Duality Theory and Abstract Algebraic Logic

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A mis padres y a mis hermanos
A Juan

Abstract

In this thesis we present the results of our research on duality theory for non-classical logics under the point of view of Abstract Algebraic Logic. Firstly, we propose an abstract Spectral-like duality and an abstract Priestley-style duality for every filter distributive finitary congruential logic with theorems. This proposal aims to unify the various dualities for concrete logics that we find in the literature, by showing the abstract template in which all of them fit. Secondly, the dual correspondence of some logical properties is examined. This serves to reveal the connection between our abstract dualities and the concrete dualities related to concrete logics. We apply those results to get new dualities for suitable expansions of a well-known logic: the implicative fragment of intuitionistic logic. Finally, we develop a new technique that can be modularly applied to simplify some of the obtained dualities.

Resumen

En esta tesis presentamos los resultados de nuestra investigación acerca de la teoría de la dualidad para lógicas no clásicas desde el punto de vista de la Lógica Algebráica Abstracta. En primer lugar, proponemos una dualidad abstracta de tipo espectral y otra dualidad abstracta de tipo Priestley para cada lógica congruencial, filtro distributiva, finitaria y con teoremas. Esta propuesta pretende unificar las distintas dualidades para lógicas no clásicas que encontramos en la literatura, mostrando el esquema abstracto en el que todas ellas encajan. En segundo lugar, la correspondencia dual de algunas propiedades lógicas es examinada. Esto sirve para revelar la conexión que existe entre nuestras dualidades abstractas y las dualidades concretas relacionadas con lógicas concretas. Aplicamos estos resultados para obtener nuevas dualidades para expansiones apropiadas de una lógica bien conocida: el fragmento implicativo de la lógica intuicionista. Finalmente, desarrollamos una nueva técnica que puede ser aplicada de forma modular para simplificar algunas de las dualidades obtenidas.

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Thanks to a Master fellowship from Obra Social Caja España, I could come to study the Master in Barcelona, and thanks to a FPU doctoral grant from the spanish Ministry of Education, Culture and Sport, I could stay in Barcelona four more years and finish my PhD. During these years I met a lot of people along the way, that made that the task that lay ahead was more enjoyable. This dissertation could not have been written without the support and friendship found at Barcelona. Pablo welcomed me so charming at the very beginning and Marta and María helped me a lot at the very end. Nuria and Manel provided me with a home, and it was so nice to have Bernat and Nuno as neighbors. I will miss so much lunch at *el office* with Luz, Pepe, Viviane, Blanca, Sergi, Luciano, Daniel, Gonçalo,... I would like to thank you all for being there, and also the people of the Logic Team and of Logos.

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In Amsterdam (and also later on in Barcelona) I had the opportunity of discussing with Alessandra, whose work has been very inspiring to this dissertation. I could also feel the great atmosphere of the ILLC, and I met many people that made the stay so great. I would like to thank all of them, and specially Margot. In Tandil (and also later on in Barcelona) I had the pleasure of working with Sergio, that also shared with me *asados* and walks. Ramon, Ismael, Felix and Carles helped to make the stay nicer. In Paris I had the chance of discussing and working with Mai, whose enthusiasm and devotion about mathematics makes any conversation with her amazing. I also enjoyed a lot the discussions with Sam, that definitely transmits the same passion about what he does. Besides, Margot and my flatmates at Télégraphe made so pleasant my stay there.

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Cuando el cocodrilo entró en mi dormitorio pensé que tampoco había que exagerar. No me refiero al cocodrilo sino a mí mismo. Ya que mi primer impulso fue alcanzar el teléfono y marcar los tres números de urgencias: policía, bomberos y ambulancia. Pero justamente semejante reacción me pareció exagerada. Puesto que soy un europeo educado en el espíritu cartesiano, siento repulsión por los extremismos, pienso de un modo racional y no sucumbo a impulsos de ningún tipo sin haberlos analizado previamente.

Así que me cubrí la cabeza con el edredón y emprendí un trabajo mental.

Primero –determiné– la aparición de un cocodrilo en mi dormitorio es un absurdo, y, según el pensamiento lógico, el absurdo sirve sólo para ser excluido del razonamiento ulterior. O sea que no había ningún cocodrilo. Tranquilizado con esta conclusión, asomé la cara por debajo del edredón, gracias a lo cual logré ver cómo el cocodrilo cortaba de un mordisco el cable del aparato telefónico, ya anteriormente devorado por él. Incluso en el caso de que alargando la mano a través de sus fauces hasta el estómago consiguiera marcar uno de los números de urgencias, la comunicación ya estaba cortada.

(Fragmento de “Un europeo”, en Mrożek, Sławomir: Juego de azar)

Contents

Introduction and Summary of Contents

The main goal of this dissertation is to show that Abstract Algebraic Logic provides the appropriate theoretical framework for developing a uniform and abstract duality theory for non-classical logics.

Algebraic Logic can be seen as the study of logics through the study of different sorts of algebra related mathematical structures. The basic point is the study of how algebra-based structures can be associated to a given logic, so they provide, in a broad sense, algebraic semantics for the logic. The notion of algebraic semantics is very natural under a mathematical point of view, as it is usually very close to the syntactic presentation of the logics. However, other semantic approaches such as Kripke-style semantics have traditionally led to more intuitive comprehension of the nature of non-classical logics. These two semantic approaches are, in many cases, two sides of the same coin. And this can be shown through the powerful tools that category theory provides us with.

Duality Theory

Duality theory usually refers to the study of categorical dualities in mathematics. *Category theory* is commonly considered an appropriate mathematical framework for the study of the relations between mathematical objects of different nature. One of the basic points of this theory is that not only the objects are taken into account, but also the morphisms that transform an object into another. For example, for an arbitrary logic, we might define a category by considering as objects the algebra related structures that are associated with the logic, and as morphisms the homomorphisms between the underlying algebras (perhaps requiring some additional conditions). A (categorical) duality is a special relation between two different categories. We should say, to be more precise, a dual equivalence of categories. This is a precise formulation of two related correspondences: one between the objects of the two categories, and the other between the morphisms between two objects of one of the categories, and the morphisms between the corresponding objects of the other category. The key point of a duality is that within these correspondences, morphisms are reversed. And this is precisely what makes a dual equivalence of categories such an interesting phenomenon under a mathematical point of view.

In relation to (mathematical) logic, the work by Stone on representation of Boolean algebras [**69**] is usually taken as the pioneering work of a fruitful field of study, that we refer to simply as *Stone/Priestley duality*. Stone studied how Boolean algebras, that are the algebraic counterpart of classical logic, can be dually described in terms of compact totally disconnected Hausdorff spaces, that are called Stone spaces (a. k. a. Boolean spaces). This yields to a dual equivalence of categories, one having algebras as objects, and the other having topological spaces as objects. And it also results in an elegant proof of completeness of classical logic with respect to truth table semantics. What makes Stone duality a powerful mathematical tool is precisely the fact that it is a dual equivalence of categories. This implies, for instance, that dual of injectivity is surjectivity (and vice versa), or duals of homomorphic images are closed sets.

Further generalizations of Stone's approach yield to *Spectral-like* and *Priestley-style* dualities for distributive lattices. Throughout these and other related dualities, we can build bridges between algebraic and Kripke-style semantics of some non-classical logics. For example, we can use extended Stone duality for Boolean algebras with operators [**58**] to prove completeness of the local and the global consequences of normal modal logics with respect to Kripke frames. Or we can use Esakia duality for Heyting algebras [**30**] to prove completeness of intuitionistic propositional logic with respect to intuitionistic Kripke frames.

One of the strengths of Stone/Priestley duality is precisely that it allows us to use topological tools in the study of logic. Of particular interest is the theory of *extended Priestley duality*. Within this theory, a modular account of a wide range of dualities for suitable expansions of distributive lattices is carried out. More precisely, Priestley duality for distributive lattices is used to get the basic building blocks over which the dualities for the expansions are built. A suitable expansion of a distributive lattice is dually represented by a suitable expansion of the dual Priestley space of the underlying distributive lattice. This theory accounts uniformly for a wide range of dualities for non-classical logics, such as the already mentioned dualities for Boolean algebras with operators.

A closely related topic that also yields to bridges between algebraic and Kripke-style semantics of non-classical logics, is the theory of *canonical extensions*. This theory consists of the study of how a given algebraic structure (that is usually lattice-based) can be embedded in a complete lattice. Specific properties, such as compactness and denseness, are required for such embedding. By constructing the canonical extensions of algebras related with non-classical logics, and then applying discrete dualities (i. e. dualities that do not involve topology) to the resulting complete algebras, completeness of non-classical logics with respect to Kripke-style semantics can also be proven. This is usually an alternative way of getting such completeness results, but there are cases when it is the only known way, as in the case of several substructural logics that were studied in [**26**].

In summary, duality theory in (mathematical) logic has been proven to be a fruitful field of study from which, among other results, completeness with respect to Kripke-style semantics of a wide range of non-classical logics has been proven. We are interested in the topological dualities that are encountered along the way, and more precisely, in the so called Spectral-like and Priestley-style dualities. We aim to develop a common framework that reports all these dualities in a uniform way.

Abstract Algebraic Logic

Abstract Algebraic Logic (AAL from now on) is a general theory of the algebraization of logic. Emphasis is put on the general process of associating logics and algebra related mathematical structures, regardless of the language and the way in which the logics are defined. Algebras, logical matrices and generalized matrices have been used to develop a general and uniform procedure for canonically associating the class of $\mathcal{S}$-algebras $\mathbb{A}\mathrm{lg}\mathcal{S}$, the class of reduced $\mathcal{S}$-algebras $\mathbb{A}\mathrm{lg}^*\mathcal{S}$ or the intrinsic variety of $\mathcal{S}$, to any logic $\mathcal{S}$. In [**35**] it is argued why $\mathbb{A}\mathrm{lg}\mathcal{S}$ should be taken as the canonical algebraic counterpart of any arbitrary logic, and it is currently considered so in AAL.

From this point of view, the study of which metalogical properties of the logics correspond with which algebraic properties of the related algebras (or algebra-based structures) is one of the main topics. These results are known as bridge theorems. For example, it is well-known that for any algebraizable logic $\mathcal{S}$, to have a deduction theorem is related to the property that the members of $\mathbb{A}\mathrm{lg}\mathcal{S}$ have uniformly equationally definable principal relative congruences.

Another topic of AAL is to classify logics according to their abstract properties or to the algebraic properties of their algebraic counterparts. Mainly two hierarchies have been studied in depth: the Frege hierarchy and the Leibniz hierarchy. The Frege hierarchy is a classification scheme of logics under four classes defined in terms of congruence properties of the algebraic counterparts of the logics. The Leibniz hierarchy presently consists of twenty different classes of logics, that form a lattice when ordered by inclusion, whose bottom element is the class of implicative logics.

The notion of closure operator is one of the key points of this field of study. It is worth mentioning that within AAL, logics are studied as deductive systems, this is, as systems concerning validity of inferences, instead of validity of formuli. Formally, a logic is a substitution invariant closure relation on a formula algebra. This approach is precisely what allows us to study in a uniform way logics that have been defined according to different methods, such as Hilbert-style presentations, natural deduction, Gentzen calculus, tableaux, algebraic semantics, relational semantics, game-theoretic semantics, etc. Moreover, this takes us to another interesting topic in AAL, namely, the study of how to define logics from classes of algebras (or algebra based structures).

In summary, although AAL is a relatively young field of study, its value for a uniform account of the study of non-classical logics has been largely proven. In particular, the canonical algebraic counterpart of any arbitrary logic has been studied. Given this abstract and general study of the algebra-based structures that are canonically associated with arbitrary logics, the following question arises: can we regard all dualities for non-classical logics under this abstract point of view, and search for an abstract and general duality theory that unifies all the results that are scattered throughout the literature?

Duality Theory and Abstract Algebraic Logic

In the late eighties, Wójcicki studied in his *Theory of logical calculi* [**73**] which abstract properties of the logics allow us to define a referential semantics for them. The concept of referential semantics aims to capture under a uniform point of view different semantic approaches such as frame semantics, Kripke-style semantics, relational semantics, etc. The basic underlying idea is that the truth values of the formuli depend on reference points. Thus models are based on non-empty sets of reference points, endowed with additional structure. Each propositional variable is assigned to a subset of points, and this assignment is extended to any formuli using the additional structure, that is always explicitly or implicitly algebra-based.

Wójcicki identified selfextensional logics as those that admit referential semantics. Selfextensional logics might be defined, in brief, as those logics for which the relation between two formuli of having the same consequences is a congruence in the formula algebra. This is one of the four classes of algebras of the Frege hierarchy, the others being congruential logics, Fregean logics and fully Fregean logics.

Wójcicki studied under an abstract perspective how referential semantic models for selfextensional logics correspond with the algebra based structures that AAL associates with such logics. The correspondence studied by Wójcicki was recently formulated by Jansana and Palmigiano as a proper categorical duality in [**56**]. More specifically, they proved that for any selfextensional logic $\mathcal{S}$, there is a dual equivalence between the category of reduced generalized $\mathcal{S}$-models and generalized morphisms between them, and the category of reduced $\mathcal{S}$-referential algebras and strict homomorphisms between them. They also studied the restriction of this duality to congruential (a. k. a. fully selfextensional or strongly selfextensional) logics. These logics are particularly well-behaved selfextensional logics. They can be described as those logics for which the property of selfextensionality *transfers* to every algebras. We do not go into details here. What we aim to highlight is that these studies already tackled the problem of developing a uniform account of the bridges that can be built between algebraic and referential semantics for non-classical logics under an abstract algebraic logic point of view.

In [**56**] it was remarked that their duality serves as a general template where a wide range of Spectral-like and Priestley-style dualities related with concrete logics can fit. But they do not inquire further into this topic, and their construction is rather far from those concrete examples. Our aim is to provide a construction as general as possible, but closer to the dualities for non-classical logics that are already well-known.

To accomplish this, the study of various recent dualities for non-classical logics that we find in the literature was a crucial step. It is worth noting that until the mid-2000s, all categories of algebras (and homomorphisms) for which Spectral-like or Priestley-style dualities had been studied were built upon lattice-based algebras, in most cases distributive. In recent literature, however, we find studies that extend the same ideas beyond the distributive lattice setting. There have been considered categories whose objects are the algebraic counterparts of certain fragments of intuitionistic logic, that do not have conjunction and disjunction at the same time or that do not have any of these connectives [**5, 6, 11, 15, 18, 19**].

It was precisely through the study of those dualities that we came up with the appropriate notions of filters upon which the general theory can be formulated in such a way that it subsumes all the related results in the literature. For the Spectral-like duality, the notion of irreducible logical filter is the basic tool. These are the meet-irreducible elements of the lattice of logical filters. For the Priestley-style duality, the issue is more involved. Optimal logical filters turn out to be the right tools, and these filters are defined through another notion of strong logical ideal, that will be introduced when appropriate.

The approach that we adopted is the following: we examined sufficient conditions for a logic in order to posses a Spectral-like or a Priestley-style duality for the class of algebras $\mathbb{A}\mathrm{lg}\mathcal{S}$, that is the one canonically associated to it according to

AAL. Obviously, from the mentioned work by Wójcicki [**73**] and by Jansana and Palmigiano [**56**], it follows that we should focus on congruential logics. Moreover, we encountered that we need to restrict ourselves to filter distributive logics. This class of logics, first studied by Czelakowski [**21**], consists of the logics for which the collection of logical filters of any algebra is a distributive lattice. Such property is satisfied by any logic with a disjunction, or with the deduction-detachment theorem, among others. Filter-distributivity comes as a natural assumption, given that distributivity, as it was already mentioned, is the ground assumption for Spectral-like and Priestley-style dualities. Further assumptions over the logics are finitarity and having theorems. Finitarity is used to get necessary separation lemmas, and having theorems is a technical requirement that we assume for convenience.

All these properties of logics have been extensively studied within AAL, and they are satisfied by many logics such as classical logic (and its fragments), intuitionistic logic (and its fragments), local consequences of modal logics,... Some well-known logics such as relevance logic R and Łukasiewicz's infinite-valued logic are not congruential (according to their usual formulation). However, in [**53–55**] several strategies have been studied according to which a logic $\mathcal{S}$ with certain properties can be endowed with a congruential logic companion $\mathcal{S}'$ so that $\mathbb{A}\mathrm{lg}\mathcal{S} = \mathbb{A}\mathrm{lg}\mathcal{S}'$. And these strategies apply, in particular, for those mentioned logics.

In the present dissertation we prove that there is a Spectral-like duality and a Priestley-style duality associated with any logic satisfying such conditions. The precise and detailed formulation of those dualities is one of the main contributions of this dissertation. Most of the dualities for non-classical logics that we encounter in the literature fit straightforwardly in our general pattern and, moreover, new dualities might potentially be studied out of it.

Those dualities for the category of $\mathcal{S}$-algebras and homomorphisms, for any filter distributive finitary congruential logic $\mathcal{S}$, are the base that supports all other results in the dissertation. Notice that we follow the spirit of AAL, and we work with a fixed but arbitrary language. Our abstract approach yields, as it also happened in [**73**] and in [**56**], to dual categories that also possess an algebraic nature. This drawback cannot be bypassed within this abstract program, since we need the arbitrary language to somehow be encoded in the dual spaces. However, the obstacle can be overcome for concrete logics. We usually refer to the dualities for which the dual categories have no explicit algebraic natura, as *elegant* dualities. Through the study of the dual properties that correspond with several properties of logics, we come up with a modular account of how the nature of the dual spaces can be substantially simplified whenever the logics are sufficiently well-behaved.

This analysis, however, is not satisfactory for some logics that are not so well-behaved, that is to say, we address this problem under a more concrete point of view. We focus on some expansions of the implicative fragment of intuitionistic logic, for which our general theory does not supply *elegant* dualities. Our contribution consists of the development of a new strategy for defining dualities for such expansions in a modular way. For this part of the dissertation, we were inspired by extended Priestley duality. In the same way as the duality for distributive lattice is taken as the cornerstone from which dualities for distributive lattice expansions are defined, we take the dualities for Hilbert algebras (that are the algebraic counterpart of the implicative fragment of intuitionistic logic) as the cornerstone from which dualities for suitable expansions of the implicative fragment of intuitionistic logic can be defined.

Summary of contents

We will now give a broad overview of the main contents of this dissertation. It has three linear parts that should preferably be read consecutively, since the first part introduces the preliminaries in which the other two are supported, and the second part presents the general theory with which the results in the third part are closely related.

Part 1. Preliminaries and Literature Survey. This part consists of three independent chapters. As its title indicates, we introduce here the preliminaries and basic notations, as well as a non-exhaustive account of the literature on duality theory for structures related with non-classical logics.

In **Chapter 1** we briefly discuss some of the mathematical background knowledge which we assume that the reader is familiar with and we introduce the notational conventions that we use throughout the dissertation. Of particular interest is § 1.6, where we revise with more detail some concepts from AAL that often appear later on.

The notion of closure operator plays a prominent role in AAL, and it is also a leading notion in this dissertation. Therefore the entire **Chapter 2** is devoted to the study of filters, ideals and separation lemmas associated with closure operators. Some of these results were already known, and others are new. Moreover, this serves as an excuse to introduce two algebraic structures with which we deal throughout this dissertation, namely *meet-distributive semilattices with top element* (distributive semilattices for short) and *Hilbert algebras*.

In **Chapter 3** we present what is meant by Stone/Priestley dualities or, according to the terminology that we introduce thereafter, by *Spectral-like* and *Priestley-style* dualities. We survey the Spectral-like and Priestley-style dualities for distributive semilattices and Hilbert algebras that we encounter in the literature. These are dualities located out of the setting of distributive lattices, and from their analysis we came up with the basic ideas for the abstract duality theory that is provided later on.

Part 2. Duality Theory for Filter Distributive Congruential Logics. This part is divided in two related chapters. We study in the first one the basic tools that we need to develop the theory of the second one.

In **Chapter 4** we argue about the interest of an abstract view of the duality theory for non-classical logics. We review previous works in this respect, and we bring in some notions such as *referential algebra*, *irreducible* and *optimal* $\mathcal{S}$-filters or $\mathcal{S}$*-semilattice* that provide us with the toolkit we need for the next chapter. This study encompasses some well-known results that were scattered throughout the literature, and some new results as well.

The bulk of this part of the dissertation is contained in **Chapter 5**, where the abstract Stone/Priestley dualities for a wide range of non-classical logics are systematically exposed. The analysis of the dual correspondence of the most important logical properties lead us to recover most of the dualities for non-classical logics that we find in the literature.

Part 3. Applications to Expansions of the Implicative Fragment of Intuitionistic Logic. Here we explore the applications of the work in Part 2 in a concrete setting, namely, we focus on Stone/Priestley dualities for logics that are expansions of the implicative fragment of intuitionistic logic. This part is again divided in two related chapters.

In **Chapter 6** we introduce several logics, all of which are expansions of the implicative fragment of intuitionistic logic. We study their properties and how the theory in Chapter 5 can be specialised for them. We show how the general theory yields to new dualities for them, but in some cases, those dualities are not *elegant*. We focus on the algebras associated with one of those troublesome logics, namely the class of Hilbert algebras with Infimum, and we study their properties in § 6.5.1.

Finally in **Chapter 7** we develop new Stone/Priestley dualities for a subclass of Hilbert algebras with Infimum. And these can be used, similarly to what is done in *extended Priestley duality*, to provide Stone/Priestley dualities for some of those troublesome logics. Therefore, the work in Part 3 can be seen as a refinement of the results of the theory in Part 2 for the particular case of expansions of the implicative fragment of intuitionistic logic.

Part 1

Preliminaries and Literature Survey

CHAPTER 1

Background and Notational Conventions

In this chapter we introduce the mathematical background that supports the main body of this dissertation. The purpose of this chapter is to fix the notation we use throughout the dissertation, and to introduce some notational conventions that should be kept in mind.

Notational issues concerning Set theory, Order and Lattice theory, Topology, Universal Algebra, Category theory and Abstract Algebraic Logic are treated in the following sections. Of special interest is the last section, where we introduce the concept of *logic* we work with.

Logics are denoted by calligraphic complexes of letters, e. g. $\mathcal{S}, \mathcal{H}, \mathcal{IPC}^+ \ldots$ Algebraic structures, in particular algebras, are denoted by combinations of boldface letters, e. g. $\mathbf{A}, \mathbf{B}, \mathbf{M}, \ldots$, and their universes (or carriers) by the corresponding light-face letters, $A, B, M, \ldots$ Classes of algebras are denoted by combinations of blackboard bold letters, with maybe additional superscripts or subscripts, e. g. $\mathbb{H}, \mathbb{DL}, \mathbb{DH}^\wedge$, and by '$\mathbb{K}$-*algebra*' we mean an algebra in the class $\mathbb{K}$. Categories are denoted by combinations of sans serif letters, with maybe additional superscripts or subscripts, e. g. $\mathsf{DL}, \mathsf{Pr}, \mathsf{H}_H, \ldots$ Dual spaces of algebras are denoted by combinations of letters beginning with a Fraktur capital letter, e. g. $\mathfrak{X}, \mathfrak{X}_1, \mathfrak{O}\mathrm{p}_\wedge(\mathbf{A}), \ldots$ The expression 'iff' is used as an abbreviation of 'if and only if'. The expression '... & ...' is used as an abbreviation of '... and ...'. When introducing formal definitions, we use ':='. The symbol '=' is used to express the fact that both sides name the same object, whereas '$\approx$' is used to build equations that may or may not be true of particular elements.

1.1. Set theory

We assume that the reader is familiar with elementary set theoretical notions such as *membership*, $x \in X$, *inclusion*, $Y \subseteq X$, *union*, $X \cup Y$, *intersection*, $X \cap Y$ and *difference*, $X \setminus Y$. By ω we denote the set of all natural numbers. For any subset $Y \subseteq X$, we use Y^c, meaning the *complement* of Y with respect to X, as an abbreviation of $X \setminus Y$. We write $Y \subseteq^\omega X$ to concisely say that $Y \subseteq X$ is a (possibly empty) finite subset of X. For X a set, $\mathcal{P}(X)$ denotes the powerset of X, i. e. the collection of all subsets of X. For an equivalence relation R on a set X, X/R denotes the quotient.

For $f : X \longrightarrow Y$ a function between sets X and Y, by $f^{-1}[\,] : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ we denote the *inverse image function*, that maps any set $U \subseteq Y$ to

$$f^{-1}[U] := \{x \in X : f(x) \in U\}.$$

Moreover, by $f[\,] : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ we denote the function that maps any set $U \subseteq X$ to

$$f[U] := \{f(x) : x \in U\}.$$

For functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, regarding composition of functions, we use the *right* composition notation, where the first function applied is the right one, as it is standard practice, and we write $g \circ f$, or sometimes gf.

We also introduce a non standard notational convention, about which we warn the reader whenever it is used: for $f : X \longrightarrow \mathcal{P}(Y)$ a function between a set X and the powerset $\mathcal{P}(Y)$, by $\widehat{f} : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ we denote the function that maps any set $U \subseteq X$ to

$$\widehat{f}(U) := \bigcap f[U] = \{y \in Y : \forall u \in U, y \in f(u)\}.$$

For $R \subseteq X \times Y$ a binary relation between sets X and Y, we use interchangeably $(x, y) \in R$ or xRy for denoting that the pair (x, y) belongs to the relation. By $R(\,) : X \longrightarrow \mathcal{P}(Y)$ we denote the function that maps any element $x \in X$ to

$$R(x) := \{y \in Y : (x, y) \in R\}.$$

By $R^{-1}(\,) : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ we denote the function that maps any set $U \subseteq Y$ to

$$R^{-1}(U) := \{x \in X : \exists y((x, y) \in R \ \& \ y \in U)\}.$$

The former should not be confused with $\Box_R : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$, that denotes the function that maps any set $U \subseteq Y$ to

$$\Box_R(U) := \{x \in X : \forall y(\text{ if } (x, y) \in R, \text{ then } y \in U)\}.$$

For binary relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, regarding composition of relations, we use again the *right* composition notation, this time against to the standard practice, and we write $S \circ R$, or sometimes SR.

1.2. Order and Lattice theory

Our main reference for Order theory is Davey and Priestley [**24**]. We assume that the reader is familiar with elementary order theoretical notions, such as *quasiorder* (set endowed with a reflexive and transitive binary relation), *partial order* or *partially ordered set* or *poset* (quasiorder where the relation is also anti-symmetric), *lattice* and *distributive lattice*. As usual, for $\langle P, \leq_P \rangle$ a poset and elements $a, b \in P$, we write $a \leq_P b$ when the pair (a, b) belongs to the relation $\leq_P$. When no confusion is possible, we identify $\langle P, \leq_P \rangle$ with P. We denote by P^∂ the dual of the poset P, namely the poset $\langle P, \geq_P \rangle$, where for all $a, b \in P$, $a \geq_P b$ if and only if $b \leq_P a$.

For $\langle P, \leq_P \rangle$ a poset, for any $a \in P$ we define $\uparrow_{\leq_P} a := \{b \in P : a \leq_P b\}$ and $\downarrow_{\leq_P} a := \{b \in P : b \leq_P a\}$ and when the context is clear, we omit the subscript using instead $\uparrow a$ and $\downarrow a$. For $U \subseteq P$ we define $\uparrow U := \bigcup\{\uparrow a : a \in U\}$ and $\downarrow U := \bigcup\{\downarrow a : a \in U\}$. Moreover, we say that $U \subseteq P$ is an *up-set* when $\uparrow U = U$, or equivalently, when for any $a \in U$, if $a \leq_P b$ for some $b \in P$ then $b \in U$. Dually, $U \subseteq P$ is a *down-set* whenever $\downarrow U = U$. For any $a \in P$ we call $\uparrow a$ (resp. $\downarrow a$) the *principal up-set* (resp. principal *down-set*) *generated by* a. Similarly, for any $U \subseteq P$, $\uparrow U$ (resp. $\downarrow U$) is said to be the *up-set* (resp. *down-set*) *generated by* U. By $\mathcal{P}^\uparrow(P)$ (resp. $\mathcal{P}^\downarrow(P)$) we denote the collection of all up-sets (resp. down-sets) of $\langle P, \leq_P \rangle$.

For $\langle P, \leq_P \rangle$ a poset, $U \subseteq P$ is *up-directed* when for any $a, b \in U$ there exists $c \in U$ such that $a, b \leq_P c$. Dually, $U \subseteq P$ is *down-directed* when for any $a, b \in U$ there exists $c \in U$ such that $c \leq_P a, b$.

Given two posets $\langle P, \leq_P \rangle$ and $\langle Q, \leq_Q \rangle$, a function $f : P \longrightarrow Q$ is said to be *order-preserving* when for all $a, b \in P$, if $a \leq_P b$, then $f(a) \leq_Q f(b)$. On the other hand, f is *order-reversing* when for all $a, b \in P$, if $a \leq_P b$, then $f(b) \leq_Q f(a)$. We say that f is an *order embedding* when for all $a, b \in P$, $a \leq_P b$ iff $f(a) \leq_Q f(b)$, and it is an *order isomorphism* when it is a surjective order-embedding.

Given a poset $\langle P, \leq_P \rangle$ and a subset $U \subseteq P$, we say that u is a *maximal element* of U (or a is maximal in U), when $a \in U$ and for all $b \in U$, $a \nleq b$. We define dually *minimal elements* on U. We denote by $\max(P)$ the collection of all maximal elements of P. An element $a \in P$ such that $b \leq_P a$ for all $b \in P$ is called the *top element* of P (notice that if it exists, it is unique), and it is usually denoted by 1^P, or simply 1. Dually, an element $a \in P$ such that $a \leq_P b$ for all $b \in P$ is called the *bottom element* of P, and it is usually denoted by 0^P, or simply 0.

Given a lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$, an element $m \in L$ is a *meet-irreducible* element of $\mathbf{L}$, when $m \neq 1$ (in case L has a top element 1) and $m = a \wedge b$ implies $m = a$ or $m = b$ for any $a, b \in L$. When m satisfies moreover the last condition generalized to arbitrary meets, m is called *completely meet-irreducible*.

We denote by $\mathcal{M}(\mathbf{L})$ and $\mathcal{M}^\infty(\mathbf{L})$ the collections of meet-irreducible and completely meet-irreducible elements of $\mathbf{L}$ respectively. *Join-irreducibles* $\mathcal{J}(\mathbf{L})$ and *completely join-irreducible* $\mathcal{J}^\infty(\mathbf{L})$ of $\mathbf{L}$ are defined dually.An element $m \in L$ is a *meet-prime element* of L, when $a \wedge b \leq m$ implies $a \leq m$ or $b \leq m$ for any $a, b \in L$. *Join-prime elements* of L are defined dually. Similarly we define *completely join-prime elements* and *completely meet-prime elements*. Clearly, (completely) meet-prime (resp. join-prime) elements are always (completely) meet-irreducible (resp. join-irreducible), and both notions coincide for distributive lattices.

1.3. Topology

Our main reference for General Topology is Engelking [**29**]. We assume that the reader is familiar with elementary topological notions, such as *topology*, *topological space*, *base*, *subbase*, *open*, *closed* and *compact* sets, *Kolmogorov* or T_0 spaces (for every pair of distinct points of X, at least one of them has an open neighborhood not containing the other) and *Hausdorff* spaces (distinct points have disjoint neighborhoods).

For $\langle X, \tau \rangle$ a topological space, we usually refer to it as X when it is clear what the topology on X is under consideration. By $\mathcal{O}(X)$ (resp. $\mathcal{C}(X)$) we denote the collection of all open (resp. closed) sets of $\langle X, \tau \rangle$. By $\mathcal{C}\ell(X)$ we denote the collection of all *clopen* sets, i. e. all sets that are open and closed. By $\mathcal{K}(X)$ we denote the collection of all compact sets and by $\mathcal{KO}(X)$ we denote the collection of all open and compact sets, i. e. all open sets whose open covers have always finite sub-covers. A topological space is called *compactly-based* provided it has a basis of open and compact sets. When moreover we have that $\langle X, \leq \rangle$ is a partial order, by $\mathcal{C}\ell\mathcal{U}(X)$ we denote the collection of all clopen sets of $\langle X, \tau \rangle$ that are up-sets of $\langle X, \leq \rangle$, and we refer to them as clopen up-sets of $\langle X, \tau \rangle$.

For $\langle X, \tau \rangle$ a topological space, the *closure* of a set $U \subseteq X$ (the smallest closed set containing U) is denoted by $\mathrm{cl}(U)$. A subset $U \subseteq X$ is *saturated* if it is an intersection of open sets. The *saturation* of a set $U \subseteq X$ (the smallest saturated set containing U) is denoted by $\mathrm{sat}(U)$. For $x \in X$, we generally write $\mathrm{cl}(x)$ and $\mathrm{sat}(x)$ in place of $\mathrm{cl}(\{x\})$ and $\mathrm{sat}(\{x\})$. An arbitrary non-empty subset $Y \subseteq X$ is called *irreducible* if $Y \subseteq U \cup V$ for closed subsets U and V implies $Y \subseteq U$ or $Y \subseteq V$. A topological space is called *sober* provided for every irreducible closed set Y, there exists a unique $x \in X$ such that $\mathrm{cl}(x) = Y$. A subset $Y \subseteq X$ is *dense* provided any non-empty open subset U of X has non-empty finite intersection with Y.

We recall that for a topological space $\langle X, \tau \rangle$, the *specialization quasiorder* of $\langle X, \tau \rangle$ is defined by

$$x \preceq_X y \quad \text{iff} \quad x \in \mathrm{cl}(y),$$

and when the space is T_0, $\preceq_X$ turns out to be a partial order, that we call the *specialization order* of $\langle X, \tau \rangle$.

If $\langle X, \tau \rangle$ is a topological space and $Y \subseteq X$ is a set, we can define a topology τ_Y on Y, that is known as the *subspace topology*, by

$$\tau_Y := \{U \cap X : U \in \tau\},$$

and the space $\langle Y, \tau_Y \rangle$ is called the *subspace* of $\langle X, \tau \rangle$ generated by Y.

1.4. Universal Algebra

Our main reference for Universal Algebra is Burris and Sankappanavar [**8**]. We assume that the reader is familiar with elementary universal algebraic notions.

A *language* (or *logical language* or *algebraic language* or *similarity type*) is a set $\mathscr{L}$ of function symbols, each with a fixed arity $n \geq 0$. Given a language $\mathscr{L}$ and a countably infinite set of propositional variables Var, the $\mathscr{L}$-formulas are defined by induction as usual:

- for each variable $x \in Var$, x is an $\mathscr{L}$-formula,
- for each connective $f \in \mathscr{L}$, with arity $n \in \omega$, and $\mathscr{L}$-formulas $\delta_1, \ldots, \delta_n$, $f(\delta_1, \ldots, \delta_n)$ is an $\mathscr{L}$-formula.

We denote by $Fm_{\mathscr{L}}$ the collection of all $\mathscr{L}$-formulas. When we consider the function symbols as the operation symbols of an algebraic similarity type, we have the algebra of terms, that is the absolutely free algebra of type $\mathscr{L}$ over a denumerable set of generators Var. We call this algebra the *algebra of formulas* or the *formula algebra* on the language $\mathscr{L}$ and we denote it by $\mathbf{Fm}_{\mathscr{L}}$.

For $\mathbf{A}$ and $\mathbf{B}$ algebras of algebraic similarity type $\mathscr{L}$, by $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ we denote the collection of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$. Any endomorphism of $\mathbf{Fm}_{\mathscr{L}}$, i. e. any function $e \in \mathrm{Hom}(\mathbf{Fm}_{\mathscr{L}}, \mathbf{Fm}_{\mathscr{L}})$ is said to be a *substitution*. By $\mathrm{Co}(\mathbf{A})$ we denote the collection of all congruences on $\mathbf{A}$. The identity congruence on $\mathbf{A}$ is denoted by $\triangle_{\mathbf{A}}$ and the identity homomorphism from $\mathbf{A}$ to $\mathbf{A}$ is denoted by $\mathrm{id}_{\mathbf{A}}$. We may omit the subscripts when the context is clear.

Given an algebra $\mathbf{A}$ on a language $\mathscr{L}$ and a subset $\mathscr{L}' \subseteq \mathscr{L}$, we call the algebra $\langle A, \{f : f \in \mathscr{L}'\}\rangle$ the $\mathscr{L}'$-reduct of $\mathbf{A}$.

Given a class of algebras $\mathbb{K}$ of type $\mathscr{L}$, we define the *equational consequence* relative to $\mathbb{K}$, denoted by $\vDash_{\mathbb{K}}$, as the relation between sets of equations and equations given by: for any $\{\delta_i : i \in I\} \cup \{\gamma_i : i \in I\} \cup \{\delta, \gamma\} \subseteq Fm_{\mathscr{L}}$

$$\{\delta_i \approx \gamma_i : i \in I\} \vDash_{\mathbb{K}} \delta \approx \gamma \quad \text{iff} \quad (\forall \mathbf{A} \in \mathbb{K})(\forall h \in \mathrm{Hom}(\mathbf{Fm}_{\mathscr{L}}, \mathbf{A})) \\ \text{if } (\forall i \in I) h(\delta_i) = h(\gamma_i), \text{ then } h(\delta) = h(\gamma).$$

1.5. Category theory

Our main reference for Category theory is Mac Lane [**61**]. We assume that the reader is familiar with elementary category-theoretic notions, such as *category*, *subcategory*, *object*, *morphism*, *composition* of morphisms (denoted by $\circ$), *identity morphism* (denoted by $\mathrm{id}_X : X \longrightarrow X$), *isomorphism* and *functor*. For composition of functors we use the *right* composition notation.

Given a category C, we construct its *dual category* C^{op} by taking objects of C as its objects, and for each morphism f in C, we take f^{op} as a morphism in C^{op}, that is defined as follows: if $f : X \longrightarrow Y$, then $f^{op} : Y \longrightarrow X$, i. e. it goes in the other direction. Composition of morphisms $f^{op} : Y \longrightarrow X$ and $g^{op} : Z \longrightarrow Y$ in C^{op} is given by $f^{op} \circ g^{op} := (gf)^{op}$. And $(\mathrm{id}_X)^{op}$ is the identity morphism for X in C^{op}.

A functor $F : \mathsf{C} \longrightarrow \mathsf{D}^{op}$ is a *contravariant functor* from C to D. A family of morphisms in D

$$\mathscr{H} := \big(h_X : F(X) \longrightarrow G(X)\big)_{X \in \mathsf{C}}$$

one for each object in C is a *natural transformation* between functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$, when for any morphism $f : X \longrightarrow Y$ in C, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "h_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "h_Y"'] & G(Y) \end{tikzcd}$$

If $\mathscr{H}$ is a natural transformation between functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ such that for each X, h_X is an isomorphism, then we call $\mathscr{H}$ a *natural isomorphism*, and we say that F and G are *naturally isomorphic*.

We say that the categories C and D are *equivalent* if there exist functors $F : \mathsf{C} \longrightarrow \mathsf{D}$ and $G : \mathsf{D} \longrightarrow \mathsf{C}$ such that GF is naturally isomorphic to the identity functor on C and FG is naturally isomorphic to the identity functor on D. If both F and G are contravariant functors, then we say that C and D are *dually equivalent*. Throughout this dissertation, we use the lax but commonly used term *duality* for referring to dual equivalences of categories. And we often say a duality for a class of objects (e. g. algebras) but we mean a duality for the appropriate category that has such class as objects.

1.6. Abstract Algebraic Logic

Our work is located in the field of Abstract Algebraic Logic (AAL for short). Our main reference for AAL is the survey by Font, Jansana and Pigozzi [**36**]. We introduce now the standard notion of *logic* in AAL. It arises, essentially, from regarding *logic* as concerning validity of inferences, instead of validity of formulas.

A crucial notion in AAL is the notion of closure operator. For X a set, a function $\mathrm{C} : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ on the power set of X is a *closure operator* when it satisfies the following conditions:

(C1) for all $Y \subseteq X$, $Y \subseteq \mathrm{C}(Y)$,
(C2) for all $Y, Y' \subseteq X$, if $Y \subseteq Y'$, then $\mathrm{C}(Y) \subseteq \mathrm{C}(Y')$,
(C3) for all $Y \subseteq X$, $\mathrm{C}(\mathrm{C}(Y)) = \mathrm{C}(Y)$.

Conditions (C1)-(C3) are known as being *extensive*, *isotone* and *idempotent* respectively. A closure operator C is *finitary* or *algebraic* (cf. Definition 5.4 in [**8**]), when:

(C4) for all $Y \subseteq X$, $\forall x \in X$, if $x \in \mathrm{C}(Y)$, then there is a finite $Y' \subseteq^{\omega} Y$ such that $x \in \mathrm{C}(Y')$.

When C is a closure operator on X and X is the carrier of an algebra $\mathbf{X}$, C is called $\mathbf{X}$-*structural*, when:

(C5) for all $Y \cup \{x\} \subseteq X$ and all $h \in \mathrm{Hom}(\mathbf{X}, \mathbf{X})$, we have $h(x) \in \mathrm{C}(h[Y])$ whenever $x \in \mathrm{C}(Y)$.

For any $x \in X$ and any $Y \subseteq X$, we use $\mathrm{C}(x)$ and $\mathrm{C}(Y, x)$ as a shorthand for $\mathrm{C}(\{x\})$ and $\mathrm{C}(Y \cup \{x\})$ respectively. For any closure operator C on X we define the *Frege relation of* C, $\mathbf{\Lambda}_{\mathrm{C}}$, as follows:

$$(x, y) \in \mathbf{\Lambda}_{\mathrm{C}} \quad \text{iff} \quad \mathrm{C}(x) = \mathrm{C}(y).$$

This relation is always an equivalence relation, but when X is the carrier of an algebra $\mathbf{X}$, $\mathbf{\Lambda}_{\mathrm{C}}$ is not necessarily a congruence on $\mathbf{X}$.

Any closure operator C on X can be transformed in a relation $\vdash_{\mathrm{C}}$ on X as follows: for all $Y \cup \{x\} \subseteq X$

$$Y \vdash_{\mathrm{C}} x \quad \text{iff} \quad x \in \mathrm{C}(Y).$$

The properties that $\vdash_{\mathrm{C}}$ inherits from those of C as a closure operator, define what is called a *closure relation* on X, i. e. a relation $\vdash_{\mathrm{C}} \subseteq \mathcal{P}(X) \times X$ such that:

(C1$'$) if $x \in X$, then $X \vdash_{\mathrm{C}} x$,
(C2$'$) if $Y \vdash_{\mathrm{C}} x$ for all $x \in X$ and $X \vdash_{\mathrm{C}} z$, then $Y \vdash_{\mathrm{C}} z$.

Clearly, any closure relation $\vdash_{\mathrm{C}}$ on X defines a closure operator $\mathrm{C}_{\vdash}$ by setting

$$x \in \mathrm{C}_{\vdash}(Y) \quad \text{iff} \quad Y \vdash_{\mathrm{C}} x.$$

When X is the carrier of an algebra $\mathbf{X}$ and $C_{\vdash}$ is a structural closure operator, we say that $\vdash_{\mathrm{C}}$ is *invariant under substitutions*. Notice that $\vDash_{\mathbb{K}}$, the equational consequence relative to a class of algebras $\mathbb{K}$, is a closure relation on the set of equations of type $\mathscr{L}$ and it is invariant under substitutions.

Following [**36**], given a logical language $\mathscr{L}$, a *logic* (or *deductive system*) in the language $\mathscr{L}$ is a pair $\mathcal{S} := \langle \mathbf{Fm}_{\mathscr{L}}, \vdash_{\mathcal{S}} \rangle$, where $\mathbf{Fm}_{\mathscr{L}}$ is the formula algebra of $\mathscr{L}$ and $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(Fm_{\mathscr{L}}) \times Fm_{\mathscr{L}}$ is a substitution-invariant closure relation on $Fm_{\mathscr{L}}$, i. e. $\vdash_{\mathcal{S}}$ is a relation such that:

(C1$'$) if $\gamma \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \gamma$,
(C2$'$) if $\Delta \vdash_{\mathcal{S}} \gamma$ for all $\gamma \in \Gamma$ and $\Gamma \vdash_{\mathcal{S}} \delta$, then $\Delta \vdash_{\mathcal{S}} \delta$,

(C3′) if $\Gamma \vdash_{\mathcal{S}} \delta$, then $e[\Gamma] \vdash_{\mathcal{S}} e(\delta)$ for all substitutions $e \in \mathrm{Hom}(\mathbf{Fm}_{\mathscr{L}}, \mathbf{Fm}_{\mathscr{L}})$ (structurality).

Equivalently, we could say that a logic in the language $\mathscr{L}$ is a pair $\langle \mathbf{Fm}_{\mathscr{L}}, \mathrm{C}_{\mathcal{S}} \rangle$, where $\mathbf{Fm}_{\mathscr{L}}$ is the formula algebra of $\mathscr{L}$ and $\mathrm{C}_{\mathcal{S}} : \mathcal{P}(Fm_{\mathscr{L}}) \longrightarrow \mathcal{P}(Fm_{\mathscr{L}})$ is a structural closure operator. Notice that $\mathrm{C}_{\mathcal{S}}$ is a shorthand for $\mathrm{C}_{\vdash_{\mathcal{S}}}$, the closure operator associated with a given substitution invariant closure relation $\vdash_{\mathcal{S}}$ on $\mathbf{Fm}_{\mathscr{L}}$.

We say that a logic $\mathcal{S}$ is *finitary* when the closure operator $\mathrm{C}_{\mathcal{S}}$ is finitary, i. e. when for all $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathscr{L}}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then there is a finite $\Gamma_0 \subseteq^{\omega} \Gamma$ such that $\Gamma_0 \vdash_{\mathcal{S}} \varphi$.

Let $\mathcal{S}$ be a logic in the language $\mathscr{L}$. We say that an algebra $\mathbf{A}$ has the same type as $\mathcal{S}$ when the logical language $\mathscr{L}$ is also the algebraic language of $\mathbf{A}$. Throughout this dissertation, when we assume that $\mathcal{S}$ is a logic and we pick an arbitrary algebra $\mathbf{A}$, if not otherwise stated, $\mathbf{A}$ is always assumed to be an algebra of the same type as $\mathcal{S}$.

The notion of logic we just defined is the standard notion of logic considered in the framework of contemporary Abstract Algebraic Logic. At a first sight, it might seem that only the so called ‘propositional’ or ‘sentential’ logics fall under the scope of this definition. Logics such as ordinary first order logic, quantifier logics or substructural logics seem to be left out. There have been, though, several approaches that accommodate all these logics in the framework of AAL (see Section 1.2 in [**36**] and its references).

Let $\mathcal{S}$ be a logic in a language $\mathscr{L}$ and let $\mathbf{A}$ be an algebra of the same type as $\mathcal{S}$. We call a subset $F \subseteq A$ an *$\mathcal{S}$-filter* of $\mathbf{A}$ when for any $h \in \mathrm{Hom}(\mathbf{Fm}_{\mathscr{L}}, \mathbf{A})$ and any $\Gamma \cup \{\delta\} \subseteq Fm_{\mathscr{L}}$ such that $\Gamma \vdash_{\mathcal{S}} \delta$:

$$\text{if } h(\gamma) \in F \text{ for all } \gamma \in \Gamma, \text{ then } h(\delta) \in F.$$

We denote by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ the collection of all $\mathcal{S}$-filters of $\mathbf{A}$, that is always a closure system. The notion of $\mathcal{S}$-filter is capital in AAL.

One of the basic topics of AAL is how to associate in a uniform way a class of algebras (or a class of algebraic structures) with an arbitrary logic $\mathcal{S}$. On the one hand, through the study of the Leibniz congruence we encounter the class $\mathbb{A}\mathrm{lg}^*\mathcal{S}$. Given an algebra $\mathbf{A}$ and a subset $F \subseteq A$, the *Leibniz congruence of F relative to* $\mathbf{A}$, denoted by $\boldsymbol{\Omega}^{\mathbf{A}}(F)$, is the greatest congruence on $\mathbf{A}$ compatible with F, that is, that does not relate elements in F with elements not in F. The class $\mathbb{A}\mathrm{lg}^*\mathcal{S}$ is defined as follows:

$$\mathbb{A}\mathrm{lg}^*\mathcal{S} = \{\mathbf{A} : (\exists F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}))\boldsymbol{\Omega}^{\mathbf{A}}(F) = \triangle_{\mathbf{A}}\}.$$

The class $\mathbb{A}\mathrm{lg}^*\mathcal{S}$ is the class of algebras that the semantics of logical matrices canonically associates with the logic $\mathcal{S}$, but this is not the class that is considered in contemporary AAL as the canonical algebraic counterpart of $\mathcal{S}$. Rather, the class $\mathbb{A}\mathrm{lg}\mathcal{S}$ is the canonical algebraic counterpart of an arbitrary logic $\mathcal{S}$ (as proposed in [**35**]). This class can be defined in more than one way, one being through the study of the Suszko congruence. Given an algebra $\mathbf{A}$ and an $\mathcal{S}$-filter $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, the *Suszko congruence of F relative to $\mathcal{S}$* is the congruence:

$$\tilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\mathcal{S}}(F) := \bigcap\{\boldsymbol{\Omega}^{\mathbf{A}}(G) : F \subseteq G \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})\}.$$

The class $\mathbb{A}\mathrm{lg}\mathcal{S}$ is defined as follows:

$$\mathbb{A}\mathrm{lg}\mathcal{S} := \{\mathbf{A} : (\exists F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}))\tilde{\boldsymbol{\Omega}}^{\mathbf{A}}_{\mathcal{S}}(F) = \triangle_{\mathbf{A}}\}.$$

The elements of $\mathbb{A}\mathrm{lg}\mathcal{S}$ are called *$\mathcal{S}$-algebras.* In Chapter 4 we provide an alternative definition of this class of algebras, that will play a crucial role throughout this dissertation.

Another class of algebras is associated with $\mathcal{S}$, whose definition is a bit more involved. Let us denote by $\tilde{\boldsymbol{\Omega}}(\mathcal{S})$ the Suszko congruence of the least $\mathcal{S}$-filter of $\mathbf{Fm}_{\mathscr{L}}$ relative to $\mathbf{Fm}_{\mathscr{L}}$. Then the *intrinsic variety* of $\mathcal{S}$ is the variety generated by the algebra $\mathbf{Fm}_{\mathscr{L}}/\tilde{\boldsymbol{\Omega}}(\mathcal{S})$, and it is denoted by $\mathbb{V}_{\mathcal{S}}$. $\tilde{\boldsymbol{\Omega}}(\mathcal{S})$ is usually called the *Tarski congruence of* $\mathcal{S}$, and

$$\mathbf{Fm}^{*}_{\mathscr{L}} := \mathbf{Fm}_{\mathscr{L}}/\tilde{\boldsymbol{\Omega}}(\mathcal{S})$$

is called the *Lindenbaum-Tarski algebra* of $\mathcal{S}$. Moreover, as it is pointed out in page 36 in [**35**], the Lindenbaum-Tarski algebra of any logic $\mathcal{S}$ is an $\mathcal{S}$-algebra. The relation between the three classes of algebras so far introduced goes as follows

$$\mathbb{A}\mathrm{lg}^{*}\mathcal{S} \subseteq \mathbb{A}\mathrm{lg}\mathcal{S} \subseteq \mathbb{V}_{\mathcal{S}}.$$

In principle the three classes can be different, and there are examples of all the possible combinations of equalities and inequalities.

Another topic of AAL is how to associate a logic with a class of algebraic structures. Let us show this by two examples.

Let $\mathbb{K}$ be a *pointed* class of algebras, i. e. a class of algebras in a language $\mathscr{L}$ with a constant term 1. The 1-*assertional logic* of $\mathbb{K}$ (or the *logic preserving truth for* $\mathbb{K}$) is the logic $\mathcal{S}^{1}_{\mathbb{K}} := \langle \mathbf{Fm}_{\mathscr{L}}, \vdash^{1}_{\mathbb{K}}\rangle$, such that for any $\delta \in Fm_{\mathscr{L}}$ and finite $\Gamma \subseteq^{\omega} Fm_{\mathscr{L}}$:

$$\begin{aligned}\Gamma \vdash^{1}_{\mathbb{K}} \delta \quad &\text{iff} \quad (\forall \mathbf{A} \in \mathbb{K})(\forall h \in \mathrm{Hom}(\mathbf{Fm}_{\mathscr{L}}, \mathbf{A})) \text{ if } h[\Gamma] \subseteq \{1^{\mathbf{A}}\}, \text{ then } h(\delta) = 1^{\mathbf{A}}\\ &\text{iff} \quad \{\gamma \approx 1 : \gamma \in \Gamma\} \vDash_{\mathbb{K}} \delta \approx 1,\end{aligned}$$

and for Γ an arbitrary set of formulas we take:

$$\Gamma \vdash^{1}_{\mathbb{K}} \delta \quad \text{iff} \quad (\exists \Gamma' \subseteq^{\omega} \Gamma)\, \Gamma' \vdash^{1}_{\mathbb{K}} \delta.$$

Let $\mathbb{K}$ be a class of *ordered* algebras. The *logic of the order* of $\mathbb{K}$ (or the *logic preserving degrees of truth for* $\mathbb{K}$) is the logic $\mathcal{S}^{\leq}_{\mathbb{K}} := \langle \mathbf{Fm}_{\mathscr{L}}, \vdash^{\leq}_{\mathbb{K}}\rangle$, such that for any $\delta \in Fm_{\mathscr{L}}$ and finite $\Gamma \subseteq^{\omega} Fm_{\mathscr{L}}$:

$$\begin{aligned}\Gamma \vdash^{\leq}_{\mathbb{K}} \delta \quad \text{iff} \quad &(\forall \mathbf{A} \in \mathbb{K})(\forall h \in \mathrm{Hom}(\mathbf{Fm}_{\mathscr{L}}, \mathbf{A}))(\forall a \in A)\\ &\qquad \text{if } (\forall \gamma \in \Gamma)\, a \leq^{\mathbf{A}} h(\gamma), \text{ then } a \leq^{\mathbf{A}} h(\delta),\end{aligned}$$

and for Γ an arbitrary set of formulas we take:

$$\Gamma \vdash^{\leq}_{\mathbb{K}} \delta \quad \text{iff} \quad (\exists \Gamma' \subseteq^{\omega} \Gamma)\, \Gamma' \vdash^{\leq}_{\mathbb{K}} \delta.$$

Notice that these logics are finitary by definition. In § 2.3 and § 2.4 we return to the topic of how logics can be defined from classes of algebraic structures. This will play an important role in Chapter 6. As a final illustration of how logics are studied within the framework of AAL, we present below several abstract properties that a logic may have. In what follows, let $\mathcal{S}$ be a logic:

- $\mathcal{S}$ satisfies the *property of conjunction (PC)* for a given formula in two variables that we write $p \wedge q$, if for all formulas $\delta, \gamma \in Fm_{\mathscr{L}}$:

$$\delta \wedge \gamma \vdash_{\mathcal{S}} \delta, \qquad \delta \wedge \gamma \vdash_{\mathcal{S}} \gamma, \qquad \delta, \gamma \vdash_{\mathcal{S}} \delta \wedge \gamma.$$

- $\mathcal{S}$ satisfies the *property of weak disjunction (PWDI)* for a given formula in two variables that we write $p \vee q$, if for all formulas $\delta, \gamma, \mu \in Fm_{\mathscr{L}}$:

$$\delta \vdash_{\mathcal{S}} \delta \vee \gamma, \qquad \delta \vdash_{\mathcal{S}} \gamma \vee \delta,$$

$$\text{if } \delta \vdash_{\mathcal{S}} \mu \;\;\&\;\; \gamma \vdash_{\mathcal{S}} \mu, \text{ then } \delta \vee \gamma \vdash_{\mathcal{S}} \mu.$$

- $\mathcal{S}$ satisfies the *property of disjunction (PDI)* for a given formula in two variables that we write $p \vee q$, if for all $\{\delta, \gamma, \mu\} \cup \Gamma \subseteq Fm_{\mathscr{L}}$:

$$\delta \vdash_{\mathcal{S}} \delta \vee \gamma, \qquad \delta \vdash_{\mathcal{S}} \gamma \vee \delta,$$

$$\text{if } \Gamma, \delta \vdash_{\mathcal{S}} \mu \;\;\&\;\; \Gamma, \gamma \vdash_{\mathcal{S}} \mu, \text{ then } \Gamma, \delta \vee \gamma \vdash_{\mathcal{S}} \mu.$$

- $\mathcal{S}$ satisfies the *(multiterm) deduction-detachment theorem (DDT)* for a given non-empty set of formulas in two variables p and q, that we denote by $\Delta(p, q)$, if for all $\{\delta, \gamma\} \cup \Gamma \subseteq Fm_{\mathscr{L}}$:

$$\Gamma, \delta \vdash_{\mathcal{S}} \gamma \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \Delta(\delta, \gamma).$$

- $\mathcal{S}$ satisfies the *uniterm deduction-detachment theorem (uDDT)* for a given formula in two variables that we write $p \to q$, if for all $\{\delta, \gamma\} \cup \Gamma \subseteq Fm_{\mathscr{L}}$:

$$\Gamma, \delta \vdash_{\mathcal{S}} \gamma \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \delta \to \gamma.$$

- $\mathcal{S}$ satisfies the *property of inconsistent element (PIE)* if there is a formula $\perp$, called the *inconsistent element*, such that for every formula $\delta \in Fm_{\mathscr{L}}$:

$$\perp \vdash_{\mathcal{S}} \delta.$$

- $\mathcal{S}$ satisfies the *property of being closed under introduction of a modality (PIM)*, for a given formula in one variable that we write $\Box p$, if for all $\{\delta\} \cup \Gamma \subseteq Fm_{\mathscr{L}}$:[1]

$$\text{if } \Gamma \vdash_{\mathcal{S}} \delta, \text{ then } \Box\Gamma \vdash_{\mathcal{S}} \Box\delta.$$

Notice that all these properties can be also stated using the closure operator $\mathrm{C}_{\mathcal{S}}$ associated with $\vdash_{\mathcal{S}}$. And similarly, they can be stated for any closure operator on any arbitrary algebra. Wójcicki refers to these conditions on closure operators as 'Tarski-style' conditions in [**73**]. For example, given a closure operator C on an algebra $\mathbf{A}$, we say that C has the deduction-detachment theorem for a given formula in two variables $x \to y$ provided for any $B \cup \{a, b\} \subseteq A$,

$$b \in \mathrm{C}(B, a) \quad \text{iff} \quad a \to b \in \mathrm{C}(B).$$

[1] Notice that we denote the set $\{\Box\gamma : \gamma \in \Gamma\}$ by $\Box\Gamma$.

CHAPTER 2

Filters and Ideals Associated with Closure Operators

In this chapter we examine different notions of filters and ideals that we can define as associated with a closure operator. We analyze the relationship that exists between these notions, and we study in detail two instances of the general theory.

More precisely, in § 2.1 we analyze the notions of C-closed subset, irreducible C-closed subset, C^d-closed subset, strong C^d-closed subset and optimal C-closed subset, from which we obtain two analogues of Birkhoff's Prime Filter Lemma, and interesting interrelationships between those notions under certain conditions over a closure operator C. In § 2.2 special attention is paid to the case when the lattice of closed subsets of a given closure operator is distributive. The discussion throughout these sections provides us with the tools required to develop the theory of Chapter 5.

We present two illustrative examples: *meet-semilattices with top element*, that are introduced in § 2.3, and *Hilbert algebras*, that are introduced in § 2.4. These algebraic structures are not only explanatory examples, but they play a fundamental role in the duality for Distributive Hilbert algebras with infimum of Chapter 7.

2.1. Filters, ideals and separation lemmas given by closure operators

From now on, let C be a closure operator on a set X (see definition in page 16). A subset $Y \subseteq X$ is called a *closed set of* C, or a C*-closed*, when $\mathrm{C}(Y) = Y$. For any $Y \subseteq X$, we call $\mathrm{C}(Y)$ the *closure of* Y (under C).

A *closure system* on a set X is any collection of subsets that contains X and is closed under non-empty intersections. Any closure system $\mathcal{C}$ on X yields an associated closure operator $\mathrm{C}_{\mathcal{C}}$, that is defined as follows:

$$\begin{aligned} \mathrm{C}_{\mathcal{C}} : \mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ Y &\longmapsto \bigcap\{C \in \mathcal{C} : Y \subseteq C\}. \end{aligned}$$

Moreover, for any closure operator C on X, the collection of all C-closeds is a closure system on X. Therefore, it is a complete lattice, in which the meet operation is given by the intersection, and the join operation is given by the closure of the union (cf. Theorem 5.2 in [**8**]).

All closure operators over finite sets are finitary. Moreover, finitary closure operators correspond with *inductive* or *algebraic* closure systems, i. e. closure systems closed under unions of non-empty chains. We are interested in finitarity, since many of the well-known logics are finitary, and so are all the logics we pay attention to in this dissertation.

Let $\mathcal{F}$ be a family of C-closeds. We say that $\mathcal{F}$ is a *closure base for* C provided the family $\mathcal{F}$ is such that $\cap$-generates the collection of all C-closed, i. e. for any C-closed Y, we have that $Y = \bigcap\{F \in \mathcal{F} : Y \subseteq F\}$.

Within the lattice of C-closeds, its meet-irreducible elements, that we call *irreducible* C*-closeds*, play an important role in the so called Spectral-like dualities. These subsets are also called C-irreducibles in the literature

Let us take a look at closure operators defined on posets. Let $\langle P, \leq\rangle$ be a poset, and let C be a closure operator on P. We are interested in the following property:

$$\text{(E1)} \qquad \{Y \subseteq X : \mathrm{C}(Y) = Y\} \subseteq \mathcal{P}^{\uparrow}(X),$$

in other words, in the case when all C-closeds are up-sets. In this case, we have the following analogue of Birkhoff's Prime Filter Lemma.

LEMMA 2.1.1. *Let P be a poset and let* C *be a finitary closure operator on P that satisfies (E1). For any* C*-closed $Y \subseteq P$ and any non-empty up-directed down-set $Z \subseteq P$, if $Y \cap Z = \emptyset$, then there is an irreducible* C*-closed $Y' \subseteq P$ such that $Y \subseteq Y'$ and $Y' \cap Z = \emptyset$.*

PROOF. Consider the set

$$\mathcal{Y} := \{Y' \subseteq P : \mathrm{C}(Y') = Y', Y \subseteq Y', Y' \cap Z = \emptyset\}.$$

This set is non-empty, since $Y \in \mathcal{Y}$. Moreover, it is closed under unions of chains. Let $\{Y_i : i \in \omega\} \subseteq \mathcal{Y}$ be a chain of elements of $\mathcal{Y}$, i. e. $Y_i \subseteq Y_{i+1}$ for all $i \in \omega$. By finitarity $\overline{Y} := \bigcup\{Y_i : i \in \omega\}$ is C-closed, and moreover $Y \subseteq \overline{Y}$ and $\overline{Y} \cap Z = \emptyset$.

Hence, by Zorn's Lemma, there is Y' a maximal element of $\mathcal{Y}$. We show that Y' is an irreducible C-closed. Clearly $Y' \neq P$, since $Z \neq \emptyset$. Let Y_1, Y_2 be C-closeds such that $Y_1 \cap Y_2 = Y'$, and suppose, towards a contradiction, that $Y_1, Y_2 \neq Y'$. Then there are $a_1 \in Y_1 \setminus Y'$ and $a_2 \in Y_2 \setminus Y'$. By maximality of Y', there are $b_1 \in \mathrm{C}(Y' \cup \{a_1\}) \cap Z$ and $b_2 \in \mathrm{C}(Y' \cup \{a_2\}) \cap Z$. Since Z is up-directed, there is $b \in Z$ such that $b_1, b_2 \leq b$, and by (E1), $b \in \mathrm{C}(Y' \cup \{a_1\}) \cap \mathrm{C}(Y' \cup \{a_2\})$. Therefore, as $Y' \cup \{a_1\} \subseteq Y_1$, $Y' \cup \{a_2\} \subseteq Y_2$ and Y_1, Y_2 are C-closeds, $b \in Y_1 \cap Y_2 = Y'$, and so $b \in Y' \cap Z \neq \emptyset$, a contradiction. □

COROLLARY 2.1.2. *Let P be a poset and let* C *be a finitary closure operator on P that satisfies (E1). For any* C*-closed $Y \subseteq P$ and any $z \notin Y$, there is an irreducible* C*-closed $Y' \subseteq P$ such that $Y \subseteq Y'$ and $z \notin Y'$.*

Notice that the previous corollary states that when C is a finitary closure operator defined on a poset and it satisfies (E1), the collection of all irreducible C-closeds is a closure base for C. This fact plays a key role in the Spectral-like dualities.

Returning to the general situation, let us move to the study of dual counterparts of closure operators. In [**72**] Wójcicki introduces one dual counterpart of C for any infinite cardinal, and he develops the theory of such operators. We focus on the one associated with $\aleph_0$, that has been also used in [**48**] for the formalization of reasoning on rejected information, and in [**41**] for the study of canonical extensions of congruential logics.

DEFINITION 2.1.3. The *(finitary) dual closure operator* of a given closure operator C on X, is the function $\mathrm{C}^d : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ defined as follows:

$$\mathrm{C}^d(Y) := \{x \in X : \exists Y' \subseteq^\omega Y \big(\bigcap_{y \in Y'} \mathrm{C}(y) \subseteq \mathrm{C}(x) \big)\}.$$

A subset $Z \subseteq X$ is called a *dual closed set of C*, or a C^d*-closed*, when $\mathrm{C}^d(Z) = Z$.

Notice that $\mathrm{C}^d(\emptyset) = \{x \in X : \mathrm{C}(x) = X\}$. Moreover, it follows from the definition that C^d is finitary, and we also have that

$$a \in \mathrm{C}(b) \quad \text{iff} \quad b \in \mathrm{C}^d(a).$$

We introduce now, motivated by the work in [**17**], a special class of C^d-closeds, that plays an important role in the so called Priestley-style dualities.

DEFINITION 2.1.4. A C^d-closed $Z \subseteq X$ is *strong* when for all $Z' \subseteq^\omega Z$ and all $X' \subseteq^\omega X$,

$$\text{if } \bigcap_{z \in Z'} \mathrm{C}(z) \subseteq \mathrm{C}(X'), \text{ then } \mathrm{C}(X') \cap Z \neq \emptyset.$$

Notice that for all $x \in X$, the C^d-closed set $\mathrm{C}^d(x)$ is strong: let $Y' \subseteq^\omega \mathrm{C}^d(x)$ and $X' \subseteq^\omega X$ be such that $\bigcap\{\mathrm{C}(y) : y \in Y'\} \subseteq \mathrm{C}(X')$. By assumption, for each $y \in Y'$ we have $y \in \mathrm{C}^d(x)$, i. e. $x \in \mathrm{C}(y)$. Therefore, $x \in \bigcap\{\mathrm{C}(y) : y \in Y'\} \subseteq \mathrm{C}(X')$, and since $x \in \mathrm{C}^d(x)$, then clearly $\mathrm{C}(X') \cap \mathrm{C}^d(x) \neq \emptyset$.

While the collection of C^d-closeds is always a closure system, the collection of strong C^d-closeds may fail to be so, but it is always closed under unions of non-empty chains.

LEMMA 2.1.5. *Let* C *be a closure operator on a set X. The collection of strong C^d-closed subsets of X is closed under unions of chains.*

PROOF. Let $\{Z_i : i \in \omega\}$ be a chain of strong C^d-closeds. Since C^d is finitary by definition, then $\overline{Z} := \bigcup\{Z_i : i \in \omega\}$ is C^d-closed. Let $X' \subseteq^\omega X$ and $Z' \subseteq^\omega \overline{Z}$ be such that $\bigcap\{\mathrm{C}(z) : z \in Z'\} \subseteq \mathrm{C}(X')$. As Z' is finite, there is $n \in \omega$ such that $Z' \subseteq Z_n$, and then by Z_n being strong C^d-closed, we get $\mathrm{C}(X') \cap Z_n \neq \emptyset$, and so $\mathrm{C}(X') \cap \overline{Z} \neq \emptyset$, as required. $\square$

This fact motivates the introduction of the following notions. For a given C-closed $Y \subseteq X$ and a given C^d-closed $Z \subseteq X$, we say that Y is *Z-maximal* when Y is a maximal element of the collection $\{Y' \subseteq X : \mathrm{C}(Y') = Y', Y' \cap Z = \emptyset\}$. Similarly, we say that Z is *Y-maximal* when it is a maximal element of the collection $\{Z' \subseteq X : \mathrm{C}^d(Z') = Z', Y \cap Z' = \emptyset\}$. Using these notions we introduce one more concept, that leads us to another analogue of Birkhoff's Prime Filter Lemma.

DEFINITION 2.1.6. A C-closed $Y \subseteq X$ is an *optimal* C-closed when there is a strong C^d-closed $Z \subseteq X$ such that Y is Z-maximal and Z is Y-maximal.

LEMMA 2.1.7. *Let* C *be a finitary closure operator on a set X. For any* C*-closed $Y \subseteq X$ and any strong C^d-closed $Z \subseteq X$, if $Y \cap Z = \emptyset$, then there is an optimal* C*-closed $Y' \subseteq X$ such that $Y \subseteq Y'$ and $Y' \cap Z = \emptyset$.*

PROOF. Consider the set

$$\mathcal{Y} := \{Y' \subseteq X : \mathrm{C}(Y') = Y', Y \subseteq Y', Y' \cap Z = \emptyset\}.$$

This set is non-empty, since $Y \in \mathcal{Y}$. Moreover, by finitarity it follows that $\mathcal{Y}$ is closed under unions of chains. Hence, by Zorn's Lemma, there is Y' a maximal element of $\mathcal{Y}$. We show that Y' is an optimal C-closed. For that, let us consider the set

$$\mathcal{Z} := \{Z' \subseteq X : \mathrm{C}^d(Z') = Z' \text{ strong }, Z \subseteq Z', Y' \cap Z' = \emptyset\}.$$

This set is again non-empty, since $Z \in \mathcal{Z}$. Moreover it is closed under unions of chains, since so is the collection of strong C^d-closeds. Hence by Zorn's Lemma, there is Z' a maximal element of $\mathcal{Z}$. By assumption Z' is Y'-maximal, so it is just left to show that Y' is Z'-maximal: on the contrary, there would be a C-closed Y'' such that $Y' \subsetneq Y''$ and $Y'' \cap Z' = \emptyset$, and this implies $Y \subsetneq Y''$ and so $Y' \subsetneq Y'' \in \mathcal{Y}$, contrary to the assumption of Y' being a maximal element of $\mathcal{Y}$. ☐

COROLLARY 2.1.8. *Let* C *be a finitary closure operator on a set* X*. For any* C*-closed* $Y \subseteq P$ *and any* $z \notin Y$*, there is an optimal* C*-closed* $Y' \subseteq P$ *such that* $Y \subseteq Y'$ *and* $z \notin Y'$*.*

PROOF. This follows from the previous lemma and the fact that if $z \notin Y$, then $\mathrm{C}^d(z) \cap Y = \emptyset$: on the contrary, there would be $b \in Y \cap \mathrm{C}^d(z)$, so $z \in \mathrm{C}(b) \subseteq Y$, contradicting the assumption. ☐

Notice that the previous corollary states that when C is a finitary closure operator defined on a set, then the collection of all optimal C-closeds is a closure base for C. This fact plays a key role in the Priestley-style dualities.

2.2. Distributivity of the lattice of C-closed subsets

Let us examine now the case when the lattice of C-closeds, for a given closure operator C on a set X, is distributive. In this case we have the following relations between optimal C-closeds and strong C^d-closeds, that are useful in the Priestley-style dualities.

LEMMA 2.2.1. *Let* C *be a finitary closure operator on a set* X *such that the lattice of* C*-closeds is distributive. For any* C*-closed* $Y \subseteq X$*,* Y *is optimal if and only if* Y^c *is strong* C^d*-closed.*

PROOF. Let $Y \subseteq X$ be a C-closed. If Y^c is a strong C^d-closed, then clearly Y is optimal, since Y is Y^c-maximal and Y^c is Y-maximal. For the converse, suppose that Y is optimal, so there is a strong C^d-closed $Z \subseteq X$ such that Y is Z-maximal and Z is Y-maximal. We show that Y^c is strong C^d-closed. If $Y = X$, then $Y^c = Z = \emptyset$, that by assumption is a strong C^d-closed. Assume, without loss of generality, that $Y \neq X$. We show that for any $Y' \subseteq^\omega Y^c$ and any $X' \subseteq^\omega X$, if $\bigcap\{\mathrm{C}(y) : y \in Y'\} \subseteq \mathrm{C}(X')$, then $\mathrm{C}(X') \cap Y^c \neq \emptyset$. This implies, for the case X' is a singleton, that Y^c is a C^d-closed, and hence, for the general case it also implies that Y^c is strong.

Let $Y' \subseteq^\omega Y^c$ and any $X' \subseteq^\omega X$ be such that $\bigcap\{\mathrm{C}(y) : y \in Y'\} \subseteq \mathrm{C}(X')$. If $Y' = \emptyset$, then the assumption implies $\mathrm{C}(X') = X$ and since Y is proper, we get $\mathrm{C}(X') \cap Y^c \neq \emptyset$. Assume, without loss of generality, that $Y' \neq \emptyset$. By Y being

Z-maximal, $Y \cap Z = \emptyset$ and for all $y \notin Y$ there is $b_y \in \mathrm{C}(Y, y) \cap Z \neq \emptyset$. Then by finitarity, for each $y \in Y'$ there is $T_y \subseteq^\omega Y$ such that $b_y \in \mathrm{C}(T_y, y) = \mathrm{C}(T_y) \sqcup \mathrm{C}(y)$, where the symbol $\sqcup$ denotes the join in the lattice of C-closeds. As T_y is finite for each $y \in Y'$, and Y' is also finite, so is $\overline{T} := \bigcup\{T_y : y \in Y'\}$ and clearly $\mathrm{C}(T_y) \subseteq \mathrm{C}(\overline{T})$ for all $y \in Y'$. Therefore $b_y \in \mathrm{C}(\overline{T}) \sqcup \mathrm{C}(y)$ for all $y \in Y'$. As the lattice of C-closeds is distributive and Y' is finite and non-empty, from the hypothesis we get:

$$\begin{aligned}\bigcap\{\mathrm{C}(b_y) : y \in Y'\} &\subseteq \bigcap\{\mathrm{C}(\overline{T}) \sqcup \mathrm{C}(y) : y \in Y'\} = \mathrm{C}(\overline{T}) \sqcup \bigcap\{\mathrm{C}(y) : y \in Y'\} \\ &\subseteq \mathrm{C}(\overline{T}) \sqcup \mathrm{C}(X') = \mathrm{C}(\overline{T} \cup X').\end{aligned}$$

By assumption $\{b_y : y \in Y'\} \subseteq^\omega Z$, Z is strong C^d-closed and $\overline{T} \cup X'$ is finite, so the previous equation implies, by definition of strong C^d-closed, that $\mathrm{C}(\overline{T} \cup X') \cap Z \neq \emptyset$. Suppose, towards a contradiction, that $\mathrm{C}(X') \cap Y^c = \emptyset$. Then $\mathrm{C}(X') \subseteq Y$, and since $\overline{T} \subseteq Y$, we get $\mathrm{C}(\overline{T} \cup X') \subseteq Y$, and then we obtain $Y \cap Z \neq \emptyset$, a contradiction. $\square$

There is another useful correspondence between the different notions so far examined, for the case when we have a closure operator defined on a poset P. We have a lemma similar to the previous one, where irreducible C-closeds play the role that optimals did, and non-empty up-directed down-sets play the role that C^d-closeds did.

LEMMA 2.2.2. *Let P be a poset and let C be a finitary closure operator on P that satisfies (E1), such that the lattice of* C*-closeds is distributive and $\mathrm{C}(p) = \uparrow p$ for all $p \in P$. For any* C*-closed $Y \subseteq X$, Y is irreducible if and only if Y^c is a non-empty up-directed down-set.*

PROOF. Let first $Y \subseteq P$ be an irreducible C-closed. By assumption, Y is an up-set, so Y^c is a down-set. As Y is proper, then Y^c is non-empty. It is just left to show that Y^c is up-directed: let $a, b \notin Y$, so $\mathrm{C}(a), \mathrm{C}(b) \nsubseteq Y$. By assumption, the lattice of C-closeds is distributive, so meet-irreducible and meet-prime elements of this lattice coincide. Then by meet-primeness of Y we get $\mathrm{C}(a) \cap \mathrm{C}(b) \nsubseteq Y$. Therefore, there is $c \in \mathrm{C}(a) \cap \mathrm{C}(b) = \uparrow a \cap \uparrow b$ such that $c \notin Y$, so $a, b \leq c \in Y^c$, as required.

Let now $Y \subseteq P$ be a C-closed such that Y^c is a non-empty up-directed down-set. We show Y is a meet-prime element of the lattice of C-closeds: since Y^c is non-empty, then Y is proper; let Y_1, Y_2 be C-closeds such that $Y_1 \cap Y_2 \subseteq Y$, and suppose, towards a contradiction, that $Y_1 \nsubseteq Y$ and $Y_2 \nsubseteq Y$. Then there are $p_1 \in Y_1 \setminus Y$ and $p_2 \in Y_2 \setminus Y$ such that $p_1, p_2 \notin Y$. Since Y^c is up-directed, there is $p \notin Y$ such that $p_1, p_2 \leq p$, so we get $p \in \mathrm{C}(p_1) \cap \mathrm{C}(p_2) \subseteq Y_1 \cap Y_2 \subseteq Y$, a contradiction. $\square$

Summarizing, we have studied separation lemmas for closure operators defined on ordered sets, and we have focused on the case when the lattice of closed subsets is distributive. In the remaining sections we consider two examples of that general theory. We study first meet semilattices with top element, and after that we consider Hilbert algebras.

2.3. Meet-semilattices with top element

We introduce now meet-semilattices with top element as an example of what has been treated in § 2.1 and § 2.2. Be aware that these algebraic structures are an important tool both in Part 2 and in 3.

DEFINITION 2.3.1. An algebra $\mathbf{M} = \langle M, \wedge, 1\rangle$ of type $(2, 0)$ is a *meet-semilattice with top element* when the binary operation $\wedge$ is idempotent, commutative, associative, and $a \wedge 1 = 1$ for all $a \in M$.

A binary relation $\leq_{\mathbf{M}}$ is defined on M such that for any $a, b \in M$:

$$a \leq_{\mathbf{M}} b \quad \text{iff} \quad a \wedge b = a.$$

This relation is indeed a partial order on M in which $a \wedge b$ is the meet of a and b, for every $a, b \in M$. We use $\leq$ for $\leq_{\mathbf{M}}$ when no confusion is possible. Meet-semilattices with top element and with an additional constant 0 that is the bottom element of that order are called *bounded meet-semilattices*. We denote by $\mathbb{S}$ and $\mathbb{BS}$ the varieties of meet-semilattices with top element and bounded meet-semilattices respectively.

Dual structures of meet semilattices with top element are usually called *join-semilattices with bottom element*, and they are defined just changing the order upside-down. Classical books on order theory or lattice theory [**24**, **49**] usually work with join-semilattices, but it is wise for us to work with meet-semilattices. All results in the rest of the section could be stated though for join-semilattices, changing the role of meets by joins and reversing the order. From now on, let $\mathbf{M} = \langle M, \wedge, 1\rangle$ be a meet-semilattice with top element. We use *'semilattice'* as an abbreviation of 'meet-semilattice with top element', not only in the present chapter but also throughout the whole dissertation.

An *order ideal* of $\mathbf{M}$ is a non-empty up-directed down-set of $\langle M, \leq\rangle$, i. e. $I \subseteq M$ is an order ideal of $\mathbf{M}$ if $I \neq \emptyset$ and for all $a, b \in M$:

- if $a \in I$ and $b \leq a$, then $b \in I$,
- if $a, b \in I$, then there is $c \in I$ such that $a, b \leq c$.

We denote by $\mathrm{Id}(\mathbf{M})$ the collection of all order ideals of $\mathbf{M}$. Notice that all principal down-sets are order ideals.

A *meet filter* of $\mathbf{M}$ is a non-empty up-set of $\langle M, \leq\rangle$ closed under the meet operation or equivalently, a non-empty down-directed up-set of $\langle M, \leq\rangle$, i. e. $F \subseteq M$ is a meet filter of $\mathbf{M}$ if $F \neq \emptyset$ and for all $a, b \in M$:

- if $a \in F$ and $a \leq b$, then $b \in F$,
- if $a, b \in F$, then $a \wedge b \in F$.

We denote by $\mathrm{Fi}_\wedge(\mathbf{M})$ the collection of all meet filters of $\mathbf{M}$. Notice that all principal up-sets are meet filters. A meet filter F is *proper* when $F \neq M$.

The collection $\mathrm{Fi}_\wedge(\mathbf{M})$ is closed under arbitrary intersections. Therefore, we may define the function $\llbracket\ \rangle\!\rangle : \mathcal{P}(M) \longrightarrow \mathcal{P}(M)$ that assigns to each subset $B \subseteq M$, the least meet filter containing B. We call $\llbracket B\rangle\!\rangle$ the *meet filter generated by* B. It is well known that for any $B \subseteq M$ and any $a \in M$:

$$a \in \llbracket B\rangle\!\rangle \quad \text{iff} \quad a = 1 \text{ or } (\exists n \in \omega)(\exists b_0, \ldots, b_n \in B)\, b_0 \wedge \cdots \wedge b_n \leq a$$

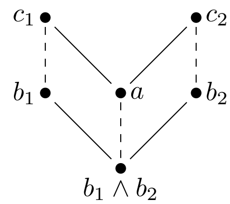


FIGURE 1. Distributivity of meet semilattices.

The map given by $[\![\ \rangle\!\rangle$ is in fact a finitary closure operator on M and $\mathrm{Fi}_\wedge(\mathbf{M})$ is the collection of all closed subsets of $[\![\ \rangle\!\rangle$. We consider the bounded lattice

$$\mathbf{Fi}_\wedge(\mathbf{M}) := \langle \mathrm{Fi}_\wedge(\mathbf{M}), \cap, \vee, M, \{1\} \rangle ,$$

where the meet operation is given by intersection and the join operation is given by the meet filter generated by the union. Meet-irreducible elements of this lattice are called *$\wedge$-irreducible meet filters* or *irreducible meet filters* when no confusion is possible. Recall that $F \in \mathrm{Fi}_\wedge(\mathbf{M})$ is a meet-irreducible element of the lattice $\mathbf{Fi}_\wedge(\mathbf{M})$ when for all $F_1, F_2 \in \mathrm{Fi}_\wedge(\mathbf{M})$, if $F = F_1 \cap F_2$ then $F = F_1$ or $F = F_2$. We denote by $\mathrm{Irr}_\wedge(\mathbf{M})$ the collection of all irreducible meet filters of $\mathbf{M}$. When $\mathbf{M}$ is moreover a lattice, irreducible meet filters are precisely *prime* filters, this is, meet filters F such that $a \in F$ or $b \in F$ whenever $a \vee b \in F$. In this case we denote the collection of irreducible/prime meet filters of $\mathbf{M}$ by $\mathrm{Pr}(\mathbf{M})$. The following proposition characterizes irreducible meet filters, and its proof can be found in Lemma 6 in [**12**].

PROPOSITION 2.3.2. *Let $\mathbf{M}$ be a semilattice and let $F \in \mathrm{Fi}_\wedge(\mathbf{M})$ be proper. The following are equivalent:*

(1) $F \in \mathrm{Irr}_\wedge(\mathbf{M})$.
(2) *For all $a, b \notin F$, there are $c \notin F$ and $f \in F$ such that $a \wedge f, b \wedge f \leq c$.*

Notice that meet filters are up-sets, so Property (E1) holds for $[\![\ \rangle\!\rangle$, and so we have the following instance of Lemma 2.1.1, that is also proved in Theorem 8 in [**12**].

LEMMA 2.3.3. *Let $\mathbf{M}$ be a semilattice, and let $F \in \mathrm{Fi}_\wedge(\mathbf{M})$ and $I \in \mathrm{Id}(\mathbf{M})$ be such that $F \cap I = \emptyset$. Then there is $G \in \mathrm{Irr}_\wedge(\mathbf{M})$ such that $F \subseteq G$ and $G \cap I = \emptyset$.*

COROLLARY 2.3.4. *Let $\mathbf{M}$ be a semilattice, and let $F \in \mathrm{Fi}_\wedge(\mathbf{M})$ be such that $a \notin F$. Then there is $G \in \mathrm{Irr}_\wedge(\mathbf{M})$ such that $F \subseteq G$ and $a \notin G$.*

DEFINITION 2.3.5. A semilattice is *distributive* (cf. Section II.5 in [**49**]) when for each $a, b_1, b_2 \in M$ with $b_1 \wedge b_2 \leq a$, there exist $c_1, c_2 \in M$ such that $b_1 \leq c_1, b_2 \leq c_2$ and $a = c_1 \wedge c_2$ (see Figure 1).

We denote by $\mathbb{DS}$ and $\mathbb{BDS}$ the classes of distributive semilattices and bounded distributive semilattices respectively.

It is well known that a semilattice $\mathbf{M}$ is distributive if and only if the lattice of meet filters $\mathbf{Fi}_\wedge(\mathbf{M})$ is distributive (for a proof, see Lemma 1 in Section II.5 of [**49**]). Since we have that $[\![a\rangle\!\rangle = {\uparrow}a$ for all $a \in M$, we obtain the following instance of Lemma 2.2.2, that is proven in Theorem 10 in [**12**], where it is also shown that it characterizes semilattices that are distributive:

THEOREM 2.3.6. *Let* $\mathbf{M}$ *be a semilattice.* $\mathbf{M}$ *is distributive if and only if for all* $F \in \mathrm{Fi}_\wedge(\mathbf{M})$,

$$F \in \mathrm{Irr}_\wedge(\mathbf{M}) \quad \textit{iff} \quad F^c \in \mathrm{Id}(\mathbf{M}).$$

Let us return to consider the closure operator $\llbracket\ \rrangle$. We denote by $\llangle\ \rrbracket$ the dual closure operator of $\llbracket\ \rrangle$. By definition, for any $B \subseteq M$ and any $a \in M$:

$$a \in \llangle B \rrbracket \quad \text{iff} \quad \exists B' \subseteq^\omega B \text{ such that } \bigcap_{b \in B'} \llbracket b \rrangle \subseteq \llbracket a \rrangle.$$

Dually closed subsets of $\llbracket\ \rrangle$, i. e. $\llangle\ \rrbracket$-closed subsets, are Frink ideals, that were introduced by Frink in [**37**]. This notion has also been considered recently in [**17**] and [**5**], and it can be defined indeed for any poset: Frink ideals are those down-sets closed under existing joins. Equivalently, we say that $I \subseteq M$ is a *Frink ideal* (or *F-ideal*) of $\mathbf{M}$ if for every $I' \subseteq^\omega I$ and $b \in M$, $\bigcap\{\uparrow a : a \in I'\} \subseteq \uparrow b$ implies $b \in I$.

In [**5**] a slightly different definition of 'Frink ideal' is given. As the authors deal with bounded distributive semilattices, they require Frink ideals to be non-empty. In the bounded case, both notions coincide, unlike the non-bounded case. We denote by $\mathrm{Id}_F(\mathbf{M})$ the collection of all F-ideals of $\mathbf{M}$. Notice that the empty set may be an F-ideal, but this happens if and only if there is no bottom element in $\mathbf{M}$. Moreover, it is easy to see that all order ideals are F-ideals, therefore:

$$\mathrm{Id}(\mathbf{M}) \subseteq \mathrm{Id}_F(\mathbf{M}).$$

Notice that for any $B \subseteq^\omega M$, $\llbracket B \rrangle = \uparrow \bigwedge B$, therefore strong dually closed subsets of $\llbracket\ \rrangle$ are the same as dually closed subsets of $\llbracket\ \rrangle$, i. e. Frink ideals.

Applying Definition 2.1.6 to $\llbracket\ \rrangle$, a meet filter $F \in \mathrm{Fi}_\wedge(\mathbf{M})$ is said to be $\wedge$*-optimal* (or simply *optimal*), when there is an F-ideal I of $\mathbf{M}$ such that:

- F is *I-maximal*, i. e. F is a maximal element of $\{G \in \mathrm{Fi}_\wedge(\mathbf{M}) : G \cap I = \emptyset\}$,
- I is *F-maximal*, i. e. I is a maximal element of $\{J \in \mathrm{Id}_F(\mathbf{M}) : F \cap J = \emptyset\}$.

In [**5**] a slightly different notion of optimal (meet) filter for bounded distributive semilattices is considered, requiring these filters to be proper. That notion coincides with ours for the bounded case, but differs from it in the general case. Recall that when the algebra has no bottom element, the empty set is an F-ideal, and so the total M is an optimal filter, and optimal filters are not necessarily proper. We denote by $\mathrm{Op}_\wedge(\mathbf{M})$ the collection of all optimal meet filters of $\mathbf{M}$. As an instance of Lemma 2.1.7, concerning optimal meet filters and Frink-ideals we have:

LEMMA 2.3.7. *Let* $\mathbf{M}$ *be a semilattice and let* $F \in \mathrm{Fi}_\wedge(\mathbf{M})$ *and* $I \in \mathrm{Id}_F(\mathbf{M})$ *be such that* $F \cap I = \emptyset$. *Then there is* $G \in \mathrm{Op}_\wedge(\mathbf{M})$ *such that* $F \subseteq G$ *and* $G \cap I = \emptyset$.

COROLLARY 2.3.8. *Let* $\mathbf{M}$ *be a semilattice and let* $F \in \mathrm{Fi}_\wedge(\mathbf{M})$ *be such that* $a \notin F$. *Then there is* $G \in \mathrm{Op}_\wedge(\mathbf{M})$ *such that* $F \subseteq G$ *and* $a \notin G$.

For distributive semilattices, we obtain the following instance of Lemma 2.2.1.

THEOREM 2.3.9. *Let* $\mathbf{M}$ *be a distributive semilattice. For any* $F \in \mathrm{Fi}_\wedge(\mathbf{M})$, $F \in \mathrm{Op}_\wedge(\mathbf{M})$ *if and only if* $F^c \in \mathrm{Id}_F(\mathbf{M})$.

From the previous theorem, Theorem 2.3.6 and the fact that $\mathrm{Id}(\mathbf{M}) \subseteq \mathrm{Id}_F(\mathbf{M})$, we obtain that for any distributive semilattice:

$$\mathrm{Irr}_\wedge(\mathbf{M}) \subseteq \mathrm{Op}_\wedge(\mathbf{M}) \subseteq \mathrm{Fi}_\wedge(\mathbf{M}).$$

Let us introduce a definition that is used later on. We say that an F-ideal is $\wedge$-*prime* (or simply *prime*) if it is a proper F-ideal $I \in \mathrm{Id}_F(\mathbf{M})$ such that for all non-empty $Y \subseteq^\omega M$, $\bigwedge Y \in I$ implies $y \in I$ for some $y \in Y$. Making use of this notion, we obtain the following corollaries of theorems 2.3.6 and 2.3.9.

COROLLARY 2.3.10. *Let* $\mathbf{M}$ *be a distributive semilattice and* $F \subseteq M$. $F \in \mathrm{Op}_\wedge(\mathbf{M})$ *if and only if* F^c *is an* $\wedge$*-prime F-ideal.*

COROLLARY 2.3.11. *Let* $\mathbf{M}$ *be a distributive semilattice and* $F \subseteq M$. $F \in \mathrm{Irr}_\wedge(\mathbf{M})$ *if and only if* F^c *is an* $\wedge$*-prime order ideal.*

Let us conclude this section by considering classes of algebras with semilattice reducts, that is, classes of algebras $\mathbb{K}$ on a given language $\mathscr{L}$ that contains a binary function symbol $\wedge$ and a constant 1 such that the $(\wedge, 1)$-reducts of the algebras in $\mathbb{K}$ are semilattices. Notice that this implies that the algebras in $\mathbb{K}$ are ordered by the order of the semilattice reduct. For such class of algebras we may provide an alternative definition of the logic of the order of $\mathbb{K}$, that recall that is denoted by $\mathcal{S}_{\mathbb{K}}^{\leq} := \langle \mathbf{Fm}, \vdash_{\mathbb{K}}^{\leq} \rangle$. For any non-empty finite set Γ of formulas and any formula δ we define:

$$\begin{aligned} \Gamma \vdash_{\mathbb{K}}^{\leq} \delta \quad &\text{iff} \quad (\forall \mathbf{A} \in \mathbb{K})(\forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))\big(\bigwedge_{\gamma\in\Gamma}^{\mathbf{A}} h(\gamma)\big) \leq h(\delta) \\ &\text{iff} \quad \vDash_{\mathbb{K}} \big(\bigwedge_{\gamma\in\Gamma} \gamma \wedge \delta\big) \approx \bigwedge_{\gamma\in\Gamma} \gamma. \end{aligned}$$

For Γ the empty set of formulas and any formula δ we have we define:

$$\emptyset \vdash_{\mathbb{K}}^{\leq} \delta \quad \text{iff} \quad \vDash_{\mathbb{K}} \delta \approx 1.$$

And for Γ an arbitrary set of formulas and any formula δ we take:

$$\Gamma \vdash_{\mathbb{K}}^{\leq} \delta \quad \text{iff} \quad (\exists \Gamma' \subseteq^\omega \Gamma)\, \Gamma' \vdash_{\mathbb{K}}^{\leq} \delta.$$

In this context, $\mathcal{S}_{\mathbb{K}}^{\leq}$ is also called the *semilattice based logic of* $\mathbb{K}$.

2.4. Hilbert algebras

As another example of what has been treated in § 2.1 and § 2.2, we introduce now Hilbert algebras. For the moment, we just present these structures under an algebraic point of view. We refer the reader to § 6.2, where we explain in detail the connexion between logic and Hilbert algebras.

DEFINITION 2.4.1. An algebra $\mathbf{A} = \langle A, \to, 1 \rangle$ of type $(2, 0)$ is a *Hilbert algebra* or $\mathbb{H}$-*algebra* (also called *positive implication algebra* in [**67**]) if for all $a, b, c \in A$:

(H1) $a \to (b \to a) = 1$,
(H2) $(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$,
(H3) if $(a \to b = 1 = b \to a)$, then $a = b$.

LEMMA 2.4.2. *Let* $\mathbf{A}$ *be a Hilbert algebra and* $a, b, c \in A$. *Then the following equalities are satisfied:*

(1) $a \to a = 1$,
(2) $1 \to a = a$,
(3) $a \to (b \to c) = b \to (a \to c)$,
(4) $a \to (b \to c) = (a \to b) \to (a \to c)$,
(5) $a \to ((a \to b) \to b) = 1$,
(6) $a \to (a \to b) = a \to b$,
(7) $((a \to b) \to b) \to b = a \to b$,
(8) $(a \to b) \to ((b \to a) \to a) = (b \to a) \to ((a \to b) \to b)$.

Let us denote by $\mathbb{H}$ the class of all Hilbert algebras. This class was extensively studied by Diego in [**25**]. It is indeed a variety, for which an equational definition is given as follows. $\mathbf{A} = \langle A, \to, 1\rangle$ is a Hilbert algebra if for all $a, b, c \in A$:

(K) $a \to a = 1$,
(H2′) $(a \to (b \to c)) = ((a \to b) \to (a \to c))$,
(H4) $1 \to a = a$,
(H5) $(a \to b) \to ((b \to a) \to a) = (b \to a) \to ((a \to b) \to b)$.

It is well known that Hilbert algebras are subalgebras of the $(\to, 1)$-reducts of Heyting algebras. It is also well known that Hilbert algebras are a subclass of the class of Implicative algebras, that was studied by Rasiowa in [**67**]. An *implicative algebra* is an algebra $\mathbf{A} = \langle A, \to, 1\rangle$ of type $(2, 0)$ such that for all $a, b, c \in A$:

(K) $a \to a = 1$,
(H3) if $(a \to b = 1 = b \to a)$, then $a = b$,
(IA1) if $(a \to b = 1 \ \&\ b \to c = 1)$, then $a \to c = 1$,
(IA2) $a \to 1 = 1$.

Let us denote by $\mathbb{IA}$ the class of implicative algebras. Hilbert algebras are precisely the implicative algebras that satisfy (H1) and (H2).

It is also well known that Hilbert algebras are a subclass of the class of $\mathbb{BCK}$-algebras, that was first introduced by Iséki in [**52**]. For our purposes, following Idziak [**51**], we define a $\mathbb{BCK}$-*algebra* as an algebra $\mathbf{A} = \langle A, \to, 1\rangle$ of type $(2, 0)$ such that for all $a, b, c \in A$:

(B) $(a \to b) \to ((b \to c) \to (a \to c)) = 1$,
(C) $a \to ((a \to b) \to b) = 1$,
(K) $a \to a = 1$,
(H3) if $(a \to b = 1 = b \to a)$, then $a = b$,
(IA2) $a \to 1 = 1$.

Let us denote by $\mathbb{BCK}$ the quasivariety of $\mathbb{BCK}$-algebras. This presentation is somewhat unusual. The most of literature concerning $\mathbb{BCK}$-algebras employs the dual notion. We opt for this presentation, as then it is easy to check that Hilbert algebras are precisely the $\mathbb{BCK}$-algebras $\mathbf{A}$ such that for all $a, b \in A$:

(H) $(a \to (a \to b)) = a \to b$.

A binary relation $\leq_{\mathbf{A}}$ is defined on any Hilbert algebra $\mathbf{A}$, such that for all $a, b \in A$:

$$a \leq_{\mathbf{A}} b \quad \text{iff} \quad a \to b = 1.$$

This relation is indeed a partial order on A, whose top element is 1. We use $\leq$ for $\leq_{\mathbf{A}}$ when no confusion is possible. Besides *order ideals* of $\mathbf{A}$, that are defined as in page 26, and *order filters* of $\mathbf{A}$, that are defined order dually, a well-known notion of filter associated with Hilbert algebras is the following. An *implicative filter* (also known as *deductive system*) of $\mathbf{A}$ is a subset $P \subseteq A$ such that for all $a, b \in A$:

- $1 \in P$,
- if $a, a \to b \in P$, then $b \in P$.

We denote by $\mathrm{Fi}_{\to}(\mathbf{A})$ the collection of all implicative filters of $\mathbf{A}$. Notice that implicative filters are up-sets, and all principal up-sets are implicative filters.

The collection $\mathrm{Fi}_{\to}(\mathbf{A})$ is closed under arbitrary intersections. Therefore, we may define the function $\langle\ \rangle : \mathcal{P}(A) \longrightarrow \mathcal{P}(A)$ that assigns to each subset $B \subseteq A$, the least implicative filter containing B. We call $\langle B \rangle$ the *implicative filter generated by B*. It is well known that for any $B \subseteq A$ and any $a \in A$:

$$a \in \langle B \rangle \quad \text{iff} \quad a = 1 \text{ or } (\exists n \in \omega)(\exists b_0, \dots, b_n \in B) \\ b_0 \to (b_1 \to (\dots (b_n \to a) \dots)) = 1.$$

The map given by $\langle\ \rangle$ is in fact a finitary closure operator and $\mathrm{Fi}_{\to}(\mathbf{A})$ is the collection of all closed subsets of $\langle\ \rangle$. We consider the bounded lattice

$$\mathbf{Fi}_{\to}(\mathbf{A}) := \langle \mathrm{Fi}_{\to}(\mathbf{A}), \cap, \vee, A, \{1\} \rangle ,$$

where the meet operation is given by intersection, and the join operation is given by the implicative filter generated by the union. It is well known [**25**] that for any Hilbert algebra $\mathbf{A}$, the lattice of implicative filters $\mathbf{Fi}_{\to}(\mathbf{A})$ is distributive. Meet-irreducible elements of this lattice are called *$\to$-irreducible implicative filters* or simply *irreducible implicative filters* when no confusion is possible. Recall that $P \in \mathrm{Fi}_{\to}(\mathbf{A})$ is a meet-irreducible element of $\mathbf{Fi}_{\to}(\mathbf{A})$ when for all $P_1, P_2 \in \mathrm{Fi}_{\to}(\mathbf{A})$, if $P = P_1 \cap P_2$ then $P = P_1$ or $P = P_2$. As the lattice $\mathbf{Fi}_{\to}(\mathbf{A})$ is distributive, meet-irreducible elements and meet-prime elements of $\mathbf{Fi}_{\to}(\mathbf{A})$ coincide, so we have that $P \in \mathrm{Fi}_{\to}(\mathbf{A})$ is an irreducible implicative filter of $\mathbf{A}$ when for all $P_1, P_2 \in \mathrm{Fi}_{\to}(\mathbf{A})$, if $P_1 \cap P_2 \subseteq P$, then $P_1 \subseteq P$ or $P_2 \subseteq P$. We denote by $\mathrm{Irr}_{\to}(\mathbf{A})$ the collection of all irreducible implicative filters of $\mathbf{A}$. The following theorem characterizes irreducible implicative filters, and its proof can be found in Lemma 2.4 in [**10**].

PROPOSITION 2.4.3. *Let $\mathbf{A}$ be a Hilbert algebra and let $P \subseteq A$. The following are equivalent:*

(1) $P \in \mathrm{Irr}_{\to}(\mathbf{A})$.
(2) *For all $a, b \notin P$ there is $c \notin P$ such that $a \to c, b \to c \in P$.*
(3) *For all $a, b \notin P$ there is $c \notin P$ such that $a, b \leq c$.*

COROLLARY 2.4.4. *Let $\mathbf{A}$ be a Hilbert algebra. Then for all $P \in \mathrm{Fi}_{\to}(\mathbf{A})$, $P \in \mathrm{Irr}_{\to}(\mathbf{A})$ if and only if $P^c \in \mathrm{Id}(\mathbf{A})$.*

Notice that the previous corollary is another instance of Lemma 2.2.2, that holds because the lattice of implicative filters is distributive, and moreover for all $a \in A$, $\langle a \rangle = {\uparrow} a$. Notice also that implicative filters are up-sets, so Property (E1) also holds for $\langle\ \rangle$, and so we have the following instance of Lemma 2.1.1, that is also proven in Theorem 2.6 in [**10**].

LEMMA 2.4.5. *Let* $\mathbf{A}$ *be a Hilbert algebra and let* $P \in \mathrm{Fi}_{\rightarrow}(\mathbf{A})$ *and* $I \in \mathrm{Id}(\mathbf{A})$ *be such that* $P \cap I = \emptyset$. *Then there is* $Q \in \mathrm{Irr}_{\rightarrow}(\mathbf{A})$ *such that* $P \subseteq Q$ *and* $Q \cap I = \emptyset$.

COROLLARY 2.4.6. *Let* $\mathbf{A}$ *be a Hilbert algebra, and let* $P \in \mathrm{Fi}_{\rightarrow}(\mathbf{A})$ *be such that* $a \notin P$. *Then there is* $Q \in \mathrm{Irr}_{\rightarrow}(\mathbf{A})$ *such that* $P \subseteq Q$ *and* $a \notin Q$.

Let us return to consider the closure operator $\langle\ \rangle$. Since for all $a \in A$, $\langle a\rangle = {\uparrow}a$, dually closed subsets of $\langle\ \rangle$ are the same as Frink ideals (see definition in page 28). However, strong dually closed subsets of $\langle\ \rangle$ provides us with a different notion of ideal. We say that $I \in \mathrm{Id}_F(\mathbf{A})$ is a *strong Frink ideal* (or *sF-ideal*) of $\mathbf{A}$ if for all $I' \subseteq^{\omega} I$ and all $B \subseteq^{\omega} A$,

$$\text{if } \bigcap_{a\in I'} {\uparrow}a \subseteq \langle B\rangle, \text{ then } \langle B\rangle \cap I \neq \emptyset.$$

Recall that we have that $\emptyset \notin \mathrm{Id}_F(\mathbf{A})$ whenever $\mathbf{A}$ has a bottom element 0. From the definition it also follows that $\emptyset \notin \mathrm{Id}_{sF}(\mathbf{A})$ whenever $\mathbf{A}$ has a bottom element 0, since in this case $A = \bigcap\{{\uparrow}a : a \in \emptyset\} \subseteq \langle 0\rangle$ but $\langle 0\rangle \cap \emptyset = \emptyset$.

In [**17**] a slightly different notion of 'strong Frink ideal' is introduced, since the authors require them to be non-empty. Both notions coincide for bounded Hilbert algebras, but not for the general case. We denote by $\mathrm{Id}_{sF}(\mathbf{A})$ the collection of all sF-ideals of $\mathbf{A}$. We already know that $\mathrm{Id}_{sF}(\mathbf{A})$ is not necessarily a closure system, but it is always an inductive family. It is easy to check that:

$$\mathrm{Id}(\mathbf{A}) \subseteq \mathrm{Id}_{sF}(\mathbf{A}) \subseteq \mathrm{Id}_F(\mathbf{A}).$$

Applying Definition 2.1.6 to $\langle\ \rangle$, an implicative filter $P \in \mathrm{Fi}_{\rightarrow}(\mathbf{A})$ is said to be $\rightarrow$*-optimal* (or simply *optimal*), when there is an sF-ideal I of $\mathbf{A}$ such that:

- P is I*-maximal*, i. e. P is a maximal element of $\{G \in \mathrm{Fi}_{\rightarrow}(\mathbf{A}) : G \cap I = \emptyset\}$,
- I is P*-maximal*, i. e. I is a maximal element of $\{J \in \mathrm{Id}_{sF}(\mathbf{A}) : F \cap J = \emptyset\}$.

In [**17**] a slightly different notion of 'optimal implicative filter' for Hilbert algebras is considered, requiring these filters to be proper. That notion coincides with ours for bounded Hilbert algebras, but differs from it for the general case. Recall that when the algebra has no bottom element, the empty set is an sF-ideal, and so the total A is an optimal implicative filter. We denote by $\mathrm{Op}_{\rightarrow}(\mathbf{A})$ the collection of all optimal implicative filters of $\mathbf{A}$. As an instance of Lemma 2.1.7, concerning optimal implicative filters and strong Frink-ideals we have:

LEMMA 2.4.7. *Let* $\mathbf{A}$ *be a Hilbert algebra and let* $P \in \mathrm{Fi}_{\rightarrow}(\mathbf{A})$ *and* $I \in \mathrm{Id}_{sF}(\mathbf{A})$ *be such that* $P \cap I = \emptyset$. *Then there is* $Q \in \mathrm{Op}_{\rightarrow}(\mathbf{A})$ *such that* $P \subseteq Q$ *and* $Q \cap I = \emptyset$.

COROLLARY 2.4.8. *Let* $\mathbf{A}$ *be a Hilbert algebra and let* $P \in \mathrm{Fi}_{\rightarrow}(\mathbf{A})$ *be such that* $a \notin P$. *Then there is* $Q \in \mathrm{Op}_{\rightarrow}(\mathbf{A})$ *such that* $P \subseteq Q$ *and* $a \notin Q$.

And since the lattice of implicative filters is distributive, the following instance of Lemma 2.2.1 also holds:

THEOREM 2.4.9. *Let* $\mathbf{A}$ *be a Hilbert algebra. For any* $P \in \mathrm{Fi}_{\rightarrow}(\mathbf{A})$, $P \in \mathrm{Op}_{\rightarrow}(\mathbf{A})$ *if and only if* $P^c \in \mathrm{Id}_{sF}(\mathbf{A})$.

From the previous theorem, Corollary 2.4.4, and the fact that $\mathrm{Id}(\mathbf{A}) \subseteq \mathrm{Id}_{sF}(\mathbf{A})$, we obtain that for any Hilbert algebra:

$$\mathrm{Irr}_{\rightarrow}(\mathbf{A}) \subseteq \mathrm{Op}_{\rightarrow}(\mathbf{A}) \subseteq \mathrm{Fi}_{\rightarrow}(\mathbf{A}).$$

We say that an F-ideal is *$\to$-prime*, if it is a proper F-ideal $I \in \mathrm{Id}_F(\mathbf{A})$ such that for all non-empty $Y \subseteq^\omega A$, $\langle Y \rangle \cap I \neq \emptyset$ implies $Y \cap I \neq \emptyset$. This notion is also considered in [**17**] under the name of 'prime'. We prefer to use '$\to$-prime' in order to avoid confusion with the notion of '$\wedge$-prime', that is usually called 'prime'. Making use of this notion, we obtain the following corollaries of Corollary 2.4.4 and Theorem 2.4.9.

