geq \begin{cases} \binom{q(X)}{2} & \text{if } q(X) \leq 2\dim X - 1 \\ 2(\dim X - 1)\, q(X) - \binom{2\dim X - 1}{2} & \text{otherwise.} \end{cases}$$

This Theorem generalizes the Castelnuovo-de Franchis inequality to higher dimensions, but in a different way than the works of Lazarsfeld, Pareschi and Popa ([31, 27]). The case $q \leq 2d-1$ can be easily deduced from the work [10] of Causin and Pirola. On the other hand, for the general case $q \geq 2d$, Theorem 2.3.9 improves several inequalities obtained by Lombardi in [28] for $d = 3, 4$ (with slightly more restrictive hypothesis than only the non-existence of fibrations).

Comparing the two methods

In the final section of the first part of the Thesis we consider a different approach that could produce the same inequalities of Theorem 2.3.9, but starting from the general Theorem 2.2.15. If this new method works, it could be extended to other C_r^j and used to find stronger inequalities for Hodge numbers other than $h^{2,0}(X)$.

This new approach depends on general computations on the cohomology algebra of Grassmannian varieties, which we have only been able to carry out in some small cases. We have observed some regularities in these computations that would give the desired result (see Conjecture 2.4.4 and Proposition 2.4.5), but we have been unable to proof that they hold in the general case.

Part II: Fibred surfaces

In the second part of the Thesis, the scope is restricted, with some exceptions, to irregular surfaces fibred over a curve. The main objective is to prove the following

Theorem 6.3.4. *Let $f: S \to B$ be a fibration of genus g, relative irregularity q_f and Clifford index c_f. If f is non-isotrivial, then*

$$q_f \leq g - c_f.$$

Recall that the genus and the Clifford index of a fibration are respectively the genus and the Clifford index of a general fibre (Definitions 3.1.5 and 3.4.1), and that the relative irregularity $q_f = q(S) - g(B)$ is the difference between the irregularities of the total space S and the base B (Definition 3.1.6). Recall also that a fibration is isotrivial if all its smooth fibres are isomorphic (Definition 3.1.4).

In order to reach such a result, we have previously studied some aspects of infinitesimal deformations of smooth curves (Section 4.1), which we have then extended to arbitrary one-dimensional families of curves (Section 4.3). We have also developed some results about adjoint images (Chapter 5), which are a very useful tool to study both infinitesimal and local deformations of varieties of Albanese general type.

Infinitesimal deformations of smooth curves

In the first section of Chaper 4 we develop some ideas about infinitesimal deformations of smooth curves introduced by Collino and Pirola in [11], specially the concept of *divisor supporting a deformation*. Let C be a smooth compact curve of genus $g \geq 2$,

$$\mathcal{C} \longrightarrow \operatorname{Spec} \mathbb{C}\left[\epsilon\right] / \left(\epsilon^2\right)$$

an infinitesimal deformation, and let $\xi \in H^1\left(C, T_C\right)$ be its Kodaira-Spencer class. It is said (Definition 4.1.8) that ξ is *supported on* an effective divisor D if and only if it belongs to the kernel of

$$H^1\left(C, T_C\right) \longrightarrow H^1\left(C, T_C\left(D\right)\right).$$

We say furthermore that ξ is *minimally* supported on D if and only if it is not supported on any other effective divisor $E < D$.

From another point of view, ξ corresponds to the extension of locally free sheaves on C

$$\xi: \quad 0 \longrightarrow N^{\vee}_{C/\mathcal{C}} \cong \mathcal{O}_C \longrightarrow \Omega^1_{\mathcal{C}|C} \longrightarrow \omega_C \longrightarrow 0,$$

and it is supported on D if and only if the pull-back sequence ξ_D splits.

$$\begin{tikzcd}
\xi_D: & 0 \arrow[r] & N^{\vee}_{C/\mathcal{C}} \arrow[r] \arrow[d, equal] & \mathcal{F}_D \arrow[r] \arrow[d, hook] & \omega_C\left(-D\right) \arrow[r] \arrow[l, dashed, bend right] \arrow[d, hook] & 0 \\
\xi: & 0 \arrow[r] & N^{\vee}_{C/\mathcal{C}} \arrow[r] & \Omega^1_{\mathcal{C}|C} \arrow[r] & \omega_C \arrow[r] & 0
\end{tikzcd}$$

In fact, this definition (in any of its equivalent forms) can be extended to infinitesimal deformations of irregular varieties of any dimension (see Definition 5.1.3).

In a more geometrical flavor, one can define the *span* $\langle D \rangle$ of D inside the bicanonical space $\mathbb{P}\left(H^0\left(C, \omega_C^{\otimes 2}\right)^{\vee}\right) = \mathbb{P}\left(H^1\left(C, T_C\right)\right)$ as the intersection of all the hyperplanes schematically containing D. Then ξ is supported on D if and only if the point $[\xi]$ lies on $\langle D \rangle$.

The main result about infinitesimal deformations of smooth curves is the following theorem (due to Ginensky [17]), which gives a lower-bound on the rank of the cup-product map

$$\cup \xi: H^0\left(C, \omega_C\right) \longrightarrow H^1\left(C, \mathcal{O}_C\right)$$

(written $\operatorname{rk} \xi$ for short) in terms of some invariants of a divisor D minimally supporting ξ.

Theorem 4.1.17. *If ξ is minimally supported on D, then*

$$\operatorname{rk}\xi \geq \deg D - 2\dim|D|\,.$$

One dimensional families of curves

After a technical interlude to introduce the *relative Ext sheaves* $\mathcal{E}\mathrm{xt}^i_f$ (Section 4.2), we devote the last section of Chaper 4 to extend to arbitrary (one-dimensional) families of curves some of the concepts and results known for infinitesimal deformations. To be precise, let $f : S \to B$ be a fibration from a smooth surface S onto an analytical smooth curve B (not necessarily compact). Assume also that f is not isotrivial.

The role of the Kodaira-Spencer class of an infinitesimal deformation is now played by the exact sequence

$$\xi: \quad 0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/B} \longrightarrow 0,$$

which serves as a definition of the *sheaf of relative differentials* $\Omega^1_{S/B}$.

We define the *relative bicanonical space* $\mathbb{P}$ as the projective bundle corresponding to the sheaf

$$\mathcal{E} = \mathcal{E}\mathrm{xt}^1_f\left(\Omega^1_{S/B}, f^*\omega_B\right),$$

that is, $\mathbb{P} = \operatorname{Proj}_{\mathcal{O}_B}\left(\operatorname{Sym}^*\mathcal{E}^\vee\right)$. The extension ξ gives a section of $\mathcal{E}$, which is not identically zero because f is not isotrivial. Hence, it induces a section $\gamma : B \to \mathbb{P}$, which sends a general point $b \in B$ to the class of the infinitesimal deformation of the fibre $C_b = f^{-1}(b)$.

Given any subscheme $\Gamma \subset S$, we define its span $\mathbb{P}_\Gamma \subseteq \mathbb{P}$ (Definition 4.3.11) in such a way that over a general point $b \in B$ it coincides with the span of the divisor $\Gamma \cap C_b$ in the context of infinitesimal deformations. As a consequence of the way we construct $\mathbb{P}_\Gamma$, it turns out to depend only on the divisorial components of Γ dominating B (Corollary 4.3.15).

Let $\mathcal{L}_\Gamma$ be the kernel of the composition

$$\Omega^1_{S/B} \xrightarrow{\alpha} \omega_{S/B} \to \omega_{S/B|\Gamma}$$

(for more details about the *relative canonical sheaf* $\omega_{S/B} = \omega_S \otimes f^*\omega_B^\vee$ and the map α see Section 3.2). Denote by ξ_Γ the pull-back sequence

$$\begin{tikzcd}
\xi_\Gamma : & 0 \arrow[r] & f^*\omega_B \arrow[r] \arrow[d, equal] & \mathcal{F}_\Gamma \arrow[r] \arrow[d, hook] & \mathcal{L}_\Gamma \arrow[r] \arrow[d, hook] & 0 \\
\xi : & 0 \arrow[r] & f^*\omega_B \arrow[r] & \Omega^1_S \arrow[r] & \Omega^1_{S/B} \arrow[r] & 0
\end{tikzcd}$$

One could be tempted to define ξ to be supported on Γ if and only if $\mathcal{L}_\Gamma$ splits. In general, this direct definition is too restrictive, so we have introduced a slightly more relaxed one (Definition 4.3.13). Although we do not reproduce it here because it is quite technical, we can informally say that ξ is supported on Γ if for a general $b \in B$, the deformation of the fibre C_b is supported on $\Gamma \cap C_b$.

One of the main properties of this definition is that it also can be characterized in terms of the span of Γ.

Proposition 4.3.14. *The deformation ξ is supported on Γ if and only if the image of $\gamma : B \to \mathbb{P}$ lies in $\mathbb{P}_\Gamma$.*

Also, in some cases, it can be characterized in terms of the splitting of the pull-back ξ_Γ.

Lemma 4.3.17. *Assume that Γ_{div}, the divisorial subscheme of Γ, satisfies*

$$\Gamma_{div} \cdot C_b < 2g - 2$$

for some smooth fibre C_b. Then ξ is supported on Γ if and only if the pull-back sequence ξ_Γ splits.

Adjoint images

In order to prove Theorem 6.3.4, we will first need to produce a subscheme supporting the fibration, or more generally, a subsheaf $\mathcal{L} \subseteq \Omega^1_{S/B}$ such that the pull-back of ξ splits. This is accomplished using adjoint images.

Adjoint images were introduced by Collino and Pirola in [11] for infinitesimal deformations of curves, and then extended to higher dimensions by Pirola and Zucconi in [34]. We devote the first sections of Chapter 5 to give an overview of the basic definitions and known results (basically, the Adjoint and Volumetric Theorems), which can be summarized as follows.

Let $\mathcal{X} \to \operatorname{Spec} \mathbb{C}[\epsilon] / (\epsilon^2)$ be an infinitesimal deformation of an irregular variety X of dimension d, and denote by $\xi \in H^1(X, T_X)$ its Kodaira-Spencer class. Let $W \subseteq H^0(X, \Omega^1_X)$ be a $(d+1)$-dimensional subspace contained in

$$\begin{aligned} K_\xi &= \ker\left(H^0\left(X, \Omega^1_X\right) \xrightarrow{\cup \xi} H^1\left(X, \mathcal{O}_X\right)\right) \\ &= \operatorname{im}\left(H^0\left(X, \Omega^1_{\mathcal{X}|X}\right) \longrightarrow H^0\left(X, \Omega^1_X\right)\right) \end{aligned}$$

with basis $w_1, \ldots, w_{d+1}$. Define $W^d \subseteq H^0(X, \omega_X)$ as the image of $\bigwedge^d W$ by wedge product, and let D_W be the base divisor of the induced sublinear system of $|\omega_X|$ (assuming $W^d \neq 0$).

By the definition of K_ξ, we can choose some infinitesimal extensions $s_i \in H^0\left(X, \Omega^1_{\mathcal{X}|X}\right)$ of the w_i. Their wedge product

$$\sigma = s_1 \wedge \cdots \wedge s_{d+1}$$

belongs to $H^0\left(X, \omega_{\mathcal{X}|X}\right)$, which is isomorphic to $H^0\left(X, \omega_X\right)$ by the Poincaré residue map. The content of the Adjoint Theorem (Theorem 5.1.4) is that $\sigma \in W^d$ if and only if ξ is supported on D_W. In this case, it is said that *the adjoint class of W vanishes.*

Our results about adjoint images concern only deformations of curves, giving numerical conditions that guarantee the existence of subspaces with vanishing adjoint class. We first deal with an infinitesimal deformation ξ of a curve C of genus g.

Theorem 5.2.7. *If $V \subseteq K_\xi$ has dimension $\dim V > \frac{g+1}{2}$, then there exists some 2-dimensional subspace $W \subseteq V$ with vanishing adjoint class.*

In order to prove it we construct the *adjoint map*, a map of vector bundles on the Grassmannian $\mathbb{G}$ of 2-dimensional subspaces of K_ξ, which vanishes at a point $W \in \mathbb{G}$ if and only if its adjoint image vanish. Then a computation of Chern classes finishes the proof.

All the discussions so far consider only infinitesimal deformations, but our setting is a global family of curves (over a compact curve B). Hence we devote the final section of Chapter 5 to construct a global version of the adjoint map. Since we need to consider 1-forms on every fibre that extend infinitesimally, we take the vector space

$$V = H^0\left(S, \Omega^1_S\right) / f^* H^0\left(B, \omega_B\right) \subseteq H^0\left(S, \Omega^1_{S/B}\right),$$

which has dimension q_f, and then consider vector subbundles of $V \otimes \mathcal{O}_B$ of rank 2. With a construction analogous to the case of infinitesimal deformations above, the *global adjoint map*, we can prove the second main result concerning adjoint images.

Theorem 5.3.4. *If*

$$q_f > \frac{g+1}{2},$$

then there exist a finite change of base $\pi : B' \to B$ and a rank-two vector subbundle $\mathcal{W} \subseteq V \otimes \mathcal{O}_{B'}$ whose associated global adjoint map vanishes identically.

Finally, after changing the base of the fibration by π, the Adjoint Theorem implies (Proposition 6.3.7) that the subsheaf we are looking for is the image of the relative evaluation map

$$f^*\mathcal{W} \longrightarrow \Omega^1_{S'/B'}.$$

In the language of supporting subschemes, this is equivalent to say that the fibration is supported on the zero locus of the above map.

Isotriviality of fibrations

The final chapter of the Thesis contains the proof of Theorem 6.3.4, part of which is based on the following structure result.

Theorem 6.3.1. *Let $f : S \to B$ be a fibration of genus g and relative irregularity $q_f \geq 2$. Suppose it is supported on an effective divisor D such that $D \cdot C < 2g-2$ and $h^0\left(C, \mathcal{O}_C\left(D_{|C}\right)\right) = 1$ for some smooth fibre C. Then, after finitely many blow-ups and a change of base, there is a different fibration $h : S \to B'$ over a curve of genus $g(B') = q_f$. In particular S is a covering of the product $B \times B'$, and both surfaces have the same irregularity.*

In fact, Theorem 6.3.1 can be considered as the most important result in Chapter 6, giving a criterion for the isotriviality of fibred surfaces.

The proof of Theorem 6.3.1 relies on the following technical proposition, whose proof is the content of Section 6.2. As a matter of language, we say that a subsheaf $\mathcal{L} \subseteq \Omega^1_{S/B}$ *lifts to* Ω^1_S if there is an injective map $\mathcal{L} \hookrightarrow \Omega^1_S$ factoring the inclusion into $\Omega^1_{S/B}$ (Definition 6.2.1). Equivalently, $\mathcal{L} \subseteq \Omega^1_{S/B}$ lifts to Ω^1_S if the pull-back sequence $\xi_{\mathcal{L}}$ is split.

Proposition 6.2.2. *Assume that $f : S \to B$ is a fibration with reduced fibres. If a rank-one subsheaf $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ satisfies* $\deg\left(\mathcal{L}_{|C_b}\right) > 0$ *for some smooth fibre C_b and lifts to Ω^1_S, then there exists an effective divisor D on S such that*

1. *the inclusions $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ and $\omega_{S/B}(-D) \hookrightarrow \omega_{S/B}$ fit into the following chain*
$$\mathcal{L} \longhookrightarrow \omega_{S/B}(-D) \longhookrightarrow \Omega^1_{S/B} \overset{\alpha}{\longhookrightarrow} \omega_{S/B},$$
2. *the injection $\omega_{S/B}(-D) \hookrightarrow \Omega^1_{S/B}$ lifts to Ω^1_S,*
3. *$D \cdot C_b < 2g-2$ for any fibre C_b,*
4. *D has no component contracted by f, and*
5. *the quotient $\Omega^1_S/\omega_{S/B}(-D)$ is isomorphic to*
$$f^*\omega_B \otimes \mathcal{O}_S(D) \otimes I_Z$$
for some finite subscheme $Z \subset S$, hence torsion-free.

Roughly speaking, Proposition 6.2.2 says that from a supporting subscheme Γ (or subsheaf of $\Omega^1_{S/B}$ lifting to Ω^1_S) which is not too big on a general fibre, we can obtain a *subdivisor* $D \subset \Gamma$ with much better properties and still supporting the deformation.

Finally, we include an alternative proof of a case of Theorem 6.3.4 which works also for local deformations (in fact, it is a local version of Theorem 6.3.1). It falls in the context of the Volumetric Theorem of Pirola and Zucconi [34], assuming that there is a map from the fibration to a trivial family of Abelian varieties. The precise statement is

Proposition 6.3.9. *Suppose that $f : S \to B$ is a fibration where the base B is a smooth, not necessarily compact curve. Assume that there is an Abelian variety A of dimension a, and a morphism $\Phi : S \to A \times B$ respecting the fibres of f and such that the image of any restriction to a fibre $\phi_b : C_b \to A$ generates A. Suppose also that the deformation is supported on a divisor $D \subset S$ such that $h^0\left(C_b, \mathcal{O}_{C_b}(D)\right) = 1$ for general $b \in B$. If $a > \frac{g+1}{2}$, then f is isotrivial.*

Part One

Hodge numbers of irregular varieties

Introduction to Part I

In the classification of higher dimensional algebraic varieties, a first step can be to decide whether the variety admits (or not) a fibration onto a variety of lower dimension. If the answer is positive, then one can reduce the problem to the study of the base and the fibres, which are of lower dimension and, somehow, eaiser than the original variety. Therefore, it is interesting to have any kind of criteria to decide the existence of fibrations whose total space is the given variety, and in particular, it is useful to know conditions on the numerical invariants of the variety (e.g. its Betti, Chern or Hodge numbers) implying that it is (or not) fibred.

A paradigmatical example is the classical Castelnuovo-de Franchis theorem, which says that an irregular surface S admits a fibration onto a curve of genus $g \geq 2$ if and only if there are two holomorphic 1-forms whose wedge product is zero. This theorem gives a numerical criterion in the spirit mentioned above: if the geometric genus $p_g(S)$ and the irregularity $q(S)$ of the surface satisfy

$$p_g(S) \leq 2q(S) - 4, \tag{1}$$

then there exist two 1-forms wedging to zero, and therefore the variety is fibred.

The Castelnuovo-de Franchis theorem suggests that, for an irregular variety X, its *higher irrational pencils* (fibrations analogous to surfaces fibred over curves of genus $g \geq 2$) are closely related to some special property of the algebra of holomorphic differential forms. In fact, let $A = \mathrm{Alb}(X)$ be its Albanese variety, and $a = \mathrm{alb}_X : X \to A$ its Albanese morphism. Since

$$a^* : H^0\left(A, \Omega^1_A\right) \xrightarrow{\cong} H^0\left(X, \Omega^1_X\right)$$

is an isomorphism, and $H^0\left(A, \Omega^k_A\right) \cong \bigwedge^k H^0\left(A, \Omega^1_A\right)$, the pull-back maps $a^* : H^0\left(A, \Omega^k_A\right) \to H^0\left(X, \Omega^k_X\right)$ are precisely the wedge product maps

$$\psi_k : \bigwedge^k H^0\left(X, \Omega^1_X\right) \longrightarrow H^0\left(X, \Omega^k_X\right).$$

Because of this interpretation, the maps ψ_k are very related to the geometry of X. In particular, Catanese [9] and Ran [35] proved independently the *Generalized Castelnuovo-de Franchis theorem* (Theorem 1.2.3), which roughly speaking says that the higher irrational pencils of X correspond to the decomposable elements in the kernels of the ψ_k. As a consequence, one obtains that a non-fibred irregular variety X must verify

$$h^{k,0}(X) > k(q(X) - k)$$

for every $k = 1, \dots, \dim X$.

Beyond the existence of decomposable elements in its kernel, the case $k = 2$ has been studied by Causin and Pirola in [10], proving in particular that ψ_2 is injective for $q \leq 2d - 1$, and also by Barja, Naranjo and Pirola in [2], where they focus on the consequences of the existence of elements of rank $2d$ (what they call *generalized Lagrangian forms*) in the kernel of ψ_2.

A completely different approach is followed by Green and Lazarsfeld in [18, 19], where they introduced the *derivative complexes* and related the higher irrational pencils to the positive-dimensional components of the *cohomological support loci* of the variety. This alternative characterization led to the following different generalization of the Castelnuovo-de Franchis inequality (1) for varieties without higher irrational pencils:

$$\chi(X, \omega_X) \geq q(X) - \dim X.$$

This inequality was first obtained by Pareschi and Popa in [31], using the Fourier-Mukai transform and the Evans-Griffith Syzygy Theorem, and later by Lazarsfeld and Popa in [27], using a completely different technique: the *BGG complex*, which aggregates all the possible derivative complexes into a complex of vector bundles on a projective space. Using a similar construction (a BGG complex for the sheaves Ω_X^p of holomorphic p-forms), Lombardi obtained in [28] more inequalities involving the Hodge numbers of varieties all whose 1-forms vanish at most at isolated points (a much more restrictive hypothesis than the non-existence of fibrations). Following the ideas in [27], we have used the BGG complex to obtain some new inequalities for the *partial Euler characteristics* of the variety.

While the derivative and BGG complexes take into account only the multiplicative structure of the algebra $\oplus_{p=0}^{d} H^0(X, \Omega_X^p)$ of holomorphic forms, we have constructed some generalizations, the *higher-rank derivative* and *Grassmannian BGG complexes*, that also capture some of the additive structure. Although we have not been able to directly relate the exactness of our complexes neither to the existence of fibrations nor to the cohomological support loci of the variety, we do have proved exactness in a few steps in terms of the degeneracy loci of a subspace $W \subseteq H^0(X, \Omega_X^1)$ (Theorem 2.2.15). This approach is based on some ideas used by Green and Lazarsfeld in [18] to prove a Kodaira-Nakano type generic vanishing theorem. As an application, we obtain sharp lower bounds for some Hodge numbers of subvarieties of Abelian varieties.

The exactness of the higher-rank derivative complexes of X can also be studied by means of the ψ_k. In fact, these maps give a natural morphism of complexes from the higher-rank derivative complexes of $A = \mathrm{Alb}(X)$ (which are exact because A is a complex torus) to those of X. Following this approach, it is possible to strength some of our general results mentioned above. In fact, we have been able to characterize the exactness of the shortest higher-rank derivative complex in terms of the kernel of ψ_2 and, as a byproduct, we have obtained a stronger lower bound for the $h^{2,0}$ of a variety without higher irrational pencils (Theorem 2.3.9). This new inequality generalizes (1) in a different way than Lazarsfeld, Pareschi and Popa, and also generalizes some inequalities proven by Lombardi [28] for threefolds and fourfolds.

1

Chapter One

Preliminaries on irregular fibrations

In this chapter we summarize the main known results relating fibrations of irregular varieties and inequalities between their Hodge numbers. We first introduce in Section 1.1 the basic notation that will be used both in this chapter and in Chapter 2. After that, we recall in Section 1.2 the Castelnuovo-de Franchis theorems, both the original version (for surfaces) and the general one (due to Catanese and Ran). Then, we devote Section 1.3 to the most basic results about generic vanishing theory and the structure of the cohomological support loci. To close the chapter, we briefly recall in Section 1.4 the construction of the BGG complex and the generalization of the Castelnuovo-de Franchis inequality obtained from it.

1.1 Definitions and notation

In this first section we set de basic notation and definitions that will be used along Part I.

Throughout Chapters 1 and 2, X will denote a complex smooth irregular projective (or more generally, compact Kähler) variety of dimension $d = \dim X$. Quite often, for the sake of brevity, we will denote by $V = H^0\left(X, \Omega^1_X\right)$ the space of holomorphic 1-forms on X.

Recall that the *irregularity* of X is the dimension of V, and it is denoted by $q\left(X\right)$ or simply by q. Note that q will always be assumed to be positive. The Hodge numbers of X will be denoted by

$$h^{i,j} = h^{i,j}\left(X\right) = \dim_{\mathbb{C}} H^j\left(X, \Omega^i_X\right),$$

and sometimes by $h^{p,j}$ or $h^{p,q}$ when no confusion may arise between the second superindex and the irregularity.

More generally, if $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, we will write

$$h^i(\mathcal{F}) = h^i(X, \mathcal{F}) = \dim_{\mathbb{C}} H^i(X, \mathcal{F})$$

for the dimesion of its i-th cohomology group.

We will denote the *holomorphic Euler-Poincaré characteristic* of X as

$$\chi(X) = \chi(X, \omega_X) = h^{d,0} - h^{d-1,0} + \cdots + (-1)^d.$$

The Albanese torus of X will be denoted by $A = \mathrm{Alb}(X)$, and the Albanese morphism will be written as $a = \mathrm{alb}_X : X \to A$. Recall that A is a q-dimensional complex torus, which is projective (i.e., an Abelian variety) if X is projective too.

Definition 1.1.1. *An irregular variety X is said to be of* maximal Albanese dimension *if $\dim a(X) = \dim X$ i.e., if the Albanese morphism is generically finite.*

If furthermore a is not surjective, i.e. $a(X) \subsetneq \mathrm{Alb}(X)$, X is said to be of Albanese general type.

These definitions can be extended to non-smooth varieties considering any desingularization.

Equivalently, a variety is of Albanese general type if it is of maximal Albanese dimension and $q(X) > \dim X$. For example, every irregular curve (i.e. of genus $g \geq 1$) is of maximal Albanese dimension, because the Albanese map is nothing but the Abel-Jacobi map. Moreover, the curves of Albanese general type are exactly the curves of genus $g \geq 2$.

For any $k = 1, \ldots, d$, let

$$\psi_k : \bigwedge^k H^0\left(X, \Omega^1_X\right) \longrightarrow H^0\left(X, \Omega^k_X\right)$$

be the map induced by wedge product. Since $a^* : H^0(A, \Omega^1_A) \to H^0(X, \Omega^1_X)$ is an isomorphism and $H^0\left(A, \Omega^k_A\right) \cong \bigwedge^k H^0(A, \Omega^1_A)$, we can identify ψ_k with the pull-back $a^* : H^0\left(A, \Omega^k_A\right) \to H^0\left(X, \Omega^k_X\right)$ of k-forms by the Albanese morphism.

We will now introduce some basic notions on fibrations of irregular varieties. Recall that a *fibration* is a surjective proper flat morphism $f : X \to Y$ of varieties which has connected fibres. If X is compact, we can remove the properness from the definition, while if Y is a (smooth) curve, the flatness is automatic.

When dealing with irregular varieties, one can consider some special classes of fibrations.

Definition 1.1.2. *A fibration $f : X \to Y$ is called* irregular *if Y is irregular. If furthermore Y is of Albanese general type, then f is said to be a* higher irrational pencil *(on X).*

Note that irregular fibrations (resp. higher irrational pencils) are higher-dimensional analogues to fibrations over non-rational curves (resp. curves of genus $g \geq 2$).

We will often deal with linear subspaces of V, hence with Grassmannian varieties. For any positive integer k, we will denote by $\mathbb{G}_k = Gr\,(k, V)$ the Grassmannian of k-dimensional subspaces of V. Recall that $\mathbb{G}_k$ is naturally a subvariety of the projective space $\mathbb{P}_k = \mathbb{P}\left(\bigwedge^k V\right)$ via the Plücker embedding.

In general, if E is any vector space and $e \in E$ is a non-zero vector, we will denote by $\mathbb{P}\,(E)$ the projective space of one-dimensional subspaces of E, and by $[e] \in \mathbb{P}\,(E)$ the point corresponding to e. With this notation, the Plücker embedding maps the subspace spanned by $v_1, \ldots, v_k \in V$ to the point $[v_1 \wedge \cdots \wedge v_k] \in \mathbb{P}_k$.

Still about Grassmannian varieties, we will denote by $S \subset \mathbb{G}_k \times V$ the tautological subbundle of $\mathbb{G}_k$, i.e., the vector bundle of rank k such that $S_W = W$ for any $W \in \mathbb{G}_k$. The tautological quotient bundle of $\mathbb{G}_k$ will be denoted by $Q = (\mathbb{G}_k \times V)\,/S$.

For some explicit computations in the cohomology algebra of $\mathbb{G}_k$, we will use the following notation for Schubert classes. Fixed a basis $\{v_1, \ldots, v_q\}$ of V, and given a non-increasing sequence

$$\lambda = (q - k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0),$$

the set

$$\Sigma_\lambda = \{W \in \mathbb{G}_k \mid \dim\left(W \cap \mathbb{C}\left\langle v_1, \ldots, v_{q-k+i-\lambda_i}\right\rangle\right) \geq i\}$$

is a closed cycle of (real) codimension $2\sum_i \lambda_i = 2\,|\lambda|$, called the *Schubert cycle* associated to λ and the chosen basis. Its cohomology class, which is independent of the choice of the basis, will be denoted by

$$\sigma_\lambda \in H^{2|\lambda|}\left(\mathbb{G}_k, \mathbb{C}\right).$$

We will also use symmetric powers of vector spaces and vector bundles. If E is a vector space (or a vector bundle over some smooth variety), we will denote by $\operatorname{Sym}^r E$ its r-th symmetric power, which is a quotient of $E^{\otimes r}$. We will denote elements in $\operatorname{Sym}^r E$ using multiplicative notation, so that if $e_1, \ldots, e_r \in E$ are arbitrary elements, we will denote by $e_1 \cdots e_r \in \operatorname{Sym}^r E$

the image of $e_1 \otimes \cdots \otimes e_r$, and by e_1^r the image of $e_1^{\otimes r} = e_1 \otimes \overset{r}{\cdots} \otimes e_1$. Since the base field has characteristic zero, we can also identify $\mathrm{Sym}^r E$ with the subspace of $E^{\otimes r}$ of symmetric tensors.

At some point, secant varieties of $\mathbb{G}_k$ inside $\mathbb{P}_k$ will appear. In general, if $Z \subset \mathbb{P}(E)$ is any projective variety, and r is any positive integer, we will denote by $\mathrm{Sec}^r(Z) \subseteq \mathbb{P}(E)$ the r-th secant variety of Z i.e., the closure of the union of the $(r-1)$-planes spanned by r independent points in Z. In particular, $\mathrm{Sec}^1(Z) = Z$ and $\mathrm{Sec}^2(Z)$ is the usual secant variety of Z. More explicitly, $\mathrm{Sec}^r(Z)$ is the closure of the set

$$\{[e_1 + \cdots + e_r] \mid [e_1], \ldots, [e_r] \in Z\}.$$

Finally, we will often use the following definition for complexes of vector spaces.

Definition 1.1.3. *We say that a complex of vector spaces*

$$0 \longrightarrow V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} \cdots \longrightarrow V_k \xrightarrow{\phi_k} \cdots$$

is exact in the first n steps *if the truncated complex*

$$0 \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n$$

is exact, or equivalently, if the (co)homology groups $H^i = \ker \phi_i / \operatorname{im} \phi_{i-1}$ vanish for $i < n$.

1.2 Castelnuovo-de Franchis theorems

It is well known that some kind of fibrations of surfaces are closely related to the structure of the algebra of differential forms, as the following result (which goes back to Castelnuovo and de Franchis) shows.

Proposition 1.2.1 (Castelnuovo-de Franchis, [5] Prop. X.9, or [4] Prop. IV.5.1)**.** *Let S be a compact surface having two linearly independent holomorphic 1-forms w_1, w_2 such that $w_1 \wedge w_2 \equiv 0$. Then there exist a fibration $f : S \to B$ onto a smooth curve B of genus $g \geq 2$, and 1-forms $\alpha_1, \alpha_2 \in H^0(B, \Omega_B^1)$ such that $w_i = f^*\alpha_i$ for $i = 1, 2$.*

Looking at the numerical invariants of S, this proposition has the following

Corollary 1.2.2 (Castelnuovo-de Franchis inequality, [4] Prop. IV.5.2)**.** *If S is a compact surface that does not admit any fibration onto a curve of genus $g \geq 2$, then*

$$p_g(S) \geq 2q(S) - 3. \tag{1.1}$$

In higher dimensions, we can take the higher irrational pencils as analogues to fibrations over curves of genus $g \geq 2$. They are related to the maps ψ_k (hence to the structure of the algebra of holomorphic forms) by the following result, which was proven independently and with very different techniques by Catanese and Ran, and is clearly a generalization of Proposition 1.2.1.

Theorem 1.2.3 (Generalized Castelnuovo-de Franchis, [9] Th. 1.14, or [35] Prop. II.1)**.** *If $w_1, \ldots, w_k \in H^0\left(X, \Omega^1_X\right)$ are linearly independent 1-forms such that $\psi_k\left(w_1 \wedge \cdots \wedge w_k\right) = 0$, then there exists a higher irrational pencil $f : X \to Y$ over a normal variety Y of dimension $\dim Y < k$ and such that $w_i \in f^* H^0\left(Y, \Omega^1_Y\right)$.*

This result motivates the following

Definition 1.2.4. *An irregular variety X is said to be* primitive *if it does not admit any higher irrational pencil.*

With this definition, Theorem 1.2.3 can be restated as "X is primitive if and only if the maps ψ_k are injective in decomposable elements".

As in the case of surfaces, Theorem 1.2.3 has consequences on the Hodge numbers of a primitive variety X.

Corollary 1.2.5. *If X is a primitive irregular variety of dimension d, then*

$$h^{k,0} \geq k\left(q\left(X\right) - k\right) + 1$$

for every $k = 1, \ldots, d$.

Note that, indeed, for $k = d = 2$ we recover the inequality (1.1).

1.3 Generic vanishing theory

We recall now the basic concepts about generic vanishing theory. The main objects are the cohomological support loci.

Definition 1.3.1. *Let X be an irregular (smooth) variety of dimension d. The* cohomological support loci *of ω_X are the sets*

$$V^i\left(X\right) = V^i\left(X, \omega_X\right) = \left\{L \in \mathrm{Pic}^0\left(X\right) \mid h^i\left(X, \omega_X \otimes L\right) \neq 0\right\},$$

where $i = 1, \ldots, d$.

The main result about the structure of these sets and their relation to the geometry of X was proved by Green and Lazarsfeld, with an important addition due to Simpson (the fact that the translations are given by torsion elements).

Theorem 1.3.2 ([19] Th. 0.1, [41])**.** *Let X be an irregular variety of dimension d, and let W be an irreducible component of some $V^i(X)$. Then*

1. *there exist a subtorus $Z \subseteq \operatorname{Pic}^0(X)$ and a torsion point $\beta \in \operatorname{Pic}^0(X)$ such that $W = \beta + Z$, and*

2. *there exists a fibration $f : X \to Y$ onto a normal variety Y of dimension $\dim Y \leq d - i$, such that (any smooth model of) Y is of maximal Albanese dimension and $Z \subseteq f^* \operatorname{Pic}^0(Y)$.*

As a corollary, they obtained the following result, previously proved also by Green and Lazarsfeld.

Theorem 1.3.3 ([18] Th. 1)**.** *For any irregular variety X of dimension d,*

$$\operatorname{codim}_{\operatorname{Pic}^0(X)} V^i(X) \geq i - (d - \dim a(X)).$$

In particular, $h^i(L) = 0$ for general L and $i < \dim a(X)$.

Clearly, the bigger the Albanese dimension of X, the stronger this result is, giving the best results when X is a variety of maximal Albanese dimension. In this case, the $V^i(X)$ also satisfy the following chain of inclusions

$$\operatorname{Pic}^0(X) \supseteq V^0(X) \supseteq V^1(X) \supseteq \ldots \supseteq V^d(X) = \{\mathcal{O}_X\}.$$

Finally, we want to recall that the cohomological support loci $V^i(X)$ are also related to the exactness of a special Koszul-like complexes. For any non-zero $v \in H^1(X, \mathcal{O}_X)$, consider the complex

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \xrightarrow{\cup v} H^1(X, \mathcal{O}_X) \xrightarrow{\cup v} \cdots \xrightarrow{\cup v} H^d(X, \mathcal{O}_X) \longrightarrow 0 \quad (1.2)$$

given by cup-product with v.

Lemma 1.3.4 ([13] Th. 1.2(3))**.** *With the above notations, if v is tangent at $\mathcal{O}_X$ to some component of $V^i(X)$, then both maps in*

$$H^{i-1}(X, \mathcal{O}_X) \xrightarrow{\cup v} H^i(X, \mathcal{O}_X) \xrightarrow{\cup v} H^{i+1}(X, \mathcal{O}_X)$$

vanish, whereas if v is not tangent to any component of $V^i(X)$, then the complex (1.2) is exact at $H^i(X, \mathcal{O}_X)$.

1.4 The BGG complex

To close this first chapter of preliminaries, we want to say a few words about the BGG complex, a complex of vector bundles on $\mathbb{P}\left(H^1\left(X, \mathcal{O}_X\right)\right)$ introduced by Lazarsfeld and Popa in [27].

For the sake of brevity, denote by $\mathbb{P} = \mathbb{P}(H^1(X, \mathcal{O}_X))$ the projective space of one-dimensional subspaces of $H^1\left(X, \mathcal{O}_X\right)$. Over a point $[v] \in \mathbb{P}$, we can consider the complex (1.2) given by successive cup product with v. Letting $[v]$ vary in $\mathbb{P}$, these complexes glue to give the linear complex

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}}\left(-d\right) \otimes H^0\left(X, \mathcal{O}_X\right) \to \mathcal{O}_{\mathbb{P}}\left(-d+1\right) \otimes H^1\left(X, \mathcal{O}_X\right) \to \cdots \\ \cdots \to \mathcal{O}_{\mathbb{P}}\left(-1\right) \otimes H^{d-1}\left(X, \mathcal{O}_X\right) \to \mathcal{O}_{\mathbb{P}} \otimes H^d\left(X, \mathcal{O}_X\right). \end{aligned} \tag{1.3}$$

Definition 1.4.1. *The complex (1.3) is called the* BGG complex *of X, and it is denoted by $BGG\left(X\right)$. The cokernel of the right-most map,*

$$\mathcal{F} = \operatorname{coker}\left(\mathcal{O}_{\mathbb{P}}\left(-1\right) \otimes H^{d-1}\left(X, \mathcal{O}_X\right) \longrightarrow \mathcal{O}_{\mathbb{P}} \otimes H^d\left(X, \mathcal{O}_X\right)\right),$$

is called the BGG sheaf *of X.*

The reason for this name is that it is quite related to the BGG correspondence introduced by Bernšteĭn, Gel′fand and Gel′fand in [6].

The exactness of $BGG\left(X\right)$ turns out to be governed by the Albanese map of X and its irregular fibrations

Theorem 1.4.2 ([27] Th. A)**.** *The complex $BGG\left(X\right)$ is exact in the first $\dim \mathrm{alb}_X\left(X\right)$ steps. Moreover, if X admits no irregular fibration, then $\mathcal{F}$ is a vector bundle on $\mathbb{P}$ of rank $\operatorname{rk}\mathcal{F} = \chi\left(X\right)$, and $BGG\left(X\right)$ is a linear resolution of $\mathcal{F}$.*

Amongst other applications, the BGG complex is used by Lazarsfeld and Popa in [27] to prove the following higher-dimensional analogue of the Castelnuovo-de Franchis inequality (1.1). The same result has been previously obtained by Pareschi and Popa in [31], using very different methods (namely, the Fourier-Mukai transform and the Evans-Griffith Syzygy Theorem).

Theorem 1.4.3 ([27] Th. C(iii))**.** *If X is an irregular variety that does not admit any irregular fibration, or, more generally, $\mathcal{O}_X \in \mathrm{Pic}^0(X)$ is an isolated point of $V^i(\omega_X)$ for every $i > 0$, then*

$$\chi(X) \geq q(X) - \dim X.$$

Note that this generalization is very different from Corollary 1.2.5 for higher dimensions, but for $\dim X = 2$ the classical inequality (1.1) is also recovered.

2

Chapter Two

Generalizations of the derivative and BGG complexes

In this chapter we develop different methods to obtain inequalities between the Hodge numbers of a smooth, compact, irregular Kähler variety, assuming suitable geometric hypothesis. In particular, we obtain some generalizations to arbitrary dimensions of some inequalities known for surfaces, threefolds and fourfolds without higher irrational pencils.

More precisely, in Section 2.1 we generalize the methods used by Lazarsfeld and Popa in [27] (mainly, the BGG complex), obtaining inequalities for the *partial Euler characteristics* of irregular varieties that do not admit fibrations over Albanese general type varieties of low dimension. We also study the functoriality of our constructions, obtaning inequalities for irregular varieties whose Albanese image is smooth and non-degenerate.

In the second section, we extend the construction of the BGG complex in order to allow not only one-dimensional subspaces. The resulting complexes are closely related to the complexes of Eagon-Northcott type (as the BGG complex is related to the Koszul complex of a 1-form). With this modification, we let the additive structure of the cohomology algebra of the variety to play a role. This extra flexibility leads to more general inequalitites between the Hodge numbers of the variety, but also need slightly stronger hypothesis (the non-existence of higher irrational pencils is apparently not enough).

Section 2.3 presents a different approach to study the shortest case of the above-mentioned complexes. Therein we obtain much stronger inequalities than with the general method, and they are valid for every irregular variety without higher irrational pencils (and even more general varieties).

Finally, in the fourth section, we present some methods that could lead to the inequalities of Section 2.3 using the results in Section 2.2. They depend on some computations on the cohomology ring of Grassmannian varieties that we have carried out in some particular cases (see Appendix A), but we have not been able to do in the general case. Hence, some parts of this last section are quite conjectural.

2.1 Partial Euler characteristics

In this section we introduce a generalization of the methods used in [27] (*truncated* BGG complexes) to obtain linear lower bounds on the *partial Euler characteristics* of an irregular variety. Later on, we compute these partial characteristics for some families of irregular varieties (suitable subvarieties of Abelian varieties), and we close the section with a study of the functorial properties of the truncated complexes to strongly improve the linear bounds for varieties whose Albanese image is geometrically non-degenerate.

2.1.1 First definitions

Definition 2.1.1. *Let X be any (compact, smooth) variety of dimension $d = \dim X$, and fix an integer $0 \leq i \leq d$. We define the i-th* partial Euler characteristic *of X as*

$$\begin{aligned}\chi_i = \chi_i(X) = \chi_i(X, \omega_X) = \\ = h^i(X, \omega_X) - h^{i+1}(X, \omega_X) + \cdots + (-1)^{d-i} h^d(X, \omega_X).\end{aligned}$$

Note that $\chi_0(X) = \chi(X)$ is the usual holomorphic Euler-Poincaré characteristic, and on the other extreme, $\chi_d(X) = 1$ and $\chi_{d-1}(X) = q(X) - 1$.

In Section 1.4 we have recalled the construction and main properties of the BGG complex of X: it is the linear complex $BGG(X)$ of vector bundles

$$\begin{aligned}0 \to \mathcal{O}_{\mathbb{P}}(-d) \otimes H^0(X, \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(-d+1) \otimes H^1(X, \mathcal{O}_X) \to \cdots \\ \cdots \to \mathcal{O}_{\mathbb{P}}(-1) \otimes H^{d-1}(X, \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}} \otimes H^d(X, \mathcal{O}_X),\end{aligned}$$

on $\mathbb{P} = \mathbb{P}_X = \mathbb{P}(H^1(X, \mathcal{O}_X)) \cong \mathbb{P}^{q-1}$, which over a point $[v] \in \mathbb{P}$ is given by cup-product with v.

Definition 2.1.2. *For any $r = 0, \ldots, d-1$ we define the r-th* BGG sheaf *of X as*

$$\mathcal{F}^r = \operatorname{coker}\left\{\mathcal{O}_{\mathbb{P}}(-r-1) \otimes H^{d-r-1}(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{\mathbb{P}}(-r) \otimes H^{d-r}(X, \mathcal{O}_X)\right\}$$

Note also that for $r = 0$ we recover the BGG sheaf introduced by Lazarsfeld and Popa.

By Theorem 1.4.2, the complex

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \otimes H^0(X, \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(-d+1) \otimes H^1(X, \mathcal{O}_X) \to \cdots$$
$$\cdots \to \mathcal{O}_{\mathbb{P}}(-k) \otimes H^{d-k}(X, \mathcal{O}_X) \to \mathcal{F}^k \to 0 \quad (2.1)$$

is exact for any $k \geq d - \dim \mathrm{alb}_X(X)$. It is possible to split the above complex into the short exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \otimes H^0(X, \mathcal{O}_X) \to \mathcal{O}_{\mathbb{P}}(-d+1) \otimes H^1(X, \mathcal{O}_X) \to \mathcal{F}^{d-1} \to 0 \quad (2.2)$$

and

$$0 \longrightarrow \mathcal{F}^{r+1} \longrightarrow \mathcal{O}_{\mathbb{P}}(-r) \otimes H^{d-r}(X, \mathcal{O}_X) \longrightarrow \mathcal{F}^r \longrightarrow 0 \quad (2.3)$$

for $r = k, k+1, \ldots, d-2$. The next result follows immediately.

Proposition 2.1.3. *If X is any irregular variety, then*

$$\chi_k(X) = \operatorname{rk} \mathcal{F}^k \geq 0 \quad \textit{for every } k \geq d - \dim \mathrm{alb}_X(X).$$

In particular, we recover that $\chi(X) \geq 0$ for any variety of maximal Albanese dimension.

2.1.2 Linear bounds

The third statement of Theorem 1.4.3 asserts that if X does not admit any irregular fibration, or more generally, if $\mathcal{O}_X$ is an isolated point of $V^i(X)$ for every $i > 0$, then $\chi \geq q(X) - d$. This result is clearly an improvement of Proposition 2.1.3 for $k = 0$, hence for X of maximal Albanese dimension. We will now prove a similar result, using the same techniques, for all the partial Euler characteristics starting from the dimension of the general fibre of the Albanese map.

Theorem 2.1.4. *Let k be an integer such that $d - \dim \mathrm{alb}_X(X) \leq k < d$. If X does not admit any irregular fibration $f : X \to Y$ with $\dim Y < d - k$, or more generally, if $\mathcal{O}_X$ is an isolated point of $V^i(X)$ for all $i > k$, then*

$$\chi_r(X) \geq (q(X) - \dim X) + r \quad (2.4)$$

for all $k \leq r < d$.

Remark 2.1.5. *Let us consider the extremal cases of k.*

1. *For $k = r = 0$ we recover the result of Lazarsfeld and Popa (Theorem 1.4.3).*

2. *The case $r = d - 1$ is automatically satisfied with equality, since by definition*

$$\chi_{d-1}(X) = q - 1 = (q - d) + (d - 1).$$

Therefore, Theorem 2.1.4 has interest only for $k \leq d - 2$.

Remark 2.1.6. *In the recent work [30], Mendes-Lopes, Pardini and Pirola prove (Theorem 1.2) that a smooth projective variety X of* Albanese general type*, dimension $d \geq 3$, irregularity $q \geq d + 1$, and without higher irrational pencils satisfies*

$$\chi_1(X) \geq q - 1.$$

This inequality improves Theorem 2.1.4 for $k = 1$, with the only extra assumption that X does not admit a fibration by curves over an Albanese general type variety.

Before proceeding with the proof, we need an auxiliary lemma. It is analogous to a result used in [27] and proved explicitly in [28], but we prefer to prove it here for the sake of completeness.

Lemma 2.1.7. *For any integer $r = d - \dim \mathrm{alb}_X(X), \ldots, d - 1$ it holds $H^i(\mathcal{F}^r(m)) = 0$ for every $m \in \mathbb{Z}$ and every $0 < i < q - d + r - 1$.*

Proof. For $r = d - 1$, the short exact sequence (2.2) (twisted by $\mathcal{O}_{\mathbb{P}}(m)$) gives the exact sequence

$$H^i(\mathcal{O}_{\mathbb{P}}(-d + 1 + m))^{\oplus h^1(\mathcal{O}_X)} \to H^i\left(\mathcal{F}^{d-1}(m)\right) \to \\ \to H^{i+1}(\mathcal{O}_{\mathbb{P}}(-d + m))^{\oplus h^0(\mathcal{O}_X)},$$

where the outer terms vanish as soon as $0 < i < i + 1 < q - 1$, that is, if $0 < i < q - 2 = q - d + r - 1$. Therefore, we obtain that $H^i\left(\mathcal{F}^{d-1}(m)\right)$ for all $m \in \mathbb{Z}$ and $0 < i < q - d + r - 1$, as wanted.

For the rest of the cases, the sequence (2.3) (again twisted by $\mathcal{O}_{\mathbb{P}}(m)$) gives the exact sequence

$$H^i(\mathcal{O}_{\mathbb{P}}(-r + m))^{\oplus h^{d-r}(\mathcal{O}_X)} \to H^i(\mathcal{F}^r(m)) \to \\ \to H^{i+1}\left(\mathcal{F}^{r+1}(m)\right) \to H^{i+1}(\mathcal{O}_{\mathbb{P}}(-r + m))^{\oplus h^{d-r}(\mathcal{O}_X)},$$

where again the outer terms vanish if $0 < i < q - 2$. Hence, for $0 < i < q - 2$ there are isomorphisms

$$H^i(\mathcal{F}^r(m)) \cong H^{i+1}\left(\mathcal{F}^{r+1}(m)\right),$$

and by descending induction over r, the rightmost cohomology group vanish for $0 < i + 1 < q - d + r$, so that $H^i(\mathcal{F}^r(m)) = 0$ if $0 < i < q - d + r - 1$ as claimed. $\square$

As Lazarsfeld and Popa did, we will use the following result on vector bundles on projective spaces (whose proof can be found, for example, in [26], Example 7.3.10).

Theorem 2.1.8 (Evans-Griffith). *Let E be a vector bundle of rank $e \geq 2$ on $\mathbb{P}^n$ such that*

$$H^i\left(\mathbb{P}^n, E(k)\right) = 0 \text{ for all } 1 \leq i \leq e-1 \text{ and every } k \in \mathbb{Z}. \tag{2.5}$$

Then E is a direct sum of line bundles.

Proof of Theorem 2.1.4. First of all, by Lemma 1.3.4, if a tangent vector $v \in H^1(X, \mathcal{O}_X) \cong T_{\mathcal{O}_X} \operatorname{Pic}^0(X)$ is not tangent (at $\mathcal{O}_X$) to any component of $V^r(X)$, then the sequence

$$H^{r-1}(X, \omega_X) \xrightarrow{\cup v} H^r(X, \omega_X) \xrightarrow{\cup v} H^{r+1}(X, \omega_X)$$

is exact, or equivalently,

$$H^{d-r-1}(X, \mathcal{O}_X) \xrightarrow{\cup v} H^{d-r}(X, \mathcal{O}_X) \xrightarrow{\cup v} H^{d-r+1}(X, \mathcal{O}_X)$$

is exact. Therefore, if $\mathcal{O}_X$ is an isolated point of $V^r(X)$ for every $r > k$, the sequence (2.1) is exact at every point, hence the differentials have constant rank, as well as each of the BGG sheaves $\mathcal{F}^r$, $r = k, \dots, d-1$, which are thus locally free.

Now Lemma 2.1.7 says that for any $r = k, \dots, d-1$, $\mathcal{F}^r$ satisfies (2.5) for every $1 \leq i \leq q-d+r-2$. Hence, if $\chi_r = \operatorname{rk} \mathcal{F}^r \leq q-d+r-1$, Theorem 2.1.8 implies that $\mathcal{F}^r$ is either a direct sum of line bundles or 0 (in case $\operatorname{rk} \mathcal{F}^r = 0$, since $\mathcal{F}^r$ is locally free). Therefore, it only remains to check that none of the $\mathcal{F}^r$ is either zero or a sum of line bundles.

