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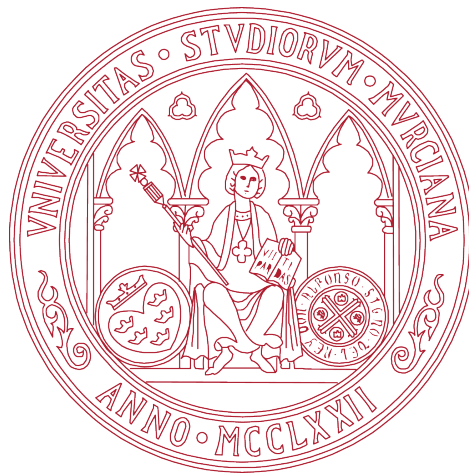
FACULTAD DE QUÍMICA

Gaugings and Other Aspects
in Supergravity

Gaugeos y Otros Aspectos
en Supergravedad

D. Jose Juan Fernández Melgarejo
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Gaugings and other aspects in supergravity



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Jose Juan Fernández Melgarejo

Supervised by Emilio Torrente Luján

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Department of Physics
University of Murcia

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Introduction

The purpose of this manuscript is the study and extension of some theoretical aspects that appear in the context of supergravity theories. These issues are gauged supergravities, black holes and non-geometric fluxes. In order to motivate the research on these topics, we will contextualize the role of supergravity and its importance in the research that is being done nowadays.

Supergravity was proposed as a unifying theory which is able to host the four interactions that, up to now, are known in the Universe. As a good candidate, it offers some appealing ingredients: it is a quantum field theory, namely, it is a quantum and relativistic theory, and includes a spin-2 particle in its spectrum. These three pillars, discovered at the end of the last century, are understood to be indispensable for any theory whose goal is the description of all the interactions.

Despite of its encouraging birth as a unifying theory of all of the interactions, supergravity become ruled out due to its problem of renormalizability. However, it keeps on considered an intriguing and useful topic to be treated by the scientific community. To really understand why it gains relevance in the field of Theoretical Physics, let us glance over the precedents of theories that tried to carry out this unifying goal. The search of a unifying theory that describes the physics of the Universe has been a challenge for humanity since long time ago. Theories of the matter composition are considered as the first attempts of unifying descriptions. During the 6th century BC some Indian philosophers formulated the first ideas about the composition of the matter¹ [3]. They claimed that there were up to 6 different classes of atoms and assigned up to 24 properties that described their ability to move, vibrate or combine with each other.

In addition, Greek philosophers, led off by Democritus, also studied the concept of ‘atom’. One remarkable fact of this thinking was the justification of any kind of interaction as contact forces between the atoms [4]. During the Classical Greece epoch, Aristotle proposed a different concept of matter composition as well as new ideas about motion and vacuum. At that time, new advances in astronomy and astrophysics were done due to observational measurements.

Despite of during the Dark Ages (from the 5th to the 15th century), not too relevant discoveries came up in Europe. However, Mathematics were particularly improved in by

¹Some historical studies refer to Mochus of Sidon, a Phoenician protophilosopher who lived in the 14th century BC as the author of the first atomic theory that inspired the atomism thinking [1,2]

Muslim scientists. Important mathematicians and physicists developed new concepts that soon reached Europe and were applied and extended. This progress contributed to the rise of an outstanding generation of physicists as Copernicus, Galileo, Newton or Descartes. All of them made contributions to several branches of Physics that result essential for the development of the theories that describe our Universe nowadays.

At this point, we consider necessary to stress some of Newton's contributions. Newton's results supposed a leap in the unification and description of diverse fields like optics or gravity. In particular, his gravitational theory [5] resulted revolutionary because of two facts: it was the first formulation of an interaction without contact and unified Galileo's work on terrestrial gravity, Kepler's laws and the phenomenon of tides by explaining them with one single law.

Michael Faraday was one of the first physicists to search an evidence of unifying theory for gravity and electromagnetism in his experiments [6]. Despite of he did not succeed in this direction, his work was crucial for the formulation of Maxwell equations in 1865. During this century and the following one, it gradually became apparent that many common examples of forces (contact forces, elasticity, viscosity, friction, pressure, ...) result from electrical interactions between the smallest particles of matter.

The beginning of the 20th century brought a trail-blazing proposal. Quantum mechanics formulation supposed a revolution in Physics and, despite of its innovating formalism, its success and predictive power were soon known and recognized. This can be reflected in Dirac's thought, who at the end of the 1920s, claimed that quantum mechanics was tested enough that 'the underlying physical laws necessary for the mathematical theory of a large part of physics and the whole of chemistry are thus completely known' [7]. The emergence of this new formalism attracted the interest of the majority of scientists, leaving the search of a unifying theory as a secondary problem.

It was after *General Relativity* (GR) when the search for the joined theory of gravity and electromagnetism recovered interest. In some sense, this happened because no other interaction but electromagnetism and gravity was expected. Some authors like Nordström, Weyl, Eddington, Kaluza, Klein or Einstein himself developed at that period some contributions that are essential for the new theories that are currently formulated.

The formulation of QED theory, together with the discoveries of the strong and weak interactions impulsed quantum mechanics as a fundamental pillar in any physical theory. This contradicted Einstein's thought, who expected its appearance as a consequence in the framework of a bigger deterministic theory. Moreover, the work done by Glashow, Weinberg and Salam [8–10] in 1968 led to the unification of the electromagnetic and the weak interaction. It seemed that quantum formulation was defeating the geometry approach of Einstein's theory. That is why people emphasized in attempting to unify the unique quantum-mechanically unexplained theory, QCD, with the new electroweak theory.

Despite of there is not a unique theory of the quantum interactions, strong and

electroweak forces coexist in the so-called *Standard Model of particle physics* (SM). This theory showed non-renormalizability problems that were solved by 't Hooft and Veltman [11]. They showed that any quantum field theory whose interactions are based on internal gauge symmetries is renormalizable. In particular, the gauge group of the SM is $SU(3) \times SU(2) \times U(1)$. One remarkable feature of this theory is the prediction of massless particles in its spectrum, which disagrees with the experimental evidences. However, this was solved by the so-called *Higgs mechanism* [12, 13], which breaks the $SU(2) \times U(1)$ in such a way that a potential is generated and the degrees of freedom are rearranged so that some particles become massive. The existence of this mechanism in Nature has been recently proven by means of the detection of the Higgs boson [14, 15].

The phenomena described by the SM happen at the scale energy of 10^2 GeV and its accuracy is excellent [16]. However, when we consider theories that describe the weak and strong interactions as sharing a common origin, the unification scale blows up to 10^{16} GeV. Several *Grand Unification Theories* (GUTs) have been formulated and, in general, only the ones that include an additional ingredient, supersymmetry, can host the bunch of particles that are already known and are not experimentally ruled out.

Supersymmetry (SUSY) is nowadays a crucial ingredient in the study of unification theories. From the theoretical point of view, it can be understood as a trick to circumvent the Coleman-Mandula theorem [17]. This no-go theorem forbids the existence of symmetries that combine spacetime and internal symmetries by means of Lie algebras. However, Haag-Lopuszanski-Shonius theorem [18] demonstrates that supersymmetry, and thus graded Lie algebras, is the only way in that spacetime and internal symmetries can be consistently mixed.

In my opinion, what it is really interesting about supersymmetry is the matching between this natural extension of the group structure of a given theory and its parallel natural consequences from the phenomenological perspective. If we have an arbitrary theory of fermions and bosons that enjoys Poincaré and internal symmetries, Coleman-Mandula theorem does not allow to include more symmetries than those two. Nevertheless, there is one remaining possibility: what if I demand my theory to be invariant under the exchange of fermions and bosons? According to the spin-statistics theorem, bosonic fields commute whereas fermionic fields anticommute. Combining the two kinds of fields into a single algebra requires the introduction of a \mathbb{Z}_2 -grading under which bosons are the even elements and fermions are the odd ones. This is not as trivial as it looks like, and some restrictions have to be imposed. In order to preserve the equilibrium between the bosonic and fermionic sectors, a bunch of new particles that are not predicted by the SM arise. Some critics use this fact to detract SUSY but, despite of this is not a matter of faith, we could think of its similarity to the moment in which Dirac predicted the antiparticles, that required the doubling of all the fermionic particles known. Supersymmetry solves more theoretical problems: it provides a suitable candidate particle to solve the problem of dark matter and cancel the quadratic divergencies of fermions and bosons to the Higgs mass.

At this point, motivated by the success of gauge theories (GR and SM) and the

pleasant scenario that SUSY provides, it seems natural to consider theories based on local supersymmetry. This is the main idea of *Supergravity theories* (SUGRA). Quickly, a good news appears: since the Poincaré generators are part of the superalgebra, requiring local supersymmetry automatically implies local Poincaré invariance, so that GR is automatically guaranteed. The first examples were soon found [19, 20]. The initial excitement over supergravity soon waned, as various failings were discovered: during the 1980s it was proven that supergravities could not be renormalized.

A more rebel idea than supergravity seemed to be required to describe quantum gravity. Then, why are we studying a theory that is already ruled out? To answer this question, a new theory has to be introduced: string theory, the most promising candidate for the theory of everything. Its origin in the context of certain scattering amplitudes for hadrons [21] does not have to do with the unifying interpretation that was first given in [22]. In this work, the authors realized that some bosonic patterns of vibrating strings had similar properties of the graviton. This led to the development of the bosonic string, which soon was generalized to supersymmetric versions. In general, strings can vibrate in many ways and be restricted to several boundary conditions. The different vibrational modes correspond to different types of elementary particles observed in Nature.

One peculiar feature of string theory is the so-called *critical dimension*: all string theories necessarily live in a number of dimensions greater than four. This is motivated by imposing Weyl invariance on the world-sheet metric of the string. That is, similarly to the 1-dimensional worldline traced by a particle in its motion, strings generate a 2-dimensional surface where a metric can be defined. In order to perform a suitable quantization of the string without breaking Poincaré invariance, Weyl invariance of the worldsheet is required. Once this is imposed, a given number of dimensions is required to avoid the conformal anomaly [23, 24]. For the bosonic strings, this critical dimension is $D = 26$, and for the rest of theories, it is $D = 10^2$. Despite of having more than 4 dimensions seems to be a problem, the additional dimensions give us more freedom to mold our original theory towards an effective theory that describes reality accurately. This is done by performing the so-called *Kaluza-Klein (KK) dimensional reductions* [26, 27], which were originally motivated to embed 4-dimensional gravity and electromagnetism in 5-dimensional spacetime. Although from the phenomenological point of view, it is required to have a 4-dimensional theory, the reductions in supergravities and string theories can be done from any arbitrary dimension to any lower dimension.

5 different supersymmetric string theories are known: type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. The discovery of T duality related IIA and IIB theories as well as the heterotic $SO(32)$ and $E_8 \times E_8$ strings. There were found more dualities that connect different regimes of the superstring theories. In addition, there is a perfect agreement between the massless modes of the superstrings and the

²The so-called non-critical string theory is a formulation is an alternative formulation that circumvents this critical dimension keeping a vanishing conformal anomaly. We refer to [25] for its treatment.

field content of certain supergravities. That is, type IIA/IIB superstrings fill the $N = 2A/2B$ SUGRA massless multiplets, type I strings fill the $N = 1$ $D = 10$ massless multiplets, and the same occurs for the heterotic strings and the $N = 1$ $D = 10$ SUGRAs, with their respective gauge multiplets. Furthermore, it was found that the respective supergravities described the weak coupling regime of the corresponding string theories.

On the other hand, new models of compactifications based on Calabi-Yau manifolds were able to host the gauge group of the SM of particles [28]. This period, known as the *first superstring revolution*, finished at the end of the 1980s with the discovery of the D-branes, solitonic objects that were found to be solutions of the corresponding supergravity theories associated to each string theory.

During the 1990s, new boosts were given to this field. Dualities provided a new understanding of the different string theories. The so-called *second superstring revolution* began with the birth of a new 11-dimensional theory, *M theory*, whose aim was the inclusion of the five string theories [29]. It is expected to recover the superstring theories by dimensional reduction of M theory as well as $D = 11$ supergravity when we calculate its low energy limit. M theory would implement the underlying idea of having a unique theory in which all the string theories are particular subcases. The BFSS model [30], a matricial approach to construct this theory, is able to obtain $D = 11$ SUGRA at low energies, but crashes at short distances. In fact, it suggests the need of a non-commutative geometry that, in principle, discards the continuum spacetime over which SUGRA is formulated.

In 1997, the work of Maldacena opened a new enigma in string theory. Studying extremal and charged black holes, he noted that the low energy excitations of the theory near the horizons behaved as the string excitations of the strings near the branes. Thus, he conjectured that string theory on a near-horizon extreme-charged black hole geometry, an $AdS \times S^2$, is equally well described by the $N = 4$ supersymmetric Yang-Mills theory. This is the so-called *AdS/CFT correspondence* [31].

At this point, we are ready to answer the question about the role of supergravity in current research. Supergravity provides a useful scenario to extract some issues and limits of string and M theory. In particular, every string theory gives rise to a supergravity theory in its low energy limit. Similarly, due to the dualities among different string theories, the conjectured 11-dimensional M theory has to reproduce Cremer-Julia-Scherk supergravity [32] as its low energy limit. This means that the results on superstrings can be projected to SUGRA and, what is more appealing, all the results in SUGRA are exploratory hints to discover their corresponding analogs in superstring theories. For example, the dimensional reductions of superstring theories and the supergravities that imply have to be consistent with the SUGRAs that can independently be constructed in any dimension. This correspondence is one of the topics treated in this document. In particular, we will study the higher-dimensional origin of gauged supergravities.

General Relativity and massive theories

Einstein's formulation of gravity completely differs from the field theoretical approach that in the 1950s and 1960s were motivated by quantum mechanics. Had we wanted to follow a logical path³, we had looked for a relativistic theory of bosonic fields, since fermionic states cannot describe long range macroscopic forces due to their impossibility to build up coherent states. Some of the first scalar theories were proposed by Nördstrom [33, 34] and Einstein [35], but all of them were ruled out by experimental evidences⁴. For states with helicity ≥ 1 , gauge invariance is required to subtract extra degrees of freedom and guarantee positive energy conditions. By imposing gauge invariance for the $h = 1$ states, Maxwell equations are obtained. Helicity 2 states provide a linearized general coordinate invariance as gauge symmetry. The additional inclusion of consistent self-interactions automatically bring us to, and uniquely to, GR and full general coordinate invariance [37–41]. Up to now, states of helicity 3 or greater cannot describe theories that include self-interactions [42].

Thus, starting from special relativity (Lorentz invariance), all plausible particles are considered and their interactions restrict our scenario to helicity 2 fields. However, Einstein performed a leap of insight in this 'logical' description; his assumptions of the equivalence principle and general coordinate invariance, though it was not the only possibility, led to General Relativity. Let us see some examples of this non-uniqueness. General coordinate invariance is the gauge symmetry of GR. The redundant description that gauge symmetry provides, motivates the existence of the so-called *Stückelberg mechanism* [43]. This procedure restores gauge invariance by including additional auxiliary fields when the symmetry is broken. It can be applied to make any lagrangian invariant under general coordinate diffeomorphisms, so that this property does not necessarily implies GR. Similarly, the principle of equivalence is not exclusive of GR. For example, Einstein-Fokker theory [36, 44] includes this feature. Thus, the real underlying principle of GR is the following: *GR is the theory of a non-trivial interacting massless helicity 2 particle*. The rest of properties are consequences of this statement.

What about the range of validity of GR? It is not UV complete theory. In particular, three different regimes are distinguished. The classical linear regime, where both non-linear effects and quantum corrections can be ignored for $r > r_S \sim \frac{M}{M_P^2}$. The quantum regime, $r < \frac{1}{M_P}$, describes the region near the singularity of the black hole, where GR has to take into account high-energy corrections. Finally, there is a middle interval, $\frac{1}{M_P} < r < r_S$, where non-linearities of the theory are valid without worrying about quantum corrections. This scenario is a plausible motivation to try to add a mass to the graviton (or any other modification of GR) and see what happens by making use of the behaviour that physics must show in the continuity of its parameter (in this case, the mass).

One of the predictions of GR is the existence of a density $\rho \sim 10^{-29}$ g/cm³, which

³Logical, in the sense of the trends of that epoch.

⁴We earnestly recommend [36] for a very pedagogical introduction to scalar gravity.

is generically thought as a dark energy contribution that implies an additional term proportional to Λ in the Einstein field equations. To give the correct vacuum energy, this constant has to be $\Lambda/M_P^2 \sim 10^{-65}$, whereas arguments from quantum field theory suggest a value up to order unity [45]. Many modifications can be constructed. For example, the inclusion of more Ricci scalar terms in the Einstein-Hilbert action, the so-called $F(R)$ theories [46–48]. However, these models do not find a solution of the fine-tuning of the experimental parameters. We could understand this problem from a different viewpoint. Sometimes, these small parameters arise as natural parameters that protects a symmetry, *i.e.*, when they are set to zero, the system enjoys a new symmetry. An example of this statement is chiral symmetry and the small fermion masses. However, the case of the Higgs mass and the cosmological constant are not justified by any obvious symmetry.

Black holes in supergravity theories

Once we have glanced some qualitative aspects and the limits of GR, we will introduce one of the most important kind of solutions of the Einstein field equations, the black holes. Since SUGRA is the framework of this work, we will focus on the issues of BH solutions in these theories.

Einstein field equations explained the anomalous perihelion of Mercury in 1915, confirming its predictive power. In the following years, Schwarzschild, Reissner, Nordstrom, Kerr and many other authors found solutions that, at a certain limit, implied the existence of a region where even light could not get away of gravitational force. This region is what it is understood as *black hole* (BH). Some of the classical properties of BHs can be similarly stated as laws of thermodynamics. However, this analogy turns out to be complete when the semi-classical Bekenstein-Hawking entropy is considered. Very appealing references of this classical treatment are [36, 49–51].

But what kind of solutions do we have when gravity is coupled to more fields? Are they similar to the ones obtained for pure gravity? What is the influence of preserving a fraction of supersymmetric charges in our solutions? We will describe BHs in some particular supergravity and string theory scenarios.

The discovery of string dualities in the *second string revolution* led to a new picture in the knowledge of solutions of the theories. In particular, S duality provided a bridge between the strong coupling limit of a given string theory and its dual theory that turned out to be weakly coupled. The result of applying dualities to black hole is a dual description of string excitations. The so-called *string-black hole correspondence* [52, 53] predicts the black hole entropy in terms of string states and gives an explanation for the final state of a Schwarzschild BH.

Supergravity reproduces the Einstein-Hilbert action coupled to a certain number of matter fields, whose specific content depends on the theory under analysis. Generically, these fields are a bunch of scalar fields (moduli), spin-1/2 fermions, spin-1 gauge fields

and N gravitinos, where the latter behave as the gauge fields of the local supersymmetry. At this point, we can wonder what are the simplest BH solutions in this scenario and their relation to the pure gravity ones. The answer is that the solution has to carry the following quantities: electric/magnetic charges and scalar hair. Due to additional magnitudes that describe a black hole, we have freedom enough to set a particular configuration in the parameter space such that BH properties become very special. This is the so-called *extremality*, that gives rise to *extremal BHs* [54–58]. These extremal BHs are stable gravitational objects with finite entropy but vanishing temperature. This means that there is a particular relation between entropy, charges and angular momentum such that the full gravitational energy comes from scalar fields, charges and rotation [59]. Extremality also implies that internal and external horizons, namely the Cauchy and event horizons, do coincide. Another feature of this family of solutions in 4-dimensional spacetime is its geometry near the event horizon. The metric in this region is described by the Bertotti-Robinson [60] solution, which shows an $AdS_2 \times S^2$ shape. Surprisingly, the radius of the AdS_2 sector coincides with the radius of S^2 and it is proportional to the square root of the BH entropy.

In general, BHs in SUGRA are surrounded by scalar fields that provide a scalar hair (the value of the moduli at infinity). Nevertheless, the BH entropy does not show a dependence on the scalar charges, showing only dependence on the asymptotic electromagnetic charges.

This blurry situation is explained by means of the so-called *attractor mechanism* [61–64], a curious phenomenon that entails the successful collaboration of extremal BHs, its dynamics, algebra and number theory⁵. The situation is described as follows: the scalar fields approach fixed values at the BH horizon, that are only determined by the charge configuration. The asymptotic values of the moduli are forgotten even though the dynamics is completely valid and the fixed point represents the equilibrium of the system. In fact, the flow of the scalars towards the horizon behaves as a gradient flow towards a fixed point, which is the minimum of a function called *black hole potential*. This black hole potential is a positive definite function in the moduli space.

For solutions that preserve the maximal amount of supersymmetry (1/2-BPS) in $\mathcal{N} = 2$ theories, all of the scalar are fixed and the entropy is shown to be proportional to the black hole potential evaluated in the horizon [67]. On the other hand, non-supersymmetric extremal solutions (non-BPS states) also exhibit this attractor behaviour. However, in this case not all of the scalar of the vector multiplets become stabilized in terms of the BH conserved charges at the event horizon. Some of them generate flat directions in the potential minimum [68]. Even though, the entropy of non-BPS BHs also depends on the dyonic charges, as in the supersymmetric case [68, 69].

Finally, it is worthy to remark the numerous bibliography that has emerged in the late times relating pure entangled qubits states in the framework of quantum informa-

⁵Actually, this mechanism also applies to non-extremal BH configurations [65, 66].

tion with extremal BHs in superstring theory [70–72].

Outline

This work is organized as follows.

- Chapter 1 presents the main features of ungauged supergravities, emphasizing in supersymmetry as one of its pillars. In the last part, we will show the different reduction schemes that we can follow to perform a dimensional compactification of an arbitrary field theory.
- After having performed a knowledge in supergravity, Chapter 2 tries to provide an understanding of supergravity from the superstring viewpoint. An overview of the main features of string theory is done in such a way that supergravity is understood as a reference that not only checks the consistency among the string theories, but also gives some insights about symmetries, solutions and many other issues that happen in string theory.
- In Chapter 3 gauged supergravities are studied, where the embedding tensor formalism appears as an innovating tool in their search.
- After that, we give a full example of how the embedding tensor scans all the possible gaugings of $D = 9$ maximal supergravity. This is done in Chapter 4. We construct the tensor hierarchy of the gauged theory and compare with our results with the ones found in the bibliography.
- Chapter 5 deals about flux compactifications and how do they motivate one of the mismatches between string theory and supergravity. The classification of gauged supergravities that the embedding tensor provides and the ones that arise under flux compactification of higher-dimensional theories do not coincide. Double field theory is used to solve this problem, at least, for all maximal and half-maximal $D = 9, 8, 7$ supergravities.
- In Chapter 6 we study black holes in $N = 2$ $D = 4$ supergravity and provide a mechanism to obtain explicit composite black hole solutions with an arbitrary number of centers and for any quadratic prepotential.
- Finally, Chapter 7 resumes and synthesizes the main results and conclusions of the work done in this thesis. Some prospects and further projects are shown as possible candidate ideas to address.
- Some appendices are included. Appendix A treats several aspects related to supergravity theories: dimensional reductions, scalar cosets and central charges. Appendix B includes notation, definitions and more results obtained in Chapter

4. Appendix C shows some technical material used in the development of the calculus of Chapter 5.

Chapter 1

Supergravity: a primer

In this chapter, we will introduce some basic aspects of supersymmetry and supergravity theories. We will show how supersymmetry restricts and casts the field content of the theory depending on its dimension. Furthermore, we will show a catalog of the higher-dimensional theories, $D = 11, 10, 9$.

1.1 SUSY: fermionic symmetries, spinors and all that

It is generally supposed that the exact or approximate symmetry groups of the world were (at least locally) isomorphic to direct products of the Poincare group and compact Lie groups. However symmetries which extend Poincare symmetry in a non trivial way were suggested in the early sixties as a way of formulating a theory of hadronic physics (see for example [73–78], also [79]) where the older, nonrelativistic, $SU(4) \supset SU(2)_s \times SU(2)_{iso-s}$ Wigner “supermultiplet” model [79] was extended to groups as $SU(6)$.

The partial success of the $SU(6)$ theories raised the possibility of a relativistic symmetry group which was not simply such a direct product. This extension of these theories to include special relativity was however very problematic ¹ A possibility of extension was to search for a larger group which includes the $SU(6)$ group and the Lorentz group as subgroups. One possible group of this kind is the $SL(6)$ group which contains $SL(2) \times SL(3)$ as a subgroup. However, it seemed to be impossible such an extension without the introduction of 36-dimensional space. Other possibilities were explored as the the $\tilde{U}(8), \tilde{U}(12)$ theories, a covariant merging of isospin and space-time symmetries, including higher dimensional gamma matrices generators.

All attempts to find such a group were clearly unsuccessful. At the same time, as it appeared set of no-go theorems [17, 80, 81], the Coleman–Mandula theorem [17]

¹ In a relativistic wave equation, the spin indices are so tightly related to the coordinates (as we may see in the Dirac equation) that it is impossible to decouple the spin variables from the coordinates even in the free Hamiltonian.

the strongest among them, which showed that the symmetry group of a consistent 4-dimensional relativistic quantum field theory with a finite number of massive particles is necessarily the direct product of the internal symmetry group and the Poincaré group. Typically, these theorems showed that a physical field theory with a finite number of definite mass particles and with an analytical S matrix with not any of these groups as symmetries, would not allow anything but trivial scattering, in the forward and backward directions.

Superalgebras, as a way of avoiding the no-go theorems, appear in particle physics for the first time in [82], where an approximate unified model of mesons, baryons, antibaryons and exotic ($qq\bar{q}\bar{q}$) mesons is presented ^{2 3}.

In 1971, Gol’fand & Likhman developed the four-dimensional Poincaré superalgebra [85]. Ramond [86] and Neveu & Schwarz [87] developed superstrings and the supersymmetric extensions of a non Lie algebra, the Virasoro algebra. Volkov & Akulov [88] and Wess & Zumino [89] wrote different realizations of supersymmetric field theories, even without being aware of the earlier work by Gol’fand & Likhman. In particular ⁴, the Wess–Zumino model [89–92] was the first widely known example of an interacting four–dimensional quantum field theory with supersymmetry. ⁵ Super-space formalism was introduced in 1974 [93]. In 1975 Rudolf Haag, Jan Lopuszanski, and Martin Sohnius published [18] a general proof that weakening the assumptions of the Coleman–Mandula theorem by allowing both commuting and anticommuting symmetry generators, there is a nontrivial extension of the Poincaré algebra, namely the supersymmetry algebra and this is the most general symmetry of the S matrix of a quantum field theory. More in detail, the theorem may be summarized as follows: the most general Lie algebra of generators of supersymmetries and ordinary symmetries of the S-matrix in a massive theory involves the following Bose type operators: the energy-momentum operators P_μ ; the generators of the homogeneous Lorentz group $M_{\mu\nu}$; and a finite number of scalar charges. It will involve in addition Fermi type operators, all of which commute with the translations and transform like spinors under the homogeneous Lorentz group.

²The term “supermultiplet”, as in “the $SU(4)$ Wigner supermultiplet”, appears for the first time in 1964, with no connection to “supersymmetry”

³Lie superalgebras appeared, though not in a central role, in some mathematical contexts in the sixties [83, 84]

⁴ At this time QCD and the full SM with their symmetries product of the Poincaré group and local Lie groups were well established and apparently there was not need for further developments.

⁵The model consists of a single chiral superfield (composed of a complex scalar and a spinor fermion) whose cubic superpotential leads to a renormalizable theory. The action of the free massless Wess–Zumino model is invariant under the transformations generated by a superalgebra allowing both commuting and anticommuting symmetry generators.

1.1.1 Clifford algebras and spinors

The transformation properties of Bose and Fermi generators under the Lorentz group imply restrictions on the number of each of these types of generators and, indirectly, on the number and signature of spacetime dimensions. These restrictions are trivial for the case of Bose generators: a vector representation in a D -dimensional spacetime has always D components, for any D . The situation is much more complicated for the Fermi generators. They carry a spinorial representation of the Lorentz group which makes convenient the detailed study of the representation theory of Clifford algebras.

Clifford algebras are relevant in Physics due to the fact that their representations can be used to construct specific representations of symmetry groups, the spinorial representations. In particular, a representation of the D -dimensional Clifford algebra can be used to construct a representation of the D -dimensional Lorentz algebra $\mathfrak{so}(1, D - 1)$. More in detail, if we define gamma matrices $\{\gamma_\mu\}_{\mu=0,\dots,D-1}$ which satisfy a Clifford algebra with associated metric $\eta_{\mu\nu} = \text{diag}(-, +, \dots, +)$,

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{I}, \quad (1.1)$$

then the matrices

$$\Sigma_{\mu\nu}^S = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \quad (1.2)$$

are generators for a spinorial representation S of the Lorentz group. The exponentiation of these generators gives a Lorentz transformation

$$\Gamma^S(\Lambda) = \exp\left(\frac{1}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}^S\right). \quad (1.3)$$

It can be shown that there is only one inequivalent irreducible representation of the Clifford algebra in D dimensions and this is $2^{\lfloor D/2 \rfloor}$ -dimensional. The elements of these $2^{\lfloor D/2 \rfloor}$ -dimensional vector representation space, where the algebra acts are the *Dirac spinors*.

Irreducible representations of Clifford algebras may lead to reducible Lorentz representations depending on the dimension of the spacetime. For instance, even dimensions allow the existence of $2^{\lfloor D/2 \rfloor - 1}$ -dimensional irreducible representations. This can be easily seen by defining a matrix γ^{D+1} ,

$$\gamma^{D+1} = i(-1)^{\frac{D-2}{4}-1}\gamma^0 \dots \gamma^{D-1}. \quad (1.4)$$

This *chirality matrix* is traceless, squares to unity, half of its eigenvalues are +1s and the other half are -1s. It is natural then to split Dirac spinors into the direct sum of the subspaces of spinors with different eigenvalues. The elements of each of these subspaces are called *Weyl spinors*, they satisfy, by definition, the Weyl or chirality condition,

$$\frac{1}{2}(1 \pm \gamma^{D+1})\chi = \chi, \quad (1.5)$$

$D \pmod{8}$	spinor irreps	real components
1, 3	M	$2^{(D-1)/2}$
2	MW	$2^{D/2-1}$
4, 8	M	$2^{D/2}$
5, 7	D	$2^{(D+1)/2}$
6	W	$2^{D/2}$

Table 1.1: Different irreducible spinorial representations depending on the dimension D and their corresponding real components.

where χ is an arbitrary spinor. The so-called *left-* and *right-handed* spinors correspond to the eigenvectors with eigenvalues $+1$ and -1 , respectively.

We can also reduce Dirac spinors using the fact that, since γ_μ satisfies 1.1, also γ_μ^* and γ_μ^T do. This implies the existence of isomorphisms relating these representations. One isomorphism, represented by C , relates

$$C\gamma_{\mu\nu}C^{-1} = -\gamma_{\mu\nu}^T. \quad (1.6)$$

The matrix C is called a *charge conjugate matrix* and allows to define a charge-conjugate spinor,

$$\hat{\lambda} = \lambda^T C. \quad (1.7)$$

We can look for spinors whose charge-conjugate spinors are proportional to their Dirac conjugate $\bar{\lambda}$ defined by

$$\bar{\lambda} \equiv i\lambda^\dagger \gamma^0. \quad (1.8)$$

That is,

$$\hat{\lambda} = \alpha \bar{\lambda} = \lambda^T C = \alpha i \lambda^\dagger \gamma^0. \quad (1.9)$$

This is a “reality” condition for the spinors. The ones that fulfill it are called *Majorana spinors*. Sometimes chirality and Majorana conditions may be simultaneously satisfied.

We schematically show in Table 1.1 some characteristics of the spinorial irreducible representations for any dimension.

1.1.2 SUSY algebras and their representations

From a mathematical point of view a Lie superalgebra is based over a Z_2 graded vector space [82–84]. The physical Bose and Fermion elements will be respectively the grade 0 and grade 1 algebra vectors.

A Lie superalgebra \mathfrak{s} satisfies the following properties:

- \mathfrak{s} is a mod 2 graded vector space over \mathbb{C} . I.e., it admits a map:

$$\text{gr} : \mathfrak{s} \longrightarrow \mathbb{Z}_2 \quad (1.10)$$

which decomposes \mathfrak{s} into $\mathfrak{s}^{(0)}$ and $\mathfrak{s}^{(1)}$ such that

$$\text{gr}(B) = 0 \pmod{2}, \quad \forall B \in \mathfrak{s}^{(0)}, \quad (1.11)$$

$$\text{gr}(F) = 1 \pmod{2}, \quad \forall F \in \mathfrak{s}^{(1)}. \quad (1.12)$$

- \mathfrak{s} is endowed with a binary operation, the bracket $\{, \}$, which is bilinear, superanticommutative and mod 2 grade additive,

This means that, given $A, B \in \mathfrak{s}$, we have $[A, B] = -[B, A]$ in all cases but one, where both A and B are Fermi in which case $[A, B] = +[B, A]$. The mod 2 grade additivity means that denoting the grades a, b, c of $A, B, C \in \mathfrak{s}$ respectively, if we have $[A, B] = C$ then $a + b = c \pmod{2}$ has to be satisfied.

$$\{A, B\} = (-1)^{1+\text{gr}(A)\text{gr}(B)} \{B, A\}, \quad (1.13)$$

$$\text{gr}(\{A, B\}) = \text{gr}(A) + \text{gr}(B). \quad (1.14)$$

- The bracket operation obeys the superJacobi identity

$$\begin{aligned} & (-1)^{1+\text{gr}(C)\text{gr}(A)} \{\{A, B\}, C\} \\ & + (-1)^{1+\text{gr}(A)\text{gr}(B)} \{\{B, C\}, A\} \\ & + (-1)^{1+\text{gr}(B)\text{gr}(C)} \{\{C, A\}, B\} = 0. \end{aligned}$$

This reduces to the ordinary Jacobi identity in all cases but one: when any two of the elements A, B, C are Fermi and the third one is Bose in which case one of the three usual Jacobi terms has its sign flipped.

The simple finite-dimensional Lie superalgebras over \mathbb{C} are fully classified [94, 95]. There are eight infinite families, a continuum $D(2|1; \alpha)$ of 17-dimensional exceptional superalgebras, and one exceptional superalgebra each in dimensions 31 and 40. The special linear $\mathfrak{sl}(m|n)$ and the orthosymplectic $\mathfrak{osp}(m|n)$ superalgebras are the most relevant ones from the physical point of view. The superalgebra $\mathfrak{osp}(4|N)$, which has as bosonic Lie algebra $\mathfrak{so}(3, 2) \times \mathfrak{so}(N)$, corresponds to the AdS superalgebra. The superconformal one is $\mathfrak{su}(2, 2|N)$, which has as Lie algebra $\mathfrak{so}(4, 2) \times \mathfrak{su}(N) \times \mathfrak{u}(1)$.

It is of interest to us superalgebras which include the Poincare group. The Poincare superalgebra (the superalgebra whose bosonic sector is strictly the Poincare algebra) is generated by $P_\mu, M_{\mu\nu}, Q_\alpha^i$ which satisfies the relations

$$\begin{aligned} [M_{\mu\nu}, M^{\rho\sigma}] &= -2\delta_{[\mu}^{[\rho} M_{\nu]}^{\sigma]}, & [P_\mu, M_{\nu\rho}] &= \eta_{\mu[\nu} P_{\rho]}, & [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, Q_\alpha^i] &= -\frac{1}{4}(\gamma_{\mu\nu})_\alpha^\beta Q_\beta^i, & [P_\mu, Q_\alpha^i] &= 0, \\ \{Q_\alpha^i, Q_\beta^j\} &= (\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \delta^{ij}. \end{aligned} \quad (1.15)$$

The last term implies that two internal fermionic transformations lead to a spacetime translation. Here it is realized basic feature of SUSY, the interplay between spacetime and some other internal symmetry.⁶

In superalgebras including the Poincare group the number of supercharges (or grade 1 generators) turns out to be a multiple of the number of real components of an irreducible spinor. This is required by Lorentz invariance itself, since in this case the components of an irreducible spinor transform into each other. Thus, the supercharges, Q_α^i , carry two indices: $i = 1, \dots, N$, where N is, in principle, an arbitrary integer, and α is an irreducible spinor index (as it is obvious from the commutation relations of the Q 's and the M 's).

Some elementary properties

Some well-known important properties can be direct and elementarily deduced from the Fermi sector of the Poincare superalgebra. Let us take as an example the simplest ($N = 1$) supersymmetric extension of the Poincare algebra, which can be written in terms of two complex Weyl spinors and their conjugates with the following anti-commutation relations:

$$\{Q_\alpha, Q_\beta\} = \{Q_\alpha^\dagger, Q_\beta^\dagger\} = 0, \quad (1.16)$$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu. \quad (1.17)$$

$$(1.18)$$

Contracting the first relation with $(\bar{\sigma}^\nu)^{\dot{\beta}\alpha}$, we have

$$4P_\nu = (\bar{\sigma}^\nu)^{\dot{\beta}\alpha} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\}. \quad (1.19)$$

In a quantum theory the superalgebra generators are operators in the Hilbert space of the system which includes bosonic and fermi states. Single particle states fall into irreducible representations of the algebra, the supermultiplets. Since the fermi generators commute with $P^\mu P_\mu$ all particles in a supermultiplet have the same mass⁷.

The time component, P_0 corresponds to the hamiltonian operator, which can be written

$$4P_0 = 4H = \sum_\alpha \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = \sum_\alpha \{Q_\alpha, Q_\alpha^\dagger\} = \sum_\alpha (Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha). \quad (1.20)$$

⁶Incidentally we observe here the spin-statistics connection at work: fermi half integer spin generators have to be anticommuting.

⁷They have in addition the same charge corresponding to any possible gauge symmetry.

The expected value of the hamiltonian in an arbitrary state $|s\rangle$ is given by,

$$\langle s|H|s\rangle = \frac{1}{4} \sum_{\alpha} \langle s|(Q_{\alpha}Q_{\alpha}^{\dagger} + Q_{\alpha}^{\dagger}Q_{\alpha})|s\rangle = \frac{1}{4} \sum_{\alpha} \sum_{s'} \langle s|Q_{\alpha}|s'\rangle \langle s'|Q_{\alpha}^{\dagger}|s\rangle + \langle s|Q_{\alpha}^{\dagger}|s'\rangle \langle s'|Q_{\alpha}|s\rangle \quad (1.21)$$

$$= \frac{1}{2} \sum_{\alpha} \sum_{s'} |\langle s'|Q_{\alpha}|s\rangle|^2, \quad (1.22)$$

$$\geq 0 \quad (1.23)$$

where we have introduced the closure relation $\mathbb{I} = \sum_{s'} |s'\rangle \langle s'|$. Thus, we conclude that in a supersymmetric quantum theory, any physical state $|s\rangle$ must have non-negative energy. The inequality saturates if the (ground or vacuum) state (denoted by $|0\rangle$) is annihilated by a SUSY generator $Q_{\alpha}|0\rangle = 0$ in this case one talks of absence of spontaneous SUSY symmetry breaking.

Since Q_{α} has spinor indices, when it acts on a bosonic state of the hilbert state it produces a spinor, fermionic state, any supermultiplet has both bosonic and fermionic states. One can show that the number of boson states is equal to the number of fermion states for each supermultiplet with nonzero energy.

Using the SUSY algebra properties one can construct [96] the corresponding algebra representations that is the detailed particle supermultiplet content. As all the particles in the supermultiplet have the same mass one can study independently the massive and massless cases. In both cases the SUSY algebra reduces to a Clifford algebra of raising and lowering anticommuting operators. Combining the Clifford algebra representation theory and maximal weight techniques one can construct the entire massive or massless multiplets applying repeatedly times “raising” Q^{\dagger} operators to a given maximal spin state. As an example, the so called massive (massless) “chiral” multiplet is formed by starting with a spin 0 state: contains a Majorana (Weyl) fermion and a complex scalar. The massive vector multiplet is formed from a spin 1/2 initial state, contains two Majorana fermions, a massive spin 1 vector and a real scalar. The massless vector multiplet turns out to be composed of a Weyl fermion and a massless spin 1 boson.

There exists a physical upper bound for N , the number spinor charges. If $N \geq 9$, massless representations necessarily contain some undesirable particles of higher spin $s \geq 5/2$. We can see in table 1.2 the spin content of all the representations whose maximum spin satisfies $s_{max} \leq 2$.

If we restrict ourselves to theories with particles of spin no higher than 2 (and not more than one “time”-like coordinate) the maximum number of supercharges that we can have is 32 and may live in dimensions no higher than 11. We will refer to these SUSY theories with the maximal number of supercharges as *maximal*. In 1, 3, 4, 5, 7, 8, 9, 11 dimensions, a SUSY algebra is classified by a positive integer N . Meanwhile in 2, 6, 10 dimensions a SUSY algebra is classified by two integers (N_R, N_L) , at least one of which is nonzero. M represents the number of left-handed SUSYs and N represents the number of right-handed SUSYs.

The simple SUSY algebra above, with $N = 1$ is invariant under a multiplication of the Q_α by a phase. The corresponding symmetry group, called $U(1)_R$, is the simplest example of an additional symmetry at the level of the supercharges. The so-called *R symmetry* is a automorphism of the fermi sector, it transforms different supercharges into each other. For extended SUSY ($N \geq 2$), it becomes a non-abelian group. Formally, it is defined as the largest subgroup of the automorphism group of the SUSY algebra that commutes with Lorentz transformations. A summary of this *R symmetry* is shown in table 1.3.

Central charges

SUSY algebras with $N \geq 2$ can be furtherly extended by adding “central charge” operators. These can be Lorentz scalars “central charges” Z^{ij} [18]. or “tensorial” *central charges* $Z_{\mu_1, \dots}^{ij}$, [97, 98]. They appear in the anticommutator of two SUSY generators as

$$\{Q^i_\alpha, Q^j_\beta\} = (\gamma_\mu C^{-1})_{\alpha\beta} P_\mu \delta^{ij} + \sum_k (\Gamma^{\mu_1 \dots \mu_k} C)_{\alpha\beta} Z_{\mu_1 \dots \mu_k}^{ij}. \quad (1.24)$$

The possible combinations of central extensions will depend on the dimension and characteristics of the theory. For example, for $D = 11$, we have [99]

$$\{Q_\alpha, Q_\beta\} = (\gamma_\mu C^{-1})_{\alpha\beta} P_\mu \delta^{ij} + \sum_n (\Gamma^{\mu_1 \mu_2} C)_{\alpha\beta} Z_{\mu_1 \mu_2} + \sum_n (\Gamma^{\mu_1 \dots \mu_5} C)_{\alpha\beta} Z_{\mu_1 \dots \mu_5}. \quad (1.25)$$

For a pure scalar central charge we have [18] ($\epsilon = i\sigma^2$):

$$\{Q^i_\alpha, Q^j_\beta\} = 2\epsilon_{\alpha\beta} Z^{ij}, \quad (1.26)$$

$$\{Q^{i\dagger}_{\dot{\alpha}}, Q^{j\dagger}_{\dot{\beta}}\} = 2\epsilon_{\dot{\alpha}\dot{\beta}} Z^{ij\dagger}, \quad (1.27)$$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu. \quad (1.28)$$

It is possible to choose a set of states $|M, Z^{ij}\rangle$ eigenstates of $P^\mu P_\mu$ and Z^{ij} . The corresponding $N \times N$ matrix (Z^{ij}) is antisymmetric in its indices. This matrix can be skew-diagonalized to $N/2$ real eigenvalues. Thus, for example, for $N = 2$ one can write

$$(Z^{ij}) = Z(\epsilon^{ij}), \quad (1.29)$$

where Z is a real parameter which can be assigned, in addition to the mass, to any corresponding multiplet. By a redefinition of the supercharges, and ensuring that all states of the supermultiplet have non-negative norm one arrives to the inequality ([100], see [96] for a simple example)

$$M \geq Z. \quad (1.30)$$

This is an example of a *Bogmol'nyi-Prasad-Sommerfeld (BPS) bound* [101, 102]. In particular, for massless states $Z = 0$. The states which saturates the inequality, $M = Z$

are of zero norm, in other terms, they are annihilated by some (half or a quarter or an eighth) of the supercharges.

The structure of the algebra unitary representations is different for the cases $M > Z$ and $M = Z$: the supermultiplets with $M = Z$ are much smaller (“short multiplets”) to those corresponding to $M > Z$ long multiplets. The short multiplets, those for which $M = Z$ are also called BPS multiplets because they are related to BPS monopoles [97, 98, 101, 102].