Corollary 2.4.10. *Let $\mathbf{A}$ be a Hilbert algebra and $P \subseteq A$. $P \in \mathrm{Op}_\to(\mathbf{A})$ if and only if P^c is a $\to$-prime sF-ideal.*

Corollary 2.4.11. *Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra and $P \subseteq A$. $P \in \mathrm{Irr}_\to(\mathbf{A})$ if and only if P^c is a $\to$-prime order ideal.*

Let us conclude this section by considering Hilbert-based classes of algebras. These are classes of algebras $\mathbb{K}$ on a given language $\mathscr{L}$ that contains a binary function symbol $\to$ and a constant 1 such that the $(\to, 1)$-reducts of the algebras in $\mathbb{K}$ are Hilbert algebras. Notice that this implies that the algebras in $\mathbb{K}$ are ordered by the order given by $\to$. For any of such class of algebras, similarly to the case of semilattice based logics, we may define the *Hilbert based logic* of $\mathbb{K}$ (see Definition 4 in [**54**]), as the logic $\mathcal{S}_{\mathbb{K}}^{\to} := \langle \mathbf{Fm}, \vdash_{\mathbb{K}}^{\to} \rangle$, defined for any non-empty finite set $\Gamma = \{\gamma_0, \ldots, \gamma_n\}$ of formulas and any formula δ:

$$\begin{aligned} \Gamma \vdash_{\mathbb{K}}^{\to} \delta \quad \text{iff} \quad & (\forall \mathbf{A} \in \mathbb{K})(\forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})) \\ & \qquad h(\gamma_0) \to^{\mathbf{A}} (h(\gamma_1) \to^{\mathbf{A}} \ldots (h(\gamma_n) \to^{\mathbf{A}} h(\delta)) \ldots) = 1^{\mathbf{A}} \\ \text{iff} \quad & \vDash_{\mathbb{K}} \gamma_0 \to (\gamma_1 \to \ldots (\gamma_n \to \delta) \ldots) \approx 1. \end{aligned}$$

For Γ the empty set of formulas and any formula δ we define:

$$\emptyset \vdash_{\mathbb{K}}^{\to} \delta \quad \text{iff} \quad \vDash_{\mathbb{K}} \delta \approx 1.$$

And for Γ an arbitrary set of formulas and any formula δ we define:

$$\Gamma \vdash_{\mathbb{K}}^{\to} \delta \quad \text{iff} \quad (\exists \Gamma' \subseteq^\omega \Gamma)\, \Gamma' \vdash_{\mathbb{K}}^{\to} \delta.$$

CHAPTER 3

Literature Survey

In this chapter we define what we refer to as *Spectral-like* and *Priestley-style* dualities, and we review in detail some of the dualities in the literature that can be seen as Spectral-like or Priestley-style dualities.

In § 3.1 we explain what we mean by *Spectral-like* and *Priestley-style* dualities. There is a vast literature on this topic, but we focus our attention on two recent results, that are used later on and whose study provided us with inspiration for our work.

On the one hand, in § 3.2 we focus on dualities for distributive meet-semilattices with top element. In § 3.2.1 we briefly present a simplified version of the Spectral-like duality for distributive meet-semilattices with top element that was studied in [**12**] by Celani. In § 3.2.2 we discuss the Priestley-style duality for distributive meet-semilattices with top element that was only sketched in [**5**] by Bezhanishvili and Jansana.

On the other hand, in § 3.3 we focus on dualities for Hilbert algebras. In § 3.3.1 we briefly present the Spectral-like duality for Hilbert algebras that was introduced in [**19**] by Celani and Montangie as a simplification of their work with Cabrer in [**15**]. Finally, in § 3.3.2 we outline the Priestley-style duality for Hilbert algebras that was studied in [**18**] by Celani and Jansana. Apart from some insignificant details in the last one, these dualities turn out to be instances of the general theory we develop in Chapter 5.

3.1. Spectral-like and Priestley-style dualities

The mathematical interest of studying Spectral-like and Priestley-style dualities goes back to Stone's duality for Boolean algebras [**69**], that properly speaking, is a dual equivalence between the category of Boolean algebras with algebraic homomorphisms, and the category of Boolean spaces and continuous maps. This duality has been generalized to distributive lattices in at least three ways (cf. [**3**] and its references).

The approach initiated by Stone himself [**69**] leads to a representation of distributive lattices in terms of spaces $\langle X, \tau\rangle$ that are sober, compactly-based and in which the collection of compact open sets is closed under finite intersections. Duals of algebraic homomorphisms are the so called *Spectral functions*, which are the maps whose inverse sends compact opens to compact opens.

A different approach initiated by Priestley [**65**] leads to a representation of distributive lattices in terms of ordered Hausdorff topological spaces that are named *Priestley spaces*. These are ordered topological spaces $\langle X, \leq, \tau\rangle$ that are compact and totally order-disconnected (whenever $a \leq b$, there exists a clopen up-set U such that $a \in U$ and $b \notin U$). Duals of algebraic homomorphisms are order-preserving continuous maps.

Recently, a third duality for Distributive Lattices, based on *Pairwise Stone spaces* has been studied in detail in [**3**]. Although we restrict our study to the first two dualities, it would be very interesting to look into the last one as well.

What makes the Stone/Priestley duality a powerful mathematical tool is that it allows us to use topology in the study of algebra (and vice versa). Many algebraic notions have their dual translation in terms of nice topological notions. It is precisely the fact that it is a *dual* equivalence of categories, i. e. the morphisms are *reversed*, which implies that dual of injectivity is surjectivity (and vice versa), duals of subalgebras are order-quotients, duals of homomorphic images are closed subsets and duals of disjoint unions are products (and vice versa).

All mentioned so far motivates the name of *Spectral-like dualities* for those dualities for which the objects of one of the categories are structures of the form $\langle X, \tau, \dots \rangle$, where $\langle X, \tau \rangle$ is a compactly-based sober topological space, and the suspension points indicate that we may have additional structure. Similarly, we use the name of *Priestley-style dualities* for those dualities for which the objects of one of the categories are structures of the form $\langle X, \tau, \leq, \dots \rangle$, where $\langle X, \tau, \leq \rangle$ is a compact totally order-disconnected ordered topological space, and the suspension points indicate again that we may have additional structure.

3.2. Duality theory for distributive semilattices

In this section we revise the results of Celani in [**12**], and the results of Bezhanishvili and Jansana in [**5**]. Special attention should be paid to the notation introduced, as it is used later on.

3.2.1. Spectral-like duality for distributive semilattices. A representation theorem for distributive semilattices can be obtained from Stone's pioneering work in [**69**], or more detailed in [**49**], where Grätzer considers distributive semilattices as the appropriate setting to discuss topological representations of distributive lattices. A duality for the category of distributive semilattices and algebraic homomorphisms (DS), was studied in [**12**], where dual objects of distributive semilattices are topological spaces called $\mathbb{DS}$-spaces. We recall that $\mathfrak{X} = \langle X, \tau \rangle$ is a $\mathbb{DS}$-*space* (Definition 14 in [**12**]) when it is a compactly-based sober topological space, that is, a topological space such that:[1]

(DS1) the collection $\mathcal{KO}(X)$ of compact open subsets forms a basis for the topology τ,

(DS2) the space $\langle X, \tau \rangle$ is sober.

In place of condition (DS2) we could also have:

(DS2′) the space $\langle X, \tau \rangle$ is T_0 and if Z is a closed subset and L is a non-empty down-directed subfamily of $\mathcal{KO}(X)$ such that $Z \cap U \neq \emptyset$ for all $U \in L$, then $Z \cap \bigcap\{U : U \in L\} \neq \emptyset$.

[1]Notice that $\mathbb{DS}$-spaces were originally defined in [**12**] as ordered topological spaces, where the order considered turns out to be precisely the dual of the specialization quasiorder of the space, that is in fact an order, since the space is T_0 (this follows from sobriety). This simple fact considerably simplifies the definition.

For any $\mathbb{DS}$-space $\mathfrak{X} = \langle X, \tau \rangle$, we consider the family

$$F(\mathfrak{X}) := \{U^c : U \in \mathcal{KO}(X)\},$$

which is closed under finite intersection by condition (DS1). In [**12**] (see also [**49**]) it is proven that $\mathfrak{X}^* := \langle F(\mathfrak{X}), \cap, X \rangle$ is a distributive semilattice, called the *Spectral-dual distributive semilattice* of $\mathfrak{X}$.

For $\mathbf{M} = \langle M, \wedge, 1 \rangle$ a given distributive semilattice, we shall consider the map $\psi_{\mathbf{M}} : M \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Irr}_{\wedge}(\mathbf{M}))$ given by:

$$\psi_{\mathbf{M}}(a) := \{P \in \mathrm{Irr}_{\wedge}(\mathbf{M}) : a \in P\}.$$

In [**12**] it is proven that $\{\psi_{\mathbf{M}}(a)^c : a \in M\}$ is a basis for a topology $\widehat{\tau}_{\mathbf{M}}$ on $\mathrm{Irr}_{\wedge}(\mathbf{M})$. Moreover the structure $\mathfrak{Irr}_{\wedge}(\mathbf{M}) := \langle \mathrm{Irr}_{\wedge}(\mathbf{M}), \widehat{\tau}_{\mathbf{M}} \rangle$ is shown to be a $\mathbb{DS}$-space, called the *dual* $\mathbb{DS}$*-space* of $\mathbf{M}$.

If $\mathfrak{X} = \langle X, \widehat{\tau} \rangle$ is a $\mathbb{DS}$-space, then it is homeomorphic to $\langle \mathrm{Irr}_{\wedge}(\mathfrak{X}^*), \widehat{\tau}_{\mathfrak{X}^*} \rangle$ by means of the map $\widehat{\varepsilon}_{\mathfrak{X}} : X \longrightarrow \mathrm{Irr}_{\wedge}(\mathfrak{X}^*)$, given by:

$$\widehat{\varepsilon}_{\mathfrak{X}}(x) := \{U \in F(\mathfrak{X}) : x \in U\}.$$

If $\mathbf{M}$ is a distributive semilattice, then it is isomorphic to $(\mathfrak{Irr}_{\wedge}(\mathbf{M}))^*$, the Spectral-dual distributive semilattice of $\mathfrak{Irr}_{\wedge}(\mathbf{M})$, by means of the map $\psi_{\mathbf{M}}$.

With respect to morphisms, duals of algebraic morphisms are not functions but relations, called meet-relations. We recall that for $\mathbb{DS}$-spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, a relation $R \subseteq X_1 \times X_2$ is a *meet-relation* when:[2]

(DSR1) $\Box_R(U) \in F(\mathfrak{X}_1)$ for all $U \in F(\mathfrak{X}_2)$,
(DSR2) $R(x)$ is a closed subset of X_2 for any $x \in X_1$.

For any meet-relation $R \subseteq X_1 \times X_2$, the map $\Box_R : \mathcal{P}(X_2) \longrightarrow \mathcal{P}(X_1)$ is an algebraic homomorphism between the distributive semilattices $\mathfrak{X}_2^*$ and $\mathfrak{X}_1^*$. For any homomorphism $h : M_1 \longrightarrow M_2$ between distributive semilattices, the relation $\overline{R}_h \subseteq \mathrm{Irr}_{\wedge}(\mathbf{M}_2) \times \mathrm{Irr}_{\wedge}(\mathbf{M}_1)$, given by:

$$(P, Q) \in \overline{R}_h \quad \text{iff} \quad h^{-1}[P] \subseteq Q,$$

is a meet-relation between the $\mathbb{DS}$-spaces $\mathfrak{Irr}_{\wedge}(\mathbf{M}_2)$ and $\mathfrak{Irr}_{\wedge}(\mathbf{M}_1)$. If $R \subseteq X_1 \times X_2$ is a meet-relation, then $(x_1, x_2) \in R$ if and only if $(\widehat{\varepsilon}_1(x_1), \widehat{\varepsilon}_2(x_2)) \in \overline{R}_{\Box_R}$, for all $x_1 \in X_1$ and $x_2 \in X_2$. If $h : M_1 \longrightarrow M_2$ is a homomorphism, then we have $\psi_2(h(a)) = \Box_{\overline{R}_h}(\psi_1(a))$ for all $a \in M_1$.

In brief, what is proven in [**11**] is that the category of distributive semilattices and homomorphisms DS is dually equivalent to the category $\mathsf{Sp}^{\mathbb{DS}}$, that has $\mathbb{DS}$-spaces as objects and meet-relations as morphisms.

3.2.2. Priestley-style duality for distributive semilattices. A different approach was followed in [**50**] and [**5**]. We focus on the work in [**5**], where two categorical dualities for categories having distributive semilattices as objects are studied and where the authors make an explicit connection between Priestley duality for

[2]Notice that this is a simplification of the original definition of meet-relation, that was more involved and obscure.

distributive lattices and the duality they provide for distributive semilattices, that is a generalization of the former.[3]

Regarding objects, their strategy consists in what follows: first they study how any distributive semilattice $\mathbf{M}$ can be embedded in a distributive lattice $\mathrm{L}(\mathbf{M})$, that they call *distributive envelope of* $\mathbf{M}$. This construction is also called in [**5**] the *free distributive lattice extension of* $\mathbf{M}$, due to a universal property that it has, and that will be stated later on in this section. We refer the reader to the Appendix A to go into details of this construction.

Once the authors have defined the distributive envelope of any distributive semilattice $\mathbf{M}$, they associate as the Priestley dual of $\mathbf{M}$, the Priestley dual of $\mathrm{L}(\mathbf{M})$, that can be described in terms of optimal meet filters of $\mathbf{M}$. But then they need to add additional structure to the dual spaces in order to recover the original semilattice. Dual objects of distributive semilattices are called generalized Priestley spaces. We recall that $\mathfrak{X} = \langle X, \tau, \leq, X_B \rangle$ is a *generalized Priestley space* (Definition 9.1 in [**5**]) when:

(DS3) $\langle X, \tau, \leq \rangle$ is a Priestley space,
(DS4) X_B is a dense subset of X,
(DS5) $X_B = \{x \in X : \{U \in \mathcal{C\ell U}^{ad}_{X_B}(X) : x \notin U\}$ is non-empty and up-directed$\}$,
(DS6) for all $x, y \in X$, $x \leq y$ iff $(\forall U \in \mathcal{C\ell U}^{ad}_{X_B}(X))$ if $x \in U$, then $y \in U$.

where $\mathcal{C\ell U}^{ad}_{X_B}(X) := \{U \in \mathcal{C\ell U}(X) : \max(U^c) \subseteq X_B\}$. Notice that the authors work with distributive semilattices that are bounded, whose dual spaces are the ones that they call generalized Priestley spaces. Only in Section 9 they briefly consider the case when there is not necessarily a bottom element, in which case dual spaces are called *-generalized Priestley spaces, and these are precisely the ones that we introduce here under the simplified name of generalized Priestley spaces. It should be noted that there is an inaccuracy in that outline: the duality they sketch for the non-bounded case works only if we modify the definition of optimal meet filter given in [**5**], and we use instead the one given in page 28.

For a given generalized Priestley space $\mathfrak{X} = \langle X, \tau, \leq, X_B \rangle$, we call X_B*-admissible clopen up-sets* the elements in $\mathcal{C\ell U}^{ad}_{X_B}(X)$. It turns out that this collection is closed under intersection, and in [**5**] it is proven that $\mathfrak{X}^\bullet := \langle \mathcal{C\ell U}^{ad}_{X_B}(X), \cap, X \rangle$ is a distributive semilattice, called the *Priestley-dual distributive semilattice* of $\mathfrak{X}$.

For a given distributive semilattice $\mathbf{M} = \langle M, \wedge, 1 \rangle$, we consider the function $\vartheta_{\mathbf{M}} : M \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Op}_\wedge(\mathbf{M}))$ given by:

$$\vartheta_{\mathbf{M}}(a) = \{P \in \mathrm{Op}_\wedge(\mathbf{M}) : a \in P\}.$$

For the bounded case, in [**5**] it is proven that $\{\vartheta_{\mathbf{M}}(a) : a \in M\} \cup \{\vartheta_{\mathbf{M}}(b)^c : b \in M\}$ is a subbasis for a Hausdorff topology $\tau_{\mathbf{M}}$ on $\mathrm{Op}_\wedge(\mathbf{M})$. Relying on the one-to-one correspondence that exists between optimal meet filters of $\mathbf{M}$ and prime filters of its distributive envelope $\mathrm{L}(\mathbf{M})$, the authors prove that the ordered topological space $\langle \mathrm{Op}_\wedge(\mathbf{M}), \tau_{\mathbf{M}}, \leq \rangle$ is order homeomorphic to the Priestley dual of $\mathrm{L}(\mathbf{M})$, and hence it is a Priestley space as well. As stated before, if we use the definition of optimal meet filter given by § 2.3, the same results hold for the general case, and moreover,

[3]In fact they define more categories, concerning the preservation of the bottom element by the morphisms. As we work in the more general setting where no bottom element is required, we do not treat these other categorical dualities.

the structure $\mathfrak{Op}_\wedge(\mathbf{M}) := \langle \mathrm{Op}_\wedge(\mathbf{M}), \tau_{\mathbf{M}}, \subseteq, \mathrm{Irr}_\wedge(\mathbf{M})\rangle$ turns out to be a generalized Priestley space, called the *dual (generalized) Priestley space* of $\mathbf{M}$.

If $\mathfrak{X} = \langle X, \tau, \leq, X_B\rangle$ is a generalized Priestley space, then it is order homeomorphic to $\langle \mathrm{Op}_\wedge(\mathfrak{X}^\bullet), \tau_{\mathfrak{X}^\bullet}, \subseteq, \mathrm{Irr}_\wedge(\mathfrak{X}^\bullet)\rangle$ by means of the map $\widehat{\xi}_{\mathfrak{X}} : X \longrightarrow \mathrm{Op}_\wedge(\mathfrak{X}^\bullet)$, given by:

$$\widehat{\xi}_{\mathfrak{X}}(x) := \{U \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X) : x \in U\}.$$

If $\mathbf{M}$ is a distributive semilattice, then it is isomorphic to $(\mathfrak{Op}_\wedge(\mathbf{M}))^\bullet$, the Priestley-dual distributive semilattice of $\mathfrak{Op}_\wedge(\mathbf{M})$, by means of the map $\vartheta_{\mathbf{M}}$.

Regarding morphisms, two different notions are considered. One is the usual notion of algebraic homomorphism, and the other is a stronger notion, called *sup-homomorphism*. These are algebraic homomorphisms $h : A_1 \longrightarrow A_2$ that preserve all existing finite suprema (including the bottom, when it exists). This is equivalent to saying that for all $n \in \omega$ and all $a_1, \dots, a_n, b \in M$:

$$\text{if } \bigcap_{i\leq n} {\uparrow} a_i \subseteq {\uparrow} b, \text{ then } \bigcap_{i\leq n} {\uparrow} h(a_i) \subseteq {\uparrow} h(b).$$

The importance of sup-homomorphism in the study of distributive semilattices is due precisely to the universal property of the distributive envelope of a distributive lattice: the distributive envelope of $\mathbf{M}$ is the unique (up to isomorphism) distributive lattice $\mathbf{L}$ such that there is a one-to-one sup-homomorphism $h : M \longrightarrow L$ such that for every distributive lattice $\mathbf{L}'$ and every sup-homomorphism $g : M \longrightarrow L'$, there exists a unique lattice homomorphism $\overline{g} : L \longrightarrow L'$ such that $g = \overline{g} \circ h$. An alternative characterization of the distributive envelope is the following (the proof can be found in Theorem 3.9 of [**5**]).

Theorem 3.2.1. *Let* $\mathbf{M}$ *be a bounded distributive semilattice. The distributive envelope* $\mathrm{L}(\mathbf{M})$ *of* $\mathbf{M}$ *is up to isomorphism the unique distributive lattice* $\mathbf{L}$ *for which there is a one-to-one sup-homomorphism* $e : M \longrightarrow L$ *such that* $e[L]$ *is join-dense in* L.

The duals of algebraic homomorphisms are called generalized Priestley morphisms while the duals of sup-homomorphisms are called functional generalized Priestley morphisms. We recall that for generalized Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, a relation $R \subseteq X_1 \times X_2$ is a *generalized Priestley morphism* (Definition 6.2 in [**5**]) when:

(DSR3) $\Box_R(U) \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_{B_1}}(X_1)$ for all $U \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_{B_2}}(X_2)$,

(DSR4) if $(x, y) \notin R$, then there is $U \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_{B_2}}(X_2)$ such that $y \notin U$ and $R(x) \subseteq U$.

We say that R is a *functional generalized Priestley morphism* (Definition 6.11 in [**5**]) when it is a generalized Priestley morphism that satisfies:

(DSF) for each $x \in X_1$ there is $x' \in X_2$ such that $R(x) = {\uparrow} x'$.

For a given generalized Priestley morphism $R \subseteq X_1 \times X_2$, we get that the map $\Box_R : \mathcal{P}(X_2) \longrightarrow \mathcal{P}(X_1)$ is an algebraic homomorphism between distributive semilattices $\mathfrak{X}_2^\bullet$ and $\mathfrak{X}_1^\bullet$. Moreover, if R is functional, then $\Box_R$ is a sup-homomorphism.

TABLE 1. Categories involved in the Priestley-style duality for distributive semilattices [**5**].

CATEGORY	OBJECTS	MORPHISMS
DS	Distributive semilattices	Algebraic homomorphisms
DS_{sup}	Distributive semilattices	Sup-homomorphisms
$\mathsf{Pr}_M^{\mathbb{DS}}$	Generalized Priestley spaces	Generalized Priestley morphisms (composition $\star$)
$\mathsf{Pr}_F^{\mathbb{DS}}$	Generalized Priestley spaces	Functional generalized Priestley morphisms (composition $\star$)

For a given homomorphism between distributive semilattices $h : M_1 \longrightarrow M_2$, the relation $R_h \subseteq \mathrm{Op}_\wedge(\mathbf{M}_2) \times \mathrm{Op}_\wedge(\mathbf{M}_1)$, given by:

$$(P, Q) \in R_h \quad \text{iff} \quad h^{-1}[P] \subseteq Q,$$

is a generalized Priestley morphism between generalized Priestley spaces $\mathfrak{Op}_\wedge(\mathbf{M}_2)$ and $\mathfrak{Op}_\wedge(\mathbf{M}_1)$. Moreover, if h is a sup-homomorphism, then R_h is functional. If $R \subseteq X_1 \times X_2$ is a generalized Priestley morphism, then $(x, y) \in R$ if and only if $(\widehat{\xi}(x), \widehat{\xi}(y)) \in R_{\Box_R}$, for all $x \in X_1$ and all $y \in X_2$. If $h : M_1 \longrightarrow M_2$ is a homomorphism, then $\vartheta_2(h(a)) = \Box_{R_h}(\vartheta_1(a))$ for all $a \in M_1$.

It should be noted that these Priestley-style dualities have a slight drawback: unfortunately, usual composition of relations does not work as composition between generalized Priestley morphisms. Instead we have that for any generalized Priestley spaces $\mathfrak{X}_1$, $\mathfrak{X}_2$ and $\mathfrak{X}_3$ and any generalized Priestley morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, the composition of R and S as morphisms between generalized Priestley spaces is $S \star R$,[4] where:

$$(x, z) \in (S \star R) \quad \text{iff} \quad \forall U \in \mathcal{C\ell U}^{ad}_{X_{B_3}}(X_3)\big(\text{ if } x \in (\Box_R \circ \Box_S)(U), \text{ then } z \in U\big).$$

Table 1 collects all the categories involved in this duality. Summarizing, in [**5**] Bezhanishvili and Jansana work out a Priestley-style duality for bounded distributive semilattices. Although it is only sketched in [**5**], their results can also be applied to get a Priestley-style duality for distributive semilattices not necessarily bounded. Then we get that DS is dually equivalent to $\mathsf{Pr}_M^{\mathbb{DS}}$ and DS_{sup} is dually equivalent to $\mathsf{Pr}_F^{\mathbb{DS}}$.

3.3. Duality theory for Hilbert algebras

In this section we revise the results of Celani, Cabrer and Montangie in [**15**], and we only sketch the results of Celani and Jansana in [**18**]. Special attention should be paid again to the notation introduced, as it is used later on.

[4]Notice that in the original paper [**5**] there is a notational inconsistency that is worth being aware of: up to page 107, where $\star$ is defined, the notation used for composition of relations is the one usually used in category theory, namely the *left* composition, where the first applied relation is the left one. From this point on, the notation used for both $\circ$ and $\star$ is the usual one for composition of relations, namely the *right* composition.

3.3.1. Spectral-like duality for Hilbert algebras. In [**15**] and [**19**] Celani et al. have studied Spectral-like dualities for two categories having Hilbert algebras as objects. In both dualities, dual objects of Hilbert algebras are ordered topological spaces called $\mathbb{H}$-spaces. We recall that $\mathfrak{X} = \langle X, \tau_\kappa \rangle$ is an $\mathbb{H}$-*space* (Definition 3.4 in [**19**]) when:[5]

(H6) κ is a basis of open and compact subsets for the topological space $\langle X, \tau_\kappa \rangle$,
(H7) for every $U, V \in \kappa$, $\mathrm{sat}(U \cap V^c) \in \kappa$,
(H8) $\langle X, \tau_\kappa \rangle$ is sober.

In place of condition (H8) we could also have:

(H8′) the space $\langle X, \tau_\kappa \rangle$ is T_0 and whenever Z is a closed subset and $\mathcal{U}$ is a non-empty down-directed subfamily of κ such that $Z \cap U \neq \emptyset$ for all $U \in \mathcal{U}$, we have $Z \cap \bigcap\{U : U \in \mathcal{U}\} \neq \emptyset$.

For any $\mathbb{H}$-space $\mathfrak{X} = \langle X, \tau \rangle$, we consider the family

$$D(\mathfrak{X}) := \{U^c : U \in \kappa\},$$

and we define on this set the operation $\Rightarrow$ such that for all $U, V \in \kappa$,

$$U^c \Rightarrow V^c := (\mathrm{sat}(U \cap V^c))^c.$$

This operation is well defined by condition (H7), and in [**15**] it is proven that the structure $\mathfrak{X}^* := \langle D(\mathfrak{X}), \Rightarrow, X \rangle$ is a Hilbert algebra, called the *Spectral-dual Hilbert algebra* of $\mathfrak{X}$.

For a given Hilbert algebra $\mathbf{A} = \langle A, \to, 1 \rangle$, we shall consider the function $\psi_\mathbf{A} : A \longrightarrow \mathcal{P}^\uparrow(\mathrm{Irr}_\to(\mathbf{A}))$, given by:

$$\psi_\mathbf{A}(a) := \{P \in \mathrm{Irr}_\to(\mathbf{A}) : a \in P\}.$$

In [**15**] it is proven that $\kappa_\mathbf{A} = \{\psi_\mathbf{A}(a)^c : a \in A\}$ is a basis for a topology $\tau_{\kappa_\mathbf{A}}$ on $\mathrm{Irr}_\to(\mathbf{A})$. Moreover the structure $\mathfrak{Irr}_\to(\mathbf{A}) := \langle \mathrm{Irr}_\to(\mathbf{A}), \tau_{\kappa_\mathbf{A}} \rangle$ is shown to be an $\mathbb{H}$-space, called the *dual* $\mathbb{H}$-*space* of $\mathbf{A}$.

If $\mathfrak{X} = \langle X, \tau_\kappa \rangle$ is an $\mathbb{H}$-space, then it is homeomorphic to $\langle \mathrm{Irr}_\to(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}} \rangle$ by means of the map $\varepsilon_\mathfrak{X} : X \longrightarrow \mathrm{Irr}_\to(\mathfrak{X}^*)$, given by:

$$\varepsilon_\mathfrak{X}(x) := \{U \in D(\mathfrak{X}) : x \in U\}.$$

If $\mathbf{A}$ is a Hilbert algebra, then it is isomorphic to $(\mathfrak{Irr}_\to(\mathbf{A}))^*$, the Spectral-dual Hilbert algebra of $\mathfrak{Irr}_\to(\mathbf{A})$, by means of the map $\psi_\mathbf{A}$.

With regard to morphisms, two different notions are considered. One is the usual notion of algebraic homomorphism, and the other is a weaker notion, called *semi-homomorphism*. These are functions $h : A_1 \longrightarrow A_2$ such that $h(1^{\mathbf{A}_1}) = 1^{\mathbf{A}_2}$ and for all $a, b \in A_1$, it holds $h(a \to^{\mathbf{A}_1} b) \leq^{\mathbf{A}_2} h(a) \to^{\mathbf{A}_2} h(b)$. The importance of semi-homomorphisms in the study of Hilbert algebras is given by the following theorem, that involves the important notion of implicative filter (see page 31 for the definition), and whose proof can be found in Theorem 3.2 in [**10**].

[5] $\mathbb{H}$-spaces were originally defined in [**15**] as ordered topological spaces with the designated basis. In [**19**] it was remarked that the order was nothing but the dual of the specialization order of the space. Therefore some conditions in the original definition of $\mathbb{H}$-spaces are redundant, and a more compact definition of such spaces is given in [**19**], where the original name is maintained for the new spaces, while the former are renamed as *ordered* $\mathbb{H}$-*spaces*.

TABLE 2. Categories involved in the Spectral-like duality for Hilbert algebras [**15**].

CATEGORY	OBJECTS	MORPHISMS
H_S	Hilbert algebras	Semi-homomorphisms
H_H	Hilbert algebras	Algebraic homomorphisms
$\mathsf{Sp}_M^{\mathbb{H}}$	$\mathbb{H}$-spaces	$\mathbb{H}$-relations
$\mathsf{Sp}_F^{\mathbb{H}}$	$\mathbb{H}$-spaces	functional $\mathbb{H}$-relations

THEOREM 3.3.1. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two Hilbert algebras. A map $h : A_1 \longrightarrow A_2$ is a semi-homomorphism if and only if $h^{-1}[F] \in \mathrm{Fi}_{\to}(\mathbf{A}_1)$ for every $F \in \mathrm{Fi}_{\to}(\mathbf{A}_2)$.*

Duals of semi-homomorphisms are called $\mathbb{H}$-relations while duals of homomorphisms are called functional $\mathbb{H}$-relations. We recall that for $\mathbb{H}$-spaces $\langle X_1, \tau_{\kappa_1}\rangle$ and $\langle X_2, \tau_{\kappa_2}\rangle$, a relation $R \subseteq X_1 \times X_2$ is an $\mathbb{H}$-*relation* (definition 3.2 in [**15**]) when:

(HR1) $\Box_R(U) \in \kappa_1$, for all $U \in \kappa_2$,
(HR2) $R(x)$ is a closed subset of X_2, for all $x \in X_1$.

We say that R is a *functional* $\mathbb{H}$-*relation* (Definition 3.3 in [**15**]) when it is an $\mathbb{H}$-relation that satisfies:

(HF) if $(x, y) \in R$, then there exists $z \in \mathrm{cl}(x)$ such that $R(z) = \mathrm{cl}(y)$.

Notice that here the authors use the adjective 'functional' associated with being the dual of an algebraic homomorphism whereas in the Priestley-style duality for distributive semilattices, the adjective 'functional' is used associated with being the dual of a sup-homomorphism.

For $R \subseteq X_1 \times X_2$ a given $\mathbb{H}$-relation, the map $\Box_R : \mathcal{P}(X_2) \times \mathcal{P}(X_1)$ is a semi-homomorphism between Hilbert algebras $\mathfrak{X}_2^*$ and $\mathfrak{X}_1^*$. Moreover, if R is functional, then $\Box_R$ is a homomorphism. For $h : A_1 \longrightarrow A_2$ a given semi-homomorphism between Hilbert algebras, the relation $R_h \subseteq \mathrm{Irr}_{\to}(\mathbf{A}_2) \times \mathrm{Irr}_{\to}(\mathbf{A}_1)$, defined by:

$$(P, Q) \in R_h \quad \text{iff} \quad h^{-1}[P] \subseteq Q,$$

is an $\mathbb{H}$-relation between $\mathbb{H}$-spaces $\mathfrak{Irr}_{\to}(\mathbf{A}_2)$ and $\mathfrak{Irr}_{\to}(\mathbf{A}_1)$. Moreover, if h is a homomorphism, then R_h is functional. If $R \subseteq X_1 \times X_2$ is a (functional) $\mathbb{H}$-relation, then $(x_1, x_2) \in R$ if and only if $(\varepsilon_1(x_1), \varepsilon_2(x_2)) \in R_{\Box_R}$, for all $x_1 \in X_1$ and $x_2 \in X_2$. If $h : A_1 \longrightarrow A_2$ is a (semi-)homomorphism, then $\psi_2(h(a)) = \Box_{R_h}(\psi_1(a))$ for all $a \in A_1$.

Table 2 collects all the categories involved in this duality. Summarizing, what is proven in [**15**] is that H_S is dually equivalent to $\mathsf{Sp}_M^{\mathbb{H}}$ and H_H is dually equivalent to $\mathsf{Sp}_F^{\mathbb{H}}$.

3.3.2. Priestley-style duality for Hilbert algebras. In [**18**], Celani and Jansana have studied Priestley-style dualities for four categories having Hilbert algebras as objects.[6] This spread of categories is because their work is based in the mentioned Priestley-style dualities for DS and DS_{sup}.

In relation to objects, their strategy consists in what follows: first they study how any Hilbert algebra $\mathbf{A}$ can be embedded in a bounded distributive semilattice $\mathrm{M}(\mathbf{A})$, so that they can associate as the Priestley dual of $\mathbf{A}$, the Priestley dual of $\mathrm{M}(\mathbf{A})$ (as it was defined in [**5**]). This construction is based on their work in [**17**] where they study how any Hilbert algebra can be embedded in an implicative semilattice (not necessarily bounded) that they call *free implicative semilattice extension of* $\mathbf{A}$ (the name is due to the universal property that this construction has). This issue concerning bounds (Hilbert algebras do not have necessarily a bottom element, while bounded distributive semilattices certainly do) makes their construction a bit involved. Their duality can be simplified if we forget about the bottom, and we use instead the duality for distributive semilattices we already presented. This follows, in fact, as a particular instance of the theory we present in Chapter 5. We encourage the reader to address § 6.2 for a full description of a Priestley-style duality for Hilbert algebras. In what follows, we just briefly present, for the sake of completeness, the definitions of dual objects and morphisms the authors give in [**18**]. We recall that $\mathfrak{X} = \langle X, \tau, \leq, B\rangle$ is an *augmented Priestley space* (Definition 5.4 in [**18**]) when:

(H9) $\langle X, \tau\rangle$ is a compact topological space,
(H10) $\langle X, \leq\rangle$ is a poset with top element t,
(H11) B is a non-empty collection of non-empty clopen up-sets of X,
(H12) for every $x, y \in X$, $x \leq y$ iff $\forall U \in B($ if $x \in U$, then $y \in U)$,
(H13) the set $X_B \cup \{t\}$ is dense in X, where

$$X_B := \{x \in X : \{U \in B : x \notin U\} \text{ is non-empty and up-directed}\},$$

(H14) for all $U, V \in B$, $({\downarrow}(U \cap V^c))^c \in B$.

Notice that from compactness and condition (H12), it follows that $\langle X, \tau, \leq\rangle$ is a Priestley-space.

For a given augmented Priestley space $\mathfrak{X} = \langle X, \tau, \leq, B\rangle$, consider the operation $\Rightarrow: B \times B \longrightarrow B$ such that for all $U, V \in B$,

$$U \Rightarrow V := ({\downarrow}(U \cap V^c))^c.$$

By condition (H14), this operation is well defined. In [**17**] it is proven that the algebra $\mathfrak{X}^\bullet := \langle B, \Rightarrow, X\rangle$ is a Hilbert algebra, called the *Priestley-dual Hilbert algebra* of $\mathfrak{X}$.

For a given Hilbert algebra $\mathbf{A} = \langle A, \to, 1\rangle$, define $\mathrm{Op}^+_\to(\mathbf{A}) := \mathrm{Op}_\to(\mathbf{A}) \cup \{A\}$. Recall that only if $\mathbf{A}$ has no bottom element, the emptyset is an strong Frink ideal, and so A is an optimal implicative filter, in which case $\mathrm{Op}^+_\to(\mathbf{A}) = \mathrm{Op}_\to(\mathbf{A})$. Consider the map $\vartheta^+_{\mathbf{A}} : A \longrightarrow \mathcal{P}^\uparrow(\mathrm{Op}^+_\to(\mathbf{A}))$ given by:

$$\vartheta^+_{\mathbf{A}}(a) := \{P \in \mathrm{Op}^+_\to(\mathbf{A}) : a \in P\}.$$

In [**17**] it is proven that $\{\vartheta^+_{\mathbf{A}}(a) : a \in A\} \cup \{\vartheta^+_{\mathbf{A}}(b)^c : b \in A\}$ is a subbasis for a topology $\tau^+_{\mathbf{A}}$ on $\mathrm{Op}^+_\to(\mathbf{A})$. Moreover it is defined the structure $\mathfrak{Op}^+_\to(\mathbf{A}) :=$

[6] In fact, they consider more categories, concerning preservation of the bottom element, that we do not consider here.

$\langle \mathrm{Op}^+_\to(\mathbf{A}), \tau^+_{\mathbf{A}}, \subseteq, \vartheta^+_{\mathbf{A}}[A]\rangle$, that is shown to be an augmented Priestley space, called the *dual (augmented) Priestley space* of $\mathbf{A}$.

If $\mathfrak{X} = \langle X, \tau, \leq, B\rangle$ is an augmented Priestley space, then we obtain that the structure $\langle \mathrm{Op}^+_\to(\mathfrak{X}^\bullet), \tau^+_{\mathfrak{X}^\bullet}, \subseteq, \vartheta^+_{\mathfrak{X}^\bullet}[B]\rangle$ is an augmented Priestley space, and the ordered topological space $\langle X, \tau, \leq\rangle$ is order homeomorphic to $\langle \mathrm{Op}^+_\to(\mathfrak{X}^\bullet), \tau^+_{\mathfrak{X}^\bullet}, \subseteq\rangle$ by means of the map $\xi^+_{\mathfrak{X}} : X \longrightarrow \mathrm{Op}^+_\to(\mathfrak{X}^\bullet)$, given by:

$$\xi^+_{\mathfrak{X}}(x) := \{U \in B : x \in U\}.$$

If $\mathbf{A}$ is a Hilbert algebra, then it is isomorphic to $(\mathrm{Op}^+_\to(\mathbf{A}))^\bullet$, the Priestley-dual Hilbert algebra of $\mathrm{Op}^+_\to(\mathbf{A})$, by means of the map $\vartheta^+_{\mathbf{A}}$.

In relation to morphisms, the following property of maps is considered. A map $h : A \longrightarrow B$ between Hilbert algebras $\mathbf{A}$ and $\mathbf{B}$ has the *sup-property* if for every $a_1, \ldots, a_n, b_0, \ldots, b_m \in A$:

$$\text{if } \bigcap_{i\leq n} \uparrow a_i \subseteq \langle\{b_0, \ldots, b_n\}\rangle, \text{ then } \bigcap_{i\leq n} \uparrow h(a_i) \subseteq \langle\{h(b_0), \ldots, h(b_m)\}\rangle.$$

Besides semi-homomorphisms and homomorphisms, two more notions are considered, namely *sup-semi-homomorphisms*, that are semi-homomorphisms with the sup-property, and *sup-homomorphisms*, that are homomorphisms with the sup-property.

Duals of semi-homomorphisms are called augmented Priestley semi-morphisms. We recall that for augmented Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, a relation $R \subseteq X_1 \times X_2$ is an *augmented Priestley semi-morphism* (Definition 5.13 in [**18**]) when:

(HR3) if $(x, y) \notin R$, then there is $U \in B_2$ such that $y \notin U$ and $R(x) \subseteq U$,
(HR4) $\Box_R(U) \in B_1$ for all $U \in B_2$.

We say that R is an *augmented Priestley morphism* if in addition satisfies:

(HF′) for every $x \in X_1$ and every $y \in X_{B_2}$, if $(x, y) \in R$, then there exists $z \in X_{B_1}$ such that $z \in \uparrow x$ and $R(z) = \uparrow y$.

that for

We say that an augmented Priestley semi-morphism is *functional* if for every $x \in X_1$, the set $R(x)$ has a least element. Notice that concerning the use of the adjective 'functional', the authors follows the terminology used in [**5**].

For a given augmented Priestley semi-morphism $R \subseteq X_1 \times X_2$, the map $\Box_R : \mathcal{P}(X_2) \longrightarrow \mathcal{P}(X_1)$ is a semi-homomorphism between Hilbert algebras $\mathfrak{X}_2^\bullet$ and $\mathfrak{X}_1^\bullet$. Moreover, if R is an augmented Priestley morphism, then $\Box_R$ is a homomorphism, and if R is functional, then $\Box_R$ has the sup-property. For any semi-homomorphism between Hilbert algebras $h : A_1 \longrightarrow A_2$, we obtain that the relation $R_h \subseteq \mathrm{Op}^+_\to(\mathbf{A}_2) \times \mathrm{Op}^+_\to(\mathbf{A}_1)$, given by

$$(P, Q) \in R_h \quad \text{iff} \quad h^{-1}[P] \subseteq Q,$$

is an augmented Priestley semi-morphism between augmented Priestley spaces $\mathfrak{O}\mathrm{p}^+_\to(\mathbf{A}_2)$ and $\mathfrak{O}\mathrm{p}^+_\to(\mathbf{A}_1)$. Moreover, if h is an homomorphism, then R_h is an augmented Priestley morphism, and if h has the sup-property, then R_h is functional. If $R \subseteq X_1 \times X_2$ is an augmented Priestley semi-morphism, then $(x, y) \in R$ if and only if $(\xi^+(x), \xi^+(y)) \in R_{\Box_R}$, for all $x \in X_1$ and $y \in X_2$. If $h : A_1 \longrightarrow A_2$ is a semi-homomorphism, then $\vartheta^+_2(h(a)) = \Box_{R_h}(\vartheta^+_1(a))$ for all $a \in A_1$.

Table 3. Categories involved in the Priestley-style duality for Hilbert algebras [**18**].

Category	Objects	Morphisms
H_S	Hilbert algebras	semi-homomorphisms
H_H	Hilbert algebras	algebraic homomorphisms
H_{supS}	Hilbert algebras	sup-semi-homomorphisms
H_{supH}	Hilbert algebras	sup-homomorphisms
$\mathsf{Pr}^{\mathbb{H}}_{sM}$	Augmented Priestley spaces	Augmented Priestley semi-morphisms (composition $\star$)
$\mathsf{Pr}^{\mathbb{H}}_{M}$	Augmented Priestley spaces	Augmented Priestley morphisms (composition $\star$)
$\mathsf{Pr}^{\mathbb{H}}_{sF}$	Augmented Priestley spaces	Functional augmented Priestley semi-morphisms (composition $\star$)
$\mathsf{Pr}^{\mathbb{H}}_{F}$	Augmented Priestley spaces	Functional augmented Priestley morphisms (composition $\star$)

As in the case of Priestley-style duality for distributive semilattices, usual composition of relations does not work as composition between augmented Priestley semi-morphisms. Instead we have that for any augmented Priestley spaces $\mathfrak{X}_1$, $\mathfrak{X}_2$ and $\mathfrak{X}_3$ and any augmented Priestley semi-morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, the composition of R and S as morphisms between augmented Priestley spaces is $S \star R$, where:

$$(x, z) \in (S \star R) \quad \text{iff} \quad \forall U \in B_3 \big(\text{ if } x \in \Box_R \circ \Box_S(U), \text{ then } z \in U \big).$$

Table 3 collects all the categories involved in these dualities. Summarizing, what is proven in [**18**] is that H_S is dually equivalent to $\mathsf{Pr}^{\mathbb{H}}_{sM}$, H_H is dually equivalent to $\mathsf{Pr}^{\mathbb{H}}_{M}$, H_{supS} is dually equivalent to $\mathsf{Pr}^{\mathbb{H}}_{sF}$ and H_{supH} is dually equivalent to $\mathsf{Pr}^{\mathbb{H}}_{F}$.

Part 2

Duality Theory for Filter Distributive Congruential Logics

CHAPTER 4

Duality Theory and Abstract Algebraic Logic: Introduction and Motivation

In this chapter we introduce our work plan for Part 2, we give the motivation and we make the first steps towards an abstract Stone/Priestley duality theory for non-classical logics under the point of view of AAL.

In § 4.1 we introduce the problem we aim to solve, and how we propose to reach it. We tackle that problem in Chapter 5, but in the remaining part of the present chapter we examine the tools we need to achieve that goal. First, we introduce in § 4.2 some concepts from Wójcicki's theory of logical calculi [**73**], that were introduced to deal with a more general problem closely related to ours. In § 4.3 we focus on closure bases for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$, the closure operator associated with the collection of $\mathcal{S}$-filters of $\mathbf{A}$. We present some results that can also be found in [**56**]. Then in § 4.4 we introduce some new concepts, from another instance of what was presented in Chapter 2, specialized in this case for the closure operator $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ defined on the poset $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle$. This completes the toolkit we need for the next chapter. Finally, relying on the work in [**41**], the $\mathcal{S}$-semilattice of $\mathbf{A}$ is introduced and studied in § 4.5, and we take a look in § 4.6 at canonical extensions for congruential logics.

4.1. Introduction and motivation

Regarding logic and theoretic computer science, Stone/Priestley duality has been used for different purposes: Rasiowa and Sikorski [**68**] applied Baire category theorem to the dual space of the Lindenbaum-Tarski algebra of the first-order logic to provide a topological proof of Gödel's completeness theorem for first-order logic. Abramsky [**1**] used Stone duality for distributive lattices to connect specification languages and denotational semantics, thus linking lambda calculus and domain theory. More recently, Gehrke, Grigorieff and Pin [**39**, **40**] studied the connection between regular languages and monoids as another case of Stone duality.

Apart from these less-known applications of duality theory in the study of logic, what is always mentioned, and rightfully so, is the work by Jónsson and Tarski [**58**, **59**] on representation of Boolean algebras with operators. As it is addressed in [**47**], this algebraic work was overlooked by modal logicians at that time, but it could immediately have been applied to give new algebraic semantics of modal logics, and even more, to prove completeness of modal logic with respect to what later became known as general Kripke frames.

Nevertheless, Jónsson and Tarski's paper ushered a fruitful field of study: the study of the relation between algebraic semantics and Kripke-style semantics of a logic via dual equivalences of categories. Although both Stone and Priestley approaches have been followed to generalize this pioneering work on representation of Boolean algebras with operators [**58**], the latter, that deals with Hausdorff (i. e. nicer) topological spaces, has been held to be advantageous [**13**, **46**], especially in view of recent developments of the theory of canonical extensions (see [**43**] and its references). The key point of this theory, called *extended Priestley duality*, is that the additional n-ary operations either preserve joins (resp. meets) in each coordinate, or send joins (resp. meets) in each coordinate to meets (resp. joins). These n-ary operations are dually represented in terms of $n+1$-ary relations satisfying certain conditions. Moreover, the theory of canonical extensions has enabled the study of the relation between algebraic and relational semantics of some well-behaved substructural logics [**26**, **70**, **71**], via discrete dualities (i. e. dualities in which no topology is involved). A modular study of the relational semantics that follows from these studies was developed by Gehrke in [**38**], where such semantic models were called *generalized Kripke frames*.

Until the mid-2000s, all categories of algebras (and homomorphisms) for which Stone/Priestley dualities were studied had as objects lattice-based algebras (i. e. algebras with a lattice reduct), in most cases distributive. Which means that all logics for which the relation between its algebraic semantics and its Kripke-style semantics had been studied via a topological duality, were logics having well-behaved conjunction and disjunction connectives. In the recent literature, we find further studies that work out dualities for logics that do not have both a conjunction and a disjunction at the same time, or that do not have any of these connectives. In other words, Spectral-like and Priestley-style dualities have been studied for categories whose objects correspond to certain ordered algebraic structures that are not lattice-based. The approach initiated by Stone has been followed in [**11**, **15**, **19**], whereas the approach initiated by Priestley has been followed in [**5**, **6**, **18**], among others.

Although these studies often have a logical motivation, their content is mathematical above all, and it happens that the connection with logic is not studied in a sufficiently explicit manner. For example, it has become *vox populi* that Stone duality for Modal algebras provides relational semantics for modal logics. But it is rather unusual to specify *which* modal logic (as a closure relation) they refer to, namely the local consequence of the referred modal logic. This connection can be made explicit by using the notions from AAL of $\mathcal{S}$-algebras and $\mathcal{S}$-filters. As these notions are defined for any arbitrary logic and any arbitrary algebra, a natural question arises: which abstract properties should a logic have in order to possess a Stone/Priestley duality for the class of algebras canonically associated with it? We aim to identify the class of logics $\mathcal{S}$ such that the category of $\mathcal{S}$-algebras and homomorphisms can be seen dually as a Spectral-like and/or a Priestley-style category.

This problem might be seen as a restriction of a more general question that was tackled by Wójcicki in [**73**]. He asked about the abstract properties that a logic should have in order to posses a frame semantics. In the next section we review in detail this work, given that the problem we tackle in this dissertation can be seen as a restriction of that addressed in [**73**] by Wójcicki.

4.2. Referential semantics and selfextensional logics

Referential algebras were introduced by Wójcicki [**73**] as a tool for studying the link between frame semantics and algebraic semantics of arbitrary logics. The underlying idea of *frame semantics* is simple and straightforward: it consists in assuming that truth values of the formulas depend on reference points. One of the possible interpretations of those reference points is that of the possible worlds, in which case frame semantics reduces to possible world semantics. Instead of dealing with *frames*, Wójcicki opts to deal with *referential algebras*. These structures are linked with *referential semantics* in the same way as Kripke frames are linked with Kripke semantics. Moreover in [**73**], it is shown that a logic has a frame semantics if, and only if, it admits referential semantics. So the only difference is the point of view: referential semantics is nothing but regarding frame semantics (or Kripke-style semantics) under an algebraic point of view. And this seems to be the suitable outlook if we aim to study the link between algebraic and frame semantics.

Given a logical language $\mathscr{L}$, an *$\mathscr{L}$-referential algebra* is a structure $\mathcal{X} = \langle X, \mathbf{B}\rangle$ where:

(1) X is a non-empty set, and
(2) $\mathbf{B}$ is an $\mathscr{L}$-algebra whose elements are subsets of X.

For any $\mathscr{L}$-referential algebra $\mathcal{X} = \langle X, \mathbf{B}\rangle$, we define the relation $\preceq_{\mathcal{X}} \subseteq X \times X$ as follows:

$$x \preceq_{\mathcal{X}} y \quad \text{iff} \quad \forall U \in B\big(\text{ if } x \in U, \text{ then } y \in U\big).$$

This relation is a quasiorder on X, and whenever $\preceq_{\mathcal{X}}$ is a partial order, the $\mathscr{L}$-referential algebra $\mathcal{X}$ is said to be *reduced*. In this case, we denote $\preceq_{\mathcal{X}}$ by $\leq_{\mathcal{X}}$, or even by $\leq$ when the context is clear.

Referential algebras are another example of structures that can be used to define logics. For instance, for any $\mathscr{L}$-referential algebra $\mathcal{X} = \langle X, \mathbf{B}\rangle$ we might define the relation $\vdash_{\mathcal{X}} \subseteq \mathcal{P}(Fm_{\mathscr{L}}) \times Fm_{\mathscr{L}}$ such that for all $\Gamma \cup \{\delta\} \subseteq Fm_{\mathscr{L}}$:

$$\Gamma \vdash_{\mathcal{X}} \delta \quad \text{iff} \quad \forall h \in \mathrm{Hom}(\mathbf{Fm}_{\mathscr{L}}, \mathbf{B}), \bigcap_{\gamma \in \Gamma} h(\gamma) \subseteq h(\delta).$$

Given a logic $\mathcal{S}$ in the language $\mathscr{L}$, and an $\mathscr{L}$-referential algebra $\mathcal{X}$, we say that $\mathcal{X}$ is an *$\mathcal{S}$-referential algebra* provided $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{X}}$. Moreover, we say that $\mathcal{S}$ *admits a (complete local) referential semantics* if there is a class of referential algebras $\mathbb{X}$ such that $\vdash_{\mathcal{S}} = \bigcap\{\vdash_{\mathcal{X}} : \mathcal{X} \in \mathbb{X}\}$.

Remark 4.2.1. It is easy to see (check Remark 5.2 in [**56**]) that for each algebraic reduct $\mathbf{B}$ of a reduced $\mathcal{S}$-referential algebra, $\mathbf{B} \in \mathbb{A}\mathrm{lg}\mathcal{S}$.

In [**73**] it is identified the abstract property of a logic that corresponds to admitting a complete local referential semantics. Wójcicki defines *selfextensional logics* as those logics $\mathcal{S}$ for which $\mathbf{\Lambda}_{\mathrm{C}_{\mathcal{S}}}$, the Frege relation of $\mathrm{C}_{\mathcal{S}}$, is a congruence of $\mathbf{Fm}$, where recall that $\mathbf{\Lambda}_{\mathrm{C}_{\mathcal{S}}} \subseteq Fm \times Fm$ is given by: for all $\gamma, \delta \in \mathbf{Fm}$

$$(\gamma, \delta) \in \mathbf{\Lambda}_{\mathrm{C}_{\mathcal{S}}} \quad \text{iff} \quad \mathrm{C}_{\mathcal{S}}(\gamma) = \mathrm{C}_{\mathcal{S}}(\delta).$$

Alternatively, we can define selfextensional logics (see Definition 2.41 in [**35**]) as those logics for which the Frege relation of $\mathrm{C}_{\mathcal{S}}$ and the Tarski congruence of $\mathcal{S}$ coincide.

Selfextensional logics are precisely those logics that admit complete and local referential semantics. An updated approach to the topic is carried out by Jansana and Palmigiano in [**56**], using modern notation and terminology that we follow. The mentioned correspondence between selfextensional logics and logics that admit referential semantics is formulated in [**56**] as a proper equivalence of categories.

These studies fit in the field of AAL, where selfextensional logics have been studied in depth (cf. [**34**, **53**–**55**] and its references). Other classes of logics can be defined using the Frege relation, and this yields to what is known as the *Frege hierarchy*. This is a classification scheme of logics under four classes defined in terms of congruence properties. Selfextensional logics are one of theses classes, the others being *Fregean*, *fully Fregean* and *congruential* (a. k. a. *fully selfextensional*) logics. The study of this classification, its structure and its relations with the *Leibniz hierarchy* started in the late 90's, and has continued to be intense in the last twenty years.[1] Almost all known selfextensional logics are congruential, and only in [**2**] it is presented an *ad hoc* example that shows that the inclusion of the former in the latter is strict.

As it is formulated by Jansana and Palmigiano in [**56**], the correspondence between selfextensional logics and logics that admit referential semantics involves a dual equivalence of categories: one being the class of referential algebras associated with the referential semantics, and the other being the class of reduced g-models that provides the algebraic semantics. Thus they provide a duality between algebraic and referential semantics, and hence we take it as an starting point of our work. Recall that we aim to identify the class of logics $\mathcal{S}$ for which a Stone/Priestley duality can be defined for $\mathbb{A}\mathrm{lg}\mathcal{S}$. We are not interested in the algebraic semantics given by the reduced g-models, rather in the algebraic semantics given by purely algebraic structures. In [**56**] it is shown that when we deal with selfextensional logics that are congruential, the correspondence can be formulated as a dual equivalence of categories, one being the class of $\mathcal{S}$-algebras (and homomorphisms), and the other being what they call *perfect* $\mathcal{S}$-referential algebras (and suitable morphisms). What they present is a general framework in which our work is placed as well. In particular, the representation theorem for congruential logics they deal with, is the same that we study in the next section. We review in detail their work in § 5.4, where we compare their results with ours.

4.3. Closure Bases and congruential logics

We define now congruential logics and we present a representation theorem for $\mathcal{S}$-algebras, when $\mathcal{S}$ is congruential. From now on, fix a language $\mathscr{L}$, let $\mathcal{S}$ be a logic in the language $\mathscr{L}$ and let $\mathbf{A}$ be an algebra of the same type. If not otherwise

[1] The *Leibniz hierarchy* is another classification scheme of logics that can be presented in at least four ways: according to either syntactic characterizations of logics, or definability characterizations, or lattice-theoretical characterizations in terms of the properties of the Leibniz congruence, or model-theoretic characterizations of the classes of reduced $\mathcal{S}$-models and reduced $\mathcal{S}$-algebras. This hierarchy has been enriched recently by the contributions of Raftery [**66**] and Cintula and Noguera [**20**], and some well-known classes of the hierarchy are, for instance, the class of implicative logics introduced by Rasiowa [**67**], or the class of protoalgebraic logics, that was first defined by Blok and Pigozzi in [**7**] and independently by Czelakowski in [**22**], and that was studied in depth by Czelakowski [**23**].

stated, all logics and algebras considered in what follows are assumed to have the type of $\mathscr{L}$.

Recall that we denote by $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ the collection of $\mathcal{S}$-filters of $\mathbf{A}$, that is a closure system. Let us denote by $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ the closure operator associated with $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. Thus for any subset $B \subseteq A$, $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$ denotes the least $\mathcal{S}$-filter of $\mathbf{A}$ that contains B.

This closure operator defines the specialization quasiorder $\leq^{\mathbf{A}}_{\mathcal{S}}$ on A associated with $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$, such that for all $a, b \in A$:

$$a \leq^{\mathbf{A}}_{\mathcal{S}} b \quad \text{iff} \quad \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a).$$

We denote by $\equiv^{\mathbf{A}}_{\mathcal{S}}$ the equivalence relation associated with $\leq^{\mathbf{A}}_{\mathcal{S}}$, i. e. $\equiv^{\mathbf{A}}_{\mathcal{S}} := \leq^{\mathbf{A}}_{\mathcal{S}} \cap \geq^{\mathbf{A}}_{\mathcal{S}}$. Notice that $\equiv^{\mathbf{A}}_{\mathcal{S}}$ is precisely $\mathbf{\Lambda}_{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}}$, the Frege relation of $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$. This relation provides a definition of the canonical class of algebras associated with $\mathcal{S}$ alternative to that presented in page 18.

DEFINITION 4.3.1. An algebra $\mathbf{A}$ is an $\mathcal{S}$-*algebra* when for every congruence θ of $\mathbf{A}$, if $\theta \subseteq \equiv^{\mathbf{A}}_{\mathcal{S}}$, then $\theta = \triangle_{\mathbf{A}}$.

In particular, from the definition of Lindenbaum-Tarski algebra (see page 18), and since the Frege relation of $\mathrm{C}_{\mathcal{S}}$ and the Tarski congruence of $\mathcal{S}$ coincide for any selfextensional logic we obtain that for any selfextensional logic $\mathcal{S}$, the Lindenbaum-Tarski algebra $\mathbf{Fm}^* = \mathbf{Fm}/\mathbf{\Lambda}_{\mathrm{C}_{\mathcal{S}}} = \mathbf{Fm}/\equiv^{\mathbf{Fm}}_{\mathcal{S}}$ is an $\mathcal{S}$-algebra. The relation $\equiv^{\mathbf{A}}_{\mathcal{S}}$ is also used to define the class of congruential logics.

DEFINITION 4.3.2. (Prop. 2.42 in [**35**]) A logic $\mathcal{S}$ is called *congruential*,[2] when for every algebra $\mathbf{A}$, $\equiv^{\mathbf{A}}_{\mathcal{S}}$ is a congruence of $\mathbf{A}$.

Notice that it follows from the definition that any congruential logic is selfextensional. When the Frege relation of a closure operator C on an algebra $\mathbf{A}$ is a congruence, we say that the structure $\langle \mathbf{A}, \mathrm{C}\rangle$ has the *congruence property* (Def. 2.39 in [**35**]). Therefore, a logic $\mathcal{S}$ is congruential, provided for any algebra $\mathbf{A}$, $\langle \mathbf{A}, \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}\rangle$ has the congruence property. Many of the well-known logics, including classical and intuitionistic propositional logics, are congruential. A sufficient condition for a selfextensional logic for being congruential is satisfying (uDDT) or (PC). Next theorem, stated in Theorem 2.2 in [**41**] without a proof, gives an alternative definition of congruentiality.

THEOREM 4.3.3. *A logic $\mathcal{S}$ is congruential if and only if for every algebra $\mathbf{A}$ of the same type:*

$$\mathbf{A} \in \mathbb{A}\mathrm{lg}\mathcal{S} \quad \textit{iff} \quad \langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle \textit{ is a poset.}$$

PROOF. Let $\mathcal{S}$ be a congruential logic. Clearly $\leq^{\mathbf{A}}_{\mathcal{S}}$ being an order implies that $\mathbf{A} \in \mathbb{A}\mathrm{lg}\mathcal{S}$. For the converse, let $\mathbf{A} \in \mathbb{A}\mathrm{lg}\mathcal{S}$. By assumption $\equiv^{\mathbf{A}}_{\mathcal{S}}$ is a congruence of $\mathbf{A}$, so by definition of $\mathcal{S}$-algebra, $\equiv^{\mathbf{A}}_{\mathcal{S}} = \triangle_{\mathbf{A}}$, and therefore, $\leq^{\mathbf{A}}_{\mathcal{S}}$ is an order.

Let now $\mathcal{S}$ be a logic such that for every algebra $\mathbf{A}$, $\mathbf{A} \in \mathbb{A}\mathrm{lg}\mathcal{S}$ if and only if $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle$ is a poset. Let $\mathbf{B}$ be any algebra of the same type of $\mathcal{S}$. We show that $\equiv^{\mathbf{B}}_{\mathcal{S}} = \triangle_{\mathbf{B}}$. By propositions 2.10 and 2.21 in [**35**], $\mathbf{B}^* := \mathbf{B}/\widetilde{\mathbf{\Omega}}_{\mathbf{B}}(\mathrm{Fi}_{\mathcal{S}}(\mathbf{B})) \in \mathbb{A}\mathrm{lg}\mathcal{S}$,

[2]We follow here the terminology used in [**41**]. Congruential logics were previously called *strongly selfextensional* [**35**] and *fully selfextensional* [**56**].

where $\widetilde{\mathbf{\Omega}}_{\mathbf{B}}(\mathrm{Fi}_{\mathcal{S}}(\mathbf{B}))$ is the Tarski congruence of $\langle \mathbf{B}, \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})\rangle$.[3] We also get that the projection map is a bilogical morphism from $\langle \mathbf{B}, \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})\rangle$ to $\langle \mathbf{B}^*, \mathrm{Fi}_{\mathcal{S}}(\mathbf{B}^*)\rangle$. by assumption $\equiv_{\mathcal{S}}^{\mathbf{B}^*} = \triangle_{\mathbf{B}^*}$, i. e. $\langle \mathbf{B}^*, \mathrm{Fi}_{\mathcal{S}}(\mathbf{B}^*)\rangle$ has the congruence property. But since this property is preserved by bilogical morphisms (Proposition 2.40 in [**35**]), we conclude that $\langle \mathbf{B}, \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})\rangle$ has the congruence property, i. e. we get that $\equiv_{\mathcal{S}}^{\mathbf{B}} = \triangle_{\mathbf{B}}$ is a congruence, as required. □

From the previous theorem we infer that for any congruential logic $\mathcal{S}$,

$$\mathbb{A}\mathrm{lg}\mathcal{S} = \{\mathbf{A} : \equiv_{\mathcal{S}}^{\mathbf{A}} = \triangle_{\mathbf{A}}\}.$$

REMARK 4.3.4. Notice that when $\mathcal{S}$ is congruential, all $\mathcal{S}$-filters are up-sets with respect to $\leq_{\mathcal{S}}^{\mathbf{A}}$, and for all $a \in A$, $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(a) = \uparrow_{\leq_{\mathcal{S}}^{\mathbf{A}}} a$. When the context is clear, we drop the subscript of $\uparrow_{\leq_{\mathcal{S}}^{\mathbf{A}}}$ as well as the subscript of $\downarrow_{\leq_{\mathcal{S}}^{\mathbf{A}}}$.

From now on we focus on congruential logics and on closure bases for $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$, as they provide us with the representation theorems for $\mathcal{S}$-algebras we are looking for. Recall that $\mathcal{F} \subseteq \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is a *closure base* for $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$ provided any $\mathcal{S}$-filter is an intersection of elements in $\mathcal{F}$.

For any closure base $\mathcal{F}$ for $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$, we define the map $\varphi_{\mathcal{F}} : A \longrightarrow \mathcal{P}^{\uparrow}(\mathcal{F})$ such that for any $a \in A$:

$$\varphi_{\mathcal{F}}(a) = \{P \in \mathcal{F} : a \in P\}.$$

For any $a \in A$, $\varphi_{\mathcal{F}}(a)^c$ denotes the set $\{P \in \mathcal{F} : a \notin P\}$. For any $B \subseteq A$, we denote:

$$\widehat{\varphi}_{\mathcal{F}}(B) := \bigcap \{\varphi_{\mathcal{F}}(b) : b \in B\} = \{P \in \mathcal{F} : B \subseteq P\}.$$

Notice that for any $B, B' \subseteq A$, we have that $\widehat{\varphi}_{\mathcal{F}}(B) \cap \widehat{\varphi}_{\mathcal{F}}(B') = \widehat{\varphi}_{\mathcal{F}}(B \cup B')$. This notation should not be confused with $\varphi_{\mathcal{F}}[B] := \{\varphi_{\mathcal{F}}(b) : b \in B\}$.

Let us denote by $\varphi_{\mathcal{F}}[\mathbf{A}]$ the algebra whose carrier is $\varphi_{\mathcal{F}}[A]$ and such that for each n-ary connective f of the language $\mathscr{L}$, and any elements $a_1, \ldots, a_n \in A$, an operation $f^{\varphi_{\mathcal{F}}[\mathbf{A}]}$ on $\varphi_{\mathcal{F}}[A]$ is defined as follows:

$$f^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}(a_1), \ldots, \varphi_{\mathcal{F}}(a_n)) := \varphi_{\mathcal{F}}(f^{\mathbf{A}}(a_1, \ldots, a_n)).$$

These operations are well defined since the map $\varphi_{\mathcal{F}}$ is injective, and injectivity of $\varphi_{\mathcal{F}}$ follows easily from $\mathcal{S}$ being congruential and $\mathcal{F}$ being a closure base for $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$: let $a, b \in A$ be such that $a \neq b$. Since $\mathcal{S}$ is congruential and $\mathbf{A}$ is an $\mathcal{S}$-algebra, we have $\triangle_{\mathbf{A}} = \equiv_{\mathcal{S}}^{\mathbf{A}}$, so from $a \neq b$ we can assume, without loss of generality, that $a \not\leq_{\mathcal{S}}^{\mathbf{A}} b$. Then $b \notin \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(a)$, and therefore there is $P \in \mathcal{F}$ such that $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(a) \subseteq P$ and $b \notin P$. We conclude that $P \in \varphi_{\mathcal{F}}(a) \setminus \varphi_{\mathcal{F}}(b)$, so $\varphi_{\mathcal{F}}(a) \neq \varphi_{\mathcal{F}}(b)$, as required. Thus $\varphi_{\mathcal{F}}[\mathbf{A}]$ is well defined and moreover $\varphi_{\mathcal{F}} \in \mathrm{Hom}(\mathbf{A}, \varphi_{\mathcal{F}}[\mathbf{A}])$.

THEOREM 4.3.5. *Let $\mathcal{S}$ be a congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be a closure base for $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$. The map $\varphi_{\mathcal{F}} : A \longrightarrow \mathcal{P}^{\uparrow}(\mathcal{F})$ is an isomorphism between*

[3] The Tarski congruence has been introduced through the study of the semantics of generalized matrices. It is defined for any pair consisting of an algebra and a closure system, as the greatest congruence on the algebra compatible with all the subsets of the closure system. The Tarski congruence can be defined also in terms of the Leibniz congruence, and it can be used to give an alternative definition of $\mathbb{A}\mathrm{lg}\mathcal{S}$, for any logic $\mathcal{S}$. For a more precise definition see Definition 1.1 in [**35**].

the algebras $\mathbf{A}$ *and* $\varphi_{\mathcal{F}}[\mathbf{A}]$*, and an order embedding between the posets* $\langle A, \leq_{\mathcal{S}}^{\mathbf{A}}\rangle$ *and* $\langle \varphi_{\mathcal{F}}[A], \subseteq\rangle$.

PROOF. By definition $\varphi_{\mathcal{F}}$ is a homomorphism of $\mathbf{A}$ onto $\varphi_{\mathcal{F}}[\mathbf{A}]$. Notice that $a \leq_{\mathcal{S}}^{\mathbf{A}} b$ implies $\varphi_{\mathcal{F}}(a) \subseteq \varphi_{\mathcal{F}}(b)$, and from this and the previous argument about injectivity, we get the required order embedding. $\square$

Previous theorem is the representation theorem we were looking for. Moreover, from it we obtain that the algebra $\varphi_{\mathcal{F}}[\mathbf{A}]$ is an $\mathcal{S}$-algebra. Therefore, we may consider the closure operator $\mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}$ associated with the closure system $\mathrm{Fi}_{\mathcal{S}}(\varphi_{\mathcal{F}}[\mathbf{A}])$.

LEMMA 4.3.6. *Let* $\mathcal{S}$ *be a congruential logic, let* $\mathbf{A}$ *be an* $\mathcal{S}$*-algebra and let* $\mathcal{F}$ *be a closure base for* $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$*. Then* $\{\varphi_{\mathcal{F}}[P] : P \in \mathcal{F}\}$ *is a closure base for* $\mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}$.

PROOF. As $\varphi_{\mathcal{F}}$ is an isomorphism between $\mathcal{S}$-algebras $\mathbf{A}$ and $\varphi_{\mathcal{F}}[\mathbf{A}]$, in particular we have that $\mathrm{Fi}_{\mathcal{S}}(\varphi_{\mathcal{F}}[\mathbf{A}]) = \{\varphi_{\mathcal{F}}[F] : F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})\}$. Consider the family $\mathcal{F}' := \{\varphi_{\mathcal{F}}[P] : P \in \mathcal{F}\}$ and let $G \in \mathrm{Fi}_{\mathcal{S}}(\varphi_{\mathcal{F}}[\mathbf{A}])$ and $a \in A$ be such that $\varphi_{\mathcal{F}}(a) \notin G$. Since $\varphi_{\mathcal{F}}$ is an isomorphism, there is $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ such that $\varphi_{\mathcal{F}}[F] = G$, so we have $a \notin F$. Then by $\mathcal{F}$ being closure base, there is $P \in \mathcal{F}$ such that $F \subseteq P$ and $a \notin P$. This implies $G = \varphi_{\mathcal{F}}[F] \subseteq \varphi_{\mathcal{F}}[P] \in \mathcal{F}'$ and $\varphi_{\mathcal{F}}(a) \notin \varphi_{\mathcal{F}}[P]$, as required. $\square$

COROLLARY 4.3.7. *Let* $\mathcal{S}$ *be a congruential logic, let* $\mathbf{A}$ *be an* $\mathcal{S}$*-algebra and let* $\mathcal{F}$ *be a closure base for* $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$*. For any* $a \in A$ *and any* $B \subseteq A$*:*

$$a \in \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B) \quad \textit{iff} \quad \widehat{\varphi}_{\mathcal{F}}(B) \subseteq \varphi_{\mathcal{F}}(a) \quad \textit{iff} \quad \varphi_{\mathcal{F}}(a) \in \mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B]).$$

PROOF. Assume first that $a \in \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)$ and let $P \in \widehat{\varphi}_{\mathcal{F}}(B)$, i.e. $B \subseteq P$. Then we have $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B) \subseteq \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(P) = P$, and so $a \in P$, i.e. $P \in \varphi_{\mathcal{F}}(a)$. For the converse, let $a \notin \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)$, then by $\mathcal{F}$ being a closure base for $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$, there is $P \in \mathcal{F}$ such that $a \notin P$ and $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B) \subseteq P$, i.e. $P \in \widehat{\varphi}_{\mathcal{F}}(B) \setminus \varphi_{\mathcal{F}}(a)$.

We show now that $a \in \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)$ if and only if $\varphi_{\mathcal{F}}(a) \in \mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B])$. Assume first that $\varphi_{\mathcal{F}}(a) \in \mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B])$. Notice that since $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)$ is an $\mathcal{S}$-filter of $\mathbf{A}$, by $\varphi_{\mathcal{F}}$ being an isomorphism, the set $\varphi_{\mathcal{F}}[\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)]$ is an $\mathcal{S}$-filter of $\varphi_{\mathcal{F}}[\mathbf{A}]$. Then from $B \subseteq \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)$ we get $\mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B]) \subseteq \mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)]) = \varphi_{\mathcal{F}}[\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)]$. Therefore, from the assumption it follows $\varphi_{\mathcal{F}}(a) \in \varphi_{\mathcal{F}}[\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)]$, and so $a \in \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)$. Assume now that $\varphi_{\mathcal{F}}(a) \notin \mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B])$. Then as $\mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B])$ is an $\mathcal{S}$-filter of $\varphi_{\mathcal{F}}[\mathbf{A}]$, from lemma 4.3.6 we get that there is $P \in \mathcal{F}$ such that $\varphi_{\mathcal{F}}(a) \notin \varphi_{\mathcal{F}}[P]$ and $\mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B]) \subseteq \varphi_{\mathcal{F}}[P]$. So from $\varphi_{\mathcal{F}}[B] \subseteq \varphi_{\mathcal{F}}[P]$, we infer $B \subseteq P$, and then $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B) \subseteq \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(P) = P$. And from $a \notin P$, we conclude $a \notin \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)$. $\square$

COROLLARY 4.3.8. *Let* $\mathcal{S}$ *be a congruential logic, let* $\mathbf{A}$ *be an* $\mathcal{S}$*-algebra and let* $\mathcal{F}$ *be a closure base for* $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$*. For any* $B, D \subseteq A$*:*

$$D \subseteq \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B) \quad \textit{iff} \quad \widehat{\varphi}_{\mathcal{F}}(B) \subseteq \widehat{\varphi}_{\mathcal{F}}(D) \quad \textit{iff} \quad \bigcup\{\varphi_{\mathcal{F}}(d) : d \in D\} \subseteq \mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B]).$$

Notice that from Corollary 4.3.7 it follows that for all $B \subseteq A$:

$$\mathrm{C}_{\mathcal{S}}^{\varphi_{\mathcal{F}}[\mathbf{A}]}(\varphi_{\mathcal{F}}[B]) = \varphi_{\mathcal{F}}[\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(B)].$$

The structure $\langle \mathcal{F}, \varphi_{\mathcal{F}}[\mathbf{A}]\rangle$ is a referential algebra, that in Section 5.6.7 of [**73**] is called the *canonical referential algebra for* $\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}$ *determined by* $\mathcal{F}$.

THEOREM 4.3.9. *Let $\mathcal{S}$ be a congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be a closure base for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$. Then $\langle \mathcal{F}, \varphi_{\mathcal{F}}[\mathbf{A}]\rangle$ is a reduced $\mathcal{S}$-referential algebra and the associated order is given by the inclusion relation.*

PROOF. By definition, $\langle \mathcal{F}, \varphi_{\mathcal{F}}[\mathbf{A}]\rangle$ is a referential algebra. We show first that $\langle \mathcal{F}, \varphi_{\mathcal{F}}[\mathbf{A}]\rangle$ is reduced. Consider the quasiorder $\preceq \subseteq \mathcal{F} \times \mathcal{F}$ of the referential algebra, which is defined as follows:

$$P \preceq Q \quad \text{iff} \quad \forall a \in A\big(\text{ if } P \in \varphi_{\mathcal{F}}(a), \text{ then } Q \in \varphi_{\mathcal{F}}(a)\big).$$

Note that using the definitions of the notions involved, it follows that this quasiorder is the inclusion relation on $\mathcal{F}$. Therefore it is a partial order and the referential algebra is reduced.

Let us show that $\langle \mathcal{F}, \varphi_{\mathcal{F}}[\mathbf{A}]\rangle$ is an $\mathcal{S}$-referential algebra. Let $\Gamma \cup \{\delta\} \subseteq Fm$ be such that $\Gamma \vdash_{\mathcal{S}} \delta$, and let $h \in \mathrm{Hom}(\mathbf{Fm}, \varphi_{\mathcal{F}}[\mathbf{A}])$. Since $\varphi_{\mathcal{F}} \in \mathrm{Hom}(\mathbf{A}, \varphi_{\mathcal{F}}[\mathbf{A}])$ is an isomorphism, there is $h' \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ such that $\varphi_{\mathcal{F}} \circ h' = h$. We have to show that $\bigcap\{h(\gamma) : \gamma \in \Gamma\} \subseteq h(\delta)$, so let $P \in \mathcal{F}$ be such that $P \in \bigcap\{h(\gamma) : \gamma \in \Gamma\} = \bigcap\{\varphi_{\mathcal{F}}(h'(\gamma)) : \gamma \in \Gamma\}$. Then $h'(\gamma) \in P$ for all $\gamma \in \Gamma$. And since $P \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ and $h' \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, from the assumption and the definition of $\mathcal{S}$-filter we obtain $h'(\delta) \in P$, so $P \in \varphi_{\mathcal{F}}(h'(\delta)) = h(\delta)$, as required. □

Notice that the previous theorem, Remark 4.2.1 and Lemma 4.3.6 imply that for any $\mathcal{S}$ congruential logic, there is a back and forth correspondence between reduced $\mathcal{S}$-referential algebras and structures of the form $\langle \mathbf{A}, \mathcal{F}\rangle$, where $\mathbf{A}$ is an $\mathcal{S}$-algebra and $\mathcal{F}$ is a closure base for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$. This correspondence between objects, first addressed by Czelakowski in [**23**], was formulated as a full-fledged duality in [**56**], for the case when the collection $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ was taken as the closure base. But this is not the closure base that properly generalizes the representation theorem on which they are based the Stone/Priestley dualities that we find in the literature. Let us consider the example of intuitionistic logic, for which the canonical class of algebras associated with are Heyting algebras. Logical filters of Heyting algebras are lattice filters. But the representation theorem on which is based Stone/Priestley duality for Heyting algebras focuses on prime lattice filters and not on all lattice filters. Therefore, for our purposes, we should not work with the whole collection of $\mathcal{S}$-filters, but rather we should identify the closure bases that provide us with a direct generalization of the mentioned representation theorem in the literature. This is what we do in the next section, where we define *irreducible* and *optimal* $\mathcal{S}$-filters, using what we studied in Chapter 2.

4.4. The closure operator $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$: irreducible and optimal logical filters

From now on, let $\mathcal{S}$ be a congruential logic and let $\mathbf{A}$ be an $\mathcal{S}$-algebra. Notice that when $\mathcal{S}$ is a finitary logic, then $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ is a finitary closure operator for any $\mathcal{S}$-algebra $\mathbf{A}$. By convenience we assume that $\mathcal{S}$ has theorems. Recall that we say that a logic $\mathcal{S}$ *has theorems* when there is at least one formula $\varphi \in Fm$ such that $\emptyset \vdash_{\mathcal{S}} \varphi$. The collection of all formulas that are theorems is denoted by $\mathrm{Thm}\mathcal{S}$. It is easy to see that when $\mathcal{S}$ has theorems, then the poset $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle$ has a top element, that we denote by $1^{\mathbf{A}}$, that is the image of any theorem of the logic by any homomorphism from $\mathbf{Fm}$ to $\mathbf{A}$. Moreover, when $\mathcal{S}$ has theorems, all $\mathcal{S}$-filters of $\mathbf{A}$

are non-empty, since they contain the top element $1^{\mathbf{A}}$: for $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ an $\mathcal{S}$-filter, $\delta \in \mathrm{Thm}\mathcal{S}$ a theorem, and $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ a homomorphism, from $\emptyset \vdash_{\mathcal{S}} \delta$ it follows $h(\delta) = 1^{\mathbf{A}} \in F$. Therefore $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\emptyset)$ is non-empty.

The usual notions of order ideal and order filter might be defined for the poset $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$. An *order ideal* (cf. definition in page 26) of $\mathbf{A}$ is a non-empty up-directed down-set of $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$. Dually, an *order filter* of $\mathbf{A}$ is a non-empty down-directed up-set of $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$. We denote by $\mathrm{Id}(\mathbf{A})$ (resp. $\mathrm{Fi}(\mathbf{A})$) the collection of all order ideals (resp. order filters) of $\mathbf{A}$. Notice that all principal down-sets (resp. up-sets) are, in particular, order ideals (resp. order filters).

Concerning the closure operator $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ defined on A, making use of what was presented in § 2.1, we get several notions of filter and ideal, as well as separation lemmas and other important results.

Recall that by definition, the $\mathcal{S}$-filters of $\mathbf{A}$ are the closed sets of $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$. We say that an $\mathcal{S}$-filter is *irreducible* when it is an irreducible $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$-closed, i. e. when it is a meet-irreducible element of the lattice of $\mathcal{S}$-filters $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})$. We denote by $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ the collection of all irreducible $\mathcal{S}$-filters of $\mathbf{A}$. Notice that $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ satisfies condition (E1) in page 22 on the poset $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$, i. e. all $\mathcal{S}$-filters are up-sets with respect to $\leq^{\mathbf{A}}_{\mathcal{S}}$. Therefore, the following instance of Lemma 2.1.1 holds for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$, when the logic is finitary.

LEMMA 4.4.1. *Let $\mathcal{S}$ be a finitary congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $P \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ and $I \in \mathrm{Id}(\mathbf{A})$ be such that $P \cap I = \emptyset$. Then there is $Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $P \subseteq Q$ and $Q \cap I = \emptyset$.*

COROLLARY 4.4.2. *Let $\mathcal{S}$ be a finitary congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $P \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ and $a \in A$ be such that $a \notin P$. Then there is $Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $P \subseteq Q$ and $a \notin Q$.*

By the previous corollary, $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ is a closure base for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ provided $\mathcal{S}$ is a finitary congruential logic. Remember that for all $a \in A$, $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) = {\uparrow} a$.

(Finitary) dually closed sets of $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ are called *$\mathcal{S}$-ideals* by Gehrke, Jansana and Palmigiano in [**41**]. For the sake of completeness, we refresh now the definition. A subset $I \subseteq A$ is an $\mathcal{S}$-ideal of $\mathbf{A}$ provided for all $I' \subseteq^{\omega} I$ and all $a \in A$

$$\text{if } \bigcap_{b \in I'} \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a), \text{ then } a \in I.$$

We denote by $\mathrm{Id}_{\mathcal{S}}(\mathbf{A})$ the collection of all $\mathcal{S}$-ideals of $\mathbf{A}$. For any $a \in A$, $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) = A$ if and only if a is the bottom element of $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$. Moreover, $\emptyset \in \mathrm{Id}_{\mathcal{S}}(\mathbf{A})$ if and only if $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$ has no bottom element. This fact should be kept in mind, because it will be repeatedly used later on.

Up to this point, our definitions of $\mathcal{S}$-filter and $\mathcal{S}$-ideal, as well as the notation introduced, coincide with those of [**41**]. However, our approach differs in what follows.

Strong dually closed sets of $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ are called *strong $\mathcal{S}$-ideals* (or *s$\mathcal{S}$-ideals*). An $\mathcal{S}$-ideal $I \in \mathrm{Id}_{\mathcal{S}}(\mathbf{A})$ is strong when for all $I' \subseteq^{\omega} I$ and all $B \subseteq^{\omega} A$:

$$\text{if } \bigcap_{b \in I'} \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B), \text{ then } \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap I \neq \emptyset.$$

We denote by $\mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ the collection of all $s\mathcal{S}$-ideals of $\mathbf{A}$. This is a new notion of ideal, that comes from a generalization of the notion of *strong Frink ideals* for Hilbert algebras, first introduced by Celani and Jansana in [**17**].

LEMMA 4.4.3. *For any congruential logic $\mathcal{S}$ and any $\mathcal{S}$-algebra $\mathbf{A}$:*

$$\mathrm{Id}(\mathbf{A}) \subseteq \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}) \subseteq \mathrm{Id}_{\mathcal{S}}(\mathbf{A}).$$

PROOF. The second inclusion is immediate, so we just have to check the first inclusion. Let us show first that any order ideal is an $\mathcal{S}$-ideal. Let $I \in \mathrm{Id}(\mathbf{A})$, $I' \subseteq^{\omega} I$ and $b \in A$ be such that $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) : a \in I'\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b)$. If $I' = \emptyset$, then $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b) = A$, so b is the bottom element and then $b \in I$ because I is a non-empty down-set. If $I' \neq \emptyset$, then there is $c \in I$ such that $a \leq^{\mathbf{A}}_{\mathcal{S}} c$ for all $a \in I'$, since I is up-directed. Therefore $c \in \bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) : a \in I'\}$ and consequently $c \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b)$, i. e. $b \leq^{\mathbf{A}}_{\mathcal{S}} c$. Since I is a down-set, we get $b \in I$. We conclude that any order ideal is an $\mathcal{S}$-ideal. Let us show now that I is strong. Let $I' \subseteq^{\omega} I$ and $B \subseteq^{\omega} A$ be such that $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) : a \in I'\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. If $I' = \emptyset$, then $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) = A$ and certainly $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap I \neq \emptyset$ since I is non-empty. If $I' \neq \emptyset$, using that I is updirected, we get $c \in I$ such that $c \in \bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) : a \in I'\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. Therefore $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap I \neq \emptyset$. We conclude that any order ideal is an $s\mathcal{S}$-ideal. □

Optimal closed subsets of $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ are called *optimal* $\mathcal{S}$-filters. Hence an $\mathcal{S}$-filter $P \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is an optimal $\mathcal{S}$-filter when there is an strong $\mathcal{S}$-ideal $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ such that P is I-maximal and I is P-maximal, i. e. P is a maximal element of the collection $\{P' \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}) : P' \cap I = \emptyset\}$ and I is a maximal element of the collection $\{I' \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}) : P \cap I' = \emptyset\}$. We denote by $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ the collection of optimal $\mathcal{S}$-filters of $\mathbf{A}$.

REMARK 4.4.4. Notice that from the definition it follows that $\emptyset$ is an $s\mathcal{S}$-ideal if and only if A is an optimal $\mathcal{S}$-filter.

This is a new notion of filter, that comes from a generalization of that of *optimal implicative filter* for Hilbert algebras that was first introduced in [**17**]. The following instance of Lemma 2.1.7 holds when the logic is finitary.

LEMMA 4.4.5. *Let $\mathcal{S}$ be a finitary congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $P \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ and $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ be such that $P \cap I = \emptyset$. Then there is $Q \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ such that $P \subseteq Q$ and $Q \cap I = \emptyset$.*

COROLLARY 4.4.6. *Let $\mathcal{S}$ be a finitary congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $P \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ and $a \in A$ be such that $a \notin P$. Then there is $Q \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ such that $P \subseteq Q$ and $a \notin Q$.*

By the previous corollary, $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ is a closure base for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ provided $\mathcal{S}$ is a finitary congruential logic. Up to this point, all the results in this section hold in general for any finitary congruential logic with theorems, and they are just instances of what was treated in Chapter 2. If we assume further properties of the logic, we get more results. In particular, we are interested in the following class of logics:

DEFINITION 4.4.7. We say that a logic $\mathcal{S}$ is *filter distributive* when for all algebras $\mathbf{A}$ (of the type of $\mathcal{S}$), $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})$ is a distributive lattice.

The class of filter distributive logics, first considered by Czelakowski in [**21**] and also studied in [**22**, **63**, **73**] and indirectly in [**27**, **28**, **57**] includes a lot of well-known logics, for example, any axiomatic extension (expansion in the same language) of the intuitionistic logic, or any logic satisfying either the (DDT) or (PDI) (more on this topic in § 5.5). When $\mathcal{S}$ is a filter distributive finitary congruential logic with theorems, the following instances of lemma 2.2.2 and 2.2.1 hold.

THEOREM 4.4.8. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems and let $\mathbf{A}$ be an $\mathcal{S}$-algebra. For any $P \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ if and only if $P^c \in \mathrm{Id}(\mathbf{A})$.*

THEOREM 4.4.9. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems and let $\mathbf{A}$ be an $\mathcal{S}$-algebra. For any $P \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, $P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ if and only if $P^c \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$.*

Precisely these theorems together with the relation between the different classes of ideals, lead us to the following relation between the different classes of filters, that holds under the assumptions of finitarity and filter distributivity.

LEMMA 4.4.10. *For any filter distributive finitary congruential logic with theorems $\mathcal{S}$ and any $\mathcal{S}$-algebra $\mathbf{A}$:*

$$\mathrm{Irr}_{\mathcal{S}}(\mathbf{A}) \subseteq \mathrm{Op}_{\mathcal{S}}(\mathbf{A}) \subseteq \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}).$$

PROOF. This follows from Lemma 4.4.3 and theorems 4.4.8 and 4.4.9. □

Lemmas 4.4.1 and 4.4.5 are crucial in Spectral-like and Priestley-style dualities respectively, as it is shown in Chapter 5. Theorems 4.4.8 and 4.4.9 are crucial as well, and might be refined making use of the following notion that generalizes the concept of *prime ideal* of a lattice.

A subset $X \subseteq A$ is a called $\mathcal{S}$-*prime* when it is a proper subset ($X \neq A$) and for all non-empty $B \subseteq^{\omega} A$,

$$\text{if } \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap X \neq \emptyset, \text{ then } B \cap X \neq \emptyset.$$

In [**17**], within the setting of Hilbert algebras, the adjective associated with this condition is *prime*, and it is usually addressed to ideals. In [**41**], *prime* is used in relation to congruential logics in a slightly different way. As this can be messy, we prefer to use $\mathcal{S}$-*prime*. The following lemma points out that $\mathcal{S}$-prime is somehow a dual notion of that of $\mathcal{S}$-filter, and it is used to prove two corollaries of theorems 4.4.8 and 4.4.9.

LEMMA 4.4.11. *Let $\mathcal{S}$ be a finitary congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $X \subseteq A$. Then $X \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ if and only if X^c is $\mathcal{S}$-prime.*

PROOF. Let $X \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. We show that X^c is $\mathcal{S}$-prime. As $\mathcal{S}$ has theorems, X is non-empty, and so X^c is proper. Let $B \subseteq^{\omega} A$ be non-empty and such that $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap X^c \neq \emptyset$. Suppose, towards a contradiction, that $B \cap X^c = \emptyset$. Then $B \subseteq X$, and therefore $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \subseteq X$, so $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap X^c = \emptyset$, a contradiction.

For the converse, let $X \subseteq A$ be such that X^c is $\mathcal{S}$-prime. If $X^c = \emptyset$, then $X = A$, that is trivially an $\mathcal{S}$-filter. Suppose $X^c \neq \emptyset$. We show that X is an $\mathcal{S}$-filter, by showing that $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X) = X$. Clearly $X \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X)$, so in order to show the other inclusion, let $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X)$. By finitarity, there is $X' \subseteq^{\omega} X$ such that $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X')$. We can assume without loss of generality that X' is non-empty. Suppose, towards a contradiction that $a \notin X$. Then from $a \in X^c$ and the fact that X^c is an $\mathcal{S}$-prime, we conclude $X' \cap X^c \neq \emptyset$, so $X' \nsubseteq X$, a contradiction. □

COROLLARY 4.4.12. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $P \subseteq A$. Then $P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ if and only if $P^c \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ and P^c is an $\mathcal{S}$-prime.*

COROLLARY 4.4.13. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $P \subseteq A$. Then $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ if and only if $P^c \in \mathrm{Id}(\mathbf{A})$ is and P^c is an $\mathcal{S}$-prime.*

Before concluding this section, we introduce one more concept that is used later on. We consider finite families of elements that *behave* like a bottom element in the following sense.

DEFINITION 4.4.14. Let $\mathcal{S}$ be a congruential logic and $\mathbf{A}$ and $\mathcal{S}$-algebra. We say that a non-empty finite set $B \subseteq^{\omega} A$ of incomparable elements with respect to $\leq^{\mathbf{A}}_{\mathcal{S}}$ is a *bottom-family* of $\mathbf{A}$ if $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) = A$.

Notice that $\emptyset \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ if and only if $\mathbf{A}$ has no bottom-family. This fact is used repeatedly later on, especially in § 4.5.

Summarizing, for any finitary congruential logic $\mathcal{S}$ (with theorems) we have a version of Birkhoff's Prime Filter Lemma for both irreducible and optimal $\mathcal{S}$-filters of $\mathbf{A}$, and so $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ and $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ are both closure bases for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$, for any $\mathcal{S}$-algebra $\mathbf{A}$. Furthermore, when the logic is filter distributive, both collections are complements of order ideals of $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$ and strong $\mathcal{S}$-ideals of $\mathbf{A}$ respectively. We use these facts to formulate the answer to the question we suggested in § 4.1. We carried out the first steps towards such answer. We have stated two representation theorems of $\mathcal{S}$-algebras that yield $\mathcal{S}$-referential algebras with interesting properties.

Now we change slightly the point of view, and we focus on the poset $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$. We already know that for any closure base $\mathcal{F}$, the poset $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$ isomorphically embeds in the poset $\langle \varphi_{\mathcal{F}}[\mathbf{A}], \subseteq \rangle$ (Theorem 4.3.5). Clearly it also embeds in the Boolean algebra given by $\mathcal{P}(\mathcal{F})$ and in the distributive lattice given by $\mathcal{P}^{\uparrow}(\mathcal{F})$. We pursue, however, to embed such poset in a smaller distributive semilattice with some nice properties. We do not intend this semilattice to run properly with all additional operations of $\mathbf{A}$, but to get a good correspondence between logical filters and logical ideals of $\mathbf{A}$ and order filters and ideals of the semilattice. This construction is important for the Priestley-style duality, as the dual space of $\mathbf{A}$ is built from the dual Priestley space of such distributive semilattice.

4.5. The $\mathcal{S}$-semilattice of A

We study now the semilattice of finitely generated $\mathcal{S}$-filters of $\mathbf{A}$. Some of the results in this section are new, and others were first proven by Gehrke, Jansana and Palmigiano in [**41**] (this will be remarked when appropriate).

This structure is called in [**41**] the $\mathcal{S}$-semilattice of $\mathbf{A}$, name that we adopt here. Different approaches to this object can be followed. In [**41**], Gehrke et al. choose to work with equivalence classes of generators of filters. In [**17**] Celani and

Jansana follow a different approach, using the concept of separating family.[4] For our purposes, we prefer to take the same approach as in [**17**], as then it is easier to see how some of our results are generalizations of results there.

Definition 4.5.1. A family $\mathcal{F} \subseteq \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ of optimal $\mathcal{S}$-filters of $\mathbf{A}$ is an *optimal $\mathcal{S}$-base* if for every $\mathcal{S}$-filter $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ and every $a \notin F$, there is $P \in \mathcal{F}$ such that $F \subseteq P$ and $a \notin P$.

Notice that for any $\mathcal{S}$-algebra $\mathbf{A}$, an optimal $\mathcal{S}$-base is nothing but a closure base for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ consisting in optimal $\mathcal{S}$-filters of $\mathbf{A}$. By Lemma 4.4.5, for any finitary congruential logic $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ is itself an optimal $\mathcal{S}$-base, and by Lemma 4.4.1, $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ is also an optimal $\mathcal{S}$-base. From now on, let $\mathcal{F}$ be an optimal $\mathcal{S}$-base.

From Theorem 4.3.5 and Lemma 4.3.6 we get that $\{\varphi_{\mathcal{F}}[P] : P \in \mathcal{F}\}$ is an optimal $\mathcal{S}$-base for $\varphi_{\mathcal{F}}[\mathbf{A}]$. Let us denote by $\mathrm{M}_{\mathcal{F}}(A)$ the closure of $\varphi_{\mathcal{F}}[A]$ under non-empty finite intersections. Notice that $\mathcal{F} \in \mathrm{M}_{\mathcal{F}}(A)$, since $\varphi_{\mathcal{F}}(1^{\mathbf{A}}) = \mathcal{F}$.

Definition 4.5.2. For any congruential logic $\mathcal{S}$ and any $\mathcal{S}$-algebra $\mathbf{A}$, the algebra $\mathrm{M}_{\mathcal{F}}(\mathbf{A}) := \langle \mathrm{M}_{\mathcal{F}}(A), \cap, \mathcal{F}\rangle$ is called the *$\mathcal{S}$-semilattice of* $\mathbf{A}$.

From Corollary 4.3.8 it follows that for closure bases $\mathcal{F}$ and $\mathcal{F}'$ for $\mathbf{A}$, $\mathrm{M}_{\mathcal{F}}(\mathbf{A})$ and $\mathrm{M}_{\mathcal{F}'}(\mathbf{A})$ are isomorphic semilattices. By convenience, we dispense with the subscript $\mathcal{F}$ of $\mathrm{M}_{\mathcal{F}}(\mathbf{A})$, $\varphi_{\mathcal{F}}$ and $\widehat{\varphi}_{\mathcal{F}}$ and we use instead $\mathrm{M}(\mathbf{A})$, φ and $\widehat{\varphi}$.

By definition, $\mathrm{M}(\mathbf{A})$ is a meet semilattice with top element, and clearly, we have that for any $U \in \mathcal{P}^{\uparrow}(\mathcal{F})$:

(E2) $\qquad U \in \mathrm{M}(\mathbf{A}) \quad \text{iff} \quad U = \widehat{\varphi}(B)$ for some non-empty $B \subseteq^{\omega} A$.

Hereinafter we will repeatedly use this property, so it is convenient to keep it in mind. We should be careful when dealing with the bottom element, so let us state the following technical lemma concerning the bottom element and bottom-families of $\mathbf{A}$:

Lemma 4.5.3. *Let $\mathcal{S}$ be a congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base:*

(1) *If $\mathbf{A}$ has a bottom element $0^{\mathbf{A}}$, then $\mathrm{M}(\mathbf{A})$ has a bottom element $0^{\mathrm{M}(\mathbf{A})} = \varphi(0^{\mathbf{A}}) = \emptyset$. So if $\emptyset \notin \mathrm{Id}_{\mathcal{S}}(\mathbf{A})$, then $\emptyset \notin \mathrm{Id}_F(\mathrm{M}(\mathbf{A}))$.*
(2) *If $\mathbf{A}$ has a bottom-family B, then $\mathrm{M}(\mathbf{A})$ has a bottom element $0^{\mathrm{M}(\mathbf{A})} = \widehat{\varphi}(B) = \emptyset$. So if $\emptyset \notin \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$, then $\emptyset \notin \mathrm{Id}_F(\mathrm{M}(\mathbf{A}))$.*
(3) *$\mathbf{A}$ has a bottom-family if and only if $\emptyset \in \mathrm{M}(\mathbf{A})$.*

Proof. (1) If $\mathbf{A}$ has a bottom element $0^{\mathbf{A}}$, then $\emptyset \notin \mathrm{Id}_{\mathcal{S}}(\mathbf{A})$. Therefore $A \notin \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ (Remark 4.4.4), so $\varphi(0^{\mathbf{A}}) = \emptyset \in \mathrm{M}(\mathbf{A})$, which is clearly the bottom element of $\mathrm{M}(\mathbf{A})$.

(2) If $\mathbf{A}$ has a bottom-family B, then $\emptyset \notin \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$. Therefore $A \notin \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ (by Remark 4.4.4 again), and there is no optimal $\mathcal{S}$-filter containing B. So $\widehat{\varphi}(B) = \emptyset \in \mathrm{M}(\mathbf{A})$, which is the bottom element of $\mathrm{M}(\mathbf{A})$.

(3) We show that if $\emptyset \in \mathrm{M}(\mathbf{A})$, then $\mathbf{A}$ has a bottom-family. Assume that $\emptyset \in \mathrm{M}(\mathbf{A})$, then by (E2), there is a non-empty $B \subseteq^{\omega} A$ such that $\emptyset = \widehat{\varphi}(B)$. We

[4] Notice that they work with a particular $\mathcal{S}$, namely the implicative fragment of intuitionistic logic.

can assume, without loss of generality, that B is a family of incomparable elements. Moreover, $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) = A$, since if there is $a \notin \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$, then using that $\mathcal{F}$ is an optimal $\mathcal{S}$-base, there is $P \in \mathcal{F}$ such that $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \subseteq P$ and $a \notin P$, so $P \in \widehat{\varphi}(B) \neq \emptyset$. Therefore B is a bottom-family of $\mathbf{A}$. The converse follows from item (2). □

Let us consider first $\mathcal{S}$-filters of $\mathbf{A}$ and meet filters of $\mathrm{M}(\mathbf{A})$. Recall that for any $\mathcal{U} \subseteq \mathrm{M}(A)$, we denote by $[\![\mathcal{U}\rangle\!\rangle$ the meet filter of $\mathrm{M}(\mathbf{A})$ generated by $\mathcal{U}$ (see definition in page 26). In particular, for any non-empty $B \subseteq^{\omega} A$, $[\![\widehat{\varphi}(B)\rangle\!\rangle$ denotes the meet filter generated by $\widehat{\varphi}(B)$. As $\widehat{\varphi}(B)$ is an element of $\mathrm{M}(\mathbf{A})$, then $[\![\widehat{\varphi}(B)\rangle\!\rangle$ is the principal up-set $\uparrow_{\mathrm{M}(\mathbf{A})} \widehat{\varphi}(B)$.

LEMMA 4.5.4. *Let $\mathcal{S}$ be a congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. For any non-empty $B, B_0, \dots, B_n \subseteq^{\omega} A$:*

$$\bigcap_{i\leq n} \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \quad \textit{iff} \quad \bigcap_{i\leq n} [\![\widehat{\varphi}(B_i)\rangle\!\rangle \subseteq [\![\widehat{\varphi}(B)\rangle\!\rangle.$$

PROOF. Assume first that $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i) : i \leq n\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$ and let $D \subseteq^{\omega} A$ be such that $\widehat{\varphi}(D) \in \bigcap\{[\![\widehat{\varphi}(B_i)\rangle\!\rangle : i \leq n\}$, i.e. $\widehat{\varphi}(B_i) \subseteq \widehat{\varphi}(D)$ for all $i \leq n$. Then by Corollary 4.3.8 for all $i \leq n$, $D \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i)$. Thus by hypothesis $D \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. and by Corollary 4.3.8 again $\widehat{\varphi}(B) \subseteq \widehat{\varphi}(D)$, hence $\widehat{\varphi}(D) \in [\![\widehat{\varphi}(B)\rangle\!\rangle$.

For the converse, we assume that $\bigcap\{[\![\widehat{\varphi}(B_i)\rangle\!\rangle : i \leq n\} \subseteq [\![\widehat{\varphi}(B)\rangle\!\rangle$. So let $a \in \bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i) : i \leq n\}$. Then for each $i \leq n$, $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i)$, and so by Corollary 4.3.7 $\widehat{\varphi}(B_i) \subseteq \varphi(a)$. This implies that $\varphi(a) \in \bigcap\{[\![\widehat{\varphi}(B_i)\rangle\!\rangle : i \leq n\}$, and so by hypothesis $\varphi(a) \in [\![\widehat{\varphi}(B)\rangle\!\rangle$, i.e. $\widehat{\varphi}(B) \subseteq \varphi(a)$. Then by Corollary 4.3.7 again we get $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. □

From the previous lemma we already get the idea of what happens here. We see that $\mathcal{S}$-filters of $\mathbf{A}$ and meet filters of $\mathrm{M}(\mathbf{A})$ are closely related. This relation becomes clearer when $\mathcal{S}$ is finitary, as it is shown in the following proposition, that was first proven in Lemmas 4.5 and 4.8 in [**41**].

PROPOSITION 4.5.5. *Let $\mathcal{S}$ be a finitary congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base:*

(1) *If F is an $\mathcal{S}$-filter of $\mathbf{A}$, then*
  (a) *$[\![\varphi[F]\rangle\!\rangle$ is a meet filter of $\mathrm{M}(\mathbf{A})$, and*
  (b) *$\varphi^{-1}\big[[\![\varphi[F]\rangle\!\rangle\big] = F$.*

(2) *If F is a meet filter of $\mathrm{M}(\mathbf{A})$, then*
  (a) *$\varphi^{-1}[F]$ is an $\mathcal{S}$-filter of $\mathbf{A}$, and*
  (b) *$[\![F \cap \varphi[A]\rangle\!\rangle = F$.*

PROOF. (1) For $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, by definition $[\![\varphi[F]\rangle\!\rangle$ is a filter of $\mathrm{M}(\mathbf{A})$, and clearly $F \subseteq \varphi^{-1}\big[[\![\varphi[F]\rangle\!\rangle\big]$. Let us show the other inclusion. Let $a \in \varphi^{-1}\big[[\![\varphi[F]\rangle\!\rangle\big]$, i.e. $\varphi(a) \in [\![\varphi[F]\rangle\!\rangle$. By definition of meet filter generated, there is $B \subseteq^{\omega} F$ such that $\widehat{\varphi}(B) \subseteq \varphi(a)$. Then by Corollary 4.3.7, $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \subseteq F$.

(2) Let now $F \in \mathrm{Fi}_{\wedge}(\mathrm{M}(\mathbf{A}))$ and let $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\varphi^{-1}[F])$. We show first that $a \in \varphi^{-1}[F]$. By finitarity, there is $B \subseteq^{\omega} \varphi^{-1}[F]$ such that $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. Then by Corollary 4.3.7, $\widehat{\varphi}(B) \subseteq \varphi(a)$. Moreover by F being a filter, since meet is given in $\mathrm{M}(\mathbf{A})$ by intersection, from $\varphi[B] \subseteq F$ we get $\widehat{\varphi}(B) \in F$, and then as F is a up-set, $\varphi(a) \in F$. Hence $a \in \varphi^{-1}[F]$, as required.

Notice that $[\![\varphi[\varphi^{-1}[F]]\rangle\!\rangle = [\![F \cap \varphi[A]\rangle\!\rangle$. Clearly $[\![F \cap \varphi[A]\rangle\!\rangle \subseteq F$. For the other inclusion we use (E2), so let a non-empty $B \subseteq^\omega A$ be such that $\widehat{\varphi}(B) \in F$. Then for all $b \in B$, $\varphi(b) \in F \cap \varphi[A]$, thus $\bigcap\{\varphi(b) : b \in B\} = \widehat{\varphi}(B) \in [\![F \cap \varphi[A]\rangle\!\rangle$. □

Previous proposition shows that the maps $[\![\varphi[\ ]\rangle\!\rangle$ and φ^{-1} give us, for any finitary congruential logic, an order isomorphism between $\mathcal{S}$-filters of $\mathbf{A}$ and meet filters of $\mathrm{M}(\mathbf{A})$:

(E3) $$\langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \subseteq\rangle \cong \langle \mathrm{Fi}_{\wedge}(\mathrm{M}(\mathbf{A})), \subseteq\rangle$$

Let us move now to consider $\mathcal{S}$-ideals of $\mathbf{A}$ and F-ideals of $\mathrm{M}(\mathbf{A})$. Recall that for any $\mathcal{U} \subseteq \mathrm{M}(A)$, we denote by $\langle\!\langle \mathcal{U}]\!]$ the Frink ideal of $\mathrm{M}(\mathbf{A})$ generated by $\mathcal{U}$ (see definition in page 28). In particular, for any $B \subseteq^\omega A$, $\langle\!\langle \widehat{\varphi}(B)]\!]$ denotes the Frink ideal generated by $\widehat{\varphi}(B)$. As $\widehat{\varphi}(B)$ is an element of $\mathrm{M}(\mathbf{A})$, $\langle\!\langle \widehat{\varphi}(B)]\!]$ is the principal down-set $\downarrow_{\mathrm{M}(\mathbf{A})} \widehat{\varphi}(B)$. Recall also that an F-ideal $I \in \mathrm{Id}_F(\mathrm{M}(\mathbf{A}))$ is $\wedge$-prime provided $\widehat{\varphi}(B) \in I$ or $\widehat{\varphi}(B') \in I$ whenever $\widehat{\varphi}(B) \cap \widehat{\varphi}(B') \in I$. Notice that for any $B \subseteq^\omega A$, $\widehat{\varphi}(B)$ is itself a meet of elements of $\mathrm{M}(\mathbf{A})$, so if I is a $\wedge$-prime F-ideal such that $\widehat{\varphi}(B) \in I$, then there is $b \in B$ such that $\varphi(b) \in I$.

Next proposition is new, and it shows that when $\mathcal{S}$ is finitary, there is also a close relation between $s\mathcal{S}$-ideals of $\mathbf{A}$ and F-ideals of $\mathrm{M}(\mathbf{A})$. By convenience, throughout the next proof, we use $\downarrow\mathcal{U}$ instead of $\downarrow_{\mathrm{M}(\mathbf{A})}\mathcal{U}$, for any $\mathcal{U} \subseteq \mathrm{M}(A)$.