Let r be the maximal integer between k and $d-1$ such that

$$\chi_r \leq q-d+r-1, \tag{2.6}$$

and thus $\mathcal{F}^r$ is either zero or sum of line bundles (if there is no such r, we are done). Note that $\chi_r \geq 0$ in any case, so that we may assume $q-d+r-1 \geq 0$. Moreover, since $\chi_{d-1} = q-1 = (q-d)+(d-1)$, we can also suppose $r < d-1$.

The inequality (2.6) implies that $H^i(\mathcal{F}^s(r)) = 0$ for every $s > r$ and every $i \geq 0$. Indeed, for $s = d-1$, taking cohomology on the short exact sequence (2.2) twisted by $\mathcal{O}_{\mathbb{P}}(r)$ we obtain the exact sequence

$$H^i\left(\mathcal{O}_{\mathbb{P}}(-d+r+1)\right)^{\oplus h^1(\mathcal{O}_X)} \to H^i\left(\mathcal{F}^{d-1}(r)\right) \to$$
$$\to H^{i+1}\left(\mathcal{O}_{\mathbb{P}}(-d+r)\right)^{\oplus h^0(\mathcal{O}_X)},$$

where the outer terms vanish because $-d+r+1 < -d+s+1 = 0$ and $-d+r \geq -q+1 > -q$. For $s < d-1$ we proceed by descending induction (as in the proof of Lemma 2.1.7) down to $s = r+1$. Again, taking cohomology on the short exact sequence (2.3) twisted by $\mathcal{O}_{\mathbb{P}}(r)$, we get

$$0 = H^i\left(\mathcal{O}_{\mathbb{P}}(-s+r)\right)^{\oplus h^{d-s}(\mathcal{O}_X)} \to H^i\left(\mathcal{F}^s(r)\right) \to \\ \to H^{i+1}\left(\mathcal{F}^{s+1}(r)\right) \to H^1\left(\mathcal{O}_{\mathbb{P}}(-s+r)\right)^{\oplus h^{d-s}(\mathcal{O}_X)} = 0,$$

where the outer terms vanish because

$$-q \leq -d+r-1 < -s+r-2 < -s+r < 0.$$

Therefore we have isomorphisms $H^i\left(\mathcal{F}^s(r)\right) \cong H^{i+1}\left(\mathcal{F}^{s+1}(r)\right)$ for every $i \geq 0$, and the second group is zero by the induction hypothesis, so that $H^i\left(\mathcal{F}^s(r)\right) = 0$ for every $i \geq 0$ and $s = r+1, \ldots, d-1$, as claimed.

We will finally show that $\mathcal{F}^r$ cannot be neither zero nor a direct sum of line bundles.

The sequence (2.3) twisted by $\mathcal{O}_{\mathbb{P}}(r)$ gives the exact sequence

$$0 = H^0\left(\mathcal{F}^{r+1}(r)\right) \longrightarrow H^{d-r}\left(X, \mathcal{O}_X\right) \longrightarrow \\ \longrightarrow H^0\left(\mathcal{F}^r(r)\right) \longrightarrow H^1\left(\mathcal{F}^{r+1}(r)\right) = 0,$$

so that $H^0\left(\mathcal{F}^r(r)\right) = H^{d-r}\left(X, \mathcal{O}_X\right)$ and $\mathcal{F}^r(r)$ is generated by global sections (by definition, $\mathcal{O}_{\mathbb{P}} \otimes H^{d-r}\left(X, \mathcal{O}_X\right) \twoheadrightarrow \mathcal{F}^r(r)$).

Therefore, in the case $\mathcal{F}^r = 0$ we would have $H^{d-r}\left(X, \mathcal{O}_X\right) = 0$, and hence $\mathcal{F}^{r+1} = 0$ (it is a subsheaf of $\mathcal{O}_{\mathbb{P}}(-r) \otimes H^{d-r}\left(X, \mathcal{O}_X\right)$). Analogously, going back through the short exact sequences (2.2) and (2.3), we would get $0 = H^{d-r-1}\left(X, \mathcal{O}_X\right) = \ldots = H^0\left(X, \mathcal{O}_X\right)$, which is clearly impossible.

Suppose then that $\mathcal{F}^r$ descomposes as $\bigoplus_{j=1}^{\chi_r} \mathcal{O}_{\mathbb{P}}(a_j)$, with $\chi_r > 0$ and $a_1 \geq a_2 \geq \ldots \geq a_{\chi_r}$. Since $\mathcal{F}^r(r) = \bigoplus_{j=1}^{\chi_r} \mathcal{O}_{\mathbb{P}}(a_j + r)$ is globally generated, it must hold that $a_j + r \geq 0$, that is to say, $a_j \geq -r$ for every $j = 1, \ldots, \chi_r$. If $a_1 > -r$, the sequence (2.3) twisted by $\mathcal{O}_{\mathbb{P}}(r-1)$ gives

$$0 = H^0\left(\mathcal{O}_{\mathbb{P}}(-1) \otimes H^{d-r}\left(X, \mathcal{O}_X\right)\right) \to \bigoplus_{j=1}^{\chi_r} H^0\left(\mathcal{O}_{\mathbb{P}}(a_j + r - 1)\right) \to \\ \to H^1\left(\mathcal{F}^{r+1}(-1)\right) = 0,$$

where the last term vanishes because of Lemma 2.1.7 (we have assumed $q - d + r > \chi_r \geq 1 > 0$). But the central group is not zero, since at least the first summand of $\mathcal{F}^r(r-1)$ has non-negative degree.

Hence, the only remaining possibility is $\mathcal{F}^r = \mathcal{O}_{\mathbb{P}}(-r)^{\oplus \chi_r}$. But in this case the sequence (2.3) splits, so that $\mathcal{F}^{r+1} \cong \mathcal{O}_{\mathbb{P}}(-r)^{\oplus \chi_{r+1}}$. This implies that $\chi_{r+1} = 0$ because $0 = H^0(\mathcal{F}^{r+1}(r)) \cong \mathbb{C}^{\chi_{r+1}}$, but the maximality of r implies that $\chi_{r+1} \geq q - d + r + 1 > q - d + r - 1 \geq 0$, discarding this last case and finishing the proof. □

2.1.3 Examples

In this section we compute the partial Euler characteristics of some families of irregular varieties, and we see that the inequalities provided by Theorem 2.1.4 do not seem to be sharp.

Complete intersections of ample divisors in Abelian varieties

As a first example, we will compute the Hodge numbers of type $h^{p,0}$ (and the partial Euler characteristics) of smooth complete intersections of ample divisors in Abelian varieties. In the case the ambient Abelian variety is simple, the hypothesis of Theorem 2.1.4 are trivially satisfied, so that we can use this first family of varieties to test how sharp are the inequalities (2.4). Indeed, we will see that they are very far from being sharp except in some special cases, and in the next section a great improvement will be obtained (Corollary 2.1.16) for varieties whose Albanese image is (geometrically) non-degenerate (in particular, the intersections of ample divisors).

Let thus A be an Abelian variety of dimension g. Remember that its Hodge numbers are $h^{p,q}(A) = \binom{g}{p}\binom{g}{q}$, so that

$$h^r(\omega_A) = h^{g,r}(A) = \binom{g}{r}.$$

and the partial Euler characteristics of A, for $r < g$, are

$$\chi_r(A) = \sum_{j=r}^{g} (-1)^{j-r} \binom{g}{r} = \binom{g-1}{r-1} = \binom{g-1}{g-r}.$$

Let now $\Theta_1, \ldots, \Theta_{q-d} \subseteq A$ be ample divisors such that the complete intersection $X = \Theta_1 \cap \ldots \cap \Theta_{q-d}$ is a smooth d-dimensional subvariety. The next theorem computes the partial Euler characteristics of X.

Theorem 2.1.9. *The partial Euler characteristics of X are*

$$\chi_k(X) = \binom{g-1}{d-k} \quad \forall\, k > 0,$$

$$\chi(X) = \sum_{\substack{I \subseteq \{1,\dots,g-d\} \\ I \neq \emptyset}} (-1)^{g-d-|I|} h^0\left(A, \mathcal{O}_A\left(\sum_{i\in I} \Theta_i\right)\right).$$

Proof. Let $\mathcal{F}_1 = \bigoplus_{i=1}^{g-d} \mathcal{O}_A(-\Theta_i)$ and let

$$\sigma : \mathcal{F}_1 \longrightarrow \mathcal{F}_0 = \mathcal{O}_A$$

be the addition of all the inclusions $\mathcal{O}_A(-\Theta_i) \hookrightarrow \mathcal{O}_A$. Clearly, the image of σ is the ideal sheaf $\mathcal{I}_X$ of X, so the cokernel of σ is precisely $\mathcal{O}_X$.

For any $1 \le r \le g-d$ denote by $\mathcal{F}_r = \bigwedge^r \mathcal{F}_1 \cong \bigoplus_{|I|=r} \mathcal{O}_A\left(-\sum_{i\in I}\Theta_i\right)$ and let

$$\mathcal{K}: \quad 0 \longrightarrow \mathcal{F}_{g-d} \longrightarrow \mathcal{F}_{g-d-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \xrightarrow{\sigma} \mathcal{F}_0 \longrightarrow 0$$

be the Koszul complex associated to σ, which is a locally free resolution of $\mathcal{O}_X$ (this can be taken as well known, but it is quite immediate to prove using Theorem 2.2.9 in the next section, with $\phi = \sigma$ and $r = 0$).

Therefore, we have $H^n(X, \mathcal{O}_X) = \mathbb{H}^n(A, \mathcal{K})$, where the second term can be computed for $n < d$ by means of the spectral sequence

$$E_1^{i,j} = H^j(A, \mathcal{F}_{g-d-i}) \Longrightarrow \mathbb{H}^n(A, \mathcal{K}).$$

But since the Θ_i are ample, $E_1^{i,j} = 0$ unless $i = g-d = \operatorname{codim} X$ or $j = g = \dim A$, and therefore $E_\infty^{i,j} = E_1^{i,j} = H^j(A, \mathcal{F}_{g-d-i})$ for all $i+j<d$. In fact, for fixed $i+j = n < d$, the only non-zero term is $E_1^{0,n} = H^n(A, \mathcal{O}_A)$. This means that

$$H^n(X, \mathcal{O}_X) = \mathbb{H}^n(A, \mathcal{K}) \cong \oplus_{i+j=n} E_\infty^{i,j} = H^n(A, \mathcal{O}_A),$$

so that $h^{0,n}(X) = h^{0,n}(A) = \binom{g}{n}$ for all $n < \dim X$, hence

$$\chi_r(X) = \chi_{g-d+r}(A) = \binom{g-1}{g-d+r-1} = \binom{g-1}{d-r}$$

for every $r > 0$.

Finally, for $r = 0$, note that

$$\chi_0(X) = \chi(X, \omega_X) = (-1)^d \chi(X, \mathcal{O}_X) = (-1)^d \sum_{r=0}^{g-d} (-1)^r \chi(A, \mathcal{F}_r),$$

and since all the Θ_i are ample,

$$\begin{aligned}\chi(A,\mathcal{F}_r) = \sum_{|I|=r}\chi\left(A,\mathcal{O}_A\left(-\sum_{i\in I}\Theta_i\right)\right) &= \\ = (-1)^g\sum_{|I|=r}\chi\left(A,\mathcal{O}_A\left(\sum_{i\in I}\Theta_i\right)\right) &= \\ = \begin{cases}\chi(A,\mathcal{O}_A) = 0 & \text{if } r=0, \\ (-1)^g\sum_{|I|=r} h^0\left(A,\mathcal{O}_A\left(\sum_{i\in I}\Theta_i\right)\right) & \text{otherwise.}\end{cases}\end{aligned}$$

□

Let us check now whether the inequalities

$$\chi_r(X) \geq (q-d)+r$$

hold, distinguishing the cases $r>0$ and $r=0$.

The case $r>0$ is straightforward, because

$$\chi_r(X) = \binom{q-1}{d-r} = \binom{q-1}{q-d+r-1} = \frac{q-d+r}{q}\binom{q}{q-d+r}$$

equals $q-d+r$ if and only if $\binom{q}{q-d+r}=q$, that is, if and only if $q-d+r=1$ or $q-d+r=q-1$. But since we are considering the case $d\geq r>0$, it turns out that the only possible cases where equality can happen are

- either $r=1$ and $q=d$, i.e. $X=A$ is the whole Abelian variety, and only happens for $\chi_1(A)$,
- or $r=d-1$, which is automatic for any variety by definition of $\chi_{d-1}(X)$.

The case $r=0$ is by far much more complicated, and we only consider two simple cases:

- $X=A$ is the whole Abelian variety. In this case, the equality always happens because $\chi(A)=0$ and $q(A)=\dim A$.
- $X=\Theta\subset A$ is an ample divisor. Now, Theorem 2.1.9 gives that $\chi(X)=h^0(A,\mathcal{O}_A(\Theta))$, so the equality $\chi=q-d=1$ happens only if X induces a principal polarization on A.

Symmetric products of curves

Another interesting family of irregular varieties are symmetric products of curves. Let C be a curve of genus $g \geq 1$, and let $C^{(d)}$ be its d-th symmetric product. It is known ([29] Example 1.1) that $h^{p,0}\left(C^{(d)}\right) = \binom{g}{p}$, and thus the partial Euler characteristics of $C^{(d)}$ are

$$\chi_r\left(C^{(d)}, \omega_{C^{(d)}}\right) = \binom{g-1}{d-r} = \binom{q-1}{d-r}$$

since $q\left(C^{(d)}\right) = g\left(C\right)$. Therefore, we obtain the same thing we obtained above (which is not a surprise, since for example $C^{(g-1)}$ is birational to a Theta divisor in $J\left(C\right)$).

Threefolds.

We know that the equality $\chi_0\left(X\right) = q\left(X\right) - \dim X$ is very difficult to obtain, except in the cases $q = d$ (Abelian varieties) and $q = d+1$ (Theta divisors). Therefore, from now on we will try to produce varieties satisfying as many equalities $\chi_r = (q-d) + r$ as possible, assuming that it will not be possible for $r = 0$. Since for $r = d$ and $r = d-1$ the equalities hold by definition, the smallest dimensional cases of interest are threefolds.

The two equalities we want to be satisfied are

$$\chi_1 = (q-3) + 1 \quad \text{and} \quad \chi_0 = q - 3.$$

The first one can be written as $h^{2,0} = 2q - 3$, analogous to the classical Castelnuovo-de Franchis inequality. The second one, which we do not expect to obtain except in the abovementioned known cases, can be written as $p_g = h^{2,0} - 2$.

We will consider the following construction. On the one hand, we will take a double covering S of a principally polarized Abelian variety (A, Θ), ramified over a smooth divisor $D \in |2\Theta|$. On the other hand, we will take C a double covering of a curve B of arbitrary genus. Both on S and C there is an action of $G = \mathbb{Z}/2\mathbb{Z}$, so we can consider the diagonal action of G in the product $S \times C$. We consider X a desingularization of the quotient $\left(S \times C\right)/G$.

The Hodge numbers of X are

- $q\left(X\right) = q\left(A\right) + g\left(B\right) = g\left(B\right) + 2$,
- $h^0\left(X, \Omega^2_X\right) = p_g\left(A\right) + q\left(A\right)g\left(B\right) = 2g\left(B\right) + 1$, and
- $p_g\left(X\right) = g\left(B\right) + \left(g\left(C\right) - g\left(B\right)\right) = g\left(C\right)$.

Hence, on the one hand, the first partial Euler characteristic is

$$\chi_1(X) = q(X) - 2 = (q(X) - 3) + 1,$$

so the wanted inequality always hold. On the other hand, the "complete" Euler characteristic is

$$\chi_0 = p_g(X) - \chi_1 = g(C) - g(B) \geq g(B) - 1 = q(X) - 3,$$

where the inequality is a consequence of Hurwitz's formula. Therefore, the equality holds if and only if the double covering $C \to B$ is étale.

Sumarizing, we have obtained a family of 3-folds satisfying both equalities $\chi_1(X) = (q(X) - 3) + 1$ and $\chi_0(X) = q(X) - 3$ for every irregularity $q(X) \geq 3$, that is, for every Euler characteristic $\chi_0 \geq 0$. However, if $\chi_0 \geq 1$, these varieties are fibred over the curve B, which has genus $q(X) - 2 = \chi_0(X) + 1 \geq 2$, so Theorem 2.1.4 does not apply and we do not obtain any interesting example. Indeed, since X is fibred over the curve C of genus $g(C) \geq 2g(B) - 1 \geq 3$, Theorem 2.1.4 does not apply for any k.

Fourfolds

We pass now to fourfolds, applying similar constructions to obtain varieties satisfying the inequalities of Theorem 2.1.4. We will see, however, that these constructions do not allow to obtain the equality for $r = 2$.

First construction

We will first try by substituting the surface S by a threefold T, which will also be a double covering of a principally polarized Abelian threefold (A, Θ). We take thus X a desingularization of $(T \times C)/G$, where

- $\pi : T \to A$ is a double covering ramified over a smooth $D \in |2\Theta|$,
- $\tau : C \to B$ is a double covering of a curve of genus $g(B) \geq 1$, and
- $G = \mathbb{Z}/2\mathbb{Z}$ acts diagonally on the product $T \times C$.

In this case, X has the following Hodge numbers

- $q(X) = q(A) + g(B) = g(B) + 3$,
- $h^0(X, \Omega_X^2) = h^{2,0}(A) + q(A)\, g(B) = 3g(B) + 3$,
- $h^0(X, \Omega_X^3) = h^{3,0}(A) + h^{2,0}(A)\, g(B) = 3g(B) + 1$, and
- $p_g(X) = g(B) + (g(C) - g(B)) = g(C)$.

Let us now compute the partial Euler characteristics:

- $\chi_2(X) = h^{2,0}(X) - q(X) + 1 = 2q(X) - 5$, which is greater than $(q(X) - 4) + 2 = q(X) - 2$ if and only if $q(X) > 3$. This inequality holds automatically since we have assumed $g(B) \geq 1$. Therefore, with this construction is impossible to achieve the equality in the case $r = 2$.

- $\chi_1(X) = h^{3,0}(X) - \chi_2(X) = g(B) = q(X) - 3 = (q(X) - 4) + 1$, so that the equality always hold for $r = 1$, as in the case of threefolds.

- $\chi_0(X) = p_g(X) - \chi_1(X) = g(C) - g(B) \geq g(B) - 1 = q(X) - 4$, with equality if and only if the covering $C \to B$ is étale, as in the three-dimensional case.

Thus, with this construction we get equalities for $r = 1$ and any irregularity $q(X) \geq 4$, as well as for $r = 0$ if the covering of curves is étale. However, also as in the case of threefolds, Theorem 2.1.4 does not apply in this case because X is fibred over C, which has genus $g(C) \geq 2$ if $\chi_0(X) \geq 1$.

Second construction

In this second construction we will keep the first surface S and we will change the curve C by a surface (also a double covering). More precisely, we will consider X a desingularization of $(S_1 \times S_2)/G$, where

- $\pi : S_1 \to A$ is a double covering of a principally polarized Abelian surface (A, Θ), ramified over $D \in |2\Theta|$,

- $\tau : S_2 \to B$ is a double covering of a surfece B (still without any further condition), and

- $G = \mathbb{Z}/2\mathbb{Z}$ acts diagonally on the product $S_1 \times S_2$.

The Hodge numbers of this second construction are

- $q(X) = q(A) + q(B) = q(B) + 2$,

- $h^0(X, \Omega^2_X) = p_g(A) + q(A)\,q(B) + p_g(B) = p_g(B) + 2q(B) + 1$,

- $h^0(X, \Omega^3_X) = 2p_g(B) + q(S_2)$, and

- $p_g(X) = p_g(S_2)$.

Consequently, the partial Euler characteristics are

- $\chi_2(X) = h^{2,0}(X) - q(X) + 1 = p_g(B) + q(B)$,
- $\chi_1(X) = h^{3,0}(X) - \chi_2(X) = p_g(B) + (q(S_2) - q(B))$, and
- $\chi_0(X) = p_g(X) - \chi_1(X) = (p_g(S_2) - p_g(B)) - (q(S_2) - q(B))$.

Firstly, it is evident that the only way to obtain the equality

$$\chi_2(X) = (q(X) - 4) + 2 = q(B)$$

is to impose $p_g(B) = 0$, which we will assume from now on.

Secondly, the equality

$$\chi_1(X) = q(S_2) - q(B) = q(X) - 3 = q(B) - 1$$

will hold if and only if $q(S_2) = 2q(B) - 1$. Supposing that the covering $\tau : S_2 \to B$ is given by a line bundle $L \in \mathrm{Pic}(B)$ and is ramified over $E \in |L^{\otimes 2}|$, then $q(S_2) = q(B) + h^1(B, L^\vee) = q(B) + h^1(B, \omega_B \otimes L)$.

Therefore, if there exists a surface B with $p_g(B) = 0$, and admitting a line bundle L such that $h^1(B, L^\vee) = q(B) - 1$ and that $H^0(B, L^{\otimes 2}) \neq 0$, it is possible to obtain a fourfold X such that $\chi_i(X) = q(X) - 4 + i$ for $i = 1, 2$.

2.1.4 Functoriality

We will now study the behaviour of the BGG complex with respect to morphisms of varieties, and its consequences on the partial Euler characteristics.

Therefore, let $f : X \to Y$ be a morphism of smooth irregular varieties of dimensions $n = \dim X$ and $m = \dim Y$. The map f induces pull-back homomorphisms $f_k^* : H^k(Y, \mathcal{O}_Y) \to H^k(X, \mathcal{O}_X)$, and in particular

$$f_1^* : H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X),$$

which in turn induces a rational map

$$\mathbb{P}_Y = \mathbb{P}\left(H^1(\mathcal{O}_Y)\right) \twoheadrightarrow \mathbb{P}'_Y = \mathbb{P}\left(f_1^*\left(H^1(\mathcal{O}_Y)\right)\right) \subseteq \mathbb{P}_X = \mathbb{P}\left(H^1(\mathcal{O}_X)\right).$$

For simplicity, we will assume from now on that f_1^* is injective, so that we can identify $H^1(Y, \mathcal{O}_Y)$ with a subspace $H^1(X, \mathcal{O}_X)$, and the rational map above is indeed an injective morphism that identifies $\mathbb{P}_Y$ with the linear subspace $\mathbb{P}'_Y \subseteq \mathbb{P}_X$.

The morphisms f_k^* give rise naturally to morphisms of sheaves on $\mathbb{P}_Y$

$$\mathcal{O}_{\mathbb{P}_Y}(-n + k) \otimes H^k(Y, \mathcal{O}_Y) \longrightarrow \mathcal{O}_{\mathbb{P}_Y}(-n + k) \otimes H^k(X, \mathcal{O}_X).$$

These morphisms are compatible with cup-product, and thus induce a morphism of complexes over $\mathbb{P}_Y$:

$$f^* : BGG\,(Y) \otimes \mathcal{O}_{\mathbb{P}_Y}\left(-\left(n-m\right)\right) \longrightarrow BGG\,(X)_{|\mathbb{P}_Y}$$

(note that, in order to make f^* into a morphism of complexes, it is necessary to twist the BGG complex of Y in order to adjust the degrees).

On the other hand, since restriction to a subvariety is always right-exact (it is the pull-back via the inclusion map), for every s we have

$$\mathcal{F}^s_{X|\mathbb{P}_Y} = \operatorname{coker}\left\{\mathcal{O}_{\mathbb{P}_Y}\left(-s-1\right) \otimes H^{n-s-1}\left(\mathcal{O}_X\right) \to \mathcal{O}_{\mathbb{P}_Y}\left(-s\right) \otimes H^{n-s}\left(\mathcal{O}_X\right)\right\}.$$

Hence, the pullback f^* induces morphisms

$$g_r : \mathcal{F}^{m-r}_Y\left(-\left(n-m\right)\right) \longrightarrow \mathcal{F}^{n-r}_{X|\mathbb{P}_Y}$$

obtained by completing the commutative diagram with exact rows

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}_Y}\left(-n+r-1\right) \otimes H^{r-1}\left(\mathcal{O}_Y\right) \arrow[r] \arrow[d, "f^*_{r-1}"'] & \mathcal{O}_{\mathbb{P}_Y}\left(-n+r\right) \otimes H^{r}\left(\mathcal{O}_Y\right) \arrow[r] \arrow[d, "f^*_r"'] & \mathcal{F}^{m-r}_Y\left(-n+m\right) \arrow[r] \arrow[d, dashed, "g_r"] & 0 \\
\mathcal{O}_{\mathbb{P}_Y}\left(-n+r-1\right) \otimes H^{r-1}\left(\mathcal{O}_X\right) \arrow[r] & \mathcal{O}_{\mathbb{P}_Y}\left(-n+r\right) \otimes H^{r}\left(\mathcal{O}_X\right) \arrow[r] & \mathcal{F}^{n-r}_{X|\mathbb{P}_Y} \arrow[r] & 0
\end{tikzcd}$$

Recall from the beginning of the section that for any positive integer $r \leq \dim \mathrm{alb}_X\left(X\right)$ it holds $\chi_{n-r}\left(X\right) = \operatorname{rk}\mathcal{F}^{n-r}_X$ (and analogously for Y, with $m = \dim Y$ instead of $n = \dim X$). Hence, whenever the following conditions hold

1. $r \leq \min\left\{\dim \mathrm{alb}_X\left(X\right), \dim \mathrm{alb}_Y\left(Y\right)\right\}$, and
2. $\operatorname{rk}\mathcal{F}^{n-r}_{X|\mathbb{P}_Y} = \operatorname{rk}\mathcal{F}^{n-r}_X$,

the morphism g_r allows to relate the partial Euler characteristics $\chi_{m-r}\left(Y\right)$ and $\chi_{n-r}\left(X\right)$. More precisely, in any situation implying that g_r is injective, it will hold in particular that $\chi_{m-r}\left(Y\right) \leq \chi_{n-r}\left(X\right)$.

One easy way to guarantee the injectivity of some g_r is imposing that $f^*_{r+1} : H^{r+1}\left(Y, \mathcal{O}_Y\right) \to H^{r+1}\left(X, \mathcal{O}_X\right)$ is injective and that both $BGG\,(Y)$ and $BGG\,(X)_{|\mathbb{P}_Y}$ are exact in the first $r+1$ steps. Indeed, in this case we have the inclusions

$$\mathcal{F}^{m-r}_Y\left(-\left(n-m\right)\right) \hookrightarrow \mathcal{O}_{\mathbb{P}_Y}\left(-n+r+1\right) \otimes H^{r+1}\left(Y, \mathcal{O}_Y\right)$$

and

$$\mathcal{F}^{n-r}_{X|\mathbb{P}_Y} \hookrightarrow \mathcal{O}_{\mathbb{P}_Y}\left(-n+r+1\right) \otimes H^{r+1}\left(X, \mathcal{O}_X\right),$$

and g_r is exactly the restriction of

$$f^*_{r+1} : \mathcal{O}_{\mathbb{P}_Y}(-n+r+1) \otimes H^{r+1}(\mathcal{O}_Y) \longrightarrow \mathcal{O}_{\mathbb{P}_Y}(-n+r+1) \otimes H^{r+1}(\mathcal{O}_X).$$

We have thus obtained the following general result:

Theorem 2.1.10. *Let $f : X \to Y$ be a morphism between smooth irregular varieties of dimensions n and m respectively, such that the pull-back $H^1(Y, \mathcal{O}_Y) \hookrightarrow H^1(X, \mathcal{O}_X)$ is injective. Let r be an integer such that both $BGG(Y)$ and $BGG(X)_{|\mathbb{P}_Y}$ are exact in the first $r+1$ first steps (from the left), and assume also that $f^*_{r+1} : H^{r+1}(Y, \mathcal{O}_Y) \hookrightarrow H^{r+1}(X, \mathcal{O}_X)$ is injective and $\operatorname{rk} \mathcal{F}^{n-r}_{X|\mathbb{P}_Y} = \operatorname{rk} \mathcal{F}^{n-r}_X$. Then*

$$\chi_{n-r}(X) \geq \chi_{m-r}(Y).$$

Remark 2.1.11. *Even in the case that both X and Y are varieties of maximal Albanese dimension, the previous Theorem does not provide bounds for the holomorphic Euler-Poincaré caracteristics except for very specific cases:*

- *If $m \leq n$, the maximal r for which the Theorem may apply is $m-1$, since $BGG(Y)$ is exact in the first m steps, and we would only obtain inequalities for $\chi_1(Y), \chi_2(Y), \ldots$*
- *If $m \geq n$, the maximal r is now n in the case $\mathcal{F}^0_{X|\mathbb{P}_Y} = 0$. And in this case, f^*_{n+1} would be injective only if $H^{n+1}(Y, \mathcal{O}_Y) = 0$.*

Let us now study some particular cases.

Morphisms such that $f^*H^1(Y, \mathcal{O}_Y) = H^1(X, \mathcal{O}_X)$

We will first focus on morphisms such that the pull-back induces an isomorphism $H^1(Y, \mathcal{O}_Y) \cong H^1(X, \mathcal{O}_X)$. The paradigmatical examples are the Albanese morphisms.

In this case, $\mathbb{P}_Y \cong \mathbb{P}_X$. Therefore, there is no restriction of the complex $BGG(X)$ and the equality $\operatorname{rk} \mathcal{F}^r_{X|\mathbb{P}_Y} = \operatorname{rk} \mathcal{F}^r_X$ is tautologically satisfied for every $r = 0, \ldots, n$. Since $BGG(X)$ (resp. $BGG(Y)$) is exact in the first $\dim \operatorname{alb}_X(X)$ (resp. $\dim \operatorname{alb}_Y(Y)$) steps, it is possible to apply Theorem 2.1.10 for any $r < \min\{\dim \operatorname{alb}_X(X), \dim \operatorname{alb}_Y(Y)\}$, as long as $f^*_{r+1} : H^{r+1}(Y, \mathcal{O}_Y) \to H^{r+1}(X, \mathcal{O}_X)$ is injective.

Definition 2.1.12. *Let A be an Abelian variety, and $X \subseteq A$ a (smooth) subvariety of dimension d. If the set $\{a \in A \,|\, X + a = X\}$ is discrete, X*

is said to be non-degenerate. *Moreover, X is said to be* geometrically non-degenerate *if the restriction induces an injection $H^d(A, \mathcal{O}_A) \hookrightarrow H^d(X, \mathcal{O}_X)$ (or equivalently $H^0\left(A, \Omega_A^d\right) \hookrightarrow H^0(X, \omega_X)$).*

Remark 2.1.13. *Every geometrically non-degenerate subvariety is non-degenerate.*

Therefore, we can apply Theorem 2.1.10 to geometrically non-degenerate subvarieties to obtain the following

Corollary 2.1.14. *If $X \subseteq A$ is a geometrically non-degenerate subvariety of an Abelian variety A, with $\dim X = n$ and $\dim A = g$, then*

$$\chi_{n-r}(X) \geq \chi_{g-r}(A) = \binom{g-1}{r}, \quad \forall\, r = 0, \ldots, n-1.$$

Surjective morphisms

Suppose now that the morphism $f : X \to Y$ is surjective, so that in particular we have $m = \dim Y \leq \dim X = n$. This case includes fibrations and (generically) finite morphisms.

Since the differential of such a morphism has generically maximal rank, all the pull-back morphisms $f^* : H^0\left(Y, \Omega_Y^k\right) \to H^0\left(X, \Omega_X^k\right)$ are injective (consider the restriction of a holomorphic k-form on Y to the open set where df is surjective). Conjugating, we obtain that

$$f_k^* : H^k(Y, \mathcal{O}_Y) \hookrightarrow H^k(X, \mathcal{O}_X)$$

is injective for every $k = 0, \ldots, m$.

On the other hand, by the functoriality of the Albanese map, one has the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, two heads, "\mathrm{alb}_X"] \arrow[d, "f"'] & \mathrm{alb}_X(X) \arrow[r, hook] \arrow[d, "\mathrm{Alb}(f)"] & \mathrm{Alb}(X) \arrow[d, "\mathrm{Alb}(f)"] \\
Y \arrow[r, two heads, "\mathrm{alb}_Y"'] & \mathrm{alb}_Y(Y) \arrow[r, hook] & \mathrm{Alb}(Y)
\end{tikzcd}$$

and taking into account that both f and $\mathrm{alb}_Y : Y \to \mathrm{alb}_Y(Y)$ are surjective, one concludes that the restriction $\mathrm{Alb}(f) : \mathrm{alb}_X(X) \to \mathrm{alb}_Y(Y)$ is also surjective. Therefore, $\dim \mathrm{alb}_X(X) \geq \dim \mathrm{alb}_Y(Y)$ and the only limitation we have is $r < \dim \mathrm{alb}_Y(Y)$.

Finally, we have to study the exactness of the complex $BGG(X)_{|\mathbb{P}_Y}$, since we want it to be exact in as many steps as possible (at least the first $\dim \mathrm{alb}_Y(Y)$ ones).

It is known ([19], or the proof of Theorem 2.1.4) that if $\mathcal{O}_X \in V^i(\omega_X)$ is an isolated point for every $i > k$, then $BGG(X)$ is exact *at every point* in the first $n-k$ steps, and the restriction $BGG(X)_{|\mathbb{P}_Y}$ will be exact too.

So far, we have proved the following

Corollary 2.1.15. *Let $f : X \to Y$ be a surjective morphism, denote by $n = \dim X$ and $m = \dim Y$, and let $r < \dim \mathrm{alb}_Y(Y)$ be any positive integer such that $\mathcal{O}_X \in V^i(X)$ is isolated for every $i > n-r+1$. Then*

$$\chi_{n-r}(X) \geq \chi_{m-r}(Y).$$

And combining this last result with Corollary 2.1.14 we obtain the final

Corollary 2.1.16. *Let X be a smooth n-dimensional irregular variety such that its Albanese image $Y = \mathrm{alb}_X(X)$ is smooth of dimension $m = \dim Y$. Assume moreover that $\mathcal{O}_X \in V^i(X)$ is isolated for every $i > n-m$ and that Y is geometrically non-degenerate. Then*

$$\chi_{n-r}(X) \geq \binom{q(X)-1}{r} \qquad \forall\, r = 1, \ldots, m-1.$$

In particular, if X is primitive (hence of maximal Albanese dimension) and its Albanese image is smooth and non-degenerate, then

$$\chi_r(X) \geq \binom{q(X)-1}{\dim(X)-r} \qquad \forall\, r = 1, \ldots, \dim X - 1.$$

Remark 2.1.17. *Note that these bounds are much stronger than the linear ones provided by Theorem 2.1.4.*

2.2 Higher-rank derivative complexes

In this section we expose a generalization of the derivative and BGG complexes, obtaining stronger inequalities for the Hodge numbers of irregular varieties X admitting *non-degenerate* subspaces $W \subseteq H^0(X, \Omega^1_X)$. The section begins with the basic definitions followed by a digression through complexes of Eagon-Northott type. These are the technical tools that will provide the main results presented next. The section is closed with the particular case of subvarieties of Abelian varieties, showing that they admit non-degenerate subspaces of any rank and hence it is possible to apply all the preceding results (as well as some variants).

2.2.1 Definitions

We first explain the construction of our main tools, which we call *higher rank derivative complex* and *Grassmannian BGG complex*. The reason of the name is that they generalize the derivative and BGG complexes to the case where more than one 1-form (or cohomology class $v \in H^1(X, \mathcal{O}_X)$) are put into the picture. However, we do not obtain them from a "derivative" setting, nor from a categorical analogue to the BGG correspondence. Instead, we construct them directly and show that they coincide with the previous ones in the case of one-dimensional subspaces.

As in the previous section, X will denote an irregular complex projective or compact Kähler variety of dimension d.

Definition 2.2.1 (Higher-rank derivative complex)**.** *Fix integers $r \geq 1$, $1 \leq n \leq \min\{r, d\}$, $0 \leq j \leq d$, and a linear subspace $W \subseteq V$. We define $C^j_{r,n,W}$ as the complex (of vector spaces)*

$$\begin{aligned} 0 \longrightarrow \operatorname{Sym}^r W \otimes H^j(X, \mathcal{O}_X) \longrightarrow \operatorname{Sym}^{r-1} W \otimes H^j\left(X, \Omega^1_X\right) \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Sym}^{r-i} W \otimes H^j\left(X, \Omega^i_X\right) \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Sym}^{r-n} W \otimes H^j(X, \Omega^n_X) \quad (2.7) \end{aligned}$$

where the maps $\mu^j_i : \operatorname{Sym}^{r-i} W \otimes H^j(X, \Omega^i_X) \to \operatorname{Sym}^{r-i-1} W \otimes H^j\left(X, \Omega^{i+1}_X\right)$ are given by

$$\mu^j_i((w_1 \cdots w_{r-i}) \otimes [\alpha]) = \sum_{t=1}^{r-i} (w_1 \cdots \widehat{w_t} \cdots w_{r-i}) \otimes [w_t \wedge \alpha].$$

Lemma 2.2.2. *The maps μ^j_i are well defined and indeed make $C^j_{r,n,W}$ into a complex.*

Proof. In order to see that the μ^j_i are well defined, consider first the linear map

$$\widetilde{\mu}^j_i : W^{\otimes(r-i)} \otimes H^j\left(X, \Omega^i_X\right) \longrightarrow \operatorname{Sym}^{r-i-1} W \otimes H^j\left(X, \Omega^{i+1}_X\right)$$

defined as

$$\widetilde{\mu}^j_i((w_1 \otimes \cdots \otimes w_{r-i}) \otimes [\alpha]) = \sum_{t=1}^{r-i} (w_1 \cdots \widehat{w_t} \cdots w_{r-i}) \otimes [w_t \wedge \alpha].$$

Clearly, $\widetilde{\mu}^j_i$ satisfies

$$\widetilde{\mu}^j_i\left(\left(w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(r-i)}\right) \otimes [\alpha]\right) = \widetilde{\mu}^j_i((w_1 \otimes \cdots \otimes w_{r-i}) \otimes [\alpha])$$

for any permutation $\sigma : \{1, \dots, r-i\} \to \{1, \dots, r-i\}$, and hence it factors through the quotient

$$W^{\otimes(r-i)} \otimes H^j\left(X, \Omega_X^i\right) \longrightarrow \operatorname{Sym}^{r-i} W \otimes H^j\left(X, \Omega_X^i\right)$$

and a map

$$\operatorname{Sym}^{r-i} W \otimes H^j\left(X, \Omega_X^i\right) \longrightarrow \operatorname{Sym}^{r-i-1} W \otimes H^j\left(X, \Omega_X^{i+1}\right)$$

which is precisely μ_i^j.

Once we know that the μ_i^j are well defined, it is an straightforward computation to check that $\mu_{i+1}^j \circ \mu_i^j = 0$. Indeed

$$\begin{aligned}
&\mu_{i+1}^j\left(\mu_i^j\left((w_1 \cdots w_{r-i}) \otimes [\alpha]\right)\right) = \\
&\qquad = \sum_{t=1}^{r-i} \mu_{i+1}^j\left((w_1 \cdots \widehat{w_t} \cdots w_{r-i}) \otimes [w_t \wedge \alpha]\right) = \\
&\qquad = \sum_{t=1}^{r-i}\left(\sum_{s=1}^{t-1}(w_1 \cdots \widehat{w_s} \cdots \widehat{w_t} \cdots w_{r-i}) \otimes [w_s \wedge w_t \wedge \alpha] + \right.\\
&\qquad\quad \left. + \sum_{s=t+1}^{r-i}(w_1 \cdots \widehat{w_t} \cdots \widehat{w_s} \cdots w_{r-i}) \otimes [w_s \wedge w_t \wedge \alpha]\right) = \\
&= \sum_{1 \le s < t \le r-i}(w_1 \cdots \widehat{w_t} \cdots \widehat{w_s} \cdots w_{r-i}) \otimes \left([w_s \wedge w_t \wedge \alpha] + [w_t \wedge w_s \wedge \alpha]\right) = 0
\end{aligned}$$

since obviously

$$[w_s \wedge w_t \wedge \alpha] + [w_t \wedge w_s \wedge \alpha] = [w_s \wedge w_t \wedge \alpha - w_s \wedge w_t \wedge \alpha] = 0.$$

□

Since for every $1 \le n' < n$ the complex $C^j_{r,n',W}$ is a truncation of $C^j_{r,n,W}$, we may assume that n is the greatest possible, that is $n = \min\{r, d\}$, and denote the complex simply by $C^j_{r,W}$.

Note that in the case of a 1-dimensional W, generated by w, we have $\operatorname{Sym}^r W \equiv \mathbb{C}\langle w^r \rangle \cong \mathbb{C}$, and $C^j_{d,\mathbb{C}\langle w\rangle}$ is nothing but the complex

$$0 \longrightarrow H^j(X, \mathcal{O}_X) \xrightarrow{\wedge w} H^j\left(X, \Omega_X^1\right) \xrightarrow{\wedge w} \dots \xrightarrow{\wedge w} H^j(X, \omega_X),$$

which is (complex-conjugate to) the *derivative complex* studied by Green and Lazarsfeld in [18].

Our main aim is to study the exactness of $C^j_{r,W}$. More precisely, we look for conditions on W which guarantee that $C^j_{r,W}$ is exact in some (say m) of its first steps, (i.e., $C^j_{r,m,W}$ is exact), because this exactness will provide several inequalities between the Hodge numbers $h^{p,j}(X)$.

At some points, we will need to consider different subspaces W. Hence, we "glue" all the complexes (2.7) with fixed $k = \dim W$ as follows. Denote by $\mathbb{G} = \mathbb{G}_k = Gr(k, V)$ the Grassmannian variety of k-dimensional subspaces of V, and by $S \subseteq V \otimes \mathcal{O}_{\mathbb{G}}$ the tautological subbundle, the vector bundle of rank k whose fibre over a point $W \in \mathbb{G}$ is precisely the subspace $W \subseteq V$.

Definition 2.2.3 (Grassmannian BGG complex)**.** *For any integers $r \geq 1$ and $0 \leq j \leq d$, the (r, j)-th* Grassmannian BGG complex *(of rank k) of X is the complex of vector bundles on $\mathbb{G}_k$*

$$
\begin{aligned}
C^j_r : 0 \longrightarrow \operatorname{Sym}^r S \otimes H^j(X, \mathcal{O}_X) \longrightarrow \operatorname{Sym}^{r-1} S \otimes H^j\left(X, \Omega^1_X\right) \longrightarrow \cdots \\
\cdots \longrightarrow \operatorname{Sym}^{r-i} S \otimes H^j\left(X, \Omega^i_X\right) \longrightarrow \cdots \\
\cdots \longrightarrow \operatorname{Sym}^{r-n} S \otimes H^j(X, \Omega^n_X)
\end{aligned}
$$

where $n = \min\{r, d\}$ and over each point $W \in \mathbb{G}_k$ it is given by (2.7). Let $\mathcal{F}^j_{r,n}$ denote the cokernel of the last map in $C^j_{r,n}$, the (n, r, j)-th Grassmannian BGG sheaf *(of rank k) of X.*

Remark 2.2.4. *If $k = 1$, then $\mathbb{G} = \mathbb{P} = \mathbb{P}(H^0(X, \Omega^1_X))$, $S = \mathcal{O}_{\mathbb{P}}(-1)$ and $\operatorname{Sym}^r S = \mathcal{O}_{\mathbb{P}}(-r)$. So taking $k = 1$ and $r = d$, the above complex is precisely (the complex-conjugate of) the BGG complex introduced by Lazarsfeld and Popa in [27]. More generally, fixing only $k = 1$ and $r < d$ we obtain all the complexes considered in the first section of this chapter (suitably twisted). In this way, the Grassmannian BGG complexes can be seen as generalizations of the former complexes, with the new feature that they capture also the additive structure of the cohomology algebra of X, and the sheaves $\mathcal{F}^j_{r,n}$ generalize the BGG sheaves introduced in Definition 2.1.2.*

The interest of studying these complexes is that, whenever they are exact at some point $W \in \mathbb{G}$, they provide some inequalities involving the Hodge numbers $h^{i,j}(X) = h^j(X, \Omega^i_X)$. These inequalities are much stronger when the complex is exact *at every point*, so that the cokernel sheaves of the maps μ^j_i are vector bundles and a deeper study of them is feasible (as we will do in Section 2.4). For example, the proof of the higher-dimensional Castelnuovo-de Franchis inequality given by Lazarsfeld and Popa in [27] is based on the fact that the BGG sheaf (the cokernel of the last map of C^0_d with $k = 1$) is an indecomposable vector bundle on $\mathbb{P}^{q-1}$.

2.2.2 Eagon-Northcott complexes

In order to study the exactness of the higher rank derivative complexes, we will strongly use some results of commutative algebra concerning a generalization of the Koszul complexes: the complexes of Eagon-Northcott type. In this section we will give an overview of them, exposing their main properties and focusing on those that will be useful for our purposes. For more detailed explanations, we refer to the books [8, 14] and the article [1].

Consider a ring R (commutative, with identity and Noetherian) and a map $\phi : G \to F$ of finitely generated free R-modules, of ranks $n = \operatorname{rk} G$ and $k = \operatorname{rk} F$. For any integer $r \geq 0$, denote by $\mathcal{C}_r(\phi)$ the complex

$$\begin{aligned}\mathcal{C}_r(\phi) : 0 \longrightarrow \bigwedge^{r} G \otimes \operatorname{Sym}^0 F \xrightarrow{\partial} \bigwedge^{r-1} G \otimes \operatorname{Sym}^1 F \xrightarrow{\partial} \cdots \\ \cdots \xrightarrow{\partial} \bigwedge^{1} G \otimes \operatorname{Sym}^{r-1} F \xrightarrow{\partial} \bigwedge^{0} G \otimes \operatorname{Sym}^{r} F \longrightarrow 0,\end{aligned}$$

where

$$\partial(g_1 \wedge \cdots \wedge g_m \otimes f) = \sum_{j=1}^{m} (-1)^{j+1} g_1 \wedge \cdots \wedge \widehat{g_j} \wedge \cdots \wedge g_m \otimes (\phi(g_j) f)$$

for every $g_1 \dots g_m \in G, f \in \operatorname{Sym}^{r-m} F$.

If $n \geq k$ and $0 \leq r \leq n-k$, then the R-dual complex $(\mathcal{C}_{n-k-r}(\phi))^\vee$ (where $(-)^\vee = \operatorname{Hom}_R(-, R)$) and $\mathcal{C}_r(\phi)$ can be spliced by a map

$$\nu_r : \bigwedge^{n-k-r} G^\vee = \left(\bigwedge^{n-k-r} G \otimes \operatorname{Sym}^0 F\right)^\vee \longrightarrow \bigwedge^{r} G \otimes \operatorname{Sym}^0 F = \left(\bigwedge^{r} G^\vee\right)^\vee$$

which is constructed as follows: choose two *orientations* $\gamma : \bigwedge^k F^\vee \xrightarrow{\cong} R$ and $\delta : \bigwedge^n G^\vee \xrightarrow{\cong} R$, define $\alpha = \left(\wedge^k \phi^\vee\right)(\gamma^{-1}(1))$ (where

$$\wedge^k \phi^\vee : \bigwedge^{k} F^\vee \longrightarrow \bigwedge^{k} G^\vee$$

is the induced map), and set

$$(\nu_r(x))(y) = \delta(x \wedge y \wedge \alpha) \qquad \forall x \in \bigwedge^{n-k-r} G^\vee, y \in \bigwedge^{r} G^\vee.$$

The resulting complex is

$$\mathcal{D}_r(\phi) : 0 \longrightarrow \left(\bigwedge^0 G \otimes \mathrm{Sym}^{n-k-r}(F)\right)^\vee \xrightarrow{\partial^\vee} \cdots$$

$$\cdots \xrightarrow{\partial^\vee} \left(\bigwedge^{n-k-r} G \otimes \mathrm{Sym}^0(F)\right)^\vee \xrightarrow{\nu_r} \bigwedge^r G \otimes \mathrm{Sym}^0(F) \xrightarrow{\partial} \cdots$$

$$\cdots \xrightarrow{\partial} \bigwedge^0 G \otimes \mathrm{Sym}^r(F) \longrightarrow 0.$$

Note that different choices of orientations lead to different ν_r differring only by multiplication by an invertible element.

The exactness of the complexes $\mathcal{C}_r(\phi)$ and $\mathcal{D}_r(\phi)$ depends on the ideals $I_i(\phi) \subseteq R$ generated by the $i \times i$ minors of ϕ.

For the case $k \leq n$, the main results are the following theorems.

Theorem 2.2.5 ([8] Th. 2.16)**.** *With the previous notations, suppose $k \leq n$ and $0 \leq r \leq n-k$. If $\mathrm{depth}(I_k(\phi)) = n-k+1$, then the complex $\mathcal{D}_r(\phi)$ is a free resolution of $R/I_k(\phi)$ if $r = 0$, and of $\mathrm{Sym}^r(\mathrm{coker}(\phi))$ if $r > 0$.*

Theorem 2.2.6 ([14] Th. A.2.10)**.** *With the previous notations, assume also that $k \leq n$ and $r \geq n-k+1$. If $\mathrm{depth}(I_i(\phi)) = n-i+1$ for every $\max\{1, n-r+1\} \leq i \leq k$, then the complex $\mathcal{C}_r(\phi)$ is a free resolution of $\mathrm{Sym}^r(\mathrm{coker}(\phi))$.*

On the other hand, for the case $k \geq n$ it holds an analogous result.

Theorem 2.2.7 ([1] Prop. 3.(3))**.** *With the previous notations, if $k \geq n$ and $\mathrm{depth}(I_i(\phi)) \geq n-i+1$ for every $1 \leq i \leq n$, then the complex $\mathcal{C}_r(\phi)$ is a free resolution of $\mathrm{Sym}^r(\mathrm{coker}(\phi))$ for every $r > 0$.*

We will now translate the previous algebraic constructions and results into geometry. Consider a map of vector bundles $\phi : G \to F$ over a smooth (or at least Cohen-Macaulay) variety X. As above, denote by $n = \mathrm{rk}\, G$ and by $k = \mathrm{rk}\, F$. The case $n \geq k$ has been studied in [25] App. B, but as far as the author is aware, the case $n < k$ has not been written anywhere, and this is the reason why we include this discussion.