1.2 Supergravity

Supergravity theories are field theories that are invariant under local supersymmetry [19, 20, 103–108], i.e. under superpoincare transformations with spacetime dependent commuting and anticommuting parameters. Because of the underlying supersymmetry algebra, the invariance under local supersymmetry implies the invariance under spacetime diffeomorphisms. Therefore these theories are necessarily theories of gravity. Supergravity, was initially proposed in 1973 by D. Volkov [88], it was quickly generalized in various dimensions and additional N supersymmetry charges. The number of supercharges in a spinor depends on the dimension and the signature of spacetime. Supergravity theories do not contain any fields that transform as symmetric tensors of rank higher than two under Lorentz transformations. Thus the limit on the number of supercharges cannot be satisfied in a spacetime of arbitrary dimension. Supergravity can be formulated, in spacetimes with Lorentz signatures, in any number of dimensions up to eleven [109].

About supermultiplets the most common that appear in supergravity are

- Gravity multiplet. The field content satisfies $s_{max} = 2$. They contain the graviton plus N gravitinos at least.
- Vector/gauge multiplet. Here $s_{max} = 1$. They exist for $N \leq 4$ theories. The gauge fields of those multiplets can gauge an extra Yang-Mills-like group that commutes with supercharges and it is not part of the superalgebra.
- Chiral multiplet. $s_{max} = 1/2$. In $D = 4$ theories, they only exist for $N = 1$. Supersymmetry requires the scalars to span a Kähler-Hodge manifold. They must transform under the gauge group defined by the vector multiplet.
- Hypermultiplets. They are the equivalent chiral multiplets for $N = 2$. They also must transform under the gauge group. In this case, the scalars must parameterize a Quaternionic Kähler manifold.
- Tensor multiplet. They include antisymmetric tensors $T_{\mu\nu\dots}$. In some cases, they can be dualized to scalars or vectors and be included in the other multiplets.

1.3 Higher-dimensional supergravities

In the next sections we are going to inspect the $D = 9, D = 10$ and $D = 11$ maximal supergravities.

1.3.1 $D = 11$ supergravity

In 1978, Cremmer, Julia and Scherk (CJS) found the classical action for an 11-dimensional supergravity theory. This remains today the only known classical 11-dimensional theory with local supersymmetry and no fields of spin higher than two.

Other 11-dimensional theories are known that are quantum-mechanically inequivalent to the CJS theory, but classically equivalent (that is, they reduce to the CJS theory when one imposes the classical equations of motion). For example, the de Wit and Nicolai $D = 11$ Supergravity with Local $SU(8)$ Invariance.

Supergravity in eleven spacetime dimensions is based on an “elfbein” e_μ^a , a Majorana gravitino field ψ_μ and a 3-rank antisymmetric gauge field $C_{\mu\nu\rho}$. Together with chiral $(2, 0)$ supergravity in $D = 6$, it is the only $Q \geq 16$ theory without a scalar field. Its full action reads

$$S = \frac{1}{2\kappa^2} \int d^{11}x e [e^{a\mu} e^{b\nu} R_{\mu\nu ab} - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma}] \quad (1.31)$$

$$- \frac{\sqrt{2}}{192} \bar{\psi}_\nu (\gamma^{\alpha\beta\gamma\delta\nu\rho} + 12\gamma^{\alpha\beta} g^{\gamma\nu} g^{\delta\rho}) \psi_\rho (F_{\alpha\beta\gamma\delta} + \tilde{F}_{\alpha\beta\gamma\delta}) \quad (1.32)$$

$$- \frac{2\sqrt{2}}{(144)^2} e^{-1} \epsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho} F_{\alpha'\beta'\gamma'\delta'} F_{\alpha\beta\gamma\delta} C_{\mu\nu\rho}], \quad (1.33)$$

where the Ricci scalar and the covariant derivative, respectively $R = R(\omega)$, $D_\nu = D_\nu(\frac{1}{2}(\omega + \tilde{\omega}))$, depend on the spinorial connection ω and its supercovariant version $\tilde{\omega}$. In components, we have for these and other quantities

$$\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab}, \quad (1.34)$$

$$\tilde{\omega}_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab} - \frac{1}{8} \bar{\psi}_\nu \gamma^{\nu\rho}{}_{\mu ab} \psi_\rho, \quad (1.35)$$

$$K_{\mu ab} = -\frac{1}{4} (\bar{\psi}_\mu \gamma_a \psi_b - \bar{\psi}_a \gamma_\mu \psi_b + \bar{\psi}_b \gamma_a \psi_\mu) + \frac{1}{8} \bar{\psi}_\nu \gamma^{\nu\rho}{}_{\mu ab} \psi_\rho, \quad (1.36)$$

$$\tilde{F}_{\mu\nu\rho\sigma} = 4\partial_{[\mu} C_{\nu\rho\sigma]} + \frac{3\sqrt{2}}{2} \bar{\psi}_{[\mu} \gamma_{\nu\rho} \psi_{\sigma]}, \quad (1.37)$$

where $\psi_b = e_b^\mu \psi_\mu$, and the covariant derivative D acts on spinors as,

$$D_\nu \psi_\nu D_\nu \psi_\nu = \partial_\mu \psi_\nu + \left(\frac{1}{8} (\omega + \tilde{\omega}) \right) \gamma^{ab} \psi_\nu. \quad (1.38)$$

The field equations can be derived in the context of the so-called 1.5 formalism, in which the spin connection is defined as a field that depends on e_μ^a and it is determined by its equation of motion meanwhile its supersymmetric variation in the action is

treated as if it were an independent field [107, 110], together with the Bianchi identities for the field strength form associated to the 3-rank form field C are ($F = dC$)

$$R_{\mu\nu} = \frac{1}{72} g_{\mu\nu} F_{\rho\sigma\lambda\tau} F^{\rho\sigma\lambda\tau} - \frac{1}{6} F_{\mu\rho\sigma\lambda} F_{\nu}{}^{\rho\sigma\lambda}, \quad (1.39)$$

$$\partial_{\mu}(eF^{\mu\nu\rho\sigma}) = \frac{1}{1152} \sqrt{2} \varepsilon^{\nu\rho\sigma\lambda\tau\alpha\beta\gamma\delta\kappa\pi} F_{\lambda\tau\alpha\beta} F_{\gamma\delta\kappa\pi}, \quad (1.40)$$

$$\partial_{[\mu} F_{\nu\rho\sigma\lambda]} = 0. \quad (1.41)$$

An alternative form for the second equation is

$$\partial_{[\mu} H_{\nu\rho\sigma\lambda\tau\alpha\beta]} = 0, \quad (1.42)$$

where $H_{\mu\nu\rho\sigma\lambda\tau\alpha}$ is the dual field strength,

$$H_{\mu\nu\rho\sigma\lambda\tau\alpha} = \frac{1}{7!} e \varepsilon_{\mu\nu\rho\sigma\lambda\tau\alpha\beta\gamma\delta\kappa} F^{\beta\gamma\delta\kappa} - \frac{1}{\sqrt{2}} F_{[\mu\nu\rho\sigma} C_{\lambda\tau\alpha]}. \quad (1.43)$$

Let us analyze the constant κ_{11}^{-2} that multiplies the lagrangian and carries dimension [mass]⁹. We can see that, in principle, it is undetermined and depends on fixing some length scale. If we apply the following shift on the fields (an R^+ symmetry):

$$e_{\mu}{}^a \rightarrow e^{-\alpha} e_{\mu}{}^a, \quad \psi_{\mu} \rightarrow e^{-\alpha/2} \psi_{\mu}, \quad C_{\mu\nu\rho} \rightarrow e^{-3\alpha} C_{\mu\nu\rho}, \quad (1.44)$$

the lagrangian rescales as

$$\mathcal{L}_{11} \rightarrow e^{-9\alpha} \mathcal{L}_{11}. \quad (1.45)$$

This is the so-called *trombone symmetry* [111] and it is manifest only at the level of the equations of motion. This scaling could be reabsorbed into a redefinition of κ_{11}^{-2} ,

$$\kappa_{11}^2 \rightarrow e^{-9\alpha} \kappa_{11}^2. \quad (1.46)$$

In many other supergravities in D dimensions we have a similar behaviour. In general, we could make the following redefinitions:

$$g_{\mu\nu} \rightarrow e^{-2\alpha} g_{\mu\nu}, \quad \mathcal{L}_D \rightarrow e^{(2-D)\alpha} \mathcal{L}_D, \quad \kappa_D^2 \rightarrow e^{(2-D)\alpha} \kappa_D^2. \quad (1.47)$$

1.3.2 $D = 10$ supergravities.

In $D = 10$ we have Majorana-Weyl (MW) irreducible spinors. The maximal supersymmetry is $N = 2$, which gives rise to two discrete and inequivalent possibilities, $N = (1, 1)$ with opposite chiralities and $N = (2, 0)$, with same chirality. They correspond to the $N = 2A$ and $N = 2B$ theories, respectively.

In addition, a half-maximal $N = 1$ $D = 10$ supergravity can be formulated. Its origin from the $N = 2$ theories will be discussed in Chapter 2.

$N = 2A$ supergravity

The $N = 2A$ 10-dimensional theory can be obtained by dimensional reduction of $D = 11$ on a circle. Its field content appears in table 1.4. The gravitinos and the two dilatinos have opposite chiralities.

The bosonic part of the Lagrangian is

$$\mathcal{L}_{2A}/e = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-\phi}|H|^2 - \frac{1}{2}\sum_{d=1,3} e^{(4-d)\phi/2}|G^{(d+1)}|^2 - \frac{1}{2}\star(dC^{(3)} \wedge dC^{(3)} \wedge B), \quad (1.49)$$

where H is the field strength associated to the (“NSNS”) 2-form B and G is the modified field strength

$$G^{(d+1)} = dC^{(d)} - dB \wedge C^{(d-2)} \quad d = 1, 3. \quad (1.50)$$

In this case we have two different \mathbb{R}^+ symmetries: one is a trombone symmetry, a symmetry of the field equations analog to the existing one in $D = 11$ supergravity and the other is a symmetry of the lagrangian, which acts on the field as follows:

$$e^\phi \rightarrow \lambda e^\phi, \quad B \rightarrow \lambda^{1/2} B, \quad C^{(1)} \rightarrow \lambda^{1/2} C^{(1)}, \quad C^{(3)} \rightarrow \lambda^{-1/4} C^{(3)}. \quad (1.51)$$

 $N = 2B$ supergravity

The field content of $N = 2B$ $D = 10$ supergravity appears in table 1.5. The rank 4 antisymmetric tensor is supposed to have a self-dual field strength. This is a $N = (2, 0)$ theory. Both gravitinos have the same chirality. Both dilatinos also have the same chirality but opposite to that of the gravitinos.

The Lagrangian of the bosonic sector is given by

$$\mathcal{L}_{2B}/e = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{-\phi}|H|^2 - \frac{1}{2}\sum_{d=0,2,4} |G^{(d+1)}|^2 - \frac{1}{2}\star(C^{(4)} \wedge dC^{(2)} \wedge B). \quad (1.53)$$

where the field strengths H and $G^{(d)}$, $d = 0, 2, 4$ are defined similarly as before. The action for the $N = 2B$ theory, due to the presence of the field strength of $C^{(4)}$ and its self-dual behaviour, has supplemented by a self-duality at the level of field equations.

The $N = 2B$ theory enjoys two symmetries. A trombone scaling symmetry and a $SL(2, \mathbb{R})$ symmetry. The former, as in the 11-dimensional case, is only realized on-shell whereas the latter is realized at the Lagrangian level and acts on the fields as follows. Considering an element

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad (1.54)$$

the fields transform as

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad B^\alpha \rightarrow (\Lambda^{-1})^\alpha_\beta B^\beta, \quad C^{(4)} \rightarrow C^{(4)}, \quad (1.55)$$

$$\psi_\mu \rightarrow \left(\frac{c\tau^* + d}{c\tau + d} \right)^{1/4} \psi_\mu, \quad \chi \rightarrow \left(\frac{c\tau^* + d}{c\tau + d} \right)^{3/4} \chi, \quad (1.56)$$

where

$$\tau \equiv C^{(0)} + ie^{-\phi}$$

and

$$B^\alpha \equiv (-B, C^{(2)}).$$

8

1.4 Maximal $D = 9$ supergravity

Next we study, in some more detail, maximal supergravity in $D = 9$. There is only one undeformed (*i.e.* ungauged, massless) maximal (*i.e.* $N = 2$, containing no dimensionful parameters in their action, apart from the overall Newton constant) 9-dimensional supergravity [112]. Both the dimensional reduction of the massless $N = 2A, d = 10$ theory and that of the $N = 2B, d = 10$ theory on a circle give the same $N = 2, D = 9$ theory⁹.

The fundamental (*electric*) fields of this theory are,

$$\left\{ e_\mu^a, \varphi, \tau \equiv \chi + ie^{-\phi}, A^I_\mu, B^i_{\mu\nu}, C_{\mu\nu\rho}, \psi_\mu, \tilde{\lambda}, \lambda, \right\}. \quad (1.57)$$

where $I = 0, \mathbf{i}$, with $\mathbf{i}, \mathbf{j}, \mathbf{k} = 1, 2$ and $i, j, k = 1, 2$ ¹⁰. The complex scalar τ parametrizes an $SL(2, \mathbb{R})/U(1)$ coset that can also be described through the symmetric $SL(2, \mathbb{R})$ matrix

$$\mathcal{M} \equiv e^\phi \begin{pmatrix} |\tau|^2 & \chi \\ \chi & 1 \end{pmatrix}, \quad \mathcal{M}^{-1} \equiv e^\phi \begin{pmatrix} 1 & -\chi \\ -\chi & |\tau|^2 \end{pmatrix}. \quad (1.58)$$

⁸Type IIB string theory breaks $SL(2)$ into its discrete subgroup $SL(2, \mathbb{Z})$. This group contains the so-called S -duality transformation that flips the sign of the dilaton ϕ in a background with vanishing axion $C^{(0)}$. Explicitly, this is done by choosing $a = b = d = 0$ and $c = 1$ in the $SL(2, \mathbb{R})$ transformation. Because of its very definition, S -duality turns out to be a non-perturbative duality relating the strong- and weak-coupling regimes. This is better justified in Chapter 2, when we establish a relation between $\langle e^\phi \rangle$ and the string coupling constant g .

⁹This is a property related to the T duality between type IIA and IIB string theories compactified on circles [113, 114] and from which the type II Buscher rules can be derived [115].

¹⁰Sometimes we need to distinguish the indices 1, 2 of the 1-forms (and their dual 6-forms) from those of the 2-forms (and their dual 5-forms). We will use boldface indices for the former and their associated gauge parameters.

The field strengths of the electric p -forms are, in our conventions ^{11,12}

$$F^I = dA^I, \quad (1.59)$$

$$H^i = dB^i + \frac{1}{2}\delta^i_{\mathbf{i}}(A^0 \wedge F^{\mathbf{i}} + A^{\mathbf{i}} \wedge F^0), \quad (1.60)$$

$$G = d[C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}}] - \varepsilon_{\mathbf{ij}}F^{\mathbf{i}} \wedge (B^j + \frac{1}{2}\delta^j_{\mathbf{j}}A^{0\mathbf{j}}), \quad (1.61)$$

and are invariant under the gauge transformations

$$\delta_\Lambda A^I = -d\Lambda^I, \quad (1.62)$$

$$\delta_\Lambda B^i = -d\Lambda^i + \delta^i_{\mathbf{i}}[\Lambda^{\mathbf{i}}F^0 + \Lambda^0F^{\mathbf{i}} + \frac{1}{2}(A^0 \wedge \delta_\Lambda A^{\mathbf{i}} + A^{\mathbf{i}} \wedge \delta_\Lambda A^0)] \quad (1.63)$$

$$\begin{aligned} \delta_\Lambda [C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}}] &= -d\Lambda - \varepsilon_{\mathbf{ij}}(F^{\mathbf{i}} \wedge \Lambda^j + \Lambda^{\mathbf{i}} \wedge H^j - \delta_\Lambda A^{\mathbf{i}} \wedge B^j \\ &\quad + \frac{1}{2}\delta^j_{\mathbf{j}}A^{0\mathbf{i}} \wedge \delta_\Lambda A^{\mathbf{j}}). \end{aligned} \quad (1.64)$$

The bosonic action is, in these conventions, given by

$$\begin{aligned} S = \int \left\{ -\star R + \frac{1}{2}d\varphi \wedge \star d\varphi + \frac{1}{2}[d\phi \wedge \star d\phi + e^{2\phi}d\chi \wedge \star d\chi] + \frac{1}{2}e^{\frac{4}{\sqrt{7}}\varphi}F^0 \wedge \star F^0 \right. \\ \left. + \frac{1}{2}e^{\frac{3}{\sqrt{7}}\varphi}(\mathcal{M}^{-1})_{\mathbf{ij}}F^{\mathbf{i}} \wedge \star F^{\mathbf{j}} + \frac{1}{2}e^{-\frac{1}{\sqrt{7}}\varphi}(\mathcal{M}^{-1})_{ij}H^i \wedge \star H^j + \frac{1}{2}e^{\frac{2}{\sqrt{7}}\varphi}G \wedge \star G \right. \\ \left. - \frac{1}{2}[G + \varepsilon_{\mathbf{ij}}A^{\mathbf{i}} \wedge (H^j - \frac{1}{2}\delta^j_{\mathbf{j}}A^{\mathbf{j}} \wedge F^0)] \wedge \{[G + \varepsilon_{\mathbf{ij}}A^{\mathbf{i}} \wedge (H^j - \frac{1}{2}\delta^j_{\mathbf{j}}A^{\mathbf{j}} \wedge F^0)] \wedge A^0 \right. \\ \left. - \varepsilon_{ij}(H^i - \delta^i_{\mathbf{i}}A^{\mathbf{i}} \wedge F^0) \wedge (B^j - \frac{1}{2}\delta^j_{\mathbf{j}}A^{0\mathbf{j}})\} \right\}. \end{aligned} \quad (1.65)$$

The kinetic term for the $SL(2, \mathbb{R})$ scalars ϕ and χ can be written in the alternative forms

$$\frac{1}{2}[d\phi \wedge \star d\phi + e^{2\phi}d\chi \wedge \star d\chi] = \frac{d\tau \wedge \star d\bar{\tau}}{2(\Im m \tau)^2} = \frac{1}{4}\text{Tr}[d\mathcal{M}\mathcal{M}^{-1} \wedge \star d\mathcal{M}\mathcal{M}^{-1}], \quad (1.66)$$

the last of which is manifestly $SL(2, \mathbb{R})$ -invariant. The Chern-Simons term of the action (the last two lines of Eq. (1.65)) can also be written in the alternative form

$$\begin{aligned} -\frac{1}{2}d[C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}} - \varepsilon_{\mathbf{ij}}A^{\mathbf{i}} \wedge B^j] \wedge \{d[C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}} - \varepsilon_{\mathbf{ij}}A^{\mathbf{i}} \wedge B^j] \wedge A^0 \\ - \varepsilon_{ij}d(B^i - \frac{1}{2}\delta^i_{\mathbf{i}}A^{0\mathbf{i}}) \wedge (B^j - \frac{1}{2}\delta^j_{\mathbf{j}}A^{0\mathbf{j}})\}, \end{aligned} \quad (1.67)$$

that has an evident 11-dimensional origin.

The equations of motion of the scalars, derived from the action above, are

¹¹We use the shorthand notation $A^{IJ} \equiv A^I \wedge A^J$, $B^{ijk} \equiv B^i \wedge B^j \wedge B^k$ etc.

¹²The relation between these fields and those of Refs. [116] and [117] are given in Appendix B.2.

$$d \star d\varphi - \frac{2}{\sqrt{7}} e^{\frac{4}{\sqrt{7}}\varphi} F^0 \wedge \star F^0 - \frac{3}{2\sqrt{7}} e^{\frac{3}{\sqrt{7}}\varphi} (\mathcal{M}^{-1})_{\mathbf{ij}} F^{\mathbf{i}} \wedge \star F^{\mathbf{j}} \\ + \frac{1}{2\sqrt{7}} e^{-\frac{1}{\sqrt{7}}\varphi} (\mathcal{M}^{-1})_{ij} H^i \wedge \star H^j - \frac{1}{\sqrt{7}} e^{\frac{2}{\sqrt{7}}\varphi} G \wedge \star G = 0, \quad (1.68)$$

$$d \left[\star \frac{d\bar{\tau}}{(\Im m \tau)^2} \right] - i \frac{d\tau \wedge \star d\bar{\tau}}{(\Im m \tau)^3} - \partial_\tau (\mathcal{M}^{-1})_{\mathbf{ij}} [F^{\mathbf{i}} \wedge \star F^{\mathbf{j}} + H^i \wedge \star H^j] = 0, \quad (1.69)$$

and those of the fundamental p -forms ($p \geq 1$), after some algebraic manipulations, take the form

$$d \left(e^{\frac{4}{\sqrt{7}}\varphi} \star F^0 \right) = -e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{\mathbf{ij}}^{-1} F^{\mathbf{i}} \wedge \star H^j + \frac{1}{2} G \wedge G, \quad (1.70)$$

$$d \left(e^{\frac{3}{\sqrt{7}}\varphi} \mathcal{M}_{\mathbf{ij}}^{-1} \star F^{\mathbf{j}} \right) = -e^{\frac{3}{\sqrt{7}}\varphi} \mathcal{M}_{\mathbf{ij}}^{-1} F^0 \wedge \star H^j + \varepsilon_{\mathbf{ij}} e^{\frac{2}{\sqrt{7}}\varphi} H^j \wedge \star G, \quad (1.71)$$

$$d \left(e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} \star H^j \right) = \varepsilon_{ij} e^{\frac{2}{\sqrt{7}}\varphi} F^{\mathbf{j}} \wedge \star G - \varepsilon_{ij} H^j \wedge G, \quad (1.72)$$

$$d \left(e^{\frac{2}{\sqrt{7}}\varphi} \star G \right) = F^0 \wedge G + \frac{1}{2} \varepsilon_{ij} H^i \wedge H^j. \quad (1.73)$$

1.4.1 Global symmetries

The theory has as (classical) global symmetry group $SL(2, \mathbb{R}) \times (\mathbb{R}^+)^2$. The $(\mathbb{R}^+)^2$ symmetries correspond to scalings of the fields, the first of which, that we will denote by α ¹³, acts on the metric and only leaves the equations of motion invariant while the second of them, which we will denote by β , leaves invariant both the metric and the action. The β rescaling corresponds to a *trombone symmetry*.

One can also discuss two more scaling symmetries γ and δ , but γ is just a subgroup of $SL(2, \mathbb{R})$ and δ is related to the other scaling symmetries by

$$\frac{4}{9}\alpha - \frac{8}{3}\beta - \gamma - \frac{1}{2}\delta = 0. \quad (1.74)$$

We will take α and β as the independent symmetries. The weights of the electric fields under all the scaling symmetries are given in Table 1.6. We can see that each of the three gauge fields A^I_μ has zero weight under *two* (linear combinations) of these three symmetries: one is a symmetry of the action, the other is a symmetry of the equations of motion only. The 1-form that has zero weight under a given rescaling is

¹³This discussion follows closely that of Ref. [117] in which the higher-dimensional origin of each symmetry is also studied. In particular, we use the same names and definitions for the scaling symmetries and we reproduce the table of scaling weights for the electric fields.

precisely the one that can be used to gauge that rescaling, but this kind of conditions are automatically taken into account by the embedding-tensor formalism and we will not have to discuss them in detail.

The action of the element of $SL(2, \mathbb{R})$ given by the matrix

$$(\Omega_j^i) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad (1.75)$$

on the fields of the theory is

$$\begin{aligned} \tau' &= \frac{a\tau + b}{c\tau + d}, & \mathcal{M}'_{ij} &= \Omega_i^k \mathcal{M}_{kl} \Omega_j^l, \\ A^{i'} &= \Omega_j^i A^j, & B^{i'} &= \Omega_j^i B^j, \\ \psi'_\mu &= e^{\frac{i}{2}l} \psi_\mu, & \lambda &= e^{\frac{3i}{2}l} \lambda, \\ \tilde{\lambda}' &= e^{-\frac{i}{2}l} \tilde{\lambda}, & \epsilon' &= e^{\frac{i}{2}l} \epsilon. \end{aligned} \quad (1.76)$$

where

$$e^{2il} \equiv \frac{c\tau^* + d}{c\tau + d}. \quad (1.77)$$

The rest of the fields ($e^a{}_\mu, \varphi, A^0{}_\mu, C_{\mu\nu\rho}$), are invariant under $SL(2, \mathbb{R})$.

We are going to label the 5 generators of these global symmetries by T_A , $A = 1, \dots, 5$. $\{T_1, T_2, T_3\}$ will be the 3 generators of $SL(2, \mathbb{R})$ (collectively denoted by $\{T_m\}$, $m = 1, 2, 3$), and T_4 and T_5 will be, respectively, the generators of the rescalings α and β . Our choice for the generators of $SL(2, \mathbb{R})$ acting on the doublets of 1-forms A^i and 2-forms B^i is

$$T_1 = \frac{1}{2}\sigma^3, \quad T_2 = \frac{1}{2}\sigma^1, \quad T_3 = \frac{i}{2}\sigma^2, \quad (1.78)$$

where the σ^m are the standard Pauli matrices, so

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = -T_1, \quad [T_3, T_1] = -T_2. \quad (1.79)$$

Then, the 3×3 matrices corresponding to generators acting (contravariantly) on the 3 1-forms A^I (and covariantly on their dual 6-forms \tilde{A}_I to be introduced later) are

$$\begin{aligned} ((T_1)_{J^I}) &= \frac{1}{2} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma^3 \end{array} \right), & ((T_2)_{J^I}) &= \frac{1}{2} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma^1 \end{array} \right), & ((T_3)_{J^I}) &= \frac{1}{2} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & i\sigma^2 \end{array} \right), \\ ((T_4)_{J^I}) &= \text{diag}(3, 0, 0), & ((T_5)_{J^I}) &= \text{diag}(1/2, -3/4, 0). \end{aligned} \quad (1.80)$$

We will sometimes denote this representation by $T_A^{(3)}$. The 2×2 matrices corresponding to generators acting (contravariantly) on the doublet of 2-forms B^i (and covariantly on their dual 5-forms \tilde{B}_i to be introduced later) are

$$\begin{aligned} ((T_1)_j{}^i) &= \frac{1}{2}\sigma^3, & ((T_2)_j{}^i) &= \frac{1}{2}\sigma^1, & ((T_3)_j{}^i) &= \frac{i}{2}\sigma^2, \\ ((T_4)_j{}^i) &= \text{diag}(3, 3), & ((T_5)_j{}^i) &= \text{diag}(-1/4, 1/2). \end{aligned} \quad (1.81)$$

We will denote this representation by $T_A^{(2)}$. The generators that act on the 3-form C (sometimes denoted by $T_A^{(1)}$) are

$$T_1 = T_2 = T_3 = 0, \quad T_4 = 3, \quad T_5 = -1/4. \quad (1.82)$$

We will also need the generators that act on the magnetic 4-form \tilde{C} (see next section), also denoted by $T_A^{(\tilde{1})}$

$$\tilde{T}_1 = \tilde{T}_2 = \tilde{T}_3 = 0, \quad \tilde{T}_4 = 6, \quad \tilde{T}_5 = 1/4. \quad (1.83)$$

We define the structure constants $f_{AB}{}^C$ by

$$[T_A, T_B] = f_{AB}{}^C T_C. \quad (1.84)$$

The symmetries of the theory are isometries of the scalar manifold $(\mathbb{R} \times SL(2, \mathbb{R})/U(1))$. The Killing vector associated to the generator T_A will be denoted by k_A and will be normalized so that their Lie brackets are given by

$$[k_A, k_B] = -f_{AB}{}^C k_C. \quad (1.85)$$

The $SL(2, \mathbb{R})/U(1)$ factor of the scalar manifold is a Kähler space with Kähler potential, Kähler metric and Kähler 1-form, respectively given by

$$\mathcal{K} = -\log \Im \tau = \phi, \quad \mathcal{G}_{\tau\tau^*} = \partial_\tau \partial_{\tau^*} \mathcal{K} = \frac{1}{4} e^{2\phi}, \quad \mathcal{Q} = \frac{1}{2i} (\partial_\tau \mathcal{K} d\tau - \text{c.c.}) = \frac{1}{2} e^\phi d\chi. \quad (1.86)$$

In general, the isometries of the Kähler metric only leave invariant the Kähler potential up to Kähler transformations :

$$\mathcal{L}_{k_m} \mathcal{K} = k_m^\tau \partial_\tau \mathcal{K} + \text{c.c.} = \lambda_m(\tau) + \text{c.c.}, \quad \mathcal{L}_{k_m} \mathcal{Q} = -\frac{i}{2} d\lambda_m, \quad (1.87)$$

where the λ_m are holomorphic functions of the coordinates that satisfy the equivariance property

$$\mathcal{L}_{k_m} \lambda_n - \mathcal{L}_{k_n} \lambda_m = -f_{mn}{}^p \lambda_p. \quad (1.88)$$

Then, for each of the $SL(2, \mathbb{R})$ Killing vectors k_m , $m = 1, 2, 3$, it is possible to find a real *Killing prepotential* or *momentum map* \mathcal{P}_m such that

$$\begin{aligned}
k_{m\tau^*} &= \mathcal{G}_{\tau^*\tau} k_m{}^\tau = i\partial_{\tau^*}\mathcal{P}_m, \\
k_m{}^\tau \partial_\tau \mathcal{K} &= i\mathcal{P}_m + \lambda_m, \\
\mathcal{L}_{k_m}\mathcal{P}_n &= -f_{mn}{}^p \mathcal{P}_p.
\end{aligned} \tag{1.89}$$

The non-vanishing components of all the Killing vectors are¹⁴

$$k_1{}^\tau = \tau, \quad k_2{}^\tau = \frac{1}{2}(1 - \tau^2), \quad k_3{}^\tau = \frac{1}{2}(1 + \tau^2), \quad k_4{}^\tau = 0, \quad k_5{}^\tau = -\frac{3}{4}\tau. \tag{1.90}$$

and

$$k_4{}^\varphi = 6/\sqrt{7}, \quad k_5{}^\varphi = \sqrt{7}/4. \tag{1.91}$$

The holomorphic functions $\lambda_m(\tau)$ take the values

$$\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{1}{2}\tau, \quad \lambda_3 = -\frac{1}{2}\tau, \tag{1.92}$$

and the momentum maps are given by:

$$\mathcal{P}_1 = \frac{1}{2}e^\phi \chi, \quad \mathcal{P}_2 = \frac{1}{4}e^\phi(1 - |\tau|^2), \quad \mathcal{P}_3 = \frac{1}{4}e^\phi(1 + |\tau|^2). \tag{1.93}$$

These objects will be used in the construction of $SL(2, \mathbb{R})$ -covariant derivatives for the fermions.

1.4.2 Magnetic fields

As it is well known, for each p -form potential with $p > 0$ one can define a *magnetic* dual which in $D = 9$ dimensions will be a $(7 - p)$ -form potential. Then, we will have magnetic 4-, 5- and 6-form potentials in the theory.

A possible way to define those potentials and identify their $(8 - p)$ -form field strengths consists in writing the equations of motion of the p -forms as total derivatives. Let us take, for instance, the equation of motion of the 3-form C Eq. (4.38). It can be written as

$$\begin{aligned}
d\frac{\partial \mathcal{L}}{\partial G} &= d\left\{ e^{\frac{2}{\sqrt{7}}\varphi} \star G - [G + \varepsilon_{ij} A^i \wedge (H^j - \frac{1}{2}\delta^j{}_j A^i \wedge F^0)] \wedge A^0 \right. \\
&\quad \left. + \frac{1}{2}\varepsilon_{ij} (H^i - \delta^i{}_i A^i \wedge F^0) \wedge (B^j - \frac{1}{2}\delta^j{}_j A^{0j}) \right\} = 0.
\end{aligned} \tag{1.94}$$

¹⁴The holomorphic and anti-holomorphic components are defined by $k = k^\tau \partial_\tau + \text{c.c.} = k^\chi \partial_\chi + k^\phi \partial_\phi$.

We can transform this equation of motion into a Bianchi identity by replacing the combination of fields on which the total derivative acts by the total derivative of a 4-form which we choose for the sake of convenience¹⁵

$$d \left[\tilde{C} - C \wedge A^0 - \frac{3}{4} \varepsilon_{ij} A^{0i} \wedge B^j \right] \equiv e^{\frac{2}{\sqrt{7}} \varphi} \star G - \left[G + \varepsilon_{ij} A^i \wedge \left(H^j - \frac{1}{2} \delta^j_{\mathbf{j}} A^{\mathbf{j}} \wedge F^0 \right) \right] \wedge A^0 \\ + \frac{1}{2} \varepsilon_{ij} \left(H^i - \delta^i_{\mathbf{i}} A^{\mathbf{i}} \wedge F^0 \right) \wedge \left(B^j - \frac{1}{2} \delta^j_{\mathbf{j}} A^{\mathbf{j}} \wedge F^0 \right), \quad (1.95)$$

where \tilde{C} will be the magnetic 4-form. This relation can be put in the form of a duality relation

$$e^{\frac{2}{\sqrt{7}} \varphi} \star G = \tilde{G}, \quad (1.96)$$

where we have defined the magnetic 5-form field strength

$$\tilde{G} \equiv d\tilde{C} + C \wedge F^0 - \frac{1}{24} \varepsilon_{ij} A^{0ij} \wedge F^0 - \varepsilon_{ij} \left(H^i - \frac{1}{2} dB^i \right) \wedge B^j. \quad (1.97)$$

The equation of motion for \tilde{C} is just the Bianchi identity of G rewritten in terms of \tilde{G} .

In a similar fashion we can define a doublet of 5-forms \tilde{B}_i with field strengths denoted by \tilde{H}_i , and a singlet and a doublet of 6-forms \tilde{A}_0, \tilde{A}_i with field strengths denoted, respectively, by \tilde{F}_0 and \tilde{F}_i . The field strengths can be chosen to have the form

$$\tilde{H}_i = d\tilde{B}_i - \delta_{ij} B^j \wedge G + \delta_{ij} \tilde{C} \wedge F^{\mathbf{j}} + \frac{1}{2} \delta_{ij} \left(A^0 \wedge F^{\mathbf{j}} + A^{\mathbf{j}} \wedge F^0 \right) \wedge C \\ + \frac{1}{2} \delta_{ij} \varepsilon_{kl} B^{jk} \wedge F^{\mathbf{l}}, \quad (1.98)$$

$$\tilde{F}_0 = d\tilde{A}_0 + \frac{1}{2} C \wedge G - \varepsilon_{ij} F^{\mathbf{i}} \wedge \left(\delta^{jk} \tilde{B}_k - \frac{2}{3} B^j \wedge C \right) \\ - \frac{1}{18} \varepsilon_{ij} A^{\mathbf{ij}} \wedge \left(\tilde{G} - F^0 \wedge C - \frac{1}{2} \varepsilon_{kl} B^k \wedge H^{\mathbf{l}} \right) \\ - \frac{1}{6} \varepsilon_{ij} A^{\mathbf{i}} \wedge \left(B^j \wedge G - C \wedge H^j - \frac{2}{3} \delta^j_{\mathbf{j}} \tilde{C} \wedge F^{\mathbf{j}} - \varepsilon_{kl} B^{jk} \wedge F^{\mathbf{l}} \right), \quad (1.99)$$

$$\tilde{F}_i = d\tilde{A}_i + \delta_{ij} \left(B^j + \frac{7}{18} \delta^j_{\mathbf{k}} A^{0\mathbf{k}} \right) \wedge \tilde{G} - \delta_i^{\mathbf{j}} F^0 \wedge \tilde{B}_j - \frac{1}{9} \delta_{ij} \left(8A^0 \wedge F^{\mathbf{j}} + A^{\mathbf{j}} \wedge F^0 \right) \wedge \tilde{C} \\ - \frac{1}{3} \delta_{ij} \varepsilon_{lm} \left(B^j + \frac{1}{3} \delta^j_{\mathbf{k}} A^{0\mathbf{k}} \right) \wedge B^{\mathbf{l}} \wedge H^{\mathbf{m}} - \frac{1}{6} \delta_{ij} \varepsilon_{kl} \left(A^0 \wedge H^j - B^j \wedge F^0 \right) \wedge A^{\mathbf{k}} \wedge B^{\mathbf{l}} \\ - \frac{1}{9} A^0 \wedge F^0 \wedge \delta_{ij} \left(\frac{7}{2} A^{\mathbf{j}} \wedge C + \delta^{\mathbf{j}}_{k\mathbf{l}m} A^{\mathbf{l}m} \wedge B^{\mathbf{k}} \right), \quad (1.100)$$

¹⁵With this definition \tilde{G} will have exactly the same form that we will obtain from the embedding tensor formalism.

and the duality relations are

$$\tilde{H}_i = e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} \star H^j, \quad (1.101)$$

$$\tilde{F}_0 = e^{\frac{4}{\sqrt{7}}\varphi} \star F^0, \quad (1.102)$$

$$\tilde{F}_i = e^{\frac{3}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} \star F^j. \quad (1.103)$$

The situation is summarized in Table 1.7. The scaling weights of the magnetic fields are given in Table 1.8.

This dualization procedure is made possible by the gauge symmetries associated to all the p -form potentials for $p > 0$ (actually, by the existence of gauge transformations with constant parameters) and, therefore, it always works for massless p -forms with $p > 0$ and generically fails for 0-form fields. However, in maximal supergravity theories at least, there is a global symmetry group that acts on the scalar manifold and whose dimension is larger than that of the scalar manifold. Therefore, there is one Noether 1-form current j_A associated to each of the generators of the global symmetries of the theory T_A . These currents are conserved on-shell, *i. e.* they satisfy

$$d \star j_A = 0,$$

on-shell, and we can define a $(d-2)$ -form potential $\tilde{A}_{(d-2)}^A$ by

$$d\tilde{A}_{(d-2)}^A = G^{AB} \star j_B,$$

where G^{AB} is the inverse Killing metric of the global symmetry group, so that the conservation law (dynamical) becomes a Bianchi identity.

Thus, while the dualization procedure indicates that for each electric p -form with $p > 0$ there is a dual magnetic $(7-p)$ -form transforming in the conjugate representation, it tells us that there are as many magnetic $(d-2)$ -form duals of the scalars as the dimension of the global group (and not of as the dimension of the scalar manifold) and that they transform in the co-adjoint representation. Actually, since there is no need to have scalar fields in order to have global symmetries, it is possible to define magnetic $(d-2)$ -form potentials even in the total absence of scalars¹⁶.

1.5 E_{11} formalism

$D = 3$ is the lowest number of spacetime dimensions for which the global symmetry is still finite. In that case, $G = E_8$, the largest finite exceptional Lie group, and the dimension of the coset $E_8/SO(16)$ is 128. For $D = 2$, the global symmetry group

¹⁶See Refs. [118, 119] for examples.

is the affine extension of the three-dimensional one, which for the maximal theory is $E_9 = E_8^+$. Similarly, for $D = 1$, we have the overextension of the 3-dimensional group, which for the maximal case is $G = E_{10} = E_8^{++}$.

Following the conjecture one step further, one can formally reduce to zero dimensions and hope to obtain the very extension $E_{11} = E_8^{+++}$, which is conjectured to be a symmetry of M theory. The key idea of the E_{11} formalism is to decompose E_{11} with respect to subgroups that coincide with the symmetry structure of supergravities G in various dimensions. The way in which E_{11} is decomposed is crucial: it will be splitted into

$$E_{11} = GL(D) \times G, \quad (1.104)$$

where $GL(D)$ is the sector associated to the symmetries of spacetime and G is the duality group in D dimensions.

This construction can be also made in the context of half-maximal theories. The general idea is just chop up the preferred Kac-Moody algebra into the gravity line ($GL(D)$ in the Dynkin diagrams) times the duality group. This completely determines the full spectrum of the theory and its possible deformations. More details about this approach can be found in [120].

However, despite of this Kac-Moody approach successfully provides the right symmetries, spectra and deformations of all possible (half-)maximal supergravities, it is not clear whether Kac-Moody algebras play a more fundamental role in supergravity and string theory. E_{11} formalism has motivated considerable literature and supplied a mechanism that infer the field content of any theory from group theoretical arguments.

N	s_{max}	$h = 2$	$h = 3/2$	$h = 1$	$h = 1/2$	$h = 0$	$h = -1/2$	$h = -1$	$h = -3/2$	$h = -2$
$N = 1$	2	1	1							
	3/2		1	1						
	1			1	1					
	1/2				1	1				
$N = 2$	2	1	2	1						
	3/2		1	2	1					
	1			1	2	1				
	1/2				1	2	1			
$N = 3$	2	1	3	3	1					
	3/2		1	3	3	1				
	1			1	3	3	1			
	1/2				1	3	3	1		
$N = 4$	2	1	4	6	4	1				
	3/2		1	4	6	4	1			
	1			1	4	6	4	1		
	1/2				1	4	6	4	1	
$N = 5$	2	1	5	10	10	5	1			
	3/2		1	5	10	10	5	1		
	1			1	5	10	10	5	1	
	1/2				1	5	10	10	5	1
$N = 6$	2	1	6	15	20	15	6	1		
	3/2		1	6	15	20	15	6	1	
	1			1	6	15	20	15	6	1
	1/2				1	6	15	20	15	6
$N = 7$	2	1	7	21	35	35	21	7	1	
	3/2		1	7	21	35	35	21	7	1
	1			1	7	21	35	35	21	7
	1/2				1	7	21	35	35	21
$N = 8$	2	1	8	28	56	70	56	28	8	1
	3/2		1	8	28	56	70	56	28	8
	1			1	8	28	56	70	56	28
	1/2				1	8	28	56	70	56

Table 1.2: Different irreducible spinorial representations depending on the dimension D and their corresponding real components.

dimension	spinor irrep	R -symmetry
10	MW	$SO(N_L) \times SO(N_R)$
9	M and D odd	$SO(N)$
8, 4	M and D even	$U(N)$
7, 5	S	$USp(N)$
6	SW	$USp(N_L) \times USp(N_R)$

Table 1.3: R symmetry summary for different dimensions and its relation to the spinorial irreducible representations.

$$N = 2A \left\{ \begin{array}{ll} g_{\mu\nu} & \text{metric} - \text{graviton} \\ \phi & \text{dilaton} \\ B_{\mu\nu} & \text{rank 2 antisymmetric} \\ C_{\mu\nu\rho}^{(3)} & \text{antisymmetric rank 3} \\ C_{\mu}^{(1)} & \text{graviphoton} \\ \psi_{\mu}^{\pm} & \text{2MW gravitinos} \\ \chi^{\pm} & \text{2MW dilatinos} \end{array} \right. \quad (1.48)$$

Table 1.4: Field Content of $N = 2A$ supergravity.

$$N = 2B \left\{ \begin{array}{ll} g_{\mu\nu} & \text{metric} - \text{graviton} \\ B_{\mu\nu}, C_{\mu\nu}^{(2)} & \text{rank 2 antisymmetric} \\ \phi & \text{scalar(dilaton)} \\ C^{(0)} & \text{scalar(axion)} \\ C_{\mu\nu\rho\sigma SD}^{(4)} & \text{self} - \text{dual antisymmetric rank 4} \\ \psi_{\mu\alpha}^I \quad I=1,2 & \text{2MW gravitinos} \\ \lambda_{\alpha}^I \quad I=1,2 & \text{2MW dilatinos} \end{array} \right. \quad (1.52)$$

Table 1.5: Field content of $N = 2B$ supergravity.

\mathbb{R}^+	e_{μ}^a	e^{φ}	e^{ϕ}	χ	A^0	A^1	A^2	B^1	B^2	C	ψ_{μ}	λ	$\tilde{\lambda}$	ϵ	\mathcal{L}
α	9/7	$6/\sqrt{7}$	0	0	3	0	0	3	3	3	9/14	-9/14	-9/14	9/14	9
β	0	$\sqrt{7}/4$	3/4	-3/4	1/2	-3/4	0	-1/4	1/2	-1/4	0	0	0	0	0
γ	0	0	-2	2	0	1	-1	1	-1	0	0	0	0	0	0
δ	8/7	$-4/\sqrt{7}$	0	0	0	2	2	2	2	4	4/7	-4/7	-4/7	4/7	8

Table 1.6: The scaling weights of the electric fields of maximal $D = 9$ supergravity.

0	1	2	3	4	5	6	7	8	9
j_A	A^I	B^i	C	\tilde{C}	\tilde{B}_i	\tilde{A}_I	$\tilde{A}_{(7)}^A$	$\tilde{A}_{(8)}$	$\tilde{A}_{(9)}$
	F^I	H^i	G	\tilde{G}	\tilde{H}_i	\tilde{F}_I	$\tilde{F}_{(8)}^A$	$\tilde{F}_{(9)}$	

Table 1.7: Electric and magnetic forms and their field strengths.