Proposition 4.5.6. *Let $\mathcal{S}$ be a finitary congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base:*

(1) *For any I $s\mathcal{S}$-ideal of $\mathbf{A}$, $\langle\!\langle \varphi[I]]\!] = \downarrow_{\mathrm{M}(\mathbf{A})}\varphi[I]$.*
(2) *If I is an $s\mathcal{S}$-ideal of $\mathbf{A}$, then*
  (a) *$\langle\!\langle \varphi[I]]\!]$ is an F-ideal of $\mathrm{M}(\mathbf{A})$, and*
  (b) *$\varphi^{-1}[\langle\!\langle \varphi[I]]\!]] = I$.*
(3) *If I is an $\mathcal{S}$-prime $s\mathcal{S}$-ideal of $\mathbf{A}$, then $\langle\!\langle \varphi[I]]\!]$ is a $\wedge$-prime F-ideal of $\mathrm{M}(\mathbf{A})$.*
(4) *If I is a $\wedge$-prime F-ideal of $\mathrm{M}(\mathbf{A})$, then*
  (a) *$\varphi^{-1}[I]$ is an $\mathcal{S}$-prime $s\mathcal{S}$-ideal of $\mathbf{A}$, and*
  (b) *$\langle\!\langle \varphi[\varphi^{-1}[I]]]\!] = I$.*

Proof. (1) Let $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$. If $I = \emptyset$ then there is nothing to prove, so assume $I \neq \emptyset$. As F-ideals are down-sets, then clearly $\downarrow\varphi[I] \subseteq \langle\!\langle \varphi[I]]\!]$, so it is enough to show that $\downarrow\varphi[I]$ is an F-ideal. By (E2), let $B_1, \dots, B_n, C \subseteq^\omega A$ be non-empty and such that $\widehat{\varphi}(B_i) \in \downarrow\varphi[I]$ for all $i \leq n$, and assume that $\bigcap\{[\![\widehat{\varphi}(B_i)\rangle\!\rangle : i \leq n\} \subseteq [\![\widehat{\varphi}(B)\rangle\!\rangle$. We show that $\widehat{\varphi}(B) \in \downarrow\varphi[I]$. If $n = 0$, then $[\![\widehat{\varphi}(B)\rangle\!\rangle = \mathrm{M}(A)$, and so $\widehat{\varphi}(B)$ is the bottom element of $\mathrm{M}(\mathbf{A})$, and since I is non-empty, then there is $a \in I$ and clearly $\widehat{\varphi}(B) \subseteq \varphi(a)$, so $\widehat{\varphi}(B) \in \downarrow\varphi[I]$. If $n \neq 0$, then by assumption, for each $i \leq n$ there is $a_i \in I$ such that $\widehat{\varphi}(B_i) \subseteq \varphi(a_i)$ and then clearly $\bigcap\{[\![\varphi(a_i)\rangle\!\rangle : i \leq n\} \subseteq [\![\widehat{\varphi}(B)\rangle\!\rangle$. Now using Lemma 4.5.4, $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a_i) : i \leq n\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$, and since $a_i \in I$ for each $i \leq n$ and I is an $s\mathcal{S}$-ideal, then $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap I \neq \emptyset$. Then for $c \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap I$, using Corollary 4.3.7 $\widehat{\varphi}(B) \subseteq \varphi(c) \in \varphi[I]$, and then $\widehat{\varphi}(B) \in \downarrow\varphi[I]$, as required.

(2) For $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$, by definition $\langle\!\langle \varphi[I]]\!]$ is an F-ideal of $\mathrm{M}(\mathbf{A})$, and clearly $I \subseteq \varphi^{-1}[\langle\!\langle \varphi[I]]\!]]$. Let us show the other inclusion. If $I = \emptyset$ then there is nothing to prove, so assume $I \neq \emptyset$. Let $a \in \varphi^{-1}[\langle\!\langle \varphi[I]]\!]]$, i.e. $\varphi(a) \in \langle\!\langle \varphi[I]]\!]$. By definition of F-ideal generated, there is $I' \subseteq^\omega I$ such that $\bigcap\{[\![\varphi(b)\rangle\!\rangle : b \in I'\} \subseteq [\![\varphi(a)\rangle\!\rangle$. As

$I \neq \emptyset$, we can assume, without loss of generality, that $I' \neq \emptyset$. Then by Lemma 4.5.4, $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b) : b \in I'\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a)$. And since I is an $\mathcal{S}$-ideal, we obtain $a \in I$, as required.

(3) Let $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ be $\mathcal{S}$-prime, and using (E2) let $B_1, B_2 \subseteq^\omega A$ be non-empty and such that $\widehat{\varphi}(B_1) \cap \widehat{\varphi}(B_2) \in \langle\!\langle \varphi[I] \rrbracket$. We use that $\widehat{\varphi}(B_1) \cap \widehat{\varphi}(B_2) = \widehat{\varphi}(B_1 \cup B_2)$ and $\langle\!\langle \varphi[I] \rrbracket = {\downarrow}\varphi[I]$. Then from $\widehat{\varphi}(B_1 \cup B_2) \in {\downarrow}\varphi[I]$, we obtain that there is $c \in I$ such that $\widehat{\varphi}(B_1 \cup B_2) \subseteq \varphi(c)$. Then by Corollary 4.3.7, $c \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_1 \cup B_2)$, so $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_1 \cup B_2) \cap I \neq \emptyset$. Moreover, since I is $\mathcal{S}$-prime, we get $(B_1 \cup B_2) \cap I \neq \emptyset$, so $B_1 \cap I \neq \emptyset$ or $B_2 \cap I \neq \emptyset$. This implies, by Corollary 4.3.7 again that either $\widehat{\varphi}(B_1) \in {\downarrow}\varphi[I]$ or $\widehat{\varphi}(B_2) \in {\downarrow}\varphi[I]$. Hence $\langle\!\langle \varphi[I] \rrbracket$ is $\wedge$-prime.

(4) Let now $I \in \mathrm{Id}_F(\mathrm{M}(\mathbf{A}))$ be $\wedge$-prime. First we show that $\varphi^{-1}[I]$ is an $\mathcal{S}$-ideal. Let $I' \subseteq^\omega \varphi^{-1}[I]$ and $a \in A$ be such that $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b) : b \in I'\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a)$. If $I' = \emptyset$, then a is the bottom element of $\mathbf{A}$, so by Lemma 4.5.3, $\varphi(a)$ is the bottom element of $\mathrm{M}(\mathbf{A})$, and any F-ideal of $\mathrm{M}(\mathbf{A})$ contains the bottom element, so $\varphi(a) \in I$ and then $a \in \varphi^{-1}[I]$. If $I' \neq \emptyset$, then by Lemma 4.5.4, $\bigcap\{\llbracket \varphi(b) \rangle\!\rangle : b \in I'\} \subseteq \llbracket \varphi(a) \rangle\!\rangle$, and by I being an F-ideal, we get $\varphi(a) \in I$, so $a \in \varphi^{-1}[I]$.

Now we show that the $\mathcal{S}$-ideal $\varphi^{-1}[I]$ is strong. Let $I' \subseteq^\omega \varphi^{-1}[I]$ and $B \subseteq^\omega A$ be such that $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b) : b \in I'\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. Since $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\emptyset) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(1) = \{1\}$, we can assume, without loss of generality, that $B \neq \emptyset$. We show that $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap \varphi^{-1}[I] \neq \emptyset$. If $I' = \emptyset$, then there is $B' \subseteq B$ such that B' is a bottom-family for $\mathbf{A}$, so $\widehat{\varphi}(B')$ is a bottom element of $\mathrm{M}(\mathbf{A})$, and it belongs to all its F-ideals, in particular $\widehat{\varphi}(B') \in I$. Now since I is $\wedge$-prime, there is $b \in B'$ such that $\varphi(b) \in I$, so $b \in \varphi^{-1}[I]$. As $b \in B$, we conclude $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap \varphi^{-1}[I] \neq \emptyset$. If $I' \neq \emptyset$, then by Lemma 4.5.4, we get $\bigcap\{\llbracket \varphi(b) \rangle\!\rangle : b \in I'\} \subseteq \llbracket \widehat{\varphi}(B) \rangle\!\rangle$. As I is an F-ideal and by assumption $\varphi(b) \in I$ for all $b \in I'$, we obtain $\widehat{\varphi}(B) \in I$. As before, primeness of I implies $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap \varphi^{-1}[I] \neq \emptyset$.

It remains to show that $\varphi^{-1}[I]$ is a $\mathcal{S}$-prime. As I is proper, $\varphi(1) \notin I$, so $\varphi^{-1}[I]$ is proper. Let $B \subseteq^\omega A$ be non-empty and such that $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap \varphi^{-1}[I] \neq \emptyset$, and let $c \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap \varphi^{-1}[I]$. As $c \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$, then by Corollary 4.3.7 $\widehat{\varphi}(B) \subseteq \varphi(c)$. Moreover, since $\varphi(c) \in I$, and I is a down-set, we get $\widehat{\varphi}(B) \in I$. Now, as I is $\wedge$-prime, there is $b \in B$ such that $\varphi(b) \in I$, so $B \cap \varphi^{-1}[I] \neq \emptyset$, as required.

Finally, we show that $\langle\!\langle \varphi[\varphi^{-1}[I]] \rrbracket = I$. Clearly the inclusion from left to right holds, so we just have to show the other inclusion. By (E2) let $B \subseteq^\omega A$ be non-empty and such that $\widehat{\varphi}(B) \in I$. Then, as I is $\wedge$-prime, there is $b \in B$, such that $\varphi(b) \in I$. So $\varphi(b) \in \varphi[\varphi^{-1}[I]]$ and as $\widehat{\varphi}(B) \subseteq \varphi(b)$ and F-ideals are down-sets, then $\widehat{\varphi}(B) \in \langle\!\langle \varphi[\varphi^{-1}[I]] \rrbracket$. $\square$

Previous proposition shows that the maps $\langle\!\langle \varphi[\ ] \rrbracket$ and φ^{-1} give us, for any finitary congruential logic, an order isomorphism between $\mathcal{S}$-prime $s\mathcal{S}$-ideals of $\mathbf{A}$ and $\wedge$-prime F-ideals of $\mathrm{M}(\mathbf{A})$:

$$\text{(E4)} \qquad \langle \mathcal{S}\text{-prime } \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}), \subseteq \rangle \cong \langle \text{prime } \mathrm{Id}_F(\mathrm{M}(\mathbf{A})), \subseteq \rangle$$

A different correspondence between certain class of $\mathcal{S}$-ideals of $\mathbf{A}$ and certain class of F-ideals of $\mathrm{M}(\mathbf{A})$ was studied in [**41**]. The authors introduce the following notion of ideal of $\mathrm{M}(\mathbf{A})$.

DEFINITION 4.5.7. An order ideal I of $\mathrm{M}(\mathbf{A})$ is an $\mathbf{A}$-*ideal* if for every $\widehat{\varphi}(B) \in I$ there exists $a \in A$ such that $\widehat{\varphi}(B) \subseteq \varphi(a) \in I$.

Notice that any $\wedge$-prime order ideal of $\mathrm{M}(\mathbf{A})$ is an $\mathbf{A}$-ideal. In Propositions 4.10 and 4.11 in [**41**] it is proven the following:

PROPOSITION 4.5.8. *Let $\mathcal{S}$ be a finitary congruential logic, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base:*

(1) *If I is a non-empty up-directed $\mathcal{S}$-ideal of $\mathbf{A}$, then $\langle\!\langle\varphi[I]]\!]$ is an $\mathbf{A}$-ideal of $\mathrm{M}(\mathbf{A})$.*
(2) *If I is an $\mathbf{A}$-ideal of $\mathrm{M}(\mathbf{A})$, then $\varphi^{-1}[I]$ is a non-empty up-directed $\mathcal{S}$-ideal of $\mathbf{A}$.*

Notice that up-directed $\mathcal{S}$-ideals are strong: let $I \in \mathrm{Id}_{\mathcal{S}}(\mathbf{A})$ be up-directed, $I' \subseteq^{\omega} I$ and $B \subseteq^{\omega} A$ such that $\bigcap\{\mathrm{C}_{\mathcal{S}}(b) : b \in I'\} \subseteq \mathrm{C}_{\mathcal{S}}(B)$. By I up-directed, there is $c \in I$ such that $b \leq^{\mathbf{A}}_{\mathcal{S}} c$ for all $b \in I'$. Therefore we obtain that $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(c) \subseteq \bigcap\{\mathrm{C}_{\mathcal{S}}(b) : b \in I'\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$, and then $c \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap I \neq \emptyset$, as required.

Let us denote by ${}_{\mathrm{ud}}\mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ the collection of all non-empty up-directed (strong) $\mathcal{S}$-ideals of $\mathbf{A}$. As a consequence of the previous proposition, the same maps that gave us (E4), provide us with an order isomorphism (stated in Proposition 4.14 in [**41**]) between non-empty up-directed strong $\mathcal{S}$-ideals of $\mathbf{A}$ and $\mathbf{A}$-ideals of $\mathrm{M}(\mathbf{A})$:

$$\langle {}_{\mathrm{ud}}\mathrm{Id}_{s\mathcal{S}}(\mathbf{A}), \subseteq\rangle \cong \langle\ \mathbf{A}\text{-ideal } \mathrm{Id}(\mathrm{M}(\mathbf{A})), \subseteq\rangle \tag{E5}$$

In [**41**] the authors are mainly interested in a restriction of (E5), where on the right-hand-side we have $\wedge$-prime order ideals of $\mathrm{M}(\mathbf{A})$. Using our terminology,[5] from Proposition 4.16 in [**41**] we get an order isomorphism between non-empty up-directed $\mathcal{S}$-prime $s\mathcal{S}$-ideals of $\mathbf{A}$ and $\wedge$-prime order ideals of $\mathrm{M}(\mathbf{A})$, given by the same maps as in (E4):

$$\langle \mathcal{S}\text{-prime } {}_{\mathrm{ud}}\mathrm{Id}_{s\mathcal{S}}(\mathbf{A}), \subseteq\rangle \cong \langle \text{prime } \mathrm{Id}(\mathrm{M}(\mathbf{A})), \subseteq\rangle \tag{E6}$$

Notice that (E6) is also a restriction of (E4). Moreover, this approach makes it clear that having order ideals on the right-hand-side corresponds with having non-empty up-directed subsets on the left-hand-side. In § 4.6 we analyze further consequences of these facts.

Up to this point, all results in the present section are valid in general for any finitary congruential logic (with theorems). If we assume besides, that $\mathcal{S}$ is filter distributive, then we get further results. Notice that next corollaries and lemmas use the assumption of filter-distributivity of the logic indirectly, when appealing to Theorem 4.4.9.

COROLLARY 4.5.9. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. Then $\mathrm{M}(\mathbf{A})$ is a distributive semilattice.*

PROOF. By assumption, the lattice $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})$ of $\mathcal{S}$-filters of $\mathbf{A}$ is distributive, and by Proposition 4.5.5, this lattice is isomorphic to $\mathbf{Fi}_{\wedge}(\mathrm{M}(\mathbf{A}))$. Then we are done, since a semilattice is distributive whenever the lattice of its meet filters is distributive. □

[5]The authors name *prime $\mathcal{S}$-ideals* what we call 'non-empty up-directed $\mathcal{S}$-prime $s\mathcal{S}$-ideals'.

LEMMA 4.5.10. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. For any non-empty $B, B_0, \ldots, B_n \subseteq^\omega A$:*

$$\bigcap_{i\leq n} \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \quad \textit{iff} \quad \widehat{\varphi}(B) \subseteq \bigcup_{i\leq n} \widehat{\varphi}(B_i).$$

PROOF. Assume $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i) : i \leq n\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$ and let $G \in \widehat{\varphi}(B)$. Then we have $B \subseteq G$, and so $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \subseteq G$. Suppose, towards a contradiction, that $G \notin \bigcup\{\widehat{\varphi}(B_i) : i \leq n\}$. Then for each $i \leq n$ there is $b_i \in B_i$ such that $b_i \notin G$. Notice that $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b_i) : i \leq n\} \subseteq \bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i) : i \leq n\}$, thus from the assumption we get $\bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b_i) : i \leq n\} \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. As G is an optimal $\mathcal{S}$-fiter, by Theorem 4.4.9 we know that G^c is an $s\mathcal{S}$-ideal, and then we obtain $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \cap G^c \neq \emptyset$, a contradiction.

For the converse, assume $\widehat{\varphi}(B) \subseteq \bigcup_{i\leq n} \widehat{\varphi}(B_i)$ and let $a \in \bigcap\{\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B_i) : i \leq n\}$. Then $\widehat{\varphi}(B_i) \subseteq \varphi(a)$ for all $i \leq n$. Using the assumption, we obtain $\widehat{\varphi}(B) \subseteq \bigcup_{i\leq n} \widehat{\varphi}(B_i) \subseteq \varphi(a)$, and then by Corollary 4.3.7, we obtain $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. □

COROLLARY 4.5.11. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. For any $a, a_0, \ldots, a_n \in A$:*

$$\bigcap_{i\leq n} \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a_i) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) \quad \textit{iff} \quad \varphi(a) \subseteq \bigcup_{i\leq n} \varphi(a_i).$$

COROLLARY 4.5.12. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ and $\mathcal{F}'$ be two optimal $\mathcal{S}$-bases. For any non-empty $B, B_0, \ldots, B_n \subseteq^\omega A$:*

$$\widehat{\varphi}_{\mathcal{F}}(B) \subseteq \bigcup_{i\leq n} \widehat{\varphi}_{\mathcal{F}}(B_i) \quad \textit{iff} \quad \widehat{\varphi}_{\mathcal{F}'}(B) \subseteq \bigcup_{i\leq n} \widehat{\varphi}_{\mathcal{F}'}(B_i).$$

PROPOSITION 4.5.13. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base:*

(1) *If F is an irreducible $\mathcal{S}$-filter of $\mathbf{A}$, then $[\![\varphi[F]\rangle\!\rangle$ is an irreducible meet filter of* $\mathrm{M}(\mathbf{A})$.
(2) *If F is an optimal $\mathcal{S}$-filter of $\mathbf{A}$, then $[\![\varphi[F]\rangle\!\rangle$ is an optimal meet filter of* $\mathrm{M}(\mathbf{A})$.
(3) *If F is an optimal meet filter of* $\mathrm{M}(\mathbf{A})$*, then $\varphi^{-1}[F]$ is an optimal $\mathcal{S}$-filter of $\mathbf{A}$.*

PROOF. (1) Let $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ be irreducible. We show that $[\![\varphi[F]\rangle\!\rangle$ is irreducible. As we already know that $[\![\varphi[F]\rangle\!\rangle$ is a meet filter of $\mathrm{M}(\mathbf{A})$, by Theorem 2.3.6 we just need to show that $[\![\varphi[F]\rangle\!\rangle^c$ is up-directed. We use (E2), so let $B_1, B_2 \subseteq^\omega A$ be non-empty and such that $\widehat{\varphi}(B_1), \widehat{\varphi}(B_2) \notin [\![\varphi[F]\rangle\!\rangle$. We show that there is $B \subseteq^\omega A$ non-empty and such that $\widehat{\varphi}(B_1), \widehat{\varphi}(B_2) \subseteq \widehat{\varphi}(B) \notin [\![\varphi[F]\rangle\!\rangle$. By assumption, there are $b_1 \in B_1$, and $b_2 \in B_2$ such that $\varphi(b_1), \varphi(b_2) \notin [\![\varphi[F]\rangle\!\rangle$. Then $b_1, b_2 \notin F$, so $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b_1), \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b_2) \nsubseteq F$. Now as F is irreducible, $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b_1) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b_2) \nsubseteq F$. Let $c \in (\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b_1) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b_2)) \setminus F$. Then $\varphi(b_1), \varphi(b_2) \subseteq \varphi(c)$, so $\widehat{\varphi}(B_1), \widehat{\varphi}(B_2) \subseteq \varphi(c)$ and moreover $\varphi(c) \notin [\![\varphi[F]\rangle\!\rangle$.

(2) Let $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ be optimal. Then by Corollary 4.4.12 F^c is an $\mathcal{S}$-prime $s\mathcal{S}$-ideal of $\mathbf{A}$, and so by Proposition 4.5.6 $\langle\!\langle\varphi[F^c]]\!]$ is a $\wedge$-prime F-ideal of $\mathrm{M}(\mathbf{A})$, and

moreover by Corollary 2.3.10 $\langle\!\langle \varphi[F^c]]\!]^c$ is an optimal meet filter of $\mathrm{M}(\mathbf{A})$. Therefore, it is enough to show that $[\![\varphi[F]\rangle\!\rangle = \langle\!\langle \varphi[F^c]]\!]^c$.

First we show the inclusion from right to left. We use (E2), so let $B \subseteq^\omega A$ be non-empty and such that $\widehat{\varphi}(B) \in \langle\!\langle \varphi[F^c]]\!]^c$. Then for all $b \in B$, $\varphi(b) \notin \langle\!\langle \varphi[F^c]]\!]$. Thus by Proposition 4.5.6, we get that $b \notin \varphi^{-1}[\langle\!\langle \varphi[F^c]]\!]] = F^c$, and therefore $\varphi(b) \in \varphi[F]$ for all $b \in B$. Thus $\bigcap\{\varphi(b) : b \in B\} = \widehat{\varphi}(B) \in [\![\varphi[F]\rangle\!\rangle$.

For the other inclusion, we use (E2) so let $B \subseteq^\omega A$ be non-empty and such that $\widehat{\varphi}(B) \in [\![\varphi[F]\rangle\!\rangle$. Then either $\widehat{\varphi}(B) = \widehat{\varphi}(1)$ or there is a non-empty $B' \subseteq^\omega F$ such that $\widehat{\varphi}(B') \subseteq \widehat{\varphi}(B)$. In the first case, as $1 \in F \neq \emptyset$, then clearly $\widehat{\varphi}(1) \notin \langle\!\langle \varphi[F^c]]\!]$, and we are done, so assume that there is non-empty $B' \subseteq^\omega F$ such that $\widehat{\varphi}(B') \subseteq \widehat{\varphi}(B)$. By Lemma 4.5.10, $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B') \subseteq F$, so $B \subseteq F$. Therefore for all $b \in B$, $b \notin F^c = \varphi^{-1}[\langle\!\langle \varphi[F^c]]\!]]$, using Proposition 4.5.6 again. Hence $\varphi(b) \notin \langle\!\langle \varphi[F^c]]\!]$ for all $b \in B$. Moreover, as by assumption $\langle\!\langle \varphi[F^c]]\!]$ is a $\wedge$-prime F-ideal, then $\widehat{\varphi}(B) \notin \langle\!\langle \varphi[F^c]]\!]$, i. e. $\widehat{\varphi}(B) \in \langle\!\langle \varphi[F^c]]\!]^c$, as required.

(3) Let $F \in \mathrm{Op}_\wedge(\mathrm{M}(\mathbf{A}))$ be optimal. Then by Corollary 2.3.10, F^c is a $\wedge$-prime F-ideal of $\mathrm{M}(\mathbf{A})$, and so by Proposition 4.5.6, $\varphi^{-1}[F^c]$ is an $\mathcal{S}$-prime $s\mathcal{S}$-ideal of $\mathbf{A}$, and moreover by Corollary 4.4.12, $\varphi^{-1}[F^c]^c$ is an optimal $\mathcal{S}$-filter of $\mathbf{A}$. Notice that $(\varphi^{-1}[F^c])^c = \varphi^{-1}[F]$. Therefore $\varphi^{-1}[F]$ is an optimal $\mathcal{S}$-filter of $\mathbf{A}$. $\square$

Previous proposition shows that for any filter distributive finitary congruential logic with theorems, (E3) restricts to an order isomorphism between optimal $\mathcal{S}$-filters of $\mathbf{A}$ and optimal meet filters of $\mathrm{M}(\mathbf{A})$:

$$\text{(E7)} \qquad \langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq\rangle \cong \langle \mathrm{Op}_\wedge(\mathrm{M}(\mathbf{A})), \subseteq\rangle$$

Summarizing, what we have seen throughout this chapter is that for any filter distributive finitary congruential logic with theorems $\mathcal{S}$, optimal $\mathcal{S}$-filters and irreducible $\mathcal{S}$-filters are two optimal $\mathcal{S}$-bases. This gives us an $\mathcal{S}$-algebra of subsets $\varphi[\mathbf{A}]$ isomorphic to $\mathbf{A}$, that is a reduced $\mathcal{S}$-referential algebra. And it allows us to embed $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle$ in a distributive semilattice $\mathrm{M}(\mathbf{A})$ with *nice* properties. We exploit these two facts in the next chapter, where Spectral-like and Priestley-style dualities for $\mathcal{S}$-algebras are studied.

4.6. Canonical extensions and Δ_1-completions for filter distributive finitary congruential logics with theorems

In this section we just intend to take a look at canonical extensions and Δ_1-completions for filter distributive finitary congruential logics with theorems. We recall that Δ_1-*completion* is an order-theoretic tool which allows for modular development of representation theory of classes of ordered algebras. When the class of algebras under consideration is not lattice-based, there may be a wide range of Δ_1-completions on hand, being *canonical extensions* one of such Δ_1-completions. Gehrke, Jansana and Palmigiano define and study in detail in [**42**] Δ_1-completions for posets, that are defined as those completions for which, simultaneously, each element is obtainable as a join of meets of elements of the original poset and as a meet of joins of elements of the original poset. For any poset P, we denote by P^δ the canonical extension of P, in the sense defined by Dunn, Gehrke and Palmigiano in [**26**].

Following a development parallel and complementary to the progress of duality theory, canonical extensions have been applied to several classes of algebras that are the algebraic counterpart of certain non-classical logics. It is a natural question accordingly to explore whether a *logic based* notion of canonical extensions can be built within the field of AAL. This is precisely the motivation of the work in [**41**], whose main results we review in § 4.6.1. Inasmuch as in Chapter 5 we introduce new tools for the study of duality theory for non-classical logics within the perspective of AAL, it is natural to ask whether such tools may be used to enhance the results of [**41**].

The remaining subsections are organized as follows: first we refresh the definition of canonical extension for finitary congruential logics with theorems and satisfying (uDDT) proposed in [**41**], and we just outline how it could be extended for any filter distributive finitary congruential logic with theorems. After that we use the notions we introduced previously in this Chapter to study a different Δ_1-completion for filter distributive finitary congruential logics with theorems, and we show that this Δ_1-completion has most of the nice properties that the canonical extension proposed in [**41**] has.

4.6.1. $\mathcal{S}$-canonical extensions. Let $\mathcal{S}$ be a finitary congruential logic with theorems and let $\mathbf{A}$ be an $\mathcal{S}$-algebra. Remember that we denote by $\mathrm{M}(\mathbf{A})$ the $\mathcal{S}$-semilattice of $\mathbf{A}$ (cf. definition in page 61), that in [**41**] is denoted by $L^{\wedge}_{\mathcal{S}}(\mathbf{A})$.

The *$\mathcal{S}$-canonical extension* of $\mathbf{A}$ (Definition 4.17 in [**41**]) is defined as the $(\mathrm{Fi}_{\wedge}(\mathrm{M}(\mathbf{A})), \mathrm{Id}(\mathrm{M}(\mathbf{A})))$-completion of $\mathrm{M}(\mathbf{A})$, and it is denoted by $\mathbf{A}^{\mathcal{S}}$. Notice that if we look at $\mathrm{M}(\mathbf{A})$ as a poset, then $\mathbf{A}^{\mathcal{S}}$ is precisely what in [**26**] Dunn et al. called *the canonical extension* of $\mathrm{M}(\mathbf{A})$, that we denote by $\mathrm{M}(\mathbf{A})^{\delta}$. Accordingly, the $\mathcal{S}$-canonical extension of $\mathbf{A}$ is a complete lattice in which $\mathrm{M}(\mathbf{A})$ embeds satisfying the usual properties of denseness and compactness. Moreover, it follows from the embedding of $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$ into $\mathrm{M}(\mathbf{A})$, that $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}} \rangle$ lives into $\mathbf{A}^{\mathcal{S}}$. But within this abstract approach, the operations in $\mathbf{A}$ are not taken into account.

Once the $\mathcal{S}$-canonical extension of $\mathbf{A}$ is defined in such a general way, the proof strategy by Gehrke, Jansana and Palmigiano in [**41**] goes as follows: first they prove (Theorem 4.20 in [**41**]) that when $\mathbf{A}^{\mathcal{S}}$ satisfies that for all $B \cup \{c\} \subseteq A^{\mathcal{S}}$:

$$((\vee, \textstyle\bigwedge)\text{-distributive law}) \qquad c \vee (\bigwedge B) = \bigwedge_{b \in B} (c \vee b),$$

then $\mathbf{A}^{\mathcal{S}}$ is (up to isomorphism) the $(\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), {}_{\mathrm{ud}}\mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$-completion of $\mathbf{A}$. After that, they show (Theorem 5.6 in [**41**]) that when $\mathcal{S}$ satisfies (uDDT), $\mathbf{A}^{\mathcal{S}}$ satisfies the $(\vee, \bigwedge)$-distributive law. They conclude that whenever $\mathcal{S}$ satisfies (uDDT),

$$\text{(Can)} \qquad \mathbf{A}^{\mathcal{S}} \text{ is the } (\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), {}_{\mathrm{ud}}\mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))\text{-completion of } \mathbf{A}.$$

Finally, they apply these results to the logic $\mathcal{H}$, the implicative fragment of intuitionistic logic, whose algebraic semantics is given by Hilbert algebras (we study this in detail in § 6.2). They use the following fact (that was extensively studied by Celani and Jansana in [**17**]): for any Hilbert algebra $\mathbf{A} = \langle A, \to, 1 \rangle$, an implication $\to'$ may be defined in $\mathrm{M}(\mathbf{A})$ such that $\langle \mathrm{M}(A), \to', \wedge, 1 \rangle$ is the free implicative semilattice extension of $\mathbf{A}$. Using this fact, the π-extension of $\to$ to $\mathbf{A}^{\mathcal{H}}$ may be defined as the π-extension of $\to'$ to $\mathrm{M}(\mathbf{A})^{\delta}$. Let us denote such operation by $\to^{\pi}$.

It follows that $\langle A^{\mathcal{H}}, \to^{\pi}, \wedge, \vee, 0, 1\rangle$ is a complete Heyting algebra, and in particular $\langle A^{\mathcal{H}}, \to^{\pi}, 1\rangle$ is a Hilbert algebra. This is, of course, a desirable property of a logic-based notion of canonical extension, and it fails dramatically for the order-theoretic notion of canonical extension, when applied to Hilbert algebras as poset expansions, as Example B.16 in Appendix B shows.

Notice that the $\mathcal{S}$-canonical extension of $\mathbf{A}$ is based on the definition of the canonical extension of meet semilattices given in [**26**]. It would be certainly desirable, that the canonical extension of a meet semilattice is completely distributive provided the semilattice is distributive.[6] If that would be true, then we could extend (Can) to any filter distributive finitary congruential logic with theorems. This would follow since filter distributivity of the logic $\mathcal{S}$ implies, by Corollary 4.5.9, that $\mathrm{M}(\mathbf{A})$ is a distributive semilattice. It would be very interesting to investigate whether distributivity of semilattices lifts to complete distributivity of their canonical extensions, but we do not go further into this topic, because it escapes the purposes of this section.

4.6.2. $s\mathcal{S}$-extensions. The main goal of [**41**] was, as stated in the introduction, to explore whether canonical extensions can be developed as a *logical* construct within AAL rather that just as a purely order-theoretical construct. Nevertheless, the notion of $\mathcal{S}$-canonical extension that is introduced in [**41**] falls mid-way between being logic-based and order-based, since, as we have already seen, it involves the collection of non-empty up-directed $\mathcal{S}$-ideals.

We pursue to study now, making use of the new notions introduced in this chapter, a Δ_1-completion that may seem *more* logic-based. The proof strategy that we follow is similar to that employed in [**41**]. From now on, let $\mathcal{S}$ be a finitary congruential logic with theorems, and let $\mathbf{A}$ be an $\mathcal{S}$-algebra.

We are interested in the $(\mathrm{Fi}_{\wedge}(\mathrm{M}(\mathbf{A})), \mathrm{Id}_F(\mathrm{M}(\mathbf{A})))$-completion of $\mathrm{M}(\mathbf{A})$, that we call *F-extension* of $\mathrm{M}(\mathbf{A})$, and that we denote by $\mathrm{M}(\mathbf{A})^F$. Recall that filter distributivity of the logic $\mathcal{S}$ implies, by Corollary 4.5.9, that $\mathrm{M}(\mathbf{A})$ is a distributive semilattice. Then by results in Appendix B, the F-extension of $\mathrm{M}(\mathbf{A})$ is an algebraic lattice, and in particular, it is completely distributive.

Now we aim to prove that the F-extension $\mathrm{M}(\mathbf{A})^F$ of $\mathrm{M}(\mathbf{A})$ is (up to isomorphism) the $(\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$-completion of $\mathbf{A}$. Notice that we have:

$$\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle \xrightarrow{\varphi} \mathrm{M}(\mathbf{A}) \xrightarrow{k} \mathrm{M}(\mathbf{A})^F,$$

where φ is the embedding of $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle$ into its $\mathcal{S}$-semilattice, defined in § 4.5, and k is the embedding of $\mathrm{M}(\mathbf{A})$ into its F-extension, defined in Appendix B. Let us define g as the composition of these two maps:

$$g := (k \circ \varphi) : \mathbf{A} \longrightarrow \mathrm{M}(\mathbf{A})^F.$$

[6]From results by Gehrke and Priestley in [**44**] and Gehrke and Vosmaer in [**45**] it follows that the canonical extension of any meet semilattice with top element $\mathbf{M}$ is the dcpo-completion of (i. e. dcpo freely generated by) $\langle \mathrm{Fi}_{\wedge}(\mathbf{M}), \supseteq, \lhd\rangle$, where $\lhd$ is a binary relation between meet filters of $\mathbf{M}$ and up-directed collections of meet filters of $\mathbf{M}$, given by $F \lhd U$ if and only if for all $I \in \mathrm{Id}(\mathbf{M})$, if $F' \cap I = \emptyset$ for all $F' \in U$, then $F \cap I = \emptyset$. Moreover, if $\mathbf{M}$ is a distributive semilattice, then $\mathbf{Fi}_{\wedge}(\mathbf{M})$ is a distributive lattice. By properties of dcpo-completions we know that the distributivity equation should lift through these completions, and hence the canonical extension of any distributive semilattice is a distributive (complete) lattice.

Similarly to what it was done in Section 4.4 in [**41**], the next lemma states some facts concerning the map g, that we need in the following theorem. Notice that all infinite meet and joins in the next proofs are referred to the complete lattice $\mathrm{M}(\mathbf{A})^F$. And recall that $\mathcal{M}^\infty(\mathrm{M}(\mathbf{A}))$ is the collection of all completely meet-irreducible elements of the semilattice $\mathrm{M}(\mathbf{A})$ (see definition in page 13).

LEMMA 4.6.1. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra and let g and k be as defined above. Then*

(1) *For all $F \in \mathrm{Fi}_\mathcal{S}(\mathbf{A})$, $\bigwedge g[F] = \bigwedge k[\llbracket \varphi[F] \rrangle]$.*
(2) *For all $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$, $\bigvee g[I] = \bigvee k[\downarrow \varphi[I]]$.*
(3) *For all $I \in \mathrm{Id}_F(\mathrm{M}(\mathbf{A}))$, I is $\wedge$-prime if and only if $\bigvee k[I] \in \mathcal{M}^\infty(\mathrm{M}(\mathbf{A})^F)$.*
(4) *For all $c \in \mathcal{M}^\infty(\mathrm{M}(\mathbf{A})^F)$, there is I_c $\wedge$-prime F-ideal of $\mathrm{M}(\mathbf{A})$ such that $c = \bigvee k[I]$.*

PROOF. Item (1) was proven in Lemma 4.18 (1) in [**42**]. The proof of (3) is similar to that of Proposition 4.19 (1) in [**42**], and (4) is a corollary of (3). It only remains to prove (2): on the one hand, from $\varphi[I] \subseteq \downarrow\varphi[I]$, we get $k[\varphi[I]] \subseteq k[\downarrow\varphi[I]]$, and so $\bigvee g[I] = \bigvee k[\varphi[I]] \leq \bigvee k[\downarrow\varphi[I]]$. On the other hand, for any $X \in \downarrow\varphi[I]$, there is some $a_X \in I$ such that $X \leq \varphi(a_X)$, and so $k(X) \leq k(\varphi(a_X))$. This implies that $\bigvee k[\downarrow\varphi[I]] = \bigvee\{k(X) : X \in \downarrow\varphi[I]\} \leq \bigvee\{k(\varphi(a)) : a \in I\} = \bigvee g[I]$, as required. □

THEOREM 4.6.2. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}$ be an $\mathcal{S}$-algebra. The F-completion $\mathrm{M}(\mathbf{A})^F$ of $\mathrm{M}(\mathbf{A})$, is (up to isomorphism) the $(\mathrm{Fi}_\mathcal{S}(\mathbf{A}), \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$-completion of $\mathbf{A}$.*

PROOF. We show that g gives us the required dense and compact embedding.

CLAIM 4.6.3. $\mathrm{M}(\mathbf{A})^F$ *is* $(\mathrm{Fi}_\mathcal{S}(\mathbf{A}), \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$*-compact.*

PROOF OF THE CLAIM. Let $F \in \mathrm{Fi}_\mathcal{S}(\mathbf{A})$ and let $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ be such that $\bigwedge g[F] \leq \bigvee g[I]$. By Lemma 4.6.1, $\bigwedge k[\llbracket\varphi[F]\rrangle] \leq \bigvee k[\downarrow\varphi[I]]$. By Proposition 4.5.6 $\downarrow\varphi[I]$ is an F-ideal of $\mathrm{M}(\mathbf{A})$. By Proposition 4.5.5 $\llbracket\varphi[F]\rrangle$ is a meet filter of $\mathrm{M}(\mathbf{A})$. Then by $(\mathrm{Fi}_\wedge(\mathrm{M}(\mathbf{A})), \mathrm{Id}_F(\mathrm{M}(\mathbf{A})))$-compactness of $\mathrm{M}(\mathbf{A})^F$ we get that $\downarrow\varphi[I] \cap \llbracket\varphi[F]\rrangle \neq \emptyset$. Then by definition of down-set generated and since meet filters are up-set, we conclude that there is $a \in I$ such that $\varphi(a) \in \llbracket\varphi[F]\rrangle$. Then from Proposition 4.5.5 again, $a \in \varphi^{-1}[\llbracket\varphi[F]\rrangle] = F$, so $F \cap I \neq \emptyset$, as required. □

CLAIM 4.6.4. $\mathrm{M}(\mathbf{A})^F$ *is* $(\mathrm{Fi}_\mathcal{S}(\mathbf{A}), \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$*-dense.*

PROOF OF THE CLAIM. First we show that the collection of $\mathrm{Fi}_\mathcal{S}(\mathbf{A})$-filter elements of $\mathrm{M}(\mathbf{A})^F$ is join-dense in $\mathrm{M}(\mathbf{A})^F$. Recall that these are the elements of the form $\bigwedge g[F]$ for some $F \in \mathrm{Fi}_\mathcal{S}(\mathbf{A})$. By $(\mathrm{Fi}_\wedge(\mathrm{M}(\mathbf{A})), \mathrm{Id}_F(\mathrm{M}(\mathbf{A})))$-denseness we have that for each $z \in \mathrm{M}(A)^F$ there is $\mathcal{X} \subseteq \mathrm{Fi}_\wedge(\mathrm{M}(\mathbf{A}))$ a collection of meet filters of $\mathrm{M}(\mathbf{A})$ such that $z = \bigvee\{\bigwedge k[F] : F \in \mathcal{X}\}$. Notice that for any $F \in \mathcal{X}$, by Lemma 4.6.1 and Proposition 4.5.5, $\varphi^{-1}[F]$ is an $\mathcal{S}$-filter of $\mathbf{A}$ and $\bigwedge k[F] = \bigwedge k[\varphi[\varphi^{-1}[F]]] = \bigwedge g[\varphi^{-1}[F]]$, so we are done.

Finally we show that the collection of $\mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$-ideal elements of $\mathrm{M}(\mathbf{A})^F$ is meet-dense in $\mathrm{M}(\mathbf{A})^F$. Recall that these are the elements of the form $\bigvee g[I]$ for some $I \in \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$. Recall also that $\mathrm{M}(\mathbf{A})^F$ is algebraic, so its completely meet irreducible elements are completely meet prime, and they meet-generate $\mathrm{M}(\mathbf{A})^F$.

Therefore, for all $z \in \mathrm{M}(A)^F$, there is $X \subseteq \mathcal{M}^\infty(\mathrm{M}(\mathbf{A})^F)$ a collection of completely meet irreducible elements of $\mathrm{M}(\mathbf{A})^F$ such that $z = \bigwedge X$. Then by Lemma 4.6.1, $z = \bigwedge\{\bigvee k[I] : I \in \mathcal{Y}\}$ for some $\mathcal{Y}$ a collection of $\wedge$-prime F-ideals of $\mathrm{M}(\mathbf{A})$. Notice that for any $I \in \mathcal{Y}$, by Lemma 4.6.1 and Proposition 4.5.6, $\varphi^{-1}[I]$ is an strong $\mathcal{S}$-ideal of $\mathbf{A}$ and $\bigvee k[I] = \bigvee k[\downarrow\varphi[\varphi^{-1}[I]]] = \bigvee g[\varphi^{-1}[I]]$, so we are done. □

Notice that we have shown that $\mathrm{M}(\mathbf{A})^F$ is a $(\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$-compact and $(\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$-dense extension of $\mathbf{A}$, we conclude that $\mathrm{M}(\mathbf{A})^F$ is, up to isomorphism, the $(\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$-completion of $\mathbf{A}$. □

Previous theorem justifies the introduction of the following definition:

DEFINITION 4.6.5. The *$s\mathcal{S}$-extension* of $\mathbf{A}$ is the $(\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \mathrm{Id}_{s\mathcal{S}}(\mathbf{A}))$-completion of $\mathbf{A}$, and it is denoted by $\mathbf{A}^{s\mathcal{S}}$.

We showed in Appendix B that the canonical extension and the F-extension of a distributive meet semilattice may not be isomorphic. This implies that the $\mathcal{S}$-canonical extension and the $s\mathcal{S}$-extension of an $\mathcal{S}$-algebra may not be isomorphic either. It would be very interesting to investigate under which conditions on $\mathcal{S}$ might the $\mathcal{S}$-canonical extension and the $s\mathcal{S}$-extension of any $\mathcal{S}$-algebra coincide, but we leave this as future work. What we do know from ${}_{\mathrm{ud}}\mathrm{Id}_{s\mathcal{S}}(\mathbf{A}) \subseteq \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$ is that the $\mathcal{S}$-canonical extension of $\mathbf{A}$ is always embeddable in the $s\mathcal{S}$-extension of $\mathbf{A}$. The main difference between both concepts is that the $s\mathcal{S}$-extension is defined for a wider class of logics, namely filter distributive finitary congruential logics with theorems, that, as we know, include finitary congruential logics with theorems and satisfying (uDDT).

We conclude this section by applying these results to the implicative fragment of intuitionistic logic $\mathcal{H}$. Recall that for any Hilbert algebra $\mathbf{A}$ and implication $\to'$ may be defined in $\mathrm{M}(\mathbf{A})$ such that $\langle M(A), \to', \wedge, 1\rangle$ is an implicative semilattice in which $\mathbf{A}$ embeds. From results in Appendix B we know that $\langle M(A)^F, (\to')^\pi, \wedge, \vee, 0, 1\rangle$ is a complete Heyting algebra, where $(\to')^\pi$ is the π-extension of $\to'$ to $\mathrm{M}(\mathbf{A})^F$. Therefore, if we define the π-extension of $\to$ to $A^{s\mathcal{H}}$ as the π-extension of $\to'$ to $\mathrm{M}(\mathbf{A})^F$, and we denote it simply by $\to^\pi$, it follows that $\langle A^{s\mathcal{H}}, \to^\pi, \wedge, \vee, 0, 1\rangle$ is a complete Heyting algebra, and in particular $\langle A^{s\mathcal{H}}, \to^\pi, 1\rangle$ is a Hilbert algebra.

In summary, the notions that we have introduced throughout this section, such as the notion of $s\mathcal{S}$-ideal, can be used to define a logic-based Δ_1-completion of $\mathcal{S}$-algebras that has at least the same nice properties as the $\mathcal{S}$-canonical extension of $\mathcal{S}$-algebras that was introduced in [**41**]. We do not go further into this topic, since we are mainly interested in using such notions for developing an abstract duality theory for $\mathcal{S}$-algebras and homomorphisms between them. This is precisely what we do in the next chapter.

CHAPTER 5

Duality Theory for Filter Distributive Congruential Logics

In Chapter 4 we introduced the toolkit we need to develop Spectral-like and a Priestley-style abstract dualities for any filter distributive finitary congruential logic with theorems. In the present chapter we expose systematically in parallel these two dualities for $\mathbb{A}\mathrm{lg}\mathcal{S}$, with $\mathcal{S}$ a fixed but arbitrary filter distributive finitary congruential logic with theorems.

In § 5.1 we prove representation theorems for $\mathcal{S}$-algebras and we introduce the definitions of $\mathcal{S}$-Spectral spaces and $\mathcal{S}$-Priestley spaces. In § 5.2 we consider morphisms, and we introduce the definitions of $\mathcal{S}$-Spectral morphism and $\mathcal{S}$-Priestley morphism. In § 5.3 they are defined the functors and the natural transformations involved in the dualities. In § 5.4 we compare our work with that of Jansana and Palmigiano in [**56**].

Notice that we do not fix any specific language, so our approach is necessarily abstract in the sense that the dual categories will necessarily *involve* a similar notion to that of $\mathcal{S}$-algebra and homomorphism between $\mathcal{S}$-algebras. This constraint can be avoided in many cases, when a concrete language is under consideration. We analyze in § 5.5 different logical properties that a logic may have, and we study how each of them corresponds with a dual property of the dual categories. Thanks to this analysis, the connection with the results in the literature is evidenced, as discussed in Chapter 6.

5.1. Duality for objects

In the present section, we use the results from Chapter 4 to present two correspondences between $\mathcal{S}$-algebras and certain classes of Spectral-like and Priestley-style spaces that we introduce later on.

Recall that for any finitary congruential logic with theorems $\mathcal{S}$, and for any $\mathcal{S}$-algebra $\mathbf{A}$, the collection of irreducible $\mathcal{S}$-filters of $\mathbf{A}$ and the collection of optimal $\mathcal{S}$-filters of $\mathbf{A}$ are both optimal $\mathcal{S}$-bases. Therefore, by our work in § 4.3 we know that the maps $\varphi_{\mathrm{Irr}_\mathcal{S}(\mathbf{A})}$ and $\varphi_{\mathrm{Op}_\mathcal{S}(\mathbf{A})}$ have some interesting properties. For convenience, let us denote the map $\varphi_{\mathrm{Irr}_\mathcal{S}(\mathbf{A})}$ by $\psi_\mathbf{A}$ and similarly let us denote the map $\varphi_{\mathrm{Op}_\mathcal{S}(\mathbf{A})}$ by $\vartheta_\mathbf{A}$, so we have:

$$
\begin{aligned}
\psi_\mathbf{A} : A &\longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Irr}_\mathcal{S}(\mathbf{A})) \qquad & \vartheta_\mathbf{A} : A &\longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Op}_\mathcal{S}(\mathbf{A}))\\
a &\longmapsto \{P \in \mathrm{Irr}_\mathcal{S}(\mathbf{A}) : a \in P\} & a &\longmapsto \{P \in \mathrm{Op}_\mathcal{S}(\mathbf{A}) : a \in P\}
\end{aligned}
$$

Recall that for any $B \subseteq A$, by $\widehat{\psi}_\mathbf{A}(B)$ we denote the set $\bigcap\{\psi_\mathbf{A}(b) : b \in B\}$, and similarly for $\widehat{\vartheta}_\mathbf{A}$. When the context is clear, we drop the subscripts of $\psi_\mathbf{A}, \widehat{\psi}_\mathbf{A}, \vartheta_\mathbf{A}, \widehat{\vartheta}_\mathbf{A}$.

Let us collect in the following two theorems what we obtained in theorems 4.3.5 and 4.3.9 and Corollary 4.3.7.

THEOREM 5.1.1. *Let $\mathcal{S}$ be a finitary congruential logic with theorems and let $\mathbf{A}$ be an $\mathcal{S}$-algebra. The map $\psi_{\mathbf{A}}$ is an isomorphism between $\mathbf{A}$ and $\psi_{\mathbf{A}}[\mathbf{A}]$. The structure $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \psi_{\mathbf{A}}[\mathbf{A}]\rangle$ is a reduced $\mathcal{S}$-referential algebra whose associated order is given by the inclusion relation. Moreover, for any $\{a\} \cup B \subseteq A$,*

$$a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \quad \textit{iff} \quad \widehat{\psi}(B) \subseteq \psi(a) \quad \textit{iff} \quad \psi(a) \in \mathrm{C}^{\psi[\mathbf{A}]}_{\mathcal{S}}(\psi[B]).$$

THEOREM 5.1.2. *Let $\mathcal{S}$ be a finitary congruential logic with theorems and let $\mathbf{A}$ be an $\mathcal{S}$-algebra. The map $\vartheta_{\mathbf{A}}$ is an isomorphism between $\mathbf{A}$ and $\vartheta_{\mathbf{A}}[\mathbf{A}]$. The structure $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \vartheta_{\mathbf{A}}[\mathbf{A}]\rangle$ is a reduced $\mathcal{S}$-referential algebra whose associated order is given by the inclusion relation. Moreover, for any $\{a\} \cup B \subseteq A$,*

$$a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \quad \textit{iff} \quad \widehat{\vartheta}(B) \subseteq \vartheta(a) \quad \textit{iff} \quad \vartheta(a) \in \mathrm{C}^{\vartheta[\mathbf{A}]}_{\mathcal{S}}(\vartheta[B]).$$

Notice that these representation theorems hold for any finitary congruential logic with theorems, not necessarily a filter distributive one. However, for getting a full duality between objects, we should assume additionally filter-distributivity of the logic. In the following subsections, we discuss first the Spectral-like dual objects of $\mathcal{S}$-algebras, and then the Priestley-style dual objects of $\mathcal{S}$-algebras. In both cases we prove the facts that motivate the definition of the dual objects before introducing such definition. For the Priestley-style duality, some results from § 4.5 about the $\mathcal{S}$-semilattice of $\mathbf{A}$ are essential.

5.1.1. Spectral-like dual objects. We assume that $\mathcal{S}$ is a filter distributive finitary congruential logic with theorems and $\mathbf{A}$ an $\mathcal{S}$-algebra. We define on $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ a topology $\tau_{\kappa_{\mathbf{A}}}$, having as basis the collection:

$$\kappa_{\mathbf{A}} := \{\psi(a)^c : a \in A\}.$$

Next proposition shows that this topology is well defined.

PROPOSITION 5.1.3. *Let $a, b \in A$, $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ and $B \subseteq A$ non-empty. Then:*

(1) $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A}) = \bigcup\{\psi(a)^c : a \in A\}$.

(2) *If $P \in \psi(a)^c \cap \psi(b)^c$, then there is an element $c \in A$ such that $P \in \psi(c)^c$ and $\psi(c)^c \subseteq \psi(a)^c \cap \psi(b)^c$.*

(3) *If $\psi(a)^c = \bigcup\{\psi(b)^c : b \in B\}$, then there is a subset $B' \subseteq^{\omega} B$ such that $\psi(a)^c = \bigcup\{\psi(b)^c : b \in B'\}$.*

PROOF. (1) By definition $\bigcup\{\psi(a)^c : a \in A\} \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. Moreover, for any $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, since P is proper, there is an element $a_P \in A$ such that $a_P \notin P$. Thus $P \in \psi(a_P)^c \subseteq \bigcup\{\psi(a)^c : a \in A\}$.

(2) Let $P \in \psi(a)^c \cap \psi(b)^c$, i. e. $a, b \notin P$. By Theorem 4.4.8, P^c is a poset ideal of $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle$, so there is $c \in A$ such that $a, b \leq^{\mathbf{A}}_{\mathcal{S}} c \notin P$. Then we have $P \in \psi(c)^c$ and moreover, since P is an up-set, $a, b \notin P'$ for all $P' \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $c \notin P'$. Hence $P \in \psi(c)^c \subseteq \psi(a)^c \cap \psi(b)^c$.

(3) Assume $\psi(a)^c = \bigcup\{\psi(b)^c : b \in B\}$, i. e. $\psi(a) = \widehat{\psi}(B)$. Then from Theorem 5.1.1, $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$, and in particular $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. Then by finitarity, there is $B' \subseteq^{\omega} B$ such that $a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B')$. But then $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B') \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a)$,

and from Theorem 5.1.1 again, we get $\psi(a) = \widehat{\psi}(B')$, i.e. $\psi(a)^c = \bigcup\{\psi(b)^c : b \in B'\}$, as required. $\square$

Item (3) of the previous proposition states that $\kappa_{\mathbf{A}}$ is a basis of open-compacts for the topology $\tau_{\kappa_{\mathbf{A}}}$. Moreover, the space $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$ is T_0: if $P \neq Q$ we can assume, without loss of generality, that there is $a \in P \setminus Q$, and so $P \notin \psi(a)^c$ and $Q \in \psi(a)^c$. Hence the space separates points. Therefore, the specialization order of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$ is a partial order, whose dual is denoted by $\leq_{\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})}$, or simply by $\leq$ when no confusion is possible. Moreover, all open (resp. closed) subsets of the space are down-sets (resp. up-sets) with respect to $\leq_{\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})}$.

PROPOSITION 5.1.4. *For any $F_1, F_2 \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$:*

$$F_1 \subseteq F_2 \quad \textit{iff} \quad \widehat{\psi}(F_2) \subseteq \widehat{\psi}(F_1).$$

PROOF. Clearly from $F_1 \subseteq F_2$ it follows that $\widehat{\psi}(F_2) \subseteq \widehat{\psi}(F_1)$. For the converse, suppose $F_1 \nsubseteq F_2$. Then there is $a \in F_1$ such that $a \notin F_2$. By Corollary 4.4.2, there is $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $F_2 \subseteq P$ and $a \notin P$, and so $F_1 \nsubseteq P$. Thus $P \in \widehat{\psi}(F_2)$ and $P \notin \widehat{\psi}(F_1)$, hence $\widehat{\psi}(F_2) \nsubseteq \widehat{\psi}(F_1)$. $\square$

PROPOSITION 5.1.5. *For any $F_0, \dots F_n \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ and $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$:*

$$\bigcap\{F_i : i \leq n\} \subseteq P \quad \textit{iff} \quad \widehat{\psi}(P) \subseteq \bigcup_{i \leq n} \widehat{\psi}(F_i).$$

PROOF. Assume first that $\bigcap\{F_i : i \leq n\} \subseteq P$ and let $Q \in \widehat{\psi}(P)$, i.e. $P \subseteq Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. By assumption $\bigcap\{F_i : i \leq n\} \subseteq Q$, and since Q is an irreducible $\mathcal{S}$-filter and the logic is filter distributive, there is $j \leq n$ such that $F_j \subseteq Q$. Now by the previous proposition $\widehat{\psi}(Q) \subseteq \widehat{\psi}(F_j) \subseteq \bigcup\{\widehat{\psi}(F_i) : i \leq n\}$.

For the converse, suppose that $\bigcap\{F_i : i \leq n\} \nsubseteq P$. Then there is an element $a \in \bigcap\{F_i : i \leq n\}$ such that $a \notin P$. So for each $i \leq n$, $F_i \nsubseteq P$, and then we have $P \in \widehat{\psi}(P)$ and $P \notin \bigcup\{\widehat{\psi}(F_i) : i \leq n\}$, hence $\widehat{\psi}(P) \nsubseteq \bigcup\{\widehat{\psi}(F_i) : i \leq n\}$. $\square$

The following propositions serve us to complete the description of the topological space $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$, as they characterize closed subsets and irreducible closed subsets of the space $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$.

PROPOSITION 5.1.6. *$U \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ is a closed subset of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$ if and only if there is $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ such that $U = \widehat{\psi}(F)$. Moreover, for all $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, $\mathrm{cl}(P) = \widehat{\psi}(P)$.*

PROOF. By definition, for any $B \subseteq A$, the subset $\widehat{\psi}(B) = \bigcap\{\psi(a) : a \in B\}$ is a closed subset of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$. For the converse, let U be a closed subset of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$, i.e. $U = \bigcap\{\psi(b) : b \in B\} = \widehat{\psi}(B)$ for some $B \subseteq A$. Since we know that $\widehat{\psi}(B) = \widehat{\psi}(\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B))$, and $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ we are done.

Let now $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. As $\widehat{\psi}(P)$ is closed and $P \in \widehat{\psi}(P)$, clearly $\mathrm{cl}(P) \subseteq \widehat{\psi}(P)$. For the converse, assume U be a closed subset such that $P \in U$. We show that $\widehat{\psi}(P) \subseteq U$. Since U is closed, there is $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ such that $U = \widehat{\psi}(F)$, and then by assumption $F \subseteq P$. Then by Proposition 5.1.4 $\widehat{\psi}(P) \subseteq \widehat{\psi}(F) = U$. We conclude that $\widehat{\psi}(P) \subseteq \mathrm{cl}(P)$, and we are done. $\square$

PROPOSITION 5.1.7. *Let U be a closed subset of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$. Then U is irreducible if and only if $U = \widehat{\psi}(P)$ for some $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$.*

PROOF. Let first $U \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ be an irreducible closed subset and, using Proposition 5.1.6, let $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ be the $\mathcal{S}$-filter such that $U = \widehat{\psi}(F)$. We show that F is a meet prime element of the lattice of $\mathcal{S}$-filters. Since irreducible closed subsets are non-empty, then F is proper. Let $F_1, F_2 \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ be such that $F_1 \cap F_2 \subseteq F$. By Proposition 5.1.5 we get $\widehat{\psi}(F) \subseteq \widehat{\psi}(F_1) \cup \widehat{\psi}(F_2)$. Now from $\widehat{\psi}(F)$ being irreducible closed subset, either $\widehat{\psi}(F) \subseteq \widehat{\psi}(F_1)$ or $\widehat{\psi}(F) \subseteq \widehat{\psi}(F_2)$, i. e. either $F_1 \subseteq F$ or $F_2 \subseteq F$. Hence, F is a meet prime element of the lattice of $\mathcal{S}$-filters, and so by filter distributivity of the logic, it is an irreducible $\mathcal{S}$-filter.

Let now $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. We show that $\widehat{\psi}(P)$ is an irreducible closed subset. Since $P \in \widehat{\psi}(P)$, then $\widehat{\psi}(P)$ is non-empty. Let V_1, V_2 be closed subsets of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$ such that $\widehat{\psi}(P) \subseteq V_1 \cup V_2$. Using Proposition 5.1.6, let $F_1, F_2 \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ be the $\mathcal{S}$-filters of $\mathbf{A}$ such that $V_1 = \widehat{\psi}(F_1)$ and $V_2 = \widehat{\psi}(F_2)$. Then we have $\widehat{\psi}(P) \subseteq \widehat{\psi}(F_1) \cup \widehat{\psi}(F_2)$. By Proposition 5.1.5 again, $F_1 \cap F_2 \subseteq P$, and since P is an irreducible $\mathcal{S}$-filter and the logic is filter distributive, then $F_1 \subseteq P$ or $F_2 \subseteq P$. Thus we obtain $\widehat{\psi}(P) \subseteq \widehat{\psi}(F_1) = V_1$ or $\widehat{\psi}(P) \subseteq \widehat{\psi}(F_2) = V_2$, as required. □

COROLLARY 5.1.8. *The space $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$ is sober.*

COROLLARY 5.1.9. *The dual of the specialization order of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$ coincides with the inclusion relation.*

PROOF. Let $P, Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. By Proposition 5.1.6 we have that $P \leq Q$ if and only if $Q \in \mathrm{cl}(P) = \widehat{\psi}(P)$ if and only if $P \subseteq Q$. □

From Theorem 5.1.1 and Corollary 5.1.9 we obtain that the dual of the specialization order of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$ coincides with the order associated with the referential algebra $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \psi[\mathbf{A}] \rangle$. Now we are ready to introduce the definition of Spectral-like dual objects of $\mathcal{S}$-algebras.

DEFINITION 5.1.10. A structure $\mathfrak{X} = \langle X, \mathbf{B} \rangle$ is an *$\mathcal{S}$-Spectral space* when:

(Sp1) $\langle X, \mathbf{B} \rangle$ is an $\mathcal{S}$-referential algebra,
(Sp2) for all $\mathcal{U} \cup \{V\} \subseteq^{\omega} B$, if $\bigcap \mathcal{U} \subseteq V$, then $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U})$,
(Sp3) $\kappa_{\mathfrak{X}} := \{U^c : U \in B\}$ is a basis of open compact subsets for a topology $\tau_{\kappa_{\mathfrak{X}}}$ on X,
(Sp4) the space $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ is sober.

We will see later on that the converse of (Sp2) follows from the other conditions. Recall that the quasiorder $\preceq$ associated with the referential algebra $\langle X, \mathbf{B} \rangle$ is given by: $x \preceq y$ if and only if for all $U \in B$, if $x \in U$ then $y \in U$. Moreover, the order $\leq$ associated with a sober topological space $\langle X, \tau \rangle$ (the dual of the specialization order of this space) is given by: $x \leq y$ if and only if $y \in \mathrm{cl}(x)$. From the definition we get that for any $\mathcal{S}$-Spectral space $\langle X, \mathbf{B} \rangle$, the quasiorder associated with the referential algebra and the order associated with the sober topological space $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ coincide. This implies, in particular, that the $\mathcal{S}$-referential algebra $\langle X, \mathbf{B} \rangle$ is reduced, and then, by Remark 4.2.1, we know that $\mathbf{B}$ is an $\mathcal{S}$-algebra. In this case, we denote the (quasi)order simply by $\leq_X$, and we drop the subscript when the context is clear.

Corollary 5.1.11. *For any filter distributive finitary congruential logic with theorems $\mathcal{S}$ and any $\mathcal{S}$-algebra $\mathbf{A}$, the structure $\mathfrak{Irr}_{\mathcal{S}}(\mathbf{A}) := \langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \psi_{\mathbf{A}}[\mathbf{A}]\rangle$ is an $\mathcal{S}$-Spectral space.*

Proof. Conditions (Sp1) and (Sp2) are stated in Theorem 5.1.1. Condition (Sp3) follows from Proposition 5.1.3 and condition (Sp4) follows from Corollary 5.1.8. □

Notice that for any $\mathcal{S}$-Spectral space $\mathfrak{X} = \langle X, \mathbf{B}\rangle$, since $\mathbf{B}$ is an $\mathcal{S}$-algebra by (Sp1), then $\mathfrak{Irr}_{\mathcal{S}}(\mathbf{B}) = \langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{B}), \psi_{\mathbf{B}}[\mathbf{B}]\rangle$ is an $\mathcal{S}$-Spectral space, for which the basis $\kappa_{\mathfrak{Irr}_{\mathcal{S}}(\mathbf{B})} = \{U^c : U \in \psi_{\mathbf{B}}[B]\}$ given by (Sp3) is precisely what we denote by $\kappa_{\mathbf{B}}$.

From now on, we focus on the structures $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ that satisfy conditions (Sp1)–(Sp3) in Definition 5.1.10. Let us call such structures *$\mathcal{S}$-pre-Spectral spaces.* For any $\mathcal{S}$-pre-Spectral space we define the map $\varepsilon_{\mathfrak{X}} : X \longrightarrow \mathcal{P}^{\uparrow}(B)$ as follows:

$$\varepsilon_{\mathfrak{X}}(x) := \{U \in B : x \in U\}.$$

And for any $Y \subseteq X$, we define:

$$\widehat{\varepsilon}_{\mathfrak{X}}(Y) := \bigcap\{\varepsilon_{\mathfrak{X}}(w) : w \in Y\} = \{U \in B : Y \subseteq U\}.$$

When the context is clear, we drop the subscript of $\varepsilon_{\mathfrak{X}}$ and $\widehat{\varepsilon}_{\mathfrak{X}}$.

Remark 5.1.12. By condition (Sp3), we obtain that for any $\mathcal{S}$-pre-Spectral space $\mathfrak{X}$, for all $x \in X$, it holds $\{U \in B : x \in U\} = \{U \in B : \mathrm{cl}(x) \subseteq U\}$. Therefore $\widehat{\varepsilon}(\mathrm{cl}(x)) = \varepsilon(x)$ for all $x \in X$.

Lemma 5.1.13. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-pre-Spectral space. If the topological space $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ is T_0, then ε is one-to-one.*

Proof. If the space $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ is T_0, then for any $x, y \in X$ such that $x \neq y$, there is $U \in B$ such that $x \in U^c$ and $y \notin U^c$. So $U \in \varepsilon(y)$ and $U \notin \varepsilon(x)$, and therefore $\varepsilon(x) \neq \varepsilon(y)$. Hence ε is one-to-one. □

Lemma 5.1.14. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-pre-Spectral space. Then for any closed subset Y of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$, $\widehat{\varepsilon}(Y) \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$, and moreover $\bigcap \widehat{\varepsilon}(Y) = Y$.*

Proof. Let $Y \subseteq X$ be a closed subset of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$, let $\Gamma \cup \{\delta\} \in Fm$ be such that $\Gamma \vdash_{\mathcal{S}} \delta$ and let $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{B})$ be such that $h(\gamma) \in \widehat{\varepsilon}(Y)$ for all $\gamma \in \Gamma$. Then by $\langle X, \mathbf{B}\rangle$ being an $\mathcal{S}$-referential algebra, we have $\bigcap\{h(\gamma) : \gamma \in \Gamma\} \subseteq h(\delta)$, and by assumption $Y \subseteq h(\gamma)$ for all $\gamma \in \Gamma$, so we get $Y \subseteq h(\delta)$, i. e. $h(\delta) \in \widehat{\varepsilon}(Y)$. This shows that $\widehat{\varepsilon}(Y)$ is an $\mathcal{S}$-filter of $\mathbf{B}$. Moreover, as Y is closed, by (Sp3) we get $\bigcap \widehat{\varepsilon}(Y) = \bigcap\{U \in B : Y \subseteq U\} = Y$. □

Remark 5.1.15. Notice that from Remark 5.1.12 and the previous lemma we obtain that for any $x \in X$, $\varepsilon(x)$ is an $\mathcal{S}$-filter. This implies that the converse of (Sp2) holds, i. e. for all $\mathcal{U} \cup \{V\} \subseteq^{\omega} B$, if $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U})$, then $\bigcap \mathcal{U} \subseteq V$. Assume $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U})$ and let $x \in \bigcap \mathcal{U}$, so $\mathcal{U} \subseteq \varepsilon(x)$. Since $\varepsilon(x)$ is an $\mathcal{S}$-filter, $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U}) \subseteq \varepsilon(x)$, and therefore by assumption $V \in \varepsilon(x)$, i. e. $x \in V$.

Corollary 5.1.16. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-pre-Spectral space. Then the order $\leq^{\mathbf{B}}_{\mathcal{S}}$ on B given by $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}$, coincides with the inclusion relation.*

PROOF. Let $U, V \in B$. By (Sp2) we get that $U \subseteq V$ implies $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U\})$, i.e. $U \leq^{\mathbf{B}}_{\mathcal{S}} V$. Let us show the converse. Suppose $U \leq^{\mathbf{B}}_{\mathcal{S}} V$, i.e. $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U\})$. We show that $U \subseteq V$. Let $x \in U$, then using Remark 5.1.12, $U \in \varepsilon(x) = \widehat{\varepsilon}(\mathrm{cl}(x))$, where $\widehat{\varepsilon}(\mathrm{cl}(x))$ is an $\mathcal{S}$-filter of $\mathbf{B}$ by Lemma 5.1.14, and so $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U\}) \subseteq \widehat{\varepsilon}(\mathrm{cl}(x)) = \varepsilon(x)$, hence $x \in V$. □

LEMMA 5.1.17. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-pre-Spectral space. Then for any irreducible closed subset Y of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$, $\widehat{\varepsilon}(Y) \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$. Hence $\varepsilon[X] \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$.*

PROOF. Let $Y \subseteq X$ be an irreducible closed subset of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$. By Lemma 5.1.14, $\widehat{\varepsilon}(Y)$ is an $\mathcal{S}$-filter of $\mathbf{B}$, so we just have to show that $\widehat{\varepsilon}(Y)$ is irreducible as an $\mathcal{S}$-filter. Since Y is non-empty, $\widehat{\varepsilon}(Y)$ is proper. Let $F_1, F_2 \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$ be such that $F_1 \cap F_2 \subseteq \widehat{\varepsilon}(Y)$ and suppose, towards a contradiction, that $F_1, F_2 \nsubseteq \widehat{\varepsilon}(Y)$. Let $U_1 \in F_1 \setminus \widehat{\varepsilon}(Y)$ and $U_2 \in F_2 \setminus \widehat{\varepsilon}(Y)$. Then $Y \nsubseteq U_1, U_2$, and since U_1, U_2 are closed subsets and Y is an irreducible closed subset, then $Y \nsubseteq U_1 \cup U_2$. Let $x \in Y \setminus U_1 \cup U_2$, so $x \in U_1^c \cap U_2^c$. By (Sp3) there is $V \in B$ such that $x \in V^c \subseteq U_1^c \cap U_2^c$. On the one hand, we have $x \notin V$, and therefore $Y \nsubseteq V$, i.e. $V \notin \widehat{\varepsilon}(Y)$. On the other hand, from $U_1, U_2 \subseteq V$, using Corollary 5.1.16 we get $V \in F_1 \cap F_2$. Since by assumption $F_1 \cap F_2 \subseteq \widehat{\varepsilon}(Y)$, we obtain $V \in \widehat{\varepsilon}(Y)$, a contradiction.

This shows that $\widehat{\varepsilon}(Y)$ is an irreducible $\mathcal{S}$-filter of $\mathbf{B}$ for any irreducible closed subset Y of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$. In particular, this holds for $\mathrm{cl}(x)$, for every $x \in X$. Then by Remark 5.1.12 we obtain $\varepsilon(x) = \widehat{\varepsilon}(\mathrm{cl}(x)) \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$. □

LEMMA 5.1.18. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-pre-Spectral space. Then:*

(1) *For any $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$, $\bigcap F = \bigcap\{V \in B : V \in F\}$ is a closed subset of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$, and moreover $\widehat{\varepsilon}(\bigcap F) = F$.*

(2) *For any $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$, $\bigcap P$ is an irreducible closed subset of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$.*

PROOF. (1) Let $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$. Clearly $\bigcap F$ is closed. Consider the set $\widehat{\varepsilon}(\bigcap F) = \{U \in B : \bigcap F \subseteq U\}$. It is immediate that $F \subseteq \widehat{\varepsilon}(\bigcap F)$, so we just have to show the other inclusion. If $F = \emptyset$, then $\widehat{\varepsilon}(\bigcap F) = \{U \in B : X \subseteq U\}$. From (Sp2) we know that for any $V \in B$, if $V = X$, then $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\emptyset)$, and so $V \in G$ for all $G \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$. Therefore, from $F = \emptyset$ we conclude that $\widehat{\varepsilon}(\bigcap F) = \emptyset$. If $F \neq \emptyset$, let $U \in \widehat{\varepsilon}(\bigcap F)$, i.e. $\bigcap F \subseteq U \in B$. Then $U^c \subseteq \bigcup\{V^c : V \in F\}$, and by (Sp3) U^c is compact, so there is $F' \subseteq^{\omega} F$, such that $U^c \subseteq \bigcup\{V^c : V \in F'\}$, i.e. $\bigcap F' \subseteq U$. Thus by (Sp2) $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(F') \subseteq F$, as required.

(2) Let $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$. Notice that as P is proper, $B \neq P = \widehat{\varepsilon}(\bigcap P)$, and therefore $\bigcap P \neq \emptyset$. Let C_1, C_2 be closed subsets of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ such that $\bigcap P \subseteq C_1 \cup C_2$. By Lemma 5.1.14 we have $\widehat{\varepsilon}(C_1), \widehat{\varepsilon}(C_2) \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$, $C_1 = \bigcap \widehat{\varepsilon}(C_1)$ and $C_2 = \bigcap \widehat{\varepsilon}(C_2)$. Suppose, towards a contradiction, that $\bigcap P \nsubseteq C_1$ and $\bigcap P \nsubseteq C_2$. Then $\widehat{\varepsilon}(C_1) \nsubseteq P$ and $\widehat{\varepsilon}(C_2) \nsubseteq P$. Now since P is an irreducible $\mathcal{S}$-filter, we obtain that $\widehat{\varepsilon}(C_1 \cup C_2) = \widehat{\varepsilon}(C_1) \cap \widehat{\varepsilon}(C_2) \nsubseteq P$, and therefore $\bigcap P \nsubseteq C_1 \cup C_2$, a contradiction. □

From lemmas 5.1.14, 5.1.17 and 5.1.18 we get that for any $\mathcal{S}$-pre-Spectral space $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ there is an order isomorphism between closed subsets of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ and $\mathcal{S}$-filters of $\mathbf{B}$ given by the maps:

$$\begin{aligned} \widehat{\varepsilon} : \mathcal{C}(X) &\longrightarrow \mathrm{Fi}_{\mathcal{S}}(\mathbf{B}) & \qquad \bigcap : \mathrm{Fi}_{\mathcal{S}}(\mathbf{B}) &\longrightarrow \mathcal{C}(X) \\ Y &\longmapsto \{U \in B : Y \subseteq U\} & F &\longmapsto \bigcap F \end{aligned}$$

that restricts to an order isomorphism between irreducible closed subsets of $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ and irreducible $\mathcal{S}$-filters of $\mathbf{B}$.

Corollary 5.1.19. *Let $\mathfrak{X} = \langle X, \mathbf{B} \rangle$ be an $\mathcal{S}$-pre-Spectral space and let $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$ and $U \in B$. Then $U \in F$ if and only if $\bigcap F \subseteq U$.*

Proof. The inclusion from left to right is immediate, so let us show the other inclusion. Assume that $\bigcap F \subseteq U$ for some $U \in B$. Notice that this implies that $F \neq \emptyset$: otherwise, we have $U = X$, and then by (Sp2), U belongs to all $\mathcal{S}$-filters of $\mathbf{B}$, in particular, $U \in F = \emptyset$, a contradiction. Then we have, using Lemma 5.1.18, that $\widehat{\varepsilon}(U) \subseteq \widehat{\varepsilon}(\bigcap F) = F$. And since $U \in \widehat{\varepsilon}(U)$, we conclude $U \in F$. ☐

Theorem 5.1.20. *Let $\mathfrak{X} = \langle X, \mathbf{B} \rangle$ be an $\mathcal{S}$-pre-Spectral space. Then the following conditions are equivalent:*

(1) *$\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ is T_0, and for every closed subset Y of X and every non-empty and down-directed $\mathcal{Q} \subseteq \{U^c : U \in B\}$, if $Y \cap V^c \neq \emptyset$ for all $V^c \in \mathcal{Q}$, then $Y \cap \bigcap\{V^c : V^c \in \mathcal{Q}\} \neq \emptyset$.*
(2) *$\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ is T_0 and the map $\varepsilon : X \longrightarrow \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$ is onto.*
(3) *$\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ is sober.*

Proof. Recall that any sober space is T_0.

(1) implies (2). Let $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$. We show that there is $x \in X$ such that $\varepsilon(x) = P$. Since P is an irreducible $\mathcal{S}$-filter, by Theorem 4.4.8 we have $P^c \in \mathrm{Id}(\mathbf{B})$, and so $\mathcal{Q}' := \{V^c : V \notin P\}$ is down-directed. Moreover, since P is proper, $\mathcal{Q}'$ is non-empty. If $P = \emptyset$, then we get that for all $U \in B$, $U \neq X$, because otherwise by (Sp2) we obtain $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(P) = P = \emptyset$. Then we have $\bigcap P \cap U^c = X \cap U^c \neq \emptyset$ for all $U \notin P$. If $P \neq \emptyset$, then using Corollary 5.1.19, we obtain that for the closed subset $\bigcap P$, $\bigcap P \cap V^c \neq \emptyset$ whenever $V \notin P$. Then in any case, by (1) there is $x \in \bigcap P \cap \bigcap\{V^c : V \notin P\} \neq \emptyset$, and clearly $\varepsilon(x) = P$.

(2) implies (3). Let Y be an irreducible closed subset of $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$. We show that there is $x \in X$ such that $Y = \mathrm{cl}(x)$. By Lemma 5.1.17, $\widehat{\varepsilon}(Y) \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$. Then by (2) there is a $x \in X$ such that $\varepsilon(x) = \widehat{\varepsilon}(Y)$. By Lemma 5.1.13 ε is one-to-one, so such x is unique. Moreover, as $\varepsilon(x) = \widehat{\varepsilon}(\mathrm{cl}(x))$, and using Lemma 5.1.18 we obtain $\mathrm{cl}(x) = \bigcap \widehat{\varepsilon}(\mathrm{cl}(x)) = \bigcap \widehat{\varepsilon}(Y) = Y$, as required.

(3) implies (1). Let $Y \subseteq X$ be a closed subset of $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ and $\mathcal{Q} \subseteq \{U^c : U \in B\}$ be non-empty and down-directed, and such that $Y \cap V^c \neq \emptyset$ for all $V^c \in \mathcal{Q}$. We show that $Y \cap \bigcap\{V^c : V^c \in \mathcal{Q}\} \neq \emptyset$. As Y is closed, then $\widehat{\varepsilon}(Y) \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. As $\mathcal{Q}$ is down-directed and non-empty, then $\downarrow\{V : V^c \in \mathcal{Q}\}$ is a poset ideal of $\langle B, \leq^{\mathbf{B}}_{\mathcal{S}} \rangle$. We claim that $\widehat{\varepsilon}(Y) \cap \downarrow\{V : V^c \in \mathcal{Q}\} = \emptyset$. Suppose, towards a contradiction, that there is $U \in \widehat{\varepsilon}(Y) \cap \downarrow\{V : V^c \in \mathcal{Q}\}$. Then there is $V^c \in \mathcal{Q}$ such that $U \subseteq V$ and $U \in \widehat{\varepsilon}(Y)$. Thus $Y \subseteq U \subseteq V$, so $Y \cap V^c = \emptyset$, contrary to the assumption.

Then we have $\widehat{\varepsilon}(Y)$, an $\mathcal{S}$-filter of $\mathbf{B}$, and $\downarrow\{V : V^c \in \mathcal{Q}\}$, a poset ideal of $\langle B, \leq^{\mathbf{B}}_{\mathcal{S}} \rangle$, such that $\widehat{\varepsilon}(Y) \cap \downarrow\{V : V^c \in \mathcal{Q}\} = \emptyset$. By the Lemma 4.4.1, there is $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$ such that $\widehat{\varepsilon}(Y) \subseteq P$ and $\downarrow\{V : V^c \in \mathcal{Q}\} \cap P = \emptyset$. Then consider the irreducible closed subset $\bigcap P$. By (3) there is $x \in X$ such that $\bigcap P = \mathrm{cl}(x)$. Since $\widehat{\varepsilon}(Y) \subseteq P$, then $\mathrm{cl}(x) = \bigcap P \subseteq \bigcap \widehat{\varepsilon}(Y) = Y$, so in particular $x \in Y$. Suppose, towards a contradiction, that $x \notin \bigcap\{V^c : V^c \in \mathcal{Q}\}$. Then there is $V^c \in \mathcal{Q}$ such that $x \notin V^c$. So $\bigcap P = \mathrm{cl}(x) \subseteq V$, and then $V^c \subseteq \bigcup\{U^c : U \in P\}$. Since V^c is compact, there is $P' \subseteq^\omega P$ such that $V^c \subseteq \bigcup\{U^c : U \in P'\}$. Thus $\bigcap P' \subseteq V$, and by condition (Sp2), we get $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(P') \subseteq P$. But then ${\downarrow}\{V : V^c \in \mathcal{Q}\} \cap P \neq \emptyset$, a contradiction. We conclude that $x \in Y \cap \bigcap\{V^c : V^c \in \mathcal{Q}\} \neq \emptyset$. □

COROLLARY 5.1.21. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-Spectral space. Then ε is a homeomorphism between the topological spaces $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ and $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{B}), \tau_{\kappa_{\mathbf{B}}}\rangle$.*

PROOF. Notice that for all $x \in X$ and all $U \in B$ we have: $x \in U$ if and only if $U \in \varepsilon(x)$ if and only if $\varepsilon(x) \in \psi_{\mathbf{B}}(U)$. Therefore, we have:

$$x \in \varepsilon^{-1}[\psi_{\mathbf{B}}(U)^c] \quad \text{iff} \quad \varepsilon(x) \in \psi_{\mathbf{B}}(U)^c \quad \text{iff} \quad U \notin \varepsilon(x) \quad \text{iff} \quad x \in U^c.$$

Thus $\varepsilon^{-1}[\psi_{\mathbf{B}}(U)^c] = U^c$ for all $U \in B$. Recall that $\kappa_{\mathbf{B}} = \{\psi_{\mathbf{B}}(U)^c : U \in B\}$ is a basis for the topology $\tau_{\kappa_{\mathbf{B}}}$ on $\mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$. Then we have that the inverse image by ε of any element of the basis $\kappa_{\mathbf{B}}$ belongs to $\kappa_{\mathfrak{X}} = \{U^c : U \in B\}$, that is a basis for $\tau_{\kappa_{\mathfrak{X}}}$. Moreover, by Lemma 5.1.13 and Theorem 5.1.20, ε is a one-to-one map onto $\mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$. We conclude that ε is a homeomorphism between $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ and $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{B}), \tau_{\kappa_{\mathbf{B}}}\rangle$. □

COROLLARY 5.1.22. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-Spectral space. Then the structure $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{B}), \psi_{\mathbf{B}}[\mathbf{B}]\rangle$ is an $\mathcal{S}$-Spectral space such that $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ and $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{B}), \tau_{\kappa_{\mathbf{B}}}\rangle$ are homeomorphic topological spaces by means of the map $\varepsilon_{\mathfrak{X}} : X \longrightarrow \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$ and moreover $\mathbf{B}$ and $\psi_{\mathbf{B}}[\mathbf{B}]$ are isomorphic $\mathcal{S}$-algebras by means of the map $\psi_{\mathbf{B}} : B \longrightarrow \psi_{\mathbf{B}}[B]$.*

Previous corollary together with Corollary 5.1.11 summarize all preceding results, and should be kept in mind for § 5.2 and § 5.3, where the duality for morphisms is studied, and the functors involved are defined. Before moving to that, let us examine Priestley-dual objects of $\mathcal{S}$-algebras.

5.1.2. Priestley-style dual objects. We assume that $\mathcal{S}$ is a filter distributive finitary congruential logic with theorems and $\mathbf{A}$ is an $\mathcal{S}$-algebra. Recall that the map $\vartheta : A \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Op}_{\mathcal{S}}(\mathbf{A}))$ assigns to each $a \in A$, the collection $\{P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A}) : a \in P\}$. We define on $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ a topology $\tau_{\mathbf{A}}$, having as subbasis the collection:

$$\{\vartheta(a) : a \in A\} \cup \{\vartheta(b)^c : b \in A\}.$$

REMARK 5.1.23. Notice that for any non-empty $B \subseteq^\omega A$ we have $\widehat{\vartheta}(B) \neq \{A\}$. Suppose, towards a contradiction, that there is a non-empty $B \subseteq^\omega A$ such that $\widehat{\vartheta}(B) = \{A\}$. This implies that A is an optimal $\mathcal{S}$-filter of $\mathbf{A}$, and therefore, by Theorem 4.4.9 and Remark 4.4.14, $\mathbf{A}$ has no bottom-family. Then $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \neq A$ and so there is $a \notin \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B)$. Then by Corollary 4.4.6 there is $P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ such that $a \notin P$ and $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \subseteq P$. In particular, $B \subseteq P$, so $P \in \widehat{\vartheta}(B) = \{A\}$, but $a \notin P$, a contradiction.

Recall that $\mathrm{M}(\mathbf{A})$ denotes the $\mathcal{S}$-semilattice of $\mathbf{A}$. For the purposes of this section, as $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ is an optimal $\mathcal{S}$-base, we assume that $\mathrm{M}(\mathbf{A})$ is the closure under non-empty finite intersections of $\vartheta[A]$. Recall that by Property (E2), any element of $\mathrm{M}(\mathbf{A})$ has the form $\widehat{\vartheta}(B)$ for some non-empty $B \subseteq^\omega A$. Moreover there is an order isomorphism between optimal $\mathcal{S}$-filters of $\mathbf{A}$ and optimal meet filters of $\mathrm{M}(\mathbf{A})$. Recall also that $\mathrm{Id}_{\mathcal{S}}(\mathbf{A})$ is always a closure system, but this is not the case for $\mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$, so we do not have a closure operator that generates the least strong $\mathcal{S}$-ideal containing a given subset. Through the mentioned correspondence between $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ and $\mathrm{Op}_{\wedge}(\mathrm{M}(\mathbf{A}))$, we can avoid the difficulties that this fact brings us by moving our proof-strategies to the $\mathcal{S}$-semilattice of $\mathbf{A}$. This is precisely what we do in the next proposition, where we show that the space $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}} \rangle$ is a Priestley space.

Proposition 5.1.24. *The structure* $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}} \rangle$ *is a Priestley space.*

Proof. *Priestley Separation Axiom.* Let $P, P' \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ be such that $P \nsubseteq P'$. So there is $a \in P \setminus P'$. Then $P \in \vartheta(a)$ and $P' \notin \vartheta(a)$, and we are done.

Compactness. We use Alexandrov Subbasis Theorem, so let $B \cup D \subseteq A$ be such that $\mathrm{Op}_{\mathcal{S}}(\mathbf{A}) \subseteq \bigcup\{\vartheta(b) : b \in B\} \cup \bigcup\{\vartheta(d)^c : d \in D\}$ and suppose, towards a contradiction, that $\mathrm{Op}_{\mathcal{S}}(\mathbf{A}) \nsubseteq \{\vartheta(b) : b \in B'\} \cup \bigcup\{\vartheta(d)^c : d \in D'\}$ for any finite $B' \subseteq^{\omega} B$ and $D' \subseteq^{\omega} D$. Without loss of generality, we can assume $D \neq \emptyset$, since $\vartheta(1) = \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$. Let F be the filter of $\mathrm{M}(\mathbf{A})$ generated by $\vartheta[D]$, i. e. $F := \llbracket \vartheta[D] \rangle\!\rangle = {\uparrow}\{\widehat{\vartheta}(D') : D' \subseteq^{\omega} D\}$. Let I be the F-ideal of $\mathrm{M}(\mathbf{A})$ generated by $\vartheta[B]$, i. e.

$$I := \langle\!\langle \vartheta[B] \rrbracket = \{\widehat{\vartheta}(C) \in \mathrm{M}_{\mathcal{F}}(A) : \exists B' \subseteq^{\omega} B \big(\bigcap\{{\uparrow}\vartheta(b) : b \in B'\} \subseteq {\uparrow}\widehat{\vartheta}(C) \big)\}.$$

We claim that $F \cap I = \emptyset$. Suppose, towards a contradiction, that $F \cap I \neq \emptyset$, so let $C \subseteq^{\omega} A$ such that $\widehat{\vartheta}(C) \in F \cap I$. On the one hand, since $D \neq \emptyset$, there is non-empty $D' \subseteq^{\omega} D$ such that $\widehat{\vartheta}(D') \subseteq \widehat{\vartheta}(C)$, and so ${\uparrow}\widehat{\vartheta}(C) \subseteq {\uparrow}\widehat{\vartheta}(D')$. On the other hand, there is $B' \subseteq^{\omega} B$ such that $\bigcap\{{\uparrow}\vartheta(b) : b \in B'\} \subseteq {\uparrow}\widehat{\vartheta}(C)$. Hence we have $\bigcap\{{\uparrow}\vartheta(b) : b \in B'\} \subseteq {\uparrow}\widehat{\vartheta}(D')$. If $B' = \emptyset$, then ${\uparrow}\widehat{\vartheta}(D') = \mathrm{M}_{\mathcal{F}}(A)$, so $\widehat{\vartheta}(D')$ is the bottom element of $\mathrm{M}(\mathbf{A})$, and so there is no optimal meet filter of $\mathrm{M}(\mathbf{A})$ containing $\widehat{\vartheta}(D')$. Using the isomorphism between optimal meet filters of $\mathrm{M}(\mathbf{A})$ and optimal $\mathcal{S}$-filters of $\mathbf{A}$ given by Proposition 4.5.13, we conclude that this implies that no optimal $\mathcal{S}$-filter of $\mathbf{A}$ includes D': if $G \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ is such that $D' \subseteq G$, then $\widehat{\vartheta}(D')$ would be an element of the optimal meet filter $\llbracket \vartheta[G] \rangle\!\rangle \in \mathrm{Fi}_{\wedge}(\mathrm{M}(\mathbf{A}))$. Therefore, we have that $\bigcup\{\vartheta(d)^c : d \in D'\}$ is a finite cover of the space, a contradiction. If $B' \neq \emptyset$. Then by Lemma 4.5.10, $\widehat{\vartheta}(D') \subseteq \bigcup\{\vartheta(b) : b \in B'\}$, so $\bigcup\{\vartheta(d)^c : d \in D'\} \cup \bigcup\{\vartheta(b) : b \in B'\}$ is a finite cover of the space, a contradiction.

We conclude that $F \cap I = \emptyset$. Then by Lemma 2.3.7 there is an optimal meet filter $P \in \mathrm{Op}_{\wedge}(\mathrm{M}(\mathbf{A}))$ such that $F \subseteq P$ and $I \cap P = \emptyset$. Then by the isomorphism between optimal meet filters of $\mathrm{M}(\mathbf{A})$ and optimal $\mathcal{S}$-filters of $\mathbf{A}$ given by Proposition 4.5.13, $\vartheta^{-1}[P]$ is an optimal $\mathcal{S}$-filter of $\mathbf{A}$. Then from $F \subseteq P$ we get $D \subseteq \vartheta^{-1}[P]$, and from $I \cap P = \emptyset$ we get $b \notin \vartheta^{-1}[P]$ for all $b \in B$. Therefore $P \notin \bigcup\{\vartheta(b) : b \in B\} \cup \bigcup\{\vartheta(d)^c : d \in D\}$, a contradiction. ☐

For the proof of the next proposition, we use that $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ is also an optimal $\mathcal{S}$-base. Recall that we denote by ψ the map that assigns to each $a \in A$, the collection $\{P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}) : a \in P\}$.

Proposition 5.1.25. *The collection* $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ *is dense in the space* $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}} \rangle$.

Proof. We show that each non-empty basic open contains an irreducible $\mathcal{S}$-filter. Let $B, D \subseteq^{\omega} A$ and suppose $P \in \bigcap\{\vartheta(b) : b \in B\} \cap \bigcap\{\vartheta(d)^c : d \in D\} \neq \emptyset$. Without loss of generality, we can assume that $B \neq \emptyset$, since $\vartheta(1) = \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$. If $D = \emptyset$, then $P \in \widehat{\vartheta}(B) \neq \emptyset$, and by Remark 5.1.23 we can assume that $P \neq A$, so there is $a \notin P$. Then by Corollary 4.4.2 there is an irreducible $\mathcal{S}$-filter $Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $a \notin Q \supseteq P$, so $Q \in \bigcap\{\vartheta(b) : b \in B\} \cap \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, as required. If $D \neq \emptyset$, then we have $\widehat{\vartheta}(B) \nsubseteq \bigcup\{\vartheta(d) : d \in D\}$, and so by Corollary 4.5.12 we get $\widehat{\psi}(B) \nsubseteq \bigcup\{\psi(d) : d \in D\}$. Thus there is $Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $B \subseteq Q$ and $d \notin Q$ for all $d \in D$, i.e. $Q \in \bigcap\{\vartheta(b) : b \in B\} \cap \bigcap\{\vartheta(d)^c : d \in D\} \cap \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, as required. □

REMARK 5.1.26. Notice that from Theorem 4.4.8 we get that for any optimal $\mathcal{S}$-filter P of $\mathbf{A}$, P is irreducible if and only if $\{a \in A : a \notin P\}$ is non-empty and up-directed in $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle$. Moreover, from Theorem 5.1.2, ϑ is an isomorphism between $\mathbf{A}$ and $\vartheta[\mathbf{A}]$. Therefore $\{a \in A : a \notin P\}$ is non-empty and up-directed in $\langle A, \leq^{\mathbf{A}}_{\mathcal{S}}\rangle$ if and only if $\{\vartheta(a) : a \notin P\}$ is also non-empty and up-directed in $\langle \vartheta[A], \subseteq\rangle$. Hence, for any $P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$

$$P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}) \quad \text{iff} \quad \{\vartheta(a) : a \in P\} \text{ is non-empty and up-directed in } \langle \vartheta[A], \subseteq\rangle.$$

Finally we prove some facts concerning clopen up-sets of the Priestley-space $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}}\rangle$. Recall that in Priestley duality for bounded distributive lattices, the collection of clopen up-sets takes a prominent role. In Priestley duality for distributive semilattices, such a role is taken by the collection of admissible clopen up-sets (see definition in page 38). In what follows we see that in the present duality admissible clopen up-sets play an important role as well.

PROPOSITION 5.1.27. *Each non-empty open up-set of* $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}}\rangle$ *is a non-empty union of non-empty finite intersections of elements of* $\vartheta[A]$.

PROOF. Let U be a non-empty open up-set of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}}\rangle$ and let $P \in U$. It is enough to show that there are $a_0, \dots, a_n \in A$, for some $n \in \omega$, such that $P \in \vartheta(a_0) \cap \dots \cap \vartheta(a_n) \subseteq U$. If $U = \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ we are done, since $\mathrm{Op}_{\mathcal{S}}(\mathbf{A}) = \vartheta(1^{\mathbf{A}})$. So suppose $U \neq \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$. Then for each $Q \notin U$, $P \nsubseteq Q$, so there is $a_Q \in P \setminus Q$. Notice that in case A is an $\mathcal{S}$-optimal filter, then $A \notin U^c$, since U is an up-set. Then we have $\bigcap\{\vartheta(a_Q) : Q \notin U\} \cap U^c = \emptyset$, and by compactness of the space we are done. □

PROPOSITION 5.1.28. *Each non-empty clopen up-set of* $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}}\rangle$ *is a non-empty finite union of non-empty finite intersections of elements of* $\vartheta[A]$.

PROOF. This follows from the previous proposition and the fact that in any Priestley space clopen up-sets are compact. □

Note that the emptyset is a clopen up-set of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}}\rangle$, and it can be described as an (empty) finite union of non-empty finite intersections of elements of $\vartheta[A]$. When $\mathbf{A}$ has a bottom element, the emptyset is moreover a non-empty finite union of non-empty finite intersections of elements of $\vartheta[A]$, since optimal $\mathcal{S}$-filters are proper, we have $\emptyset = \vartheta(0^{\mathbf{A}})$. Recall that for any poset P, by $\max(P)$ we denote the collection of maximal elements of P. And $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$-admissible clopen up-sets of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}}\rangle$ are the clopen up-sets $U \subseteq \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ such that $\max(U^c) \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$.

PROPOSITION 5.1.29. *For any clopen up-set* U *of* $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}}\rangle$, *if* $U = \widehat{\vartheta}(B)$ *for some non-empty* $B \subseteq^{\omega} A$, *then* $\max(U^c) \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$.

Proof. Let U be a clopen up-set such that $\widehat{\vartheta}(B) = U$ for some non-empty $B \subseteq^{\omega} A$. If $U = \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$, then we are done, since $\max(U^c) = \emptyset$ and this set is trivially included in $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. Suppose $U \neq \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ and let $P \in \max(U^c) = \max(\widehat{\vartheta}(B)^c)$. We show that P is an irreducible $\mathcal{S}$-filter. By assumption $B \nsubseteq P$, so there is $b \in B \setminus P$. Then by Lemma 4.4.1, there is $Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$ such that $b \notin Q$ and $P \subseteq Q$. This implies $B \nsubseteq Q$, so we have $Q \in \widehat{\vartheta}(B)^c = U^c$ and $P \subseteq Q$. Since P is a maximal element of U^c, we conclude $P = Q$, i. e. P is an irreducible $\mathcal{S}$-filter, as required. $\square$

The converse of the previous proposition also holds. Notice that we move again to the $\mathcal{S}$-semilattice of $\mathbf{A}$ to prove it.

Proposition 5.1.30. *For any clopen up-set U of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \subseteq, \tau_{\mathbf{A}} \rangle$, whenever $\max(U^c) \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, then there is a non-empty $B \subseteq^{\omega} A$ such that $U = \widehat{\vartheta}(B)$.*

Proof. Let U be a clopen up-set such that $\max(U^c) \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. Let us consider first the case when $U = \emptyset$. Then since irreducible $\mathcal{S}$-filters are proper, A is not an optimal $\mathcal{S}$-filter, otherwise we would have $A \in \max(\mathrm{Op}_{\mathcal{S}}(\mathbf{A})) = \max(U^c) \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. Then $\mathbf{A}$ has a bottom-family $D \subseteq^{\omega} A$, and thus $\widehat{\vartheta}(D) = \emptyset = U$, so we are done.

Let now U be non-empty. Then by Proposition 5.1.28, there are non-empty $B_0, \dots, B_n \subseteq^{\omega} A$, for some $n \in \omega$, such that

$$U = \bigcup \{\widehat{\vartheta}(B_i) : i \leq n\}.$$

On the one hand, consider the set $F := \bigcap \{[\![\widehat{\vartheta}(B_i)\rangle\!\rangle : i \leq n\}$, that is a meet filter of the $\mathcal{S}$-semilattice of $\mathbf{A}$. On the other hand, consider the set $J := \langle\!\langle \{\widehat{\vartheta}(B_i) : i \leq n\}]\!]$, that is a Frink ideal of the $\mathcal{S}$-semilattice of $\mathbf{A}$.

We claim that $F \cap J \neq \emptyset$. Suppose not, then by Lemma 2.3.7, there is $P \in \mathrm{Op}_{\wedge}(\mathrm{M}(\mathbf{A}))$ an optimal meet filter of the $\mathcal{S}$-semilattice of $\mathbf{A}$, such that $F \subseteq P$ and $P \cap J = \emptyset$. Then by definition of J, $\widehat{\vartheta}(B_i) \notin P$ for all $i \leq n$. Therefore for each $i \leq n$, there is $b_i \in B_i$ such that $\vartheta(b_i) \notin P$, i. e. $b_i \notin \vartheta^{-1}[P]$. Then $B_i \nsubseteq \vartheta^{-1}[P]$ for all $i \leq n$. Recall that by the isomorphism between optimal $\mathcal{S}$-filters of $\mathbf{A}$ and optimal meet filters of $\mathrm{M}(\mathbf{A})$ (Proposition 4.5.13), $\vartheta^{-1}[P] \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ is an optimal $\mathcal{S}$-filter of $\mathbf{A}$. Thus we have $\vartheta^{-1}[P] \notin \bigcup \{\widehat{\vartheta}(B_i) : i \leq n\} = U$. Let $Q \in \max(U^c)$ be such that $\vartheta^{-1}[P] \subseteq Q$. This implies that $P = [\![\vartheta[\vartheta^{-1}[P]]\rangle\!\rangle \subseteq [\![\vartheta[Q]\rangle\!\rangle$. By assumption $Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, so by Proposition 4.5.13 again, $[\![\vartheta[Q]\rangle\!\rangle$ is an irreducible meet filter of $\mathrm{M}(\mathbf{A})$. Then from $F \subseteq P$ we get $F \subseteq [\![\vartheta[Q]\rangle\!\rangle$. Since $[\![\vartheta[Q]\rangle\!\rangle$ is irreducible, by definition of F we get $\widehat{\vartheta}(B_i) \in [\![\vartheta[Q]\rangle\!\rangle$ for some $i \leq n$. By definition of meet filter generated either $\widehat{\vartheta}(B_i) = \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ or there is non-empty $Q' \subseteq^{\omega} Q$ such that $\widehat{\vartheta}(Q') \subseteq \widehat{\vartheta}(B_i)$. If $\widehat{\vartheta}(B_i) = \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$, then $B_i = \{1^{\mathbf{A}}\}$, and clearly $B_i \subseteq Q$. If there is non-empty $Q' \subseteq^{\omega} Q$ such that $\widehat{\vartheta}(Q') \subseteq \widehat{\vartheta}(B_i)$, then by Lemma 4.5.10, $B_i \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(Q') \subseteq Q$. So in either case we get $B_i \subseteq Q$, and this implies $Q \in \widehat{\vartheta}(B_i) \subseteq U$, a contradiction.

We conclude that $F \cap J \neq \emptyset$. Then by (E2), let a non-empty $B \subseteq^{\omega} A$ be such that $\widehat{\vartheta}(B) \in F \cap J$. On the one hand, we have $\widehat{\vartheta}(B) \in [\![\widehat{\vartheta}(B_i)\rangle\!\rangle$ for all $i \leq n$, so $\widehat{\vartheta}(B_i) \subseteq \widehat{\vartheta}(B)$ for all $i \leq n$. On the other hand, by definition of Frink ideal generated by a subset, $\bigcap\{[\![\widehat{\vartheta}(B_i)\rangle\!\rangle : i \leq n\} \subseteq [\![\widehat{\vartheta}(B)\rangle\!\rangle$, and so by Lemma 4.5.10 we obtain $\widehat{\vartheta}(B) \subseteq \bigcup\{\widehat{\vartheta}(B_i) : i \leq n\}$. We conclude that $U = \widehat{\vartheta}(B)$ as required. □

Now we are ready to introduce the definition of Priestley-dual objects of $\mathcal{S}$-algebras.

DEFINITION 5.1.31. A structure $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$ is an *$\mathcal{S}$-Priestley space* when:

(Pr1) $\langle X, \mathbf{B}\rangle$ is a reduced $\mathcal{S}$-referential algebra, whose associated order is denoted by $\leq$,
(Pr2) for all $\mathcal{U} \cup \{V\} \subseteq^{\omega} B$, $\bigcap \mathcal{U} \subseteq V$ iff $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U})$,
(Pr3) $\langle X, \tau\rangle$ is a compact space,
(Pr4) B is a family of clopen up-sets for $\langle X, \tau, \leq\rangle$ that contains X,
(Pr5) the set $X_B := \{x \in X : \{U \in B : x \notin U\}$ is non-empty and up-directed$\}$ is dense in $\langle X, \tau\rangle$.

From now on let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. By conditions (Pr1), (Pr3) and (Pr4) we obtain that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, the space $\langle X, \tau, \leq\rangle$ is a Priestley space, and by condition (Pr2) we obtain that for all $U, V \in B$:

$$U \subseteq V \quad \text{iff} \quad V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U).$$

Therefore, the order $\leq^{\mathbf{B}}_{\mathcal{S}}$ on B coincides with the inclusion relation on B. Moreover, concerning the bottom element and bottom-families, we have the following lemma, that is used later on:

LEMMA 5.1.32. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space.*

(1) $\mathbf{B}$ *has a bottom element if and only if $\emptyset \in B$.*
(2) $\mathbf{B}$ *has a bottom-family if and only if there is $D \subseteq^{\omega} B$ such that $\bigcap D = \emptyset$.*

PROOF. (1) Clearly if $\emptyset \in B$, then $\emptyset$ is the bottom element of $\mathbf{B}$. For the converse, assume that U is the bottom element of $\mathbf{B}$, and suppose, towards a contradiction, that there is $x \in U \cap X_B$. Then by condition (Pr5), there is $V \in B$ such that $x \notin V$, but since U is the bottom element, then $U \subseteq V$. This implies $x \in V$, a contradiction. Hence we obtain that $U \cap X_B = \emptyset$, and then from denseness given by (Pr5), $U = \emptyset$ as required.

For (2), by (Pr2) it follows the implication from right to left. For the converse, assume that $\mathbf{B}$ has a bottom-family $D \subseteq^{\omega} B$, and suppose, towards a contradiction, that there is $x \in \bigcap D$. Since D is finite, by denseness we can assume, without loss of generality, that $x \in X_B$. Then by condition (Pr5) there is $V \in B$ such that $x \notin V$. But by assumption there is $U \in D$ such that $U \subseteq V$, and from $x \in U$ it follows $x \in V$, a contradiction. □

COROLLARY 5.1.33. *For any filter distributive finitary congruential logic with theorems $\mathcal{S}$ and any $\mathcal{S}$-algebra $\mathbf{A}$, the structure $\mathfrak{Op}_{\mathcal{S}}(\mathbf{A}) := \langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \vartheta[\mathbf{A}]\rangle$ is an $\mathcal{S}$-Priestley space.*

PROOF. Conditions (Pr1) and (Pr2) are stated in Theorem 5.1.2. Condition (Pr3) was shown in Proposition 5.1.24. Condition (Pr4) follows from the definition of $\tau_{\mathbf{A}}$, and from $\vartheta(1^{\mathbf{A}}) = \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$. Condition (Pr5) follows from Remark 5.1.26 and Proposition 5.1.25. □

Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Note that since $\mathbf{B}$ is an algebra of subsets of X, this provides us with an alternative characterization of the $\mathcal{S}$-semilattice of $\mathbf{B}$: let $B^{\cap}$ be the closure of B under non-empty finite intersections. Then clearly $\mathbf{B}^{\cap} := \langle B^{\cap}, \cap, X\rangle$ is isomorphic to $\mathrm{M}(\mathbf{B})$, the $\mathcal{S}$-semilattice of $\mathbf{B}$. Let $B^{\cap\cup}$ be the closure of $B^{\cap}$ under non-empty finite unions. It also follows that $\mathbf{B}^{\cap\cup} := \langle B^{\cap\cup}, \cap, \cup, X\rangle$ is isomorphic to $\mathrm{L}(\mathrm{M}(\mathbf{B}))$, the distributive envelope of $\mathrm{M}(\mathbf{B})$ (see Appendix A for the definition). Hence we have that:

$$\mathbf{B}^{\cap} \cong \mathrm{M}(\mathbf{B}),$$
$$\mathbf{B}^{\cap\cup} \cong \mathrm{L}(\mathrm{M}(\mathbf{B})).$$

For convenience we take such isomorphisms as the identity and we identify $B^{\cap}$ with $\mathrm{M}(B)$ and $B^{\cap\cup}$ with $\mathrm{L}(\mathrm{M}(B))$. Now we examine some properties of $\mathcal{S}$-Priestley spaces, that lead us to an alternative definition of $\mathcal{S}$-Priestley spaces, that provides us with a better understanding of the structure of these spaces.

PROPOSITION 5.1.34. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. A subset $U \subseteq X$ is a non-empty open up-set of $\langle X, \tau, \leq\rangle$ if and only if it is a non-empty union of non-empty finite intersections of elements of B.*

PROOF. Let U be a non-empty open up-set of $\langle X, \tau, \leq\rangle$. When $U = X$ there is nothing to prove, as $X \in B$ by (Pr4), so assume that $U \neq X$. As U is non-empty, let $x \in U$. Because U is an up-set, we have that for all $y \notin U$, $x \not\leq y$. Then by (Pr1), since the $\mathcal{S}$-referential algebra $\langle X, \mathbf{B}\rangle$ is reduced, for all $y \notin U$ there is $V^x_y \in B$ such that $x \in V^x_y$ and $y \notin V^x_y$. Then we have a closed set U^c and open sets $\{(V^x_y)^c : y \notin U\}$ such that $U^c \subseteq \bigcup\{(V^x_y)^c : y \notin U\}$. Now by compactness of the space given by (Pr3), there are $y_0, \ldots, y_n \notin U$, for some $n \in \omega$, such that $U^c \subseteq (V^x_{y_0})^c \cup \cdots \cup (V^x_{y_n})^c$. Hence $V^x_{y_0} \cap \cdots \cap V^x_{y_n} \subseteq U$. Notice that by construction, $x \in V^x_{y_0} \cap \cdots \cap V^x_{y_n}$, therefore we get

$$U \subseteq \bigcup_{x\in U} (V^x_{y_0} \cap \cdots \cap V^x_{y_n}) \subseteq U.$$

Thus, as U is non-empty, U is a non-empty union of non-empty finite intersections of elements of B, as required. □

PROPOSITION 5.1.35. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. A subset $U \subseteq X$ is a non-empty clopen up-set of $\langle X, \tau, \leq\rangle$ if and only if it is a non-empty finite union of non-empty finite intersections of elements of B.*

PROOF. It follows from the previous proposition and compactness of the space. □

Notice also that the emptyset is a clopen up-set that can be trivially described as an (empty) finite union of non-empty finite intersections of elements of B.

COROLLARY 5.1.36. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then the collection $B \cup \{U^c : U \in B\}$ is a subbasis for $\langle X, \tau\rangle$.*

PROOF. This follows from the previous proposition and the fact that for any Priestley space $\langle X, \tau, \leq\rangle$, $\{U \setminus V : U, V \in \mathcal{C}\ell\mathcal{U}(X)\}$ is a basis of the space. □

The next proposition highlights that previous fact is strongly connected with the property of the $\mathcal{S}$-referential algebra $\langle X, \mathbf{B}\rangle$ of being reduced.

PROPOSITION 5.1.37. *For any $\mathcal{S}$-referential algebra $\langle X, \mathbf{B}\rangle$ augmented with a topology τ and an order $\leq$ on X, if $\langle X, \tau, \leq\rangle$ is a Priestley space, $X \in B$ and $\mathcal{C\ell U}(X)$ is the closure of B under finite unions of non-empty finite intersections of elements of B, then $\langle X, \mathbf{B}\rangle$ is reduced.*

PROOF. Assume that $\langle X, \tau, \leq\rangle$ is a Priestley space, $X \in B$ and $\mathcal{C\ell U}(X)$ is the closure of B under finite unions of non-empty finite intersections. We show that $\langle X, \mathbf{B}\rangle$ is reduced, by showing that $\leq$ is the quasiorder $\preceq$ associated with the referential algebra.

Let first $x, y \in X$ be such that $x \leq y$. As elements of B are up-sets, it follows that for all $V \in B$, if $x \in V$ then $y \in V$. Let now $x, y \in X$ be such that $x \nleq y$. Then by totally order disconnectedness of the space, there is U a clopen up-set such that $x \in U$ and $y \notin U$. Then there are non-empty $\mathcal{U}_0 \dots, \mathcal{U}_n \subseteq^\omega B$ finite subsets, for some $n \in \omega$, such that $x \in \bigcap \mathcal{U}_1 \cup \cdots \cup \bigcap \mathcal{U}_n = U$. So there is $i \leq n$ such that $x \in \bigcap \mathcal{U}_i$ and $y \notin \bigcap \mathcal{U}_i$. And then there is $V \in \mathcal{U}_i \subseteq B$ such that $x \in V$ and $y \notin V$.