For any $r \geq 0$, one can construct complexes of vector bundles $\mathcal{C}_r(\phi)$ and $\mathcal{D}_r(\phi)$, whose stalks over a point $p \in X$ are the $\mathcal{C}_r(\phi_p)$ and $\mathcal{D}_r(\phi_p)$ associated to the map of $\mathcal{O}_{X,p}$-free modules $\phi_p : G_p \to F_p$. More explicitly, they have the shape

$$\mathcal{C}_r(\phi) \; : \; 0 \; \to \; \bigwedge^r G \; \to \; \bigwedge^{r-1} G \otimes F \cdots \; \to \; G \otimes \mathrm{Sym}^{r-1} F \; \to \; \mathrm{Sym}^r F$$

and, if $r \leq n-k$ (in case $k \leq n$),

$$\begin{aligned}\mathcal{D}_r(\phi) : 0 \to \bigwedge^{n} G \otimes \bigwedge^{k} F^\vee \otimes \operatorname{Sym}^{n-k-r} F^\vee \to \\ \to \bigwedge^{n-1} G \otimes \bigwedge^{k} F^\vee \otimes \operatorname{Sym}^{n-k-r-1} F^\vee \cdots \to \bigwedge^{k+r+1} G \otimes \bigwedge^{k} F^\vee \otimes F^\vee \to \\ \to \bigwedge^{k+r} G \otimes \bigwedge^{k} F^\vee \xrightarrow{\nu_r} \bigwedge^{r} G \to \bigwedge^{r-1} G \otimes F \to \cdots \to G \otimes \operatorname{Sym}^{r-1} F \to \\ \to \operatorname{Sym}^r F.\end{aligned}$$

The maps

$$\bigwedge^{i} G \otimes \operatorname{Sym}^{r-i} F \longrightarrow \bigwedge^{i-1} G \otimes \operatorname{Sym}^{r-i+1} F$$

are given by

$$(g_1 \wedge \cdots \wedge g_i) \otimes f \mapsto \sum_{j=1}^{i} (-1)^{j-1} (g_1 \wedge \cdots \wedge \widehat{g_j} \wedge \cdots \wedge g_i) \otimes (\phi(g_j) f),$$

the maps

$$\bigwedge^{n-i+1} G \otimes \bigwedge^{k} F^\vee \otimes \operatorname{Sym}^{r-i+1} F^\vee \longrightarrow \bigwedge^{n-i} G \otimes \bigwedge^{k} F^\vee \otimes \operatorname{Sym}^{r-i} F^\vee$$

are the duals of the previous ones twisted by the line bundle $\bigwedge^n G \otimes \bigwedge^k F^\vee$ (so that there is no need to choose the orientations γ and δ, which by the way, may not exist globally), and the maps

$$\nu_r : \bigwedge^{k+r} G \otimes \bigwedge^{k} F^\vee \longrightarrow \bigwedge^{r} G$$

are induced by $\bigwedge^k \phi : \bigwedge^k G \to \bigwedge^k F$.

Definition 2.2.8. *For the sake of simplicity, define*

$$EN_r(\phi) = \begin{cases} \mathcal{D}_r(\phi) & \text{if } r \leq n-k, \text{ or} \\ \mathcal{C}_r(\phi) & \text{if } r \geq n-k+1. \end{cases}$$

We call these complexes the Eagon-Northcott complexes *associated to ϕ.*

In this geometric setting, the exactness of these complexes is governed by the degeneracy loci of ϕ. For any positive integer i, denote by

$$Z_i = Z_i(\phi) = \{p \in X | rk(\phi(p) : G \otimes \mathbb{C}(p) \to F \otimes \mathbb{C}(p)) < i\}$$

the locus where ϕ has rank smaller than i. Locally at a point $p \in X$, the Z_i are (set-theoretically) the zero loci of the ideals $I_i(\phi_p)$, and since X is smooth (or at least Cohen-Macaulay) we have the equality

$$\operatorname{codim}_p Z_i = \operatorname{depth} I_i(\phi_p).$$

Therefore, we can translate Theorems 2.2.5, 2.2.6 and 2.2.7 into the following

Theorem 2.2.9. *With the preceding notations, assume that*

1. *either* $k \leq n$ *and*
   - $\operatorname{codim} Z_k = n - k + 1$ *if* $r \leq n - k$, *or*
   - $\operatorname{codim} Z_i \geq n - i + 1$ *for every* $i = \max\{1, n - r + 1\}, \ldots, k$ *if* $r \geq n - k + 1$,
2. *or* $k \geq n$ *and* $\operatorname{codim} Z_i \geq n - i + 1$ *for all* $i = 1, \ldots, n$.

Then $EN_r(\phi)$ *is a locally free resolution of* $\operatorname{Sym}^r(\operatorname{coker}\phi)$ *for every* $r \geq 1$, *and of* $\mathcal{O}_{Z_k}$ *if* $r = 0$ *and* $k \leq n$.

2.2.3 Exactness of $C^j_{r,W}$

We will now use the previous results on Eagon-Northcott complexes to study the exactness of the complexes $C^j_{r,W}$. The approach we follow is analogous to the method used by Green and Lazarsfeld in Section 3 (A Nakano-type generic vanishing theorem) of [18], suitably adapted to subspaces of rank $k > 1$.

So fix a k-dimensional subspace $W \subseteq H^0(X, \Omega^1_X)$, and consider the following complex of sheaves on X,

$$\begin{aligned}\mathcal{C}_{r,W} : 0 \longrightarrow \operatorname{Sym}^r W \otimes \mathcal{O}_X \longrightarrow \operatorname{Sym}^{r-1} W \otimes \Omega^1_X \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Sym}^{r-i} W \otimes \Omega^i_X \longrightarrow \cdots \longrightarrow \operatorname{Sym}^{r-n} W \otimes \Omega^n_X,\end{aligned}$$

where the maps $\mu_i : \operatorname{Sym}^{r-i} W \otimes \Omega^i_X \to \operatorname{Sym}^{r-i-1} W \otimes \Omega^{i+1}_X$ are defined as in Definition 2.2.1, and we also assume $n = \min\{r, d\}$. Clearly, its global sections form the complex $C^0_{r,W}$, and in general, $C^j_{r,W} = H^j(X, \mathcal{C}_{r,W})$ is the complex obtained by applying the j-th sheaf cohomology functor. Denote by $K^i = \mathrm{Sym}^{r-i} W \otimes \Omega^i_X$ the i-th term of $\mathcal{C}_{r,W}$, and by $\mathcal{H}^i = \mathcal{H}^i(\mathcal{C}_{r,W})$ its i-th cohomology sheaf.

There are two spectral sequences, $'E$ and $''E$, both abutting to the hypercohomology of $\mathcal{C}_{r,W}$, starting at

$$'E_1^{i,j} = H^j\left(X, K^i\right) = \mathrm{Sym}^{r-i} W \otimes H^j\left(X, \Omega^i_X\right) \quad \text{and} \quad ''E_2^{i,j} = H^i\left(X, \mathcal{H}^j\right), \tag{2.8}$$

respectively. Note that the rows of $'E_1$ are precisely the complexes $C^j_{r,W}$ whose exactness we want to determine. Hence, the combined study of these two spectral sequences will lead to some results in the wanted direction.

We start with a generalization of Proposition 3.7 in [18], whose proof is analogous (but notationally more complicated).

Proposition 2.2.10. *For any $W \in \mathbb{G}_k$, the spectral sequence $'E$ degenerates at $'E_2$, i.e. $'E_2 = {}'E_\infty$.*

Proof. We will denote by $A^{i,j}(X)$ the vector space of $\mathcal{C}^\infty$ differential forms of type (i,j), and will identify each cohomology class $[b] \in H^j(X, \Omega^i_X)$ with its only harmonic representative $b \in A^{i,j}(X)$. We will also use the following result ([42] Proposition 6.17): if $b \in A^{i,j}(X)$ is both ∂- and $\bar\partial$-closed, and either ∂- or $\bar\partial$-exact, then $b = \partial\bar\partial c = -\bar\partial\partial c$ for some $c \in A^{i-1,j-1}(X)$.

Fix a basis $\{w_1, \dots, w_k\}$ of W, so that any $b \in \mathrm{Sym}^{r-i} W \otimes H^j(X, \Omega^i_X)$ may be uniquely written as

$$b = \sum_{|J|=r-i} w_J \otimes [b_J]$$

where $J = \{1 \le j_1 \le j_2 \le \cdots \le j_{r-i} \le k\}$, $w_J = w_{j_1} \cdots w_{j_{r-i}} \in \mathrm{Sym}^{r-i} W$ and $b_J \in A^{i,j}(X)$ is harmonic.

Firstly, we will show that the differential d_2 of $'E_2$ vanishes on every $'E_2^{i,j}$. By definition, any class in $'E_2^{i,j}$ is represented by some

$$b = \sum_{|J|=r-i} w_J \otimes [b_J] \in \ker \mu^j_i$$

that is, such that

$$\sum_{|J|=r-i} \sum_{s=1}^{r-i} w_{J-\{j_s\}} \otimes [w_{j_s} \wedge b_J] = \sum_{|J'|=r-i-1} w_{J'} \otimes \left[\sum_{j=1}^k w_j \wedge b_{J' \cup \{j\}}\right] = 0$$

where $J - \{j_s\}$ and $J' \cup \{j\}$ should be understood as operations on *multisets*. This last sum is zero if and only if all the classes $\left[\sum_{j=1}^k w_j \wedge b_{J' \cup \{j\}}\right]$ vanish in $H^j\left(X, \Omega_X^{i+1}\right) \cong H_{\bar{\partial}}^{i+1,j}(X)$ (viewed as Dolbeault's cohomology), so we can assume that all the $\sum_{j=1}^k w_j \wedge b_{J' \cup \{j\}}$ are $\bar{\partial}$-exact. Since they are also both ∂ and $\bar{\partial}$-closed (because so are the w_j and the b_J), there exist forms $c_{1,J'} \in A^{i,j-1}(X)$ such that

$$\sum_{j=1}^k w_j \wedge b_{J' \cup \{j\}} = \bar{\partial}\partial c_{1,J'}, \tag{2.9}$$

and $d_2(b)$ is represented by

$$\begin{aligned}\mu_{i+1}^{j-1}\left(\sum_{|J'|=r-i-1} w_{J'} \otimes \partial c_{1,J'}\right) &= \sum_{|J'|=r-i-1} \sum_{s=1}^{r-i-1} w_{J'-\{j'_s\}} \otimes \left(w_{j'_s} \wedge \partial c_{1,J'}\right) = \\ &= \sum_{|J''|=r-i-2} w_{J''} \otimes \left(\sum_{j=1}^k w_j \wedge \partial c_{1,J'' \cup \{j\}}\right).\end{aligned}$$

Therefore, in order to see that $d_2(b) = 0$, we only need to check that all the $a_{J''} = \sum_{j=1}^k w_j \wedge \partial c_{1,J''\cup\{j\}}$ are $\bar{\partial}$-exact (thus representing the zero class in $H_{\bar{\partial}}^{i+2,j-1}(X) \cong H^{j-1}\left(X, \Omega_X^{i+2}\right)$). On the one hand, note that

$$a_{J''} = -\partial\left(\sum_{j=1}^k w_j \wedge c_{1,J''\cup\{j\}}\right),$$

so they are ∂-exact, and hence ∂-closed. On the other hand, using equation (2.9) we obtain

$$\bar{\partial} a_{J''} = -\sum_{j=1}^k w_j \wedge \bar{\partial}\partial c_{1,J''\cup\{j\}} = -\sum_{1 \le j < l \le k} (w_j \wedge w_l + w_l \wedge w_j) \wedge b_{J''\cup\{j,l\}} = 0,$$

so $a_{J''} = \bar{\partial}\partial c_{2,J''}$ for some $c_{2,J''} \in A^{i+1,j-2}(X)$. In particular, it is $\bar{\partial}$-exact and hence $d_2(b) = 0$, as wanted.

Now we have to show that all the subsequent differentials d_m also vanish. Assume inductively that for any $2 \le l < m$ we have $d_l = 0$, and that for any b as above we can find differential forms $c_{l,J_l} \in A^{i+l-1,j-l}(X)$ such that $\bar{\partial}\partial c_{l,J_l} = \sum_{j=1}^k w_j \wedge \partial c_{l-1,J_l\cup\{j\}}$ for every multisubset J_l of $\{1,\ldots,r\}$ of cardinality $r-i-l$. Then, as before, the image $d_m(b)$ is the class in ${}'E_m^{i+m,j-m+1} = {}'E_2^{i+m,j-m+1}$ of

$$\sum_{|J_m|=r-i-m} w_{J_m} \otimes \left(\sum_{j=1}^k w_j \wedge \partial c_{m-1,J_m\cup\{j\}}\right).$$

As above, the forms $\sum_{j=1}^{k} w_j \wedge \partial c_{m-1,J_m\cup\{j\}}$ are ∂-exact and $\bar{\partial}$-closed, so there exist forms c_{m,J_m} as in the induction hypothesis, and in particular $d_r(b) = 0$ because they are $\bar{\partial}$-exact. $\square$

Suppose now that there is some integer N such that $\mathcal{H}^j = 0$ for all $j < N$, or more generally $''E_2^{i,j} = H^i(X, \mathcal{H}^j) = 0$ for $i + j < N$. Then, by (2.8), we would have $\mathbb{H}^m(X, \mathcal{C}_{r,W}) = 0$ for $m < N$. Looking at the other spectral sequence, it must hold $'E_\infty^{i,j} = {}'E_2^{i,j} = 0$ for all $i + j < N$. But $'E_2^{i,j}$ is precisely the cohomology of $H^j(\mathcal{C}_{r,W}) = C_{r,W}^j$ at the i-th step, so we get that $C_{r,W}^j$ is exact in the first $N - j$ steps. In particular, $C_{r,W}^0$ would be exact in the first N steps.

Therefore, we will next try to answer the next

Question 2.2.11. *Fixed N, under which hypothesis on W can we assure $H^i(X, \mathcal{H}^j) = 0$ for $i + j < N$?*

For this purpose, we will first try to identify the sheaves $\mathcal{H}^j$. Consider the map

$$\phi : T_X = \left(\Omega_X^1\right)^\vee \longrightarrow W^\vee \otimes \mathcal{O}_X$$

dual to the evaluation map $ev : W \otimes \mathcal{O}_X \to \Omega_X^1$, and denote $\mathcal{K} = \operatorname{coker}(\phi)$. For any $i = 1, \ldots, k$, let

$$\begin{aligned} Z_i = Z_i(W) = \{p \in X \mid \operatorname{rk}(\phi(p) : T_X \otimes \mathbb{C}(p) \to W^\vee) < i\} = \\ = \left\{p \in X \mid \operatorname{rk}\left(ev(p) : W \to \Omega_X^1 \otimes \mathbb{C}(p)\right) < i\right\} \end{aligned}$$

be the locus where the forms in W span a subspace of dimension $< i$ of the cotangent space, or where the kernel of the evaluation map has dimension greater than $k - i$. Clearly, $\mathcal{K}$ is supported on Z_k, the locus where ϕ is not surjective.

Definition 2.2.12 (Non-degenerate subspace). *We say that a subspace $W \subseteq H^0(X, \Omega_X^1)$ is* non-degenerate *if*

$$\operatorname{codim} Z_i \geq d - i + 1 \qquad \forall\, 1 \leq i \leq \min\{k, d\}.$$

Remark 2.2.13. *We can define W to be non-degenerate* in degree r *if*

- $\operatorname{codim} Z_k = d - k + 1$ *in the case $r \leq d - k$, or*
- $\operatorname{codim} Z_j \geq d - j + 1$ *for $j = \max\{1, d - r + 1\}, \ldots, \min\{k, d\}$ in the case $r \geq d - k + 1$.*

Equivalently, W is non-degenerate in degree r if ϕ satisfies the hypothesis of Theorem 2.2.9 for the fixed r. Therefore, a non-degenerate subspace is non-degenerate in every degree, but not conversely.

Although this definition is more precise and could lead to better results in some cases, we prefer the original one because of its simplicity.

The motivation of Definition 2.2.12 (or its generalization of Remark 2.2.13) is that Theorem 2.2.9 allows to identify the cohomology sheaves $\mathcal{H}^i$ of $\mathcal{C}_{r,W}$ for non-degenerate W.

Lemma 2.2.14. *Fix any $r \geq 1$, and assume that W is non-degenerate (at least in degree r). Then $\mathcal{H}^i\left(\mathcal{C}_{r,W}\right) = \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}\left(\mathrm{Sym}^r \mathcal{K}, \mathcal{O}_X\right)$ for all $0 \leq i < r$.*

Proof. Consider the r-th Eagon-Northcott complex $EN_r(\phi)$ associated to ϕ, whose last r steps look like

$$\begin{aligned} EN_r(\phi) : \ldots \longrightarrow \left(\Omega^r_X\right)^\vee \longrightarrow \left(\Omega^{r-1}_X\right)^\vee \otimes W^\vee \longrightarrow \cdots \\ \cdots \longrightarrow \left(\Omega^1_X\right)^\vee \otimes \mathrm{Sym}^{r-1} W^\vee \longrightarrow \mathcal{O}_X \otimes \mathrm{Sym}^r W^\vee. \end{aligned}$$

By Theorem 2.2.9, the non-degeneracy of W implies that $EN_r(\phi)$ is (the beginning of) a locally free resolution of $\mathrm{Sym}^r \mathcal{K}$, so we can compute

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}\left(\mathrm{Sym}^r \mathcal{K}, \mathcal{O}_X\right) = \mathcal{H}^i\left(\mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(EN_r, \mathcal{O}_X\right)\right).$$

But clearly the first r steps of $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}\left(EN_r, \mathcal{O}_X\right)$ form the complex $\mathcal{C}_{r,W}$, and the claim follows. $\square$

We now focus on the case $k \leq d$, where some well-known properties of the $\mathcal{E}\mathrm{xt}$ sheaves lead to a first result:

Theorem 2.2.15. *If W is non-degenerate, then the complex*

$$\begin{aligned} C^j_{r,W} : 0 \longrightarrow \mathrm{Sym}^r W \otimes H^j\left(X, \mathcal{O}_X\right) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{Sym}^{r-i} W \otimes H^j\left(X, \Omega^i_X\right) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{Sym}^{r-n} W \otimes H^j\left(X, \Omega^n_X\right) \end{aligned}$$

is exact at least in the first $d-k-j+1$ steps.

Proof. For a general coherent sheaf $\mathcal{F}$ on X we have (see [23] Prop. 1.6.6)

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}\left(\mathcal{F}, \mathcal{O}_X\right) = 0 \qquad \forall\, i < \operatorname{codim} \operatorname{Supp} \mathcal{F}.$$

Since $\operatorname{Supp} \mathrm{Sym}^r \mathcal{K} = \operatorname{Supp} \mathcal{K} = Z_k$ has codimension at least $d-k+1$ because W is non-degenerate, we obtain

$$\mathcal{H}^j\left(\mathcal{C}_{r,W}\right) = \mathcal{E}\mathrm{xt}^j_{\mathcal{O}_X}\left(\mathrm{Sym}^r \mathcal{K}, \mathcal{O}_X\right) = 0$$

for all $j \leq d-k$. Therefore, the second spectral sequence in (2.8) satisfies $''E_2^{i,j} = 0$ for all i and all $j \leq d-k$. Since $''E_2^{i,j}$ abuts to the hypercohomology of $\mathcal{C}_r$, this implies that $\mathbb{H}^n\left(X, \mathcal{C}_{r,W}\right) = 0$ for all $n \leq d-k$. Recalling that the first spectral sequence $'E_1^{i,j}$ degenerates at $'E_2$ (Proposition 2.2.10), and it also abuts to the hypercohomology of $\mathcal{C}_{r,W}$, this implies that $'E_2^{i,j} = 0$ for all $i+j \leq d-k$. But $'E_2^{i,j}$ is precisely the cohomology of the complex $C_{r,W}^j$ at the i-th step, so the claim follows. □

Some examples and results of the next section suggest that the complex $C_{r,W}^j$ should be exact under weaker hypothesis, and even for some $k > d$. To obtain such a result we should study the cohomology of the sheaves $\mathcal{H}^i = \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}\left(\mathrm{Sym}^r \mathcal{K}, \mathcal{O}_X\right)$, which may vanish even if the sheaves do not. For instance, in general, the kernel of the first map of $C_{r,W}^0$,

$$\mu_0^0 : \mathrm{Sym}^r W \longrightarrow \mathrm{Sym}^{r-1} W \otimes H^0\left(X, \Omega_X^1\right)$$

is $H^0\left(X, \mathcal{H}\mathrm{om}\left(\mathrm{Sym}^r \mathcal{K}, \mathcal{O}_X\right)\right)$, which must always vanish because μ_0^0 is always injective. Furthermore, according to Theorem 2.3.3, $'E_2^{1,0}$ vanishes for generic W and even k if X is not fibred over an Albanese general type variety of dimension at most $\frac{k}{2}$ (more generally, if X has no generalized Lagrangian form of rank $\frac{k}{2}$).

Moreover, as the following example shows, the spectral sequence $''E_2$ is not degenerated in general. Therefore, even if the cohomologies $''E_2^{i,j}$ of $\mathcal{H}^i$ do not vanish, the limit groups $''E_\infty^{i,j}$ might anyway vanish, so the previous Theorem is not sharp.

Example 2.2.16. *Consider $C_1, C_2 \subset \mathbb{P}^2$ two smooth plane curves of degree 4 (hence of genus 3) intersecting transversely in 16 points $p_1, \ldots, p_{16}$, and let $X = C_1 \times C_2$ with projections $\pi_i : X \to C_i$. Fix a basis $\{\eta_1, \eta_2, \eta_3\}$ of $H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\right)$ and denote by α_i and β_i its restrictions to C_1 and C_2 respectively, which can be thought as differential forms since $\omega_{C_i} \cong \mathcal{O}_{C_i}(1)$ by adjunction. Finally, let $w_i = \pi_1^*\alpha_i + \pi_2^*\beta_i \in H^0\left(X, \Omega_X^1\right)$, let $W \subset H^0\left(X, \Omega_X^1\right)$ be the vector space spanned by the w_i, and consider the case $r = 2$:*

$$\mathcal{C}_{2,W} : 0 \longrightarrow \mathrm{Sym}^2 W \otimes \mathcal{O}_X \longrightarrow W \otimes \Omega_X^1 \longrightarrow \omega_X \longrightarrow 0. \tag{2.10}$$

The situation is explicit enough to compute most of the objects above. An immediate computation shows that $Z_1 = \emptyset$ and $Z_2 = \{P_1, \ldots, P_{16}\}$, where $P_i = (p_i, p_i)$, so W is non-degenerate. Moreover, a complete description of the first spectral sequence $'E_1$ can be carried out to find that $'E_2^{i,j} = 0$ for all i, j except for $'E_2^{0,2} \cong \mathbb{C}^{37}$, $'E_2^{1,1} \cong \mathbb{C}^{18}$ and $'E_2^{1,0} \cong \mathbb{C}^3$. This implies that $\mathbb{H}^1\left(X, \mathcal{C}_{2,W}\right) \cong \mathbb{C}^3$, $\mathbb{H}^2\left(X, \mathcal{C}_{2,W}\right) \cong \mathbb{C}^{55}$, and all the other hypercohomology groups vanish.

As for the second spectral sequence, we start computing the cohomology sheaves $\mathcal{H}^i$ of (2.10). The last map is surjective, hence $\mathcal{H}^2 = 0$. $\mathcal{H}^1$ is supported on Z_2, and the transversality of C_1 and C_2 implies that each stalk $\mathcal{H}^1_{P_i}$ is a three-dimensional vector space, so that $H^0(X, \mathcal{H}^1) \cong \mathbb{C}^{48}$ and the rest of the cohomology groups are zero. This computation is enough to show that $''E_2$ is not degenerate, since if it was, the group $H^0(X, \mathcal{H}^1) \cong \mathbb{C}^{48}$ would be a summand of $\mathbb{H}^1(X, \mathcal{C}_{2,W}) \cong \mathbb{C}^3$, which is clearly impossible.

Remark 2.2.17. *The previous example also shows that there is no obvious relation between the exactness of the complexes $C^0_{r,W}$ and the transversality of W to the cohomological support loci $V^i(X, \omega_X)$, which in this case are $V^1 = \pi_1^* \operatorname{Pic}^0(C_1) \cup \pi_2^* \operatorname{Pic}^0(C_2)$ and $V^2 = \{\mathcal{O}_X\}$.*

We now turn to the numerical consequences of Theorem 2.2.15.

Corollary 2.2.18. *If X admits a non-degenerate subspace of dimension $k\,(\leq d)$, then*

$$\sum_{i=0}^{p} (-1)^{p-i} \binom{r-i+k-1}{k-1} h^{i,j}(X) \geq 0 \tag{2.11}$$

for every $p \leq \min\{d-k-j+1, r\}$. In particular

$$h^{p,j}(X) \geq \sum_{i=0}^{p-1} (-1)^{p-i-1} \binom{p-i+k-1}{k-1} h^{i,j}(X)$$

for every $p + j \leq d - k + 1$.

Proof. The first inequality is a direct consequence of Theorem 2.2.15, and the second one is the particularization to the case $r = p$. □

And computing a little bit more we find the next (more explicit) result.

Corollary 2.2.19. *If X admits a non-degenerate subspace of dimension $k \leq d$, then*

$$h^{p,j}(X) \geq \binom{k}{p} h^{0,j}(X)$$

for every $p \leq k$ and $p \leq d-k-j+1$, and therefore

$$h^{p,j}(X) \geq \binom{k}{p}\binom{k}{j}$$

if $p, j \leq k$ and $p + j \leq d - k + 1$.

Proof. It is a consequence of the identity

$$\sum_{n=0}^{\min\{A,B\}} (-1)^{B-i} \binom{A}{n} \binom{A+B-n-1}{B-n} \quad = \quad \begin{cases} 1 & \text{if } B=0 \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

which holds for any non-negative integers A, B and can be easily proved by looking at the coefficient of x^B in the expansion of the right-hand side of

$$1 = \frac{(1+x)^A}{(1+x)^A} = \left(\sum_{n=0}^{A} \binom{A}{n} x^n \right) \left(\sum_{m\geq 0} (-1)^m \binom{A+m-1}{m} x^m \right).$$

Indeed, denote by $M_{p,j} = \sum_{i=0}^{p} (-1)^{p-i} \binom{p-i+k-1}{k-1} h^{i,j}(X)$, the right-hand side of (2.11) with $r = p$, and compute

$$\begin{aligned} \sum_{i=0}^{p} \binom{k}{p-i} M_{i,j} &= \\ &= \sum_{i=0}^{p} \binom{k}{p-i} \sum_{m=0}^{i} (-1)^{i-m} \binom{i-m+k-1}{k-1} h^{m,j}(X) = \\ &= \sum_{m=0}^{p} \left(\sum_{i=m}^{p} (-1)^{i-m} \binom{k}{p-i} \binom{i-m+k-1}{i-m} \right) h^{m,j}(X) = h^{p,j}(X), \end{aligned}$$

where the last equality follows from (2.12) because

$$\begin{aligned} \sum_{i=m}^{p} (-1)^{i-m} \binom{k}{p-i} \binom{i-m+k-1}{i-m} &= \\ &= \sum_{n=0}^{p-m} (-1)^{p-m-n} \binom{k}{n} \binom{p-n-m+k-1}{p-n-m} \end{aligned}$$

and $p - m \leq p \leq k$. Therefore,

$$0 \leq \sum_{i=1}^{p} \binom{k}{p-i} A_{i,j} = h^{p,j}(X) - \binom{k}{p} M_{0,j} = h^{p,j}(X) - \binom{k}{p} h^{0,j}(X),$$

as wanted. The second statemet follows at once from the first statement applied to $h^{0,j}(X) = h^{j,0}(X)$. □

2.2.4 Subvarieties of Abelian varieties

We now focus on subvarieties of Abelian varieties, showing that in this case generic subspaces $W \subseteq H^0(X, \Omega_X^1)$ are non-degenerate (Proposition 2.2.20) and then applying the preceding results. After that, we expose a different approach by considering the case $W = H^0(X, \Omega_X^1)$.

Proposition 2.2.20. *Let X be a smooth subvariety of an Abelian variety A such that $V = H^0(X, \Omega_X^1) \cong H^0(A, \Omega_A^1)$. Then, for every $k = 1, \ldots, q(X)$, the non-degenerate subspaces $W \in Gr(k, V)$ form a non-empty Zariski-open subset.*

Proof. Since non-degeneracy is an open condition, we only need to construct a non-degenerate subspace of any dimension k. We will proceed by induction over k.

A one-dimensional subspace $W = \mathbb{C}\langle w \rangle$ is non-degenerate if and only if $\operatorname{codim} Z_1 \geq d$. Since $Z_1 = Z(w)$ is the set of zeroes of any generator w, W is non-degenerate if and only if w vanishes (at most) at isolated points. To prove that generic elements $w \in V$ satisfy that, let us consider the incidence variety

$$I = \{(x, [w]) \in X \times \mathbb{P}(V) \mid w(x) = 0\} \subseteq X \times \mathbb{P}(V).$$

The first projection makes I into a projective bundle of fibre $\mathbb{P}^{q-d-1}$ (where as usual, $d = \dim X$ and $q = q(X)$). Indeed, the fibre over any $x \in X$ is (the projectivization of) the set of 1-forms vanishing at x. Since the tangent space $T_{X,x}$ injects into $T_{A,x}$, the set of 1-forms vanishing at x is the annihilator $T_{X,x}^{\perp}$ inside $T_{A,x}^{\vee} \cong V$, which has dimension $q - d$. In particular, I is irreducible of dimension $(q - d - 1) + d = q - 1$.

Consider now the second projection $I \to \mathbb{P}(V)$. It is clear that the fibre over a point $[w]$ is the zero set $Z(w)$, so we want to see that a general fibre has dimension at most 0. If I dominates $\mathbb{P}(V) \cong \mathbb{P}^{q-1}$, the general fibre has dimension $(q-1) - (q-1) = 0$. If otherwise I does not dominate $\mathbb{P}(V)$, the general fibre is empty (that is, a generic 1-form does not vanish at any point). In any case, we are done.

For the inductive step, note first that if we have two nested subspaces $W' \subseteq W \subseteq V$, then $Z_i(W) \subseteq Z_i(W')$ for every $i = 1, \ldots, \dim W'$. Therefore, if W' is non-degenerate and $k = \dim W = \dim W' + 1$, then

$$\operatorname{codim} Z_i(W) \geq \operatorname{codim} Z_i(W') \geq d - i + 1$$

for every $i = 1, \ldots, k - 1$, and W will be non-degenerate as soon as $\operatorname{codim} Z_k(W) \geq d - k + 1$.

Fix a non-degenerate subspace W' of dimension $k-1$ (it exists by the induction hypothesis), so that in particular $\operatorname{codim} Z_{k-1}(W') \geq d-k+2$, and let $X' = X - Z_{k-1}(W')$ be the open set where the evaluation $W' \to T_{X,x}^{\vee}$ is injective. For any $x \in X'$ denote by $W'_x \subseteq T_{X,x}^{\vee}$ the image of the evaluation, and by $E_x \subseteq T_{X,x}$ the subspace of tangent vectors annihilated by W'_x, which has dimension $\dim T_{X,x} - \dim W'_x = d-k+1$. Consider the new incidence variety

$$I_k = \{(x, W) \mid x \in X', W = W' + \mathbb{C}\langle w\rangle, E_x \subseteq \ker w(x)\} \subseteq X' \times \mathbb{P}(V/W').$$

Note that the condition $E_x \subseteq \ker w(x)$ is independent of the choice of the complement $\mathbb{C}\langle w\rangle$ of W' in W, so I_k is well defined. As for $k=1$, the first projection makes I_k into a $\mathbb{P}^{q-d-1}$-bundle, so I_k is irreducible of dimension $q-1$. Indeed, the fibre over a point $x \in X'$ is the projectivization of

$$\begin{aligned}\{w + W' \in V/W' \mid E_x \subseteq \ker w\} = \\ = \{w \in V \mid E_x \subseteq \ker w\}/W' = E_x^{\perp}/W' \cong \mathbb{C}^{q-d},\end{aligned}$$

where the annihilator $E_x^{\perp}$ is taken in V, that is, it is the kernel of the restriction map $V \twoheadrightarrow E_x^{\vee}$, the map dual to the composition of inclusions $E_x \subseteq T_{X,x} \subseteq T_{A,x} = V^{\vee}$.

As for the second projection, the fibre over $W = \mathbb{C}\langle w\rangle + W' \in \mathbb{P}(V/W')$ is the set

$$\begin{aligned}\{x \in X' \mid E_x \subseteq \ker w(x)\} = \{x \in X' \mid w(x) \in W'_x\} = \\ = Z_k(W) \cap X' = Z_k(W) - Z_{k-1}(W'),\end{aligned}$$

and for W generic its dimension is either zero (if the second projection is not dominant) or

$$\dim I_k - \dim \mathbb{P}(V/W') = (q-1) - (q-(k-1)-1) = k-1.$$

Since the dimension of $Z_{k-1}(W')$ is at most $k-2$, we can conclude that $\dim Z_k(W) \leq k-1$ for W generic containing W', finishing the proof. □

Remark 2.2.21. *Note that the only property we have used is that the tangent spaces $T_{X,x}$ inject into the tangent space of the Abelian variety at every point. Therefore, the same result holds true for étale coverings of subvarieties of Abelian varieties.*

Therefore we can apply Corollaries 2.2.18 and 2.2.19 for any $k \leq d$ to obtain in particular the next inequality.

Corollary 2.2.22. *If X is a subvariety of an Abelian variety A such that $H^0(X, \Omega^1_X) = H^0(A, \Omega^1_A)$, and $p, j \geq 0$ satisfy $\max\{p, j\} \leq d + 1 - (p + j)$, then*

$$h^{p,j}(X) \geq \binom{d+1-(p+j)}{p}\binom{d+1-(p+j)}{j}.$$

If X is a subvariety of an Abelian variety A such that the restriction induces an equality $H^0(X, \Omega^1_X) = H^0(A, \Omega^1_A)$ as in the hypothesis of Proposition 2.2.20, it is also useful to consider the extremal case $k = q$, that is, $W = H^0(X, \Omega^1_X)$ is the whole space of holomorphic 1-forms. In this case, the cokernel $\mathcal{K}$ of the previous section is simply the normal bundle $N_{X/A}$. Since it is a vector bundle, so is $\operatorname{Sym}^r \mathcal{K}$, and hence $\mathcal{E}xt^i_{\mathcal{O}_X}(\operatorname{Sym}^r \mathcal{K}, \mathcal{O}_X) = 0$ for every $i > 0$. Therefore, the second spectral sequence $''E$ is degenerate at $''E_2$, and its only possibly non-zero terms are $''E_2^{i,0} = H^i\left(X, \operatorname{Sym}^r N^\vee_{X/A}\right)$. This leads to the following

Proposition 2.2.23. *Let $X \subseteq A$ be a subvariety of an Abelian variety such that $H^0(X, \Omega^1_X) = H^0(A, \Omega^1_A)$. If for some positive integers r, N the normal bundle $N_{X/A}$ satisfies $H^i\left(X, \operatorname{Sym}^r N^\vee_{X/A}\right) = 0$ for all $i < N$, then for every $j < N$ the complex*

$$\begin{aligned} 0 \longrightarrow \left(\operatorname{Sym}^r H^0\left(X, \Omega^1_X\right)\right) \otimes H^j(X, \mathcal{O}_X) \longrightarrow \cdots \\ \cdots \to \left(\operatorname{Sym}^{r-i} H^0\left(X, \Omega^1_X\right)\right) \otimes H^j\left(X, \Omega^i_X\right) \to \cdots \\ \cdots \longrightarrow \left(\operatorname{Sym}^{r-N+j} H^0\left(X, \Omega^1_X\right)\right) \otimes H^j\left(X, \Omega^{N-j}_X\right) \end{aligned}$$

is exact.

Proof. Let $V = H^0(X, \Omega^1_X)$. By the previous discussion, since $\operatorname{Sym}^r N_{X/A}$ is locally free, the spectral sequence

$$''E_2^{i,j} = H^i\left(X, \mathcal{E}xt^j_{\mathcal{O}_X}\left(\operatorname{Sym}^r N_{X/A}, \mathcal{O}_X\right)\right) \Rightarrow \mathbb{H}^n(X, \mathcal{C}_{r,V})$$

is degenerate and gives $\mathbb{H}^i(X, \mathcal{C}_{r,V}) = H^i\left(X, \operatorname{Sym}^r N^\vee_{X/A}\right) = 0$ for any $i < N$. These vanishings, combined with Proposition 2.2.10, imply the vanishing of $'E_2^{i,j}$ for all $i + j < N$. Recalling that $'E_2^{i,j}$ is the cohomology of $C^j_{r,V}$ at the i-th step, the claim follows directly. □

Corollary 2.2.24. *If X, r and N are as in the above Proposition, then*

$$\sum_{i=0}^{p} (-1)^{p-i} \binom{r-i+q(X)-1}{q(X)-1} h^{i,j}(X) \geq 0 \tag{2.13}$$

for all $p \leq N$.

The main drawback of Proposition 2.2.23 is the difficulty to check the vanishing of $H^i\left(X, \operatorname{Sym}^r N^\vee_{X/A}\right)$.

Example 2.2.25. *Let $D_1, \ldots, D_c \subseteq A$ be ample divisors on an Abelian variety such that the partial intersections $X_i = D_1 \cap \ldots \cap D_i$ are smooth, and let $X = X_c$. Then $H^i\left(X, \operatorname{Sym}^r N^\vee_{X/A}\right) = 0$ for every positive r and $i < \dim X = q(A) - c$, and X satisfies with equality all except one (the case $p = \dim X$) of the inequalities of Corollary 2.2.24.*

In fact, using a double induction (both on c and r) it is possible to show more generally that

$$H^i\left(X, \left(\operatorname{Sym}^r N^\vee_{X/A}\right)(-D)\right) = 0$$

for any positive r and $i < \dim X$, where D is either zero or an ample divisor on A. For the induction step one only needs to take cohomology on the exact sequences

$$\begin{aligned} 0 \longrightarrow \left(\operatorname{Sym}^r N^\vee_{Y/A|X}\right)(-D) \longrightarrow \left(\operatorname{Sym}^r N^\vee_{X/A}\right)(-D) \longrightarrow \\ \longrightarrow \left(\operatorname{Sym}^{r-1} N^\vee_{X/A}\right)(-D_c - D) \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow \left(\operatorname{Sym}^r N^\vee_{Y/A}\right)(-D_c - D) \longrightarrow \left(\operatorname{Sym}^r N^\vee_{Y/A}\right)(-D) \longrightarrow \\ \longrightarrow \operatorname{Sym}^r N^\vee_{Y/A|X} \longrightarrow 0, \end{aligned}$$

where $Y = X_{c-1}$.

On the other hand, also by induction on c, combining the cohomologies of the exact sequences

$$0 \longrightarrow \Omega^i_X \longrightarrow \Omega^i_{Y|X} \longrightarrow \Omega^{i-1}_X(-D_c) \longrightarrow 0$$

and

$$0 \longrightarrow \Omega^i_Y(-D_c) \longrightarrow \Omega^i_Y \longrightarrow \Omega^i_{Y|X} \longrightarrow 0$$

with the Kodaira-Nakano vanishing theorem, one obtains

$$h^{i,j}(X) = h^{i,j}(A) = \binom{q(A)}{i}\binom{q(A)}{j}$$

as long as $i + j < \dim X$. Substituting these values in the left-hand side of (2.13) one obtains 0 (after applying the identity (2.12)).

2.3 Improved bounds for $h^{2,0}(X)$

In this section we consider the Grassmannian BGG complex

$$C_2^0 : 0 \longrightarrow \operatorname{Sym}^2 S \longrightarrow S \otimes H^0\left(X, \Omega_X^1\right) \longrightarrow \mathcal{O}_{\mathbb{G}} \otimes H^0\left(X, \Omega_X^2\right) \qquad (2.14)$$

over a Grasmannian variety of even-dimensional subspaces of $H^0(X, \Omega_X^1)$, and use it to obtain lower bounds on $h^{2,0}(X)$. In particular we improve the results of the previous sections for varieties without higher irrational pencils, improving some results of Lazarsfeld and Popa [27] and Lombardi [28] for threefolds and fourfolds.

The main result is Theorem 2.3.3, which shows that the exactness of (2.14) at a general point is related to the existence of bivectors of small rank in the kernel of $\psi_2 : \bigwedge^2 H^0(X, \Omega_X^1) \to H^0(X, \Omega_X^2)$. We start defining such notion.

Definition 2.3.1. *Let V be any vector space. An element $v \in \bigwedge^2 V$ is said to have* rank $2k$ *if it can be written as*

$$v = v_1 \wedge v_2 + \cdots + v_{2k-1} \wedge v_{2k}$$

for some linearly independent elements $v_1, \ldots, v_{2k} \in V$.

Remark 2.3.2. *If we represent v as an antisymmetric $q \times q$ matrix A with respect to some fixed basis of V, then the rank of v coincides with the rank of A (which is always even). In particular, any element $v \in \bigwedge^2 V$ has rank at most q, and the elements of rank 2 are precisely the (non-zero) decomposable elements. More generally, the set of bivectors of rank at most $2m$ is the cone over $\operatorname{Sec}^m(Gr(2, V)) \subseteq \mathbb{P}\left(\bigwedge^2 V\right)$.*

We present now our main result.

Theorem 2.3.3. *Fix a positive integer $k \leq \frac{q}{2}$. If every non-zero element in $\ker \psi_2$ has rank bigger than $2k$, then the complex (2.14) on $\mathbb{G}_{2k}$ is generically exact.*

Proof. Set $V = H^0(X, \Omega_X^1)$. By the previous remark, the hypothesis is equivalent to $\mathbb{P}(\ker \psi_2) \cap \operatorname{Sec}^k(\mathbb{G}_2) = \emptyset$. In this case, the rational map $\pi = \mathbb{P}(\psi_2) : \mathbb{P}\left(\bigwedge^2 V\right) \dashrightarrow \mathbb{P}(H^0(X, \Omega_X^2))$ restricts to a *morphism*

$$\pi_k = \pi_{|\operatorname{Sec}^k(\mathbb{G}_2)} : \operatorname{Sec}^k(\mathbb{G}_2) \longrightarrow \mathbb{P}\left(H^0\left(X, \Omega_X^2\right)\right)$$

which is finite onto its image. Indeed, if it is not the case, then there exists a curve $C \subseteq \operatorname{Sec}^k(\mathbb{G}_2)$ such that $\pi(C) = p$ is just a point. Such a curve is thus contained in the linear space $\pi^{-1}(p)$, which contains $\mathbb{P}(\ker\psi_2)$ as a hyperplane, and hence C should intersect it, contradicting the fact that π_k is defined everywhere in $\mathrm{Sec}^k(\mathbb{G}_2)$.

Now suppose that the complex (2.14) is not exact at a point $W \in \mathbb{G}_{2k}$, i.e. the complex of vector spaces

$$C^0_{2,W} : 0 \longrightarrow \mathrm{Sym}^2 W \xrightarrow{\mu_0^0} W \otimes H^0\left(X, \Omega^1_X\right) \xrightarrow{\mu_1^0} H^0\left(X, \Omega^2_X\right) \qquad (2.15)$$

is not exact. Fix $\{w_1, \ldots, w_{2k}\}$ any base of W. Since

$$\mu_0^0(w_i w_j) = w_i \otimes w_j + w_j \otimes w_i$$

for every i, j, and the elements $\{w_i \otimes w_j\}_{i,j=1}^{2k}$ are linearly independent in $W \otimes H^0(X, \Omega^1_X)$, μ_0^0 is clearly injective, identifying $\mathrm{Sym}^2 W$ with the subspace of $W \otimes H^0(X, \Omega^1_X)$ spanned by

$$\{w_i \otimes w_i\}_{1 \le i \le k} \cup \{w_i \otimes w_j + w_j \otimes w_i\}_{1 \le i < j \le k}.$$

Therefore, the lack of exactness of (2.15) must come from the central term, that is, there exist some 1-forms $\alpha_1 \ldots \alpha_{2k} \in H^0(X, \Omega^1_X)$ such that $\sum_{i=1}^{2k} w_i \otimes \alpha_i \notin \mathrm{im}\,\mu_0^0$ but

$$\mu_1^0\left(\sum_{i=1}^{2k} w_i \otimes \alpha_i\right) = \psi_2\left(\sum_{i=1}^{2k} w_i \wedge \alpha_i\right) = 0.$$

By substracting a suitable element from $\mu_0^0\left(\mathrm{Sym}^2 W\right)$, we can assume furthermore that $\alpha_i \notin \mathbb{C}\langle w_1, \ldots, w_i\rangle$ for every i. In particular, we may assume that $\alpha_{2k} \notin W$.

Consider now the arc of curve $C \subseteq \mathrm{Sec}^k(\mathbb{G}_2)$ parametrized by

$$\gamma(t) = [(w_1 - t\alpha_2) \wedge (w_2 + t\alpha_1) + \cdots + (w_{2k-1} - t\alpha_{2k}) \wedge (w_{2k} + t\alpha_{2k-1})],$$

with t varying in an open neighbourhood of $0 \in \mathbb{C}$. Let

$$p = \gamma(0) = [w_1 \wedge w_2 + \ldots + w_{2k-1} \wedge w_{2k}].$$

The tangent vector to C at p (to the branch of C given by the image of a neighbourhood of $t = 0$) is the class of

$$v = \sum_{i=1}^{2k} w_i \wedge \alpha_i$$

in $T_{\mathbb{P}(\bigwedge^2 V),p} = \left(\bigwedge^2 V\right)/\mathbb{C}\left\langle w_1 \wedge w_2 + \ldots + w_{2k-1} \wedge w_{2k}\right\rangle$. Since $\alpha_{2k} \notin W$, this class is clearly non zero. However, its image by the differential or π_k is precisely the class of

$$\psi_2\left(\sum_{i=1}^{2k} w_i \wedge \alpha_i\right) = 0$$

in $T_{\mathbb{P}(H^0(X,\Omega_X^2)),\pi(p)} = H^0\left(X, \Omega_X^2\right)/\mathbb{C}\left\langle \psi_2\left(w_1 \wedge w_2 + \ldots + w_{2k-1} \wedge w_{2k}\right)\right\rangle$, so π_k is ramified at p. Since the general point of $\mathbb{P}\left(\bigwedge^2 W\right)$ is of the form $[w_1 \wedge w_2 + \ldots + w_{2k-1} \wedge w_{2k}]$ for some basis of W, we see that π_k ramifies at every point in $\mathbb{P}\left(\bigwedge^2 W\right)$.

To finish the proof, note that $\operatorname{Sec}^k(\mathbb{G}_2)$ is the union of all the $\mathbb{P}\left(\bigwedge^2 W\right)$ as W varies in $\mathbb{G}_{2k}$, so if (2.14) were not exact for a general (and hence for any) $W \in \mathbb{G}_{2k}$, then π_k would be ramified all over $\operatorname{Sec}^k(\mathbb{G}_2)$, contradicting the fact that it is finite. □

Now an easy dimension count gives our inequality.

Corollary 2.3.4. *If there is no non-zero element of rank $2k \leq q$ in $\ker \psi_2$, then*

$$h^{2,0}(X) \geq 2rq - \binom{2r+1}{2}$$

for all $1 \leq r \leq k$.

Proof. By Theorem 2.3.3, for every $1 \leq r \leq k$, the complex (2.14) over any $\mathbb{G} = \mathbb{G}_{2r}$ is generically exact. Let $W \in \mathbb{G}_{2r}$ be such that

$$0 \longrightarrow \operatorname{Sym}^2 W \longrightarrow W \otimes H^0\left(X, \Omega_X^1\right) \longrightarrow H^0\left(X, \Omega_X^2\right)$$

is exact. The cokernel of the last map has dimension

$$\begin{aligned}\dim H^0\left(X, \Omega_X^2\right) - \dim\left(W \otimes H^0\left(X, \Omega_X^1\right)\right) + \dim\left(\operatorname{Sym}^2 W\right) = \\ = h^{2,0}(X) - 2rq + \binom{2r+1}{2},\end{aligned}$$

which must be non-negative, giving the desired inequality. □

Remark 2.3.5. *The case $k = 1$ is the classical Castelnuovo-de Franchis inequality. The case $k = 2$ has been already considered in [2] and [27], where the same inequality is obtained.*

Remark 2.3.6. *From the proof of Theorem 2.3.3 we deduce that $C^0_{2,W}$ is exact (where $\dim W = 2k$) if and only if the image of $\bigwedge^2 W$ by ψ_2 is not contained in the ramification locus of*

$$\mathbb{P}(\psi_2)_{|\operatorname{Sec}^k(\mathbb{G}_2)} : \operatorname{Sec}^k(\mathbb{G}_2) \longrightarrow \mathbb{P}\left(H^0\left(X, \Omega_X^2\right)\right).$$

The existence of low-rank elements in the kernel of ψ_2 can be related to the existence of higher irrational pencils on X, and this will give us a more geometric hypothesis to apply Corollary 2.3.4.

Lemma 2.3.7. *If $v \in \ker \psi_2$ has rank $2k > 0$, $k < d$, then there exists a higher irrational pencil $f : X \to Y$ with $\dim Y \leq k$.*

Proof. The proof relies on Theorem 1.2.3. By this theorem, it suffices to find a decomposable element $v_1 \wedge \cdots \wedge v_{k+1}$ in the kernel of ψ_{k+1}. Writing $v = v_1 \wedge v_2 + \ldots v_{2k-1} \wedge v_{2k}$ with the v_i linearly independent, it is immediate that the element $v_1 \wedge v_3 \wedge \ldots \wedge v_{2k-1} \wedge v_{2k}$, obtained by wedging v with $v_1 \wedge v_3 \wedge \ldots \wedge v_{2k-3}$, maps to zero by ψ_{k+1} because $\psi_2(v) = 0$. $\square$

We immediately obtain the next

Corollary 2.3.8. *If X does not admit any irrational pencil, then*

$$h^{2,0}(X) \geq 2rq - \binom{2r+1}{2}$$

for all $1 \leq r \leq \min\left\{\frac{q}{2}, \dim X - 1\right\}$.

Proof. Simply observe that Lemma 2.3.7 allows us to apply Corollary 2.3.4 for any $k \leq \dim X - 1$. $\square$

And taking the maximum over all the possible r, we get the final result.

Theorem 2.3.9. *Let X be an irregular variety without higher irrational pencils. Then it holds*

$$h^{2,0}(X) \geq \begin{cases} \binom{q(X)}{2} & \text{if } q(X) \leq 2\dim X - 1 \\ 2(\dim X - 1)\, q(X) - \binom{2\dim X - 1}{2} & \text{otherwise.} \end{cases} \tag{2.16}$$

Remark 2.3.10. *The inequality in the case $q \leq 2d - 1$ was already obtained by Causin and Pirola in [10], while the case $q \geq 2d$ is new for high dimension (although for $d = 2$ it is nothing but the classical Castelnuovo-de Franchis inequality for surfaces without irrational pencils, and for $d = 3$ it coincides with a bound given in [27]). Furthermore, it says that for fixed dimension and big irregularity, $h^{2,0}$ behaves asymptotically at least as $2(d-1)q$. For threefolds, this bound coincides with the one proven (with slightly more restrictive hypothesis) by Lombardi in [28], but improves his results in dimension four.*

2.4 Comparison of the two methods

In this final section we compare the results of Sections 2.2 and 2.3. After a first insight, we make a break to talk about computation of Chern classes of symmetric powers of vector bundles. Finally, we compute some explicit cases and check whether the first intuition was right or not.

2.4.1 A first (naive) approach

In the previous section we proved (Theorem 2.3.3) that if X does not admit any higher irrational pencil, then the complex

$$0 \longrightarrow \operatorname{Sym}^2 W \longrightarrow W \otimes H^0\left(X, \Omega^1_X\right) \longrightarrow H^0\left(X, \Omega^2_X\right) \tag{2.17}$$

is exact for a generic even-dimensional subspace $W \subseteq H^0\left(X, \Omega^1_X\right)$, of dimension $\dim W = 2k' < 2\dim X$, and this exactness gives the inequalities (Corollary 2.3.8)

$$h^{2,0}(X) \geq 2k'q(X) - \binom{2k'+1}{2} \qquad \forall\, k' < d. \tag{2.18}$$

Using Theorem 2.2.15 for the case $r = 2, j = 0$, one only obtains the exactness of (2.17) if W is non-degenerate of dimension $k \leq d-1$.