\mathbb{R}^+	\tilde{C}	\tilde{B}_2	\tilde{B}_1	\tilde{A}_2	\tilde{A}_1	\tilde{A}_0
α	6	6	6	9	9	6
β	1/4	-1/2	+1/4	0	+3/4	-1/2
γ	0	1	-1	1	-1	0
δ	4	6	6	6	6	8

Table 1.8: *The scaling weights of the magnetic fields of maximal $D = 9$ supergravity can be determined by requiring that the sum of the weights of the electric and magnetic potentials equals that of the Lagrangian. The scaling weights of the 7-, 8- and 9-forms can be determined in the same way after we find the entities they are dual to (Noether currents, embedding-tensor components and constraints, see Appendix B).*

Chapter 2

Stringy motivation of Supergravity

Supergravity is equivalent to the low-energy limit of string theory. The description of any string theory at the regime $\alpha' \rightarrow 0$, a characteristic constant in string theories, can be described by means of a supergravity theory, so that any of the different string theory have their corresponding associated supergravities. In this chapter we will introduce the bosonic string and supersymmetric string theories.

We will explicitly see how this limit is taken properly and how SUGRA naturally emerges. We will also review the main features of the dualities and their important implications at the level of string theory and supergravity. Finally, we will show some remarks about extended objects, emphasizing in the D-branes.

2.1 String theory

The origin of string theory dates from the late 1960s. Experimental data about the strong interaction implied the existence of new resonances. Most of them looked to follow a certain relation between their masses and their spin. This behaviour proposed by Regge [121]

$$M^2 = \frac{J}{\alpha'} + \alpha_0, \quad (2.1)$$

where M is the mass, J is the spin of the particle and $\alpha' \simeq 1 \text{ GeV}$ is known as the *Regge slope*. The scattering amplitude for two hadrons \rightarrow two hadrons ($1, 2 \rightarrow 3, 4$) has to show certain symmetry in the momenta k_i , for $i = 1, \dots, 4$. In terms of the Mandelstam variables, the amplitude amounts to the interchange of $t \leftrightarrow s$, where

$$s = -(k_1 + k_2)^2, \quad t = -(k_2 + k_3)^2, \quad u = -(k_1 + k_3)^2. \quad (2.2)$$

At high energies, the amplitude of the exchange of a spin- J particle of mass M is proportional to

$$A(s, t) \propto \frac{(-s)^J}{t - M^2}, \quad (2.3)$$

which diverges for $J > 1$. Thus, a proposal for a dual amplitude was made by Veneziano [21],

$$A(s, t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}, \quad (2.4)$$

where Γ is the standard Euler Γ -function and $\alpha(s) = \alpha(0) + \alpha's$. Soon, it was realized that these amplitudes could be interpreted as the ones for quantum relativistic strings [122]. The dynamics of a relativistic string with tension T can be formulated by assigning coordinates $\sigma^a = (\sigma, \tau)$ to the 2-dimensional worldsheet Σ . The action is given by the surface that the worldsheet sweeps out in the spacetime,

$$S_{\text{string}} = -T \int_{\Sigma} d^2\sigma \sqrt{|g_{ij}|}, \quad (2.5)$$

where $X^\mu(\sigma)$, $\mu = 0, \dots, D-1$ are the spacetime coordinates of the string, and $|g_{ij}|$ stands for the determinant of the induced metric g_{ij} on the worldvolume (the pullback of the spacetime metric $g_{\mu\nu}$)

$$g_{ij} \equiv \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}(X). \quad (2.6)$$

This is the *Nambu-Goto action*. T is a constant with dimensions M^2 (equivalent to mass per unit length) and, for historical connotations, it is expressed in terms of the Regge slope α' ,

$$T = \frac{1}{2\pi\alpha'}. \quad (2.7)$$

We can define characteristic length (ℓ_s) and mass (m_s) of the string,

$$\ell_s = \sqrt{\alpha'}, \quad m_s = \frac{1}{\sqrt{\alpha'}}. \quad (2.8)$$

In addition, an action containing quadratic derivative terms of X^μ can be constructed by introducing an auxiliary field γ_{ij} , the so-called *Polyakov action* [123],

$$S_P = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{|\gamma|} \gamma^{ij} \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}(X). \quad (2.9)$$

We can eliminate γ_{ij} from the action by means of its equation of motion,

$$\gamma_{ij} = 2 \frac{g_{ij}}{g_k^k}, \quad (2.10)$$

and we recover the Nambu-Goto action. The Polyakov action is invariant under Weyl rescalings of the worldsheet metric,

$$\gamma_{ij} \rightarrow \Omega^2(\sigma) \gamma_{ij}, \quad (2.11)$$

which is a crucial requirement for the quantization of the theory.

The equation of motion is the 2-dimensional wave equation

$$\frac{\partial}{\partial\sigma^-}\frac{\partial}{\partial\sigma^+}X^\mu(\tau,\sigma)=0, \quad (2.12)$$

where $\sigma^\pm \equiv \tau \pm \sigma$. For the closed string, periodic boundary conditions are required,

$$X^\mu(\tau,0)=X^\mu(\tau,\ell_s), \quad (2.13)$$

whereas for open strings, Neumann (+) or Dirichlet (−) boundary conditions can be chosen,

$$\frac{\partial}{\partial\sigma^-}X^\mu(\tau,\sigma)=\pm\frac{\partial}{\partial\sigma^-}X^\mu(\tau,\sigma), \quad \sigma=0,\ell_s. \quad (2.14)$$

An additional Einstein-Hilbert term can be added to the Polyakov action,

$$-\frac{\phi_0}{4}\int d^2\sigma\sqrt{|\gamma|}R(\gamma). \quad (2.15)$$

It is Weyl invariant and does not modify the equations of motion because the 2-dimensional Einstein-Hilbert lagrangian is just the curvature 2-form, which is locally a total derivative ($R_{ab} = \epsilon_{ab}d\omega$). Actually, this term is the constant ϕ_0 times the *Euler characteristic*¹. ϕ_0 is the vacuum expectation value (VEV) of the dilaton, a scalar field present in all string theories and $g \equiv e^\phi$ can be interpreted as the *string coupling constant*.

Furthermore, the string worldsheet action, written as a non-linear σ -model, can be generalized to describe the coupling to all background fields associated with the massless modes. The coupling to an antisymmetric background field, the Kalb-Ramond 2-form $B_{\mu\nu}$ can be added,

$$\frac{T}{2}\int_\Sigma B = \frac{T}{2}\int_\Sigma d^2\sigma\epsilon^{ij}\partial_iX^\mu\partial_jX^\nu B_{\mu\nu}. \quad (2.16)$$

In analogy with the coupling to the metric, this term is the integral of the pullback of the 2-form over the worldsheet. It can be interpreted as the generalization of the coupling to the Maxwell vector field to a charged point-particle, but now our charges are vectors carried by the strings. Thus, the full non-linear σ -model is

$$S = -\frac{T}{2}\int_\Sigma d^2\sigma\sqrt{|\gamma|}[(\gamma^{ij}g_{\mu\nu}(X) + \epsilon^{ij}B_{\mu\nu}(X))\partial_iX^\mu\partial_jX^\nu - \alpha'\phi(X)R(\gamma)]. \quad (2.17)$$

The generators of the conformal symmetry that the worldsheet enjoys are the Virasoro generators. They satisfy the infinite-dimensional Virasoro algebra,

$$[L_m, L_n]_{\text{PB}} = i(m-n)L_{m+n}, \quad (2.18)$$

¹The Euler characteristic χ is defined as $\chi = 2 - 2g - b - c$, where g is the number of genus, b is the number of boundaries and c is the number of crosscaps.

where $[\circ, \circ]_{\text{PB}}$ is a Poisson bracket and the L_m, \tilde{L}_m generators²

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \quad (2.19)$$

encode the Fourier modes $\alpha_k^\mu, \tilde{\alpha}_k^\mu$ that span the worldsheet coordinates $X^\mu(\alpha_k^\mu, \tilde{\alpha}_k^\mu)$.

New results arise when we quantize these Fourier modes and keep the conformal invariance simultaneously. One of them is the condition that the theory has to be defined in a 26-dimensional spacetime. This can be solved by performing suitable KK dimensional reductions and getting lower-dimensional theory. The next one is the appearance of a tachyon and a spin-2 vibration modes. While the presence of a tachyon is problematic, the appearance of a spin-2 particle is one of the motivations of considering string theory as a realistic description of Quantum Gravity. Despite of a spin-2 particle is essential for a quantum gravity, it gets very problematic for a description of the strong interaction. The absence of fermions and the presence of the tachyon in the spectrum motivates the consideration of a more general action to have matter included as well.

2.2 Superstring theory

The supersymmetric theory that generalizes the Polyakov action in Minkowski spacetime is the theory of D 2-dimensional scalar multiplets (X^μ, ψ^μ) coupled to a 2-dimensional auxiliary multiplet (e^a_i, χ_i) whose action is [124, 125]

$$S = -\frac{T}{2} \int_{\Sigma} e \left[\gamma^{ij} \partial_i X^\mu \partial_j X_\mu - i \bar{\psi}^\mu \not{D} \psi_\mu + 2 \bar{\chi}_i \rho^j \rho^i \psi^\mu \partial_j X_\mu + \frac{1}{2} (\bar{\chi}_i \rho^j \rho^i \chi_j) (\bar{\psi}^\mu \psi_\mu) \right], \quad (2.20)$$

where $i, j = 1, 2$ are indices on the worldsheet and μ, ν are spacetime indices. Both ψ^μ and χ_i are real spinors and ρ^i is the 2-dimensional realization of gamma matrices. The local worldsheet supersymmetric transformations are

$$\delta_\epsilon X^\mu = \bar{\epsilon} \psi^\mu, \quad \delta_\epsilon e^a_i = -2i \bar{\epsilon} \rho^a \chi_i, \quad (2.21)$$

$$\delta_\epsilon \psi^\mu = i(\partial_i X^\mu + \frac{1}{4} \bar{\chi}_i \psi^\mu) \rho^i \epsilon, \quad \delta_\epsilon \chi_i = \tilde{D}_i \epsilon, \quad (2.22)$$

where

$$\tilde{D}_i \epsilon = (\partial_i + \tilde{\omega}_i \rho_3) \epsilon = [\partial_i + \omega_i \rho_3 + i(\bar{\chi}_i \rho_3 \rho^j \chi_j) \rho_3] \epsilon, \quad (2.23)$$

ρ^3 is the chiral matrix and ω_i is the spin connection defined on the worldsheet.

Weyl invariance is kept and, by means of their equations of motion, the zweibein e^a and the worldsheet gravitino χ^a , can be decoupled and eliminated from the action.

²The \tilde{L}_m generators are defined in terms of the $\tilde{\alpha}_m^\mu$ oscillators in analogy with the L_m and α_m^μ .

The resulting action in terms of (X^μ, ψ^μ) is the so-called *Ramon-Neveu-Scharwz (RNS) model*,

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma [\eta^{ij} \partial_i X^\mu \partial_j X_\mu - i\bar{\psi}^\mu \not{\partial} \psi_\mu] . \quad (2.24)$$

For open superstrings, Neumann and Dirichlet conditions can be chosen for the X^μ s, whereas for the ψ^μ s, the possible boundary conditions are

- Ramond (R) $\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$ and $\psi_+^\mu(\tau, 2\pi\ell_s) = \psi_-^\mu(\tau, 2\pi\ell_s)$,
- Neveu-Schwarz (NS) $\psi_+^\mu(\tau, 0) = \psi_-^\mu(\tau, 0)$ and $\psi_+^\mu(\tau, 2\pi\ell_s) = -\psi_-^\mu(\tau, 2\pi\ell_s)$.

For closed superstrings, $\sigma \sim \sigma + 2\pi\ell$, so for each component ψ_+^μ and ψ_-^μ , we can independently have

- Ramond (R) boundary conditions (periodic) $\psi_\pm^\mu(\tau, 0) = \psi_\pm^\mu(\tau, 2\pi\ell_s)$,
- Neveu-Schwarz (NS) boundary conditions(antiperiodic) $\psi_\pm^\mu(\tau, 0) = -\psi_\pm^\mu(\tau, 2\pi\ell_s)$

Supersymmetry in the spacetime of the theory requires additional restrictions to the worldsheet supersymmetry [103]. The alternative *Green-Schwarz (GS) formulation* [126] already incorporates this by construction. A new local symmetry, the so-called κ -*symmetry*, arises in this formulation and, due to the fermionic nature of its transformation parameter, it halves the number of degrees of freedom in the action. Thus, we can use this κ -symmetry and invariance under reparametrization to gauge away all the possible degrees of freedom and impose a condition so that the bosonic and fermionic degrees of freedom coincide. In this direction, a general result is obtained in [127] that provides the condition for having a supersymmetric worldsheet or, for extended objects, a supersymmetric worldvolume. If M is the number of real components of the minimal spinor in the D -dimensional spacetime and N is a natural number, for an object with p extended dimensions, supersymmetry is guaranteed if

$$NM = 4(D - p - 1) . \quad (2.25)$$

Table 2.2 resumes the possible cases for which this is satisfied.

The model that incorporates these improvements is the *Green-Schwarz action*,

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{|\gamma|} \gamma^{ij} (\partial_i X^\mu \delta_\mu^a - i\bar{\theta}^I \gamma^a \partial_i \theta^I) (\partial_j X^\nu \delta_\nu^b - i\bar{\theta}^J \gamma^b \partial_j \theta^J) \eta_{ab} + T \int_{\Sigma} \Omega_2 . \quad (2.26)$$

θ^I are the anticommuting spacetime spinors and $I, J = 1, \dots, N$. This action can be understood as a generalization of a superparticle action plus the addition of a Wess-Zumino term that makes κ -symmetry to be preserved. The Wess-Zumino term

D	M	$p = -1$	$p = 0$	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
2	1	8						
3	2	12	8	4				
4	4	16	12	8	4			
5	8		16		8			
6	8			16		8		
7	16				16			
8	16	32				16		
9	16		32				16	
10	16			32				16
11	32				32			

Table 2.1: Classification of all the possible supersymmetric objects and their respective supercharges, just taking into account the scalar multiplets. The entries are the product MN , so that N is trivially obtained. Adapted from [127].

is the integral of a 2-form Ω_2 such that $\Omega_3 = d\Omega_2$ is Poincaré- and supersymmetry-invariant [128]:

$$\Omega_2 = -idX^\mu \delta_\mu^a \wedge (\bar{\theta}^1 \Gamma_a \theta^1 - \bar{\theta}^2 \Gamma_a d\theta^2) + (\bar{\theta}^1 \Gamma_a d\theta^1) \wedge (\bar{\theta}^1 \Gamma_a d\theta^2). \quad (2.27)$$

In addition, it can be shown that superstring theories are Poincaré invariant only in the critical dimension $D = 10$. This is allowed by the supersymmetry condition (2.25), as we can check in table 2.2. Further details about this model are treated in [36].

The zoo of different superstrings that live in these supersymmetric theories and the boundary conditions that restrict them imply a systematic study and classification. Following this aim, it is useful to introduce a new indicator that helps us to group them, the worldvolume fermion number F (defined modulo 2). It splits the Ramond (R) and Neveu-Schwarz (NS) sectors into R_\pm and NS_\pm subsectors with respect to the operator $e^{i\pi F}$. By combining these new subsectors properly, we can construct consistent and tachyon-free string theories. This mechanism is known as the *Gliozzi-Scherk-Olive (GSO) projection* [103]. The different possibilities are:

- Type IIB₊. $R_+R_+ \oplus R_+NS_+ \oplus NS_+R_+ \oplus NS_+NS_+$, whose massless spectrum coincides with the $N = 2B_+$ $D = 10$ SUGRA multiplet with self-dual 4-form (1.53).
- Type IIB₋. $R_-R_- \oplus R_-NS_+ \oplus NS_+R_- \oplus NS_+NS_+$, whose massless spectrum coincides with the $N = 2B_-$ $D = 10$ SUGRA multiplet with antiself-dual 4-form and opposite chirality fermions than type IIB₋.
- Type IIA₊₋. $R_+R_- \oplus R_+NS_+ \oplus NS_+R_- \oplus NS_+NS_+$, whose massless spectrum coincides with the $N = 2A_{+-}$ $D = 10$ SUGRA multiplet (1.49).

Theory	NSNS b	RR b	Chiral f	Non-chiral	Vector superm
Type IIA	$g_{\mu\nu}, B_{\mu\nu}, \phi$	$C^{(1)}, C^{(3)}$		ψ_μ, λ	
Type IIB	$J_{\mu\nu}, B_{\mu\nu}, \varphi$	$C^{(0)}, C^{(2)}, C^{(4\pm)}$	$\xi_\mu^{i(\mp)}, \chi^{i(\pm)}$		
Type I	$J_{\mu\nu}, \varphi$	$C^{(2)}$	$\xi_\mu^{i(\pm)}, \chi^{i(\mp)}$		(V_μ^I, η^I)
Heterotic	$g_{\mu\nu}, B_{\mu\nu}, \varphi$	$C^{(2)}$	$\psi_\mu^{i(\pm)}, \lambda^{i(\mp)}$		(V_μ^I, η^I)

Table 2.2: Massless sector of the 10-dimensional superstring theories.

- Type IIA $_{-+}$. $R_-R_+ \oplus R_-NS_+ \oplus NS_+R_+ \oplus NS_+NS_+$, whose massless spectrum coincides with the $N = 2A_{-+}$ $D = 10$ SUGRA multiplet. The sign of the CS term is different from the previous one.
- Open superstrings. They arise from the combination of two subsectors: $R_+ \oplus NS_+$ or $R_- \oplus NS_+$. Both massless spectra correspond to $N = 1$ $D = 10$ vector supermultiplet V_μ^{IJ}, χ , where χ is a Majorana-Weyl gaugino. Open superstrings need a closed superstring sector, which is constructed by truncating the type IIB theory properly. The result is the type I $_{\pm}$ $SO(32)$ superstring, whose massless modes correspond to $N = 1_{\pm}$ $D = 10$ SUGRA coupled to $SO(32)$ vector multiplets.

In addition, we can construct another bosonic theory that includes a tachyon.

- Heterotic string. Constructed by joining the right-moving sector of the closed type II superstring with the left-moving fields of the closed bosonic strings. Two versions with different gauge groups exist. These local symmetries are $SO(32)$ and $E_8 \times E_8$. The massless modes coincide with those of $N = 1_{\pm}$ $D = 10$.

In table 2.2 we show the massless sectors of all these supersymmetric theories.

2.2.1 Interactions and backgrounds

Open string theories require a closed string sector for consistency. Scattering amplitudes for particles are conveniently calculated by means of Feynman diagrams in which there is a one-to-one correspondence between a graph and a contribution to the amplitude. For strings, its analog is given in terms of Riemann surfaces. The most convenient way to obtain scattering amplitudes is through the path integral methods over all embeddings X^μ and all worldsheet metrics γ_{ij} with boundary data given by vertex operators. Without vertex operators, we get vacuum amplitudes, given by

$$\mathcal{Z} = \int \mathcal{D}X \mathcal{D}\gamma e^{-S_P - S_{\text{Euler}}}, \quad (2.28)$$

where S_P is the Polyakov action (2.9) and S_{Euler} is the topological term (2.15). Since any 2-dimensional surface is characterized by the Euler characteristic χ , the path integral can be rewritten as

$$\mathcal{Z} = \sum_t (e^{\phi_0})^{-\chi(t)} \int_{\Sigma_t} \mathcal{D}X \mathcal{D}\gamma e^{-S_P}, \quad (2.29)$$

where t runs over Σ_t , the space of surfaces with topology t . Thus, we associate each topology with a loop order given by $-\chi(t)$. This sum can be understood as a perturbative series expansion in which e^{ϕ_0} is interpreted as the *string coupling constant* g ,

$$g \equiv e^{\phi_0} . \quad (2.30)$$

2.3 Supergravity from the string viewpoint

We will motivate supergravity as the result of taking the limit of a given string theory, *i.e.*, as an effective theory of superstrings. Supergravity coincides with the low-energy effective description of string theory. For each superstring theory, a supergravity is given. We mean by low-energy limit the $\alpha' \rightarrow 0$ limit, *i.e.*, the one in which the string length can be ignored and a theory of particles (a field theory) emerges. Since we are interested in the low-energy limit, only the massless modes will be relevant. Thus, an obvious way to find this field theory is to compute the string amplitudes for these massless modes, take the limit $\alpha' \rightarrow 0$ and then construct a theory that reproduces those amplitudes. Despite of just by supersymmetry arguments some superstring theories can be matched to their supergravity analogs [103, 129–136], we will use scaling invariance arguments to find this parallelism.

In order to simplify our calculation, we will focus on the so-called *common sector*, spanned by the NSNS fields $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$. This sector is shared by all the superstring theories, as Table 2.2 shows. Its effective action in the *string frame*, that will be defined below, is given by

$$S = \frac{g^2}{16\pi G_N^{(d)}} \int d^D x \sqrt{|g|} e^{-2\phi} \left[R - 4(\partial\phi)^2 + \frac{1}{2 \cdot 3!} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] , \quad (2.31)$$

where $H_{\mu\nu\rho} = 3! \partial_{[\mu} B_{\nu\rho]}$. The factor $e^{-2\phi}$ is associated to the genus-0 (tree level) origin of these terms.

However, (2.17) has a problem in the coupling of the string to the dilaton. This action breaks Weyl invariance due to the topological nature of the last term. Since Weyl invariance is absolutely necessary for the consistency and quantization of the theory, we can wonder what backgrounds $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$ guarantee this scaling invariance. This is a way of interpreting the background fields as coupling functions, and then the question can be reformulated in terms of the β -functions. That is, the vanishing of the β -functions associated to them are the constraints that the fields have to satisfy to

preserve Weyl invariance [137], namely,

$$\beta_{\mu\nu}(g) = \alpha' \left[R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \phi + \frac{1}{4} H_\mu^{\alpha\beta} H_{\nu\alpha\beta} \right] + \mathcal{O}(\alpha'^2), \quad (2.32)$$

$$\beta_{\mu\nu}(B) = \frac{\alpha'}{2} e^{2\phi} \nabla^\rho (e^{-2\phi} H_{\rho\mu\nu}) + \mathcal{O}(\alpha'^2), \quad (2.33)$$

$$\beta(\phi) = \frac{d-26}{6} - \frac{\alpha'}{2} \left[\nabla^2 \phi - (\partial\phi)^2 - \frac{1}{4} R - \frac{1}{48} H^2 \right] + \mathcal{O}(\alpha'^2). \quad (2.34)$$

If we express the equations of motion in terms of the β -functions, we have

$$\frac{16\pi G_N^{(d)} e^{2(\phi-\phi_0)}}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}} \sim \frac{1}{\alpha'} [\beta_{\mu\nu}(g) - 4g_{\mu\nu}\beta(\phi)] + \mathcal{O}(\alpha'^2), \quad (2.35)$$

$$\frac{16\pi G_N^{(d)} e^{2(\phi-\phi_0)}}{\sqrt{|g|}} \frac{\delta S}{\delta \phi} \sim -\frac{16}{\alpha'} \beta(\phi) + \mathcal{O}(\alpha'^2), \quad (2.36)$$

$$\frac{16\pi G_N^{(d)} e^{2(\phi-\phi_0)}}{\sqrt{|g|}} \frac{\delta S}{\delta B^{\mu\nu}} \sim -\frac{1}{\alpha'} \beta_{\mu\nu}(B) + \mathcal{O}(\alpha'^2). \quad (2.37)$$

Thus, we conclude that the equations of motion are equivalent to the vanishing of the β -functions. The metric that appears in the action (2.31) is the same one as that to which the string couples in (2.17), and is called the *string-frame metric*. A conformal scaling

$$g_{\mu\nu} = e^{\frac{4}{d-2}\phi} g_{E\mu\nu} \quad (2.38)$$

in terms of the so-called *Einstein-frame metric*, eliminates the factor $e^{-2\phi}$. In this Einstein frame, the action is rewritten as

$$S = \frac{1}{16\pi G_N^{(d)}} \int d^D x \sqrt{|g_E|} \left[R_E + \frac{4}{D-2} (\partial\phi)^2 + \frac{1}{2 \cdot 3!} e^{-\frac{8}{D-2}\phi} H_{\mu\nu\rho} H^{\mu\nu\rho} \right]. \quad (2.39)$$

However, the identification of the field theories with the string modes is ambiguous, since the supergravity theories are unique up to field redefinitions. For a consistent identification, these field redefinitions have to coincide with the duality transformations that apply to the string modes. That is, in order to fully establish the link between SUGRA fields and string modes, we need to ensure that the formers are related by these dualities as the latter are.

2.4 Dualities

The main idea of dualities is that, in general, the global symmetries of the effective field theories (SUGRAs) correspond to *dualities* of the string theories. This was the statement proposed in [138], but a more precise reformulation can be done. We have to understand any of the relations of the different effective theories as the corresponding

dualities of their respective string theories. It is important to notice that these relations among supergravities are, in general, described by global symmetries of a theory, but in other cases we can establish relations even between different effective field theories.

Some of these dualities are essentially perturbative and the worldsheet approach is valid to be studied. For instance, *T duality* [139], that relates string theories compactified on circles of radius R and dual radius $R' = 1/R$, is an exact symmetry at all orders in string perturbation theory [140]. However, the so-called *S duality*, is non-perturbative in the string coupling constant and cannot be studied using the standard worldsheet approach. Its non-perturbative nature has to do with the inversion of the dilaton it implies. Finally, *U duality* is another duality that includes S and T duality and is directly related to the existence of the so-called M theory.

One remarkable difference is that supergravity theories enjoy duality groups that are continuous, whereas in string theory quantum effects break them into discrete subgroups. For example, the S duality group is, in general, associated to $SL(2, \mathbb{R})$ and gets broken into $SL(2, \mathbb{Z})$ by charge quantization [141]. Thus, we have to find a suitable translator to decode the different supergravity field configurations and their symmetries in terms of superstring theory and vice versa. Moreover, S duality can be a useful tool to provide new non-perturbative solutions in both scenarios.

At this point, it is clarifying to show an explicit and detailed example of how a duality works. For simplicity, we will perform a T duality transformation to the bosonic string.

2.4.1 T duality

The bosonic string

We will restrict to the string common sector. We will follow [36, 142]. Since T duality relates different theories compactified on a circle, we will choose the effective action (2.31) as the one on which we will perform the dimensional reduction. We will get a $D = \hat{D} - 1$ dimensional theory that will enjoy this duality. Let us assume the following standard KK reduction ansatz,

$$\hat{e}_{\hat{\mu}}^{\hat{a}} = \begin{pmatrix} e_{\mu}^a & kA_{\mu} \\ 0 & k \end{pmatrix}, \quad \phi = \hat{\phi} - \frac{1}{2} \ln k, \quad (2.40)$$

$$\hat{B}_{\hat{\mu}\hat{\nu}} = B_{\mu\nu} - A_{[\mu}B_{\nu]}, \quad \hat{B}_{\hat{\mu}\hat{z}} = B_{\mu}, \quad (2.41)$$

where we use the same notation as in Appendix A. After integrating over the compact coordinate, the reduced effective action is

$$S \sim \int d^D x \sqrt{|g|} e^{-2\phi} \left[R - 4(\partial\phi)^2 + \frac{1}{2 \cdot 3!} H^2 + (\partial \log k)^2 - \frac{1}{4} k^2 F^2(A) - \frac{1}{4} k^{-2} F^2(B) \right], \quad (2.42)$$

where $F(A)$ and $F(B)$ are the field strengths of the vector fields A_μ and B_μ , respectively. We can check the invariance of this action under the transformation rules

$$A_\mu \rightarrow B_\mu, \quad B_\mu \rightarrow A_\mu, \quad k \rightarrow k^{-1}, \quad (2.43)$$

so that the KK scalar gets inverted and the KK vector and the winding vector are interchanged. Two interpretations can be done: first, we compactify a string background, T dualize it, and decompactify it into a different background. Second, we have two different compactifications of a given background; these compactifications give the same D -dimensional background and thus, are dual.

The way in that these two backgrounds are related is described by an isometry. These field relations are known as *Buscher's rules* [143–145].

$$\begin{aligned} \hat{e}'^a{}_z &= \mp \frac{\hat{e}^a{}_z}{\hat{g}_{zz}}, & \hat{e}'^a{}_\mu &= \hat{e}^a{}_\mu - \frac{\hat{g}_{\mu z} \pm \hat{B}_{\mu z}}{\hat{g}_{zz}}, \\ \hat{B}'_{\mu z} &= \frac{\hat{g}_{\mu z}}{\hat{g}_{zz}}, & \hat{B}'_{\mu\nu} &= \hat{B}_{\mu\nu} + 2 \frac{\hat{g}_{[\mu|z|} \hat{B}_{\nu]z}}{\hat{g}_{zz}}, \\ \hat{\phi}' &= \hat{\phi} - \frac{1}{2} \ln |\hat{g}_{zz}|. \end{aligned} \quad (2.44)$$

Now, let us study T duality applied to the σ -model (2.17), without considering the dilaton term, since it does not play any relevant role in this classic approach. Let us assume (2.17) with hatted fields running over hatted indices. Then, decomposing the \hat{D} -dimensional fields into D -dimensional fields using (2.40), we have

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{|\gamma|} [\gamma^{ij} g_{ij} - k^2 F^2] + \frac{T}{2} \int d^2\sigma \sqrt{|\gamma|} \epsilon^{ij} [B_{ij} + A_i B_j - 2F_i B_j], \quad (2.45)$$

where g_{ij} , B_{ij} , A_i , B_i are the pullbacks of the D -dimensional metric, KR 2-form, KK and winding vectors respectively. F_i is the field strength of the Z coordinate,

$$F_i = \partial_i Z + A_i, \quad (2.46)$$

which reflects the shift invariance

$$\delta_\Lambda Z = -\Lambda(x), \quad (2.47)$$

$$\delta_\Lambda A_\mu = \partial_\mu \Lambda. \quad (2.48)$$

This invariance implies the following conserved current

$$P_z{}^i = T(k^2 F^i - \star B^i), \quad (2.49)$$

whose associated magnetic-like conserved current is

$$W_z{}^i = T \star F^i - \star A^i. \quad (2.50)$$

Their associated charges are the momentum of the string in the compact dimension and the winding number, respectively. Now, if we perform a Poincaré duality transformation on the Z coordinate, $Z \rightarrow Z'$, by using the Bianchi identity of F_i and its equation of motion, we have

$$S' = -\frac{T}{2} \int d^2\sigma \sqrt{|\gamma|} [\gamma^{ij} g_{ij} - k^{-2} F'^2] + \frac{T}{2} \int d^2\sigma \sqrt{|\gamma|} \epsilon^{ij} [B_{ij} + B_i A_j - 2F'_i A_j], \quad (2.51)$$

where

$$F'_i = \partial_i Z' + B_i. \quad (2.52)$$

This action coincides with the original one when we make the field replacements (2.43). We find its conserved currents $P_{Z'}^i$ and $W_{Z'}^i$, which are closely related to those of the original theory,

$$P_{Z'}^i = W_Z^i, \quad W_{Z'}^i = P_Z^i. \quad (2.53)$$

Thus, we summarize that T duality inverts the compactification radius and interchanges momentum modes with winding modes, leaving invariant the mass spectrum and performing a parity transformation on the right-moving modes.

For type II superstrings, this parity transformation changes the chirality of the spinors and the overall result is that the $N = (1, 1)$ type IIA theory can be mapped into the $N = (2, 0)$ type IIB version. This relation holds for any value of the radius, in particular it relates the limits $R \rightarrow 0$ and $R \rightarrow \infty$. For the case of $N = 2A$ and $N = 2B$ supergravity theories, there is a discrete symmetry relating the two supergravity theories when both are reduced to 9 dimensions [146]. A generalization of the Buscher's rules can be established [115, 116] when we perform dimensional reductions from $N = 2A$ and $N = 2B$ to $D = 9$ and identify the same fields from the two different reduction schemes [36],

$$\begin{aligned} \hat{J}_{\mu\nu} &= \hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu z} \hat{g}_{\nu z} - \hat{B}_{\mu z} \hat{B}_{\nu z}}{\hat{g}_{zz}}, & \hat{J}_{\mu y} &= \frac{\hat{B}_{\mu z}}{\hat{g}_{zz}}, \\ \hat{B}_{\mu\nu} &= \hat{B}_{\mu\nu} + \frac{\hat{g}_{\mu z} \hat{B}_{\nu z} - \hat{B}_{\mu z} \hat{g}_{\nu z}}{\hat{g}_{zz}}, & \hat{B}_{\mu y} &= \frac{\hat{g}_{\mu z}}{\hat{g}_{zz}}, \\ \hat{\varphi} &= \hat{\phi} - \frac{1}{2} \ln |\hat{g}_{zz}|, & \hat{J}_{yy} &= \frac{1}{\hat{g}_{zz}}, \end{aligned} \quad (2.54)$$

$$\begin{aligned} \hat{C}_{\mu_1 \dots \mu_{2n}}^{(2n)} &= \hat{C}_{\mu_1 \dots \mu_{2n} z}^{(2n+1)} + 2n \hat{B}_{[\mu_1 | z] \hat{C}_{\mu_2 \dots \mu_{2n}] z}^{(2n-1)} - 2n(2n-1) \frac{\hat{B}_{[\mu_1 | z] \hat{g}_{\mu_2 | z] \hat{C}_{\mu_3 \dots \mu_{2n}] z}^{(2n-1)}}{g_{zz}}, \\ \hat{C}_{\mu_1 \dots \mu_{2n-1} y}^{(2n)} &= -\hat{C}_{\mu_1 \dots \mu_{2n-1} z}^{(2n-1)} + (2n-1) \frac{\hat{g}_{[\mu_1 | z] \hat{C}_{\mu_2 \dots \mu_{2n-1}] z}^{(2n-1)}}{g_{zz}}. \end{aligned}$$

On the other hand, T duality effects on the heterotic superstrings result in the transformation laws of the heterotic whose gauge group is $E_8 \times E_8$ into the heterotic theory with $SO(32)$ as a gauge group, and vice versa [147].

T duality in type I string theory is even more subtle. We can obtain the effective action of type I by considering type IIB and truncating it using one of its \mathbb{Z}_2 symmetries plus the inclusion of an O9-plane and 32 D9-branes.³ The T duality between type IIB and type IIA theories implies the existence of the so-called *type I'* [148], which can be interpreted as a rotation of the space where we compactify. This implies the interchange of Neumann and Dirichlet boundary conditions for certain coordinates.

The examples of T-duality that we have discussed are only the tip of a mathematical iceberg: there exist additional dualities known as *mirror symmetries*, in which different 10-dimensional string theories compactified on Calabi-Yau manifolds are related to each other [149].

2.4.2 S duality

String theory also possesses non-perturbative dualities in which the strong-coupling regime of a theory is related to the weak-coupling limit of another (or even the same) theory. This kind of duality is called S duality and, as we argued before, it has to do with the inversion of the dilaton and its role as the coupling constant of the theory. Reference [150] is a well-known and useful review in this topic. We will compare some particular string actions to explicitly show this duality and its consequences.

Let us compare the effective actions of the heterotic $SO(32)$ and type I superstring theories. The former is

$$S_h = \frac{g_h^2}{16\pi G_{\text{Nh}}^{(10)}} \int d^{10}x \sqrt{|g|} e^{-2\phi} \left[R - 4(\partial\phi)^2 + \frac{1}{2 \cdot 3!} H^2 - \frac{1}{4} \alpha' F_{\mu\nu}^I F^{I\mu\nu} \right], \quad (2.55)$$

whereas the type I action is

$$S_I = \frac{g_I^2}{16\pi G_{\text{NI}}^{(10)}} \int d^{10}x \sqrt{|\mathcal{J}|} \left\{ e^{-2\varphi} [R(\mathcal{J}) - 4(\partial\varphi)^2] + \frac{1}{2 \cdot 3!} (G^{(3)})^2 + \frac{\alpha'}{4} e^{-\varphi} \text{Tr}(F^2) \right\}. \quad (2.56)$$

$J_{\mu\nu}$ is the metric and $G^{(3)}$ is the field strength of the RR potential $C^{(2)}$. Upon inspection, we can check that both actions are transformed into each other under the mapping

$$\mathcal{J}_{\mu\nu} = e^{-\phi} g_{\mu\nu}, \quad \varphi = -\phi, \quad C_{\mu\nu}^{(2)} = B_{\mu\nu}. \quad (2.57)$$

Since the VEV of exponential of the dilaton corresponds to the string coupling constant, this suggests that the strong and weak coupling regimes of the heterotic and type I superstrings are mapped into each other [151].

Another example can be found in the effective action of type IIB strings written in

³Reference [36] pedagogically shows how to do it.

the Einstein frame,

$$S_B = \frac{g_B}{16\pi G_{\text{NB}}^{(10)}} \int d^{10}x \sqrt{|\mathcal{J}_E|} \left[R(\mathcal{J}_E) + \frac{1}{4} \text{Tr} (\partial \mathcal{M} \mathcal{M}^{-1})^2 + \frac{1}{2 \cdot 3!} \vec{\mathcal{H}}^T \mathcal{M}^{-1} \vec{\mathcal{H}} \right. \\ \left. + \frac{1}{4 \cdot 5!} F^2 - \frac{1}{2^7 \cdot 3^3} \frac{1}{\sqrt{|\mathcal{J}_E|}} \epsilon D \vec{\mathcal{H}}^T \eta \vec{\mathcal{H}} \right], \quad (2.58)$$

where \mathcal{M} is a scalar matrix,

$$\mathcal{M} = \frac{1}{\text{Im}(\tau)} \begin{pmatrix} |\tau|^2 & \text{Re}(\tau) \\ \text{Re}(\tau) & 1 \end{pmatrix}, \quad (2.59)$$

that depends on the dilaton and the RR scalar field by means of the complex scalar

$$\tau \equiv C^{(0)} + ie^{-\varphi}. \quad (2.60)$$

$\vec{\mathcal{H}}$ is the field strength of $\vec{\mathcal{B}} = (C^{(2)}, \mathcal{B})$ and $F = 5(\partial D - \vec{\mathcal{B}}^T \eta \vec{\mathcal{H}})$ is the modified field strength of $D = C^{(4)} - 3\mathcal{B}C^{(2)}$. This action is invariant under $SL(2, \mathbb{R})$ transformations, which act on the complex scalar as follows:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad ad - bc = 1. \quad (2.61)$$

In particular, if $a = d = 0$ and $b = -c = 1$, this transformation implies $\phi \rightarrow -\phi$, so that the strong coupling limit of type IIB superstring theory is actually dual to its own weak coupling regime [138].

Thus, we have proven the link between the strong and weak-coupling limits of type IIB theory.

2.4.3 U duality and M theory

The strong coupling limit of type IIA string theory is even more exciting. To study this regime, let us oxidize the $N = 2A$ action and analyze its relation to $D = 11$ SUGRA (1.33). If we arrange the fields of the $N = 2A$ action (1.49) as follows,

$$\hat{g}_{\mu\nu} = e^{-\frac{1}{6}\phi} g_{\mu\nu} + e^{\frac{4}{3}\phi} C_\mu C_\nu, \quad \hat{g}_{\mu z} = e^{\frac{4}{3}\phi} C_\mu, \\ \hat{g}_{zz} = e^{\frac{4}{3}\phi}, \quad (2.62)$$

$$\hat{C}^{(3)} = C^{(3)} + B^{(2)} \wedge (dz + C^{(1)}), \quad (2.63)$$

we fill the field content of $D = 11$ supergravity. In fact, substituting these fields by the ones of the genuine action, we obtain that

$$\mathcal{L}_{11} = \mathcal{L}_{2A} \wedge dz. \quad (2.64)$$

That is, we notice that $N = 2A$ supergravity is a KK dimensionally-reduced theory of $D = 11$ supergravity on a circle. The already explained role of the exponential of

the dilaton allows to relate the radius of the 11th dimension to the string coupling constant. Now, let us give an expression for the compactified radius R_{11} and study some of its limits. Following [36], the radius of the compactified 11th dimension is naturally measured by the metric

$$R_{11} = \frac{1}{2\pi} \lim_{r \rightarrow \infty} \int \sqrt{|\hat{g}_{zz}|} dz = \frac{\ell_{\text{Planck}}^{(11)}}{2\pi} e^{\frac{2}{3}\phi_0} = \frac{\ell_{\text{Planck}}^{(11)}}{2\pi} g_A^{2/3}. \quad (2.65)$$

From the prefactors of both sides of (2.64), we have

$$\ell_{\text{Planck}}^{(11)} g_A^{2/3} G_{\text{NA}}^{(10)} = G_N^{(11)}. \quad (2.66)$$

On the other hand, Dirac quantization and consistency arguments on extended objects imply

$$G_{N\ 2A}^{(10)} = 8\pi^6 g_A^2 (\alpha'^4), \quad (2.67)$$

so that defining the 11-dimensional Planck length as

$$16\pi G_N^{(11)} \equiv \frac{(\ell_{\text{Planck}}^{(11)})^9}{2\pi}, \quad (2.68)$$

we have

$$\ell_{\text{Planck}}^{(11)} = 2\pi \ell_s g_A^{1/3}. \quad (2.69)$$

This implies

$$R_{11} = \ell_s g_A. \quad (2.70)$$

Now it is easy to see that the strong-coupling limit of the type IIA theory, $g_A \rightarrow \infty$ coincides with the decompactification limit in which a new dimension becomes macroscopic [29]. This new theory is the so-called *M theory* [152]: it is defined to be the theory that has $D = 11$ supergravity as its low-energy limit. In a same fashion, it is argued [153] that the strong coupling of the heterotic $E_8 \times E_8$ theory is also related to this intriguing theory, but this time the compactification is done under an interval.

Thus, one expects a generalization when considering M theory reductions on a n -torus T^n . Indeed, it has been conjectured [138] that M theory on a torus is invariant under the duality groups given in table 2.4.3. This duality group of M theory consists of a combination of T and S dualities of string theory and is called the *U-duality* group. For example, M theory on a T^2 , would have an $SL(2, \mathbb{Z})$ U duality group. That is, M theory on different backgrounds that are related by $SL(2, \mathbb{Z})$ transformations are equivalent. As in the other dualities, these U duality groups will reappear as global symmetry groups of the corresponding supergravity theories.

D	U duality	T duality
9	$SL(2, \mathbb{Z})$	
8	$SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$	$SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$
7	$SL(5, \mathbb{Z})$	$SO(3, 3; \mathbb{Z})$
6	$SO(5, 5; \mathbb{Z})$	$SO(4, 4; \mathbb{Z})$

Table 2.3: U duality groups of M theory on T^{11-D} and T duality groups of IIA and IIB string theory on T^{10-D} for $6 \leq D \leq 9$ dimensions.

2.5 Extended objects

In the spectrum of the string theories, not only we find excitations of the so-called fundamental string itself but, for instance, making use of dualities, also extended objects appear as solitonic states in the spectrum. These extended states have a worldvolume action, analog to the worldsheet action of a string. The presence of these objects changes the boundary conditions and the orientability of the theory. New theories can be built by compactifying or adding these extended objects and breaking thus Lorentz invariance or supersymmetry. That is, since these states can generate backgrounds that modify the number of preserved supersymmetries, some truncations to half-maximal theories can be done by tuning certain configurations of these objects.

D -brane worldvolume metrics arise, together with a suitable field configuration, as solutions of the SUGRA field equations. In fact, it was a breakthrough in string theory when it was realized that these Dp -branes (p counts the number of spatial dimensions over that the worldvolume is expanded) could be identified as the hyperplanes on which open strings can end [154].

Let us briefly introduce the so-called p -brane a -model [155], which is the simplest model that entails the fundamental features of a supergravity. For further details, we submit to [36].