We conclude that for all $x, y \in X$, $x \leq y$ if and only for all $V \in B$, if $x \in V$ then $y \in V$. Hence $\leq = \preceq$. Since $\leq$ is a partial order, it follows that the referential algebra $\langle X, \mathbf{B}\rangle$ is reduced. □

From previous results we come up with the following corollary, that provides an alternative definition of $\mathcal{S}$-Priestley spaces.

COROLLARY 5.1.38. *A structure $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$ is an $\mathcal{S}$-Priestley space if and only if the following conditions are satisfied:*

- (Pr1′) *$\langle X, \mathbf{B}\rangle$ is an $\mathcal{S}$-referential algebra, whose associated quasiorder is denoted by $\leq$,*
- (Pr2) *for all $\mathcal{U} \cup \{V\} \subseteq^\omega B$, $\bigcap \mathcal{U} \subseteq V$ iff $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U})$,*
- (Pr3′) *$\langle X, \tau, \leq\rangle$ is a Priestley space, and $B \cup \{U^c : U \in B\}$ is a subbasis for it*
- (Pr4′) *$X \in B$ and $\mathcal{C\ell U}(X) = B^{\cap\cup} \cup \{\emptyset\}$,*
- (Pr5) *the set $X_B := \{x \in X : \{U \in B : x \notin U\}$ is non-empty and up-directed$\}$ is dense in $\langle X, \tau\rangle$.*

In the same way that we stablished that clopen up-sets of X are the elements of $\mathbf{B}^{\cap\cup}$, we prove now that X_B-admissible clopen up-sets are the elements of $\mathbf{B}^{\cap}$. Notice that in the following proofs we use the well-known correspondence between the elements of a Priestley space and the prime meet filters of the lattice of its clopen up-sets. Recall also that for convenience we identify $\mathrm{M}(\mathbf{B})$ and $\mathbf{B}^{\cap}$.

PROPOSITION 5.1.39. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then for any non-empty finite intersection U of elements of B, we have that $\max(U^c) \subseteq X_B$.*

PROOF. Let $U \in \mathrm{M}(B)$ be a non-empty finite intersection of elements of B, and let $x \in \max(U^c)$. We show that $x \in X_B$. Let $F_x := \{V \in \mathrm{M}(B) : x \in V\}$. This set is a meet filter of $\mathrm{M}(\mathbf{B})$ and by hypothesis $U \notin F_x$. By Lemma 2.3.3, there is $Q \in \mathrm{Irr}_\wedge(\mathrm{M}(\mathbf{B}))$ such that $F_x \subseteq Q$ and $U \notin Q$. By Proposition A.8 in Appendix A, $Q' := [\![Q\rangle\!\rangle_{\mathrm{L}(\mathrm{M}(\mathbf{B}))}$ is an optimal filter of $\mathrm{L}(\mathrm{M}(\mathbf{B}))$. By Proposition 5.1.35, Q' can also be seen as a prime filter of the lattice of clopen up-sets of

X. Therefore, by Priestley duality for distributive lattices, there is $y \in X$ such that $Q' = \{W \in \mathrm{L}(\mathrm{M}(B)) : y \in W\}$. Moreover, by Proposition A.8 again, $Q = Q' \cap \mathrm{M}(B) = \{W \in \mathrm{M}(B) : y \in W\}$. Then since Q is irreducible, we obtain that $Q^c = \{W \in \mathrm{M}(B) : y \notin W\}$ is up-directed.

We claim that $\{W \in B : y \notin W\}$ is up-directed. Let $W_1, W_2 \in B$ be such that $y \notin W_1, W_2$. As Q^c is up-directed, there is $W \in \mathrm{M}(B)$ such that $y \notin W$ and $W_1, W_2 \subseteq W$. By definition of $\mathrm{M}(B)$, there are $U_0, \ldots, U_n \in B$, for some $n \in \omega$ such that $W = U_0 \cap \cdots \cap U_n$. It follows that there is $i \leq n$ with $y \notin U_i \in B$, and clearly $W_1, W_2 \subseteq U_i$, as required.

From the claim and since $y \notin U$, we have that $\{W \in B : y \notin W\}$ is non-empty and up-directed, hence by (Pr5), $y \in X_B$. Now we claim that $x \leq y$. On the contrary, by (Pr1) there is $W \in B$ such that $x \in W$ and $y \notin W$. But then $W \in F_x \subseteq Q$, and so $y \in W$, a contradiction. Finally, by x being maximal in U^c, we obtain $x = y \in X_B$, as required. □

Previous proposition establishes that non-empty finite intersections of elements of B are X_B-admissible clopen up-sets of X. The following proposition shows that the converse also holds.

Proposition 5.1.40. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Each X_B-admissible clopen up-set U of X is a non-empty finite intersection of elements of B.*

Proof. Let $U \in \mathcal{C}\ell\mathcal{U}(X)$ be a clopen up-set of X such that $\max(U^c) \subseteq X_B$. Let us consider first the case when $U = \emptyset$. Then by assumption $\max(X) \subseteq X_B$, which implies by condition (Pr5) that for each $x \in \max(X)$ there is $U \in B$ such that $x \notin U$. Since the elements of B are up-sets, this implies that for each $x \in X$ there is $U \in B$ such that $x \notin U$. Therefore, there is $\mathcal{U} \subseteq B$ such that $\bigcap \mathcal{U} = \emptyset$. Now since the elements of B are clopens, $\bigcup\{U^c : U \in \mathcal{U}\}$ is an open cover of the space, and so by compactness given by (Pr5), there is a finite subcover $U_0, \ldots, U_n$. Hence, $U = \emptyset = U_0 \cap \cdots \cap U_n$ is a non-empty finite intersection of elements of B, as required.

Let now U be non-empty. By Proposition 5.1.35 we know that there are $V_0, \ldots, V_n \in \mathrm{M}(B)$, for some $n \in \omega$, such that $U = V_0 \cup \cdots \cup V_n$. Let us consider the set $G := \bigcap\{\uparrow_{\mathrm{M}(\mathbf{B})} V_i : i \leq n\}$, which is a meet filter of $\mathrm{M}(\mathbf{B})$. And let $I := \langle\!\langle \{V_0, \ldots, V_n\}]\!]_{\mathrm{M}(\mathbf{B})}$ be the F-ideal of $\mathrm{M}(\mathbf{B})$ generated by $\{V_0, \ldots, V_n\}$.

We claim that $G \cap I \neq \emptyset$. Suppose, towards a contradiction, that $G \cap I = \emptyset$. Then by Lemma 2.3.7, there is $P \in \mathrm{Op}_\wedge(\mathrm{M}(\mathbf{B}))$, such that $G \subseteq P$ and $P \cap I = \emptyset$. By Proposition A.8 in Appendix A, $P' := [\![P \rangle\!\rangle_{\mathrm{L}(\mathrm{M}(\mathbf{B}))}$ is an optimal filter of $\mathrm{L}(\mathrm{M}(\mathbf{B}))$. By Proposition 5.1.35, P' can also be seen as a prime filter of the lattice of clopen up-sets of X. Therefore, by Priestley duality for distributive lattices, there is $x \in X$ such that $P' = \{W \in \mathrm{L}(\mathrm{M}(B)) : x \in W\}$. Moreover, by Proposition A.8 again, $P' \cap \mathrm{M}(B) = \{W \in \mathrm{M}(B) : x \in W\} = P$. From $P \cap I = \emptyset$ we obtain that $x \notin V_i$ for all $i \leq n$, and so $x \notin U$. Now let $y \in \max(U^c)$ be such that $x \leq y$. Then $y \notin V_i$ for all $i \leq n$, and by hypothesis $y \in X_B$, so the collection $\{W \in B : y \notin W\}$ is up-directed. So there is $W \in B$ such that $y \notin W$ and $V_i \subseteq W$ for all $i \leq n$. This implies, since W is an up-set, that $x \notin W$. Moreover, from the definition of G we get that $W \in G \subseteq P$, so $x \in W$, a contradiction.

From the claim, we get $W \in G \cap I \neq \emptyset$. By definition of G, $V_i \subseteq W$ for each $i \leq n$, so $U = \bigcup\{V_i : i \leq n\} \subseteq W$. By definition of F-ideal generated, we know that $\bigcap\{\uparrow V_i : i \leq n\} \subseteq \uparrow W$. We only have to show that $W \cap X_B \subseteq \bigcup\{V_i : i \leq n\}$, because from this fact and denseness of the space, since U is open, it follows that $W \subseteq \bigcup\{V_i : i \leq n\} = U$, and this completes the proof of $U = W \in \mathrm{M}(B)$.

Suppose, towards a contradiction, that there is $z \in W \cap X_B$ such that $z \notin V_i$ for all $i \leq n$. Then $V_i \in \{W \in B : z \notin W\}$ for each $i \leq n$. Notice that by (Pr5) we know that this collection is up-directed. Thus there is $V' \in B$ such that $z \notin V'$ and $V_i \subseteq V'$ for each $i \leq n$. Hence we get $V' \in \bigcap\{\uparrow V_i : i \leq n\} \subseteq \uparrow W$, so $W \subseteq V'$, and then from $z \in W$ it follows $z \in V'$, a contradiction. □

Previous propositions shed light on our construction: they reveal the following connection between $\mathcal{S}$-Priestley spaces and generalized Priestley spaces.

THEOREM 5.1.41. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then $\langle X, \tau, \leq, X_B\rangle$ is a generalized Priestley space.*

PROOF. We just need to check that conditions (DS5) and (DS6), given in page 38 are satisfied by the structure $\langle X, \tau, \leq, X_B\rangle$. By propositions 5.1.40 and 5.1.39, condition (DS6) reduces to the following condition: for all $x, y \in X$

$$x \leq y \quad \text{iff} \quad \forall U \in \mathrm{M}(B)\big(\text{ if } x \in U, \text{ then } y \in U\big).$$

And this follows straightforwardly from $\langle X, \mathbf{B}\rangle$ being reduced (condition (Pr1)). For condition (DS5) we have to show that:

$$X_B = \{x \in X : \{U \in \mathrm{M}(B) : x \notin U\} \text{ is non-empty and up-directed}\}.$$

Let first $x \in X_B$, so by (Pr5), $\{V \in B : x \notin V\}$ is non-empty and up-directed. We only have to show that $\{U \in \mathrm{M}(B) : x \notin U\}$ is up-directed, so let $U_1, U_2 \in \mathrm{M}(B)$ be such that $x \notin U_1, U_2$. By definition, U_1 and U_2 are intersections of non-empty finite subsets of B, thus by assumption there are $V_1, V_2 \in B$ such that $U_i \subseteq V_i$ and $x \notin V_i$ for $i \in \{1, 2\}$. Now by hypothesis, there is $W \in B$ such that $V_1, V_2 \subseteq W$ and $x \notin W$, and as $W \in \mathrm{M}(B)$ we are done.

For the converse, let $x \in X$ be such that $\{U \in \mathrm{M}(B) : x \notin U\}$ is non-empty and up-directed. So by definition of $\mathrm{M}(B)$, there is $V \in B$ such that $U \subseteq V$ and $x \notin V$. Hence $\{V \in B : x \notin V\}$ is non-empty. We only have to show that this collection is also up-directed, so let $V_1, V_2 \in B$ be such that $x \notin V_1, V_2$. By hypothesis, there is $U \in \mathrm{M}(B)$ such that $V_1, V_2 \subseteq U$ and $x \notin U$. And then by definition of $\mathrm{M}(B)$, there is $V \in B$ such that $U \subseteq V$ and $x \notin V$, so we are done. □

We aim to prove that the correspondence between X_B-admissible clopen up-sets of X and elements of $\mathbf{B}^{\cap}$ is in fact an isomorphism between distributive semilattices. For proving this we need some results from Priestley duality for distributive semilattices. Since in [**5**] it was studied a Priestley duality for bounded distributive semilattices, and only outlined how their results can be generalized for the non-bounded case, we present in Appendix A several results from [**5**] but generalized for the case of distributive semilattices with top element. We encourage the reader that is not familiar with such duality to read that appendix before continuing with the reading.

For any $\mathcal{S}$-Priestley space $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$, we define the map $\xi_{\mathfrak{X}} : X \longrightarrow \mathcal{P}^{\uparrow}(B)$ as follows:

$$\xi_{\mathfrak{X}}(x) := \{U \in B : x \in U\}.$$

Notice that this definition is analogous to that of ε given in page 77. For any $Y \subseteq X$, we use the following notation:

$$\widehat{\xi}_{\mathfrak{X}}(Y) := \bigcap\{\xi_{\mathfrak{X}}(w) : w \in Y\} = \{U \in B : Y \subseteq U\}.$$

When the context is clear, we drop the subscript of $\xi_{\mathfrak{X}}$ and $\widehat{\xi}_{\mathfrak{X}}$.

LEMMA 5.1.42. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then ξ is one-to-one.*

PROOF. This follows easily from (Pr1) and (Pr3): let $x, y \in X$ be such that $x \neq y$. We can assume, without loss of generality, that $x \nleq y$. Then since $\leq$ is the order associated with the reduced referential algebra $\langle X, \mathbf{B}\rangle$, by definition of this order, there is $U \in B$ such that $x \in U$ and $y \notin U$. Therefore $\xi(x) \neq \xi(y)$. □

PROPOSITION 5.1.43. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. For any non-empty $\mathcal{U}, \mathcal{V} \subseteq^{\omega} B$:*

$$\bigcap \mathcal{V} \subseteq \bigcup \mathcal{U} \quad \textit{iff} \quad \bigcap_{U \in \mathcal{U}} \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U) \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V}).$$

PROOF. Assume first that $\bigcap \mathcal{V} \subseteq \bigcup \mathcal{U}$ and let $U' \in \bigcap\{\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U) : U \in \mathcal{U}\}$. By condition (Pr2) we get $\bigcup \mathcal{U} \subseteq U'$, and then by assumption $\bigcap \mathcal{V} \subseteq U'$. It follows from condition (Pr2) again that $U' \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V})$.

Assume now that $\bigcap\{\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U) : U \in \mathcal{U}\} \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V})$. We show that $\bigcap \mathcal{V} \cap X_B \subseteq \bigcup \mathcal{U}$, and then the claim follows from denseness and from $\bigcup \mathcal{U}$ being clopen. Let $x \in \bigcap \mathcal{V} \cap X_B$ and suppose, towards a contradiction, that $x \notin \bigcup \mathcal{U}$. Then using condition (Pr5), there is $U' \in B$ such that $\bigcup \mathcal{U} \subseteq U'$ and $x \notin U'$. Thus $U' \in \bigcap\{\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U) : U \in \mathcal{U}\}$, and so by assumption $U' \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V})$, that by (Pr2) implies $\bigcap \mathcal{V} \subseteq U'$. As $x \in \bigcap \mathcal{V}$, we get $x \in U'$, a contradiction. □

COROLLARY 5.1.44. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. For any non-empty $\mathcal{U}, \mathcal{V} \subseteq^{\omega} B$:*

$$\bigcap \mathcal{U} = \bigcap \mathcal{V} \quad \textit{iff} \quad \widehat{\vartheta}_{\mathbf{B}}(\mathcal{U}) = \widehat{\vartheta}_{\mathbf{B}}(\mathcal{V}).$$

PROOF. From the previous proposition and Lemma 4.5.10 we get $\bigcap \mathcal{U} \subseteq V$ if and only if $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(V) \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U})$ if and only if $\widehat{\vartheta}_{\mathbf{B}}(\mathcal{U}) \subseteq \vartheta_{\mathbf{B}}(V)$. Therefore, we get $\bigcap \mathcal{U} \subseteq \bigcap \mathcal{V}$ if and only if $\widehat{\vartheta}_{\mathbf{B}}(\mathcal{U}) \subseteq \bigcap\{\vartheta_{\mathbf{B}}(V) : V \in \mathcal{V}\} = \widehat{\vartheta}_{\mathbf{B}}(\mathcal{V})$. And hence, $\bigcap \mathcal{U} = \bigcap \mathcal{V}$ if and only if $\widehat{\vartheta}_{\mathbf{B}}(\mathcal{U}) = \widehat{\vartheta}_{\mathbf{B}}(\mathcal{V})$. □

PROPOSITION 5.1.45. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then for any $x \in X$, $\xi(x) \in \mathrm{Op}_{\mathcal{S}}(\mathbf{B})$.*

PROOF. First we show that $\xi(x)$ is an $\mathcal{S}$-filter of $\mathbf{B}$. Let $\Gamma \cup \{\delta\} \subseteq Fm$ be such that $\Gamma \vdash_{\mathcal{S}} \delta$ and let $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{B})$ be such that $h(\gamma) \in \xi(x)$ for all $\gamma \in \Gamma$, i. e. $x \in h(\gamma)$ for all $\gamma \in \Gamma$. Then by condition (Pr1) and definition of $\mathcal{S}$-referential algebra, we obtain $\bigcap\{h(\gamma) : \gamma \in \Gamma\} \subseteq h(\delta)$, and therefore $x \in h(\delta)$ i. e. $h(\delta) \in \xi(x)$.

Notice that if $\xi(x) = B$, then by Lemma 5.1.32, $\mathbf{B}$ has no bottom-family: otherwise there is $\mathcal{V} \subseteq^{\omega} B$ such that $\bigcap \mathcal{V} = \emptyset$, but by assumption $x \in \bigcap \mathcal{V}$. Therefore, by definition of bottom-family, it follows that $\emptyset$ is an strong $\mathcal{S}$-ideal of $\mathbf{B}$, and so $\xi(x) = B$ is an optimal $\mathcal{S}$-filter of $\mathbf{B}$.

Suppose now that $\xi(x) \neq B$. First we show that $\xi(x)^c$ is an $\mathcal{S}$-ideal of $\mathbf{B}$, where $\xi(x)^c = \{U \in B : x \notin U\}$. Let $V \in B$ and $U_1, \ldots, U_n \in \xi(x)^c$, for some $n \in \omega$, be such that $\bigcap\{\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U_i) : i \leq n\} \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(V)$. For $n = 0$ the hypothesis turns into $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(V) = B$. This implies that V is the bottom element of $\mathbf{B}$. Thus, by Lemma 5.1.32, $V = \emptyset$ and then clearly $x \notin V$. For $n \neq 0$, by Proposition 5.1.43 we get $V \subseteq \bigcup\{U_i : i \leq n\}$, and then from $x \notin U_i$ for all $i \leq n$ we get $x \notin V$. We conclude that $\xi(x)^c \in \mathrm{Id}_{\mathcal{S}}(\mathbf{B})$.

Now we show that $\xi(x)^c$ is a strong $\mathcal{S}$-ideal of $\mathbf{B}$. Let $\mathcal{V} \subseteq^{\omega} B$ and let $U_1, \ldots, U_n \in \xi(x)^c$, for some $n \in \omega$, be such that $\bigcap\{\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U_i) : i \leq n\} \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V})$. Recall that since $\mathcal{S}$ is assumed to have theorems, $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\emptyset) = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(1^{\mathbf{B}}) = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(X) \neq \emptyset$. Then if $\mathcal{V} = \emptyset$, by $\xi(x)^c$ being $\mathcal{S}$-ideal, the assumption implies that $\xi(x)^c = B$. Therefore $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V}) \cap \xi(x)^c \neq \emptyset$, and we are done, so assume $\mathcal{V} \neq \emptyset$. For $n = 0$ the hypothesis turns into $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V}) = B$. This implies that there is $\mathcal{V}' \subseteq \mathcal{V}$ that is a bottom-family for $\mathbf{B}$. Thus $\bigcap \mathcal{V}' = \emptyset$, and then there is $V \in \mathcal{V}$ such that $x \notin V$, i. e. $V \in \xi(x)^c$. For $n \neq 0$, by Proposition 5.1.43 we get $\bigcap \mathcal{V} \subseteq \bigcup\{U_i : i \leq n\}$, and then from $x \notin U_i$ for all $i \leq n$ we get that there is $V \in \mathcal{V}$ such that $x \notin V$, i. e. $V \in \xi(x)^c$. From either case we get that $\mathcal{V} \cap \xi(x)^c \neq \emptyset$, and so $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{V}) \cap \xi(x)^c \neq \emptyset$. Thus, we have shown that $\xi(x)^c$ is an strong $\mathcal{S}$-ideal, and by Theorem 4.4.9 we conclude that $\xi(x)$ is an optimal $\mathcal{S}$-filter. $\square$

PROPOSITION 5.1.46. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then for any $x \in X_B$, $\xi(x) \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$.*

PROOF. This follows from the previous proposition, Theorem 4.4.8 and condition (Pr5), that states that for any $x \in X_B$, $\xi(x)^c$ is non-empty and up-directed, i. e. an order ideal of $\mathbf{B}$. $\square$

Let us show now that the bijection between X_B-admissible clopen up-sets of X and the elements of $\mathbf{B}^{\cap}$ (i. e. the elements of $\mathrm{M}(\mathbf{B})$) given by propositions 5.1.40 and 5.1.39 is an isomorphism of distributive semilattices. On the one hand, from Theorem 5.1.41 and Priestley duality for distributive semilattices we know that $\boldsymbol{\mathcal{C}\ell\mathcal{U}}^{ad}_{X_B}(X) := \langle \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X), \cap, X\rangle$ is a distributive semilattice. On the other hand, we have the $\mathcal{S}$-semilattice of $\mathbf{B}$, that is also distributive. Let us define the map $g : \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X) \longrightarrow \mathrm{M}(B)$, such that for any non-empty $\mathcal{U} \subseteq^{\omega} B$:

$$g(\bigcap \mathcal{U}) := \bigcap\{\vartheta_{\mathbf{B}}(U) : U \in \mathcal{U}\} = \widehat{\vartheta}_{\mathbf{B}}(\mathcal{U}).$$

THEOREM 5.1.47. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. The map g is an isomorphism between $\boldsymbol{\mathcal{C}\ell\mathcal{U}}^{ad}_{X_B}(X)$ and $\mathrm{M}(\mathbf{B})$.*

PROOF. Injectivity of g follows from Corollary 5.1.44 and surjectivity follows immediately from the definition of the $\mathcal{S}$-semilattice of $\mathbf{B}$. Finally, meets are preserved, since $g(\bigcap \mathcal{U} \cap \bigcap \mathcal{V}) = \widehat{\vartheta}_{\mathbf{B}}(\mathcal{U} \cup \mathcal{V}) = \widehat{\vartheta}_{\mathbf{B}}(\mathcal{U}) \cap \widehat{\vartheta}_{\mathbf{B}}(\mathcal{V}) = g(\bigcap \mathcal{U}) \cap g(\bigcap \mathcal{V})$, and $g(X) = g(\bigcap\{X\}) = \vartheta_{\mathbf{B}}(X) = \mathrm{Op}_{\mathcal{S}}(\mathbf{B})$, that recall that is the top element of $\mathrm{M}(\mathbf{B})$. $\square$

Summarizing, we have that for any $\langle X, \tau, \mathbf{B}\rangle$ $\mathcal{S}$-Priestley space, $\langle X, \tau, \leq, X_B\rangle$ is a generalized Priestley space, whose dual distributive semilattice is isomorphic to $\mathrm{M}(\mathbf{B})$. Table 4 collects all these results. Recall that when the algebra $\mathbf{A}$ has a

Table 4. Priestley duality for $\mathcal{S}$-algebras – Summary.

Algebras	Dual spaces
$\mathbf{A}$	$\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \vartheta_{\mathbf{A}}[\mathbf{A}]\rangle$
$\mathrm{M}(\mathbf{A})$	$\langle \mathrm{Op}_{\wedge}(\mathrm{M}(\mathbf{A})), \tau_{\mathrm{M}(\mathbf{A})}, \subseteq, \mathrm{Irr}_{\wedge}(\mathrm{M}(\mathbf{A}))\rangle \cong \langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})\rangle$
$\mathrm{L}(\mathrm{M}(\mathbf{A}))$	$\langle \mathrm{Op}_{\wedge}(\mathrm{L}(\mathrm{M}(\mathbf{A}))), \tau_{\mathrm{L}(\mathrm{M}(\mathbf{A}))}, \subseteq\rangle \cong \langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq\rangle$

bottom element, so do $\mathrm{M}(\mathbf{A})$ and $\mathrm{L}(\mathrm{M}(\mathbf{A}))$. In this case, the optimal meet filters of $\mathrm{L}(\mathrm{M}(\mathbf{A}))$ are the prime filters defined as usual.

Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. By Priestley duality for distributive semilattices we know that for any $P \in \mathrm{Op}_{\wedge}(\boldsymbol{\mathcal{C}\ell\mathcal{U}}^{ad}_{X_B}(X))$ there is $x \in X$ such that $P = \{U \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X) : x \in U\}$. Then we can translate this and obtain that for any $P \in \mathrm{Op}_{\wedge}(\mathrm{M}(\mathbf{B}))$ there is $x \in X$ such that $P = \{U \in \mathrm{M}(B) : x \in U\}$. We use this fact to prove that the map $\xi : X \longrightarrow \mathcal{P}^{\uparrow}(B)$ is onto $\mathrm{Op}_{\mathcal{S}}(\mathbf{B})$.

Proposition 5.1.48. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then the map $\xi : X \longrightarrow \mathcal{P}^{\uparrow}(B)$ is onto* $\mathrm{Op}_{\mathcal{S}}(\mathbf{B})$.

Proof. Let P be an optimal $\mathcal{S}$-filter of $\mathbf{B}$. Then by Proposition 4.5.13, $[\![P\rangle\!\rangle_{\mathrm{M}(\mathbf{B})}$ is an optimal meet filter of $\mathrm{M}(\mathbf{B})$. By Priestley duality for distributive semilattices, there is $x \in X$ such that $[\![P\rangle\!\rangle_{\mathrm{M}(\mathbf{B})} = \{U \in \mathrm{M}(B) : x \in U\}$. Then by Proposition 4.5.5, $P = [\![P\rangle\!\rangle_{\mathrm{M}(\mathbf{B})} \cap B = \{U \in B : x \in U\} = \xi(x)$. $\square$

Corollary 5.1.49. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then ξ is an order homeomorphism between ordered topological spaces $\langle X, \tau, \leq\rangle$ and $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{B}), \tau_{\mathbf{B}}, \subseteq\rangle$.*

Proof. Notice that for all $x \in X$ and all $U \in B$ we have: $x \in U$ if and only if $U \in \xi(x)$ if and only if $\xi(x) \in \vartheta_{\mathbf{B}}(U)$. Thus $\xi^{-1}[\vartheta_{\mathbf{B}}(U)] = U$ and moreover:

$$x \in \xi^{-1}[\vartheta_{\mathbf{B}}(U)^c] \quad \text{iff} \quad \xi(x) \in \vartheta_{\mathbf{B}}(U)^c \quad \text{iff} \quad U \notin \xi(x) \quad \text{iff} \quad x \in U^c.$$

Therefore $\xi^{-1}[\vartheta_{\mathbf{B}}(U)^c] = U^c$. From condition (Pr1) it follows that ξ is order preserving. As ξ is one-to-one, onto (Proposition 5.1.48), and its inverse sends subbasic opens of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{B}), \tau_{\mathbf{B}}\rangle$ to subbasic opens of $\langle X, \tau\rangle$, we conclude that ξ is an homeomorphism, as required (notice that we use that inverse map preserve intersections, so the previous condition implies that the inverse of ξ sends basic opens to basic opens). $\square$

Corollary 5.1.50. *Let $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space. Then the structure $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{B}), \tau_{\mathbf{B}}, \vartheta_{\mathbf{B}}[\mathbf{B}]\rangle$ is an $\mathcal{S}$-Priestley space such that $\langle X, \tau\rangle$ and $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{B}), \tau_{\mathbf{B}}\rangle$ are homeomorphic topological spaces by means of the map $\xi_{\mathfrak{X}} : X \longrightarrow \mathrm{Op}_{\mathcal{S}}(\mathbf{B})$, that moreover is an order isomorphism between $\langle X, \leq\rangle$ and $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{B}), \subseteq\rangle$. Furthermore $\mathbf{B}$ and $\vartheta_{\mathbf{B}}[\mathbf{B}]$ are isomorphic $\mathcal{S}$-algebras by means of the map $\vartheta_{\mathbf{B}} : B \longrightarrow \vartheta_{\mathbf{B}}[B]$.*

Previous corollary together with Corollary 5.1.33 summarize all preceding results, and should be kept in mind for § 5.2 and § 5.3, where the duality for morphisms is studied, and the functors involved are defined.

5.2. Duality for morphisms

In the present section we use the results from Chapter 4 and from § 5.1 to present two dual correspondences between algebraic homomorphisms between $\mathcal{S}$-algebras and certain classes of relations between $\mathcal{S}$-Spectral spaces and $\mathcal{S}$-Priestley spaces. The approach for the Spectral-like duality is similar to that by Celani et al. in [**15**]. For the Priestley-style duality, we follow the same line as Bezhanishvili and Jansana follow in [**5**]. Let us begin with a basic fact concerning algebraic homomorphisms and $\mathcal{S}$-filters that is used later on. A proof can be found in Proposition 1.19 in [**35**].

LEMMA 5.2.1. *Let $\mathcal{S}$ be a logic, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be $\mathcal{S}$-algebras and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$ be an algebraic homomorphism between them. Then for any $\mathcal{S}$-filter F of $\mathbf{A}_2$, $h^{-1}[F]$ is an $\mathcal{S}$-filter of $\mathbf{A}_1$.*

From now on let $\mathcal{S}$ be a finitary congruential logic with theorems, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be $\mathcal{S}$-algebras and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$ be a homomorphism between them. We define a binary relation $R_h \subseteq \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_2) \times \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_1)$ by:

$$(P, Q) \in R_h \quad \text{iff} \quad h^{-1}[P] \subseteq Q.$$

We denote the restriction of R_h to $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_2) \times \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_1)$ by $\overline{R}_h$. These are the relations that we use to represent h. Recall that for the relation R_h we may consider the function $\Box_{R_h} : \mathcal{P}(\mathrm{Op}_{\mathcal{S}}(\mathbf{A}_1)) \longrightarrow \mathcal{P}(\mathrm{Op}_{\mathcal{S}}(\mathbf{A}_2))$ given by:

$$\Box_{R_h}(U) := \{Q \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_2) : R_h(Q) \subseteq U\}.$$

Similarly, for $\overline{R}_h$, the restriction of R_h to $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_2) \times \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_1)$, we may consider the function $\Box_{\overline{R}_h} : \mathcal{P}(\mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_1)) \longrightarrow \mathcal{P}(\mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_2))$ given by:

$$\Box_{\overline{R}_h}(U) := \{Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_2) : \overline{R}_h(Q) \subseteq U\}.$$

Let us examine in detail the properties of the relations R_h and $\overline{R}_h$. Notice that, for convenience, we denote by ϑ_i and ψ_i the maps $\vartheta_{\mathbf{A}_i}$ and $\psi_{\mathbf{A}_i}$ respectively.

PROPOSITION 5.2.2. *Let $\mathcal{S}$ be a finitary congruential logic with theorems, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be $\mathcal{S}$-algebras and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$. For all $a \in A_1$:*

(1) $R_h^{-1}(\vartheta_1(a)^c) = \vartheta_2(h(a))^c$.
(2) $\overline{R}_h^{-1}(\psi_1(a)^c) = \psi_2(h(a))^c$.

PROOF. For (1), first we show that $R_h^{-1}(\vartheta_1(a)^c) \subseteq \vartheta_2(h(a))^c$, so we take $P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_2)$ be such that $P \in R_h^{-1}(\vartheta_1(a)^c)$, i. e. $h^{-1}[P] \subseteq Q$ for some $Q \notin \vartheta_1(a)$. Then from $a \notin Q$ we get $a \notin h^{-1}[P]$, i. e. $h(a) \notin P$ so $P \in \vartheta_2(h(a))^c$. For the converse, let $P \in \vartheta_2(h(a))^c$, i. e. $a \notin h^{-1}[P]$. As P is an $\mathcal{S}$-filter of $\mathbf{A}_2$, by Lemma 5.2.1 we know that $h^{-1}[P]$ is an $\mathcal{S}$-filter of $\mathbf{A}_1$. Then by Corollary 4.4.6 there is an optimal $\mathcal{S}$-filter Q of $\mathbf{A}_1$ such that $a \notin Q \supseteq h^{-1}[P]$. So we have $Q \in \vartheta_1(a)^c$ and $Q \in R_h(P)$, hence $P \in R_h^{-1}(\vartheta_1(a)^c)$.

(2) The proof is similar to that of item (1), using Corollary 4.4.2 instead of Corollary 4.4.6. □

PROPOSITION 5.2.3. *Let $\mathcal{S}$ be a finitary congruential logic with theorems, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be $\mathcal{S}$-algebras and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$. For all $a \in A_1$:*

(1) $\Box_{R_h}(\vartheta_1(a)) = \vartheta_2(h(a))$.

(2) $\Box_{\overline{R}_h}(\psi_1(a)) = \psi_2(h(a))$.

PROOF. (1) First we show that $\Box_{R_h}(\vartheta_1(a)) \subseteq \vartheta_2(h(a))$, so let $P \in \Box_{R_h}(\vartheta_1(a))$, i. e. $R_h(P) \subseteq \vartheta_1(a)$. Suppose, towards a contradiction, that $P \notin \vartheta_2(h(a))$. Then by item (1) of Proposition 5.2.2 we get that $P \in R_h^{-1}(\vartheta_1(a)^c)$, so there is $Q \in R_h(P)$ such that $Q \notin \vartheta_1(a)$, a contradiction. For the converse, let $P \in \vartheta_2(h(a))$, so $a \in h^{-1}[P]$. Then for any $Q \in R_h(P)$, from $h^{-1}[P] \subseteq Q$ and the hypothesis we get $a \in Q$, i. e. $Q \in \vartheta_1(a)$. This implies that $R_h(P) \subseteq \vartheta_1(a)$, i. e. $P \in \Box_{R_h}(\vartheta_1(a))$, as required.

(2) The proof is similar to that of item (1), using item (2) of Proposition 5.2.2 instead of item (1). $\Box$

Notice that in the statement of the next corollary, by '$\Box_{R_h} \in \mathrm{Hom}(\vartheta_1[\mathbf{A}_1], \vartheta_2[\mathbf{A}_2])$' we mean that the restriction of the function $\Box_{R_h}$ to $\vartheta_1[A_1]$ is an homomorphism from $\vartheta_1[\mathbf{A}_1]$ to $\vartheta_2[\mathbf{A}_2]$. Similarly for '$\Box_{\overline{R}_h} \in \mathrm{Hom}(\psi_1[\mathbf{A}_1], \psi_2[\mathbf{A}_2])$'. We keep using this abuse of notation, but we should retain in mind what it refers to.

COROLLARY 5.2.4. *Let $\mathcal{S}$ be a finitary congruential logic with theorems, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be $\mathcal{S}$-algebras and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$. Then:*

(1) $\Box_{R_h} \in \mathrm{Hom}(\vartheta_1[\mathbf{A}_1], \vartheta_2[\mathbf{A}_2])$.
(2) $\Box_{\overline{R}_h} \in \mathrm{Hom}(\psi_1[\mathbf{A}_1], \psi_2[\mathbf{A}_2])$.

PROOF. (1) Let f be an n-ary connective of the language and let $a_i \in A_1$ for any $i \leq n$. We have to show that $\Box_{R_h}\big(f^{\vartheta_1[\mathbf{A}_1]}(\vartheta_1(a_1), \ldots, \vartheta_1(a_n))$ is equal to

$$f^{\vartheta_2[\mathbf{A}_2]}(\Box_{R_h}(\vartheta_1(a_1)), \ldots, \Box_{R_h}(\vartheta_1(a_n))).$$

Using the definition of $\vartheta_1[\mathbf{A}_1]$ and $\vartheta_2[\mathbf{A}_2]$, item (1) in Proposition 5.2.3, and the fact that h is a homomorphism between $\mathbf{A}_1$ and $\mathbf{A}_2$, we get:

$$\begin{aligned}
\Box_{R_h}(f^{\vartheta_1[\mathbf{A}_1]}(\vartheta_1(a_1), \ldots, \vartheta_1(a_n))) &= \Box_{R_h}(\vartheta_1(f^{\mathbf{A}_1}(a_1, \ldots, a_n))) \\
&= \vartheta_2(h(f^{\mathbf{A}_1}(a_1, \ldots, a_n))) \\
&= \vartheta_2(f^{\mathbf{A}_2}(h(a_1), \ldots, h(a_n))) \\
&= f^{\vartheta_2[\mathbf{A}_2]}(\vartheta_2(h(a_1)), \ldots, \vartheta_2(h(a_n))) \\
&= f^{\vartheta_2[\mathbf{A}_2]}(\Box_{R_h}(\vartheta_1(a_1)), \ldots, \Box_{R_h}(\vartheta_1(a_n))).
\end{aligned}$$

(2) The proof is similar to that of item (1), using the definition of $\psi_1[\mathbf{A}_1]$ and $\psi_2[\mathbf{A}_2]$, and item (2) in Proposition 5.2.3 instead of item (1). $\Box$

Notice that Corollary 5.2.4 gives us two analogous representation theorems for h, that hold for any finitary congruential logic, not necessarily a filter distributive one. However, for getting a full duality between morphisms, we should assume additionally filter-distributivity of the logic. In the following subsections, we discuss first the Spectral-dual morphisms of homomorphisms between $\mathcal{S}$-algebras, and then the Priestley-dual morphisms. In both cases, we prove the facts that motivate the definition of the dual morphisms before introducing such definition.

5.2.1. Spectral-like dual morphisms. Assume that $\mathcal{S}$ is a filter distributive finitary congruential logic with theorems, $\mathbf{A}_1$ and $\mathbf{A}_2$ are two $\mathcal{S}$-algebras, and $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$ is a homomorphism between them.

PROPOSITION 5.2.5. *For any $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_2)$, $\overline{R}_h(P)$ is a closed subset of the space $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_1), \tau_{\kappa_{\mathbf{A}_1}} \rangle$.*

PROOF. Notice that for any $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_2)$, we have that the subset $\overline{R}_h(P) = \{Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_1) : h^{-1}[P] \subseteq Q\}$ coincides with $\widehat{\psi}_1(h^{-1}[P])$, and since h is an algebraic homomorphism, by Lemma 5.2.1 we get that $h^{-1}[P]$ is an $\mathcal{S}$-filter of $\mathbf{A}_1$. Then by Proposition 5.1.6 we conclude that $\overline{R}_h(P)$ is closed subset of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_1), \tau_{\kappa_{\mathbf{A}_1}} \rangle$. $\square$

We introduce now the definition of the morphisms between $\mathcal{S}$-Spectral spaces, that are the Spectral-dual morphisms of homomorphisms between $\mathcal{S}$-algebras.

DEFINITION 5.2.6. Let $\mathfrak{X}_1 = \langle X_1, \mathbf{B}_1 \rangle$ and $\mathfrak{X}_2 = \langle X_2, \mathbf{B}_2 \rangle$ be two $\mathcal{S}$-Spectral spaces. A relation $R \subseteq X_1 \times X_2$ is an *$\mathcal{S}$-Spectral morphism* when:

(SpR1) $\Box_R \in \mathrm{Hom}(\mathbf{B}_2, \mathbf{B}_1)$,
(SpR2) $R(x)$ is a closed subset of $\langle X_2, \tau_{\kappa_{\mathfrak{X}_2}} \rangle$ for all $x \in X_1$.

Notice that for any $\mathcal{S}$-Spectral morphism $R \subseteq X_1 \times X_2$ between $\mathcal{S}$-Spectral spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, we have that for all $U \in B_2$:

$$R^{-1}(U^c) = \{x \in X_1 : \exists y \notin U, (x, y) \in R\} = \{x \in X_1 : R(x) \nsubseteq U\} = (\Box_R(U))^c.$$

COROLLARY 5.2.7. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be $\mathcal{S}$-algebras and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$. Then $\overline{R}_h$ is an $\mathcal{S}$-Spectral morphism between $\mathcal{S}$-Spectral spaces $\mathfrak{Irr}_{\mathcal{S}}(\mathbf{A}_2)$ and $\mathfrak{Irr}_{\mathcal{S}}(\mathbf{A}_1)$.*

PROOF. (SpR1) follows from Corollary 5.2.4 and (SpR2) follows from Proposition 5.2.5. $\square$

PROPOSITION 5.2.8. *For any $\mathcal{S}$-Spectral space $\mathfrak{X} = \langle X, \mathbf{B} \rangle$, the order associated with the $\mathcal{S}$-referential algebra $\langle X, \mathbf{B} \rangle$ is an $\mathcal{S}$-Spectral morphism.*

PROOF. Recall that we denote by $\leq$ the order associated with the $\mathcal{S}$-referential algebra $\langle X, \mathbf{B} \rangle$, that coincides with the dual of the specialization order of the space $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$. Therefore, for all $x \in X$, $\uparrow x = \mathrm{cl}(x)$, which is a closed subset of $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$, hence condition (SpR2) is satisfied by $\leq$. Notice also that $\Box_{\leq}(Y) = \{x \in X : \uparrow x \subseteq Y\}$. As the elements of B are closed subsets of $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$, they are up-sets with respect to the order $\leq$, so for all $U \in B$, $\Box_{\leq}(U) = U$. Therefore $\Box_{\leq}$ is the identity map from B to B, and so $\Box_{\leq} \in \mathrm{Hom}(\mathbf{B}, \mathbf{B})$ and condition (SpR1) is also satisfied by $\leq$. Hence the relation $\leq \subseteq X \times X$ is an $\mathcal{S}$-Spectral morphism. $\square$

5.2.2. Priestley-style dual morphisms. Assume that $\mathcal{S}$ is a filter distributive finitary congruential logic with theorems, $\mathbf{A}_1$ and $\mathbf{A}_2$ are two $\mathcal{S}$-algebras, and $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$ is a homomorphism between them.

PROPOSITION 5.2.9. *For any $P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_2)$ and $Q \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_1)$ such that $(P, Q) \notin R_h$, there is $a \in A_1$ such that $Q \notin \vartheta(a)$ and $R_h \subseteq \vartheta(a)$.*

PROOF. From $(P, Q) \notin R_h$ we get $h^{-1}[P] \nsubseteq Q$, so there is $a \in A$ such that $a \in h^{-1}[P]$ and $a \notin Q$. This implies that $Q \notin \vartheta(a)$ and for all $Q' \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_1)$ such that $(P, Q') \in R_h$, $a \in Q'$. Therefore $R_h(P) \subseteq \vartheta(a)$ and we are done. $\square$

We introduce now the definition of the morphisms between $\mathcal{S}$-Priestley spaces, that are the Priestley-dual morphisms of homomorphisms between $\mathcal{S}$-algebras.

DEFINITION 5.2.10. Let $\mathfrak{X}_1 = \langle X_1, \tau_1, \mathbf{B}_1\rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_2, \mathbf{B}_2\rangle$ be two $\mathcal{S}$-Priestley spaces. A relation $R \subseteq X_1 \times X_2$ is an *$\mathcal{S}$-Priestley morphism* when:

(PrR1) $\Box_R \in \mathrm{Hom}(\mathbf{B}_2, \mathbf{B}_1)$,
(PrR2) if $(x, y) \notin R$, then there is $U \in B_2$ such that $y \notin U$ and $R(x) \subseteq U$.

COROLLARY 5.2.11. *Let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems, let $\mathbf{A}_1$ and $\mathbf{A}_2$ be $\mathcal{S}$-algebras and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$. Then R_h is an $\mathcal{S}$-Priestley morphism between $\mathcal{S}$-Priestley spaces $\mathfrak{Op}_{\mathcal{S}}(\mathbf{A}_2)$ and $\mathfrak{Op}_{\mathcal{S}}(\mathbf{A}_1)$.*

PROOF. (PrR1) follows from Corollary 5.2.4 and (PrR2) follows from Proposition 5.2.9. □

Recall that in Theorem 5.1.41 we proved that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, the structure $\langle X, \tau, \leq, X_B\rangle$ is a generalized Priestley space. Analogously, in the next theorem we show how $\mathcal{S}$-Priestley morphisms and generalized Priestley morphisms are related:

THEOREM 5.2.12. *Let $R \subseteq X_1 \times X_2$ be an $\mathcal{S}$-Priestley morphism between $\mathcal{S}$-Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Then R is a generalized Priestley morphism between generalized Priestley spaces $\langle X_1, \tau_1, \leq_1, X_{B_1}\rangle$ and $\langle X_2, \tau_2, \leq_2, X_{B_2}\rangle$.*

PROOF. We just need to check that condition (DSR3) holds, as (DSR4) follows directly from (PrR2) and Proposition 5.1.39. So let $U \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_{B_2}}(X_2)$. By Proposition 5.1.40 there are $U_0, \ldots, U_n \in B_2$ such that $U = U_0 \cap \cdots \cap U_n$. Then we have that $\Box_R(U) = \{x \in X : R(x) \subseteq U_0 \cap \cdots \cap U_n\} = \Box_R(U_0) \cap \cdots \cap \Box_R(U_n)$. And then by (PrR1) and Proposition 5.1.39, $\Box_R(U) \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_{B_1}}(X_1)$, as required. □

PROPOSITION 5.2.13. *For any $\mathcal{S}$-Priestley space $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$, the order associated with the $\mathcal{S}$-referential algebra $\langle X, \mathbf{B}\rangle$ is an $\mathcal{S}$-Spectral morphism.*

PROOF. Recall that we denote the order associated with the $\mathcal{S}$-referential algebra $\langle X, \mathbf{B}\rangle$ by $\leq$. As the referential algebra is reduced, for any $x, y \in X$ such that $x \nleq y$, there is $U \in B$ such that $x \in U$ and $y \notin U$. Moreover, as B is a family of clopen up-sets, for every $z \in {\uparrow}x$ we get $z \in U$. Therefore ${\uparrow}x \subseteq U$, hence condition (PrR2) is satisfied by $\leq$. Notice also that $\Box_{\leq}(Y) = \{x \in X : {\uparrow}x \subseteq Y\}$. As the elements of B are up-sets with respect to $\leq$, for all $U \in B$ we have $\Box_{\leq}(U) = U$. Therefore $\Box_{\leq}$ is the identity map from B to B, and so $\Box_{\leq} \in \mathrm{Hom}(\mathbf{B}, \mathbf{B})$ and condition (SpR1) is also satisfied by $\leq$. Hence the relation $\leq \subseteq X \times X$ is an $\mathcal{S}$-Priestley morphism. □

5.3. Categorical dualities

In the present section we conclude the presentation of the dualities, by showing the functors and the natural transformations involved in them. From now on, let $\mathcal{S}$ be a filter distributive finitary congruential logic with theorems. Clearly $\mathcal{S}$-algebras and homomorphisms between them form a category, that we denote by $\mathsf{Alg}\mathcal{S}$. Before proving the two categorical dualities for $\mathsf{Alg}\mathcal{S}$, we need to show that

$\mathcal{S}$-Spectral spaces and $\mathcal{S}$-Spectral morphisms form a category, and that $\mathcal{S}$-Priestley spaces and $\mathcal{S}$-Priestley morphisms form a category as well.

THEOREM 5.3.1. *Let $\mathfrak{X}_1 = \langle X_1, \mathbf{B}_1\rangle$, $\mathfrak{X}_2 = \langle X_2, \mathbf{B}_2\rangle$ and $\mathfrak{X}_3 = \langle X_3, \mathbf{B}_3\rangle$ be $\mathcal{S}$-Spectral spaces and let $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be $\mathcal{S}$-Spectral morphisms. Then:*

(1) *The $\mathcal{S}$-Spectral morphism $\leq_2 \subseteq X_2 \times X_2$ satisfies:*

$$\leq_2 \circ R = R \text{ and } S \circ \leq_2 = S,$$

(2) *$S \circ R \subseteq X_1 \times X_3$ is an $\mathcal{S}$-Spectral morphism.*

PROOF. (1) By (SpR2), $R(x)$ is closed subset of $\langle X_2, \tau_{\kappa_{\mathfrak{X}_2}}\rangle$ for any $x \in X_1$. As closed subsets are up-sets with respect to $\leq_2$ (the dual of the specialization order), it follows that $\leq_2 \circ R = R$.

Let us show that $S \circ \leq_2 = S$. Let $x \leq_2 y$ and $(y, z) \in S$, and suppose, towards a contradiction, that $z \notin S(x)$. Then by $\{V^c : V \in B_3\}$ being a basis for $\langle X_3, \tau_{\kappa_{\mathfrak{X}_3}}\rangle$ (condition (Sp3)), there is $V \in B_3$ such that $z \in V^c$ and $S(x) \cap V^c = \emptyset$. Then we have $S(x) \subseteq V$, so $x \in \Box_S(V)$. Moreover, as $\Box_S(V) \in B_2$ (by condition (SpR1)), it follows that $y \in \Box_S(V)$, and therefore $S(y) \subseteq V$. Then by assumption, from $(y, z) \in S$ we get $z \in V$, a contradiction. This proves that $S \circ \leq_2 \subseteq S$. The other inclusion is immediate.

(2) It is easy to see that $\Box_{S\circ R} = \Box_R \circ \Box_S$. Therefore, since $\Box_S \in \mathrm{Hom}(\mathbf{B}_3, \mathbf{B}_2)$ and $\Box_R \in \mathrm{Hom}(\mathbf{B}_2, \mathbf{B}_1)$, we conclude that $\Box_{S\circ R} \in \mathrm{Hom}(\mathbf{B}_3, \mathbf{B}_1)$, and then condition (SpR1) is satisfied by $S \circ R$.

We prove that condition (SpR2) is also satisfied by $S \circ R$. Let $y \in X_3$ and $x \in X_1$ be such that $y \notin (S \circ R)(x)$. We show that there is $V \in B_3$ such that $y \in V^c$ and $V^c \cap (S \circ R)(x) = \emptyset$. This implies, by $\kappa_{\mathfrak{X}_3} = \{V^c : V \in B_3\}$ being a basis, that $(S \circ R)(x)$ is a closed subset of $\langle X_3, \tau_{\kappa_{\mathfrak{X}_3}}\rangle$.

Notice that for any $z \in R(x)$, $y \notin S(z)$. By condition (SpR2) on $\langle X_3, \mathbf{B}_3\rangle$, $S(z)$ is closed. And then by (Sp3), there is $V_z \in B_3$ such that $y \in V_z^c$ and $S(z) \subseteq V_z$, so $z \in \Box_S(V_z)$. This implies that:

$$R(x) \cap \bigcap \{\Box_S(V)^c : y \in V^c \in \kappa_{\mathfrak{X}_3}\} = \emptyset.$$

Now as $\kappa_{\mathfrak{X}_3}$ is a basis for $\langle X_3, \tau_{\kappa_{\mathfrak{X}_3}}\rangle$, the set $\{V^c : y \in V^c \in \kappa_{\mathfrak{X}_3}\}$ is down-directed, and then so is the set $\{\Box_S(V)^c : y \in V^c \in \kappa_{\mathfrak{X}_3}\}$. Then by Theorem 5.1.20, we conclude that there is $V^c \in \kappa_{\mathfrak{X}_3}$ such that $y \in V^c$ and $R(x) \cap \Box_S(V)^c = \emptyset$, i.e. $R(x) \subseteq \Box_S(V)$. This implies that $x \in \Box_R \circ \Box_S(V) = \Box_{S\circ R}(V)$ so $(S \circ R)(x) \subseteq V$, as required. □

COROLLARY 5.3.2. *$\mathcal{S}$-Spectral spaces and $\mathcal{S}$-Spectral morphisms form a category.*

PROOF. For an $\mathcal{S}$-Spectral space $\mathfrak{X}$, Proposition 5.2.8 shows that the order $\leq_X$ defined on X is an $\mathcal{S}$-Spectral morphism. Then by item (1) in Theorem 5.3.1, $\leq_X$ is the identity morphism for $\mathfrak{X}$. By item (2) in Theorem 5.3.1, relational composition works as composition between $\mathcal{S}$-Spectral morphisms. □

For the Priestley-style category, we obtain similar results, except that set-theoretic relational composition does not work as composition in the new category, but we have to define a new composition between $\mathcal{S}$-Priestley morphisms. For $\mathcal{S}$-Priestley spaces $\mathfrak{X}_1, \mathfrak{X}_2$ and $\mathfrak{X}_3$ and $\mathcal{S}$-Priestley morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, let $(S \star R) \subseteq X_1 \times X_3$ be the relation given by:

$$\begin{aligned}(x,z) \in (S \star R) \quad &\text{iff} \quad \forall U \in B_3 \big(\text{ if } x \in \Box_R \circ \Box_S(U), \text{ then } z \in U \big) \\ &\text{iff} \quad \forall U \in B_3 \big(\text{ if } (S \circ R)(x) \subseteq U, \text{ then } z \in U \big).\end{aligned}$$

Notice that from the definition of $\star$ it follows that $\Box_{(S \star R)} = \Box_R \circ \Box_S$.

THEOREM 5.3.3. *Let $\langle X_1, \tau_1, \mathbf{B}_1 \rangle$, $\langle X_2, \tau_2, \mathbf{B}_2 \rangle$ and $\langle X_3, \tau_3, \mathbf{B}_3 \rangle$ be $\mathcal{S}$-Priestley spaces and let $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be $\mathcal{S}$-Priestley morphisms. Then:*

(1) *The $\mathcal{S}$-Priestley morphism $\leq_2 \subseteq X_2 \times X_2$ satisfies:*

$$\leq_2 \circ R = R \text{ and } S \circ \leq_2 = S,$$

(2) *$(S \star R) \subseteq X_1 \times X_3$ is an $\mathcal{S}$-Priestley morphism.*

PROOF. (1) First we show that $\leq_2 \circ R = R$. Let $y \in R(x)$ and $y \leq_2 z$, and suppose, towards a contradiction, that $z \notin R(x)$. By (PrR2) there is $U \in B_2$ such that $R(x) \subseteq U$ and $z \notin U$. Then by assumption $y \in U$, and since U is an up-set, we get $z \in U$, a contradiction. Hence we have $\leq_2 \circ R \subseteq R$. The other inclusion is immediate.

Now we show that $S \circ \leq_2 = S$. Let $x \leq_2 y$ and $z \in S(y)$, and suppose, towards a contradiction, that $z \notin S(x)$. By (PrR2) again, there is $U \in B_3$ such that $S(x) \subseteq U$ and $z \notin U$. Then we have $x \in \Box_S(U)$ and by (PrR1) we get $\Box_S(U) \in B_2$. In particular $\Box_S(U)$ is an up-set, thus $y \in \Box_S(U)$. Then $S(y) \subseteq U$, and therefore $z \in U$, a contradiction. Hence we have $S \circ \leq_2 = S$. The other inclusion is immediate.

(2) Conditions (PrR1) and (PrR2) follow easily from the definition of $\star$. $\Box$

COROLLARY 5.3.4. *$\mathcal{S}$-Priestley spaces and $\mathcal{S}$-Priestley morphisms form a category.*

PROOF. For an $\mathcal{S}$-Priestley space $\mathfrak{X}$, Proposition 5.2.13 shows that the order $\leq_X$ defined on X is an $\mathcal{S}$-Priestley morphism. We claim that for any $\mathcal{S}$-Priestley spaces $\langle X_1, \tau_1, \mathbf{B}_1 \rangle$ and $\langle X_2, \tau_2, \mathbf{B}_2 \rangle$, and $\mathcal{S}$-Priestley morphism $R \subseteq X_1 \times X_2$, we have $\leq_2 \circ R = \leq_2 \star R$. The inclusion from left to right follow by definition. From the other inclusion, let $(x,z) \in (\leq_2 \star R)$ and suppose, towards a contradiction that $(x,z) \notin \leq_2 \circ R$. By item (1) in Theorem 5.3.3 we know that $\leq_2 \circ R = R$, and then from the hypothesis and (Pr2), there is $U \in B_2$ such that $R(x) \subseteq U$ and $z \notin U$. But since $(\leq_2 \circ R)(x) = R(x)$, we conclude $(x,z) \notin (\leq_2 \star R)$, a contradiction.

Hence by item (1) in Theorem 5.3.3 we obtain that $\leq_X$ is the identity morphism for $\mathfrak{X}$. By item (2) in Theorem 5.3.3, composition of $\mathcal{S}$-Priestley morphisms is given by $\star$ (associativity of $\star$ follows easily). $\Box$

For any filter distributive finitary congruential logic with theorems $\mathcal{S}$, let $\mathsf{Sp}\mathcal{S}$ be the category of $\mathcal{S}$-Spectral spaces and $\mathcal{S}$-Spectral morphisms, and let $\mathsf{Pr}\mathcal{S}$ be the category of $\mathcal{S}$-Priestley spaces and $\mathcal{S}$-Priestley morphisms. We summarize in Table 5 all the categories so far considered.

TABLE 5. Categories involved in the dualities for $\mathcal{S}$ a filter distributive finitary congruential logic with theorems.

CATEGORY	OBJECTS	MORPHISMS
$\mathsf{Alg}\mathcal{S}$	$\mathcal{S}$-algebras	algebraic homomorphisms
$\mathsf{Sp}\mathcal{S}$	$\mathcal{S}$-Spectral spaces	$\mathcal{S}$-Spectral morphisms
$\mathsf{Pr}\mathcal{S}$	$\mathcal{S}$-Priestley spaces	$\mathcal{S}$-Priestley morphisms (comp $\star$)

Once we have defined the categories, we need to build the contravariant functors and the natural isomorphisms involved in the dualities. Let us examine first the Spectral-like duality, and then we move to the Priestley-style duality.

5.3.1. Spectral-like duality. Let us start with the functors for the Spectral-like duality. On the one hand, we consider the functor $\mathfrak{Irr}_{\mathcal{S}} : \mathsf{Alg}\mathcal{S} \longrightarrow \mathsf{Sp}\mathcal{S}$ such that for any $\mathcal{S}$-algebras $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2$ and any $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$:

$$\mathfrak{Irr}_{\mathcal{S}}(\mathbf{A}) := \langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \psi_{\mathbf{A}}[\mathbf{A}] \rangle,$$
$$\mathfrak{Irr}_{\mathcal{S}}(h) := \overline{R}_h \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_2) \times \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}_1).$$

Recall that a topology $\tau_{\kappa_{\mathbf{A}}}$ is defined on $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, taking $\kappa_{\mathbf{A}} := \{\psi_{\mathbf{A}}(a)^c : a \in A\}$ as a basis, for $\psi_{\mathbf{A}} : A \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Irr}_{\mathcal{S}}(\mathbf{A}))$ given by $\psi_{\mathbf{A}}(a) := \{P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}) : a \in P\}$. And by definition $(P, Q) \in \overline{R}_h$ if and only if $h^{-1}[P] \subseteq Q$.

Clearly, for $\mathrm{id}_{\mathbf{A}} : A \longrightarrow A$, the identity morphism for $\mathbf{A}$ in $\mathsf{Alg}\mathcal{S}$, we obtain $\overline{R}_{\mathrm{id}_{\mathbf{A}}} = \subseteq$, and this is precisely the identity morphism for $\mathfrak{Irr}_{\mathcal{S}}(\mathbf{A})$ in $\mathsf{Sp}\mathcal{S}$. Moreover, it follows from from definition that for $\mathcal{S}$-algebras $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ and homomorphisms $f \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$ and $g \in \mathrm{Hom}(\mathbf{A}_2, \mathbf{A}_3)$, $\overline{R}_{g \circ f} = \overline{R}_f \circ \overline{R}_g$. Therefore, by corollaries 5.1.11 and 5.2.7, the functor $\mathfrak{Irr}_{\mathcal{S}}$ is well defined.

On the other hand, we consider the functor $(\ )^* : \mathsf{Sp}\mathcal{S} \longrightarrow \mathsf{Alg}\mathcal{S}$ such that for any $\mathcal{S}$-Spectral spaces $\mathfrak{X}, \mathfrak{X}_1, \mathfrak{X}_2$ and any $\mathcal{S}$-Spectral morphism $R \subseteq X_1 \times X_2$:

$$\mathfrak{X}^* := \mathbf{B},$$
$$R^* := \Box_R : B_2 \longrightarrow B_1.$$

We recall that for all $U \in B_2$, $\Box_R(U) := \{x \in X_1 : R(x) \subseteq U\}$. For $\leq_X \subseteq X \times X$, the identity morphism for $\mathfrak{X}$ in $\mathsf{Sp}\mathcal{S}$, $\Box_{\leq_X} = \mathrm{id}_{\mathbf{B}}$, that is precisely the identity morphism for $\mathbf{B}$ in $\mathsf{Alg}\mathcal{S}$. Moreover, it follows by definition that for $\mathcal{S}$-Spectral spaces $\mathfrak{X}_1, \mathfrak{X}_2$ and $\mathfrak{X}_3$, and $\mathcal{S}$-Spectral morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, $\Box_{S \circ R} = \Box_R \circ \Box_S$. Therefore, by Remark 4.2.1 and definition of $\mathcal{S}$-Spectral morphism, the functor $(\ )^*$ is well defined.

In order to complete the duality, we need to define two natural isomorphisms, one between the identity functor on $\mathsf{Alg}\mathcal{S}$ and $(\mathfrak{Irr}_{\mathcal{S}}(\ ))^*$, and the other between the identity functor on $\mathsf{Sp}\mathcal{S}$ and $\mathfrak{Irr}_{\mathcal{S}}((\ )^*)$. Consider first the family of morphisms in $\mathsf{Alg}\mathcal{S}$:

$$\Psi_{\mathcal{S}} = \big(\psi_{\mathbf{A}} : A \longrightarrow \psi_{\mathbf{A}}[A]\big)_{\mathbf{A} \in \mathsf{Alg}\mathcal{S}}$$

THEOREM 5.3.5. *$\Psi_{\mathcal{S}}$ is a natural isomorphism between the identity functor on $\mathsf{Alg}\mathcal{S}$ and $(\mathfrak{Irr}_{\mathcal{S}}(\ ))^*$.*

Proof. Let $\mathbf{A}_1, \mathbf{A}_2$ be $\mathcal{S}$-algebras and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$. We prove that $\Box_{R_h} \circ \psi_1 = \psi_2 \circ h$. For $a \in A_1$ and $P \in \Box_{R_h}(\psi_1(a))$, we have $R_h(P) \subseteq \psi_1(a)$. It follows that $h(a) \in P$, so, $P \in \psi_2(h(a))$. For $P' \in \psi_2(h(a))$, we have $h(a) \in P'$. It follows that $R_h(P') \subseteq \psi_1(a)$, so $P' \in \Box_{R_h}(\psi_1(a))$.

From this we have that $\Psi_{\mathcal{S}}$ is a natural transformation, and since for any $\mathcal{S}$-algebra $\mathbf{A}$, we have that $\psi_{\mathbf{A}}$ is an isomorphism from $\mathbf{A}$ to $\psi_{\mathbf{A}}[\mathbf{A}]$, we conclude that $\Psi_{\mathcal{S}}$ is a natural isomorphism. $\Box$

Clearly, what we have is that for any $\mathcal{S}$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ and any homomorphism $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$, the following diagram commutes:

$$\begin{tikzcd} A_1 \arrow[r, "\psi_1"] \arrow[d, "h"'] & \psi_1[A_1] \arrow[d, "\Box_{\overline{R}_h}"] \\ A_2 \arrow[r, "\psi_2"'] & \psi_2[A_2] \end{tikzcd}$$

Before stating the other natural isomorphism, we need to do some work. Recall that for any $\mathcal{S}$-Spectral space $\mathfrak{X} = \langle X, \mathbf{B} \rangle$, we proved that the function $\varepsilon_{\mathfrak{X}} : X \longrightarrow \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$ is a homeomorphism between topological spaces $\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ and $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{B}), \tau_{\kappa_{\mathbf{B}}} \rangle$. This map encodes the natural isomorphism we are looking for, but since morphisms in $\mathsf{Sp}\mathcal{S}$ are relations, we need to give a relation associated with this map. We define the relation $E_{\mathfrak{X}} \subseteq X \times \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})$ given by:

$$(x, P) \in E_{\mathfrak{X}} \quad \text{iff} \quad \varepsilon_{\mathfrak{X}}(x) \subseteq P.$$

Proposition 5.3.6. *$E_{\mathfrak{X}}$ is an $\mathcal{S}$-Spectral morphism.*

Proof. We have to show that $\Box_{E_{\mathfrak{X}}} \in \mathrm{Hom}(\psi_{\mathbf{B}}[\mathbf{B}], \mathbf{B})$. Notice that for all $\psi_{\mathbf{B}}(b) \in \psi_{\mathbf{B}}[B]$, we have:

$$\begin{aligned} \Box_{E_{\mathfrak{X}}}(\psi_{\mathbf{B}}(b)) &= \{x \in X : \forall y \in X\big((x, \varepsilon_{\mathfrak{X}}(y)) \in E_{\mathfrak{X}} \Rightarrow \varepsilon_{\mathfrak{X}}(y) \in \psi_{\mathbf{B}}(b)\big)\} \\ &= \{x \in X : \forall y \in X\big(\varepsilon_{\mathfrak{X}}(x) \subseteq \varepsilon_{\mathfrak{X}}(y) \Rightarrow b \in \varepsilon_{\mathfrak{X}}(y)\big)\} \\ &= \{x \in X : b \in \varepsilon_{\mathfrak{X}}(x)\} = b. \end{aligned}$$

Therefore $\Box_{E_{\mathfrak{X}}} = \psi_{\mathbf{B}}^{-1}$. And since $\mathbf{B}$ and $\psi_{\mathbf{B}}[\mathbf{B}]$ are isomorphic $\mathcal{S}$-algebras by means of the map $\psi_{\mathbf{B}}$, it follows that $\Box_{E_{\mathfrak{X}}} \in \mathrm{Hom}(\psi_{\mathbf{B}}[\mathbf{B}], \mathbf{B})$. This proves that condition (SpR1) is satisfied by $E_{\mathfrak{X}}$. Moreover, this also proves that $E_{\mathfrak{X}}$ is an isomorphism in the category $\mathsf{Sp}\mathcal{S}$.

Notice that for each $x \in X$, we have $E_{\mathfrak{X}}(x) = {\uparrow}\varepsilon_{\mathfrak{X}}(x) = \mathrm{cl}(\varepsilon_{\mathfrak{X}}(x))$, which is a principal up-set of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{B}), \subseteq \rangle$, and so a closed subset of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{B}), \tau_{\kappa_{\mathbf{B}}} \rangle$. Therefore condition (SpR2) is also satisfied by $E_{\mathfrak{X}}$. $\Box$

Consider the family of morphisms in $\mathsf{Sp}\mathcal{S}$:

$$\Sigma_{\mathcal{S}} = \big(E_{\mathfrak{X}} \subseteq X \times \mathrm{Irr}_{\mathcal{S}}(\mathbf{B})\big)_{\mathfrak{X} \in \mathsf{Sp}\mathcal{S}}$$

Theorem 5.3.7. *$\Sigma_{\mathcal{S}}$ is a natural transformation between the identity functor on $\mathsf{Sp}\mathcal{S}$ and $\mathfrak{Irr}_{\mathcal{S}}((\ )^*)$.*

PROOF. Let $\mathfrak{X}_1 = \langle X_1, \mathbf{B}_1 \rangle$ and $\mathfrak{X}_2 = \langle X_2, \mathbf{B}_2 \rangle$ be two $\mathcal{S}$-Spectral spaces and let $R \subseteq X_1 \times X_2$ be an $\mathcal{S}$-Spectral morphism. For convenience, we denote $\varepsilon_{\mathfrak{X}_1}$ and $\varepsilon_{\mathfrak{X}_2}$ by ε_1 and ε_2 respectively. First we show that:

$$(x, y) \in R \quad \text{iff} \quad (\varepsilon_1(x), \varepsilon_2(y)) \in \overline{R}_{\Box_R}.$$

Let first $x \in X_1$ and $y \in X_2$ be such that $(x, y) \in R$ and let $U \in B_2$. Notice that we have:

$$U \in \Box_R^{-1}[\varepsilon_1(x)] \quad \text{iff} \quad \Box_R(U) \in \varepsilon_1(x) \quad \text{iff} \quad x \in \Box_R(U) \quad \text{iff} \quad R(x) \subseteq U.$$

Thus if $U \in \Box_R^{-1}[\varepsilon_1(x)]$, then $R(x) \subseteq U$, and since $(x, y) \in R$, we obtain $y \in U$, i. e. $U \in \varepsilon_2(y)$, and therefore $(\varepsilon_1(x), \varepsilon_2(y)) \in \overline{R}_{\Box_R}$. For the converse, let $x \in X_1$ and $y \in X_2$ be such that $(\varepsilon_1(x), \varepsilon_2(y)) \in \overline{R}_{\Box_R}$ and suppose, towards a contradiction, that $y \notin R(x)$. Since R is an $\mathcal{S}$-Spectral morphism, $R(x)$ is closed, so there is $V \in B_2$ such that $y \in V^c$ and $V^c \cap R(x) = \emptyset$. Then $R(x) \subseteq V$, so $x \in \Box_R(V)$, and then $\Box_R(V) \in \varepsilon_1(x)$. Thus by hypothesis $V \in \varepsilon_2(y)$, so $y \in V$, a contradiction.

The equivalence that we just proved implies that $\overline{R}_{\Box_R} \circ E_{\mathfrak{X}_1} = E_{\mathfrak{X}_2} \circ R$. Thus $\Sigma_{\mathcal{S}}$ is a natural equivalence. Moreover, as $E_{\mathfrak{X}}$ is an isomorphism for each $\mathcal{S}$-Spectral space $\mathfrak{X}$, then $\Sigma_{\mathcal{S}}$ is a natural isomorphism in $\mathsf{Sp}\mathcal{S}$. $\Box$

THEOREM 5.3.8. *The categories* $\mathsf{Alg}\mathcal{S}$ *and* $\mathsf{Sp}\mathcal{S}$ *are dually equivalent by means of the contravariant functors* $\mathfrak{Irr}_{\mathcal{S}}$ *and* $(\ )^*$ *and the natural equivalences* $\Psi_{\mathcal{S}}$ *and* $\Sigma_{\mathcal{S}}$.

5.3.2. Priestley-style duality. Let us move now to the other duality. We begin by considering the functors involved on it. On the one hand, we consider the functor $\mathfrak{Op}_{\mathcal{S}} : \mathsf{Alg}\mathcal{S} \longrightarrow \mathsf{Pr}\mathcal{S}$ such that for any $\mathcal{S}$-algebras $\mathbf{A}$, $\mathbf{A}_1$ and $\mathbf{A}_2$ and any homomorphism $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$:

$$\mathfrak{Op}_{\mathcal{S}}(\mathbf{A}) := \langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \vartheta[\mathbf{A}] \rangle,$$
$$\mathfrak{Op}_{\mathcal{S}}(h) := R_h \subseteq \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_2) \times \mathrm{Op}_{\mathcal{S}}(\mathbf{A}_1).$$

Recall that the topology $\tau_{\mathbf{A}}$ is defined taking $\{\vartheta_{\mathbf{A}}(a) : a \in A\} \cup \{\vartheta_{\mathbf{A}}(b)^c : b \in A\}$ as a subbasis, for $\vartheta_{\mathbf{A}} : A \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Op}_{\mathcal{S}}(\mathbf{A}))$ given by $\vartheta_{\mathbf{A}}(a) := \{P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A}) : a \in P\}$. By definition $(P, Q) \in R_h$ if and only if $h^{-1}[P] \subseteq Q$.

Clearly, for $\mathrm{id}_{\mathbf{A}} : A \longrightarrow A$, the identity morphism for $\mathbf{A}$ in $\mathsf{Alg}\mathcal{S}$, we get $R_{\mathrm{id}_{\mathbf{A}}} = \subseteq$, and this is exactly the identity morphism for $\mathfrak{Op}_{\mathcal{S}}(\mathbf{A})$ in $\mathsf{Pr}\mathcal{S}$. Moreover, it follows from definition that for $\mathcal{S}$-algebras $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ and homomorphisms $f \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$ and $g \in \mathrm{Hom}(\mathbf{A}_2, \mathbf{A}_3)$, $R_{g \circ f} = R_f \star R_g$. Therefore, using corollaries 5.1.33 and 5.2.11, we conclude that the functor $\mathfrak{Op}_{\mathcal{S}}$ is well defined.

On the other hand, we consider the functor $(\ )^\bullet : \mathsf{Pr}\mathcal{S} \longrightarrow \mathsf{Alg}\mathcal{S}$ such that for any $\mathcal{S}$-Priestley spaces $\mathfrak{X}$, $\mathfrak{X}_1$, $\mathfrak{X}_2$ and any $\mathcal{S}$-Priestley morphism $R \subseteq X_1 \times X_2$:

$$\mathfrak{X}^\bullet := \mathbf{B},$$
$$R^\bullet := \Box_R : B_2 \longrightarrow B_1.$$

We recall that for all $U \in B_2$, $\Box_R(U) := \{x \in X_1 : R(x) \subseteq U\}$. Clearly, for $\leq_X \subseteq X \times X$, the identity morphism for $\mathfrak{X}$ in $\mathsf{Pr}\mathcal{S}$, $\Box_{\leq_X} = \mathrm{id}_{\mathbf{B}}$, that is the identity morphism for $\mathbf{B}$ in $\mathsf{Alg}\mathcal{S}$. Moreover, it follows by definition that for any $\mathcal{S}$-Priestley spaces $\mathfrak{X}_1$, $\mathfrak{X}_2$ and $\mathfrak{X}_3$, and any $\mathcal{S}$-Priestley morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq$

$X_2 \times X_3$, we have $\Box_{S\star R} = \Box_R \circ \Box_S$. Therefore, by Remark 4.2.1 and definition of $\mathcal{S}$-Priestley morphism, the functor $(\ )^\bullet$ is also well defined.

In order to complete the duality, we need to define two natural isomorphisms, the one between the identity functor on $\mathsf{Alg}\mathcal{S}$ and $(\mathfrak{Op}_{\mathcal{S}}(\ ))^\bullet$, and the other between the identity functor on $\mathsf{Pr}\mathcal{S}$ and $\mathfrak{Op}_{\mathcal{S}}((\ )^\bullet)$. Consider first the family of morphisms in $\mathsf{Alg}\mathcal{S}$:

$$\Theta_{\mathcal{S}} : (\vartheta_{\mathbf{A}} : A \longrightarrow \vartheta_{\mathbf{A}}[A])_{\mathbf{A}\in\mathsf{Alg}\mathcal{S}}$$

THEOREM 5.3.9. *$\Theta_{\mathcal{S}}$ is a natural isomorphism between the identity functor on $\mathsf{Alg}\mathcal{S}$ and $(\mathfrak{Op}_{\mathcal{S}}(\ ))^\bullet$.*

PROOF. Let $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{A}\mathrm{lg}\mathcal{S}$ and let $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$. It is enough to show that $\Box_{R_h} \circ \vartheta_1 = \vartheta_2 \circ h$. For $a \in A_1$ and $P \in \Box_{R_h}(\vartheta_1(a))$, we have $R_h(P) \subseteq \vartheta_1(a)$. It follows that $h(a) \in P$, so $P \in \vartheta_2(h(a))$. For $P' \in \vartheta_2(h(a))$, we have $h(a) \in P'$. It follows that $R_h(P') \subseteq \vartheta_1(a)$, so $P' \in \Box_{R_h}(\vartheta_1(a))$.

From this we have that $\Theta_{\mathcal{S}}$ is a natural transformation, and since ϑ_1 is an isomorphism from $\mathbf{A}_1$ to $\vartheta_1[\mathbf{A}_1]$, we conclude that $\Theta_{\mathcal{S}}$ is a natural isomorphism. $\Box$

In other words, we have that for any $\mathcal{S}$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ and any homomorphism $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$, the following diagram commutes:

$$\begin{tikzcd} A_1 \arrow[r, "\vartheta_1"] \arrow[d, "h"'] & \vartheta_1[A_1] \arrow[d, "\Box_{R_h}"] \\ A_2 \arrow[r, "\vartheta_2"'] & \vartheta_2[A_2] \end{tikzcd}$$

Before formulating the other natural isomorphism, we need again to do some work. Recall that for any $\mathcal{S}$-Priestley space $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$, we define the map $\xi_{\mathfrak{X}} : X \longrightarrow \mathrm{Op}_{\mathcal{S}}(\mathbf{B})$ that is a homeomorphism between topological spaces $\langle X, \tau\rangle$ and $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{B}), \tau_{\mathbf{B}}\rangle$. This map encodes the natural isomorphism we are looking for, but since morphisms in $\mathsf{Pr}\mathcal{S}$ are relations, we need to give a relation associated with this map. We define the relation $T_{\mathfrak{X}} \subseteq X \times \mathrm{Op}_{\mathcal{S}}(\mathbf{B})$ given by:

$$(x, P) \in T_{\mathfrak{X}} \quad \text{iff} \quad \xi_{\mathfrak{X}}(x) \subseteq P.$$

PROPOSITION 5.3.10. *$T_{\mathfrak{X}}$ is an $\mathcal{S}$-Priestley morphism.*

PROOF. We have to show that $\Box_{T_{\mathfrak{X}}} \in \mathrm{Hom}(\vartheta_{\mathbf{B}}[\mathbf{B}], \mathbf{B})$. Notice that for all $\vartheta_{\mathbf{B}}(b) \in \vartheta_{\mathbf{B}}[B]$, we have:

$$\begin{aligned} \Box_{T_{\mathfrak{X}}}(\vartheta_{\mathbf{B}}(b)) &= \{x \in X : \forall y \in X\big((x, \xi_{\mathfrak{X}}(y)) \in T_{\mathfrak{X}} \Rightarrow \xi_{\mathfrak{X}}(y) \in \vartheta_{\mathbf{B}}(b)\big)\} \\ &= \{x \in X : \forall y \in X\big(\xi_{\mathfrak{X}}(x) \subseteq \xi_{\mathfrak{X}}(y) \Rightarrow b \in \xi_{\mathfrak{X}}(y)\big)\} \\ &= \{x \in X : b \in \xi_{\mathfrak{X}}(x)\} = b. \end{aligned}$$

Therefore $\Box_{T_{\mathfrak{X}}} = \vartheta_{\mathbf{B}}^{-1}$. And since $\mathbf{B}$ and $\vartheta_{\mathbf{B}}[\mathbf{B}]$ are isomorphic $\mathcal{S}$-algebras by means of the map $\vartheta_{\mathbf{B}}$, it follows that $\Box_{T_{\mathfrak{X}}} \in \mathrm{Hom}(\vartheta_{\mathbf{B}}[\mathbf{B}], \mathbf{B})$. This proves that condition (PrR1) is satisfied by $T_{\mathfrak{X}}$. Moreover, this also proves that $T_{\mathfrak{X}}$ is an isomorphism of $\mathsf{Pr}\mathcal{S}$.

We show now that condition (PrR2) is also satisfied by $T_{\mathfrak{X}}$. Notice that for each $x \in X$, we have $T_{\mathfrak{X}}(x) = {\uparrow}\xi_{\mathfrak{X}}(x)$. Let $x \in X$ and $P \in \mathrm{Op}_{\mathcal{S}}(\mathbf{B})$ be such that

$(x, P) \notin T_{\mathfrak{X}}$. We have to show that there is $U \in B$ such that $P \notin \vartheta_{\mathbf{B}}(U)$ and $T_{\mathfrak{X}}(x) \subseteq \vartheta_{\mathbf{B}}(U)$. By definition of $T_{\mathfrak{X}}$, we have that $\xi_{\mathfrak{X}}(x) \nsubseteq P$, so there is $U \in B$ such that $U \in \xi_{\mathfrak{X}}(x)$ and $U \notin P$. Hence $P \notin \vartheta_{\mathbf{B}}(U)$ and $\xi_{\mathfrak{X}}(x) \in \vartheta_{\mathbf{B}}(U)$. Now since $T_{\mathfrak{X}}(x) = {\uparrow}\xi_{\mathfrak{X}}(x)$, we obtain that $T_{\mathfrak{X}}(x) \subseteq \vartheta_{\mathbf{B}}(U)$, as required. $\square$

Consider now the family of morphisms in $\mathsf{Pr}\mathcal{S}$:

$$\Xi_{\mathcal{S}} = \big(T_{\mathfrak{X}} \subseteq X \times \mathrm{Op}_{\mathcal{S}}(\mathbf{B})\big)_{\mathfrak{X} \in \mathsf{Pr}\mathcal{S}}$$

THEOREM 5.3.11. *$\Xi_{\mathcal{S}}$ is a natural transformation between the identity functor on $\mathsf{Pr}\mathcal{S}$ and $\mathfrak{O}\mathrm{p}_{\mathcal{S}}((\ )^{\bullet})$.*

PROOF. Let $\mathfrak{X}_1 = \langle X_1, \tau_1, \mathbf{B}_1\rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_2, \mathbf{B}_2\rangle$ be two $\mathcal{S}$-Priestley spaces and let $R \subseteq X_1 \times X_2$ be an $\mathcal{S}$-Priestley morphism. For convenience, we denote $\xi_{\mathfrak{X}_1}$ and $\xi_{\mathfrak{X}_2}$ by ξ_1 and ξ_2 respectively. First we show that:

$$(x, y) \in R \quad \text{iff} \quad (\xi_1(x), \xi_2(y)) \in R_{\Box_R}.$$

Let $x \in X_1$ and $y \in X_2$ be such that $(x, y) \in R$, and let $U \in B_2$. Notice that we have:

$$U \in \Box_R^{-1}[\xi_1(x)] \quad \text{iff} \quad \Box_R(U) \in \xi_1(x) \quad \text{iff} \quad x \in \Box_R(U) \quad \text{iff} \quad R(x) \subseteq U.$$

Thus if $U \in \Box_R^{-1}[\xi_1(x)]$, then $R(x) \subseteq U$, and since $(x, y) \in R$, we obtain $y \in U$, i. e. $U \in \xi_2(y)$, and therefore $(\xi_1(x), \xi_2(y)) \in R_{\Box_R}$. For the converse, let $x \in X_1, y \in X_2$ be such $(\xi_1(x), \xi_2(y)) \in R_{\Box_R}$ and suppose, towards a contradiction, that $y \notin R(x)$. Since R is an $\mathcal{S}$-Priestley morphism, by (PrR1), there is $U \in B_2$ such that $y \notin U$ and $R(x) \subseteq U$. From previous equivalences we obtain $U \in \Box^{-1}[\xi_1(x)]$. But then from the hypothesis $U \in \xi_2(y)$, so $y \in U$, a contradiction.

The equivalence that we just proved implies that $R_{\Box_R} \star T_{\mathfrak{X}_1} = T_{\mathfrak{X}_2} \star R$. Thus $\Xi_{\mathcal{S}}$ is a natural equivalence. Moreover, as $T_{\mathfrak{X}}$ is an isomorphism for each $\mathcal{S}$-Priestley space $\mathfrak{X}$, then $\Xi_{\mathcal{S}}$ is a natural isomorphism in $\mathsf{Pr}\mathcal{S}$. $\square$

THEOREM 5.3.12. *The categories $\mathsf{Alg}\mathcal{S}$ and $\mathsf{Pr}\mathcal{S}$ are dually equivalent by means of the contravariant functors $\mathfrak{O}\mathrm{p}_{\mathcal{S}}$ and $(\ )^{\bullet}$ and the natural equivalences $\Theta_{\mathcal{S}}$ and $\Xi_{\mathcal{S}}$.*

5.4. Comparison with another duality for congruential logics

As it was already mentioned in § 4.2, the back and forth correspondences between objects that underly our dualities are two particular cases of a more general correspondence, that can be formulated for any selfextensional logic. In [**56**] Jansana and Palmigiano pointed out that this general correspondence serves as a general template where a wide range of Stone/Priestley dualities related with concrete logics can fit. The theory we developed in the present chapter consisted precisely in making this assertion more precise, by showing *how* such dualities do fit in such general correspondence. We emphasize that we are indebted with the work in [**56**], because it served as an inspiration and as an starting point of our theory.

In [**56**] a different case of such general correspondence is also studied, and it is used to characterize the congruential logics among the selfextensional ones, as those logics $\mathcal{S}$ for which $\mathsf{Alg}\mathcal{S}$ is dually equivalent to the category $\mathsf{PRA}_{\mathcal{S}}$. From now on, let $\mathcal{S}$ be a congruential logic.

We recall that an $\mathcal{S}$-referential algebra $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ is *perfect* (see definition in Section 5.1 in [**56**]) when:

(P1) $\langle X, \preceq\rangle$ is a complete lattice, where $\preceq$ is the quasiorder associated with the referential algebra,
(P2) for all $\mathcal{U} \cup \{V\} \subseteq B$, if $\bigcap \mathcal{U} \subseteq V$, then $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U})$,
(P3) $B \subseteq \{\uparrow x : x \in X\}$,
(P4) $\{x : \uparrow x \in B\}$ is join-dense in X.

Notice that for any $\mathcal{S}$-algebra $\mathbf{A}$, $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is trivially a closure base for $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$. So results in § 4.3 can be applied for $\mathcal{F} = \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. In particular, we have that the map $\varphi_{\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})}$ is an isomorphism between algebras $\mathbf{A}$ and $\varphi_{\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})}[\mathbf{A}]$, and by Theorem 4.3.9, it follows that $\langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \varphi_{\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})}[\mathbf{A}]\rangle$ is a reduced $\mathcal{S}$-referential algebra. And then it is easy to prove (see Lemma 5.5 in [**56**]) that $\langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}), \varphi_{\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})}[\mathbf{A}]\rangle$ is a perfect $\mathcal{S}$-referential algebra.

This is the definition of the dual space of $\mathcal{S}$-algebras, for any $\mathcal{S}$ a congruential logic. Notice that in this general case, no topology is considered. It is precisely the assumption of finitarity, what enables us to *topologize* the dual space. This is what it was done in Section 5.2 in [**56**], where the previous correspondence between objects is specialized for the case when $\mathcal{S}$ is a finitary congruential logic. In this case, we have that an $\mathcal{S}$-referential algebra $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ is *f-perfect* when:[1]

(Pf1) $\langle X, \leq\rangle$ is an algebraic lattice, where $\leq$ is the quasiorder associated with the referential algebra,
(Pr2) for all $\mathcal{U} \cup \{V\} \subseteq^{\omega} B$, if $\bigcap \mathcal{U} \subseteq V$, then $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{U})$,
(Pr3′) $B \cup \{U^c : U \in B\}$ is a subbasis for a topology τ on X such that $\langle X, \tau, \leq\rangle$ is a Priestley space,
(Pf3) $B \subseteq \{\uparrow x \in \mathcal{C}\ell(X) : x \in \mathcal{K}(X)\}$,
(P4) $\{x : \uparrow x \in B\}$ is join-dense in X.

Recall that $\mathcal{K}(X)$ denotes the collection of compact subsets of X and $\mathcal{C}\ell(X)$ the collection of clopen subsets of X. Notice that there are some similarities between the previous definition and the definition of $\mathcal{S}$-Priestley space (for a clearer comparison, take into account the characterization of $\mathcal{S}$-Priestley spaces given in Corollary 5.1.38 in page 86), but the comparison does not go further. The approach in [**56**] regarding morphisms differs substantially from ours. A *morphism* between perfect $\mathcal{S}$-referential algebras $\langle X_1, \mathbf{B}_1\rangle$ and $\langle X_2, \mathbf{B}_2\rangle$ is a map $h : X_1 \longrightarrow X_2$ such that $h^{-1} : B_2 \longrightarrow B_1$ is a homomorphism between $\mathcal{S}$-algebras $\mathbf{B}_2$ and $\mathbf{B}_1$.

Recall that we remarked in Lemma 5.2.1 that it is a well-known fact that for any logic $\mathcal{S}$, and any homomorphism between $\mathcal{S}$-algebras $h \in \mathrm{Hom}(\mathbf{A}_1, \mathbf{A}_2)$, the set $h^{-1}[F]$ is an $\mathcal{S}$-filter of $\mathbf{A}_1$ for all $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}_2)$. Then it is easy to prove (see Proposition 4.3 in [**56**]) that h^{-1} is a morphism between $\mathcal{S}$-referential algebras $\langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}_2), \varphi_2[\mathbf{A}_2]\rangle$ and $\langle \mathrm{Fi}_{\mathcal{S}}(\mathbf{A}_1), \varphi_1[\mathbf{A}_1]\rangle$, in the sense defined above.

We have just reviewed the definition of the dual morphisms of homomorphisms between $\mathcal{S}$-algebras, for any (finitary) congruential logic $\mathcal{S}$. Notice that there is a

[1]This is defined in Theorem 5.11 in [**56**], although no name is given for such structures.

fundamental difference between this approach and ours, as in [**56**] duals of homomorphisms are maps, whereas we need to consider relations, instead of maps. This is because we do not have in general that the inverse image of an optimal (resp. irreducible) $\mathcal{S}$-filter by a homomorphism between $\mathcal{S}$-algebras is an optimal (resp. irreducible) $\mathcal{S}$-filter. But the correspondence between morphisms is, in essence, the same in both approaches.

Finally, for any congruential logic, $\mathsf{PRA}_{\mathcal{S}}$ is the category of perfect $\mathcal{S}$-referential algebras and morphisms between them. Theorem 5.9 in [**56**] states that for any congruential logic $\mathcal{S}$, $\mathbb{A}\mathrm{lg}\mathcal{S}$ and $\mathsf{PRA}_{\mathcal{S}}$ are dually equivalent. This duality is clearly less restrictive than ours, since it applies for any congruential logic, but it is not connected so directly with the various Spectral-like and Priestley-style dualities that we already mentioned, that indeed follow as particular cases of our general theory, as we show in Chapter 6. Nevertheless, an indirect connection with Priestley duality for bounded distributive lattices, Stone duality for bounded distributive lattices and Stone duality for Boolean algebras with operators is pointed out in [**56**]. Similar results follow for our dualities from our work in § 5.5, as it will be pointed out where appropriate.

5.5. Dual correspondence of some logical properties

In this section we examine how the correspondences between objects presented in § 5.1 can be refined depending on the properties of the logic under consideration. Recall that in § 1.6 we already introduced several abstract properties of logics, as they are studied within AAL. These properties are particularly interesting for our purposes, because they may be connected with properties of the consequence operator associated with the logical filters. In case a good connection exists, we talk about *transfer theorems*. Let $\mathcal{S}$ be a logic and let Φ be one of those properties, relative to some term. We say that the property Φ *transfers to every algebra* if for every algebra $\mathbf{A}$ (of the same type as $\mathcal{S}$) the closure operator $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$ has Φ relative to the same term.

Note that since we do not fix any concrete logical language, we obtain dualities between categories that both have algebraic nature. This inelegance cannot be avoided within such abstract program. Nevertheless, it might be dodged when we fix a concrete logical language and a concrete logic. Some steps towards this direction are carried out in the Chapter 7, using results from the present section.

In what follows we study, for a given filter distributive finitary congruential logic with theorems $\mathcal{S}$, which properties of the dual spaces correspond with which properties of the logic. Given the abstraction of our general approach, we are allowed to carry out this study in a modular fashion, treating each property independently. Afterwards we might combine these results, as it is indicated when appropriate.

The following subsections are organized as follows: we treat one by one the logical properties that we introduced in § 1.6. First we examine, for each property Φ relative to some term t, the corresponding property of the closure operator $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}$. After that we suggest a Spectral-dual and a Priestley-dual of Φ. For some of those properties we obtain moreover that these dual conditions imply the corresponding property of the logic. Finally we obtain how the term t is represented in each case in the referential algebra $\mathbf{B}$. The results for the Spectral-like and for the Priestley-style dualities are collected in separate tables at the end of the section. Throughout this section we assume that $\mathcal{S}$ is a filter distributive finitary congruential logic with theorems and $\mathbf{A}$ is an $\mathcal{S}$-algebra.

5.5.1. Property of Conjunction. Let $\mathcal{S}$ be a logic that satisfies (PC) for a term $p \wedge q$. We say that *(PC) transfers to every algebra*, if for every algebra $\mathbf{A}$ and every $a, b \in A$:

$$\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a \wedge^{\mathbf{A}} b) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a, b).$$

It is well known that for any logic, (PC) transfers to every algebra (see page 50 in [**35**]).

LEMMA 5.5.1. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PC), and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. Then for all $a, b \in A$, $\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b) = \varphi_{\mathcal{F}}(a \wedge^{\mathbf{A}} b)$.*

PROOF. Notice that, since (PC) transfers to every algebra, for all $a, b \in A$ we have $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a \wedge^{\mathbf{A}} b) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a, b)$. Then we get that for any $P \in \mathcal{F}$:

$$a, b \in P \quad \text{iff} \quad \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a, b) \subseteq P \quad \text{iff} \quad \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a \wedge^{\mathbf{A}} b) \subseteq P \quad \text{iff} \quad a \wedge^{\mathbf{A}} b \in P.$$

Now notice that by definition $\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b) = \{P \in \mathcal{F} : a, b \in P\}$ and $\varphi_{\mathcal{F}}(a \wedge^{\mathbf{A}} b) = \{P \in \mathcal{F} : a \wedge^{\mathbf{A}} b \in P\}$, so we are done. $\square$

Notice that by associativity of intersection, the previous lemma implies that for any non-empty $B \subseteq^{\omega} A$, $\bigcap\{\varphi_{\mathcal{F}}(b) : b \in B\} = \varphi_{\mathcal{F}}(\bigwedge^{\mathbf{A}} B)$. Recall that we defined the $\mathcal{S}$-semilattice of $\mathbf{A}$ as the closure of $\varphi_{\mathcal{F}}[A]$ under non-empty finite intersections. Therefore, if $\mathcal{S}$ satisfies (PC), then $\mathbf{A}$ and $\mathrm{M}(\mathbf{A})$ are isomorphic.

Let us consider first the Spectral-like duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (PC). In the proof of the following proposition we use Lemma 5.5.1 when $\mathcal{F}$ is $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$.

PROPOSITION 5.5.2. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PC). For all $U \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, if U is an open compact subset of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$, then there is $a \in A$ such that $U = \psi(a)^c$.*

PROOF. If $U = \emptyset$, then $U = \psi(1^{\mathbf{A}})^c = \emptyset$ and we are done. Since $\kappa_{\mathbf{A}} = \{\psi(a)^c : a \in A\}$ is a basis for $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, for any U open set there is $B \subseteq A$ such that $U = \bigcup\{\psi(b)^c : b \in B\}$. Thus for any non-empty open and compact U, there are $b_0, \ldots, b_n \in A$, for some $n \in \omega$, such that $U = \psi(b_0)^c \cup \cdots \cup \psi(b_n)^c = (\psi(b_0) \cap \cdots \cap \psi(b_n))^c$. Now we use Lemma 5.5.1, and we get $U = (\psi(b_0 \wedge^{\mathbf{A}} \cdots \wedge^{\mathbf{A}} b_n))^c$, as required. $\square$

COROLLARY 5.5.3. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PC). Then the collection of all open compact sets of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$ is included in $\kappa_{\mathbf{A}}$.*

From the previous corollary we get the idea that for any $\mathcal{S}$-Spectral space $\langle X, \mathbf{B} \rangle$, the Spectral-dual of (PC) is the property of $\kappa_{\mathfrak{X}}$ being the collection of open compact subsets of the space. Let us check now that this condition is enough for recovering the conjunction.

PROPOSITION 5.5.4. *Let $\mathfrak{X} = \langle X, \mathbf{B} \rangle$ be an $\mathcal{S}$-Spectral space such that $\mathcal{KO}(X) = \kappa_{\mathfrak{X}}$. Then for all $U, V \in B$, $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U, V) = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U \cap V)$.*

PROOF. First notice that the hypothesis implies that $\kappa_{\mathfrak{X}}$ is closed under finite unions, i. e. B is closed under finite intersections. Now let $U, V \in B$. On the one hand, notice that $\bigcap \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U,V) \subseteq U \cap V$, since for any $x \in \bigcap \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U,V)$, as $U, V \in \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U,V)$, then $x \in U, V$ and so $x \in U \cap V$. Then by Corollary 5.1.19 we have $U \cap V \in \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U,V)$, and thus $\mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U \cap V) \subseteq \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U,V)$. On the other hand, from $U \cap V \subseteq U, V$ we get $U, V \in \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U \cap V)$, and therefore $\mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U,V) \subseteq \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U \cap V)$. □

THEOREM 5.5.5. *Let $\mathcal{S}$ be a logic such that for any $\mathcal{S}$-Spectral space $\mathfrak{X} = \langle X, \mathbf{B} \rangle$ it holds that $\mathcal{KO}(X) = \kappa_{\mathfrak{X}}$. Then $\mathcal{S}$ satisfies (PC).*

PROOF. Recall that the Lindenbaum-Tarski algebra $\mathbf{Fm}^* = \mathbf{Fm}/\equiv_{\mathcal{S}}^{\mathbf{Fm}}$ is an $\mathcal{S}$-algebra, so $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{Fm}^*), \psi[\mathbf{Fm}^*] \rangle$ is an $\mathcal{S}$-Spectral space. For any variable $p \in Fm$, we denote by $\overline{p}$ its equivalence class in $\mathbf{Fm}^*$, i. e. $\overline{p} := p/\equiv_{\mathcal{S}}^{\mathbf{Fm}}$. Let $p, q \in Var$. By assumption, there is $\rho \in Fm$ such that $\psi(\overline{p}) \cap \psi(\overline{q}) = \psi(\overline{\rho})$. Moreover, by Proposition 5.5.4 we have that

$$\mathrm{C}_{\mathcal{S}}^{\psi[\mathbf{Fm}^*]}\left(\psi(\overline{p}), \psi(\overline{q})\right) = \mathrm{C}_{\mathcal{S}}^{\psi[\mathbf{Fm}^*]}\left(\psi(\overline{p}) \cap \psi(\overline{q})\right).$$

Then by Corollary 4.3.7 we obtain $\mathrm{C}_{\mathcal{S}}^{\mathbf{Fm}^*}(\overline{p}, \overline{q}) = \mathrm{C}_{\mathcal{S}}^{\mathbf{Fm}^*}(\overline{\rho})$. Recall that by Proposition 2.21 in [**35**] the projection map is a bilogical morphism, and then we obtain using Proposition 1.4 in [**35**] that $\mathrm{C}_{\mathcal{S}}(p,q) = \mathrm{C}_{\mathcal{S}}(\rho)$. By structurality of the logic $\mathcal{S}$, we get that there is a formula $\rho'(p,q)$ in at most the variables p and q such that $\mathrm{C}_{\mathcal{S}}(p,q) = \mathrm{C}_{\mathcal{S}}(\rho'(p,q))$. By structurality again we get that for any formulas $\delta, \mu \in Fm$, $\mathrm{C}_{\mathcal{S}}(\delta, \mu) = \mathrm{C}_{\mathcal{S}}(\rho'(\delta, \mu))$. Hence $\mathcal{S}$ satisfies (PC) forthe term ρ'. □

COROLLARY 5.5.6. *Let $\mathcal{S}$ be a logic. Then $\mathcal{S}$ satisfies (PC) if and only if for any $\mathcal{S}$-Spectral space $\mathfrak{X} = \langle X, \mathbf{B} \rangle$ it holds that $\mathcal{KO}(X) = \kappa_{\mathfrak{X}}$.*

Let us consider now the Priestley-style duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (PC). In the proof of the following proposition we use Lemma 5.5.1 when $\mathcal{F}$ is $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$.

PROPOSITION 5.5.7. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PC). For all $U \subseteq \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$, if U is an $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$-admissible clopen up-set of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq \rangle$, then there is $a \in A$ such that $U = \vartheta(a)$.*

PROOF. Let $U \subseteq \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ be a clopen up-set of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq \rangle$ such that $\max(U^c) \subseteq \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$. Then by Proposition 5.1.30, there is non-empty $B \subseteq^{\omega} A$ and such that $U = \widehat{\vartheta}(B)$. Then by Lemma 5.5.1 $U = \vartheta(\bigwedge^{\mathbf{A}} B)$, as required. □

COROLLARY 5.5.8. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PC). Then the collection of $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$-admissible clopen up-sets of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq \rangle$ is included in $\vartheta[A]$.*

From the previous corollary we get the idea that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B} \rangle$, the Priestley-dual of (PC) is the property of B being the collection of X_B-admissible clopen up-sets. Let us check now that this conditions is enough for recovering the conjunction.

PROPOSITION 5.5.9. *Let $\langle X, \tau, \mathbf{B} \rangle$ be an $\mathcal{S}$-Priestley space such that B is the collection of X_B-admissible clopen up-sets $\mathcal{C\ell U}^{ad}_{X_B}(X)$. Then for all $U, V \in B$, $\mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U,V) = \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(U \cap V)$.*

PROOF. First notice that the hypothesis implies that B is closed under finite intersections. Now let $U, V \in B$. Moreover, by (Pr2) we get $\bigcap\{U, V\} \subseteq U \cap V$ if and only if $U \cap V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U, V)$. So we are done. $\square$

THEOREM 5.5.10. *Let $\mathcal{S}$ be a logic such that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, $\mathcal{C\ell U}^{ad}_{X_B}(X) = B$. Then $\mathcal{S}$ satisfies (PC).*

PROOF. The proof is similar to that of Theorem 5.5.5. $\square$

COROLLARY 5.5.11. *Let $\mathcal{S}$ be a logic. Then $\mathcal{S}$ satisfies (PC) if and only if for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$ it holds that $\mathcal{C\ell U}^{ad}_{X_B}(X) = B$.*

5.5.2. Property of Disjunction. Let $\mathcal{S}$ be a logic that satisfies (PWDI) for a term $p \vee q$. We say that *(PWDI) transfers to every algebra*, if for every algebra $\mathbf{A}$, and every $a, b \in A$:

$$\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a \vee^{\mathbf{A}} b) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b).$$

Let now $\mathcal{S}$ be a logic that satisfies (PDI) for a term $p \vee q$. We say that *(PDI) transfers to every algebra*, if for every algebra $\mathbf{A}$, and every $\{a, b\} \cup X \subseteq A$:

$$\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a \vee^{\mathbf{A}} b) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, b).$$

LEMMA 5.5.12. *If a logic $\mathcal{S}$ is filter distributive and satisfies (PWDI), then it satisfies (PDI).*

PROOF. Let $\mathcal{S}$ be a filter distributive logic that satisfies (PWDI) for $p \vee q$ and let $\mathbf{A}$ be an algebra of the same type as $\mathcal{S}$. We denote by $\sqcup$ the join in $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})$, that is a distributive lattice by assumption. Then we have that for all $\{a, b\} \cup X \subseteq A$:

$$\begin{aligned}\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, b) &= \big(\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X) \sqcup \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a)\big) \cap \big(\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X) \sqcup \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a)\big) \\ &= \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X) \sqcup \big(\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b)\big) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X) \sqcup \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a \vee^{\mathbf{A}} b) \\ &= \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a \vee^{\mathbf{A}} b).\end{aligned}$$

$\square$

By the previous lemma, we conclude that for our purposes it is enough to consider only the property (PDI). It is well known that (PDI) transfers to every algebra (see Theorem 2.52 in [**35**]). Moreover (PDI) implies filter-distributivity of the logic (see [**21**]).

LEMMA 5.5.13. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PDI), and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. Then for all $a, b \in A$, $\varphi_{\mathcal{F}}(a) \cup \varphi_{\mathcal{F}}(b) = \varphi_{\mathcal{F}}(a \vee^{\mathbf{A}} b)$.*

PROOF. Notice that since (PDI) transfers to every algebra, for all $a, b \in A$ we have $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a \vee^{\mathbf{A}} b) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(a) \cap \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(b)$. This implies, on the one hand, that $a, b \leq^{\mathbf{A}}_{\mathcal{S}} a \vee^{\mathbf{A}} b$, therefore for any $P \in \mathcal{F}$, if $a \in P$ or $b \in P$, as P is an up-set, thus $a \vee^{\mathbf{A}} b \in P$. On the other hand, we also have that $\uparrow a \cap \uparrow b \subseteq \uparrow(a \vee^{\mathbf{A}} b)$. Therefore, for any $P \in \mathcal{F}$ such that $a \notin P$ and $b \notin P$, as P^c is an strong $\mathcal{S}$-ideal by Theorem 4.4.9, we get $a \vee^{\mathbf{A}} b \notin P$. Hence we conclude that $\varphi_{\mathcal{F}}(a) \cup \varphi_{\mathcal{F}}(b) = \varphi_{\mathcal{F}}(a \vee^{\mathbf{A}} b)$. $\square$

Notice that by associativity of union, the previous lemma implies that for any non-empty $B \subseteq^{\omega} A$, $\bigcup\{\varphi_{\mathcal{F}}(b) : b \in B\} = \varphi_{\mathcal{F}}(\bigvee^{\mathbf{A}} B)$.

Let us consider first the Spectral-like duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (PDI). In the proof of the following corollary we use Lemma 5.5.13, when $\mathcal{F}$ is $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$.

COROLLARY 5.5.14. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PDI). Then $\kappa_{\mathbf{A}}$ is closed under finite intersections.*

PROOF. Since $\kappa_{\mathbf{A}} = \{\psi(a)^c : a \in A\}$, by Lemma 5.5.13 we get that for all $a, b \in A$, $\psi(a)^c \cap \psi(b)^c = (\psi(a) \cup \psi(b))^c = \psi(a \vee^{\mathbf{A}} b)^c \in \kappa_{\mathbf{A}}$. □

From the previous corollary we get the idea that for any $\mathcal{S}$-Spectral space $\langle X, \mathbf{B}\rangle$, the Spectral-dual of (PDI) is the property of $\kappa_{\mathfrak{X}}$ being closed under intersection. Let us check now that this condition is enough for recovering the disjunction in the algebras of the dual.

PROPOSITION 5.5.15. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-Spectral space such that $\kappa_{\mathfrak{X}}$ is closed under finite intersections. Then for all $U, V \in B$, $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U) \cap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(V) = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U \cup V)$.*

PROOF. Notice that the hypothesis implies that B is closed under finite unions. Then for any $W \in B$ we have:

$$\begin{aligned} W \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U) \cap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(V) \ \text{ iff } \ &(U \subseteq W \ \& \ V \subseteq W) \ \text{ iff } \ U \cup V \subseteq W \\ \text{ iff } \ &W \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U \cup V). \end{aligned}$$

□

COROLLARY 5.5.16. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-Spectral space such that $\kappa_{\mathfrak{X}}$ is closed under finite intersections. Then for all $\{U, V\} \cup \mathcal{W} \subseteq B$, $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U) \cap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, V) = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U \cup V)$.*

PROOF. This follows from filter distributivity of $\mathcal{S}$ and the previous proposition, as we have:

$$\begin{aligned} \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U) \cap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, V) &= \big(\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}) \sqcup \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U)\big) \cap \big(\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}) \sqcup \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(V)\big) \\ &= \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}) \sqcup \big(\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U) \cap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(V)\big) = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}) \sqcup \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U \cup V) \\ &= \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U \cup V). \end{aligned}$$

□

Let us consider now the Priestley-style duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (PDI). In the proof of the following corollary we use Lemma 5.5.13, when $\mathcal{F}$ is $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$.

COROLLARY 5.5.17. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PDI). Then $\vartheta[A]$ is closed under finite unions.*

From the previous corollary we get the idea that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, the Priestley-dual of (PDI) is the property of B being closed under union. As for the Spectral-like duality, it follows straightforwardly that this condition is enough in each case for recovering the disjunction in the algebras of the dual.

COROLLARY 5.5.18. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space such that B is closed under finite unions. Then for all $U, V \in B$, $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U) \cap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(V) = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(U \cup V)$.*

COROLLARY 5.5.19. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space such that B is closed under finite unions. Then for all $\{U, V\} \cup \mathcal{W} \subseteq B$, $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U) \cap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, V) = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U \cup V)$.*

To conclude, let us consider the case when $\mathcal{S}$ satisfies both (PC) and (PDI). Then it is well known that all $\mathcal{S}$-algebras have a distributive lattice reduct (see Proposition 2.8 in [**41**]) and $\mathcal{S}$-filters are the same as order filters. In this case, by corollaries 5.5.8 and 5.5.17 and Proposition 5.1.28 we know the following: if $\mathbf{A}$ has a bottom element, then $\vartheta[A]$ is the collection of clopen up-sets of $\langle \mathrm{Op}_{\mathcal{S}}(\mathbf{A}), \tau_{\mathbf{A}}, \leq\rangle$. Since in this case optimal $\mathcal{S}$-filters coincide with prime filters, what we obtain is precisely what Priestley duality for bounded distributive lattices gives us. Notice that if no bottom element is assumed, we still need to deal with $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$-admissible clopen up-sets for recovering the algebra from the space. This collection coincides with all clopen up-sets when the algebra has a bottom element, but excludes the emptyset when the algebra has no bottom element.