The first thing we would like to mention is that the hypothesis used in the two sections are quite different. Indeed, if the Albanese map of the variety is ramified, there is no obvious relation between existence of non-degenerate subspaces and the non-existence of fibrations over varieties of Albanese general type.

As for the inequalities, if $q(X) \geq 2\dim X$, the best inequality of (2.18) is obtained for $k' = d-1$, hence $k = 2d-2$. Such an inequality is impossible to obtain with Theorem 2.2.15, since it requires $k \leq d-1$, which is very far away from $k = 2d-2$.

However, it seems possible to obtain a better bound with stronger hypothesis. Suppose that Theorem 2.2.15 holds for $r = 2, j = 0$ and *every* $W \in \mathbb{G} = Gr\left(k, H^0\left(X, \Omega^1_X\right)\right)$ for some $k \leq d-1$. In this case, the Grassmannian BGG complex on $\mathbb{G}$

$$0 \to \operatorname{Sym}^2 S \to S \otimes H^0\left(X, \Omega^1_X\right) \to H^0\left(X, \Omega^2_X\right) \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{F}^0_{2,2} \to 0 \tag{2.19}$$

is *everywhere* exact, so the cokernel $\mathcal{F} = \mathcal{F}^0_{2,2}$ is also a vector bundle. If we were able to compute the (total) Chern class of $\mathcal{F}$, $c(\mathcal{F})$, we would obtain estimates on $\operatorname{rk}\mathcal{F}$ which in turn will give lower bounds on

$$h^{0,2}(X) = \operatorname{rk}(\mathcal{F}) + kq - \binom{k+1}{2}.$$

Suppose for a moment that the Chern class of degree $\dim \mathbb{G} = k(q-k)$ of $\mathcal{F}$ was non-zero. This would imply that $\mathcal{F}$ has rank at least $k(q-k)$, and therefore

$$h^{2,0}(X) \geq k(q-k) + kq - \binom{k+1}{2} = 2kq - \left(k^2 + \binom{k+1}{2}\right), \quad (2.20)$$

which has the same asymptotic behaviour as (2.16). Furhtermore, since $k^2 + \binom{k+1}{2} < \binom{2k+1}{2}$, we would obtain a slightly stronger bound.

The problem is now reduced to compute the Chern class

$$c(\mathcal{F}) = \frac{c\left(H^0\left(X, \Omega_X^2\right) \otimes \mathcal{O}_{\mathbb{G}}\right) c\left(\operatorname{Sym}^2 S\right)}{c\left(S \otimes H^0\left(X, \Omega_X^1\right)\right)} = c\left(\operatorname{Sym}^2 S\right) c(Q)^q, \quad (2.21)$$

(since $c(S)^{-1} = c(Q)$), which turns out to be very complicated in general. Indeed, although the power $c(Q)^q$ is easy to describe in terms of the Schubert classes of $\mathbb{G}$, the formula for the Chern class of a symmetric power of some vector bundle E depends on the rank of E, and we do not know of any explicit computation even in the (rather concrete) case of tautological bundles over a Grassmannian.

In Appendix A.2 we will compute the Chern classes $c_i(\mathcal{F})$ for some low values of k and q. Unfortunately, the maximal Chern class vanishes in all these cases, so the bound (2.20) is not true at all. However, we will observe the particularly surprising fact that the lower bounds for $h^{2,0}(X)$ obtained by this new method coincide with (2.18) with $k' = k$.

2.4.2 Chern Classes of Symmetric Powers

We make now a break to say a few words about symmetric powers of a vector bundle, and more precisely, about the computation of their Chern classes from the Chern classes of the original bundle. Hence, let M denote any complex manifold, and E a vector bundle on M of rank $k = \operatorname{rk} E$ with Chern classes $c_i = c_i(E)$. The problem we are concerned with is to find the universal formulas $P_i(k, r, c_1, \ldots, c_k, \ldots)$ such that

$$c_i(\operatorname{Sym}^r E) = P_i(\operatorname{rk} E, r, c_1, c_2, \ldots).$$

For fixed rank k and exponent r, it is easy (though very tedious) to obtain the explicit formulas. Indeed, one only has to express a symmetric polynomial in k variables as a polynomial in the elemenatary symmetric polynomials, and there are plenty of software packages that can help with this computation. However, we would like to have general formulas valid for (at least) arbitrary k, and it is very desirable to have results also for arbitrary r. Unfortunately, we have not find any such results in the literature, probably because the hidden combinatorics is very complicated. In order to illustrate the difficulty of the general computation, we include a proof of the simplest case.

Proposition 2.4.1. *For any $r \geq 1$ and any vector bundle E it holds*

$$c_1\left(\operatorname{Sym}^r E\right) = \binom{\operatorname{rk} E + r - 1}{r-1} c_1\left(E\right),$$

that is, $P_1 = \binom{\operatorname{rk} E+r-1}{r-1} c_1$.

Proof. By the splitting principle, we know that if the total Chern class of E is $c\left(E\right) = \prod_{i=1}^{k}\left(1+x_i\right)$, the Chern class of $\operatorname{Sym}^r E$ is

$$c\left(\operatorname{Sym}^r E\right) = \prod_{1\leq i_1\leq\ldots\leq i_r\leq k}\left(1+\sum_{j=1}^{r} x_{i_j}\right).$$

Therefore, taking the parts of degree 1 we obtain

$$c_1\left(\operatorname{Sym}^r E\right) = \sum_{1\leq i_1\leq\ldots\leq i_r\leq k}\left(\sum_{j=1}^{r} x_{i_j}\right),$$

which is clearly a symmetric polynomial in the $x_1, \ldots, x_k$, hence it is a multiple of the elementary symmetric polynomial $x_1 + \cdots + x_k = c_1\left(E\right)$. To find the scalar A such that

$$\sum_{1\leq i_1\leq\ldots\leq i_r\leq k}\left(\sum_{j=1}^{r} x_{i_j}\right) = A\left(x_1 + \cdots + x_k\right)$$

one only has to find the sum of all the coefficients of the x_i. That of the right-hand side is clearly kA, while the sum of the coefficients of the left-hand side is r times the number of summands, which is $\binom{k+r-1}{k-1}$. Hence we have

$$A = \frac{r}{k}\binom{k+r-1}{k-1} = \binom{k+r-1}{k},$$

as claimed. $\square$

The approach of the preceding proof does not work so easily for higher Chern classes. For example, only for c_2 one would have

$$\begin{aligned} c_2\left(\operatorname{Sym}^r E\right) &= \sum_{I\neq J}\left(\sum_{i\in I} x_i\right)\left(\sum_{j\in J} x_j\right) = \\ &= A\left(\sum_{i=1}^{k} x_i\right)^2 + B\left(\sum_{i<j} x_i x_j\right) = A c_1^2 + B c_2, \quad (2.22) \end{aligned}$$

where I and J denote multisets of size r with elements in $\{1, \ldots, k\}$.

The sum of the coefficients of the first term of the second row of (2.22) is $k^2 A + \binom{k}{2} B$, which must equal r^2 (the sum of the coefficients of each summand in the first row) times the number of couples of multiindices $\{I \neq J\}$. That is,

$$k^2 A + \binom{k}{2} B = r^2 \binom{\binom{k+r-1}{r}}{2}, \tag{2.23}$$

which is not enough to determine A and B, and we have not found any other easy equality involving A, B, r and k.

However, we do not really need such general formulas, since we are only concerned with the cases $r = 2$ and $r = 3$ (the latter will be useful at the very end of the chapter). Fixing r, the size of the multisets I, J, does actually simplify the underlying combinatorial problem (although we still have an arbitrary number k of indeterminates). Indeed, let us continue the last computation with $r = 2$.

Proposition 2.4.2. *For any vector bundle E of rank k it holds*

$$c_2\left(\operatorname{Sym}^2 E\right) = \frac{(k+2)(k-1)}{2} c_1(E)^2 + (k+2)\, c_2(E).$$

Proof. We first rewrite (2.22) as

$$c_2(\operatorname{Sym}^r E) = A c_1^2 + B c_2 = A\left(\sum_{i=1}^{k} x_i^2\right) + (2A + B)\left(\sum_{i<j} x_i x_j\right).$$

Since we are considering $r = 2$, the multisets of indices are of the form $I = \{1 \leq i_1 \leq i_2 \leq k\}, J = \{1 \leq j_1 \leq j_2 \leq k\}$.

Let us first compute A as the number of times the monomial x_1^2 appears in

$$\sum_{I \neq J}\left(\sum_{i \in I} x_i\right)\left(\sum_{j \in J} x_j\right). \tag{2.24}$$

Hence we must consider all multiindices containing at least one 1, and take into account the multiplicity of the 1.

- The first case is $I = J = \{1 \leq 1\}$, which must be discarded because we need $I \neq J$.
- The second case is $I = \{1 \leq 1\}$ and $J = \{1 \leq j\}$ with $j > 1$. In this case the summands are

$$\left(\sum_{i \in I} x_i\right)\left(\sum_{j \in J} x_j\right) = 2x_1(x_1 + x_j),$$

hence each one contributes with 2 monomials x_1^2, and there are $k-1$ of them (as many as indices j satisfying $1 < j \leq k$). Therefore, the total contribution is $2\,(k-1)$.

- The last case is $I = \{1 \leq i\}$ and $J = \{1 \leq j\}$ with $1 < i < j$. Each summand contributes with exactly one x_1^2, and there are $\binom{k-1}{2}$ of them.

Hence, adding up all the contributions, we finally obtain

$$A = 2\,(k-1) + \binom{k-1}{2} = \frac{(k+2)\,(k-1)}{2},$$

We can now compute B with equation (2.23), but we want to expose an explicit computation (requiring much more subcases than the case of A) which gives an idea on how the code in Appendix A.1 works. We first calculate $2A + B$ as the number of times $x_1 x_2$ appears in (2.24):

- $I = \{1 \leq 1\}\,, J = \{1 \leq 2\}$. There is only one summand contributing with 2 monomials.
- $I = \{1 \leq 1\}\,, J = \{2 \leq 2\}$. There is only one summand contributing with 4 monomials.
- $I = \{1 \leq 1\}\,, J = \{2 \leq j\}$ with $j > 2$. There are $k-2$ summands, each contributing with 2 monomials. Hence the total contribution is $2\,(k-2)$.
- $I = \{1 \leq 2\}\,, J = \{1 \leq j\}$ with $j > 2$. There are $k-2$ such summands, each contributing with one monomial. Total: $k-2$.
- $I = \{1 \leq 2\}\,, J = \{2 \leq 2\}$. Only one summand contributing with 2 monomials.
- $I = \{1 \leq 2\}\,, J = \{2 \leq j\}$ with $j > 2$. There are $k-2$ summands, each contributing with one monomial.
- $I = \{1 \leq i\}\,, J = \{2 \leq 2\}$ with $i > 2$. There are $k-2$ summands, each contributing with 2 monomials.
- $I = \{1 \leq i\}\,, J = \{2 \leq j\}$ with $i, j > 2$. There are $(k-2)^2$ such summands, each contributing with only one monomial.

Adding up all the contributions, we obtain

$$2A + B = (k-2)^2 + 6\,(k-2) + 8 = k^2 + 2k.$$

Hence $B = k + 2$ and the proof is done. $\square$

It is clear from the previous proof that the computations become more and more complicated when the exponent r or the degree i of the Chern class grow. With the help of a computer program in Singular (see Appendix A.1) we have computed the smallest cases:

Proposition 2.4.3. *Let E be a vector bundle of rank k. Then the following formulas hold*

$$\begin{aligned}
c_1\left(\operatorname{Sym}^2 E\right) &= (k+1)\, c_1(E) \\
c_2\left(\operatorname{Sym}^2 E\right) &= \frac{(k+2)(k-1)}{2} c_1(E)^2 + (k+2)\, c_2(E) \\
c_3\left(\operatorname{Sym}^2 E\right) &= \frac{(k+3)(k-1)(k-2)}{6} c_1(E)^3 + \\
&\quad + \left(k^2+2k-4\right) c_1(E)\, c_2(E) + (k+4)\, c_3(E) \\
c_1\left(\operatorname{Sym}^3 E\right) &= \binom{k+2}{2} c_1(E) \\
c_2\left(\operatorname{Sym}^3 E\right) &= \frac{(k+2)(k-1)\left(k^2+5k+8\right)}{8} c_1(E)^2 + \binom{k+3}{2} c_2(E) \\
c_3\left(\operatorname{Sym}^3 E\right) &= \frac{(k-1)\left(k^5+10k^4+37k^3+40k^2-84k-192\right)}{48} c_1(E)^3 + \\
&\quad + \frac{(k+3)\left(k^3+5k^2+6k-16\right)}{4} c_1(E)\, c_2(E) + \\
&\quad + \frac{(k+3)(k+6)}{2} c_3(E)
\end{aligned}$$

2.4.3 Bounds from non-vanishing of Chern classes

We go now back to the specific case of (2.21). The computation is carried out in the cohomology of the Grassmannian variety $\mathbb{G} = Gr(k,q)$ of k-dimensional subspaces of $V = H^0\left(X, \Omega^1_X\right) \cong \mathbb{C}^q$. Fixed a basis $\{v_1, \ldots, v_q\}$ of V and given a non-increasing sequence $\lambda = (q-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0)$, the set

$$\Sigma_\lambda = \left\{ W \in \mathbb{G} \mid \dim\left(W \cap \mathbb{C}\left\langle v_1, \ldots, v_{q-k+i-\lambda_i}\right\rangle\right) \geq i \right\}$$

is a closed cycle of (real) codimension $2\sum_i \lambda_i = 2|\lambda|$, called the *Schubert cycle* associated to λ and the chosen basis.

Let $\sigma_\lambda \in H^{2|\lambda|}(\mathbb{G}, \mathbb{C})$ denote its cohomology class, which is independent of the choice of the basis. Then it is known (see for instance [20] pp. 410–411) that

$$c(S) = 1 - \sigma_1 + \sigma_{1,1} - \cdots + (-1)^k \sigma_{\underbrace{1,\ldots,1}_{k}}$$

and

$$c(Q) = 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_{q-k}.$$

Since we do not know a closed formula for every $c_i\left(\operatorname{Sym}^2 S\right)$, we cannot compute the total Chern class $c\left(\operatorname{Sym}^2 S\right)$ unless we fix $k = \operatorname{rk} S$. Moreover, although it is possible to give a quite explicit expression of the power $c(Q)^q$ as a linear combination of Schubert classes valid for any q, we have not found a way to carry out the product $c\left(\operatorname{Sym}^2 S\right) c(Q)^q$ and look for which coefficients vanish and which ones do not.

Therefore, we have been forced to make explicit computations fixing both $k = 2, 3, 4$ and $q = k+1, \ldots, 12$, the complete results of which are included in Appendix A.2. From these computations, it is clear that the Chern classes of $\mathcal{F}$ of highest degree vanish, hence the bounds (2.20) are out of reach with this last method. Furthermore, there is some pattern in the Schubert classes whose coefficient is non-zero, which leads us to formulate the following

Conjecture 2.4.4. *Let $\mu = (q-k-1, q-k-2, q-k-3, \ldots, q-2k)$ (or $(q-k-1, q-k-2, \ldots, 1, 0, \ldots)$ if $q < 2k$). The coefficient in $c(\mathcal{F})$ of the Schubert class σ_λ is zero for every λ not contained*[1] *in μ, while the coefficient of σ_μ is non-zero.*

The computations in Appendix A.2 prove this Conjecture for small k and q, but we do not know of any method to prove it for all the possible values.

Proposition 2.4.5. *If the Grassmannian BGG complex (2.19) of an irregular variety X is everywhere exact and Conjecture 2.4.4 holds, then*

$$h^{2,0}(X) \geq \begin{cases} \binom{q}{2} & \text{if } q \leq 2k, \\ 2kq - \binom{2k+1}{2} & \text{if } q \geq 2k. \end{cases}$$

Proof. Computing the codimension of σ_μ we obtain

$$\operatorname{rk}(\mathcal{F}) \geq \begin{cases} \binom{q-k}{2} & \text{if } q \leq 2k, \\ \frac{k(2q-3k-1)}{2} & \text{if } q \geq 2k, \end{cases}$$

and adding it to $kq - \binom{k+1}{2}$ we obtain precisely the bound

$$h^{2,0} \geq \begin{cases} \binom{q}{2} & \text{if } q \leq 2k, \\ 2kq - \binom{2k+1}{2} & \text{if } q \geq 2k. \end{cases}$$

□

[1] We say that a partition $(\lambda_1 \geq \ldots \geq \lambda_k)$ is contained in $(\mu_1 \geq \ldots \geq \mu_k)$ if $\lambda_i \leq \mu_i$ for all i.

Remark 2.4.6. *The above bound is exactly the lower bound of Theorem 2.3.9, but obtained in a very different way.*

2.4.4 Bounds from positivity of Chern Classes

We close both this section and the chapter exploring a different approach to obtain inequalities among the Hodge numbers of certain irregular varieties. Although the method can be used with any of the complexes C_r^j, we will focus on the case C_3^0, since it leads to more inequalitites involving $h = h^{2,0}$ and q which we can compare with the previous ones.

Consider thus the complex C_3^0 over the Grasmannian $\mathbb{G}_k$ for some k,

$$\begin{aligned} C_3^0: \quad & 0 \to \operatorname{Sym}^3 S \to \operatorname{Sym}^2 S \otimes H^0\left(X, \Omega_X^1\right) \to \\ & \to S \otimes H^0\left(X, \Omega_X^2\right) \to H^0\left(X, \Omega_X^3\right) \otimes \mathcal{O}_{\mathbb{G}} \to \mathcal{G} = \mathcal{F}_{3,3}^0 \to 0, \end{aligned} \tag{2.25}$$

and assume that it is exact as a sequence of *sheaves* on $\mathbb{G}$. As in the previous discussion, we do not know of better (geometric) hypothesis to be put directly on the variety X and guaranteeing the exactness of (2.25) (in the flavor of Theorem 1.4.2).

Since $\mathcal{G}$ is generated by global sections (it is a quotient of a trivial bundle), all its Chern classes must be represented by effective cycles, and this gives some inequalities involving h, q and k (the rank of S).

Without using the global generation, one can truncate the complex after $S \otimes H^0\left(X, \Omega_X^2\right)$ and use that the cokernel must have non-negative rank. This gives

$$kh - \binom{k+1}{2} q + \binom{k+2}{3} \geq 0,$$

or equivalently

$$h \geq \frac{1}{k}\left(q\binom{k+1}{2} - \binom{k+2}{3}\right) = \frac{k+1}{2} q - \frac{(k+2)(k+1)}{6}. \tag{2.26}$$

This inequality is better than $h \geq kq - \binom{k+1}{2}$ (the one obtained from the exactness of some $C_{2,W}^0$) if and only if $q < \frac{2}{3}(k+1)$. Since $k \leq q$ by definition, $q < \frac{2}{3}(k+1)$ implies $3k < 2k+2$ and hence $k < 2$. Furthermore, for $k = 1$, (2.26) is equivalent to $h \geq kq - \binom{k+1}{2} = q - 1$, so we do not get any improvement by considering only the ranks.

In order to use the global generation, we compute the lower terms of

$$c(\mathcal{G}) = \frac{c\left(\operatorname{Sym}^2 S\right)^q}{c(S)^h c\left(\operatorname{Sym}^3 S\right)}.$$

Writing $A_i = c_i(\mathrm{Sym}^2 S)$, $B_i = c_i(\mathrm{Sym}^3 S)$, and denoting by D_i the component in $H^i(\mathbb{G}, \mathbb{C})$ of $c_i(\mathrm{Sym}^3 S)^{-1}$, we have

$$\begin{aligned} c\,(Q) &= 1 + \sigma_1 + \sigma_2 + \sigma_3 + \cdots \\ c\left(\mathrm{Sym}^2 S\right) &= 1 + A_1 + A_2 + A_3 + \cdots \\ c\left(\mathrm{Sym}^3 S\right) &= 1 + B_1 + B_2 + B_3 + \cdots \\ c\left(\mathrm{Sym}^3 S\right)^{-1} &= 1 + D_1 + D_2 + D_3 + \cdots \end{aligned}$$

and

$$c\,(\mathcal{G}) = (1 + \sigma_1 + \sigma_2 + \cdots)^h\,(1 + A_1 + A_2 + \cdots)^q\,(1 + D_1 + D_2 + \cdots)\,. \tag{2.27}$$

From the identity $(1 + B_1 + B_2 + B_3 + \cdots)(1 + D_1 + D_2 + D_3 + \cdots) = 1$ we can recover recursively the D_i from the B_i as

$$D_i = -B_i - B_{i-1}D_1 - B_{i-2}D_2 - \cdots - B_1 D_{i-1} \qquad \forall\, i \geq 1.$$

Denote also by $a_\lambda, b_\lambda, d_\lambda, g_\lambda \in \mathbb{Q}[k]$ the coefficients of the Schubert class σ_λ in $c\left(\mathrm{Sym}^2 S\right), c\left(\mathrm{Sym}^3 S\right), c\left(\mathrm{Sym}^3 S\right)^{-1}$ and $c\,(\mathcal{G})$, respectively. Then, the family of inequalities we want to describe as explicitly as possible is $\{g_\lambda \geq 0\}$.

Inequality from $c_1(\mathcal{G}) \geq 0$

From the formula (2.27) we obtain

$$c_1\,(\mathcal{G}) = h\sigma_1 + qA_1 + D_1 = (h + qa_1 + d_1)\,\sigma_1$$

so $g_1 = h + qa_1 + d_1$, and the first inequality we obtain is $h \geq -qa_1 - d_1$. Our objective now is to determine a_1 and d_1.

From Proposition 2.4.3, we obtain

$$c_1\left(\mathrm{Sym}^2 S\right) = (k+1)\, c_1\,(S) = -\,(k+1)\,\sigma_1$$

and

$$c_1\left(\mathrm{Sym}^3 S\right) = \binom{k+2}{2} c_1\,(S) = -\binom{k+2}{2}\sigma_1.$$

Hence, $a_1 = -\,(k+1)$ and $d_1 = -b_1 = \binom{k+2}{2}$, so that we obtain the inequality

$$h \geq -qa_1 - d_1 = q(k+1) - \binom{k+2}{2}. \tag{2.28}$$

Note that this inequality is the same that we would have obtained from the exactness of $C^0_{2,W}$ for some W of dimension $k+1$. Hence, the assumption of exactness for every subspace of a certain dimension gives the same result that the exactness for only one subspace of bigger dimension.

Inequality from $c_2(\mathcal{G}) \geq 0$

From the formula (2.27) we obtain

$$\begin{aligned}
c_2(\mathcal{G}) &= h\sigma_2 + qA_2 + D_2 + \\
&\quad + \binom{h}{2}\sigma_1^2 + \binom{q}{2}A_1^2 + hq\sigma_1 A_1 + (h\sigma_1 + qA_1)D_1 = \\
&= (h + qa_2 + d_2)\sigma_2 + (qa_{1,1} + d_{1,1})\sigma_{1,1} + \\
&\quad + \left(\binom{h}{2} + \binom{q}{2}a_1^2 + hqa_1 + (h + qa_1)d_1\right)\sigma_1^2 = \\
&= \left(h + qa_2 + d_2 + \binom{h}{2} + \binom{q}{2}a_1^2 + hqa_1 + (h + qa_1)d_1\right)\sigma_2 + \\
&\quad + \left(qa_{1,1} + d_{1,1} + \binom{h}{2} + \binom{q}{2}a_1^2 + hqa_1 + (h + qa_1)d_1\right)\sigma_{1,1}
\end{aligned}$$

(because $\sigma_1^2 = \sigma_2 + \sigma_{1,1}$). Therefore,

$$g_2 = h + qa_2 + d_2 + \binom{h}{2} + \binom{q}{2}a_1^2 + hqa_1 + (h + qa_1)d_1$$

and

$$g_{1,1} = qa_{1,1} + d_{1,1} + \binom{h}{2} + \binom{q}{2}a_1^2 + hqa_1 + (h + qa_1)d_1.$$

We now have to determine $a_2, a_{1,1}, d_2$ and $d_{1,1}$.

Using the formulas of Proposition 2.4.3 we obtain

$$\begin{aligned}
c_2(\operatorname{Sym}^2 S) &= \frac{1}{2}(k+2)(k-1)c_1(S)^2 + (k+2)c_2(S) = \\
&= \frac{1}{2}(k+2)(k-1)\sigma_1^2 + (k+2)\sigma_{1,1} = \\
&= \frac{1}{2}(k+2)(k-1)\sigma_2 + \left(\frac{1}{2}(k+2)(k-1) + (k+2)\right)\sigma_{1,1}.
\end{aligned}$$

Therefore, $a_2 = \frac{1}{2}(k+2)(k-1)$ and $a_{1,1} = \binom{k+2}{2}$.

The same computation for $\operatorname{Sym}^3 S$ gives

$$\begin{aligned}
c_2(\operatorname{Sym}^3 S) &= \frac{1}{8}(k+2)(k-1)(k^2+5k+8)c_1(S)^2 + \binom{k+3}{2}c_2(S) = \\
&= \frac{1}{8}(k+2)(k-1)(k^2+5k+8)\sigma_1^2 + \binom{k+3}{2}\sigma_{1,1} = \\
&= \frac{1}{8}(k+2)(k-1)(k^2+5k+8)\sigma_2 + \\
&\quad + \left(\frac{1}{8}(k+2)(k-1)(k^2+5k+8) + \binom{k+3}{2}\right)\sigma_{1,1}.
\end{aligned}$$

Simplifying a bit more, we get

$$b_2 = \frac{1}{8}(k+2)(k-1)(k^2+5k+8)$$

and

$$b_{1,1} = \frac{1}{8}(k+2)(k+1)(k^2+3k+4) = \binom{\binom{k+2}{2}+1}{2}.$$

Finally, $D_2 = -B_2 - B_1 D_1$ implies that

$$\begin{aligned} d_2\sigma_2 + d_{1,1}\sigma_{1,1} = -(b_2\sigma_2 + b_{1,1}\sigma_{1,1}) - (b_1\sigma_1)(d_1\sigma_1) = \\ = -(b_2 + b_1 d_1)\sigma_2 - (b_{1,1} + b_1 d_1)\sigma_{1,1}, \end{aligned}$$

and therefore

$$d_2 = -(b_2 + b_1 d_1) = \frac{1}{8}(k+3)(k+2)(k^2+k+4)$$

and

$$d_{1,1} = -(b_{1,1} + b_1 d_1) = 3\binom{k+3}{4}.$$

Summing up all the results so far we obtain the inequalities

$$\begin{aligned} g_2 = \frac{1}{2}h^2 - \left(q(k+1) - \binom{k+2}{2} - \frac{1}{2}\right) h + \\ + \left(\binom{q}{2}(k+1)^2 - \frac{1}{2}q\,(k+2)\left(k^2+k+2\right) + \right. \\ \left. + \frac{1}{8}\,(k+3)\,(k+2)\left(k^2+k+4\right)\right) \geq 0 \end{aligned}$$

and

$$\begin{aligned} g_{1,1} = \frac{1}{2}h^2 - \left(q(k+1) - \binom{k+2}{2} + \frac{1}{2}\right) h + \\ + \left(\binom{q}{2}(k+1)^2 - qk\binom{k+2}{2} + 3\binom{k+3}{4}\right) \geq 0. \end{aligned}$$

Viewing g_2 and $g_{1,1}$ as quadratic polynomials in h, we can compute their roots formally, which are (for g_2 and $g_{1,1}$ respectively)

$$\alpha_\pm = \left(q(k+1) - \binom{k+2}{2} - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{8(q-k)-15}$$

and

$$\beta_{\pm} = \left(q(k+1) - \binom{k+2}{2} + \frac{1}{2}\right) \pm \frac{1}{2}.$$

First of all, note that $\beta_{\pm}$ are consecutive integers, so $g_{1,1} \geq 0$ holds for any integers h, k, q and it does not give any inequality at all.

Secondly, the roots $\alpha_{\pm}$ are not defined if $8(q-k) - 15 < 0$, which is equivalent to $k \geq q-1$ (both q and k are integers). Therefore, for $k \geq q-1$ we again do not obtain any inequality. Assuming $k \leq q-2$, $g_2 \geq 0$ implies that either $h \geq \alpha_+$ or $h \leq \alpha_-$. But since $\alpha_- < q(k+1) - \binom{k+2}{2}$ and we already know that $h \geq q(k+1) - \binom{k+2}{2}$ (inequality (2.28)), the option $h \leq \alpha_-$ is impossible, and we only obtain

Proposition 2.4.7. *If X is an irregular variety and $k \leq q(X) - 2$ is such that (2.25) is an exact sequence of sheaves on $\mathbb{G}_k$, then*

$$h^{2,0}(X) \geq q(k+1) - \binom{k+2}{2} + \frac{1}{2}\left(\sqrt{8q - (8k+15)} - 1\right). \qquad (2.29)$$

Remark 2.4.8. *In the case $k = 1$, the inequality (2.29) concides with the results of Lombardi [28] for threefolds.*

Appendix One

EXPLICIT COMPUTATIONS

A.1 COMPUTING $c_n(\mathrm{Sym}^r E)$ FOR E OF ARBITRARY RANK.

In this first section of the Appendix we present a code in Singular that helps to express the Chern classes of $\mathrm{Sym}^r E$ as polynomials

$$c_n(\mathrm{Sym}^r E) = P_n(\mathrm{rk}\, E, r, c_1, \dots, c_k, \dots)$$

in the Chern classes $c_i = c_i(E)$.

More precisely, if we denote by $x_1, \dots, x_k$ the formal Chern roots of E, the $c_n(\mathrm{Sym}^r E)$ are symmetric polynomials in the x_i, hence they admit unique expressions as polynomials in the *monomial symmetric polynomials*

$$m_{(\alpha_1,\dots,\alpha_k)}(x_1, \dots, x_k) = \frac{1}{N_\alpha} \sum_{\sigma \in \mathfrak{S}_k} x_1^{\alpha_{\sigma(1)}} \cdots x_k^{\alpha_{\sigma(k)}},$$

where $\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_k)$ is any non-increasing sequence such that $n = \sum_{j=1}^k \alpha_j$, $\mathfrak{S}_k$ is the symmetric group of permutations of k letters, and N_α is the number of permutations σ such that $\alpha_{\sigma(i)} = \alpha_i$ for all i.

The following code computes the coefficient of a given m_α in $c_n(\mathrm{Sym}^r E)$. After that, it is easy to express $c_n(\mathrm{Sym}^r E)$ as a polynomial in the *elementary symmetric polynomials*, i.e., the Chern classes $c_i(E)$.

The code needs the library `"general.lib"`, and works over any ring of characteristic 0 with at least the variable `k` (e.g. `ring R=0,k,dp`).

The main procedure is `coe`. The inputs are `Exp` and `r`, the non-increasing sequence α of exponents and the exponent r of the symmetric power, respectively. The output is the corresponding coefficient of m_α in $c_n(\mathrm{Sym}^r E)$.

For example, if we call `coe(intvec(2),2)` and `coe(intvec(1,1),2)`, we obtain the polynomials $\frac{1}{2}(k+2)(k-1)$ and k^2+2k, which are respectively the coefficients of $m_2 = \sum_{i=1}^k x_i^2$ and $m_{1,1} = \sum_{i<j} x_i x_j$ in $c_2\left(\mathrm{Sym}^2 E\right)$ found in the proof of Proposition 2.4.2.

```
proc coe(intvec Exp, int r){
 int i,j;
 int n=size(Exp);
 int d=sum(Exp);
 list C;
 for(i=1;i<=n;i=i+1){
  for(j=1;j<=Exp[i];j=j+1){
   C=C+list(i);
  }
 }
 matrix M[d][n];
 intvec V=0:r;
 V[r]=1;
 poly res=c_rec(poly(0),C,poly(1),M,d,r,n,V);
 for(i=1;i<=n;i=i+1){
  for(j=2;j<=Exp[i];j=j+1){
   res=res/j;
  }
 }
 return(res);
}


proc c_rec(poly res, list C, poly aux, matrix M, int rrow,
int r, int n, intvec lrow){
 if(rrow==0){
  poly A=perm(M,C);
  return(res+A*aux);
 }
 if(lrow[1]==-1){
  return(res);
 }else{
  int i,j,m,s;
  poly aux2;
  for(i=rrow; i>=0; i=i-1){
   for(j=0; j<i; j=j+1){
    for(m=1; m<=n; m=m+1){
     M[rrow-j,m]=0;
    }
    for(m=1; m<=r; m=m+1){
     if(lrow[m]!=0){
```

```
      M[rrow-j,lrow[m]]=M[rrow-j,lrow[m]]+1;
     }
    }
   }
   s=r;
   for(j=1;j<=r;j=j+1){
    if(lrow[j]!=0){
     s=s-1;
    }
   }
   aux2=binom(binom(k-n+s-1,s),i);
   res=c_rec(res,C,aux*aux2,M,rrow-i,r,n,next_seq(lrow,n));
  }
  return(res);
 }
}


proc perm(matrix M, list C){
 int R=size(C);
 if(R==0){
  return(poly(1));
 }else{
  poly aux=0;
  for(int i=1; i<=R; i=i+1){
   aux=aux+M[R,C[i]]*perm(M,delete(C,i));
  }
  return(aux);
 }
}


proc binom(poly p, int m){
 if(m==0){
  return(poly(1));
 }else{
  return(p*binom(p-1,m-1)/m);
 }
}
```

```
proc next_seq(intvec V, int n){
 int r=size(V);
 int i=r;
 int a;
 while(i>0){
  if(V[i]==n){
   i=i-1;
  }else{
   a=i;
   i=-1;
  }
 }
 if(i==0){
  return((-1):r);
 }else{
  i=a;
  V[i]=V[i]+1;
  for(int j=i+1;j<=r;j=j+1){
   V[j]=V[i];
  }
  return(V);
 }
}
```

A.2 Computations of $c(\mathcal{F}_{2,2})$

This second part of the Appendix contains the explicit computations of the total Chern class of the sheaf $\mathcal{F} = \mathcal{F}_{2,2}$ appearing in Section 2.4. We present them here in order to support Conjecture 2.4.4. As we said, we have computed only the cases $q = 2, 3, 4$ and $k = q + 1, \dots, 12$, which we believe are more than enough to illustrate the vanishing of the higher Chern classes of $\mathcal{F}$. The computations have been carried out with Macaulay2, using the package SchurRings.

$\boldsymbol{k=2}$

In this case,

$$c\left(\operatorname{Sym}^2 S\right) = 1 + 3c_1\left(S\right) + 2c_1\left(S\right)^2 + 4c_2\left(S\right) + 4c_1\left(S\right)c_2\left(S\right) = 1 - 3\sigma_1 + \left(6\sigma_{1,1} + 2\sigma_2\right) - 4\sigma_{2,1}.$$

And multiplying by the powers of $c\left(Q\right)$ we obtain

$$\begin{aligned}
c\left(\operatorname{Sym}^2 S\right)c\left(Q\right)^3 =& 1\\
c\left(\operatorname{Sym}^2 S\right)c\left(Q\right)^4 =& 1 + \sigma_1\\
c\left(\operatorname{Sym}^2 S\right)c\left(Q\right)^5 =& 1 + 2\sigma_1 + \left(\sigma_{1,1} + 2\sigma_2\right) + \sigma_{2,1}\\
c\left(\operatorname{Sym}^2 S\right)c\left(Q\right)^6 =& 1 + 3\sigma_1 + \left(3\sigma_{1,1} + 5\sigma_2\right) + \left(6\sigma_{2,1} + 5\sigma_3\right) + \left(3\sigma_{2,2} + 6_{3,1}\right) + 6\sigma_{3,2}\\
c\left(\operatorname{Sym}^2 S\right)c\left(Q\right)^7 =& 1 + 4\sigma_1 + \left(6\sigma_{1,1} + 9\sigma_2\right) + \left(17\sigma_{2,1} + 14\sigma_3\right) + \left(14\sigma_{2,2} + 28\sigma_{3,1} + 14\sigma_4\right) + \left(28\sigma_{3,2} + 24\sigma_{4,1}\right) + \left(14\sigma_{3,3} + 28\sigma_{4,2}\right) +\\
&+ 14\sigma_{4,3}\\
c\left(\operatorname{Sym}^2 S\right)c\left(Q\right)^8 =& 1 + 5\sigma_1 + \left(10\sigma_{1,1} + 14\sigma_2\right) + \left(36\sigma_{2,1} + 28\sigma_3\right) + \left(40\sigma_{2,2} + 78\sigma_{3,1} + 42\sigma_4\right) + \left(110\sigma_{3,2} + 120\sigma_{4,1} + 42\sigma_5\right) +\\
&+ \left(84\sigma_{3,3} + 180\sigma_{4,2} + 120\sigma_{5,1}\right) + \left(168\sigma_{4,3} + 180\sigma_{5,2}\right) + \left(84\sigma_{4,4} + 168\sigma_{5,3}\right) + 84\sigma_{5,4}\\
c\left(\operatorname{Sym}^2 S\right)c\left(Q\right)^9 =& 1 + 6\sigma_1 + \left(15\sigma_{1,1} + 20\sigma_2\right) + \left(65\sigma_{2,1} + 48\sigma_3\right) + \left(90\sigma_{2,2} + 171\sigma_{3,1} + 90\sigma_4\right) + \left(306\sigma_{3,2} + 333\sigma_{4,1} + 132\sigma_5\right) +\\
&+ \left(300\sigma_{3,3} + 648\sigma_{4,2} + 495\sigma_{5,1} + 132\sigma_6\right) + \left(810\sigma_{4,3} + 990\sigma_{5,2} + 495\sigma_{6,1}\right) + \left(594\sigma_{4,4} + 1320\sigma_{5,3} + 990\sigma_{6,2}\right) +\\
&+ \left(1188\sigma_{5,4} + 1320\sigma_{6,3}\right) + \left(594\sigma_{5,5} + 1188\sigma_{6,4}\right) + 594\sigma_{6,5}\\
c\left(\operatorname{Sym}^2 S\right)c\left(Q\right)^{10} =& 1 + 7\sigma_1 + \left(21\sigma_{1,1} + 27\sigma_2\right) + \left(106\sigma_{2,1} + 75\sigma_3\right) + \left(175\sigma_{2,2} + 326\sigma_{3,1} + 165\sigma_4\right) + \left(700\sigma_{3,2} + 748\sigma_{4,1} + 297\sigma_5\right) +\\
&+ \left(825\sigma_{3,3} + 1771\sigma_{4,2} + 1375\sigma_{5,1} + 429\sigma_6\right) + \left(2706\sigma_{4,3} + 3388\sigma_{5,2} + 2002\sigma_{6,1} + 429\sigma_7\right) +\\
&+ \left(2475\sigma_{4,4} + 5643\sigma_{5,3} + 5005\sigma_{6,2} + 2002\sigma_{7,1}\right) + \left(6600\sigma_{5,4} + 8580\sigma_{6,3} + 5005\sigma_{7,2}\right) +\\
&+ \left(4719\sigma_{5,5} + 10725\sigma_{6,4} + 8580\sigma_{7,3}\right) + \left(9438\sigma_{6,5} + 10725\sigma_{7,4}\right) + \left(4719\sigma_{6,6} + 9438\sigma_{7,5}\right) + 4719\sigma_{7,6}
\end{aligned}$$

$$
\begin{aligned}
c\left(\mathrm{Sym}^2 S\right) c\left(Q\right)^{11} =& 1+8\sigma_1+(28\sigma_{1,1}+35\sigma_2)+(161\sigma_{2,1}+110\sigma_3)+(308\sigma_{2,2}+561\sigma_{3,1}+275\sigma_4)+(1408\sigma_{3,2}+1474\sigma_{4,1}+572\sigma_5)+\\
&+(1925\sigma_{3,3}+4092\sigma_{4,2}+3146\sigma_{5,1}+1001\sigma_6)+(7315\sigma_{4,3}+9152\sigma_{5,2}+5577\sigma_{6,1}+1430\sigma_7)+\\
&+(7865\sigma_{4,4}+18018\sigma_{5,3}+16588\sigma_{6,2}+8008\sigma_{7,1}+1430\sigma_8)+(25168\sigma_{5,4}+34034\sigma_{6,3}+24024\sigma_{7,2}+8008\sigma_{8,1})+\\
&+(22022\sigma_{5,5}+51909\sigma_{6,4}+50050\sigma_{7,3}+24024\sigma_{8,2})+(58201\sigma_{6,5}+78650\sigma_{7,4}+50050\sigma_{8,3})+\\
&+(40898\sigma_{6,6}+94380\sigma_{7,5}+78650\sigma_{8,4})+(81796\sigma_{7,6}+94380\sigma_{8,5})+(40898\sigma_{7,7}+81796\sigma_{8,6})+40898\sigma_{8,7}
\end{aligned}
$$

$$
\begin{aligned}
c\left(\mathrm{Sym}^2 S\right) c\left(Q\right)^{12} =& 1+9\sigma_1+(36\sigma_{1,1}+44\sigma_2)+(232\sigma_{2,1}+154\sigma_3)+(504\sigma_{2,2}+903\sigma_{3,1}+429\sigma_4)+(2583\sigma_{3,2}+2652\sigma_{4,1}+1001\sigma_5)+\\
&+(4004\sigma_{3,3}+8424\sigma_{4,2}+6370\sigma_{5,1}+2002\sigma_6)+(17160\sigma_{4,3}+21294\sigma_{5,2}+12948\sigma_{6,1}+3432\sigma_7)+\\
&+(21021\sigma_{4,4}+48048\sigma_{5,3}+44460\sigma_{6,2}+22386\sigma_{7,1}+4862\sigma_8)+\\
&+(77077\sigma_{5,4}+105248\sigma_{6,3}+77922\sigma_{7,2}+31824\sigma_{8,1}+4862\sigma_9)+\\
&+(78078\sigma_{5,5}+186186\sigma_{6,4}+188760\sigma_{7,3}+111384\sigma_{8,2}+31824\sigma_{9,1})+\\
&+(245388\sigma_{6,5}+348348\sigma_{7,4}+272272\sigma_{8,3}+111384\sigma_{9,2})+(208208\sigma_{6,6}+501930\sigma_{7,5}+510510\sigma_{8,4}+272272\sigma_{9,3})+\\
&+(546546\sigma_{7,6}+758472\sigma_{8,5}+510510\sigma_{9,4})+(379236\sigma_{7,7}+884884\sigma_{8,6}+758472\sigma_{9,5})+\\
&+(758472\sigma_{8,7}+884884\sigma_{9,6})+(379263\sigma_{8,8}+758472\sigma_{9,7})+379236\sigma_{9,8}
\end{aligned}
$$

$k = 3$

In this case

$$c(\mathrm{Sym}^2 S) = 1-4\sigma_1+(10\sigma_{1,1}+5\sigma_2)-(20\sigma_{1,1,1}+15\sigma_{2,1}+2\sigma_3)+(30\sigma_{2,1,1}+10\sigma_{2,2}+6\sigma_{3,1})-(20\sigma_{2,2,1}+12\sigma_{3,1,1}+4\sigma_{3,2})+8\sigma_{3,2,1}.$$

And multiplying by the powers of $c\left(Q\right)$ we obtain

$$c\left(\mathrm{Sym}^2 S\right) c\left(Q\right)^{4} = 1$$

$$c\left(\mathrm{Sym}^2 S\right) c\left(Q\right)^{5} = 1+\sigma_1$$

$$
\begin{aligned}
c\left(\operatorname{Sym}^2 S\right) c\left(Q\right)^6 =& 1+2\sigma_1+(\sigma_{1,1}+2\sigma_2)+\sigma_{2,1}\\
c\left(\operatorname{Sym}^2 S\right) c\left(Q\right)^7 =& 1+3\sigma_1+(3\sigma_{1,1}+5\sigma_2)+(\sigma_{1,1,1}+6\sigma_{2,1}+5\sigma_3)+(2\sigma_{2,1,1}+3\sigma_{2,2}+6\sigma_{3,1})+(\sigma_{2,2,1}+2\sigma_{3,1,1}+3\sigma_{3,2})+\sigma_{3,2,1}\\
c\left(\operatorname{Sym}^2 S\right) c\left(Q\right)^8 =& 1+4\sigma_1+(6\sigma_{1,1}+9\sigma_2)+(4\sigma_{1,1,1}+17\sigma_{2,1}+14\sigma_3)+(12\sigma_{2,1,1}+14\sigma_{2,2}+28\sigma_{3,1}+14\sigma_4)+\\
&+(12\sigma_{2,2,1}+20\sigma_{3,1,1}+28\sigma_{3,2}+28\sigma_{4,1})+(4\sigma_{2,2,2}+24\sigma_{3,2,1}+14\sigma_{3,3}+20\sigma_{4,1,1}+28\sigma_{4,2})+\\
&+(8\sigma_{3,2,2}+12\sigma_{3,3,1}+24\sigma_{4,2,1}+14\sigma_{4,3})+(4\sigma_{3,3,2}+8\sigma_{4,2,2}+12\sigma_{4,3,1})+4\sigma_{4,3,2}\\
c\left(\operatorname{Sym}^2 S\right) c\left(Q\right)^9 =& 1+5\sigma_1+(10\sigma_{1,1}+14\sigma_2)+(10\sigma_{1,1,1}+36\sigma_{2,1}+28\sigma_3)+(39\sigma_{2,1,1}+40\sigma_{2,2}+78\sigma_{3,1}+42\sigma_4)+\\
&+(55\sigma_{2,2,1}+87\sigma_{3,1,1}+110\sigma_{3,2}+120\sigma_{4,1}+42\sigma_5)+\\
&+(30\sigma_{2,2,2}+155\sigma_{3,2,1}+84\sigma_{3,3}+135\sigma_{4,1,1}+180\sigma_{4,2}+120\sigma_{5,1})+\\
&+(90\sigma_{3,2,2}+126\sigma_{3,3,1}+255\sigma_{4,2,1}+168\sigma_{4,3}+135\sigma_{5,1,1}+180\sigma_{5,2})+\\
&+(90\sigma_{3,3,2}+150\sigma_{4,2,2}+252\sigma_{4,3,1}+84\sigma_{4,4}+255\sigma_{5,2,1}+168\sigma_{5,3})+\\
&+(30\sigma_{3,3,3}+180\sigma_{4,3,2}+126\sigma_{4,4,1}+150\sigma_{5,2,2}+252\sigma_{5,3,1}+84\sigma_{5,4})+(60\sigma_{4,3,3}+90\sigma_{4,4,2}+180\sigma_{5,3,2}+126\sigma_{5,4,1})+\\
&+(30\sigma_{4,4,3}+60\sigma_{5,3,3}+90\sigma_{5,4,2})+30\sigma_{5,4,3}\\
c\left(\operatorname{Sym}^2 S\right) c\left(Q\right)^{10} =& 1+6\sigma_1+(15\sigma_{1,1}+20\sigma_2)+(20\sigma_{1,1,1}+65\sigma_{2,1}+48\sigma_3)+(95\sigma_{2,1,1}+90\sigma_{2,2}+171\sigma_{3,1}+90\sigma_4)+\\
&+(170\sigma_{2,2,1}+306\sigma_{3,2}+333\sigma_{4,1}+132\sigma_5)+\\
&+(125\sigma_{2,2,2}+600\sigma_{3,2,1}+300\sigma_{3,3}+515\sigma_{4,1,1}+648\sigma_{4,2}+495\sigma_{5,1}+132\sigma_6)+\\
&+(480\sigma_{3,2,2}+640\sigma_{3,3,1}+1290\sigma_{4,2,1}+810\sigma_{4,3}+770\sigma_{5,1,1}+990\sigma_{5,2}+495\sigma_{6,1})+\\
&+(655\sigma_{3,3,2}+1065\sigma_{4,2,2}+1750\sigma_{4,3,1}+594\sigma_{4,4}+1980\sigma_{5,2,1}+1320\sigma_{5,3}+770\sigma_{6,1,1}+990\sigma_{6,2})+\\
&+(330\sigma_{3,3,3}+1840\sigma_{4,3,2}+1320\sigma_{4,4,1}+1650\sigma_{5,2,2}+2860\sigma_{5,3,1}+1188\sigma_{5,4}+1980\sigma_{6,2,1}+1320\sigma_{6,3})+\\
&+(990\sigma_{4,3,3}+1485\sigma_{4,4,2}+3025\sigma_{5,3,2}+2640\sigma_{5,4,1}+594\sigma_{5,5}+1650\sigma_{6,2,2}+2860\sigma_{6,3,1}+1188\sigma_{6,4})+\\
&+(990\sigma_{4,4,3}+1650\sigma_{5,3,3}+2970\sigma_{5,4,2}+1320\sigma_{5,5,1}+3025\sigma_{6,3,2}+2640\sigma_{6,4,1}+594\sigma_{6,5})+\\
&+(330\sigma_{4,4,4}+1980\sigma_{5,4,3}+1485\sigma_{5,5,2}+1650\sigma_{6,3,3}+2970\sigma_{6,4,2}+1320\sigma_{6,5,1})+\\
&+(660\sigma_{5,4,4}+990\sigma_{5,5,3}+1980\sigma_{6,4,3}+1485\sigma_{6,5,2})+(330\sigma_{5,5,4}+660\sigma_{6,4,4}+990\sigma_{6,5,3})+330\sigma_{6,5,4}
\end{aligned}
$$