We assume a generic supergravity: gravity coupled to one scalar and to a $(p+1)$ -form potential as follows:

$$S = \frac{1}{16\pi G_N^{(d)}} \int d^D x \sqrt{|g|} \left\{ R + 2(\partial\varphi)^2 + \frac{(-1)^{p+1}}{2 \cdot (p+2)!} e^{-2a\varphi} F_{(p+2)}^2 \right\}, \quad (2.71)$$

where $F_{(p+2)} = dA_{(p+1)}$ is the field strength of the $(p+1)$ -form potential under which the p -brane is charged. The equations of motion are

$$G_{\mu\nu} + 2T_{\mu\nu}^\phi - \frac{1}{2} e^{-2a\varphi} T_{\mu\nu}^{A_{(p+1)}} = 0, \quad (2.72)$$

$$\nabla^2 \varphi + \frac{(-1)^{p+1}}{4 \cdot (p+2)!} a e^{-2a\varphi} F_{(p+2)}^2 = 0, \quad (2.73)$$

$$\nabla_\mu \left(e^{-2a\varphi} F_{(p+2)}^{\mu\nu_1 \dots \nu_{p+1}} \right) = 0, \quad (2.74)$$

where

$$T_{\mu\nu}^{A(p+1)} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_{(p)}}{\delta g^{\mu\nu}} = \frac{(-1)^p}{(p+1)!} \left[F_{(p+2)\mu}{}^{\rho_1 \dots \rho_{(p+1)}} F_{(p+2)\mu\rho_1 \dots \rho_{(p+1)}} - \frac{1}{2(p+2)} g_{\mu\nu} F_{(p+2)}^2 \right]$$

is the corresponding energy-momentum tensor associated to the term of the action associated to the p -form. Their general solution, by assuming that there are additional translational isometries in the directions z_1, \dots, z_q , is given by

$$\begin{aligned} ds^2 &= (e^{-2a\varphi} H^{-2})^{\frac{1}{p+1}} [W dt^2 - d\vec{y}_p^2] \\ &\quad - (e^{-2a\varphi} H^{-2})^{-\frac{1}{p+1}} [d\vec{z}_q^2 + W^{-1} d\rho^2 + \rho^2 d\Omega_{(\delta-2)}^2], \\ e^{-2a\varphi} &= H^{2x}, & A_{ty^1 \dots y^p} &= \alpha(H^{-1} - 1), \\ H &= 1 + \frac{h}{\rho^{\delta-3}}, & W &= 1 + \frac{\omega}{\rho^{\delta-3}}, \\ \omega &= h \left[1 - \frac{a^2}{4x} \alpha^2 \right], & x &= \frac{(a^2/2)c}{1 + (a^2/2)c}, \\ c &= \frac{(p+1) + (\tilde{p}+1)}{(p+1)(\tilde{p}+1)}, \end{aligned} \tag{2.75}$$

where $\delta = d - (p+q) > 3$. The parallel coordinates t, y_1, \dots, y_p span the worldvolume of the brane, and the rest are transverse to the brane. H is a harmonic function in the $(\delta-1)$ transverse dimensions that do not suffer any isometry,

$$\Delta_{(\delta-1)} H = 0. \tag{2.76}$$

In addition, we can wonder about the magnetic dual of this a -model. When we write the dual field strength and dilaton field,

$$F_{(p+2)} = e^{-2\varphi\tilde{p}} F_{(\tilde{p}+2)}, \quad \varphi_p = -\varphi_{\tilde{p}}, \tag{2.77}$$

where $\tilde{p} = D - p - 4$, the dilaton gets inverted. Then, due to its role as the coupling constant, new solutions take place that are different to the dual solutions that can be obtained from (2.75). The more general solution is

$$\begin{aligned} ds^2 &= (e^{+2a\varphi} H^{-2})^{\frac{1}{\tilde{p}+1}} [W dt^2 - d\vec{y}_p^2] \\ &\quad - (e^{+2a\varphi} H^{-2})^{-\frac{1}{\tilde{p}+1}} [d\vec{z}_q^2 + W^{-1} d\rho^2 + \rho^2 d\Omega_{(\tilde{\delta}-2)}^2], \\ e^{-2a\varphi} &= H^{-2x}, & F_{(p+2)z_1 \dots z_q \psi_1 \dots \psi_{(\tilde{\delta}-2)}} &= (\tilde{\delta}-3)\alpha h \Omega_{\psi_1 \dots \psi_{(\tilde{\delta}-2)}}^{(\tilde{\delta}-2)}, \\ H &= 1 + \frac{h}{\rho^{\tilde{\delta}-3}}, & W &= 1 + \frac{\omega}{\rho^{\tilde{\delta}-3}}, \\ \omega &= h \left[1 - \frac{a^2}{4x} \alpha^2 \right], & x &= \frac{(a^2/2)c}{1 + (a^2/2)c}, \\ c &= \frac{(p+1) + (\tilde{p}+1)}{(p+1)(\tilde{p}+1)}, \end{aligned} \tag{2.78}$$

IIB	S-dual IIB
NS1	D1
NS5	D5
D1	NS1
D3	D3
D5	NS5
D7	$\widetilde{D7}$

Table 2.4: IIB branes and S duality acting on them. Under $SL(2, \mathbb{Z})$, D3-branes are singlet, and (NS1, D1) and (NS5, D5) are doublets. D7-branes transforms to its magnetic dual, the $\widetilde{D7}$ -brane. [156].

where $\tilde{\delta} = d - (\tilde{p} + q)$ and $\Omega^{(n)}$ is the volume form of the n -sphere.

As long as point-particles (D0-branes) can be coupled to 1-form potentials and strings (D1-branes) are charged with respect to 2-form potentials, Dp -branes can be charged electrically under RR gauge potentials $C^{(p+1)}$ and magnetically under $C^{(7-p)}$. Since D-branes also couple to NSNS potentials and we only have the 2-form $B_{\mu\nu}$, the resulting branes are the NS1- (the fundamental string) and the NS5-brane. Thus, we can easily infer the existence of a $M2$ - and a $M5$ -brane in 11-dimensional SUGRA. The $M2$ -brane is charged with respect to the 3-form potential $C_{\mu\nu\rho}$ whereas its dual $M5$ -brane gets charged with respect to the corresponding 6-form potential dual to $C_{\mu\nu\rho}$.

On the other hand, one also can introduce *orientifold planes*, Op -planes, which are objects with negative tension. If in a string theory, worldsheet parity is combined with a discrete symmetry of the spacetime, then the quotient is called an *orbifold*. Although they are similar in other respects to Dp -branes, they are not dynamical objects. They are attached to the fixed points of the spacetime orbifold and cannot translate or oscillate.

Some important issues have to be taken into account to add D-branes and O-planes consistently to superstring theories.

- Type IIA (IIB) theory admits only Dp -branes and Op -planes with p even (odd).
- Despite of Dp -branes and Op -planes are charged with respect to RR $(p + 1)$ -potentials, one has to introduce a 9 (10)-form potential for the D8 (D9)-brane and the O8 (O9)-plane.
- Dp -branes carry a unit of positive or negative RR charge that equals its tension, $q_{Dp} = \pm T_{Dp}$.
- Having a single Dp -brane or Op -planes in our configuration halves the supersymmetry of the theory.

Let us see an example of the branes that can live in a given theory. Due to its duality richness, let us see what branes can be coupled to the massless fields of the

type IIB theory. In the NSNS sector there is only the Kalb-Ramond 2-form, so we can have the NS1- and NS5-branes, as we said. The RR sector is spanned by the $C^{(0)}$, $C^{(2)}$ and $C^{(4)}$. Thus, the D(-1), D1 and D3 branes electrically interact with them, whereas D7, D5 and D3 become magnetically charged. The D(-1) solution is a particular type of *instanton* and it is localized both in space and time. We show in table 2.5 how S-duality acts on these BPS objects, which is a good approach to study non-perturbative configurations.

Chapter 3

Gauged supergravities

In any supergravity theory, we can implement deformations in such a way that part of the global symmetry becomes gauged. We will study now how to gauge a given supergravity theory, *i.e.*, we will choose a subgroup $G_0 \subset G$ and promote it to a local symmetry. To do this, we will use a covariant formalism that preserves and guarantees the covariance of the final theory. This formalism is the so-called *embedding tensor* (ET).

3.1 Motivation

The discovery of the relation between RR $(p + 1)$ -form potentials in 10-dimensional type II supergravity theories and D-branes [154] made it possible to associate most of the fields of the string low-energy effective field theories (supergravity theories in general) to extended objects (*branes*) of diverse kinds: fundamental, Dirichlet, solitonic, Kaluza-Klein etc. This association has been fruitfully used in two directions: to infer the existence of new supergravity fields from the known existence in the String Theory of a given brane or string state and *vice versa*. Thus, the knowledge of the existence of Dp -branes with large values of p made it necessary to learn how to deal consistently with the magnetic duals of the RR fields that were present in the standard formulations of the supergravity theories constructed decades before, because in general it is impossible to dualize and rewrite the theory in terms of the dual magnetic fields. The existence of NSNS $(p + 1)$ -forms in the supergravity theories that could also be dualized made it necessary to include solitonic branes dual to the fundamental ones (strings, basically). It was necessary to include all the objects and fields that could be reached from those already known by U-duality transformations and this effort led to the discovery of new branes and the introduction of the *democratic* formulations of the type II supergravities [157] dealing simultaneously with all the relevant electric and magnetic supergravity fields in a consistent way.

The search for all the extended states of string theory has motivated the search for all the fields that can be consistently introduced in the corresponding supergravity

theories, a problem that has no simple answer for the d -, $(d-1)$ and $(d-2)$ -form fields, which are not the duals of electric fields already present in the standard formulation, at least in any obvious way. The branes that would couple to them can play important rôles in String Theory models, which makes this search more interesting.

As mentioned before, U-duality arguments have been used to find new supergravity fields but U-duality can only reach new fields belonging to the same orbits as the known fields. To find other possible fields, a systematic study of the possible consistent supersymmetry transformation rules for p -forms has been carried out in the 10-dimensional maximal supergravities in refs. [157–162] but this procedure is long and not systematic. The conjectured E_{11} symmetry [163–165] can be used to determine the bosonic extended field content of maximal supergravity in different dimensions¹. These results have been recently used to construct the U-duality-covariant Wess-Zumino terms of all possible branes in all dimensions [167, 168]. In this approach supersymmetry is not explicitly taken into account, only through the U-duality group.

Another possible systematic approach to this problem (that does not take supersymmetry into account explicitly either) is provided by the embedding-tensor formalism². This formalism, introduced in refs. [172–176] allows the study of the most general deformations of field theories and, in particular, of supergravity theories [118, 119, 177–183]. One of the main features of this formalism is that it requires the systematic introduction of new higher-rank potentials which are related by Stückelberg gauge transformations. This structure is known as the *tensor hierarchy* of the theory [175, 176, 182, 184–186] and can be taken as the (bosonic) extended field content of the theory. In Supergravity Theories one may need to take into account additional constraints on the possible gaugings, but, if the gauging is allowed by supersymmetry, then gauge invariance will require the introduction of all the fields in the associated tensor hierarchy and, since gauge invariance is a *sine qua non* condition for supersymmetry, the tensor hierarchy will be automatically compatible with supersymmetry. Furthermore, if we set to zero all the deformation parameters (gauge coupling constants, Romans-like mass parameters [187] etc.) the fields that we have introduced will remain in the undeformed theory.

This formalism, therefore, provides another systematic way of finding the extended field content of Supergravity Theories. However, it cannot be used in the most interesting cases, $N = 1, d = 11$ and $N = 2A, B, d = 10$ Supergravity, because these theories cannot be gauged because they do not have 1-forms ($N = 1, d = 11$ and $N = 2B, d = 10$) or the 1-form transforms under the only (Abelian) global symmetry ($N = 2A, d = 10$). Only $N = 2A, d = 10$ can be deformed through the introduction of Romans' mass parameter, but the consistency of this deformation does not seem to require the introduction of any higher-rank potentials. The dimensional reduction to $d = 9$ of these theories, though, has 3 vector fields, and their embedding tensor

¹Smaller Kač-Moody algebras can be used in supergravities with smaller number of supercharges such as $N = 2$ theories in $d = 4, 5, 6$ dimensions [166].

²For recent reviews see refs. [169–171].

formalism can be used to study all its possible gaugings and find its extended field content.

3.2 Embedding tensor: the troika

The role of the vector fields are crucial in the gauging procedure. The vectors A_μ^M of the ungauged theory transform under global G transformations ξ^α and Abelian gauge symmetry $U(1)^{n_V}$, where n_V is the number of vector fields in the theory,

$$\delta_\xi A_\mu^M = -\xi^\alpha (t_\alpha)_N^M A_\mu^N, \quad \delta_\Lambda A_\mu^M = \partial_\mu \Lambda^M, \quad (3.1)$$

where $M = 1, \dots, n_V$ is an index of the fundamental representation and $\alpha = 1, \dots, \dim G$ is an index of the adjoint representation.

As we said, our aim is to promote a subgroup $G_0 \subset G$ to be local. Let us assume a subset of generators $X_M \subset \mathfrak{g} = \text{Lie } G$ spanned by the candidates to be gauged. We implement this local dependence by introducing the corresponding covariant derivative

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - g A_\mu^M X_M. \quad (3.2)$$

The choice of what and how the generators have to be chosen to preserve the covariance of the theory is done by means of a matrix ϑ_M^α , which describes the explicit embedding of G_0 into G and is restricted to transform covariantly. ϑ_M^α is a constant $n_V \times \dim G$ matrix. Thus, the relation between the global and the gauge candidate generators is

$$X_M = \vartheta_M^\alpha t_\alpha \in \mathfrak{g}. \quad (3.3)$$

Since ϑ_M^α is constant, this is nothing but a linear combination of the generators of G and thus, the dimension of the gauge group is the rank of ϑ_M^α .

The new covariant derivatives guarantee the covariance of the theory under the G symmetry group. However, only when we choose a particular gauge group G_0 , *i.e.*, a particular ϑ_M^α , G gets broken. Thus, considering that the fields transform as follows,

$$\delta_\Lambda V = \Lambda^M X_M \mathcal{V}, \quad \delta_\Lambda A_\mu^M = D_\mu \Lambda^M, \quad (3.4)$$

the gauge invariance of the theory is not guaranteed. Gauge transformations act on the embedding tensor in this way:

$$\delta_\Lambda \vartheta_M^\alpha = \Lambda^N X_{NM}^P \vartheta_P^\alpha - \Lambda^N X_{N\beta}^\alpha \vartheta_M^\beta \quad (3.5)$$

$$= \Lambda^N \vartheta_N^\beta (t_{\beta M}^P \vartheta_P^\alpha - f_{\beta\gamma}^\alpha \vartheta_M^\gamma). \quad (3.6)$$

If we demand gauge invariance of the ET, we obtain a set of second-order constraints in ϑ_M^α , the so-called *quadratic constraints* (QC). Contracting (3.5) with a generator

t_α , we obtain a constraint that guarantees the closure of the algebra of the gauge generators,

$$[X_M, X_N] = -X_{MN}{}^P X_P, \quad (3.7)$$

where

$$X_{MN}{}^P = \vartheta_M{}^\alpha (t_\alpha)_N{}^P. \quad (3.8)$$

So, gauge invariance implies the closure of the gauge generators, *i.e.*, constraint (3.5) is more restrictive than (3.7).

Let us split the gauge generators as follows,

$$X_{MN}{}^P = X_{[MN]}{}^P + Z^P{}_{MN}, \quad (3.9)$$

where $Z^P{}_{MN} = X_{(MN)}{}^P$. Thus, from (3.7), we can check that

$$Z^P{}_{MN} X_P = 0. \quad (3.10)$$

However, if we define $X_{[MN]}{}^P$ to be the structure constants, we realize that

$$X_{[MN]}{}^P X_{[QP]}{}^R + X_{[QM]}{}^P X_{[NP]}{}^R + X_{[NQ]}{}^P X_{[MP]}{}^R = -Z^R{}_{P[Q} X_{MN]}{}^P. \quad (3.11)$$

That is, Jacobi identity is satisfied upon contracting with X_R , due to the condition (3.10). This is enough in order to the QC (3.5) be satisfied.

There also exists a *linear constraint* (LC) arising from supersymmetry. Since we want to keep SUSY unspoiled, $\vartheta_M{}^\alpha$ gets restricted by dimensional and group-theoretical arguments to live in certain representations. That is, SUSY kills some of the representations of the embedding tensor. As we will see in the following chapter, this restriction appears explicitly when we study the closure of the supersymmetric transformations of the fields.

As we said, the embedding tensor lives in

$$\vartheta_M{}^\alpha : R_{V^*} \otimes R_{adj} = V' \otimes \mathfrak{g}_0 = \theta_1 \oplus \theta_2 \oplus \cdots \oplus \theta_k, \quad (3.12)$$

where V' is the conjugate representation of the fundamental V , \mathfrak{g}_0 is the adjoint representation and θ_i are several irreps. Then, the linear constraint, which can be schematically represented as

$$\mathbb{P}\vartheta = 0, \quad (3.13)$$

restricts the r.h.s. of (3.12). In table 3.2 we have the resulting representations of the embedding tensor in maximal theories. For half-maximal supergravities, the structure is similar. In some cases, this linear constraint can be inferred even at the very beginning of the procedure, throughout the bosonic sector. For instance, in $D = 4$ $N = 8$ [176], the constraint

$$X_{(MN)}{}^P \Omega_{K)P} = \text{Samtleben} : 2008pe0, \quad \Omega_{MN} = \begin{pmatrix} 0 & \not\propto \\ \not\propto & 0 \end{pmatrix}, \quad (3.14)$$

D	G_0	H	# scalars	vectors	ϑ
9	$\mathbb{R}^+ \times SL(2)$	$SO(2)$	3	$\mathbf{1}_{+4} + \mathbf{2}_{-3}$	$\mathbf{2}_{+3} + \mathbf{3}_{-4}$
8	$SL(2) \times SL(3)$	$SO(2) \times SO(3)$	7	$(\mathbf{2}, \mathbf{3}')$	$(\mathbf{2}, \mathbf{3}) + (\mathbf{2}, \mathbf{6}')$
7	$SL(5)$	$SO(5)$	14	$\mathbf{10}'$	$\mathbf{15} + \mathbf{40}'$
6	$S(5, 5)$	$SO(5) \times SO(5)$	25	$\mathbf{16}$	$\mathbf{144}$
5	$E_{6(6)}$	$USp(8)$	42	$\mathbf{27}'$	$\mathbf{351}$
4	$E_{7(7)}$	$SU(8)$	70	$\mathbf{56}$	$\mathbf{912}$

Table 3.1: Embedding tensor representations in maximal supergravities for different dimensions.

guarantees the supersymmetry of the action.

Thus, the classification of all the possible gaugings of a given theory reduces to the search and analysis of solutions of the QC and the LC, where the former is, in general, arithmetically difficult to solve. Moreover, the counting of inequivalent gaugings is also a non-trivial problem to be solved, where the heaviest group theory machinery has to be used to identify the different orbits.

3.3 Deformed tensor gauge algebra

Up to this point, we have introduced additional couplings by means of the covariant derivative. This automatically implies that the field strengths have to include this covariant derivative in order to ‘feel’ the new non-Abelian nature of the gauge group. A priori, we could assume a natural ansatz

$$F_{\mu\nu}{}^M = 2\partial_{[\mu}A_{\nu]}{}^M + gX_{[NP]}{}^M A_\mu{}^N A_\nu{}^P, \quad (3.15)$$

but this is not enough. The condition (3.10) ensures that the QC is satisfied. However, this is not compatible with (3.15), since it does not transform covariantly,

$$\delta_\Lambda F_{\mu\nu}{}^M = -g\Lambda^P X_{PN}{}^M + 2gZ^M{}_{PQ} (\Lambda^P F_{\mu\nu}{}^Q - A_{[\mu}{}^P \delta A_{\nu]}{}^Q), \quad (3.16)$$

Only when $Z^M{}_{PQ}$ vanishes, it transforms properly.

The condition of keeping G covariance is the responsible of this situation. In some sense, we are performing a redundant description of the gauge in terms of the n_V generators X_M . In general, since the dimension of the gauge group is smaller than that of the global symmetry group, n_V , not all of the X_M generators are linearly independent. For some cases, we can split the vector fields into

- $A_\mu{}^m$, which transform in the adjoint of G_0 ,
- $A_\mu{}^i$, which transform in some representation of G_0 ,

D	G	1	2	3	4
9	GL(2)	$\mathbf{1}^{-4} + \mathbf{2}^{+3}$	$\mathbf{2}^{-1}$	$\mathbf{1}^{+2}$	$\mathbf{1}^{-2}$
8	SL(2)×SL(3)	$(\mathbf{2}, \mathbf{3}')$	$(\mathbf{1}, \mathbf{3})$	$(\mathbf{2}, \mathbf{1})$	$(\mathbf{1}, \mathbf{3}')$
7	SL(5)	$\mathbf{10}'$	$\mathbf{5}$	$\mathbf{5}'$	
6	SO(5,5)	$\mathbf{16}_c$	$\mathbf{10}$	$\mathbf{10}_s$	
5	E ₆₍₆₎	$\mathbf{27}'$	$\mathbf{27}$		
4	E ₇₍₇₎	$\mathbf{56}$	$\mathbf{133}$		
3	E ₈₍₈₎	$\mathbf{248}$			

Table 3.2: The p -form content being part of the Lagrangian of gauged maximal supergravities.

where the gauge transformation of the fields are

$$\begin{aligned}
\delta A_\mu{}^M &= D_\mu \Lambda^M + \dots - Z^{MI} \Xi_{\mu I}, \\
\delta B_{\mu\nu I} &= 2D_{[\mu} \Xi_{\nu]} + \dots - Z_{IA} \Sigma_{\mu\nu}{}^A, \\
\delta C_{\mu\nu\rho}{}^A &= 3D_{[\mu} \Sigma_{\nu\rho}{}^A + \dots - Z^{Aa} \Upsilon_{\mu\nu\rho a}, \\
\vdots &= \quad \quad \quad \vdots
\end{aligned} \tag{3.21}$$

Thus, we summarize that not only covariant derivatives are necessary as new ingredients to gauge a theory, but also Stückelberg-like couplings between p -forms and $(p+1)$ -forms are necessary, especially to construct suitable field transformations and guarantee the covariance of the field strengths.

Another consequence of the new gaugings is that the field strength does not satisfy the standard Bianchi identity, but it satisfies the so-called *deformed Bianchi identity*,

$$\begin{aligned}
3D_{[\mu} \mathcal{F}_{\nu\rho]}{}^M &= Z^{MI} \mathcal{F}_{\mu\nu\rho I}, \\
4D_{[\mu} \mathcal{F}_{\nu\rho\lambda]} I &= \dots + Z_{IA} \mathcal{F}_{\mu\nu\rho\lambda}{}^A, \\
5D_{[\mu} \mathcal{F}_{\nu\rho\lambda\sigma]} A &= Z^{Aa} \mathcal{F}_{\mu\nu\rho\lambda\sigma a}, \\
\vdots &= \quad \quad \quad \vdots
\end{aligned}$$

A more detailed analysis of the higher rank tensor gauge transformations allows us to determine the full field content of the theory, including the D - and $(D-1)$ -forms, which are non-propagating fields. In table 3.3, we summarize the p -form content of maximal supergravities [175, 182].

3.4 The Lagrangian

Once we have studied the impact of the gaugings in the group structure of supergravities, let us focus on the Lagrangian. Apart from covariantizing the derivatives and

replace the Abelian field strengths by the fully covariant ones, we need to modify the topological terms of the ungauged version.

We have seen how the Stückelberg coupling connects the p - and $(p + 1)$ -forms throughout the field strengths of the former. The presence of these fields could be, in principle, problematic, since they do not appear in the ungauged theory and they could imply new equations of motion.

However, these contributions combine into first order equations of motions (of the magnetic fields), which show nothing but the fact that they are the on-shell dual fields of the ungauged theory. They enter as Lagrange multipliers-like equations in the Lagrangian.

It is important to point out the conceptual difference between this situation and the so-called *democratic formulations* of supergravities [157], in which all the dual fields are introduced in the action in an egalitarian way and the duality relations must be added by hand.

Once the gaugings are properly implemented in the theory, local supersymmetry invariance of the Lagrangian has to be imposed. The SUSY variations of the new Stückelberg couplings of the field strengths have to be cancelled by new terms of the lagrangian. If we schematically have a generic term

$$\mathcal{F}^{(p)}\mathcal{F}_{(p)} = F^{(p)}F_{(p)} + 2ZC_{(p)}F^{(p)} + ZZC^{(p)}C_{(p)}, \quad (3.22)$$

then, its SUSY variation at first order is given by

$$\delta_\epsilon(\mathcal{F}^{(p)}\mathcal{F}_{(p)}) = 2 \left[F^{(p)}\delta_\epsilon F_{(p)} + F^{(p)}Z\delta_\epsilon C_{(p)} + \delta_\epsilon F^{(p)}ZC_{(p)} + ZZC^{(p)}\delta_\epsilon C_{(p)} \right]. \quad (3.23)$$

The first term also appears in the ungauged theory and does not imply any problem. The rest of the terms depend on the embedding tensor. In addition,

$$\begin{aligned} F^{(p)} &\sim d(\bar{\epsilon}\gamma\cdots\gamma\lambda), \\ \delta_\epsilon C^{(p)} &\sim \bar{\epsilon}\gamma\cdots\gamma\lambda. \end{aligned} \quad (3.24)$$

The only possibility to cancel these terms is by means of fermionic mass terms,

$$\mathcal{L}_{\text{fm}} = g \left(\bar{\psi}_\mu{}^a A^{\mu\nu}{}_{ab} \psi_\nu{}^b + \bar{\chi}^m B^\mu{}_{ab} \psi_\mu{}^b + \bar{\chi}^m C_{ab} \chi^b \right) + h.c., \quad (3.25)$$

where ψ_μ and χ are the gravitinos and 1/2-fermions, respectively. The indices a, b are spinorial and the tensors $A^{\mu\nu}{}_{ab}$, $B^\mu{}_{ab}$ and C_{ab} depend, by construction, on the embedding tensor and may depend on the scalar fields. However, the presence of these new terms require the modification of the supersymmetric rules of the fermions⁴:

$$\begin{aligned} \delta\psi_\mu{}^a &= \delta_0\psi_\mu{}^a + A_\mu{}^{ab}\epsilon_b, \\ \delta\chi^a &= \delta_0\chi^a + B^{ab}\epsilon_b, \end{aligned} \quad (3.27)$$

⁴For instance,

$$\delta_\epsilon(\bar{\psi}_\mu{}^a A^{\mu\nu}{}_{ab} \psi_\nu{}^b) \sim 2(\bar{\psi}_\mu{}^a A^{\mu\nu}{}_{ab} \delta_\epsilon \psi_\nu{}^b) \sim 2(\bar{\psi}_\mu{}^a A^{\mu\nu}{}_{ab} D_\mu \epsilon^b) + \dots. \quad (3.26)$$

The only cancellation of this term arises from $\delta_\epsilon(\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho)$, but there is not any contribution proportional to ϑ_M^α . Thus, we need to modify the supersymmetric rules of the fermion fields.

where the δ_0 represents the supersymmetric transformation of the ungauged theory. These terms are known as *fermion shifts* and, as happened before, they solve a problem but they create a new one. New terms proportional to ϑ^2 appear from the action of the fermion shifts on (3.25). This requires the addition of a scalar potential, which schematically has the form

$$\mathcal{L}_{pot} = -eV = -e(B_\mu{}^{ab}B^\mu{}_{ab} - A_{\mu\nu}{}^{ab}A^{\mu\nu}{}_{ab}). \quad (3.28)$$

This scalar potential supports dS and AdS vacua. V can be rewritten in terms of the embedding tensor. In general, it can be expressed into the form

$$V = V^{MN}{}_{\alpha\beta}\vartheta_M{}^\alpha\vartheta_N{}^\beta, \quad (3.29)$$

where $V^{MN}{}_{\alpha\beta}$ is a scalar dependent matrix.

The tensors $A_{\mu\nu}{}^{ab}, B_\mu{}^{ab}, C_{ab}$ depend on the ET and transform under the maximal compact subgroup H of G . These two features necessarily imply the appearance of the scalar matrix V , which as we saw, is the link between elements that transform under G and others that transform under H . Then, these tensor A, B, C are obtained from $\vartheta_M{}^\alpha$ by acting on any of its fundamental G indices with V . This leads to define the so-called *T-tensor*,

$$T_a{}^m \equiv \vartheta_M{}^\alpha V^M{}_a V_\alpha{}^b. \quad (3.30)$$

where a, m are indices of a certain representation of H . Then, every irreducible component of $\vartheta_M{}^\alpha$ branches into one or more H -irreducible components of T ,

$$\vartheta = \theta_1 \oplus \theta_2 \oplus \dots \rightarrow T = (t_{11} \oplus t_{12} \oplus \dots) \oplus (t_{11} \oplus t_{12} \oplus \dots) \oplus \dots. \quad (3.31)$$

These irreducible representations correspond to these tensors A, B, C ⁵. For example, in $D = 4$ $N = 8$ supergravity, the maximal compact subgroup is $H = SU(8)$ and the embedding tensor lives in the **912** representation of $G = E_7$. Decomposing this representation into irreps of $SU(8)$, we have

$$\vartheta_M{}^\alpha \rightarrow T_a{}^b \rightarrow (A^{ab}, A_{ab}, B^{ab}, B_{ab}) \quad (3.32)$$

$$\mathbf{912} \rightarrow \mathbf{36} + \overline{\mathbf{36}} + \mathbf{420} + \overline{\mathbf{420}}. \quad (3.33)$$

These are the main aspects of gauged supergravities and the embedding tensor formalism. The next chapter will show an exhaustive study of all of the gauged supergravities in 9 dimensions, using the ET to scan all the possibilities.

⁵The Lorentz index of these tensors arises from the internal structure of gamma matrices that is required for any case.

Chapter 4

Gaugings in $N = 2$ $D = 9$ supergravity

4.1 Introduction

We use the embedding tensor method to construct the most general maximal gauged (massive) supergravity in $d = 9$ dimensions and to determine its extended field content. Some gaugings of the maximal $d = 9$ supergravity have been obtained in the past by generalized dimensional reduction [188] of the 10-dimensional theories with respect to the $SL(2, \mathbb{R})$ global symmetry of the $N = 2B$ theory [116, 189, 190] or other rescaling symmetries [191]¹. All these possibilities were systematically and separately studied in Ref. [117], taking into account the dualities that relate the possible deformation parameters introduced with the generalized dimensional reductions. However, the possible combinations of deformations were not studied, and, as we will explain, some of the higher-rank fields are associated to the constraints on the combinations of deformations. Furthermore, we do not know if other deformations, with no higher-dimensional origin (such as Romans' massive deformation of the $N = 2A, d = 10$ supergravity) are possible.

Our goal in this chapter will be to make a systematic study of all these possibilities using the embedding-tensor formalism plus supersymmetry to identify the extended-field content of the theory, finding the rôle played by the possible 7-, 8- and 9-form potentials, and compare the results with the prediction of the E_{11} approach. We expect to get at least compatible results, as in the $N = 2, d = 4, 5, 6$ cases studied in [119] and [166].

This chapter is organized as follows: in Section 4.2 we review the undeformed maximal 9-dimensional supergravity and its global symmetries. In Section 4.3 we study the possible deformations of the theory using the embedding-tensor formalism and checking the closure of the local supersymmetry algebra for each electric p -form of the theory. In Section 4.4 we summarize the results of the previous section describing the possible

¹An $SO(2)$ -gauged version of the theory was directly constructed in Ref. [192].

deformations and the constraints they must satisfy. We discuss the relations between those results and the possible 7- 8- and 9-form potentials of the theory and how these results compare with those obtained in the literature using the E_{11} approach. Section 4.5 contains our conclusions. Our conventions are briefly discussed in Appendix B.1. The Noether currents of the undeformed theory are given in Appendix B.3. A summary of our results for the deformed theory (deformed field strengths, gauge transformations and covariant derivatives, supersymmetry transformations etc.) is contained in Appendix B.4.

4.2 Maximal $d = 9$ supergravity: the undeformed theory

There is only one undeformed (*i.e.* ungauged, massless) maximal (*i.e.* $N = 2$, containing no dimensionful parameters in their action, apart from the overall Newton constant) 9-dimensional supergravity [112]. Both the dimensional reduction of the massless $N = 2A, d = 10$ theory and that of the $N = 2B, d = 10$ theory on a circle give the same undeformed $N = 2, d = 9$ theory, a property related to the T duality between type IIA and IIB string theories compactified on circles [113, 114] and from which the type II Buscher rules can be derived [115].

The fundamental (*electric*) fields of this theory are,

$$\left\{ e_{\mu}{}^a, \varphi, \tau \equiv \chi + ie^{-\phi}, A^I_{\mu}, B^i_{\mu\nu}, C_{\mu\nu\rho}, \psi_{\mu}, \tilde{\lambda}, \lambda, \right\}. \quad (4.1)$$

where $I = 0, \mathbf{i}$, with $\mathbf{i}, \mathbf{j}, \mathbf{k} = 1, 2$ and $i, j, k = 1, 2^2$. The complex scalar τ parametrizes an $SL(2, \mathbb{R})/U(1)$ coset that can also be described through the symmetric $SL(2, \mathbb{R})$ matrix

$$\mathcal{M} \equiv e^{\phi} \begin{pmatrix} |\tau|^2 & \chi \\ \chi & 1 \end{pmatrix}, \quad \mathcal{M}^{-1} \equiv e^{\phi} \begin{pmatrix} 1 & -\chi \\ -\chi & |\tau|^2 \end{pmatrix}. \quad (4.2)$$

The undeformed field strengths of the electric p -forms are, in our conventions^{3,4}

²Sometimes we need to distinguish the indices 1, 2 of the 1-forms (and their dual 6-forms) from those of the 2-forms (and their dual 5-forms). We will use boldface indices for the former and their associated gauge parameters.

³We use the shorthand notation $A^{IJ} \equiv A^I \wedge A^J$, $B^{ijk} \equiv B^i \wedge B^j \wedge B^k$ etc.

⁴The relation between these fields and those of Refs. [116] and [117] are given in Appendix B.2.

$$F^I = dA^I, \quad (4.3)$$

$$H^i = dB^i + \frac{1}{2}\delta^i_{\mathbf{i}}(A^0 \wedge F^{\mathbf{i}} + A^{\mathbf{i}} \wedge F^0), \quad (4.4)$$

$$G = d[C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}}] - \varepsilon_{\mathbf{ij}}F^{\mathbf{i}} \wedge (B^j + \frac{1}{2}\delta^j_{\mathbf{j}}A^{0\mathbf{j}}), \quad (4.5)$$

and are invariant under the undeformed gauge transformations

$$\delta_{\Lambda}A^I = -d\Lambda^I, \quad (4.6)$$

$$\delta_{\Lambda}B^i = -d\Lambda^i + \delta^i_{\mathbf{i}}[\Lambda^{\mathbf{i}}F^0 + \Lambda^0F^{\mathbf{i}} + \frac{1}{2}(A^0 \wedge \delta_{\Lambda}A^{\mathbf{i}} + A^{\mathbf{i}} \wedge \delta_{\Lambda}A^0)], \quad (4.7)$$

$$\begin{aligned} \delta_{\Lambda}[C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}}] &= -d\Lambda - \varepsilon_{\mathbf{ij}}(F^{\mathbf{i}} \wedge \Lambda^j + \Lambda^{\mathbf{i}} \wedge H^j - \delta_{\Lambda}A^{\mathbf{i}} \wedge B^j \\ &\quad + \frac{1}{2}\delta^j_{\mathbf{j}}A^{0\mathbf{i}} \wedge \delta_{\Lambda}A^{\mathbf{j}}). \end{aligned} \quad (4.8)$$

The bosonic action is, in these conventions, given by

$$\begin{aligned} S &= \int \left\{ -\star R + \frac{1}{2}d\varphi \wedge \star d\varphi + \frac{1}{2}[d\phi \wedge \star d\phi + e^{2\phi}d\chi \wedge \star d\chi] + \frac{1}{2}e^{\frac{4}{\sqrt{7}}\varphi}F^0 \wedge \star F^0 \right. \\ &\quad + \frac{1}{2}e^{\frac{3}{\sqrt{7}}\varphi}(\mathcal{M}^{-1})_{\mathbf{ij}}F^{\mathbf{i}} \wedge \star F^{\mathbf{j}} + \frac{1}{2}e^{-\frac{1}{\sqrt{7}}\varphi}(\mathcal{M}^{-1})_{ij}H^i \wedge \star H^j + \frac{1}{2}e^{\frac{2}{\sqrt{7}}\varphi}G \wedge \star G \\ &\quad - \frac{1}{2}[G + \varepsilon_{\mathbf{ij}}A^{\mathbf{i}} \wedge (H^j - \frac{1}{2}\delta^j_{\mathbf{j}}A^{\mathbf{j}} \wedge F^0)] \wedge \{[G + \varepsilon_{\mathbf{ij}}A^{\mathbf{i}} \wedge (H^j - \frac{1}{2}\delta^j_{\mathbf{j}}A^{\mathbf{j}} \wedge F^0)] \wedge A^0 \\ &\quad \left. - \varepsilon_{ij}(H^i - \delta^i_{\mathbf{i}}A^{\mathbf{i}} \wedge F^0) \wedge (B^j - \frac{1}{2}\delta^j_{\mathbf{j}}A^{0\mathbf{j}})\} \right\}. \end{aligned} \quad (4.9)$$

The kinetic term for the $SL(2, \mathbb{R})$ scalars ϕ and χ can be written in the alternative forms

$$\frac{1}{2}[d\phi \wedge \star d\phi + e^{2\phi}d\chi \wedge \star d\chi] = \frac{d\tau \wedge \star d\bar{\tau}}{2(\Im m\tau)^2} = \frac{1}{4}\text{Tr}[d\mathcal{M}\mathcal{M}^{-1} \wedge \star d\mathcal{M}\mathcal{M}^{-1}], \quad (4.10)$$

the last of which is manifestly $SL(2, \mathbb{R})$ -invariant. The Chern-Simons term of the action (the last two lines of Eq. (4.9)) can also be written in the alternative form

$$\begin{aligned} &-\frac{1}{2}d[C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}} - \varepsilon_{\mathbf{ij}}A^{\mathbf{i}} \wedge B^j] \wedge \{d[C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}} - \varepsilon_{\mathbf{ij}}A^{\mathbf{i}} \wedge B^j] \wedge A^0 \\ &\quad - \varepsilon_{ij}d(B^i - \frac{1}{2}\delta^i_{\mathbf{i}}A^{0\mathbf{i}}) \wedge (B^j - \frac{1}{2}\delta^j_{\mathbf{j}}A^{0\mathbf{j}})\}, \end{aligned} \quad (4.11)$$

that has an evident 11-dimensional origin.

The equations of motion of the scalars, derived from the action above, are

$$\begin{aligned} d \star d\varphi - \frac{2}{\sqrt{7}} e^{\frac{4}{\sqrt{7}}\varphi} F^0 \wedge \star F^0 - \frac{3}{2\sqrt{7}} e^{\frac{3}{\sqrt{7}}\varphi} (\mathcal{M}^{-1})_{ij} F^i \wedge \star F^j \\ + \frac{1}{2\sqrt{7}} e^{-\frac{1}{\sqrt{7}}\varphi} (\mathcal{M}^{-1})_{ij} H^i \wedge \star H^j - \frac{1}{\sqrt{7}} e^{\frac{2}{\sqrt{7}}\varphi} G \wedge \star G = 0, \end{aligned} \quad (4.12)$$

$$d \left[\star \frac{d\bar{\tau}}{(\Im m \tau)^2} \right] - i \frac{d\tau \wedge \star d\bar{\tau}}{(\Im m \tau)^3} - \partial_\tau (\mathcal{M}^{-1})_{ij} [F^i \wedge \star F^j + H^i \wedge \star H^j] = 0, \quad (4.13)$$

and those of the fundamental p -forms ($p \geq 1$), after some algebraic manipulations, take the form

$$d \left(e^{\frac{4}{\sqrt{7}}\varphi} \star F^0 \right) = -e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} F^i \wedge \star H^j + \frac{1}{2} G \wedge G, \quad (4.14)$$

$$d \left(e^{\frac{3}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} \star F^j \right) = -e^{\frac{3}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} F^0 \wedge \star H^j + \varepsilon_{ij} e^{\frac{2}{\sqrt{7}}\varphi} H^j \wedge \star G, \quad (4.15)$$

$$d \left(e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} \star H^j \right) = \varepsilon_{ij} e^{\frac{2}{\sqrt{7}}\varphi} F^j \wedge \star G - \varepsilon_{ij} H^j \wedge G, \quad (4.16)$$

$$d \left(e^{\frac{2}{\sqrt{7}}\varphi} \star G \right) = F^0 \wedge G + \frac{1}{2} \varepsilon_{ij} H^i \wedge H^j. \quad (4.17)$$

4.2.1 Global symmetries

The undeformed theory has as (classical) global symmetry group $SL(2, \mathbb{R}) \times (\mathbb{R}^+)^2$. The $(\mathbb{R}^+)^2$ symmetries correspond to scalings of the fields, the first of which, that we will denote by α^5 , acts on the metric and only leaves the equations of motion invariant while the second of them, which we will denote by β , leaves invariant both the metric and the action. The β rescaling corresponds to the so-called *trombone symmetry* which may not survive to higher-derivative string corrections.

One can also discuss two more scaling symmetries γ and δ , but γ is just a subgroup of $SL(2, \mathbb{R})$ and δ is related to the other scaling symmetries by

$$\frac{4}{9}\alpha - \frac{8}{3}\beta - \gamma - \frac{1}{2}\delta = 0. \quad (4.18)$$

We will take α and β as the independent symmetries. The weights of the electric fields under all the scaling symmetries are given in Table 4.2.1. We can see that each

⁵This discussion follows closely that of Ref. [117] in which the higher-dimensional origin of each symmetry is also studied. In particular, we use the same names and definitions for the scaling symmetries and we reproduce the table of scaling weights for the electric fields.

\mathbb{R}^+	e_μ^a	e^φ	e^ϕ	χ	A^0	A^1	A^2	B^1	B^2	C	ψ_μ	λ	$\tilde{\lambda}$	ϵ	\mathcal{L}
α	9/7	$6/\sqrt{7}$	0	0	3	0	0	3	3	3	9/14	-9/14	-9/14	9/14	9
β	0	$\sqrt{7}/4$	3/4	-3/4	1/2	-3/4	0	-1/4	1/2	-1/4	0	0	0	0	0
γ	0	0	-2	2	0	1	-1	1	-1	0	0	0	0	0	0
δ	8/7	$-4/\sqrt{7}$	0	0	0	2	2	2	2	4	4/7	-4/7	-4/7	4/7	8

Table 4.1: The scaling weights of the electric fields of maximal $d = 9$ supergravity.

of the three gauge fields A^I_μ has zero weight under *two* (linear combinations) of these three symmetries: one is a symmetry of the action, the other is a symmetry of the equations of motion only. The 1-form that has zero weight under a given rescaling is precisely the one that can be used to gauge that rescaling, but this kind of conditions are automatically taken into account by the embedding-tensor formalism and we will not have to discuss them in detail.

The action of the element of $SL(2, \mathbb{R})$ given by the matrix

$$(\Omega_j^i) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1, \quad (4.19)$$

on the fields of the theory is

$$\begin{aligned} \tau' &= \frac{a\tau + b}{c\tau + d}, & \mathcal{M}'_{ij} &= \Omega_i^k \mathcal{M}_{kl} \Omega_j^l, \\ A^{i'} &= \Omega_j^i A^j, & B^{i'} &= \Omega_j^i B^j, \\ \psi'_\mu &= e^{\frac{i}{2}l} \psi_\mu, & \lambda &= e^{\frac{3i}{2}l} \lambda, \\ \tilde{\lambda}' &= e^{-\frac{i}{2}l} \tilde{\lambda}, & \epsilon' &= e^{\frac{i}{2}l} \epsilon. \end{aligned} \quad (4.20)$$

where

$$e^{2il} \equiv \frac{c\tau^* + d}{c\tau + d}. \quad (4.21)$$

The rest of the fields ($e^a_\mu, \varphi, A^0_\mu, C_{\mu\nu\rho}$), are invariant under $SL(2, \mathbb{R})$.