5.5.3. Deduction-Detachment Theorem. Let $\mathcal{S}$ be a logic that satisfies (DDT) for a non-empty set of formulas in two variables $\Delta(p, q)$. We say that *(DDT) transfers to every algebra*, if for every algebra $\mathbf{A}$, and every $\{a, b\} \cup X \subseteq A$:

$$b \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a) \quad \text{iff} \quad \Delta^{\mathbf{A}}(a, b) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X).$$

Let now $\mathcal{S}$ be a logic that satisfies (uDDT) for a term $p \to q$. We say that *(uDDT) transfers to every algebra*, if for every algebra $\mathbf{A}$, and every $\{a, b\} \cup X \subseteq A$:

$$b \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a) \quad \text{iff} \quad a \to^{\mathbf{A}} b \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X).$$

It is well known that (DDT) transfers to every algebra (see Theorem 2.48 in [**35**]). Moreover (DDT) implies filter-distributivity of the logic (see [**21**]).

LEMMA 5.5.20. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (uDDT) and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. Then for all $a, b \in A$, $(\downarrow(\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c))^c = \varphi_{\mathcal{F}}(a \to^{\mathbf{A}} b)$.*

PROOF. Since (uDDT) transfers to every algebra, for any $\{a, b\} \cup X \subseteq A$ we have $b \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X, a)$ if and only if $a \to^{\mathbf{A}} b \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(X)$. Let first $P \in \varphi_{\mathcal{F}}(a \to^{\mathbf{A}} b)$, and suppose, towards a contradiction, that $P \notin (\downarrow(\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c))^c$. Then it follows that $P \in \downarrow(\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c)$, and so there is $Q \in \mathcal{F}$ such that $P \subseteq Q$, $Q \in \varphi_{\mathcal{F}}(a)$ and $Q \notin \varphi_{\mathcal{F}}(b)$. By assumption, from $P \subseteq Q$ we get $a \to^{\mathbf{A}} b \in Q$, and then by (uDDT) we obtain $b \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(Q, a)$. Since $a \in Q$, then $b \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(Q, a) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(Q) = Q$, a contradiction. We conclude that $P \in (\downarrow(\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c))^c$, as required.

Let now $P \in \mathcal{F}$ be such that $P \notin \varphi_{\mathcal{F}}(a \to^{\mathbf{A}} b)$, i.e. $a \to^{\mathbf{A}} b \notin P$. By (uDDT) we get that $b \notin \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(P, a)$. Then by definition of optimal $\mathcal{S}$-base, there is $Q \in \mathcal{F}$ such that $b \notin Q$ and $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(P, a) \subseteq Q$. So, we have $a \in Q$, $P \subseteq Q$ and $b \notin Q$, i.e. $Q \in \varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c$, and so $P \in \downarrow(\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c)$. Therefore $P \notin (\downarrow((\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c))^c)$, as required. ☐

Notice that when the logic $\mathcal{S}$ satisfies (DDT) for $\Delta(p, q)$, the following generalization of the previous lemma also holds: for all $a, b \in A$:

$$(\downarrow(\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c))^c = \widehat{\varphi}_{\mathcal{F}}(\Delta^{\mathbf{A}}(a, b)).$$

Let us consider first the Spectral-like duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (uDDT). In the proof of the following corollary we use Lemma 5.5.20, when $\mathcal{F}$ is $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$.

COROLLARY 5.5.21. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (uDDT). Then for all $W_1, W_2 \in \kappa_{\mathbf{A}}$, $\downarrow(W_1^c \cap W_2) \in \kappa_{\mathbf{A}}$.*

PROOF. Since $\kappa_{\mathbf{A}} = \{\psi(a)^c : a \in A\}$, by Lemma 5.5.20 we get that for all $a, b \in A$, $\downarrow((\psi(a)^c)^c \cap \psi(b)^c) = \downarrow(\psi(a) \cap \psi(b)^c) = \psi(a \to^{\mathbf{A}} b)^c \in \kappa_{\mathbf{A}}$. □

From the previous corollary we get the idea that for any $\mathcal{S}$-Spectral space $\langle X, \mathbf{B}\rangle$, the Spectral-dual of (uDDT) is the property of $\kappa_{\mathfrak{X}}$ being closed under $\downarrow((\ )^c \cap (\ ))$. Let us check now that this condition is enough for recovering the implication in the algebras of the dual.

PROPOSITION 5.5.22. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-Spectral space such that for any $W_1, W_2 \in \kappa_{\mathfrak{X}}$, it holds that $\downarrow(W_1^c \cap W_2) \in \kappa_{\mathfrak{X}}$. Then for all $\{U, V\} \cup \mathcal{W} \subseteq B$:*

$$V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U) \quad \textit{iff} \quad (\downarrow(U \cap V^c))^c \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}).$$

PROOF. Assume first that $(\downarrow(U \cap V^c))^c \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})$. By Corollary 5.1.19, it is enough to show that $\bigcap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U) \subseteq V$, so let $x \in \bigcap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U)$ and suppose, towards a contradiction, that $x \notin V$. Since $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U)$, $x \in U$. By hypothesis $(\downarrow(U \cap V^c))^c \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U)$, so $x \in (\downarrow(U \cap V^c))^c$, i. e. $x \notin \downarrow(U \cap V^c)$. But from $x \notin V$ and $x \in U$ we get $x \in U \cap V^c \subseteq \downarrow(U \cap V^c)$, a contradiction.

Assume now that $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U)$. By Corollary 5.1.19, it is enough to show that $\bigcap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}) \subseteq (\downarrow(U \cap V^c))^c$. Let $x \in \bigcap \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})$ and suppose, towards a contradiction, that $x \notin (\downarrow(U \cap V^c))^c$. Then there is $y \in U \cap V^c$ such that $x \leq y$. Let $\varepsilon(y) = \{W \in B : y \in W\}$, that is an irreducible $\mathcal{S}$-filter of $\mathbf{B}$ by Lemma 5.1.17, and let $W \in \mathcal{W}$. Clearly $W \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})$, so by assumption $x \in W$. As W is an up-set, then $y \in W$, and so $W \in \varepsilon(y)$. Therefore $\mathcal{W} \subseteq \varepsilon(y)$. Moreover, since $y \in U$, we also have $U \in \varepsilon(y)$. And since $\varepsilon(y)$ is an $\mathcal{S}$-filter, then $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U) \subseteq \varepsilon(y)$. Now by hypothesis $V \in \varepsilon(y)$, i. e. $y \in V$, a contradiction. □

We can show that, under the assumption of $\mathcal{S}$ being protoalgebraic, we can also find the conditions over the dual space that make the logic to have (DDT).[2] This result is supported in the following theorem due to Czelakowski.

THEOREM 5.5.23 (Theorem 2.6.8 in [**23**]). *Let $\mathcal{S}$ be a protoalgebraic logic. Then $\mathcal{S}$ satisfies (DDT) if and only if for any $\mathcal{S}$-algebra $\mathbf{A}$, the lattice of $\mathcal{S}$-filters $\mathbf{Fi}_{\mathcal{S}}(\mathbf{A})$ is infinitely meet-distributive over its compact elements, i. e. for any $B \subseteq^{\omega} A$ and any $\{G_i : i \in I\} \subseteq \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$:*

$$\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \sqcup \bigcap_{i \in I} G_i = \bigcap_{i \in I} (\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) \sqcup G_i).$$

THEOREM 5.5.24. *Let $\mathcal{S}$ be a protoalgebraic logic such that for any $\mathcal{S}$-Spectral space $\mathfrak{X} = \langle X, \mathbf{B}\rangle$, $\downarrow(W_1^c \cap W_2) \in \kappa_{\mathfrak{X}}$ for all $W_1, W_2 \in \kappa_{\mathfrak{X}}$. Then $\mathcal{S}$ satisfies (DDT).*

[2] Recall that a logic $\mathcal{S}$ is *protoalgebraic*, following the definition of Block and Pigozzi [**7**], when for any $\mathrm{C}_{\mathcal{S}}$-closed set of formulas $\Gamma \subseteq Fm$ and any formulas $\delta, \mu \in Fm$, if $(\delta, \mu) \in \mathbf{\Omega}^{\mathbf{Fm}}(\Gamma)$, then $\Gamma, \delta \vdash_{\mathcal{S}} \mu$ and $\Gamma, \mu \vdash_{\mathcal{S}} \delta$. Remind that $\mathbf{\Omega}^{\mathbf{Fm}}(\Gamma)$ is the Leibniz congruence of Γ relative to $\mathbf{Fm}$.

Proof. Let $\mathcal{S}$ be protoalgebraic and let $\mathbf{A}$ be an $\mathcal{S}$-algebra. By Theorem 5.5.23, it is enough to show that $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$ is infinitely meet-distributive over its compact elements. By Theorem 5.1.1, the representation theorem for $\mathcal{S}$-algebras, and Corollary 5.1.11, we know that for any $\mathcal{S}$-algebra $\mathbf{A}$ there is an $\mathcal{S}$-Spectral space $\langle X, \mathbf{B}\rangle$ such that $\mathbf{A}$ is isomorphic to $\mathbf{B}$. Therefore, it is enough to show that for any $\mathcal{S}$-Spectral space $\langle X, \mathbf{B}\rangle$, $\mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$ is infinitely meet-distributive over its compact elements.

So let $\langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-Spectral space, let $\{G_i : i \in I\} \subseteq \mathrm{Fi}_{\mathcal{S}}(\mathbf{B})$ and let $U_1, \dots, U_n \subseteq^{\omega} B$. We show that

$$\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U_1, \dots, U_n\}) \sqcup \bigcap\{G_i : i \in I\} = \bigcap\{\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U_1, \dots, U_n\} \cup G_i) : i \in I\}.$$

Notice that the inclusion from left to right is immediate by finitarity of the logic, so we just have to show the other inclusion. Let $V \in \bigcap\{\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U_1, \dots, U_n\} \cup G_i) : i \in I\}$. Then for each $i \in I$ we have that $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U_1, \dots, U_n\} \cup G_i)$. For any $W_1, W_2 \in B$, let us denote $(\downarrow(W_1 \cap W_2^c))^c$ by $W_1 \Rightarrow W_2$. Then for each $i \in I$, by assumption we get $U_1 \Rightarrow (\dots (U_n \Rightarrow V) \dots) \in G_i$. Thus $U_1 \Rightarrow (\dots (U_n \Rightarrow V) \dots) \in \bigcap\{G_i : i \in I\}$. Recall that $\bigcap\{G_i : i \in I\} = \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\bigcap\{G_i : i \in I\})$ is an $\mathcal{S}$-filter of $\mathbf{B}$. So by assumption again we conclude

$$\begin{aligned} V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U_1, \dots, U_n\} \cup \bigcap\{G_i : i \in I\}) &= \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}\big(\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U_1, \dots, U_n\}) \cup \bigcap\{G_i : i \in I\}\big) \\ &= \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\{U_1, \dots, U_n\}) \sqcup \bigcap\{G_i : i \in I\}. \end{aligned}$$

□

Let us consider now the Priestley-style duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (uDDT). In the proof of the following corollary we use Lemma 5.5.20, when $\mathcal{F}$ is $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$.

Corollary 5.5.25. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (uDDT). Then for all $a, b \in A$, $(\downarrow(\vartheta(a) \cap \vartheta(b)^c))^c = \vartheta(a \to^{\mathbf{A}} b) \in \vartheta[A]$.*

From the previous corollary we get the idea that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, the Priestley-dual of (uDDT) is the property of B being closed under $(\downarrow((\ ) \cap (\ )^c))^c$. Let us check now that this condition is enough for recovering the implication.

Proposition 5.5.26. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space such that $U, V \in B$, $(\downarrow(U \cap V^c))^c \in B$. Then for all $\{U, V\} \cup \mathcal{W} \subseteq B$:*

$$V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U) \quad \textit{iff} \quad (\downarrow(U \cap V^c))^c \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}).$$

Proof. Assume first that $(\downarrow(U \cap V^c))^c \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})$. Then as the logic $\mathcal{S}$ is finitary, there is $\mathcal{W}' \subseteq^{\omega} \mathcal{W}$ a finite subset such that $(\downarrow(U \cap V^c))^c \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}')$. Thus by (Pr2), $\bigcap \mathcal{W}' \subseteq (\downarrow(U \cap V^c))^c$. We show that $U \cap \bigcap \mathcal{W}' \subseteq V$, so let $x \in U \cap \bigcap \mathcal{W}'$ and suppose, towards a contradiction, that $x \notin V$. On the one hand $x \in U$. Moreover $x \in \bigcap \mathcal{W}' \subseteq (\downarrow(U \cap V^c))^c$. i. e. $x \notin \downarrow(U \cap V^c)$. But from $x \notin V$ and $x \in U$ we get $x \in U \cap V^c \subseteq \downarrow(U \cap V^c)$, a contradiction. We conclude that $U \cap \bigcap \mathcal{W}' \subseteq V$, and thus by (Pr2), $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}') \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})$.

Assume now that $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}, U)$. Then by finitarity again, there is $\mathcal{W}' \subseteq^{\omega} \mathcal{W}$ a finite subset such that $V \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}', U)$. We show that $\bigcap \mathcal{W}' \subseteq (\downarrow(U \cap V^c))^c$. Let

$x \in \bigcap \mathcal{W}'$ and suppose, towards a contradiction, that $x \notin (\downarrow(U \cap V^c))^c$. Then there is $y \in U \cap V^c$ such that $x \leq y$. Let $\xi(y) = \{W \in B : y \in W\}$, that is an optimal $\mathcal{S}$-filter of $\mathbf{B}$ by Proposition 5.1.45, and let $W \in \mathcal{W}'$. By assumption $x \in W$ and since W is an up-set, $y \in W$, i.e. $W \in \xi(y)$. Therefore $\mathcal{W}' \subseteq \xi(y)$ and moreover, since $y \in U$, $U \in \xi(y)$. Furthermore, as $\xi(y)$ is an $\mathcal{S}$-filter $\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}', U) \subseteq \xi(y)$, so by hypothesis $V \in \xi(y)$, i.e. $y \in V$, a contradiction. Thus we conclude that $\bigcap \mathcal{W}' \subseteq (\downarrow(U \cap V^c))^c$, and then by (Pr2), $(\downarrow(U \cap V^c))^c \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}') \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})$, as required. □

THEOREM 5.5.27. *Let $\mathcal{S}$ be a protoalgebraic logic such that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, $\downarrow(U \cap V^c) \in B$ for all $U, V \in B$. Then $\mathcal{S}$ satisfies (DDT).*

PROOF. The proof is similar to that of Theorem 5.5.24 □

It would also be very interesting to study in detail the case when $\mathcal{S}$ satisfies (DDT) for $\Delta(p, q)$, but this is not our aim here since it would take too long. The case when $\Delta(p, q)$ is a finite subset of formulas seems to be simpler that the general case, because then by Lemma 5.5.20 we obtain that for any $\mathcal{S}$-algebra $\mathbf{A}$, for any optimal $\mathcal{S}$-base $\mathcal{F}$ and for any $a, b \in A$, $(\downarrow(\varphi_{\mathcal{F}}(a) \cap \varphi_{\mathcal{F}}(b)^c))^c$ belongs to $\mathrm{M}(\mathbf{A})$, the $\mathcal{S}$-semilattice of $\mathbf{A}$, that for the Priestley-style duality can be dually defined as the collection of X_B-admissible clopen up-sets. We leave this as future work.

5.5.4. Property of Inconsistent element. Let $\mathcal{S}$ be a logic that satisfies (PIE) for $\perp$. We say that *(PIE) transfers to every algebra*, if for every algebra $\mathbf{A}$ the element $\perp^{\mathbf{A}} \in A$, called the *inconsistent element*, is such that for every $a \in A$:

$$a \in \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\perp^{\mathbf{A}}).$$

It is immediate that (PIE) transfers to every algebra. Moreover, if $\mathcal{S}$ satisfies (PIE) for $\perp$, then for any $\mathcal{S}$-algebra $\mathbf{A}$, $\perp^{\mathbf{A}}$ is the bottom element of $\mathbf{A}$, and for convenience, we denote it by $0^{\mathbf{A}}$.

LEMMA 5.5.28. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PIE) and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. Then for all $a \in A$, $\varphi_{\mathcal{F}}(0^{\mathbf{A}}) \subseteq \varphi_{\mathcal{F}}(a)$.*

PROOF. Notice that, since (PIE) transfers to every algebra, we have that $\mathbf{A}$ has a bottom element $0^{\mathbf{A}}$. Then we have that for any $P \in \mathcal{F}$:

$$P \in \varphi_{\mathcal{F}}(0^{\mathbf{A}}) \quad \text{iff} \quad 0^{\mathbf{A}} \in P \quad \text{iff} \quad A \subseteq P.$$

Recall that when $\mathbf{A}$ has a bottom element, then $\emptyset \notin \mathrm{Id}_{s\mathcal{S}}(\mathbf{A})$, so optimal $\mathcal{S}$-filters are proper. In particular, all elements of $\mathcal{F}$ are proper, and we get $\varphi_{\mathcal{F}}(0^{\mathbf{A}}) = \emptyset$, so it follows trivially that $\varphi_{\mathcal{F}}(0^{\mathbf{A}}) = \emptyset \subseteq \varphi_{\mathcal{F}}(a)$ for all $a \in A$. □

Let us consider first the Spectral-like duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (PIE). In the proof of the next corollary we use Lemma 5.5.28 when $\mathcal{F}$ is $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$.

COROLLARY 5.5.29. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PIE). Then $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$ is compact.*

PROOF. Let $0^{\mathbf{A}}$ be the inconsistent element of A. By Lemma 5.5.28 $\psi(0^{\mathbf{A}}) = \emptyset$, and therefore $X \in \kappa_{\mathbf{A}} = \{\psi(a)^c : a \in A\}$, that is a collection of open compact elements. Hence, in particular X is compact. □

From the previous corollary we get the idea that for any $\mathcal{S}$-Spectral space $\langle X, \mathbf{B}\rangle$, the Spectral-dual of (PIE) is the property of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ being compact. Let us check now that this condition is enough for recovering the inconsistent element.

COROLLARY 5.5.30. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-Spectral space such that $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ is compact. Then there is $W \in B$ such that for all $U \in B$, $U \in \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(W)$.*

PROOF. By hypothesis we get that $\emptyset \in B$. Moreover, since $\emptyset \subseteq U$ for all $U \in B$, by definition of $\mathrm{C}_{\mathcal{S}}^{\mathbf{B}}$ we get $U \in \mathrm{C}_{\mathcal{S}}^{\mathbf{B}}(\emptyset)$. Hence $\emptyset$ is the required inconsistent element. $\square$

THEOREM 5.5.31. *Let $\mathcal{S}$ be a logic such that for any $\mathcal{S}$-Spectral space $\mathfrak{X} = \langle X, \mathbf{B}\rangle$, the space $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ is compact. Then $\mathcal{S}$ satisfies (PIE).*

PROOF. Recall that the Lindenbaum-Tarski algebra $\mathbf{Fm}^* = \mathbf{Fm}/\equiv_{\mathcal{S}}^{\mathbf{Fm}}$ is an $\mathcal{S}$-algebra, so the structure $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{Fm}^*), \psi[\mathbf{Fm}^*]\rangle$ is an $\mathcal{S}$-Spectral space. For any formula $\mu \in Fm$, we denote by $\overline{\mu}$ its equivalence class in $\mathbf{Fm}^*$. By assumption and Corollary 5.5.30 there is $\delta \in Fm$ such that for all $\mu \in Fm$, $\psi(\overline{\mu}) \in \mathrm{C}_{\mathcal{S}}^{\psi[\mathbf{Fm}^*]}(\psi(\overline{\delta}))$. Then by Corollary 4.3.7 we obtain that for all $\mu \in Fm$, $\overline{\mu} \in \mathrm{C}_{\mathcal{S}}^{\mathbf{Fm}^*}(\overline{\delta})$. Using again that the projection map is a bilogical morphism, we get that for all $\mu \in Fm$, $\mu \in \mathrm{C}_{\mathcal{S}}(\delta)$. It is immediate that $\mathcal{S}$ satisfies (PIE) for δ. $\square$

COROLLARY 5.5.32. *Let $\mathcal{S}$ be a logic. Then $\mathcal{S}$ satisfies (PIE) if and only if for any $\mathcal{S}$-Spectral space $\mathfrak{X} = \langle X, \mathbf{B}\rangle$, the space $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ is compact.*

Let us consider now the Priestley-style duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (PIE). In the proof of the next corollary we use Lemma 5.5.28 when $\mathcal{F}$ is $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$.

COROLLARY 5.5.33. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PIE) and let $0^{\mathbf{A}}$ be the inconsistent element of $\mathbf{A}$. Then $\vartheta(0^{\mathbf{A}}) = \emptyset \in \vartheta[A]$.*

THEOREM 5.5.34. *Let $\mathcal{S}$ be a logic such that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, $\emptyset \in B$. Then $\mathcal{S}$ satisfies (PIE).*

PROOF. The proof is similar to that of Theorem 5.5.31. $\square$

COROLLARY 5.5.35. *Let $\mathcal{S}$ be a logic. Then $\mathcal{S}$ satisfies (PIE) if and only if for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$ it holds that $\emptyset \in B$.*

Observe that both in the Spectral-like and in the Priestley-style duality, when the logic $\mathcal{S}$ satisfies (PIE), we have that $\emptyset$ is the inconsistent element in $\mathbf{B}$, so the inconsistent element in the referential algebra $\mathbf{B}$ is represented by the emptyset in both cases.

To conclude, consider now the Priestley-style duality when $\mathcal{S}$ satisfies both (PC) and (PIE). Then we know that $\mathbf{B} = \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$, so in this case Corollary 5.5.30 is equivalent to say that $\max(X) \subseteq X_B$, or in other words, $\downarrow X_B = X$. In fact this property corresponds, in general, with the property of the $\mathcal{S}$-algebras of having a bottom-family.

5.5.5. Property of being closed under introduction of a modality. Let $\mathcal{S}$ be a logic that satisfies (PIM) for a term $\Box p$. We say that *(PIM) transfers to every algebra*, if for every algebra $\mathbf{A}$, and every $X \subseteq A$:

$$\Box^{\mathbf{A}}(\mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(X)) \subseteq \mathrm{C}_{\mathcal{S}}^{\mathbf{A}}(\Box^{\mathbf{A}} X).$$

For convenience, let us denote $\Box^{\mathbf{A}}$ as $\Box$. It is well known that (PIM) transfers to every algebra (see Proposition 2.56 in [**35**]). Notice that this implies that for any algebra $\mathbf{A}$ and any $B, B' \subseteq A$:

$$\text{if } \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B'), \text{ then } \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\Box[B]) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\Box[B']).$$

Suppose that $f : A \longrightarrow A$ is a map such that for any $B, B' \subseteq^{\omega} A$ finite subsets:

$$\text{if } \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(B'), \text{ then } \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(f[B]) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(f[B']).$$

Notice that this condition implies that f preserves the top element of $\mathbf{A}$. Moreover, we may extend f in a unique way to the $\mathcal{S}$-semilattice of $\mathbf{A}$ as follows. Let $\widehat{f} : \mathrm{M}(\mathbf{A}) \longrightarrow \mathrm{M}(\mathbf{A})$ be such that

$$\widehat{f}(\widehat{\varphi}(B)) := \widehat{\varphi}(f[B]).$$

By definition $\widehat{f}$ is well defined, and moreover it is an homomorphism between distributive semilattices. By either Spectral-like or Priestley-style duality for distributive semilattices, we already know how to *dualize* it by a relation. Thus we could take such relation as the dual of f. This is precisely what we do in detail in what follows, for the case where the function f is precisely $\Box$. It would be very interesting to investigate whether the same method could be generalized for any n-ary f satisfying a similar property.

LEMMA 5.5.36. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PIM) and let $\mathcal{F}$ be an optimal $\mathcal{S}$-base. Then for all $a \in A$, $\varphi_{\mathcal{F}}(\Box(a)) = \Box_{\widetilde{R}_{\Box}}(\varphi_{\mathcal{F}}(a))$, where $\widetilde{R}_{\Box} \subseteq \mathcal{F} \times \mathcal{F}$ is given by:*

$$(P, Q) \in \widetilde{R}_{\Box} \quad \textit{iff} \quad \Box^{-1}[P] \subseteq Q.$$

PROOF. First we show that for any $F \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$, $\Box^{-1}[F] \in \mathrm{Fi}_{\mathcal{S}}(\mathbf{A})$. From (PIM) we have that $\Box(\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\Box^{-1}[F])) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\Box(\Box^{-1}[F]))$, but since $\Box(\Box^{-1}[F]) \subseteq F$, then we get $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\Box(\Box^{-1}[F])) \subseteq \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(F) = F$. Thus $\Box(\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\Box^{-1}[F])) \subseteq F$, and so $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\Box^{-1}[F]) \subseteq \Box^{-1}[F]$. We conclude that the set $\Box^{-1}[F] = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}}(\Box^{-1}[F])$ is an $\mathcal{S}$-filter of $\mathbf{A}$.

Let us prove now the statement of the lemma. By definition we have that $\varphi_{\mathcal{F}}(\Box(a)) = \{P \in \mathcal{F} : \Box(a) \in P\}$. Let first $P \in \varphi_{\mathcal{F}}(\Box(a))$. We show that $P \in \Box_{\widetilde{R}_{\Box}}(\varphi_{\mathcal{F}}(a)) = \{P \in \mathcal{F} : \widetilde{R}_{\Box}(P) \subseteq \varphi_{\mathcal{F}}(a)\}$. Let $Q \in \widetilde{R}_{\Box}(P)$, i. e. $\Box^{-1}[P] \subseteq Q$. By assumption $\Box(a) \in P$, so $a \in \Box^{-1}[P] \subseteq Q$. Hence $Q \in \varphi_{\mathcal{F}}(a)$, as required. Let now $P \in \Box_{\widetilde{R}_{\Box}}(\varphi_{\mathcal{F}}(a))$, i. e. $\widetilde{R}_{\Box}(P) \subseteq \varphi_{\mathcal{F}}(a)$, and suppose, towards a contradiction, that $\Box(a) \notin P$. Then $a \notin \Box^{-1}[P]$, that is an $\mathcal{S}$-filter. Therefore by definition of optimal $\mathcal{S}$-base, there is $Q \in \mathcal{F}$ such that $a \notin Q$ and $\Box^{-1}[P] \subseteq Q$. Then $Q \in \widetilde{R}_{\Box}(P) \setminus \varphi_{\mathcal{F}}(a)$, a contradiction. $\Box$

Let us consider first the Spectral-like duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (PIM). In the proof of the following corollaries we use Lemma 5.5.36, when $\mathcal{F}$ is $\mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$.

COROLLARY 5.5.37. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PIM). For all $P \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A})$, $\overline{R}_{\Box}(P)$ is a closed set of $\langle \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$.*

Proof. Notice that by definition $\overline{R}_{\Box}(P) = \{Q \in \mathrm{Irr}_{\mathcal{S}}(\mathbf{A}) : \Box^{-1}[P] \subseteq Q\} = \widehat{\psi}(\Box^{-1}[P])$. Therefore, by Proposition 5.1.6, since $\Box^{-1}[P]$ is an $\mathcal{S}$-filter, $\overline{R}_{\Box}(P)$ is closed. $\Box$

From the previous corollary we get the idea that for any $\mathcal{S}$-Spectral space $\langle X, \mathbf{B}\rangle$, the Spectral-dual of (PIM) is the following pair or properties: $\kappa_{\mathfrak{X}}$ is closed under $(\Box_{\overline{R}_{\Box}}((\ )^c))^c$ and $\overline{R}_{\Box}(\ )$ maps elements of the dual space to closed subsets. Let us check now that this condition is enough for recovering the operation $\Box$ in the algebras of the dual.

Proposition 5.5.38. *Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{S}$-Spectral space and let $\overline{R} \subseteq X \times X$ be a binary relation such that $\Box_{\overline{R}}(U) \in B$ for all $U \in B$ and $\overline{R}(x)$ is a closed set of $\langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ for all $x \in X$. Then for all $\mathcal{W} \subseteq B$:*

$$\Box_{\overline{R}}[\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})] \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\Box_{\overline{R}}[\mathcal{W}]).$$

Proof. Let $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})$. We show that $\Box_{\overline{R}}(U) \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\Box_{\overline{R}}[\mathcal{W}])$. As $\mathcal{S}$ is finitary, there is $\mathcal{W}' \subseteq^{\omega} \mathcal{W}$ a finite subset such that $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}')$. We claim that $\bigcap \Box_{\overline{R}}[\mathcal{W}'] \subseteq \Box_{\overline{R}}(U)$. Let $x \in \bigcap \Box_{\overline{R}}[\mathcal{W}']$, so by definition $\overline{R}(x) \subseteq W$ for all $W \in \mathcal{W}'$. Since by hypothesis $\overline{R}(x)$ is closed, by Lemma 5.1.14 we obtain that $\widehat{\varepsilon}(\overline{R}(x)) = \{U \in B : \overline{R}(x) \subseteq U\}$ is an $\mathcal{S}$-filter of $\mathbf{B}$. Therefore, as $W \in \widehat{\varepsilon}(\overline{R}(x))$ for all $W \in \mathcal{W}'$, from $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}')$ we get $U \in \widehat{\varepsilon}(\overline{R}(x))$, i.e. $\overline{R}(x) \subseteq U$, and therefore $x \in \Box_{\overline{R}}(U)$.

Now from the claim and (Sp2) we get $\Box_{\overline{R}}(U) \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\Box_{\overline{R}}[\mathcal{W}'])$, and hence $\Box_{\overline{R}}(U) \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\Box_{\overline{R}}[\mathcal{W}])$, as required. $\Box$

Let us consider now the Priestley-style duality for $\mathcal{S}$-algebras, when $\mathcal{S}$ satisfies (PIM). In the proof of the following corollary we use Lemma 5.5.36, when $\mathcal{F}$ is $\mathrm{Op}_{\mathcal{S}}(\mathbf{A})$.

Corollary 5.5.39. *Let $\mathbf{A}$ be an $\mathcal{S}$-algebra for a logic $\mathcal{S}$ that satisfies (PIM). For all $P, Q \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$, if $\Box^{-1}[P] \nsubseteq Q$, then there is $a \in A$ such that $Q \notin \vartheta(a)$ and $R_{\Box}(P) \subseteq \vartheta(a)$.*

Proof. Let $P, Q \in \mathrm{Op}_{\mathcal{S}}(\mathbf{A})$ be such that $\Box^{-1}[P] \nsubseteq Q$, and let $a \in \Box^{-1}[P] \setminus Q$. Let $Q' \in R_{\Box}(P)$, i.e. $\Box^{-1}[P] \subseteq Q'$. Then clearly by hypothesis $a \in Q'$, i.e. $Q' \in \vartheta(a)$. Therefore $R_{\Box}(P) \subseteq \vartheta(a)$ but $Q \notin \vartheta(a)$, as required. $\Box$

From the previous corollary we get the idea that for any $\mathcal{S}$-Priestley space $\langle X, \tau, \mathbf{B}\rangle$, the Priestley-dual of (PIM) is $\mathbf{B}$ being closed under $\Box_{R_{\Box}}$. Let us see now that this condition is enough to recover the operation $\Box$ in the algebras of the dual.

Proposition 5.5.40. *Let $\langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{S}$-Priestley space and let $R \subseteq X \times X$ be a binary relation such that $\Box_R(U) \in B$ for all $U \in B$. Then for all $\mathcal{W} \subseteq B$:*

$$\Box_R[\mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})] \subseteq \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\Box_R[\mathcal{W}]).$$

Proof. Let $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W})$. We show that $\Box_R(U) \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\Box_R[\mathcal{W}])$. As $\mathcal{S}$ is finitary, there is $\mathcal{W}' \subseteq^{\omega} \mathcal{W}$ a finite subset such tat $U \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\mathcal{W}')$. Then by condition (Pr2) $\bigcap \mathcal{W}' \subseteq U$. We claim that $\bigcap \Box_R[\mathcal{W}'] \subseteq \Box_R(U)$. Let $x \in \bigcap \Box_R[\mathcal{W}']$, so $x \in \Box_R(W) = \{y \in X : R(y) \subseteq W\}$ for all $W \in \mathcal{W}'$. Then $R(x) \subseteq \bigcap \mathcal{W}'$, that by hypothesis is included in U, so $R(x) \subseteq U$, i.e. $x \in \Box_R(U)$.

TABLE 6. Correspondence for the Spectral-like duality.

PROPERTY	DUAL CORRESPONDENCE	REPRESENTATION IN $\mathbf{B}$
(PC)	$\mathcal{KO}(X) = \kappa_{\mathfrak{X}}$	$\mathbf{B}$ closed under $\cap$
(PDI)	$\kappa_{\mathfrak{X}}$ closed under $\cap$	$\mathbf{B}$ closed under $\cup$
(uDDT)	$\kappa_{\mathfrak{X}}$ closed under $\downarrow((\ )^c \cap (\ ))$	$\mathbf{B}$ closed under $(\downarrow((\ ) \cap (\ )^c))^c$
(PIE)	$\langle X, \tau_{\kappa_{\mathfrak{X}}} \rangle$ is compact	$\emptyset \in B$
(PIM)	$\kappa_{\mathfrak{X}}$ is closed under $(\Box_{\overline{R}_\Box}((\ )^c))^c$ and $\overline{R}_\Box(\ )$ maps elements of the dual space to closed subsets	$\mathbf{B}$ closed under $\Box_{\overline{R}_\Box}$

From the claim and (Pr2) we get $\Box_R(U) \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\Box_R[\mathcal{W}'])$, and hence $\Box_R(U) \in \mathrm{C}^{\mathbf{B}}_{\mathcal{S}}(\Box_R[\mathcal{W}])$, as required. $\Box$

Observe that both in the Spectral-like and in the Priestley-style duality, when the logic $\mathcal{S}$ satisfies (PIM), we have that $\Box$ in the referential algebra $\mathbf{B}$ is represented by $\Box_{R_\Box}$.

It would be very interesting to make a deep study into the correspondence theory between properties that $\Box$ may satisfy, and properties of its dual relation. For example, it is easy to see that for any logic $\mathcal{S}$ satisfying (PIM), the property that for any $\Gamma \subseteq Fm$:

$$\Box\Gamma \vdash \gamma \text{ for all } \gamma \in \Gamma.$$

corresponds with the dual relation being reflexive. Similarly, the property that for any $\Gamma \subseteq Fm$:

$$\Box\Gamma \vdash \Box\Box\gamma \text{ for all } \gamma \in \Gamma.$$

corresponds with the dual relation being transitive. These results generalize the well-known results of correspondence theory for normal modal logics. We do not go further into this topic, but we leave it as future work.

5.5.6. Summary of results. Tables 6 and 7 summarize what we have seen throughout this section. Notice that as a preliminary conclusion we could say that the Spectral-like duality allows us to carry out a smoother modular analysis than the Priestley-style duality.

Table 7. Correspondence for the Priestley-style duality.

Property	Dual correspondence	Representation in $\mathbf{B}$
(PC)	$\mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X) = B$	$\mathbf{B}$ closed under $\cap$
(PDI)	$\mathbf{B}$ closed under $\cup$	$\mathbf{B}$ closed under $\cup$
(uDDT)	$\mathbf{B}$ closed under $(\downarrow((\ ) \cap (\ )^c))^c$	$\mathbf{B}$ closed under $(\downarrow((\ ) \cap (\ )^c))^c$
(PIE)	$\emptyset \in B$	$\emptyset \in B$
(PIM)	$\mathbf{B}$ closed under $\Box_{R_\Box}$	$\mathbf{B}$ closed under $\Box_{R_\Box}$

Part 3

Applications to Expansions of the Implicative Fragment of Intuitionistic Logic

CHAPTER 6

Filter Distributive and Congruential Expansions of the Implicative Fragment of Intuitionistic Logic

In this chapter we define several filter distributive finitary congruential logics with theorems that are expansions of the implicative fragment of intuitionistic logic, and we study how our results from Chapter 5 can be applied to obtain dualities for such logics. In § 6.1 we motivate our study of such logics, in relation with extended Priestley duality for distributive lattices expansions. We explain why, instead of keep dealing with an abstract framework, we restrict ourselves to a more concrete setting. What we do, from an algebraic point of view, is looking at expansions of Hilbert algebras. More precisely, we consider varieties of algebras with a Hilbert algebra reduct.

In § 6.2 we recall how $\mathcal{H}$, the implicative fragment of intuitionistic logic, is axiomatized, and we study Spectral-like and Priestley-style dualities for it as a particular case of our theory in Chapter 5. So we recover, on the one hand, the Spectral-like duality for Hilbert algebras that Celani et al. studied in [**15**] (see § 3.3.1). On the other hand, we obtain a Priestley-style duality for Hilbert algebras that slightly simplifies the one presented by Celani and Jansana in [**18**] (see § 3.3.2).

In § 6.3 we consider a class of algebras expanded with a modal operator, that yields a modal expansion of $\mathcal{H}$ for which dualities can be studied following our general approach. In § 6.4 we do the same for a class of algebras expanded with a supremum, that yields an expansion of $\mathcal{H}$ with a disjunction, that fits in the framework of our theory. In § 6.5 we consider classes of algebras expanded with a conjunction. One of them corresponds to the implicative-conjunctive fragment of intuitionistic logic, and it is suitable for our general theory. The others are wider classes of algebras for which the general theory is not completely satisfactory, so we aim to develop new tools that yield *nice* dualities for such algebras. In § 6.5.1 we study in depth *Distributive Hilbert algebras with infimum*, and in the next chapter we develop Spectral-like and Priestley-style dualities for these algebras.

Unfortunately, the class of Distributive Hilbert algebras with infimum is not associated with any logic, but we consider in § 6.6 and § 6.7 other classes related with it, that yield expansions of $\mathcal{H}$ for which the general theory is neither satisfactory, but for which the results in Chapter 7 can be applied to get new dualities.

6.1. Introduction and motivation

Extended Priestley duality provided inspiration for our work on applications of what was studied in Chapter 5. From an algebraic point of view, the main idea behind extended Priestley duality is the following: from Priestley duality for distributive lattices, Priestley-style dualities for expansions of distributive lattices shall be developed just by *expanding* Priestley spaces. And this can be done in a modular way when the expansions of the distributive lattices are *nice* enough. Let us review this with more detail, and translate these ideas into logic.

Extended Priestley duality provides a uniform approach to Priestley-style dualities for a wide range of distributive lattice-based algebras. It generalizes the work by Jónsson and Tarski [**58**, **59**] on Boolean algebras with operators. A *distributive lattice expansion*, is a structure $\mathbf{A} = \langle A, \wedge, \vee, f : f \in \mathcal{F}\rangle$, such that $\langle A, \wedge, \vee\rangle$ is a distributive lattice. We denote by $\mathbb{DLE}$ the class of distributive lattice expansions. A *bounded distributive lattice expansion* is a structure $\mathbf{A} = \langle A, \wedge, \vee, 1, 0, f : f \in \mathcal{F}\rangle$ such that $\langle A, \wedge, \vee, 1, 0\rangle$ is a bounded distributive lattice, in which 1 is the top element and 0 is the bottom element.

Goldblatt [**46**] develops a general duality theory for bounded $\mathbb{DLE}$'s in which the additional operations are normal operators or normal dual operators. We recall that for $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ an arbitrary bounded lattice, an n-ary function $f : L^n \longrightarrow L$ is an *operator* (resp. *dual operator*) provided f preserves non-empty finite joins (resp. meets) in each coordinate. Moreover, f is a *normal operator* (resp. normal dual operator) provided f is an operator (resp. dual operator) that preserves arbitrary finite joins (resp. meets) in each coordinate.

In [**46**], Priestley duality for distributive lattices is used to get the basic building block over which the dual spaces are constructed. The additional normal (dual) n-ary operators are dually represented by additional $n+1$-ary relations on the dual Priestley space. This theory can be generalized to distributive lattice expansions in which the additional operations are (dual) quasioperators, this is precisely what is known as *extended Priestley duality.* We recall that for an arbitrary bounded lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$, an n-ary function $f : L^n \longrightarrow L$ is a *(ϵ)-quasioperator* (resp. *dual (ϵ)-quasioperator*) provided there is an n-tuple $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_i \in \{1, \partial\}$ for each $i \leq n$, such that $f : L^\epsilon \longrightarrow L$ is an operator (resp. dual operator). Notice that by L^ϵ we denote $L^{\epsilon_1} \times \cdots \times L^{\epsilon_n}$.

Translating these ideas into logic, from this theory we get Kripke-style semantics for logics that are expansions of the conjunctive-disjunctive fragment of classical logic. Extended Priestley duality and what can be called *extended Spectral-like duality* that can be worked out in a similar way, fit well into the framework developed in Chapter 5. Let $\mathbb{K}$ be a variety of bounded distributive lattices with quasioperators and dual quasioperators in the language $\mathscr{L}$. The points of the dual Priestley space of an algebra $\mathbf{A} \in \mathbb{K}$, as well as the points of its Spectral dual space, are the prime filters of the distributive lattice reduct of $\mathbf{A}$. These are the irreducible $\mathcal{S}^{\leq}_{\mathbb{K}}$-filters of $\mathbf{A}$, which in this case coincide with the optimal $\mathcal{S}^{\leq}_{\mathbb{K}}$-filters of $\mathbf{A}$, where recall that the finitary logic $\mathcal{S}^{\leq}_{\mathbb{K}}$ is the *semilattice based logic of* $\mathbb{K}$, that we defined in page 29 as follows: for Γ any a non-empty finite set of formulas and any formula δ:

$$\Gamma \vdash^{\leq}_{\mathbb{K}} \delta \quad \text{iff} \quad (\forall \mathbf{A} \in \mathbb{K})(\forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))\ \Big(\bigwedge_{\gamma \in \Gamma}^{\mathbf{A}} h(\gamma)\Big) \leq h(\delta).$$

For Γ the empty set of formulas and any formula δ:

$$\emptyset \vdash^{\leq}_{\mathbb{K}} \delta \quad \text{iff} \quad \vDash_{\mathbb{K}} \delta \approx 1.$$

And for Γ any arbitrary set of formulas and any formula δ:

$$\Gamma \vdash^{\leq}_{\mathbb{K}} \delta \quad \text{iff} \quad (\exists \Gamma' \subseteq^{\omega} \Gamma)\, \Gamma' \vdash^{\leq}_{\mathbb{K}} \delta.$$

By definition $\mathcal{S}^{\leq}_{\mathbb{K}}$ is finitary. It follows from Proposition 3.13 in [**53**] that it is congruential, and also that $\mathbb{A}\mathrm{lg}\mathcal{S}^{\leq}_{\mathbb{K}} = \mathbb{K}$ (given that $\mathbb{K}$ is a variety). Moreover it is an expansion of the logic $\mathcal{S}^{\leq}_{\mathbb{BDL}}$, where $\mathbb{BDL}$ stands for the variety of bounded distributive lattices. Furthermore, from Lemma 3.8 in [**53**] we get that for any $\mathbf{A} \in \mathbb{K}$, the $\mathcal{S}^{\leq}_{\mathbb{K}}$-filters of $\mathbf{A}$ are the lattice filters. This implies that $\mathcal{S}^{\leq}_{\mathbb{K}}$ is filter distributive, since the lattice of filters of any distributive lattice is a distributive lattice. Thus, $\mathcal{S}^{\leq}_{\mathbb{K}}$ fits in the framework developed in Chapter 5. Moreover $\mathcal{S}^{\leq}_{\mathbb{K}}$ satisfies (PC) and (PDI), so we shall simplify the definition of the dual spaces according to what was investigated in § 5.5.

We are interested in applying a similar strategy in a different setting, using what we studied in Chapter 5. But instead of keeping a fully abstract approach, we restrict ourselves to the following case: we investigate dualities for classes of algebras that correspond to filter distributive finitary congruential logics with theorems that expand the implicative fragment of intuitionistic logic $\mathcal{H}$. We want to emphasize the similarities between the two approaches: in what follows $\mathbb{A}\mathrm{lg}\mathcal{H}$ will play the role that $\mathbb{BDL}$ did in extended Priestley duality. One of the main contributions of this approach is that it allows us to tackle dualities for varieties of distributive lattice expansions that do not fall under the scope of extended Priestley duality.

6.2. The implicative fragment of intuitionistic logic

Let $\mathcal{H}$ be the implicative fragment of intuitionistic logic, i. e. the logic $\mathcal{H} := \langle \mathbf{Fm}, \vdash_{\mathcal{H}} \rangle$ in the language $(\to, 1)$ of type $(2, 0)$, where $\vdash_{\mathcal{H}}$ is the restriction of the intuitionistic logic (as a closure relation) to the formulas in the language $(\to, 1)$. The logic $\mathcal{H}$ can be presented in a Hilbert-style calculus by the following axioms and rules:

(A1) $\vdash_{\mathcal{H}} \beta \to (\gamma \to \beta)$,
(A2) $\vdash_{\mathcal{H}} (\gamma \to (\beta \to \delta)) \to ((\gamma \to \beta) \to (\gamma \to \delta))$,
(MP) $\gamma, \gamma \to \beta \vdash_{\mathcal{H}} \beta$.

The logic $\mathcal{H}$ is the least finitary logic that satisfies (uDDT) for $p \to q$.[1] Clearly $\mathcal{H}$ has theorems, as for any $\gamma \in Fm$, $\gamma \to \gamma \in \mathrm{Thm}\mathcal{H}$. It is well known that the logic $\mathcal{H}$ is equal to the 1-assertional logic $\mathcal{S}^{1}_{\mathbb{H}}$ of $\mathbb{H}$, where $\mathbb{H}$ is the variety of Hilbert algebras (definition in page 18). And for any Hilbert algebra $\mathbf{A}$, we have $\mathrm{Fi}_{\mathcal{H}}(\mathbf{A}) = \mathrm{Fi}_{\to}(\mathbf{A})$.

Theorem 6.2.1. *The logics $\mathcal{H}$, $\mathcal{S}^{\to}_{\mathbb{H}}$ and $\mathcal{S}^{\leq}_{\mathbb{H}}$ are equal.*

Proof. Assume first that $\Gamma \vdash_{\mathcal{H}} \delta$. Then either δ is a theorem, in which case for every $\mathbf{A} \in \mathbb{H}$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ we have that $h(\delta) = 1^{\mathbf{A}}$, or there are $\gamma_0, \ldots, \gamma_n \in \Gamma$ such that $\gamma_0, \ldots, \gamma_n \vdash_{\mathcal{H}} \delta$. In the last case, by (uDDT) we get $\vdash_{\mathcal{H}} \gamma_0 \to (\gamma_1 \to (\ldots(\gamma_n \to \delta)\ldots))$. So if $\mathbf{A} \in \mathbb{H}$ and $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, then $h(\gamma_0 \to (\gamma_1 \to (\ldots(\gamma_n \to \delta)\ldots))) = 1^{\mathbf{A}}$, and therefore $\Gamma \vdash^{\to}_{\mathbb{H}} \delta$. And if $a \in A$ is

[1] This follows from results in Porębska and Wroński [**64**] and it is also remarked in Corollary 2.4.3 in [**73**].

such that $a \leq^{\mathbf{A}} h(\gamma_i)$ for every $i \leq n$, it easily follows that $a \leq^{\mathbf{A}} h(\delta)$. We show it by induction on n. If $n = 0$, then we have $\vdash_{\mathcal{H}} \gamma_0 \to \delta$ and $a \leq^{\mathbf{A}} h(\gamma_0)$. This implies $h(\gamma_0) \to^{\mathbf{A}} h(\delta) = 1^{\mathbf{A}}$, and by definition of the order on a Hilbert algebra, we obtain $h(\gamma_0) \leq^{\mathbf{A}} h(\delta)$, and so we are done. Suppose now that the hypothesis holds for n and let $\gamma_0, \dots, \gamma_n, \gamma_{n+1} \vdash_{\mathcal{H}} \delta$. Then by (uDDT) $\gamma_0, \dots, \gamma_n \vdash_{\mathcal{H}} \gamma_{n+1} \to \delta$, therefore, if $a \in A$ is such that $a \leq^{\mathbf{A}} h(\gamma_i)$ for every $i \leq n+1$, by the induction hypothesis it follows that $a \leq h(\gamma_{n+1}) \to^{\mathbf{A}} h(\delta)$. Hence, since $a \leq^{\mathbf{A}} h(\gamma_{n+1})$ we obtain $a \leq^{\mathbf{A}} h(\delta)$. We conclude that $\Gamma \vdash^{\leq}_{\mathbb{H}} \delta$.

For the converse, assume first that $\Gamma \vdash^{\to}_{\mathbb{H}} \delta$. Then either for every $\mathbf{A} \in \mathbb{H}$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ we have $h(\delta) = 1^{\mathbf{A}}$, or there are $\gamma_0, \dots, \gamma_n \in \Gamma$ such that for every $\mathbf{A} \in \mathbb{H}$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $h(\gamma_0 \to (\gamma_1 \to (\dots (\gamma_n \to \delta) \dots))) = 1^{\mathbf{A}}$. This implies $\vdash_{\mathcal{H}} \gamma_0 \to (\gamma_1 \to (\dots (\gamma_n \to \delta) \dots))$, and by (uDDT) we have $\Gamma \vdash_{\mathcal{H}} \delta$.

Finally, assume that $\Gamma \vdash^{\leq}_{\mathbb{H}} \delta$. Then there is $\Gamma' \subseteq^{\omega} \Gamma$ such that $\Gamma' \vdash^{\leq}_{\mathbb{H}} \delta$. Suppose, towards a contradiction, that $\Gamma' \nvdash_{\mathcal{H}} \delta$. Then there is a Hilbert algebra $\mathbf{A}$ and $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$ such that $h(\gamma) = 1^{\mathbf{A}}$ for all $\gamma \in \Gamma'$ and $h(\delta) \neq 1^{\mathbf{A}}$. Then $1^{\mathbf{A}} \leq^{\mathbf{A}} h(\gamma)$ for all $\gamma \in \Gamma'$ but $1^{\mathbf{A}} \nleq^{\mathbf{A}} h(\delta)$, contrary to the assumption. We conclude $\Gamma' \vdash_{\mathcal{H}} \delta$, and therefore $\Gamma \vdash_{\mathcal{H}} \delta$. □

The previous theorem implies, by Proposition 7 in [**54**], that $\mathcal{H}$ is selfextensional, and then by Theorem 4.46 in [**35**] it follows that $\mathcal{H}$ is congruential. Moreover, since $\mathcal{H}$ satisfies (uDDT), then it is filter distributive.

Thereupon, we are in a framework in which the theory exhibited in Chapter 5 can be straightforwardly applied. Let us briefly review how the definitions of $\mathcal{H}$-Spectral space and $\mathcal{H}$-Priestley space might be simplified using what we studied in § 5.5. In what follows, let us denote by $\Rightarrow$ the binary operation $(\downarrow((\ ) \cap (\ )^c))^c$. First we prove two useful propositions.

Proposition 6.2.2. *Let $\langle X, \mathbf{B}\rangle$ be a reduced referential algebra such that $X \in B$, $B \subseteq \mathcal{P}^{\uparrow}(X)$ and B is closed under $\Rightarrow$. Then for any $U_0, \dots, U_n, V \in B$,*

$$\bigcap\{U_i : i \leq n\} \subseteq V \quad \textit{iff} \quad U_0 \Rightarrow (\dots (U_n \Rightarrow V) \dots) = X.$$

Proof. Assume first that $\bigcap\{U_i : i \leq n\} \subseteq V$ and suppose, towards a contradiction, that there is $x \in X$ such that $x \notin U_0 \Rightarrow (\dots (U_n \Rightarrow V))$. Then by definition of $\Rightarrow$, there is $x_0 \geq x$ such that $x_0 \in U_0$ and $x_0 \notin U_1 \Rightarrow (\dots (U_n \Rightarrow V))$. Similarly, we get that there is $x_1 \geq x_0$ such that $x_1 \in U_1$ and $x_1 \notin U_0 \Rightarrow (\dots (U_n \Rightarrow V))$. Iterating this process n times we obtain $x \leq x_0 \leq \dots x_n$ such that $x_i \in U_i$ and $x_n \notin V$. By assumption U_i is an up-set for all $i \leq n$, so $x_n \in \bigcap\{U_i : i \leq n\}$. But then by assumption we have $x \in V$, a contradiction.

Assume now that $U_0 \Rightarrow (\dots (U_n \Rightarrow V)) = X$ and let $x \in \bigcap\{U_i : i \leq n\}$. Notice that for each $i \leq n$, if $x \in U_i \Rightarrow Y$ for some $Y \subseteq X$, implies $x \notin U_i \cap Y^c$. Therefore, by hypothesis we get $x \in Y$. Thus, for any $i \leq n$, from $x \in U_i \Rightarrow (\dots (U_n \Rightarrow V) \dots)$ we obtain $x \in U_{i+1} \Rightarrow (\dots (U_n \Rightarrow V) \dots)$, and hence $x \in V$, as required. □

Proposition 6.2.3. *Let $\langle X, \mathbf{B}\rangle$ be a reduced referential algebra such that $X \in B$, $B \subseteq \mathcal{P}^{\uparrow}(X)$ and B is closed under $\Rightarrow$. Then $\langle X, \langle B, \Rightarrow, X\rangle\rangle$ is an $\mathcal{H}$-referential algebra.*

Proof. As the referential algebra is reduced, $\langle X, \leq\rangle$ is a poset. It is easy to see that then $\langle \mathcal{P}^{\uparrow}(X), \Rightarrow, X\rangle$ is a Hilbert algebra. From the hypothesis we obtain that $\overline{\mathbf{B}} := \langle B, \Rightarrow, X\rangle$ is a Hilbert algebra, as it is a subalgebra of the Hilbert algebra $\langle \mathcal{P}^{\uparrow}(X), \Rightarrow, X\rangle$. We show that for all $\Gamma \cup \{\delta\} \subseteq Fm$, $\Gamma \vdash_{\mathcal{H}} \delta$ implies that for all $h \in \mathrm{Hom}(\mathbf{Fm}, \overline{\mathbf{B}})$, $\bigcap\{h(\gamma) : \gamma \in \Gamma\} \subseteq h(\delta)$.

Assume that $\Gamma \vdash_{\mathcal{H}} \delta$ and let $h \in \mathrm{Hom}(\mathbf{Fm}, \overline{\mathbf{B}})$. Then by hypothesis we get $h(\delta) \in \mathrm{C}^{\overline{\mathbf{B}}}_{\mathcal{H}}(h[\Gamma])$. Then as $\mathcal{H}$-filters are implicative filters, by definition and finitarity, there are $\gamma_1, \dots, \gamma_n \in \Gamma$ such that $h(\gamma_1) \Rightarrow (\dots(h(\gamma_n) \Rightarrow h(\delta))\dots) = X$. Then by the previous proposition, this implies $h(\gamma_1) \cap \dots \cap h(\gamma_n) \subseteq h(\delta)$, and so $\bigcap\{h(\gamma) : \gamma \in \Gamma\} \subseteq h(\delta)$. $\square$

Now we focus on the Spectral-like duality for Hilbert algebras. For the sake of completeness, we retype now the definition of $\mathbb{H}$-space that we already stated in § 3.3.1. A structure $\mathfrak{X} = \langle X, \tau_\kappa\rangle$ is an $\mathbb{H}$-*space* when:

(H6) κ is a basis of open compact subsets for the topological space $\langle X, \tau_\kappa\rangle$,
(H7) for every $U, V \in \kappa$, $\mathrm{sat}(U \cap V^c) \in \kappa$,
(H8) $\langle X, \tau_\kappa\rangle$ is sober.

THEOREM 6.2.4. *For $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ an $\mathcal{H}$-Spectral space, the structure $\mathfrak{X}' = \langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ is an $\mathbb{H}$-space such that $\langle X, (\mathfrak{X}')^*\rangle = \mathfrak{X}$.*

PROOF. Let $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ be an $\mathcal{H}$-Spectral space. Recall that since $\mathcal{H}$ satisfies (uDDT), from Corollary 5.5.21 it follows that $\mathbf{B} = \langle B, \Rightarrow, X\rangle$ where $U \Rightarrow V = (\mathrm{sat}(U \cap V^c))^c$ for all $U, V \in B$. Therefore $\mathfrak{X}'$ satisfies condition (H7). From condition (Sp3) of the definition of $\mathcal{H}$-Spectral space it follows condition (H6) and from condition (Sp3) it follows condition (H8).

Recall that for the $\mathbb{H}$-space $\mathfrak{X}'$, the algebra $(\mathfrak{X}')^*$ is defined as $\langle D(\mathfrak{X}'), \Rightarrow, X\rangle$, where $D(\mathfrak{X}) := \{U^c : U \in \kappa_{\mathfrak{X}}\}$ and $U \Rightarrow V := (\mathrm{sat}(U \cap V^c))^c$ for all $U, V \in D(\mathfrak{X}')$. So we have $D(\mathfrak{X}') = B$ and clearly $\mathbf{B} = (\mathfrak{X}')^*$. Hence $\langle X, (\mathfrak{X}')^*\rangle = \mathfrak{X}$. $\square$

THEOREM 6.2.5. *For $\mathfrak{X} = \langle X, \tau_\kappa\rangle$ an $\mathbb{H}$-space, the structure $\overline{\mathfrak{X}} = \langle X, \mathfrak{X}^*\rangle$ is an $\mathcal{H}$-Spectral space such that $\langle X, \tau_{\kappa_{\overline{\mathfrak{X}}}}\rangle = \mathfrak{X}$.*

PROOF. Let $\mathfrak{X} = \langle X, \tau_\kappa\rangle$ an $\mathbb{H}$-space. As sobriety implies T_0, from condition (H8) it follows that $\overline{\mathfrak{X}}$ is a reduced referential algebra. Moreover, the order associated with it coincides with the dual of the specialization order of the space, so from condition (H6) it follows that all elements of $D(\mathfrak{X})$ are up-sets and clearly $X \in D(\mathfrak{X})$. Therefore, as condition (H7) implies that $D(\mathfrak{X})$ is closed under $\Rightarrow$, we conclude, by Proposition 6.2.3, that $\overline{\mathfrak{X}}$ is an $\mathcal{H}$-referential algebra, i. e. condition (Sp1) holds. Moreover, by Proposition 6.2.2 it follows condition (Sp2). And clearly conditions (Sp3) and (Sp4) follow straightforwardly.

By definition $\kappa = \{U^c : U \in D(\mathfrak{X})\}$ is a basis of open compacts for $\tau_{\kappa_{\overline{\mathfrak{X}}}}$, therefore this topology is equal to τ_κ and thus $\langle X, \tau_{\kappa_{\overline{\mathfrak{X}}}}\rangle = \mathfrak{X}$. $\square$

Regarding morphisms, recall the definition of $\mathbb{H}$-relation that we already stated in § 3.3.1. Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be two $\mathbb{H}$-spaces. A relation $R \subseteq X_1 \times X_2$ is an $\mathbb{H}$-*relation* when:

(HR1) $\Box_R(U) \in \kappa_1$ for all $U \in \kappa_2$,
(HR2) $R(x)$ is a closed subset of X_2, for all $x \in X_1$.

Moreover, R is said to be *functional* when:

(HF) if $(x, y) \in R$, then there exists $z \in \mathrm{cl}(x)$ such that $R(z) = \mathrm{cl}(y)$

The following proposition will allow us to compare this notion with that of $\mathcal{H}$-Spectral morphism.

PROPOSITION 6.2.6. *Let $R \subseteq X_1 \times X_2$ be an $\mathcal{H}$-Spectral morphism between $\mathcal{H}$-Spectral spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Then for all $x_1 \in X_1$ and all $x_2 \in X_2$ such that $(x_1, x_2) \in R$, there is $z \in X_1$ such that $x_1 \leq z$ and $R(z) = \mathrm{cl}(x_2)$.*

PROOF. Let $x_1 \in X_1$ and $x_2 \in X_2$ be such that $x_2 \in R(x_1)$. Then by Spectral-like duality for $\mathcal{H}$, $\varepsilon(x_2) \in \overline{R}_{\Box_R}(\varepsilon(x_1))$, i.e. $\Box_R^{-1}[\varepsilon(x_1)] \subseteq \varepsilon(x_2)$, where $\varepsilon(x_1)$ is an irreducible $\mathcal{H}$-filter of $\mathbf{B}_1$ and $\varepsilon(x_2)$ is an irreducible $\mathcal{H}$-filter of $\mathbf{B}_2$. Since $\Box_R$ is order preserving, from $\varepsilon(x_2)^c$ being up-directed we get that $\Box_R[\varepsilon(x_2)^c]$ is also up-directed, and so $I := \downarrow\Box_R[\varepsilon(x_2)^c]$ is an order ideal of $\mathbf{B}_1$. Let $F := \langle \varepsilon(x) \cup \Box_R[\varepsilon(y)] \rangle$.

We claim that $F \cap I = \emptyset$. Suppose, towards a contradiction, that there is $U \in F \cap I$. Then, using the definition of implicative filter generated, we get that there are $V \in \varepsilon(x_1)$, $W \in \varepsilon(x_2)$ and $W' \notin \varepsilon(x_2)$ such that $U \subseteq \Box_R(W')$ and $V \Rightarrow (\Box_R(W) \Rightarrow U) = X$. Then we get $V \Rightarrow (\Box_R(W) \Rightarrow \Box_R(W')) = X$, and since $V \in \varepsilon(x_1)$, that is an $\mathcal{H}$-filter, then $\Box_R(W) \Rightarrow \Box_R(W') \in \varepsilon(x_1)$. By hypothesis we have that $\Box_R(W) \Rightarrow \Box_R(W') = \Box_R(W \Rightarrow W')$, then from the assumption we obtain $W \Rightarrow W' \in \Box_R^{-1}(\varepsilon(x_1)) \subseteq \varepsilon(x_2)$. And since $W \in \varepsilon(x_2)$, that is an $\mathcal{H}$-filter, we obtain $W' \in \varepsilon(x_2)$, a contradiction.

From the claim, by Lemma 2.3.3 there is an irreducible $\mathcal{H}$-filter G such that $F \subseteq G$ and $I \cap G = \emptyset$. But then by definition of F and I we get $\varepsilon(x_1) \subseteq G$ and $\Box_R^{-1}[G] = \varepsilon(x_2)$. Now by Spectral-like duality for $\mathcal{H}$, there is $z \in X_1$ such that $\varepsilon(z) = G$. But then we obtain $x_1 \leq z$ and $R(z) = \mathrm{cl}(x_2)$, as required. $\Box$

THEOREM 6.2.7. *For $R \subseteq X_1 \times X_2$ an $\mathcal{H}$-Spectral morphism between $\mathcal{H}$-Spectral spaces $\mathfrak{X}_1 = \langle X_1, \mathbf{B}_1 \rangle$ and $\mathfrak{X}_2 = \langle X_2, \mathbf{B}_2 \rangle$, R is a functional $\mathbb{H}$-relation between $\mathbb{H}$-spaces $\langle X_1, \tau_{\kappa_{\mathfrak{X}_1}} \rangle$ and $\langle X_2, \tau_{\kappa_{\mathfrak{X}_2}} \rangle$.*

PROOF. This follows from Proposition 6.2.6. $\Box$

THEOREM 6.2.8. *For $R \subseteq X_1 \times X_2$ a functional $\mathbb{H}$-relation between $\mathbb{H}$-spaces $\mathfrak{X}_1 = \langle X_1, \tau_{\kappa_1} \rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_{\kappa_2} \rangle$, R is an $\mathcal{H}$-Spectral morphism between $\mathcal{H}$-Spectral spaces $\langle X_1, \mathfrak{X}_1^* \rangle$ and $\langle X_2, \mathfrak{X}_2^* \rangle$.*

PROOF. This follows from the duality studied in [**15**]. $\Box$

Let us move now to the Priestley-style duality. Taking inspiration from the Spectral-like case, we come up with the following definition.

DEFINITION 6.2.9. A structure $\mathfrak{X} = \langle X, \tau, \leq, B \rangle$ is a $\mathbb{H}$-*Priestley space* when:

(H9) $\langle X, \tau \rangle$ is a compact topological space,
(H10′) $\langle X, \leq \rangle$ is a poset,
(H11′) B is a collection of clopen up-sets of X that contains X,
(H12) for every $x, y \in X$, $x \leq y$ iff $\forall U \in B($ if $x \in U$, then $y \in U)$,
(H13′) the set $X_B := \{x \in X : \{U \in B : x \notin U\}$ is non-empty and up-directed $\}$ is dense in X,
(H14) for all $U, V \in B$, $(\downarrow(U \cap V^c))^c \in B$.

For a given $\mathbb{H}$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, B \rangle$ we define a binary operation $\Rightarrow$ on B such that for all $U, V \in B$:

$$U \Rightarrow V := (\downarrow(U \cap V^c))^c.$$

By condition (H14) this operation is well-defined, and it is easy to check that $\mathbf{B}^\bullet = \langle B, \Rightarrow, X\rangle$ is a Hilbert algebra, that we call the Priestley-dual Hilbert algebra of $\mathfrak{X}$.

THEOREM 6.2.10. *For $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$ an $\mathcal{H}$-Priestley space, the structure $\mathfrak{X}' = \langle X, \tau, \leq, B\rangle$ is an $\mathbb{H}$-Priestley space such that $\langle X, \tau, \mathbf{B}^\bullet\rangle = \mathfrak{X}$.*

PROOF. Let $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$ be an $\mathcal{H}$-Priestley space. Recall that since $\mathcal{H}$ satisfies (uDDT), from Corollary 5.5.25 it follows that $\mathbf{B} = \langle B, \Rightarrow, X\rangle$, where $U \Rightarrow V = (\downarrow(U \cap V^c))^c$. Therefore $\mathfrak{X}'$ satisfies condition (H14). From condition (Pr3) in the definition of $\mathcal{H}$-Priestley space, they follow conditions (H9) and (H10$'$), from condition (Pr4) it follows condition (H11$'$), from condition (Pr5) it follows condition (H13$'$), from condition (Pr1) it follows condition (H12). From the definition it follows easily that $\langle X, \tau, \mathbf{B}^\bullet\rangle = \mathfrak{X}$. $\square$

THEOREM 6.2.11. *For $\mathfrak{X} = \langle X, \tau, \leq, B\rangle$ an $\mathbb{H}$-Priestley space, the structure $\overline{\mathfrak{X}} = \langle X, \tau, \mathbf{B}^\bullet\rangle$ is an $\mathcal{H}$-Priestley space.*

PROOF. By condition (H12), the referential algebra $\langle X, \mathbf{B}^\bullet\rangle$ is reduced. By condition (H11$'$) B is a family of up-sets, so from condition (H14) and Proposition 6.2.3 we conclude that $\langle X, \mathbf{B}^\bullet\rangle$ is a reduced $\mathcal{H}$-referential algebra, so condition (Pr1) holds. Moreover, by Proposition 6.2.2 it follows condition (Pr2). And clearly the rest of conditions also follow. $\square$

This definition of the Priestley-dual space of a Hilbert algebra can be viewed as a simplification of the one presented in § 3.3.2, i. e. the notion of $\mathbb{H}$-Priestley space is a simplification of the notion of augmented Priestley space, that was introduced by Celani and Jansana in [**18**]. We will repeatedly use $\mathbb{H}$-Priestley spaces in Chapter 7. With regard to morphisms, again the Spectral-like case provided us with inspiration for the following definition:

DEFINITION 6.2.12. For $\mathbb{H}$-Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, a relation $R \subseteq X_1 \times X_2$, is an $\mathbb{H}$-*Priestley morphism* when:

(HR3) if $(x, y) \notin R$, then there is $U \in B_2$ such that $y \notin U$ and $R(x) \subseteq U$,
(HR4) $\Box_R(U) \in B_1$ for all $U \in B_2$.

Moreover, R is said to be *functional* when:

(HF$'$) for every $x \in X_1$ and every $y \in X_{B_2}$, if $(x, y) \in R$, then there exists $z \in X_{B_1}$ such that $z \in \uparrow x$ and $R(z) = \uparrow y$.

Notice that the definition of $\mathbb{H}$-Priestley morphism is similar to that of *augmented Priestley semi-morphisms* introduced by Celani and Jansana in [**18**], and functional $\mathbb{H}$-Priestley morphisms are what they called there *augmented Priestley morphisms*.

PROPOSITION 6.2.13. *Let $R \subseteq X_1 \times X_2$ be an $\mathcal{H}$-Priestley morphism between $\mathcal{H}$-Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Then for all $x_1 \in X_1$ and all $x_2 \in X_{B_2}$ such that $(x_1, x_2) \in R$, there is $z \in X_{B_1}$ such that $x_1 \leq z$ and $R(z) = \uparrow y$.*

PROOF. The proof is similar to that of Proposition 6.2.6. $\square$

THEOREM 6.2.14. *For $R \subseteq X_1 \times X_2$ a functional $\mathbb{H}$-Priestley morphism between $\mathbb{H}$-Priestley spaces $\mathfrak{X}_1 = \langle X_1, \tau_1, \leq_1, B_1 \rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_2, \leq_2, B_2 \rangle$, R is an $\mathcal{H}$-Priestley morphism between $\mathcal{H}$-Priestley spaces $\langle X_1, \tau_1, \mathbf{B}_1^\bullet \rangle$ and $\langle X_2, \tau_2, \mathbf{B}_2^\bullet \rangle$.*

PROOF. This follows from the Priestley-style duality for $\mathcal{H}$-algebras studied in Chapter 5. $\square$

THEOREM 6.2.15. *For $R \subseteq X_1 \times X_2$ an $\mathcal{H}$-Priestley morphism between $\mathcal{H}$-Priestley spaces $\langle X_1, \tau_1, \mathbf{B}_1 \rangle$ and $\langle X_2, \tau_2, \mathbf{B}_2 \rangle$, R is a functional $\mathbb{H}$-Priestley morphism between $\mathbb{H}$-Priestley spaces $\langle X_1, \tau_1, \leq_1, B_1 \rangle$ and $\langle X_2, \tau_2, \leq_2, B_2 \rangle$.*

PROOF. This follows from Proposition 6.2.13. $\square$

This concludes the review of the Spectral-like duality and the Priestley-style duality for $\mathcal{H}$, in relation to our work in Chapter 5. In the following sections we consider several filter distributive finitary congruential logics with theorems that are expansions of $\mathcal{H}$, and we pay attention to the Spectral-like and Priestley-style dualities for these logics. We review, when appropriate, the dualities in the literature. Moreover, we carry out analyses similar to what we have done in this section, in order to get simplified definitions of the dual spaces of the corresponding algebras.

6.3. Modal expansions

Let us focus on the language $(\to, \Box, 1)$, of type $(2, 1, 0)$.

DEFINITION 6.3.1. An algebra $\mathbf{A} = \langle A, \to, \Box, 1 \rangle$ of type $(2, 1, 0)$ is a *modal Hilbert algebra* or an *$\mathbb{H}^\Box$-algebra* if $\langle A, \to, 1 \rangle$ is a Hilbert algebra and for all $a, b \in A$:

($\mathrm{H}^\Box 1$) $\Box 1 = 1$,
($\mathrm{H}^\Box 2$) $\Box(a \to b) \to (\Box a \to \Box b) = 1$.

Let us denote by $\mathbb{H}^\Box$ the variety of modal Hilbert algebras. It follows from the study of selfextensional logics with implication in [**54**], that $\mathcal{S}^{\to}_{\mathbb{H}^\Box}$, the Hilbert based logic of $\mathbb{H}^\Box$, is finitary and congruential. Moreover, it satisfies (uDDT) for $p \to q$, $\mathbb{A}\mathrm{lg}\mathcal{S}^{\to}_{\mathbb{H}^\Box} = \mathbb{H}^\Box$, and for any modal Hilbert algebra $\mathbf{A}$, the collection of implicative filters of $\mathbf{A}$ is the collection of $\mathcal{S}^{\to}_{\mathbb{H}^\Box}$-filters of $\mathbf{A}$. Thus the logic $\mathcal{S}^{\to}_{\mathbb{H}^\Box}$ is filter distributive.

Consider the logic $\mathcal{H}^\Box$, that has all axioms and rules of $\mathcal{H}$ applied to the formulas of the language $(\to, \Box, 1)$, together with the following list of axioms:

(A$\Box$1) $\vdash_{\mathcal{H}^\Box} \Box^n \gamma$ for every substitution instance of a theorem of $\mathcal{H}$ and for every $n \in \omega$,
(A$\Box$2) $\vdash_{\mathcal{H}^\Box} \Box^n(\Box(\gamma \to \beta) \to (\Box\gamma \to \Box\beta))$ for all formulas γ, β and for every $n \in \omega$.

It is immediate that the logic $\mathcal{H}^\Box$ satisfies (uDDT) with respect to the formula $p \to q$, because it is an axiomatic expansion of $\mathcal{H}$. In order to show that the logics $\mathcal{H}^\Box$ and $\mathcal{S}^{\to}_{\mathbb{H}^\Box}$ are equal, we need the following lemmas:

LEMMA 6.3.2. *For every formula γ in the language $(\to, \Box, 1)$, if $\vdash_{\mathcal{H}^\Box} \gamma$, then $\vdash_{\mathcal{H}^\Box} \Box\gamma$.*

Proof. Let $\Gamma := \{\gamma : \vdash_{\mathcal{H}^\Box} \Box\gamma\}$. We show that this set contains all axioms and is closed under modus ponens. It is clear by (A$\Box$2) that if δ is an axiom, then $\Box\delta$ is an axiom, and therefore $\delta \in \Gamma$ for all axioms δ. Suppose that $\delta, \delta \to \beta \in \Gamma$, so $\vdash_{\mathcal{H}^\Box} \Box\delta$ and $\vdash_{\mathcal{H}^\Box} \Box(\delta \to \beta)$. Then by (A$\Box$2) for $n = 1$ we have the axiom $\vdash_{\mathcal{H}^\Box} \Box(\delta \to \beta) \to (\Box\delta \to \Box\beta)$. It follows by (MP) that $\vdash_{\mathcal{H}^\Box} \Box\delta \to \Box\beta$, and then by hypothesis again $\vdash_{\mathcal{H}^\Box} \Box\beta$, so $\beta \in \Gamma$, as required. $\Box$

Lemma 6.3.3. *For every formula γ,*

$$\vdash_{\mathcal{H}^\Box} \gamma \quad \textit{iff} \quad (\forall \mathbf{A} \in \mathbb{H}^\Box)(\forall h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A}))h(\gamma) = 1^{\mathbf{A}}.$$

Proof. The direction from left to right follows easily, we only have to check that every axiom takes value $1^{\mathbf{A}}$ in every valuation on an arbitrary modal Hilbert algebra $\mathbf{A}$, and that taking value $1^{\mathbf{A}}$ is preserved by modus ponens.

To prove the other direction, assume $\nvdash_{\mathcal{H}^\Box} \gamma$. Let us consider the following congruence θ on the formula algebra $\mathbf{Fm}$:

$$(\beta, \delta) \in \theta \quad \text{iff} \quad \vdash_{\mathcal{H}^\Box} \beta \to \delta \text{ and } \vdash_{\mathcal{H}^\Box} \delta \to \beta.$$

Notice that for every equation $\beta \approx \delta$ defining the variety $\mathbb{H}^\Box$, we have that $\beta \to \delta$ and $\delta \to \beta$ are theorems of $\mathcal{H}^\Box$. This implies that $\mathbf{Fm}/\theta \in \mathbb{H}^\Box$. Moreover, for any $\beta \in Fm$, β is a theorem of $\mathcal{H}^\Box$ if and only if $(\beta, 1) \in \theta$ (we use 1 as a shorthand for $\delta \to \delta$ for any $\delta \in Fm$). Let $\pi : \mathbf{Fm} \longrightarrow \mathbf{Fm}/\theta$ be the canonical natural map. Then $\pi(\beta) = 1/\theta$ if and only if β is a theorem of $\mathcal{H}^\Box$. So since by hypothesis $\nvdash_{\mathcal{H}^\Box} \gamma$, we get $\pi(\gamma) \neq 1/\theta$. This proves the direction from right to left. $\Box$

Theorem 6.3.4. *The logics $\mathcal{H}^\Box$ and $\mathcal{S}^{\to}_{\mathbb{H}^\Box}$ are equal.*

Proof. Assume first that $\Gamma \vdash_{\mathcal{H}^\Box} \delta$. Then we know that either δ is a theorem of $\mathcal{H}^\Box$, in which case for every $\mathbf{A} \in \mathbb{H}^\Box$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $h(\delta) = 1^{\mathbf{A}}$, or there are $\gamma_0, \ldots, \gamma_n \in \Gamma$ such that $\gamma_0, \ldots, \gamma_n \vdash_{\mathcal{H}^\Box} \delta$. By (uDDT) it follows that $\vdash_{\mathcal{H}^\Box} \gamma_0 \to (\gamma_1 \to (\ldots(\gamma_n \to \delta)\ldots))$. Then for every $\mathbf{A} \in \mathbb{H}^\Box$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $h(\gamma_0 \to (\gamma_1 \to (\ldots(\gamma_n \to \delta)\ldots))) = 1^{\mathbf{A}}$, and so $\Gamma \vdash^{\to}_{\mathbb{H}^\Box} \delta$.

For the converse, assume $\Gamma \vdash^{\to}_{\mathbb{H}^\Box} \delta$. Then either δ is a theorem of $\mathcal{S}^{\to}_{\mathbb{H}^\Box}$, in which case for every $\mathbf{A} \in \mathbb{H}^\Box$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $h(\delta) = 1^{\mathbf{A}}$, or there are $\gamma_0, \ldots, \gamma_n \in \Gamma$ such that $\gamma_0, \ldots, \gamma_n \vdash^{\to}_{\mathbb{H}^\Box} \delta$. In the last case, for every $\mathbf{A} \in \mathbb{H}^\Box$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $h(\gamma_0) \to^{\mathbf{A}} h((\gamma_1) \to^{\mathbf{A}} (\ldots(h(\gamma_n) \to^{\mathbf{A}} h(\delta))\ldots)) = 1^{\mathbf{A}}$. Then by Lemma 6.3.3 and since h is an homomorphism, the previous fact implies that $\vdash_{\mathcal{H}^\Box} \gamma_0 \to (\gamma_1 \to (\ldots(\gamma_n \to \delta)\ldots))$. And so, using (MP) again, we obtain $\Gamma \vdash_{\mathcal{H}^\Box} \delta$. $\Box$

As the axioms of $\mathcal{H}^\Box$ are closed under the addiction of $\Box$ by (A$\Box$2), an easy induction over the length of the proofs shows that $\mathcal{H}^\Box$ satisfies (PIM), so our theory in Chapter 5 can be applied to get dualities for it, and these dualities can be refined using the correspondences studied in § 5.5. Notice that for any modal Hilbert algebra $\mathbf{A} = \langle A, \to, \Box, 1\rangle$, the operation $\Box$ is a semi-homomorphism from the Hilbert algebra reduct $\langle A, \to, 1\rangle$ to itself. From the Spectral-like and Priestley-style dualities presented in § 6.2 we know how to *dualize* semi-homomorphisms. Therefore, in regard to objects, we can build dualities for modal Hilbert algebras as follows: for the Spectral-like duality, the Spectral-dual of a modal Hilbert algebra is the Spectral-dual of its Hilbert algebra reduct, augmented with a binary relation that is an $\mathbb{H}$-relation. For the Priestley-style duality, the Priestley-dual of a modal Hilbert algebra is the Priestley-dual of its Hilbert algebra reduct, augmented with a binary relation that is an $\mathbb{H}$-Priestley relation.

In summary, $\mathcal{S}_{\mathbb{H}^\Box}^{\rightarrow}$ is a filter distributive finitary congruential logic with theorems, so our theory of Chapter 5 might be applied directly to it. But if we carry out a more detailed analysis of the logic, we discover that, analogously as in extended Priestley duality, we can build the dualities for $\mathbb{H}^\Box$ from the ones for $\mathbb{H}$, and so we obtain dual spaces with no explicit algebraic structure. It should be further investigated how to dispense with the algebraic structure of the dual morphisms, but we leave this as future work.

6.4. Expansions with a disjunction

Let us consider now the language $(\rightarrow, \vee, 1)$ of type $(2, 2, 0)$.

DEFINITION 6.4.1. An algebra $\mathbf{A} = \langle A, \rightarrow, \vee, 1\rangle$ of type $(2, 2, 0)$ is a *Hilbert algebra with supremum* or an $\mathbb{H}^\vee$*-algebra* if $\langle A, \rightarrow, 1\rangle$ is a Hilbert algebra, $\langle A, \vee\rangle$ is a join-semilattice and $\rightarrow$ and $\vee$ define the same order, i. e. for all $a, b \in A$:

$$a \rightarrow b = 1 \quad \text{iff} \quad a \vee b = b.$$

Let us denote by $\mathbb{H}^\vee$ the class of Hilbert algebras with supremum. This class of algebras was studied by Busneag and Ghita in [**9**] and more recently by Celani and Montangie in [**19**] and. It is easy to check that $\mathbb{H}^\vee$-algebras are $\mathbb{BCK}$*-join-semilattices* (or $\mathbb{BCK}^\vee$*-algebras*).This class was studied by Idziak in [**51**], where he proves that $\mathbb{BCK}^\vee$ is indeed a variety. It follows that $\mathbb{H}^\vee$ is also a variety, for which an equational definition is given as follows. $\mathbf{A} = \langle A, \rightarrow, \vee, 1\rangle$ is a Hilbert algebra with supremum if $\langle A, \rightarrow, 1\rangle$ is a Hilbert algebra, $\langle A, \vee\rangle$ is a join-semilattice and for all $a, b \in A$:

($\mathrm{H}^\vee 1$) $a \rightarrow (a \vee b) = 1$,
($\mathrm{H}^\vee 2$) $(a \rightarrow b) \rightarrow ((a \vee b) \rightarrow b) = 1$.

Then it is easy to check that the $(\rightarrow, \vee, 1)$-reduct of any Heyting algebra is a Hilbert algebra with supremum.

Let us consider the Hilbert based logic $\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow}$ of $\mathbb{H}^\vee$. From the general theory in [**54**] it follows that this logic is finitary and congruential. Moreover it satisfies (uDDT) for $p \rightarrow q$, $\mathbb{A}\mathrm{lg}\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow} = \mathbb{H}^\vee$, and for any Hilbert algebra with supremum $\mathbf{A}$, the collection of implicative filters of $\mathbf{A}$ is the collection of $\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow}$-filters of $\mathbf{A}$. Thus the logic $\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow}$ is also filter distributive. We show now that it has the property of disjunction.

LEMMA 6.4.2. *For any $\delta, \gamma \in Fm$, $\delta \vdash_{\mathbb{H}^\vee}^{\rightarrow} \delta \vee \gamma$ and $\gamma \vdash_{\mathbb{H}^\vee}^{\rightarrow} \delta \vee \gamma$.*

PROOF. Notice that for every $\mathbb{H}^\vee$-algebra $\mathbf{A}$, and every $a, b \in A$, it holds $a \leq a \vee^{\mathbf{A}} b$, i. e. $a \rightarrow^{\mathbf{A}} (a \vee^{\mathbf{A}} b) = 1^{\mathbf{A}}$. This implies that for every $\mathbf{A} \in \mathbb{H}^\vee$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $h(\delta) \rightarrow^{\mathbf{A}} h(\delta \vee \gamma) = 1^{\mathbf{A}}$. Hence we obtain $\delta \vdash_{\mathbb{H}^\vee}^{\rightarrow} \delta \vee \gamma$. The proof of the other statement is similar. $\Box$

LEMMA 6.4.3. *For any $\delta, \gamma, \mu \in Fm$, if $\delta \vdash_{\mathbb{H}^\vee}^{\rightarrow} \mu$ and $\gamma \vdash_{\mathbb{H}^\vee}^{\rightarrow} \mu$, then $\delta \vee \gamma \vdash_{\mathbb{H}^\vee}^{\rightarrow} \mu$.*

PROOF. Assume that $\delta \vdash_{\mathbb{H}^\vee}^{\rightarrow} \mu$ and $\gamma \vdash_{\mathbb{H}^\vee}^{\rightarrow} \mu$. Then by definition of $\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow}$ we have that for every $\mathbf{A} \in \mathbb{H}^\vee$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, $h(\delta) \rightarrow^{\mathbf{A}} h(\mu) = 1^{\mathbf{A}}$ and $h(\gamma) \rightarrow^{\mathbf{A}} h(\mu) = 1^{\mathbf{A}}$. This implies that $h(\delta) \leq^{\mathbf{A}} h(\mu)$ and $h(\gamma) \leq^{\mathbf{A}} h(\mu)$. Therefore $h(\delta \vee \gamma) = h(\delta) \vee^{\mathbf{A}} h(\gamma) \leq^{\mathbf{A}} h(\mu)$, i. e. $h(\delta \vee \gamma) \rightarrow^{\mathbf{A}} h(\mu) = 1^{\mathbf{A}}$. As this holds for every $\mathbb{H}^\vee$algebra $\mathbf{A}$ and every $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$, we obtain $\delta \vee \gamma \vdash_{\mathbb{H}^\vee}^{\rightarrow} \mu$, as required. □

COROLLARY 6.4.4. *The logic $\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow}$ satisfies (PWDI) and (PDI) for $p \vee q$.*

PROOF. From lemmas 6.4.2 and 6.4.3 it follows that $\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow}$ satisfies (PWDI), and since the logic is filter distributive, by Lemma 5.5.12 this implies that the logic satisfies (PDI). □

Let $\mathcal{H}^\vee$ be the implicative-disjunctive fragment of intuitionistic logic, i. e. the logic $\mathcal{H}^\vee := \langle \mathbf{Fm}, \vdash_{\mathcal{H}^\vee} \rangle$ in the language $(\rightarrow, \vee, 1)$, where $\vdash_{\mathcal{H}^\vee}$ is the restriction of intuitionistic logic to the formulas of the language $(\rightarrow, \vee, 1)$. The logic $\mathcal{H}^\vee$ can be presented in a Hilbert-style calculus by the following axioms and rules (see Lemma 2.4.6 in [**73**]):

(A1) $\vdash_{\mathcal{H}^\vee} \beta \rightarrow (\gamma \rightarrow \beta)$,
(A2) $\vdash_{\mathcal{H}^\vee} (\gamma \rightarrow (\beta \rightarrow \delta)) \rightarrow ((\gamma \rightarrow \beta) \rightarrow (\gamma \rightarrow \delta))$,
(A6) $\vdash_{\mathcal{H}^\vee} \gamma \rightarrow (\gamma \vee \beta)$,
(A7) $\vdash_{\mathcal{H}^\vee} \gamma \rightarrow (\beta \vee \gamma)$,
(A8) $\vdash_{\mathcal{H}^\vee} (\gamma \rightarrow \delta) \rightarrow ((\beta \rightarrow \delta) \rightarrow ((\gamma \vee \beta) \rightarrow \delta))$,
(MP) $\gamma, \gamma \rightarrow \beta \vdash_{\mathcal{H}^\vee} \beta$.

It follows from results by Porębska and Wroński in [**64**], that $\mathcal{H}^\vee$ is the least finitary logic in the language $(\rightarrow, \vee, 1)$ that satisfies (uDDT) for $p \rightarrow q$ and satisfies (PDI) for $p \vee q$.[2]

THEOREM 6.4.5. *The logics $\mathcal{H}^\vee$ and $\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow}$ ar equal.*

PROOF. Since $\mathcal{S}_{\mathbb{H}^\vee}^{\rightarrow}$ is finitary and satisfies (uDDT) and (PDI), and $\mathcal{H}^\vee$ is the least finitary logic satisfying (uDDT) and (PCI), it follows that $\vdash_{\mathcal{H}^\vee} \subseteq \vdash_{\mathbb{H}^\vee}^{\rightarrow}$. For the converse, let $\mathscr{L} = (\rightarrow, \vee, 1)$ the and let $\mathscr{L}' = (\rightarrow, \wedge, \vee, 0, 1)$ be the language of intuitionistic logic. let $\Gamma \cup \{\delta\} \subseteq Fm_{\mathscr{L}}$ be such that $\Gamma \nvdash_{\mathcal{H}^\vee} \delta$. Then as $\mathcal{H}^\vee$ is a fragment of the intuitionistic logic $\mathcal{IPC}$, we have that $\Gamma \nvdash_{\mathcal{IPC}} \delta$. Hence there is a Heyting algebra $\mathbf{A} = \langle A, \rightarrow, \wedge, \vee, 0^{\mathbf{A}}, 1^{\mathbf{A}} \rangle$ and a homomorphism $h \in \mathrm{Hom}(\mathbf{Fm}_{\mathscr{L}'}, \mathbf{A})$ such that $h(\gamma) = 1^{\mathbf{A}}$ for all $\gamma \in \Gamma$ and $h(\delta) \neq 1^{\mathbf{A}}$. This implies that for any $n \in \omega$ and any $\gamma_0, \ldots, \gamma_n \in \Gamma$, $h(\gamma_0 \rightarrow (\gamma_1 \rightarrow (\ldots (\gamma_n \rightarrow \delta) \ldots))) \neq 1^{\mathbf{A}}$.

Recall that the $(\rightarrow, \vee, 1)$-reduct of $\mathbf{A}$, that we denote by $\mathbf{A}' = \langle A, \rightarrow, \vee, 1^{\mathbf{A}} \rangle$ is an $\mathbb{H}$–algebra. Moreover, h is a homomorphism from $\mathbf{Fm}_{\mathscr{L}}$ to $\mathbf{A}'$. So by definition of the Hilbert based logic we conclude that $\Gamma \nvdash_{\mathbb{H}^\vee}^{\rightarrow} \delta$, as required. □

Given that the logic $\mathcal{H}^\vee$ is a filter distributive finitary congruential logic with theorems, and it satisfies (uDDT) and (PDI), our theory of § 5.5 can be applied to it, as we did in § 6.2:

DEFINITION 6.4.6. A structure $\mathfrak{X} = \langle X, \tau_\kappa \rangle$ is an $\mathbb{H}^\vee$-*Spectral space* when $\langle X, \tau_\kappa \rangle$ is an $\mathbb{H}$-space and:

[2]This was also remarked by Wójcicki in [**73**].

($\mathrm{H}^{\vee}3$) κ is closed under finite intersections.

DEFINITION 6.4.7. A structure $\mathfrak{X} = \langle X, \tau, \leq, B\rangle$ is an $\mathbb{H}^{\vee}$-*Priestley space* when $\langle X, \tau, \leq, B\rangle$ is an $\mathbb{H}$-Priestley space and:

($\mathrm{H}^{\vee}4$) B is closed under finite unions.

In [**19**] Celani and Montangie studied a Spectral-like duality for $\mathbb{H}^{\vee}$-algebras, and the definition of dual spaces they came up with is precisely that of $\mathbb{H}^{\vee}$-Spectral spaces. In regard to morphisms, we refer the reader to [**19**], where the duality is studied in detail. For the Priestley-style duality, the definition of $\mathbb{H}^{\vee}$-Priestley space is new, but it works analogously to the Spectral-like case.

In summary, $\mathcal{S}^{\rightarrow}_{\mathbb{H}^{\vee}}$ is a filter distributive finitary congruential logic with theorems, that is the implicative-disjunctive fragment of intuitionistic logic. It falls under the scope of our theory in Chapter 5. Moreover $\mathcal{S}^{\rightarrow}_{\mathbb{H}^{\vee}}$ satisfies (uDDT) and (PDI). So we can use the correspondences studied in § 5.5 to put aside the algebraic structure of the definition of the dual spaces. Although we did not go into details, it is remarkable that in this case we can put aside also the algebraic structure in the definition of dual morphisms.

6.5. Expansions with a conjunction

Let us consider the language $(\rightarrow, \wedge, 1)$ of type $(2, 2, 0)$. In this section we study mainly two logics defined in this language. One is the implicative-conjunctive fragment of intuitionistic logic, that is a well-known logic for which Spectral-like and Priestley-style have been already studied. We will show that these results follow from our general theory. And the other is a weaker logic with some interesting properties. We will study in detail the class of algebras associated with it, since in Chapter 7 we develop new Spectral-like and Priestley-style dualities for a subclass of such algebras.

Let $\mathcal{H}^{\wedge}$ be the implicative-conjunctive fragment of intuitionistic logic, i. e. the logic $\mathcal{H}^{\wedge} := \langle \mathbf{Fm}, \vdash_{\mathcal{H}^{\wedge}}\rangle$ in the language $(\rightarrow, \wedge, 1)$, where $\vdash_{\mathcal{H}^{\wedge}}$ is the restriction of intuitionistic logic to the formulas of the language $(\rightarrow, \wedge, 1)$. The logic $\mathcal{H}^{\wedge}$ is presented in a Hilbert-style calculus by the following axioms and rules:

(A1) $\vdash_{\mathcal{H}^{\wedge}} \beta \rightarrow (\gamma \rightarrow \beta)$,
(A2) $\vdash_{\mathcal{H}^{\wedge}} (\gamma \rightarrow (\beta \rightarrow \delta)) \rightarrow ((\gamma \rightarrow \beta) \rightarrow (\gamma \rightarrow \delta))$,
(A3) $\vdash_{\mathcal{H}^{\wedge}} (\gamma \wedge \beta) \rightarrow \beta$,
(A4) $\vdash_{\mathcal{H}^{\wedge}} (\gamma \wedge \beta) \rightarrow \gamma$,
(A5) $\vdash_{\mathcal{H}^{\wedge}} (\gamma \rightarrow \beta) \rightarrow ((\gamma \rightarrow \delta) \rightarrow (\gamma \rightarrow (\beta \wedge \delta)))$,
(MP) $\gamma, \gamma \rightarrow \beta \vdash_{\mathcal{H}^{\wedge}} \beta$.

From results by Porębska and Wroński in [**64**], it follows that $\mathcal{H}^{\wedge}$ is the least finitary logic satisfying (PC) and (uDDT).[3] Moreover, as it satisfies (uDDT), then it is a filter distributive logic. Clearly $\mathcal{H}^{\wedge}$ has theorems, as for any $\gamma \in Fm$, $\gamma \rightarrow \gamma \in \mathrm{Thm}\mathcal{H}^{\wedge}$. It is also well known that $\mathcal{H}^{\wedge}$ is the 1-assertional logic of $\mathbb{IS}$, the variety of implicative semilattices.

DEFINITION 6.5.1. An *implicative semilattice* or $\mathbb{IS}$-*algebra* (also called *Hertz algebra* or *Browerian semilattice*) is an algebra $\mathbf{A} = \langle A, \rightarrow, \wedge, 1\rangle$ of type $(2, 2, 0)$

[3] This was also remarked in Lemma 2.4.5 in [**73**].

such that $\langle A, \wedge, 1\rangle$ is a meet-semilattice with top element 1, and $\to$ is the right residuum of $\wedge$, i.e. for all $a, b, c \in A$:

$$a \wedge c \leq_{\mathbf{A}} b \quad \text{iff} \quad c \leq_{\mathbf{A}} a \to b,$$

where the partial order $\leq_{\mathbf{A}}$ is the one associated with the semilattice.

Let us denote by $\mathbb{IS}$ and $\mathbb{BIS}$ the classes of implicative semilattices and bounded implicative semilattices respectively. Nemitz studied $\mathbb{IS}$ in [**62**] from an algebraic point of view, and later Köhler studied it in [**60**] from a logical point of view. It is well known that the $(\wedge, 1)$-reduct of an implicative semilattice is a distributive semilattice, and the $(\to, 1)$-reduct is a Hilbert algebra, whose order associated coincides with $\leq_{\mathbf{A}}$. Moreover, for any implicative semilattice $\mathbf{A}$, $\mathrm{Fi}_{\wedge}(\mathbf{A}) = \mathrm{Fi}_{\to}(\mathbf{A}) = \mathrm{Fi}_{\mathcal{H}^{\wedge}}(\mathbf{A})$.

It is also well known that implicative semilattices can also be obtained as the subalgebras of the $(\to, \wedge, 1)$-reducts of Heyting algebras. Implicative semilattices are indeed a variety, for which an equational definition is given as follows. $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ is an $\mathbb{IS}$-algebra when for all $a, b, c \in A$:

(K) $a \to a = 1$,
(IS1) $(a \to b) \wedge b = b$,
(IS2) $a \wedge (a \to b) = a \wedge b$,
(IS3) $a \to (b \wedge c) = (a \to c) \wedge (a \to b)$.

Theorem 6.5.2. *The logics $\mathcal{H}^{\wedge}$, $\mathcal{S}^{\to}_{\mathbb{IS}}$ and $\mathcal{S}^{\leq}_{\mathbb{IS}}$ are equal.*

Proof. The proof is similar to that of Theorem 6.2.1, and it is based on the fact that $\mathcal{H}^{\wedge}$ satisfies (uDDT) for $p \to q$. □

The previous theorem implies, by Proposition 7 in [**54**], that $\mathcal{H}^{\wedge}$ is selfextensional, and then by Theorem 4.46 in [**35**] it follows that $\mathcal{H}^{\wedge}$ is congruential.

As the logic $\mathcal{H}^{\wedge}$ is a filter distributive finitary congruential logic with theorems, and it satisfies (uDDT) and (PC), our theory of Chapter 5 can be again applied to $\mathcal{H}^{\wedge}$, as we did in § 6.2, § 6.3 and § 6.4.

More specifically, for the Spectral-like duality, we get analogues of propositions 6.2.3 and 6.2.6 that allow us to dispense with the algebraic structure in the definitions of dual spaces and dual morphisms. This leads us to recover the Spectral-like duality for $\mathbb{IS}$-algebras that have been studied in the literature, as particular cases of our general theory.

In [**11**] Celani studied a Spectral-like duality for $\mathbb{IS}$-algebras, where dual objects are topological spaces called $\mathbb{IS}$-spaces. We recall that $\mathfrak{X} = \langle X, \tau\rangle$ is an $\mathbb{IS}$-*space* (Definition 4.1 in [**11**]) when $\langle X, \tau\rangle$ is a $\mathbb{DS}$-space (see definition in page 36) and:[4]

(IS4) for any $U, V \in \mathcal{KO}(X)$, $\mathrm{sat}(U \cap V^c) \in \mathcal{KO}(X)$.

Recall that for a $\mathbb{DS}$-space $\mathfrak{X}$ (see definition in page 36), we denote by $F(\mathfrak{X})$ the set $\{U^c : U \in \mathcal{KO}(X)\}$. Then for any $\mathbb{IS}$-space $\mathfrak{X} = \langle X, \tau\rangle$, the algebra $\mathfrak{X}^* := \langle F(\mathfrak{X}), \Rightarrow, \cap, X\rangle$ is an $\mathbb{IS}$-algebra, where $\Rightarrow$ is defined as $U \Rightarrow V := (\mathrm{sat}(U \cap V^c))^c$ for all $U, V \in F(\mathfrak{X}')$.

[4]Notice that what we present here is a simplification of the original definition, that involves the notion of $\mathbb{IS}$-frame.

With respect to morphisms, duals of algebraic morphisms between $\mathbb{IS}$-algebras are binary relations called in [**11**] $\mathbb{IS}$-morphisms. We recall that for $\mathbb{IS}$-spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, a relation $R \subseteq X_1 \times X_2$ is a $\mathbb{IS}$-*morphism* (or *functional meet-relation*) when:

(DSR1) $\Box_R(U) \in F(\mathfrak{X}_1)$ for all $U \in F(\mathfrak{X}_2)$,
(DSR2) $R(x)$ is a closed subset of X_2 for any $x \in X_1$,
(HF) for all $x_1 \in X_1$ and $x_2 \in X_2$ such that $(x_1, x_2) \in R$, there is $z \in X_1$ such that $z \in \mathrm{cl}(x_1)$ and $R(x_2) = \mathrm{cl}(x_2)$.

THEOREM 6.5.3. *For $\mathfrak{X} = \langle X, \mathbf{B}\rangle$ an $\mathcal{H}^{\wedge}$-Spectral space, the structure $\mathfrak{X}' = \langle X, \tau_{\kappa_{\mathfrak{X}}}\rangle$ is an $\mathbb{IS}$-space such that $\langle X, (\mathfrak{X}')^*\rangle = \mathfrak{X}$. Moreover, for $R \subseteq X_1 \times X_2$ an $\mathcal{H}^{\wedge}$-Spectral morphism between $\mathcal{H}^{\wedge}$-Spectral spaces $\mathfrak{X}_1 = \langle X_1, \mathbf{B}_1\rangle$ and $\mathfrak{X}_2 = \langle X_2, \mathbf{B}_2\rangle$, R is an $\mathbb{IS}$-morphism between $\mathbb{IS}$-spaces $\langle X_1, \tau_{\kappa_{\mathfrak{X}_1}}\rangle$ and $\langle X_2, \tau_{\kappa_{\mathfrak{X}_2}}\rangle$.*

THEOREM 6.5.4. *For $\mathfrak{X} = \langle X, \tau\rangle$ an $\mathbb{IS}$-space, the structure $\overline{\mathfrak{X}} = \langle X, \mathfrak{X}^*\rangle$ is an $\mathcal{H}^{\wedge}$-Spectral space such that $\langle X, \tau_{\kappa_{\overline{\mathfrak{X}}}}\rangle = \mathfrak{X}$. Moreover, for $R \subseteq X_1 \times X_2$ an $\mathbb{IS}$-morphism between $\mathbb{IS}$-spaces $\mathfrak{X}_1 = \langle X_1, \tau_1\rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_2\rangle$, R is an $\mathcal{H}^{\wedge}$-Spectral morphism between $\mathcal{H}^{\wedge}$-Spectral spaces $\langle X_1, \mathfrak{X}_1^*\rangle$ and $\langle X_2, \mathfrak{X}_2^*\rangle$.*

On the other hand, for the Priestley-style duality, from analogues of propositions 6.2.3 and 6.2.13 we figure out how to dispense with the algebraic structure in the definitions of dual spaces and dual morphisms. And this leads us to recover as particular cases of our general theory, the Priestley-style duality for $\mathbb{IS}$-algebras that Bezhanishvili and Jansana studied in [**6**]. Dual objects were called there generalized Esakia spaces.[5] We recall that $\mathfrak{X} = \langle X, \tau, \leq, X_B\rangle$ is a *generalized Esakia space* when $\langle X, \tau, \leq, X_B\rangle$ is a generalized Priestley space (see definition in page 38) and:

(IS5) for all $U, V \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$, $(\downarrow(U \cap V^c))^c \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$.

Strictly speaking, the definition of generalized Esakia spaces in [**6**] involves, instead of condition (IS5), the following equivalent and nicer condition:

(Es) $\downarrow U$ is clopen for every Esakia clopen U,

where $U \subseteq X$ is an *Esakia clopen* if and only if $U = \bigcup\{(U_i \cap V_i^c) : i \leq n\}$ for some $n \in \omega$, and $U_i, V_i \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$ for all $i \leq n$.

For a given generalized Esakia space $\mathfrak{X} = \langle X, \tau, \leq, X_B\rangle$ we define a binary operation $\Rightarrow$ on $\mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$ such that for all $U, V \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$:

$$U \to V := (\downarrow(U \cap V^c))^c.$$

Then we get that $\mathfrak{X}^{\bullet} := \langle \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X), \Rightarrow, \cap, X\rangle$ is an $\mathbb{IS}$-algebra, that we call the Priestley-dual implicative semilattice of $\mathfrak{X}$.

In relation to morphisms, duals of homomorphisms between $\mathbb{IS}$-algebras are binary relations called generalized Esakia morphisms in [**6**]. We recall that for generalized Esakia spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, a relation $R \subseteq X_1 \times X_2$ is an *generalized Esakia morphism* when:

[5] Actually, Bezhanishvili and Jansana work with bounded $\mathbb{IS}$-algebras, and their duals are what they call *generalized Esakia spaces*. Similarly to what we presented for the Priestley-style duality for $\mathbb{DS}$-algebras, their work can be extended to $\mathbb{IS}$-algebras that do not have necessarily bottom. For simplicity we use the same name, but it should be kept in mind that the context we work in is broader than the one in [**6**].

(DSR3) $\Box_R(U) \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_{B_1}}(X_1)$ for all $U \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_{B_2}}(X_2)$,

(DSR4) if $(x, y) \notin R$, then there is $U \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_{B_2}}(X_2)$ such that $y \notin U$ and $R(x) \subseteq U$.

(HF$'$) for every $x \in X_1$ and every $y \in X_{B_2}$, if $(x, y) \in R$, then there exists $z \in X_{B_1}$ such that $z \in {\uparrow}x$ and $R(z) = {\uparrow}y$.

THEOREM 6.5.5. *For $\mathfrak{X} = \langle X, \tau, \mathbf{B}\rangle$ an $\mathcal{H}^{\wedge}$-Priestley space, the structure $\mathfrak{X}' = \langle X, \tau, \leq, X_B\rangle$ is a generalized Esakia space such that $\langle X, \tau, (\mathfrak{X}')^{\bullet}\rangle = \mathfrak{X}$. Moreover, for $R \subseteq X_1 \times X_2$ an $\mathcal{H}^{\wedge}$-Priestley morphism between $\mathcal{H}^{\wedge}$-Priestley spaces $\langle X_1, \tau_1, \mathbf{B}_1\rangle$ and $\langle X_2, \tau_2, \mathbf{B}_2\rangle$, R is a generalized Esakia morphism between generalized Esakia spaces $\langle X_1, \tau_1, \leq_1, X_{B_1}\rangle$ and $\langle X_2, \tau_2, \leq_2, X_{B_2}\rangle$.*

THEOREM 6.5.6. *For $\mathfrak{X} = \langle X, \tau, \leq, X_B\rangle$ a generalized Esakia space, the structure $\langle X, \tau, \mathfrak{X}^{\bullet}\rangle$ is an $\mathcal{H}^{\wedge}$-Priestley space such that $\langle X, \tau, \leq, X_{\mathfrak{X}^{\bullet}}\rangle = \mathfrak{X}$. Moreover, for $R \subseteq X_1 \times X_2$ an generalized Esakia morphism between generalized Esakia spaces $\mathfrak{X}_1 = \langle X_1, \tau_1, \leq_1, X_{B_1}\rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_2, \leq_2, X_{B_2}\rangle$, R is an $\mathcal{H}^{\wedge}$-Priestley morphism between $\mathcal{H}^{\wedge}$-Priestley spaces $\langle X_1, \tau_1, \mathfrak{X}_1^{\bullet}\rangle$ and $\langle X_2, \tau_2, \mathfrak{X}_2^{\bullet}\rangle$.*

Summarizing, from our general theory we recover the Spectral-like and Priestley-style dualities for $\mathbb{IS}$-algebras that we find in the literature. Let us change now the subject and consider a different logic defined in the language $(\to, \wedge, 1)$.

DEFINITION 6.5.7. An algebra $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ of type $(2, 2, 0)$ is a *Hilbert algebra with infimum* or an $\mathbb{H}^{\wedge}$-*algebra* if $\langle A, \to, 1\rangle$ is a Hilbert algebra, $\langle A, \wedge, 1\rangle$ is a semilattice with top element 1, and $\to$ and $\wedge$ define the same order, i. e. for all $a, b \in A$:

$$a \to b = 1 \quad \text{iff} \quad a \wedge b = a.$$

EXAMPLE 6.5.8. In any semilattice $\langle A, \wedge, 1\rangle$ it is possible to define a structure of $\mathbb{H}^{\wedge}$-algebra considering the *implication $\to$ defined by the order*:

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if otherwise.} \end{cases}$$

Let us denote by $\mathbb{H}^{\wedge}$ the class of Hilbert algebras with infimum. In [**32**] Figallo et al. prove that $\mathbb{H}^{\wedge}$ is a variety. It is not difficult to see that $\mathbb{H}^{\wedge}$-algebras are in particular $\mathbb{BCK}$-*meet-semilattices* or $\mathbb{BKC}^{\wedge}$-*algebras*. Idziak studied $\mathbb{BKC}^{\wedge}$-algebras in [**51**], and we note that from his work it also follows that $\mathbb{H}^{\wedge}$ is a variety. In fact, $\mathbb{H}^{\wedge}$-algebras are precisely $\mathbb{BKC}^{\wedge}$-algebras that satisfy condition (H) (see page 30). An equational definition of $\mathbb{H}^{\wedge}$ is given as follows. $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ is an $\mathbb{H}^{\wedge}$-algebra when $\langle A, \to, 1\rangle$ is a Hilbert algebra, $\langle A, \wedge, 1\rangle$ is a semilattice with top element 1, and for all $a, b, c \in A$:

(H$^{\wedge}$1) $a \wedge (a \to b) = a \wedge b$,

(H$^{\wedge}$2) $(a \to (b \wedge c)) \to ((a \to b) \wedge (a \to c)) = 1$.

In [**32**] Figallo et al. also prove that implicative semilattices are the $\mathbb{H}^{\wedge}$-algebras that satisfy the following equation:

(PA) $a \to (b \to (a \wedge b)) = 1$.

The following example from [**32**] shows that the inclusion is strict.

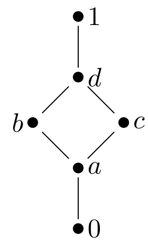


FIGURE 2. Example of a Hilbert algebra with infimum that is not an implicative semilattice.

EXAMPLE 6.5.9. Consider the lattice in Figure 2 as a semilattice $\langle A, \wedge, 1\rangle$, where $A = \{0, a, b, c, d, 1\}$, and let $\to$ be the implication defined on A by the order (cf. Example 6.5.8). Then we have that $\mathbf{A} := \langle A, \to, \wedge, 1\rangle$ is an $\mathbb{H}^{\wedge}$-algebra, that is not an implicative semilattice because (PA) fails: $b \to (c \to (b \wedge c)) = b \to (c \to a) = b \to a = a \neq 1$.