$$
\begin{aligned}
c\left(\mathrm{Sym}^2 S\right) c\left(Q\right)^{11} = & 1 + 7\sigma_{1} + (21\sigma_{1,1} + 27\sigma_{2}) + (35\sigma_{1,1,1} + 106\sigma_{2,1} + 75\sigma_{3}) + (195\sigma_{2,1,1} + 175\sigma_{2,2} + 325\sigma_{3,1} + 165\sigma_{4}) + \\
& + (420\sigma_{2,2,1} + 626\sigma_{3,1,1} + 700\sigma_{3,2} + 748\sigma_{4,1} + 297\sigma_{5}) + \\
& + (385\sigma_{2,2,2} + 1757\sigma_{3,2,1} + 825\sigma_{3,3} + 1474\sigma_{4,1,1} + 1771\sigma_{4,2} + 1375\sigma_{5,1} + 429\sigma_{6}) + \\
& + (1771\sigma_{3,2,2} + 2277\sigma_{3,3,1} + 4543\sigma_{4,2,1} + 2706\sigma_{4,3} + 2739\sigma_{5,1,1} + 3388\sigma_{5,2} + 2002\sigma_{6,1} + 429\sigma_{7}) + \\
& + (2981\sigma_{3,3,2} + 4774\sigma_{4,2,2} + 7623\sigma_{4,3,1} + 2475\sigma_{4,4} + 8778\sigma_{5,2,1} + 5643\sigma_{5,3} + 4004\sigma_{6,1,1} + 5005\sigma_{6,2} + 2002\sigma_{7,1}) + \\
& + (1925\sigma_{3,3,3} + 10384\sigma_{4,3,2} + 7260\sigma_{4,4,1} + 9394\sigma_{5,2,2} + 16038\sigma_{5,3,1} + \\
& \qquad + 6600\sigma_{5,4} + 13013\sigma_{6,2,1} + 8580\sigma_{6,3} + 4004\sigma_{7,1,1} + 5005\sigma_{7,2}) + \\
& + (7315\sigma_{4,3,3} + 10802\sigma_{4,4,2} + 22209\sigma_{5,3,2} + 19503\sigma_{5,4,1} + 4719\sigma_{5,5} + \\
& \qquad + 14014\sigma_{6,2,2} + 24453\sigma_{6,3,1} + 10725\sigma_{6,4} + 13013\sigma_{7,2,1} + 8580\sigma_{7,3}) + \\
& + (9779\sigma_{4,4,3} + 16170\sigma_{5,3,3} + 29425\sigma_{5,4,2} + 14157\sigma_{5,5,1} + 34034\sigma_{6,3,2} + \\
& \qquad + 31746\sigma_{6,4,1} + 9438\sigma_{6,5} + 14014\sigma_{7,2,2} + 24453\sigma_{7,3,1} + 10725\sigma_{7,4}) + \\
& + (4719\sigma_{4,4,4} + 27412\sigma_{5,4,3} + 22022\sigma_{5,5,2} + 25025\sigma_{6,3,3} + 48048\sigma_{6,4,2} + \\
& \qquad + 28314\sigma_{6,5,1} + 4719\sigma_{6,6} + 34034\sigma_{7,3,2} + 31746\sigma_{7,4,1} + 9438\sigma_{7,5}) + \\
& + (14157\sigma_{5,4,4} + 22022\sigma_{5,5,3} + 45045\sigma_{6,4,3} + 44044\sigma_{6,5,2} + \\
& \qquad + 14157\sigma_{6,6,1} + 25025\sigma_{7,3,3} + 48048\sigma_{7,4,2} + 28314\sigma_{7,5,1} + 4719\sigma_{7,6}) + \\
& + (14157\sigma_{5,5,4} + 23595\sigma_{6,4,4} + 44044\sigma_{6,5,3} + 22022\sigma_{6,6,2} + 45045\sigma_{7,4,3} + 44044\sigma_{7,5,2} + 14157\sigma_{7,6,1}) + \\
& + (4719\sigma_{5,5,5} + 28314\sigma_{6,5,4} + 22022\sigma_{6,6,3} + 23595\sigma_{7,4,4} + 44044\sigma_{7,5,3} + 22022\sigma_{7,6,2}) + \\
& + (9438\sigma_{6,5,5} + 14157\sigma_{6,6,4} + 28314\sigma_{7,5,4} + 22022\sigma_{7,6,3}) + (4719\sigma_{6,6,5} + 9438\sigma_{7,5,5} + 14157\sigma_{7,6,4}) + 4719\sigma_{7,6,5}
\end{aligned}
$$

$$
\begin{aligned}
c\left(\operatorname{Sym}^2 S\right) c\left(Q\right)^{12} = & 1 + 8\sigma_1 + (28\sigma_{1,1} + 35\sigma_2) + (56\sigma_{1,1,1} + 161\sigma_{2,1} + 110\sigma_3) + (357\sigma_{2,1,1} + 308\sigma_{2,2} + 561\sigma_{3,1} + 275\sigma_4) + \\
& + (896\sigma_{2,2,1} + 1308\sigma_{3,1,1} + 1408\sigma_{3,2} + 1474\sigma_{4,1} + 572\sigma_5) + \\
& + (980\sigma_{2,2,2} + 4304\sigma_{3,2,1} + 1925\sigma_{3,3} + 3530\sigma_{4,1,1} + 4092\sigma_{4,2} + 3146\sigma_{5,1} + 1001\sigma_6) + \\
& + (5208\sigma_{3,2,2} + 6510\sigma_{3,3,1} + 12840\sigma_{4,2,1} + 7315\sigma_{4,3} + 7644\sigma_{5,1,1} + 9152\sigma_{5,2} + 5577\sigma_{6,1} + 1430\sigma_7) + \\
& + (10311\sigma_{3,3,2} + 16296\sigma_{4,2,2} + 25375\sigma_{4,3,1} + 7865\sigma_{4,4} + 29120\sigma_{5,2,1} + \\
& \qquad + 18018\sigma_{5,3} + 13650\sigma_{6,1,1} + 16588\sigma_{6,2} + 8008\sigma_{7,1} + 1430\sigma_8) + \\
& + (8008\sigma_{3,3,3} + 42112\sigma_{4,3,2} + 28600\sigma_{4,4,1} + 37856\sigma_{5,2,2} + 63336\sigma_{5,3,1} + \\
& \qquad + 25168\sigma_{5,4} + 53144\sigma_{6,2,1} + 34034\sigma_{6,3} + 19656\sigma_{7,1,1} + 24024\sigma_{7,2} + 8008\sigma_{8,1}) + \\
& + (36036\sigma_{4,3,3} + 52338\sigma_{4,4,2} + 107562\sigma_{5,3,2} + 92664\sigma_{5,4,1} + 22022\sigma_{5,5} + 69888\sigma_{6,2,2} + \\
& \qquad + 120393\sigma_{6,3,1} + 51909\sigma_{6,4} + 77168\sigma_{7,2,1} + 50050\sigma_{7,3} + 19656\sigma_{8,1,1} + 24024\sigma_{8,2}) + \\
& + (58344\sigma_{4,4,3} + 96096\sigma_{5,3,3} + 173316\sigma_{5,4,2} + 82940\sigma_{5,5,1} + 206661\sigma_{6,3,2} + 192192\sigma_{6,4,1} + \\
& \qquad + 58201\sigma_{6,5} + 101920\sigma_{7,2,2} + 177450\sigma_{7,3,1} + 78650\sigma_{7,4} + 77168\sigma_{8,2,1} + 50050\sigma_{8,3}) + \\
& + (35035\sigma_{4,4,4} + 201344\sigma_{5,4,3} + 161876\sigma_{5,5,2} + 188188\sigma_{6,3,3} + 362934\sigma_{6,4,2} + 220220\sigma_{6,5,1} + \\
& \qquad + 40898\sigma_{6,6} + 305760\sigma_{7,3,2} + 291720\sigma_{7,4,1} + 94380\sigma_{7,5} + 101920\sigma_{8,2,2} + 177450\sigma_{8,3,1} + 78650\sigma_{8,4}) + \\
& + (132132\sigma_{5,4,4} + 205920\sigma_{5,5,3} + 429000\sigma_{6,4,3} + 433290\sigma_{6,5,2} + 156156\sigma_{6,6,1} + 280280\sigma_{7,3,3} + \\
& \qquad + 552552\sigma_{7,4,2} + 357500\sigma_{7,5,1} + 81796\sigma_{7,6} + 305760\sigma_{8,3,2} + 291720\sigma_{8,4,1} + 94380\sigma_{8,5}) + \\
& + (174174\sigma_{5,5,4} + 291291\sigma_{6,4,4} + 559416\sigma_{6,5,3} + 312312\sigma_{6,6,2} + 656656\sigma_{7,4,3} + 704704\sigma_{7,5,2} + \\
& \qquad + 312312\sigma_{7,6,1} + 40898\sigma_{7,7} + 280280\sigma_{8,3,3} + 552552\sigma_{8,4,2} + 357500\sigma_{8,5,1} + 81796\sigma_{8,6}) + \\
& + (81796\sigma_{5,5,5} + 487487\sigma_{6,5,4} + 416416\sigma_{6,6,3} + 450450\sigma_{7,4,4} + 912912\sigma_{7,5,3} + \\
& \qquad + 624624\sigma_{7,6,2} + 156156\sigma_{7,7,1} + 656656\sigma_{8,4,3} + 704704\sigma_{8,5,2} + 312312\sigma_{8,6,1} + 40898\sigma_{8,7}) + \\
& + (245388\sigma_{6,5,5} + 390390\sigma_{6,6,4} + 800800\sigma_{7,5,4} + 832832\sigma_{7,6,3} + \\
& \qquad + 312312\sigma_{7,7,2} + 450450\sigma_{8,4,4} + 912912\sigma_{8,5,3} + 624624\sigma_{8,6,2} + 156156\sigma_{8,7,1}) + \\
& + (245388\sigma_{6,6,5} + 408980\sigma_{7,5,5} + 780780\sigma_{7,6,4} + 416416\sigma_{7,7,3} + 800800\sigma_{8,5,4} + 832832\sigma_{8,6,3} + 312312\sigma_{8,7,2}) + \\
& + (81796\sigma_{6,6,6} + 490776\sigma_{7,6,5} + 390390\sigma_{7,7,4} + 408980\sigma_{8,5,5} + 780780\sigma_{8,6,4} + 416416\sigma_{8,7,3}) + \\
& + (163592\sigma_{7,6,6} + 245388\sigma_{7,7,5} + 490776\sigma_{8,6,5} + 390390\sigma_{8,7,4}) + (81796\sigma_{7,7,6} + 163592\sigma_{8,6,6} + 245388\sigma_{8,7,5}) + \\
& + 81796\sigma_{8,7,6}
\end{aligned}
$$

$k = 4$

In this case

$$\begin{aligned} c(\mathrm{Sym}^2 S) = 1 - 5\sigma_1 + (15\sigma_{1,1} + 9\sigma_2) - (35\sigma_{1,1,1} + 34\sigma_{2,1} + 7\sigma_3) + (70\sigma_{1,1,1,1} + 84\sigma_{2,1,1} + 35\sigma_{2,2} + 28\sigma_{3,1} + 2\sigma_4) - \\ - (168\sigma_{2,1,1,1} + 105\sigma_{2,2,1} + 70\sigma_{3,1,1} + 35\sigma_{3,2} + 8\sigma_{4,1}) + (210\sigma_{2,2,1,1} + 70\sigma_{2,2,2} + 140\sigma_{3,1,1,1} + 105\sigma_{3,2,1} + 14\sigma_{3,3} + 20\sigma_{4,1,1} + 10\sigma_{4,2}) - \\ - (140\sigma_{2,2,2,1} + 210\sigma_{3,2,1,1} + 70\sigma_{3,2,2} + 42\sigma_{3,3,1} + 40\sigma_{4,1,1,1} + 30\sigma_{4,2,1} + 4\sigma_{4,3}) + \\ + (140\sigma_{3,2,2,1} + 84\sigma_{3,3,1,1} + 28\sigma_{3,3,2} + 60\sigma_{4,2,1,1} + 20\sigma_{4,2,2} + 12\sigma_{4,3,1}) - (56\sigma_{3,3,2,1} + 40\sigma_{4,2,2,1} + 24\sigma_{4,3,1,1} + 8\sigma_{4,3,2}) + 16\sigma_{4,3,2,1}. \end{aligned}$$

And multiplying by the powers of $c\,(Q)$ we obtain

$$c\left(\mathrm{Sym}^2 S\right) c\,(Q)^5 = 1$$

$$c\left(\mathrm{Sym}^2 S\right) c\,(Q)^6 = 1 + \sigma_1$$

$$c\left(\mathrm{Sym}^2 S\right) c\,(Q)^7 = 1 + 2\sigma_1 + (\sigma_{1,1} + 2\sigma_2) + \sigma_{2,1}$$

$$c\left(\mathrm{Sym}^2 S\right) c\,(Q)^8 = 1 + 3\sigma_1 + (3\sigma_{1,1} + 5\sigma_2) + (\sigma_{1,1,1} + 6\sigma_{2,1} + 5\sigma_3) + (2\sigma_{2,1,1} + 3\sigma_{2,2} + 6\sigma_{3,1}) + (\sigma_{2,2,1} + 2\sigma_{3,1,1} + 3\sigma_{3,2}) + \sigma_{3,2,1}$$

$$\begin{aligned} c\left(\mathrm{Sym}^2 S\right) c\,(Q)^9 = & 1 + 4\sigma_1 + (6\sigma_{1,1} + 9\sigma_2) + (4\sigma_{1,1,1} + 17\sigma_{2,1} + 14\sigma_3) + (\sigma_{1,1,1,1} + 12\sigma_{2,1,1} + 14\sigma_{2,2} + 28\sigma_{3,1} + 14\sigma_4) + \\ & + (3\sigma_{2,1,1,1} + 12\sigma_{2,2,1} + 20\sigma_{3,1,1} + 28\sigma_{3,2} + 28\sigma_{4,1}) + \\ & + (3\sigma_{2,2,1,1} + 4\sigma_{2,2,2} + 5\sigma_{3,1,1,1} + 24\sigma_{3,2,1} + 14\sigma_{3,3} + 20\sigma_{4,1,1} + 28\sigma_{4,2}) + \\ & + (\sigma_{2,2,2,1} + 6\sigma_{3,2,1,1} + 8\sigma_{3,2,2} + 12\sigma_{3,3,1} + 5\sigma_{4,1,1,1} + 24\sigma_{4,2,1} + 14\sigma_{4,3}) + \\ & + (2\sigma_{3,2,2,1} + 3\sigma_{3,3,1,1} + 4\sigma_{3,3,2} + 6\sigma_{4,2,1,1} + 8\sigma_{4,2,2} + 12\sigma_{4,3,1}) + (\sigma_{3,3,2,1} + 2\sigma_{4,2,2,1} + 3\sigma_{4,3,1,1} + 4\sigma_{4,3,2}) + \sigma_{4,3,2,1} \end{aligned}$$

$$
\begin{aligned}
c\left(\mathrm{Sym}^2 S\right) c\left(Q\right)^{10} = & 1 + 5\sigma_1 + (10\sigma_{1,1} + 14\sigma_2) + (10\sigma_{1,1,1} + 36\sigma_{2,1} + 28\sigma_3) + (5\sigma_{1,1,1,1} + 39\sigma_{2,1,1} + 40\sigma_{2,2} + 78\sigma_{3,1} + 42\sigma_4) + \\
& + (20\sigma_{2,1,1,1} + 55\sigma_{2,2,1} + 87\sigma_{3,1,1} + 110\sigma_{3,2} + 120\sigma_{4,1} + 42\sigma_5) + \\
& + (30\sigma_{2,2,1,1} + 30\sigma_{2,2,2} + 45\sigma_{3,1,1,1} + 155\sigma_{3,2,1} + 84\sigma_{3,3} + 135\sigma_{4,1,1} + 180\sigma_{4,2} + 120\sigma_{5,1}) + \\
& + (20\sigma_{2,2,2,1} + 85\sigma_{3,2,1,1} + 90\sigma_{3,2,2} + 126\sigma_{3,3,1} + 70\sigma_{4,1,1,1} + 255\sigma_{4,2,1} + 168\sigma_{4,3} + 135\sigma_{5,1,1} + 180\sigma_{5,2}) + \\
& + (5\sigma_{2,2,2,2} + 60\sigma_{3,2,2,1} + 70\sigma_{3,3,1,1} + 90\sigma_{3,3,2} + 140\sigma_{4,2,1,1} + \\
& \qquad + 150\sigma_{4,2,2} + 252\sigma_{4,3,1} + 84\sigma_{4,4} + 70\sigma_{5,1,1,1} + 255\sigma_{5,2,1} + 168\sigma_{5,3}) + \\
& + (15\sigma_{3,2,2,2} + 60\sigma_{3,3,2,1} + 30\sigma_{3,3,3} + 100\sigma_{4,2,2,1} + 140\sigma_{4,3,1,1} + \\
& \qquad + 180\sigma_{4,3,2} + 126\sigma_{4,4,1} + 140\sigma_{5,2,1,1} + 150\sigma_{5,2,2} + 252\sigma_{5,3,1} + 84\sigma_{5,4}) + \\
& + (15\sigma_{3,3,2,2} + 20\sigma_{3,3,3,1} + 25\sigma_{4,2,2,2} + 120\sigma_{4,3,2,1} + 70\sigma_{4,4,1,1} + \\
& \qquad + 60\sigma_{4,3,3} + 90\sigma_{4,4,2} + 100\sigma_{5,2,2,1} + 140\sigma_{5,3,1,1} + 180\sigma_{5,3,2} + 126\sigma_{5,4,1}) + \\
& + (5\sigma_{3,3,3,2} + 30\sigma_{4,3,2,2} + 40\sigma_{4,3,3,1} + 60\sigma_{4,4,2,1} + 30\sigma_{4,4,3} + \\
& \qquad + 25\sigma_{5,2,2,2} + 120\sigma_{5,3,2,1} + 70\sigma_{5,4,1,1} + 60\sigma_{5,3,3} + 90\sigma_{5,4,2}) + \\
& + (10\sigma_{4,3,3,2} + 15\sigma_{4,4,2,2} + 20\sigma_{4,4,3,1} + 30\sigma_{5,3,2,2} + 40\sigma_{5,3,3,1} + 60\sigma_{5,4,2,1} + 30\sigma_{5,4,3}) + \\
& + (5\sigma_{4,4,3,2} + 10\sigma_{5,3,3,2} + 15\sigma_{5,4,2,2} + 20\sigma_{5,4,3,1}) + 5\sigma_{5,4,3,2}
\end{aligned}
$$

$$
\begin{aligned}
c\left(\mathrm{Sym}^2 S\right) c\,(Q)^{11} =\,&1 + 6\sigma_1 + (15\sigma_{1,1} + 20\sigma_2) + (20\sigma_{1,1,1} + 65\sigma_{2,1} + 48\sigma_3) + (15\sigma_{1,1,1,1} + 95\sigma_{2,1,1} + 90\sigma_{2,2} + 171\sigma_{3,1} + 90\sigma_4) + \\
&+ (74\sigma_{2,1,1,1} + 170\sigma_{2,2,1} + 260\sigma_{3,1,1} + 306\sigma_{3,2} + 333\sigma_{4,1} + 132\sigma_5) + \\
&+ (144\sigma_{2,2,1,1} + 125\sigma_{2,2,2} + 206\sigma_{3,1,1,1} + 600\sigma_{3,2,1} + 300\sigma_{3,3} + 515\sigma_{4,1,1} + 648\sigma_{4,2} + 495\sigma_{5,1} + 132\sigma_6) + \\
&+ (135\sigma_{2,2,2,1} + 516\sigma_{3,2,1,1} + 480\sigma_{3,2,2} + 640\sigma_{3,3,1} + 411\sigma_{4,1,1,1} + \\
&\qquad +1290\sigma_{4,2,1} + 810\sigma_{4,3} + 770\sigma_{5,1,1} + 990\sigma_{5,2} + 495\sigma_{6,1}) + \\
&+ (55\sigma_{2,2,2,2} + 525\sigma_{3,2,2,1} + 568\sigma_{3,3,1,1} + 655\sigma_{3,3,2} + 1116\sigma_{4,2,1,1} + 1065\sigma_{4,2,2} + \\
&\qquad +1750\sigma_{4,3,1} + 594\sigma_{4,4} + 616\sigma_{5,1,1,1} + 1980\sigma_{5,2,1} + 1320\sigma_{5,3} + 770\sigma_{6,1,1} + 990\sigma_{6,2}) + \\
&+ (220\sigma_{3,2,2,2} + 736\sigma_{3,3,2,1} + 330\sigma_{3,3,3} + 1170\sigma_{4,2,2,1} + 1560\sigma_{4,3,1,1} + 1840\sigma_{4,3,2} + 1320\sigma_{4,4,1} + \\
&\qquad +1716\sigma_{5,2,1,1} + 1650\sigma_{5,2,2} + 2860\sigma_{5,3,1} + 1188\sigma_{5,4} + 616\sigma_{6,1,1,1} + 1980\sigma_{6,2,1} + 1320\sigma_{6,3}) + \\
&+ (330\sigma_{3,3,2,2} + 396\sigma_{3,3,3,1} + 495\sigma_{4,2,2,2} + 2073\sigma_{4,3,2,1} + 1188\sigma_{4,4,1,1} + 990\sigma_{4,3,3} + 1485\sigma_{4,4,2} + 1815\sigma_{5,2,2,1} + \\
&\qquad +2552\sigma_{5,3,1,1} + 3025\sigma_{5,3,2} + 2640\sigma_{5,4,1} + 594\sigma_{5,5} + 1716\sigma_{6,2,1,1} + 1650\sigma_{6,2,2} + 2860\sigma_{6,3,1} + 1188\sigma_{6,4}) + \\
&+ (220\sigma_{3,3,3,2} + 935\sigma_{4,3,2,2} + 1188\sigma_{4,3,3,1} + 1683\sigma_{4,4,2,1} + 990\sigma_{4,4,3} + 770\sigma_{5,2,2,2} + 3410\sigma_{5,3,2,1} + 2376\sigma_{5,4,1,1} + \\
&\qquad +1650\sigma_{5,3,3} + 2970\sigma_{5,4,2} + 1320\sigma_{5,5,1} + 1815\sigma_{6,2,2,1} + 2552\sigma_{6,3,1,1} + 3025\sigma_{6,3,2} + 2640\sigma_{6,4,1} + 594\sigma_{6,5}) + \\
&+ (55\sigma_{3,3,3,3} + 660\sigma_{4,3,3,2} + 770\sigma_{4,4,2,2} + 1188\sigma_{4,4,3,1} + 330\sigma_{4,4,4} + \\
&\qquad +1540\sigma_{5,3,2,2} + 1980\sigma_{5,3,3,1} + 3366\sigma_{5,4,2,1} + 1188\sigma_{5,5,1,1} + 1980\sigma_{5,4,3} + 1485\sigma_{5,5,2} + \\
&\qquad +770\sigma_{6,2,2,2} + 3410\sigma_{6,3,2,1} + 2376\sigma_{6,4,1,1} + 1650\sigma_{6,3,3} + 2970\sigma_{6,4,2} + 1320\sigma_{6,5,1}) + \\
&+ (165\sigma_{4,3,3,3} + 660\sigma_{4,4,3,2} + 396\sigma_{4,4,4,1} + 1100\sigma_{5,3,3,2} + 1540\sigma_{5,4,2,2} + \\
&\qquad +2376\sigma_{5,4,3,1} + 1683\sigma_{5,5,2,1} + 660\sigma_{5,4,4} + 990\sigma_{5,5,3} + 1540\sigma_{6,3,2,2} + \\
&\qquad +1980\sigma_{6,3,3,1} + 3366\sigma_{6,4,2,1} + 1188\sigma_{6,5,1,1} + 1980\sigma_{6,4,3} + 1485\sigma_{6,5,2}) + \\
&+ (165\sigma_{4,4,3,3} + 220\sigma_{4,4,4,2} + 275\sigma_{5,3,3,3} + 1320\sigma_{5,4,3,2} + 770\sigma_{5,5,2,2} + 792\sigma_{5,4,4,1} + 1188\sigma_{5,5,3,1} + \\
&\qquad +330\sigma_{5,5,4} + 1100\sigma_{6,3,3,2} + 1540\sigma_{6,4,2,2} + 2376\sigma_{6,4,3,1} + 1683\sigma_{6,5,2,1} + 660\sigma_{6,4,4} + 990\sigma_{6,5,3}) + \\
&+ (55\sigma_{4,4,4,3} + 330\sigma_{5,4,3,3} + 440\sigma_{5,4,4,2} + 660\sigma_{5,5,3,2} + 396\sigma_{5,5,4,1} + \\
&\qquad +275\sigma_{6,3,3,3} + 1320\sigma_{6,4,3,2} + 770\sigma_{6,5,2,2} + 792\sigma_{6,4,4,1} + 1188\sigma_{6,5,3,1} + 330\sigma_{6,5,4}) + \\
&+ (110\sigma_{5,4,4,3} + 165\sigma_{5,5,3,3} + 220\sigma_{5,5,4,2} + 330\sigma_{6,4,3,3} + 440\sigma_{6,4,4,2} + 660\sigma_{6,5,3,2} + 396\sigma_{6,5,4,1}) + \\
&+ (55\sigma_{5,5,4,3} + 110\sigma_{6,4,4,3} + 165\sigma_{6,5,3,3} + 220\sigma_{6,5,4,2}) + 55\sigma_{6,5,4,3}
\end{aligned}
$$

$$
\begin{aligned}
c\left(\operatorname{Sym}^2 S\right) c\left(Q\right)^{12} =& 1 + 7\sigma_1 + (21\sigma_{1,1} + 27\sigma_2) + (35\sigma_{1,1,1} + 106\sigma_{2,1} + 75\sigma_3) + (35\sigma_{1,1,1,1} + 195\sigma_{2,1,1} + 175\sigma_{2,2} + 325\sigma_{3,1} + 165\sigma_4) + \\
&+ (204\sigma_{2,1,1,1} + 420\sigma_{2,2,1} + 626\sigma_{3,1,1} + 700\sigma_{3,2} + 748\sigma_{4,1} + 297\sigma_5) + \\
&+ (483\sigma_{2,2,1,1} + 385\sigma_{2,2,2} + 670\sigma_{3,1,1,1} + 1757\sigma_{3,2,1} + 825\sigma_{3,3} + 1474\sigma_{4,1,1} + 1771\sigma_{4,2} + 1375\sigma_{5,1} + 429\sigma_6) + \\
&+ (574\sigma_{2,2,2,1} + 2065\sigma_{3,2,1,1} + 1771\sigma_{3,2,2} + 2277\sigma_{3,3,1} + 1596\sigma_{4,1,1,1} + \\
&\qquad + 4543\sigma_{4,2,1} + 2706\sigma_{4,3} + 2739\sigma_{5,1,1} + 3388\sigma_{5,2} + 2002\sigma_{6,1} + 429\sigma_7) + \\
&+ (315\sigma_{2,2,2,2} + 2695\sigma_{3,2,2,1} + 2790\sigma_{3,3,1,1} + 2981\sigma_{3,3,2} + 5397\sigma_{4,2,1,1} + 4774\sigma_{4,2,2} + 7623\sigma_{4,3,1} + \\
&\qquad + 2475\sigma_{4,4} + 2982\sigma_{5,1,1,1} + 8778\sigma_{5,2,1} + 5643\sigma_{5,3} + 4004\sigma_{6,1,1} + 5005\sigma_{6,2} + 2002\sigma_{7,1}) + \\
&+ (1540\sigma_{3,2,2,2} + 4720\sigma_{3,3,2,1} + 1925\sigma_{3,3,3} + 7336\sigma_{4,2,2,1} + 9432\sigma_{4,3,1,1} + 10384\sigma_{4,3,2} + 7260\sigma_{4,4,1} + 10479\sigma_{5,2,1,1} + \\
&\qquad + 9394\sigma_{5,2,2} + 16038\sigma_{5,3,1} + 6600\sigma_{5,4} + 4368\sigma_{6,1,1,1} + 13013\sigma_{6,2,1} + 8580\sigma_{6,3} + 4004\sigma_{7,1,1} + 5005\sigma_{7,2}) + \\
&+ (2946\sigma_{3,3,2,2} + 3325\sigma_{3,3,3,1} + 4270\sigma_{4,2,2,2} + 16584\sigma_{4,3,2,1} + 9150\sigma_{4,4,1,1} + 7315\sigma_{4,3,3} + \\
&\qquad + 10802\sigma_{4,4,2} + 14497\sigma_{5,2,2,1} + 19926\sigma_{5,3,1,1} + 22209\sigma_{5,3,2} + 19503\sigma_{5,4,1} + 4719\sigma_{5,5} + \\
&\qquad + 15561\sigma_{6,2,1,1} + 14014\sigma_{6,2,2} + 24453\sigma_{6,3,1} + 10725\sigma_{6,4} + 4368\sigma_{7,1,1,1} + 13013\sigma_{7,2,1} + 8580\sigma_{7,3}) + \\
&+ (2667\sigma_{3,3,3,2} + 10524\sigma_{4,3,2,2} + 12726\sigma_{4,3,3,1} + 17540\sigma_{4,4,2,1} + 9779\sigma_{4,4,3} + 8505\sigma_{5,2,2,2} + \\
&\qquad + 35592\sigma_{5,3,2,1} + 24660\sigma_{5,4,1,1} + 16170\sigma_{5,3,3} + 29425\sigma_{5,4,2} + 14157\sigma_{5,5,1} + 21658\sigma_{6,2,2,1} + 30420\sigma_{6,3,1,1} + \\
&\qquad + 34034\sigma_{6,3,2} + 31746\sigma_{6,4,1} + 9438\sigma_{6,5} + 15561\sigma_{7,2,1,1} + 14014\sigma_{7,2,2} + 24453\sigma_{7,3,1} + 10725\sigma_{7,4}) + \\
&+ (1001\sigma_{3,3,3,3} + 10353\sigma_{4,3,3,2} + 11516\sigma_{4,4,2,2} + 17255\sigma_{4,4,3,1} + 4719\sigma_{4,4,4} + \\
&\qquad + 28203\sigma_{5,3,3,1} + 47900\sigma_{5,4,2,1} + 18018\sigma_{5,5,1,1} + 27412\sigma_{5,4,3} + 22022\sigma_{5,5,2} + \\
&\qquad + 12740\sigma_{6,2,2,2} + 54600\sigma_{6,3,2,1} + 40170\sigma_{6,4,1,1} + 25025\sigma_{6,3,3} + 48048\sigma_{6,4,2} + 28314\sigma_{6,5,1} + \\
&\qquad + 4719\sigma_{6,6} + 21658\sigma_{7,2,2,1} + 30420\sigma_{7,3,1,1} + 34034\sigma_{7,3,2} + 31746\sigma_{7,4,1} + 9438\sigma_{7,5}) + \\
&+ (4004\sigma_{4,3,3,3} + 7722\sigma_{4,4,4,2} + 14462\sigma_{4,4,3,2} + 8580\sigma_{4,4,4,1} + 23058\sigma_{5,3,3,2} + 31602\sigma_{5,4,2,2} + \\
&\qquad + 48440\sigma_{5,4,3,1} + 36036\sigma_{5,5,2,1} + 14157\sigma_{5,4,4} + 22022\sigma_{5,5,3} + 34944\sigma_{6,3,2,2} + 43680\sigma_{6,3,3,1} + \\
&\qquad + 78260\sigma_{6,4,2,1} + 36036\sigma_{6,5,1,1} + 45045\sigma_{6,4,3} + 44044\sigma_{6,5,2} + 14157\sigma_{6,6,1} + 12740\sigma_{7,2,2,2} + \\
&\qquad + 54600\sigma_{7,3,2,1} + 40170\sigma_{7,4,1,1} + 25025\sigma_{7,3,3} + 48048\sigma_{7,4,2} + 28314\sigma_{7,5,1} + 4719\sigma_{7,6}) + \\
&+ \cdots
\end{aligned}
$$

$$
\begin{aligned}
\cdots &+ (6006\sigma_{4,4,3,3} + 9009\sigma_{5,3,3,3} + 40719\sigma_{5,4,3,2} + 24024\sigma_{5,5,2,2} + 22734\sigma_{5,3,2,2} + 25740\sigma_{5,4,4,1} + 39039\sigma_{5,5,3,1} + \\
&\qquad +14157\sigma_{5,5,4} + 35763\sigma_{6,3,3,2} + 51688\sigma_{6,4,2,2} + 79625\sigma_{6,4,3,1} + 72072\sigma_{6,5,2,1} + 18018\sigma_{6,6,1,1} + 23595\sigma_{6,4,4} + 44044\sigma_{6,5,3} + \\
&\qquad +22022\sigma_{6,6,2} + 34944\sigma_{7,3,2,2} + 43680\sigma_{7,3,3,1} + 78260\sigma_{7,4,2,1} + 36036\sigma_{7,5,1,1} + 45045\sigma_{7,4,3} + 44044\sigma_{7,5,2} + 14157\sigma_{7,6,1}) + \\
&+ (4004\sigma_{4,4,4,3} + 17017\sigma_{5,4,3,3} + 23166\sigma_{5,4,4,2} + 33033\sigma_{5,5,3,2} + 25740\sigma_{5,5,4,1} + 4719\sigma_{5,5,5} + 14014\sigma_{6,3,3,3} + \\
&\qquad +66976\sigma_{6,4,3,2} + 48048\sigma_{6,5,2,2} + 42900\sigma_{6,4,4,1} + 78078\sigma_{6,5,3,1} + 36036\sigma_{6,6,2,1} + 28314\sigma_{6,5,4} + 22022\sigma_{6,6,3} + \\
&\qquad +35763\sigma_{7,3,3,2} + 51688\sigma_{7,4,2,2} + 79625\sigma_{7,4,3,1} + 72072\sigma_{7,5,2,1} + 18018\sigma_{7,6,1,1} + 23595\sigma_{7,4,4} + 44044\sigma_{7,5,3} + 22022\sigma_{7,6,2}) + \\
&+ (1001\sigma_{4,4,4,4} + 12012\sigma_{5,4,4,3} + 14014\sigma_{5,5,3,3} + 23166\sigma_{5,5,4,2} + 8580\sigma_{5,5,5,1} + 28028\sigma_{6,4,3,3} + 38610\sigma_{6,4,4,2} + \\
&\qquad +66066\sigma_{6,5,3,2} + 24024\sigma_{6,6,2,2} + 51480\sigma_{6,5,4,1} + 39039\sigma_{6,6,3,1} + 9438\sigma_{6,5,5} + 14157\sigma_{6,6,4} + 14014\sigma_{7,3,3,3} + \\
&\qquad +66976\sigma_{7,4,3,2} + 48048\sigma_{7,5,2,2} + 42900\sigma_{7,4,4,1} + 78078\sigma_{7,5,3,1} + 36036\sigma_{7,6,2,1} + 28314\sigma_{7,5,4} + 22022\sigma_{7,6,3}) + \\
&+ (3003\sigma_{5,4,4,4} + 12012\sigma_{5,5,4,3} + 7722\sigma_{5,5,5,2} + 20020\sigma_{6,4,4,3} + 28028\sigma_{6,5,3,3} + 46332\sigma_{6,5,4,2} + \\
&\qquad +66066\sigma_{7,5,3,2} + 24024\sigma_{7,6,2,2} + 51480\sigma_{7,5,4,1} + 39039\sigma_{7,6,3,1} + 9438\sigma_{7,5,5} + 14157\sigma_{7,6,4}) + \\
&+ (3003\sigma_{5,5,4,4} + 4004\sigma_{5,5,5,3} + 5005\sigma_{6,4,4,4} + 24024\sigma_{6,5,4,3} + 14014\sigma_{6,6,3,3} + 15444\sigma_{6,5,5,2} + 23166\sigma_{6,6,4,2} + \\
&\qquad +8580\sigma_{6,6,5,1} + 20020\sigma_{7,4,4,3} + 28028\sigma_{7,5,3,3} + 46332\sigma_{7,5,4,2} + 33033\sigma_{7,6,3,2} + 17160\sigma_{7,5,5,1} + 25740\sigma_{7,6,4,1} + 4719\sigma_{7,6,5}) + \\
&+ (1001\sigma_{5,5,5,4} + 6006\sigma_{6,5,4,4} + 8008\sigma_{6,5,5,3} + 12012\sigma_{6,6,4,3} + 7722\sigma_{6,6,5,2} + \\
&\qquad +5005\sigma_{7,4,4,4} + 24024\sigma_{7,5,4,3} + 14014\sigma_{7,6,3,3} + 15444\sigma_{7,5,5,2} + 23166\sigma_{7,6,4,2} + 8580\sigma_{7,6,5,1}) + \\
&+ (2002\sigma_{6,5,5,4} + 3003\sigma_{6,6,4,4} + 4004\sigma_{6,6,5,3} + 6006\sigma_{7,5,4,4} + 8008\sigma_{7,5,5,3} + 12012\sigma_{7,6,4,3} + 7722\sigma_{7,6,5,2}) + \\
&\qquad +33033\sigma_{6,6,3,2} + 17160\sigma_{6,5,5,1} + 25740\sigma_{6,6,4,1} + 4719\sigma_{6,6,5} + 28028\sigma_{7,4,3,3} + 38610\sigma_{7,4,4,2} + \\
&+ (1001\sigma_{6,6,5,4} + 2002\sigma_{7,5,5,4} + 3003\sigma_{7,6,4,4} + 4004\sigma_{7,6,5,3}) + 1001\sigma_{7,6,5,4}
\end{aligned}
$$

II

Part Two

Fibred surfaces

Introduction to Part II

In the classification of smooth algebraic surfaces it is natural to study its possible fibrations over curves, trying to relate the geometry of the surface to the properties of the fibres and the base. For varieties of small Kodaira dimension there are canonical fibrations (e.g. Mori and Iitaka fibrations), and for surfaces S satisfying $p_g(S) \leq 2q(S) - 4$ one can consider the fibration provided by the classical Castelunovo-de Franchis theorem. In general, up to birational equivalence, every algebraic surface admits a fibration because it has an algebraic function (resolve its indeterminacy and then take the Stein factorization).

In this Thesis we focus on the relations between the isotriviality of a fibration and its numerical invariants. Denote from now on by $f: S \to B$ a fibration from a compact surface S to a compact curve B, that is, a morphism with connected fibres. Denote also by F a general fibre of f. The first invariants one can consider are the Hodge numbers of S, B and F: the genus of B and F, the geometric genus and the irregularity of S, and also $h^{1,1}(S)$, which is not a birational invariant. Frome these "primary" invariants, one can define other ones (see Chapter 3 for the precise definitions): the relative irregularity q_f, the relative (topological and holomorphic) Euler characteristics e_f and χ_f, the self-intersection of the relative canonical divisor K_f^2,...

There are some direct relations between these invariants and the isotriviality of f. For example, it always holds

$$\chi_f \geq 0,$$

with equality if and only if f is isotrivial and smooth (i.e. locally trivial). In another direction, Beauville showed in its Appendix to [12] that

$$0 \leq q_f \leq g,$$

and the equality $q_f = g$ holds if and only if S is birational to $B \times F$ (f is *trivial*).

As a consequence of the work of Serrano [40] (Corollary 6.1.4), non-trivial isotrivial fibrations verify

$$q_f \leq \frac{g+1}{2}.$$

Xiao conjectured that this last equality holds also for non-isotrivial fibrations, but he was able to prove that only in the case $B = \mathbb{P}^1$ ([44]). In fact, the conjecture is false, because Pirola gave a counterexample in [33]. In the general case, Xiao proved in [43] the weaker inequality

$$q_f \leq \frac{5g+1}{6}.$$

In fact, he obtained this inequality after proving the *slope inequality*

$$\lambda_f \geq 4 - \frac{4}{g},$$

where the *slope* $\lambda_f = K_f^2/\chi_f$ is defined for any non-locally trivial fibration.

Another invariant of a fibration, introduced by Konno in [24], is the *Clifford* index of f, defined simply as the Clifford index of a general fibre (which is in fact the maximum of the Clifford indexes of the smooth fibres). By using this new invariant, Konno, and more recently Barja and Stoppino in [3], obtain some strengthenings of the original slope inequality.

From a different point of view, Serrano studies the properties of the sheaf $\Omega^1_{S/B}$ of *relative differentials* and its double dual $\Delta_{S/B}$. In fact, in [38] he obtains an explicit description of $\Delta_{S/B}$ in terms of the relative canonical sheaf $\omega_{S/B}$ and the singular fibres, which allows him to study its Zariski decomposition and to relate the isotriviality of f to the Iitaka dimension of $\Delta_{S/B}$. In his later work [39], Serrano shows that if f is relatively minimal and non-isotrivial, then S is birational to

$$\operatorname{Proj}\left(\bigoplus_{n\geq 0} H^0\left(S, \Delta_{S/B}^{\otimes n}\right)\right),$$

the canonical model of the pair $\left(S, \Delta_{S/B}\right)$ as defined by Sakai in [36].

A third, different approach to study fibred surfaces (and particularly the abovementioned conjecture posed by Xiao) is followed by Pirola in [33]. In this work, Pirola considers the Albanese map of S, which in fact is a map from S to a locally trivial family of Abelian varieties $\mathcal{A}$ over B with fibre the kernel A of the induced map $\operatorname{Alb}(S) \to J(B)$. With this setting, he characterizes the failure of Xiao's conjecture by the non-constancy of the generalized Abel-Jacobi map from B to the *primitive first intermediate Jacobian* of A.

In this second part of the Thesis we use tools from all the previous approaches, as well as the *adjoint images* introduced by Collino and Pirola in [11], to prove the following inequality for non-isotrivial fibrations.

Theorem 6.3.4. *Let $f : S \to B$ be a fibration of genus g, relative irregularity q_f and Clifford index c_f. If f is non-isotrivial, then*

$$q_f \leq g - c_f.$$

Meanwhile, we extend to global families some constructions and results about infinitesimal deformations of smooth curves (Chapter 4), and obtain an structure result for some special fibrations (Theorem 6.3.1). All these new concepts and results seem to be generalizable to one-dimensional families of irregular varieties of any dimension, but we have not explored yet this more general possibility because it exceeds the scope of our initial objective (the study of Xiao's conjecture).

Chapter Three 3

PRELIMINARIES ON FIBRED SURFACES

In this chapter we recall some of the most basic definitions and results concerning fibrations of complex algebraic varieties, paying special atention to the case of an irregular surface fibred over a non-rational curve.

In the first section we introduce the fundamental definitions and the most general results. The second section is devoted to study the relationship between the sheaf of relative differentials and the relative canonical bundle of a fibred surface, which will be very useful in Chapters 4 and 6. The third section focuses on how a fibration of irregular varieties behaves under the Albanese functor, and the last section is a summary on numerical invariants that will not be used in the sequel, but are strongly related to the isotriviality of a fibred surface.

3.1 Basic preliminaries

We start with the fundamental definitions.

Definition 3.1.1. *A* fibration, *or* fibre space, *is a surjective, flat, proper morphism $f : X \to Y$ with connected fibres, where X and Y are smooth varieties such that* $\dim X > \dim Y$.

- *The variety X is called the* total space *of the fibration, and Y is the* base.
- *The fibration is called* Kähler *(resp.* projective*) if the total space X is Kähler (resp.projective).*
- *The fibre over a point $y \in Y$ is the subscheme*

$$X_y = X \times_Y \operatorname{Spec} \mathbb{C}(y)\,,$$

which is always proper.

- *A point $x \in X$ is* regular *if the differential $df_p : T_{X,x} \to T_{Y,f(x)}$ is surjective. Otherwise, x is called* critical.
- *A point $y \in Y$ is called a* regular value *if all the points in $f^{-1}(x)$ are regluar. Otherwise, if y is the image of some critical point, it is called a* critical value.

Remark 3.1.2. *If the base Y is a smooth curve, any surjective morphism is automatically flat, hence we can drop this condition from the definition of a fibration.*

If the total space X is compact, the morphism is automatically proper.

We now collect some basic results about fibrations, which are well stablished in the literature.

Theorem 3.1.3. *1. (Regular value theorem) The fibre X_y is smooth if and only if $y \in Y$ is a regular value.*

2. (Generic smoothness) The critical values form a proper Zariski-closed subset of Y.

3. For any smooth fibre $F = X_y$, the normal bundle $N_{F/X} = (T_{X|F})/T_F$ is trivial, of rank $\dim Y$. *More intrinsically, $N_{F/X} \cong T_{Y,y} \otimes \mathcal{O}_F$ is the trivial bundle with fibre the tangent space of Y at y.*

4. (Ehresmann's theorem) Locally over the regular values, the fibration f is differentiably trivial. In particular, all the smooth fibres are diffeomorphic.

Definition 3.1.4. *A fibration $f : X \to Y$ is* isotrivial *if all the smooth fibres are isomorphic. If furthermore X is birational to the product $X_y \times Y$ (where X_y is any smooth fibre), f is called* trivial. *An isotrivial fibration whose fibres are all smooth is called either a* fibre bundle *or a* locally trivial *fibration. A non-isotrivial fibration is* strongly non-isotrivial *if the smooth fibres are not even birationally equivalent.*

We are mostly concerned about fibred projective surfaces. Therefore, from now on, S (resp. B) will denote a smooth projective complex surface (resp. curve), and $f : S \to B$ will be a surjective morphism with connected fibres. By all the previous considerations, f is automatically a fibration (flat and proper). Since the fibres are curves, we will denote them by C_b (instead of S_b), for any $b \in B$. Also, since two smooth curves are birational if and only if they are isomorphic, fibred surfaces are strongly non-isotrivial if and only if they are non-isotrivial. Furthermore, the general fibres are diffeomorphic compact Riemann surfaces, hence they have the same genus.

Definition 3.1.5. *The* genus *of the fibration is $g = g(C_b)$, the genus of any smooth fibre C_b.*

The genus is one of the basic numerical invariants of the fibration, but not the only one we consider.

Definition 3.1.6. *The* relative irregularity *of f is*

$$q_f = q(S) - g(B),$$

the difference between the irregularities of S and B.

Since f is surjective, the pull-back of 1-forms is an injection

$$f^* : H^0(B, \omega_B) \hookrightarrow H^0\left(S, \Omega^1_S\right),$$

and the relative irregularity is the dimension of the quotient space

$$V = V_f := H^0\left(S, \Omega^1_S\right) / f^* H^0(B, \omega_B).$$

Lemma 3.1.7. *For any smooth fibre C, the composition*

$$H^0\left(S, \Omega^1_S\right) \longrightarrow H^0\left(C, \Omega^1_{S|C}\right) \longrightarrow H^0(C, \omega_C)$$

factors through an injective map $V \hookrightarrow H^0(C, \omega_C)$.

In particular, $\operatorname{rk} f_* \Omega^1_S = \dim H^0\left(C, \Omega^1_{S|C}\right) \geq q_f$.

Corollary 3.1.8. *Any fibration satisfies the inequalities*

$$0 \leq q_f \leq g.$$

Lemma 3.1.7 also implies that "V gets bigger under base change". More precisely, let $B' \to B$ be a finite morphism, $S' = \widetilde{S \times_B B'}$ the minimal desingularization of the fibre product, and $f' : S' \to B'$ the induced fibration. The pull-back morphism $H^0(S, \Omega^1_S) \to H^0(S', \Omega^1_{S'})$ clearly sends $f^* H^0(B, \omega_B)$ into $(f')^* H^0(B', \omega_{B'})$, so there is a natural pull-back map

$$V \longrightarrow V' := H^0\left(S', \Omega^1_{S'}\right) / (f')^* H^0(B', \omega_{B'}). \tag{3.1}$$

Corollary 3.1.9. *The map (3.1) is injective.*

Proof. Let $b' \in B'$ be a point where π is not ramified, and let $b = \pi(b')$, so that C_b and $C_{b'}$ are isomorphic. Clearly, the inclusion $V \hookrightarrow H^0(C_b, \omega_{C_b})$ factors as $V \to V' \hookrightarrow H^0\left(C_{b'}, \omega_{C_{b'}}\right)$, hence $V \to V'$ must be injective. □

Another immediate consequence of Lemma 3.1.7 concerns the infinitesimal deformations of the smooth fibres induced by f.

Definition 3.1.10. *For any regular value b, let $\xi_b \in H^1(C_b, T_{C_b})$ be the class of the extension*

$$0 \longrightarrow N^{\vee}_{C_b/X} \cong \mathcal{O}_{C_b} \longrightarrow \Omega^1_{S|C_b} \longrightarrow \omega_{C_b} \longrightarrow 0.$$

Denote by ∂_{ξ_b} the connecting homomorphism

$$H^0(C_b, \omega_{C_b}) \longrightarrow H^1(C_b, \mathcal{O}_{C_b}),$$

which can be identified with the cup-product with ξ_b.

Corollary 3.1.11. *For any smooth fibre $C = C_b$, the vector space V is contained in the kernel of ∂_{ξ_b}.*

From Theorem 3.1.3 we deduce that the general fibres of a fibration are smooth, but the finitely many possible singular fibres can behave very badly (they can have several irreducible components, some of them even non-reduced). However, after finitely many blowing-ups and a change of base, we can make things slightly better.

Lemma 3.1.12 ([4] Th. III.10.3)**.** *Let $f : S \to B$ be any fibration. There always exists a composition of blow-ups $\pi : S_1 \to S$ and a finite morphism $B' \to B$ such that $f' : S' = \widetilde{S_1 \times_B B'} \to B'$ has reduced fibres.*

Remark 3.1.13. *The composition of blow-ups π of Lemma 3.1.12 can be chosen to be an isomorphism over the regular points of f. Equivalently, the necessary blow-ups are performed at the critical points of f and the subsequent exceptional divisors.*

We have seen in Part I that some kinds of fibrations can be recognized from properties of the algebra of cohomology of the total space. We close this section with a version of the Generalized Castelnuovo-de Franchis Theorem specially adapted to our case.

Theorem 3.1.14 (Castelnuovo-de Franchis, [9] 1.9)**.** *Let S be a compact complex surface, and $\alpha_1, \dots, \alpha_k \in H^0(S, \Omega^1_S)$ linearly independent holomorphic 1-forms such that $\alpha_i \wedge \alpha_j = 0$ for every $1 \le i, j \le k$. Then there exists a fibration $f : S \to B$ over a curve such that $\alpha_1, \dots, \alpha_k \in f^*H^0(B, \omega_B)$, and hence $g(B) \ge k$.*

3.2 $\Omega^1_{S/B}$ AND $\omega_{S/B}$

In this section we focus on the properties of the sheaves $\Omega^1_{S/B}$ and $\omega_{S/B}$, which will frequently appear in all the following chapters.

As a first observation, the surjectivity of f implies that the dual

$$f^*\omega_B \longrightarrow \Omega^1_S \tag{3.2}$$

of the tangent map of f is an injection of sheaves. This allows us to make the following

Definition 3.2.1. *The cokernel of the map (3.2) is the sheaf of* relative differentials *of f, and it is denoted by $\Omega^1_{S/B}$.*

Its dual (that is, the kernel of the tangent map) is called the relative tangent *sheaf, and it is denoted by $T_{S/B}$.*

Define also the Jacobian ideal sheaf *of f as*

$$J := \operatorname{im}(T_f : T_S \longrightarrow f^*T_B) \otimes f^*\omega_B \subseteq \mathcal{O}_S.$$

It is the ideal of a subscheme Z supported on the critical points of f. Denote by Z_d the divisorial component of Z, and by Z_p the subscheme supported on points.

It is immediate to see that both $\Omega^1_{S/B}$ and $T_{S/B}$ are locally free away from Z, and that their restrictions to any smooth fibre C are precisely the canonical and tangent bundles of C, respectively.

Remark 3.2.2. *In [38], Serrano defined a sheaf (also denoted by J) which is essentially our Jacobian ideal sheaf, but without the twisting by $f^*\omega_B$.*

Next we introduce the second sheaf mentioned in the title of the section.

Definition 3.2.3. *The line bundle $\omega_{S/B} = \omega_S \otimes (f^*\omega_B)^\vee$ is the* relative canonical sheaf *of the fibration.*

The adjunction formula says that the canonical sheaf of any smooth curve C contained in S is

$$\omega_S(C)_{|C} = \omega_S \otimes \mathcal{O}_C(C) = \omega_S \otimes N_{C/S}.$$

In the case $C = C_b$ is a smooth fibre, the normal bundle is trivial, isomorphic to $T_{B,b} \otimes \mathcal{O}_C$, which turns out to be $(f^*\omega_B)^\vee_{|C}$. Therefore, the restriction of $\omega_{S/B}$ to C is also naturally the canonical sheaf of C.