We are going to label the 5 generators of these global symmetries by T_A , $A = 1, \dots, 5$. $\{T_1, T_2, T_3\}$ will be the 3 generators of $SL(2, \mathbb{R})$ (collectively denoted by $\{T_m\}$, $m = 1, 2, 3$), and T_4 and T_5 will be, respectively, the generators of the rescalings α and β . Our choice for the generators of $SL(2, \mathbb{R})$ acting on the doublets of 1-forms A^i and 2-forms B^i is

$$T_1 = \frac{1}{2}\sigma^3, \quad T_2 = \frac{1}{2}\sigma^1, \quad T_3 = \frac{i}{2}\sigma^2, \quad (4.22)$$

where the σ^m are the standard Pauli matrices, so

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = -T_1, \quad [T_3, T_1] = -T_2. \quad (4.23)$$

Then, the 3×3 matrices corresponding to generators acting (contravariantly) on the 3 1-forms A^I (and covariantly on their dual 6-forms \tilde{A}_I to be introduced later) are

$$\begin{aligned} ((T_1)_{J^I}) &= \frac{1}{2} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma^3 \end{array} \right), & ((T_2)_{J^I}) &= \frac{1}{2} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \sigma^1 \end{array} \right), & ((T_3)_{J^I}) &= \frac{1}{2} \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & i\sigma^2 \end{array} \right), \\ ((T_4)_{J^I}) &= \text{diag}(3, 0, 0), & ((T_5)_{J^I}) &= \text{diag}(1/2, -3/4, 0). \end{aligned} \quad (4.24)$$

We will sometimes denote this representation by $T_A^{(3)}$. The 2×2 matrices corresponding to generators acting (contravariantly) on the doublet of 2-forms B^i (and covariantly on their dual 5-forms \tilde{B}_i to be introduced later) are

$$\begin{aligned} ((T_1)_{j^i}) &= \frac{1}{2}\sigma^3, & ((T_2)_{j^i}) &= \frac{1}{2}\sigma^1, & ((T_3)_{j^i}) &= \frac{i}{2}\sigma^2, \\ ((T_4)_{j^i}) &= \text{diag}(3, 3), & ((T_5)_{j^i}) &= \text{diag}(-1/4, 1/2). \end{aligned} \quad (4.25)$$

We will denote this representation by $T_A^{(2)}$. The generators that act on the 3-form C (sometimes denoted by $T_A^{(1)}$) are

$$T_1 = T_2 = T_3 = 0, \quad T_4 = 3, \quad T_5 = -1/4. \quad (4.26)$$

We will also need the generators that act on the magnetic 4-form \tilde{C} (see next section), also denoted by $T_A^{(\tilde{1})}$

$$\tilde{T}_1 = \tilde{T}_2 = \tilde{T}_3 = 0, \quad \tilde{T}_4 = 6, \quad \tilde{T}_5 = 1/4. \quad (4.27)$$

We define the structure constants f_{AB}^C by

$$[T_A, T_B] = f_{AB}^C T_C. \quad (4.28)$$

The symmetries of the theory are isometries of the scalar manifold $(\mathbb{R} \times SL(2, \mathbb{R})/U(1))$. The Killing vector associated to the generator T_A will be denoted by k_A and will be normalized so that their Lie brackets are given by

$$[k_A, k_B] = -f_{AB}^C k_C. \quad (4.29)$$

The $SL(2, \mathbb{R})/U(1)$ factor of the scalar manifold is a Kähler space with Kähler potential, Kähler metric and Kähler 1-form, respectively given by

$$\mathcal{K} = -\log \Im m \tau = \phi, \quad \mathcal{G}_{\tau\tau^*} = \partial_\tau \partial_{\tau^*} \mathcal{K} = \frac{1}{4} e^{2\phi}, \quad \mathcal{Q} = \frac{1}{2i} (\partial_\tau \mathcal{K} d\tau - \text{c.c.}) = \frac{1}{2} e^\phi d\chi. \quad (4.30)$$

In general, the isometries of the Kähler metric only leave invariant the Kähler potential up to Kähler transformations :

$$\mathcal{L}_{k_m} \mathcal{K} = k_m^\tau \partial_\tau \mathcal{K} + \text{c.c.} = \lambda_m(\tau) + \text{c.c.}, \quad \mathcal{L}_{k_m} \mathcal{Q} = -\frac{i}{2} d\lambda_m, \quad (4.31)$$

where the λ_m are holomorphic functions of the coordinates that satisfy the equivariance property

$$\mathcal{L}_{k_m} \lambda_n - \mathcal{L}_{k_n} \lambda_m = -f_{mn}^p \lambda_p. \quad (4.32)$$

Then, for each of the $SL(2, \mathbb{R})$ Killing vectors k_m , $m = 1, 2, 3$, it is possible to find a real *Killing prepotential* or *momentum map* \mathcal{P}_m such that

$$\begin{aligned} k_{m\tau^*} &= \mathcal{G}_{\tau^*\tau} k_m^\tau = i\partial_{\tau^*} \mathcal{P}_m, \\ k_m^\tau \partial_\tau \mathcal{K} &= i\mathcal{P}_m + \lambda_m, \end{aligned} \quad (4.33)$$

$$\mathcal{L}_{k_m} \mathcal{P}_n = -f_{mn}^p \mathcal{P}_p.$$

The non-vanishing components of all the Killing vectors are⁶

$$k_1^\tau = \tau, \quad k_2^\tau = \frac{1}{2}(1 - \tau^2), \quad k_3^\tau = \frac{1}{2}(1 + \tau^2), \quad k_4^\tau = 0, \quad k_5^\tau = -\frac{3}{4}\tau. \quad (4.34)$$

and

$$k_4^\varphi = 6/\sqrt{7}, \quad k_5^\varphi = \sqrt{7}/4. \quad (4.35)$$

The holomorphic functions $\lambda_m(\tau)$ take the values

$$\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{1}{2}\tau, \quad \lambda_3 = -\frac{1}{2}\tau, \quad (4.36)$$

and the momentum maps are given by:

$$\mathcal{P}_1 = \frac{1}{2}e^\phi \chi, \quad \mathcal{P}_2 = \frac{1}{4}e^\phi(1 - |\tau|^2), \quad \mathcal{P}_3 = \frac{1}{4}e^\phi(1 + |\tau|^2). \quad (4.37)$$

These objects will be used in the construction of $SL(2, \mathbb{R})$ -covariant derivatives for the fermions.

4.2.2 Magnetic fields

As it is well known, for each p -form potential with $p > 0$ one can define a *magnetic* dual which in $d - 9$ dimensions will be a $(7 - p)$ -form potential. Then, we will have magnetic 4-, 5- and 6-form potentials in the theory.

A possible way to define those potentials and identify their $(8 - p)$ -form field strengths consists in writing the equations of motion of the p -forms as total derivatives. Let us take, for instance, the equation of motion of the 3-form C Eq. (4.17). It can be written as

⁶The holomorphic and anti-holomorphic components are defined by $k = k^\tau \partial_\tau + \text{c.c.} = k^\chi \partial_\chi + k^\phi \partial_\phi$.

$$\begin{aligned}
d \frac{\partial \mathcal{L}}{\partial G} = d \left\{ e^{\frac{2}{\sqrt{7}}\varphi} \star G - [G + \varepsilon_{ij} A^i \wedge (H^j - \frac{1}{2} \delta^j_j A^j \wedge F^0)] \wedge A^0 \right. \\
\left. + \frac{1}{2} \varepsilon_{ij} (H^i - \delta^i_i A^i \wedge F^0) \wedge (B^j - \frac{1}{2} \delta^j_j A^{0j}) \right\} = 0.
\end{aligned} \tag{4.38}$$

We can transform this equation of motion into a Bianchi identity by replacing the combination of fields on which the total derivative acts by the total derivative of a 4-form which we choose for the sake of convenience⁷

$$\begin{aligned}
d \left[\tilde{C} - C \wedge A^0 - \frac{3}{4} \varepsilon_{ij} A^{0i} \wedge B^j \right] \equiv e^{\frac{2}{\sqrt{7}}\varphi} \star G - [G + \varepsilon_{ij} A^i \wedge (H^j - \frac{1}{2} \delta^j_j A^j \wedge F^0)] \wedge A^0 \\
+ \frac{1}{2} \varepsilon_{ij} (H^i - \delta^i_i A^i \wedge F^0) \wedge (B^j - \frac{1}{2} \delta^j_j A^{0j}),
\end{aligned} \tag{4.39}$$

where \tilde{C} will be the magnetic 4-form. This relation can be put in the form of a duality relation

$$e^{\frac{2}{\sqrt{7}}\varphi} \star G = \tilde{G}, \tag{4.40}$$

where we have defined the magnetic 5-form field strength

$$\tilde{G} \equiv d\tilde{C} + C \wedge F^0 - \frac{1}{24} \varepsilon_{ij} A^{0ij} \wedge F^0 - \varepsilon_{ij} (H^i - \frac{1}{2} dB^i) \wedge B^j. \tag{4.41}$$

The equation of motion for \tilde{C} is just the Bianchi identity of G rewritten in terms of \tilde{G} .

In a similar fashion we can define a doublet of 5-forms \tilde{B}_i with field strengths denoted by \tilde{H}_i , and a singlet and a doublet of 6-forms \tilde{A}_0, \tilde{A}_i with field strengths denoted, respectively, by \tilde{F}_0 and \tilde{F}_i . The field strengths can be chosen to have the form

⁷With this definition \tilde{G} will have exactly the same form that we will obtain from the embedding tensor formalism.

0	1	2	3	4	5	6	7	8	9
j_A	A^I	B^i	C	\tilde{C}	\tilde{B}_i	\tilde{A}_I	$\tilde{A}_{(7)}^A$	$\tilde{A}_{(8)}$	$\tilde{A}_{(9)}$
	F^I	H^i	G	\tilde{G}	\tilde{H}_i	\tilde{F}_I	$\tilde{F}_{(8)}^A$	$\tilde{F}_{(9)}$	

Table 4.2: Electric and magnetic forms and their field strengths.

$$\begin{aligned} \tilde{H}_i &= d\tilde{B}_i - \delta_{ij}B^j \wedge G + \delta_{ij}\tilde{C} \wedge F^j + \frac{1}{2}\delta_{ij}(A^0 \wedge F^j + A^j \wedge F^0) \wedge C \\ &+ \frac{1}{2}\delta_{ij}\varepsilon_{kl}B^{jk} \wedge F^l, \end{aligned} \quad (4.42)$$

$$\begin{aligned} \tilde{F}_0 &= d\tilde{A}_0 + \frac{1}{2}C \wedge G - \varepsilon_{ij}F^i \wedge \left(\delta^{jk}\tilde{B}_k - \frac{2}{3}B^j \wedge C \right) \\ &- \frac{1}{18}\varepsilon_{ij}A^{ij} \wedge \left(\tilde{G} - F^0 \wedge C - \frac{1}{2}\varepsilon_{kl}B^k \wedge H^l \right) \\ &- \frac{1}{6}\varepsilon_{ij}A^i \wedge \left(B^j \wedge G - C \wedge H^j - \frac{2}{3}\delta^j_{\mathbf{j}}\tilde{C} \wedge F^{\mathbf{j}} - \varepsilon_{kl}B^{jk} \wedge F^l \right), \end{aligned} \quad (4.43)$$

$$\begin{aligned} \tilde{F}_1 &= d\tilde{A}_1 + \delta_{ij}(B^j + \frac{7}{18}\delta^j_{\mathbf{k}}A^{0\mathbf{k}}) \wedge \tilde{G} - \delta_i^j F^0 \wedge \tilde{B}_j - \frac{1}{9}\delta_{ij}(8A^0 \wedge F^j + A^j \wedge F^0) \wedge \tilde{C} \\ &- \frac{1}{3}\delta_{ij}\varepsilon_{lm}(B^j + \frac{1}{3}\delta^j_{\mathbf{k}}A^{0\mathbf{k}}) \wedge B^l \wedge H^m - \frac{1}{6}\delta_{ij}\varepsilon_{kl}(A^0 \wedge H^j - B^j \wedge F^0) \wedge A^{\mathbf{k}} \wedge B^l \\ &- \frac{1}{9}A^0 \wedge F^0 \wedge \delta_{ij}\left(\frac{7}{2}A^j \wedge C + \delta^j_{\mathbf{k}}\varepsilon_{lm}A^{\mathbf{lm}} \wedge B^k\right), \end{aligned} \quad (4.44)$$

and the duality relations are

$$\tilde{H}_i = e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} \star H^j, \quad (4.45)$$

$$\tilde{F}_0 = e^{\frac{4}{\sqrt{7}}\varphi} \star F^0, \quad (4.46)$$

$$\tilde{F}_1 = e^{\frac{3}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} \star F^j. \quad (4.47)$$

The situation is summarized in Table 4.2.2. The scaling weights of the magnetic fields are given in Table 4.2.2.

This dualization procedure is made possible by the gauge symmetries associated to all the p -form potentials for $p > 0$ (actually, by the existence of gauge transformations with constant parameters) and, therefore, it always works for massless p -forms with

\mathbb{R}^+	\tilde{C}	\tilde{B}_2	\tilde{B}_1	\tilde{A}_2	\tilde{A}_1	\tilde{A}_0
α	6	6	6	9	9	6
β	1/4	-1/2	+1/4	0	+3/4	-1/2
γ	0	1	-1	1	-1	0
δ	4	6	6	6	6	8

Table 4.3: The scaling weights of the magnetic fields of maximal $d = 9$ supergravity can be determined by requiring that the sum of the weights of the electric and magnetic potentials equals that of the Lagrangian. The scaling weights of the 7-, 8- and 9-forms can be determined in the same way after we find the entities they are dual to (Noether currents, embedding-tensor components and constraints, see Section 4.4).

$p > 0$ and generically fails for 0-form fields. However, in maximal supergravity theories at least, there is a global symmetry group that acts on the scalar manifold and whose dimension is larger than that of the scalar manifold. Therefore, there is one Noether 1-form current j_A associated to each of the generators of the global symmetries of the theory T_A . These currents are conserved on-shell, *i. e.* they satisfy

$$d \star j_A = 0,$$

on-shell, and we can define a $(d - 2)$ -form potential $\tilde{A}_{(d-2)}^A$ by

$$d\tilde{A}_{(d-2)}^A = G^{AB} \star j_B,$$

where G^{AB} is the inverse Killing metric of the global symmetry group, so that the conservation law (dynamical) becomes a Bianchi identity.

Thus, while the dualization procedure indicates that for each electric p -form with $p > 0$ there is a dual magnetic $(7-p)$ -form transforming in the conjugate representation, it tells us that there are as many magnetic $(d - 2)$ -form duals of the scalars as the dimension of the global group (and not of as the dimension of the scalar manifold) and that they transform in the co-adjoint representation. Actually, since there is no need to have scalar fields in order to have global symmetries, it is possible to define magnetic $(d - 2)$ -form potentials even in the total absence of scalars⁸.

According to these general arguments, which are in agreement with the general results of the embedding-tensor formalism [118, 119, 184, 186], we expect a triplet of 7-form potentials $\tilde{A}_{(7)}^m$ associated to the $SL(2, \mathbb{R})$ factor of the global symmetry group [116] and two singlets $\tilde{A}_{(7)}^4, \tilde{A}_{(7)}^5$ associated to the rescalings α, β (see Table 4.2.2).

Finding or just determining the possible magnetic $(d - 1)$ - and d -form potentials in a given theory is more complicated. In the embedding-tensor formalism it is natural to expect as many $(d - 1)$ -form potentials as deformation parameters (embedding-tensor components, mass parameters etc.) can be introduced in the theory since the rôle of the $(d - 1)$ -forms in the action is that of being Lagrange multipliers enforcing their

⁸See Refs. [118, 119] for examples.

constancy⁹. The number of deformation parameters that can be introduced in this theory is, as we are going to see, very large, but there are many constraints that they have to satisfy to preserve gauge and supersymmetry invariance. Furthermore, there are many Stückelberg shift symmetries acting on the possible $(d - 1)$ -form potentials. Solving the constraints leaves us with the independent deformation parameters that we can denote by m_{\sharp} and, correspondingly, with a reduced number of $(d - 1)$ -form potentials $\tilde{A}_{(d-1)}^{\sharp}$ on which only a few Stückelberg symmetries (or none at all) act¹⁰.

The d -form field strengths $\tilde{F}_{(d)}^{\sharp}$ are related to the scalar potential of the theory through the expression [118, 119, 184, 186]

$$\tilde{F}_{(d)}^{\sharp} = \frac{1}{2} \star \frac{\partial V}{\partial m_{\sharp}}. \quad (4.48)$$

Thus, in order to find the possible 8-form potentials of this theory we need to study its independent consistent deformations m_{\sharp} . We will consider this problem in the next section.

In the embedding-tensor formalism, the d -form potentials are associated to constraints of the deformation parameters since they would be the Lagrange multipliers enforcing them in the action [181]. If we do not solve any of the constraints there will be many d -form potentials but there will be many Stückelberg symmetries acting on them as well. Thus, only a small number of *irreducible* constraints that cannot be solved¹¹ and of associated d -forms may be expected in the end, but we have to go through the whole procedure to identify them. This identification will be one of the main results of the following section.

However, this is not the end of the story for the possible 9-forms. As it was shown in Ref. [119] in 4- 5- and 6-dimensional cases, in the ungauged case one can find more d -forms with consistent supersymmetric transformation rules than predicted by the embedding-tensor formalism. Those additional fields are predicted by the Kač-Moody approach [166]. However, after gauging, the new fields do not have consistent, independent, supersymmetry transformation rules to all orders in fermions¹², and have to be combined with other d -forms, so that, in the end, only the number of d -forms predicted by the embedding-tensor formalism survive.

This means that the results obtained via the embedding-tensor formalism for the 9-forms have to be interpreted with special care and have to be compared with the results obtained with other approaches.

⁹The embedding-tensor formalism gives us a reason to introduce the $(d - 1)$ -form potentials based on the deformation parameters but the $(d - 1)$ -form potentials do not disappear when the deformation parameters are set equal to zero.

¹⁰The $(d - 1)$ -form potentials that “disappear” when we solve the constraints are evidently associated to the gauge-fixing of the missing Stückelberg symmetries.

¹¹In general, the quadratic constraints cannot be used to solve some deformation parameters in terms of the rest. For instance, in this sense, if a and b are two of them, a constraint of the form $ab = 0$ cannot be solved and we can call it *irreducible*.

¹²The insufficiency of first-order in fermions checks was first noticed in Ref. [161].

The closure of the local supersymmetry algebra needs to be checked on all the fields in the tensor hierarchy predicted by the embedding-tensor formalism and, in particular, on the 9-forms to all orders in fermions. However, given that gauge invariance is requirement for local supersymmetry invariance, we expect consistency in essentially all cases with the possible exception of the 9-forms, according to the above discussion. In the next section we will do this for the electric fields of the theory.

4.3 Deforming the maximal $d = 9$ supergravity

In this section we are going to study the possible deformations of $d = 9$ supergravity, starting from its possible gaugings using the embedding-tensor formalism and constructing the corresponding tensor hierarchy [172–176, 184, 186] up to the 4-form potentials.

If we denote by $\Lambda^I(x)$ the scalar parameters of the gauge transformations of the 1-forms A^I and by α^A the constant parameters of the global symmetries, we want to promote

$$\alpha^A \longrightarrow \Lambda^I(x)\vartheta_I^A, \quad (4.49)$$

where ϑ_I^A is the *embedding tensor*, in the transformation rules of all the fields, and we are going to require the theory to be covariant under the new local transformations using the 1-forms as gauge fields.

To achieve this goal, starting with the transformations of the scalars, the successive introduction of higher-rank p -form potentials is required, which results in the construction of a tensor hierarchy. Most of these fields are already present in the supergravity theory or can be identified with their magnetic duals but this procedure allows us to introduce consistently the highest-rank fields (the d -, $(d - 1)$ - and $(d - 2)$ -form potentials), which are not dual to any of the original electric fields. Actually, as explained in Section 4.2.2, the highest-rank potentials are related to the symmetries (Noether currents), the independent deformation parameters and the constraints that they satisfy, but we need to determine these, which requires going through this procedure checking the consistency with gauge and supersymmetry invariance at each step.

Thus, we are going to require invariance under the new gauge transformations for the scalar fields and we are going to find that we need new couplings to the gauge 1-form fields (as usual). Then we will study the modifications of the supersymmetry transformation rules of the scalars and fermion fields which are needed to ensure the closure of the local supersymmetry algebra on the scalars. Usually we do not expect modifications in the bosons' supersymmetry transformations, but the fermions' transformations need to be modified by replacing derivatives and field strengths by covariant derivatives and covariant field strengths and, furthermore, by adding *fermion shifts*. The local supersymmetry algebra will close provided that we impose certain constraints on the embedding tensor components and on the fermion shifts.

Repeating this procedure on the 1-forms (which requires the coupling to the 2-forms) etc. we will find a set of constraints that we can solve, determining the independent components of the deformation tensors¹³ and the fermions shifts. Some constraints (typically quadratic in deformation parameters) have to be left unsolved and we will have to take them into account towards the end of this procedure.

As a result we will identify the independent deformations of the theory and the constraints that they satisfy. From this we will be able to extract information about the highest-rank potentials in the tensor hierarchy.

4.3.1 The 0-forms φ, τ

Under the global symmetry group, the scalars transform according to

$$\delta_\alpha \varphi = \alpha^A k_A^\varphi, \quad \delta_\alpha \tau = \alpha^A k_A^\tau, \quad (4.50)$$

where the α^A are the constant parameters of the transformations, labeled by $A = 1, \dots, 5$, and where k_A^φ and k_A^τ are the corresponding components of the Killing vectors of the scalar manifold, given in Eq. (4.35) (Eq. (4.34)).

According to the general prescription Eq. (4.49), we want to gauge these symmetries making the theory invariant under the local transformations

$$\delta_\Lambda \varphi = \Lambda^I \vartheta_I^A k_A^\varphi, \quad \delta_\Lambda \tau = \Lambda^I \vartheta_I^A k_A^\tau, \quad (4.51)$$

where $\Lambda^I(x)$, $I = 0, \mathbf{1}, \mathbf{2}$, are the 0-form gauge parameters of the 1-form gauge fields A^I and ϑ_I^A is the embedding tensor.

To construct gauge-covariant field strengths for the scalars it is enough to replace their derivatives by covariant derivatives.

Covariant derivatives

The covariant derivatives of the scalars have the standard form

$$\mathfrak{D}\varphi = d\varphi + A^I \vartheta_I^A k_A^\varphi, \quad \mathfrak{D}\tau = d\tau + A^I \vartheta_I^A k_A^\tau, \quad (4.52)$$

and they transform covariantly provided that the 1-form gauge fields transform as

$$\delta_\Lambda A^I = -\mathfrak{D}\Lambda^I + Z^I{}_i \Lambda^i, \quad (4.53)$$

where the Λ^i , $i = 1, 2$, are two possible 1-form gauge parameters and $Z^I{}_i$ is a possible new deformation parameter that must satisfy the orthogonality constraint

$$\vartheta_I^A Z^I{}_i = 0. \quad (4.54)$$

¹³As we are going to see, besides the embedding tensor, one can introduce many other deformation tensors.

Furthermore, it is necessary that the embedding tensor satisfies the standard quadratic constraint

$$\vartheta_I^A T_{AJ}^K \vartheta_K^C - \vartheta_I^A \vartheta_J^B f_{AB}^C = 0, \quad (4.55)$$

that expresses the gauge-invariance of the embedding tensor.

As a general rule, all the deformation tensors have to be gauge-invariant and we can anticipate that we will have to impose the constraint that expresses the gauge-invariance of Z^I_i , namely

$$X_{JK}^I Z^K_i - X_{J_i}^j Z^I_j = 0, \quad (4.56)$$

where

$$X_{IJ}^K \equiv \vartheta_I^A T_{AJ}^K, \quad X_{J_i}^j \equiv \vartheta_J^A T_{A_i}^j. \quad (4.57)$$

Supersymmetry transformations of the fermion fields

We will assume for simplicity that the supersymmetry transformations of the fermion fields in the deformed theory have essentially the same form as in the undeformed theory but covariantized (derivatives and field strengths) and, possibly, with the addition of fermion shifts which we add in the most general form:

$$\begin{aligned} \delta_\epsilon \psi_\mu &= \mathfrak{D}_\mu \epsilon + f \gamma_\mu \epsilon + k \gamma_\mu \epsilon^* + \frac{i}{8 \cdot 2!} e^{-\frac{2}{\sqrt{7}} \varphi} \left(\frac{5}{7} \gamma_\mu \gamma^{(2)} - \gamma^{(2)} \gamma_\mu \right) F^0 \epsilon \\ &\quad - \frac{1}{8 \cdot 2!} e^{\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} \left(\frac{5}{7} \gamma_\mu \gamma^{(2)} - \gamma^{(2)} \gamma_\mu \right) (F^1 - \tau F^2) \epsilon^* \\ &\quad - \frac{i}{8 \cdot 3!} e^{-\frac{1}{2\sqrt{7}} \varphi} \left(\frac{3}{7} \gamma_\mu \gamma^{(3)} + \gamma^{(3)} \gamma_\mu \right) (H^1 - \tau H^2) \epsilon^* \\ &\quad - \frac{1}{8 \cdot 4!} e^{\frac{1}{\sqrt{7}} \varphi} \left(\frac{1}{7} \gamma_\mu \gamma^{(4)} - \gamma^{(4)} \gamma_\mu \right) G \epsilon, \end{aligned} \quad (4.58)$$

$$\begin{aligned} \delta_\epsilon \tilde{\lambda} &= i \mathfrak{D} \varphi \epsilon^* + \tilde{g} \epsilon + \tilde{h} \epsilon^* - \frac{1}{\sqrt{7}} e^{-\frac{2}{\sqrt{7}} \varphi} F^0 \epsilon^* - \frac{3i}{2 \cdot 2! \sqrt{7}} e^{\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} (F^1 - \tau^* F^2) \epsilon \\ &\quad - \frac{1}{2 \cdot 3! \sqrt{7}} e^{-\frac{1}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} (H^1 - \tau^* H^2) \epsilon - \frac{i}{4! \sqrt{7}} e^{\frac{1}{\sqrt{7}} \varphi} G \epsilon^*, \end{aligned} \quad (4.59)$$

$$\begin{aligned} \delta_\epsilon \lambda &= -e^\phi \mathfrak{D} \tau \epsilon^* + g \epsilon + h \epsilon^* - \frac{i}{2 \cdot 2!} e^{\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} (F^1 - \tau F^2) \epsilon \\ &\quad + \frac{1}{2 \cdot 3!} e^{-\frac{1}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} (H^1 - \tau H^2) \epsilon. \end{aligned} \quad (4.60)$$

In these expressions, $f, k, g, h, \tilde{g}, \tilde{h}$ are six functions of the scalars and deformation parameters to be determined, the covariant field strengths have the general form predicted by the tensor hierarchy (to be determined) and the covariant derivatives of the scalars have the forms given above. Furthermore, in $\delta_\epsilon \psi_\mu$, $\mathfrak{D}_\mu \epsilon$ stands for the Lorentz- and gauge-covariant derivative of the supersymmetry parameter, which turns out to be given by

$$\mathfrak{D}_\mu \epsilon \equiv \left\{ \nabla_\mu + \frac{i}{2} \left[\frac{1}{2} e^\phi \mathfrak{D}_\mu^5 \chi + A^I{}_\mu \vartheta_I{}^m \mathcal{P}_m \right] + \frac{9}{14} \gamma_\mu \mathbb{A}^I \vartheta_I{}^4 \right\} \epsilon \quad (4.61)$$

where \mathcal{P}_m 1, 2, 3 are the momentum maps of the holomorphic Killing vectors of $SL(2, \mathbb{R})$, defined in Eq. (4.33) and given in Eq. (4.37), ∇_μ is the Lorentz-covariant derivative and

$$\mathfrak{D}_\mu^5 \chi \equiv \partial_\mu \chi - \frac{3}{4} A^I{}_\mu \vartheta_I{}^5 \chi \quad (4.62)$$

is the derivative of χ covariant only with respect to the β rescalings. it can be checked that $\mathfrak{D}_\mu \epsilon$ transforms covariantly under gauge transformations if and only if the embedding tensor satisfies the standard quadratic constraint Eq. (4.55).

An equivalent expression for it is

$$\mathfrak{D}_\mu \epsilon = \left\{ \nabla_\mu + \frac{i}{2} \left[\frac{1}{2} e^\phi \mathfrak{D}_\mu \chi - A^I{}_\mu \vartheta_I{}^m \Im \lambda_m \right] + \frac{9}{14} \gamma_\mu \mathbb{A}^I \vartheta_I{}^4 \right\} \epsilon, \quad (4.63)$$

where the λ_m , $m = 1, 2, 3$, of $SL(2, \mathbb{R})$ and defined in Eq. (4.33) and given in Eq. (4.36) and where now

$$\mathfrak{D}_\mu \chi \equiv \partial_\mu \chi + A^I{}_\mu \vartheta_I{}^A k_A \chi, \quad (4.64)$$

is the total covariant derivative of χ (which is invariant under both the α and β scaling symmetries as well as under $SL(2, \mathbb{R})$).

The actual form of the $(p + 1)$ -form field strengths will not be needed until the moment in which study the closure of the supersymmetry algebra on the corresponding p -form potential.

Closure of the supersymmetry algebra on the 0-forms φ, τ

We assume that the supersymmetry transformations of the scalars are the same as in the undeformed theory

$$\delta_\epsilon \varphi = -\frac{i}{4} \bar{\epsilon} \tilde{\lambda}^* + \text{h.c.}, \quad (4.65)$$

$$\delta_\epsilon \tau = -\frac{1}{2} e^{-\phi} \bar{\epsilon}^* \lambda. \quad (4.66)$$

To lowest order in fermions, the commutator of two supersymmetry transformations gives

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \varphi = \xi^\mu \mathfrak{D}_\mu \varphi + \Re(\tilde{h})b - \Im(\tilde{g})c + \Re(\tilde{g})d, \quad (4.67)$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \tau = \xi^\mu \mathfrak{D}_\mu \tau + e^{-\phi} [g(c - id) - ihb], \quad (4.68)$$

where ξ^μ is one of the spinor bilinears defined in Appendix B.1.1 that clearly plays the rôle of parameter of the general coordinate transformations and a, b, c, d are the scalar bilinears defined in the same appendix.

In the right hand side of these commutators, to lowest order in fermions, we expect a general coordinate transformation (the Lie derivative \mathcal{L}_ξ of the scalars with respect to ξ^μ) and a gauge transformation which has the form of Eq. (4.51) for the scalars. Therefore, the above expressions should be compared with

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \varphi = \mathcal{L}_\xi \varphi + \Lambda^I \vartheta_I^A k_A^\varphi, \quad (4.69)$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \tau = \mathcal{L}_\xi \tau + \Lambda^I \vartheta_I^A k_A^\tau, \quad (4.70)$$

from which we get the relations

$$\Re(\tilde{h})b - \Im(\tilde{g})c + \Re(\tilde{g})d = (\Lambda^I - a^I) \vartheta_I^A k_A^\varphi, \quad (4.71)$$

$$g(c - id) - ihb = e^\phi (\Lambda^I - a^I) \vartheta_I^A k_A^\tau, \quad (4.72)$$

which would allow us to determine the fermion shift functions if we knew the gauge parameters Λ^I . In order to determine the Λ^I s we have to close the supersymmetry algebra on the 1-forms. In these expressions and in those that will follow, we use the shorthand notation

$$a^I \equiv \xi^\mu A^I_\mu, \quad b^i{}_\mu \equiv \xi^\nu B^i{}_{\nu\mu}, \quad c_{\mu\nu} \equiv \xi^\rho C_{\rho\mu\nu}, \quad \text{etc.} \quad (4.73)$$

4.3.2 The 1-forms A^I

The next step in this procedure is to consider the 1-forms that we just introduced to construct covariant derivatives for the scalars.

The 2-form field strengths F^I

The gauge transformations of the 1-forms are given in Eq. (4.53) and we first need to determine their covariant field strengths. A general result of the embedding-tensor formalism tells us that we need to introduce 2-form potentials in the covariant field

strengths. In this case only have the $SL(2, \mathbb{R})$ doublet B^i at our disposal and, therefore, the 2-form field strengths have the form

$$F^I = dA^I + \frac{1}{2}X_{JK}{}^I A^J \wedge A^K + Z^I{}_i B^i, \quad (4.74)$$

where $X_{JK}{}^I$ has been defined in Eq. (4.57) and $Z^I{}_i$ is precisely the deformation tensor we introduced in Eq. (4.53). F^I will transform covariantly under Eq. (4.53) if simultaneously the 2-forms B^i transform according to

$$\delta_\Lambda B^i = -\mathfrak{D}\Lambda^i - 2h_{IJ}{}^i [\Lambda^I F^J + \frac{1}{2}A^I \wedge \delta_\Lambda A^J] + Z^i \Lambda, \quad (4.75)$$

where $h_{IJ}{}^i$ and Z^i are two possible new deformation tensors the first of which must satisfy the constraint

$$X_{(JK)}{}^I + Z^I{}_i h_{JK}{}^i = 0, \quad (4.76)$$

while Z^i must satisfy the orthogonality constraint

$$Z^I{}_i Z^i = 0. \quad (4.77)$$

Both of them must satisfy the constraints that express their gauge invariance:

$$X_{Ij}{}^i h_{JK}{}^j - 2X_{I(J}{}^L h_{K)L}{}^i = 0, \quad (4.78)$$

$$X_I Z^i - X_{Ij}{}^i Z^j = 0, \quad (4.79)$$

where

$$X_I \equiv \vartheta_I{}^A T_A^{(1)}. \quad (4.80)$$

Closure of the supersymmetry algebra on the 1-forms A^I

We assume, as we are doing with all the bosons, that the supersymmetry transformations of the 1-forms of the theory are not deformed by the gauging, so they take the form

$$\delta_\epsilon A^0{}_\mu = \frac{i}{2} e^{\frac{2}{\sqrt{7}}\varphi} \bar{\epsilon} \left(\psi_\mu - \frac{i}{\sqrt{7}} \gamma_\mu \tilde{\lambda}^* \right) + \text{h.c.}, \quad (4.81)$$

$$\delta_\epsilon A^1{}_\mu = \frac{i}{2} \tau^* e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left(\bar{\epsilon}^* \psi_\mu - \frac{i}{4} \bar{\epsilon} \gamma_\mu \lambda + \frac{3i}{4\sqrt{7}} \bar{\epsilon}^* \gamma_\mu \tilde{\lambda}^* \right) + \text{h.c.}, \quad (4.82)$$

$$\delta_\epsilon A^2{}_\mu = \frac{i}{2} e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left(\bar{\epsilon}^* \psi_\mu - \frac{i}{4} \bar{\epsilon} \gamma_\mu \lambda + \frac{3i}{4\sqrt{7}} \bar{\epsilon}^* \gamma_\mu \tilde{\lambda}^* \right) + \text{h.c.} \quad (4.83)$$

The commutator of two of them gives, to lowest order in fermions,

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^0{}_\mu &= \xi^\nu F^0{}_{\nu\mu} - \mathfrak{D}_\mu \left(e^{\frac{2}{\sqrt{7}}\varphi} b \right) + \frac{2}{\sqrt{7}} e^{\frac{2}{\sqrt{7}}\varphi} \left\{ \left[\Re(\tilde{h}) - \sqrt{7} \Im(f) \right] \xi_\mu \right. \\
&\quad \left. + \left[\Re(\tilde{g}) - \sqrt{7} \Im(k) \right] \sigma_\mu + \left[\Im(\tilde{g}) - \sqrt{7} \Re(k) \right] \rho_\mu \right\}, \tag{4.84}
\end{aligned}$$

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^1{}_\mu &= \xi^\nu F^1{}_{\nu\mu} - \partial_\mu \left[e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} (\chi d + e^{-\phi} c) \right] \\
&\quad - A^I{}_\mu \left[\left(\frac{1}{2} \vartheta_I^1 - \frac{3}{4} \vartheta_I^5 \right) e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} (\chi d + e^{-\phi} c) + \frac{1}{2} (\vartheta_I^2 + \vartheta_I^3) e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} d \right] \\
&\quad - 2e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left\{ \chi \left[\Im(k) + \frac{3}{4\sqrt{7}} \Re(\tilde{g}) - \frac{1}{4} \Re(g) \right] + e^{-\phi} \left[-\Re(k) - \frac{3}{4\sqrt{7}} \Im(\tilde{g}) - \frac{1}{4} \Im(g) \right] \right\} \xi_\mu \\
&\quad - 2e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left\{ \chi \left[-\Re(f) - \frac{3}{4\sqrt{7}} \Im(\tilde{h}) + \frac{1}{4} \Im(h) \right] + e^{-\phi} \left[-\Im(f) - \frac{3}{4\sqrt{7}} \Re(\tilde{h}) - \frac{1}{4} \Re(h) \right] \right\} \rho_\mu \\
&\quad - 2e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left\{ \chi \left[\Im(f) + \frac{3}{4\sqrt{7}} \Re(\tilde{h}) - \frac{1}{4} \Re(h) \right] + e^{-\phi} \left[-\Re(f) - \frac{3}{4\sqrt{7}} \Im(\tilde{h}) - \frac{1}{4} \Im(h) \right] \right\} \sigma_\mu, \tag{4.85}
\end{aligned}$$

and

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^2{}_\mu &= \xi^\nu F^2{}_{\nu\mu} - \partial_\mu \left(e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} d \right) \\
&\quad - A^I{}_\mu \left[\frac{1}{2} (\vartheta_I^2 - \vartheta_I^3) e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} (\chi d + e^{-\phi} c) - \frac{1}{2} \vartheta_I^1 e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} d \right] \\
&\quad - 2e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left[\Im(k) + \frac{3}{4\sqrt{7}} \Re(\tilde{g}) - \frac{1}{4} \Re(g) \right] \xi_\mu \\
&\quad - 2e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left[-\Re(f) - \frac{3}{4\sqrt{7}} \Im(\tilde{h}) + \frac{1}{4} \Im(h) \right] \rho_\mu \\
&\quad - 2e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left[\Im(f) + \frac{3}{4\sqrt{7}} \Re(\tilde{h}) - \frac{1}{4} \Re(h) \right] \sigma_\mu, \tag{4.86}
\end{aligned}$$

where σ_μ and ρ_μ are spinor bilinears defined in Appendix B.1.1.

The closure of the local supersymmetry algebra requires the commutators to take the form

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^I{}_\mu = \mathcal{L}_\xi A^I{}_\mu - \mathfrak{D}_\mu \Lambda^I + Z^I{}_i \Lambda^i{}_\mu, \tag{4.87}$$

which will only happen if gauge parameters Λ^I are given by

$$\begin{aligned}
\Lambda^0 &= a^0 + e^{\frac{2}{\sqrt{7}}\varphi} b, \\
\Lambda^1 &= a^1 + e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} (\chi d + e^{-\phi} c), \\
\Lambda^2 &= a^2 + e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} d,
\end{aligned} \tag{4.88}$$

and the 1-form gauge parameters $\Lambda^i{}_\mu$ satisfy the relations

$$\begin{aligned}
& \left[\Re(\tilde{h}) - \sqrt{7} \Im(f) \right] \xi_\mu + \left[\Re(\tilde{g}) - \sqrt{7} \Im(k) \right] \sigma_\mu + \left[\Im(\tilde{g}) - \sqrt{7} \Re(k) \right] \rho_\mu \\
&= \frac{\sqrt{7}}{2} e^{-\frac{2}{\sqrt{7}}\varphi} Z^0{}_i \left[\Lambda^i{}_\mu - (b^i{}_\mu - h_{IJ}{}^i a^I A^J{}_\mu) \right], \tag{4.89}
\end{aligned}$$

$$\begin{aligned}
& \left\{ \chi \left[\Im(k) + \frac{3}{4\sqrt{7}} \Re(\tilde{g}) - \frac{1}{4} \Re(g) \right] + e^{-\phi} \left[-\Re(k) - \frac{3}{4\sqrt{7}} \Im(\tilde{g}) - \frac{1}{4} \Im(g) \right] \right\} \xi_\mu \\
&+ \left\{ \chi \left[-\Re(f) - \frac{3}{4\sqrt{7}} \Im(\tilde{h}) + \frac{1}{4} \Im(h) \right] + e^{-\phi} \left[-\Im(f) - \frac{3}{4\sqrt{7}} \Re(\tilde{h}) - \frac{1}{4} \Re(h) \right] \right\} \rho_\mu \\
&+ \left\{ \chi \left[\Im(f) + \frac{3}{4\sqrt{7}} \Re(\tilde{h}) - \frac{1}{4} \Re(h) \right] + e^{-\phi} \left[-\Re(f) - \frac{3}{4\sqrt{7}} \Im(\tilde{h}) - \frac{1}{4} \Im(h) \right] \right\} \sigma_\mu, \\
&= -\frac{1}{2} e^{+\frac{3}{2\sqrt{7}}\varphi - \frac{1}{2}\phi} Z^1{}_i \left[\Lambda^i{}_\mu - (b^i{}_\mu - h_{IJ}{}^i a^I A^J{}_\mu) \right], \tag{4.90}
\end{aligned}$$

$$\begin{aligned}
& \left[\Im(k) + \frac{3}{4\sqrt{7}} \Re(\tilde{g}) - \frac{1}{4} \Re(g) \right] \xi_\mu + \left[-\Re(f) - \frac{3}{4\sqrt{7}} \Im(\tilde{h}) + \frac{1}{4} \Im(h) \right] \rho_\mu \\
&+ \left[\Im(f) + \frac{3}{4\sqrt{7}} \Re(\tilde{h}) - \frac{1}{4} \Re(h) \right] \sigma_\mu, \\
&= -\frac{1}{2} e^{+\frac{3}{2\sqrt{7}}\varphi - \frac{1}{2}\phi} Z^2{}_i \left[\Lambda^i{}_\mu - (b^i{}_\mu - h_{IJ}{}^i a^I A^J{}_\mu) \right]. \tag{4.91}
\end{aligned}$$

Using the values of the parameters Λ^I that we just have determined in the relations Eqs. (4.71) and (4.72) we can determine some of the fermions shifts:

$$\Re(\tilde{h}) = \vartheta_0{}^A k_A{}^\varphi e^{\frac{2}{\sqrt{7}}\varphi}, \tag{4.92}$$

$$\tilde{g} = (\vartheta_1{}^A \tau^* + \vartheta_2{}^A) k_A{}^\varphi e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi}, \tag{4.93}$$

$$h = i\vartheta_0{}^A k_A{}^\tau e^{\frac{2}{\sqrt{7}}\varphi + \phi}, \tag{4.94}$$

$$g = \vartheta_1{}^A k_A{}^\tau e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi}. \tag{4.95}$$

As a matter of fact, g is overdetermined: we get two different expressions for it that give the same value if and only if

$$(\vartheta_1^A \tau + \vartheta_2^A) k_A^\tau = 0, \quad (4.96)$$

which, upon use of the explicit expressions of the holomorphic Killing vectors k_A^τ in Section 4.2.1, leads to the following linear constraints on the components of the embedding tensor:

$$\begin{aligned} \vartheta_2^2 + \vartheta_2^3 &= 0, \\ \vartheta_1^2 + \vartheta_1^3 + 2\vartheta_2^1 - \frac{3}{2}\vartheta_2^5 &= 0, \\ \vartheta_2^2 - \vartheta_2^3 - 2\vartheta_1^1 + \frac{3}{2}\vartheta_1^5 &= 0, \\ \vartheta_1^2 - \vartheta_1^3 &= 0. \end{aligned} \quad (4.97)$$

These constraints allow us to express 4 of the 15 components of the embedding tensor in terms of the remaining 11, but we are only going to do this after we take into account the constraints that we are going to find in the closure of the local supersymmetry algebra on the doublet of 2-forms B^i .

The values of $g, h, \tilde{g}, \tilde{h}$ and the above constraints are compatible with those of the primary deformations found in Ref. [117].

4.3.3 The 2-forms B^i

In the previous subsection we have introduced a doublet of 2-forms B^i with given gauge transformations to construct the 2-form field strengths F^I . We now have to construct their covariant field strengths and check the closure of the local supersymmetry algebra on them.