In [**33**] Figallo et al. provide an axiomatization of the 1-assertional logic of $\mathbb{H}^{\wedge}$, that we denote by $\mathcal{S}^1_{\mathbb{H}^{\wedge}}$. The logic $\mathcal{S}^1_{\mathbb{H}^{\wedge}} := \langle \mathbf{Fm}, \vdash^1_{\mathbb{H}^{\wedge}}\rangle$ in the language $(\to, \wedge, 1)$ of type $(2, 2, 0)$, is presented in a Hilbert-style calculus by the following axioms and rules:

(A1) $\vdash^1_{\mathbb{H}^{\wedge}} \beta \to (\gamma \to \beta)$,
(A2) $\vdash^1_{\mathbb{H}^{\wedge}} (\gamma \to (\beta \to \delta)) \to ((\gamma \to \beta) \to (\gamma \to \delta))$,
(A3) $\vdash^1_{\mathbb{H}^{\wedge}} (\gamma \wedge \beta) \to \beta$,
(A$\wedge$1) $\vdash^1_{\mathbb{H}^{\wedge}} (\gamma \wedge (\gamma \to \beta)) \to (\gamma \wedge \beta)$,
(A$\wedge$2) $\vdash^1_{\mathbb{H}^{\wedge}} (\gamma \wedge \beta) \to (\beta \wedge \gamma)$,
(A$\wedge$3) $\vdash^1_{\mathbb{H}^{\wedge}} ((\gamma \wedge \beta) \wedge \delta) \to ((\gamma \wedge \delta) \wedge \beta)$,
(MP) $\gamma, \gamma \to \beta \vdash^1_{\mathbb{H}^{\wedge}} \beta$,
(AB) $\gamma \to \beta \vdash^1_{\mathbb{H}^{\wedge}} \gamma \to (\gamma \wedge \beta)$.

Notice that $\mathcal{S}^1_{\mathbb{H}^{\wedge}}$ is an expansion of $\mathcal{H}$, but it is not an axiomatic expansion, as the rule (AB) cannot be derived from any collection of axioms and the only rule of (MP).

Clearly $\mathcal{S}^1_{\mathbb{H}^{\wedge}}$ is finitary and has theorems as for any $\gamma \in Fm$, $\gamma \to \gamma \in \mathrm{Thm}\mathcal{S}^1_{\mathbb{H}^{\wedge}}$. Moreover it satisfies (PC) for $p \wedge q$. In [**33**] it is claimed erroneously that $\mathcal{S}^1_{\mathbb{H}^{\wedge}}$ satisfies (uDDT) for $p \to q$. Notice that from (A1), (A$\wedge$2), (MP) and (AB), by isotonicity (condition (C2$'$) in page 16) it follows $\gamma \vdash^1_{\mathbb{H}^{\wedge}} \beta \to (\gamma \wedge \beta)$. Then if $\mathcal{S}^1_{\mathbb{H}^{\wedge}}$ would have the deduction theorem, it would follow $\vdash^1_{\mathbb{H}^{\wedge}} \gamma \to (\beta \to (\beta \wedge \gamma))$, and this implies, in particular, that (PA) holds for any $\mathbb{H}^{\wedge}$-algebra, a contradiction.

Let us show that $\mathcal{S}^1_{\mathbb{H}^{\wedge}}$ is not congruential. We use Definition 4.3.2 and we show that there is an algebra $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ of type $(2, 2, 0)$ such that $\equiv^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}}$ is not a congruence of $\mathbf{A}$. Take the algebra $\mathbf{A}$ defined in Example 6.5.9. Notice that $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}}(b) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}}(c) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}}(a) = \uparrow a$. Therefore $a \equiv^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}} b \equiv^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}} c$. However $b \to c \not\equiv^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}} a \to b$, since $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}}(b \to c) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}}(c) = \uparrow a$ but $\mathrm{C}^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}}(a \to b) = \mathrm{C}^{\mathbf{A}}_{\mathcal{S}^1_{\mathbb{H}^{\wedge}}}(1) = \{1\}$.

Logical filters of the logic $\mathcal{S}^1_{\mathbb{H}^\wedge}$ are also studied by Figallo et al. in [**32**]. For any $\mathbb{H}^\wedge$-algebra $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$, a subset $H \subseteq A$ is an *absorbent filter* of $\mathbf{A}$ if for all $a, b \in A$:

- H is an implicative filter of $\langle A, \to, 1\rangle$,
- if $b \in H$, then $a \to (a \wedge b) \in H$.

We denote by $\mathrm{Ab}(\mathbf{A})$ the collection of all absorbent filters of $\mathbf{A}$. It is easy to prove that all absorbent filters are meet filters of $\langle A, \wedge, 1\rangle$: let $a, b \in H \in \mathrm{Ab}(\mathbf{A})$. Clearly H is an up-set and moreover $a \to (a \wedge b) \in H$. Since H is a implicative filter, then $a \wedge b \in H$. Notice that $\mathrm{Ab}(\mathbf{A})$ is closed under arbitrary intersections, so for $B \subseteq A$ we may consider the least absorbent filter containing B. But we do not have on hand an alternative characterization of the absorbent filter generated by a set. And we do not know whether the lattice of absorbent filters is distributive or not. However, later on we prove a proposition (see page 143) that sheds light on these filters.

The logic $\mathcal{S}^1_{\mathbb{H}^\wedge}$ does not have the properties that are required for the application of our theory in Chapter 5. However, a different logic that has such properties can be defined from $\mathbb{H}^\wedge$-algebras, namely $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$, the Hilbert based logic of $\mathbb{H}^\wedge$.

By the general theory in [**54**], $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$ is finitary and congruential. Moreover it satisfies (uDDT) for $p \to q$, $\mathbb{A}\mathrm{lg}\mathcal{S}^{\to}_{\mathbb{H}^\wedge} = \mathbb{H}^\wedge$, and for any Hilbert algebra with infimum $\mathbf{A}$, the collection of implicative filters of $\mathbf{A}$ is the collection of $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$-filters of $\mathbf{A}$. Thus the logic $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$ is also filter distributive.

The logic $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$ does not satisfy (PC) for $p \wedge q$ though. This follows easily from Example 6.5.9. From it we get that $\not\models_{\mathbb{H}^\wedge} \gamma \to (\delta \to (\gamma \wedge \delta)) \approx 1$, and this implies that $\gamma, \delta \nvdash^{\to}_{\mathbb{H}^\wedge} \gamma \wedge \delta$. Hence $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$ does not satisfy (PC) for $p \wedge q$, and so we cannot apply the correspondences studied in § 5.5 in order to dispense with the algebraic structure in the dual spaces of $\mathbb{H}^\wedge$-algebras. We obtain, from our theory in Chapter 5, Spectral-like and Priestley-style dualities for $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$, but they are not *elegant* dualities.

We can consider even one more logic, the logic of the order $\mathcal{S}^{\leq}_{\mathbb{H}^\wedge}$ of $\mathbb{H}^\wedge$, that is also the semilattice based logic of $\mathbb{H}^\wedge$. By definition, this logic is a finitary and congruential logic with theorems, it satisfies (PC), $\mathbb{A}\mathrm{lg}\mathcal{S}^{\leq}_{\mathbb{H}^\wedge} = \mathbb{H}^\wedge$ and for any $\mathbb{H}^\wedge$-algebra $\mathbf{A}$, the order filters of $\mathbf{A}$ are the $\mathcal{S}^{\leq}_{\mathbb{H}^\wedge}$-filters of $\mathbf{A}$. By results reported below (see Example 6.5.12) we know that the logic $\mathcal{S}^{\leq}_{\mathbb{H}^\wedge}$ is not filter distributive, so it does not fall under the scope or our study.

Notice that, unlike the case of implicative semilattices, for $\mathbb{H}^\wedge$-algebras we have that $\mathcal{S}^1_{\mathbb{H}^\wedge}$, $\mathcal{S}^{\leq}_{\mathbb{H}^\wedge}$ and $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$ are three different logics, and moreover, the relation between them goes as follows:

$$\vdash^{\to}_{\mathbb{H}^\wedge} \subsetneq \vdash^{\leq}_{\mathbb{H}^\wedge} \subsetneq \vdash^{1}_{\mathbb{H}^\wedge}.$$

Theorem 6.5.10. *Let $\mathscr{L} = (\to, \wedge, 1, \dots)$ be a language and let $\mathbb{K}$ be a quasi-variety of $\mathscr{L}$-algebras such that $\langle A, \to, \wedge, 1\rangle$ is an $\mathbb{H}^\wedge$-algebra for any $\mathbf{A} \in \mathbb{K}$. Then $\mathcal{S}^{\to}_{\mathbb{K}} = \mathcal{S}^{\leq}_{\mathbb{K}}$ if and only if*

$$(\forall \mathbf{A} \in \mathbb{K})(\forall a, b, c \in A)\ a \wedge b \leq^{\mathbf{A}} c \quad \textit{iff} \quad b \leq^{\mathbf{A}} a \to c,$$

where $\leq^{\mathbf{A}}$ is the order associated with the $\mathbb{H}^\wedge$-algebra $\langle A, \to, \wedge, 1\rangle$.

PROOF. Assume first that $\mathcal{S}^{\to}_{\mathbb{K}} = \mathcal{S}^{\leq}_{\mathbb{K}}$. Then by the general theory in [**54**] and [**53**], $\mathcal{S} = \mathcal{S}^{\to}_{\mathbb{K}} = \mathcal{S}^{\leq}_{\mathbb{K}}$ is congruential and satisfies (PC) and (uDDT). This implies that $\mathrm{Fi}_{\mathcal{S}}(\mathbf{A}) = \mathrm{Fi}_{\to}(\mathbf{A}) = \mathrm{Fi}_{\wedge}(\mathbf{A})$ for all $\mathbf{A} \in \mathbb{K}$. Let $\mathbf{A} \in \mathbb{K}$ and $a, b, c \in A$. We have $\langle \{a, b\} \rangle = [\![\{a, b\}\rangle\!\rangle = {\uparrow}(a \wedge b)$. Then

$$\begin{aligned} a \wedge b \leq^{\mathbf{A}} c \quad &\text{iff} \quad c \in {\uparrow}(a \wedge b) = [\![\{a, b\}\rangle\!\rangle = \langle \{a, b\} \rangle \quad \text{iff} \quad a \to c \in \mathrm{Fi}_{\to}(b) = {\uparrow} b \\ &\text{iff} \quad b \leq^{\mathbf{A}} a \to c. \end{aligned}$$

For the converse, let us assume that for all $\mathbf{A} \in \mathbb{K}$ and all $a, b, c \in A$ we have that $a \wedge b \leq^{\mathbf{A}} c$ if and only if $b \leq^{\mathbf{A}} a \to c$. Let $\gamma_1, \ldots, \gamma_n, \delta \in Fm$, let $\mathbf{A} \in \mathbb{K}$ and let $h \in \mathrm{Hom}(\mathbf{Fm}, \mathbf{A})$. Notice that by definition of the order on $\mathbf{A}$, we have that $h(\gamma_1) \to^{\mathbf{A}} (h(\gamma_2) \to^{\mathbf{A}} (\ldots (h(\gamma_n) \to^{\mathbf{A}} h(\delta)) \ldots)) \neq 1^{\mathbf{A}}$ if and only if $h(\gamma_1) \nleq^{\mathbf{A}} h(\gamma_2) \to^{\mathbf{A}} (\ldots (h(\gamma_n) \to^{\mathbf{A}} h(\delta)) \ldots)$. By assumption, we can use the residuation law $n-1$ times and we get $h(\gamma_1) \nleq^{\mathbf{A}} h(\gamma_2) \to^{\mathbf{A}} (\ldots (h(\gamma_n) \to^{\mathbf{A}} h(\delta)) \ldots)$ if and only if $h(\gamma_1) \wedge \cdots \wedge h(\gamma_n) \nleq^{\mathbf{A}} h(\delta)$. Hence, we conclude that $\mathcal{S}^{\to}_{\mathbb{K}} = \mathcal{S}^{\leq}_{\mathbb{K}}$, as required. $\square$

The problem that we originally addressed in the early stages of our research, was to get *elegant* Spectral-like and Priestley-style dualities for $\mathcal{S}^{\to}_{\mathbb{H}^{\wedge}}$, i. e. for the variety of Hilbert algebras with infimum. But for the moment we have only been able to find a solution for a subclass of $\mathbb{H}^{\wedge}$, namely the Hilbert algebras with infimum with the additional property that the semilattice reduct is distributive. These algebras are called *distributive Hilbert algebras with infimum.* We devote Chapter 7 to expound Spectral-like and Priestley-style dualities for such algebras. Distributive Hilbert algebras with infimum do not form a variety, not even a quasi-variety. So they are not the algebraic counterpart of any logic that we could define form them. However, it turns out that the dualities for Hilbert algebras with infimum can be restricted to dualities for other classes that do relate with interesting logics. We address this issue in § 7.6. For the moment, let us introduce the subclass of $\mathbb{H}^{\wedge}$ for which we develop the dualities in Chapter 7.

6.5.1. Distributive Hilbert algebras with infimum.

DEFINITION 6.5.11. A $\mathbb{H}^{\wedge}$-algebra $\mathbf{A} = \langle A, \to, \wedge, 1 \rangle$ is a *distributive* $\mathbb{H}^{\wedge}$-algebra or a $\mathbb{DH}^{\wedge}$-*algebra* when the underlying semilattice $\langle A, \wedge, 1 \rangle$ is distributive.

Let us denote by $\mathbb{DH}^{\wedge}$ the class of distributive Hilbert algebras with infimum. Notice that the algebra defined in Example 6.5.9 is in fact a $\mathbb{DH}^{\wedge}$-algebra. Therefore, that example shows that implicative semilattices are strictly included in $\mathbb{DH}^{\wedge}$. Moreover, the following example shows that $\mathbb{DH}^{\wedge}$ is strictly included in $\mathbb{H}^{\wedge}$.

EXAMPLE 6.5.12. Consider the lattice N5 in Figure 3 as a semilattice $\langle A, \wedge, 1 \rangle$, where $A = \{0, a, b, c, 1\}$, and let $\to$ be the implication defined on A by the order (cf. Example 6.5.8). Then we have that $\mathbf{A} := \langle A, \to, \wedge, 1 \rangle$ is an $\mathbb{H}^{\wedge}$-algebra, that is obviously not distributive.

The relations between the four classes of algebras in the language $(\to, \wedge, 1)$ so far considered are:

$$\mathbb{IS} \subsetneq \mathbb{DH}^{\wedge} \subsetneq \mathbb{H}^{\wedge} \subsetneq \mathbb{BKC}^{\wedge}$$

We focus now on filters and ideals of $\mathbb{DH}^{\wedge}$-algebras. However, all the definitions and several lemmas stated in what follows hold in general for $\mathbb{H}^{\wedge}$-algebras, not

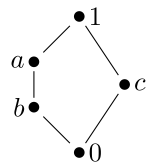


Figure 3. Example of a Hilbert algebra with infimum that is not distributive – N5 with the order given by implication.

necessarily distributive. We state them in the more general form when possible, and so we bring out which properties related with distributivity are essential to get which results. From now on, let $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ be an $\mathbb{H}^{\wedge}$-algebra, where we have the order $\leq$ given by:

$$a \leq b \quad \text{iff} \quad a \to b = 1 \quad \text{iff} \quad a \wedge b = a.$$

We focus on the two underlying structures of any $\mathbb{H}^{\wedge}$-algebra: the semilattice and the Hilbert algebra.

Concerning the underlying semilattice $\langle A, \wedge, 1\rangle$, we may consider the collections of meet filters $\mathrm{Fi}_{\wedge}(\mathbf{A})$ (definition in page 26), irreducible meet filters $\mathrm{Irr}_{\wedge}(\mathbf{A})$ (definition in page 27), optimal meet filters $\mathrm{Op}_{\wedge}(\mathbf{A})$ (definition in page 28), order ideals $\mathrm{Id}(\mathbf{A})$ (definition in page 26), and F-ideals $\mathrm{Id}_F(\mathbf{A})$ (definition in page 28), that we introduced in § 2.3, and that yield the following lemmas and corollaries that we retype here for the sake of completeness:

Lemma 2.3.3: Let $\mathbf{A}$ be an $\mathbb{H}^{\wedge}$-algebra, and let $F \in \mathrm{Fi}_{\wedge}(\mathbf{A})$ and $I \in \mathrm{Id}(\mathbf{A})$ be such that $F \cap I = \emptyset$. Then there is $G \in \mathrm{Irr}_{\wedge}(\mathbf{A})$ such that $F \subseteq G$ and $G \cap I = \emptyset$.

Theorem 2.3.6: Let $\mathbf{A}$ be a $\mathbb{DH}^{\wedge}$-algebra. Then for all $F \in \mathrm{Fi}_{\wedge}(\mathbf{A})$, $F \in \mathrm{Irr}_{\wedge}(\mathbf{A})$ if and only if $F^c \in \mathrm{Id}(\mathbf{A})$.

Lemma 2.3.7: Let $\mathbf{A}$ be an $\mathbb{H}^{\wedge}$-algebra and let $F \in \mathrm{Fi}_{\wedge}(\mathbf{A})$ and $I \in \mathrm{Id}_F(\mathbf{A})$ be such that $F \cap I = \emptyset$. Then there is $G \in \mathrm{Op}_{\wedge}(\mathbf{A})$ such that $F \subseteq G$ and $G \cap I = \emptyset$.

Theorem 2.3.9: Let $\mathbf{A}$ be a $\mathbb{DH}^{\wedge}$-algebra. For any $F \in \mathrm{Fi}_{\wedge}(\mathbf{A})$, $F \in \mathrm{Op}_{\wedge}(\mathbf{A})$ if and only if $F^c \in \mathrm{Id}_F(\mathbf{A})$.

Notice that the separation lemmas hold in general for any $\mathbb{H}^{\wedge}$-algebra, but the correspondences between filters and ideals hold only for the distributive ones.

Concerning the underlying Hilbert algebra $\langle A, \to, 1\rangle$, we may consider the collections of implicative filters $\mathrm{Fi}_{\to}(\mathbf{A})$ (definition in page 31), irreducible implicative filters $\mathrm{Irr}_{\to}(\mathbf{A})$ (definition in page 31), optimal implicative filters $\mathrm{Op}_{\to}(\mathbf{A})$ (definition in page 32), order ideals $\mathrm{Id}(\mathbf{A})$ (definition in page 26), and strong F-ideals $\mathrm{Id}_{sF}(\mathbf{A})$ (definition in page 32), that we introduced in § 2.4, and that yield the following lemmas and corollaries that we retype here for the sake of completeness:

Lemma 2.4.5: Let $\mathbf{A}$ be an $\mathbb{H}^{\wedge}$-algebra, and let $F \in \mathrm{Fi}_{\to}(\mathbf{A})$ and $I \in \mathrm{Id}(\mathbf{A})$ be such that $F \cap I = \emptyset$. Then there is $G \in \mathrm{Irr}_{\to}(\mathbf{A})$ such that $F \subseteq G$ and $G \cap I = \emptyset$.

Corollary 2.4.4: Let $\mathbf{A}$ be a $\mathbb{H}^\wedge$-algebra. Then for all $F \in \mathrm{Fi}_\to(\mathbf{A})$, $F \in \mathrm{Irr}_\to(\mathbf{A})$ if and only if $F^c \in \mathrm{Id}(\mathbf{A})$.

Lemma 2.4.7: Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra and let $F \in \mathrm{Fi}_\to(\mathbf{A})$ and $I \in \mathrm{Id}_{sF}(\mathbf{A})$ be such that $F \cap I = \emptyset$. Then there is $G \in \mathrm{Op}_\to(\mathbf{A})$ such that $F \subseteq G$ and $G \cap I = \emptyset$.

Theorem 2.4.9: Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra. For any $F \in \mathrm{Fi}_\to(\mathbf{A})$, $F \in \mathrm{Op}_\to(\mathbf{A})$ if and only if $F^c \in \mathrm{Id}_{sF}(\mathbf{A})$.

Notice that distributivity is not needed in any of the previous lemmas and corollaries. Now we focus on the relations between all these notions. Recall that for any $\mathbb{H}^\wedge$-algebra, the order defined by the meet and the order defined by the implication coincide. This fact implies a strong link between the two operations, that is reflected in the following propositions.

PROPOSITION 6.5.13. *Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra. Then any meet filter of $\mathbf{A}$ is an implicative filter of $\mathbf{A}$.*

PROOF. Let $F \in \mathrm{Fi}_\wedge(\mathbf{A})$. Since F is a non-empty up-set, clearly $1 \in F$. Let $a, a \to b \in F$. Then $a \wedge b = a \wedge (a \to b) \in F$ so since $a \wedge b \leq b$ and F is an up-set, we obtain $b \in F$. ☐

COROLLARY 6.5.14. *Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra. Then for all $B \subseteq A$, $\langle B \rangle \subseteq \llbracket B \rrbracket$.*

LEMMA 6.5.15. *Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra. Then $a \to (b \to c) \leq (a \wedge b) \to c$ for all $a, b, c \in A$.*

PROOF. Let $a, b, c \in A$. From $a \wedge b \leq a$ we get $a \to (b \to c) \leq (a \wedge b) \to (b \to c)$. From $a \wedge b \leq b$ we get $b \to c \leq (a \wedge b) \to c$, and so $(a \wedge b) \to (b \to c) \leq (a \wedge b) \to ((a \wedge b) \to c) = (a \wedge b) \to c$, and we are done. ☐

PROPOSITION 6.5.16. *Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra. Then*

$$\mathrm{Irr}_\to(\mathbf{A}) \cap \mathrm{Fi}_\wedge(\mathbf{A}) \subseteq \mathrm{Irr}_\wedge(\mathbf{A}).$$

PROOF. Let $F \in \mathrm{Irr}_\to(\mathbf{A}) \cap \mathrm{Fi}_\wedge(\mathbf{A})$ and let $F_1, F_2 \in \mathrm{Fi}_\wedge(\mathbf{A})$ be such that $F_1 \cap F_2 = F$. Since F_1, F_2, F are implicative filters, and F is $\to$-irreducible, we get $F_1 = F$ of $F_2 = F$, therefore F is $\wedge$-irreducible meet filter. ☐

PROPOSITION 6.5.17. *Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra. Then*

$$\mathrm{Op}_\to(\mathbf{A}) \cap \mathrm{Fi}_\wedge(\mathbf{A}) \subseteq \mathrm{Op}_\wedge(\mathbf{A}).$$

PROOF. Let $F \in \mathrm{Op}_\to(\mathbf{A}) \cap \mathrm{Fi}_\wedge(\mathbf{A})$. By Theorem 2.4.9 we know that F^c is sF-ideal, so it is in particular an F-ideal. Let us show that it is $\wedge$-prime: let $B \subseteq^\omega A$ be such that $\bigwedge B \in P^c$. We show that $B \cap P^c \neq \emptyset$. Suppose, towards a contradiction, that $B \cap P^c = \emptyset$. Then $B \subseteq P$, and since P is a meet filter by assumption, we get $\bigwedge B \in P$, a contradiction.

Hence, we have shown that F^c is a $\wedge$-prime F-ideal. Thus by Corollary 2.3.10 we conclude that F is an $\wedge$-optimal meet filter. ☐

For the case when the underlying semilattice is distributive, we find stronger links between these collections of filters.

PROPOSITION 6.5.18. *Let $\mathbf{A}$ be an $\mathbb{H}^\wedge$-algebra. Then the underlying semilattice is distributive if and only if $\mathrm{Irr}_\wedge(\mathbf{A}) \subseteq \mathrm{Irr}_\to(\mathbf{A})$.*

PROOF. Assume first that $\mathbf{A}$ is distributive and let $P \in \mathrm{Irr}_\wedge(\mathbf{A})$. On the one hand we have that $P \in \mathrm{Fi}_\rightarrow(\mathbf{A})$. On the other hand, by Theorem 2.3.6, $P^c \in \mathrm{Id}(\mathbf{A})$. Then by Corollary 2.4.4, we conclude that $P \in \mathrm{Irr}_\rightarrow(\mathbf{A})$.

Assume now that $\mathrm{Irr}_\wedge(\mathbf{A}) \subseteq \mathrm{Irr}_\rightarrow(\mathbf{A})$. Then by Corollary 2.4.4 we have that for all $P \in \mathrm{Irr}_\wedge(\mathbf{A})$, P^c is an order ideal. By Theorem 10 in [**12**], this implies that the underlying semilattice is distributive, so $\mathbf{A}$ is a $DH^\wedge$-algebra, as required. □

COROLLARY 6.5.19. *Let* $\mathbf{A}$ *be a* $\mathbb{DH}^\wedge$*-algebra. Then*

$$\mathrm{Irr}_\rightarrow(\mathbf{A}) \cap \mathrm{Fi}_\wedge(\mathbf{A}) = \mathrm{Irr}_\wedge(\mathbf{A}).$$

PROOF. This follows from propositions 6.5.18 and 6.5.16. □

PROPOSITION 6.5.20. *Let* $\mathbf{A}$ *be a* $\mathbb{DH}^\wedge$*-algebra. Then any* $\wedge$*-prime F-ideal is an sF-ideal.*

PROOF. Assume that $\mathbf{A}$ is distributive and let $I \in \mathrm{Id}_F(\mathbf{A})$ be $\wedge$-prime. Let $I' \subseteq^\omega I$ and let $B \subseteq^\omega A$ be such that $\bigcap\{\uparrow a : a \in I'\} \subseteq \langle B\rangle$. We show that $\langle B\rangle \cap I \neq \emptyset$. Recall that for any $C \subseteq^\omega A$ we have $\langle C\rangle \subseteq [\![C\rangle\!\rangle = \uparrow \bigwedge C$. Thus from the hypothesis we have $\bigcap\{\uparrow a : a \in I'\} \subseteq [\![B\rangle\!\rangle = \uparrow \bigwedge B$. And by I being an F-ideal of $\mathbf{A}$, we get $\bigwedge B \in I$. Now we use that I is $\wedge$-prime, so there is $b \in B$ such that $b \in I$. We conclude that $\langle B\rangle \cap I \neq \emptyset$, as required. □

COROLLARY 6.5.21. *Let* $\mathbf{A}$ *be a* $\mathbb{DH}^\wedge$*-algebra. Then* $\mathrm{Op}_\wedge(\mathbf{A}) \subseteq \mathrm{Op}_\rightarrow(\mathbf{A})$.

PROOF. This follows from Corollary 2.3.10, Proposition 6.5.20 and Theorem 2.4.9. □

The relation between $\wedge$-irreducibles and $\rightarrow$-irreducibles is shown in Proposition 6.5.18 to characterize $\mathbb{DH}^\wedge$-algebras, but it remains as an open question whether the inclusion in last corollary characterizes $\mathbb{DH}^\wedge$-algebras or not.

COROLLARY 6.5.22. *Let* $\mathbf{A}$ *be a* $\mathbb{DH}^\wedge$*-algebra. Then*

$$\mathrm{Op}_\rightarrow(\mathbf{A}) \cap \mathrm{Fi}_\wedge(\mathbf{A}) = \mathrm{Op}_\wedge(\mathbf{A}).$$

PROOF. This follows from Propositions 6.5.17 and Corollary 6.5.21. □

For the non-distributive case, it may happen that the equalities in corollaries 6.5.19 and 6.5.22 fail, as the Example 6.5.23 shows.

EXAMPLE 6.5.23. Consider the $\mathbb{H}^\wedge$-algebra $\mathbf{A}$ given in Example 6.5.12 (N5 as a semilattice, with the implication defined by the order). On the one hand, we have that $\mathrm{Fi}_\wedge(\mathbf{A})$ is the collection of all principal up-sets and so $\mathrm{Irr}_\wedge(\mathbf{A}) = \{\uparrow a, \uparrow b, \uparrow c, \uparrow 1\}$. On the other hand, we have that $\mathrm{Fi}_\rightarrow(\mathbf{A})$ is the collection of all principal up-sets together with $F_a := \{a, c, 1\}$ and $F_{ab} := \{a, b, c, 1\}$. The lattice of implicative filters of $\mathbf{A}$ is represented in Figure 4.

Clearly $\mathrm{Irr}_\rightarrow(\mathbf{A}) = \{\uparrow 0, F_{ab}, F_a, \uparrow b, \uparrow c\}$. Hence as $\uparrow a \in \mathrm{Irr}_\wedge(\mathbf{A}) \setminus \mathrm{Irr}_\rightarrow(\mathbf{A})$, we have an example of a $\mathbb{H}^\wedge$-algebra that is not distributive and for which it holds $\mathrm{Irr}_\wedge(\mathbf{A}) \nsubseteq \mathrm{Irr}_\rightarrow(\mathbf{A}) \cap \mathrm{Fi}_\wedge(\mathbf{A})$, and also $\mathrm{Op}_\wedge(\mathbf{A}) \nsubseteq \mathrm{Op}_\rightarrow(\mathbf{A}) \cap \mathrm{Fi}_\wedge(\mathbf{A})$.

Notice that we should restrict ourselves to search dualities for distributive $\mathbb{H}^\wedge$-algebras, precisely because of the possible failure of the inclusions in corollaries 6.5.19 and 6.5.22 when the underlying semilattice is not distributive.

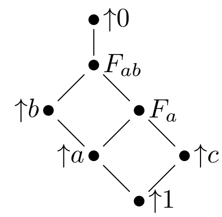


FIGURE 4. Lattice of implicative filters of N5 with the implication given by the order.

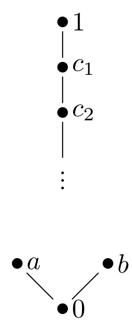


FIGURE 5. Example of a distributive Hilbert algebra with infimum.

Let us present one more example extracted from [**5**], that shows that for the distributive case, some of the mentioned inclusions may be strict.

EXAMPLE 6.5.24. Consider the semilattice in Figure 5 that is a distributive semilattice $\langle A, \wedge, 1\rangle$, where $A = \{0, a, b, 1\} \cup \{c_i : i \in \omega\}$, and let $\to$ be the implication defined on A by the order (cf. Example 6.5.8). Then we have that $\mathbf{A} := \langle A, \to, \wedge, 1\rangle$ is a $\mathbb{DH}^{\wedge}$-algebra.

Let us denote by F_{ab} the implicative filter $\uparrow(\{a, b\}) = \{a, b, 1\} \cup \{c_i : i \in \omega\}$, and by F_c the meet filter $\{c_i : i \in \omega\} \cup \{1\}$. It is easy to see that $\mathrm{Fi}_\wedge(\mathbf{A})$ is the collection of all principal up-sets together with F_c. Moreover, $\mathrm{Fi}_\to(\mathbf{A})$ is $\mathrm{Fi}_\wedge(\mathbf{A})$ together with F_{ab}. It is not difficult to check that $F_{ab} \in \mathrm{Irr}_\to(\mathbf{A})$. But since F_{ab} is not closed under meet, $F_{ab} \notin \mathrm{Fi}_\wedge(\mathbf{A})$. Hence, we have:

$$\mathrm{Irr}_\wedge(\mathbf{A}) \subsetneq \mathrm{Irr}_\to(\mathbf{A}),$$

$$\mathrm{Op}_\wedge(\mathbf{A}) \subsetneq \mathrm{Op}_\to(\mathbf{A}),$$

$$\mathrm{Fi}_\wedge(\mathbf{A}) \subsetneq \mathrm{Fi}_\to(\mathbf{A}).$$

Moreover, we have that $F_c \in \mathrm{Op}_\to(\mathbf{A})$, since $\downarrow\{a, b\}$ is an sF-ideal of $\mathbf{A}$, but $F_c \notin \mathrm{Irr}_\to(\mathbf{A})$, since we have $F_c = \uparrow a \cap \uparrow b$ but $F_c \neq \uparrow a, \uparrow b$. Similarly, $F_c \in \mathrm{Op}_\wedge(\mathbf{A})$, since it is closed under meet, but $F_c \notin \mathrm{Irr}_\wedge(\mathbf{A})$. Finally, notice that $\mathbf{A}$ is bounded, so $\emptyset \notin \mathrm{Id}_{sF}(\mathbf{A})$. Therefore $\uparrow 0 = A \notin \mathrm{Op}_\to(\mathbf{A})$, but clearly $A \in \mathrm{Fi}_\to(\mathbf{A})$. Similarly $A \in \mathrm{Fi}_\wedge(\mathbf{A})$ but $A \notin \mathrm{Op}_\wedge(\mathbf{A})$. Hence, we have:

$$\mathrm{Irr}_\to(\mathbf{A}) \subsetneq \mathrm{Op}_\to(\mathbf{A}) \subsetneq \mathrm{Fi}_\to(\mathbf{A}),$$

$$\mathrm{Irr}_\wedge(\mathbf{A}) \subsetneq \mathrm{Op}_\wedge(\mathbf{A}) \subsetneq \mathrm{Fi}_\wedge(\mathbf{A}).$$

Let us conclude this section with a proposition concerning absorbent filters that will be useful later.

PROPOSITION 6.5.25. *Let* $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ *be an* $\mathbb{H}^\wedge$*-algebra. For any meet filter* F *of* $\langle A, \wedge, 1\rangle$, $F \in \mathrm{Ab}(\mathbf{A})$ *if and only if* $\langle F \cup {\uparrow}a\rangle$ *is a meet filter of* $\langle A, \wedge, 1\rangle$ *for all* $a \in A$.

PROOF. Let $F \in \mathrm{Ab}(\mathbf{A})$ and let $a \in A$. If $a \in F$ there is nothing to prove, so suppose $a \notin F$. Recall that $\langle F \cup {\uparrow}a\rangle$ denotes the implicative filter of $\langle A, \to, 1\rangle$ generated by $F \cup {\uparrow}a$.

We claim that $\langle F \cup {\uparrow}a\rangle$ is a meet filter. We show that $b \wedge c \in \langle F \cup {\uparrow}a\rangle$ for any $b, c \in \langle F \cup {\uparrow}a\rangle$, so let $b, c \in \langle F \cup {\uparrow}a\rangle$. As $F \cup {\uparrow}a \neq \emptyset$, we can assume that there are $b_0, \dots, b_n, c_0, \dots, c_m \in F \cup {\uparrow}a$ such that $b_0 \to (\dots(b_n \to b)\dots) = 1$ and $c_0 \to (\dots(c_m \to c)\dots) = 1$. By Lemma 6.5.15, this implies $(b_0 \wedge \dots \wedge b_n) \to b = 1$ and $(c_0 \wedge \dots \wedge c_m) \to c = 1$. Then we have $b_0 \wedge \dots \wedge b_n \wedge c_0 \wedge \dots \wedge c_m \leq b \wedge c$. Since $b_0, \dots, b_n, c_0, \dots, c_m \in F \cup {\uparrow}a$ and F and ${\uparrow}a$ are both closed under meets, then we have $d_1 \in F$ and $d_2 \in {\uparrow}a$ such that $b_0 \wedge \dots \wedge b_n \wedge c_0 \wedge \dots \wedge c_m = d_1 \wedge d_2 \leq b \wedge c$. Moreover, by definition of absorbent filter, $d_2 \to (d_1 \wedge d_2) \in F \subseteq \langle F \cup {\uparrow}a\rangle$. And since $d_2 \in {\uparrow}a \subseteq \langle F \cup {\uparrow}a\rangle$, by definition of implicative filter we obtain $d_1 \wedge d_2 \in \langle F \cup {\uparrow}a\rangle$. Now since $\langle F \cup {\uparrow}a\rangle$ is an up-set, we conclude $b \wedge c \in \langle F \cup {\uparrow}a\rangle$.

For the converse, let $F \in \mathrm{Fi}_\wedge(\mathbf{A})$ be such that for all $a \in A$, $\langle F \cup {\uparrow}a\rangle$ is a meet filter. We show that F is absorbent. Let $b \in F$ and $a \in A$. We show that $a \to (a \wedge b) \in F$. Notice first that $\langle F \cup {\uparrow}a\rangle = \langle F \cup \{a\}\rangle$. Then, as $a \in \langle F \cup {\uparrow}a\rangle$ and $b \in F$ we get by hypothesis that $a \wedge b \in \langle F \cup {\uparrow}a\rangle$. Now we use the definition of implicative filter generated, and we get that there are $c_0, \dots, c_n \in F$, for some $n \in \omega$, such that $c_0 \to (c_1 \to (\dots(c_n \to (a \to (a \wedge b)))\dots)) = 1$. But this implies $a \to (a \wedge b) \in F$, as required. $\square$

In brief, we have studied the properties of the different collections of filters and ideals for $\mathbb{DH}^\wedge$-algebras, and these results will be used in Chapter 7, where Spectral-like and Priestley-style dualities for categories that have $\mathbb{DH}^\wedge$-algebras as objects are studied in detail. Before moving to this topic, let us review some other filter distributive and congruential expansions of $\mathcal{H}$ for which the mentioned dualities could also be applied, as it will be outlined in § 7.6.

6.6. Expansions with a conjunction and a disjunction

Let us concentrate now on the language $(\to, \wedge, \vee, 1)$ of type $(2, 2, 2, 0)$. A well-known logic defined in this language is $\mathcal{IPC}^+$, the positive (intuitionistic) logic, that is, the negation-less fragment of intuitionistic logic. The logic $\mathcal{IPC}^+ := \langle \mathbf{Fm}, \vdash_{\mathcal{IPC}^+}\rangle$ in the language $(\to, \wedge, \vee, 1)$ can be presented in a Hilbert-style calculus by the following axioms and rules:

(A1) $\vdash_{\mathcal{IPC}^+} \beta \to (\gamma \to \beta)$,
(A2) $\vdash_{\mathcal{IPC}^+} (\gamma \to (\beta \to \delta)) \to ((\gamma \to \beta) \to (\gamma \to \delta))$,
(A3) $\vdash_{\mathcal{IPC}^+} (\gamma \wedge \beta) \to \beta$,
(A4) $\vdash_{\mathcal{IPC}^+} (\gamma \wedge \beta) \to \gamma$,
(A5) $\vdash_{\mathcal{IPC}^+} (\gamma \to \beta) \to ((\gamma \to \delta) \to (\gamma \to (\beta \wedge \delta)))$,

(A6) $\vdash_{\mathcal{IPC}^+} \gamma \to (\gamma \vee \beta)$,
(A7) $\vdash_{\mathcal{IPC}^+} \gamma \to (\beta \vee \gamma)$,
(A8) $\vdash_{\mathcal{IPC}^+} (\gamma \to \delta) \to ((\beta \to \delta) \to ((\gamma \vee \beta) \to \delta))$,
(MP) $\gamma, \gamma \to \beta \vdash_{\mathcal{IPC}^+} \beta$.

It is well known that $\mathcal{IPC}^+$ is the 1-assertional logic of the variety of relatively pseudo-complemented lattices, that coincides with $\mathbb{A}\mathrm{lg}\mathcal{IPC}^+$.

DEFINITION 6.6.1. A *relatively pseudo-complemented lattice* or *generalized Heyting algebra* ($g\mathbb{H}e$-algebra for short) is an algebra $\mathbf{A} = \langle A, \to, \wedge, \vee, 1\rangle$ of type $(2, 2, 2, 0)$ such that $\langle A, \wedge, \vee, 1\rangle$ is a lattice with top element 1 and $\to$ is the right residuum of $\wedge$, i. e. for all $a, b, c \in A$:

$$a \wedge c \leq b \quad \text{iff} \quad c \leq a \to b.$$

Let us denote by $g\mathbb{H}e$ the class of relatively pseudo-complemented lattices and by $\mathbb{H}e$ the class of bounded relatively pseudo-complemented lattices, i. e. the class of *Heyting algebras*. It is well known that the lattice reduct of any $g\mathbb{H}e$-algebra is distributive. Since implicative semilattices are a variety, it follows that $g\mathbb{H}e$ and $\mathbb{H}e$ are also varieties.

THEOREM 6.6.2. *The logics $\mathcal{IPC}^+$, $\mathcal{S}^{\to}_{g\mathbb{H}e}$ and $\mathcal{S}^{\leq}_{g\mathbb{H}e}$ are equal.*

PROOF. The proof is similar to that of Theorem 6.2.1, using that $\mathcal{IPC}^+$ satisfies (uDDT). □

Again we obtain that $\mathcal{IPC}^+$ is a filter distributive finitary congruential logic with theorems, that satisfies (uDDT), (PC) and (PDI), so our theory from Chapter 5 can be applied to it, and similarly as we did with the implicative-conjunctive fragment of intuitionistic logic, the dualities obtained from the general theory can be refined to dispense with the algebraic structures in the dual side. Notice that relatively pseudo-complemented lattices are an example of distributive lattices expanded with a binary quasioperator. And in fact extended Priestley-duality (or what could be called extended Spectral-duality) can be applied to them, in order to obtain the mentioned dualities.

In [**13**] Celani and Cabrer consider another class of algebras in this language, that they call *(bounded) distributive lattices with implication* or $\mathbb{DLI}$-algebras. These are bounded distributive lattices expanded with a normal dual $(\partial, 1)$-quasioperator $\to$, i. e. algebras $\mathbf{A} = \langle A, \to, \wedge, \vee, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and for all $a, b, c \in A$:

(IA2) $(a \to 1) = 1$,
(DLI0) $(0 \to a) = 1$,
(DLI1) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$,
(H^L1) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.

Extended Priestley duality shall be applied to this class of algebras, and this is precisely what Celani and Cabrer do, in order to study some of their subvarieties in a modular way, getting relational semantics for the algebraic counterpart of certain fuzzy logics, such as $\mathbb{MTL}$-algebras or $\mathbb{MV}$-algebras. Moreover, Heyting algebras are contained in $\mathbb{DLI}$, as well as weakly Heyting algebras introduced by Celani and Jansana in [**16**], where a study of Priestley-style duality for them yields relational semantics for strict implication fragments of some normal modal logics. We are not interested in $\mathbb{DLI}$ and its subvarieties, since Priestley-style duality for them is already known, but we show in what follows that the variety we are interested in does not include and is not included in $\mathbb{DLI}$.

Notice that $\mathbb{DLI}$-algebras are lattices expanded with an implication that preserves the order in the second coordinate and reverses the order in the first coordinate. The class of algebras that we introduce now shares the same property.

Let us focus on a different logic defined in the language $(\to, \wedge, \vee, 1)$. We consider the Hilbert based logic of the following class of algebras:

DEFINITION 6.6.3. An algebra $\mathbf{A} = \langle A, \to, \wedge, \vee, 1\rangle$ of type $(2, 2, 2, 0)$ is a *Hilbert algebra with lattice structure* or a $\mathbb{H}^L$*-algebra* if:

(1) $\langle A, \to, 1\rangle$ is a Hilbert algebra,
(2) $\langle A, \wedge, \vee, 1\rangle$ is a lattice with top element 1,
(3) $\to$ and $\wedge$ define the same order, i. e. for all $a, b \in A$:

$$a \to b = 1 \quad \text{iff} \quad a \wedge b = a.$$

Let us denote by $\mathbb{H}^L$ the class of Hilbert algebras with lattice structure, and by $\mathbb{H}^L_0$ the class of bounded Hilbert algebras with lattice structure, i. e. $\mathbb{H}^L$-algebras with an additional constant, that is interpreted as the bottom element of the underlying lattice.

EXAMPLE 6.6.4. The $\mathbb{H}^\wedge$-algebra considered in Example 6.5.12 (see page 138), that recall is the lattice N5 with the implication given by the order, can also be seen as an $\mathbb{H}^L$-algebra.

Note that $\mathbb{H}^L$-algebras are a subclass of $\mathbb{BCK}$-lattices (or $\mathbb{BCK}^L$-algebras). These algebras were studied by Idziak in [**51**], where he shows that they form a variety. In fact, $\mathbb{H}^L$-algebras are those $\mathbb{BCK}^L$-algebras that satisfy condition (H) (see page 30), and an equational definition of $\mathbb{H}^L$ is given as follows. $\mathbf{A} = \langle A, \to, \wedge, \vee, 1\rangle$ is an $\mathbb{H}^L$-algebra if for all $a, b, c \in A$:

(1) $\langle A, \to, 1\rangle$ is a Hilbert algebra,
(2) $\langle A, \wedge, \vee, 1\rangle$ is a lattice with top element 1,
(H$^\wedge$1) $a \wedge (a \to b) = a \wedge b$,
(H$^\wedge$2) $(a \to (b \wedge c)) \to ((a \to b) \wedge (a \to c)) = 1$,
(H^{L}1) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.

The variety $g\mathbb{H}e$ of relatively pseudo-complemented lattices is strictly included in the variety of $\mathbb{H}^L$-algebras. A relevant equation is again (PA) (see page 135). Heyting algebras are precisely those bounded $\mathbb{H}^L$-algebras that satisfy (PA). Moreover, the inclusion is strict. The $\mathbb{H}^\wedge$-algebra considered in Example 6.5.9 (see page 136) shall also be seen as an $\mathbb{H}^L$-algebra, that is obviously not a Heyting algebra. In [**32**] we find a different example:

EXAMPLE 6.6.5. Consider the lattice in Figure 6, that is distributive, and let $\to$ be the implication defined on A by the table in Figure 7. Then we have that $\mathbf{A} = \langle A, \to, \wedge, 1, 0\rangle$ is a bounded $\mathbb{H}^L$-algebra, but is not a Heyting algebra, since $a \to 0 = b \neq f = \max\{z : a \wedge z \leq 0\}$.

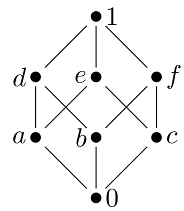


FIGURE 6. Example of a distributive Hilbert algebra with lattice structure.

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	b	1	b	f	1	1	f	1
b	e	e	1	e	1	e	1	1
c	b	d	b	1	d	1	1	1
d	0	e	b	c	1	e	f	1
e	b	d	b	f	d	1	f	1
f	0	a	b	e	d	e	1	1
1	0	a	b	c	d	e	f	1

FIGURE 7. Example of a distributive Hilbert algebra with lattice structure – definition of the implication.

As in the case of Hilbert algebras with infimum, the 1-assertional logic of $\mathbb{H}^L$ is not congruential. Therefore we focus on the Hilbert based logic of $\mathbb{H}^L$, that we denote by $\mathcal{S}^{\to}_{\mathbb{H}^L}$. By construction, this logic is finitary and congruential. Moreover it satisfies (uDDT) for $p \to q$, $\mathbb{A}\mathrm{lg}\mathcal{S}^{\to}_{\mathbb{H}^L} = \mathbb{H}^L$, and for any Hilbert algebra with lattice structure $\mathbf{A}$, the collection of implicative filters of $\mathbf{A}$ is the collection of $\mathcal{S}^{\to}_{\mathbb{H}^L}$-filters of $\mathbf{A}$. Therefore the logic $\mathcal{S}^{\to}_{\mathbb{H}^L}$ is also filter distributive. Similarly to the case of the Hilbert based logic of $\mathbb{H}^\wedge$, it turns out that the logic $\mathcal{S}^{\to}_{\mathbb{H}^L}$ does not satisfy (PC) for $p \wedge q$. Notice that the $(\to, \wedge, 1)$-reduct of any $\mathbb{H}^L$-algebra is an $\mathbb{H}^\wedge$-algebra. Therefore, the logic $\mathcal{S}^{\to}_{\mathbb{H}^L}$ is also an expansion of the logic $\mathcal{S}^{\to}_{\mathbb{H}^\wedge}$.

We could think on a different logic defined from $\mathbb{H}^L$, the semilattice based logic of $\mathbb{H}^L$, that we denote by $\mathcal{S}^{\leq}_{\mathbb{H}^L}$. From the general theory (see [**53**] and [**55**]) it follows that $\mathcal{S}^{\leq}_{\mathbb{H}^L}$ is a finitary congruential logic with theorems that satisfies (PC). Furthermore, $\mathbb{A}\mathrm{lg}\mathcal{S}^{\leq}_{\mathbb{H}^L} = \mathbb{H}^L$, and for every $\mathbb{H}^L$-algebra $\mathbf{A}$, the order filters of $\mathbf{A}$ are the $\mathcal{S}^{\leq}_{\mathbb{H}^L}$-filters of $\mathbf{A}$. Hence the logic $\mathcal{S}^{\leq}_{\mathbb{H}^L}$ is not filter distributive, so it is out of reach of our study.

Just like the case of $\mathbb{H}^\wedge$-algebras, we have not been able to get *elegant* Spectral-like and Priestley-style dualities for $\mathbb{H}^L$, but for a subclass of $\mathbb{H}^L$. Nevertheless, that subclass turns out to be an interesting variety, from both a logical and an algebraic point of view.

DEFINITION 6.6.6. An $\mathbb{H}^L$-algebra $\mathbf{A} = \langle A, \to, \wedge, \vee, 1\rangle$ is a *Hilbert algebra with distributive lattice structure* or an $\mathbb{H}^{DL}$*-algebra* when the underlying lattice $\langle A, \wedge, \vee\rangle$ is distributive.

Let us denote by $\mathbb{H}^{DL}$ the class of Hilbert algebras with distributive lattice structure, that by definition is a variety, because distributivity in this setting is an equational condition, unlike in the semilattice setting, where it is not equational. Bounded $\mathbb{H}^{DL}$-algebras (or $\mathbb{H}_0^{DL}$-algebras following a similar notation) have been studied by Figallo et al. in [**31**], under the name of *distributive Hilbert algebras*, and also by Celani and Cabrer in [**14**], under the name of *Hilbert implications over bounded distributive lattices*.

Notice that the algebra defined in Example 6.5.12 (see page 138) is an $\mathbb{H}^L$-algebra that is obviously not an $\mathbb{H}^{DL}$-algebra. Hence, $\mathbb{H}^{DL}$-algebras are strictly included in $\mathbb{H}^L$. Moreover, the algebra considered in Example 6.5.9 is in fact a $\mathbb{H}^{DL}$-algebra. Therefore, that example shows that relatively pseudo-complemented lattices are strictly included in $\mathbb{H}^{DL}$. It also shows that bounded $\mathbb{H}^{DL}$-algebras are not included in $\mathbb{DLI}$: (DLI1) fails since $1 = a \to (b \wedge c) \neq (a \to b) \wedge (a \to c) = a$. This implies, in particular, that $\mathbb{H}^{DL}$-algebras is not a variety of distributive lattices expanded with (dual) quasioperators.

PROPOSITION 6.6.7. *Heyting algebras are precisely those algebras that are both $\mathbb{DLI}$-algebras and $\mathbb{H}_0^L$-algebras.*

PROOF. We already know that Heyting algebras are $\mathbb{DLI}$-algebras and bounded Hilbert algebras with lattice structure. So we just need to show the other inclusion. Let $\mathbf{A} = \langle A, \wedge, \vee, \to, 1, 0\rangle$ be a $\mathbb{DLI}$-algebra and an $\mathbb{H}_0^L$-algebra. We just need to show that the residuation law holds.

Let first $a, b, c \in A$ be such that $a \wedge c \leq b$. By (DLI1) we have $a \to (a \wedge c) = (a \to a) \wedge (a \to c) = a \to c$. Since $\langle A, \to, 1\rangle$ is a Hilbert algebra, $\to$ is order preserving in the second coordinate, so we get $a \to (a \wedge c) \leq a \to b$. Moreover, by (H1) we have $c \leq a \to c$. And putting all these equations together, we obtain $c \leq a \to b$.

Let now $a, b, c \in A$ be such that $c \leq a \to b$. Then by definition of $\wedge$, we have that $a \wedge c \leq a \wedge (a \to b)$. Now by (H^1) $a \wedge (a \to b) = a \wedge b$, and since $a \wedge b \leq b$, we conclude $a \wedge c \leq b$, as required. $\square$

We conclude with the following example, that witnesses that $\mathbb{DLI}$-algebras are not included in bounded $\mathbb{BCK}^L$-algebras. This implies, in particular, that $\mathbb{DLI}$-algebras are not included in bounded $\mathbb{H}^{DL}$-algebras.

EXAMPLE 6.6.8. Consider the lattice in Figure 8, that is distributive, and let $\to$ be the implication defined on A by the table in Figure 9. It is easy to check that $\mathbf{A}$ is a $\mathbb{DLI}$-algebra. Notice that $a \wedge ((a \to 0) \to 0) = 0 \neq a$, therefore ($\mathbb{BKC}$^2) fails and so $\mathbf{A}$ is not a $\mathbb{BKC}^{\wedge}$-algebra, nor a $\mathbb{DH}^{\wedge}$-algebra.

Summarizing, some of the relations between the classes of algebras in the language $(\to, \wedge, \vee, 1)$ so far considered are:

$$g\mathbb{H}e \subsetneq \mathbb{H}^{DL} \subsetneq \mathbb{H}^L \subsetneq \mathbb{BCK}^L$$

$$\mathbb{H}e \subsetneq \mathbb{H}_0^{DL} \subsetneq \mathbb{H}_0^L \subsetneq \mathbb{BCK}_0^L$$

All these varieties are distributive lattice expansions, but only relatively pseudo-complemented lattices and Heyting algebras are distributive lattices expanded with

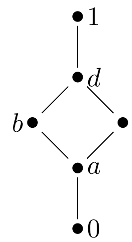


FIGURE 8. Example of a weakly Heyting algebra.

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	0	0	1	1	1
c	0	0	b	1	1
1	0	0	b	c	1

FIGURE 9. Example of a weakly Heyting algebra – definition of the implication.

(dual) quasioperators. Hence, the rest of varieties does not fall under the scope of extended Priestley duality.

Since $\mathbb{H}^{DL}$ is a variety, unlike the case of $\mathbb{DH}^{\wedge}$-algebras, we shall consider its 1-assertional logic, its Hilbert based logic or its semilattice based logic. As it was to be expected, the 1-assertional logic of $\mathbb{H}^{DL}$ does not have *nice* properties, but the Hilbert based logic of $\mathbb{H}^{DL}$ does.

The Hilbert based logic of $\mathbb{H}^{DL}$, that we denote by $\mathcal{S}^{\to}_{\mathbb{H}^{DL}}$, is by construction a filter distributive finitary congruential logic with theorems. We have that $\mathbb{A}\mathrm{lg}\mathcal{S}^{\to}_{\mathbb{H}^{DL}} = \mathbb{H}^{DL}$ and for every $\mathbb{H}^{DL}$-algebra $\mathbf{A}$, the implicative filters of $\mathbf{A}$ are the $\mathcal{S}^{\to}_{\mathbb{H}^{DL}}$-filters of $\mathbf{A}$. Moreover $\mathcal{S}^{\to}_{\mathbb{H}^{DL}}$ satisfies (uDDT) for $p \to q$. However, by Example 6.5.9 we get that $\gamma, \delta \nvdash^{\to}_{\mathbb{H}^{DL}} \delta$, and therefore $\mathcal{S}^{\to}_{\mathbb{H}^{DL}}$ does not satisfy (PC) for $p \wedge q$.

We could think on the semilattice based logic of $\mathbb{H}^{DL}$, that we denote by $\mathcal{S}^{\leq}_{\mathbb{H}^{DL}}$. The logic is finitary, congruential and has theorems by construction. Moreover $\mathbb{A}\mathrm{lg}\mathcal{S}^{\leq}_{\mathbb{H}^{DL}} = \mathbb{H}^{DL}$ and for every $\mathbb{H}^{DL}$-algebra $\mathbf{A}$, the order filters of $\mathbf{A}$ are the $\mathcal{S}^{\leq}_{\mathbb{H}^{DL}}$-filter of $\mathbf{A}$. Hence $\mathcal{S}^{\leq}_{\mathbb{H}^{DL}}$ is filter distributive, and moreover it satisfies (PC), but it does not satisfy (uDDT). We do not go further into details, and we leave the in-depth study of this logic as future work.

For none of the logics $\mathcal{S}^{\to}_{\mathbb{H}^{DL}}$ and $\mathcal{S}^{\leq}_{\mathbb{H}^{DL}}$ we can dispense immediately with the algebraic structure in the dual spaces of $\mathbb{H}^{DL}$-algebras. However, we will see in § 7.6 that the dualities for Distributive Hilbert algebras with infimum that we present in Chapter 7 can be easily restricted to get dualities for $\mathcal{S}^{\to}_{\mathbb{H}^{DL}}$, in which in the definition of the dual spaces there is no explicit mention to any algebraic structure. This is even more interesting, given that $\mathbb{H}^{DL}$ is a variety of distributive lattice expansions for which extended Priestley duality cannot be applied.

6.7. More expansions

We can get things even more complicated and think of other varieties of Hilbert based algebras whose Hilbert based logics are suitable for our analysis. We mention just two of them, one is defined in the language $(\to, \wedge, \to', 1)$ of type $(2, 2, 2, 0)$, and the other is defined in the language $(\to, \wedge, \vee, \to', 1)$ of type $(2, 2, 2, 2, 0)$.

It is well known that there are distributive lattices that are not relatively pseudo-complemented. Therefore, there are distributive semilattices that cannot be turned into implicative semilattices. Notice that for any distributive semilattice $\langle A, \wedge, 1\rangle$, at most one operation $\to'$ can be defined on A such that $\langle A, \to', \wedge, 1\rangle$ turns out to be an implicative semilattice. However, *a priori* there is no such restriction over the number of operations $\to$ that can be defined on A such that $\langle A, \to, \wedge, 1\rangle$ turns out to be an $\mathbb{H}^{\wedge}$-algebra. Let us show this situation by an example.

Example 6.7.1. Let $\langle A, \wedge, 1\rangle$ be the distributive semilattice given by Figure 5 (see page 142). On the one hand, we already know that the implication $\to$ defined on A by the order (see 6.5.8) is such that $\langle A, \to, \wedge, 1\rangle$ is a $\mathbb{DH}^{\wedge}$-algebra but it is not an implicative semilattice. Consider now a new implication $\to'$ defined on A as follows:

$$x \to' y = \begin{cases} 1 & \text{if } x \leq y, \\ a & \text{if } x = b \text{ and } y = 0, \\ b & \text{if } x = a \text{ and } y = 0, \\ y & \text{otherwise.} \end{cases}$$

It is easy to check that $\langle A, \to', \wedge, 1\rangle$ is an implicative semilattice. Moreover, it is also a $\mathbb{DH}^{\wedge}$-algebra, that is evidently different from $\langle A, \to, \wedge, 1\rangle$.

Previous remarks motivate the study of the following classes of algebras:

Definition 6.7.2. An algebra $\mathbf{A} = \langle A, \to, \wedge, \to', 1\rangle$ of type $(2, 2, 2, 0)$ is an *implicative Hilbert algebra with infimum* or $\mathbb{IH}^{\wedge}$-*algebra* if:

(1) $\langle A, \to, \wedge, 1\rangle$ is an $\mathbb{H}^{\wedge}$-algebra,
(2) $\langle A, \to', \wedge, 1\rangle$ is an implicative semilattice.

Definition 6.7.3. An algebra $\mathbf{A} = \langle A, \to, \wedge, \vee, \to', 1\rangle$ of { $(2, 2, 2, 2, 0)$ is an *implicative Hilbert algebra with lattice structure* or $\mathbb{IH}^{L}$-*algebra* if:

(1) $\langle A, \to, \wedge, \vee, 1\rangle$ is an $\mathbb{H}^{L}$-algebra,
(2) $\langle A, \to', \wedge, \vee, 1\rangle$ is a relatively pseudo-complemented lattice.

For any implicative semilattice $\langle A, \to, \wedge, 1\rangle$, it follows that $\langle A, \to, \wedge, \to, 1\rangle$ is an $\mathbb{IH}^{\wedge}$-algebra, and similarly for any relatively pseudo-complemented lattice. Example 6.7.1 shows that not all $\mathbb{IH}^{\wedge}$-algebras have this form, i. e. it shows that there are $\mathbb{IH}^{\wedge}$-algebras $\langle A, \to, \wedge, \to', 1\rangle$ for which $\to$ and $\to'$ are different.

It follows from the definition that $\mathbb{IH}^{\wedge}$-algebras form a variety and $\mathbb{IH}^{L}$-algebras form a variety as well. Like in previous cases, the Hilbert based logics of $\mathbb{IH}^{\wedge}$ and $\mathbb{IH}^{L}$ (taking $\to$ as the main connective) are filter distributive finitary congruential logics with theorems. The key point is that the dualities for Distributive Hilbert algebras with infimum that we present in Chapter 7 can also be restricted to get dualities for these logics. We do not go further into this, as the reader shall already figure out how these and other classes of algebras could be defined in the same way.

Table 8 summarizes the classes of algebras so far considered. The filter distributive and congruential expansions of $\mathcal{H}$ they are related with are also listed. All classes of algebras, except for $\mathbb{H}^{\Box}$ and $\mathbb{H}^{\vee}$, have a $(\to, \wedge, 1)$-reduct that is a Hilbert algebra with infimum. Except for $\mathbb{DH}^{\wedge}$, all the classes of algebras in Table 8 are known to be varieties.

Notice that we only consider the classes of algebras that are not bounded, but a similar table with the corresponding bounded algebras could be given.

Language	Class of algebras	Description	Logic	Properties of the logic
$(\to, \Box, 1)$	$\mathbb{H}^{\Box}$	—	$\mathcal{H}^{\Box} = \mathcal{S}^{\to}_{\mathbb{H}^{\Box}}$	(uDDT) and (PIM)
$(\to, \vee, 1)$	$\mathbb{H}^{\vee}$	$\langle A, \vee, 1\rangle$ is a join-semilattice	$\mathcal{H}^{\vee} = \mathcal{S}^{\to}_{\mathbb{H}^{\vee}}$	(uDDT) and (PDI)
$(\to, \wedge, 1)$	$\mathbb{H}^{\wedge}$	$\langle A, \wedge, 1\rangle$ is a meet-semilattice	$\mathcal{S}^{\to}_{\mathbb{H}^{\wedge}}$	(uDDT)
	$\mathbb{DH}^{\wedge}$	$\langle A, \wedge, 1\rangle$ is a distributive meet-semilattice	—	—
	$\mathbb{IS}$	$\langle A, \wedge, 1\rangle$ is an implicative semilattice	$\mathcal{H}^{\wedge} = \mathcal{S}^{\to}_{\mathbb{IS}} = \mathcal{S}^{\leq}_{\mathbb{IS}}$	(uDDT) and (PC)
$(\to, \wedge, \vee, 1)$	$\mathbb{H}^{L}$	$\langle A, \wedge, \vee, 1\rangle$ is a lattice	$\mathcal{S}^{\to}_{\mathbb{H}^{L}}$	(uDDT) and (PDI)
	$\mathbb{H}^{DL}$	$\langle A, \wedge, \vee, 1\rangle$ is a distributive lattice	$\mathcal{S}^{\to}_{\mathbb{H}^{DL}}$	(uDDT) and (PDI)
			$\mathcal{S}^{\leq}_{\mathbb{H}^{DL}}$	(PC) and (PDI)
	$g\mathbb{H}e$	$\langle A, \wedge, \vee, 1\rangle$ is a relatively pseudo-complemented lattice	$\mathcal{IPC}^{+} = \mathcal{S}^{\to}_{g\mathbb{H}} = \mathcal{S}^{\leq}_{g\mathbb{H}}$	(uDDT), (PC) and (PDI)
$(\to, \wedge, \to', 1)$	$\mathbb{IH}^{\wedge}$	$\langle A, \to', \wedge, 1\rangle$ is an implicative semilattice	$\mathcal{S}^{\to}_{\mathbb{IH}^{\wedge}}$	(uDDT) for $p \to q$
			$\mathcal{S}^{\leq}_{\mathbb{IH}^{\wedge}}$	(PC)
$(\to, \wedge, \vee, \to', 1)$	$\mathbb{IH}^{L}$	$\langle A, \to', \wedge, \vee, 1\rangle$ is relatively pseudo-complemented lattice	$\mathcal{S}^{\to}_{\mathbb{IH}^{L}}$	(PDI) and (uDDT) for $p \to q$
			$\mathcal{S}^{\leq}_{\mathbb{IH}^{L}}$	(PC) and (PDI)

Table 8. Algebras with a reduct that is a Hilbert Algebra and their logics.

CHAPTER 7

Duality theory for Distributive Hilbert Algebras with infimum

In this chapter we present Priestley-style and Spectral-like dualities for the class of $\mathbb{DH}^{\wedge}$-algebras, and we apply these results to tackle several problems.

Recall that in § 6.5.1 we reviewed the toolkit we need to carry out this objective. In what follows we use such toolkit, as well as the dualities for Hilbert algebras (cf. § 6.2) and for distributive semilattices (cf. § 3.2) to develop the mentioned Spectral-like and Priestley-style dualities for $\mathbb{DH}^{\wedge}$-algebras.

We expose systematically both dualities in parallel. In § 7.1 we prove representation theorems for $\mathbb{DH}^{\wedge}$-algebras and we introduce the definitions of $\mathbb{DH}^{\wedge}$-Spectral spaces and $\mathbb{DH}^{\wedge}$-Priestley spaces. In § 7.2 we consider morphisms, and we introduce the definition of $\mathbb{DH}^{\wedge}$-Spectral morphisms and $\mathbb{DH}^{\wedge}$-Priestley morphisms. In § 7.3 we define the functors and the natural transformations involved in the dualities.

In § 7.4 we study how the different notions of filters can be characterized within the Spectral-like duality for $\mathbb{DH}^{\wedge}$-algebras. We use those results in § 7.5, where we compare both dualities, and we show the functors involved in the equivalence of the Spectral-like and the Priestley-style dual categories.

Finally in § 7.6 we explain how the same strategy followed for the dualities for $\mathbb{DH}^{\wedge}$-algebras can yield dualities for other classes of algebras that were already introduced in § 6.6 and § 6.7. In particular, in § 7.6.1 we show how the Spectral-like and Priestley-style dualities for implicative semilattices that we find in the literature can be obtained as a particular case of the dualities for $\mathbb{DH}^{\wedge}$-algebras. Moreover, in § 7.6.3–§ 7.6.4 we outline how the dualities for $\mathbb{DH}^{\wedge}$-algebras yield *elegant* Spectral-like and Priestley-style dualities for some filter distributive finitary congruential logics with theorems for which our theory in Chapter 5 does not lead us to *elegant* dualities.

7.1. Dual objects

In this section we use what we know about duality theory for distributive semilattices and Hilbert algebras (cf. § 3.2 and § 6.2) to develop two correspondences between $\mathbb{DH}^{\wedge}$-algebras and certain classes of Spectral-like and Priestley-style spaces that we introduce later on.

From now on, let $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ be a $\mathbb{DH}^{\wedge}$-algebra. As we already mentioned, throughout this chapter the reader should keep in mind that the implication is taken as the *main* operation on the $\mathbb{DH}^{\wedge}$-algebra, whereas the conjunction is taken as the *additional* operation. Hence, the Spectral-like duality for $\mathbb{DH}^{\wedge}$-algebras that we study here is built upon the Spectral-like duality for Hilbert algebras of [**15**] that we reviewed in § 3.3.1. And the Priestley-style duality for $\mathbb{DH}^{\wedge}$-algebras is built upon the Priestley-style duality for Hilbert algebras that we obtained in § 6.2, that is a simplification of the one in [**17**]. More specifically, the posets $\langle \mathrm{Irr}_{\to}(\mathbf{A}), \subseteq\rangle$ and $\langle \mathrm{Op}_{\to}(\mathbf{A}), \subseteq\rangle$ play a crucial role, as well as the maps:

$$\begin{aligned} \psi_{\mathbf{A}} : A &\longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Irr}_{\to}(\mathbf{A})) & \vartheta_{\mathbf{A}} : A &\longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Op}_{\to}(\mathbf{A})) \\ a &\longmapsto \{P \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \in P\} & a &\longmapsto \{P \in \mathrm{Op}_{\to}(\mathbf{A}) : a \in P\} \end{aligned}$$

that we already know that are isomorphisms between the Hilbert algebras $\langle A, \to, 1\rangle$, and $\langle \psi_{\mathbf{A}}[A], \Rightarrow, A\rangle$ and $\langle \vartheta_{\mathbf{A}}[A], \Rightarrow, A\rangle$ respectively (cf. theorems 5.1.1 and 5.1.2). When the context is clear, we drop the subscripts of $\psi_{\mathbf{A}}$ and $\vartheta_{\mathbf{A}}$.

The next proposition gives us the representation theorem for $\mathbf{A}$ based on the collection of irreducible implicative filters of $\mathbf{A}$. Notice that we use $\uparrow$ (resp. $\downarrow$) for the up-set (resp. down-set) generated by a set in the poset $\langle \mathrm{Irr}_{\to}(\mathbf{A}), \subseteq\rangle$.

PROPOSITION 7.1.1. *For any $\mathbb{DH}^{\wedge}$-algebra $\mathbf{A}$:*

(1) $\uparrow(\psi(a) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) = \psi(a)$.
(2) $\psi(a \wedge b) = \uparrow(\psi(a) \cap \psi(b) \cap \mathrm{Irr}_{\wedge}(\mathbf{A}))$.

PROOF. For (1), the inclusion from left to right is immediate, since $\psi(a)$ is an up-set. Let us show the inclusion from right to left. Let $P \in \psi(a)$, i. e. $a \in P$ for $P \in \mathrm{Irr}_{\to}(\mathbf{A})$. Then by Corollary 2.4.4, P^c is an order ideal. And since $a \notin P^c$, there are an order ideal P^c and a meet filter $\uparrow a$ such that $\uparrow a \cap P^c = \emptyset$. By Lemma 2.3.3 there exists $Q \in \mathrm{Irr}_{\wedge}(\mathbf{A})$, such that $\uparrow a \subseteq Q$ and $P^c \cap Q = \emptyset$. Therefore we have $a \in Q \subseteq P$ for $Q \in \mathrm{Irr}_{\wedge}(\mathbf{A})$, i. e. $Q \in \psi(a) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})$ and $Q \subseteq P$, hence $P \in \uparrow(\psi(a) \cap \mathrm{Irr}_{\wedge}(\mathbf{A}))$. For (2), notice that $\psi(a) \cap \psi(c) \cap \mathrm{Irr}_{\wedge}(\mathbf{A}) = \psi(a \wedge c) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})$. Now using item (1), it follows that $\uparrow(\psi(a \wedge c) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) = \psi(a \wedge c)$. □

Let us define a new binary operation $\sqcap$ on $\psi[A]$ as follows:

$$\psi(a) \sqcap \psi(b) := \uparrow(\psi(a) \cap \psi(b) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})).$$

By the previous proposition we obtain that $\mathbf{A}$ is isomorphic to the algebra

$$\psi[\mathbf{A}] := \langle \psi[A], \Rightarrow, \sqcap, \mathrm{Irr}_{\to}(\mathbf{A})\rangle.$$

An alternative representation theorem for $\mathbf{A}$, based on the collection of optimal implicative filters of $\mathbf{A}$, is obtained from the following proposition. It should be kept in mind that in this case we use $\uparrow$ (resp. $\downarrow$) for the up-set (resp. down-set) generated by a set in the poset $\langle \mathrm{Op}_{\to}(\mathbf{A}), \subseteq\rangle$.

PROPOSITION 7.1.2. *For any $\mathbb{DH}^{\wedge}$-algebra $\mathbf{A}$:*

(1) $\uparrow(\vartheta(a) \cap \mathrm{Op}_{\wedge}(\mathbf{A})) = \vartheta(a)$.
(2) $\vartheta(a \wedge b) = \uparrow(\vartheta(a) \cap \vartheta(b) \cap \mathrm{Op}_{\wedge}(\mathbf{A}))$.

PROOF. For (1), the inclusion from left to right is immediate, since $\vartheta(a)$ is an up-set. Let us show the inclusion from right to left: let $P \in \vartheta(a)$, i. e. $a \in P$ for $P \in \mathrm{Op}_{\to}(\mathbf{A})$. Then by Theorem 2.4.9, P^c is an sF-ideal. And since $a \notin P^c$, there are an F-ideal P^c and a meet filter $\uparrow a$ such that $\uparrow a \cap P^c = \emptyset$. By Lemma 2.3.7 there exists $Q \in \mathrm{Op}_{\wedge}(\mathbf{A})$, with $\uparrow a \subseteq Q$ and $P^c \cap Q = \emptyset$. Therefore we have $a \in Q \subseteq P$ for $Q \in \mathrm{Op}_{\wedge}(\mathbf{A})$, i. e. $Q \in \vartheta(a) \cap \mathrm{Op}_{\wedge}(\mathbf{A})$ and $Q \subseteq P$, hence $P \in \uparrow(\vartheta(a) \cap \mathrm{Op}_{\wedge}(\mathbf{A}))$. For (2), notice that $\vartheta(a) \cap \vartheta(c) \cap \mathrm{Op}_{\wedge}(\mathbf{A}) = \vartheta(a \wedge c) \cap \mathrm{Op}_{\wedge}(\mathbf{A})$. Now using item (1), it follows that $\uparrow(\vartheta(a) \cap \vartheta(c) \cap \mathrm{Op}_{\wedge}(\mathbf{A})) = \uparrow(\vartheta(a \wedge c) \cap \mathrm{Op}_{\wedge}(\mathbf{A})) = \vartheta(a \wedge c)$. □

As before, let us define a new binary operation $\sqcap$ on $\vartheta[A]$ as follows:

$$\vartheta(a) \sqcap \vartheta(b) := \uparrow(\vartheta(a) \cap \vartheta(b) \cap \mathrm{Op}_{\wedge}(\mathbf{A})).$$

In this case we get that $\mathbf{A}$ is isomorphic to the algebra

$$\vartheta[\mathbf{A}] := \langle \vartheta[A], \Rightarrow, \sqcap, \mathrm{Op}_{\to}(\mathbf{A})\rangle.$$

Once we got the representation theorems, we need to introduce topologies for characterizing dual objects. At this point both dualities differ substantially, and that is why we discuss them in different subsections.

7.1.1. Spectral-like dual objects. Recall that within the Spectral-like duality for Hilbert algebras reviewed in § 3.3.1, we define on $\mathrm{Irr}_{\to}(\mathbf{A})$ a topology $\tau_{\kappa_{\mathbf{A}}}$, having as basis the collection

$$\kappa_{\mathbf{A}} := \{\psi(a)^c : a \in A\},$$

and we obtain that the structure $\langle \mathrm{Irr}_{\to}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$ is an $\mathbb{H}$-space (see definition in page 41). Furthermore, the dual of the specialization order of the space coincides with the inclusion relation on $\mathrm{Irr}_{\to}(\mathbf{A})$. And for all $U \subseteq \mathrm{Irr}_{\to}(\mathbf{A})$, $\mathrm{cl}(U) = \uparrow U$ and $\mathrm{sat}(U) = \downarrow U$, where $\uparrow$ (resp. $\downarrow$) are the up-set (resp. down-set) generated with respect to the dual of the specialization order.

Let us consider the subspace of $\langle \mathrm{Irr}_{\to}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$ generated by $\mathrm{Irr}_{\wedge}(\mathbf{A})$. As $\kappa_{\mathbf{A}}$ is a basis for $\mathrm{Irr}_{\to}(\mathbf{A})$, then we have that

$$\widehat{\kappa}_{\mathbf{A}} := \{U \cap \mathrm{Irr}_{\wedge}(\mathbf{A}) : U \in \kappa_{\mathbf{A}}\} = \{\psi(a)^c \cap \mathrm{Irr}_{\wedge}(\mathbf{A}) : a \in A\}$$

is a basis for the induced topology on $\mathrm{Irr}_{\wedge}(\mathbf{A})$, that we denote by $\tau_{\widehat{\kappa}_{\mathbf{A}}}$. Notice that for each $a \in A$,

$$\psi(a)^c \cap \mathrm{Irr}_{\wedge}(\mathbf{A}) = \{F \in \mathrm{Irr}_{\wedge}(\mathbf{A}) : a \notin F\}.$$

We should recall now the Spectral-like duality for distributive semilattices presented in § 3.2.1. From that duality it follows that $\langle \mathrm{Irr}_{\wedge}(\mathbf{A}), \tau_{\widehat{\kappa}_{\mathbf{A}}}\rangle$ is a $\mathbb{DS}$-space (see definition in page 36), and so it is compactly-based and sober. In order to complete the characterization of the Spectral-like dual spaces of $\mathbb{DH}^{\wedge}$-algebras, we just need the following proposition.

Proposition 7.1.3. *For any non-empty subset $B \subseteq A$ and any $c \in A$, if $\mathrm{cl}(\bigcap\{\psi(b) : b \in B\} \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \subseteq \psi(c)$, then there are $n \in \omega$ and $b_0, \dots, b_n \in B$, such that:*

$$\mathrm{cl}(\psi(b_0) \cap \cdots \cap \psi(b_n) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \subseteq \psi(c).$$

Proof. Assume that $\mathrm{cl}(\bigcap\{\psi(b) : b \in B\} \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \subseteq \psi(c)$. We claim that $c \in [\![B\rangle\!\rangle$. Suppose, towards a contradiction, that $c \notin [\![B\rangle\!\rangle$. Then by Corollary 2.3.4 there is $G \in \mathrm{Irr}_{\wedge}(\mathbf{A})$ such that $[\![B\rangle\!\rangle \subseteq G$ and $c \notin G$. So $B \subseteq G$ and thus $G \in \bigcap\{\psi(b) : b \in B\} \cap \mathrm{Irr}_{\wedge}(\mathbf{A}) \subseteq \mathrm{cl}(\bigcap\{\psi(b) : b \in B\} \cap \mathrm{Irr}_{\wedge}(\mathbf{A}))$. And then from the hypothesis it follows that $G \in \psi(c)$ and so $c \in G$, a contradiction

Now if $c = 1$ then $\psi(c) = \mathrm{Irr}_{\to}(\mathbf{A})$ and there is nothing to prove. So assume $c \neq 1$. Since $c \in [\![B\rangle\!\rangle$ and $c \neq 1$, there are $n \in \omega$ and $b_0, \dots, b_n \in B$ such that $(b_0 \wedge \cdots \wedge b_n) \to c = 1$, i. e. $b_0 \wedge \cdots \wedge b_n \leq c$. So $\psi(b_0) \cap \cdots \cap \psi(b_n) \cap \mathrm{Irr}_{\wedge}(\mathbf{A}) = \psi(b_0 \wedge \cdots \wedge b_n) \cap \mathrm{Irr}_{\wedge}(\mathbf{A}) \subseteq \psi(c)$, and since $\psi(c)$ is an up-set, we obtain that $\mathrm{cl}(\psi(b_0) \cap \cdots \cap \psi(b_n) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \subseteq \psi(c)$, as required. $\square$

DEFINITION 7.1.4. A structure $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$ is a $\mathbb{DH}^{\wedge}$-*Spectral space* when:

(DH^1) $\langle X, \tau_\kappa \rangle$ is an $\mathbb{H}$-space,
(DH^2) $\widehat{X} \subseteq X$ generates a sober subspace of $\langle X, \tau_\kappa \rangle$,
(DH^3) $U^c = \mathrm{cl}(U^c \cap \widehat{X})$, for all $U \in \kappa$,
(DH^4) $\mathrm{cl}(U^c \cap V^c \cap \widehat{X})^c \in \kappa$, for any $U, V \in \kappa$,
(DH^5) for any $U, V \in \kappa$ and $\mathcal{W} \subseteq \kappa$ non-empty, if $\mathrm{cl}(\bigcap\{W^c : W \in \mathcal{W}\} \cap \widehat{X}) \subseteq U^c$, then $\mathrm{cl}(W_0^c \cap \cdots \cap W_n^c \cap \widehat{X}) \subseteq U^c$ for some $W_0, \ldots, W_n \in \mathcal{W}$ and some $n \in \omega$.

Recall that for any $\mathbb{H}$-space $\langle X, \tau_\kappa \rangle$, sobriety implies that the space is T_0, so the specialization quasiorder turns out to be an order whose dual is denoted by $\leq_X$, or simply by $\leq$. Moreover, by condition (DH^3) we get that for any $\mathbb{DH}^{\wedge}$-Spectral space $\langle X, \widehat{X}, \tau_\kappa \rangle$, we have $X = \mathrm{cl}(X \cap \widehat{X}) = \mathrm{cl}(\widehat{X})$, and $\{U \cap \widehat{X} : U \in \kappa\}$ is a basis for the subspace of X generated by $\widehat{X}$, that we may denote simply by $\widehat{X}$.

COROLLARY 7.1.5. *Let $\mathbf{A} = \langle A, \to, \wedge, 1 \rangle$ be a $\mathbb{DH}^{\wedge}$-algebra. Then*

$$\mathfrak{Irr}(\mathbf{A}) := \langle \mathrm{Irr}_{\to}(\mathbf{A}), \mathrm{Irr}_{\wedge}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle$$

is a $\mathbb{DH}^{\wedge}$-Spectral space.

PROOF. Condition (DH^1) follows from Spectral-like duality for Hilbert algebras (see § 3.3.1). Condition (DH^2) follows from Spectral-like duality for distributive semilattices (see § 3.2.1) and the fact that $\mathrm{Irr}_{\wedge}(\mathbf{A}) \subseteq \mathrm{Irr}_{\to}(\mathbf{A})$ given by Proposition 6.5.16. Conditions (DH^3) and (DH^4) follow from Proposition 7.1.1, and condition (DH^5) follows from Proposition 7.1.3. □

REMARK 7.1.6. Concerning a $\mathbb{DH}^{\wedge}$-Spectral space $\langle X, \widehat{X}, \tau_\kappa \rangle$, we have to be careful when using complements, since we are working with two topological spaces at the same time, namely, the main space $\langle X, \tau_\kappa \rangle$ and the subspace generated by $\widehat{X}$. We establish now the following convention: complements are always referred to the biggest set X.

PROPOSITION 7.1.7. *Let $\langle X, \widehat{X}, \tau_\kappa \rangle$ be a $\mathbb{DH}^{\wedge}$-Spectral space. Then $\mathcal{KO}(\widehat{X}) = \{U \cap \widehat{X} : U \in \kappa\}$.*

PROOF. First we show that $\mathcal{KO}(\widehat{X})$ is included in $\{U \cap \widehat{X} : U \in \kappa\}$. Let $W \in \mathcal{KO}(\widehat{X})$, so by definition of subspace generated and using that κ is a basis for τ_κ, we get $W = \bigcup\{V \cap \widehat{X} : V \in \mathcal{V}\}$ for some set $\mathcal{V} \subseteq \kappa$. Since W is compact, there are $V_0, \ldots, V_n \in \mathcal{V}$, for some $n \in \omega$, such that $W = (V_0 \cap \widehat{X}) \cup \cdots \cup (V_n \cap \widehat{X}) = (V_0^c \cap \cdots \cap V_n^c)^c \cap \widehat{X}$. Notice that for each $i \leq n$, we have that V_i^c is closed, so $\mathrm{cl}(V_0^c \cap \cdots \cap V_n^c \cap \widehat{X}) \subseteq V_i^c$. We obtain that $\mathrm{cl}(V_0^c \cap \cdots \cap V_n^c \cap \widehat{X}) \subseteq V_0^c \cap \cdots \cap V_n^c$, and then $V_0^c \cap \cdots \cap V_n^c \cap \widehat{X} \subseteq \mathrm{cl}(V_0^c \cap \cdots \cap V_n^c \cap \widehat{X}) \cap \widehat{X}$. Clearly the reverse inclusion also holds, so $V_0^c \cap \cdots \cap V_n^c \cap \widehat{X} = \mathrm{cl}(V_0^c \cap \cdots \cap V_n^c \cap \widehat{X}) \cap \widehat{X}$ and then $(V_0^c \cap \cdots \cap V_n^c)^c \cap \widehat{X} = (\mathrm{cl}(V_0^c \cap \cdots \cap V_n^c \cap \widehat{X}))^c \cap \widehat{X}$. Therefore $W = \mathrm{cl}(V_0^c \cap \cdots \cap V_n^c \cap \widehat{X})^c \cap \widehat{X}$. By condition (DH^4), $\mathrm{cl}(V_0^c \cap \cdots \cap V_n^c \cap \widehat{X})^c \in \kappa$. Thus we obtain, $W = V \cap \widehat{X}$ for some $V \in \kappa$, as required.

Now we show the reverse inclusion. We just have to show that for any $U \in \kappa$, the set $U \cap \widehat{X}$ is compact in $\widehat{X}$. Let $U \in \kappa$ and consider $W := U \cap \widehat{X}$. Suppose that $W = \bigcup\{V \cap \widehat{X} : V \in \mathcal{V}\}$ for some non-empty $\mathcal{V} \subseteq \kappa$. We claim that

$\bigcap\{V^c : V \in \mathcal{V}\} \cap \widehat{X} \subseteq U^c$. Let $x \in \bigcap\{V^c : V \in \mathcal{V}\} \cap \widehat{X}$, i.e. $x \in \widehat{X}$ and $x \notin V$ for all $V \in \mathcal{V}$. Therefore $x \notin \bigcup\{V \cap \widehat{X} : V \in \mathcal{V}\} = U \cap \widehat{X}$, and since $x \in \widehat{X}$, then $x \in U^c$. As U^c is an up-set, it follows from the claim that $\mathrm{cl}(\bigcap\{V^c : V \in \mathcal{V}\} \cap \widehat{X}) \subseteq U^c$. And then by condition (DH^5), there are $V_0, \ldots, V_n \in \mathcal{V}$, for some $n \in \omega$, such that $\mathrm{cl}(V_0^c \cap \cdots \cap V_n^c \cap \widehat{X}) \subseteq U^c$. So $U \subseteq (V_0^c \cap \cdots \cap V_n^c \cap \widehat{X})^c = (V_0^c \cap \widehat{X})^c \cup \cdots \cup (V_n^c \cap \widehat{X})^c$. Therefore:

$$\begin{aligned} W = U \cap \widehat{X} &\subseteq ((V_0^c \cap \widehat{X})^c \cup \cdots \cup (V_n^c \cap \widehat{X})^c) \cap \widehat{X} \\ &= ((V_0^c \cap \widehat{X})^c \cap \widehat{X}) \cup \cdots \cup ((V_n^c \cap \widehat{X})^c \cap \widehat{X}) = (V_0 \cap \widehat{X}) \cup \cdots \cup (V_n \cap \widehat{X}) \subseteq W. \end{aligned}$$

and thus $W = (V_0 \cap \widehat{X}) \cup \cdots \cup (V_n \cap \widehat{X})$, so W is compact. □

Corollary 7.1.8. *Let $\langle X, \widehat{X}, \tau_\kappa\rangle$ be a $\mathbb{DH}^\wedge$-Spectral space. Then the subspace generated by $\widehat{X}$ is a $\mathbb{DS}$-space.*

Proof. Recall that $\{U \cap \widehat{X} : U \in \kappa\}$ is a basis for the subspace generated by $\widehat{X}$. By definition $\widehat{X}$ is sober, and it is T_0 since the space $\langle X, \tau_\kappa\rangle$ is T_0. Moreover, by Proposition 7.1.7, the subspace $\widehat{X}$ is compactly-based, so we are done. □

Similarly as it is done when dealing with $\mathbb{H}$-spaces, for any $\mathbb{DH}^\wedge$-Spectral space $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$, we define the family $D(\mathfrak{X}) := \{U^c : U \in \kappa\}$, that is a collection of closed elements of X, and we also define the binary operation $\Rightarrow$ on it, such that for all $U, V \in \kappa$:

$$U^c \Rightarrow V^c := (\mathrm{sat}(U \cap V^c))^c.$$

The structure $\langle D(\mathfrak{X}), \Rightarrow, X\rangle$ turns out to be a Hilbert algebra. Moreover, the map $\varepsilon_{\mathfrak{X}} : X \longrightarrow \mathcal{P}^{\uparrow}(D(\mathfrak{X}))$ given by

$$\varepsilon_{\mathfrak{X}}(x) := \{U \in D(\mathfrak{X}) : x \in U\}$$

is a map onto the collection of irreducible implicative filters of the Hilbert algebra $\langle D(\mathfrak{X}), \Rightarrow, X\rangle$. Let $\sqcap$ be the binary operation on $D(\mathfrak{X})$, such that for all $U, V \in \kappa$:

$$U^c \sqcap V^c := \mathrm{cl}(U^c \cap V^c \cap \widehat{X}).$$

By condition (DH^4), we obtain that $D(\mathfrak{X})$ is closed under $\sqcap$. Let us show that $\langle D(\mathfrak{X}), \sqcap, X\rangle$ is isomorphic to the dual distributive semilattice of the $\mathbb{DS}$-space $\widehat{X}$.

Recall that for the $\mathbb{DS}$-space $\widehat{X}$ given by Corollary 7.1.8, the family $F(\mathfrak{X}) := \{U^c : U \in \mathcal{KO}(\widehat{X})\}$ is closed under finite intersections and moreover the structure $\langle F(\mathfrak{X}), \cap, \widehat{X}\rangle$ is the dual distributive semilattice of $\widehat{X}$. Consider the map $f : D(\mathfrak{X}) \longrightarrow F(\mathfrak{X})$, given by

$$f(U) := U \cap \widehat{X}.$$

Clearly f is a surjective map, and by condition (DH^3), $U = \mathrm{cl}(U \cap \widehat{X})$ for all $U \in D(\mathfrak{X})$. Thus f is also injective. Moreover, from $U, V \in D(\mathfrak{X})$ being up-sets, it follows that $f(U) \cap f(V) = U \cap V \cap \widehat{X} = \mathrm{cl}(U \cap V \cap \widehat{X}) \cap \widehat{X} = f(U \sqcap V)$. Hence f is an isomorphism between $\langle D(\mathfrak{X}), \sqcap, X\rangle$ and $\langle F(\mathfrak{X}), \cap, \widehat{X}\rangle$.

Theorem 7.1.9. *Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$ be a $\mathbb{DH}^\wedge$-Spectral space. Then $\mathfrak{X}^* := \langle D(\mathfrak{X}), \Rightarrow, \sqcap, X\rangle$ is a $\mathbb{DH}^\wedge$-algebra.*

PROOF. We just need to show that for all $U, V \in D(\mathfrak{X})$ the following condition holds:

$$U \Rightarrow V = X \quad \text{iff} \quad U \sqcap V = U.$$

Recall that by definition of $\Rightarrow$, we have that $U \Rightarrow V = X$ if and only if $U \subseteq V$. Assume that $U \subseteq V$. Then by (DH^3), $U \sqcap V = \mathrm{cl}(U \cap V \cap \widehat{X}) = \mathrm{cl}(U \cap \widehat{X}) = U$. Assume now that $U = U \sqcap V = \mathrm{cl}(U \cap V \cap \widehat{X})$, and let $P \in U$. We show that $P \in V$. By assumption, there is $Q \in U \cap V \cap \widehat{X}$ such that $Q \subseteq P$. In particular, $Q \in V$ and since V is an up-set and $Q \subseteq P$, we obtain $P \in V$, as required. $\square$

COROLLARY 7.1.10. *Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$ be a $\mathbb{DH}^{\wedge}$-Spectral space. Then $\varepsilon_{\mathfrak{X}}[\widehat{X}] = \mathrm{Irr}_\wedge(\mathfrak{X}^*)$.*

PROOF. For all $x \in \widehat{X}$ we have that $f[\varepsilon_{\mathfrak{X}}(x)] = \{U \cap \widehat{X} : x \in U, U^c \in \kappa\} = \{U : x \in U, U^c \in \mathcal{KO}(\widehat{X})\}$, therefore by Spectral-like duality for distributive semilattices, $f[\varepsilon_{\mathfrak{X}}[\widehat{X}]]$ is the collection of all irreducible meet filters of $\langle F(\mathfrak{X}), \cap, \widehat{X} \rangle$, and then, by the isomorphism given by f, we obtain that $\varepsilon_{\mathfrak{X}}[\widehat{X}] = \mathrm{Irr}_\wedge(\mathfrak{X}^*)$. $\square$

THEOREM 7.1.11. *Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$ be a $\mathbb{DH}^{\wedge}$-Spectral space. Then*

$$\mathfrak{Irr}(\mathfrak{X}^*) := \langle \mathrm{Irr}_\rightarrow(\mathfrak{X}^*), \mathrm{Irr}_\wedge(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}} \rangle$$

is a $\mathbb{DH}^{\wedge}$-Spectral space such that $\langle X, \tau_\kappa \rangle$ and $\langle \mathrm{Irr}_\rightarrow(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}} \rangle$ are homeomorphic topological spaces by means of the map $\varepsilon_{\mathfrak{X}} : X \longrightarrow \mathrm{Irr}_\rightarrow(\mathfrak{X}^*)$ and moreover $\varepsilon_{\mathfrak{X}}[\widehat{X}] = \mathrm{Irr}_\wedge(\mathfrak{X}^*)$.

PROOF. By Spectral-like duality for Hilbert algebras we know that $\varepsilon_{\mathfrak{X}}$ is a homeomorphism between the topological spaces $\langle X, \tau_\kappa \rangle$ and $\langle \mathrm{Irr}_\rightarrow(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}} \rangle$. Moreover by Corollary 7.1.10, $\varepsilon_{\mathfrak{X}}[\widehat{X}] = \mathrm{Irr}_\wedge(\mathfrak{X}^*)$. $\square$

THEOREM 7.1.12. *Let $\mathbf{A} = \langle A, \rightarrow, \wedge, 1 \rangle$ be a $\mathbb{DH}^{\wedge}$-algebra. Then*

$$(\mathfrak{Irr}(\mathbf{A}))^* := \langle D(\mathfrak{Irr}(\mathbf{A})), \Rightarrow, \sqcap, \mathrm{Irr}_\rightarrow(\mathbf{A}) \rangle$$

is a $\mathbb{DH}^{\wedge}$-algebra isomorphic to $\mathbf{A}$ by means of the map $\psi_{\mathbf{A}} : A \longrightarrow D(\mathfrak{Irr}(\mathbf{A}))$.

PROOF. By Spectral-like duality for Hilbert algebras we know that $\psi_{\mathbf{A}}$ is an isomorphism between Hilbert algebras $\langle A, \rightarrow, 1 \rangle$ and $\langle D(\mathfrak{Irr}(\mathbf{A})), \Rightarrow, \mathrm{Irr}_\rightarrow(\mathbf{A}) \rangle$. It only remains to show that for all $a, c \in A$, $\psi_{\mathbf{A}}(a) \sqcap \psi_{\mathbf{A}}(c) = \psi_{\mathbf{A}}(a \wedge c)$. This follows from the definition of $\sqcap$ and from Proposition 7.1.1, since $\psi_{\mathbf{A}}(a) \sqcap \psi_{\mathbf{A}}(c) = \mathrm{cl}(\psi_{\mathbf{A}}(a) \cap \psi_{\mathbf{A}}(c) \cap \mathrm{Irr}_\wedge(\mathbf{A})) = \psi_{\mathbf{A}}(a \wedge c)$. $\square$

The previous theorem, together with Corollary 7.1.5 and theorems 7.1.9 and 7.1.11 summarize all preceding results, and should be kept in mind for § 7.2 and § 7.3, where the duality for morphisms is studied, and the functors between the categories we will be interested in are defined. Before moving to that, let us examine Priestley-dual objects of $\mathbb{DH}^{\wedge}$-algebras

7.1.2. Priestley-style dual objects. Recall that within the Priestley-style duality for Hilbert algebras that we developed in § 6.2, we define on $\mathrm{Op}_\rightarrow(\mathbf{A})$ a topology $\tau_{\mathbf{A}}$, having as subbasis the collection:

$$\{\vartheta(a) : a \in A\} \cup \{\vartheta(b)^c : b \in A\},$$

and we obtain that the structure $\langle \mathrm{Op}_{\to}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, \vartheta[A]\rangle$ is an $\mathbb{H}$-Priestley space (see definition in page 126). Furthermore, the dense subset given by condition (H13) is precisely $\mathrm{Irr}_{\to}(\mathbf{A})$.

Let us consider the subspace of $\langle \mathrm{Op}_{\to}(\mathbf{A}), \tau_{\mathbf{A}}\rangle$ generated by $\mathrm{Op}_{\wedge}(\mathbf{A})$. By definition,

$$\{\vartheta(a) \cap \mathrm{Op}_{\wedge}(\mathbf{A}) : a \in A\} \cup \{\vartheta(b)^c \cap \mathrm{Op}_{\wedge}(\mathbf{A}) : b \in A\}$$

is a subbasis for the induced topology on $\mathrm{Op}_{\wedge}(\mathbf{A})$, that we denote by $\widehat{\tau}_{\mathbf{A}}$.

We should recall now the Priestley-style duality for distributive semilattices presented in § 3.2.2. From that it follows that $\langle \mathrm{Op}_{\wedge}(\mathbf{A}), \widehat{\tau}_{\mathbf{A}}, \subseteq\rangle$ is a Priestley space, such that for any clopen up-set W of that space, W is $\mathrm{Irr}_{\wedge}(\mathbf{A})$-admissible if and only if $W = \vartheta(a) \cap \mathrm{Op}_{\wedge}(\mathbf{A})$ for some $a \in A$. Recall that $\mathrm{Irr}_{\wedge}(\mathbf{A})$-admissible clopen up-sets of $\langle \mathrm{Op}_{\wedge}(\mathbf{A}), \widehat{\tau}_{\mathbf{A}}, \subseteq\rangle$ are subsets $W \subseteq \mathrm{Op}_{\wedge}(\mathbf{A})$ such that $\mathrm{Op}_{\wedge}(\mathbf{A}) \setminus W \subseteq {\downarrow}(\mathrm{Irr}_{\wedge}(\mathbf{A}) \setminus W)$.

Now we are ready to introduce the definition of Priestley-style dual objects of $\mathbb{DH}^{\wedge}$-algebras.

DEFINITION 7.1.13. A structure $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ is a $\mathbb{DH}^{\wedge}$-*Priestley space* when:

- (DH^6) $\langle X, \tau, \leq, B\rangle$ is an $\mathbb{H}$-Priestley space,
- (DH^7) $\widehat{X} \subseteq X$ generates a compact subspace,
- (DH^8) $U = {\uparrow}(U \cap \widehat{X})$, for any $U \in B$,
- (DH^9) ${\uparrow}(U \cap V \cap \widehat{X}) \in B$, for any $U, V \in B$,
- (DH^10) W is $\widehat{X} \cap X_B$-admissible clopen up-set of $\widehat{X}$ iff $W = U \cap \widehat{X}$ for some $U \in B$.

Recall that for any $\mathbb{H}$-Priestley space $\langle X, \tau, \leq, B\rangle$ (see Definition 6.2.9 in page 126), the set $X_B := \{x \in X : \{U \in B : x \notin U\}$ is non-empty and up-directed$\}$ is the dense subset of $\langle X, \tau\rangle$ given by condition (H13), and it follows from Corollary 5.1.36 that $B \cup \{U^c : U \in B\}$ is a subbasis of the Priestley space $\langle X, \tau, \leq\rangle$. Moreover, by condition (DH^7) we get that for any $\mathbb{DH}^{\wedge}$-Priestley space $\langle X, \tau, \leq, B, \widehat{X}\rangle$, the family $\{U \cap \widehat{X} : U \in B\} \cup \{U^c \cap \widehat{X} : U \in B\}$ is a subbasis for the subspace of X generated by $\widehat{X}$. We may denote this subspace by $\langle \widehat{X}, \widehat{\tau}\rangle$, or simply by $\widehat{X}$. Let us denote $\{U \cap \widehat{X} : U \in B\}$ by $\widehat{B}$.

COROLLARY 7.1.14. *Let* $\mathbf{A} = \langle A, \to, \wedge, 1\rangle$ *be a* $\mathbb{DH}^{\wedge}$-*algebra. Then*

$$\mathfrak{Op}(\mathbf{A}) := \langle \mathrm{Op}_{\to}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, \vartheta_{\mathbf{A}}[A], \mathrm{Op}_{\wedge}(\mathbf{A})\rangle$$

is a $\mathbb{DH}^{\wedge}$-*Priestley space.*

PROOF. Condition (DH^6) follows from Priestley-style duality for $\mathbb{H}$-algebras (see § 6.2). Conditions (DH^7) and (DH^10) follow from Priestley-duality for distributive semilattices (see § 3.2.2) and the fact that $\mathrm{Op}_{\wedge}(\mathbf{A}) \subseteq \mathrm{Op}_{\to}(\mathbf{A})$ given by Corollary 6.5.21. Finally conditions (DH^8) and (DH^9) follow from Proposition 7.1.2. □

REMARK 7.1.15. Concerning a $\mathbb{DH}^{\wedge}$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$, we have to be careful again when using complements, since we are dealing with two ordered topological spaces at the same time, namely the main space $\langle X, \leq, \tau\rangle$, and the subspace of it generated by $\widehat{X}$, equipped with the inherited order. As before, we establish now the following convention: complements are always referred to the biggest set X.

PROPOSITION 7.1.16. *Let $\langle X, \tau, \leq, B, \widehat{X}\rangle$ be a $\mathbb{DH}^\wedge$-Priestley space. Then the set $X_B \cap \widehat{X}$ is dense in $\widehat{X}$.*

PROOF. It is enough to show that for every non-empty basic open K of $\widehat{X}$, there is $x \in X_B \cap \widehat{X}$ such that $x \in K$. Let K be a non-empty basic open of $\widehat{X}$. By definition of subbasis, there are $U_0, \dots, U_n, V_0, \dots, V_m \in B$ such that $K = U_0 \cap \dots \cap U_n \cap V_0^c \cap \dots \cap V_m^c \cap \widehat{X}$. By assumption $U_0 \cap \dots \cap U_n \cap V_0^c \cap \dots \cap V_m^c \cap \widehat{X} \neq \emptyset$ so $U_0 \cap \dots \cap U_n \cap \widehat{X} \nsubseteq V_0 \cup \dots \cup V_m$. Then we also have $\uparrow(U_0 \cap \dots \cap U_n \cap \widehat{X}) \nsubseteq V_0 \cup \dots \cup V_m$. By (DH^9), $U := \uparrow(U_0 \cap \dots \cap U_n \cap \widehat{X}) \in B$. Then we have $U \nsubseteq V_0 \cup \dots \cup V_m$, and since X_B is dense in $\langle X, \tau\rangle$, and B is a family of clopen up-sets, there is $x \in X_B$ such that $x \in U$ and $x \notin V_0 \cup \dots \cup V_m$. As $x \in X_B$, the collection $\{W \in B : x \notin W\}$ is up-directed, so there is $W \in B$ such that $V_j \subseteq W$, for all $j \leq m$, and $x \notin W$. By definition of U, there is $x' \in U_0 \cap \dots \cap U_n \cap \widehat{X}$ such that $x' \leq x$. Since W is an up-set, it follows $x' \notin W$. Therefore, we have $x' \in W^c \cap \widehat{X}$. Thus by (DH^10), there is $z \in W^c \cap \widehat{X} \cap X_B$ such that $x' \leq z$. As U_i are up-sets for all $i \leq n$, it follows $z \in U_0 \cap \dots \cap U_n$. Moreover, since $V_j \subseteq W$ for all $j \leq m$, it follows $z \notin V_j$, for all $j \leq m$. Hence we have $z \in X_B$ such that $z \in U_0 \cap \dots \cap U_n \cap V_0^c \cap \dots \cap V_m^c \cap \widehat{X}$, as required. $\square$

Recall that for a $\mathbb{DH}^\wedge$-Priestley space $\langle X, \tau, \leq, B, \widehat{X}\rangle$, the X_B-admissible clopen up-sets of X are the clopen up-sets $U \in \mathcal{C\ell U}(X)$ such that $\max(U^c) \subseteq X_B$. Similarly, the $X_B \cap \widehat{X}$-admissible clopen up-sets of $\widehat{X}$ are the clopen up-sets $V \in \mathcal{C\ell U}(\widehat{X})$ such that $\widehat{X} \setminus V \subseteq \downarrow(\widehat{X} \cap X_B \setminus V)$.

PROPOSITION 7.1.17. *Let $\langle X, \tau, \leq, B, \widehat{X}\rangle$ be a $\mathbb{DH}^\wedge$-Priestley space and let $x \in \widehat{X}$. Then $x \in X_B$ if and only if the collection of $X_B \cap \widehat{X}$-admissible clopen up-sets W of $\widehat{X}$ such that $x \notin W$ is non-empty and up-directed.*

PROOF. Notice that from condition (DH^10), this proposition can be restated as follows: for any $x \in \widehat{X}$, $x \in X_B$ if and only if $\{U \cap \widehat{X} : x \notin U \in B\}$ is non-empty and up-directed.

Let $x \in X_B \cap \widehat{X}$. Then as $x \in X_B$, by condition (H13′) in Definition 6.2.9, the collection $\{U \in B : x \notin U\}$ is non-empty and up-directed. From this it clearly follows the claim.

Let now $x \in \widehat{X}$ and assume that $\{U \cap \widehat{X} : x \notin U \in B\}$ is non-empty and up-directed. On the one hand, this clearly implies that $\{U \in B : x \notin U\}$ is non-empty. On the other hand, let $U_1, U_2 \in B$ be such that $x \notin U_1, U_2$. Then $x \notin U_1 \cap \widehat{X}$ and $x \notin U_2 \cap \widehat{X}$, so by assumption, there is $V \in B$ such that $U_1 \cap \widehat{X}, U_2 \cap \widehat{X} \subseteq V \cap \widehat{X}$ and $x \notin V$. And since V is an up-set, we have $\uparrow(U_1 \cap \widehat{X}), \uparrow(U_2 \cap \widehat{X}) \subseteq V$, and by condition (DH^8), this implies that $U_1, U_2 \subseteq V$, for $x \notin V$. We conclude that $\{U \in B : x \notin U\}$ is up-directed, and therefore, by definition of $\mathbb{H}$-Priestley space, $x \in X_B$. $\square$

Proposition 7.1.18. *Let $\langle X, \tau, \leq, B, \widehat{X}\rangle$ be a $\mathbb{DH}^{\wedge}$-Priestley space. Then for any $x, y \in \widehat{X}$, $x \leq y$ if and only if for every $X_B \cap \widehat{X}$-admissible clopen up-set W of $\widehat{X}$, $x \in W$ implies $y \in W$.*

Proof. This follows from condition (DH$^\wedge$10) and condition (H13$'$) in the definition of $\mathbb{H}$-Priestley space. □

Corollary 7.1.19. *Let $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ be a $\mathbb{DH}^{\wedge}$-Priestley space. Then $\langle \widehat{X}, \widehat{\tau}, \leq, X_B \cap \widehat{X}\rangle$ is a generalized Priestley space.*

Proof. As Priestley separation axiom is inherited by subspaces, and $\langle \widehat{X}, \widehat{\tau}\rangle$ is compact by definition, $\langle \widehat{X}, \widehat{\tau}, \leq\rangle$ is a Priestley space, so condition (DS3) holds. By Proposition 7.1.16, condition (DS4) holds. By Proposition 7.1.17, condition (DS5) holds, and by Proposition 7.1.18, condition (DS6) also holds. □

Similarly as when dealing with $\mathbb{H}$-Priestley spaces, for any $\mathbb{DH}^{\wedge}$space $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$, we define a binary operation $\Rightarrow$ on B,such that for all $U, V \in B$:

$$U \Rightarrow V := (\downarrow(U \cap V^c))^c.$$

The structure $\langle B, \Rightarrow, X\rangle$ turns out to be a Hilbert algebra. Moreover, the map $\xi_{\mathfrak{X}} : X \longrightarrow \mathcal{P}^{\uparrow}(B)$ given by

$$\xi_{\mathfrak{X}}(x) := \{U \in B : x \in U\}$$

is a map onto the collection of optimal implicative filters of the Hilbert algebra $\langle B, \Rightarrow, X\rangle$. Let $\sqcap$ be the binary operation on B such that for all $U, V \in B$:

$$U \sqcap V := \uparrow(U \cap V \cap \widehat{X}).$$

By condition (DH$^\wedge$9) B is closed under $\sqcap$. Let us show that $\langle B, \sqcap, X\rangle$ is isomorphic to the dual distributive semilattice of the generalized Priestley space $\widehat{\mathfrak{X}} := \langle \widehat{X}, \widehat{\tau}, \leq, X_B \cap \widehat{X}\rangle$.

Recall that for the generalized Priestley space $\widehat{\mathfrak{X}}$ given by Corollary 7.1.19, the collection of $X_B \cap \widehat{X}$-admissible clopen up-sets of $\widehat{X}$ is closed under finite intersections and moreover the structure $\langle \mathcal{C}\ell\mathcal{U}^{ad}_{X_B\cap\widehat{X}}(\widehat{X}), \cap, \widehat{X}\rangle$ is the dual distributive semilattice of $\widehat{\mathfrak{X}}$. By condition (DH$^\wedge$10) we know that this collection is precisely $\widehat{B} = \{U \cap \widehat{X} : U \in B\}$, therefore $\langle \widehat{B}, \cap, \widehat{X}\rangle$ is a distributive semilattice. Consider the map $g : B \longrightarrow \widehat{B}$, given by:

$$g(U) := U \cap \widehat{X}.$$

Clearly g is a surjective map, and by condition (DH$^\wedge$8), for all $U \in B$, $U = \uparrow(U \cap \widehat{X})$. Thus g is also injective. Moreover, from $U, V \in B$ being up-sets, it follows that $g(U) \cap g(V) = U \cap V \cap \widehat{X} = \uparrow(U \cap V \cap \widehat{X}) \cap \widehat{X} = g(U \sqcap V)$. Hence g is an isomorphism between $\langle B, \sqcap, X\rangle$ and $\langle \widehat{B}, \cap, \widehat{X}\rangle$.