Remark 3.2.4. *The sheaves $\Omega^1_{S/B}, T_{S/B}$ and $\omega_{S/B}$ can be analogously defined for any fibration, but some of the properties we list below may not have an analogous version in higher dimensions.*

We summarize now the main properties relating $\Omega^1_{S/B}$, $T_{S/B}$, J and $\omega_{S/B}$, but we need a little bit more of notation. Let $\{E_i\}$ be the set of irreducible components of the singular fibers, and let ν_i be the multiplicity of E_i as a component of the corresponding singular fibre.

Lemma 3.2.5 ([38] Lemma 1.1)**.** *Some properties of J, $\Omega^1_{S/B}$ and $T_{S/B}$.*

1. *The relative tangent sheaf $T_{S/B}$ is an invertible sheaf, whose inverse is*
$$\left(\Omega^1_{S/B}\right)^{\vee\vee} = T^{\vee}_{S/B} \cong \omega_{S/B}\left(-\sum_i (\nu_i - 1) E_i\right).$$

2. *$J^{\vee\vee} \cong \mathcal{O}_S\left(-\sum_i (\nu_i - 1) E_i\right)$. Therefore $Z_d = \sum_i (\nu_i - 1) E_i$ and*
$$J = J^{\vee\vee} \otimes I_{Z_p} = I_{Z_p}\left(-\sum_i (\nu_i - 1) E_i\right).$$

3. *$\mathcal{O}_{Z_p} \cong J^{\vee\vee}/J$ has length*
$$c_2(S) + c_1\left(T_{S/B}\right) c_1\left(J^{\vee} \otimes f^*\omega_B\right).$$

Lemma 3.2.6. *The sheaves $\Omega^1_{S/B}$ and $\omega_{S/B}$ fit into the exact sequence*
$$0 \longrightarrow \left(f^*\omega_B\left(Z_d\right)\right)_{|Z_d} \longrightarrow \Omega^1_{S/B} \xrightarrow{\alpha} \omega_{S/B} \longrightarrow \omega_{S/B|Z} \longrightarrow 0.$$
In particular, if all the fibres are reduced, then $Z = Z_p$, α is injective and $\Omega^1_{S/B} \cong \omega_{S/B} \otimes J$ is torsion-free. In general, $\omega_{S/B} \otimes J$ is the quotient of $\Omega^1_{S/B}$ by its torsion subsheaf.

Proof. Let us first recall the construction of the map $\alpha : \Omega^1_{S/B} \to \omega_{S/B}$. Twisting by $f^*\omega_B$ the exact sequence defining $\Omega^1_{S/B}$ one gets
$$0 \longrightarrow \left(f^*\omega_B\right)^{\otimes 2} \longrightarrow f^*\omega_B \otimes \Omega^1_S \longrightarrow f^*\omega_B \otimes \Omega^1_{S/B} \longrightarrow 0.$$
Wedge product induces a map $\widetilde{\beta} : f^*\omega_B \otimes \Omega^1_S \to \omega_S$ that sends $\left(f^*\omega_B\right)^{\otimes 2}$ to zero. Therefore, $\widetilde{\beta}$ induces a map $\beta : f^*\omega_B \otimes \Omega^1_{S/B} \to \omega_S$. The map α is

β twisted by $(f^*\omega_B)^\vee$. Denoting by $\widetilde{\alpha}$ the corresponding twist of $\widetilde{\beta}$, we get the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & f^*\omega_B \arrow[r] \arrow[d] & \Omega^1_S \arrow[r] \arrow[d, "\widetilde{\alpha}"'] & \Omega^1_{S/B} \arrow[r] \arrow[d, "\alpha"] & 0 \\
& 0 \arrow[r] & \omega_{S/B} \arrow[r] & \omega_{S/B} \arrow[r] & 0
\end{tikzcd}$$

The snake lemma says that $\operatorname{coker}\alpha = \operatorname{coker}\widetilde{\alpha}$ and $\ker\alpha = (\ker\widetilde{\alpha})\,/f^*\omega_B$, so it is enough to study the map $\widetilde{\alpha}$. But $\widetilde{\alpha}$ is exactly the tangent map T_f twisted by ω_S, so by the definition of J, Z and Z_d we get

$$\ker\widetilde{\alpha} = \ker(T_f)\otimes\omega_S = T_{S/B}\otimes\omega_{S/B}\otimes f^*\omega_B = f^*\omega_B\,(Z_d)$$

and

$$\operatorname{coker}\widetilde{\alpha} = \operatorname{coker}(T_f)\otimes\omega_S = (\mathcal{O}_S/J)\otimes\omega_{S/B} = \omega_{S/B|Z}.$$

To conclude, just note that the inclusion $f^*\omega_B \hookrightarrow \ker\widetilde{\alpha}$ is induced by the natural map $\mathcal{O}_S \hookrightarrow \mathcal{O}_S\,(Z_d)$, so

$$\ker\alpha = f^*\omega_B\otimes(\mathcal{O}_S\,(Z_d)\,/\mathcal{O}_S) = (f^*\omega_B\,(Z_d))_{|Z_d}\,.$$

□

Remark 3.2.7. *Taking global sections on the exact sequence*

$$0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/B} \longrightarrow 0$$

we obtain that there is a natural inclusion $V \hookrightarrow H^0\left(S,\Omega^1_{S/B}\right)$. *Hence* V *can be seen as a subspace of the global sections of either* $\Omega^1_{S/B}$ *or* $f_*\Omega^1_{S/B}$. *Restricting the evaluation map we obtain*

$$V\otimes\mathcal{O}_B \longrightarrow f_*\Omega^1_{S/B},$$

which composed with $f_*\alpha$ *gives*

$$V\otimes\mathcal{O}_B \longrightarrow f_*\omega_{S/B}.$$

Over a general regular value $b\in B$, *both maps agree with the inclusion of Lemma 3.1.7, and are therefore injective maps of sheaves.*

Since the base of our fibration is a curve, $\omega_{S/B}$ has some specially nice properties. First of all, it works as a relative dualizing sheaf.

Lemma 3.2.8 ([4] Th. III.12.3)**.** *The relative canonical sheaf $\omega_{S/B}$ is the relative dualizing sheaf, that is, for every locally free sheaf $\mathcal{F}$ on S, there is a natural isomorphism*

$$f_*\left(\mathcal{F}^\vee \otimes \omega_{S/B}\right) \xrightarrow{\cong} \left(R^1 f_* \mathcal{F}\right)^\vee .$$

Another good property of $\omega_{S/B}$ follows from the work [15] of Fujita, where the direct image of the relative canonical sheaf is studied. The most interesing results for our purposes are the next ones.

Theorem 3.2.9 ([15] Th. 3.1)**.** *Let $f : X \to B$ be a Kähler fibration over a curve. Then*

$$f_*\omega_{X/B} \cong \mathcal{O}_B^{\oplus h} \oplus \mathcal{E},$$

where

- *$h = h^1\left(B, f_*\omega_X\right)$, and*
- *$\mathcal{E}$ is locally free and such that $H^1\left(B, \mathcal{E} \otimes \omega_B\right) = 0$.*

Corollary 3.2.10 (of the proof of Theorem 3.2.9)**.** *In the case that $X = S$ is a surface, the trivial part $\mathcal{O}_B^{\oplus h}$ is precisely the image of the inclusion*

$$V \otimes \mathcal{O}_B \longrightarrow f_*\omega_{S/B},$$

in Remark 3.2.7. In particular, $h = q_f$.

3.3 The irregular case

We devote this section to discuss some facts about the morphism on Albanese varieties induced by a fibration between irregular varieties, and their implications on the fibration itself.

Given any fibration $f : X \to Y$, the universal property of the Albanese map gives a morphism between the respective Albanese varieties

$$a_f = \mathrm{Alb}\,(f) : \mathrm{Alb}\,(X) \to \mathrm{Alb}\,(Y)\,.$$

Moreover, if X_y is any smooth fibre, the inclusion $\iota_y : X_y \hookrightarrow X$ induces another morphism $a_y : \mathrm{Alb}\,(X_y) \to \mathrm{Alb}\,(X)$, which fits with the previous one into the following commutative diagram

$$\begin{tikzcd}
X_y \arrow[r, hook, "\iota_y"] \arrow[d, "a_{X_y}"'] & X \arrow[r, two heads, "f"] \arrow[d, "a_X"'] & Y \arrow[d, "a_Y"] \\
\mathrm{Alb}\,(X_y) \arrow[r, "a_y"] \arrow[rr, bend right, "h"'] & \mathrm{Alb}\,(X) \arrow[r, two heads, "a_f"] & \mathrm{Alb}\,(Y)
\end{tikzcd}$$

(where the vertical arrows are the corresponding Albanese morphisms).

Of course, the composition $h = a_f \circ a_y$ is the constant map to the point $a_Y(y)$. Therefore, the image $a_y(\mathrm{Alb}(X_y))$ is contained in (a suitable translate of) the kernel of a_f, and the rigidity of Abelian subvarieties implies that all these images are isomorphic.

Let us focus now in the case of a fibred surface $f : S \to B$. If $C = C_b$ is a general (smooth) fibre, the above diagram looks now like

$$\begin{tikzcd} C \arrow[r, hook, "\iota_b"] \arrow[d, hook, "a_C"'] & S \arrow[r, two heads, "f"] \arrow[d, "a_S"'] & B \arrow[d, hook, "a_B"] \\ J(C) \arrow[r, "a_b"] \arrow[rr, bend right, "h"'] & \mathrm{Alb}(S) \arrow[r, two heads, "a_f"] & J(B) \end{tikzcd}$$

By the above discussion, the image $a_b(J(C))$ of the Jacobian variety of C is contained in $a_f^{-1}(a_B(b)) \cong A := \ker a_f$. In fact, $\ker a_f$ is connected, so it is an Abelian variety of dimension q_f. Furthermore, Lemma 3.1.7 implies that the map $J(C) \to A$ is surjective, because its cotangent map at any point is precisely the inclusion $V \hookrightarrow H^0(C, \omega_C)$.

Let us consider the extremal cases of the inequalities $0 \leq q_f \leq g$. On the one hand, if $q_f = g$, then all the Jacobian varieties $J(C_b)$ are isogenous to A. Since the set of Abelian varieties isogenous to A is discrete, all the $J(C_b)$ must be isomorphic. In fact, with a little of care, Beauville showed that also the principal polarizations coincide, hence f is isotrivial by Torelli's theorem. This is the starting point of the proof of a much stronger result (Lemma in the Appendix of [12]), which asserts that a fibration with $q_f = g$ is in fact trivial (birational to a product).

On the other hand, if $q_f = 0$, then a_f is an isogeny, but since it has also connected fibres, it is indeed an isomorphism and f is simply the Albanese map of S.

As for the remaining cases, the inequalities $0 < q_f < g$ imply that the maps $J(C_b) \to A$ are not zero and have positive-dimensional kernel, hence the Jacobian varieties $J(C_b)$ are not simple. This restricts the possible fibres of such a fibration to the union of countably many closed subvarieties of the moduli space $\mathcal{M}_g$ of curves of genus g.

To close this section, note that the Albanese image $a_S(S)$ is contained in $\mathcal{A} := a_f^{-1}(a_B(B))$, which is a trivial fibre bundle over $a_B(B) \cong B$, with constant fibre A. In this way, taking any contractible open subset $U \subset B$, we can trivialize $\mathcal{A}_U$ and obtain a map

$$\Phi : f^{-1}(U) \xrightarrow{a_S} \mathcal{A}_U \cong A \times U \xrightarrow{p} A$$

from any “tubular” open set of S to the Abelian variety A. This is the setting of the Volumetric Theorem ([34] Th. 1.5.3), which will appear a few times in the forthcoming chapters.

3.4 More numerical invariants

We have already introduced the genus g and the relative irregularity q_f of a fibration $f : S \to B$. In this section we introduce a few more numerical invariants, summarizing their main properties and their relations with the isotriviality of a fibration of genus $g \geq 2$. Denote by C any smooth fibre.

From a topological point of view, we can consider the *relative Euler characteristic*

$$e_f := e(S) - e(B)\, e(C) = e(S) - 4\,(g(B) - 1)\,(g - 1)\,.$$

It is always non-negative, and can be written as a sum of non-negative quantities associated to the singular fibres ([5] Prop. X.10). Therefore $e_f = 0$ if and only if f has no singular fibres (i.e. f is smooth).

Taking into account the complex structure, we can first consider the *self-intersection of the relative canonical sheaf*,

$$K_f^2 := c_1\left(\omega_{S/B}\right)^2 = K_S^2 - 8\,(g(B) - 1)\,(g - 1)\,.$$

As e_f, it is always non-negative , and moreover, if $K_f^2 = 0$ then f is isotrivial.

We can also define the *relative holomorphic Euler characteristic* as

$$\begin{aligned} \chi_f &:= \chi(\mathcal{O}_S) - \chi(\mathcal{O}_B)\,\chi(\mathcal{O}_C) = \\ &= \chi(\mathcal{O}_S) - (g(B) - 1)\,(g - 1) = \deg f_*\omega_{S/B}. \end{aligned}$$

This three invariants satisfy a relative version of the Noether's formula:

$$K_f^2 + e_f = 12\chi_f.$$

Hence, $\chi_f \geq 0$, and the equality implies that f is smooth and isotrivial (i.e., f is a fibre bundle).

In the case that f is not a fibre bundle, Xiao introduced in [43] the *slope* of f, defined as

$$\lambda_f := \frac{K_f^2}{\chi_f}.$$

In the same work, Xiao proved the so called *slope inequality*

$$\lambda_f \geq 4 - \frac{4}{g} \tag{3.3}$$

for a non-locally trivial fibration f of genus $g \geq 2$. This inequality had been proven before by Horikawa and Persson for hyperelliptic fibrations, and by Cornalba and Harris for semistable fibrations.

The last invariant we want to introduce is the Clifford index of a fibration, which plays a central role in the main result of this part of the Thesis (Theorem 6.3.4).

Definition 3.4.1 ([24] Def. 1.1)**.** *Given a fibration $f : S \to B$, its* Clifford index *is defined as*

$$c_f = \max\{\mathrm{Cliff}\,(C_b) \mid C_b = f^{-1}\,(b) \ \ \textit{is smooth}\},$$

which is attained for b ranging in a non-empty Zariski-open set.

The Clifford index allows to obtain several improvements of the inequality (3.3), for example, as those obtained by Konno in [24] and by Barja and Stoppino in [3].

Chapter Four 4

Deformations of smooth curves

In this chapter we deal with several aspects of deformations of (smooth) compact curves. In the first section we explore the relation between non-trivial infinitesimal deformations of a curve and its bicanonical embedding. The second section is simply a summary of the definition and main properties of the relative Ext sheaves, extracted from the first chapter of [7] (which is in turn taken from [21]). Finally, in the third section we use the relative Ext sheaves as a tool to construct global analogues of some concepts of the first section, considering arbitrary (local, compact) one-dimensional families of deformations (possibly with singular fibres).

4.1 Infinitesimal deformations

As mentioned above, we devote this section to study the relation between a non-trivial infinitesimal deformation of a smooth curve and the geometry of its bicanonical embedding. More precisely, we introduce the notion of a deformation being supported on an effective divisor, which is closely related to the span of the divisor in the bicanonical space. We also give upper and lower bounds for the rank of a deformation in terms of the numerical invariants of a supporting divisor (degree and dimension of the associated complete linear series), finding some relations with the Clifford index of the curve.

Almost all the definitions in this section are taken from [11], and most of the results (as well as some ideas of the proofs) already appeared in [17]. However, the latter contains some unaccuracies, hence we have preferred to include new versions of the results that are useful to our situation, adapting also some definitions and rewriting some proofs.

Let C be a smooth curve of genus $g \geq 2$. An infinitesimal deformation of C is a proper flat morphism $\mathcal{C} \to \Delta$ over the spectrum of the dual numbers $\Delta = \operatorname{Spec} \mathbb{C}[\epsilon] / (\epsilon^2)$, such that the special fibre (over $\operatorname{Spec} \mathbb{C}(\epsilon)$) is isomorphic to C.

Since the conormal sheaf $N^\vee_{C/\mathcal{C}}$ is trivial with fibre $T^\vee_{\Delta,0}$, the exact sequence

$$0 \longrightarrow N^\vee_{C/\mathcal{C}} \longrightarrow \Omega^1_{\mathcal{C}|C} \longrightarrow \omega_C \longrightarrow 0 \tag{4.1}$$

determines an extension class $\xi \in H^1(C, T_C) \otimes T^\vee_{\Delta,0}$, which is nothing but the Kodaira-Spencer class of the deformation. Indeed, by choosing a generator of $T^\vee_{\Delta,0} \cong \mathbb{C}$, we can (and will very often do) think of ξ as an element of $H^1(C, T_C)$.

Throughout all the section we will assume that the deformation is not trivial, that is $\mathcal{C} \ncong C \times \Delta$, or equivalently, $\xi \neq 0$.

Denote by

$$\mathbb{P} = \mathbb{P}\left(H^0\left(C, \omega_C^{\otimes 2}\right)^\vee\right) = \mathbb{P}\left(H^1(C, T_C)\right)$$

the bicanonical space of C, and by $\phi_2 : C \hookrightarrow \mathbb{P}$ the bicanonical embedding of C. Since $\xi \neq 0$, it determines a point $[\xi] \in \mathbb{P}$, which is in fact well defined, independently of the chosen isomorphism $T^\vee_{\Delta,0} \cong \mathbb{C}$.

Cup-product with ξ induces a map

$$\partial_\xi = \cup\, \xi : H^0(C, \omega_C) \longrightarrow H^1(C, \mathcal{O}_C)$$

which coincides with the connecting homomorphism in the exact sequence of cohomology obtained form (4.1).

Definition 4.1.1. *We define the* rank *of ξ as the rank of ∂_ξ, and denote it as*

$$\operatorname{rk} \xi = \operatorname{rk} \partial_\xi.$$

Remark 4.1.2. *If C is non-hyperelliptic, the map*

$$\begin{aligned} H^1(C, T_C) &\longrightarrow \operatorname{Hom}\left(H^0(C, \omega_C), H^1(C, \mathcal{O}_C)\right) \\ \xi &\longmapsto \partial_\xi \end{aligned}$$

is injective, hence no information is lost when considering ∂_ξ instead of ξ. However, if C is hyperelliptic, the above map is not injective, hence we may have $\operatorname{rk} \xi = 0$ even if $\xi \neq 0$. This exception is a manifestation of the failure of the infinitesimal Torelli Theorem for hyperelliptic curves.

From now on, until the end of the section, $D = \sum_{i=1}^{k} n_i p_i$ will be an effective divisor on C, of degree $d = \deg D = \sum_{i=1}^{k} n_i$. We will also denote by $r = r(D) = h^0(C, \mathcal{O}_C(D)) - 1$ the dimension of its complete linear series. The divisor D induces the exact sequences

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0 \tag{4.2}$$

and

$$0 \longrightarrow T_C \longrightarrow T_C(D) \longrightarrow T_C(D)_{|D} \longrightarrow 0, \tag{4.3}$$

with connecting homomorphisms

$$\partial_{\mathcal{O},D} : H^0(D, \mathcal{O}_D(D)) \longrightarrow H^1(C, \mathcal{O}_C)$$

and

$$\partial_{T,D} : H^0\left(D, T_C(D)_{|D}\right) \longrightarrow H^1(C, T_C).$$

We will denote their images as

$$\sigma_D = \operatorname{im} \partial_{\mathcal{O},D} = \ker\left(H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, \mathcal{O}_C(D))\right)$$

and

$$\tau_D = \operatorname{im} \partial_{T,D} = \ker\left(H^1(C, T_C) \longrightarrow H^1(C, T_C(D))\right).$$

Lemma 4.1.3. *Keeping the above notations,*

1. $\dim \sigma_D = d - r$ *and*

2. *if* $d < 2g - 2$*, then* $\dim \tau_D = d$.

Proof. 1. From the exact sequence of cohomology of (4.2) and the definition of σ_D, the sequence

$$0 \longrightarrow H^0(\mathcal{O}_C) \longrightarrow H^0(\mathcal{O}_C(D)) \longrightarrow H^0(\mathcal{O}_D(D)) \longrightarrow \sigma_D \longrightarrow 0$$

is exact. Then

$$\dim \sigma_D = h^0(\mathcal{O}_D(D)) - h^0(\mathcal{O}_C(D)) + 1 = d - r.$$

2. The beginning of the cohomology sequence of (4.3) is

$$0 \longrightarrow H^0(T_C) \longrightarrow H^0(T_C(D)) \longrightarrow H^0\left(T_C(D)_{|D}\right) \xrightarrow{\partial_{T,D}} H^1(T_C).$$

Since $\deg T_C \leq \deg T_C(D) = 2-2g+d < 0$, the first two terms vanish and the connecting homomorphism $\partial_{T,D}$ is injective. Therefore

$$\dim \tau_D = h^0\left(T_C(D)_{|D}\right) = h^0(\mathcal{O}_D) = \operatorname{length} \mathcal{O}_D = \deg D = d.$$

$\square$

Remark 4.1.4. *If $D = p$ consists of a single point, then both σ_p and τ_p are one-dimensional, and the maps $p \mapsto \sigma_p \in \mathbb{P}\left(H^0(C, \omega_C)^\vee\right)$ and $p \mapsto \tau_p \in \mathbb{P}$ are precisely the canonical and bicanonical maps, respectively.*

Definition 4.1.5. *With the above notations, define the* span *of D as*

$$\langle D\rangle = \mathbb{P}(\tau_D) \subseteq \mathbb{P}.$$

The next lemma is quite elementary (almost immediate), but will clarify the construction of the global analogue of $\langle D\rangle \subseteq \mathbb{P}$.

Lemma 4.1.6. *The ideal sheaf $\mathcal{J}_D \subset \mathcal{O}_{\mathbb{P}}$ of $\langle D\rangle$ is the image of the map*

$$H^0\left(C, \omega_C^{\otimes 2}(-D)\right) \otimes \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}}.$$

Proof. Since $\langle D\rangle$ is a linear variety, its ideal sheaf is generated by linear forms, so we only need to see that the space $H^0(\mathbb{P}, \mathcal{J}_D(1))$ of linear equations defining $\langle D\rangle$ is $H^0\left(C, \omega_C^{\otimes 2}(-D)\right)$. From the structural sequence

$$0 \to \mathcal{J}_D \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\langle D\rangle} \to 0$$

of $\langle D\rangle$ we obtain the exact sequence

$$0 \to H^0(\mathcal{J}_D(1)) \to H^0(\mathcal{O}_{\mathbb{P}}(1)) \cong H^0\left(C, \omega_C^{\otimes 2}\right) \to H^0\left(\mathcal{O}_{\langle D\rangle}(1)\right) \cong \tau_D^\vee.$$

The last map is the dual of the inclusion $\tau_D \hookrightarrow H^1(C, T_C)$, and therefore

$$H^0(\mathbb{P}, \mathcal{J}_D(1)) \cong \left(H^1(C, T_C(D))\right)^\vee \cong H^0\left(C, \omega_C^{\otimes 2}(-D)\right)$$

because $H^1(C, T_C) \to H^1(C, T_C(D))$ is surjective. □

Remark 4.1.7. *One interpretation of the previous lemma is that $\langle D\rangle$ is the intersection of all the hyperplanes in $\mathbb{P}$ that cut out on C at least the divisor D, which justifies the name* span. *In symbols:*

$$\langle D\rangle = \langle \phi_2(D)\rangle = \bigcap_{\substack{H \in \mathbb{P}^\vee \\ \phi_2^*(H) \geq D}} H$$

We now relate the divisor D and the deformation ξ, starting with a basic definition.

Definition 4.1.8. *We say that the deformation ξ is* supported on D *if and only if $\xi \in \tau_D$.*

Furthermore, if ξ is not supported on any strictly smaller effective divisor $D' < D$, we say that ξ is minimally supported on D.

Remark 4.1.9. *The reason for the name is double. On the one hand, $\xi \in \tau_D$ means that there is a Laurent tail of a meromorphic vector bundle $\eta \in H^0\left(D, T_C(D)_{|D}\right)$ supported on D, such that $\xi = \partial_{T,D}(\eta)$. On the other hand, $\xi \in \tau_D$ if and only if $[\xi] \in \langle D\rangle$, that is, if $[\xi]$ is supported on the span of D in $\mathbb{P}$.*

Remark 4.1.10. *Observe that if D has the smallest degree among the divisors supporting ξ, then ξ is minimally supported on D, but not conversely. Indeed, ξ being minimally supported on a divisor D means that it is not possible to remove some point of D and still support ξ, but there is no reason for D to have minimal degree.*

Remark 4.1.11. *One could equivalently define ξ to be supported on the divisor D if and only if the top row in the following pull-back diagram is split.*

$$\begin{tikzcd}
\xi_D: & 0 \arrow[r] & N^\vee_{C/\mathcal{C}} \arrow[r] \arrow[d, equal] & \mathcal{F}_D \arrow[r] \arrow[d, hook] & \omega_C(-D) \arrow[r] \arrow[l, dashed, bend right=0] \arrow[d, hook] & 0 \\
\xi: & 0 \arrow[r] & N^\vee_{C/\mathcal{C}} \arrow[r] & \Omega^1_{\mathcal{C}|C} \arrow[r] & \omega_C \arrow[r] & 0
\end{tikzcd}$$

Indeed, τ_D is the kernel of the map $H^1(C, T_C) \to H^1(C, T_C(D))$, which can be identified with the pull-back

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C, \mathcal{O}_C) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C(-D), \mathcal{O}_C) \\
& & \\
\xi & \longmapsto & \xi_D
\end{array}$$

However, for non-infinitesimal deformations (specially when the base is a compact curve) the splitting of the analogous pull-back is not always equivalent to the natural extension of Definition 4.1.8 (see Definition 4.3.13, Proposition 4.3.14 and Lemma 4.3.17).

Lemma 4.1.12. *Suppose ξ is supported on D. Then*

1. $H^0(C, \omega_C(-D)) \subseteq \ker \partial_\xi$, *and*
2. $\operatorname{im} \partial_\xi \subseteq \sigma_D$.

Proof. Any 1-form $w \in H^0(C, \omega_C)$ induces (in fact, it is equivalent to) a map

$$w \cup : T_C \longrightarrow \mathcal{O}_C,$$

and its restriction to D, $w_{|D} \in H^0\left(D, \omega_{C|D}\right)$, induces

$$w_{|D} \cup : T_C(D)_{|D} \longrightarrow \mathcal{O}_D(D),$$

such that the diagram

$$\begin{tikzcd}
H^0\left(D, T_C(D)_{|D}\right) \arrow[r, "\partial_{T,D}"] \arrow[d, "w_{|D}\cup"'] & H^1(C, T_C) \arrow[d, "w\cup"] \\
H^0(D, \mathcal{O}_D(D)) \arrow[r, "\partial_{\mathcal{O},D}"] & H^1(C, \mathcal{O}_C)
\end{tikzcd}$$

is commutative.

Now, as we said in Remark 4.1.9, ξ is supported on D if and only if there is $\eta \in H^0\left(D, T_C(D)_{|D}\right)$ such that $\xi = \partial_{T,D}(\eta)$. Taking such an η, we can compute

$$\partial_\xi(w) = w \cup \xi = w \cup (\partial_{T,D}(\eta)) = \partial_{\mathcal{O},D}\left(w_{|D} \cup \eta\right) \tag{4.4}$$

Since $\sigma_D = \operatorname{im} \partial_{\mathcal{O},D}$, the second claim follows immediately.

As for the first claim, $w \in H^0(C, \omega_C(-D))$ if and only if $w_{|D} = 0$, which by (4.4) implies $\partial_\xi(w) = 0$ and the proof is done. Alternatively, the fact that ξ_D is split implies that all the sections of $\omega_C(-D)$ lift to sections of $\Omega^1_{\mathcal{C}|C}$, and hence belong to the kernel of ∂_ξ. □

Remark 4.1.13. *Lemma 4.1.12 implies that if ξ is supported on a divisor D, the value of $\partial_\xi(w)$ depends only on the restriction $w_{|D}$. Furthermore, the proof shows that ∂_ξ can be factored as*

$$H^0(C, \omega_C) \overset{rest_D}{\longrightarrow} H^0\left(D, \omega_{C|D}\right) \overset{\cup\eta}{\longrightarrow} H^0(D, \mathcal{O}_D(D)) \overset{\partial_{\mathcal{O},D}}{\longrightarrow} H^1(C, \mathcal{O}_C),$$

where $\eta \in H^0\left(D, T_C(D)_{|D}\right)$ is a preimage of ξ by $\partial_{T,D}$.

Corollary 4.1.14. *If ξ is supported on D, then* $\operatorname{rk}\xi \leq \deg D - r(D)$.

Proof. It follows immediately from the facts that $\operatorname{im}\partial_\xi \subseteq \sigma_D$ (Lemma 4.1.12) and $\dim \sigma_D = d - r$ (Lemma 4.1.3). □

After these preliminary results, we give the main Theorem of the section, which gives a *lower* bound for the rank of a deformation in terms of a supporting divisor.

Definition 4.1.15. *Given any divisor D, we define its* Clifford index *as*

$$\operatorname{Cliff}(D) = \deg D - 2r(D).$$

Remark 4.1.16. *With the above definition, the Clifford index of the curve C is*

$$\mathrm{Cliff}\,(C) = \min\left\{\mathrm{Cliff}\,(D) \mid h^0\left(\mathcal{O}_C\left(D\right)\right), h^1\left(\mathcal{O}_C\left(D\right)\right) \geq 2\right\}.$$

The following is essentially the main statement of Theorem 2.5 in [17], with a slightly modified proof.

Theorem 4.1.17. *If ξ is minimally supported on D, then*

$$\mathrm{rk}\,\xi \geq \deg D - 2r\left(D\right) = \mathrm{Cliff}\left(D\right).$$

Proof. Let $\eta \in H^0\left(D, T_C\left(D\right)_{|D}\right)$ be such that $\partial_{T,D}\left(\eta\right) = \xi$. From the factorization of Remark 4.1.13, and decomposing the restriction $rest_D$ as

$$H^0\left(C, \omega_C\right) \twoheadrightarrow W = H^0\left(C, \omega_C\right)/H^0\left(C, \omega_C\left(-D\right)\right) \hookrightarrow H^0\left(D, \omega_{C|D}\right),$$

we obtain the following commutative diagram

$$\begin{tikzcd}
H^0\left(\omega_C\right) \arrow[rrr, "\partial_\xi"] \arrow[d] & & & H^1\left(\mathcal{O}_C\right) \\
W \arrow[r, hook] \arrow[rrru, "\psi"] \arrow[rrr, bend right, "\widetilde{\psi}"'] & H^0\left(\omega_{C|D}\right) \arrow[rr, "\cup\eta"] & & H^0\left(\mathcal{O}_D\left(D\right)\right) \arrow[u, "\partial_{\mathcal{O},D}"']
\end{tikzcd}$$

Claim: If ξ is minimally supported on D, then $\cup\,\eta$ is an isomorphism.

Assuming the claim for a moment, the proof finishes as follows. Clearly, $\mathrm{rk}\,\xi = \mathrm{rk}\,\psi = \dim W - \dim\ker\psi$. On the one hand, by Riemann-Roch, $\dim W = d - r$. On the other hand, since $\cup\,\eta$ is injective

$$\ker\psi \cong \ker\left(\partial_{\mathcal{O},D|\,\mathrm{im}\,\widetilde{\psi}}\right) \subseteq \ker\partial_{\mathcal{O},D},$$

and hence $\dim\ker\psi \leq \dim\ker\partial_{\mathcal{O},D} = r$. Summing up, we finally obtain

$$\mathrm{rk}\,\xi \geq (d - r) - r = d - 2r.$$

Proof of the Claim: Let $D = \sum_{i=1}^k n_i p_i$, with $p_i \neq p_j$ for $i \neq j$, and for each i let z_i be a local coordinate centered at p_i. Then $\mathcal{O}_D \cong \bigoplus_{i=1}^k \mathbb{C}\left[z_i\right]/\left(z_i^{n_i}\right)$, and therefore

$$H^0\left(D, T_C\left(D\right)_{|D}\right) \cong \bigoplus_{i=1}^k \frac{\mathbb{C}\left[z_i\right]}{\left(z_i^{n_i}\right)}\left\langle \frac{1}{z_i^{n_i}}\frac{\partial}{\partial z_i}\right\rangle \cong \bigoplus_{i=1}^k\bigoplus_{j=1}^{n_i}\mathbb{C}\left\langle\frac{1}{z_i^j}\frac{\partial}{\partial z_i}\right\rangle.$$

Hence, there exist scalars $\eta_{ij} \in \mathbb{C}$ such that

$$\eta = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \eta_{ij} \frac{1}{z_i^j} \frac{\partial}{\partial z_i},$$

and the minimality of D implies that $\eta_{in_i} \neq 0$ for all $i = 1, \dots, k$. Indeed, ξ is supported on $D' = \sum_{i=1}^{k} m_i p_i < D$ if and only if it can be written as $\xi = \partial_{T,D'}(\eta')$ for some

$$\eta' = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \eta'_{ij} \frac{1}{z_i^j} \frac{\partial}{\partial z_i}.$$

But the map $T_C(D') \xrightarrow{+(D-D')} T_C(D)$ induces a commutative diagram

$$\begin{tikzcd}
H^0\left(D', T_C(D')_{|D'}\right) \cong \bigoplus_{i=1}^{k} \frac{\mathbb{C}[z_i]}{(z_i^{m_i})} \left\langle \frac{1}{z_i^{m_i}} \frac{\partial}{\partial z_i} \right\rangle \arrow[d, "\alpha"'] \arrow[dr, "\partial_{T,D'}"] & \\
& H^1(C, T_C) \\
H^0\left(D, T_C(D)_{|D}\right) \cong \bigoplus_{i=1}^{k} \frac{\mathbb{C}[z_i]}{(z_i^{n_i})} \left\langle \frac{1}{z_i^{n_i}} \frac{\partial}{\partial z_i} \right\rangle \arrow[ur, "\partial_{T,D}"'] &
\end{tikzcd}$$

where

$$\alpha\left(\frac{1}{z_i^{m_i}} \frac{\partial}{\partial z_i}\right) = z_i^{n_i - m_i} \left(\frac{1}{z_i^{n_i}} \frac{\partial}{\partial z_i}\right) = \frac{1}{z_i^{m_i}} \frac{\partial}{\partial z_i}.$$

Therefore, if $\eta_{in_i} = 0$ for some i, then ξ would be supported on $D - p_i$, contradicting the minimality of D.

Let us now compute the expression of $\cup\,\eta$ in these coordinates to show that $\eta_{in_i} \neq 0$ for all i implies that it is an isomorphism. $H^0\left(D, \omega_{C|D}\right)$ and $H^0(D, \mathcal{O}_D(D))$ can be written explicitly as

$$H^0\left(D, \omega_{C|D}\right) \cong \bigoplus_{i=1}^{k} \frac{\mathbb{C}[z_i]}{(z_i^{n_i})} \left\langle dz_{i|D} \right\rangle \cong \bigoplus_{i=1}^{k} \bigoplus_{j=0}^{n_i - 1} \mathbb{C} \left\langle z_i^j dz_{i|D} \right\rangle, \tag{4.5}$$

and

$$H^0(D, \mathcal{O}_D(D)) \cong \bigoplus_{i=1}^{k} \frac{\mathbb{C}[z_i]}{(z_i^{n_i})} \left\langle \frac{1}{z_i^{n_i}} \right\rangle \cong \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{n_i} \mathbb{C} \left\langle \frac{1}{z_i^j} \right\rangle. \tag{4.6}$$

Since

$$\left(z_i^l dz_{i|D}\right) \cup \left(\frac{1}{z_i^m} \frac{\partial}{\partial z_i}\right) = \left(\frac{1}{z_i^{m-l}}\right)$$

(which is zero if $l \geq m$) and $\left(z_i^l dz_{i|D}\right) \cup \left(\frac{1}{z_j^m}\frac{\partial}{\partial z_j}\right) = 0$ for $i \neq j$, we have

$$\left(z_i^l dz_{i|D}\right) \cup \eta = \sum_{j=1}^{n_i - l} \eta_{i,j+l} \frac{1}{z_i^j}.$$

Hence, in the basis of (4.5) and (4.6), $\cup\,\eta$ is given by a matrix of the form

$$\begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_k \end{pmatrix},$$

where each block M_i has the form

$$M_i = \begin{pmatrix} \eta_{i1} & \eta_{i2} & \dots & \eta_{in_i} \\ \vdots & \vdots & \iddots & \vdots \\ \eta_{i,n_i-1} & \eta_{in_i} & \dots & 0 \\ \eta_{in_i} & 0 & \dots & 0 \end{pmatrix}.$$

It is now clear that $\cup\,\eta$ is an isomorphism if and only if $\eta_{in_i} \neq 0$ for every i, and the theorem is proved. □

4.2 Relative Ext sheaves

Since they will play a central role in the next section, we wish to recall the definition and some of the main properties of the relative Ext sheaves, which can be found in the first chapter of [7].

Definition 4.2.1 (Relative ext sheaves, [7] Def. 1.1.1)**.** *Given a morphism of schemes (or more generally, of ringed spaces) $f : X \to Y$, and an $\mathcal{O}_X$-module $\mathcal{F}$, we define $\mathcal{E}\mathrm{xt}^p_f(\mathcal{F}, -)$ as the p-th right derived functor of the left-exact functor $f_*\,\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F}, -)$.*

Example 4.2.2 ([7], Def.-Remark 1.1.2)**.** *Some particular cases:*

1. *If $Y = \operatorname{Spec}\mathbb{C}$ is a point, then $\mathcal{E}\mathrm{xt}^p_f(\mathcal{F}, -) = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, -)$, the global* Ext *functor. If furthermore $\mathcal{F} = \mathcal{O}_X$, $\mathcal{E}\mathrm{xt}^p_f(\mathcal{O}_X, -) = H^p(X, -)$ is the usual sheaf cohomology.*

2. *If f is the identity (hence $Y = X$), then $\mathcal{E}\mathrm{xt}^p_f(\mathcal{F}, -) = \mathcal{E}\mathrm{xt}^p_{\mathcal{O}_X}(\mathcal{F}, -)$ is the usual local* $\mathcal{E}$xt *functor.*

3. *If $\mathcal{F} = \mathcal{O}_X$, then $f_* \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X, -) = f_*$ is the usual push-forward functor, so that $\mathcal{E}\mathrm{xt}^p_f(\mathcal{O}_X, -) = R^p f_*$ are the higher-direct image functors.*

Theorem 4.2.3. *Some properties:*

1. *([7] Th. 1.1.3) For any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, $\mathcal{E}\mathrm{xt}_f(\mathcal{F}, \mathcal{G})$ is the sheaf associated to the presheaf*

$$U \mapsto \mathrm{Ext}^p_{\mathcal{O}_{f^{-1}(U)}}\left(\mathcal{F}_{|f^{-1}(U)}, \mathcal{G}_{|f^{-1}(U)}\right).$$

   In particular, for any open subset $W \subseteq Y$,

$$\mathcal{E}\mathrm{xt}^p_f(\mathcal{F}, \mathcal{G})_{|f^{-1}(W)} \cong \mathcal{E}\mathrm{xt}^p_f\left(\mathcal{F}_{|f^{-1}(W)}, \mathcal{G}_{|f^{-1}(W)}\right).$$

2. *([7] Th. 1.1.4) If $\mathcal{L}$ and $\mathcal{N}$ are locally free sheaves of finite rank on X and Y, respectively, then*

$$\begin{aligned}\mathcal{E}\mathrm{xt}^p_f(\mathcal{F} \otimes \mathcal{L}, - \otimes f^*\mathcal{N}) \cong \mathcal{E}\mathrm{xt}^p_f(\mathcal{F}, - \otimes \mathcal{L}^\vee \otimes f^*\mathcal{N}) \cong \\ \cong \mathcal{E}\mathrm{xt}^p_f(\mathcal{F}, - \otimes \mathcal{L}^\vee) \otimes \mathcal{N}.\end{aligned}$$

3. *([7] Th. 1.1.5) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, and $\mathcal{G}$ is another $\mathcal{O}_X$-module, then there is a long exact sequence*

$$\begin{aligned}\cdots \longrightarrow \mathcal{E}\mathrm{xt}^{p-1}_f(\mathcal{F}', \mathcal{G}) \longrightarrow \\ \longrightarrow \mathcal{E}\mathrm{xt}^p_f(\mathcal{F}'', \mathcal{G}) \longrightarrow \mathcal{E}\mathrm{xt}^p_f(\mathcal{F}, \mathcal{G}) \longrightarrow \mathcal{E}\mathrm{xt}^p_f(\mathcal{F}', \mathcal{G}) \longrightarrow \\ \longrightarrow \mathcal{E}\mathrm{xt}^{p+1}_f(\mathcal{F}'', \mathcal{G}) \longrightarrow \cdots\end{aligned}$$

4. *(Local to global spectral sequence, [7] Th. 1.2.1) Suppose $g : Y \to Z$ is another morphism, and denote $h = g \circ f$. For any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ there is a spectral sequence*

$$E_2^{p,q} = R^p g_* \mathcal{E}\mathrm{xt}^q_f(\mathcal{F}, \mathcal{G}) \Rightarrow \mathcal{E}\mathrm{xt}^{p+q}_h(\mathcal{F}, \mathcal{G}).$$

5. *(Coherence, [7] Th. 1.3.1) If f is projective and $\mathcal{F}$, $\mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then $\mathcal{E}\mathrm{xt}^p_f(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{O}_Y$-module.*

4.3 Global constructions

In this last section we consider a non-isotrivial fibration of a smooth surface over a smooth (not necessarily compact) curve, considered as a one-dimensional family of curves. We extend some of the constructions of Section 4.1 in order to obtain geometric tools to study the family, and more explicitly, to develop the notion of supporting divisor (or subscheme, more generally). As mentioned in the previous section, relative Ext sheaves are an essential tool, as well as some results of Chapter 3 relating the sheaf of relative differentials and the relative dualizing sheaf.

Let $f : S \to B$ be a non-isotrivial fibration of a smooth surface S over a smooth curve B. For any $b \in B$, let $C_b = S \times_B \operatorname{Spec} \mathbb{C}(b)$ be the fibre over b. Denote by $B^o \subseteq B$ the open set of regular values, so that C_b is smooth if and only if $b \in B^o$, and denote also by $S^o = f^{-1}(B^o)$. We will assume that the generic (smooth) fibres have genus $g \geq 2$.

For every smooth fibre $C = C_b$, the fibration f induces an infinitesimal deformation, whose Kodaira-Spencer class $\xi_b \in H^1(C, T_C) \otimes T^\vee_{B,b}$ is the extension class of

$$0 \longrightarrow N^\vee_{C/S} = \mathcal{O}_C \otimes T^\vee_{B,b} \longrightarrow \Omega^1_{S|C} \longrightarrow \omega_C \longrightarrow 0,$$

obtained by restricting the sequence

$$\xi : \quad 0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/B} \longrightarrow 0 \tag{4.7}$$

defining the *sheaf of relative differentials* $\Omega^1_{S/B}$.

Since the fibration f is not isotrivial, $\xi_b \neq 0$ for general $b \in B^o$ and hence we can consider the point $[\xi_b] \in \mathbb{P}_b := \mathbb{P}(H^1(C_b, T_{C_b}))$. Furthermore, if $D \subset S$ is any divisor, we can also ask whether ξ_b is supported on $D_b = D_{|C_b}$, and if the answer is positive, what consequences for the fibration f does it have.

The aim of this section is to *glue* the constructions of Section 4.1 to the case of a smooth surface fibred over a curve (that is, a one-dimensional family of curves), extending them also to the singular fibres. Some of the ideas used here also appear in [37].

The first object to globalize is the ambient space: the vector space $H^1(C, T_C) \cong H^0\left(C, \omega_C^{\otimes 2}\right)^\vee$ and its projectivization.

Definition 4.3.1. *Let $\mathcal{E}$ be the sheaf on B defined as*

$$\mathcal{E} = \mathcal{E}\mathrm{xt}^1_f\left(\Omega^1_{S/B}, f^*\omega_B\right) \cong \mathcal{E}\mathrm{xt}^1_f\left(\Omega^1_{S/B}, \mathcal{O}_S\right) \otimes \omega_B,$$

and let

$$\mathbb{P} = \operatorname{Proj}_{\mathcal{O}_B}\left(\operatorname{Sym}^* \mathcal{E}^\vee\right)$$

be the associated projective bundle, with projection $\pi : \mathbb{P} \to B$.

Note that $\mathcal{E}^\vee$ is torsion free over a smooth curve, hence it is locally free and $\mathbb{P}$ is actually a projective bundle.

Recall that the *relative tangent sheaf* is defined as

$$T_{S/B} = \left(\Omega^1_{S/B}\right)^\vee = \mathcal{H}\mathrm{om}_{\mathcal{O}_S}\left(\Omega^1_{S/B}, \mathcal{O}_S\right).$$

Lemma 4.3.2. *There is an injection*

$$R^1 f_* T_{S/B} \otimes \omega_B \hookrightarrow \mathcal{E}$$

which is an isomorphism over B^o. *In particular, for any regular value* $b \in B^o$ *there is a natural isomorphism*

$$\mathcal{E} \otimes \mathbb{C}(b) \cong H^1\left(C_b, T_{C_b}\right) \otimes T^\vee_{B,b}.$$

Proof. The injection is obtained directly from the local-global spectral sequence

$$R^p f_* \,\mathcal{E}\mathrm{xt}^q_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right) \Longrightarrow \mathcal{E}\mathrm{xt}^{p+q}_f\left(\Omega^1_{S/B}, f^*\omega_B\right),$$

since the beginning of the corresponding five-term exact sequence is

$$R^1 f_*\left(\mathcal{H}\mathrm{om}_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right)\right) \hookrightarrow \mathcal{E}\mathrm{xt}^1_f\left(\Omega^1_{S/B}, f^*\omega_B\right),$$

and clearly

$$R^1 f_*\left(\mathcal{H}\mathrm{om}_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right)\right) \cong R^1 f_*\left(T_{S/B} \otimes f^*\omega_B\right) \cong R^1 f_* T_{S/B} \otimes \omega_B$$

by the projection formula.

As for the statement about the regular values, Lemma 3.2.6 implies that $\Omega^1_{S/B|S^o} \cong \omega_{S/B|S^o}$ and $\left(\Omega^1_{S/B|S^o}\right)^\vee = T_{S/B|S^o}$ are both locally free. Therefore, using Theorem 4.2.3 we get

$$\begin{aligned}\mathcal{E}\mathrm{xt}^1_f\left(\Omega^1_{S/B|S^o}, \mathcal{O}_{S^o}\right) \otimes \omega_{B^o} \cong \mathcal{E}\mathrm{xt}^1_f\left(\mathcal{O}_{S^o}, T_{S/B|S^o}\right) \otimes \omega_{B^o} \cong \\ \cong \left(R^1 f_* T_{S/B|S^o}\right) \otimes \omega_{B^o} = \left(R^1 f_* T_{S/B} \otimes \omega_B\right)_{|B^o}.\end{aligned}$$

Finally, $T_{S/B|C_b} = T_{C_b}$ for any smooth fibre, and since the relative dimension of f is 1, the base-change map $\mathcal{E} \otimes \mathbb{C}(b) \to H^1\left(C_b, T_{C_b}\right) \otimes T^\vee_{B,b}$ is an isomorphism for every $b \in B^o$. □

By the previous Lemma, the fibres of $\mathbb{P}$ over the regular values are isomorphic to the bicanonical spaces of the fibres, as wanted. The twisting by ω_B (or $T^\vee_{B,b}$) may seem strange, but it is indeed absolutely natural, as mentioned at the beginning of Section 4.1. Moreover, although we could forget about the ω_B because of the isomorphism $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E} \otimes T_B)$, it is convenient to keep it in order to simplify the next construction.

We define now a morphism $\gamma : B \to \mathbb{P}$ (in fact, a section of $\pi : \mathbb{P} \to B$), which maps every regular value $b \in B^o$ to $[\xi_b]$. Recall that the fibration $f : S \to B$ defines an element $\xi \in \operatorname{Ext}^1_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right)$ (the extension class of (4.7)). Now, the spectral sequence

$$E_2^{p,q} = H^p\left(B, \mathcal{E}\mathrm{xt}^q_f\left(\Omega^1_{S/B}, f^*\omega_B\right)\right) \Longrightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right) \tag{4.8}$$

gives the map

$$\rho : \operatorname{Ext}^1_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right) \longrightarrow H^0\left(B, \mathcal{E}\mathrm{xt}^1_f\left(\Omega^1_{S/B}, f^*\omega_B\right)\right) = H^0\left(B, \mathcal{E}\right).$$

Lemma 4.3.3. *The map ρ is an isomorphism.*

Proof. By the five-term exact sequence associated to the spectral sequence (4.8), we have

$$\ker \rho = H^1\left(B, f_* \mathcal{H}\mathrm{om}_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right)\right)$$

and

$$\operatorname{coker} \rho \subseteq H^2\left(B, f_* \mathcal{H}\mathrm{om}_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right)\right).$$

Since $\dim B = 1$, it is clear that $\operatorname{coker}\rho = 0$. It remains to show that $\ker\rho = 0$, and we will directly show that $f_* \mathcal{H}\mathrm{om}_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right) = 0$. Indeed,

$$f_* \mathcal{H}\mathrm{om}_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right) \cong \left(f_* T_{S/B}\right) \otimes \omega_B.$$

Since $T_{S/B}$ is torsion-free, so is $f_*T_{S/B}$, and since the base B is a curve, $f_*T_{S/B}$ is a vector bundle of rank $h^0\left(C_b, T_{S/B|C_b}\right)$ for general $b \in B$. In particular, if $b \in B^o$ is a regular value, then $T_{S/B|C_b} \cong T_{C_b}$, which has no sections because we are assuming $g\left(C_b\right) \geq 2$. Therefore, $f_*T_{S/B} = 0$ and the proof is done. $\square$

Because of Lemma 4.3.3, we can identify ξ with a section $\rho(\xi)$ of $\mathcal{E}$, which by construction maps any regular value $b \in B^o$ to the Kodaira-Spencer class ξ_b of the deformation of C_b. Since we assumed the fibration f to be non-isotrivial, $\rho(\xi)$ does not vanish identically and induces the wanted section $\gamma : B \to \mathbb{P}$.

Remark 4.3.4. *We can construct $\gamma : B \to \mathbb{P}$ more formally as follows. Consider the evaluation of $\rho(\xi)$*

$$\mathbb{C}\left\langle \rho(\xi)\right\rangle \otimes \mathcal{O}_B \cong \mathcal{O}_B \longrightarrow \mathcal{E},$$

and let $\mathcal{M} \subseteq \mathcal{O}_B$ be the image of its dual $\mathcal{E}^\vee$. According to [22], Proposition II.7.12, the surjection $\mathcal{E}^\vee \twoheadrightarrow \mathcal{M}$ corresponds to a map $\gamma : B \to \mathbb{P}$ such that $\gamma^* \mathcal{O}_{\mathbb{P}}(1) = \mathcal{M}$, and it is easy to see that it is the section we want.

The next step is to construct (up to blowing-up some points of S contained in the singular fibres) a B-morphism $\phi : S \to \mathbb{P}$ (i.e. such that $\pi \circ \phi = f$) inducing the bicanonical map on the smooth fibres. Following Proposition II.7.12 in [22] as in the previous remark, it must correspond to a line bundle $\mathcal{L}_\phi$ on S and a map $f^*\mathcal{E}^\vee \to \mathcal{L}_\phi$, surjective at least on $S^o = f^{-1}(B^o)$.