The 3-form field strengths H^i

In general we need to introduce 3-form potentials to construct the covariant 3-form field strengths and, since in maximal 9-dimensional supergravity, we only have C at our disposal, the 3-form field strengths will be given by

$$H^i = \mathfrak{D}B^i - h_{IJ}{}^i A^I \wedge dA^J - \frac{1}{3} X_{[IJ}{}^L h_{K]L}{}^i A^{IJK} + Z^i C, \quad (4.98)$$

and they transform covariantly under the gauge transformations of the 1- and 2-forms that we have previously determined provided if the 3-form C transforms as

$$\delta_\Lambda C = -\mathfrak{D}\Lambda + g_{Ii} \left[-\Lambda^I H^i - F^I \wedge \Lambda^i + \delta_\Lambda A^I \wedge B^i - \frac{1}{3} h_{JK}{}^i A^{IJ} \wedge \delta_\Lambda A^K \right] + Z \tilde{\Lambda}. \quad (4.99)$$

where g_{Ii} and Z are two possible new deformation parameters. g_{Ii} must satisfy the constraint

$$2h_{IJ}{}^i Z^J{}_j + X_{Ij}{}^i + Z^i g_{Ij} = 0, \quad (4.100)$$

while Z must satisfy the orthogonality constraint

$$Z^i Z = 0. \quad (4.101)$$

Both must be gauge-invariant, which implies the constraints

$$X_{IJ}{}^L g_{Li} + X_{Ii}{}^j g_{Jj} - X_{Ij} g_{Ii} = 0, \quad (4.102)$$

$$(X_I - \tilde{X}_I)Z = 0, \quad (4.103)$$

where

$$\tilde{X}_I \equiv \vartheta_I{}^A T_A^{(\tilde{1})}. \quad (4.104)$$

Using the constraints obeyed by the deformation parameters and the explicit form of the 2-form field strengths F^I we can rewrite the 3-form field strengths in the useful form

$$H^i = \mathfrak{D}B^i - h_{IJ}{}^i A^I \wedge F^J + \frac{1}{6} X_{[IJ}{}^L h_{K]L}{}^i A^{IJK} - \frac{1}{2} X_{Ij}{}^i A^I \wedge B^j + Z^i (C - \frac{1}{2} g_{Ij} A^I \wedge B^j). \quad (4.105)$$

Closure of the supersymmetry algebra on the 2-forms B^i

In the undeformed theory, the supersymmetry transformation rules for the 2-forms are

$$\begin{aligned} \delta_\epsilon B^1 &= \tau^* e^{\frac{1}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left[\bar{\epsilon}^* \gamma_{[\mu} \psi_{\nu]} - \frac{i}{8} \bar{\epsilon} \gamma_{\mu\nu} \lambda - \frac{i}{8\sqrt{7}} \bar{\epsilon}^* \gamma_{\mu\nu} \tilde{\lambda}^* \right] \\ &\quad - \delta^1_{\mathbf{i}} (A^0{}_{[\mu} \delta_\epsilon A^{\mathbf{i}}{}_{\nu]} + A^{\mathbf{i}}{}_{[\mu} \delta_\epsilon A^0{}_{\nu]}) + \text{h.c.}, \end{aligned} \quad (4.106)$$

$$\begin{aligned} \delta_\epsilon B^2 &= e^{\frac{1}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left[\bar{\epsilon}^* \gamma_{[\mu} \psi_{\nu]} - \frac{i}{8} \bar{\epsilon} \gamma_{\mu\nu} \lambda - \frac{i}{8\sqrt{7}} \bar{\epsilon}^* \gamma_{\mu\nu} \tilde{\lambda}^* \right] \\ &\quad - \delta^2_{\mathbf{i}} (A^0{}_{[\mu} \delta_\epsilon A^{\mathbf{i}}{}_{\nu]} + A^{\mathbf{i}}{}_{[\mu} \delta_\epsilon A^0{}_{\nu]}) + \text{h.c.} \end{aligned} \quad (4.107)$$

The last terms in both transformations are associated to the presence of derivatives of A^1 and A^2 in the field strengths of B^1 and B^2 in the undeformed theory (see Eq. (4.4)). In the deformed theory, the terms $-(A^0 \wedge dA^{\mathbf{i}} + A^{\mathbf{i}} \wedge dA^0)$ are replaced by more general

couplings $-h_{IJ}{}^i A^I \wedge dA^J$ and, therefore, it would be natural to replace the last terms in $\delta_\epsilon B^i{}_{\mu\nu}$ by

$$-2h_{IJ}{}^i A^I{}_{[\mu} \delta_\epsilon A^J{}_{\nu]}. \quad (4.108)$$

In the commutator of two supersymmetry transformations on the 2-forms, these terms give the right contributions to the terms $-2h_{IJ}{}^i \Lambda^I F^J$ of the gauge transformations (see Eq. (4.75)). However, these terms must receive other contributions in order to be complete and it turns out that the only terms of the form $-2h_{IJ}{}^i \Lambda^I F^J$ that can be completed are precisely those of the undeformed theory, which correspond to

$$h_{i0}{}^j = -\frac{1}{2}\delta_i^j. \quad (4.109)$$

In order to get more general $h_{IJ}{}^i$'s it would be necessary to deform the fermions' supersymmetry rules, something we will not do here. Furthermore, the structure of the Chern-Simons terms of the field strengths is usually determined by the closure of the supersymmetry algebra at higher orders in fermions and it is highly unlikely that a more general structure of the Chern-Simons terms will be allowed by supersymmetry. Therefore, from now on, we will set $h_{IJ}{}^i$ to the above value and we will set the values of the deformation tensors in the Chern-Simons terms of the higher-rank field strengths, to the values of the undeformed theory. Using the above value of $h_{IJ}{}^i$ in the constraints in which it occurs will help us to solve them, sometimes completely, as we will see. Nevertheless, we will keep using the notation $h_{IJ}{}^i$ for convenience.

Using the identity

$$\begin{aligned} \xi^\rho H^i{}_{\rho\mu\nu} - 2h_{IJ}{}^i A^I{}_\mu \mathcal{L}_\xi A^J{}_\nu &= \mathcal{L}_\xi B^i{}_{\mu\nu} - 2\mathfrak{D}_{[\mu}(b^i{}_{\nu]} - h_{IJ}{}^i a^I A^J{}_{\nu]}) \\ &\quad - 2h_{IJ}{}^i a^I F^J{}_{\mu\nu} \\ &\quad + Z^i \left(c_{\mu\nu} - g_{Ij} a^I B^j{}_{\mu\nu} + \frac{2}{3} g_{Jj} h_{IK}{}^j a^I A^{JK}{}_{\mu\nu} \right), \end{aligned} \quad (4.110)$$

we find that the local supersymmetry algebra closes on the B^i 's in the expected form (to lowest order in fermions)

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] B^i{}_{\mu\nu} = \mathcal{L}_\xi B^i{}_{\mu\nu} + \delta_\Lambda B^i{}_{\mu\nu}, \quad (4.111)$$

where $\delta_\Lambda B^i{}_{\mu\nu}$ is the gauge transformation given in Eq. (4.75) in which the 0-form gauge parameters Λ^I are as in Eqs. (4.88), the 1-form gauge parameters $\Lambda^i{}_\mu$ are given by

$$\Lambda^i{}_\mu = \lambda^i{}_\mu + b^i{}_\mu - h_{IJ}{}^i a^I A^J{}_\mu, \quad (4.112)$$

where

$$\begin{aligned}\lambda^1{}_\mu &\equiv e^{\frac{1}{2\sqrt{7}}\varphi + \frac{1}{2}\phi}(\chi\sigma_\mu - e^{-\phi}\rho_\mu), \\ \lambda^2{}_\mu &\equiv e^{\frac{1}{2\sqrt{7}}\varphi}\sigma_\mu,\end{aligned}\tag{4.113}$$

and the shift term is given by

$$\begin{aligned}Z^1 [\Lambda_{\mu\nu} - (c_{\mu\nu} - g_{Ij}a^I B^j{}_{\mu\nu} + \frac{2}{3}g_{Jj}h_{IK}{}^j a^I A^{JK}{}_{\mu\nu})] \\ = e^{\frac{1}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left[\left(\frac{1}{2}\Im(g) - 4\Re(k) + \frac{1}{2\sqrt{7}}\Im(\tilde{g}) \right) \chi \right. \\ \left. - \left(\frac{1}{2}\Re(g) + 4\Im(k) - \frac{1}{2\sqrt{7}}\Re(\tilde{g}) \right) e^{-\phi} \right] \xi_{\mu\nu},\end{aligned}\tag{4.114}$$

$$\begin{aligned}Z^2 [\Lambda_{\mu\nu} - (c_{\mu\nu} - g_{Ij}a^I B^j{}_{\mu\nu} - \frac{2}{3}g_{Jj}h_{IK}{}^j a^I A^{JK}{}_{\mu\nu})] \\ = e^{\frac{1}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left(\frac{1}{2}\Im(g) - 4\Re(k) + \frac{1}{2\sqrt{7}}\Im(\tilde{g}) \right) \xi_{\mu\nu}\end{aligned}\tag{4.115}$$

Now, let us analyze the constraints that involve $h_{IJ}{}^i$. From those that only involve the embedding tensor we find seven linear constraints that imply those in Eqs. (4.97) and that can be used to eliminate seven components of the embedding tensor:

$$\begin{aligned}\vartheta_2{}^1 &= 0, & \vartheta_1{}^2 &= \frac{3}{4}\vartheta_2{}^5, & \vartheta_1{}^3 &= \frac{3}{4}\vartheta_2{}^5, \\ \vartheta_1{}^1 &= \frac{3}{2}\vartheta_1{}^5, & \vartheta_2{}^2 &= \frac{3}{4}\vartheta_1{}^5, & \vartheta_2{}^3 &= -\frac{3}{4}\vartheta_1{}^5, \\ \vartheta_0{}^4 &= -\frac{1}{6}\vartheta_0{}^5,\end{aligned}\tag{4.116}$$

leaving the eight components (a triplet of $SL(2, \mathbb{R})$ in the upper component, a singlet and two doublets of $SL(2, \mathbb{R})$ in the lower components)

$$\vartheta_0{}^m, \quad m = 1, 2, 3, \quad \vartheta_0{}^5, \quad \vartheta_i{}^4, \quad \vartheta_i{}^5, \quad \mathbf{i} = \mathbf{1}, \mathbf{2},\tag{4.117}$$

as the only independent ones. These components correspond to the eight deformation parameters of the primary deformations studied in Ref. [117]. More precisely, the relation between them are

$$\begin{aligned}\vartheta_0{}^m &= m_m, \quad (m = 1, 2, 3) & \vartheta_1{}^4 &= -m_{11}, & \vartheta_1{}^5 &= \tilde{m}_4, \\ \vartheta_0{}^5 &= -\frac{16}{3}m_{\text{IIB}}, & \vartheta_2{}^4 &= m_{\text{IIA}}, & \vartheta_2{}^5 &= m_4.\end{aligned}\tag{4.118}$$

From the constraints that relate $h_{IJ}{}^i$ to $Z^I{}_i, Z^i$ and g_{Ii} we can determine all these tensors, up to a constant ζ , in terms of the independent components of the embedding tensor:

$$\begin{aligned}
Z^i_j &= \vartheta_0^m (T_m)_j^i - \frac{3}{4} \vartheta_0^5 \delta_j^1 \delta_1^i, & Z^0_i &= 3\vartheta_1^4 + \frac{1}{2} \vartheta_1^5, \\
g_{0i} &= 0, & g_{ij} &= \varepsilon_{ij}.
\end{aligned} \tag{4.119}$$

The constant ζ is the coefficient of a Chern-Simons term in the 4-form field strength and, therefore, will be completely determined by supersymmetry.

Finally, using all these results in Eqs. (4.89-4.91) we find

$$k = -\frac{9i}{14} e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} (\vartheta_1^4 \tau + \vartheta_2^4), \tag{4.120}$$

$$\Im(f) = \frac{3}{28} \vartheta_0^5 e^{\frac{2}{\sqrt{7}}\varphi}, \tag{4.121}$$

$$\begin{aligned}
\Re(f) + \frac{3}{4\sqrt{7}} \Im(\tilde{h}) &= \frac{1}{4} e^{\frac{2}{\sqrt{7}}\varphi + \phi} \left\{ \frac{1}{2} (\vartheta_0^2 + \vartheta_0^3) + (\vartheta_0^1 - \frac{3}{4} \vartheta_0^5) \chi \right. \\
&\quad \left. - \frac{1}{2} (\vartheta_0^2 - \vartheta_0^3) |\tau|^2 \right\},
\end{aligned} \tag{4.122}$$

which determines almost completely all the fermion shifts. We find that, in order to determine completely $\Re(f)$ and $\Im(\tilde{h})$, separately, one must study the closure of the supersymmetry algebra on the fermions of the theory or on the bosons at higher order in fermions. The result is

$$\Re(f) = \frac{1}{14} e^{\frac{2}{\sqrt{7}}\varphi} \vartheta_0^m \mathcal{P}_m, \tag{4.123}$$

$$\Im(\tilde{h}) = \frac{4}{\sqrt{7}} e^{\frac{2}{\sqrt{7}}\varphi} \vartheta_0^m \mathcal{P}_m. \tag{4.124}$$

All these results are collected in Appendix B.4.

4.3.4 The 3-form C

In the next step we are going to consider the last of the fundamental, electric p -forms of the theory, the 3-form C , whose gauge transformation is given in Eq. (4.99).

The 4-form field strength G

The 4-form field strength G is given by

$$G = \mathfrak{D}C - g_{Ii} \left(F^I - \frac{1}{2} Z^I_j B^j \right) \wedge B^i - \frac{1}{3} h_{IK}^i g_{Ji} A^{IJ} \wedge dA^K + Z\tilde{C}, \tag{4.125}$$

and it is covariant under general gauge transformations provided that the 4-form \tilde{C} transforms as

$$\begin{aligned}
\delta_\Lambda \tilde{C} &= -\mathfrak{D}\tilde{\Lambda} - \tilde{g}_I [\Lambda^I G + C \wedge \delta_\Lambda A^I + F^I \wedge \Lambda + \frac{1}{12} g_{Ji} h_{KL}{}^i A^{JK} \wedge \delta_\Lambda A^L] \\
&\quad - \tilde{g}_{ij} [2H^i \wedge \Lambda^j - B^i \wedge \delta_\Lambda B^j + 2h_{IJ}{}^i B^j \wedge A^I \wedge \delta_\Lambda A^J] \\
&\quad - \tilde{g}_{IJK} [3\Lambda^I F^{JK} + 2(F^I - Z^I{}_i B^i) \wedge A^J \wedge \delta_\Lambda A^K - \frac{1}{4} X_{LM}{}^J A^{ILM} \wedge \delta_\Lambda A^K] \\
&\quad + Z^i \tilde{\Lambda}_i,
\end{aligned} \tag{4.126}$$

where the new deformation tensors that we have introduced, $\tilde{g}_I, \tilde{g}_{ij} = -\tilde{g}_{ji}$ and $\tilde{g}_{IJK} = \tilde{g}_{(IJK)}$, are subject to the constraints

$$g_{I[i} Z^I{}_{j]} + Z \tilde{g}_{ij} = 0, \tag{4.127}$$

$$X_I + g_{Ii} Z^i + Z \tilde{g}_I = 0, \tag{4.128}$$

$$h_{(IJ}{}^i g_{K)i} - Z \tilde{g}_{IJK} = 0, \tag{4.129}$$

plus the constraints that express the gauge invariance of the new deformation parameters

$$\tilde{X}_I \tilde{g}_J - X_{IJ}{}^K \tilde{g}_K = 0, \tag{4.130}$$

$$\tilde{X}_I \tilde{g}_{ij} - 2X_{I[i}{}^k \tilde{g}_{k|j]} = 0, \tag{4.131}$$

$$\tilde{X}_I \tilde{g}_{JKL} - 3X_{I(J}{}^M \tilde{g}_{KL)M} = 0. \tag{4.132}$$

Closure of the supersymmetry algebra on the 3-form C

Taking into account the form of $\delta_\epsilon C_{\mu\nu\rho}$ in the undeformed case and the form of the field strength G , we arrive at the following Ansatz for the supersymmetry transformation of the 3-form C :

$$\delta_\epsilon C_{\mu\nu\rho} = -\frac{3}{2} e^{-\frac{1}{\sqrt{7}}\varphi} \bar{\epsilon} \gamma_{[\mu\nu} \left(\psi_{\rho]} + \frac{i}{6\sqrt{7}} \tilde{\lambda}^* \right) + \text{h.c.} + 3\delta_\epsilon A^I{}_{[\mu} \left(g_{Ii} B^i{}_{\nu\rho]} + \frac{2}{3} h_{IJ}{}^i g_{Ki} A^{JK}{}_{\nu\rho]} \right). \tag{4.133}$$

The last two terms are written in terms of the tensors g_{Ii} and $h_{IJ}{}^i$. In the undeformed theory these tensors have values which are determined by supersymmetry (at orders in fermions higher than we are considering here) and that cannot be changed in the

deformed theory, as we already discussed when we considered the 2-forms for h_{IJ}^i . Thus, h_{IJ}^i is given by Eq. (4.109) and g_{Ii} is given by Eqs. (4.119) with $\zeta = +1$

Using the identity

$$\begin{aligned}
& \xi^\sigma G_{\sigma\mu\nu\rho} + 3\mathcal{L}_\xi A^I{}_{[\mu} [g_{Ii} B^i{}_{|\nu\rho]} + \frac{2}{3} h_{IJ}^i g_{Ki} A^{JK}{}_{|\nu\rho}] = \\
& = \mathcal{L}_\xi C_{\mu\nu\rho} - 3\mathfrak{D}_{[\mu} [(c_{|\nu\rho]} - g_{Ij} a^I B^j{}_{|\nu\rho]} + \frac{2}{3} g_{Jj} h_{IK}^j a^I A^{JK}{}_{|\nu\rho}]] \\
& + g_{Ii} [-a^I H^i{}_{\mu\nu\rho} - 3F^I{}_{[\mu\nu]} (b^i{}_{|\rho]} - h_{JK}^i a^J A^K{}_{|\rho}]] \\
& + Z \left\{ \tilde{c}_{\mu\nu\rho} - \tilde{g}_I a^I C_{\mu\nu\rho} + 3\tilde{g}_{ij} B^i{}_{[\mu\nu]} (b^j{}_{|\rho]} - h_{JK}^j a^J A^K{}_{|\rho]} - 12\tilde{g}_{IJK} a^I A^J{}_{[\mu} \partial_\nu A^K{}_{\rho]} \right. \\
& \left. + 3h_{IJ}^i \tilde{g}_{ij} a^I A^J{}_{[\mu} B^j{}_{\nu\rho]} - \frac{1}{4} (h_{IJ}^i g_{Ki} \tilde{g}_L + 3X_{JK}{}^M \tilde{g}_{ILM}) a^I A^{JKL}{}_{\mu\nu\rho} \right\}, \tag{4.134}
\end{aligned}$$

one can see that the local supersymmetry algebra closes into a general coordinate transformation plus a gauge transformation of C of the form Eq. (4.99) with

$$\Lambda_{\mu\nu} = e^{\frac{1}{\sqrt{7}}\varphi} \xi_{\mu\nu} + (c_{\mu\nu} - g_{Ij} a^I B^j{}_{\mu\nu} - \frac{2}{3} g_{Jj} h_{IK}^j a^I A^{JK}{}_{\mu\nu}), \tag{4.135}$$

and with the identification

$$\begin{aligned}
Z \left\{ \tilde{\Lambda}_{\mu\nu\rho} - \tilde{c}_{\mu\nu\rho} + \tilde{g}_I a^I C_{\mu\nu\rho} + 3\tilde{g}_{ij} B^i{}_{[\mu\nu]} (b^j{}_{|\rho]} - h_{JK}^j a^J A^K{}_{|\rho]} - 12\tilde{g}_{IJK} a^I A^J{}_{[\mu} \partial_\nu A^K{}_{\rho]} \right. \\
\left. - 3\tilde{g}_{ij} h_{IJ}^i a^I A^J{}_{[\mu} B^j{}_{\nu\rho]} + \frac{1}{4} (\tilde{g}_L g_{Ki} h_{IJ}^i + 3\tilde{g}_{ILN} X_{JK}{}^N) a^I A^{JKL}{}_{\mu\nu\rho} \right\} \\
= 6e^{-\frac{1}{\sqrt{7}}\varphi} \left[\Im(f) + \frac{1}{6\sqrt{7}} \Re(\tilde{h}) \right] \zeta_{\mu\nu\rho}. \tag{4.136}
\end{aligned}$$

Comparing Eq. (4.135) with Eqs. (4.114) and (4.115) we find that

$$Z^1 = X_2 = 3\vartheta_2^4 - \frac{1}{4}\vartheta_2^5, \quad Z^2 = -X_1 = -3\vartheta_1^4 + \frac{1}{4}\vartheta_1^5. \tag{4.137}$$

To make further progress it is convenient to compute the 5-form \tilde{G} since it will contain the tensors $\tilde{g}_I, \tilde{g}_{ij}, \tilde{g}_{IJK}$ that appear in the above expression. These tensors cannot be deformed (just as it happens with h_{IJ}^i) and their values can be found by comparing the general form of \tilde{G} with the value found by duality, Eq. (4.41).

The generic form of the magnetic 5-form field strength \tilde{G} is

$$\begin{aligned}
\tilde{G} & = \mathfrak{D}\tilde{C} - \tilde{g}_J [(F^J - Z^J{}_j B^j) \wedge C + \frac{1}{12} g_{Kj} h_{MN}{}^j A^{JKM} \wedge dA^N] \\
& + 2\tilde{g}_{ij} (H^i - \frac{1}{2}\mathfrak{D}B^i) \wedge B^j - \tilde{g}_{JKL} (A^J \wedge dA^{KL} + \frac{3}{4} X_{MN}{}^L A^{JMN} \wedge dA^K) \\
& + Z^i \tilde{B}_i, \tag{4.138}
\end{aligned}$$

and comparing this generic expression with Eq. (4.41) we find that

$$\tilde{g}_I = -\delta_I^0, \quad \tilde{g}_{ij} = -\frac{1}{2}\varepsilon_{ij}, \quad \tilde{g}_{IJK} = 0. \quad (4.139)$$

Plugging these values into the constraints that involve Z Eqs. (4.101),(4.103), and (4.127-4.129) we find that it must be related to ϑ_0^5 by

$$Z = -\frac{3}{4}\vartheta_0^5, \quad (4.140)$$

and that ϑ_0^5 must satisfy the two doublets of quadratic constraints

$$\vartheta_{\mathbf{i}}^4 \vartheta_0^5 = 0, \quad (4.141)$$

$$\vartheta_{\mathbf{i}}^5 \vartheta_0^5 = 0. \quad (4.142)$$

Plugging our results into all the other constraints between deformation tensors, we find that all of them are satisfied provided that the quadratic constraints

$$\varepsilon^{\mathbf{ij}} \vartheta_{\mathbf{i}}^4 \vartheta_{\mathbf{j}}^5 = 0, \quad (4.143)$$

$$\vartheta_0^m (12\vartheta_{\mathbf{i}}^4 + 5\vartheta_{\mathbf{i}}^5) = 0, \quad (4.144)$$

$$\vartheta_{\mathbf{j}}^4 (\vartheta_0^m T_m)_{\mathbf{i}}^{\mathbf{j}} = 0, \quad (4.145)$$

are also satisfied. This set of irreducible quadratic constraints that cannot be used to solve some deformation parameters in terms of the rest in an analytic form, and to which the 9-form potentials of the theory may be associated as explained in Section 4.2.2 is one of our main results.

4.4 Summary of results and discussion

In the previous section we have constructed order by order in the rank of the p -forms the supersymmetric tensor hierarchy of maximal 9-dimensional supergravity, up to $p = 3$, which covers all the fundamental fields of the theory.

As it usually happens in all maximal supergravity theories, all the deformation parameters can be expressed in terms of components of the embedding tensor. Furthermore, we have shown that gauge invariance and local supersymmetry allow for one triplet, two doublets and one singlet of independent components of the embedding tensor

$$\vartheta_0^m, \quad m = 1, 2, 3, \quad \vartheta_0^5, \quad \vartheta_{\mathbf{i}}^4, \quad \vartheta_{\mathbf{i}}^5, \quad \mathbf{i} = \mathbf{1}, \mathbf{2}. \quad (4.146)$$

\mathbb{R}^+	j_1	$j_2 - j_3$	$j_2 + j_3$	j_4	j_5
α	0	0	0	0	0
β	0	+3/4	-3/4	0	0
γ	0	-2	+2	0	0
δ	0	0	0	0	0

Table 4.4: Weights of the Noether currents

They can be identified with the deformation parameters studied in Ref. [117]:

$$\begin{aligned}
\vartheta_0^m &= m_m, \quad (m = 1, 2, 3) & \vartheta_{\mathbf{1}}^4 &= -m_{11}, & \vartheta_{\mathbf{1}}^5 &= \tilde{m}_4, \\
\vartheta_0^5 &= -\frac{16}{3}m_{\text{IIB}}, & \vartheta_{\mathbf{2}}^4 &= m_{\text{IIA}}, & \vartheta_{\mathbf{2}}^5 &= m_4.
\end{aligned} \tag{4.147}$$

This proves, on the one hand, that no more deformations are possible and, on the other hand, that all the deformations of maximal 9-dimensional supergravity have a higher-dimensional origin, as shown in Ref. [117].

Furthermore, we have also shown that it is not possible to give non-zero values to all the deformation parameters at the same time, since they must satisfy the quadratic constraints

$$\vartheta_0^m (12\vartheta_{\mathbf{i}}^4 + 5\vartheta_{\mathbf{i}}^5) \equiv \mathcal{Q}^m_{\mathbf{i}} = 0, \tag{4.148}$$

$$\vartheta_{\mathbf{i}}^4 \vartheta_0^5 \equiv \mathcal{Q}^4_{\mathbf{i}} = 0, \tag{4.149}$$

$$\vartheta_{\mathbf{i}}^5 \vartheta_0^5 \equiv \mathcal{Q}^5_{\mathbf{i}} = 0, \tag{4.150}$$

$$\vartheta_{\mathbf{j}}^4 (\vartheta_0^m T_m)_{\mathbf{i}}^{\mathbf{j}} \equiv \mathcal{Q}_{\mathbf{i}} = 0, \tag{4.151}$$

$$\varepsilon^{\mathbf{ij}} \vartheta_{\mathbf{i}}^4 \vartheta_{\mathbf{j}}^5 \equiv \mathcal{Q} = 0, \tag{4.152}$$

all of which are related to gauge invariance.

Using these results, we can now apply the arguments developed in Section 4.2.2 to relate the number of symmetries (Noether currents), deformation parameters, and quadratic constraints to the numbers (and symmetry properties) of 7-, 8- and 9-forms of the theory. Our results can be compared with those presented in Ref. [167] (Table 6) and Ref. [168] (Table 3) and found from E_{11} level decomposition.

Associated to the symmetry group of the equations of motion of the theory, $SL(2, \mathbb{R}) \times \mathbb{R}^2$ there are 5 Noether currents j_A that fit into one triplet and two singlets of $SL(2, \mathbb{R})$ and are explicitly given in Appendix B.3. Their weights are given in Table 4.4. They

\mathbb{R}^+	ϑ_0^1	$\vartheta_0^2 - \vartheta_0^3$	$\vartheta_0^2 + \vartheta_0^3$	$\vartheta_1^4, \vartheta_1^5$	$\vartheta_1^4, \vartheta_2^5$	ϑ_0^5
α	-3	-3	-3	0	0	-3
β	-1/2	-5/4	1/4	3/4	0	-1/2
γ	0	2	-2	-1	1	0
δ	0	0	0	-2	-2	0

Table 4.5: Weights of the embedding tensor components

\mathbb{R}^+	\mathcal{Q}_1^1	\mathcal{Q}_2^1	\mathcal{Q}_1^{2-3}	\mathcal{Q}_2^{2-3}	\mathcal{Q}_1^{2+3}	\mathcal{Q}_2^{2+3}	$\mathcal{Q}_1^4, \mathcal{Q}_1^5$	$\mathcal{Q}_2^4, \mathcal{Q}_2^5$	\mathcal{Q}_1	\mathcal{Q}_2	\mathcal{Q}
α	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3	0
β	1/4	-1/2	-1/2	-5/4	1	1/4	1/4	-1/2	1/4	-1/2	3/4
γ	-1	1	1	3	-3	-1	-1	1	-1	1	0
δ	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-4

Table 4.6: Weights of quadratic constraints components.

can be dualized as explained in Section 4.2.2 into a triplet and two singlets of 7-forms $\tilde{A}_{(7)}$ whose weights are given in Table 4.4. In Refs. [167, 168] the β rescaling has not been considered. As mentioned before, it corresponds to the so-called trombone symmetry which may not survive to higher-derivative string corrections. The associated 7-form singlet $\tilde{A}_{(7)}^5$ does not appear in their analysis. The weights assigned in those references to the fields correspond to one third of the weight of the α rescaling in our conventions.

Associated to each of the $SL(2, \mathbb{R})$ multiplets of independent embedding-tensor components there is a dual multiplet of 8-forms $\tilde{A}_{(8)}$ (i.e. one triplet, two doublets and one singlet) whose weights are given in Table 4.4. The doublet and singlet associated to the gauging of the trombone symmetry using the doublet and singlet of 1-forms are missing in Refs. [167, 168], but the rest of the 8-forms and their weights are in perfect agreement with those obtained from E_{11} . Given the amount of work that it takes to determine which are the independent components of the embedding tensor allowed by supersymmetry, this is a quite non-trivial test of the consistency of the E_{11} and the embedding-tensor approaches.

Finally, associated to each of the quadratic constraints that the components of the embedding tensor must satisfy $\mathcal{Q}_i^m, \mathcal{Q}_i^4, \mathcal{Q}_i^5, \mathcal{Q}_i, \mathcal{Q}$ there is a 9-form potential $\tilde{A}_{(9)}$. The weights of these potentials are given in Table 4.4. If we set to zero the embedding-tensor components associated to the trombone symmetry ϑ_A^5 , the only constraints which are not automatically solved are

$$\mathcal{Q}_i^m = 12\vartheta_0^m \vartheta_i^4 = 0, \quad \mathcal{Q}_i = \vartheta_j^4 (\vartheta_0^m T_m)_i{}^j = 0. \quad (4.153)$$

The first of these constraints can be decomposed into a quadruplet and a doublet: rewriting \mathcal{Q}_i^m in the equivalent form

\mathbb{R}^+	$\tilde{A}_{(7)}^m$	$\tilde{A}_{(7)}^4$	$\tilde{A}_{(7)}^5$	$\tilde{A}_{(8)}^m$	$\tilde{A}_{(8)}^{4i}$	$\tilde{A}_{(8)}^{5i}$	$\tilde{A}_{(8)}^4$	$\tilde{A}_{(9)m}^i$	$\tilde{A}_{(9)4}^i$	$\tilde{A}_{(9)5}^i$	$\tilde{A}_{(9)}^i$	$\tilde{A}_{(9)}$
α	9	9	9	12	9	9	12	12	12	12	12	9
δ	8	8	8	8	2	2	8	10	10	10	10	12

Table 4.7: Weights of the 7-, 8- and 9-form fields.

$$\mathcal{Q}_{i(jk)} = \vartheta_i^4 (\vartheta_0^m T_m)_j{}^1 \varepsilon_{kl}, \quad (4.154)$$

the quadruplet corresponds to the completely symmetric part $\mathcal{Q}_{(ijk)}$ and the doublet to

$$\varepsilon^{jk} \mathcal{Q}_{j(ki)} = -\mathcal{Q}_i, \quad (4.155)$$

which is precisely the other doublet. Therefore, we get the quadruplet and one doublet of 9-forms with weight 4 under $\alpha/3$, while one more doublet is found in Refs. [167, 168]

This situation is similar to the one encountered in the $N = 2$ theories in $d = 4, 5, 6$ dimensions [119]. In those cases, the Kač-Moody (here E_{11}) approach predicts one doublet of d -form potentials more than the embedding-tensor formalism [166]. However, it can be seen that taking the undeformed limit of the results obtained in the embedding-tensor formalism, one additional doublet of d -forms arises because some Stückelberg shifts proportional to deformation tensors that could be used to eliminate them, now vanish. Furthermore, the local supersymmetry algebra closes on them as independent fields.

By analogy with what happens in the $N = 2$ theories in $d = 4, 5, 6$ dimensions, the same mechanism can make our results compatible with those of the E_{11} approach (up to the trombone symmetry): we expect the existence of two independent doublets of 9-forms in the undeformed theory but we also expect new Stückelberg transformations in the deformed theory such that one a combination of them is independent and the supersymmetry algebra closes.

This possibility (and the exclusion of any further 9-forms) can only be proven by the direct exploration of all the possible candidates to 9-form supersymmetry transformation rules, to all orders in fermions, something that lies outside the boundaries of this work.

4.5 Concluding remarks

In this chapter we have applied the embedding-tensor formalism to the study of the most general deformations (*i.e.* gaugings and massive deformations) of maximal 9-dimensional supergravity. We have used the complete global $SL(2, \mathbb{R}) \times \mathbb{R}^2$ symmetry of its equations of motion, which includes the so-called *trombone symmetry*. We have found the constraints that the deformation parameters must satisfy in order to preserve

both gauge and supersymmetry invariance (the latter imposed through the closure of the local supersymmetry algebra to lowest order in fermions). We have used most of the constraints to express some components of the deformation tensors in terms of a few components of the embedding tensor which we take to be independent and which are given in Eq. (4.146). At that point we have started making contact with the results of Ref. [117], since those independent components are precisely the 8 possible deformations identified there. All of them have a higher-dimensional origin discussed in detail in Ref. [117]. The field strengths, gauge transformations and supersymmetry transformations of the deformed theory, written in terms of the independent deformation tensors, are collected in Appendix B.4.

The 8 independent deformation tensors are still subject to quadratic constraints, given in Eq. (4.148), but those constraints cannot be used to express analytically some of them in terms of the rest, and, therefore, we must keep the 8 deformation parameters and we must enforce these irreducible quadratic constraints.

In Section 4.4 we have used our knowledge of the global symmetries (and corresponding Noether 1-forms), the independent deformation tensors and the irreducible quadratic constraints of the theory, together with the general arguments of Section 4.2.2 to determine the possible 7-, 8- and 9-forms of the theory (Table 4.4), which are dual to the Noether currents, independent deformation tensors and irreducible quadratic constraints. We have compared this spectrum of higher-rank forms with the results of Refs. [167, 168], based on E_{11} level decomposition. We have found that, in the sector unrelated to the trombone symmetry, which was excluded from that analysis, the embedding-tensor formalism predicts one doublet of 9-forms less than the E_{11} approach. However, both predictions are not contradictory: the extra doublet of 9-forms may not survive the deformations on which the embedding-tensor formalism is built: new 9-form Stückelberg shifts proportional to the deformation parameters may occur that can be used to eliminate it so only one combination of the two 9-form doubles survives. This mechanism is present in the $N = 2$ $d = 4, 5, 6$ theories [119], although the physics behind it is a bit mysterious.

We can conclude that we have satisfactorily identified the extended field content (the tensor hierarchy) of maximal 9-dimensional supergravity and, furthermore, that all the higher-rank fields have an interpretation in terms of symmetries and gaugings. This situation is in contrast with our understanding of the extended field content of the maximal 10-dimensional supergravities ($N = 2A, B$) for which the E_{11} approach can be used to get a prediction of the higher-rank forms (which turns out to be correct [159–161]) but the embedding-tensor approach apparently cannot be used¹⁴ for this end. This seems to preclude an interpretation for the 9- and 10-form fields in terms of symmetries and gaugings¹⁵, at least if we insist in the standard construction of the tensor hierarchy that starts with the gauging of global symmetries. Perhaps a more

¹⁴In the $N = 2B$ case there are no 1-forms to be used as gauge fields and in the $N = 2A$ case the only 1-form available is not invariant under the only rescaling symmetry available.

¹⁵The 8-form fields are dual to the Noether currents of the global symmetries.

general point of view is necessary.

Chapter 5

Duality orbits of non-geometric fluxes

In this chapter we are going to perform a SS dimensional reduction to the NSNS sector of supergravity in order to understand the meaning of geometric fluxes. We will motivate the existence of additional fluxes that are not captured by this standard reduction scheme and we will present some constructions that try to solve this problem. In particular, we will focus on *double field theory* (DFT), which possesses the T duality group transformation as a symmetry of the action. We will show how, by doing a SS dimensional reduction in that scenario, we obtain all the fluxes (geometric and non-geometric) that are in a one-to-one correspondence with the maximal and half-maximal gauged supergravity in $D = 9, 8, 7$ supergravities. That is, DFT seems to provide a suitable scenario to uplift non-geometric orbits in an extended geometrical sense.

5.1 Flux compactification: a primer

5.1.1 Geometric fluxes

Let us briefly introduce the geometric fluxes origin from Scherk-Schwarz (SS) compactifications of supergravities. We will follow [193–195].

Let us consider the common NSNS sector of supergravity, spanned by a D -dimensional metric $\hat{g}_{\hat{\rho}\hat{\nu}}$, a 2-form field $\hat{B}_{\hat{\rho}\hat{\nu}}$ and a dilaton $\hat{\phi}$, whose action is given by (2.31). All of the fields will depend on the D coordinates and, since we will compactify this theory to $d = D - n$ dimensions, we will split the coordinates as follows:

$$x^{\hat{\mu}} = (x^{\mu}, y^m), \quad (5.1)$$

where y^m , $m = 1, \dots, n$ are compact space directions and x^{μ} , $\mu = 1, \dots, d$ are spacetime directions. The fields must be decomposed into representations of the symmetry group

of the lower-dimensional theory,

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \hat{g}_{\mu\nu} + \hat{g}_{pq}\hat{A}^p_\mu\hat{A}^q_\nu & \hat{A}^p_\mu\hat{g}_{pm} \\ \hat{g}_{mp}\hat{A}^p_\nu & \hat{g}_{\mu\nu} \end{pmatrix}, \quad (5.2)$$

$$\hat{b}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} \hat{b}_{\mu\nu} - \frac{1}{2}(\hat{A}^p_\mu\hat{V}_{p\nu} - \hat{A}^p_\nu\hat{V}_{p\mu}) + \hat{A}^p_\mu\hat{A}^q_\nu\hat{b}_{pq} & \hat{V}_{n\mu} - \hat{b}_{np}\hat{A}^p_\mu \\ -\hat{V}_{m\nu} + \hat{b}_{mp}\hat{A}^p_\nu & \hat{b}_{\mu\nu} \end{pmatrix}, \quad (5.3)$$

where \hat{A}^m_μ and $\hat{V}_{m\mu}$ are vector fields and \hat{g}_{mn} and \hat{b}_{mn} are symmetric and antisymmetric scalar matrices, respectively.

Now we have to give a reduction ansatz, expressing the dependence of these D -dimensional fields on the effective fields that will live in d dimensions (unhatted). These fields cannot depend on the compact coordinates, so we propose the following ansatz

$$\hat{g}_{\mu\nu} = g_{\mu\nu}(x), \quad \hat{g}_{mn} = u^a_m(y)u^a_n(y)g_{ab}(x), \quad (5.4)$$

$$\hat{b}_{\mu\nu} = b_{\mu\nu}(x), \quad \hat{b}_{mn} = u^a_m(y)u^a_n(y)b_{ab}(x) + v_{mn}(y), \quad (5.5)$$

$$\hat{A}^m_\mu = u_a^m(y)A^a_\mu(x), \quad \hat{V}_{m\mu} = u^a_m(y)V_{a\mu}(x), \quad (5.6)$$

$$\hat{\phi} = \phi(x). \quad (5.7)$$

Thus, we are left with a d -dimensional metric and a 2-form plus $2n$ vector fields, A^a_μ and $V_{a\mu}$, and $n^2 + 1$ scalar fields (g_{ab}, b_{ab}, ϕ). The y -dependent elements $u^a_m(y)$ and $v_{mn}(y)$ carry the deformation of the compactified manifold, and they have to combine in such a way that there is not y -dependence in the effective action.

As we argued in Section A.4, the SS reduction of the gauge transformation parameters implies new contributions to the gauge transformations of the effective fields. For a detailed discussion, we refer to [193, 195]. Schematically, if we have a D -dimensional gauge parameter

$$\hat{\lambda}^{\hat{\mu}} = (\epsilon^\mu, \Lambda^m), \quad (5.8)$$

and an arbitrary vector field

$$\hat{V}^{\hat{\mu}} = (V^\mu(x), u_a^m(y)V^a(x)), \quad (5.9)$$

the effective Lie derivative gets modified. Namely, if

$$\hat{\mathcal{L}}_{\hat{\lambda}}\hat{V}^{\hat{\mu}} = \hat{\lambda}^{\hat{\nu}}\partial_{\hat{\nu}}\hat{V}^{\hat{\lambda}} - \hat{V}^{\hat{\nu}}\partial_{\hat{\nu}}\hat{\lambda}^{\hat{\lambda}} \quad (5.10)$$

is the D -dimensional Lie derivative, the (unhatted) effective Lie derivative results

$$\mathcal{L}_\epsilon V^a = \hat{\mathcal{L}}_\epsilon V^a + f_{bc}^a \Lambda^b V^c, \quad (5.11)$$

where

$$f_{ab}^c = u_a^m \partial_m u_b^n u_n^c - u_b^m \partial_m u_a^n u_n^c, \quad (5.12)$$

which are the same structure constants as in (A.31). These structure constants are known as *metric fluxes* due to the role that u plays on the definition of the lower-dimensional metric.

Inspired by the $O(n, n)$ invariance towards we are approaching, we can rearrange the fields and gauge parameters into $O(n, n)$ representatives,

$$\xi = (\epsilon_\mu, \epsilon^\mu, \Lambda^A), \quad (5.13)$$

$$\Lambda^A = (\lambda_a, \lambda^a), \quad (5.14)$$

$$A^A{}_\mu = (V_{a\mu}, A^a{}_\mu), \quad (5.15)$$

$$M_{AB} = \begin{pmatrix} g_{ab} & -g^{ac}b_{cb} \\ b_{ac}g^{cb} & g_{ab} - b_{ac}g^{cd}b_{db} \end{pmatrix}, \quad (5.16)$$

where indices $A, B = 1, \dots, 2n$ are raised and lowered by means of the metric

$$\eta_{AB} = \begin{pmatrix} 0 & \delta^a{}_b \\ \delta_a{}^b & 0 \end{pmatrix}. \quad (5.17)$$

The gauge transformations of the effective fields result modified and their dependence on the compact manifold is reflected in the structure constants f_{ABC} ,

$$\delta_\xi g_{\mu\nu} = \mathcal{L}_\epsilon g_{\mu\nu}, \quad (5.18)$$

$$\delta_\xi b_{\mu\nu} = \mathcal{L}_\epsilon b_{\mu\nu} + (\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu), \quad (5.19)$$

$$\delta_\xi A^A{}_\mu = \mathcal{L}_\epsilon A^A{}_\mu - \partial_\mu \Lambda^A + f_{BC}{}^A \Lambda^B A^C{}_\mu, \quad (5.20)$$

$$\delta_\xi M_{AB} = \mathcal{L}_\epsilon M_{AB} + f_{AC}{}^D \Lambda^C M_{DB} + f_{BC}{}^D \Lambda^C M_{AD}. \quad (5.21)$$

The structure constants f_{ABC} have non-vanishing components

$$f_{ab}{}^c = u_a{}^m \partial_m u_b{}^n u^c{}_n - u_b{}^m \partial_m u_a{}^n u^c{}_n, \quad (5.22)$$

$$f_{abc} = 3(\partial_{[a} u_{bc]} + f_{[ab}{}^d u_{c]d}), \quad (5.23)$$

so that $f_a{}^{bc} = f^{abc} = 0$.

Substituting the ansatz (5.7) into the SUGRA action (2.31), we have

$$S = \int d^d x \sqrt{|g|} e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{4} M_{AB} F^{A\mu\nu} F^B{}_{\mu\nu} - \frac{1}{12} G_{\mu\nu\rho} G^{\mu\nu\rho} + \frac{1}{8} D_\mu M_{AB} D^\mu M^{AB} - V \right), \quad (5.24)$$

where

$$F^A{}_{\mu\nu} = 2\partial_{[\mu} A^A{}_{\nu]} - f_{BC}{}^A A^B{}_\mu A^C{}_\nu, \quad (5.25)$$

$$G_{\mu\nu\rho} = 3\partial_{[\mu} b_{\nu\rho]} - f_{ABC} A^A{}_\mu A^B{}_\nu A^C{}_\rho + 3\partial_{[\mu} A^A{}_{\nu} A_{|A|\rho]}, \quad (5.26)$$

and the covariant derivative is

$$D_\mu M_{AB} = \partial_\mu M_{AB} - f_{AD}{}^C A^D{}_\mu M_{CB} - f_{AD}{}^C A^D{}_\mu M_{AC}. \quad (5.27)$$

Analogous to the results of Section A.4, a scalar potential naturally arises,

$$V = \frac{1}{4} f_{DA}{}^C f_{CB}{}^D M^{AB} + \frac{1}{12} f_{AC}{}^E f_{BD}{}^F M^{AB} M^{CD} M_{EF} + \frac{1}{6} f_{ABC} f^{ABC}. \quad (5.28)$$

The structure constants that have appeared as a consequence of the dimensional reduction of the 2-form and the metric are called *geometric fluxes* due to their geometrical reduction origin. In the literature, they are usually denoted as

$$H_{abc} \equiv f_{abc}, \quad \omega_{ab}{}^c \equiv f_{ab}{}^c. \quad (5.29)$$

Beyond geometric fluxes

If, in the ansatz (5.7), we choose

$$g_{ab} = \delta_{ab}, \quad b_{ab} = 0, \quad (5.30)$$

the twist matrices u and v can be understood as the background fields associated to the vielbein and the 2-form that live in the compact space. Since T duality exchanges metric and 2-form components by means of the Buscher's rules (2.54), these geometric fluxes can be transformed into each other as well. Let us study a simple setting of these fluxes to see explicitly how this applies [196].

Let us consider a compactification on a 3-torus with a non-trivial 2-form, *e.g.*,

$$g_{mn} = \delta_{mn}, \quad b_{23} = Cy^1, \quad (5.31)$$

whose associated twist matrices are

$$u_m{}^a = \delta_m{}^a, \quad v_{23} = Cy^1. \quad (5.32)$$

The corresponding fluxes are

$$H_{123} = C, \quad \omega_{12}{}^3 = \omega_{23}{}^1 = \omega_{31}{}^2 = 0. \quad (5.33)$$

Since these backgrounds enjoy isometries in the y^2 and y^3 directions, we can perform T duality transformations on these directions. So, applying (2.54), we get certain g_{mn} and b_{mn} ,

$$ds^2 = g_{mn} dy^m dy^n = (dy^1)^2 + (dy^2)^2 + (dy^3 + Cy^1 dy^2)^2, \quad b_{mn} = 0, \quad (5.34)$$

which imply the following fluxes:

$$H_{123} = \omega_{23}{}^1 = \omega_{31}{}^2 = 0, \quad \omega_{12}{}^3 = C. \quad (5.35)$$

By simple inspection, we notice that these fluxes still can be T dualized in the direction y^2 . Again, using the Buscher's rules, they transform into

$$ds^2 = g_{mn} dy^m dy^n = (dy^1)^2 + \frac{1}{1 + (Cy^1)^2} [(dy^2)^2 + (dy^3)^2], \quad b_{23} = -\frac{Cy^1}{1 + (Cy^1)^2}. \quad (5.36)$$

The non-vanishing component of 2-form is associated to a new flux, which in the literature is called $Q_1^2 3$. If we tried to do another T duality transformation in the direction y^1 , so that

$$H_{abc} \xleftarrow{T_c} \omega_{ab}{}^c \xleftarrow{T_b} Q_a{}^{bc} \xleftarrow{T_a} R^{abc}, \quad (5.37)$$

we could not do it, since we are running out of isometries.