Theorem 7.1.20. *Let $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ be a $\mathbb{DH}^{\wedge}$-Priestley space. Then $\mathfrak{X}^{\bullet} = \langle B, \Rightarrow, \sqcap, X\rangle$ is a $\mathbb{DH}^{\wedge}$-algebra.*

Proof. We just need to show that both $\Rightarrow$ and $\sqcap$ define the same order, i. e. we have to show that for all $U, V \in B$:

$$U \Rightarrow V = X \quad \text{iff} \quad U \sqcap V = U.$$

Notice that for all $U \in B$, $g^{-1}[U \cap \widehat{X}] = {\uparrow}(U \cap \widehat{X})$. By definition we have that $U \Rightarrow V = X$ if and only if $U \subseteq V$. First assume $U \subseteq V$. Then using (DH^8) we get $U \sqcap V = {\uparrow}(U \cap V \cap \widehat{X}) = {\uparrow}(U \cap \widehat{X}) = U$. For the converse, assume that $U = U \sqcap V = {\uparrow}(U \cap V \cap \widehat{X})$ and let $x \in U$. Then there is $y \in U \cap V \cap \widehat{X}$ such that $y \leq x$. But then since V is an up-set, we get $x \in V$, so $U \subseteq V$, as required. $\square$

COROLLARY 7.1.21. *Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$ be a $\mathbb{DH}^{\wedge}$-Spectral space. Then $\xi_{\mathfrak{X}}[\widehat{X}] = \mathrm{Op}_\wedge(\mathfrak{X}^\bullet)$.*

PROOF. Notice that for all $x \in \widehat{X}$ we have $g[\xi_{\mathfrak{X}}(x)] = \{U \cap \widehat{X} : x \in U \in B\} = \{V \in \widehat{B} : x \in V\}$, therefore by Priestley-style duality for distributive semilattices, $g[\xi_{\mathfrak{X}}[\widehat{X}]]$ is the collection of all optimal meet filters of $\langle \widehat{B}, \cap, \widehat{X} \rangle$, and then, by the isomorphism given by g, we obtain that $\xi_{\mathfrak{X}}[\widehat{X}] = \mathrm{Op}_\wedge(\mathfrak{X}^\bullet)$. $\square$

THEOREM 7.1.22. *Let $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X} \rangle$ be a $\mathbb{DH}^{\wedge}$-Priestley space. Then*

$$\mathfrak{Op}(\mathfrak{X}^\bullet) := \langle \mathrm{Op}_\rightarrow(\mathfrak{X}^\bullet), \tau_{\mathfrak{X}^\bullet}, \subseteq, \vartheta_{\mathfrak{X}^\bullet}[B], \mathrm{Op}_\wedge(\mathfrak{X}^\bullet) \rangle$$

is a $\mathbb{DH}^{\wedge}$-Priestley space such that the structures $\langle X, \tau, \leq \rangle$ and $\langle \mathrm{Op}_\rightarrow(\mathfrak{X}^\bullet), \tau_{\mathfrak{X}^\bullet}, \subseteq \rangle$ are order-homeomorphic topological spaces by means of the map $\xi_{\mathfrak{X}} : X \longrightarrow \mathrm{Op}_\rightarrow(\mathfrak{X}^\bullet)$ and moreover $\xi_{\mathfrak{X}}[X_B] = \mathrm{Irr}_\rightarrow(\mathfrak{X}^\bullet)$ and $\xi_{\mathfrak{X}}[\widehat{X}] = \mathrm{Op}_\wedge(\mathfrak{X}^\bullet)$.

PROOF. By Priestley-style duality for Hilbert algebras we know that $\xi_{\mathfrak{X}}$ is an order-homeomorphism between $\langle X, \tau, \leq \rangle$ and $\langle \mathrm{Op}_\rightarrow(\mathfrak{X}^\bullet), \tau_{\mathfrak{X}^\bullet}, \subseteq \rangle$, and $\xi_{\mathfrak{X}}[X_B] = \mathrm{Irr}_\rightarrow(\mathfrak{X}^\bullet)$. Moreover, by Corollary 7.1.21, $\xi_{\mathfrak{X}}[\widehat{X}] = \mathrm{Op}_\wedge(\mathfrak{X}^\bullet)$. $\square$

THEOREM 7.1.23. *Let $\mathbf{A} = \langle A, \rightarrow, \wedge, 1 \rangle$ be a $\mathbb{DH}^{\wedge}$-algebra. Then:*

$$\vartheta_{\mathbf{A}}[\mathbf{A}] := \langle \vartheta_{\mathbf{A}}[A], \Rightarrow, \sqcap, \mathrm{Op}_\rightarrow(\mathbf{A}) \rangle$$

is a $\mathbb{DH}^{\wedge}$-algebra isomorphic to $\mathbf{A}$ by means of the map $\vartheta_{\mathbf{A}} : A \longrightarrow \vartheta_{\mathbf{A}}[A]$.

PROOF. By Priestley-style duality for Hilbert algebras, we know that $\vartheta_{\mathbf{A}}$ is an isomorphism between Hilbert algebras $\langle A, \rightarrow, 1 \rangle$ and $\langle \vartheta_{\mathbf{A}}[A], \Rightarrow, \mathrm{Op}_\rightarrow(\mathbf{A}) \rangle$. So we just need to show that for all $b, c \in A$, $\vartheta_{\mathbf{A}}(b) \sqcap \vartheta_{\mathbf{A}}(c) = \vartheta_{\mathbf{A}}(b \wedge c)$. This follows from the definition and Proposition 7.1.2, since $\vartheta_{\mathbf{A}}(b) \sqcap \vartheta_{\mathbf{A}}(c) = {\uparrow}(\vartheta_{\mathbf{A}}(b) \cap \vartheta_{\mathbf{A}}(c) \cap \mathrm{Op}_\wedge(\mathbf{A})) = \vartheta_{\mathbf{A}}(b \wedge c)$. $\square$

The previous theorem together with theorems 7.1.11 and 7.1.20, and Corollary 7.1.14 summarize all preceding results, and should be kept in mind for the next sections, where the duality for morphisms is studied, and the functors involved are defined.

7.2. Dual morphisms

In the present section we study two dual correspondences concerning two different notions of morphisms between $\mathbb{DH}^{\wedge}$-algebras and certain classes of relations between $\mathbb{DH}^{\wedge}$-Spectral spaces and $\mathbb{DH}^{\wedge}$-Priestley spaces respectively that we introduce later on.

We follow an approach similar to that for Hilbert algebras by Celani et al. in [**15**] and we focus on two different morphisms between $\mathbb{DH}^{\wedge}$-algebras. One is the usual notion of algebraic homomorphism (preserving the constant and the operations), and the other is a weaker notion, similar to that of semi-homomorphism between Hilbert algebras:

Let $\mathbf{A}_1 = \langle A_1, \rightarrow_1, \wedge_1, 1_1\rangle$ and $\mathbf{A}_2 = \langle A_2, \rightarrow_2, \wedge_2, 1_2\rangle$ be two $\mathbb{DH}^{\wedge}$-algebras. A map $h : A_1 \longrightarrow A_2$ is a *meet-semi-homomorphism* (or $\wedge$*-semi-homomorphism*) when for all $a, b \in A_1$:

- $h(1_1) = 1_2$,
- $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$,
- $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$.

When h satisfies moreover $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$ for all $a, b \in A$, then h is called a *meet-homomorphism* or $\wedge$*-homomorphism*, or simply *homomorphism* when no confusion is possible.

Notice that $\wedge$-semi-homomorphisms are semi-homomorphisms between the respective Hilbert algebra reducts, and they are also homomorphisms between the respective distributive semilattice reducts. From now on let $\mathbf{A}_1 = \langle A_1, \rightarrow_1, \wedge_1, 1_1\rangle$ and $\mathbf{A}_2 = \langle A_2, \rightarrow_2, \wedge_2, 1_2\rangle$ be two $\mathbb{DH}^{\wedge}$-algebras and let $h : A_1 \longrightarrow A_2$ be a $\wedge$-semi-homomorphism. As in the case of the study of Hilbert algebras, $\wedge$-semi-homomorphisms are relevant, because they are the maps whose inverse map sends implicative filters to implicative filters. It also follows that the inverse map of a $\wedge$-semi-homomorphism sends meet filters to meet filters:

Lemma 7.2.1. *For any $P \in \mathrm{Fi}_{\rightarrow}(\mathbf{A}_2)$, $h^{-1}[P] \in \mathrm{Fi}_{\rightarrow}(\mathbf{A}_1)$. Moreover, if $P \in \mathrm{Fi}_{\wedge}(\mathbf{A}_2)$, then $h^{-1}[P] \in \mathrm{Fi}_{\wedge}(\mathbf{A}_1)$.*

For any $\wedge$-semi-homomorphism $h : A_1 \longrightarrow A_2$, we define a binary relation $R_h \subseteq \mathrm{Op}_{\rightarrow}(\mathbf{A}_2) \times \mathrm{Op}_{\rightarrow}(\mathbf{A}_1)$ by:

$$(P, Q) \in R_h \quad \text{iff} \quad h^{-1}[P] \subseteq Q.$$

We denote the restriction of R_h to $\mathrm{Irr}_{\rightarrow}(\mathbf{A}_2) \times \mathrm{Irr}_{\rightarrow}(\mathbf{A}_1)$ by $\overline{R}_h$. These are the relations that are used to represent h. Recall that for the relation R_h we may consider the function $\Box_{R_h} : \mathcal{P}(\mathrm{Op}_{\rightarrow}(\mathbf{A}_1)) \longrightarrow \mathcal{P}(\mathrm{Op}_{\rightarrow}(\mathbf{A}_2))$ given by:

$$\Box_{R_h}(U) := \{Q \in \mathrm{Op}_{\rightarrow}(\mathbf{A}_2) : R_h(Q) \subseteq U\}.$$

And regarding the relation $\overline{R}_h$, we may consider a different map, the function $\Box_{\overline{R}_h} : \mathcal{P}(\mathrm{Irr}_{\rightarrow}(\mathbf{A}_1)) \longrightarrow \mathcal{P}(\mathrm{Irr}_{\rightarrow}(\mathbf{A}_2))$, given by:

$$\Box_{\overline{R}_h}(U) := \{Q \in \mathrm{Irr}_{\rightarrow}(\mathbf{A}_2) : \overline{R}_h(Q) \subseteq U\}.$$

Let us examine in detail the properties of the relations R_h and $\overline{R}_h$. Notice that, for convenience, we denote by ϑ_i and ψ_i the maps $\vartheta_{\mathbf{A}_i}$ and $\psi_{\mathbf{A}_i}$ respectively. Similarly, we use $\kappa_i, \tau_{\kappa_i}$ and τ_i instead of $\kappa_{\mathbf{A}_i}$, $\tau_{\kappa_{\mathbf{A}_i}}$ and $\tau_{\mathbf{A}_i}$ respectively. The next proposition gives us the two representation theorems for h:

Proposition 7.2.2. *For any $\wedge$-semi-homomorphism $h : A_1 \longrightarrow A_2$:*

(1) *$\Box_{R_h}(\vartheta_1(a)) = \vartheta_2(h(a))$ for all $a \in A$.*
(2) *$\Box_{\overline{R}_h}(\psi_1(a)) = \psi_2(h(a))$ for all $a \in A$.*

Proof. For (1), we have from the definition of $\Box_{R_h}$ that:

$$\begin{aligned}\Box_{R_h}(\vartheta_1(a)) &= \{P \in \mathrm{Op}_{\rightarrow}(\mathbf{A}_2) : R_h(P) \subseteq \vartheta_1(a)\} \\ &= \{P \in \mathrm{Op}_{\rightarrow}(\mathbf{A}_2) : \forall Q \in \mathrm{Op}_{\rightarrow}(\mathbf{A}_1)\big(\text{ if } h^{-1}[P] \subseteq Q, \text{then } a \in Q\big)\}.\end{aligned}$$

Notice that from Corollary 2.4.8 we obtain that for any $P' \in \mathrm{Fi}_{\to}(\mathbf{A}_1)$, $a \in P'$ if and only if for all $Q \in \mathrm{Op}_{\to}(\mathbf{A}_1)$, $a \in Q$ whenever $P' \subseteq Q$. And by Lemma 7.2.1 we know that $h^{-1}[P] \in \mathrm{Fi}_{\to}(\mathbf{A}_1)$ for all $P \in \mathrm{Op}_{\to}(\mathbf{A}_2)$, so what we have is $\Box_{R_h}(\vartheta_1(a)) = \{P \in \mathrm{Op}_{\to}(\mathbf{A}_2) : a \in h^{-1}[P]\} = \{P \in \mathrm{Op}_{\to}(\mathbf{A}_2) : h(a) \in P\} = \vartheta_2(h(a))$.

For (2) we proceed similarly, since we have from the definition of $\Box_{\overline{R}_h}$ that:

$$\begin{aligned}\Box_{\overline{R}_h}(\psi_1(a)) &= \{P \in \mathrm{Irr}_{\to}(\mathbf{A}_2) : \overline{R}_h(P) \subseteq \psi_1(a)\}\\ &= \{P \in \mathrm{Irr}_{\to}(\mathbf{A}_2) : \forall Q \in \mathrm{Irr}_{\to}(\mathbf{A}_1)\big(\text{ if } h^{-1}[P] \subseteq Q, \text{ then } a \in Q\big)\}.\end{aligned}$$

And from Corollary 2.4.6 we obtain that for any $P' \in \mathrm{Fi}_{\to}(\mathbf{A}_1)$, $a \in P'$ if and only if for all $Q \in \mathrm{Irr}_{\to}(\mathbf{A}_1)$, $z \in Q$ whenever $P' \subseteq Q$. Then using again Lemma 7.2.1, as $h^{-1}[P] \in \mathrm{Fi}_{\to}(\mathbf{A}_1)$ for all $P \in \mathrm{Irr}_{\to}(\mathbf{A}_2)$, what we get is $\Box_{\overline{R}_h}(\psi_1(a)) = \{P \in \mathrm{Irr}_{\to}(\mathbf{A}_2) : a \in h^{-1}[P]\} = \{P \in \mathrm{Irr}_{\to}(\mathbf{A}_2) : h(a) \in P\} = \psi_2(h(a))$. $\Box$

It follows from the previous proposition that the restriction of $\Box_{R_h}$ to $\vartheta_1[A_1]$ is a $\wedge$-semi-homomorphism between $\vartheta_1[\mathbf{A}_1]$ and $\vartheta_2[\mathbf{A}_2]$. Similarly the restriction of $\Box_{\overline{R}_h}$ to $\psi_1[A_1]$ turns out to be a $\wedge$-semi-homomorphism between $\psi_1[\mathbf{A}_1]$ and $\psi_2[\mathbf{A}_2]$. Moreover, when h is a $\wedge$-homomorphism, then so are the respective restrictions of $\Box_{R_h}$ and $\Box_{\overline{R}_h}$. Hence Proposition 7.2.2 gives us two analogous representation theorems for h. In the following subsections, we discuss first the Spectral-like duals of $\wedge$-semi-homomorphisms and $\wedge$-homomorphisms, and then the Priestley-style duals. In both cases we prove the facts that motivate the definition of dual morphisms before introducing such definition.

7.2.1. Spectral-like dual morphisms. Recall that within the Spectral-like duality for Hilbert algebras reviewed in § 3.3.1, it is proven that the relation $\overline{R}_h$ is an $\mathbb{H}$-relation, whenever h is a semi-homomorphism between Hilbert algebras, and it is functional provided h is a homomorphism of Hilbert algebras. We just need the following proposition to complete the characterization of Spectral-duals of morphisms between $\mathbb{DH}^{\wedge}$-algebras:

PROPOSITION 7.2.3. *Let $P \in \mathrm{Irr}_{\wedge}(\mathbf{A}_2)$. Then $\overline{R}_h(P) = {\uparrow}(\overline{R}_h(P) \cap \mathrm{Irr}_{\wedge}(\mathbf{A}_1))$.*

PROOF. By definition $\overline{R}_h(P)$ is an up-set, so we just have to show the inclusion from left to right. Let $Q \in \overline{R}_h(P)$, i. e. $h^{-1}[P] \subseteq Q$. By Lemma 7.2.1 we know that $h^{-1}[P] \in \mathrm{Fi}_{\wedge}(\mathbf{A}_1)$. Moreover, as $Q \in \mathrm{Irr}_{\to}(\mathbf{A}_1)$, Q^c is an order ideal. Then from $h^{-1}[P] \cap Q^c = \emptyset$ and Lemma 2.3.3 we get that there is $Q' \in \mathrm{Irr}_{\wedge}(\mathbf{A}_1)$ such that $h^{-1}[P] \subseteq Q'$ and $Q' \cap Q^c = \emptyset$. Then Q' is the required element such that $Q' \in \overline{R}_h(P) \cap \mathrm{Irr}_{\wedge}(\mathbf{A}_1)$ and $Q' \subseteq Q$. $\Box$

DEFINITION 7.2.4. Let $\mathfrak{X}_1 = \langle X_1, \widehat{X}_1, \tau_{\kappa_1}\rangle$ and $\mathfrak{X}_2 = \langle X_2, \widehat{X}_2, \tau_{\kappa_2}\rangle$ be two $\mathbb{DH}^{\wedge}$-Spectral spaces. A relation $R \subseteq X_1 \times X_2$ is an $\mathbb{DH}^{\wedge}$-*Spectral morphism* when:

(DH^R1) R is an $\mathbb{H}$-relation between $\mathbb{H}$-spaces $\langle X_1, \tau_{\kappa_1}\rangle$ and $\langle X_2, \tau_{\kappa_2}\rangle$,

(DH^R2) for every $x \in \widehat{X}_1$, $R(x) = \mathrm{cl}(R(x) \cap \widehat{X}_2)$.

Moreover, R is said to be *functional* when it is moreover a functional $\mathbb{H}$-relation, i. e. when it satisfies the condition:

(HF) if $(x, y) \in R$, then there exists $z \in \mathrm{cl}(x)$ such that $R(z) = \mathrm{cl}(y)$.

Recall that for any $\mathbb{H}$-relation $R \subseteq X_1 \times X_2$ between $\mathbb{H}$-spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, the map $\Box_R : \mathcal{P}(X_2) \longrightarrow \mathcal{P}(X_1)$ is a semi-homomorphism between the Hilbert algebras $\langle D(\mathfrak{X}_2), \Rightarrow_2, X_2\rangle$ and $\langle D(\mathfrak{X}_1), \Rightarrow_1, X_1\rangle$. Moreover, $\Box_R$ is a homomorphism provided R is functional.

Corollary 7.2.5. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two $\mathbb{DH}^{\wedge}$-algebras and let $h : A_1 \longrightarrow A_2$ be a $\wedge$-semi-homomorphism between them. Then $\overline{R}_h$ is a $\mathbb{DH}^{\wedge}$-Spectral morphism between $\mathbb{DH}^{\wedge}$-Spectral spaces $\mathfrak{Irr}_{\rightarrow}(\mathbf{A}_2)$ and $\mathfrak{Irr}_{\rightarrow}(\mathbf{A}_1)$. Moreover, if h is a $\wedge$-homomorphism, then $\overline{R}_h$ is a functional.*

Proof. Condition (DH^R1) follows from Spectral-like duality for Hilbert algebras. Condition (DH^R2) follows from Proposition 7.2.3. Moreover, when h is a $\wedge$-homomorphism, and condition (HF) follows again from Spectral-like duality for Hilbert algebras. $\Box$

Example 7.2.6. Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$ be a $\mathbb{DH}^{\wedge}$-Spectral space. Recall that we denote by $\leq$ the dual of the specialization order of the space $\langle X, \tau_\kappa\rangle$. By Spectral-like duality for Hilbert algebras it follows that $\leq$ is a functional $\mathbb{H}$-relation between the $\mathbb{H}$-space $\langle X, \tau_\kappa\rangle$ and itself. It is, in fact, the identity morphism for X. Notice that we have for all $x \in \widehat{X}$, $\uparrow x = \uparrow(\uparrow x \cap \widehat{X})$. Therefore $\leq$ also satisfies condition (DH^R2), and so it is a $\mathbb{DH}^{\wedge}$-Spectral functional morphism.

Theorem 7.2.7. *Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be two $\mathbb{DH}^{\wedge}$-Spectral spaces and let $R \subseteq X_1 \times X_2$ be a $\mathbb{DH}^{\wedge}$-Spectral morphism between them. Then $\Box_R$ is a $\wedge$-semi-homomorphism between the $\mathbb{DH}^{\wedge}$-algebras $\langle D(\mathfrak{X}_2), \Rightarrow_2, \sqcap_2, X_2\rangle$ and $\langle D(\mathfrak{X}_1), \Rightarrow_1, \sqcap_1, X_1\rangle$.*

Proof. We only need to show that $\Box_R$ preserves meets, i. e. that $\Box_R(U \sqcap_2 V) = \Box_R(U) \sqcap_1 \Box_R(V)$ for all $U, V \in D(\mathfrak{X}_2)$. By definition, this is equivalent to show that for all $U, V \in D(\mathfrak{X}_2)$:

$$\Box_R(\mathrm{cl}(U \cap V \cap \widehat{X}_2)) = \mathrm{cl}(\Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1).$$

First we show the inclusion from left to right. Let $x \in \Box_R(\mathrm{cl}(U \cap V \cap \widehat{X}_2))$. By condition (DH^3) we know that $\Box_R(\mathrm{cl}(U\cap V\cap \widehat{X}_2)) = \mathrm{cl}(\Box_R(\mathrm{cl}(U\cap V\cap \widehat{X}_2))\cap \widehat{X}_1)$. Then there is $y \in \widehat{X}_1$ such that $y \in \Box_R(\mathrm{cl}(U \cap V \cap \widehat{X}_2))$ and $y \leq x$. By definition we have $R(y) \subseteq \mathrm{cl}(U\cap V\cap \widehat{X}_2)$. We show $R(y) \subseteq U\cap V$: let $z \in R(y)$, then there is $z' \in U \cap V \cap \widehat{X}_2$ such that $z' \leq z$. Since U, V are up-sets, then we have $z \in U \cap V$. We conclude $R(y) \subseteq U \cap V$, i. e. $y \in \Box_R(U)$ and $y \in \Box_R(V)$. Since, by assumption $y \in \widehat{X}_1$, we have $y \in \Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1$. Now using that $y \leq x$, we obtain $x \in \mathrm{cl}(\Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1)$, as required.

Let us show now the reverse inclusion. since $\Box_R(\mathrm{cl}(U \cap V \cap \widehat{X}_2))$ is an up-set, it is enough to show that $\Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1 \subseteq \Box_R(\mathrm{cl}(U \cap V \cap \widehat{X}_2))$. So we take $x \in \Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1$, i. e. $R(x) \subseteq U \cap V$ and $x \in \widehat{X}_1$. We show $R(x) \subseteq \mathrm{cl}(U \cap V \cap \widehat{X}_2)$. Let $y \in R(x)$. By condition (DH^R2) we know that $R(x) = \mathrm{cl}(R(x) \cap \widehat{X}_2)$. Then there is $y' \in R(x) \cap \widehat{X}_2$ such that $y' \leq y$. Then since $R(x)|subseteq Y \cap V$, we have $y' \in U \cap V$, so $y' \in U \cap V \cap \widehat{X}_2$. Therefore $y \in \mathrm{cl}(U \cap V \cap \widehat{X}_2)$. Hence $R(x) \subseteq \mathrm{cl}(U \cap V \cap \widehat{X}_2)$, i. e. $x \in \Box_R(\mathrm{cl}(U \cap V \cap \widehat{X}_2))$, as required. $\Box$

COROLLARY 7.2.8. *Let $R \subseteq X_1 \times X_2$ be a $\mathbb{DH}^{\wedge}$-Spectral functional morphism between $\mathbb{DH}^{\wedge}$-Spectral spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Then $\Box_R$ is a homomorphism between $\mathbb{DH}^{\wedge}$-algebras $\langle D(\mathfrak{X}_2), \Rightarrow_2, \sqcap_2, X_2\rangle$ and $\langle D(\mathfrak{X}_1), \Rightarrow_1, \sqcap_1, X_1\rangle$.*

Corollary 7.2.5 and Theorem 7.2.7 summarize the main results concerning $\wedge$-semi-homomorphisms and their duals. Corollaries 7.2.5 and 7.2.8 do the same concerning $\wedge$-homomorphisms. These results should be kept in mind for § 7.3.1, where the functors involved are defined. Before moving to that, let us examine Priestley-duals of $\wedge$-semi-homomorphisms and $\wedge$-homomorphisms.

7.2.2. Priestley-style dual morphisms. In regard to morphisms between $\mathbb{DH}^{\wedge}$-Priestley spaces, we follow the same strategy as in the previous subsection. Recall that when we developed Priestley-style duality for Hilbert algebras in § 6.2, we proved that the relation R_h is an $\mathbb{H}$-Priestley morphism, whenever h is a semi-homomorphism between Hilbert algebras. Moreover if h is a homomorphism between Hilbert algebras, then R_h is functional. The following proposition is the only result required to complete the characterization of Priestley-duals of morphisms between $\mathbb{DH}^{\wedge}$-algebras.

PROPOSITION 7.2.9. *Let $P \in \mathrm{Op}_{\wedge}(\mathbf{A}_2)$. Then $R_h(P) = \uparrow(R_h(P) \cap \mathrm{Op}_{\wedge}(\mathbf{A}_1))$.*

PROOF. By definition, R_h is an up-set, so we just have to show the inclusion from left to right. Let $Q \in R_h(P)$, i. e. $h^{-1}[P] \subseteq Q$. By Lemma 7.2.1 we know that $h^{-1}[P] \in \mathrm{Fi}_{\wedge}(\mathbf{A}_1)$. Moreover, as $Q \in \mathrm{Op}_{\rightarrow}(\mathbf{A}_1)$, Q^c is an sF-ideal, and in particular it is an F-ideal. Then from $h^{-1}[P] \cap Q^c = \emptyset$ and Lemma 2.3.7, we get that there is $Q' \in \mathrm{Op}_{\wedge}(\mathbf{A}_1)$ such that $h^{-1}[P] \subseteq Q'$ and $Q' \cap Q^c = \emptyset$. Then Q' is the required element such that $Q' \in R_h(P) \cap \mathrm{Op}_{\wedge}(\mathbf{A}_1)$ and $Q' \subseteq Q$. $\Box$

DEFINITION 7.2.10. Let $\mathfrak{X}_1 = \langle X_1, \tau_1, \leq_1, B_1, \widehat{X}_1\rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_2, \leq_2, B_2, \widehat{X}_2\rangle$ be two $\mathbb{DH}^{\wedge}$-Priestley spaces. A relation $R \subseteq X_1 \times X_2$ is a $\mathbb{DH}^{\wedge}$-*Priestley morphism* when:

(DH$^{\wedge}$R3) R is an $\mathbb{H}$-Priestley morphism between $\mathbb{H}$-Priestley spaces $\langle X_1, \tau_1, \leq_1, B_1\rangle$ and $\langle X_2, \tau_2, \leq_2, B_2\rangle$,

(DH$^{\wedge}$R4) for every $x \in \widehat{X}_1$, $R(x) = \uparrow(R(x) \cap \widehat{X}_2)$.

Moreover, R is said to be *functional* when it is moreover a functional $\mathbb{H}$-Priestley morphism, i. e. when it satisfies the condition:

(HF′) for every $x \in X_1$ and every $y \in X_{B_2}$, if $(x, y) \in R$, then there exists $z \in X_{B_1}$ such that $z \in \uparrow x$ and $R(z) = \uparrow y$.

Recall that for any $\mathbb{H}$-Priestley relation $R \subseteq X_1 \times X_2$ between $\mathbb{H}$-Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$, the map $\Box_R : \mathcal{P}(X_2) \longrightarrow \mathcal{P}(X_1)$ is a semi-homomorphism between Hilbert algebras $\langle B_2, \Rightarrow_2, X_2\rangle$ and $\langle B_1, \Rightarrow_1, X_1\rangle$. Moreover $\Box_R$ is a homomorphism whenever R is functional.

COROLLARY 7.2.11. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two $\mathbb{DH}^{\wedge}$-algebras and let $h : A_1 \longrightarrow A_2$ be a $\wedge$-semi-homomorphism between them. Then R_h is a $\mathbb{DH}^{\wedge}$-Priestley morphism between $\mathbb{DH}^{\wedge}$-Priestley spaces $\mathfrak{Op}(\mathbf{A}_2)$ and $\mathfrak{Op}(\mathbf{A}_1)$. Moreover, if h is a $\wedge$-homomorphism, then R_h is a functional.*

PROOF. Condition (DH^R3) follows from Priestley-style duality for Hilbert algebras. Condition (DH^R4) follows from Proposition 7.2.9. Moreover, when h is a $\wedge$-homomorphism, condition (IS5) follows again from Priestley-style duality for Hilbert algebras. $\square$

EXAMPLE 7.2.12. Similarly to the Spectral-like case, we have that the order of any $\mathbb{DH}^{\wedge}$-Priestley space is a functional $\mathbb{DH}^{\wedge}$-Priestley morphism. Let $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ be a $\mathbb{DH}^{\wedge}$-Priestley space. By Priestley-style duality for Hilbert algebras it follows that $\leq$ is an $\mathbb{H}$-Priestley functional morphism between the $\mathbb{H}$-Priestley space $\langle X, \tau, \leq, B\rangle$ and itself. Notice that for all $x \in \widehat{X}$, we have $\uparrow x = \uparrow(\uparrow x \cap \widehat{X})$. Therefore $\leq$ also satisfies condition (DH^R4), and thus $\leq$ is a $\mathbb{DH}^{\wedge}$-Priestley functional morphism.

THEOREM 7.2.13. *Let $R \subseteq X_1 \times X_2$ be a $\mathbb{DH}^{\wedge}$-Priestley morphism between $\mathbb{DH}^{\wedge}$-Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Then $\Box_R$ is a $\wedge$-semi-homomorphism between the $\mathbb{DH}^{\wedge}$-algebras $\langle B_2, \Rightarrow_2, \sqcap_2, X_2\rangle$ and $\langle B_1, \Rightarrow_1, \sqcap_1, X_1\rangle$.*

PROOF. We just need to show that $\Box_R$ preserves meets, i. e. that $\Box_R(U \sqcap_2 V) = \Box_R(U) \sqcap_1 \Box_R(V)$ for all $U, V \in B_2$. By definition, it is equivalent to show that for all $U, V \in B_2$:

$$\Box_R(\uparrow(U \cap V \cap \widehat{X}_2)) = \uparrow(\Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1).$$

First we show the inclusion from left to right. Let $x \in \Box_R(\uparrow(U \cap V \cap \widehat{X}_2))$. By condition (DH^8) we know that $\Box_R(\uparrow(U \cap V \cap \widehat{X}_2)) = \uparrow(\Box_R(\uparrow(U \cap V \cap \widehat{X}_2)) \cap \widehat{X}_1)$. Then there is $y \in \widehat{X}_1$ such that $y \in \Box_R(\uparrow(U \cap V \cap \widehat{X}_2))$ and $y \leq x$. By definition we have $R(y) \subseteq \uparrow(U \cap V \cap \widehat{X}_2)$. We show that $R(y) \subseteq U \cap V$: let $z \in R(y)$, then there is $z' \in U \cap V \cap \widehat{X}_2$ such that $z' \leq z$. Since U, V are up-sets, then we have $z \in U \cap V$. We conclude $R(y) \subseteq U \cap V$, i. e. $y \in \Box_R(U)$ and $y \in \Box_R(V)$. Moreover, by assumption $y \in \widehat{X}_1$, then we have $y \in \Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1$. Now using that $y \leq x$, we obtain $x \in \uparrow(\Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1)$, as required.

For the converse, since $\Box_R(\uparrow(U \cap V \cap \widehat{X}_2))$ is an up-set, it is enough to show that $\Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1 \subseteq \Box_R(\uparrow(U \cap V \cap \widehat{X}_2))$. Let $x \in \Box_R(U) \cap \Box_R(V) \cap \widehat{X}_1$, i. e. $R(x) \subseteq U \cap V$ and $x \in \widehat{X}_1$. We show $R(x) \subseteq \uparrow(U \cap V \cap \widehat{X}_2)$. Let $y \in R(x)$. By condition (DH^R4) we know that $R(x) = \uparrow(R(x) \cap \widehat{X}_2)$. Then there is $y' \in R(x) \cap \widehat{X}_2$ such that $y' \leq y$. Then since $R(x) \subseteq U \cap V$, we have $y' \in U \cap V$, so $y' \in U \cap V \cap \widehat{X}_2$. Therefore $y \in \uparrow(U \cap V \cap \widehat{X}_2)$. Hence $R(x) \subseteq \uparrow(U \cap V \cap \widehat{X}_2)$, i. e. we obtain $x \in \Box_R(\uparrow(U \cap V \cap \widehat{X}_2))$, as required. $\square$

COROLLARY 7.2.14. *Let $R \subseteq X_1 \times X_2$ be a $\mathbb{DH}^{\wedge}$-Priestley functional morphism between $\mathbb{DH}^{\wedge}$-Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Then $\Box_R$ is a homomorphism between $\mathbb{DH}^{\wedge}$-Priestley-algebras $\langle B_2, \Rightarrow_2, \sqcap_2, X_2\rangle$ and $\langle B_1, \Rightarrow_1, \sqcap_1, X_1\rangle$.*

Corollary 7.2.11 and Theorem 7.2.13 summarize the main results concerning $\wedge$-semi-homomorphisms and their duals. Corollaries 7.2.11 and 7.2.14 do the same concerning $\wedge$-homomorphisms. These results should be kept in mind for § 7.3.2, where the functors involved are defined.

7.3. Categorical dualities

In the present section we conclude the presentation of the dualities, by showing the functors and the natural transformations involved in them. Clearly we have that $\mathbb{DH}^{\wedge}$-algebras and $\wedge$-semi-homomorphisms form a category and similarly for $\mathbb{DH}^{\wedge}$-algebras and $\wedge$-homomorphisms. We denote these categories by $\mathsf{DH}^{\wedge}_S$ and $\mathsf{DH}^{\wedge}_H$ respectively. We will prove in the present section that there are two categories with $\mathbb{DH}^{\wedge}$-Spectral spaces as objects that are dually equivalent to $\mathsf{DH}^{\wedge}_S$ and $\mathsf{DH}^{\wedge}_H$ respectively. In a like manner we prove that there are two categories with $\mathbb{DH}^{\wedge}$-Priestley spaces as objects that are dually equivalent to $\mathsf{DH}^{\wedge}_S$ and $\mathsf{DH}^{\wedge}_H$ respectively. The first thing to do is to show that $\mathbb{DH}^{\wedge}$-Spectral spaces and $\mathbb{DH}^{\wedge}$-Spectral morphisms are indeed a category, and that $\mathbb{DH}^{\wedge}$-Priestley spaces and $\mathbb{DH}^{\wedge}$-Priestley morphisms form a category as well.

THEOREM 7.3.1. *Let $\langle X_1, \widehat{X}_1, \tau_{\kappa_1}\rangle$, $\langle X_2, \widehat{X}_2, \tau_{\kappa_2}\rangle$ and $\langle X_3, \widehat{X}_3, \tau_{\kappa_3}\rangle$ be three $\mathbb{DH}^{\wedge}$-Spectral spaces and let $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be $\mathbb{DH}^{\wedge}$-Spectral morphisms. Then:*

(1) *The $\mathbb{DH}^{\wedge}$-Spectral morphism $\leq_2 \subseteq X_2 \times X_2$ satisfies:*

$$\leq_2 \circ R = R \text{ and } S \circ \leq_2 = S,$$

(2) *$S \circ R \subseteq X_1 \times X_3$ is a $\mathbb{DH}^{\wedge}$-Spectral morphism,*

(3) *if R, S are functional, then $S \circ R$ is functional.*

PROOF. (1) This has been proven for $\mathbb{H}$-relations, so it holds in particular for $\mathbb{DH}^{\wedge}$-Spectral morphisms.

For (2), by Spectral-like duality for Hilbert algebras we get that $S \circ R$ is an $\mathbb{H}$-relation, so condition (DH^R1) is satisfied by $S \circ R$. We just have to show that $S \circ R$ satisfies condition (DH^R2), i. e. we have to show that for all $x \in \widehat{X}_1$:

$$(S \circ R)(x) = \mathrm{cl}(S \circ R(x) \cap \widehat{X}_3).$$

Let $x \in \widehat{X}_1$. First we prove that $(S \circ R)(x)$ is an up-set: let $(x, z) \in S \circ R$ and $z \leq_3 w$ for some $w \in X_3$. We show that $w \in S \circ R(x)$. By definition there is $y \in X_2$ such that $y \in R(x)$ and $z \in S(y)$. By condition (DH^R2) for R, we have $R(x) = \mathrm{cl}(R(x) \cap \widehat{X}_2)$. Then, there is $y' \in R(x) \cap \widehat{X}_2$ such that $y' \leq_2 y$. Now since $S \circ \leq_2 = S$, we have $z \in S(y')$. And since $y' \in \widehat{X}_2$, by condition (DH^R2) of S, $S(y') = \mathrm{cl}(S(y') \cap \widehat{X}_3)$. Then, there is $z' \in S(y') \cap \widehat{X}_3$ such that $z' \leq_3 z \leq_3 w$. Therefore, we have $w \in S(y')$, and since $(x, y') \in R$, then $(x, w) \in S \circ R$.

From $(S \circ R)(x)$ being an up-set, it is immediate that $\mathrm{cl}((S \circ R)(x) \cap \widehat{X}_3) \subseteq (S \circ R)(x)$. For the other inclusion, let $(x, z) \in S \circ R$. By a similar argument as before, we conclude that there is $z' \in (S \circ R)(x) \cap \widehat{X}_3$ such that $z' \leq z$, therefore $z \in \mathrm{cl}((S \circ R)(x) \cap \widehat{X}_3)$.

(3) follows from item (2) and Spectral-like duality for Hilbert algebras. □

COROLLARY 7.3.2. *$\mathbb{DH}^{\wedge}$-Spectral spaces and $\mathbb{DH}^{\wedge}$-Spectral morphisms form a category. $\mathbb{DH}^{\wedge}$-Spectral spaces and $\mathbb{DH}^{\wedge}$-Spectral functional morphisms form a category as well.*

PROOF. For a $\mathbb{DH}^{\wedge}$-Spectral space $\mathfrak{X}$, Example 7.2.6 shows that the order $\leq$ on X is a $\mathbb{DH}^{\wedge}$-Spectral morphism. Then by item (1) in Theorem 7.3.1, it is the identity morphism on $\mathfrak{X}$. By item (2) in Theorem 7.3.1, relational composition works as composition between $\mathbb{DH}^{\wedge}$-Spectral morphisms. $\square$

For the Priestley-style categories, we obtain similar results, except that relational composition does not work as composition in the respective categories. We have to define a new composition between $\mathbb{DH}^{\wedge}$-Priestley morphisms. For $\mathbb{DH}^{\wedge}$-Priestley spaces $\mathfrak{X}_1, \mathfrak{X}_2$ and $\mathfrak{X}_3$ and $\mathbb{DH}^{\wedge}$-Priestley morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, let $(S \star R) \subseteq X_1 \times X_3$ be the relation given by:

$$\begin{aligned}(x,z) \in (S \star R) \quad &\text{iff} \quad \forall U \in B_3 \big(\text{ if } x \in \Box_R \circ \Box_S(U), \text{ then } z \in U \big)\\ &\text{iff} \quad \forall U \in B_3 \big(\text{ if } (S \circ R)(x) \subseteq U, \text{ then } z \in U \big).\end{aligned}$$

THEOREM 7.3.3. *Let $\mathfrak{X}_1 = \langle X_1, \tau_1, \leq_1, B_1, \widehat{X}_1 \rangle$, $\mathfrak{X}_2 = \langle X_2, \tau_2, \leq_2, B_2, \widehat{X}_2 \rangle$ and $\mathfrak{X}_3 = \langle X_3, \tau_3, \leq_3, B_3, \widehat{X}_3 \rangle$ be three $\mathbb{DH}^{\wedge}$-Priestley spaces and let $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be two $\mathbb{DH}^{\wedge}$-Priestley morphisms. Then:*

(1) *The $\mathbb{DH}^{\wedge}$-Priestley morphism $\leq_2 \subseteq X_2 \times X_2$ satisfies:*

$$\leq_2 \circ R = R \text{ and } S \circ \leq_2 = S.$$

(2) *$(S \star R) \subseteq X_1 \times X_3$ is a $\mathbb{DH}^{\wedge}$-Priestley morphism,*

(3) *If R, S are functional, then $(S \star R) \subseteq X_1 \times X_3$ is functional.*

PROOF. (1) This holds for $\mathbb{H}$-Priestley morphisms, so it holds in particular for $\mathbb{DH}^{\wedge}$-Priestley morphisms.

For (2), by Priestley-style duality for Hilbert algebras we get that $(S \star R)$ is an $\mathbb{H}$-Priestley morphism, so condition (DH^R3) is satisfied by $(S \star R)$. We just have to show that $(S \star R)$ satisfies condition (DH^R4), i. e. we have to show that for all $x \in \widehat{X}_1$:

$$(S \star R)(x) = {\uparrow}((S \star R)(x) \cap \widehat{X}_3).$$

The inclusion from right to left is immediate, since $(S \star R)(x)$ is an up-set by definition. We show the reverse inclusion, that we will see that follows from Theorem 7.1.22, Proposition 7.2.9 and the definition of $\star$. Notice that by definition we have that for all $x \in X_1, z \in X_3$:

$$(x,z) \in (S \star R) \quad \text{iff} \quad (\xi_1(x), \xi_3(z)) \in R_{\Box_R \circ \Box_S}.$$

So let $x \in \widehat{X}_1$ and $z \in (S \star R)(x)$. On the one hand, we already know that $\xi_1(x) \in \mathrm{Op}_{\wedge}(\mathfrak{X}_1^{\bullet})$. On the other hand, by assumption $\xi_3(z) \in R_{\Box_R \circ \Box_S}(\xi_1(x))$. Since $\Box_R \circ \Box_S$ is a $\mathbb{DH}^{\wedge}$-Priestley semi-homomorphism, by Proposition 7.2.9, we obtain that $R_{\Box_R \circ \Box_S}(\xi_1(x)) = {\uparrow}(R_{\Box_R \circ \Box_S}(\xi_1(x)) \cap \mathrm{Op}_{\wedge}(\mathfrak{X}_3^{\bullet}))$. So we know that there is $Q \in R_{\Box_R \circ \Box_S}(\xi_1(x)) \cap \mathrm{Op}_{\wedge}(\mathfrak{X}_3^{\bullet})$ such that $Q \subseteq \xi_3(z)$. By Theorem 7.1.22, $\mathrm{Op}_{\wedge}(\mathfrak{X}_3^{\bullet}) = \xi_3[\widehat{X}_3]$, so there is $z' \in \widehat{X}_3$ such that $Q = \xi_3(z')$. Then we have $\xi_3(z') \subseteq \xi_3(z)$ and $(\xi_1(x), \xi_3(z')) \in R_{\Box_R \circ \Box_S}$. So by the definition of $\star$ we obtain $z' \in (S \star R)(x) \cap \widehat{X}_3$, and from ξ_3 being an order homeomorphism, we get $z' \leq z$. Therefore $(S \star R)(x) = {\uparrow}((S \star R)(x) \cap \widehat{X}_3)$, as required.

(3) follows from item (2) and Priestley-style duality for Hilbert algebras. $\square$

COROLLARY 7.3.4. *$\mathbb{DH}^{\wedge}$-Priestley spaces and $\mathbb{DH}^{\wedge}$-Priestley morphisms form a category. $\mathbb{DH}^{\wedge}$-Priestley spaces and $\mathbb{DH}^{\wedge}$-Priestley functional morphisms from a category as well.*

TABLE 9. Categories involved in the dualities for distributive Hilbert algebras with infimum.

CATEGORY	OBJECTS	MORPHISMS
$\mathsf{DH}_S^{\wedge}$	$\mathbb{DH}^{\wedge}$-algebras	$\wedge$-semi-homomorphisms
$\mathsf{DH}_H^{\wedge}$	$\mathbb{DH}^{\wedge}$-algebras	$(\wedge)$-homomorphisms
$\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$	$\mathbb{DH}^{\wedge}$-Spectral spaces	$\mathbb{DH}^{\wedge}$-Spectral morphisms
$\mathsf{Sp}_F^{\mathbb{DH}^{\wedge}}$	$\mathbb{DH}^{\wedge}$-Spectral spaces	$\mathbb{DH}^{\wedge}$-Spectral functional morphisms
$\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$	$\mathbb{DH}^{\wedge}$-Priestley spaces	$\mathbb{DH}^{\wedge}$-Priestley morphisms (comp $\star$)
$\mathsf{Pr}_F^{\mathbb{DH}^{\wedge}}$	$\mathbb{DH}^{\wedge}$-Priestley spaces	$\mathbb{DH}^{\wedge}$-Priestley functional morphisms (comp $\star$)

PROOF. For a $\mathbb{DH}^{\wedge}$-Priestley space $\mathfrak{X}$, Example 7.2.12 shows that the order $\leq$ on X is a $\mathbb{DH}^{\wedge}$-Priestley morphism. It is not difficult to check that for any $\mathbb{DH}^{\wedge}$-Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$ and $\mathbb{DH}^{\wedge}$-Priestley morphism $R \subseteq X_1 \times X_2$, we have $\leq_2 \circ R = \leq_2 \star R$ and $R \circ \leq_1 = R \star \leq_1$. Then by item (1) in Theorem 7.3.3 we obtain that $\leq$ is the identity morphism for $\mathfrak{X}$ By item (2) in Theorem 7.3.3, the operation $\star$ gives composition between $\mathbb{DH}^{\wedge}$-Priestley morphisms (associativity of $\star$ follows easily). $\square$

Let $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$ be the category of $\mathbb{DH}^{\wedge}$-Spectral spaces and $\mathbb{DH}^{\wedge}$-Spectral morphisms, and let $\mathsf{Sp}_F^{\mathbb{DH}^{\wedge}}$ be the category of $\mathbb{DH}^{\wedge}$-Spectral spaces and $\mathbb{DH}^{\wedge}$-Spectral functional morphisms. Let $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$ be the category of $\mathbb{DH}^{\wedge}$-Priestley spaces and $\mathbb{DH}^{\wedge}$-Priestley morphisms, and let $\mathsf{Pr}_F^{\mathbb{DH}^{\wedge}}$ be the category of $\mathbb{DH}^{\wedge}$-Priestley spaces and $\mathbb{DH}^{\wedge}$-Priestley functional morphisms. We summarize in Table 9 all the categories we have so far considered.

Once we have defined all the categories, we need to build the contravariant functors and the natural isomorphisms involved in the dualities. Let us examine first the Spectral-like duality, and then we move to the Priestley-style duality.

7.3.1. Spectral-like dualities. Let us start looking at the functors for the Spectral-like dualities. We consider first the functor $\mathfrak{Irr} : \mathsf{DH}_S^{\wedge} \longrightarrow \mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$ such that for any $\mathbb{DH}^{\wedge}$-algebras $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2$ and any $\wedge$-semi-homomorphism $h : A_1 \longrightarrow A_2$:

$$\mathfrak{Irr}(\mathbf{A}) := \langle \mathrm{Irr}_{\to}(\mathbf{A}), \mathrm{Irr}_{\wedge}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}} \rangle,$$
$$\mathfrak{Irr}(h) := \overline{R}_h \subseteq \mathrm{Irr}_{\to}(\mathbf{A}_2) \times \mathrm{Irr}_{\to}(\mathbf{A}_1).$$

We recall that $\tau_{\kappa_{\mathbf{A}}}$ is a topology on $\mathrm{Irr}_{\to}(\mathbf{A})$ having as basis $\kappa_{\mathbf{A}} := \{\psi_{\mathbf{A}}(a)^c : a \in A\}$, for $\psi_{\mathbf{A}} : A \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Irr}_{\to}(\mathbf{A}))$ given by $\psi_{\mathbf{A}}(a) := \{P \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \in P\}$, and by definition $(P, Q) \in \overline{R}_h$ if and only if $h^{-1}[P] \subseteq Q$.

Clearly, for the identity morphism $\mathrm{id}_{\mathbf{A}} : A \longrightarrow A$ for $\mathbf{A}$ in $\mathsf{DH}_S^{\wedge}$, it holds that $\overline{R}_{\mathrm{id}_{\mathbf{A}}} = \subseteq$, and this is precisely the identity morphism for $\mathfrak{Irr}(\mathbf{A})$ in $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$. Moreover, it follows by definition that for $\mathbb{DH}^{\wedge}$-algebras $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ and $\wedge$-semi-homomorphisms $f : A_1 \longrightarrow A_2$ and $g : A_2 \longrightarrow A_3$, $\overline{R}_{g \circ f} = \overline{R}_f \circ \overline{R}_g$. Therefore, by

Spectral-like duality for Hilbert algebras and corollaries 7.1.5 and 7.2.5, the functor $\mathfrak{Irr}$ is well defined.

On the other hand, we consider the functor $(\ )^* : \mathsf{Sp}_M^{\mathbb{DH}^\wedge} \longrightarrow \mathsf{DH}_S^\wedge$ such that for any $\mathbb{DH}^\wedge$-Spectral spaces $\mathfrak{X}, \mathfrak{X}_1, \mathfrak{X}_2$ and any $\mathbb{DH}^\wedge$-Spectral morphism $R \subseteq X_1 \times X_2$:

$$\mathfrak{X}^* := \langle D(\mathfrak{X}), \Rightarrow, \sqcap, X \rangle,$$
$$R^* := \Box_R : D(\mathfrak{X}_2) \longrightarrow D(\mathfrak{X}_1).$$

We recall that by definition $D(\mathfrak{X}) := \{U^c : U \in \kappa\}$, and for all $U \in D(\mathfrak{X}_2)$, $\Box_R(U) := \{x \in X_1 : R(x) \subseteq U\}$.

Obviously, for the identity morphism $\leq \subseteq X \times X$ for $\mathfrak{X}$ in $\mathsf{Sp}_M^{\mathbb{DH}^\wedge}$, we get $\Box_\leq = \mathrm{id}_{\mathfrak{X}^*}$, and this is precisely the identity morphism for $\mathfrak{X}^*$ in $\mathsf{DH}_S^\wedge$. Furthermore, it follows by definition that for $\mathbb{DH}^\wedge$-Spectral spaces $\mathfrak{X}_1, \mathfrak{X}_2$, and $\mathfrak{X}_3$, and $\mathbb{DH}^\wedge$-Spectral morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, $\Box_{S \circ R} = \Box_R \circ \Box_S$. Therefore by Spectral-like duality for Hilbert algebras and theorems 7.1.9 and 7.2.7 the functor $(\ )^*$ is well defined.

In order to complete the dualities, we need to define two natural isomorphisms, the one between the identity functor on $\mathsf{DH}_S^\wedge$ and $(\mathfrak{Irr}(\ ))^*$, and the other between the identity functor on $\mathsf{Sp}_M^{\mathbb{DH}^\wedge}$ and $\mathfrak{Irr}((\ )^*)$. Consider first the family of morphisms in $\mathsf{DH}_S^\wedge$:

$$\Psi := \big(\psi_{\mathbf{A}} : A \longrightarrow D(\mathfrak{Irr}(\mathbf{A}))\big)_{\mathbf{A} \in \mathsf{DH}_S^\wedge}$$

Theorem 7.3.5. *Ψ is a natural isomorphism between the identity functor on $\mathsf{DH}_S^\wedge$ and $(\mathfrak{Irr}(\ ))^*$.*

Proof. Let $\mathbf{A}_1, \mathbf{A}_2$ be two $\mathbb{DH}^\wedge$-algebras and let $h : A_1 \longrightarrow A_2$ be a $\wedge$-semi-homomorphism between them. By Spectral-like duality for Hilbert algebras we get that $\Box_{\overline{R}_h} \circ \psi_1 = \psi_2 \circ h$. From this we have that Ψ is a natural transformation, and by Theorem 7.1.12 we get that for all $\mathbf{A} \in \mathsf{DH}_S^\wedge$, $\psi_{\mathbf{A}}$ is an isomorphism, so we conclude that Ψ is a natural isomorphism. $\Box$

Clearly, what we have is that for any $\mathbb{DH}^\wedge$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ and any $\wedge$-semi-homomorphism $h : A_1 \longrightarrow A_2$, the following diagram commutes:

$$\begin{tikzcd} A_1 \arrow[r, "\psi_1"] \arrow[d, "h"'] & D(\mathfrak{Irr}(\mathbf{A}_1)) \arrow[d, "\Box_{\overline{R}_h}"] \\ A_2 \arrow[r, "\psi_2"'] & D(\mathfrak{Irr}(\mathbf{A}_2)) \end{tikzcd}$$

We need to do some preparatory work before enunciating the other natural isomorphism. Recall that for any $\mathbb{DH}^\wedge$-Spectral space $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$, we define the map $\varepsilon_{\mathfrak{X}} : X \longrightarrow \mathrm{Irr}_\to(\mathfrak{X}^*)$, that by Theorem 7.1.11 is a homeomorphism between the topological spaces $\langle X, \tau_\kappa \rangle$ and $\langle \mathrm{Irr}_\to(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}} \rangle$. This map encodes the natural isomorphism we are looking for, but since morphisms in $\mathsf{Sp}_M^{\mathbb{DH}^\wedge}$ are relations, we need to give a relation associated with this map. We define the relation $E_{\mathfrak{X}} \subseteq X \times \mathrm{Irr}_\to(\mathfrak{X}^*)$ given by:

$$(x, P) \in E_{\mathfrak{X}} \quad \text{iff} \quad \varepsilon_{\mathfrak{X}}(x) \subseteq P.$$

Proposition 7.3.6. *$E_{\mathfrak{X}}$ is a $\mathbb{DH}^\wedge$-Spectral functional morphism.*

PROOF. From the Spectral-like duality for Hilbert algebras, we know that $E_{\mathfrak{X}}$ is a functional $\mathbb{H}$-relation, so we just have to check that condition (DH$^{\wedge}$R2) is satisfied. Let $x \in \widehat{X}$. It is immediate that $\mathrm{cl}(E_{\mathfrak{X}}(x) \cap \mathrm{Irr}_{\wedge}(\mathfrak{X}^*)) \subseteq E_{\mathfrak{X}}(x)$, so we just have to check the other inclusion. Let $P \in E_{\mathfrak{X}}(x)$, i. e. $\varepsilon_{\mathfrak{X}}(x) \subseteq P$. By Corollary 7.1.10 we know that $\varepsilon_{\mathfrak{X}}(x) \in \mathrm{Irr}_{\wedge}(\mathfrak{X}^*)$ and clearly $\varepsilon_{\mathfrak{X}}(x) \in E_{\mathfrak{X}}(x)$. Therefore $P \in \mathrm{cl}(E_{\mathfrak{X}}(x) \cap \mathrm{Irr}_{\wedge}(\mathfrak{X}^*))$, as required. $\square$

Consider now the family of morphisms in $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$:

$$\Sigma := \left(E_{\mathfrak{X}} \subseteq X \times \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^*)\right)_{\mathfrak{X} \in \mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}}$$

THEOREM 7.3.7. *Σ is a natural isomorphism between the identity functor on $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$ and $\mathfrak{Irr}((\ )^*)$.*

PROOF. Let $\mathfrak{X}_1, \mathfrak{X}_2$ be two $\mathbb{DH}^{\wedge}$-Spectral spaces and let $R \subseteq X_1 \times X_2$ be a $\mathbb{DH}^{\wedge}$-Spectral morphism between them. By Spectral-like duality for Hilbert algebras we get that $(x, y) \in R$ if and only if $(\varepsilon_{\mathfrak{X}_1}(x), \varepsilon_{\mathfrak{X}_2}(y)) \in \overline{R}_{\Box_R}$, and from this it follows that $\overline{R}_{\Box_R} \circ E_{\mathfrak{X}_1} = E_{\mathfrak{X}_2} \circ R$. Thus Σ is a natural equivalence. Moreover, by Theorem 7.1.11 we have that the map $\varepsilon_{\mathfrak{X}}$ is an homeomorphism between $\langle X, \tau_\kappa\rangle$ and $\langle \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}}\rangle$ such that $\varepsilon_{\mathfrak{X}}[\widehat{X}] = \mathrm{Irr}_{\wedge}(\mathfrak{X}^*)$. It follows that $E_{\mathfrak{X}}$ is an isomorphism in $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$, and then Σ is a natural isomorphism in $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$. $\square$

COROLLARY 7.3.8. *The categories $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$ and $\mathsf{DH}_S^{\wedge}$ are dually equivalent by means of the contravariant functors $\mathfrak{Irr}$ and $(\ )^*$ and the natural equivalences Ψ and Σ. Similarly, the categories $\mathsf{Sp}_F^{\mathbb{DH}^{\wedge}}$ and $\mathsf{DH}_H^{\wedge}$ are dually equivalent by means of the restrictions of the functors $\mathfrak{Irr}$ and $(\ )^*$ and the restrictions of the natural equivalences Ψ and Σ.*

7.3.2. Priestley-style dualities. Let us move now to the other dualities, namely the ones involving $\mathbb{DH}^{\wedge}$-Priestley spaces. We start considering the functors: We define the functor $\mathfrak{Op} : \mathsf{DH}_S^{\wedge} \longrightarrow \mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$ such that for any $\mathbb{DH}^{\wedge}$-algebras $\mathbf{A}, \mathbf{A}_1, \mathbf{A}_2$ and any $\wedge$-semi-homomorphism $h : A_1 \longrightarrow A_2$:

$$\mathfrak{Op}(\mathbf{A}) := \langle \mathrm{Op}_{\rightarrow}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, \psi_{\mathbf{A}}[A], \mathrm{Op}_{\wedge}(\mathbf{A})\rangle,$$
$$\mathfrak{Op}(h) := R_h \subseteq \mathrm{Op}_{\rightarrow}(\mathbf{A}_2) \times \mathrm{Op}_{\rightarrow}(\mathbf{A}_1).$$

We recall that $\tau_{\mathbf{A}}$ is a topology on $\mathrm{Op}_{\rightarrow}(\mathbf{A})$ that is defined from the subbasis $\{\vartheta_{\mathbf{A}}(a) : a \in A\} \cup \{\vartheta_{\mathbf{A}}(b)^c : b \in A\}$, for $\vartheta_{\mathbf{A}} : A \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Op}_{\rightarrow}(\mathbf{A}))$ given by $\vartheta_{\mathbf{A}}(a) := \{P \in \mathrm{Op}_{\rightarrow}(\mathbf{A}) : a \in P\}$, and $(P, Q) \in R_h$ if and only if $h^{-1}[P] \subseteq Q$.

It should be clear that for the identity morphism $\mathrm{id}_{\mathbf{A}} : A \longrightarrow A$ for $\mathbf{A}$ in $\mathsf{DH}_S^{\wedge}$, we have $R_{\mathrm{id}_{\mathbf{A}}} = \subseteq$, that is the identity morphism for $\mathfrak{Op}(\mathbf{A})$ in $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$. Furthermore, it follows by definition that for $\mathbb{DH}^{\wedge}$-algebras $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ and $\wedge$-semi-homomorphisms $f : A_1 \longrightarrow A_2$ and $g : A_2 \longrightarrow A_3$, $R_{g \circ f} = R_f \circ R_g$. Therefore, by Priestley-style duality for Hilbert algebras and corollaries 7.1.14 and 7.2.11, the functor $\mathfrak{Op}$ is well defined.

Besides, we define the functor $(\ )^{\bullet} : \mathsf{Pr}_M^{\mathbb{DH}^{\wedge}} \longrightarrow \mathsf{DH}_S^{\wedge}$, that for any $\mathbb{DH}^{\wedge}$-Priestley spaces $\mathfrak{X}, \mathfrak{X}_1, \mathfrak{X}_2$ and any $\mathbb{DH}^{\wedge}$-Priestley morphism $R \subseteq X_1 \times X_2$:

$$\mathfrak{X}^{\bullet} := \langle S, \Rightarrow, \sqcap, X\rangle,$$
$$R^{\bullet} := \Box_R : B_2 \longrightarrow B_1.$$

We recall that for all $U \in B_2$, $\Box_R(U) := \{x \in X_1 : R(x) \subseteq U\}$. It is immediate that for the identity morphism $\leq \subseteq X \times X$ for $\mathfrak{X}$ in $\mathsf{Pr}_M^{\mathbb{DH}^\wedge}$, we have $\Box_{\leq} = \mathrm{id}_{\mathfrak{X}^\bullet}$, that is the identity morphism for $\mathfrak{X}^\bullet$ in $\mathsf{DH}_S^\wedge$. Moreover, it follows by definition that for $\mathbb{DH}^\wedge$-Spectral spaces $\mathfrak{X}_1, \mathfrak{X}_2$, and $\mathfrak{X}_3$, and $\mathbb{DH}^\wedge$-Spectral morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, $\Box_{S\circ R} = \Box_R \circ \Box_S$. Thus by Priestley-style duality for Hilbert algebras and theorems 7.1.20 and 7.2.13 the functor $(\ )^\bullet$ is well defined.

For completing the dualities, we need to define two natural isomorphisms, the one between the identity functor on $\mathsf{DH}_S^\wedge$ and $(\mathfrak{O}\mathrm{p}(\ ))^\bullet$, and the one between the identity functor on $\mathsf{Pr}_M^{\mathbb{DH}^\wedge}$ and $\mathfrak{O}\mathrm{p}((\ )^\bullet)$. Consider first the family of morphisms in $\mathsf{DH}_S^\wedge$:

$$\Theta := \big(\vartheta_{\mathbf{A}} : A \longrightarrow \vartheta_{\mathbf{A}}[A]\big)_{\mathbf{A}\in\mathsf{DH}_S^\wedge}$$

THEOREM 7.3.9. *Θ is a natural isomorphism between the identity functor on $\mathsf{DH}_S^\wedge$ and $(\mathfrak{O}\mathrm{p}(\ ))^\bullet$.*

PROOF. Let $\mathbf{A}_1, \mathbf{A}_2$ be two $\mathbb{DH}^\wedge$-algebras and let $h : A_1 \longrightarrow A_2$ be a $\wedge$-semi-homomorphism between them. By Priestley-style duality for Hilbert algebras we get that $\Box_{R_h} \circ \vartheta_1 = \vartheta_2 \circ h$. From this we have that Θ is a natural transformation, and by Theorem 7.1.23 we get that for all $\mathbf{A} \in \mathsf{DH}_S^\wedge$, $\vartheta_{\mathbf{A}}$ is an isomorphism, so we conclude that Θ is a natural isomorphism. $\Box$

What we obtain is that for any $\mathbb{DH}^\wedge$-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ and any $\wedge$-semi-homomorphism $h : A_1 \longrightarrow A_2$, the following diagram commutes:

$$\begin{tikzcd} A_1 \arrow[r, "\vartheta_1"] \arrow[d, "h"'] & \vartheta_1[A_1] \arrow[d, "\Box_{R_h}"] \\ A_2 \arrow[r, "\vartheta_2"'] & \vartheta_2[A_2] \end{tikzcd}$$

Before presenting the other natural isomorphism, we need again to do some work. Recall that for any $\mathbb{DH}^\wedge$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ we define the map $\xi_{\mathfrak{X}} : X \longrightarrow \mathrm{Op}_{\to}(\mathfrak{X}^\bullet)$, that by Theorem 7.1.22 is an order homeomorphism between ordered topological spaces $\langle X, \tau, \leq\rangle$ and $\langle \mathrm{Op}_{\to}(\mathfrak{X}^\bullet), \tau_{\mathfrak{X}^\bullet}, \subseteq\rangle$. As in the Spectral-like case, this map encodes the natural isomorphism we are looking for, but since morphisms in $\mathsf{Pr}_M^{\mathbb{DH}^\wedge}$ are relations, we need to give a relation associated with that map. We define the relation $T_{\mathfrak{X}} \subseteq X \times \mathrm{Op}_{\to}(\mathfrak{X}^\bullet)$ given by:

$$(x, P) \in T_{\mathfrak{X}} \quad \text{iff} \quad \xi_{\mathfrak{X}}(x) \subseteq P.$$

that turns out to be a $\mathbb{DH}^\wedge$-Priestley functional morphism.

PROPOSITION 7.3.10. *$T_{\mathfrak{X}}$ is a $\mathbb{DH}^\wedge$-Priestley functional morphism.*

PROOF. By Priestley-style duality for Hilbert algebras $T_{\mathfrak{X}}$ is an $\mathbb{H}$-Priestley functional morphism, so we just have to check that condition (DH$^\wedge$R4) is satisfied. Let $x \in \widehat{X}$. It is immediate that $\uparrow(T_{\mathfrak{X}}(x) \cap \mathrm{Op}_\wedge(\mathfrak{X}^\bullet)) \subseteq T_{\mathfrak{X}}(x)$, so we just have to check the other inclusion. Let $P \in T_{\mathfrak{X}}(x)$, i.e. $\xi_{\mathfrak{X}}(x) \subseteq P$. As $x \in \widehat{X}$, by Theorem 7.1.22 we know that $\xi_{\mathfrak{X}}(x) \in \mathrm{Op}_\wedge(\mathfrak{X}^\bullet)$ and clearly $\xi_{\mathfrak{X}}(x) \in T_{\mathfrak{X}}(x)$. Therefore $P \in \uparrow(T_{\mathfrak{X}}(x) \cap \mathrm{Op}_\wedge(\mathfrak{X}^\bullet))$, as required. $\Box$

Consider the family of morphisms in $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$:

$$\Xi := \left(T_{\mathfrak{X}} \subseteq X \times \mathrm{Op}_{\to}(\mathfrak{X}^{\bullet})\right)_{\mathfrak{X}\in \mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}}$$

THEOREM 7.3.11. *Ξ is a natural isomorphism between the identity functor on $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$ and $\mathfrak{O}\mathrm{p}((\ )^{\bullet})$.*

PROOF. Let $\mathfrak{X}_1, \mathfrak{X}_2$ be two $\mathbb{DH}^{\wedge}$-Priestley spaces and let $R \subseteq X_1 \times X_2$ be a $\mathbb{DH}^{\wedge}$-Priestley morphism between them. From Priestley-style duality for Hilbert algebra we get that $(x, y) \in R$ if and only if $(\xi_{\mathfrak{X}_1}(x), \xi_{\mathfrak{X}_2}(x)) \in R_{\Box_R}$, and from this it follows that $R_{\Box_R} \star T_{\mathfrak{X}_1} = T_{\mathfrak{X}_2} \star R$. Thus Ξ is a natural equivalence. Moreover, by Theorem 7.1.22 we have that the map $\xi_{\mathfrak{X}}$ is an order homeomorphism between $\langle X, \tau, \leq\rangle$ and $\langle \mathrm{Op}_{\to}(\mathfrak{X}^{\bullet}), \tau_{\mathfrak{X}^{\bullet}}, \subseteq\rangle$ such that $\xi_{\mathfrak{X}}[\widehat{X}] = \mathrm{Op}_{\wedge}(\mathfrak{X}^{\bullet})$. It follows that $T_{\mathfrak{X}}$ is an isomorphism in $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$, and then Ξ is a natural isomorphism in $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$. ☐

COROLLARY 7.3.12. *The categories $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$ and $\mathsf{DH}_S^{\wedge}$ are dually equivalent by means of the contravariant functors $\mathfrak{O}\mathrm{p}$ and $(\ )^{\bullet}$ and the natural equivalences Θ and Ξ. Similarly, the categories $\mathsf{Pr}_F^{\mathbb{DH}^{\wedge}}$ and $\mathsf{DH}_H^{\wedge}$ are dually equivalent by means of the restrictions of the functors $\mathfrak{O}\mathrm{p}$ and $(\ )^{\bullet}$ and the restrictions of the natural equivalences Θ and Ξ.*

7.4. Spectral-like duality: topological characterization of filters

In the present section we focus on the Spectral-like duality for $\mathbb{DH}^{\wedge}$-algebras, and we study the dual of notions such as implicative filter, irreducible implicative filter, optimal implicative filter, meet filter, irreducible meet filter, optimal meet filter and absorbent filter. We will use those results in the following section, where we compare the Spectral-like and the Priestley-style dualities for $\mathbb{DH}^{\wedge}$-algebras

From now on, let $\mathfrak{X} = \langle X, \widehat{X}, \tau_{\kappa}\rangle$ be a $\mathbb{DH}^{\wedge}$-Spectral space, and consider the following maps:

$$f : \mathcal{C}(X) \longrightarrow \mathrm{Fi}_{\to}(\mathfrak{X}^{*}) \qquad\qquad g : \mathrm{Fi}_{\to}(\mathfrak{X}^{*}) \longrightarrow \mathcal{C}(X)$$

$$C \longmapsto \{U^c \in D(\mathfrak{X}) : C \subseteq U^c\} \qquad\qquad F \longmapsto \bigcap\{U^c : U^c \in F\}$$

where recall that $\mathcal{C}(X)$ denotes the collection of closed subsets of $\langle X, \tau_{\kappa}\rangle$, and $\mathfrak{X}^{*} = \langle D(\mathfrak{X}), \Rightarrow, \sqcap, X\rangle$. In [**15**] Celani et al. show that these maps are well defined, and moreover in Proposition 5.1 it is proven the following.

PROPOSITION 7.4.1. *The maps f and g establish a dual order isomorphism between $\langle \mathcal{C}(X), \subseteq\rangle$ and $\langle \mathrm{Fi}_{\to}(\mathfrak{X}^{*}), \subseteq\rangle$.*

Let us denote by $\mathcal{C}^{\mathrm{Irr}}(X)$ the collection of all irreducible closed subsets of X. By sobriety, we know that there is a one-to-one correspondence between the elements of X and the irreducible closed subsets of X, given by the map sending each element to its topological closure:

$$\mathrm{cl} : X \longrightarrow \mathcal{C}^{\mathrm{Irr}}(X)$$
$$x \longmapsto \mathrm{cl}(x)$$

Therefore, as irreducible closed subsets are precisely the irreducible elements of the lattice of closed subsets, ordered by reverse inclusion, the next proposition follows straightforwardly.

PROPOSITION 7.4.2. *There is a dual order isomorphism between* $\langle \mathcal{C}^{\mathrm{Irr}}(X), \subseteq\rangle$ *and* $\langle \mathrm{Irr}_{\to}(\mathfrak{X}^*), \subseteq\rangle$ *given by the maps* f *and* g.

Let us introduce now a new concept, that captures the dual property of being an optimal implicative filter.

DEFINITION 7.4.3. A closed subset $C \in \mathcal{C}(X)$ is *optimal* when for all $\mathcal{V}, \mathcal{U} \subseteq^{\omega} \kappa$, if $\bigcap \mathcal{U} \subseteq \bigcup \mathcal{V}$ and $C \cap U \neq \emptyset$ for all $U \in \mathcal{U}$, then $C \cap \bigcup \mathcal{V} \neq \emptyset$.

We denote by $\mathcal{C}^{\mathrm{Op}}(X)$ the collection of all optimal closed subsets of X.

LEMMA 7.4.4. *Let* $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$ *be an* $\mathbb{DH}^{\wedge}$*-Spectral space and let* $U_1, \dots, U_n \in \kappa$ *and* $V_1, \dots, V_m \in \kappa$ *for some* $n, m \in \omega$. *Then:*

$$V_1^c \cap \dots \cap V_m^c \subseteq U_1^c \cup \dots \cup U_n^c \quad \textit{iff} \quad {\uparrow}U_1^c \cap \dots \cap {\uparrow}U_n^c \subseteq \langle\{V_1^c, \dots, V_m^c\}\rangle.$$

PROOF. Assume first that ${\uparrow}U_1^c \cap \dots \cap {\uparrow}U_n^c \subseteq \langle\{V_1^c, \dots, V_m^c\}\rangle$, and suppose, towards a contradiction that there is $x \in \left(V_1^c \cap \dots \cap V_m^c\right) \setminus \left(U_1^c \cap \dots \cup U_n^c\right)$. By κ being a basis, there is $W \in \kappa$ such that $U_1^c \cap \dots \cup U_n^c \subseteq W^c$ and $x \notin W^c$. But then by assumption $W^c \in \langle\{V_1^c, \dots, V_m^c\}\rangle$, so in particular we have that $V_1^c \Rightarrow (\dots(V_m^c \Rightarrow W^c)\dots) = X$. Notice that for any $V, U \in \kappa$, if $x \in V^c$ and $x \in V^c \Rightarrow U^c = ({\downarrow}(V^c \cap U))^c$, then $x \in U^c$. Therefore, from $x \in V_1^c \cap \dots \cap V_m^c$ and $x \in V_1^c \Rightarrow (\dots(V_m^c \Rightarrow W^c)\dots)$ we obtain $x \in W^c$, a contradiction.

For the converse, assume that $V_1^c \cap \dots \cap V_m^c \subseteq U_1^c \cup \dots \cup U_n^c$, and suppose, towards a contradiction that ${\uparrow}U_1^c \cap \dots \cap {\uparrow}U_n^c \nsubseteq \langle\{V_1^c, \dots, V_m^c\}\rangle$. So we take $W^c \in \left({\uparrow}U_1^c \cap \dots \cap {\uparrow}U_n^c\right) \setminus \langle\{V_1^c, \dots, V_m^c\}\rangle$. On the one hand, we have $U_1^c \cup \dots \cup U_n^c \subseteq W^c$, so by assumption we get $V_1^c \cap \dots \cap V_m^c \subseteq W^c$. On the other hand, we have $V_1^c \Rightarrow (\dots(V_m^c \Rightarrow W^c)\dots) \neq X$. By convenience, for each $j \leq m$, let $Z_j := (V_j^c \Rightarrow (\dots(V_m^c \Rightarrow W^c)\dots))^c$. Then there is $x \notin V_1^c \Rightarrow Z_2^c = ({\downarrow}(V_1^c \cap Z_2))^c$, so there is $x_1 \in V_1^c \cap Z_2$ such that $x \leq x_1$. Similarly we obtain that for each $3 \leq j \leq m$, there is $x_{j-1} \in V_{v-1}^c \cap Z_j$ such that $x_{j-1} \leq x_j$. Therefore, we get $x \leq x_1 \leq \dots \leq x_m$ such that $x_j \in V_j^c$ for each $2 \leq j \leq m-1$ and $x_m \in W$. Now since closed subsets are up-sets, we have $x_m \in V_j^c$ for all $j \leq m$, hence $x_m \in \left(V_1^c \cap \dots \cap V_m^c\right) \setminus W^c$, a contradiction. □

COROLLARY 7.4.5. *Let* $\mathcal{U} \subseteq \kappa$ *be non-empty. The F-ideal of* $\mathfrak{X}^*$ *generated by* $\{U^c : U \in \mathcal{U}\}$ *is* $\{W^c \in D(\mathfrak{X}) : (\exists n \in \omega)(\exists U_1, \dots, U_n \in \mathcal{U})\, W^c \subseteq U_1^c \cup \dots \cup U_n^c\}$.

PROOF. Let $\mathcal{U} \subseteq \kappa$ and let

$$Z := \{W^c \in D(\mathfrak{X}) : (\exists n \in \omega)(\exists U_1, \dots, U_n \in \mathcal{U})\, W^c \subseteq U_1^c \cup \dots \cup U_n^c\}.$$

Clearly Z is included in the Frink ideal generated by $\{U^c : U \in \mathcal{U}\}$. For the reverse inclusion, let V^c be in that Frink ideal, so there are $n \in \omega$ and $U_1, \dots, U_n \in \mathcal{U}$ such that ${\uparrow}U_1^c \cap \dots \cap {\uparrow}U_n^c \subseteq {\uparrow}V^c$. By the previous lemma we get $V^c \subseteq U_1^c \cup \dots \cup U_n^c$, and so $V^c \in Z$, as required. □

PROPOSITION 7.4.6. *There is a dual order isomorphism between* $\langle \mathcal{C}^{\mathrm{Op}}(X), \subseteq\rangle$ *and* $\langle \mathrm{Op}_{\to}(\mathfrak{X}^*), \subseteq\rangle$ *given by the maps* f *and* g.

PROOF. First we show that for C an optimal closed subset of X, $f(C)$ is an $\to$-optimal implicative filter of $\mathfrak{X}^*$, by showing that $D(\mathfrak{X}) \setminus f(C)$ is a strong Frink ideal of $\mathfrak{X}^*$. Let $U_1^c, \dots, U_n^c \notin f(C)$ be such that ${\uparrow}U_1^c \cap \dots \cap {\uparrow}U_n^c \subseteq {\uparrow}W^c$ for some

$W \in \kappa$. Recall that $\uparrow W^c = \langle \{W^c\} \rangle$, so by Lemma 7.4.4 we get $W^c \subseteq U_1^c \cup \cdots \cup U_n^c$. Suppose, towards a contradiction, that $W^c \in f(C)$, i. e. $C \cap W = \emptyset$. As by assumption $C \cap U_i \neq \emptyset$ for all $i \leq n$ and since C is optimal closed, we get $U_1 \cap \cdots \cap U_n \nsubseteq W$, a contradiction.

Let now $U_1^c, \ldots, U_n^c \notin f(C)$ be such that $\uparrow U_1^c \cap \cdots \cap \uparrow U_n^c \subseteq \langle \{W_1^c, \ldots, W_m^c\} \rangle$ for some $W_1, \ldots, W_m \in \kappa$. From Lemma 7.4.4 we have $W_1^c \cap \cdots \cap W_m^c \subseteq U_1^c \cup \cdots \cup U_n^c$. Suppose, towards a contradiction, that $\langle \{W_1^c, \ldots, W_m^c\} \rangle \subseteq f(C)$. Then we obtain that for all $i \leq m$, $W_i^c \in f(C)$. Therefore $C \subseteq W_1^c \cap \cdots \cap W_m^c$, and since C is optimal closed, then $W_1^c \cap \cdots \cap W_m^c \nsubseteq U_1^c \cup \cdots \cup U_n^c$, a contradiction. We conclude that $\langle \{W_1^c, \ldots, W_m^c\} \rangle \nsubseteq f(C)$, and so $D(\mathfrak{X}) \setminus f(C)$ is a strong F-ideal, as required.

Now we show that for any $\to$-optimal implicative filter P of $\mathfrak{X}^*$, the subset $g(P)$ is an optimal closed subset of X. Let $\mathcal{U}, \mathcal{V} \subseteq^{\omega} \kappa$ be such that $g(P) \cap U \neq \emptyset$ for all $U \in \mathcal{U}$ and $\bigcap \mathcal{U} \subseteq \bigcup \mathcal{V}$. Then from Lemma 7.4.4 we get $\bigcap \{\uparrow U^c : U \in \mathcal{U}\} \subseteq \langle \mathcal{V} \rangle$. Notice that for all $U \in \kappa$, $U^c \in P$ if and only if $g(P) \cap U = \emptyset$. Therefore, by assumption we have $U^c \notin P$ for all $U \in \mathcal{U}$. Suppose, towards a contradiction, that $g(P) \subseteq \bigcup \mathcal{V}$. Then $V^c \in P$ for all $V \in \mathcal{V}$, and so $\langle \mathcal{V} \rangle \subseteq P$. Now since P is an $\to$-optimal implicative filter, we get $\bigcap \{\uparrow U^c : U \in \mathcal{U}\} \nsubseteq \langle \mathcal{V} \rangle$, a contradiction. We conclude, using Proposition 7.4.1 that f and g give us the required dual order isomorphism. □

We introduce one more concept, that captures the dual property of being a meet filter.

DEFINITION 7.4.7. A closed subset $C \in \mathcal{C}(X)$ is a $\wedge$-*closed* subset when $C = \uparrow(C \cap \widehat{X})$.

We denote by $\mathcal{C}_{\wedge}(X)$ the collection of all $\wedge$-closed subsets of X. Similarly, we denote by $\mathcal{C}_{\wedge}^{\mathrm{Irr}}(X)$ the collection of all irreducible $\wedge$-closed subsets, and by $\mathcal{C}_{\wedge}^{\mathrm{Op}}(X)$ the collection of all optimal $\wedge$-closed subsets.

PROPOSITION 7.4.8. *There is a dual order isomorphism between* $\langle \mathcal{C}_{\wedge}(X), \subseteq \rangle$ *and* $\langle \mathrm{Fi}_{\wedge}(\mathfrak{X}^*), \subseteq \rangle$ *given by the maps* f *and* g.

PROOF. First we show that for C a $\wedge$-closed subset of X, $f(C)$ is a meet filter of $\mathfrak{X}^*$. Since $f(C)$ is an implicative filter, it is an up-set, so we just have to show that it is closed under the meet operation. Let $U_1^c, U_2^c \in f(C)$. Recall that $U_1^c \sqcap U_2^c = \mathrm{cl}(U_1^c \cap U_2^c \cap \widehat{X}) = \uparrow(U_1^c \cap U_2^c \cap \widehat{X})$. So it only remains to show that $C \subseteq \uparrow(U_1^c \cap U_2^c \cap \widehat{X})$. By assumption $C \subseteq U_1^c \cap U_2^c$, so $C \cap \widehat{X} \subseteq U_1^c \cap U_2^c \cap \widehat{X}$, and therefore, using that C is $\wedge$-closed, we get $C = \uparrow(C \cap \widehat{X}) \subseteq \uparrow(U_1^c \cap U_2^c \cap \widehat{X})$, as required.

Now we show that for F a meet filter of $\mathfrak{X}^*$, $g(F)$ is a $\wedge$-closed subset of X. Since closed subsets are up-sets, and we already know that $g(F)$ is closed, we just need to show that $g(F) \subseteq \uparrow(g(F) \cap \widehat{X})$. Let $x \in g(F)$. Then $\mathrm{cl}(x) \subseteq g(F)$, and then $F \subseteq f(\mathrm{cl}(x))$. Since $\mathrm{cl}(x)$ is an irreducible closed subset, then by Proposition 7.4.2 $f(\mathrm{cl}(x)) \in \mathrm{Irr}_{\to}(\mathfrak{X}^*)$, and so $f(\mathrm{cl}(x))^c$ is an order ideal such that $F \cap f(\mathrm{cl}(x))^c = \emptyset$. Then by Lemma 2.3.3, there is $G \in \mathrm{Irr}_{\wedge}(\mathfrak{X}^*)$ such that $F \subseteq G$ and $G \cap f(\mathrm{cl}(x))^c = \emptyset$. Now since $\mathrm{Irr}_{\wedge}(\mathfrak{X}^*) \subseteq \mathrm{Irr}_{\to}(\mathfrak{X}^*)$, from Proposition 7.4.2 again, we get that there is $x' \in X$ such that $f(\mathrm{cl}(x')) = G$. Notice that $f(\mathrm{cl}(x')) = \{U \in D(\mathfrak{X}) : \mathrm{cl}(x') \subseteq U\} = \{U \in D(\mathfrak{X}) : x' \in U\} = \varepsilon_{\mathfrak{X}}(x)$. Then by Corollary 7.1.10 we have that $\mathrm{Irr}_{\wedge}(\mathfrak{X}^*) =$

$\varepsilon_{\mathfrak{X}}[\widehat{X}]$, so we conclude that $x' \in \widehat{X}$. Now from $F \subseteq G = f(\mathrm{cl}(x'))$ we get $\mathrm{cl}(x') \subseteq g(F)$, so $x' \in g(F)$, and from $G \cap f(\mathrm{cl}(x))^c = \emptyset$, we obtain $f(\mathrm{cl}(x')) \subseteq f(\mathrm{cl}(x))$ and so $\mathrm{cl}(x) \subseteq \mathrm{cl}(x')$, i. e. $x' \leq x$. Hence $x \in \uparrow(g(F) \cap \widehat{X})$, as required. □

COROLLARY 7.4.9. *There is a dual order isomorphism between* $\langle \mathcal{C}^{\mathrm{Irr}}_{\wedge}(X), \subseteq \rangle$ *and* $\langle \mathrm{Irr}_{\wedge}(\mathfrak{X}^*), \subseteq \rangle$ *given by the maps* f *and* g.

PROOF. This follows from propositions 7.4.2, 7.4.8 and the fact that $\mathrm{Irr}_{\wedge}(\mathfrak{X}^*) = \mathrm{Fi}_{\wedge}(\mathfrak{X}^*) \cap \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^*)$ given in Corollary 6.5.19. □

COROLLARY 7.4.10. *There is a dual order isomorphism between* $\langle \mathcal{C}^{\mathrm{Op}}_{\wedge}(X), \subseteq \rangle$ *and* $\langle \mathrm{Op}_{\wedge}(\mathfrak{X}^*), \subseteq \rangle$ *given by the maps* f *and* g.

PROOF. This follows from propositions 7.4.6, 7.4.8 and the fact that $\mathrm{Op}_{\wedge}(\mathfrak{X}^*) = \mathrm{Fi}_{\wedge}(\mathfrak{X}^*) \cap \mathrm{Op}_{\rightarrow}(\mathfrak{X}^*)$ given in Corollary 6.5.22. □

Finally we identify what is the dual property of being an absorbent filter (see definition in page 137).

DEFINITION 7.4.11. A closed subset $C \subseteq \mathcal{C}(X)$ is *absorbent* when for all $U \in \kappa$, $C \cap U^c = \mathrm{cl}(C \cap U^c \cap \widehat{X})$.

We denote by $\mathcal{C}^{\mathrm{Ab}}(X)$ the collection of all absorbent closed subsets of X.

PROPOSITION 7.4.12. *There is a dual order isomorphism between* $\langle \mathcal{C}^{\mathrm{Ab}}(X), \subseteq \rangle$ *and* $\langle \mathrm{Ab}(\mathfrak{X}^*), \subseteq \rangle$ *given by the maps* f *and* g.

PROOF. First we show that for C an absorbent closed subset of X, $f(C)$ is an absorbent filter of $\mathfrak{X}^*$. Using the definition, we show that for any $U^c \in f(C)$ and any $V^c \in D(\mathfrak{X})$, we have $V^c \Rightarrow (U^c \sqcap V^c) \in f(C)$. By definition of f, it is enough to show that $C \subseteq V^c \Rightarrow (U^c \sqcap V^c)$. By hypothesis $C \cap V^c \subseteq \mathrm{cl}(C \cap V^c \cap \widehat{X})$ and by assumption $C \subseteq U^c$. Then we have $C \cap V^c \subseteq \mathrm{cl}(U^c \cap V^c \cap \widehat{X}) = U^c \sqcap V^c$. Thus $C \cap V^c \cap (U^c \sqcap V^c)^c = \emptyset$, and since C is closed, it is an up-set, and this implies $C \cap \mathrm{sat}(V^c \cap (U^c \sqcap V^c))^c = \emptyset$, i. e. $C \subseteq (\mathrm{sat}(V^c \cap (U^c \sqcap V^c)^c))^c = V^c \Rightarrow (U^c \sqcap V^c)$, as required.

Now we show that for F an absorbent filter of $\mathfrak{X}^*$, $g(F)$ is an absorbent closed subset of X, so let $U \in \kappa$. If $U^c \in F$, then $g(F) \cap U^c = g(F)$, and since F is a meet filter, by Proposition 7.4.8 we know that $g(F)$ is $\wedge$-closed, and therefore $g(F) \cap U^c = g(F) = \mathrm{cl}(g(F) \cap \widehat{X}) = \mathrm{cl}(g(F) \cap U^c \cap \widehat{X})$. Assume that $U^c \notin F$. Then by Proposition 6.5.25, the implicative filter $\langle F \cup \uparrow U^c \rangle$ is a meet filter. We show that $g(F) \cap U^c \subseteq \mathrm{cl}(g(F) \cap U^c \cap \widehat{X})$, since the reverse inclusion is immediate. Let $x \in g(F) \cap U^c$. Then for the irreducible closed subset $\mathrm{cl}(x)$ we have that $F \cup U^c \subseteq f(\mathrm{cl}(x))$ and by Proposition 7.4.2, $f(\mathrm{cl}(x))$ is an irreducible implicative filter of $\mathfrak{X}^*$. So by 2.4.11 we have that $D(\mathfrak{X}) \setminus f(\mathrm{cl}(x)))$ is an order ideal of $\mathfrak{X}^*$, and clearly it is disjoint from the meet filter $\langle F \cup \uparrow U^c \rangle$. Therefore, by Lemma 2.3.3 there is $F' \in \mathrm{Irr}_{\wedge}(\mathfrak{X}^*)$ such that $\langle F \cup \uparrow U^c \rangle \subseteq F'$ and $F' \subseteq f(\mathrm{cl}(x))$. Since F' is irreducible meet filter, there is $y \in \widehat{X}$ such that $f(\mathrm{cl}(y)) = F'$. On the one hand, we have $U^c \cup F \subseteq F'$, and this implies $y \in g(F) \cap U^c \cap \widehat{X}$. On the other hand, we have $f(\mathrm{cl}(y)) = F' \subseteq f(\mathrm{cl}(x))$, and so $y \leq x$. We conclude that $x \in \uparrow(g(F) \cap U^c \cap \widehat{X}) = \mathrm{cl}(g(F) \cap U^c \cap \widehat{X})$, as required. □

TABLE 10. Spectral-duals of filters of $\mathbb{DH}^{\wedge}$-algebras.

FILTERS	SPECTRAL-DUAL CLOSED SUBSETS
$\mathrm{Fi}_{\to}(\mathbf{A})$	closed subsets of the dual space $\mathcal{C}(X)$
$\mathrm{Irr}_{\to}(\mathbf{A})$	irreducible closed subsets of the dual space $\mathcal{C}^{\mathrm{Irr}}(X)$
$\mathrm{Op}_{\to}(\mathbf{A})$	optimal closed subsets of the dual space $\mathcal{C}^{\mathrm{Op}}(X)$, i. e. $C \in \mathcal{C}(X)$ such that for all $\mathcal{V}, \mathcal{U} \subseteq^{\omega} \kappa$, if $\bigcap \mathcal{U} \subseteq \bigcup \mathcal{V}$ and $C \cap U \neq \emptyset$ for all $U \in \mathcal{U}$, then $C \cap \bigcup \mathcal{V} \neq \emptyset$
$\mathrm{Fi}_{\wedge}(\mathbf{A})$	$\wedge$-closed subsets of the dual space $\mathcal{C}_{\wedge}(X)$, i. e. $C \in \mathcal{C}(X)$ such that $C = \uparrow(C \cap \widehat{X})$
$\mathrm{Irr}_{\wedge}(\mathbf{A})$	irreducible $\wedge$-closed subsets of the dual space $\mathcal{C}^{\mathrm{Irr}}_{\wedge}(X)$
$\mathrm{Op}_{\wedge}(\mathbf{A})$	optimal $\wedge$-closed subsets of the dual space $\mathcal{C}^{\mathrm{Op}}_{\wedge}(X)$
$\mathrm{Ab}(\mathbf{A})$	absorbent closed subsets of the dual space $\mathcal{C}^{\mathrm{Ab}}(X)$, i. e. $C \in \mathcal{C}(X)$ such that for all $U \in \kappa$, $C \cap U^c = \mathrm{cl}(C \cap U^c \cap \widehat{X})$

Let us summarize the results in this section in Table 10, where $\mathbf{A}$ denotes an arbitrary $\mathbb{DH}^{\wedge}$-algebra.

7.5. Comparison between both dualities

In the present section we carry out a comparison between the Spectral-like and the Priestley-style dualities for $\mathsf{DH}^{\wedge}_{H}$. It is remarkable that, while Priestley and Spectral dual objects of distributive lattices are built on the same base set, namely the set of prime filters of the distributive lattice, this is not the case for weaker settings such as distributive semilattices, Hilbert algebras or $\mathbb{DH}^{\wedge}$-algebras. In particular, $\mathbb{DH}^{\wedge}$-Spectral spaces are built on the set of irreducible implicative filters of the $\mathbb{DH}^{\wedge}$-algebra, whereas $\mathbb{DH}^{\wedge}$-Priestley spaces are built on the set of optimal implicative filters of the $\mathbb{DH}^{\wedge}$-algebra. This makes the comparison between both dualities interesting.

The category $\mathsf{DH}^{\wedge}_{S}$ is dually equivalent to the categories $\mathsf{Sp}^{\mathbb{DH}^{\wedge}}_{M}$ and $\mathsf{Pr}^{\mathbb{DH}^{\wedge}}_{M}$. Therefore these two latter categories are equivalent. We proceed in what follows to explicitly define the functors from one category to the other, that are obtained as the concatenation of both dual constructions, passing through the algebraic one. The definition of the functor from $\mathsf{Pr}^{\mathbb{DH}^{\wedge}}_{M}$ to $\mathsf{Sp}^{\mathbb{DH}^{\wedge}}_{M}$ is relatively simple to obtain. However the definition of the functor that goes the other way around is considerably more involved. Similarly, as $\mathsf{DH}^{\wedge}_{H}$ is dually equivalent to the categories $\mathsf{Sp}^{\mathbb{DH}^{\wedge}}_{F}$ and $\mathsf{Pr}^{\mathbb{DH}^{\wedge}}_{F}$, these two later categories are again equivalent. And we also explicitly define the functors that witness such equivalence. The situation is analogous to the case of distributive semilattices, for which the *easy* direction was pointed out by Bezhanishvili and Jansana in [**5**].

7.5.1. From the Priestley-style duality to the Spectral-like duality. Let $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ be a $\mathbb{DH}^{\wedge}$-Priestley space. Recall that the dual of $\mathfrak{X}$ is the algebra $\mathfrak{X}^{\bullet} := \langle B, \Rightarrow, \sqcap, X\rangle$ and the Spectral dual of $\mathfrak{X}^{\bullet}$ is $\mathfrak{Irr}(\mathfrak{X}^{\bullet}) := \langle \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^{\bullet}), \tau_{\kappa_{\mathfrak{X}^{\bullet}}}, \mathrm{Irr}_{\wedge}(\mathfrak{X}^{\bullet})\rangle$. We aim to provide a more transparent construction of this structure that is equivalent to $\mathfrak{X}$. We recall that the dense subset X_B is given by $\{x \in X : \{U \in B : x \notin U\}$ is non-empty and up-directed$\}$. Let us consider the collection

$$\kappa_B := \{X_B \setminus U : U \in B\}.$$

First we show that κ_B is a basis for a topological space on X_B. Let $U, V \in B$ and $x \in (X_B \setminus U) \cap (X_B \setminus V)$. Then $x \in X_B$ and $x \notin U, V$. By condition (H13$'$), there exists $W \in B$ such that $x \notin W$ and $U, V \subseteq W$. Therefore $x \in X_B \setminus W \subseteq (X_B \setminus U) \cap (X_B \setminus V)$, as required.

Thus we take κ_B as a basis for a topology τ_{κ_B} on X_B. We claim that the structure

$$\langle X_B, X_B \cap \widehat{X}, \tau_{\kappa_B}\rangle$$

is a $\mathbb{DH}^{\wedge}$-Spectral space

In page 161 we introduced the map $\xi_{\mathfrak{X}} : X \longrightarrow \mathrm{Op}_{\rightarrow}(\mathfrak{X}^{\bullet})$ given by $\xi_{\mathfrak{X}}(x) := \{U \in B : x \in U\}$ that satisfies $\xi_{\mathfrak{X}}[X_B] = \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^{\bullet})$. Moreover, notice that for each $U \in B$,

$$\begin{aligned}\xi_{\mathfrak{X}}[X_B \setminus U] &= \{\xi_{\mathfrak{X}}(x) : x \in X_B \setminus U\} = \{\xi_{\mathfrak{X}}(x) : x \in X_B, U \notin \xi_{\mathfrak{X}}(x)\}\\ &= \{P \in \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^{\bullet}) : U \notin P\} = \mathfrak{X}^{\bullet} \setminus \psi_{\mathfrak{X}^{\bullet}}(U),\end{aligned}$$

where recall that the map $\psi_{\mathfrak{X}^{\bullet}} : B \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Irr}_{\rightarrow}(\mathfrak{X}^{\bullet}))$, introduced in page 73, is given by $\psi_{\mathfrak{X}^{\bullet}}(U) := \{P \in \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^{\bullet}) : U \in P\}$.

From this fact it follows that $\xi_{\mathfrak{X}}$ is a continuous and open function between $\langle X_B, \tau_{\kappa_B}\rangle$ and $\langle \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^{\bullet}), \tau_{\kappa_{\mathfrak{X}^{\bullet}}}\rangle$, and hence it is a homeomorphism. Moreover, by Theorem 7.1.22 and Corollary 6.5.19 we obtain that $\xi_{\mathfrak{X}}[X_B \cap \widehat{X}] = \mathrm{Irr}_{\wedge}(\mathfrak{X}^{\bullet})$.

Corollary 7.5.1. *For any $\mathbb{DH}^{\wedge}$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$, $\xi_{\mathfrak{X}}$ is a homeomorphism between $\langle X_B, \tau_{\kappa_B}\rangle$ and $\langle \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^{\bullet}), \tau_{\kappa_{\mathfrak{X}^{\bullet}}}\rangle$ such that $\xi_{\mathfrak{X}}[X_B \cap \widehat{X}] = \mathrm{Irr}_{\wedge}(\mathfrak{X}^{\bullet})$.*

Corollary 7.5.2. *For any $\mathbb{DH}^{\wedge}$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$, the structure $\mathfrak{X}_B := \langle X_B, X_B \cap \widehat{X}, \tau_{\kappa_B}\rangle$ is a $\mathbb{DH}^{\wedge}$-Spectral space.*

Now we move to consider morphisms. We claim that for any $\mathbb{DH}^{\wedge}$-Spectral morphism $R \subseteq X_1 \times X_2$, the relation

$$R \cap (X_{B_1} \times X_{B_2}) \subseteq X_{B_1} \times X_{B_2}$$

is a $\mathbb{DH}^{\wedge}$-Spectral morphism. The following lemma concerning dual spaces of Hilbert algebras is all we need to get our claim.

Lemma 7.5.3. *Let $R \subseteq X_1 \times X_2$ a (functional) $\mathbb{H}$-Priestley morphism between $\mathbb{H}$-Priestley spaces $\mathfrak{X}_1 = \langle X_1, \tau_1, \leq_1, B_1\rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_2, \leq_2, B_2\rangle$. Then the relation $R \cap (X_{B_1} \times X_{B_2})$ is a (functional) $\mathbb{H}$-relation between $\mathbb{H}$-spaces $\langle X_{B_1}, \tau_{\kappa_1}\rangle$ and $\langle X_{B_2}, \tau_{\kappa_2}\rangle$.*

PROOF. As $\kappa_i = \{X_{B_i} \backslash U : U \in B_i\}$, from condition (HR3) it follows condition (HR1). For any $x \in X_{B_1}$, $R(x) \cap X_{B_2}$ is closed in X_{B_2}, since by (HR4), for each $y \in X_{B_2}$ such that $y \notin R(x)$, there is an open $U \in \kappa_2$ such that $y \in U$ and $R(x) \cap U = \emptyset$. Hence condition (HR2) also holds, and so $R \cap (X_{B_1} \times X_{B_2})$ is an $\mathbb{H}$-relation. Finally, from condition (HF′) it immediately follows that it is functional. □

PROPOSITION 7.5.4. *Let $R \subseteq X_1 \times X_2$ a $\mathbb{DH}^\wedge$-Priestley morphism between $\mathbb{DH}^\wedge$-Priestley spaces $\mathfrak{X}_1$ and $\mathfrak{X}_2$. Then $R \cap (X_{B_1} \times X_{B_2})$ is a $\mathbb{DH}^\wedge$-Spectral morphism between $\mathbb{DH}^\wedge$-Spectral spaces $\mathfrak{X}_{B_1}$ and $\mathfrak{X}_{B_2}$.*

PROOF. This follows from Lemma 7.5.3 and the fact that condition (DH^R4) follows immediately from condition (DH^R2). □

We define the functor $\mathfrak{F} : \mathsf{Pr}_M^{\mathbb{DH}^\wedge} \longrightarrow \mathsf{Sp}_M^{\mathbb{DH}^\wedge}$ such that for any $\mathbb{DH}^\wedge$-Priestley spaces $\mathfrak{X}, \mathfrak{X}_1, \mathfrak{X}_2$ and any $\mathbb{DH}^\wedge$-Priestley morphism $R \subseteq X_1 \times X_2$:

$$\mathfrak{F}(\mathfrak{X}) := \mathfrak{X}_B = \langle X_B, \widehat{X} \cap X_B, \tau_{\kappa_B} \rangle,$$
$$\mathfrak{F}(R) := R \cap (X_{B_1} \times X_{B_2}).$$

where notice that $\mathfrak{F}(\leq_X) = \leq_{X_B} = \leq_{\mathfrak{F}(X)}$, so the functor preserves the identity morphism for $\mathsf{Pr}_M^{\mathbb{DH}^\wedge}$. Let us check that it also preserves composition of morphisms in $\mathsf{Pr}_M^{\mathbb{DH}^\wedge}$.

LEMMA 7.5.5. *Let $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$ be $\mathbb{H}$-Priestley spaces and $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be $\mathbb{H}$-Priestley morphisms. Then for any $x \in X_{B_1}$ and any $z \in X_{B_3}$:*

$$(x, z) \in (S \star R) \cap (X_{B_1} \times X_{B_3}) \quad \textit{iff} \quad (x, z) \in S \circ R.$$

PROOF. Recall that for any $x_1 \in X_1$ and $x_3 \in X_3$, $x_3 \in (S \star R)(x_1)$ if and only if for all $U \in B_3$, if $S \circ R(x_1) \subseteq U$, then $x_3 \in U$. So in particular $S \circ R(x_1) \subseteq (S \star R)(x_1)$, and so one of the directions is straightforward. For the converse, let $x \in X_{B_1}$ and $z \in X_{B_3}$ be such that $z \notin S \circ R(x)$. Then as $S \circ R(x) \cap X_{B_3}$ is closed in X_{B_3} by (HR2), there is a basic open that contains z and is disjoint from it. So by definition of τ_3 and denseness of X_{B_3} in X_3, there is $U \in B_3$ such that $S \circ R \subseteq U$ and $x \notin U$, hence $z \notin (S \star R)(x)$. □

By the previous lemma we get that for any $\mathbb{DH}^\wedge$-Priestley morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, $\mathfrak{F}(S \star R) = \mathfrak{F}(S) \circ \mathfrak{F}(R)$, so the functor preserves composition of morphisms, and hence it is well defined.

7.5.2. From the Spectral-like duality to the Priestley-style duality. Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$ be a $\mathbb{DH}^\wedge$-Spectral space. Recall that the dual of $\mathfrak{X}$ is the algebra $\mathfrak{X}^* := \langle D(\mathfrak{X}), \Rightarrow, \sqcap, X \rangle$, and the Priestley dual of $\mathfrak{X}^*$ is $\mathfrak{O}\mathrm{p}(\mathfrak{X}^*) := \langle \mathrm{Op}_\rightarrow(\mathfrak{X}^*), \tau_{\mathfrak{X}^*}, \subseteq, \vartheta_{\mathfrak{X}^*}[D(\mathfrak{X})], \mathrm{Op}_\wedge(\mathfrak{X}^*) \rangle$. As before, we aim to provide a more transparent construction of this structure equivalent to $\mathfrak{X}$. For getting a $\mathbb{DH}^\wedge$-Priestley space out of it, we need to add extra points. Notice that for each $x \in X$, $\mathrm{cl}(x) \in \mathcal{C}^{\mathrm{Irr}}(X)$ and all irreducible closeds have this form. For each $C \in \mathcal{C}^{\mathrm{Op}}(X) \setminus \mathcal{C}^{\mathrm{Irr}}(X)$, we add a new point x_C to the collection X. Then we obtain the collections

$$X' := X \cup \{x_C : C \in \mathcal{C}^{\mathrm{Op}}(X) \setminus \mathcal{C}^{\mathrm{Irr}}(X)\},$$
$$\widehat{X}' := \widehat{X} \cup \{x_C : C \in \mathcal{C}^{\mathrm{Op}}_{\wedge}(X) \setminus \mathcal{C}^{\mathrm{Irr}}_{\wedge}(X)\}.$$

For convenience, sometimes we refer to x by $x_{\mathrm{cl}(x)}$. So it is clear that all the elements of X' and $\widehat{X}'$ have the form x_C for some $C \in \mathcal{C}^{\mathrm{Op}}(X)$. Notice that X' can be defined for any $\mathbb{H}$-space $\langle X, \tau_\kappa\rangle$. Moreover, an order can be defined on X' as follows. For each $x_C, x_{C'} \in X'$:

$$x_C \leq x_{C'} \quad \text{iff} \quad C' \subseteq C.$$

Notice that this order extends the dual of the specialization order of X. Let us consider the map $\eta : \kappa \longrightarrow \mathcal{P}(X')$ given by:

$$\eta(U) := \{x_C \in X' : C \subseteq U^c\}.$$

For each $U \in \kappa$, we denote by $\eta(U)^c$ the set $X' \setminus \eta(U)$. Consider the topology τ' on X' having as subbasis the collection:

$$\{\eta(U) : U \in \kappa\} \cup \{\eta(U)^c : U \in \kappa\}.$$

We claim that the structure

$$\langle X', \tau', \leq, \eta[\kappa], \widehat{X}'\rangle$$

is a $\mathbb{DH}^{\wedge}$-Priestley space. We prove the claim by showing that there is an order homeomorphism h between $\langle X', \tau', \leq\rangle$, and $\langle \mathrm{Op}_{\rightarrow}(\mathfrak{X}^*), \tau_{\mathfrak{X}^*}, \subseteq\rangle$, such that $h[\eta[\kappa]] = \vartheta_{\mathfrak{X}^*}[D(\mathfrak{X})]$ and $h[\widehat{X}'] = \mathrm{Op}_{\wedge}(\mathfrak{X}^*)$.

Recall that by definition, the dual of the $\mathbb{DH}^{\wedge}$-Spectral space $\mathfrak{X}$ is the algebra $\mathfrak{X}^* := \langle D(\mathfrak{X}), \Rightarrow, \sqcap, X\rangle$, and the map $\vartheta_{\mathfrak{X}^*} : D(\mathfrak{X}) \longrightarrow \mathcal{P}^{\uparrow}(\mathrm{Op}_{\rightarrow}(\mathfrak{X}^*))$ introduced in page 73 is given by $\vartheta_{\mathfrak{X}^*}(U) := \{P \in \mathrm{Op}_{\rightarrow}(\mathfrak{X}^*) : U \in P\}$.

And recall also the map $f : \mathcal{C}(X) \longrightarrow \mathrm{Fi}_{\rightarrow}(\mathfrak{X}^*)$, defined in page 174, that assigns to each closed C, the set $\{U^c \in D(\mathfrak{X}) : C \subseteq U^c\}$. We define a map $h : X' \longrightarrow \mathrm{Fi}_{\rightarrow}(\mathfrak{X}^*)$ such that:

$$h(x_C) := f(C) = \{U^c \in D(\mathfrak{X}) : C \subseteq U^c\}.$$

By Proposition 7.4.6 and the definition of X' we know that h is well defined, and that it is in fact an isomorphism, such that for each $P \in \mathrm{Op}_{\rightarrow}(\mathfrak{X}^*)$, $h^{-1}(P) = x_{g(P)}$, where recall that $g : \mathrm{Op}_{\rightarrow}(\mathfrak{X}^*) \longrightarrow \mathcal{C}(X)$ was defined in page 174. Moreover, from the definition of the order in X', we get that h is order preserving. Notice that for any $P \in \mathrm{Op}_{\rightarrow}(\mathfrak{X}^*)$:

$$P \in h[\eta(U)] \quad \text{iff} \quad x_{g(P)} \in \eta(U) \quad \text{iff} \quad g(P) \subseteq U^c \quad \text{iff} \quad U^c \in P \quad \text{iff} \quad P \in \vartheta_{\mathfrak{X}^*}(U^c).$$

Therefore, $h[\eta(U)] = \vartheta_{\mathfrak{X}^*}(U^c)$ for every $U \in \kappa$. From this fact it follows that h is a continuos and open function between $\langle X', \tau'\rangle$ and $\langle \mathrm{Op}_{\rightarrow}(\mathfrak{X}^*), \tau_{\mathfrak{X}^*}\rangle$, and hence it is a homeomorphism. It also follows that $h[\eta[\kappa]] = \vartheta_{\mathfrak{X}^*}[D(\mathfrak{X})]$. Furthermore, by Corollary 7.4.10 we conclude that $h[\widehat{X}'] = \mathrm{Op}_{\wedge}(\mathfrak{X}^*)$.

COROLLARY 7.5.6. *For any $\mathbb{DH}^{\wedge}$-Spectral space $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$, h is an order homeomorphism between $\langle X', \tau', \leq\rangle$ and $\langle \mathrm{Op}_{\rightarrow}(\mathfrak{X}^*), \tau_{\mathfrak{X}^*}, \subseteq\rangle$, such that $h[\widehat{X}'] = \mathrm{Op}_{\wedge}(\mathfrak{X}^*)$ and $h[\eta[\kappa]] = \vartheta_{\mathfrak{X}^*}[D(\mathfrak{X})]$.*

THEOREM 7.5.7. *For any $\mathbb{DH}^{\wedge}$-Spectral space $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$, the structure*

$$\mathfrak{X}' := \langle X', \tau', \leq, \eta[\kappa], \widehat{X}' \rangle$$

is a $\mathbb{DH}^{\wedge}$-Priestley space. Moreover, X is the dense subset $X'_{\eta[\kappa]}$.