Lemma 4.3.5. *There is a natural morphism of sheaves on S*

$$f^*\mathcal{E}^\vee \longrightarrow \left(\Omega^1_{S/B}\right)^{\vee\vee} \otimes \omega_{S/B} \otimes f^*T_B,$$

which is surjective on S^o and induces the bicanonical map on any smooth fibre.

Proof. We have seen above (Lemma 4.3.2) that there is an injective map of sheaves $R^1 f_* T_{S/B} \otimes \omega_B \hookrightarrow \mathcal{E}$. Dualizing we get a map

$$\mathcal{E}^\vee \longrightarrow \left(R^1 f_* T_{S/B} \otimes \omega_B\right)^\vee \cong f_* \left(\left(\Omega^1_{S/B}\right)^{\vee\vee} \otimes \omega_{S/B}\right) \otimes T_B, \tag{4.9}$$

where in the last isomorphism we have used relative duality together with the fact (Lemma 3.2.8) that $\omega_{S/B}$ is the relative dualizing sheaf. Moreover, Lemma 4.3.2 also implies that the map (4.9) is an isomorphism on B^o. Pulling back this map to S and composing with the "relative evaluation map"

$$f^*\left(f_*\left(\left(\Omega^1_{S/B}\right)^{\vee\vee} \otimes \omega_{S/B}\right) \otimes T_B\right) \longrightarrow \left(\Omega^1_{S/B}\right)^{\vee\vee} \otimes \omega_{S/B} \otimes f^*T_B,$$

we obtain the map we wanted:

$$f^*\mathcal{E}^\vee \longrightarrow \mathcal{L}_\phi := \left(\Omega^1_{S/B}\right)^{\vee\vee} \otimes \omega_{S/B} \otimes f^*T_B. \tag{4.10}$$

Since $\Omega^1_{S/B|S^o} \cong \omega_{S/B|S^o}$ (Lemma 3.2.6), it holds $\Omega^1_{S/B|C_b} \cong \omega_{S/B|C_b} \cong \omega_{C_b}$, hence it is immediate that $\mathcal{L}_{\phi|C_b} \cong \omega_{C_b}^{\otimes 2} \otimes T_{B,b}$. Finallly, since the bicanonical sheaf of a smooth fibre is globally generated, the map (4.10) is surjective on S^o, so that the induced rational map ϕ is defined on every smooth fibre and restricts to its bicanonical embedding. $\square$

Remark 4.3.6. *Note that in this construction it has appeared an "extra" f^*T_B. This happens because of the choice of the sheaf $\mathcal{E}$ instead of*

$\mathcal{E}\mathrm{xt}_f^1\left(\Omega^1_{S/B}, \mathcal{O}_S\right)$, *which has the same associated projective bundle, but with different tautological sheaf* $\mathcal{O}(1)$. *The only important consequence of this extra factor is that for any regular value* $b \in B^o$,

$$\phi_b^* \mathcal{O}_{\mathbb{P}_b}(1) = \omega_{C_b}^{\otimes 2} \otimes T_{B,b}$$

instead of simply the bicanonical sheaf (where $\mathbb{P}_b = \pi^{-1}(b)$ *is the fibre of* $\mathbb{P}$ *over* b, *and* $\phi_b = \phi_{|C_b} : C_b \to \mathbb{P}_b$*).*

Finally, we present a way to globalize the span of a divisor on a fibre. Instead of considering only divisors on the surface, we will start from a more general point of view, taking into account any closed subscheme $\Gamma \subset S$. Recall the natural map $\alpha : \Omega^1_{S/B} \to \omega_{S/B}$ of Lemma 3.2.6.

Definition 4.3.7. *For any closed subscheme* $\Gamma \subset S$ *with ideal sheaf* $\mathcal{I}_\Gamma$, *define*

$$\mathcal{L}_\Gamma = \alpha^{-1}\left(\omega_{S/B} \otimes \mathcal{I}_\Gamma\right) = \ker\left(\Omega^1_{S/B} \longrightarrow \omega_{S/B} \longrightarrow \omega_{S/B|\Gamma}\right),$$

and

$$\mathcal{E}_\Gamma = \mathcal{E}\mathrm{xt}_f^1\left(\mathcal{L}_\Gamma, f^*\omega_B\right).$$

The inclusion $\mathcal{L}_\Gamma \subseteq \Omega^1_{S/B}$ induces maps of sheaves

$$\mathcal{E} \longrightarrow \mathcal{E}_\Gamma \quad \text{and its dual} \quad \mathcal{E}_\Gamma^\vee \longrightarrow \mathcal{E}^\vee. \tag{4.11}$$

Lemma 4.3.8. *The map* $\mathcal{E}_\Gamma^\vee \to \mathcal{E}^\vee$ *is injective. Moreover, if* $\Gamma_2 \subseteq \Gamma_1$ *are two nested closed subschemes, then the induced map* $\mathcal{E}_{\Gamma_1}^\vee \to \mathcal{E}_{\Gamma_2}^\vee$ *is injective.*

Proof. By replacing B by some open subset (removing the singular fibres and those containing isolated or embedded points of Γ), we may assume that $\Omega^1_{S/B} = \omega_{S/B}$ and that $\Gamma = \Gamma_{div}$ is an effective divisor. Then the map $\mathcal{E}_\Gamma^\vee \to \mathcal{E}^\vee$ is simply the natural map

$$\left(f_*\omega_{S/B}^{\otimes 2}(-\Gamma)\right) \otimes T_B \longrightarrow \left(f_*\omega_{S/B}^{\otimes 2}\right) \otimes T_B,$$

which is clearly injective.

Going back to the original (complete) fibration, the kernel of $\mathcal{E}_\Gamma^\vee \to \mathcal{E}^\vee$ must be supported on a closed subset, hence it is a torsion sheaf. But $\mathcal{E}_\Gamma^\vee$ is torsion-free, so the kernel is zero and the claim is proved.

As for the second asertion, just note that the map $\mathcal{E}_{\Gamma_1}^\vee \hookrightarrow \mathcal{E}^\vee$ factors through $\mathcal{E}_{\Gamma_1}^\vee \to \mathcal{E}_{\Gamma_2}^\vee$. $\square$

Using the sheaf $\mathcal{E}_\Gamma$ we construct now a subvariety $\mathbb{P}_\Gamma \subseteq \mathbb{P}$ with the property that $\rho(\xi)$ belongs to the kernel of $H^0(\mathcal{E}) \to H^0(\mathcal{E}_\mathcal{L})$ if and only if the image of γ is contained in $\mathbb{P}_\Gamma$. In this way, we generalize the notion of *a deformation being supported on a divisor* on a smooth fibre.

Composing the pull-back of $\mathcal{E}_\Gamma^\vee \to \mathcal{E}^\vee$ by π with the natural surjection $\pi^*\mathcal{E}^\vee \to \mathcal{O}_\mathbb{P}(1)$ we obtain a map

$$\mu_\Gamma(1) : \pi^*\mathcal{E}_\Gamma^\vee \longrightarrow \mathcal{O}_\mathbb{P}(1).$$

Definition 4.3.9. *We define $\widetilde{\mathbb{P}_\Gamma} \subseteq \mathbb{P}$ as the closed subscheme whose sheaf of ideals $\mathcal{J}_\Gamma$ is the image of*

$$\mu_\Gamma : \pi^*\mathcal{E}_\Gamma^\vee \otimes \mathcal{O}_\mathbb{P}(-1) \longrightarrow \mathcal{O}_\mathbb{P}.$$

The subscheme $\widetilde{\mathbb{P}_\Gamma}$ is a first generalization of the span of a divisor on a fibre. However, it is not fine enough for us, since it may contain several irreducible components which do not dominate B and hence cannot contain the curve of deformations $\gamma(B)$.

Lemma 4.3.10. *$\widetilde{\mathbb{P}_\Gamma}$ contains a unique irreducible component $\mathbb{P}_\Gamma$ dominating B. The fibre of $\mathbb{P}_\Gamma$ over a general point $b \in B^o$ is precisely the span of $\Gamma_{|C_b}$ (in the sense of Definition 4.1.5). Moreover, if $\Gamma' \subseteq S$ is another subscheme with the same components as Γ dominating B, then $\mathbb{P}_{\Gamma'} = \mathbb{P}_\Gamma$.*

Proof. Let $D \subset S$ be the union of the (divisorial) components of Γ that dominate B, and let $U \subseteq B^o$ be the open set such that $\Gamma_{|f^{-1}(U)} = D_{|f^{-1}(U)}$ (the complement in B^o of the image of the components of Γ not dominating B). Then, as we have shown in the proof of Lemma 4.3.8,

$$\mathcal{L}_{\Gamma|f^{-1}(U)} \cong \left(\omega_{S/B}(-D)\right)_{|f^{-1}(U)}.$$

and

$$\mathcal{E}_{\Gamma|U}^\vee \cong f_*\left(\omega_{S/B}^{\otimes 2}(-D)\right)_{|U} \otimes T_U.$$

Let $V \subseteq U$ be the open set where the function $b \mapsto h^0\left(C_b, \omega_{C_b}^{\otimes 2}(-D_b)\right)$ is constant. For any $b \in V$, the base-change map gives an isomorphism

$$\mathcal{E}_\Gamma^\vee \otimes \mathbb{C}(b) \xrightarrow{\cong} H^0\left(C_b, \omega_{C_b}^{\otimes 2}(-D_b)\right) \otimes T_{B,b}.$$

Therefore, the map μ_Γ restricts to

$$\mu_{\Gamma|\mathbb{P}_b} : H^0\left(C_b, \omega_{C_b}^{\otimes 2}(-D_b)\right) \otimes T_{B,b} \otimes \mathcal{O}_{\mathbb{P}_b}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}_b},$$

which coincides with the map in Lemma 4.1.6 (the twisting by $T_{B,b}$ is explained in Remark 4.3.6). This shows that the fibres of $\widetilde{\mathbb{P}_\Gamma}$ over any $b \in V$ are the spans of $D_b = \Gamma_b$ and all of them have the same dimension. Hence $\widetilde{\mathbb{P}_\Gamma} \cap \pi^{-1}(V)$ is irreducible, and we define $\mathbb{P}_\Gamma$ to be its closure in $\mathbb{P}$.

The last assertion follows because $\mathbb{P}_\Gamma$ is determined by the components of Γ dominating B. $\square$

Definition 4.3.11 (Span of a subscheme)**.** *Given a subscheme $\Gamma \subset S$, we define its* span *as the subvariety $\mathbb{P}_\Gamma$ of Lemma 4.3.10.*

Remark 4.3.12. *1. By Lemma 4.3.10, the span of a subscheme is determined only by its divisorial components not contained in fibres.*

2. Because of this reason, the span of Γ may not contain its image $\phi(\Gamma)$ by the relative bicanonical map.

Definition 4.3.13. *Analogously to the case of an infinitesimal deformation, we say that the extension $\xi \in \operatorname{Ext}^1_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right) \cong H^0(B, \mathcal{E})$, or also the fibration f,* is supported on *a subscheme $\Gamma \subset S$ if it is mapped to zero by the map*

$$H^0(B, \mathcal{E}) \longrightarrow H^0(B, \mathcal{E}_\Gamma)$$

associated to (4.11).

As in the infinitesimal case, being supported on a subscheme Γ is related to its span in the bicanonical embedding.

Proposition 4.3.14. *The deformation ξ is supported on Γ if and only if the image of $\gamma : B \to \mathbb{P}$ lies in $\mathbb{P}_\Gamma$.*

Proof. Since $\mathbb{P}_\Gamma$ is the only component of $\widetilde{\mathbb{P}_\Gamma}$ dominating B, and $\gamma(B)$ dominates B, the statement is equivalent to prove that ξ is supported on Γ if and only if $\gamma(B) \subseteq \widetilde{\mathbb{P}_\Gamma}$. To this aim, consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^1_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right) \arrow[r, "\iota^*"] \arrow[d, "\rho"'] & \operatorname{Ext}^1_{\mathcal{O}_S}\left(\mathcal{L}_\Gamma, f^*\omega_B\right) \arrow[d, "\rho_\Gamma"] \\
H^0(B, \mathcal{E}) \arrow[r, "\widetilde{\iota^*}"] & H^0(B, \mathcal{E}_\Gamma)
\end{tikzcd}$$

where the vertical maps are given by the corresponding local-global spectral sequences, and the horizontal ones are induced by the inclusion of sheaves $\iota : \mathcal{L}_\Gamma \hookrightarrow \Omega^1_{S/B}$.

We want to show that $\gamma(B) \subseteq \widetilde{\mathbb{P}_\Gamma}$ if and only if the section

$$\widetilde{\xi_\Gamma} := \rho_\Gamma(\iota^*(\xi)) = \widetilde{\iota^*}(\rho(\xi)) \in H^0\left(B, \mathcal{E}_\Gamma\right)$$

is zero. Recall that the morphism $\gamma : B \to \mathbb{P}$ was defined by the evaluation of $\rho\left(\xi\right)$, so that (see Remark 4.3.4)

$$\gamma^*\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{M} = \operatorname{im}\left(\mathcal{E}^\vee \longrightarrow \mathcal{O}_B\right) \subseteq \mathcal{O}_B.$$

Recall also that the ideal sheaf $\mathcal{J}_\Gamma$ of $\widetilde{\mathbb{P}_\Gamma}$ is the image of the composition

$$\pi^*\mathcal{E}_\Gamma^\vee(-1) \longrightarrow \pi^*\mathcal{E}^\vee(-1) \longrightarrow \mathcal{O}_{\mathbb{P}},$$

so $\gamma^*\mathcal{J}_\Gamma(1)$ is generated by the image of the composition

$$\mathcal{E}_\Gamma^\vee \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{M} \hookrightarrow \mathcal{O}_B.$$

But this composition is dual to the composition of the evaluation of $\rho\left(\xi\right)$ and the map $\mathcal{E} \to \mathcal{E}_\Gamma$, which is precisely the evaluation of $\widetilde{\xi_\Gamma}$.

Therefore, $\widetilde{\xi_\Gamma} = 0$ if and only if the map $\mathcal{E}_\Gamma^\vee \to \mathcal{M}$ vanishes. By the previous discussion, this is equivalent to the vanishing of $\gamma^*\mathcal{J}_\Gamma \to \mathcal{O}_B$, which means precisely that the image of γ is (schematically) contained in $\widetilde{\mathbb{P}_\Gamma}$, finishing the proof. □

Corollary 4.3.15. *If $\Gamma, \Gamma' \subset S$ are two subschemes with exactly the same components dominating B, then ξ is supported on Γ if and only if it is supported on Γ'.*

Proof. It is a consequence of Proposition 4.3.14, since $\mathbb{P}_\Gamma = \mathbb{P}_{\Gamma'}$ because of Lemma 4.3.10. □

We will also need to take care of changes of base.

Lemma 4.3.16. *Let $p : B' \to B$ be a finite morphism, let S' be the minimal desingularization of $S \times_B B'$, and consider the induced commutative diagram*

$$\begin{tikzcd} S' \arrow[r, "p'"] \arrow[d, "f'"'] & S \arrow[d, "f"] \\ B' \arrow[r, "p"] & B \end{tikzcd}$$

Suppose that ξ is supported on a divisor D. Then the extension class ξ' corresponding to the fibration f' is supported on $D' = \left(p'\right)^* D$.

Proof. Denote by $\mathcal{E}', \mathbb{P}'$ and $\mathbb{P}'_{D'}$ the obvious analogues of $\mathcal{E}, \mathbb{P}$ and $\mathbb{P}_D$ attached to the fibration f' and the divisor D'.

Let $b' \in B'$ be a point such that p is not ramified at b'. Then there is an analytic open neighbourhood $U' \subseteq B'$ of b' such that $p_{|U'}$ gives an isomorphism with an open subset $U \subset B$. Furthermore, the restrictions of f' and f to $(f')^{-1}(U')$ and $f^{-1}(U)$ respectively are naturally isomorphic, hence

$$\xi'_{|U'} = p^* \xi_{|U}$$

as sections of $\mathcal{E}'_{|U'} \overset{p^*}{\cong} \mathcal{E}_{|U}$, and also $\mathbb{P}'_{D'|U'} \overset{p^*}{\cong} \mathbb{P}_{D|U}$. Since ξ is supported on D, it follows that $\xi'_{|U'}$ is supported on $D'_{|U'}$. To finish, note that the union of the U' (as b' ranges over the points where p is not ramified) is a Zariski open subset of B'. □

Proposition 4.3.14 shows that Definition 4.3.13 is the correct geometric analogue to the infinitesimal one (Definition 4.1.8). However, ξ being supported on Γ is not in general equivalent to the splitting of the pull-back sequence

$$\xi_\Gamma : 0 \longrightarrow f^*\omega_B \longrightarrow \mathcal{F}_\Gamma \longrightarrow \mathcal{L}_\Gamma \longrightarrow 0. \tag{4.12}$$

Therefore, the global analogue of Remark 4.1.11 is not equivalent to Definition 4.3.13 for a general subscheme Γ. Fortunately, the two notions are equivalent in some cases, as the following Lemma shows, and these cases are more than enough for our purposes.

Lemma 4.3.17. *If $\Gamma_{div} \cdot C_b < 2g-2$ for some fibre C_b, then ξ is supported on Γ if and only if the pull-back sequence (4.12) splits.*

Proof. In this case, the beginning of the five-term exact sequence associated to the local to global spectral sequence is

$$0 \longrightarrow H^1(B, f_* \mathcal{H}\mathrm{om}(\mathcal{L}_\Gamma, f^*\omega_B)) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{L}_\Gamma, f^*\omega_B) \xrightarrow{\rho_\Gamma} H^0(B, \mathcal{E}_\Gamma)$$

By definition, ξ is supported on Γ if and only if $\xi_\Gamma \in \mathrm{Ext}^1_{\mathcal{O}_S}(\mathcal{L}, f^*\omega_B)$ belongs to the kernel of ρ_Γ. Therefore, it is enough to show that

$$H^1(B, f_* \mathcal{H}\mathrm{om}(\mathcal{L}_\Gamma, f^*\omega_B)) = 0.$$

Indeed, analogously to the proof of Lemma 4.3.3, we will show that the sheaf $f_* \mathcal{H}\mathrm{om}(\mathcal{L}_\Gamma, f^*\omega_B)$ is zero.

First of all, $f_* \mathcal{H}\mathrm{om}(\mathcal{L}_\Gamma, f^*\omega_B) = (f_*\mathcal{L}_\Gamma^\vee) \otimes \omega_B$, so it will be enough to prove that $f_*\mathcal{L}_\Gamma^\vee = 0$. The dual $\mathcal{L}_\Gamma^\vee$ is torsion-free, so its direct image $f_*\mathcal{L}_\Gamma^\vee$ is also torsion-free, hence it is a vector bundle. Therefore, we will be done if we see that $(f_*\mathcal{L}_\Gamma^\vee) \otimes \mathbb{C}(b) = 0$ for general b.

As in the proof of Lemma 4.3.10, for a general smooth fibre C_b we have $\mathcal{L}_{\Gamma|C_b} = \left(\omega_{S/B} \otimes \mathcal{I}_\Gamma\right)_{|C_b} = \omega_{C_b}(-\Gamma_{|C_b})$ and

$$(f_*\mathcal{L}_\Gamma^\vee) \otimes \mathbb{C}(b) = H^0\left(C_b, T_{C_b}(\Gamma_{|C_b})\right).$$

To finish, the second term vanishes because the hypothesis $\Gamma \cdot C_b < 2g-2$ is equivalent to $\deg\left(T_{C_b}(\Gamma_{|C_b})\right) < 0$. □

Remark 4.3.18. *It is very likely that almost all the constructions and results of this section (except the relative bicanonical map of Lemma 4.3.5) can be generalized with minor changes to one-dimensional families of varieties of any dimension.*

Chapter Five 5

Adjoint images

The main topic of this chapter are *adjoint images*, which have proved to be a useful tool to study both infinitesimal and local deformations of irregular varieties. They were introduced in the study of curves by Collino and Pirola in [10], and then extended to higher-dimensional varieties by Pirola an Zucconi in [34]. The aim of this chapter is to construct a further generalization of adjoint images to the case of arbitrary (one-dimensional) families of irregular varieties, which in particular allows us to deal with compact surfaces fibred over curves.

The first section of the chapter is devoted to introduce the main definitions and known results about adjoint images, which will be used in the sequel. In the second section we extend a construction made in section 1.3 of [10], which gives us the existence of subspaces with vanishing adjoint class under suitable numerical hypothesis. Most of the definitions and results appearing therein are valid for infinitesimal deformations of irregular varieties of any dimension (sometimes the restriction that they have no higher-irrational pencils is needed), but we have restricted ourselves to the case of curves because it is our primary interest. Finally, the third (and last) section deals with the global setting, where the base of a family is a compact curve B.

5.1 Adjoint images and infinitesimal deformations

In this section we introduce the theory of adjoint images. Although in the rest of the chapter we basically deal with curves, some of the forthcoming constructions and results also work for higher dimensions. Hence, we have choosen to present adjoint images in their most general form, for varieties of arbitrary dimension. We start recalling the basic definitions (generalizing also Definition 4.1.8 to higher-dimensional varieties) and the two main results: the Adjoint Theorem (Theorem 5.1.4) and the Volumetric Theorem (Theorem 5.1.5). The Adjoint Theorem was first proven by Collino and Pirola for curves ([10] Th. 1.1.8), and then it was generalized to arbitrary dimensions by Pirola and Zucconi ([34] Th. 1.5.1). The Volumetric Theorem concerns local families of varieties (it is not valid for infinitesimal deformations), and was proven by Pirola and Zucconi ([34] Th. 1.5.3).

Let X be a smooth projective variety of dimension d. For any integer $k = 1, \dots, d$ we consider the map

$$\psi_k : \bigwedge^k H^0\left(X, \Omega^1_X\right) \longrightarrow H^0\left(X, \Omega^k_X\right)$$

given by wedge product (for $k = 1$ it is simply the identity). Given a linear subspace $W \subseteq H^0\left(X, \Omega^1_X\right)$, we define

$$W^k = \psi_k\left(\bigwedge^k W\right) \subseteq H^0\left(X, \Omega^k_X\right).$$

In particular, for $k = d$, $W^d \subseteq H^0\left(X, \omega_X\right)$. Hence, if $W^d \neq 0$, it induces a linear subsystem $\left|W^d\right| \subseteq |\omega_X|$ of the canonical linear series. In this case, denote by D_W the common components to all divisors in $\left|W^d\right|$, that is, the base divisor of the linear series.

Consider now an infinitesimal deformation $\mathcal{X} \to \Delta$ of X (where as usual $\Delta = \operatorname{Spec} \mathbb{C}\left[\epsilon\right] / \left(\epsilon^2\right)$ is the spectrum of the dual numbers). As in the case of curves, considered in the previous chapter, the deformation is equivalent to the class $\xi \in \operatorname{Ext}^1_{\mathcal{O}_X}\left(\Omega^1_X, \mathcal{O}_X \otimes T^\vee_{\Delta,0}\right) \cong H^1\left(X, T_X\right) \otimes T^\vee_{\Delta,0} \cong H^1\left(X, T_X\right)$ of the extension

$$0 \longrightarrow N^\vee_{X/\mathcal{X}} = \mathcal{O}_X \otimes T^\vee_{\Delta,0} \cong \mathcal{O}_X \longrightarrow \Omega^1_{\mathcal{X}|X} \longrightarrow \Omega^1_X \longrightarrow 0.$$

The corresponding connecting homomorphism

$$\partial_\xi = \cup \xi : H^0\left(X, \Omega^1_X\right) \longrightarrow H^1\left(X, \mathcal{O}_X\right) \otimes T^\vee_{\Delta,0}$$

is given by cup-product with ξ. Denote by

$$K_\xi = \ker \partial_\xi = \operatorname{im}\left(H^0\left(X, \Omega^1_{\mathcal{X}|X}\right) \longrightarrow H^0\left(X, \Omega^1_X\right)\right)$$

the subspace of 1-forms on X that are liftable to the infinitesimal deformation $\mathcal{X}$, and assume $\dim K_\xi \geq d + 1$ (in particular, $q\left(X\right) \geq d + 1$).

Consider now $d + 1$ linearly independent 1-forms $\mathcal{B} = \left\{\eta_1, \dots, \eta_{d+1}\right\}$ in K_ξ, and let $W \subseteq K_\xi$ be the linear subspace spanned by the η_i. Taking any liftings $s_i \in H^0\left(X, \Omega^1_{\mathcal{X}|X}\right)$, the wedge product $s_1 \wedge \cdots \wedge s_{d+1}$ gives a section of

$$\bigwedge^{d+1} \Omega^1_{\mathcal{X}|X} = \omega_{\mathcal{X}|X} \cong \omega_X \otimes T^\vee_{\Delta,0} \cong \omega_X.$$

The composed isomorphism is the Lie contraction with $\frac{\partial}{\partial\epsilon}$, a chosen generator of $T_{\Delta,0}$, somehow analogous to the Poincaré residue. The image $w = w_{\mathcal{B}} \in H^0(X, \omega_X)$ of $s_1 \wedge \cdots \wedge s_{d+1}$ is called an *adjoint image* of the η_i. This definition clearly depends on the choice of the liftings s_i, but the difference between any two adjoint images is a linear combination of the d-fold wedge products

$$\zeta_i = \eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_{d+1} \in H^0(X, \omega_X).$$

Therefore, the class $[w]$ of w modulo the linear subspace $W^d \subseteq H^0(X, \omega_X)$ is actually well-defined, and we call it the *adjoint class* of $\mathcal{B} = \{\eta_1, \cdots, \eta_{d+1}\}$.

Furthermore, if we take a different basis of W or a different generator of $T_{\Delta,0}$, the two adjoint classes will differ by the product of a non-zero scalar (the determinant of the change of basis), so the notion which is truly intrinsical of the subspace W is the *vanishing* of the adjoint class:

Definition 5.1.1 (Vanishing adjoint image of a subspace)**.** *Given a $(d+1)$-dimensional subspace $W \subseteq K_\xi$, we say that its adjoint image* vanishes *if $[w] = 0$ for some (hence any) choice of basis of W.*

Remark 5.1.2. *If the adjoint class of $\mathcal{B} = \{\eta_1, \ldots, \eta_{d+1}\}$ is zero, it is possible to find representatives $s_i \in H^0\left(X, \Omega^1_{\mathcal{X}|X}\right)$ such that*

$$s_1 \wedge \cdots \wedge s_{d+1} = 0 \in H^0\left(X, \Omega^{d+1}_{\mathcal{X}|X}\right),$$

and not only is a linear combination of the

$$\sigma_i = s_1 \wedge \cdots \wedge \widetilde{s_i} \wedge \cdots \wedge s_{d+1}.$$

Indeed, if the adjoint class of $\mathcal{B}$ is zero, there exist scalars $a_i \in \mathbb{C}$ such that

$$s_1 \wedge \cdots \wedge s_{d+1} = d\epsilon \wedge \left(\sum_{i=1}^{d+1} a_i \sigma_i\right)$$

(where $d\epsilon$ is a generator of $T^\vee_{\Delta,0}$). Defining new liftings as $\widetilde{s_i} = s_i + (-1)^i a_i d\epsilon$ it is immediate that

$$\widetilde{s_1} \wedge \cdots \wedge \widetilde{s_{d+1}} = 0.$$

We will now relate the adjoint images of a subspace to some properties of the infinitesimal deformation.

As in the case of curves, an effective divisor D on X induces the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_X(D) \longrightarrow T_X(D)_{|D} \longrightarrow 0$$

and the connecting homomorphism

$$\partial_{T,D} : H^0\left(D, T_X(D)_{|D}\right) \longrightarrow H^1(X, T_X).$$

Following the analogy with curves, one can make the following

Definition 5.1.3. *We say that the deformation* ξ is supported *on* D *if and only if*

$$\xi \in \operatorname{im}(\partial_{T,D}) = \ker\left(H^1(X, T_X) \longrightarrow H^1(X, T_X(D))\right),$$

or equivalently, if the pull-back sequence

$$\begin{tikzcd} \xi_D : & 0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{F}_D \arrow[r] \arrow[d, hook] & \Omega^1_X(-D) \arrow[r] \arrow[d, hook] & 0 \\ & 0 \arrow[r] & \mathcal{O}_X \arrow[r] & \Omega^1_{\mathcal{X}|X} \arrow[r] & \Omega^1_X \arrow[r] & 0 \end{tikzcd}$$

splits.

We are now ready to state the main known result concerning adjoint images:

Theorem 5.1.4 (Adjoint Theorem, [34] Th. 1.5.1, [10] Th. 1.1.8 for curves)**.** *Let* $W \subseteq K_\xi \subseteq H^0(X, \omega_X)$ *be a* $(d+1)$*-dimensional subspace such that* $W^d \neq 0$*, and let* $D = D_W$ *be the base locus of the corresponding linear series* $|W^d| \subseteq |\omega_X|$*. If the adjoint image of* W *vanishes, then* ξ *is supported on* D*, i.e.*

$$\xi \in \ker\left(H^1(X, T_X) \longrightarrow H^1(X, T_X(D))\right).$$

While all the previous considerations concern infinitesimal deformations, we give now another result of Pirola and Zucconi (the *Volumetric Theorem*, [34] Th. 1.5.3) about *local* families of irregular varieties.

Let thus B be an open analytic curve (e.g. the unit disk), and let $\pi : \mathcal{X} \to B$ be a smooth family of d-dimensional varieties. Assume also that the varieties $X_b = \pi^{-1}(b)$ (for b varying in B) are not birational (such a family is called *strongly non-isotrivial* in [34]).

Furthermore, let A be an Abelian variety and suppose there is a morphism $\Phi : \mathcal{X} \to A \times B$ such that $p_2 \circ \Phi = \pi$ (where $p_2 : A \times B \to B$ is the projection onto the second factor). We can think of such a Φ as a family of morphisms $\phi_b : X_b \to A$ from the fibres of π to the fixed Abelian variety A. Given a $(d+1)$-dimensional subspace $W \subseteq H^0(A, \Omega^1_A)$, denote by $W_b = \phi_b^* W \subseteq H^0\left(X_b, \Omega^1_{X_b}\right)$ its pull-back to X_b, and by $[w_b]$ one of the adjoint classes of W_b.

Theorem 5.1.5 (Volumetric Theorem, [34] Theorem 1.5.3)**.** *Keep the above notations and assume still that the family π is strongly non-isotrivial. Suppose also that for some $b_0 \in B$, $\phi_{b_0} : X_{b_0} \to A$ is birational onto its image Y, and that Y generates A as a group. Then for generic $(d+1)$-dimensional $W \subseteq H^0(A, \Omega^1_A)$ and generic $b \in B$, the adjoint class $[w_b]$ does not vanish.*

We would like to give a sketch the proof of the Volumetric Theorem 5.1.5, since some ideas contained in it have inspired some results of the next chapter.

First of all, after taking an infinite, étale covering $\rho : \mathcal{X}' \to \mathcal{X}$ it is possible to define a map

$$\mathcal{X}' \longrightarrow H^0\left(A, \Omega^1_A\right)^\vee.$$

Furthermore, if $W \subseteq H^0(A, \Omega^1_A)$ is a generic subspace of dimension $d+1$, then the composition $\Psi : \mathcal{X}' \to W^\vee$ of the above map with the projection $H^0(A, \Omega^1_A)^\vee \to W^\vee$ is one-to one on every fibre.

Take now a basis $\{w_1, \ldots, w_{d+1}\}$ of W, and denote by Ψ_i the components of Ψ with respect to that basis. Let $\eta_i = \Psi^* w_i = d\Psi_i \in H^0(\mathcal{X}', \Omega^1_{\mathcal{X}'})$ denote the pull-backs of the basis to $\mathcal{X}'$, so that (up to shrinking B) the restrictions $\eta_{i|X'_b} \in H^0\left(X'_b, \Omega^1_{X'_b}\right)$ form a basis of $\rho^* W_b$ for all b.

In this setting, if the generic adjoint class $[w_b]$ vanishes, then (up to shrinking B again if necessary) the $a_i d\epsilon$ of Remark 5.1.2 glue to give *holomorphic* 1-forms $a_i(t)\,dt$ on B, where t is a coordinate on B. Modifying the η_i as

$$\widetilde{\eta_i} = \eta_i + (-1)^i a_i(t)\,dt \in H^0\left(\mathcal{X}', \Omega^1_{\mathcal{X}'}\right)$$

we obtain $d+1$ forms on the total space $\mathcal{X}'$ whose wedge product is zero. But this wedge product is the pull-back of a volume form on $W^\vee$ by the modified morphism $\widetilde{\Psi} : \mathcal{X}' \to W^\vee$, whose coordinates are given by

$$\widetilde{\Psi}_i = \Psi_i + (-1)^i a_i.$$

This implies that the image Y of $\widetilde{\Psi}$ has dimension d (one less than $\mathcal{X}'$), and the rest of the hypothesis of the Theorem imply that every fibre X'_b

maps birationally to Y. To finish, a monodromy argument shows that the isotriviality of the étale covering $\mathcal{X}'$ implies the isotriviality of $\mathcal{X}$.

5.2 The case of curves

Consider now an infinitesimal deformation ξ of a smooth curve C of genus $g \geq 2$. The aim of this section is to give a numerical condition on ξ that guarantees the existence of a 2-dimensional subspace $W \subseteq K_\xi \subseteq H^0(C, \omega_C)$ with vanishing adjoint image. Some of the definitions and results of this section are inspired by the study of special deformations carried out by Collino and Pirola in [10], Section 1.3.

In order to do that, we define a vector bundle $\mathcal{A}$ on $\mathbb{G} = Gr(2, K_\xi)$ together with a section $\nu \in H^0(\mathbb{G}, \mathcal{A})$, which we call the *adjoint bundle* and *adjoint map* respectively. The main issue is to show that if K_ξ is big enough, then ν vanishes at some point $W \in \mathbb{G}$.

We first write down more precisely and intrinsically the construction of the adjoint images. For the sake of simplicity, denote by $K = K_\xi$. Given any subspace $W \subseteq K$, denote by $\widetilde{W} \subseteq H^0\left(C, \Omega^1_{\mathcal{C}|C}\right)$ its preimage, so that we have the following exact sequence

$$0 \longrightarrow T^\vee_{\Delta,0} \longrightarrow \widetilde{W} \longrightarrow W \longrightarrow 0,$$

from which we obtain the presentation

$$T^\vee_{\Delta,0} \otimes \widetilde{W} \xrightarrow{\wedge} \bigwedge^2 \widetilde{W} \longrightarrow \bigwedge^2 W \longrightarrow 0.$$

Wedge product induces also a map

$$\bigwedge^2 \widetilde{W} \longrightarrow H^0\left(C, \Omega^2_{\mathcal{C}|C}\right) \cong T^\vee_{\Delta_0} \otimes H^0(C, \omega_C),$$

and it is clear that the image of $T^\vee_{\Delta,0} \otimes \widetilde{W}$ maps precisely to $T^\vee_{\Delta,0} \otimes W$. Hence, there is a well-defined map

$$\nu_W : \bigwedge^2 W \longrightarrow T^\vee_{\Delta,0} \otimes \left(H^0(C, \omega_C)/W\right) \tag{5.1}$$

completing the diagram below.

$$\begin{tikzcd}
T^\vee_{\Delta,0} \otimes \widetilde{W} \arrow[r] \arrow[d] & \bigwedge^2 \widetilde{W} \arrow[r] \arrow[d] & \bigwedge^2 W \arrow[r] \arrow[d, dashed, "\nu_W"] & 0 \\
T^\vee_{\Delta,0} \otimes W \arrow[r, hook] & T^\vee_{\Delta,0} \otimes H^0(C, \omega_C) \arrow[r] & T^\vee_{\Delta,0} \otimes \left(H^0(C, \omega_C)/W\right) \arrow[r] & 0
\end{tikzcd} \tag{5.2}$$

Definition 5.2.1. *We call the map ν_W in (5.1) the* adjoint map associated to W.

Remark 5.2.2. *Note that this construction is valid for any subspace W of dimension* at least *2. If we restrict ourselves to the case* $\dim W = 2$*, then the choice of a basis $\mathcal{B} = \{w_1, w_2\}$ of W gives a generator $e_{\mathcal{B}} = w_1 \wedge w_2$ of $\bigwedge^2 W$, and the adjoint class $[w_{\mathcal{B}}]$ defined in the previous section is precisely $\nu_W (e_{\mathcal{B}})$.*

Remark 5.2.3. *The above construction can be easily generalized to higher-dimensional varieties, giving a map*

$$\nu_W : \bigwedge^{d+1} W \longrightarrow T^\vee_{\Delta,0} \otimes \left(H^0 (X, \omega_X) / W^d\right)$$

for any subspace $W \subseteq K_\xi$ of dimension at least $d+1$.

Let us now focus on the case $\dim W = 2$. Let $\mathbb{G} = Gr(2, K)$ be the Grassmannian variety of 2-dimensional subspaces of K. For any vector space E, denote by $E_{\mathbb{G}} = E \otimes \mathcal{O}_{\mathbb{G}}$ the trivial vector bundle with fibre E. As customary, denote by $S \subseteq K_{\mathbb{G}}$ and $Q = K_{\mathbb{G}}/S$ the tautological subbundle and quotient bundle. Note that since $K \subseteq H^0 (C, \omega_C)$, the tautological subbundle S injects in $H^0 (C, \omega_C)_{\mathbb{G}}$ and the quotient is also a vector bundle (of rank $g-2$).

Lemma 5.2.4. *The adjoint maps ν_W depend holomorphically on $W \in \mathbb{G}$. More precisely, there exists a map of vector bundles*

$$\nu : \bigwedge^2 S \longrightarrow T^\vee_{\Delta,0} \otimes \left(H^0 (C, \omega_C)_{\mathbb{G}} / S\right).$$

such that $\nu \otimes \mathbb{C}(W) = \nu_W$.

Proof. The proof is quite immediate. One only has to mimick the construction of the ν_W replacing W by the tautological subbundle S.

Denote by $\widetilde{S} \subseteq H^0 \left(C, \Omega^1_{\mathcal{C}|C}\right)_{\mathbb{G}}$ the preimage of $S \subseteq K_{\mathbb{G}}$ by the natural projection $\pi : H^0 \left(C, \Omega^1_{\mathcal{C}|C}\right) \to K$, which is a vector bundle of rank 3 and fits into the exact sequence

$$0 \longrightarrow T^\vee_{\mathbb{G}} \longrightarrow \widetilde{S} \longrightarrow S \longrightarrow 0,$$

(where $T = T^\vee_{\Delta,0}$). The analogue to the diagram (5.2) is

$$\begin{tikzcd}
T^\vee \otimes \widetilde{S} \arrow[r] \arrow[d] & \bigwedge^2 \widetilde{S} \arrow[r] \arrow[d] & \bigwedge^2 S \arrow[r] \arrow[d, dashed, "\nu"] & 0 \\
T^\vee \otimes S \arrow[r, hook] & T^\vee \otimes H^0 (C, \omega_C)_{\mathbb{G}} \arrow[r] & T^\vee \otimes (H^0 (C, \omega_C)_{\mathbb{G}} / S) \arrow[r] & 0
\end{tikzcd}$$

where the central vertical arrow is also given by wedge product and the isomorphism $\Omega^2_{\mathcal{C}|C} \cong T^\vee \otimes \omega_C$. It is immediate to check that the map ν gives the adjoint map ν_W at any point W. □

Definition 5.2.5. *We call the map ν constructed in the previous Lemma simply the* adjoint map *of the deformation ξ. It can be seen as a section of the vector bundle*

$$\mathcal{A} = T^\vee_{\Delta,0} \otimes \bigwedge^2 S^\vee \otimes \left(H^0\left(C, \omega_C\right)_{\mathbb{G}} / S\right),$$

which we call the adjoint bundle.

Remark 5.2.6. *Unlike the adjoint map associated to a fixed subspace, the extension of Definition 5.2.5 to higher-dimensional varieties is not so straightforward. We can consider the Grassmannian $\mathbb{G} = Gr\left(d+1, K_\xi\right)$ and its tautological subbundle S. Then the adjoint map should go from the line bundle $\bigwedge^{d+1} S$ to something like $T^\vee_{\Delta,0} \otimes \left(H^0\left(X, \omega_X\right)_{\mathbb{G}} / S^d\right)$, where S^d has to be understood as the image of $\bigwedge^d S$ in $H^0\left(X, \omega_X\right)_{\mathbb{G}}$ by the wedge product map ψ_d.*

The problem arises with this last object, since it is not necessarily a vector bundle. However, if X does not admit a higher irrational pencil, the construction carries over without any problem. Indeed, the Generalized Castelnuovo-de Franchis Theorem (Theorem 1.2.3) implies that the map

$$\bigwedge^d S \longrightarrow H^0\left(X, \omega_X\right)_{\mathbb{G}}$$

is everywhere injective, so $\bigwedge^d S \cong S^d$ and the quotient $H^0\left(X, \omega_X\right)_{\mathbb{G}} / S^d$ is a vector bundle of rank $p_g\left(X\right) - \left(d+1\right)$.

We are now ready to state and prove the main result of this section.

Theorem 5.2.7. *If $V \subseteq K_\xi$ has dimension $\dim V > \frac{g+1}{2}$, then there exists some 2-dimensional subspace $W \subseteq V$ such that $\nu_W = 0$.*

Proof. Let $\mathbb{G}_V = Gr\left(2, V\right) \subseteq \mathbb{G}$ be the subvariety of $\mathbb{G}$ consisting of the 2-dimensional subspaces of K contained in V, which is in turn a Grassmannian variety. Furthermore, the tautological subbundle S_V of $\mathbb{G}_V$ is the restriction of S, and the adjoint map ν restricts to

$$\nu_V : \bigwedge^2 S_V \longrightarrow T^\vee \otimes \left(H^0\left(C, \omega_C\right)_{\mathbb{G}_V} / S_V\right)$$

(as above, we have simplified $T = T_{\Delta,0}$) which is a section of

$$\mathcal{A}_V = T^\vee \otimes \bigwedge^2 S_V^\vee \otimes \left(H^0\left(C,\omega_C\right)_{\mathbb{G}_V}/S_V\right) = \mathcal{A}_{|V}.$$

Denoting by $Z = Z(\nu) \subseteq \mathbb{G}$ the zero locus of ν, and by Z_V the zero locus of ν_V, it is clear that $Z_V = Z \cap \mathbb{G}_V$.

With these notations, the theorem says that $Z_V \neq \emptyset$. In order to prove that, we will compute the top Chern class of $\mathcal{A}_V$ and show that it does not vanish. This is enough, since if a vector bundle admits a nowhere vanishing section, then its top Chern class is zero.

First of all, our only hypothesis is equivalent to $\dim V \geq \frac{g}{2}+1$. Hence

$$\operatorname{rk}\mathcal{A}_V = g-2 \leq 2(\dim V - 2) = \dim \mathbb{G}_V,$$

so it is indeed possible that $c_{g-2}(\mathcal{A}_V) \neq 0$.

Secondly, up to the trivial twisting by $T^\vee$, $\mathcal{A}_V$ is the globally generated bundle

$$\mathcal{G} = H^0\left(C,\omega_C\right)_{\mathbb{G}_V}/S_V$$

twisted by the line bundle $\bigwedge^2 S_V^\vee \cong \mathcal{O}_{\mathbb{G}_V}(1)$, which is the very ample line bundle inducing the Plücker embedding.

Therefore, we can use the formula (see [16] Remark 3.2.3.(b))

$$c_k(E\otimes L) = \sum_{i=0}^{k} \binom{r-k+i}{r-k} c_{k-i}(E)\, c_1(L)^i$$

to compute the Chern classes of a vector bundle E of rank r twisted by a line bundle L (on any variety), which for $k=r$ reduces to

$$c_r(E\otimes L) = \sum_{i=0}^{r} c_{r-i}(E)\, c_1(L)^i.$$

Summing up, since all the Chern classes of $\mathcal{G}$ are represented by zero or effective cycles (because it is globally generated), we obtain

$$\begin{aligned} c_{g-2}(\mathcal{A}_V) = \sum_{i=0}^{r} c_{r-i}(\mathcal{G})\, c_1\left(\mathcal{O}_{\mathbb{G}_V}(1)\right)^i = \\ = c_1\left(\mathcal{O}_{\mathbb{G}_V}(1)\right)^{g-2} + (\text{effective classes}) \neq 0 \end{aligned}$$

because $\mathcal{O}_{\mathbb{G}_V}(1)$ is very ample. $\square$

Corollary 5.2.8. *If $V \subseteq K_\xi$ has dimension greater than $\frac{g+1}{2}$, then there exists a two-dimensional subspace $W \subseteq V$ whose adjoint class vanishes.*

Remark 5.2.9. *It is possible to get similar results for higher-dimensional varieties without higher irrational pencils (see Remark 5.2.6). The same proof goes over as soon as the rank of the adjoint bundle is not greater than the dimension of the Grassmannian variety. In symbols, we need*

$$p_g - (d+1) \leq (d+1)\left(\dim V - d - 1\right),$$

which in particular implies (since $\dim V \leq q$*)*

$$p_g \leq (d+1)(q-d).$$

For higher dimensions this inequality becomes a quite restrictive condition (combined with the non-existence of higher irrational pencils). For example, the only surfaces to which this method could be applied are those satisfying

$$2q - 3 \leq p_g \leq 3(q-2),$$

where the first inequality is the Castelnuovo-de Franchis inequality.

5.3 Global adjoint

In this last section we extend the previous constructions to the case of a fibration over a compact curve. As in the previous sections, we stick to the case when the fibres are curves, though some constructions and results can be carried over to some cases with higher-dimensional fibres.

Therefore, let $f : S \to B$ be a fibration of a surface S over a curve B, and denote by

$$V = V_f = H^0\left(S, \Omega^1_S\right)/f^* H^0\left(B, \omega_B\right),$$

which has dimension q_f, the relative irregularity of f.

According to Lemma 3.1.7, V naturally injects into $H^0(C, \omega_C)$ for any smooth fibre C of f. Furthermore, if $\xi \in H^1(C, T_C)$ is the infinitesimal deformation of C induced by f, then V is contained in the kernel K_ξ of the cup-product map

$$\cup \xi : H^0(C, \omega_C) \to H^1(C, \mathcal{O}_C).$$

In the previous section we constructed the adjoint map associated to any subspace of K_ξ. We restrict now to a slightly less general version, considering only subspaces W of V.

All the injections $V \subseteq H^0(C, \omega_C)$ for smooth fibres glue together into an inclusion of vector bundles

$$V_B = V \otimes \mathcal{O}_B \hookrightarrow f_* \omega_{S/B} \tag{5.3}$$

whose cokernel $\mathcal{G}$ is locally free (see Theorem 3.2.9 and Corollary 3.2.10, due to Fujita). The results of Fujita [15] say moreover that the inclusion splits (so $f_*\omega_{S/B} \cong V_B \oplus \mathcal{G}$) and $\mathcal{G}$ has some good cohomological properties, but we will not use them in the sequel.

The inclusion (5.3) can be alternatively constructed as follows. First of all, wedge product gives a natural map $H^0(S, \Omega^1_S) \otimes \omega_B \to f_*\omega_S$. Clearly $(f^* H^0(B, \omega_B)) \otimes \omega_B$ maps to zero, so there is an induced map

$$V \otimes \omega_B \longrightarrow f_*\omega_S = \left(f_*\omega_{S/B}\right) \otimes \omega_B.$$

Since it is injective over a generic $b \in B$, it is everywhere injective (as a map of sheaves), and cancelling the twist by ω_B we obtain the inclusion (5.3).

Denote now by $\mathbb{G} = Gr(2, V)$ the Grassmannian of 2-planes of V, and by $S_V \subseteq V \otimes \mathcal{O}_{\mathbb{G}}$ the tautological subbundle. Consider the product $Y = B \times \mathbb{G}$, and denote by $p_1 : Y \to B$ and $p_2 : Y \to \mathbb{G}$ the natural projections. The variety Y is the Grassmann bundle of 2-dimensional subspaces of V_B, and $\mathcal{S} = p_2^* S_V$ is the corresponding tautological subbundle. Clearly, $\mathcal{S}$ is a vector subbundle[1] of $V_Y = V \otimes \mathcal{O}_Y = p_1^* V_B$, hence also of $p_1^* f_* \omega_{S/B}$.

Denote by $\widetilde{\mathcal{S}} \subseteq H^0(S, \Omega^1_S) \otimes \mathcal{O}_Y$ the natural preimage of $\mathcal{S}$, so that

$$0 \longrightarrow H^0(B, \omega_B) \xrightarrow{f^*} \widetilde{\mathcal{S}} \longrightarrow \mathcal{S} \longrightarrow 0$$

is an exact sequence of vector bundles on Y. Therefore, we obtain the following presentation of $\bigwedge^2 \mathcal{S}$,

$$\widetilde{\mathcal{S}} \otimes H^0(B, \omega_B) \longrightarrow \bigwedge^2 \widetilde{\mathcal{S}} \longrightarrow \bigwedge^2 \mathcal{S} \longrightarrow 0 \tag{5.4}$$

The wedge product $\bigwedge^2 H^0(S, \Omega^1_S) \to H^0(S, \omega_S) = H^0(B, f_*\omega_S)$ and the evaluation map $H^0(B, f_*\omega_S) \otimes \mathcal{O}_Y \cong H^0(Y, p_1^* f_* \omega_S) \otimes \mathcal{O}_Y \to p_1^* f_* \omega_S$ give a map of vector bundles on Y

$$\widetilde{\nu} : \bigwedge^2 \widetilde{\mathcal{S}} \longrightarrow p_1^* f_* \omega_S.$$

[1] By a *vector subbundle* of a vector bundle V we mean a locally free subsheaf whose quotient is also locally free.

Clearly, this map sends the image of $\widetilde{\mathcal{S}} \otimes H^0(B, \omega_B)$ into the subsheaf $\mathcal{S} \otimes p_1^* \omega_B$. Hence, according to equation (5.4), $\widetilde{\nu}$ induces a well-defined map of vector bundles on Y:

$$\nu : \bigwedge^2 \mathcal{S} \longrightarrow (p_1^* f_* \omega_S) / (\mathcal{S} \otimes p_1^* \omega_B). \tag{5.5}$$

Definition 5.3.1 (Global Adjoint Map)**.** *We call the map ν in (5.5) the* global adjoint map *of the fibration f.*

Remark 5.3.2. *It is clear from the construction that if $C = f^{-1}(b)$ is a smooth fibre of f, the restriction $\nu_{|\{b\}\times\mathbb{G}}$ coincides with the adjoint map constructed in Definition 5.2.5, restricted to the Grassmannian subvariety $Gr(2, V)$.*

To close both this section and the chapter, we give a result analogous to Theorem 5.2.7 and Corollary 5.2.8. Instead of 2-dimensional vector subspaces $W \subseteq K_\xi$, we will consider vector subbundles of rank two $\mathcal{W} \subseteq V \otimes \mathcal{O}_B$. Such a vector subbundle defines a section

$$\eta_{\mathcal{W}} : B \longrightarrow Y$$

of p_1, such that $\eta_{\mathcal{W}}(b)$ is the subspace $\mathcal{W} \otimes \mathbb{C}(b) \subseteq V$. Conversely, given any section $\eta : B \to Y$ of p_1, it defines the vector subbundle

$$\mathcal{W}_\eta = \eta^* \mathcal{S} \hookrightarrow \eta^*(V \otimes \mathcal{O}_Y) = V \otimes \mathcal{O}_B.$$

Clearly, the assignations $\mathcal{W} \mapsto \eta_{\mathcal{W}}$ and $\eta \mapsto \mathcal{W}_\eta$ are mutually inverse, giving a one-to-one correspondence between the sets of vector subbundles of $V \otimes \mathcal{O}_B$ of rank 2 and the sections of $p_1 : Y \to B$.