Coming back to our d -dimensional theory, we see that our embedding tensor admits as a universal part a full $O(n, n)$ 3-form f_{MNP} [197]. When f_{MNP} is decomposed into irreducible representations of $GL(n) \subset O(n, n)$, one realizes that the geometric fluxes arising from $\{H_{mnp}, \omega_{mn}{}^p\}$ are only half of the total number of components of the 3-form. Let us focus on the $D = 4$ case, in which we reduce over a T^6 . In this case, we have a $SL(2)$ doublet $f_{\alpha MNP}$, $\alpha = (+, -)$. Thus, the dimensional reduction only gives half of the purely electric deformations. But let us remember that twisted compactifications are invariant under $GL(6) \subset O(6, 6)$ subgroup of diffeomorphisms and gauge transformations (in this case of the B field). So, if we want to generate the full **220** representation of $O(6, 6)$ (the f_{MNP} representation), we need to expand the T duality prescription imposed by the Buscher's rules. That is, we can perform a T duality transformation along directions in which there is any isometry. This would complete the duality chain (5.37).

The first T duality transformation T_c was discussed and mentioned before, it is the one relating *metric* and *gauge fluxes*. The second one, T_b , produces the so-called Q fluxes, which describe locally geometric backgrounds despite of not being globally well-defined. The last T duality, T_a , generates the R fluxes, and since there are no isometries in this direction, there does not exist even a local description for these background fluxes. On the other hand, if we compute the number of these new backgrounds, we can check that the $O(6, 6)$ representation of f_{MNP} is perfectly filled. Even more, the different $GL(6)$ representations that appear in the decomposition perfectly match with the representations of the $\{H, \omega, Q, R\}$ fluxes,

$$\begin{array}{ccccccccc} \mathbf{220} & \longrightarrow & \mathbf{20} & \oplus & (\mathbf{6} \oplus \mathbf{84}) & \oplus & (\mathbf{6}' \oplus \mathbf{84}') & \oplus & \mathbf{20}' \\ f_{MNP} & & H_{abc} & & \omega_{ab}{}^c & & Q_a{}^{bc} & & R^{abc} \end{array} \quad (5.38)$$

We are, thus, in position to say that some embedding tensor configurations will not be obtained by dimensional reduction, *i.e.*, these gaugings will not have a higher-dimensional origin.

In [193, 196], it is shown that despite of 10-dimensional supergravities are connected by dualities, the lower-dimensional theories obtained by SS compactifications do not enjoy these dualities. They construct an argument to justify the necessary existence of new fluxes by studying the closure of the algebra of the 12-dimensional gauge algebra generated by $\{Z_m, X^m\}$, $m = 1, \dots, 6$. Z_m are the 6 KK generators corresponding to the internal coordinate transformations $\delta x^m = \Lambda^m$ and X^m are the generators associated to the internal gauge transformation of the KR form, $\delta B_{mn} = 2\partial_{[m}\Lambda_{n]}$. The

algebra is

$$[Z_m, Z_n] = \omega_{mn}{}^p Z_p + H_{mnp} X^p, \quad (5.39)$$

$$[Z_m, X^n] = -\omega_{mp}{}^n X^p + Q_m{}^{np} Z_p, \quad (5.40)$$

$$[X^m, X^n] = Q_p{}^{mn} X^p + R^{mnp} Z_p. \quad (5.41)$$

That is, we are able to keep the symmetries of the g and b fields by means of the new fluxes. In fact, without these fluxes, we would not close the algebra, especially for the generators related to the 2-form.

Focusing on the chain of dualities (5.37), T duality would allow to transform a single non-geometric flux into a geometric one. However, a configuration of both geometric and non-geometric fluxes turned on simultaneously such that T duality fully is not capable of converting all the non-geometric fluxes into geometric is a special situation. This kind of setting is called *duality orbit of a non-geometric flux* and is treated in section 5, where we explicitly show that a standard SS reduction is not able to reproduce it. This fact turns out to wonder whether we need extra ingredients in our compactification procedure to get these additional fluxes.

Thus, we realize that T duality is going to be crucial in the development and inclusion of these non-geometric backgrounds. Indeed, the way in that these fluxes have emerged suggests a new framework in which T duality becomes a truly symmetry of the genuine theory, instead of appearing after the compactification.

5.2 T duality covariant constructions

Several approaches have been developed to solve the problem of getting non-geometric fluxes in a natural and covariant formalism. We can distinguish three different trends. The first one is the *doubled geometry*, in which the local patches that define the background geometry are slightly modified. Another possibility is the so-called *generalized complex geometry*, which is defined on a manifold whose bundle structure is extended to include new elements. Finally, there exists the *double field theory* formalism, which suggests the doubling of spatial coordinates, associating the new ones to their corresponding dual winding modes.

Despite of the different approaches under which these theories are built, their aim is the same: to be able to host T duality as a global symmetry by construction.

5.2.1 Doubled Geometry

The gist of this approach is the way in which the local patches of the manifold are related. In general, we need the patches that span the manifold to be glued by charts in which diffeomorphisms and gauge transformations are included. However, T duality transformations are required to be included. Such a manifold defined in this way is called *T-fold* [198, 199].

In [200], $O(n, n)$ duality twist reductions have been performed by making use of this T-fold structure. Later on, dimensional reduction over twisted doubled tori were performed to include non-geometric fluxes configurations in [201–203].

5.2.2 Generalized Complex Geometry

The starting point of this approach appears at the level of the bundle structure. The main idea is the treatment of the tangent and cotangent space at the same level, without distinguishing them. In its original formulation [204], a new bundle is constructed by joining both spaces,

$$X + \xi \in TM_n \oplus T^*M_n. \quad (5.42)$$

The underlying physical idea is the relation between complex and symplectic geometry in string theory. It is supposed that they are each other's *mirror geometries* from the point of view of particular cases of T duality.

This generalized bundle induces a natural metric \mathcal{I} ,

$$\mathcal{I}(X + \xi, Y + \eta) \equiv \frac{1}{2}(\iota_Y \xi + \iota_X \eta), \quad (5.43)$$

where $\iota_Y \xi \equiv Y^m \xi_m$. In the coordinate basis (∂_m, dx^m) , it is realized by

$$\mathcal{I} = \frac{1}{2} \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{1}_n & 0 \end{pmatrix}. \quad (5.44)$$

Thus, a *generalized almost-complex structure* on this bundle is defined as an endomorphism J ,

$$J : TM_n \oplus T^*M_n \longrightarrow TM_n \oplus T^*M_n \quad (5.45)$$

such that $J^2 = -\mathbb{1}_{2n}$ and $J^T \mathcal{I} J = \mathcal{I}$.

Following the parallelism of an almost-complex structure, a generalized Lie bracket can be defined. This is the so-called *Courant bracket*, which is defined as

$$[X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi). \quad (5.46)$$

Later in the context of another T dual construction, double field theory, this bracket is still preserved. This bracket includes the suitable transformations for a closed 2-form, guaranteeing thus the minimal requirements discussed above. Interesting monographies are [205–210].

5.2.3 Double field theory

Double field theory (DFT) can be understood as a T duality invariant formulation of the low-energy sector of string theory. That is, it contains T duality as a symmetry of

the theory by construction. This is achieved by doubling the spacetime coordinates and associating the winding modes of the strings to the new dual coordinates introduced. Its original version was developed to describe the dynamics of closed strings on tori [211]. However, due to the successful development of a background independent version [212], people quickly started to perform SS reductions over different manifolds. These dimensional compactifications done in a DFT scenario allowed to obtain the gaugings associated to the electric sector of $N = 4$ $D = 4$ supergravity. This is how DFT and non-geometric fluxes got to know one another: a friendship had just started.

We will show here the main features of DFT to find its relation with gauged supergravities. Let us introduce the necessary ingredients of DFT and some notation. For a D -dimensional spacetime with d non-compact spacetime coordinates and n compact dimensions ($D = d + n$), the fields depend on coordinates

$$X^M = (\tilde{x}_i, x^i) = (\tilde{x}_\mu, \tilde{y}_m, x^\mu, y^m), \quad (5.47)$$

where $M = 1, \dots, 2D$ is an $O(D, D)$ index. The $2D$ coordinates can be splitted into the genuine D spacetime coordinates x^i and their dual coordinates, \tilde{x}_i . In addition, the i index can be splitted into extended and compact coordinates, $i = \mu, m$, where $\mu = 1, \dots, d$ represents extended coordinates and $m = 1, \dots, n$ runs over the compactified coordinates.

In its original formulation, DFT was restricted to satisfy the level matching condition

$$L_0 - \bar{L}_0 = 0, \quad (5.48)$$

arising for closed string theory. The implications of this condition translates to the fact that the fields and gauge parameters of DFT must be annihilated by the differential operator

$$\partial_i \tilde{\partial}^i \bullet = 0. \quad (5.49)$$

A background independent action is constructed [212] under a stronger restriction: (5.49) must hold not only for any field or gauge parameter, but for any product of them. This is the so-called *strong constraint* (SC). If we define a generalized field \mathcal{E}_{ij} in terms of the metric and the 2-form,

$$\mathcal{E}_{ij} \equiv g_{ij} + b_{ij}, \quad (5.50)$$

and a T duality invariant scalar field d ,

$$e^{-2d} \equiv \sqrt{|g|} e^{-2\phi}, \quad (5.51)$$

the background independent action is

$$S = \int d^d x d^d \tilde{x} e^{-2d} \left[-\frac{1}{4} g^{ik} g^{jl} \mathcal{D}^p \mathcal{E}_{kl} \mathcal{D}_p \mathcal{E}_{ij} + \frac{1}{4} g^{kl} \left(\mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \tilde{\mathcal{D}}^j \mathcal{E}_{ki} \tilde{\mathcal{D}}^i \mathcal{E}_{lj} \right) \right. \\ \left. + \mathcal{D}^i d \tilde{\mathcal{D}}^j \mathcal{E}_{ij} + \tilde{\mathcal{D}}^i d \mathcal{D}^j \mathcal{E}_{ji} \right]. \quad (5.52)$$

The derivative operators \mathcal{D}^i , $\tilde{\mathcal{D}}^i$ are defined as

$$\mathcal{D}_i \equiv \frac{\partial}{\partial x^i} - \mathcal{E}_{ik} \frac{\partial}{\partial \tilde{x}_k}, \quad \tilde{\mathcal{D}}_i \equiv \frac{\partial}{\partial x^i} + \mathcal{E}_{ik} \frac{\partial}{\partial \tilde{x}_k}. \quad (5.53)$$

This action is invariant under the $O(D, D)$ T duality group, which acts on the fields as follows:

$$\mathcal{E}'(X') = \frac{a\mathcal{E}(X) + b}{c\mathcal{E}(X) + d}, \quad d'(X') = d(X), \quad X' = hX, \quad (5.54)$$

where h is

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D, D), \quad h^T \eta h = \eta \text{ with } \eta = \begin{pmatrix} 0 & \mathbb{1}_D \\ \mathbb{1}_D & 0 \end{pmatrix}. \quad (5.55)$$

This action can be rewritten in terms of the so-called *generalized metric*, \mathcal{H}_{MN} . This is a $2D \times 2D$ symmetric matrix constructed from the $D \times D$ matrices g_{ij} and b_{ij} , with the remarkable property that it transforms as an $O(D, D)$ tensor,

$$\mathcal{H} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix}, \quad (5.56)$$

Under $h \in O(D, D)$ transformations, the fields transform as

$$\mathcal{H}_{MN}(X) \rightarrow h_M^P h_N^Q \mathcal{H}_{PQ}(hX), \quad d(X) \rightarrow d(hX), \quad (5.57)$$

For cases in which h corresponds to a T duality transformation, it reproduces the corresponding Buscher's rules (2.54) for $\{g_{ij}, b_{ij}, \phi\}$. In fact, it has been shown that these transformation rules allow the possibility of performing a T duality transformation in non-isometric directions [199, 200, 202, 213]. Then, in terms of this generalized metric formulation, the is rewritten as

$$S = \int d^d x d^d \tilde{x} e^{-2d} \left(\frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{PQ} \partial_N \mathcal{H}_{PQ} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{PQ} \partial_Q \mathcal{H}_{MP} \right. \\ \left. - 2 \partial_M d \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \partial_M d \partial_N d \right). \quad (5.58)$$

Gauge invariance of the action and the closure of the algebra of DFT happens upon the weak (WC) and strong (SC) versions of (5.59), which in $O(D, D)$ indices are rewritten, respectively, as

$$\partial_M \partial^M \bullet = 0, \quad \partial_M \bullet \partial^M \bullet = 0, \quad (5.59)$$

where \bullet refers to any field and/or gauge parameter. Gauge transformations of the fields $\{\mathcal{H}, d\}$ are driven by the transformation rules of \mathcal{E}_{ij} ,

$$\delta_\xi \mathcal{H}^{MN} = \xi^P \partial_P \mathcal{H}^{MN} + (\partial^M \xi_P - \partial_P \xi^M) \mathcal{H}^{PN} + (\partial^N \xi_P - \partial_P \xi^N) \mathcal{H}^{MP}, \quad (5.60)$$

$$\delta \xi d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M. \quad (5.61)$$

This motivated the definition of a generalized Lie derivative, $\hat{\mathcal{L}}_\xi$ such that, for an arbitrary $O(D, D)$ tensor $V^M{}_N$,

$$\hat{\mathcal{L}}_\xi V^M{}_N = \xi^P \partial_P V^M{}_N + (\partial^M \xi_P - \partial_P \xi^M) V^P{}_N - (\partial_N \xi^P - \partial^P \xi_N) V^M{}_P. \quad (5.62)$$

Then, the field transformations are rewritten as

$$\delta_\xi \mathcal{H}^{MN} = \hat{\mathcal{L}}_\xi \mathcal{H}^{MN}, \quad (5.63)$$

$$\delta_\xi d = \hat{\mathcal{L}}_\xi d. \quad (5.64)$$

Upon the SC, this generalized Lie derivative (and thus the gauge transformations) close under the so-called *Courant* or *C-bracket*,

$$[\xi_1, \xi_2]_C{}^M = 2\xi_{[1}^M \partial_N \xi_2^M] - \xi_{[1}^N \partial^M \xi_2]N. \quad (5.65)$$

Therefore, the SC results essential in this DFT development. However, some deficiencies to this formulation arose, especially motivated by the results of [214, 215]. In these works, some of the gaugings of $N = 4$ $D = 4$ SUGRA were obtained by SS reductions of DFT, but not all of them. Indeed, the gaugings associated to non-geometric fluxes could be geometrized by performing suitable T duality transformations as the ones shown before. This, together with the presence of the constraint (5.59), which guarantees the existence of a rotation such that the fields and the parameters do not depend on the dual coordinates, led to think about a new reformulation of DFT in which the SC (5.59) would be less restrictive and genuine non-geometric fluxes would be captured.

Thus, in ref. [216], DFT was formulated without imposing any constraint at the very beginning, but leaving as new constraints the requirements of DFT: gauge invariance of the action, the closure of the generalized Lie derivatives and the generalized Jacobi identities. Of course, these three new constraints are automatically satisfied when (5.59) is imposed, but the aim of this work was to perform SS reductions coexisting with the 3 new consistency constraints. When the SS compactification is done on the theory as well as on the constraints, it is shown that (5.59) is a sufficient but not a necessary condition for the 3 lower-dimensional constraints get satisfied. In particular, they find a less restrictive condition under which the 3 consistency constraints are fulfilled. These relaxed constraints are

$$\partial_M \partial^M \star = 0, \quad \partial_M \star \partial^M \star = 0, \quad (5.66)$$

where \star now means any effective (that is, living in the lower-dimensional theory) field and/or gauge parameter. That is, while (5.59) is required not only for the lower-dimensional fields but also for the fields of the higher-dimensional theory, the new constraints (5.66) are only imposed on fields living in the lower-dimensional theory. Moreover, not only the 3 consistency constraints (gauge invariant action, closure of the

gauge transformations, Courant-like Jacobi identities) are satisfied, but an additional term that is killed by (5.59) can be added to the action,

$$\int d^d x d^d \tilde{x} e^{-2d} \frac{1}{2} \partial_M \mathcal{E}^a P \partial^M \mathcal{E}^b S_{ab} \eta^{PQ}. \quad (5.67)$$

Actually, this term becomes crucial for matching fluxes and gaugings, as we will verify in the following section.

5.3 Duality orbits of non-geometric fluxes

As we have commented in the last section, compactifications in duality covariant constructions such as generalized geometry and double field theory have proven to be suitable frameworks to reproduce gauged supergravities containing non-geometric fluxes. However, it is a priori unclear whether these approaches only provide a reformulation of old results, or also contain new physics. To address this question, we classify the T- and U-duality orbits of gaugings of (half-)maximal supergravities in dimensions seven and higher. It turns out that all orbits have a geometric supergravity origin in the maximal case, while there are non-geometric orbits in the half-maximal case. We show how the latter are obtained from compactifications of double field theory. Some technical material used in the development of this chapter can be found in Appendix C.1. The results of this chapter were first obtained in refs. [217–219].

5.3.1 Introduction

In the context of half-maximal [179] and maximal [180] gauged supergravities, not only does supersymmetry tightly organize the ungauged theory, but also it strictly determines the set of possible deformations (*i.e.* gaugings). As we have seen in chapter 3, the development of the so-called embedding tensor formalism has enabled one to formally describe all the possible deformations in a single universal formulation, which therefore completely restores duality covariance. Unfortunately, not all the deformations have a clear higher-dimensional origin, in the sense that they can be obtained by means of a certain compactification of ten or eleven dimensional supergravity.

One of the most interesting open problems concerning flux compactifications is to reproduce, by means of a suitable flux configuration, a given lower-dimensional gauged supergravity theory. Although this was done in particular cases (see for example [220, 221]), an exhaustive analysis remains to be done. This is due to fact that, on the one hand we lack a classification of the possible gauging configurations allowed in gauged supergravities and, on the other hand, only a limited set of compactification scenarios are known. Typically, to go beyond the simplest setups one appeals to dualities. The paradigmatic example [196] starts by applying T-dualities to a simple toroidal background with a non-trivial two-form generating a single H_{abc} flux. By T-dualizing this setup, one can construct a chain of T-dualities leading to new backgrounds (like

twisted-tori or T-folds) and generating new (dual) fluxes, like the so-called $Q_a{}^{bc}$ and R^{abc} . It is precisely by following duality covariance arguments in the lower-dimensional effective description that non-geometric fluxes [196] were first introduced in order to explain the mismatch between particular flux compactifications and generic gauged supergravities.

Here we would like to emphasize that all these (a priori) different T-duality connected flux configurations by definition lie in the same orbit of gaugings, and therefore give rise to the same lower-dimensional physics. In order to obtain a different gauged supergravity, one should consider more general configurations of fluxes, involving for example combinations of geometric and non-geometric fluxes, that can never be T-dualised to a frame in which the non-geometric fluxes vanish. For the sake of clarity, we depict this concept in figure 5.1.

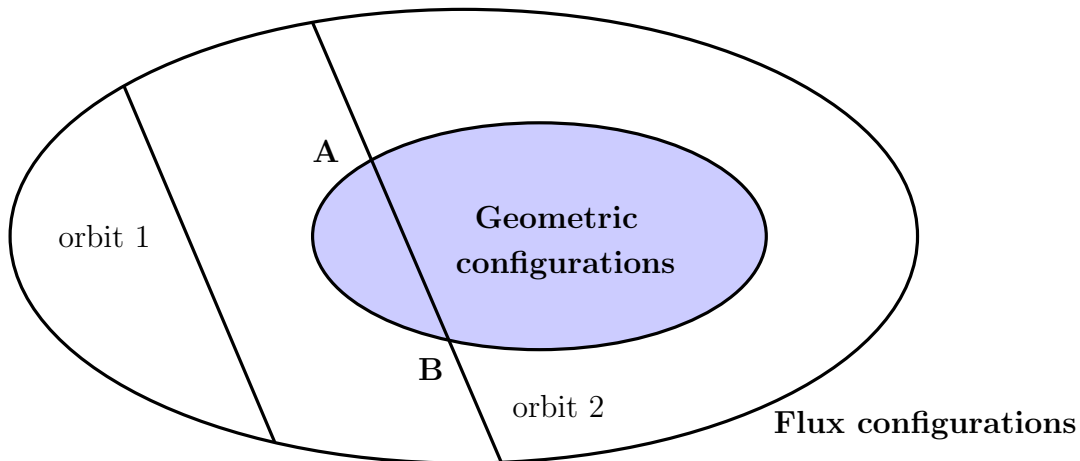


Figure 5.1: The space of flux configurations sliced into duality orbits (vertical lines). Moving along a given orbit corresponds to applying dualities to a certain flux configuration and hence it does not imply any physical changes in the lower-dimensional effective description. Geometric fluxes only constitute a subset of the full configuration space. Given an orbit, the physically relevant question is whether (orbit 2 between A and B) or not (orbit 1) this intersects the geometric subspace. We refer to a given point in an orbit as a representative.

Non-geometric fluxes are the inevitable consequence of string dualities, and only a theory which promotes such dualities to symmetries could have a chance to describe them together with geometric fluxes and to understand their origin in a unified way. From the viewpoint of the lower-dimensional effective theory, it turns out that half-maximal and maximal gauged supergravities give descriptions which are explicitly covariant with respect to T- and U-duality respectively. This is schematically depicted in table 5.1, even though only restricted to the cases we will address in this work.

In recent years, a new proposal aiming to promote T-duality to a fundamental symmetry in field theory has received increasing interest. It is named Double Field

D	T-duality	U-duality
9	$O(1, 1)$	$\mathbb{R}^+ \times SL(2)$
8	$O(2, 2) = SL(2) \times SL(2)$	$SL(2) \times SL(3)$
7	$O(3, 3) = SL(4)$	$SL(5)$

Table 5.1: The various T- and U-duality groups in $D > 6$. These turn out to coincide with the global symmetry groups of half-maximal and maximal supergravities respectively.

Theory (DFT) [211] since T-duality invariance requires a doubling of the spacetime coordinates, by supplementing them with dual coordinates associated to the stringy winding modes, whose dynamics can become important in the compactified theory. Recently it has been pointed out how to obtain gaugings of $\mathcal{N} = D = 4$ supergravity by means of twisted double torus reductions of DFT [214, 215], even though at that stage, the so-called *weak* and *strong* constraints imposed for consistency of DFT represented a further restriction that prevented one from describing the most general gaugings that solve the Quadratic Constraints (QC) of gauged supergravity.

Subsequently, an indication has been given that gauge consistency of DFT does not need the weak and strong constraints [216]. Following this direction, we could wonder whether relaxing these constraints can provide a higher-dimensional origin for all gaugings of extended supergravity in DFT. Our aim in the present work is to assess to what extent DFT can improve our description of non-geometric fluxes by giving a higher-dimensional origin to orbits which do not follow from standard supergravity compactifications. We will call such orbits of gaugings *non-geometric* (in figure 5.1 they are represented by orbit 1).

As a starting point for this investigation, we will address the problem in the context of maximal and half-maximal gauged supergravities in seven dimensions and higher, where the global symmetry groups are small enough to allow for a general classification of orbits, without needing to consider truncated sectors. We will show that in the half-maximal supergravities in seven and higher-dimensions, where the classifications of orbits can be done exhaustively, *all* the orbits (including geometric and non-geometric) admit an uplift to DFT, through Scherk-Schwarz (SS) [188] compactifications on appropriate backgrounds. We provide explicit backgrounds for every orbit, and discuss their (un)doubled nature. The result is that truly doubled DFT provides the appropriate framework to deal with orbits that can not be obtained from supergravity. In contrast, in maximal supergravities in eight and higher-dimensions, all orbits are geometric and hence can be obtained without resorting to DFT.

5.3.2 Orbits from double field theory

While toroidal compactifications of DFT lead to half-maximal ungauged supergravities, SS compactifications on more general double spaces are effectively described by gauged supergravities like the ones we will analyse in the next sections. If the internal space is restricted in such a way that there always exists a frame without dual coordinate dependence, the only orbits allowed in the effective theory are those admitting representatives that can be obtained from compactifications of ten dimensional supergravity. This is not the most general case, and we will show that some orbits require the compact space to be truly doubled, capturing information of both momentum and winding modes.

Recently in ref. [216], a new set of solutions to the constraints for DFT has been found. For these solutions the internal dependence of the fields is not dynamical, but fixed. The constraints of DFT restrict the dynamical external space to be undoubled, but allows for a doubling of the internal coordinates as long as the QC for the gaugings are satisfied. Interestingly, these are exactly the constraints needed for consistency of gauged supergravity, so there is a priori no impediment to uplift any orbit to DFT in this situation. In fact, in the following sections we show that all the orbits in half-maximal $D = 7, 8$ gauged supergravities can be reached from twisted double tori compactifications of DFT.

DFT and (half-)maximal gauged supergravities

In the SS procedure, the coordinates X^M are split into external directions $\mathbb{X} = (\tilde{x}_i, x^i)$ and compact internal $\mathbb{Y} = (\tilde{y}_i, y^i)$ coordinates. The former set contains pairs of $O(D, D)$ dual coordinates, while the latter one contains pairs of $O(n, n)$ dual coordinates, with $d = D + n$. This means that if a given coordinate is external (internal), its dual must also be external (internal), so the effective theory is formally a (gauged) DFT. The SS procedure is then defined in terms of a reduction ansatz, that specifies the dependence of the fields in (\mathbb{X}, \mathbb{Y})

$$\mathcal{H}_{MN}(\mathbb{X}, \mathbb{Y}) = U(\mathbb{Y})^A{}_M \widehat{\mathcal{H}}(\mathbb{X})_{AB} U(\mathbb{Y})^B{}_N, \quad d(\mathbb{X}, \mathbb{Y}) = \widehat{d}(\mathbb{X}) + \lambda(\mathbb{Y}). \quad (5.68)$$

Here the hatted fields $\widehat{\mathcal{H}}$ and \widehat{d} are the dynamical fields in the effective theory, parameterizing perturbations around the background, which is defined by $U(\mathbb{Y})$ and $\lambda(\mathbb{Y})$. The matrix U is referred to as the *twist matrix*, and must be an element of $O(n, n)$. It contains a DFT T-duality index M , and another index A corresponding to the T-duality group of the effective theory. When DFT is evaluated on the reduction ansatz, the twists generate the gaugings of the effective theory

$$f_{ABC} = 3\eta_{D[A} (U^{-1})^M{}_B (U^{-1})^N{}_{C]} \partial_M U^D{}_M, \quad (5.69)$$

$$\xi_A = \partial_M (U^{-1})^M{}_A - 2(U^{-1})^M{}_A \partial_M \lambda, \quad (5.70)$$

where f_{ABC} and ξ_A build the generalized structure constants of the gauge group in the lower-dimensional theory.

Although U and λ are \mathbb{Y} dependent quantities, the gaugings are forced to be constants in order to eliminate the \mathbb{Y} dependence from the lower dimensional theory. When the external-internal splitting is performed, namely $d = D + n$, the dynamical fields are written in terms of their components which are a D -dimensional metric, a D -dimensional 2-form, $2n$ D -dimensional vectors and n^2 scalars. These are the degrees of freedom of half-maximal supergravities. Since these fields are contracted with the gaugings, one must make sure that after the splitting the gaugings have vanishing Lorentzian indices, and this is achieved by stating that the twist matrix is only non-trivial in the internal directions. Therefore, although formally everything is covariantly written in terms of $O(d, d)$ indices A, B, C, \dots , the global symmetry group is actually broken to $O(n, n)$. We will not explicitly show how this splitting takes place, and refer to [215] for more details. In this work, for the sake of simplicity, we will restrict to the case $\xi_A = 0$, which should be viewed as a constraint for λ . Also we will restrict to $O(n, n)$ global symmetry groups, without additional vector fields.

There are two possible known ways to restrict the fields and gauge parameters in DFT, such that the action is gauge invariant and the gauge algebra closes. On the one hand, the so-called weak and strong constraints can be imposed

$$\partial_M \partial^M A = 0, \quad \partial_M A \partial^M B = 0, \quad (5.71)$$

where A and B generically denote products of (derivatives of) fields and gauge parameters. When this is the case, one can argue [212] that there is always a frame in which the fields do not depend on the dual coordinates. On the other hand, in the SS compactification scenario, it is enough to impose the weak and strong constraints only on the external space (*i.e.*, on hatted quantities)

$$\partial_M \partial^M \hat{A} = 0, \quad \partial_M \hat{A} \partial^M \hat{B} = 0, \quad (5.72)$$

and impose QC for the gaugings

$$f_{E[AB} f^E{}_{C]D} = 0. \quad (5.73)$$

This second option is more natural for our purposes, since these constraints exactly coincide with those of half-maximal gauged supergravities¹ (which are undoubled theories in the external space, and contain gaugings satisfying the QC).

Notice that if a given U produces a solution to the QC, any T-dual U will also. Therefore, it is natural to define the notion of *twist orbits* as the sets of twist matrices connected through T-duality transformations. If a representative of a twist orbit generates a representative of an orbit of gaugings, one can claim that the twist orbit will generate the entire orbit of gaugings. Also, notice that if a twist matrix satisfies the weak and strong constraints, any representative of its orbit will, so one can define the notions of undoubled and truly doubled twist orbits.

¹We are working under the assumption that the structure constants not only specify the gauging, but all couplings of the theory. Reproducing the correct structure constants therefore implies reproducing the full theory correctly, as has been proven in $D = 4$ and $D = 10$ [214, 215, 222].

Non-geometry VS weak and strong constraint violation

Any half-maximal supergravity can be uplifted to the maximal theory whenever the following constraint holds²

$$f_{ABC} f^{ABC} = 0 . \quad (5.74)$$

This constraint plays the role of an orthogonality condition between geometric and non-geometric fluxes. Interestingly, the constraint (5.74) evaluated in terms of the twist matrix U and λ can be rewritten as follows (by taking relations (5.69) and (5.70) into account)

$$f_{ABC} f^{ABC} = -3 \partial_D U^A{}_P \partial^D (U^{-1})^P{}_A - 24 \partial_D \lambda \partial^D \lambda + 24 \partial_D \partial^D \lambda . \quad (5.75)$$

The RHS of this equation is zero whenever the background defined by U and λ satisfies the weak and strong constraints. This immediately implies that any background satisfying weak and strong constraints defines a gauging which is upliftable to the maximal theory. Conversely, if an orbit of gaugings in half-maximal supergravity does not satisfy the extra constraint (5.74), the RHS of this equation must be non-vanishing, and then the strong and weak constraint must be relaxed. In conclusion, the orbits of half-maximal supergravity that do not obey the QC of the maximal theory require truly doubled twist orbits, and are therefore genuinely non-geometric. This point provides a concrete criterion to label these orbits as non-geometric. Also, notice that these orbits will never be captured by non-geometric flux configurations obtained by T-dualizing a geometric background³.

For the sake of clarity, let us briefly review the definitions that we use. A twist orbit is non-geometric if it doesn't satisfy the weak/strong constraint, and geometric if it does. Therefore, the notion of geometry that we consider is local, and we will not worry about global issues (given that the twist matrix is taken to be an element of the global symmetry group, the transition functions between coordinate patches are automatically elements of $O(n, n)$). On the other hand an orbit of gaugings is geometric if it contains a representative that can be obtained from 10 dimensional supergravity (or equivalently from a geometric twist orbit), and it is non-geometric if it does not satisfy the constraints of maximal supergravity.

We have now described all the necessary ingredients to formally relate dimensional reductions of DFT and the orbits of half-maximal gauged supergravities. In particular, in what follows we will:

1. Provide a classification of all the orbits of gaugings in maximal and half-maximal supergravities in $D \geq 7$.

² $D = 4$ half-maximal supergravity is slightly different because its global symmetry group features an extra $SL(2)$ factor; for full details, see [223, 224].

³However, we would like to stress that, in general, it is not true that an orbit satisfying the QC constraints of maximal supergravity (5.74) is necessarily generated by an undoubled twist orbit. An example can be found at the end of section 5.3.4.

2. Explore mechanisms to generate orbits of gaugings from twists, satisfying
 - $U(\mathbb{Y}) \in O(n, n)$
 - Constant f_{ABC}
 - $f_{E[AB}f^E{}_{C]D} = 0$
3. Show that in the half-maximal theories all the orbits of gaugings can be obtained from twist orbits in DFT.
4. Show that in the half-maximal theories the orbits that satisfy the QC of maximal supergravity admit a representative with a higher-dimensional supergravity origin. For these we provide concrete realisations in terms of unbouled backgrounds in DFT. Instead, the orbits that fail to satisfy (5.74) require, as we argued, truly doubled twist orbits for which we also provide concrete examples.
5. Show that there is a degeneracy in the space of twist orbits giving rise to the same orbit of gaugings. Interestingly, in some cases a given orbit can be obtained either from undoubled or truly doubled twist orbits.

In the next sections we will classify all the orbits in (half-)maximal $D \geq 7$ supergravities, and provide the half-maximal ones with concrete uplifts to DFT, explicitly proving the above points.

Parameterizations of the duality twists

Here we would like to introduce some notation that will turn out to be useful in the uplift of orbits to DFT. We start by noting the double internal coordinates as $\mathbb{Y}^A = (\tilde{y}_a, y^a)$ with $a = 1, \dots, n$. As we saw, the SS compactification of DFT is defined by the twists $U(\mathbb{Y})$ and $\lambda(\mathbb{Y})$. The duality twist $U(\mathbb{Y})$ is not generic, but forced to be an element of $O(n, n)$, so we should provide suitable parameterisations. One option is the *light-cone* parameterisation, where the metric of the (internal) global symmetry group is taken to be of the form

$$\eta_{AB} = \begin{pmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix}. \quad (5.76)$$

The most general form of the twist matrix is then given by

$$U(\mathbb{Y}) = \begin{pmatrix} e & 0 \\ 0 & e^{-T} \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & 0 \\ -B & \mathbf{1}_n \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \beta \\ 0 & \mathbf{1}_n \end{pmatrix}, \quad (5.77)$$

with $e \in GL(n)$ and B and β are generic $n \times n$ antisymmetric matrices. When $\beta = 0$, $e = e(y^a)$ and $B = B(y^a)$, the matrix e can be interpreted as a n -dimensional internal vielbein and B as a background 2-form for the n -dimensional internal Kalb-Ramond field b . Whenever the background is of this form, we will refer to it as geometric

(notice that this still does not determine completely the background, which receives deformations from scalar fluctuations). In this case the gaugings take the simple form

$$\begin{aligned} f_{abc} &= 3(e^{-1})^\alpha_{[a}(e^{-1})^\beta_b(e^{-1})^\gamma_{c]}\partial_{[\alpha}B_{\beta\gamma]} , \\ f^a{}_{bc} &= 2(e^{-1})^\beta_{[b}(e^{-1})^\gamma_{c]}\partial_\beta e^a{}_\gamma , \\ f^{ab}{}_c &= f^{abc} = 0 . \end{aligned} \tag{5.78}$$

If we also turn on a $\beta(y^a)$, the relation of e , B and β with the internal g and b is less trivial, and typically the background will be globally well defined up to $O(n, n)$ transformations mixing the metric and the two-form (this is typically called a T-fold). In this case, we refer to the background as locally geometric but globally non-geometric, and this situation formally allows for non-vanishing $f^{ab}{}_c$ and f^{abc} . Finally, if the twist matrix is a function of \tilde{y}_a , we refer to the background as locally non-geometric. Notice however, that if it satisfies the weak and strong constraints, one would always be able to rotate it to a frame in which it is locally geometric, and would therefore belong to an undoubled orbit.

Alternatively, one could also define the *cartesian* parametrisation of the twist matrix, by taking the metric of the (internal) global symmetry group to be of the form

$$\eta_{AB} = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix} . \tag{5.79}$$

This formulation is related to the light-cone parametrisation through a $SO(2n)$ transformation, that must also rotate the coordinates. In this case the relation between the components of the twist matrix and the internal g and b is non-trivial. We will consider the $O(n, n)$ twist matrix to contain a smaller $O(n-1, n-1)$ matrix in the directions $(y^2, \dots, y^n, \tilde{y}_2, \dots, \tilde{y}_n)$ fibred over the flat directions (y^1, \tilde{y}_1) . We have seen that this typically leads to constant gaugings.

Of course these are not the most general parameterisations and ansatz, but they will serve our purposes of uplifting all the orbits of half-maximal supergravity to DFT. Interesting works on how to generate gaugings from twists are [201].

5.3.3 U-duality orbits of maximal supergravities

Following the previous discussion of DFT and its relevance for generating duality orbits, we turn to the actual classification of these. In particular, we start with orbits under U-duality of gaugings of maximal supergravity. Moreover, we will demonstrate that all such orbits do have a higher-dimensional supergravity origin.

Starting with the highest dimension for maximal supergravity, $D = 11$, no known deformation is possible here. Moreover, in $D = 10$ maximal supergravities, the only possible deformation occurs in what is known as massive IIA supergravity⁴ [187]. It

⁴Throughout this section we will not consider the trombone gaugings giving rise to theories without an action principle, as discussed in *e.g.* [117, 191, 225, 226].

consists of a Stückelberg-like way of giving a mass to the 2-form B_2 . Therefore, such a deformation cannot be interpreted as a gauging. The string theory origin of this so-called Romans' mass parameter is nowadays well understood as arising from D8-branes [154]. Furthermore, its DFT uplift has been constructed in ref. [227]. Naturally, the structure of possible orbits becomes richer when going to lower dimensions. In what follows we will perform the explicit classification in dimensions nine and eight.

Orbits and origin of the $D = 9$ maximal case

Maximal $D = 9$ gauged supergravity

The maximal (ungauged) supergravity in $D = 9$ [112] can be obtained by reducing either massless type IIA or type IIB supergravity in ten dimensions on a circle. The global symmetry group of this theory is

$$G_0 = \mathbb{R}^+ \times \text{SL}(2) .$$

Note that G_0 is the global symmetry of the action and hence it is realised off-shell, whereas the on-shell symmetry has an extra \mathbb{R}^+ with respect to which the Lagrangian has a non-trivial scaling weight. This is normally referred to as the *trombone symmetry*. As a consequence, the on-shell symmetry contains three independent rescalings [117, 195], which we summarise in table 5.2. The full field content consists of the following

ID	e_μ^a	A_μ	A_μ^1	A_μ^2	$B_{\mu\nu}^1$	$B_{\mu\nu}^2$	$C_{\mu\nu\rho}$	e^φ	χ	e^ϕ	ψ_μ	$\lambda, \tilde{\lambda}$	\mathcal{L}
α	$\frac{9}{7}$	3	0	0	3	3	3	$\frac{6}{\sqrt{7}}$	0	0	$\frac{9}{14}$	$-\frac{9}{14}$	9
β	0	$\frac{1}{2}$	$-\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{\sqrt{7}}{4}$	$-\frac{3}{4}$	$\frac{3}{4}$	0	0	0
γ	0	0	1	-1	1	-1	0	0	2	-2	0	0	0
δ	$\frac{8}{7}$	0	2	2	2	2	4	$-\frac{4}{\sqrt{7}}$	0	0	$\frac{4}{7}$	$-\frac{4}{7}$	8

Table 5.2: The scaling weights of the nine-dimensional fields. As already anticipated, only three rescalings are independent since they are subject to the following constraint: $8\alpha - 48\beta - 18\gamma - 9\delta = 0$. As the scaling weight of the Lagrangian \mathcal{L} shows, β and γ belong to the off-shell symmetries, whereas α and δ can be combined into a trombone symmetry and an off-shell symmetry.

objects which arrange themselves into irrep's of $\mathbb{R}^+ \times \text{SL}(2)$:

$$9\text{D} : \underbrace{e_\mu^a, A_\mu, A_\mu^i, B_{\mu\nu}^i, C_{\mu\nu\rho}, \varphi, \tau = \chi + i e^{-\phi}}_{\text{bosonic dof's}} ; \underbrace{\psi_\mu, \lambda, \tilde{\lambda}}_{\text{fermionic dof's}} , \quad (5.80)$$

where μ, ν, \dots denote nine-dimensional curved spacetime, a, b, \dots nine-dimensional flat spacetime and i, j, \dots fundamental $\text{SL}(2)$ indices respectively.

The general deformations of this theory have been studied in detail in ref. [218], where both embedding tensor deformations and gaugings of the trombone symmetry

ID	θ^i	κ^{ij}	gauging
1		diag(1, 1)	SO(2)
2	(0, 0)	diag(1, -1)	SO(1, 1)
3		diag(1, 0)	\mathbb{R}_γ^+
4	(1, 0)	diag(0, 0)	\mathbb{R}_β^+

Table 5.3: All the U-duality orbits of consistent gaugings in maximal supergravity in $D = 9$. For each of them, the simplest representative is given. The subscripts β and γ refer to the rescalings summarised in table 5.2.

have been considered. For the present scope we shall restrict ourselves to the first ones. The latter ones would correspond to the additional mass parameters m_{IIB} and (m_{11}, m_{IIA}) in refs [117, 218], which give rise to theories without an action principle.

The vectors of the theory $\{A_\mu, A_\mu{}^i\}$ transform in the $V' = \mathbf{1}_{(+4)} \oplus \mathbf{2}_{(-3)}$ of $\mathbb{R}^+ \times \text{SL}(2)$, where the \mathbb{R}^+ scaling weights are included as well⁵. The resulting embedding tensor deformations live in the following tensor product

$$\mathfrak{g}_0 \otimes V = \mathbf{1}_{(-4)} \oplus 2 \cdot \mathbf{2}_{(+3)} \oplus \mathbf{3}_{(-4)} \oplus \mathbf{4}_{(+3)}. \quad (5.81)$$

The Linear Constraint (LC) projects out the $\mathbf{4}_{(+3)}$, the $\mathbf{1}_{(-4)}$ and one copy of the $\mathbf{2}_{(+3)}$ since they would give rise to inconsistent deformations. As a consequence, the consistent gaugings are parameterised by embedding tensor components in the $\mathbf{2}_{(+3)} \oplus \mathbf{3}_{(-4)}$. We will denote these allowed deformations by θ^i and $\kappa^{(ij)}$.

The closure of the gauge algebra and the antisymmetry of the brackets impose the following Quadratic Constraints (QC)

$$\epsilon_{ij} \theta^i \kappa^{jk} = 0, \quad \mathbf{2}_{(-1)} \quad (5.82)$$

$$\theta^{(i} \kappa^{jk)} = 0. \quad \mathbf{4}_{(-1)} \quad (5.83)$$

The $\mathbb{R}^+ \times \text{SL}(2)$ orbits of solutions to the QC

The QC (5.82) and (5.83) turns out to be very simple to solve; after finding all the solutions, we studied the duality orbits, *i.e.* classes of those solutions which are connected via a duality transformation. The resulting orbits of consistent gaugings in this case are presented in table 5.3.3.

Higher-dimensional geometric origin

The four different orbits of maximal $D = 9$ theory have the following higher-dimensional origin in terms of geometric compactifications [228]:

⁵The \mathbb{R}^+ factor in the global symmetry is precisely the combination $(\frac{4}{3}\alpha - \frac{3}{2}\delta)$ of the different rescalings introduced in ref. [117].

- **Orbits 1 – 3:** These come from reductions of type IIB supergravity on a circle with an $\text{SL}(2)$ twist.
- **Orbit 4:** This can be obtained from a reduction of type IIA supergravity on a circle with the inclusion of an \mathbb{R}_β^+ twist.

Orbits and origin of the $D = 8$ maximal case

Maximal $D = 8$ gauged supergravity

The maximal (ungauged) supergravity in $D = 8$ [229] can be obtained by reducing eleven-dimensional supergravity on a T^3 . The global symmetry group of this theory is

$$G_0 = \text{SL}(2) \times \text{SL}(3) .$$

The full field content consists of the following objects which arrange themselves into irrep's of $\text{SL}(2) \times \text{SL}(3)$:

$$8D : \quad \underbrace{e_\mu^a, A_\mu^{\alpha m}, B_{\mu\nu m}, C_{\mu\nu\rho}, L_m^I, \phi, \chi}_{\text{bosonic dof's}} ; \underbrace{\psi_\mu, \chi_I}_{\text{fermionic dof's}} , \quad (5.84)$$

where μ, ν, \dots denote eight-dimensional curved spacetime, a, b, \dots eight-dimensional flat spacetime, m, n, \dots fundamental $\text{SL}(3)$, I, J, \dots fundamental $\text{SO}(3)$ and α, β, \dots fundamental $\text{SL}(2)$ indices respectively. The six vector fields $A_\mu^{\alpha m}$ in (5.84) transform in the $V' = (\mathbf{2}, \mathbf{3}')$. There are eleven group generators, which can be expressed in the adjoint representation \mathfrak{g}_0 .