PROOF. The first statement follows from the previous corollary and Corollary 7.1.14. Let us prove that X is the dense subset $X'_{\eta[\kappa]}$ given by the definition of $\mathbb{DH}^{\wedge}$-Priestley space. Recall that

$$X'_{\eta[\kappa]} = \{x_C \in X' : \{\eta(U) : U \in \kappa, x_C \notin \eta(U)\} \text{ is non-empty and up-directed}\}.$$

First we show that $X \subseteq X'_{\eta[\kappa]}$. This follows from κ being a basis for τ. Let $x \in X$. Notice that $\{\eta(U) : U \in \kappa, x \notin \eta(U)\} = \{\eta(U) : U \in \kappa, \mathrm{cl}(x) \nsubseteq U^c\}$. By κ being a basis, there is $U \in \kappa$ such that $x \in U$. Then $\mathrm{cl}(x) \nsubseteq U^c$ and this implies that the set $\{\eta(U) : U \in \kappa, x \in \eta(U)\}$ is non-empty. Let $U, V \in \kappa$ be such that $x \notin \eta(U), \eta(V)$. Then $\mathrm{cl}(x) \subseteq U^c, V^c$, and so $x \in U \cap V$. Since κ is a basis for τ, there is $W \in \kappa$ such that $x \in W \subseteq U \cap V$. And this implies $\eta(U), \eta(V) \subseteq \eta(W)$ and $\mathrm{cl}(x) \nsubseteq W^c$. Hence $\{\eta(U) : U \in \kappa, x \in \eta(U)\}$ is up-directed.

In order to prove the reverse inclusion, let $x_C \in X'_{\eta[\kappa]}$. We show that C is an irreducible closed subset of X, as from this it follows that $x_C = x_{\mathrm{cl}(x_C)} \in X$. Let C_1, C_2 be two closed subsets of X. Assume $C \subseteq C_1 \cup C_2$ and suppose, towards a contradiction, that $C \nsubseteq C_1$ and $C \nsubseteq C_2$. Then we have $x_{C_1} \nleq x_C$, and using that $\mathfrak{Op}(\mathfrak{X}^*)$ is a $\mathbb{DH}^{\wedge}$-Priestley space, by condition (H12) there is $U_1 \in \kappa$ such that $x_{C_1} \in \eta(U_1)$ and $x_C \notin \eta(U_1)$. Similarly we have $x_{C_2} \nleq x_C$, and then there is $U_2 \in \kappa$ such that $x_{C_2} \in \eta(U_2)$ and $x_C \notin \eta(U_2)$. Thus by hypothesis, from $x_C \in X'_{\eta[\kappa]}$ we obtain that there is $W \in \kappa$ such that $\eta(U_1), \eta(U_2) \subseteq \eta(W)$ and $x_C \notin \eta(W)$. This implies, on the one hand, that $C_1, C_2 \subseteq W^c$, and on the other hand, that $C \nsubseteq W^c$, a contradiction. $\square$

The previous theorem gives us how the functor we are looking for acts on the objects. Now we move to morphisms. In fact, we focus on dual morphisms of homomorphisms between Hilbert algebras, as from defining an $\mathbb{H}$-Priestley morphism out of an $\mathbb{H}$-relation, it follows straightforwardly the result we need. From now on, let $\mathfrak{X}_1 = \langle X_1, \tau_{\kappa_1} \rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_{\kappa_2} \rangle$ be two $\mathbb{H}$-spaces and let $R \subseteq X_1 \times X_2$ be an $\mathbb{H}$-relation between them. We define the relation $\overline{R} \subseteq X_1' \times X_2'$ as follows:

$$(x_{C_1}, x_{C_2}) \in \overline{R} \quad \text{iff} \quad (\forall V \in \kappa_2) \text{ if } R[C_1] \subseteq V^c, \text{ then } C_2 \subseteq V^c,$$

where recall that $R[C_1] = \bigcup\{R[x] : x \in C_1\}$. We claim that $\overline{R}$ is an $\mathbb{H}$-Priestley morphism between the $\mathbb{H}$-Priestley spaces $\langle X_1', \tau_1', \leq, \eta_1[\kappa_1] \rangle$ and $\langle X_2', \tau_2', \leq, \eta_2[\kappa_2] \rangle$. In order to show this, we prove first some useful lemmas:

LEMMA 7.5.8. *Let $R \subseteq X_1 \times X_2$ be an $\mathbb{H}$-relation between $\mathbb{H}$-spaces $\mathfrak{X}_1 = \langle X_1, \tau_{\kappa_1} \rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_{\kappa_2} \rangle$. Then for all $x_1 \in X_1$ and $x_2 \in X_2$:*

$$(x_1, x_2) \in R \quad \textit{iff} \quad (x_1, x_2) \in \overline{R}.$$

Moreover for all $x_{C_1} \in X_1'$ and all $x_2 \in R[C_1]$, it holds that $(x_{C_1}, x_2) \in \overline{R}$.

PROOF. Recall that for each $x \in X$, $x = x_{\mathrm{cl}(x)}$, so for the first statement, the inclusion from left to right is immediate. For the converse, assume $(x_1, x_2) \in \overline{R}$ and suppose, towards a contradiction, that $(x_1, x_2) \notin R$. So we have $x_2 \notin R(x_1)$, that is a closed subset by (HR2), and then there is $U \in \kappa_2$ such that $x_2 \in U$ and $R(x_1) \subseteq U^c$. From $x_2 \notin U^c$, we get $\mathrm{cl}(x_2) \nsubseteq U^c$, and then by assumption, $R(\mathrm{cl}(x_1)) \nsubseteq U^c$. From $R(x_1) \subseteq U^c$, we get $x_1 \in \Box_R(U^c)$, that is closed by (HR1), and so $\mathrm{cl}(x_1) \subseteq \Box_R(U^c)$. It follows that $R[\mathrm{cl}(x_1)] \subseteq U^c$, a contradiction. The second statement follows easily. $\Box$

LEMMA 7.5.9. *Let $R \subseteq X_1 \times X_2$ be a functional $\mathbb{H}$-relation between $\mathbb{H}$-spaces $\mathfrak{X}_1 = \langle X_1, \tau_{\kappa_1}\rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_{\kappa_2}\rangle$. Then for all $x_{C_1} \in X_1'$ and any $U \in \kappa_2$:*

$$\overline{R}(x_{C_1}) \subseteq \eta_2(U) \quad \textit{iff} \quad R[C_1] \subseteq U^c.$$

PROOF. Assume first that $\overline{R}(x_{C_1}) \subseteq \eta_2(U)$ and let $y \in R[C_1]$. Then by Lemma 7.5.8 we have $(x_{C_1}, y) \in \overline{R}$, and then by assumption $y \in \eta_2(U)$. Thus by definition of η_2, $\mathrm{cl}(y) \subseteq U^c$, and hence $y \in U^c$, as required. For the converse, assume $R[C_1] \subseteq U^c$ and let $x_{C_2} \in \overline{R}(x_{C_1})$. Then by definition of $\overline{R}$, $C_2 \subseteq U^c$, i.e. $x_{C_2} \in \eta_2(U)$, as required. $\Box$

LEMMA 7.5.10. *For $R \subseteq X_1 \times X_2$ a (functional) $\mathbb{H}$-relation between $\mathbb{H}$-spaces $\mathfrak{X}_1 = \langle X_1, \tau_{\kappa_1}\rangle$ and $\mathfrak{X}_2 = \langle X_2, \tau_{\kappa_2}\rangle$, the relation $\overline{R}$ is a (functional) $\mathbb{H}$-Priestley morphism between the $\mathbb{H}$-Priestley spaces $\langle X_1', \tau_1', \leq, \eta_1[\kappa_1]\rangle$ and $\langle X_2', \tau_2', \leq, \eta_2[\kappa_2]\rangle$.*

PROOF. First we show that condition (HR3) holds, i.e. that $\Box_{\overline{R}}(\eta_2(U)) = \eta_1(\Box_R(U^c)^c)$ for all $U \in \kappa_2$. Let $U \in \kappa_2$. Recall that by (HR1), $(\Box_R(U^c))^c \in \kappa_1$. Notice that this follows from Lemma 7.5.9, as we have:

$$\begin{aligned} x_{C_1} \in \Box_{\overline{R}}(\eta_2(U)) \quad &\text{iff} \quad \overline{R}[x_{C_1}] \subseteq \eta_2(U) \quad \text{iff} \quad R[C_1] \subseteq U^c \\ &\text{iff} \quad C_1 \subseteq \Box_R(U^c) \quad \text{iff} \quad x_{C_1} \in \eta_1((\Box_R(U^c))^c). \end{aligned}$$

Now we show that condition (HR4) also holds, i.e. that if $(x_{C_1}, x_{C_2}) \notin \overline{R}$, then there is $U \in \kappa_2$ such that $x_{C_2} \notin \eta_2(U)$ and $\overline{R}(x_{C_1}) \subseteq \eta_2(U)$. Let $x_{C_1} \in X_1'$ and $x_{C_2} \in X_2'$ and assume $(x_{C_1}, x_{C_2}) \notin \overline{R}$. Then by definition of $\overline{R}$, there is $U \in \kappa_2$ such that $R[C_1] \subseteq U^c$ and $C_2 \nsubseteq U^c$. This implies $x_{C_2} \notin \eta_2(U)$ and $\overline{R}(x_{C_1}) \subseteq \eta_2(U)$, so we are done. Finally, condition (HF$'$) follows similarly from condition (HF). $\Box$

COROLLARY 7.5.11. *For $R \subseteq X_1 \times X_2$ a (functional) $\mathbb{DH}^\wedge$-Spectral morphism between $\mathbb{DH}^\wedge$-Spectral spaces $\mathfrak{X}_1 = \langle X_1, \widehat{X}_1, \tau_{\kappa_1}\rangle$ and $\mathfrak{X}_2 = \langle X_2, \widehat{X}_2, \tau_{\kappa_2}\rangle$, the relation $\overline{R}$ is a (functional) $\mathbb{DH}^\wedge$-Priestley morphism between the $\mathbb{DH}^\wedge$-Priestley spaces $\mathfrak{X}_1'$ and $\mathfrak{X}_2'$.*

PROOF. This follows from Lemma 7.5.10 and the fact that condition (DH$^\wedge$R4) follows immediately from condition (DH$^\wedge$R2). $\Box$

We are finally ready to define the functor $\mathfrak{G} : \mathsf{Sp}_M^{\mathbb{DH}^\wedge} \longrightarrow \mathsf{Pr}_M^{\mathbb{DH}^\wedge}$ such that for any $\mathbb{DH}^\wedge$-Spectral spaces $\mathfrak{X}, \mathfrak{X}_1, \mathfrak{X}_2$ and any $\mathbb{DH}^\wedge$-Spectral morphism $R \subseteq X_1 \times X_2$:

$$\begin{aligned} \mathfrak{G}(\mathfrak{X}) &:= \mathfrak{X}' = \langle X', \tau', \leq, \eta[\kappa], \widehat{X}'\rangle, \\ \mathfrak{G}(R) &:= \overline{R}. \end{aligned}$$

Clearly from (H12) it follows that $\mathfrak{G}(\leq_X) = \leq_{X'} = \leq_{\mathfrak{G}(\mathfrak{X})}$, so the functor preserves the identity morphism in $\mathsf{Sp}_M^{\mathbb{DH}^\wedge}$. We prove that $\mathfrak{G}$ preserves composition of morphisms in $\mathsf{Sp}_M^{\mathbb{DH}^\wedge}$ as well.

LEMMA 7.5.12. *Let $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$ be $\mathbb{H}$-spaces and $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$ be $\mathbb{H}$-relations. Then for any $x_{C_1} \in X_1'$ and any $U \in \kappa_2$,*

$$(S \circ R)[C_1] \subseteq U^c \quad \textit{iff} \quad (\overline{S} \circ \overline{R})(x_{C_1}) \subseteq \eta_2(U).$$

PROOF. Notice that by definition $(S \circ R)[C_1] \nsubseteq U^c$ if and only if $R[C_1] \nsubseteq \Box_B(U^c)$, and this holds if and only if there is $x_{C_2} \in \overline{R}(x_{C_1})$ such that $S[C_2] \nsubseteq U^c$. But this is equivalent to having $x_{C_2} \in \overline{R}(x_{C_1})$ and $x_{C_3} \in \overline{S}(x_{C_2})$ such that $C_3 \nsubseteq U^c$, i.e. $(\overline{S} \circ \overline{R})(x_{C_1}) \nsubseteq \eta_3(U)$, as required. $\Box$

By the previous lemma we get that for any $\mathbb{DH}^{\wedge}$-Spectral morphisms $R \subseteq X_1 \times X_2$ and $S \subseteq X_2 \times X_3$, $\mathfrak{G}(S \circ R) = \mathfrak{G}(S) \star \mathfrak{G}(R)$, since we have:

$$\begin{aligned}(x_{C_1}, x_{C_2}) \notin \overline{(S \circ R)} \quad &\text{iff} \quad \exists U \in \kappa_3\big((S \circ R)[C_1] \subseteq U^c \ \ \& \ \ C_2 \nsubseteq U^c\big)\\ &\text{iff} \quad \exists U \in \kappa_3\big((\overline{S} \circ \overline{R})[x_{C_1}] \subseteq \eta_3(U) \ \ \& \ \ x_{C_2} \notin \eta_3(U)\big)\\ &\text{iff} \quad (x_{C_1}, x_{C_2}) \notin (\overline{S} \star \overline{R}).\end{aligned}$$

Hence, the functor $\mathfrak{G}$ preserves composition of morphisms, and hence it is well defined.

7.5.3. Categorical equivalence. We finally introduce the natural isomorphisms involved in the equivalence of the categories $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$ and $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$. Let us consider first the endofunctor $\mathfrak{F}\mathfrak{G} : \mathsf{Sp}_M^{\mathbb{DH}^{\wedge}} \longrightarrow \mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$.

THEOREM 7.5.13. *$\mathfrak{F}\mathfrak{G}$ is the identity functor on $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$.*

PROOF. This follows easily from Theorem 7.5.7, where we proved that $X'_{\eta[\kappa]} = X$. This implies that $\widehat{X}' \cap X'_{\eta[\kappa]} = \widehat{X}' \cap X = \widehat{X}$ and it also follows that $\tau_{\kappa_{\eta[\kappa]}} = \tau_\kappa$, therefore $\mathfrak{F}\mathfrak{G}(\mathfrak{X}) = \langle X'_{\eta[\kappa]}, \widehat{X}' \cap X'_{\eta[\kappa]}, \tau_{\kappa_{\eta[\kappa]}}\rangle = \mathfrak{X}$. $\Box$

For each $\mathbb{DH}^{\wedge}$-Spectral space $\mathfrak{X}$, let $R_X \subseteq X \times X$ be the order associated with the space $\langle X, \tau_\kappa\rangle$. We know that R_X is the identity morphism for X, and so it is an isomorphism in $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$. Consider the family of morphisms in $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$:

$$\Phi := \big(R_X \subseteq X \times \mathfrak{F}\mathfrak{G}(\mathfrak{X})\big)_{\mathfrak{X} \in \mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}}$$

COROLLARY 7.5.14. *Φ is a natural isomorphism between the identity functor on $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$ and $\mathfrak{F}\mathfrak{G}$.*

The previous corollary gives us one of the required natural isomorphisms. Let us move to consider the endofunctor $\mathfrak{G}\mathfrak{F} : \mathsf{Pr}_M^{\mathbb{DH}^{\wedge}} \longrightarrow \mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$. We need to show that for each $\mathfrak{X} \in \mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$ there is an isomorphism S_X between $\mathfrak{X}$ and $\mathfrak{G}\mathfrak{F}(\mathfrak{X})$ such that for each $R \subseteq X_1 \times X_2 \in \mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$, $S_{X_2} \star R = \mathfrak{G}\mathfrak{F}(R) \star S_{X_1}$.

LEMMA 7.5.15. *Let $\mathfrak{X}$ be a $\mathbb{DH}^{\wedge}$-Priestley space. For any $x \in X$, we have that $\uparrow x \cap X_B \in \mathcal{C}^{\mathrm{Op}}(X_B')$. Moreover, for each $C \in \mathcal{C}^{\mathrm{Op}}(X_B')$, there is a unique $x \in X$ such that $\uparrow x \cap X_B = C$.*

PROOF. First we check that $\uparrow x \cap X_B$ is closed in X_B'. Let $y \in X_B$ be such that $y \notin \uparrow x \cap X_B$. Recall that basic opens of X_B have the form $X_B \setminus U$ for some $U \in B$. From $x \neq y$ and (H12) we get that there is $U \in B$ such that $x \in U$ and $y \notin U$. Hence $y \in X_B \setminus U$ and $(\uparrow x \cap X_B) \cap (X_B \setminus U) = \emptyset$, so we are done.

We show now that $\uparrow x \cap X_B$ is an optimal closed subset. By (H13) we get that it is non-empty. Let $V, U_0, \dots, U_n \in B$, be such that $(X_B \setminus U_0) \cap \cdots \cap (X_B \setminus U_n) \subseteq X_B \setminus V$, and for all $i \le n$, $(\uparrow x \cap X_B) \cap (X_B \setminus U_i) \neq \emptyset$. Suppose, towards a contradiction, that $(\uparrow x \cap X_B) \cap (X_B \setminus V) = \emptyset$. From the assumption, we get that $x \notin U_i$ for all $i \le n$, since each U_i is an up-set. Suppose that $x \notin V$, then as V is X_B-admissible, there is $y \in X_B$ such that $x \le y \notin V$. But then we get $y \in (\uparrow x \cap X_B) \cap (X_B \setminus V)$, a contradiction. Therefore $x \in V$. So we have $x \in V \cap U_0^c \cap \cdots \cap U_n^c$, that is open. By density, there is $z \in X_B \cap (V \cap U_0^c \cap \cdots \cap U_n^c)$, so $z \in V$ and $z \notin U_i$ for all $i \le n$. But then, since $z \notin X_B \setminus V$ by assumption there is $i \le n$ such that $z \notin X_B \setminus U_i$, and so $z \in U_i$, a contradiction. We conclude that $\uparrow x \cap X_B$ is an optimal closed subset of X'_B.

Consider now a closed $C \in \mathcal{C}^{\mathrm{Op}}(X'_B)$, and suppose, towards a contradiction, that $\bigcap\{V \in B : C \subseteq V\} \cap \bigcap\{U^c : U \in B, C \nsubseteq U\} = \emptyset$. Then since the elements of B are clopens, by compactness of X, there are $V_0, \dots, V_n, U_0, \dots, U_m \in B$ such that $C \subseteq V_i$, $C \nsubseteq U_j$, for all $i \le n$ and $j \le m$, and $V_0 \cap \cdots \cap V_n \cap U_0^c \cap \cdots \cap U_m^c = \emptyset$. Let $V := V_0 \cap \cdots \cap V_n$. Clearly $C \subseteq V$. We have $V \subseteq U_0 \cup \cdots \cup U_m$, and so $(X_B \setminus U_0) \cap \cdots \cap (X_B \setminus U_m) \subseteq X_B \setminus V$. By assumption, $C \cap (X_B \setminus U_j) \neq \emptyset$, for all $j \le m$. Then by C being optimal closed $C \cap (X_B \setminus V) \neq \emptyset$, a contradiction. We conclude that there is $x \in \bigcap\{V \in B : C \subseteq V\} \cap \bigcap\{U^c : U \in B, C \nsubseteq U\}$. Clearly $\uparrow x \cap X_B = C$. Let $x, x' \in \bigcap\{V \in B : C \subseteq V\} \cap \bigcap\{U^c : U \in B, C \nsubseteq U\}$. So x and x' belong to the same elements of B. This implies by condition (H12) that they are equal. Therefore the $x \in X$ such that $\uparrow x \cap X_B = C$ is unique. □

The previous lemma gives us the following bijection between elements of X and elements of X'_B: each $x \in X$ corresponds with $x_{\uparrow x \cap X_B} \in X'_B$, and all elements of X'_B are of this form. The following lemma is then easy to prove.

Lemma 7.5.16. *Let $\mathfrak{X}$ be a $\mathbb{DH}^{\wedge}$-Priestley space. For all $U \in B$, $\eta(X_B \setminus U) = U$.*

According to this lemma, the map η gives us a one-to-one correspondence between the subbasis of X, $\{U : U \in B\} \cup \{V^c : V \in B\}$, and the subbasis of X'_B, $\{\eta(X_B \setminus U) : U \in B\} \cup \{\eta(X_B \setminus V)^c : V \in B\}$. Thus the spaces X and X'_B are homeomorphic. Similarly, for any $\mathbb{DH}^{\wedge}$-Priestley morphism $R \subseteq X_1 \times X_2$, it follows that $(y_1, y_2) \in R$ if and only if $(x_{\uparrow y_1 \cap X_{B_1}}, x_{\uparrow y_2 \cap X_{B_2}}) \in \overline{R}$. For each $\mathbb{DH}^{\wedge}$-Priestley space $\mathfrak{X}$, let $S_X \subseteq X \times X'_B$ be the relation given by:

$$(x, x_{\uparrow y \cap X_B}) \in S_X \quad \text{iff} \quad x \le y.$$

It is easy to check that S_X is a $\mathbb{DH}^{\wedge}$-Priestley morphism, and it is in fact an isomorphism in $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$. Consider the family of morphisms in $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$:

$$\Psi := (S_X \subseteq X \times \mathfrak{G}\mathfrak{F}(\mathfrak{X}))_{\mathfrak{X} \in \mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}}$$

Theorem 7.5.17. *Ψ is a natural isomorphism between the identity functor on $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$ and $\mathfrak{G}\mathfrak{F}$.*

Corollary 7.5.18. *The categories $\mathsf{Pr}_M^{\mathbb{DH}^{\wedge}}$ and $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$ are equivalent by means of the functors $\mathfrak{F}$ and $\mathfrak{G}$ and the natural equivalences Ψ and Φ. Moreover $\mathfrak{F}\mathfrak{G}$ is the identity functor on $\mathsf{Sp}_M^{\mathbb{DH}^{\wedge}}$.*

7.6. Extending the method to other classes of algebras

In this section we aim to highlight that what has been presented in § 7.1–§ 7.3 is not only a new topological duality for $\mathbb{DH}^{\wedge}$-algebras, but moreover, it is a strategy for getting new topological dualities for other classes of algebras (and logics). In particular, we focus on the algebras introduced in Chapter 6, that correspond to filter distributive and congruential expansions of $\mathcal{H}$, the implicative fragment of intuitionistic logic. This fact is in line with our motivation: in the same way as extended Priestley duality provides a general method from which many dualities for lattice-based algebras follow, we claim that from our duality for $\mathbb{DH}^{\wedge}$-algebras we shall abstract a general method from which other dualities for $\mathbb{DH}^{\wedge}$*-based algebras* follow as well.

First we abstract the mentioned strategy in a very informal way, and then we focus on the classes of algebras exhibited in § 6.7 and we indicate how the same pattern can be followed to get topological dualities for these classes of algebras.

As for the general strategy, let us concentrate on the Spectral-like duality, but keep in mind that what follows could also be stated for the Priestley-style duality. Let $\mathbb{K}$ and $\widehat{\mathbb{K}}$ be two classes of algebras in the languages $\mathscr{L}$ and $\widehat{\mathscr{L}}$, for which Spectral-like dualities are already known. Let us call the dual spaces of the algebras in $\mathbb{K}$ and $\widehat{\mathbb{K}}$, $\mathbb{K}$-Spectral spaces and $\widehat{\mathbb{K}}$-Spectral spaces respectively. For convenience, assume that $\mathscr{L}$ and $\widehat{\mathscr{L}}$ are disjoint. Finally, let $\mathbb{K}'$ be a class of algebras in the language $\mathscr{L} \cup \widehat{\mathscr{L}}$, such that:

- the $\mathscr{L}$-reducts of $\mathbb{K}'$-algebras are $\mathbb{K}$-algebras, so for any $\mathbb{K}'$-algebra $\mathbf{A}$, let $\langle X_{\mathbf{A}}, \tau_{\kappa_{\mathbf{A}}}, \dots \rangle$ be its dual $\mathbb{K}$-Spectral space,
- the $\widehat{\mathscr{L}}$-reducts of $\mathbb{K}'$-algebras are $\widehat{\mathbb{K}}$-algebras, so for any $\mathbb{K}'$-algebra $\mathbf{A}$, let $\langle \widehat{X}_{\mathbf{A}}, \tau_{\widehat{\kappa}_{\mathbf{A}}}, \dots \rangle$ be its dual $\widehat{\mathbb{K}}$-Spectral space,

Under this general situation, if we have that $\widehat{X}_{\mathbf{A}} \subseteq X_{\mathbf{A}}$ and the subspace of $\langle X_{\mathbf{A}}, \tau_{\kappa_{\mathbf{A}}} \rangle$ generated by $\widehat{X}_{\mathbf{A}}$ is precisely $\langle \widehat{X}_{\mathbf{A}}, \tau_{\widehat{\kappa}_{\mathbf{A}}} \rangle$, then we claim that a not so complicated Spectral-like duality for $\mathbb{K}'$ might be built from the dualities for $\mathbb{K}$ and $\widehat{\mathbb{K}}$, in such a way that for any $\mathbb{K}'$-algebra $\mathbf{A}$, its dual $\mathbb{K}'$-Spectral space shall be defined as the dual $\mathbb{K}$-Spectral space of its $\mathscr{L}$-reduct augmented with a subset that satisfies certain conditions. It would be very interesting to study this thoroughly and get, for any class $\mathbb{K}$ a full characterization of the classes of algebras the strategy might be applied for, but this should be studied somewhere else. In what follows we briefly treat the case when $\mathbb{K}$ is the variety of Hilbert algebras $\mathbb{H}$.

7.6.1. Dualities for Implicative Semilattices. In the present section we show that the topological dualities for implicative semilattices studied in [**11**] and [**6**] can be easily obtained as an instance of the ones that we obtained in the present chapter. Recall that we also obtained such dualities as a particular case of the theory developed in Chapter 5 (see § 6.5). We refer the reader to § 6.5 for the definitions of $\mathbb{IS}$-space, $\mathbb{IS}$-morphism, generalized Esakia space and generalized Esakia morphism.

Let us begin with the Spectral-like duality. We show that the collection of $\mathbb{IS}$-spaces can alternatively be presented as the collection of $\mathbb{DH}^{\wedge}$-Spectral spaces $\mathfrak{X} = \langle X, \widehat{X}, \tau_{\kappa} \rangle$ that satisfy the following condition:

(IS) $\widehat{X} = X$.

Proposition 7.6.1. *Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$ be a $\mathbb{DH}^\wedge$-Spectral space that satisfies (IS). Then $\langle X, \tau_\kappa \rangle$ is an $\mathbb{IS}$-space.*

Proof. By condition (DH^1), $\langle X, \tau_\kappa \rangle$ it is a compactly based sober topological space, by conditions (DH^2), (DH^4) and (IS), $\mathcal{KO}(X) = \mathcal{KO}(\widehat{X}) = \kappa$ is a basis for the space, and then by condition (DH^1), the basis is closed under the operation $\mathrm{sat}((\ ) \cap (\ )^c)$. $\square$

Proposition 7.6.2. *Let $\langle X, \tau \rangle$ be an $\mathbb{IS}$-space. Then $\overline{\mathfrak{X}} = \langle X, X, \tau_{\mathcal{KO}(X)} \rangle$ is a $\mathbb{DH}^\wedge$-Spectral space.*

Proof. By assumption $\langle X, \tau_{\mathcal{KO}(X)} \rangle$ is an $\mathbb{H}$-space. Clearly $X \subseteq X$ generates a sober subspace. Moreover, we can easily prove that for all $\{U, V\} \cup \mathcal{W} \subseteq \mathcal{KO}(X)$. As U^c is closed, it is immediate that $\mathrm{cl}(U^c \cap X) = U^c$, so condition (DH^3) is satisfied. Also $\mathrm{cl}(U^c \cap V^c \cap X)^c = U \cup V \in \mathcal{KO}(X)$, so condition (DH^4) is satisfied. Finally, if $\mathrm{cl}(\bigcap\{W^c : W \in \mathcal{W}\} \cap X) \subseteq U^c$, as U is compact, there are $W_0, \ldots, W_n \in \mathcal{W}$, for some $n \in \omega$, such that $W_0^c \cap \cdots \cap W_n^c = \mathrm{cl}(W_0^c \cap \cdots \cap W_n^c \cap X) \subseteq U^c$. So condition (DH^5) is also satisfied. $\square$

After a careful review of the definition of $\mathbb{IS}$-morphisms, we realize that conditions (DS1), (DS2) correspond with condition (DH^R1), that appears in the definition of $\mathbb{DH}^\wedge$-Spectral morphisms, with $\mathcal{KO}(X)$ playing the role of κ. In fact, (DH^R2) is redundant, under the assumption and condition (DH^R1). Clearly condition (HF) is the same in both definitions. Therefore, from the correspondence for objects it follows easily the correspondence for morphisms.

Corollary 7.6.3. *The category of $\mathbb{IS}$-spaces and functional meet-relations is equivalent to the category of $\mathbb{DH}^\wedge$-Spectral spaces satisfying (IS) and $\mathbb{DH}^\wedge$-Spectral functional morphisms.*

Let us move now to the Priestley-style duality. We show that the collection of generalized Esakia spaces can be presented as the collection of $\mathbb{DH}^\wedge$-Priestley spaces $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X} \rangle$ that satisfy condition (IS).

Proposition 7.6.4. *Let $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X} \rangle$ be a $\mathbb{DH}^\wedge$-Priestley space that satisfies (IS). Then $\langle X, \tau, \leq, X_B \rangle$ is a generalized Esakia space.*

Proof. By Corollary 7.1.19 we know that $\langle \widehat{X}, \widehat{\tau}, \leq, X_B \cap \widehat{X} \rangle$ is a generalized Priestley space, but by assumption, this structure is the same as $\langle X, \tau, \leq, X_B \rangle$. Notice that the collection of all X_B-admissible clopen up-sets of X is precisely B. Then condition (IS5), that implies that the down-set generated by any Esakia clopen is clopen, follows easily from condition (DH^6). $\square$

Proposition 7.6.5. *Let $\langle X, \tau, \leq, X_B \rangle$ be a generalized Esakia space. Then $\overline{\mathfrak{X}} = \langle X, \tau, \leq, \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X), X \rangle$ is a $\mathbb{DH}^\wedge$-Priestley space.*

Proof. By assumption $\langle X, \tau, \leq, \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X) \rangle$ is an generalized Priestley space whose dense subset is X_B. Clearly $X \subseteq X$ generates a compact subspace. Moreover, for all $U, V \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$: as U is up-set, it follows $\uparrow(U \cap X) = U$, so condition (DH^8) is satisfied. Since $\mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$ is closed under finite intersections, we have $\uparrow(U \cap V \cap X) \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$, so condition (DH^9) is also satisfied. Finally, for every

W clopen up-set, we have, by definition of X_B-admissible, that $W \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$ if and only if $W^c \subseteq {\downarrow}(W^c \cap X_B)$. Therefore, condition (DH^10) is satisfied. $\square$

If we examine in detail the definition of generalized Esakia morphism, we realize that conditions (DS3) and (DS4) correspond with condition (DH^R3), and condition (DH^R4) is redundant under the assumption and condition (DH^R3). Therefore, from the correspondence for objects it follows again easily the correspondence for morphisms.

Corollary 7.6.6. *The category of generalized Esakia spaces and generalized Esakia morphisms is equivalent to the category of $\mathbb{DH}^{\wedge}$-Priestley spaces satisfying (IS) and $\mathbb{DH}^{\wedge}$-Priestley functional morphisms.*

7.6.2. Dualities for Hilbert algebras with distributive lattice structure. Recall that $\mathbb{H}^{DL}$-algebras are given in the language $(\to, \wedge, \vee, 1)$ of type $(2, 2, 2, 0)$. A $\mathbb{H}^{DL}$-algebras is an algebra $\mathbf{A} = \langle A, \to, \wedge, \vee, 1\rangle$, such that $\langle A, \to, 1\rangle$ is a Hilbert algebra, $\langle A, \wedge, \vee\rangle$ is a distributive lattice, and moreover the implication and the lattice define the same order. We pursue to get Spectral-like and Priestley-style dualities for $\mathbb{H}^{DL}$-algebras. From a careful analysis of the Spectral-like duality for $\mathbb{DH}^{\wedge}$-algebras, and bringing up the well-known Stone duality for distributive lattices, we deduce the following definition of the Spectral-like dual spaces of $\mathbb{H}^{DL}$-algebras.

Definition 7.6.7. A structure $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$ is an $\mathbb{H}^{DL}$*-Spectral space* when $\mathfrak{X}$ is a $\mathbb{DH}^{\wedge}$-Spectral space and:

(DH^{L}1) $\mathrm{cl}((U^c \cup V^c) \cap \widehat{X})^c \in \kappa$, for any $U, V \in \kappa$.

Similarly, for the Priestley-style dual spaces of $\mathbb{H}^{DL}$-algebras, we get the following definition from the analysis of the Spectral-like duality for $\mathbb{DH}^{\wedge}$-algebras and the well-known Priestley duality for distributive lattices.

Definition 7.6.8. A structure $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ is an $\mathbb{H}^{DL}$*-Priestley space* when $\mathfrak{X}$ is a $\mathbb{DH}^{\wedge}$-Priestley space and:

(DH^{L}2) ${\uparrow}((U \cup V) \cap \widehat{X}) \in B$, for any $U, V \in B$.

Notice that for any $\mathbb{H}^{DL}$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$, by condition (DH^8), we could rewrite condition (DH^{L}2) as follows:

(H$^\vee$4) $U \cup V \in B$, for any $U, V \in B$.

In the same way, for any $\mathbb{H}^{DL}$-Spectral space $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$, by condition (DH^3), we could rewrite condition (DH^{L}1) as follows:

(H$^\vee$3) $U \cap V \in \kappa$, for any $U, V \in \kappa$.

This yields a different formulation of the definitions of $\mathbb{H}^{DL}$-Spectral and $\mathbb{H}^{DL}$-Priestley spaces:

Proposition 7.6.9. *A structure $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$ is an* $\mathbb{H}^{DL}$-Spectral space *when $\mathfrak{X}$ is a $\mathbb{DH}^{\wedge}$-Spectral space and $\langle X, \tau_\kappa\rangle$ is a $\mathbb{H}^{\vee}$-Spectral space*

Proposition 7.6.10. *A structure $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ is an* $\mathbb{H}^{DL}$-Priestley space *when $\mathfrak{X}$ is a $\mathbb{DH}^{\wedge}$-Priestley space and $\langle X, \tau, \leq, B\rangle$ is a $\mathbb{H}^{\vee}$-Priestley space.*

From our work in § 7.1 it is easy to prove the following equivalences between objects. Some work should be done to encompass morphisms and get a full categorical duality, but we leave this as future work.

Theorem 7.6.11. *Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa \rangle$ be an $\mathbb{H}^{DL}$-Spectral space. Then $\mathfrak{X}^* := \langle D(\mathfrak{X}), \Rightarrow, \sqcap, \cup, X\rangle$ is an $\mathbb{H}^{DL}$-algebra and $\mathfrak{Irr}(\mathfrak{X}^*) := \langle \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^*), \mathrm{Irr}_{\wedge}(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}}\rangle$ is an $\mathbb{H}^{DL}$-Spectral space such that $\langle X, \tau_\kappa\rangle$ and $\langle \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}}\rangle$ are homeomorphic topological spaces by means of the map $\varepsilon_{\mathfrak{X}}$ and moreover $\varepsilon_{\mathfrak{X}}[\widehat{X}] = \mathrm{Irr}_{\wedge}(\mathfrak{X}^*)$.*

Theorem 7.6.12. *Let $\mathbf{A} = \langle A, \rightarrow, \wedge, \vee, 1\rangle$ be an $\mathbb{H}^{DL}$-algebra. Then $\mathfrak{Irr}(\mathbf{A}) := \langle \mathrm{Irr}_{\rightarrow}(\mathbf{A}), \mathrm{Irr}_{\wedge}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$ is an $\mathbb{H}^{DL}$-Spectral space and $(\mathfrak{Irr}(\mathbf{A}))^* := \langle D(\mathfrak{Irr}(\mathbf{A})), \Rightarrow, \sqcap, \cup, \mathrm{Irr}_{\rightarrow}(\mathbf{A})\rangle$ is an $\mathbb{H}^{DL}$-algebra isomorphic to $\mathbf{A}$ by means of the map $\psi_{\mathbf{A}}$.*

Note that this is another example of the modular nature of Stone/Priestley duality theory for filter distributive finitary congruential logics with theorems. We should mention that in [**14**], Celani and Cabrer follow a strategy alike the one presented above. More precisely, they study a duality for $\mathbb{DH}_0^L$-algebras, but they combine the Spectral-like duality for Hilbert algebras with the Priestley duality for bounded distributive lattices. We could also combine the Priestley-style duality for Hilbert algebras and the Spectral-like duality for distributive semilattices, in order to get another duality for $\mathbb{DH}^{\wedge}$-algebras.

7.6.3. Dualities for Implicative Hilbert algebras with infimum. Recall that in this case we deal with the language $(\rightarrow, \wedge, \rightarrow', 1)$ of type $(2, 2, 2, 0)$. We aim to get Spectral-like and Priestley-style dualities for $\mathbb{IH}^{\wedge}$-algebras. Recall that $\mathbf{A} = \langle A, \rightarrow, \wedge, \rightarrow', 1\rangle$ is an $\mathbb{IH}^{\wedge}$-algebra when $\langle A, \rightarrow, 1\rangle$ is a Hilbert algebra, $\langle A, \rightarrow', \wedge, 1\rangle$ is an implicative semilattice, and moreover $\rightarrow$ and $\rightarrow'$ define the same order on A. In order to get a Spectral-like duality for $\mathbb{IH}^{\wedge}$-algebras, we focus on the Spectral-like duality for $\mathbb{DH}^{\wedge}$-algebras together with the Spectral-like duality for implicative semilattices presented in § 6.5, and we get the following definition.

Definition 7.6.13. A structure $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$ is an $\mathbb{IH}^{\wedge}$-*Spectral space* when $\mathfrak{X}$ is a $\mathbb{DH}^{\wedge}$-Spectral space and:

(IH^1) $(\mathrm{cl}((\mathrm{sat}_{\widehat{X}}(U \cap V^c))^c \cap \widehat{X}))^c \in \kappa$, for any $U, V \in \kappa$.

Now for any $\mathbb{IH}^{\wedge}$-Spectral space $\mathfrak{X}$, we can define an operation $\Rightarrow'$ on $D(\mathfrak{X})$ such that for all $U, V \in \kappa$:

$$U^c \Rightarrow' V^c := \mathrm{cl}((\mathrm{sat}_{\widehat{X}}(U \cap V^c))^c \cap \widehat{X}).$$

On the other hand, we get the definition of Priestley-style dual spaces of $\mathbb{IH}^{\wedge}$-algebras from the Priestley-style duality for $\mathbb{DH}^{\wedge}$-algebras and from the Priestley-style duality for implicative semilattices presented in § 6.5.

Definition 7.6.14. A structure $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ is an $\mathbb{IH}^{\wedge}$-*Priestley space* when $\mathfrak{X}$ is a $\mathbb{DH}^{\wedge}$-Priestley space and:

(IH^2) $\uparrow((\downarrow(U \cap V^c))^c \cap \widehat{X}) \in B$, for any $U, V \in B$.

And again, for any $\mathbb{IH}^{\wedge}$-Priestley space $\mathfrak{X}$, we can define an operation $\Rightarrow'$ on B such that for all $U, V \in B$:

$$U \Rightarrow' V := \uparrow((\downarrow(U \cap V^c))^c \cap \widehat{X}).$$

These definitions are a bit more elaborated than the previous ones. Notice that condition (IH^1) involves the closure of a subset in $\langle X, \tau_\kappa\rangle$ and the saturation of a subset in the subspace generated by $\widehat{X}$. In any case, the dualities work as usual, although some work should be done again in order to accommodate morphisms.

THEOREM 7.6.15. *Let $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$ be an $\mathbb{IH}^{\wedge}$-Spectral space. Then $\mathfrak{X}^* := \langle D(\mathfrak{X}), \Rightarrow, \sqcap, \Rightarrow', X\rangle$ is an $\mathbb{IH}^{\wedge}$-algebra and $\mathfrak{Irr}(\mathfrak{X}^*) := \langle \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^*), \mathrm{Irr}_{\wedge}(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}}\rangle$ is an $\mathbb{IH}^{\wedge}$-Spectral space such that $\langle X, \tau_\kappa\rangle$ and $\langle \mathrm{Irr}_{\rightarrow}(\mathfrak{X}^*), \tau_{\kappa_{\mathfrak{X}^*}}\rangle$ are homeomorphic topological spaces by means of the map $\varepsilon_{\mathfrak{X}}$ and moreover $\varepsilon_{\mathfrak{X}}[\widehat{X}] = \mathrm{Irr}_{\wedge}(\mathfrak{X}^*)$.*

THEOREM 7.6.16. *Let $\mathbf{A} = \langle A, \rightarrow, \wedge, \vee, 1\rangle$ be an $\mathbb{IH}^{\wedge}$-algebra. Then $\mathfrak{Irr}(\mathbf{A}) := \langle \mathrm{Irr}_{\rightarrow}(\mathbf{A}), \mathrm{Irr}_{\wedge}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}\rangle$ is an $\mathbb{IH}^{\wedge}$-Spectral space and $(\mathfrak{Irr}(\mathbf{A}))^* := \langle D(\mathfrak{Irr}(\mathbf{A})), \Rightarrow, \sqcap, \Rightarrow', \mathrm{Irr}_{\rightarrow}(\mathbf{A})\rangle$ is an $\mathbb{IH}^{\wedge}$-algebra isomorphic to $\mathbf{A}$ by means of the map $\psi_{\mathbf{A}}$.*

7.6.4. Dualities for Implicative Hilbert algebras with lattice structure. Finally, we briefly mention the case when we deal with the language $(\rightarrow, \wedge, \vee, \rightarrow', 1)$ of type $(2, 2, 2, 2, 0)$. Recall that $\mathbf{A} = \langle A, \rightarrow, \wedge, \vee, \rightarrow', 1\rangle$ is a $\mathbb{IH}^L$-algebra when $\langle A, \rightarrow, 1\rangle$ is a Hilbert algebra, $\langle A, \rightarrow', \wedge, \vee, 1\rangle$ is a relatively pseudo-complemented lattice, and both $\rightarrow$ and $\rightarrow'$ define the same order. The definition of the Spectral dual objects of $\mathbb{IH}^L$-algebras arises from the Spectral-like duality for $\mathbb{H}^{DL}$-algebras and the Spectral-like duality for implicative semilattices.

DEFINITION 7.6.17. A structure $\mathfrak{X} = \langle X, \widehat{X}, \tau_\kappa\rangle$ is a $\mathbb{IH}^L$-*Spectral space* when $\mathfrak{X}$ is a $\mathbb{H}^{DL}$-Spectral space and:

(IH^1) $(\mathrm{cl}((\mathrm{sat}_{\widehat{X}}(U \cap V^c))^c \cap \widehat{X}))^c \in \kappa$, for any $U, V \in \kappa$.

As in previous subsection, for any $\mathbb{IH}^L$-Spectral space $\mathfrak{X}$, we can define an operation $\Rightarrow'$ on $D(\mathfrak{X})$ such that for all $U, V \in \kappa$:

$$U^c \Rightarrow' V^c := \mathrm{cl}((\mathrm{sat}_{\widehat{X}}(U \cap V^c))^c \cap \widehat{X}).$$

Regarding Priestley-style dual spaces of $\mathbb{IH}^L$-algebras, from the Priestley-style duality for $\mathbb{H}^{DL}$-algebras and the Priestley-style duality for implicative semilattices, we get the following definition:

DEFINITION 7.6.18. A structure $\mathfrak{X} = \langle X, \tau, \leq, B, \widehat{X}\rangle$ is a $\mathbb{IH}^L$-*Priestley space* when $\mathfrak{X}$ is a $\mathbb{H}^{DL}$-Priestley space and:

(IH^2) $\uparrow((\downarrow(U \cap V^c))^c \cap \widehat{X}) \in B$, for any $U, V \in B$.

And for any $\mathbb{IH}^L$-Priestley space $\mathfrak{X}$, we can define an operation $\Rightarrow'$ on B such that for all $U, V \in B$:

$$U \Rightarrow' V := \uparrow((\downarrow(U \cap V^c))^c \cap \widehat{X}).$$

Likewise, we get analogous of theorems 7.6.15 and 7.6.16. Summarizing, we have shown how we can use the dualities for $\mathbb{DH}^{\wedge}$-algebras to get new dualities for other classes of algebras. And this is very interesting since those classes of algebras are the algebraic counterpart of some interesting filter distributive finitary congruential logics with theorems, all of which are expansions of the implicative fragment of intuitionistic logic.

Summary and Conclusions

In this dissertation we aimed to show that Abstract Algebraic Logic provides the appropriate theoretical framework for developing a uniform duality theory for non-classical logics. We focused on the so called Stone/Priestley dualities, and we studied them under an abstract point of view. It would be interesting to explore also the third line that was mentioned in § 3.1, namely the so-called pairwise Stone-type dualities.

We have captured a sufficient set of conditions for a logic $\mathcal{S}$ to get a Spectral-like or a Priestley-style dual for $\mathsf{Alg}\mathcal{S}$. Such conditions are: being congruential, filter distributive, finitary, and having theorems. Under these assumptions on $\mathcal{S}$, we have identified the collections of $\mathcal{S}$-filters that can be used for developing the abstract dualities: irreducible $\mathcal{S}$-filters are used to build up the Spectral-like duality, and optimal $\mathcal{S}$-filters are the ones used for the Priestley-style duality. We have had a quick look at how these notions can also be used for an abstract study of the theory of canonical extensions for non-classical logics. Further questions on this topic, such as how to encompass canonical extensions of substructural logics, or the possibility of an abstract Sahlqvist theory, are left as future work.

We have obtained two abstract categorical dualities, in which most of the Stone/Priestley dualities for non-classical logics that we encounter in the literature fit. Due to the abstract character of our approach, we cannot avoid, in general, to have an algebraic structure on the dual side. However, we have analyzed which dual properties correspond with the best-known logical properties, such as having a conjunction or a deduction theorem. This allows us to dispense with the algebraic structure in the dual side, when the logic is sufficiently well behaved. Furthermore, this analysis is both interesting for duality theory and for AAL. On the one hand, it confirms the strength of duality theory, that can be developed in a modular way, even outside of the distributive lattice setting. On the other hand, in the same way than bridge theorems are studied in AAL between properties of the logic and properties of its algebraic semantics, our results can be regarded as bridge theorems between properties of the logic and properties of its Kripke-style semantics. In this sense, we have carried out only the first steps, and we left as future work to investigate the dual correspondence for more logical properties, such us (DDT), n-ary modal operators, etc.

The entire second part of the dissertation was devoted to extract concrete results from our general theory. To do this, instead of keeping our abstract approach, we focused on a single filter distributive finitary congruential logic with theorems, namely the implicative fragment of intuitionistic logic, and we tackled the problem of getting Spectral/Priestley dualities for extensions of such logic.

Aside from several new dualities that follow more or less straightforwardly from the general case, we have studied new Spectral-like and Priestley-style dualities for distributive Hilbert algebras with infimum. From those dualities, new Spectral-like and Priestley-style dualities for a wide range of expansions of the implicative fragment of intuitionistic logic follow. From our work, Kripke-style semantics for such logics could also follow. Such relational structures should be analyzed in more detail, but we leave this as future work.

For distributive Hilbert algebras with infimum, the case that we have studied in more detail, we also have compared both dualities in § 7.5, and we have given the Spectral-like characterization of the different classes of filters in § 7.4, but other algebraic questions about this class of algebras remain open, such as the dual characterization of subalgebras or of homomorphic images. It would be also interesting to study in depth the other mentioned classes of algebras, as well as the outlined dualities for them.

A more ambitious project would involve identigying not only sufficient but also necessary conditions that make our dualities work. On the other hand, it would be very interesting to explore the development of an abstract theory of Spectral-like and Priestley-style dualities for *nice* expansions of $\mathcal{H}$, the implicative fragment of intuitionistic logic. Or even more ambitious, formulating an abstract theory of Spectral-like and Priestley-style dualities for *nice* expansions of $\mathcal{S}$, for $\mathcal{S}$ an arbitrary filter distributive finitary congruential logic with theorems.

APPENDIX A

The distributive envelope of a distributive meet-semilattice with top element

In this Appendix we present more detailed what was only outlined in Section 9 in [**5**], namely how the Priestley-style duality for bounded distributive semilattices presented there can be modified accordingly to obtain a Priestley-style duality for distributive semilattices with top element.

From now on, let $\mathbf{M} = \langle M, \wedge, 1\rangle$ be a distributive semilattice with top element. The distributive envelope of $\mathbf{M}$ may be described, in brief, as the semilattice of finitely generated F-ideals of $\mathbf{M}$. As it is done in [**5**] and also in § 4.5, we follow an alternative approach, and construct the distributive envelope of $\mathbf{M}$ from a separating family for $\mathbf{M}$. As an instance of Definition 4.5.1 we obtain the following definition.

Definition A.1. A family $\mathcal{F} \subseteq \mathrm{Op}_{\wedge}(\mathbf{M})$ of optimal meet filters of $\mathbf{M}$ is a *separating family* for $\mathbf{M}$ if for every meet filter $F \in \mathrm{Fi}_{\wedge}(\mathbf{M})$ and every $a \notin F$, there is $P \in \mathcal{F}$ such that $F \subseteq P$ and $a \notin P$.

By Lemma 2.3.7, $\mathrm{Op}_{\wedge}(\mathbf{M})$ is itself a separating family for $\mathbf{M}$, and by Lemma 2.3.3, $\mathrm{Irr}_{\wedge}(\mathbf{M})$ is also a separating family for $\mathbf{M}$. For any separating family $\mathcal{F}$ for $\mathbf{M}$, we define the map $\sigma_{\mathcal{F}} : M \longrightarrow \mathcal{P}^{\uparrow}(\mathcal{F})$ as follows:

$$\sigma_{\mathcal{F}}(a) = \{P \in \mathcal{F} : a \in P\}.$$

The following representation theorem for semilattices that goes back to Stone is stated in [**5**].

Theorem A.2. *Let* $\mathbf{M}$ *be a distributive semilattice and* $\mathcal{F}$ *a separating family for* $\mathbf{M}$. *The map* $\sigma_{\mathcal{F}}$ *is an isomorphism between* $\mathbf{M}$ *and* $\sigma_{\mathcal{F}}[\mathbf{M}] := \langle \sigma_{\mathcal{F}}[M], \cap, \mathcal{F}\rangle$.

Let us denote by $\mathrm{L}_{\mathcal{F}}(M)$ the closure of $\sigma_{\mathcal{F}}[M]$ under non-empty finite unions. By definition $\mathrm{L}_{\mathcal{F}}(M)$ is also closed under non-empty finite intersections, since $\sigma_{\mathcal{F}}(a) \cap \sigma_{\mathcal{F}}(b) = \sigma_{\mathcal{F}}(a \wedge b)$ for all $a, b \in M$, and moreover $\mathcal{F} \in \mathrm{L}_{\mathcal{F}}(M)$ since $\sigma_{\mathcal{F}}(1) = \mathcal{F}$.

Definition A.3. The algebra $\mathrm{L}_{\mathcal{F}}(\mathbf{M}) := \langle \mathrm{L}_{\mathcal{F}}(M), \cap, \cup, \mathcal{F}\rangle$ is called the *distributive envelope of* $\mathbf{M}$.

It follows that $\mathrm{L}_{\mathcal{F}}(\mathbf{M})$ is a distributive lattice with top element. Moreover, the next theorem provides an abstract characterization of $\mathrm{L}_{\mathcal{F}}(\mathbf{M})$ by a universal property. The proof is similar to that in Theorem 5.8 in [**4**]. Recall that sup-homomorphisms are the algebraic homomorphisms that preserve existing suprema (see definition in page 39).

THEOREM A.4. *Let* $\mathbf{M}$ *be a distributive semilattice and let* $\mathcal{F}$ *be a separating family for* $\mathbf{M}$. *The distributive envelope* $\mathrm{L}_{\mathcal{F}}(\mathbf{M})$ *is, up to isomorphism, the unique distributive lattice* $\mathbf{L}$ *for which there is a one-to-one sup-homomorphism* $h : M \longrightarrow L$ *such that for any distributive lattice* $\mathbf{L}'$ *and any one-to-one sup-homomorphism* $h' : M \longrightarrow L'$ *there is a unique one-to-one lattice homomorphism* $k : L \longrightarrow L'$ *with* $k \circ h = h'$.

By the previous theorem, we know that for separating families $\mathcal{F}$ and $\mathcal{F}'$ for $\mathbf{M}$, $\mathrm{L}_{\mathcal{F}}(\mathbf{M})$ and $\mathrm{L}_{\mathcal{F}'}(\mathbf{M})$ are isomorphic lattices. For convenience, we dispense with the subscript $\mathcal{F}$ of $\mathrm{L}_{\mathcal{F}}(\mathbf{M})$, $\mathrm{L}_{\mathcal{F}}(M)$ and $\sigma_{\mathcal{F}}$, and we use $\mathrm{L}(\mathbf{M})$, $\mathrm{L}(M)$ and σ instead. Clearly we have that for each $U \in \mathcal{P}^{\uparrow}(\mathcal{F})$:

$$U \in \mathrm{L}(M) \quad \text{iff} \quad U = \bigcup_{b \in B} \sigma(b) \text{ for some non-empty } B \subseteq^{\omega} M. \tag{E8}$$

The following technical lemma concerns the case when the distributive envelope is bounded.

LEMMA A.5. *Let* $\mathbf{M}$ *be a distributive semilattice. Then:*

(1) $\mathbf{M}$ *has a bottom element if and only if* $\emptyset \in \mathrm{L}(\mathbf{M})$.
(2) $\emptyset \in \mathrm{Id}_F(\mathbf{M})$ *if and only if* $\emptyset \in \mathrm{Id}_F(\mathrm{L}(\mathbf{M}))$.

PROOF. (1) If $\mathbf{M}$ has a bottom element $0^{\mathbf{M}}$, then $\emptyset \notin \mathrm{Id}_F(\mathbf{M})$. Therefore $A \notin \mathrm{Op}_{\wedge}(\mathbf{M})$ and so $\sigma(0^{\mathbf{M}}) = \emptyset \in \mathrm{L}(\mathbf{M})$. For the converse, suppose $\emptyset \in \mathrm{L}(\mathbf{M})$. Then by (E8) there is $B \subseteq^{\omega} M$ non-empty and such that $\bigcup\{\sigma(b) : b \in B\} = \emptyset$. Let $c := \bigwedge B$. Then by assumption $\sigma(c) = \emptyset$, and this implies that $A \notin \mathrm{Op}_{\wedge}(\mathbf{M})$, so $\mathbf{M}$ has a bottom element $0^{\mathbf{M}}$ that clearly coincides with c.

(2) This follows from item (1) and definition of F-ideals of semilattices and lattices. □

Notice that the previous lemma involves F-ideals of the lattice $\mathrm{L}(\mathbf{M})$. These ideals are defined as in page 28. For any distributive lattice with top element $\mathbf{L} = \langle L, \wedge, \vee, 1\rangle$, it follows by definition that:

- if $\mathbf{L}$ has a bottom element, then the Frink ideals and the order ideals of $\mathbf{L}$ coincide, i. e. $\mathrm{Id}_F(\mathbf{L}) = \mathrm{Id}(\mathbf{L})$, and
- if $\mathbf{L}$ has no bottom element, then $\mathrm{Id}_F(\mathbf{L}) = \mathrm{Id}(\mathbf{L}) \cup \{\emptyset\}$.

It turns out that, when dealing with lattices with top but not necessarily bottom, F-ideals are the right tools to work with instead of working with order ideals. Similarly, we should work with optimal meet filters of the lattice $\mathrm{L}(\mathbf{M})$ instead of working with prime meet filters. For any distributive lattice with top element $\mathbf{L} = \langle L, \wedge, \vee, 1\rangle$, it follows by definition that:

- if $\mathbf{L}$ is has a bottom element, then optimal and prime meet filters of $\mathbf{L}$ coincide, i. e. $\mathrm{Op}_{\wedge}(\mathbf{L}) = \mathrm{Pr}(\mathbf{L})$, and
- if $\mathbf{L}$ has no bottom element, then $\mathrm{Op}_{\wedge}(\mathbf{L}) = \mathrm{Pr}(\mathbf{L}) \cup \{A\}$.

In the rest of the appendix we present several results about the distributive envelope of a distributive meet-semilattice with top element and about Priestley duality for these structures. There results generalize the ones stated in [**5**] for bounded distributive meet-semilattices. The main difference is precisely that they involve Frink ideals and optimal meet filters of lattices, instead of order ideals and prime meet filters.

Lemma A.6. *Let* $\mathbf{M}$ *be a distributive semilattice with top element. For any non-empty* $B, B_0, \ldots, B_n \subseteq^\omega M$*:*

$$\bigcap_{i\leq n} \llbracket B_i \rangle\!\rangle \subseteq \llbracket B \rangle\!\rangle \quad \textit{iff} \quad \sigma(\bigwedge B) \subseteq \bigcup_{i\leq n} \sigma(\bigwedge B_i) \quad \textit{iff} \quad \bigcap_{i\leq n} \llbracket \sigma(\bigwedge B_i) \rangle\!\rangle \subseteq \llbracket \sigma(\bigwedge B) \rangle\!\rangle.$$

Corollary A.7 (Lemma 3.2 in [**5**]). *Let* $\mathbf{M}$ *be a distributive semilattice with top element. For any* $a, a_0, \ldots, a_n \subseteq^\omega M$*:*

$$\bigcap_{i\leq n} {\uparrow} a_i \subseteq {\uparrow} a \quad \textit{iff} \quad \sigma(a) \subseteq \bigcup_{i\leq n} \sigma(a_i) \quad \textit{iff} \quad \bigcap_{i\leq n} {\uparrow}\sigma(a_i) \subseteq {\uparrow}\sigma(a).$$

Proposition A.8 (Lemma 3.10 in [**5**]). *Let* $\mathbf{M}$ *be a distributive semilattice with top element.*

(1) *If* F *is a meet filter of* $\mathbf{M}$*, then*
   (a) $\llbracket \sigma[F] \rangle\!\rangle$ *is a meet filter of* $\mathrm{L}(\mathbf{M})$*, and*
   (b) $\sigma^{-1}[\llbracket \sigma[F] \rangle\!\rangle] = F$.
(2) *If* F *is a meet filter of* $\mathrm{L}(\mathbf{M})$*, then* $\sigma^{-1}[F]$ *is a meet filter of* $\mathbf{M}$.
(3) *If* F *is an optimal meet filter of* $\mathbf{M}$*, then* $\llbracket \sigma[F] \rangle\!\rangle$ *is optimal.*
(4) *If* F *is an optimal meet filter of* $\mathrm{L}(\mathbf{M})$*, then*
   (a) $\sigma^{-1}[F]$ *is optimal, and*
   (b) $\llbracket \sigma[\sigma^{-1}[F]] \rangle\!\rangle = F$.

The previous proposition shows that the maps $\llbracket \sigma[\;] \rangle\!\rangle$ and σ^{-1} give us an order isomorphism between optimal meet filters of $\mathbf{M}$ and optimal meet filters of $\mathrm{L}(\mathbf{M})$:[1]

$$\langle \mathrm{Op}_\wedge(\mathbf{M}), \subseteq \rangle \cong \langle \mathrm{Op}_\wedge(\mathrm{L}(\mathbf{M})), \subseteq \rangle$$

Proposition A.9 (Lemma 3.12, Theorem 4.3 and Corollary 4.4 in [**5**]). *Let* $\mathbf{M}$ *be a distributive semilattice with top element. Then:*

(1) *If* I *is an F-ideal of* $\mathbf{M}$*, then*
   (a) $\langle\!\langle \sigma[I] \rrbracket$ *is an F-ideal of* $\mathrm{L}(\mathbf{M})$*, and*
   (b) $\sigma^{-1}[\langle\!\langle \sigma[I] \rrbracket] = I$.
(2) *If* I *is a prime F-ideal of* $\mathbf{M}$*, then* $\langle\!\langle \sigma[I] \rrbracket$ *is prime.*
(3) *If* I *is an F-ideal of* $\mathrm{L}(\mathbf{M})$*, then*
   (a) $\sigma^{-1}[I]$ *is an F-ideal of* $\mathbf{M}$*, and*
   (b) $\langle\!\langle \sigma[\sigma^{-1}[I]] \rrbracket = I$.
(4) *If* I *is a prime F-ideal of* $\mathrm{L}(\mathbf{M})$*, then* $\sigma^{-1}[I]$ *is prime.*

The previous proposition shows that the maps $\langle\!\langle \sigma[\;] \rrbracket$ and σ^{-1} give us an order isomorphism between F-ideals of $\mathbf{M}$ and F-ideals of $\mathrm{L}(\mathbf{M})$,[2] that restricts to an isomorphism between prime F-ideals of $\mathbf{M}$ and prime F-ideals of $\mathrm{L}(\mathbf{M})$:

$$\langle \mathrm{Id}_F(\mathbf{M}), \subseteq \rangle \cong \langle \mathrm{Id}_F(\mathrm{L}(\mathbf{M})), \subseteq \rangle$$

$$\langle \text{prime } \mathrm{Id}_F(\mathbf{M}), \subseteq \rangle \cong \langle \text{prime } \mathrm{Id}_F(\mathrm{L}(\mathbf{M})), \subseteq \rangle$$

[1] Recall that the latter are precisely prime filters of $\mathrm{L}(\mathbf{M})$ when $\mathbf{M}$ is bounded.

[2] Recall that the latter are precisely order ideals of $\mathrm{L}(\mathbf{M})$ when $\mathbf{M}$ is bounded.

In conclusion, we have seen that the properties of the distributive envelope of a bounded distributive meet-semilattice that were studied in [**5**] also hold for the distributive envelope of a distributive meet-semilattice with top element. To carry this out, however, the notion of optimal meet filter of bounded distributive meet-semilattices that is used in [**5**] has to be modified according to what was exposed in § 2.3.

APPENDIX B

The F-extension of a distributive meet-semilattice with top element

Our main reference for this Appendix is [**42**], where the theory of Δ_1-completions of posets is presented. We study in detail in what follows the properties of a particular Δ_1-completion of distributive semilattices with top element. We assume that the reader is familiar with the theory of canonical extensions.

Recall that for any poset P, a *completion* of P is an embedding of P in a complete lattice, i. e. it is a pair $(e, \mathbf{Q})$ such that $\mathbf{Q}$ is a complete lattice and $e : P \longrightarrow Q$ is an order embedding. For convenience, we usually take e as the identity.

Let P be a poset, let $\mathbf{Q}$ be a completion of P and let $\mathcal{F}$ and $\mathcal{I}$ be standard collections of up-sets and down-sets respectively, i. e. collections of up-sets (resp. down-sets) that contain the principal up-sets (resp. principal down-sets).

We call $\mathcal{F}$*-filter elements* of $\mathbf{Q}$ and $\mathcal{I}$*-ideal elements* of $\mathbf{Q}$ the elements in the following two sets, respectively:

$$\mathbb{F}^{\mathcal{F}}(\mathbf{Q}) = \{c \in \mathbf{Q} : c = \bigwedge_{\mathbf{Q}} F, F \in \mathcal{F}\},$$

$$\mathbb{I}^{\mathcal{I}}(\mathbf{Q}) = \{c \in \mathbf{Q} : c = \bigvee_{\mathbf{Q}} I, I \in \mathcal{I}\}.$$

A completion $\mathbf{Q}$ of P is $(\mathcal{F}, \mathcal{I})$*-dense* provided $\mathbb{F}^{\mathcal{F}}(\mathbf{Q})$ is join-dense in $\mathbf{Q}$ and $\mathbb{I}^{\mathcal{I}}(\mathbf{Q})$ is meet-dense in $\mathbf{Q}$. A completion $\mathbf{Q}$ of P is $(\mathcal{F}, \mathcal{I})$*-compact* provided for all $F \in \mathcal{F}$ and all $I \in \mathcal{I}$:

$$\bigwedge_{\mathbf{Q}} F \leq \bigvee_{\mathbf{Q}} I \quad \text{iff} \quad F \cap I \neq \emptyset.$$

The property of $(\mathcal{F}, \mathcal{I})$-compactness implies *weakly* $(\mathcal{F}, \mathcal{I})$*-compactness*, that holds whenever for all $F \in \mathcal{F}$ and all $I \in \mathcal{I}$:

$$\text{if } \bigwedge_{\mathbf{Q}} F \leq \bigvee_{\mathbf{Q}} I, \text{ then } (\exists X \subseteq^{\omega} F)(\exists Y \subseteq^{\omega} I) \bigwedge_{\mathbf{Q}} X \leq \bigvee_{\mathbf{Q}} Y.$$

Moreover, if $\mathcal{F}$ and $\mathcal{I}$ are algebraic closure systems, by Proposition 5.14 in [**42**], if $\mathbf{Q}$ is $(\mathcal{F}, \mathcal{I})$-compact and $(\mathcal{F}, \mathcal{I})$-dense, then $\mathbf{Q}$ is *compact*, i. e. for all $X, Y \subseteq P$

$$\text{if } \bigwedge_{\mathbf{Q}} X \leq \bigvee_{\mathbf{Q}} Y, \text{ then } (\exists X' \subseteq^{\omega} X)(\exists Y' \subseteq^{\omega} Y) \bigwedge_{\mathbf{Q}} X' \leq \bigvee_{\mathbf{Q}} Y'.$$

An $(\mathcal{F}, \mathcal{I})$*-completion* of P (Definition 5.9 in [**42**]) is a completion of P that is $(\mathcal{F}, \mathcal{I})$-compact and $(\mathcal{F}, \mathcal{I})$-dense.

From now on, let $\mathbf{M} = \langle M, \wedge, 1 \rangle$ be a distributive meet-semilattice with top element. Recall that the *canonical extension* of $\mathbf{M}$ is defined in [**26**] by Dunn.

et al. as the $(\mathrm{Fi}_\wedge(\mathbf{M}), \mathrm{Id}(\mathbf{M}))$-completion of $\mathbf{M}$ and it is customarily denoted by $\mathbf{M}^\delta$. We focus on a different Δ_1-completion of $\mathbf{M}$, namely, the $(\mathrm{Fi}_\wedge(\mathbf{M}), \mathrm{Id}_F(\mathbf{M}))$-completion of $\mathbf{M}$. For short, let us call it the *F-extension* of $\mathbf{M}$. The main result in this appendix is that the F-extension of $\mathbf{M}$ is (up to isomorphism) the canonical extension of $\mathrm{L}(\mathbf{M})$. Recall that we denote by $\mathrm{L}(\mathbf{M})$ the distributive envelope of $\mathbf{M}$, and following [**44**], for any distributive lattice $\mathbf{L}$ we denote by $\mathbf{L}^\delta$ its canonical extension. Notice that we have:

$$\mathbf{M} \xrightarrow{\sigma} \mathrm{L}(\mathbf{M}) \xrightarrow{m} \mathrm{L}(\mathbf{M})^\delta,$$

where σ is the embedding of $\mathbf{M}$ into its distributive envelope, defined in Appendix A, and m is the canonical embedding of $\mathrm{L}(\mathbf{M})$ into its canonical extension $\mathrm{L}(\mathbf{M})^\delta$. Let us define:

$$k := (m \circ \sigma) : \mathbf{M} \longrightarrow \mathrm{L}(\mathbf{M})^\delta.$$

It is easy to prove the following lemmas, in which all infinite joins and meets are referred to $\mathrm{L}(\mathbf{M})^\delta$.

Lemma B.1. *For all $F \in \mathrm{Fi}_\wedge(\mathbf{M})$, $\bigwedge k[F] = \bigwedge m[\uparrow\sigma[F]]$.*

Proof. Since $\sigma[F] \subseteq \uparrow\sigma[F]$, then $m[\sigma[F]] \subseteq m[\uparrow\sigma[F]]$ and so $\bigwedge m[\uparrow\sigma[F]] \leq \bigwedge m[\sigma[F]] = \bigwedge k[F]$. For the converse, let $x \in \mathrm{L}(M)$ be such that $x \in \uparrow\sigma[F]$. Then there is $a_x \in F \subseteq M$ such that $\sigma(a_x) \leq x$. Since m is order preserving, then $k(a_x) = m(\sigma(a_x)) \leq m(x)$. We conclude that $\bigwedge k[F] \leq \bigwedge m[\uparrow\sigma[F]]$, as required. $\square$

Lemma B.2. *For all $I \in \mathrm{Id}_F(\mathbf{M})$, $\bigvee k[I] = \bigvee m[\langle\!\langle\sigma[I]\rangle\!\rangle]$.*

Proof. Notice that since m preserves finite joins, we have that for all $X \subseteq^\omega I$, $m((\bigcup\{\sigma(x) : x \in X\}) = \bigvee m[\sigma[X]]$. By definition of Frink ideal generated we have that $\langle\!\langle\sigma[I]\rangle\!\rangle = \{\bigcup\{\sigma(x) : x \in X\} : X \subseteq^\omega I\}$. Then we get

$$\bigvee m[\langle\!\langle\sigma[I]\rangle\!\rangle] = \bigvee_{X \subseteq^\omega I} m(\bigcup\{\sigma(x) : x \in X\}) = \bigvee_{X \subseteq^\omega I} \bigvee_{x \in X} m(\sigma(x)) = \bigvee k[I].$$

$\square$

Lemma B.3. *For any meet filter F of $\mathrm{L}(\mathbf{M})$, $F \in \mathrm{Pr}(\mathrm{L}(\mathbf{M}))$ if and only if $\bigwedge m[F] \in \mathcal{J}^\infty(\mathrm{L}(\mathbf{M})^\delta)$.*

Proof. Let $F \in \mathrm{Fi}_\wedge(\mathrm{L}(\mathbf{M}))$ and assume that $F \in \mathrm{Pr}(\mathrm{L}(\mathbf{M}))$. We show that $\bigwedge m[F]$ is completely join irreducible, i. e. that for all $Y \subseteq \mathrm{L}(M)^\delta$, if $\bigwedge m[F] \leq \bigvee Y$, then there is $y \in Y$ such that $\bigwedge m[F] \leq y$. By denseness, we can assume that all elements in Y are closed. So let $\{F_s : s \in S\} \subseteq \mathrm{Fi}_\wedge(\mathrm{L}(\mathbf{M}))$ and assume that $\bigwedge m[F] \leq \bigvee_{s\in S} \bigwedge m[F_s]$. Suppose, towards a contradiction, that for all $s \in S$, $\bigwedge m[F] \nleq \bigwedge m[F_s]$. Then for each $s \in S$ there is $x_s \in F_s$ such that $x_s \notin F$. By F prime filter, for all $S' \subseteq^\omega S$, $\bigvee_{s\in S'} x_s \notin F$. And then by compactness $\bigvee_{s\in S} x_s \notin F$, so $\bigwedge m[F] \nleq \bigvee_{s\in S} m(x_s)$. But since for each $s \in S$, we have that $x_s \in F_s$, then $\bigwedge m[F_s] \leq m(x_s)$, and then by the hypothesis $\bigwedge m[F] \leq \bigvee_{s\in S} \bigwedge m[F_s] \leq \bigvee_{s\in S} m(x_s)$, a contradiction.

Let now $F \in \mathrm{Fi}_\wedge(\mathrm{L}(\mathbf{M}))$ and assume that $\bigwedge m[F] \in \mathcal{J}^\infty(\mathrm{L}(\mathbf{M})^\delta)$. We show that for all $x_1, x_2 \in \mathrm{L}(M)$, if $x_1 \cup x_2 \in F$ then $x_1 \in F$ or $x_2 \in F$. Recall that any element $x \in \mathrm{L}(M)$ is of the form $x = \bigcup\{\sigma(a) : a \in A\}$ for some $A \subseteq^\omega M$. So, for $x_1, x_2 \in \mathrm{L}(M)$, $x_1 \cup x_2 = \bigcup\{\sigma(c) : c \in C\}$ for some $C \subseteq^\omega M$. So let $C \subseteq^\omega M$ and assume that $\bigcup\{\sigma(c) : c \in C\} \in F$. We show that there is $c \in C$ such that $\sigma(c) \in F$. Using that m preserves finite joins, we get $\bigwedge m[F] \leq m(\bigcup\{\sigma(c) : c \in C\}) = \bigvee m[\sigma[C]]$. By hypothesis $\bigwedge m[F]$ is completely join irreducible, so there is $c \in C$ such that $\bigwedge m[F] \leq m(\sigma(c))$. Then by compactness we obtain $\sigma(c) \in F$, as required. □

Lemma B.4. *For all $c \in \mathcal{J}^\infty(\mathrm{L}(\mathbf{M})^\delta)$, there is $F \in \mathrm{Pr}(\mathrm{L}(\mathbf{M}))$ such that $c = \bigwedge m[F]$.*

Proof. This follows as a corollary of Lemma B.3. □

From previous lemmas and the relations between filters and ideals of $\mathbf{M}$ and $\mathrm{L}(\mathbf{M})$ that were presented in Appendix A, we get the following theorem.

Theorem B.5. *The canonical extension $\mathrm{L}(\mathbf{M})^\delta$ of $\mathrm{L}(\mathbf{M})$ is (up to isomorphism) the $(\mathrm{Fi}_\wedge(\mathbf{M}), \mathrm{Id}_F(\mathbf{M}))$-completion of $\mathbf{M}$.*

Proof. We show that k gives us the required dense and compact embedding.

Claim B.6. $\mathrm{L}(\mathbf{M})^\delta$ *is* $(\mathrm{Fi}_\wedge(\mathbf{M}), \mathrm{Id}_F(\mathbf{M}))$*-compact.*

Proof of the claim. Let $F \in \mathrm{Fi}_\wedge(\mathbf{M})$ and $I \in \mathrm{Id}_F(\mathbf{M})$ and suppose that $\bigwedge k[F] \leq \bigvee k[I]$. Then by lemmas B.1 and B.2, $\bigwedge m[\uparrow\sigma[F]] \leq \bigvee m[\langle\!\langle \sigma[I] \rrbracket]$. And then since $\uparrow\sigma[F] \in \mathrm{Fi}_\wedge(\mathrm{L}(\mathbf{M}))$ and $\langle\!\langle \sigma[I] \rrbracket \in \mathrm{Id}(\mathrm{L}(\mathbf{M}))$, by compactness we get that there is $x \in \uparrow\sigma[F] \cap \langle\!\langle \sigma[I] \rrbracket \neq \emptyset$. So there is $a \in F$ such that $\sigma(a) \leq x \in \langle\!\langle \sigma[I] \rrbracket$, and so $\sigma(a) \in \langle\!\langle \sigma[I] \rrbracket$. Then from results in Appendix A, $a \in \sigma^{-1}[\langle\!\langle \sigma[I] \rrbracket] = I$, so $F \cap I \neq \emptyset$, as required. □

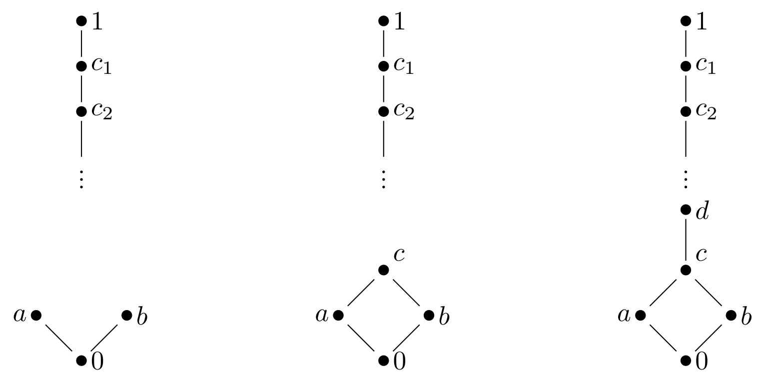
Claim B.7. $\mathrm{L}(\mathbf{M})^\delta$ *is* $(\mathrm{Fi}_\wedge(\mathbf{M}), \mathrm{Id}_F(\mathbf{M}))$*-dense.*

Proof of the claim. First we show that $\mathbb{I}^{\mathrm{Id}_F(\mathbf{M})}(\mathrm{L}(\mathbf{M})^\delta)$ is meet-dense in $\mathrm{L}(\mathbf{M})^\delta$. By denseness we have that for each $z \in \mathrm{L}(M)^\delta$ there is $\mathcal{Y} \subseteq \mathrm{Id}(\mathrm{L}(\mathbf{M}))$ such that $z = \bigwedge\{\bigvee m[I] : I \in \mathcal{Y}\}$. By Lemma B.2 and results in Appendix A, $\bigvee m[I] = \bigvee k[\sigma^{-1}[I]]$, and by results in Appendix A, $\sigma^{-1}[I] \in \mathrm{Id}_F(\mathbf{M})$, so we are done.

Now we show that $\mathbb{F}^{\mathrm{Fi}_\wedge(\mathbf{M})}(\mathrm{L}(\mathbf{M})^\delta)$ is join-dense in $\mathrm{L}(\mathbf{M})^\delta$. Recall that $\mathrm{L}(\mathbf{M})^\delta$ is an algebraic lattice, so every completely join irreducible element is completely join prime. Therefore, for all $z \in \mathrm{L}(M)^\delta$, $z = \bigvee Y$ for some $Y \subseteq \mathcal{J}^\infty(\mathrm{L}(\mathbf{M})^\delta)$. Then by Lemma B.4, $z = \bigvee\{\bigwedge m[F] : F \in \mathcal{X}\}$ for some $\mathcal{X} \subseteq \mathrm{Pr}(\mathrm{L}(\mathbf{M}))$. By Lemma B.1, $\bigwedge m[F] = \bigwedge k[\sigma^{-1}[F]]$ for each $F \in \mathcal{X}$, and by results in Appendix A, $\sigma^{-1}[F] \in \mathrm{Op}_\wedge(\mathbf{M}) \subseteq \mathrm{Fi}_\wedge(\mathbf{M})$, so we are done. □

We have shown that the canonical extension $\mathrm{L}(\mathbf{M})^\delta$ of $\mathrm{L}(\mathbf{M})$ is a completion of $\mathbf{M}$ that is $(\mathrm{Fi}_\wedge(\mathbf{M}), \mathrm{Id}_F(\mathbf{M}))$-compact and $(\mathrm{Fi}_\wedge(\mathbf{M}), \mathrm{Id}_F(\mathbf{M}))$-dense. We conclude that $\mathrm{L}(\mathbf{M})^\delta$ is, up to isomorphism, the $(\mathrm{Fi}_\wedge(\mathbf{M}), \mathrm{Id}_F(\mathbf{M}))$-completion of $\mathbf{M}$. □

From now on we consider k as the identity map and we denote the F-extension of $\mathbf{M}$ simply by $\mathbf{M}^F$. Moreover, we denote the collection of all $\mathrm{Fi}_\wedge(\mathbf{M})$-filter elements of $\mathbf{M}^F$ by $\mathcal{C}(\mathbf{M}^F)$ (or simply $\mathcal{C}$), and we call its elements *closed elements*. Similarly



$\mathbf{M}$ $\quad\quad$ $\mathbf{M}^\delta \cong \mathrm{L}(\mathbf{M})$ $\quad\quad$ $\mathbf{M}^F \cong \mathrm{L}(\mathbf{M})^\delta$

FIGURE 10. Example of a distributive semilattice for which the canonical extension and the F-extension are different.

$\mathcal{O}(\mathbf{M}^F)$ (or simply $\mathcal{O}$) denotes the collection of all $\mathrm{Id}_F(\mathbf{M})$-ideal elements of $\mathbf{M}^F$, that are called *open elements*.[1]

The following example shows that the canonical extension and the F-extension of a distributive semilattice may not be isomorphic. However, from the fact that $\mathrm{Id}(\mathbf{M}) \subseteq \mathrm{Id}_F(\mathbf{M})$ for any distributive semilattice $\mathbf{M}$, it always holds that $\mathbf{M}^\delta$, the canonical extension of $\mathbf{M}$, is embeddable in $\mathbf{M}^F$.

EXAMPLE B.8. We consider again the distributive semilattice $\mathbf{M}$ that we introduced in Example 6.5.24 (see Figure 10). On the one hand, all ideals of $\mathbf{M}$ are principal ideals, and the only filter of $\mathbf{M}$ that is not a principal filter is $F_c = \{1\} \cup \{c_i : i \in \omega\}$. From the general theory we get that the canonical extension $\mathbf{M}^\delta$ of $\mathbf{M}$ is the lattice obtained by adding the point c as shown in Figure 10. The distributive envelope $\mathrm{L}(\mathbf{M})$ of $\mathbf{M}$ was studied in detail in [**5**], and in this case it turns out to be isomorphic to $\mathbf{M}^\delta$. Then the F-extension of $\mathbf{M}$, that by Theorem B.5 is the canonical extension of $\mathrm{L}(\mathbf{M})$ is the complete distributive lattice obtained by adding the point d as shown in Figure 10. Hence $\mathbf{M}^\delta$ and $\mathbf{M}^F$ are not isomorphic. Moreover, we see that $\mathbf{M}^\delta$ embeds into $\mathbf{M}^F$.

Notice that for the F-extension, the usual arguments about duality (when arguing for the underlying posets) are not valid, since complements of filters are not necessarily F-ideals. What it is still true is that finite meets and joins are preserved, since for all $I \in \mathrm{Id}_F(\mathbf{M})$, I is closed under existing finite joins, and for all $F \in \mathrm{Fi}_\wedge(\mathbf{M})$, F is closed under existing finite meets. Furthermore $\mathrm{Fi}_\wedge(\mathbf{M})$ and $\mathrm{Id}_F(\mathbf{M})$ are algebraic closure systems, so compactness holds. Let us verify one more technical issue, taking special care of the bounds.

LEMMA B.9. *Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be distributive semilattices. Then:*

[1]Notice that a similar notation is used in the rest of the dissertation for open and closed subsets of a topological space. This should not lead to confusion, as it is only in the remaining part of the appendix that open and closed elements of a completion appear.

(1) $\mathrm{Fi}_\wedge(\mathbf{M}_1 \times \mathbf{M}_2) = \mathrm{Fi}_\wedge(\mathbf{M}_1) \times \mathrm{Fi}_\wedge(\mathbf{M}_2)$.
(2) $\mathrm{Id}_F(\mathbf{M}_1 \times \mathbf{M}_2) = \mathrm{Id}_F(\mathbf{M}_1) \times \mathrm{Id}_F(\mathbf{M}_2)$.

PROOF. Notice that $\mathbf{M}_1 \times \mathbf{M}_2$ has a bottom element if and only if both $\mathbf{M}_1$ and $\mathbf{M}_2$ have a bottom element. As we assume that the semilattices have a top element, meet filters are non-empty. It is well known that the equality for meet filters holds. We just have to check the equality for F-ideals.

Let us first show that for any $I_1 \in \mathrm{Id}_F(\mathbf{M}_1)$ and $I_2 \in \mathrm{Id}_F(\mathbf{M}_2)$, it holds that $I_1 \times I_2 := \{(a_1, a_2) : a_1 \in I_1, a_2 \in I_2\} \in \mathrm{Id}_F(\mathbf{M}_1 \times \mathbf{M}_2)$. Consider first the case $I_1, I_2 \neq \emptyset$. Then let $(a_{1_i}, a_{2_i}) \in I_1 \times I_2$, with $i \leq n$ for some $n \in \omega$, and suppose $\bigcap\{\uparrow(a_{1_i}, a_{2_i}) : i \leq n\} \subseteq \uparrow(b_1, b_2)$ for some $(b_1, b_2) \in \mathbf{M}_1 \times \mathbf{M}_2$. We show that $b_1 \in I_1$ (the other case is analogous). Suppose, towards a contradiction, that $\bigcap\{\uparrow a_{1_i} : i \leq n\} \nsubseteq \uparrow b_1$. Then there is $z \geq a_{1_i}$ for all $i \leq n$ such that $z \ngeq b_1$. Therefore $(z, 1) \in \bigcap\{\uparrow(a_{1_i}, a_{2_i}) : i \leq n\}$ and $(z, 1) \notin \uparrow(b_1, b_2)$, contrary to the assumption. Thus we obtain $\bigcap\{\uparrow a_{1_i} : i \leq n\} \subseteq \uparrow b_1$, and since I_1 is an F-ideal, then $b_1 \in I_1$. Similarly we obtain $b_2 \in I_2$, and therefore $(b_1, b_2) \in I_1 \times I_2$, as required. Assume now that $I_1 = \emptyset$ (the case $I_2 = \emptyset$ follows analogously). By assumption $I_1 \times I_2 = \emptyset$, and we know that $\mathbf{M}_1$ has no bottom element. By the remark above, this implies that $\mathbf{M}_1 \times \mathbf{M}_2$ has no bottom element, and so $I_1 \times I_2 = \emptyset$ is a Frink-ideal of $\mathbf{M}_1 \times \mathbf{M}_2$.

For the other inclusion, let $I \in \mathrm{Id}_F(\mathbf{M}_1 \times \mathbf{M}_2)$. We show that $I = I_1 \times I_2$ for some $I_1 \in \mathrm{Id}_F(\mathbf{M}_1)$ and $I_2 \in \mathrm{Id}_F(\mathbf{M}_2)$. Consider first the case $I \neq \emptyset$. Notice that for this case $I = \pi_1(I) \times \pi_2(I)$, where $\pi_1(I) := \{a \in M_1 : \exists b \in M_2((a, b) \in I)\}$ and $\pi_2(I) := \{b \in M_2 : \exists a \in M_1((a, b) \in I)\}$, as if we take $(a_1, a_2), (b_1, b_2) \in I$, from $\uparrow(a_1, a_2) \cap \uparrow(b_1, b_2) \subseteq \uparrow(a_1, b_2)$ and I being a Frink ideal it follows $(a_1, b_2) \in I$. Thus if we show that $\pi_1(I) \in \mathrm{Id}_F(\mathbf{M}_1)$ and $\pi_2(I) \in \mathrm{Id}_F(\mathbf{M}_2)$ we are done. Let $(a_i, b_i) \in I$, for some $i \leq n$ be such that $\bigcap\{\uparrow a_i : i \leq n\} \subseteq \uparrow c$ for some $c \in M_1$. We show that $c \in \pi_1(I)$. Clearly, the assumption implies that $\bigcap\{\uparrow(a_i, b_i) : i \leq n\} \subseteq \uparrow(c, \bigwedge\{b_i : i \leq n\})$. Therefore, since I is an F-ideal, we get $(c, \bigwedge\{b_i : i \leq n\}) \in I$, and so $c \in \pi_1(I)$, as required. We have shown that $\pi_1(I)$ is an F-ideal of $\mathbf{M}_1$, and the proof for $\pi_2(I)$ is analogous. Assume now that $I = \emptyset$. Then $\mathbf{M}_1 \times \mathbf{M}_2$ has no bottom element, and then either $\mathbf{M}_1$ or $\mathbf{M}_2$ have no bottom element. Assume, without loss of generality, that $\mathbf{M}_1$ has no bottom element. Then $\emptyset \in \mathrm{Id}_F(\mathbf{M}_1)$. Let $I_2 \in \mathrm{Id}_F(\mathbf{M}_2)$ be any F-ideal of $\mathbf{M}_2$. Then clearly $I = \emptyset \times I_2$, for F-ideals $\emptyset$ and I_2 of $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively, so we are done. $\Box$

Let us have a look at the extensions of order preserving and order reversing maps to the canonical extension and the F-extension of a distributive semilattices with top element.

For $f : \mathbf{M}_1 \longrightarrow \mathbf{M}_2$ an order preserving map between distributive meet semilattices $\mathbf{M}_1$ and $\mathbf{M}_2$, we define the σ and the π extension of f to the F-extension of $\mathbf{M}_1$ and $\mathbf{M}_2$, such that for any $u \in M_1^F$:

$$f^\sigma(u) := \bigvee \big\{ \bigwedge\{f(p) : x \leq p, p \in M_1\} : x \leq u, x \in \mathcal{C}_1 \big\},$$

$$f^\pi(u) := \bigwedge \big\{ \bigvee\{f(p) : p \leq y, p \in M_1\} : u \leq y, y \in \mathcal{O}_1 \big\}.$$

It is easy to check that the following statements hold:

LEMMA B.10. *For all* $x \in \mathcal{C}_1$, $y \in \mathcal{O}_1$, $u \in M_1^F$*:*

(1) $f^\sigma(x) = \bigwedge\{f(p) : x \leq p, p \in M_1\}$.
(2) f^σ *sends closed elements to closed elements.*
(3) f^σ *extends* f.
(4) f^σ *is order preserving on* $\mathcal{C}_1$.
(5) $f^\sigma(u) = \bigvee\{f^\sigma(x) : x \leq u, x \in \mathcal{C}_1\}$.

PROOF. (1) follows immediately from f being order preserving. For (2), we show first that $\uparrow\{f(p) : x \leq p \in M_1\} \in \mathrm{Fi}_\wedge(\mathbf{M}_2)$. Clearly this set is an up-set, so it is just left to show that it is closed under meets. Let $a, b \in \uparrow\{f(p) : x \leq p \in M_1\}$, so there are $p_1, p_2 \in M_1$, such that $x \leq p_1, p_2$, $f(p_1) \leq a$ and $f(p_2) \leq b$. As $x = \bigwedge F$ for some $F \in \mathrm{Fi}_\wedge(\mathbf{M}_1)$ and (by compactness) $p_1, p_2 \in F$, so $p_1 \wedge p_2 \in F$. Then $x \leq p_1 \wedge p_2$. By f order preserving $f(p_1 \wedge p_2) \leq f(p_1) \wedge f(p_2)$ and since $f(p_1) \wedge f(p_2) \leq a \wedge b$, then $a \wedge b \in \uparrow\{f(p) : x \leq p \in M_1\}$, as required. So we have $f^\sigma(x) = \bigwedge\{f(p) : x \leq p \in M_1\} = \bigwedge \uparrow\{f(p) : x \leq p \in M_1\}$, and we are done. (3)-(5) follow easily. □

LEMMA B.11. *For all* $x \in \mathcal{C}_1$, $y \in \mathcal{O}_1$, $u \in M_1^F$*:*

(1) $f^\pi(y) = \bigvee\{f(p) : p \leq y, p \in M_1\}$.
(2) f^π *sends open elements to open elements.*
(3) f^π *extends* f.
(4) f^π *is order preserving on* $\mathcal{O}_1$.
(5) $f^\pi(u) = \bigwedge\{f^\pi(y) : u \leq y, y \in \mathcal{O}_1\}$.

PROOF. (1) is immediate. For (2), let $I \in \mathrm{Id}_F(\mathbf{M}_1)$ be the F-ideal such that $\bigvee I = y$. We show that $\bigvee \llangle f[I] \rrbracket = \bigvee\{f(p) : p \leq y, p \in M_1\}$. Notice that by compactness, we get that $\bigvee\{f(p) : p \leq y, p \in M_1\} = \bigvee f[I]$. Therefore, since $f[I] \subseteq \llangle f[I] \rrbracket$, we just have to show that $\bigvee \llangle f[I] \rrbracket \leq \bigvee\{f(p) : p \leq y, p \in M_1\}$. Let $z \in \llangle f[I] \rrbracket$, so there are $a_0, \ldots, a_n \in I$ such that $\bigcap\{\uparrow f(a_i) : i \leq n\} \subseteq \uparrow z$. This implies, by definition of the distributive envelope, that $z \leq \bigvee\{f(a_i) : i \leq n\} \leq \bigvee f[I]$. Thus $\bigvee \llangle f[I] \rrbracket \leq \bigvee\{f(p) : p \leq y, p \in M_1\}$, and we are done. (3)-(5) follow easily. □

Notice that for order reversing maps we cannot argue by duality, due to the lack of symmetry between meet filters an F-ideals of meet semilattices. For $g : \mathbf{M}_1 \longrightarrow \mathbf{M}_2$ an order reversing map between distributive meet semilattice $\mathbf{M}_1$ and $\mathbf{M}_2$, we define the σ and the π extension of g to the F-extension of $\mathbf{M}_1$ and $\mathbf{M}_2$ as follows: for all $u \in M_1^F$

$$g^\sigma(u) := \bigvee \Big\{ \bigwedge\{g(p) : p \leq y, p \in M_1\} : u \leq y, y \in \mathcal{O}_1 \Big\},$$

$$g^\pi(u) := \bigwedge \Big\{ \bigvee\{g(p) : x \leq p, p \in M_1\} : x \leq u, x \in \mathcal{C}_1 \Big\}.$$

It is easy to check that the following statements hold, that are analogues of the ones in lemmas B.10 and B.11.

LEMMA B.12. *For all* $x \in \mathcal{C}_1$, $y \in \mathcal{O}_1$, $u \in M_1^F$*:*

(1) $g^\sigma(y) = \bigwedge\{g(p) : p \leq y, p \in M_1\}$.
(2) g^σ *sends open elements to closed elements.*
(3) g^σ *extends* f.

(4) g^σ *is order reversing on* $\mathcal{O}_1$.
(5) $g^\sigma(u) = \bigvee\{f^\sigma(y) : u \leq y, x \in \mathcal{O}_1\}$.

Proof. (1) is immediate. For (2), let $I \in \mathrm{Id}_F(\mathbf{M}_1)$ be the F-ideal such that $\bigvee I = y$. We show that $\bigwedge [\![g[I]]\!\rangle\!\rangle = \bigwedge\{g(p) : p \leq y, p \in M_1\}$. Notice that by compactness, we get that $\bigwedge\{g(p) : p \leq y, p \in M_1\} = \bigwedge g[I]$. Therefore, since $g[I] \subseteq [\![g[I]]\!\rangle\!\rangle$, we just have to show that $\bigwedge [\![g[I]]\!\rangle\!\rangle \leq \bigwedge\{g(p) : p \leq y, p \in M_1\}$: Let $z \in [\![f[I]]\!\rangle\!\rangle$, so there are $a_0, \dots, a_n \in I$ such that $g(a_0) \wedge \cdots \wedge g(a_n) \leq z$. Then $\bigwedge g[I] \leq z$ and we are done. (3)-(5) follow easily. $\square$

Lemma B.13. *For all* $x \in \mathcal{C}_1$, $y \in \mathcal{O}_1$, $u \in M_1^F$:
(1) $g^\pi(x) = \bigvee\{g(p) : x \leq p, p \in M_1\}$.
(2) g^π *sends closed elements to open elements.*
(3) g^π *extends* f.
(4) g^π *is order reversing on* $\mathcal{C}_1$.
(5) $g^\pi(u) = \bigwedge\{f^\pi(x) : x \leq u, x \in \mathcal{C}_1\}$.

Proof. (1) is immediate. For (2), we show first that $\downarrow\{g(p) : x \leq p \in M_1\}$ is an F-ideal of $\mathbf{M}_2$. Clearly the set $\downarrow\{g(p) : x \leq p \in M_1\}$ is a down-set. Let $a, b \in \downarrow\{g(p) : x \leq p \in M_1\}$ and $c \in M_2$ and suppose that $\uparrow a \cap \uparrow b \subseteq \uparrow c$. We have to show that $c \in \downarrow\{g(p) : x \leq p \in M_1\}$. By assumption there are $p_1, p_2 \in M_1$, $x \leq p_1, p_2$ such that $a \leq g(p_1)$ and $b \leq f(p_2)$. Since g is order reversing, then $g(p_1), g(p_2) \leq g(p_1 \wedge p_2)$. Then $\uparrow g(p_1) \cap \uparrow g(p_2) \subseteq \uparrow c$, and then $g(p_1 \wedge p_2) \in \uparrow c$. Thus $c \leq g(p_1 \wedge p_2) \in \{g(p) : x \leq p \in M_1\}$, as required. So we have $g^\pi(x) = \bigvee\{g(p) : x \leq p \in M_1\} = \bigvee \downarrow\{g(p) : x \leq p \in M_1\}$, and we are done. (3)-(5) follow easily. $\square$

Once we have studied how order preserving and order reversing maps between distributive semilattices can be extended to the F-extensions, we want to apply this to the study of implicative semilattices. It is well known that the canonical extension of the semilattice reduct of an implicative semilattice $\mathbf{N} = \langle N, \to, \wedge, 1\rangle$, augmented with the π-extension of the implication, is a Heyting algebra [**26**].

Recall that from Lemma B.9 we obtain that the π extension of a binary function $f : \mathbf{M}_1 \times \mathbf{M}_2 \longrightarrow \mathbf{M}_3$ that is order preserving in the second coordinate and order reversing in the first coordinate is given by:

$$f^\pi(u, v) := \bigwedge\Big\{\bigvee\{f(p, q) : x \leq p, q \leq y, p \in M_1, q \in M_2\} : x \leq u, v \leq y, x \in \mathcal{C}_1, y \in \mathcal{O}_2\Big\}$$

We aim to show that the F-extension of the semilattice reduct of an implicative semilattice $\mathbf{N} = \langle N, \to, \wedge, 1\rangle$, augmented with the π-extension of the implication, is also a Heyting algebra, and so it is in particular an implicative semilattice.

Let $\mathbf{N} = \langle N, \to, \wedge, 1\rangle$ be an implicative semilattice. Recall that by definition $\to$ is the right residuum of $\wedge$, so for all $a, b, c \in N$:

$$a \wedge c \leq b \quad \text{iff} \quad c \leq a \to b.$$

Let $\mathbf{N}^\delta = \langle N^\delta, \wedge^{\mathbf{N}^\delta}, \vee^{\mathbf{N}^\delta}, 0^{\mathbf{N}^\delta}, 1^{\mathbf{N}^\delta}\rangle$ be the canonical extension of the semilattice reduct of $\mathbf{N}$, and let $\mathbf{N}^F = \langle N^F, \wedge^{\mathbf{N}^F}, \vee^{\mathbf{N}^F}, 0^{\mathbf{N}^F}, 1^{\mathbf{N}^F}\rangle$ be the F-extension of the semilattice reduct of $\mathbf{N}$. We already know that $\wedge^{\mathbf{N}^\delta}$ coincides with the σ-extension of $\wedge$ in $\mathbf{N}^\delta$ and $\vee^{\mathbf{N}^\delta}$ coincides with the σ-extension of $\vee$ in $\mathbf{N}^\delta$.

LEMMA B.14. *For any implicative semilattice* $\mathbf{N}$*,* $\wedge^{\mathbf{N}^F}$ *coincides with the* σ*-extension of* $\wedge$ *in* $\mathbf{N}^F$*.*

PROOF. Let us denote the σ-extension of $\wedge$ in $\mathbf{N}^F$ by $\wedge^\sigma$. First we show that $\wedge^{\mathbf{N}^F}$ and $\wedge^\sigma$ coincide for closed elements. Let $p_1, p_2 \in \mathcal{C}$, and let $F_1, F_2 \in \mathrm{Fi}_\wedge(\mathbf{N})$ be the filters such that $p_1 = \bigwedge F_1$ and $p_2 = \bigwedge F_2$. On the one hand, by commutativity of $\wedge^{\mathbf{N}^F}$ we have:

$$p_1 \wedge^{\mathbf{N}^F} p_2 = \bigwedge F_1 \wedge^{\mathbf{N}^F} \bigwedge F_2 = \bigwedge\{a_1 \wedge^{\mathbf{N}^F} a_2 : a_1 \in F_1, a_2 \in F_2\}.$$

On the other hand, since $\wedge$ is order preserving, we have:

$$p_1 \wedge^\sigma p_2 = \bigwedge\{a_1 \wedge a_2 : p_1 \leq a_1 \in N, p_2 \leq a_2 \in N\}.$$

And since $a \wedge b = a \wedge^{\mathbf{N}^F} b$ for all $a, b \in N$, using compactness we obtain $p_1 \wedge^{\mathbf{N}^F} p_2 = p_1 \wedge^\sigma p_2$. Notice that $p_1 \wedge^{\mathbf{N}^F} p_2 = \bigwedge \uparrow\{a_1 \wedge^{\mathbf{N}^F} a_2 : a_1 \in F_1, a_2 \in F_2\}$ and moreover $\uparrow\{a_1 \wedge^{\mathbf{N}^F} a_2 : a_1 \in F_1, a_2 \in F_2\} \in \mathrm{Fi}_\wedge(\mathbf{N})$, so $p_1 \wedge^{\mathbf{N}^F} p_2$ is also a closed element of $\mathbf{N}^F$.

Now we show that for all $x_1, x_2 \in N^F$, $x_1 \wedge^{\mathbf{N}^F} x_2 = x_1 \wedge^\sigma x_2$. On the one hand, by $(\mathrm{Fi}_\wedge(\mathbf{N}), \mathrm{Id}_F(\mathbf{N}))$-denseness, we have:

$$x_1 \wedge^{\mathbf{N}^F} x_2 = \bigvee\{p \in \mathcal{C} : p \leq (x_1 \wedge^{\mathbf{N}^F} x_2)\}.$$

On the other hand, since $\wedge$ is order preserving, by Lemma B.10 we have:

$$x_1 \wedge^\sigma x_2 = \bigvee\{p_1 \wedge^\sigma p_2 : p_1 \leq x_1, p_2 \leq x_2, p_1, p_2 \in \mathcal{C}\}.$$

From above we conclude that $x_1 \wedge^{\mathbf{N}^F} x_2 = x_1 \wedge^\sigma x_2$, as required. $\square$

THEOREM B.15. *For any implicative semilattice* $\mathbf{N}$*, the* π*-extension of* $\to$ *in* $\mathbf{N}^F$ *is the right residuum of* $\wedge^{\mathbf{N}^F}$*.*

PROOF. Let $u, v, w \in N^F$. By simplicity we will denote $\wedge^{\mathbf{N}^F} = \wedge^\sigma$ by $\wedge$. First we show that $v \leq u \to^\pi w$ implies $u \wedge v \leq w$.

Let us prove first the easy case: let $s, t \in \mathcal{C}$ and $y \in \mathcal{O}$ and suppose $s \wedge t \leq y$. Recall that by s, t closed, then $s \wedge t = \bigwedge \uparrow\{p \wedge q, p, q \in N, s \leq p, t \leq q\}$ is also closed, and so by $(\mathrm{Fi}_\wedge(\mathbf{N}), \mathrm{Id}_F(\mathbf{N}))$-compactness, there are $p, q \in N$ with $s \leq p$, $t \leq q$ and $p \wedge q \leq y$. By $\mathbf{N}$ being an implicative semilattice (by residuation) we have $q \leq p \to (p \wedge q)$. Notice that $s \to^\pi y = \bigvee\{p' \to q' : p', q' \in N, s \leq p', q' \leq y\}$. Therefore, we have $t \leq q \leq p \to (p \wedge q)$ for $s \leq p$ and $p \wedge q \leq y$. Hence $t \leq s \to^\pi y$, as required.

For the general case, suppose that $u \wedge v \leq w$. We show that $v \leq u \to^\pi w$. Recall that $u \to^\pi w = \bigwedge\{s \to^\pi y : s \in \mathcal{C}, s \leq u, w \leq y \in \mathcal{O}\}$ and moreover by $(\mathrm{Fi}_\wedge(\mathbf{N}), \mathrm{Id}_F(\mathbf{N}))$-denseness, $v = \{\bigvee t \in \mathcal{C} : t \leq v\}$. Let $s, t \in \mathcal{C}$ and $y \in \mathcal{O}$ be such that $s \leq u$, $t \leq v$ and $w \leq y$. If we show that $t \leq s \to^\pi y$, we are done. Using the assumption we get that $s \wedge t \leq u \wedge v \leq w \leq y$. Then by the easy case $t \leq s \to^\pi y$, as required.

For the converse, assume that $v \leq u \to^\pi w$. We show that $u \wedge v \leq w$. By $(\mathrm{Fi}_\wedge(\mathbf{N}), \mathrm{Id}_F(\mathbf{N}))$-denseness, $w = \bigwedge\{y \in \mathcal{O} : w \leq y\}$, and moreover $u \wedge v = \bigvee\{s \wedge t : s, t \in \mathcal{C}, s \leq u, t \leq v\}$. Let $y \in \mathcal{O}, s, t \in \mathcal{C}$ be such that $s \leq u$, $t \leq v$ and $w \leq y$. If we show that $s \wedge t \leq y$, we are done. By assumption $t \leq v \leq u \to^\pi w = \bigwedge\{s \to^\pi y : s \in \mathcal{C}, s \leq u, w \leq y \in \mathcal{O}\} \leq s \to^\pi y$. Recall that by definition $s \to^{\pi} y = \bigvee\{p \to r : p, r \in N, s \leq p, r \leq y\}$. Let us define the set $X := \{p \to r : p, r \in N, s \leq p, r \leq y\}$. Using that $\bigvee X = \bigvee \langle\!\langle X \rrbracket$, and $(\mathrm{Fi}_{\wedge}(\mathbf{N}), \mathrm{Id}_F(\mathbf{N}))$-compactness, we obtain that there are $p_0, \ldots, p_n, r_0, \ldots, r_n \in N$ such that $s \leq p_i, r_i \leq y$ for all $i \leq n$ and $t \leq \bigvee\{p_i \to^{\pi} r_i : i \leq n\}$.

Let $p := \bigwedge\{p_i : i \leq n\} \in N$. From $p \leq p_i$, since $\to$ is order reversing in the first coordinate and order preserving in the second one, we obtain $p_i \to r_i \leq p \to r_i$ for all $i \leq n$, and so $t \leq \bigvee\{p_i \to r_i : i \leq n\} \leq \bigvee\{p \to r_i : i \leq n\}$. Now by residuation, for all $i \leq n$ we have $p \wedge (p \to r_i) \leq r_i \leq y$. Then $\bigvee\{p \wedge (p \to r_i) : i \leq n\} \leq y$. Using that $\mathbf{N}^F$ is a distributive lattice, we get that $\bigvee\{p \wedge (p \to r_i) : i \leq n\} = p \wedge \bigvee\{p \to r_i : i \leq n\}$. Therefore, since $t \leq \bigvee\{p \to r_i : i \leq n\}$ and $s \leq p = p_0 \wedge \cdots \wedge p_n$, we get $s \wedge t \leq y$, as required. $\square$

In conclusion, we have shown that for any distributive meet-semilattice with top element $\mathbf{M}$, the F-extension $\mathbf{M}^F$ of $\mathbf{M}$ is (up to isomorphism) the canonical extension of the distributive envelope of $\mathbf{M}$. Moreover, for any implicative semilattice $\mathbf{N}$, the F-extension of $\mathbf{N}$ augmented with the π extension of the implication is a Heyting algebra. This is an important result for defining a logic-based notion of canonical extension of Hilbert algebras. It is worth noticing that the canonical extension of a Hilbert algebra $\mathbf{A} = \langle A, \to, 1\rangle$, extended with the π extension of the implication, may fail to be a Hilbert algebra. This is shown in following example.

Example B.16. Consider again the distributive semilattice $\mathbf{M}$ in Figure 10 (see page 200). Let $\to$ be the implication given by the order in $\mathbf{M}$, and so let $\overline{\mathbf{M}} := \langle M, \to, 1\rangle$ be the resulting Hilbert algebra. We know that the canonical extension of $\overline{\mathbf{M}}$ is $\mathbf{M}^{\delta}$. By definition of the π extension of $\to$ in $\overline{\mathbf{M}}^{\delta}$, we have:

$$c \to^{\pi} c = \bigvee^{\mathbf{M}^{\delta}}\{p \to q : c \leq p, q \leq c, p, q \in M\} = c \neq 1,$$

but then $\langle M^{\delta}, \to^{\pi}, 1^{\mathbf{M}^{\delta}}\rangle$ is not a Hilbert algebra since condition (K) fails.

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(H$^\vee$4)	B is closed under finite unions	132
(IA1)	if $(a \to b = 1$ & $b \to c = 1)$, then $a \to c = 1$	30
(IA2)	$a \to 1 = 1$	30
(IH$^\wedge$1)	$\mathrm{cl}((\mathrm{sat}_{\widehat{X}}(U \cap V^c))^c \cap \widehat{X})^c \in \kappa$, for any $U, V \in \kappa$	189
(IH$^\wedge$2)	$\uparrow((\downarrow(U \cap V^c))^c \cap \widehat{X}) \in B$, for any $U, V \in B$	189
(IS)	$\widehat{X} = X$	186
(IS1)	$(a \to b) \wedge b = b$	133
(IS2)	$a \wedge (a \to b) = a \wedge b$	133
(IS3)	$a \to (b \wedge c) = (a \to c) \wedge (a \to b)$	133
(IS4)	for any $U, V \in \mathcal{KO}(X)$, $\mathrm{sat}(U \cap V^c) \in \mathcal{KO}(X)$	133
(IS5)	for all $U, V \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$, $(\downarrow(U \cap V^c))^c \in \mathcal{C}\ell\mathcal{U}^{ad}_{X_B}(X)$	134
(K)	$a \to a = 1$	30
(MP)	$\gamma, \gamma \to \beta \vdash \beta$	123
(P1)	$\langle X, \preceq \rangle$ is a complete lattice, where $\preceq$ is the quasiorder associated with the referential algebra	103

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