Now, given a vector subbundle $\mathcal{W}$ as above, we can consider the restriction $\nu_{\mathcal{W}}$ of the adjoint map ν to the curve $\eta_{\mathcal{W}}(B) \cong B$, which can be seen as a map of vector bundles on B:

$$\nu_{\mathcal{W}} : \bigwedge^2 \mathcal{W} \longrightarrow (f_* \omega_S) / (\mathcal{W} \otimes \omega_B). \tag{5.6}$$

Definition 5.3.3 (Global Adjoint Map associated to a subbundle)**.** *We call the map $\nu_{\mathcal{W}}$ in equation (5.6) the* global adjoint map associated to *the subbundle $\mathcal{W}$.*

We are now ready to state the wanted global result.

Theorem 5.3.4. *If*

$$q_f > \frac{g+1}{2},$$

then there exist a finite base change $\pi : B' \to B$ and a rank-two vector subbundle $\mathcal{W} \subseteq V \otimes \mathcal{O}_{B'}$ whose associated global adjoint map vanishes identically.

Proof. Let $Z \subseteq Y$ be the zero set of the global adjoint map ν, which is an analytic subvariety. By Remark 5.3.2, for any regular value b, the set $Z_b = Z \cap (\{b\} \times \mathbb{G})$ is the vanishing set of the adjoint map of C_b, which is non-empty by Corollary 5.2.8. Therefore, there is a component of Z dominating B, hence it is possible to choose a curve $\widetilde{B} \subseteq Z$ dominating B. Let $\mu : B' \to \widetilde{B}$ be the normalization of $\widetilde{B}$, and define the covering π as the composition $p_1 \circ \mu : B' \to B$. As for the vector subbundle, let $\eta : B' \to B' \times_B Y \cong B' \times \mathbb{G}$ be the section induced from the map $B' \to \widetilde{B} \to Y$, and let $\mathcal{W} = \mathcal{W}_\eta$. Since the image of η is contained in the zero locus of the adjoint map associated to the fibration $S' = S \times_B B' \to B'$, (see next remark), it is almost tautological that the globa adjoint map associated to $\mathcal{W}$ vanishes identically. $\square$

Remark 5.3.5 (Global Adjoint Maps and base change). *Consider any finite base change $\pi : B' \to B$. Denote by $f' : S' = S \times_B B' \to B'$ the resulting fibration, and by $V' = V_{f'} = H^0(S', \Omega^1_{S'}) / (f')^* H^0(B', \omega'_B)$ the corresponding space of relative 1-forms. Define also $\mathbb{G}' = Gr(2, V')$ and $Y' = B' \times \mathbb{G}'$, and let ν' be the global adjoint map of f'.*

According to Corollary 3.1.9, V injects into V'. Therefore, $B' \times \mathbb{G}$ is naturally a subvariety of Y'. Furthermore, the pull-back (by $\pi \times \mathrm{id}_{\mathbb{G}}$) of ν is the restriction to $B' \times \mathbb{G}$ of ν'. Hence, the zero locus of ν' always contains the preimage of the zero locus of ν.

6

Chapter Six

On a conjecture of Xiao

In this last chapter we prove the main results of the second part of the Thesis. The first one is Theorem 6.3.1, a structure result for some fibred surfaces that seems to be generalizable to higher-dimensional fibrations over a curve. As an application, we prove Theorem 6.3.4. We postpose the proofs to Section 6.3, starting the chapter with a summary of the main known results concerning the triviality and isotriviality of a fibration on a surface, and their implications to the relative irregularity and the genus of the fibres (Section 6.1). After that, Section 6.2 contains a technical result that will simplify the final proofs, allowing us to write them more neatly.

Through the whole chapter, $f : S \to B$ will denote a fibration from a smooth surface S to a smooth curve B. We will denote by $g = g(C)$ the genus of any smooth fibre C, and $q_f = q(S) - g(B)$ denote the relative irregularity of the fibration. In Sections 6.2 and 6.3 we assume furthermore that $g \geq 2$.

6.1 State of the art

In this section we expose a summary of the main known results relating the numerical invariants g and q_f to the isotriviality of the fibration. We also motivate Xiao's conjecture and the correction proposed after it was shown to be false.

To begin with, a fundamental result relating the numerical invariants of a fibration and its isotriviality was given by Beauville in his appendix to [12]. It gives a first restriction on the relative irregularity of any fibration, and characterizes those fibrations that are birational to a product.

Theorem 6.1.1 ([12], Lemma in the Appendix)**.** *Any fibration $f : S \to B$ satisfies*

$$0 \leq q_f \leq g.$$

Futhermore, the equality $q_f = g$ holds if and only if S is birational to a product $B \times C$, and in particular the fibration is isotrivial.

Therefore, non-trivial fibrations must satisfy $0 \le q_f < g$. A more detailed analysis of the isotrivial case is carried out by Serrano in [40], which has as a consequence that non-trivial isotrivial fibrations satisfy an inequality much stronger than $q_f < g$.

Theorem 6.1.2 ([40] 1.1)**.** *If $f : S \to B$ is an isotrivial fibration with general fibre C, then there exist a smooth curve B' and a finite group G acting algebraically both on B' and C, such that S is birational to $(B' \times C)/G$ (with the diagonal action), $B \cong B'/G$, and the diagram*

$$\begin{tikzcd} S \arrow[r, dashed, "\sim"] \arrow[d, "f"'] & (B' \times C)/G \arrow[d] \\ B \arrow[r, "\cong"] & B'/G \end{tikzcd}$$

commutes.

Proposition 6.1.3 ([40] Prop. 2.2)**.** *If S is birational to a quotient $(B' \times C)/G$ as in the above theorem, then*

$$q(S) = g(B'/G) + g(C/G).$$

Corollary 6.1.4. *If $f : S \to B$ is a non-trivial isotrivial fibration, then*

$$0 \le q_f \le \frac{g+1}{2}.$$

Proof. With the notation of Theorem 6.1.2, Proposition 6.1.3 implies that $q_f = g(C/G)$. Since the fibration is not trivial, $|G| \ge 2$ and hence the map $C \to C/G$ has degree at least 2. This immediately implies that

$$q_f = g(C/G) \le \frac{g(C)+1}{2} = \frac{g+1}{2}.$$

□

For non-isotrivial fibrations, the only general upper bound for q_f is given by Xiao in [43].

Theorem 6.1.5 ([43] Section 3, Cor. 3)**.** *If $f : S \to B$ is a non-trivial fibration, then*

$$q_f \le \frac{5g+1}{6}. \tag{6.1}$$

However, according to some comments made by Xiao in his later work [44], there is little hope for the inequality (6.1) to be sharp, since the methods used to prove it are not very accurate. In fact, in [44] he consider the special case in which the base is $B \cong \mathbb{P}^1$, obtaining the same lower-bound known for non-trivial isotrivial fibrations:

Theorem 6.1.6 ([44] Th. 1)**.** *If $f : S \to \mathbb{P}^1$ is a non-isotrivial fibration, then*

$$q\left(S\right) = q_f \leq \frac{g+1}{2}.$$

In view of this result, Xiao conjectured that the same bound should hold for every non-trivial fibration, and he provided several examples attaining the equality. However, the conjecture was shown to be false by Pirola. In fact, in [33], Theorem 2, he provided a non-isotrivial fibration with fibres of genus $g = 4$ and relative irregularity $q_f = 3 \not\leq \frac{5}{2} = \frac{g+1}{2}$. The same method can be applied to other cases, giving different counterexamples for even g and satisfying

$$q_f = \frac{g}{2} + 1 = \frac{g+1}{2} + \frac{1}{2}.$$

The fact that the only known counterexamples fail by just $\frac{1}{2}$ motivates the following version of the conjecture.

Conjecture 6.1.7. *For any non-trivial fibration $f : S \to B$ one has*

$$q_f \leq \frac{g}{2} + 1,$$

or equivalently

$$q_f \leq \left\lceil \frac{g+1}{2} \right\rceil.$$

Remark 6.1.8. *Note that for odd values of g, Conjecture 6.1.7 is equivalent to the original conjecture posed by Xiao.*

Note also that for $g \leq 1$, the conjecture and all the theorems above are trivially satisfied, since they are all equivalent to Theorem 6.1.1. Hence we shall assume from now on that the fibration f has genus $g \geq 2$.

6.2 A technical result

Before going through the proof of the main theorems, we need a technical result (Proposition 6.2.2) about inclusions $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ or $\mathcal{L} \hookrightarrow \omega_{S/B}$ lifting to Ω^1_S (see Definition 6.2.1 below). It will allow us to improve the properties of any subscheme supporting a global deformation, relating the liftings to Ω^1_S with the framework of Section 4.3.

Since this notion will appear very often through the rest of the section, we make first the next

Definition 6.2.1. *We say that a rank-one subsheaf $\mathcal{L}$ of $\Omega^1_{S/B}$ (resp. $\omega_{S/B}$)* lifts to Ω^1_S *if the inclusion can be factored as an injection $\mathcal{L} \hookrightarrow \Omega^1_S$ followed by the natural projection $\Omega^1_S \to \Omega^1_{S/B}$ (resp. the same projection composed with $\alpha : \Omega^1_{S/B} \to \omega_{S/B}$).*

Recall the natural map $\alpha : \Omega^1_{S/B} \to \omega_{S/B}$ defined in Lemma 3.2.6, which is injective if and only if f has reduced fibres. More generally, it is an isomorphism over the open set of regular points of f. Recall also the notation

$$\mathcal{L}_\Gamma = \ker\left(\Omega^1_{S/B} \to \omega_{S/B|\Gamma}\right)$$

introduced in Definition 4.3.7 for any subscheme $\Gamma \subseteq S$.

As we have done repeatedly in the previous chapters, we denote by $\xi \in \operatorname{Ext}^1_{\mathcal{O}_S}\left(\Omega^1_{S/B}, f^*\omega_B\right)$ the extension class of the sequence

$$0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/B} \longrightarrow 0.$$

More generally, if $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ is any subsheaf, $\xi_{\mathcal{L}} \in \operatorname{Ext}^1_{\mathcal{O}_S}\left(\mathcal{L}, f^*\omega_B\right)$ will denote the extension class of the pull-back sequence

$$\begin{tikzcd}
0 \arrow[r] & f^*\omega_B \arrow[r] \arrow[d, equal] & \mathcal{F}_{\mathcal{L}} \arrow[r] \arrow[d, hook] & \mathcal{L} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & f^*\omega_B \arrow[r] & \Omega^1_S \arrow[r] & \Omega^1_{S/B} \arrow[r] & 0
\end{tikzcd}$$

In the case $\mathcal{L} = \mathcal{L}_\Gamma$ for some subscheme Γ, we simply write ξ_Γ (resp. $\mathcal{F}_\Gamma$) instead of $\xi_{\mathcal{L}_\Gamma}$ (resp. $\mathcal{F}_{\mathcal{L}_\Gamma}$).

We are now ready to state the announced

Proposition 6.2.2. *Let $f : S \to B$ be a fibration with reduced fibres. If a rank-one subsheaf $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ lifts to Ω^1_S and satisfies $\deg\left(\mathcal{L}_{|C_b}\right) > 0$ for some smooth fibre C_b, then there exists an effective divisor D on S such that*

1. *the inclusions $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ and $\omega_{S/B}(-D) \hookrightarrow \omega_{S/B}$ fit into the following chain*
$$\mathcal{L} \hookrightarrow \omega_{S/B}(-D) \hookrightarrow \Omega^1_{S/B} \overset{\alpha}{\hookrightarrow} \omega_{S/B},$$

2. *the injection* $\omega_{S/B}(-D) \hookrightarrow \Omega^1_{S/B}$ *lifts to* Ω^1_S*,*

3. $D \cdot C_b < 2g - 2$ *for any fibre* C_b*,*

4. D *has no component contracted by* f*, and*

5. *the quotient* $\Omega^1_S / \omega_{S/B}(-D)$ *is isomorphic to*

$$f^*\omega_B \otimes \mathcal{O}_S(D) \otimes I_Z$$

for some finite subscheme $Z \subset S$*, hence torsion-free.*

Proof. We proceed in several steps.

Step 1: Obtaining a first divisor E *satisfying 1, 2 and 3.*

We first show that the double dual $\mathcal{L}^{\vee\vee}$ still injects into $\Omega^1_{S/B}$ and lifts to Ω^1_S. Indeed, on the one hand, the lifting $\mathcal{L} \hookrightarrow \Omega^1_S$ induces an injective map $\mathcal{L}^{\vee\vee} \hookrightarrow \Omega^1_S$. On the other hand, $\mathcal{L}$ also injects in $\omega_{S/B}$ because α is injective, hence there is a second injection $\mathcal{L}^{\vee\vee} \hookrightarrow \omega_{S/B}$. Both injections fit into the commutative diagram

$$\begin{tikzcd}
 & \mathcal{L}^{\vee\vee} \arrow[dl, hook'] \arrow[d, hook, dashed] \arrow[dr, hook] & \\
\Omega^1_S \arrow[r, two heads] & \Omega^1_{S/B} \arrow[r, "\alpha"] & \omega_{S/B}
\end{tikzcd}$$

so that the composition $\mathcal{L}^{\vee\vee} \hookrightarrow \Omega^1_S \to \Omega^1_{S/B}$ must still be injective, as claimed, and it clearly lifts to Ω^1_S by construction.

Therefore, we have the sequence of nested sheaves

$$\mathcal{L} \hookrightarrow \mathcal{L}^{\vee\vee} \hookrightarrow \Omega^1_{S/B} \xhookrightarrow{\alpha} \omega_{S/B}.$$

But $\mathcal{L}^{\vee\vee}$ is a locally free (reflexive of rank one) subsheaf of $\omega_{S/B}$, hence of the form $\omega_{S/B}(-E)$ for a unique effective divisor E. As for the inequality $E \cdot C_b < 2g-2$ for some (any) fibre C_b, it follows directly from the hypothesis $\deg\left(\mathcal{L}_{|C_b}\right) > 0$.

Step 2: Removing the vertical components.

As a previous step, we see that ξ is supported on E. To this aim, consider the pull-back diagram

$$\begin{tikzcd}
\zeta_E : 0 \arrow[r] & f^*\omega_B \arrow[r] \arrow[d, equal] & \widetilde{\mathcal{F}_E} \arrow[r] \arrow[d, hook] & \omega_{S/B}(-E) \arrow[r] \arrow[d, hook] \arrow[dl, hook', "\lambda"'] & 0 \\
\xi : 0 \arrow[r] & f^*\omega_B \arrow[r] & \Omega^1_S \arrow[r] & \Omega^1_{S/B} \arrow[r] & 0
\end{tikzcd}$$

where the diagonal arrow λ is the lifting. Since $\widetilde{\mathcal{F}_E}$ is the pullback of

$$\begin{tikzcd} & \omega_{S/B}(-E) \arrow[d, hook] \\ \Omega^1_S \arrow[r, two heads] & \Omega^1_{S/B} \end{tikzcd}$$

the universal property implies that λ factors through an injective map $\omega_{S/B}(-E) \hookrightarrow \widetilde{\mathcal{F}_E}$, hence ζ_E splits. Now, completing the diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}_E \arrow[r] \arrow[d, dashed] & \Omega^1_{S/B} \arrow[r] \arrow[d, hook, "\alpha"] & \omega_{S/B|E} \arrow[d, equal] \\ 0 \arrow[r] & \omega_{S/B}(-E) \arrow[r] & \omega_{S/B} \arrow[r] & \omega_{S/B|E} \end{tikzcd}$$

we obtain an injective map $\iota_E : \mathcal{L}_E \hookrightarrow \omega_{S/B}(-E)$, and

$$\xi_E : 0 \longrightarrow f^*\omega_B \longrightarrow \mathcal{F}_E \longrightarrow \mathcal{L}_E \longrightarrow 0$$

is the pull-back of ζ_E by ι_E. Therefore, ξ_E is split, and thus ξ is supported on E (recall Definition 4.3.13).

Denote by $E' \leq E$ the divisor obtained by removing from E the components contracted by f. Corollary 4.3.15 says that ξ is also supported on E'. Furthermore, since $E' \cdot C_b = E \cdot C_b < 2g - 2$ for any fibre C_b, Lemma 4.3.17 implies that

$$\xi_{E'} : 0 \longrightarrow f^*\omega_B \longrightarrow \mathcal{F}_{E'} \longrightarrow \mathcal{L}_{E'} \longrightarrow 0$$

is also split. Hence $\mathcal{L}_{E'} \hookrightarrow \Omega^1_{S/B}$ lifts to Ω^1_S, and analogously as we showed in Step 1, $\mathcal{L}_{E'}^{\vee\vee} \hookrightarrow \Omega^1_{S/B}$ also lifts to Ω^1_S. To finish, we prove that $\mathcal{L}_{E'}^{\vee\vee} \cong \omega_{S/B}(-E')$, so that E' will satisfy conditons 1 through 4. Indeed, the injection $\iota_{E'} : \mathcal{L}_{E'} \hookrightarrow \omega_{S/B}(-E')$ and its double dual $\iota_{E'}^{\vee\vee} : \mathcal{L}_{E'}^{\vee\vee} \hookrightarrow \omega_{S/B}(-E')$ are isomorphisms away from the critical points of f. But the critical points form a set of codimension 2 because f has reduced fibres, hence $\iota_{E'}^{\vee\vee}$ is an isomorphism, as wanted.

Step 3: Removing the torsion of the cokernel.

Up to now, we have an effective divisor E' satisfying conditions 1 through 4. In particular, $\omega_{S/B}(-E')$ lifts to Ω^1_S. Denote by $\mathcal{M}_0 \subseteq \Omega^1_S$ its image, and by $\widetilde{\mathcal{K}}$ the quotient $\Omega^1_S/\mathcal{M}_0$. Let $\mathcal{T}$ be the torsion subsheaf of $\widetilde{\mathcal{K}}$, and $\mathcal{K} = \widetilde{\mathcal{K}}/\mathcal{T}$ its torsion-free quotient. Finally, let $\mathcal{M}$ be the kernel of the composition of surjections $\Omega^1_S \twoheadrightarrow \widetilde{\mathcal{K}} \twoheadrightarrow \mathcal{K}$. We want to see that $\mathcal{M}$ is isomorphic to $\omega_{S/B}(-D)$ for some divisor $0 \leq D \leq E'$.

We first show that $\mathcal{M}$ is locally free. Clearly, it is torsion-free, and the inclusion $\mathcal{M} \hookrightarrow \Omega^1_S$ factors as $\mathcal{M} \hookrightarrow \mathcal{M}^{\vee\vee} \hookrightarrow \Omega^1_S$. Consider now the exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{M}^{\vee\vee} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] & 0 \\
 & 0 \arrow[r] & \Omega^1_S \arrow[r, equal] \arrow[d] & \Omega^1_S \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

where we have used the snake lemma to identify the cokernel of the first row and the kernel of the last row. On the one hand, both $\mathcal{M}$ and $\mathcal{M}^{\vee\vee}$ have rank one, so $\mathcal{G}$ is a torsion sheaf and, on the other hand, $\mathcal{G}$ is torsion free since $\mathcal{K}$ is. Therefore $\mathcal{G} = 0$ and $\mathcal{M} \cong \mathcal{M}^{\vee\vee}$ is locally free.

To finish, the composition

$$\mathcal{M} \hookrightarrow \Omega^1_S \longrightarrow \omega_{S/B}$$

is injective. Indeed, the image $\widetilde{\mathcal{M}}$ is of rank 1 because $\mathcal{M}_0 \subseteq \mathcal{M}$ and the image of $\mathcal{M}_0$ is $\omega_{S/B}(-E')$, so the kernel of $\mathcal{M} \to \omega_{S/B}$ is a rank-zero subsheaf of a torsion-free sheaf, hence zero. Therefore,

$$\mathcal{M} \cong \widetilde{\mathcal{M}} = \omega_{S/B}(-D)$$

with $D \leq E'$ because by construction $\omega_{S/B}(-E') \subseteq \widetilde{\mathcal{M}}$.

For the other asertion about $\mathcal{K} = \Omega^1_S/\omega_{S/B}(-D)$, we first compute the Chern class

$$c_1(\mathcal{K}) = c_1\left(\Omega^1_S\right) - c_1\left(\omega_{S/B}(-D)\right) = c_1\left(f^*\omega_B \otimes \mathcal{O}_S(D)\right).$$

Since $\mathcal{K}$ is torsion-free, this means that $\mathcal{K} \cong f^*\omega_B \otimes \mathcal{O}_S(D) \otimes L \otimes I_Z$ for some finite subscheme $Z \subset S$ and some $L \in \mathrm{Pic}^0(S)$.

Consider now the diagram of exact rows

$$\begin{tikzcd} 0 \arrow[r] & \omega_{S/B}(-D) \arrow[r] \arrow[d, equal] & \Omega^1_S \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \omega_{S/B}(-D) \arrow[r] & \omega_{S/B} \arrow[r] & \omega_{S/B|D} \arrow[r] & 0 \end{tikzcd} \tag{6.2}$$

Since f has reduced fibres, the map $\alpha : \Omega^1_{S/B} \to \omega_{S/B}$ is injective and its cokernel is supported on the finite subscheme Z' of critical points of f (Lemma 3.2.6). Hence the central map in (6.2) has kernel $f^*\omega_B$ and cokernel $\omega_{S/B|Z'}$, and the snake lemma leads to the exact sequence

$$0 \longrightarrow f^*\omega_B \longrightarrow \mathcal{K} \longrightarrow \omega_{S/B|D} \longrightarrow \omega_{S/B|Z'} \longrightarrow 0.$$

The first map corresponds to a section

$$\sigma \in H^0(S, \mathcal{O}_S(D) \otimes L \otimes I_Z) \subset H^0(S, \mathcal{O}_S(D) \otimes L)$$

whose zero scheme is D. Indeed, the zero scheme $Z(\sigma)$ is contained in D, and coincides with it outside the finite subscheme Z'. This implies that $L \cong \mathcal{O}_S$ and we are done.

$\square$

Remark 6.2.3 (About Step 3 in the proof of Proposition 6.2.2)**.** *If a subsheaf of the form $\mathcal{M}_0 = \omega_{S/B}(-E') \subseteq \omega_{S/B}$ lifts to Ω^1_S, there is an easy geometric interpretation of the support of the divisor E': it is the locus where $\mathcal{M}_0 \subseteq \Omega^1_S$ is not transverse to $f^*\omega_B$, that is*

$$\begin{aligned} \mathrm{Supp}\, E' &= \left\{p \mid \mathrm{im}\left(\left(f^*\omega_B \oplus \mathcal{M}_0\right)_p \to \Omega^1_{S,p}\right) \neq \Omega^1_{S,p}\right\} \\ &= \left\{p \mid \mathrm{im}\left(\left(f^*\omega_B \oplus \mathcal{M}_0\right) \otimes \mathbb{C}(p) \to \Omega^1_S \otimes \mathbb{C}(p)\right) \neq \Omega^1_S \otimes \mathbb{C}(p)\right\}. \end{aligned}$$

The failure of the transversality at some regular point $p \in E'$ may occur either because the images of $\mathcal{M}_0 \otimes \mathbb{C}(p)$ and $(f^*\omega_B) \otimes \mathbb{C}(p)$ in $\Omega^1_S \otimes \mathbb{C}(p)$ coincide, or because $\mathcal{M}_0 \otimes \mathbb{C}(p)$ maps to zero in $\Omega^1_S \otimes \mathbb{C}(p)$. The first case means that not all local sections of $\mathcal{M}_0$ vanish at p, but their values are proportional to pull-backs of 1-forms on B, while the second case means that all local sections of $\mathcal{M}_0$ vanish at p. An immediate computation in local coordinates shows that if the second case happens along some components E'_0 of E', the quotient sheaf $\Omega^1_S/\mathcal{M}_0$ would have torsion supported on E'_0. The last step in the proof of Proposition 6.2.2 replaces E' by $D = E' - E'_0$.

6.3 Main results

This is the very last section in the Thesis, and it is devoted to prove Theorems 6.3.1 and 6.3.4 about the isotriviality of fibred surfaces. The first one is a general result, while the second one is an inequality for the relative irregularity of a fibred surface in terms of its Clifford index. We have included several different proofs for some parts, using different techniques and generalizable to different situations.

Theorem 6.3.1. *Let $f : S \to B$ be a fibration of genus g and relative irregularity $q_f \geq 2$. Suppose it is supported on an effective divisor D such that $D \cdot C < 2g - 2$ and $h^0\left(C, \mathcal{O}_C\left(D_{|C}\right)\right) = 1$ for some smooth fibre C. Then, after finitely many blow-ups and a change of base, there is a different fibration $h : S \to B'$ over a curve of genus $g\left(B'\right) = q_f$. In particular S is a covering of the product $B \times B'$, and both surfaces have the same irregularity.*

Proof. By Lemma 3.1.12, after blowing-up some points and a change of base, we can assume that f has reduced fibres and still satisfies the rest of the hypotheses. Also, the new fibration is isotrivial if and only if the original one was.

Now, $\deg\left(\mathcal{L}_{D|C}\right) = 2g - 2 - (D \cdot C) > 0$ for a general fibre C because $D \cdot C < 2g - 2$. The same inequality gives, by Lemma 4.3.17, that the inclusion $\mathcal{L}_D \hookrightarrow \Omega^1_{S/B}$ lifts to Ω^1_S. Applying Proposition 6.2.2, we can replace D by a subdivisor (still called D for simplicity) and assume that $\omega_{S/B}(-D)$ lifts to Ω^1_S, D has no component contracted by f and that the cokernel $\mathcal{K} = \mathcal{K}_D$ of the lifting is torsion-free, isomorphic to $f^*\omega_B \otimes \mathcal{O}_S(D) \otimes I_Z$ for some finite subscheme $Z \subset S$. Since we have replaced D by a subdivisor, the fact that $h^0\left(C, \mathcal{O}_C\left(D_{|C}\right)\right) = 1$ for some smooth fibre C does not change.
Claim: $h^0\left(S, \omega_{S/B}(-D)\right) \geq q_f$. Indeed, it follows from the exact sequence

$$0 \longrightarrow \omega_{S/B}(-D) \longrightarrow \Omega^1_S \longrightarrow \mathcal{K} \longrightarrow 0$$

that $h^0\left(\omega_{S/B}(-D)\right) \geq h^0\left(\Omega^1_S\right) - h^0(\mathcal{K}) = q(S) - h^0(\mathcal{K})$, so it is enough to prove that $h^0(\mathcal{K}) = g(B)$.

Since f has reduced fibres, $\Omega^1_{S/B}$ is a subsheaf of $\omega_{S/B}$ (Lemma 3.2.6) and the sequence

$$0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \omega_{S/B}$$

is exact. Applying the snake lemma to the diagram of exact rows

$$\begin{tikzcd}
0 \arrow[r] & \omega_{S/B}(-D) \arrow[r] \arrow[d, equal] & \Omega^1_S \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \omega_{S/B}(-D) \arrow[r] & \omega_{S/B} \arrow[r] & \omega_{S/B|D} \arrow[r] & 0
\end{tikzcd}$$

we get that the kernel of $\mathcal{K} \to \omega_{S/B|D}$ is also $f^*\omega_B$. Therefore, taking direct images, we obtain the following exact sequence of sheaves on B

$$0 \longrightarrow \omega_B \longrightarrow f_*\mathcal{K} \longrightarrow f_*\omega_{S/B|D}.$$

Since $\mathcal{K} \cong f^*\omega_B \otimes \mathcal{O}_S(D) \otimes I_Z$ is torsion-free and $D_{|C}$ is rigid for a general fibre C,

$$f_*\mathcal{K} = \omega_B \otimes f_*\left(\mathcal{O}_S(D) \otimes I_Z\right)$$

is a vector bundle (torsion-free over a curve) of rank one. Therefore, the cokernel of the injection $\omega_B \hookrightarrow f_*\mathcal{K}$ must be a torsion subsheaf of $f_*\omega_{S/B|D}$. But the latter is torsion-free because D has no component contracted by f (see Lemma 6.3.2 below), so the injection $\omega_B \hookrightarrow f_*\mathcal{K}$ is in fact an isomorphism, and

$$h^0(S,\mathcal{K}) = h^0(B, f_*\mathcal{K}) = h^0(B,\omega_B) = g(B),$$

finishing the proof of the claim.

Since the lifting of $\omega_{S/B}(-D)$ to Ω^1_S is a line bundle $\mathcal{L}$, the wedge product of any two of its sections is zero. Therefore, since we have just seen that $h^0(\mathcal{L}) \geq q_f \geq 2$, the Castelnuovo-de Franchis Theorem 3.1.14 implies the existence of the fibration $h : S \to B'$ over a curve B' of genus $g(B') \geq q_f$.

It remains to show that $g(B') = q_f$, which follows from the last structural statement. In fact, the two fibrations give a covering π completing the diagram

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[d, dashed, "\pi"] \arrow[dr, "h"] & \\ B & B \times B' \arrow[l] \arrow[r] & B' \end{tikzcd}$$

Since π is surjective, $q(S) \geq q(B \times B') = g(B) + g(B')$, hence $g(B') \leq q_f$, and the proof is finished. $\square$

Lemma 6.3.2. *If $f : S \to B$ is any fibration, D is an effective divisor on S without components contracted by f, and L is any line bundle on S, then $f_*\left(L_{|D}\right)$ is a torsion-free sheaf on B.*

Proof. We have to show that, given any open subset $U \subseteq B$ and any non-zero section $\alpha \in H^0\left(U, f_*\left(L_{|D}\right)\right) = H^0\left(f^{-1}(U), L_{|D}\right)$, the condition

$$\phi\alpha = 0 \in H^0\left(U, f_*\left(L_{|D}\right)\right) = H^0\left(f^{-1}(U), L_{|D}\right)$$

for some $\phi \in H^0(U, \mathcal{O}_B)$ implies that $\phi = 0$.

Let $p \in D \cap f^{-1}(U)$ be any point, $R = \mathcal{O}_{S,p}$ and $T = \mathcal{O}_{B,f(p)}$ the local rings at p and $f(p)$, and $\mathfrak{m}_R$ and $\mathfrak{m}_T$ the corresponding maximal ideals.

Recall that both R and T are factorial rings because S and B are smooth varieties, and also that f induces an injection $f^* : T \hookrightarrow R$ (because it is surjective).

Let $d \in \mathfrak{m}_R$ be a local equation for D near p, which has no factors in $f^*\mathfrak{m}_T$ because D has no component contracted by f. Let

$$\widetilde{\alpha} \in L_p \cong R$$

be a germ of section of L at p that restricts to the germ of α in

$$L_{|D,p} \cong R/\langle d \rangle .$$

The condition $\phi\alpha = 0$ means that $(f^*\phi_p)\,\widetilde{\alpha} \in \langle d \rangle$. But the factoriality of R and the fact that d has no factors in $f^*\mathfrak{m}_T$ imply that either $\widetilde{\alpha} \in \langle d \rangle$ or $f^*\phi_p = 0$. But the first condition cannot happen for every $p \in D \cap f^{-1}(U)$, since it would imply that $\alpha = 0$. Hence we obtain that for some p we have $\phi_p = 0$, and therefore $\phi = 0$, as wanted. □

Remark 6.3.3. *If the generalizations pointed out in Remark 4.3.18 do actually work, it is also very reasonable to expect higher-dimensional analogues of Proposition 6.2.2 and Theorem 6.3.1, obtaining some extra structure for one-dimensional compact families of irregular varieties such that the fibre-wise deformations are supported on rigid divisors.*

We use Theorem 6.3.1 to prove the following result. Recall (Definition 3.4.1) that the Clifford index of the fibration is defined as

$$c_f = \max\{\mathrm{Cliff}\,(C_b) \mid C_b \text{ is smooth}\}.$$

Theorem 6.3.4. *Let $f : S \to B$ be a fibration of genus g, relative irregularity q_f and Clifford index c_f. If f is non-isotrivial, then*

$$q_f \leq g - c_f.$$

Remark 6.3.5. *Theorem 6.3.4 can be interpreted as the most general case of Conjecture 6.1.7. In fact, a general curve of genus g has Clifford index $\left\lfloor \frac{g-1}{2} \right\rfloor$. Hence if the fibres are general in moduli, Theorem 6.3.4 says that the fibration satisfies*

$$q_f \leq g - \left\lfloor \frac{g-1}{2} \right\rfloor = \left\lceil \frac{g+1}{2} \right\rceil,$$

which coincides with the bound predicted in Conjecture 6.1.7.

Observe that if f is not trivial and $q_f > 0$, the fibres cannot have very general moduli. Indeed, in this case, the fibres must have decomposable Jacobian varieties, and such curves form a countable union of closed subsets $\mathcal{Z}$ in the moduli space $\mathcal{M}_g$. Anyway, the locus of curves with maximal Clifford index is open in $\mathcal{M}_g$, hence it intersects all but (at most) finitely many of the components of $\mathcal{Z}$ and Conjecture 6.1.7 holds for general non-isotrivial fibrations.

In order to improve Theorem 6.3.4, it would be interesting to study the incidence relations between the components of $\mathcal{Z}$ and the strata of $\mathcal{M}_g$ given by the Clifford index.

Remark 6.3.6. *Although in general our bound is better than the general one (6.1) proved by Xiao, for small c_f our Theorem is worse. As a extremal case, if the general fibres are hyperelliptic, $c_f = 0$ and Theorem 6.3.4 has no content at all. But in this special case we can prove that the strong inequality $q_f \leq \frac{g+1}{2}$ holds using the rigidity results of Pirola in [32].*

In order to prove Theorem 6.3.4, we first need to produce (modulo change of base) a subsheaf $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ admitting a lifting to Ω^1_S. Using Theorem 5.3.4, we may obtain a rank-two vector bundle $\mathcal{W} \subseteq V \otimes \mathcal{O}_B$ with vanishing adjoint map. We show now in Proposition 6.3.7 how such a $\mathcal{W}$ leads to the existence of the wanted subsheaf $\mathcal{L}$, giving two essentially different proofs. In the first proof we use the Adjoint Theorem 5.1.4 and the results on global deformations of Section 4.3, while in the second one we directly construct the lifting $\mathcal{L} \hookrightarrow \Omega^1_S$ along the ideas in the proof of the Volumetric Theorem 5.1.5.

Recall from Section 4.3 the projective bundle $\pi : \mathbb{P} \to B$ associated to the sheaf $\mathcal{E}\mathrm{xt}^1_f \left(\Omega^1_{S/B}, f^*\omega_B\right)$, the subvariety $\mathbb{P}_\Gamma \subseteq \mathbb{P}$ associated to any subscheme $\Gamma \subset S$, and the section $\gamma : B \to \mathbb{P}$, which is defined for any non-isotrivial fibration and a generic value $b \in B$ to the class of ξ_b. Recall also that ξ is supported on Γ if and only if the image of γ lies on $\mathbb{P}_\Gamma$.

Proposition 6.3.7. *If $\mathcal{W} \subseteq V \otimes \mathcal{O}_B$ is a rank-two vector subbundle whose associated adjoint map vanishes, then the subsheaf*

$$\mathcal{L} = \mathrm{im}\left(f^*\mathcal{W} \to \Omega^1_{S/B}\right) \subseteq \Omega^1_{S/B}$$

lifts to Ω^1_S and $\deg\left(\mathcal{L}_{|C}\right) > 0$ for a general fibre C.

First proof of Proposition 6.3.7. If f is isotrivial, then the sequence

$$\xi : 0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/B} \longrightarrow 0$$

is split and there is nothing to prove. Hence, we may assume from now on that f is not isotrivial.

There exist an effective divisor $D \subset S$ and a non-empty open subset $U \subseteq B$ of regular values such that

$$\mathcal{L}_{|f^{-1}(U)} \cong \omega_{S/B}(-D)_{|f^{-1}(U)}.$$

Indeed, the image sheaf

$$\widetilde{\mathcal{L}} = \operatorname{im}\left(\mathcal{L} \hookrightarrow \Omega^1_{S/B} \xrightarrow{\alpha} \omega_{S/B}\right) \subseteq \omega_{S/B}$$

is of the form $\widetilde{\mathcal{L}} = \omega_{S/B} \otimes I_\Gamma$ for some subscheme $\Gamma \subset S$. Then it is enough to take D to be the divisorial part of Γ, and U to be the complement in B of the image of the critical points of f and the embedded or isolated points of Γ.

Then, for any $b \in U$ we have that C_b is smooth and $\mathcal{L}_{|C_b} \cong \omega_{C_b}(-D_b)$ (where $D_b = D_{|C_b}$). Furthermore, by construction, D_b is the base locus of the linear subsystem $|W_b| \subseteq |\omega_{C_b}|$ of the canonical system of C_b given by $W_b \subseteq H^0(C_b, \omega_{C_b})$, the fibre of $\mathcal{W}$ over b. This implies in particular that $\deg D_b < 2g-2$, hence $\deg\left(\mathcal{L}_{|C_b}\right) > 0$.

Now, by the Adjoint Theorem 5.1.4, the deformation ξ_b is supported on D_b. Since this happens for b varying on a non-empty subset of B, it implies that $\gamma(B)$ is contained in $\mathbb{P}_D$, that is, ξ is supported on D (by Proposition 4.3.14). According to Corollary 4.3.15, ξ is also supported on Γ, and by Lemma 4.3.17, the pullback ξ_Γ is split and $\mathcal{L}_\Gamma \subseteq \Omega^1_{S/B}$ lifts to Ω^1_S.

Finally, since the composition $\mathcal{L} \hookrightarrow \Omega^1_{S/B} \to \omega_{S/B|\Gamma}$ vanishes by definition, there is an inclusion $\mathcal{L} \subseteq \mathcal{L}_\Gamma$ and hence $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ also lifts to Ω^1_S. $\square$

Second proof of Proposition 6.3.7. The fact that $\deg\left(\mathcal{L}_{|C}\right) > 0$ for a general fibre C follows, as above, from the fact that $\mathcal{L}_{|C} \hookrightarrow \omega_C$ is generated by two linearly independent 1-forms on C.

Recall the injection of sheaves on B

$$V \otimes \mathcal{O}_B \hookrightarrow f_*\Omega^1_{S/B} \tag{6.3}$$

introduced in Remark 3.2.7. We now construct an injective map of vector bundles $\mathcal{W} \hookrightarrow f_*\Omega^1_S$ completing the diagram

$$\begin{tikzcd}
\mathcal{W} \arrow[r, hook] \arrow[d, dashed] & V \otimes \mathcal{O}_B \arrow[d, hook] \\
f_*\Omega^1_S \arrow[r] & f_*\Omega^1_{S/B}
\end{tikzcd}$$

in such a way that the image of the composition $f^*\mathcal{W} \hookrightarrow f^*f_*\Omega^1_S \to \Omega^1_S$ is a sheaf $\mathcal{L}_0$ of rank 1. In fact, we construct a subsheaf $\mathcal{G} \hookrightarrow f_*\Omega^1_S$ isomorphic to $\mathcal{W}$ and satisfying the last property. We build $\mathcal{G}$ giving its sections over any sufficiently small affine open subset $U \subseteq B$, which will be a free $H^0(U, \mathcal{O}_B)$-module of rank 2. More precisely, let $U \subset B$ be any open affine subset where $\mathcal{W}$ is trivial, and let $s_1, s_2 \in H^0(U, \mathcal{W})$ be two sections giving the isomorphism $\mathcal{O}_U^{\oplus 2} \cong \mathcal{W}_{|U}$. Let $\widetilde{\mathcal{W}} \subseteq H^0(S, \Omega^1_S) \otimes \mathcal{O}_B$ be the preimage of $\mathcal{W}$ by the natural projection $H^0(S, \Omega^1_S) \otimes \mathcal{O}_B \to V \otimes \mathcal{O}_B$, and let

$$\widetilde{s_i} \in H^0\left(U, \widetilde{\mathcal{W}}\right) \subseteq H^0\left(U, H^0\left(S, \Omega^1_S\right) \otimes \mathcal{O}_B\right)$$

be any preimages of the s_i.

The vanishing of the adjoint map of $\mathcal{W}$ means that the image of $\bigwedge^2 \widetilde{\mathcal{W}}$ in $f_*\omega_S$ lies in the subsheaf $\mathcal{W} \otimes \omega_B$. This implies that the wedge product $\widetilde{s_1} \wedge \widetilde{s_2}$ belongs to $H^0(U, \mathcal{W} \otimes \omega_B) \cong H^0(U, \omega_B) \otimes_{\mathbb{C}} \mathbb{C}\langle s_1, s_2\rangle$, and therefore there are uniquely determined $\beta_i \in H^0(U, \omega_B)$ such that

$$\widetilde{s_1} \wedge \widetilde{s_2} = \widetilde{s_1} \wedge f^*\beta_2 - \widetilde{s_2} \wedge f^*\beta_1,$$

or equivalently

$$(\widetilde{s_1} - f^*\beta_1) \wedge (\widetilde{s_2} - f^*\beta_2) = 0. \tag{6.4}$$

Define $\sigma_i = \widetilde{s_i} - f^*\beta_i$, and let $\mathcal{G}_{|U} \subseteq (f_*\Omega^1_S)_{|U}$ be the subsheaf generated by σ_1, σ_2. The uniqueness of the β_i implies that the σ_i are independent of the choice of the $\widetilde{s_i}$. Furthermore, if s'_1, s'_2 is another pair of trivializing sections of $\mathcal{W}_{|U}$, the σ'_i obtained imposing (6.4) will be $\mathcal{O}_B(U)$-linear combinations of the σ_i. Therefore, $\mathcal{G}_{|U}$ is well-defined. Moreover, if U' is another open affine subset trivializing $\mathcal{W}$, the definitions of $\mathcal{G}_{|U}$ and $\mathcal{G}_{|U'}$ must agree on $U \cap U'$ by uniqueness, so there is a well defined locally free sheaf $\mathcal{G} \hookrightarrow f_*\Omega^1_S$ of rank two, and it is clearly isomorphic to $\mathcal{W}$, as wanted.

Furthermore, since local sections of $\mathcal{G}$ wedge to zero, clearly the subsheaf $\mathcal{L}_0 \subseteq \Omega^1_S$ generated by $f^*\mathcal{G}$ is of rank one.

We finally show that $\mathcal{L}_0$ maps isomorphically to $\mathcal{L}$. It is obvious by construction that there is a surjective map $\mathcal{L}_0 \twoheadrightarrow \mathcal{L}$. But this map must also be injective, since both sheaves have the same rank and $\mathcal{L}_0$ is torsion-free (it is a subsheaf of a locally free sheaf). □

We can now proceed with the

Proof of Theorem 6.3.4. Suppose, looking for a contradiction, that the fibration $f : S \to B$ is non-isotrivial and that $q_f > g - c_f$. If f' is the fibration obtained after finitely mainy blow-ups at points and a change of base, it is still non-isotrivial, and Corollary 3.1.9 gives that $q_{f'} \geq q_f$. Hence we may apply Lemma 3.1.12 and assume in addition that f has reduced fibres.

Since in general it holds that $c_f \leq \frac{g-1}{2}$, our assumptions imply in particular that $q_f > \frac{g+1}{2}$. Hence, by Theorem 5.3.4 we may assume (possibly after another change of base) that there exists a vector subbundle $\mathcal{W} \subseteq V \otimes \mathcal{O}_B$ of rank 2 and with vanishing adjoint map. Now Proposition 6.3.7 gives a subsheaf $\mathcal{L} \subseteq \Omega^1_{S/B}$ such that the inclusion lifts to Ω^1_S. By Proposition 6.2.2, we may assume that $\mathcal{L} \cong \omega_{S/B}(-D)$ for some effective divisor D such that $d = D \cdot C < 2g - 2$ for any fibre C.

We consider now two cases:

Case 1: The divisor D is relatively rigid, that is $h^0(C, \mathcal{O}_C(D)) = 1$ for some smooth fibre $C = C_b$. In this case we can apply Theorem 6.3.1 to obtain a new fibration $h : S \to B'$ over a curve of genus $g(B') = q_f$ (after a change of base). Let $\phi : C \to B'$ be the restriction of h to the smooth fibre C. Applying Riemann-Hurwitz we obtain

$$2g - 2 \geq \deg \phi\,(2q_f - 2).$$

But we have from the beginning that $q_f > \frac{g+1}{2}$, so that $2q_f - 2 > g - 1$, and thus

$$2(g-1) > \deg \phi\,(g-1).$$

It follows that $\deg \phi = 1$ (recall that we have assumed $g \geq 2$ from the beginning), so every smooth fibre is isomorphic to B' and hence f is isotrivial.

Case 2: The divisor D moves on any smooth fibre, i.e. $h^0(C_b, \mathcal{O}_{C_b}(D)) \geq 2$ for every regular value $b \in B$.

After a further change of base, we may assume that D consists of d sections of f (possibly with multiplicities), and the new fibration is still supported on D (Lemma 4.3.16). Then we can replace D by a minimal subdivisor $D' \leq D$ such that ξ is still supported on D'. Since D is composed by sections of f, this implies that for general $b \in B$, the deformation ξ_b is minimally supported on $D_{|C_b}$. Note that this might not be true if the supporting divisors were not a union of sections, as different points of $D_{|C_b}$ lying on the same irreducible component of D may be redundant.

If this new D is relatively rigid, the proof finishes as in Case 1. Otherwise, if still $h^0(C_b, \mathcal{O}_{C_b}(D)) \geq 2$ for generic $b \in B$, we may use Theorem 4.1.17 to obtain

$$\operatorname{rk} \xi_b \geq \operatorname{Cliff}\left(D_{|C_b}\right) = c_f. \tag{6.5}$$

But $V \subseteq \ker \partial_{\xi_b} = K_{\xi_b}$, so that $\operatorname{rk} \xi_b = g - \dim K_{\xi_b} \leq g - q_f$, and the inequality (6.5) implies that

$$g - q_f \geq c_f,$$

which contradicts our very first hypothesis.

□

Remark 6.3.8. *Note that, whenever we can produce a relatively rigid divisor D supporting the fibration, the inequality $q_f > \frac{g+1}{2}$ is enough (together with the structure Theorem 6.3.1) to prove that the fibration f is isotrivial, while the stronger inequality $q_f > g - c_f$ is used only if it is impossible to find such a D (allowing arbitrary changes of base). Hence, all possible counterexamples to Xiao's original conjecture must fall into this second case.*

We wish to close this final section with Proposition 6.3.9, which gives an alternative proof of Case 1 in the proof of Theorem 6.3.4. This Proposition uses the Volumetric Theorem 5.1.5 instead of Theorem 6.3.1, hence applies for non-necessarily compact families. On the contrary, the compactness of the surface is crucial in Theorem 6.3.1, since it uses the Castelnuovo-de Franchis Theorem.

Proposition 6.3.9. *Suppose that $f : S \to B$ is a fibration where the base B is a smooth, not necessarily compact curve. Assume that there is an Abelian variety A of dimension a, and a morphism $\Phi : S \to A \times B$ respecting the fibres of f and such that the image of any restriction to a fibre $\phi_b : C_b \to A$ generates A. Suppose also that the deformation is supported on a divisor $D \subset S$ such that $h^0(C_b, \mathcal{O}_{C_b}(D)) = 1$ for general $b \in B$. If $a > \frac{g+1}{2}$, then f is isotrivial.*

Remark 6.3.10. *If we start from a fibration with compact B, we may take A to be the kernel of $a_f : \mathrm{Alb}(S) \to J(B)$, which has dimension $a = q_f$. After replacing B by an open disk, the Albanese map gives a morphism Φ as above (for more details, see Section 3.3). Hence, Proposition 6.3.9 gives indeed a new proof of the first case in the proof of Theorem 6.3.4 above.*

Proof of Proposition 6.3.9. Take any $b \in B$ such that C_b is smooth, and let $\widetilde{C_b}$ be the image of $\phi_b : C_b \to A$. Since $\widetilde{C_b}$ generates A, it has genus $g' \geq \dim A = a > \frac{g+1}{2}$. This implies, by Riemann-Hurwitz, that ϕ_b is birational onto its image for any regular value $b \in B$.

If f is not isotrivial, the Volumetric Theorem 5.1.5 implies that, for a general fibre $C = C_b$, the adjoint class of a generic 2-dimensional subspace

$$W \subseteq V := H^0\left(A, \Omega^1_A\right) \subseteq H^0\left(C, \omega_C\right)$$

is non-zero.

However, we show now that, for *every* fibre, the adjoint class of *every* 2-dimensional subspace of V vanishes, which finishes the proof. Fix any regular value $b \in B$ and denote by $C = C_b$ the corresponding fibre, by $\xi = \xi_b$ the infinitesimal deformation induced by f, and by $D = D_{|C}$ the restriction of the global divisor. Let also $K = K_\xi$ be the kernel of ∂_ξ. Since ξ is supported on D, Lemma 4.1.12 gives the inclusion $H^0\left(C, \omega_C\left(-D\right)\right) \subseteq K$, which is in fact an equality. Indeed, on the one hand we have

$$\dim H^0\left(C, \omega_C\left(-D\right)\right) = g - \deg D$$

because D is rigid, while on the other hand it holds

$$\dim K = g - \operatorname{rk} \xi = g - \deg D$$

because of the combination of Corollary 4.1.14 and Theorem 4.1.17. Therefore, $V \subseteq K = H^0\left(C, \omega_C\left(-D\right)\right)$.

Now, according to Remark 4.1.11, the upper sequence in

$$\begin{tikzcd}
\xi_D : & 0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, equal] & \mathcal{F}_D \arrow[r] \arrow[d, hook] & \omega_C\left(-D\right) \arrow[r] \arrow[d, hook] \arrow[l, dashed, bend right] & 0 \\
\xi : & 0 \arrow[r] & \mathcal{O}_C \arrow[r] & \Omega^1_{S|C} \arrow[r] & \omega_C \arrow[r] & 0
\end{tikzcd}$$

is split, giving a lifting $\omega_C\left(-D\right) \hookrightarrow \Omega^1_{S|C}$ such that every pair of elements of $H^0\left(C, \omega_C\left(-D\right)\right) \subseteq H^0\left(C, \Omega^1_{S|C}\right)$ wedge to zero (they are sections of the same sub-line bundle of $\Omega^1_{S|C}$), which finishes the proof. $\square$

Remark 6.3.11. *In the above proof, to show that the images $\widetilde{C_b}$ are all isomorphic it is only necessary to use the Volumetric Theorem 5.1.5. The inequality $a > \frac{g+1}{2}$ is only used, combined with Riemann-Hurwitz, to show that the maps ϕ_b are birational. Therefore, if we drop the inequality $a > \frac{g+1}{2}$ from the hypothesis (but still keep that the deformations are supported on rigid divisors), the same proof shows that the fibres C_b are coverings of a fixed curve $\widetilde{C_b}$.*

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