The embedding tensor Θ then lives in the representation $\mathfrak{g}_0 \otimes V$, which can be decomposed into irreducible representations as

$$\mathfrak{g}_0 \otimes V = 2 \cdot (\mathbf{2}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{6}') \oplus (\mathbf{2}, \mathbf{15}) \oplus (\mathbf{4}, \mathbf{3}) . \quad (5.85)$$

The LC restricts the embedding tensor to the $(\mathbf{2}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{6}')$ [170]. It is worth noticing that there are two copies of the $(\mathbf{2}, \mathbf{3})$ irrep in the above composition; the LC imposes a relation between them [171]. This shows that, for consistency, gauging some $\text{SL}(2)$ generators implies the necessity of gauging some $\text{SL}(3)$ generators as well. Let us denote the allowed embedding tensor irrep's by $\xi_{\alpha m}$ and $f_\alpha^{(mn)}$ respectively.

The quadratic constraints (QC) then read [217, 230]

$$\epsilon^{\alpha\beta} \xi_{\alpha p} \xi_{\beta q} = 0 , \quad (\mathbf{1}, \mathbf{3}') \quad (5.86)$$

$$f_{(\alpha}{}^{np} \xi_{\beta)p} = 0 , \quad (\mathbf{3}, \mathbf{3}') \quad (5.87)$$

$$\epsilon^{\alpha\beta} (\epsilon_{mqr} f_\alpha{}^{qn} f_\beta{}^{rp} + f_\alpha{}^{np} \xi_{\beta m}) = 0 . \quad (\mathbf{1}, \mathbf{3}') \oplus (\mathbf{1}, \mathbf{15}) \quad (5.88)$$

Any solution to the QC (5.86), (5.87) and (5.88) specifies a consistent gauging of a subgroup of $\text{SL}(2) \times \text{SL}(3)$ where the corresponding generators are given by

$$(X_{\alpha m})_\beta{}^\gamma = \delta_\alpha^\gamma \xi_{\beta m} - \frac{1}{2} \delta_\beta^\gamma \xi_{\alpha m} , \quad (5.89)$$

$$(X_{\alpha m})_n{}^p = \epsilon_{mnq} f_\alpha{}^{qp} - \frac{3}{4} \left(\delta_m^p \xi_{\alpha n} - \frac{1}{3} \delta_n^p \xi_{\alpha m} \right) . \quad (5.90)$$

ID	f_+^{mn}	f_-^{mn}	ξ_{+m}	ξ_{-m}	gauging
1	diag(1, 1, 1)				SO(3)
2	diag(1, 1, -1)				SO(2, 1)
3	diag(1, 1, 0)	diag(0, 0, 0)	(0, 0, 0)	(0, 0, 0)	ISO(2)
4	diag(1, -1, 0)				ISO(1, 1)
5	diag(1, 0, 0)				CSO(1, 0, 2)
6	diag(0, 0, 0)	diag(0, 0, 0)	(1, 0, 0)	(0, 0, 0)	Solv ₂ × Solv ₃
7	diag(1, 1, 0)				
8	diag(1, -1, 0)	diag(0, 0, 0)	(0, 0, 1)	(0, 0, 0)	Solv ₂ × Solv ₃
9	diag(1, 0, 0)				
10	diag(1, -1, 0)	$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\frac{2}{9}(0, 0, 1)$	(0, 0, 0)	Solv ₂ × SO(2) × Nil ₃ (2)

Table 5.4: All the U-duality orbits of consistent gaugings in maximal supergravity in $D = 8$. For each of them, the simplest representative is given. We denote by $Solv_2 \subset SL(2)$ and $Solv_3 \subset SL(3)$ a solvable algebra of dimension 2 and 3 respectively. To be more precise, $Solv_2$ identifies the Borel subgroup of $SL(2)$ consisting of 2×2 upper-triangular matrices. $Solv_3$, instead, is a Bianchi type V algebra.

The $SL(2) \times SL(3)$ orbits of solutions to the QC

We exploited an algebraic geometry tool called the Gianni-Trager-Zacharias (GTZ) algorithm [231]. This algorithm has been computationally implemented by the SINGULAR project [232] and it consists in the primary decomposition of ideals of polynomials. After finding all the solutions to the QC by means of the algorithm mentioned above, one has to group together all the solutions which are connected through a duality transformation, thus obtaining a classification of such solutions in terms of duality orbits. The resulting orbits of consistent gaugings⁶ in this case are presented in table 5.3.3.

Higher-dimensional geometric origin

- **Orbits 1 – 5:** These stem from reductions of eleven-dimensional supergravity on a three-dimensional group manifold of type A in the Bianchi classification [233]. The special case in orbit 1 corresponds to a reduction over an SO(3) group manifold and it was already studied in ref. [229].
- **Orbit 6:** This can be obtained from a reduction of maximal nine-dimensional supergravity on a circle with the inclusion of an \mathbb{R}^+ twist inside the global symmetry group.

⁶Recently, also the possible vacua of the different theories have been analysed [217]. It was found that only **orbit 3** has maximally symmetric vacua.

- **Orbits 7 – 9:** These can come from the same reduction from $D = 9$ but upon inclusion of a more general $\mathbb{R}^+ \times \text{SL}(2)$ twist.
- **Orbit 10:** This orbit seems at first sight more complicated to be obtained from a dimensional reduction owing to its non-trivial $\text{SL}(2)$ angles. Nevertheless, it turns out that one can land on this orbit by compactifying type IIB supergravity on a circle with an $\text{SL}(2)$ twist and then further reducing on another circle with $\mathbb{R}^+ \times \text{SL}(2)$ twist given by the residual little group leaving invariant the intermediate nine-dimensional deformation.

Remarks on the $D = 7$ maximal case

The general deformations of the maximal theory in $D = 7$ are constructed and presented in full detail in ref. [178]. For the present aim we only summarise here a few relevant facts.

The global symmetry group of the theory is $\text{SL}(5)$. The vector fields $A_\mu^{MN} = A_\mu^{[MN]}$ transform in the $\mathbf{10}'$ of $\text{SL}(5)$, where we denote by M a fundamental $\text{SL}(5)$ index. The embedding tensor Θ takes values in the following irreducible components

$$\mathbf{10} \otimes \mathbf{24} = \mathbf{10} \oplus \mathbf{15} \oplus \mathbf{40}' \oplus \mathbf{175}. \quad (5.91)$$

The LC restricts the embedding tensor to the $\mathbf{15} \oplus \mathbf{40}'$, which can be parameterised by the following objects

$$Y_{(MN)}, \quad \text{and} \quad Z^{[MN],P} \quad \text{with} \quad Z^{[MN,P]} = 0. \quad (5.92)$$

The generators of the gauge algebra can be written as follows

$$(X_{MN})_P{}^Q = \delta_{[M}^Q Y_{N]P} - 2 \epsilon_{MNP RS} Z^{RS,Q}, \quad (5.93)$$

or, identically, if one wants to express them in the $\mathbf{10}$,

$$(X_{MN})_{PQ}{}^{RS} = 2 (X_{MN})_{[P}{}^{[R} \delta_{Q]}^S]. \quad (5.94)$$

The closure of the gauge algebra and the antisymmetry of the brackets imply the following QC

$$Y_{MQ} Z^{QN,P} + 2 \epsilon_{MRSTU} Z^{RS,N} Z^{TU,P} = 0, \quad (5.95)$$

which have different irreducible pieces in the $\mathbf{5}' \oplus \mathbf{45}' \oplus \mathbf{70}'$. Unfortunately, in this case, both the embedding tensor deformations and the quadratic constraints reach a level of complexity that makes an exhaustive and general analysis difficult. Such analysis lies beyond the scope of our work.

5.3.4 T-duality orbits of half-maximal supergravities

After the previous section on maximal supergravities, we turn our attention to theories with half-maximal supersymmetry. In particular, in this section we will classify the orbits under T-duality of all gaugings of half-maximal supergravity. We will only consider the theories with duality groups $\mathbb{R}^+ \times \text{SO}(d, d)$ in $D = 10 - d$, which places a restriction on the number of vector multiplets. For these theories we will classify all duality orbits, and find a number of non-geometric orbits. Furthermore, we demonstrate that double field theory does yield a higher-dimensional origin for all of them.

Starting from $D = 10$ half-maximal supergravity without vector multiplets, it can be seen that there is no freedom to deform this theory, rendering this case trivial. In $D = 9$, instead, we have the possibility of performing an Abelian gauging inside $\mathbb{R}^+ \times \text{SO}(1, 1)$, which will depend on one deformation parameter. However, this is precisely the parameter that one expects to generate by means of a twisted reduction from $D = 10$. This immediately tells us that non-geometric fluxes do not yet appear in this theory. In order to find the first non-trivial case, we will have to consider the $D = 8$ case.

Orbits and origin of the $D = 8$ half-maximal case

Half-maximal $D = 8$ gauged supergravity

Half-maximal supergravity in $D = 8$ is related to the maximal theory analysed in the previous section by means of a \mathbb{Z}_2 truncation. The action of such a \mathbb{Z}_2 breaks $\text{SL}(2) \times \text{SL}(3)$ into $\mathbb{R}^+ \times \text{SL}(2) \times \text{SL}(2)$, where $\text{SL}(2) \times \text{SL}(2) = \text{O}(2, 2)$ can be interpreted as the T-duality group in $D = 8$ as shown in table 5.1. The embedding of $\mathbb{R}^+ \times \text{SL}(2)$ inside $\text{SL}(3)$ is unique and it determines the following branching of the fundamental representation

$$\begin{aligned} \mathbf{3} &\longrightarrow \mathbf{1}_{(+2)} \oplus \mathbf{2}_{(-1)} , \\ m &\longrightarrow (\bullet, i) , \end{aligned}$$

where the \mathbb{R}^+ direction labeled by \bullet is parity even, whereas i is parity odd, such as the other $\text{SL}(2)$ index α . In the following we will omit all the \mathbb{R}^+ weights since they do not play any role in the truncation.

The embedding tensor of the maximal theory splits in the following way

$$\begin{aligned} (\mathbf{2}, \mathbf{3}) &\longrightarrow \cancel{(\mathbf{2}, \mathbf{1})} \oplus (\mathbf{2}, \mathbf{2}) , \\ (\mathbf{2}, \mathbf{6}') &\longrightarrow \cancel{(\mathbf{2}, \mathbf{1})} \oplus (\mathbf{2}, \mathbf{2}) \oplus \cancel{(\mathbf{2}, \mathbf{3})} , \end{aligned}$$

where all the crossed irrep's are projected out because of \mathbb{Z}_2 parity. This implies that the consistent embedding tensor deformations of the half-maximal theory can be described by two objects which are doublets with respect to both $\text{SL}(2)$'s. Let us denote them by $a_{\alpha i}$ and $b_{\alpha i}$. This statement is in perfect agreement with the Kac-Moody analysis

performed in ref. [181]. The explicit way of embedding $a_{\alpha i}$ and $b_{\alpha i}$ inside $\xi_{\alpha m}$ and $f_{\alpha}{}^{mn}$ is given by

$$f_{\alpha}{}^{i\bullet} = f_{\alpha}{}^{\bullet i} = \epsilon^{ij} a_{\alpha j} , \quad (5.96)$$

$$\xi_{\alpha i} = 4b_{\alpha i} . \quad (5.97)$$

The QC given in (5.86), (5.87) and (5.88) are decomposed according to the following branching

$$\begin{aligned} (\mathbf{1}, \mathbf{3}') &\longrightarrow (\mathbf{1}, \mathbf{1}) \oplus \cancel{(\mathbf{1}, \mathbf{2})} , \\ (\mathbf{3}, \mathbf{3}') &\longrightarrow (\mathbf{3}, \mathbf{1}) \oplus \cancel{(\mathbf{3}, \mathbf{2})} , \\ (\mathbf{1}, \mathbf{15}) &\longrightarrow (\mathbf{1}, \mathbf{1}) \oplus \cancel{2 \cdot (\mathbf{1}, \mathbf{2})} \oplus 2 \cdot (\mathbf{1}, \mathbf{3}) \oplus \cancel{(\mathbf{1}, \mathbf{4})} . \end{aligned}$$

As a consequence, one expects the set of \mathbb{Z}_2 even QC to consist of 3 singlets, a $(\mathbf{3}, \mathbf{1})$ and 2 copies of the $(\mathbf{1}, \mathbf{3})$. By plugging (5.96) and (5.97) into (5.86), (5.87) and (5.88), one finds

$$\epsilon^{\alpha\beta} \epsilon^{ij} b_{\alpha i} b_{\beta j} = 0 , \quad (\mathbf{1}, \mathbf{1}) \quad (5.98)$$

$$\epsilon^{\alpha\beta} \epsilon^{ij} a_{\alpha i} b_{\beta j} = 0 , \quad (\mathbf{1}, \mathbf{1}) \quad (5.99)$$

$$\epsilon^{\alpha\beta} \epsilon^{ij} a_{\alpha i} a_{\beta j} = 0 , \quad (\mathbf{1}, \mathbf{1}) \quad (5.100)$$

$$\epsilon^{ij} a_{(\alpha i} b_{\beta)j} = 0 , \quad (\mathbf{3}, \mathbf{1}) \quad (5.101)$$

$$\epsilon^{\alpha\beta} a_{\alpha(i} b_{\beta j)} = 0 . \quad (\mathbf{1}, \mathbf{3}) \quad (5.102)$$

With respect to what we expected from group theory, we seem to be finding a $(\mathbf{1}, \mathbf{3})$ less amongst the even QC. This could be due to the fact that \mathbb{Z}_2 even QC can be sourced by quadratic expressions in the odd embedding tensor components that we truncated away. After the procedure of turning off all of them, the two $(\mathbf{1}, \mathbf{3})$'s probably collapse to the same constraint or one of them vanishes directly.

The above set of QC characterises the consistent gaugings of the half-maximal theory which are liftable to the maximal theory, and hence they are more restrictive than the pure consistency requirements of the half-maximal theory. In order to single out only these we need to write down the expression of the gauge generators and impose the closure of the algebra. The gauge generators in the $(\mathbf{2}, \mathbf{2})$ read

$$(X_{\alpha i})_{\beta j}{}^{\gamma k} = \frac{1}{2} \delta_{\beta}^{\gamma} \epsilon_{ij} \epsilon^{kl} a_{\alpha l} + \delta_{\alpha}^{\gamma} \delta_j^k b_{\beta i} - \frac{3}{2} \delta_{\beta}^{\gamma} \delta_i^k b_{\alpha j} + \frac{1}{2} \delta_{\beta}^{\gamma} \delta_j^k b_{\alpha i} + \epsilon_{\alpha\beta} \epsilon^{\gamma\delta} \delta_j^k b_{\delta i} . \quad (5.103)$$

The closure of the algebra generated by (5.103) implies the following QC

$$\epsilon^{\alpha\beta} \epsilon^{ij} (a_{\alpha i} a_{\beta j} - b_{\alpha i} b_{\beta j}) = 0 , \quad (\mathbf{1}, \mathbf{1}) \quad (5.104)$$

$$\epsilon^{\alpha\beta} \epsilon^{ij} (a_{\alpha i} b_{\beta j} + b_{\alpha i} b_{\beta j}) = 0 , \quad (\mathbf{1}, \mathbf{1}) \quad (5.105)$$

$$\epsilon^{ij} a_{(\alpha i} b_{\beta)j} = 0 , \quad (\mathbf{3}, \mathbf{1}) \quad (5.106)$$

$$\epsilon^{\alpha\beta} a_{\alpha(i} b_{\beta j)} = 0 . \quad (\mathbf{1}, \mathbf{3}) \quad (5.107)$$

To facilitate the mapping of gaugings $a_{\alpha i}$ and $b_{\alpha i}$ with the more familiar f_{ABC} and ξ_A in the DFT language, we have written a special section in the appendix C.2. The mapping is explicitly given in (C.16).

ID	$a_{\alpha i}$	$b_{\alpha i}$	gauging
1	$\text{diag}(\cos \alpha, 0)$	$\text{diag}(\sin \alpha, 0)$	$\text{Solv}_2 \times \text{SO}(1, 1)$
2	$\text{diag}(1, 1)$	$\text{diag}(-1, -1)$	$\text{SL}(2) \times \text{SO}(1, 1)$
3	$\text{diag}(1, -1)$	$\text{diag}(-1, 1)$	

Table 5.5: All the T-duality orbits of consistent gaugings in half-maximal supergravity in $D = 8$. For each of them, the simplest representative is given. Solv_2 refers again to the solvable subgroup of $\text{SL}(2)$ as already explained in the caption of table 5.3.3.

The $\text{O}(2, 2)$ orbits of solutions to the QC

After solving the QC given in (5.104), (5.105), (5.106) and (5.107) again with the aid of SINGULAR, we find a 1-parameter family of T-duality orbits plus two discrete ones. The results are all collected in table 5.5.

Higher-dimensional geometric origin

The possible higher-dimensional origin of the three different orbits is as follows:

- **Orbit 1:** This orbit can be obtained by performing a two-step reduction of type I supergravity. In the first step, by reducing a circle, we can generate an $\mathbb{R}^+ \times \text{SO}(1, 1)$ gauging of half-maximal $D = 9$ supergravity. Subsequently, we reduce such a theory again on a circle with the inclusion of a new twist commuting with the previous deformation. Also, these orbits include a non-trivial ξ_A gauging, so we will not address it from a DFT perspective.
- **Orbits 2 – 3:** These do not seem to have any obvious geometric higher-dimensional origin in supergravity. In fact, they do not satisfy the extra constraints (5.74), so one can only hope to reproduce them from truly doubled twist orbits in DFT.

Therefore we find that, while the half-maximal orbits in $D = 9$ all have a known geometric higher-dimensional origin, this is not the case for the latter two orbits in $D = 8$. We have finally detected the first signals of non-geometric orbits.

Higher-dimensional DFT origin

As mentioned, the **orbits 2** and **3** lack of a clear higher-dimensional origin. Here we would like to provide a particular twist matrix giving rise to these gaugings. We chose to start in the cartesian framework, and propose the following form for the $\text{SO}(2, 2)$ twist matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(m y^1 + n \tilde{y}_1) & 0 & \sinh(m y^1 + n \tilde{y}_1) \\ 0 & 0 & 1 & 0 \\ 0 & \sinh(m y^1 + n \tilde{y}_1) & 0 & \cosh(m y^1 + n \tilde{y}_1) \end{pmatrix}. \quad (5.108)$$

This is in fact an element of $SO(1,1)$ lying in the directions (\tilde{y}_2, y^2) , fibred over the double torus (\tilde{y}_1, y^1) . Here, the coordinates are written in the cartesian formulation, so we must rotate this in order to make contact with the light-cone case.

For this twist matrix, the weak and strong constraints in the light-cone formulation read $(m+n)(m-n) = 0$, while the QC are always satisfied. The gaugings are constant, and when written in terms of $a_{\alpha i}$ and $b_{\alpha i}$ we find

$$a_{\alpha i} = -b_{\alpha i} = \text{diag} \left(-\frac{m+n}{2\sqrt{2}}, \frac{m-n}{2\sqrt{2}} \right), \quad (5.109)$$

so **orbit 2** is obtained by choosing $m = 0$, $n = -2\sqrt{2}$, and **orbit 3** by choosing $m = -2\sqrt{2}$, $n = 0$. Notice that in both cases the twist orbit is truly doubled, so we find the first example of an orbit of gaugings without a clear supergravity origin, that finds an uplift to DFT in a truly doubled background.

Orbits and origin of the $D = 7$ half-maximal case

Half-maximal $D = 7$ gauged supergravity

A subset of half-maximal gauged supergravities is obtained from the maximal theory introduced in section 5.3.3 by means of a \mathbb{Z}_2 truncation. Thus, we will in this section perform this truncation and carry out the orbit analysis in the half-maximal theory. As we already argued before, this case is not only simpler, but also much more insightful from the point of view of understanding T-duality in gauged supergravities and its relation to DFT.

The action of our \mathbb{Z}_2 breaks⁷ $SL(5)$ into $\mathbb{R}^+ \times SL(4)$. Its embedding inside $SL(5)$ is unique and it is such that the fundamental representation splits as follows

$$\mathbf{5} \longrightarrow \mathbf{1}_{(+4)} \oplus \mathbf{4}_{(-1)}. \quad (5.110)$$

After introducing the following notation for the indices in the \mathbb{R}^+ and in the $SL(4)$ directions

$$M \longrightarrow (\diamond, m), \quad (5.111)$$

we assign an even parity to the \diamond direction and odd parity to m directions.

The embedding tensor of the maximal theory splits according to

$$\mathbf{15} \longrightarrow \mathbf{1} \oplus \mathbf{4} \oplus \mathbf{10}, \quad (5.112)$$

$$\mathbf{40}' \longrightarrow \mathbf{4}' \oplus \mathbf{6} \oplus \mathbf{10}' \oplus \mathbf{20}, \quad (5.113)$$

⁷The \mathbb{Z}_2 element with respect to which we are truncating is the following $USp(4) = SO(5)$ element

$$\alpha = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}$$

projecting out half of the supercharges.

where again, as in section 5.3.4, all the crossed irrep's are projected out because of \mathbb{Z}_2 parity. This implies that the embedding tensor of the half-maximal theory lives in the $\mathbf{1} \oplus \mathbf{6} \oplus \mathbf{10} \oplus \mathbf{10}'$ and hence it is described by the following objects

$$\theta, \xi_{[mn]}, M_{(mn)}, \tilde{M}^{(mn)}. \quad (5.114)$$

This set of deformations agrees with the decomposition $D_8^{+++} \rightarrow A_3 \times A_6$ given in ref. [181]. The objects in (5.114) are embedded in Y and Z in the following way

$$Y_{\diamond\diamond} = \theta, \quad (5.115)$$

$$Y_{mn} = \frac{1}{2} M_{mn}, \quad (5.116)$$

$$Z^{mn,\diamond} = \frac{1}{8} \xi^{mn}, \quad (5.117)$$

$$Z^{m\diamond,n} = -Z^{\diamond m,n} = \frac{1}{16} \tilde{M}^{mn} + \frac{1}{16} \xi^{mn}, \quad (5.118)$$

where for convenience we defined $\xi^{mn} = \frac{1}{2} \epsilon^{mnpq} \xi_{pq}$.

Now we will obtain the expression of the gauge generators of the half-maximal theory by plugging the expressions (5.115) – (5.118) into (5.93). We find

$$(X_{mn})_p{}^q = \frac{1}{2} \delta_{[m}^q M_{n]p} - \frac{1}{4} \epsilon_{mnp r} \left(\tilde{M} + \xi \right)^{r q}, \quad (5.119)$$

which extends the expression given in ref. [234] by adding an antisymmetric part to \tilde{M} proportional to ξ . Note that the ξ term is also the only one responsible for the trace of the gauge generators which has to be non-vanishing in order to account for \mathbb{R}^+ gaugings.

The presence of such a term in the expression (5.119) has another consequence: the associated structure constants that one writes by expressing the generators in the $\mathbf{6}$ $(X_{mn})_{pq}{}^{rs}$ will not be automatically antisymmetric in the exchange between mn and pq . This implies the necessity of imposing the antisymmetry by means of some extra QC⁸.

The QC of the maximal theory are branched into

$$\mathbf{5}' \longrightarrow \mathbf{1} \oplus \cancel{\mathbf{4}}, \quad (5.120)$$

$$\mathbf{45}' \longrightarrow \cancel{\mathbf{4}} \oplus \mathbf{6} \oplus \mathbf{15} \oplus \mathbf{20}, \quad (5.121)$$

$$\mathbf{70}' \longrightarrow \mathbf{1} \oplus \cancel{\mathbf{4}} \oplus \cancel{\mathbf{4}} \oplus \mathbf{10}' \oplus \mathbf{15} \oplus \cancel{\mathbf{36}}. \quad (5.122)$$

⁸The QC which ensure the antisymmetry of the gauge brackets are given by $(X_{mn})_{pq}{}^{rs} X_{rs} + (mn \leftrightarrow pq) = 0$, where X is given in an arbitrary representation.

By substituting the expressions (5.115) – (5.118) into the QC (5.95), one finds

$$\theta \xi_{mn} = 0, \quad (\mathbf{6}) \quad (5.123)$$

$$\left(\tilde{M}^{mp} + \xi^{mp} \right) M_{pq} = 0, \quad (\mathbf{1} \oplus \mathbf{15}) \quad (5.124)$$

$$M_{mp} \xi^{pn} - \xi_{mp} \left(\tilde{M}^{pn} + \xi^{pn} \right) = 0, \quad (\mathbf{1} \oplus \mathbf{15}) \quad (5.125)$$

$$\theta \tilde{M}^{mn} = 0. \quad (\mathbf{10}') \quad (5.126)$$

Based on the Kac-Moody analysis performed in ref. [181], the QC constraints of the half-maximal theory should only impose conditions living in the $\mathbf{1} \oplus \mathbf{6} \oplus \mathbf{15} \oplus \mathbf{15}$. The problem is then determining which constraint in the $\mathbf{1}$ is already required by the half-maximal theory and which is not.

By looking more carefully at the constraints (5.123) – (5.126), we realise that the traceless part of (5.124) exactly corresponds to the Jacobi identities that one gets from the closure of the algebra spanned by the generators (5.119), whereas the full (5.125) has to be imposed to ensure antisymmetry of the gauge brackets. Since there is only one constraint in the $\mathbf{6}$, we do not have ambiguities there⁹.

We are now able to write down the set of QC of the half-maximal theory:

$$\theta \xi_{mn} = 0, \quad (\mathbf{6}) \quad (5.127)$$

$$\left(\tilde{M}^{mp} + \xi^{mp} \right) M_{pq} - \frac{1}{4} \left(\tilde{M}^{np} M_{np} \right) \delta_q^m = 0, \quad (\mathbf{15}) \quad (5.128)$$

$$M_{mp} \xi^{pn} + \xi_{mp} \tilde{M}^{pn} = 0, \quad (\mathbf{15}) \quad (5.129)$$

$$\epsilon^{mnpq} \xi_{mn} \xi_{pq} = 0. \quad (\mathbf{1}) \quad (5.130)$$

We are not really able to confirm whether (5.127) is part of the QC of the half-maximal theory, in the sense that there appears a top-form in the $\mathbf{6}$ from the D_8^{+++} decomposition but it could either be a tadpole or a QC. This will however not affect our further discussion, in that we only consider orbits of gaugings in which $\theta = 0$. The extra QC required in order for the gauging to admit an uplift to maximal supergravity are

$$\tilde{M}^{mn} M_{mn} = 0, \quad (\mathbf{1}) \quad (5.131)$$

$$\theta \tilde{M}^{mn} = 0. \quad (\mathbf{10}') \quad (5.132)$$

The $O(3,3)$ orbits of solutions to the QC in the $\mathbf{10} \oplus \mathbf{10}'$

The aim of this section is to solve the constraints summarised in (5.127), (5.128), (5.129) and (5.130). We will start by considering the case of gaugings only involving

⁹We would like to stress that the parameter θ within the half-maximal theory is a consistent deformation, but it does not correspond to any gauging and hence QC involving it cannot be derived as Jacobi identities or other consistency constraints coming from the gauge algebra.

ID	$M_{mn}/\cos\alpha$	$\tilde{M}^{mn}/\sin\alpha$	range of α	gauging
1	diag(1, 1, 1, 1)	diag(1, 1, 1, 1)	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \text{SO}(4), & \alpha \neq \frac{\pi}{4}, \\ \text{SO}(3), & \alpha = \frac{\pi}{4}. \end{cases}$
2	diag(1, 1, 1, -1)	diag(1, 1, 1, -1)	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	SO(3, 1)
3	diag(1, 1, -1, -1)	diag(1, 1, -1, -1)	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \text{SO}(2,2), & \alpha \neq \frac{\pi}{4}, \\ \text{SO}(2,1), & \alpha = \frac{\pi}{4}. \end{cases}$
4	diag(1, 1, 1, 0)	diag(0, 0, 0, 1)	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	ISO(3)
5	diag(1, 1, -1, 0)	diag(0, 0, 0, 1)	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	ISO(2, 1)
6	diag(1, 1, 0, 0)	diag(0, 0, 1, 1)	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \text{CSO}(2, 0, 2), & \alpha \neq \frac{\pi}{4}, \\ \mathfrak{f}_1 \text{ (Solv}_6), & \alpha = \frac{\pi}{4}. \end{cases}$
7	diag(1, 1, 0, 0)	diag(0, 0, 1, -1)	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\begin{cases} \text{CSO}(2, 0, 2), & \alpha < \frac{\pi}{4}, \\ \text{CSO}(1, 1, 2), & \alpha > \frac{\pi}{4}, \\ \mathfrak{g}_0 \text{ (Solv}_6), & \alpha = \frac{\pi}{4}. \end{cases}$
8	diag(1, 1, 0, 0)	diag(0, 0, 0, 1)	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\mathfrak{h}_1 \text{ (Solv}_6)$
9	diag(1, -1, 0, 0)	diag(0, 0, 1, -1)	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \text{CSO}(1, 1, 2), & \alpha \neq \frac{\pi}{4}, \\ \mathfrak{f}_2 \text{ (Solv}_6), & \alpha = \frac{\pi}{4}. \end{cases}$
10	diag(1, -1, 0, 0)	diag(0, 0, 0, 1)	$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$	$\mathfrak{h}_2 \text{ (Solv}_6)$
11	diag(1, 0, 0, 0)	diag(0, 0, 0, 1)	$-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$	$\begin{cases} \mathfrak{l} \text{ (Nil}_6(3)), & \alpha \neq 0, \\ \text{CSO}(1, 0, 3), & \alpha = 0. \end{cases}$

Table 5.6: All the T-duality orbits of consistent gaugings in half-maximal supergravity in $D = 7$. Any value of α parameterises inequivalent orbits. More details about the non-semisimple gauge algebras \mathfrak{f}_1 , \mathfrak{f}_2 , \mathfrak{h}_1 , \mathfrak{h}_2 , \mathfrak{g}_0 and \mathfrak{l} are given in appendix C.1.

the $\mathbf{10} \oplus \mathbf{10}'$. This restriction is motivated by flux compactification, as we will try to argue later on.

The only non-trivial QC are the following

$$\tilde{M}^{mp} M_{pn} - \frac{1}{4} \left(\tilde{M}^{pq} M_{pq} \right) \delta_n^m = 0, \quad (5.133)$$

which basically implies that the matrix product between M and \tilde{M} , which in principle lives in the $\mathbf{1} \oplus \mathbf{15}$, has to be pure trace. We made use of a GL(4) transformation in order to reduce M to pure signature; as a consequence, the QC (5.133) imply that \tilde{M} is diagonal as well [235]. This results in a set of eleven 1-parameter orbits¹⁰ of solutions to the QC which are given in table 5.6.

As we will see later, some of these consistent gaugings in general include non-zero non-geometric fluxes, but at least in some of these cases one will be able to dualise the

¹⁰We would like to point out that the extra discrete generator η of $O(3, 3)$ makes sure that, given a certain gauging with M and \tilde{M} , it lies in the same orbit as its partner with the role of M and $-\tilde{M}$ interchanged.

given configuration to a perfectly geometric background.

Higher-dimensional geometric origin

Ten-dimensional heterotic string theory compactified on a T^3 gives rise to a half-maximal supergravity in $D = 7$ where the $\text{SL}(4) = \text{SO}(3, 3)$ factor in the global symmetry of this theory can be interpreted as the T-duality group. The set of generalized fluxes which can be turned on here is given by

$$\{f_{abc}, f_{ab}{}^c, f_a{}^{bc}, f^{abc}\} \equiv \{H_{abc}, \omega_{ab}{}^c, Q_a{}^{bc}, R^{abc}\}, \quad (5.134)$$

where $a, b, c = 1, 2, 3$.

These are exactly the objects that one obtains by decomposing a three-form of $\text{SO}(3, 3)$ with respect to its $\text{GL}(3)$ subgroup. The number of independent components of the above fluxes (including traces of ω and Q) amounts to $1+9+9+1 = 20$, which is the number of independent components of a three-form of $\text{SO}(3, 3)$. Nevertheless, the three-form representation is not irreducible since the Hodge duality operator in $3+3$ dimensions squares to 1. This implies that one can always decompose it in a self-dual (SD) and anti-self-dual (ASD) part

$$\mathbf{10} \oplus \mathbf{10}' \text{ of } \text{SL}(4) \quad \longleftrightarrow \quad \mathbf{10}_{\text{SD}} \oplus \mathbf{10}_{\text{ASD}} \text{ of } \text{SO}(3, 3), \quad (5.135)$$

such that the matching between the embedding tensor deformations (M_{mn}, \tilde{M}^{mn}) and the generalized fluxes given in (5.134) now perfectly works. The explicit mapping between vectors of $\text{SO}(3, 3)$ expressed in light-cone coordinates and two-forms of $\text{SL}(4)$ can be worked out by means of the $\text{SO}(3, 3)$ 't Hooft symbols $(G_A)^{mn}$ (see Appendix C.2). This gives rise to the following dictionary between the M and \tilde{M} -components and the fluxes given in (5.134)

$$M = \text{diag} (H_{123}, Q_1{}^{23}, Q_2{}^{31}, Q_3{}^{12}) , \quad \tilde{M} = \text{diag} (R^{123}, \omega_{23}{}^1, \omega_{31}{}^2, \omega_{12}{}^3) . \quad (5.136)$$

The QC given in equations (5.127)-(5.130) enjoy a symmetry in the exchange

$$(M, \xi) \xrightarrow{\eta} (-\tilde{M}, -\xi) . \quad (5.137)$$

The discrete \mathbb{Z}_2 transformation η corresponds to the following $\text{O}(3, 3)$ element with determinant -1

$$\eta = \begin{pmatrix} 0 & \mathbf{1}_3 \\ \mathbf{1}_3 & 0 \end{pmatrix}, \quad (5.138)$$

which can be interpreted as a triple T-duality exchanging the three compact coordinates y^a with the corresponding winding coordinates \tilde{y}_a in the language of DFT.

Now we have all the elements to analyze the higher dimensional origin of the orbits classified in table 5.6.

- **Orbits 1 – 3:** These gaugings are non-geometric for every $\alpha \neq 0$; for $\alpha = 0$, they correspond to coset reductions of heterotic string theory. See *e.g.* the S^3 compactification in ref. [236] giving rise to the $SO(4)$ gauging. This theory was previously obtained in ref. [237] as $\mathcal{N} = 2$ truncation of a maximal supergravity in $D = 7$.
- **Orbits 4 – 5:** For any value of α we can always dualise these representatives to the one obtained by means of a twisted T^3 reduction with H and ω fluxes.
- **Orbits 6 – 7:** For any $\alpha \neq 0$ these orbits could be obtained from supergravity compactifications on locally-geometric T-folds, whereas for $\alpha = 0$ it falls again in a special case of the reductions described for orbits 4 and 5.
- **Orbits 8 – 11:** For any value of α , these orbits always contain a geometric representative involving less general H and ω fluxes.

To summarise, in the half-maximal $D = 7$ case, we encounter a number of orbits which do not have an obvious higher-dimensional origin. To be more precise, these are orbits 1, 2 and 3 for $\alpha \neq 0$. The challenge in the next subsection will be to establish what DFT can do for us in order to give these orbits a higher-dimensional origin. Again, before reading the following subsections we refer to the section 5.3.2 for a discussion of what we mean by light-cone and cartesian formulations.

Higher-dimensional DFT origin

First of all we would like to show here how to capture the gaugings that only involve (up to duality rotations) fluxes H_{abc} and $\omega_{ab}{}^c$. For this, we start from the light-cone formulation, and propose the following Ansatz for a *globally geometric twist* (involving e and B and physical coordinates y)

$$e = \begin{pmatrix} 1 & 0 & \frac{\omega_1}{\omega_3} \sin(\omega_1 \omega_3 y^2) \\ 0 & \cos(\omega_2 \omega_3 y^1) & -\frac{\omega_2}{\omega_3} \cos(\omega_1 \omega_3 y^2) \sin(\omega_2 \omega_3 y^1) \\ 0 & \frac{\omega_3}{\omega_2} \sin(\omega_2 \omega_3 y^1) & \cos(\omega_1 \omega_3 y^2) \cos(\omega_2 \omega_3 y^1) \end{pmatrix}, \quad (5.139)$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & H y^1 \cos(\omega_1 \omega_3 y^2) \\ 0 & -H y^1 \cos(\omega_1 \omega_3 y^2) & 0 \end{pmatrix}, \quad (5.140)$$

$$\lambda = -\frac{1}{2} \log(\cos(\omega_1 \omega_3 y^2)). \quad (5.141)$$

This is far from being the most general ansatz, but it serves our purposes of reaching a large family of geometric orbits. The parameters ω_i can be real, vanishing or imaginary, since U is real and well-behaved in these cases. The QC, weak and strong constraints are all automatically satisfied, and the gaugings read

$$M = \text{diag}(H, 0, 0, 0), \quad \tilde{M} = \text{diag}(0, \omega_1^2, \omega_2^2, \omega_3^2). \quad (5.142)$$

From here, by choosing appropriate values of the parameters the **orbits 4, 5, 8, 10** and **11** can be obtained. Indeed these are geometric as they only involve gauge and (geo)metric fluxes.

Secondly, in order to address the remaining orbits, we consider an $\text{SO}(2,2)$ twist U_4 embedded in $\text{O}(3,3)$ in the following way

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix}, \quad U_4 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \lambda = 0. \quad (5.143)$$

This situation is analog to the $\text{SO}(1,1)$ twist considered in the $D = 8$ case, but with a more general twist. Working in the cartesian formulation, one can define the generators and elements of $\text{SO}(2,2)$ as

$$[t_{IJ}]_K{}^L = \delta_{[I}^L \eta_{J]K}, \quad U_4 = \exp(t_{IJ} \phi^{IJ}), \quad (5.144)$$

where the rotations are generated by t_{12} and t_{34} , and the boosts by the other generators. Also, we take $\phi^{IJ} = \alpha^{IJ} y^1 + \beta^{IJ} \tilde{y}_1$ to be linear.

From the above $\text{SO}(2,2)$ duality element one can reproduce the following orbits employing a *locally geometric twist* (including e , B and β but only depending on y , usually referred to as a T-fold):

- **Orbit 6** can be obtained by taking

$$(6) \quad \alpha^{12} = -\beta^{12} = -\frac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha), \quad \alpha^{34} = -\beta^{34} = -\frac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha).$$

and all other vanishing.

- **Orbits 7 and 9** can be obtained by the following particular identifications

$$\phi^{14} = \phi^{23}, \quad \phi^{12} = \phi^{34} \quad \text{and} \quad \phi^{13} = \phi^{24}.$$

$$(7) \quad \alpha^{14} = -\beta^{14} = -\frac{1}{\sqrt{2}} \sin \alpha, \quad \alpha^{12} = -\beta^{12} = -\frac{1}{\sqrt{2}} \cos \alpha, \quad \alpha^{13} = \beta^{13} = 0,$$

$$(9) \quad \alpha^{14} = -\beta^{14} = -\frac{1}{\sqrt{2}} \sin \alpha, \quad \alpha^{12} = \beta^{12} = 0, \quad \alpha^{13} = \beta^{13} = -\frac{1}{\sqrt{2}} \cos \alpha.$$

All these backgrounds satisfy both the weak and the strong constraints and hence they admit a locally geometric description. This is in agreement with the fact that the simplest representative of **orbits 6, 7 and 9** given in table 5.6 contains H , ω and Q fluxes but no R flux.

Finally, one can employ the same $\text{SO}(2,2)$ duality elements with different identifications to generate the remaining orbits with a *non-geometric twist* (involving both y and \tilde{y} coordinates):

- **Orbits 1, 3** can be again obtained by considering an $\text{SO}(2) \times \text{SO}(2)$ twist with arbitrary ϕ^{12} and ϕ^{34} :

$$(1) \quad \alpha^{12} = -2\sqrt{2}(\cos\alpha + \sin\alpha), \quad \beta^{34} = 2\sqrt{2}(\cos\alpha - \sin\alpha), \quad \alpha^{34} = \beta^{12} = 0,$$

$$(3) \quad \alpha^{34} = -2\sqrt{2}(\cos\alpha + \sin\alpha), \quad \beta^{12} = 2\sqrt{2}(\cos\alpha - \sin\alpha), \quad \alpha^{12} = \beta^{34} = 0.$$

- **Orbit 2** can be obtained by means of a different $\text{SO}(2,2)$ twist built out of the two rotations and two boosts subject to the following identification

$$\phi^{14} = \phi^{23}, \quad \phi^{12} = \phi^{34}. \quad (5.145)$$

$$(2) \quad \alpha^{14} = \beta^{12} = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha), \quad \alpha^{12} = -\beta^{14} = -\frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha).$$

These backgrounds violate both the weak and the strong constraints for $\alpha \neq 0$. This implies that these backgrounds are truly doubled and they do not even admit a locally geometric description.

Finally, let us also give an example of degeneracy in twist orbits-space reproducing the same orbit of gaugings. The following twist

$$\phi^{12} = \phi^{13}, \quad \phi^{34} = \phi^{24}, \quad \phi^{23} = \phi^{14} = 0 \quad (5.146)$$

$$(6) \quad \alpha^{13} = -\frac{1}{\sqrt{2}}(\cos\alpha + \sin\alpha), \quad \beta^{24} = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha), \quad \alpha^{24} = \beta^{13} = 0,$$

also reproduces the **orbit 6**, but in this case through a non-geometric twist. What happens in this case is that although the twist matrix does not satisfy the weak/strong constraints, the contractions in (5.75) cancel.

5.4 Concluding remarks

In this chapter we have provided a litmus test to the notion of non-geometry, by classifying the explicit orbits of consistent gaugings of different supergravity theories, and considering the possible higher-dimensional origins of these. The results turn out to be fundamentally different for the cases of U-duality orbits of maximal supergravities, and T-duality orbits of half-maximal theories.

In the former case we have managed to explicitly classify all U-duality orbits in dimensions $8 \leq D \leq 11$. This led to zero, one, four and ten discrete orbits in dimensions $D = 11, 10, 9$ and 8 , respectively, with different associated gauge groups. Remarkably, we have found that all of these orbits have a higher-dimensional origin via some geometric compactification, be it twisted reductions or compactifications on group manifolds or coset spaces. In our parlance, we have therefore found that all U-duality orbits are geometric. The structure of U-duality orbits is therefore dramatically different from the sketch of figure 1 in the introduction. Although a full classification of all orbits

in lower-dimensional cases becomes increasingly cumbersome, we are not aware of any examples that are known to be non-geometric. It could therefore hold in full generality that all U-duality orbits are necessarily geometric.

This is certainly not the case for T-duality orbits of gaugings of half-maximal supergravities. In this case, we have provided the explicit classification in dimensions $7 \leq D \leq 10$ (where in $D = 7$ we have only included three-form fluxes). The numbers of distinct families of orbits in this case are zero, one, three and eleven in dimensions $D = 10, 9, 8$ and 7 , respectively, which includes both discrete and one-parameter orbits. A number of these orbits do not have a higher-dimensional origin in terms of a geometric compactification. Such cases are **orbits 2** and **3** in $D = 8$ and **orbits 1, 2** and **3** in $D = 7$ for $\alpha \neq 0$. Indeed, these are exactly the orbits that do not admit an uplift to the maximal theory. As proven in section 5.3.2, all such orbits necessarily violate the weak and/or strong constraints, and therefore need truly doubled backgrounds. Thus, the structure of T-duality orbits is very reminiscent of figure 1 in the introduction. Given the complications that already arise in these simpler higher-dimensional variants, one can anticipate that the situation will be similar in four-dimensional half-maximal supergravity.

Fortunately, the formalism of double field theory seems tailor-made to generate additional T-duality orbits of half-maximal supergravity. Building on the recent generalization of the definition of double field theory [216], we have demonstrated that all T-duality orbits, including the non-geometric ones in $D = 7, 8$, can be generated by a twisted reduction of double field theory. We have explicitly provided duality twists for all orbits. For locally-geometric orbits the twists only depend on the physical coordinates y , while for the non-geometric orbits these necessarily also include \tilde{y} . Again, based on our exhaustive analysis in higher-dimensions, one could conjecture that also in lower-dimensional theories, all T-duality orbits follow from this generalized notion of double field theory.

At this point we would like to stress once more that a given orbit of gaugings can be generated from different twist orbits. Therefore, there is a degeneracy in the space of twist orbits giving rise to a particular orbit of gaugings. Interestingly, as it is the case of **orbit 6** in $D = 7$ for instance, one might find two different twist orbits reproducing the same orbit of gaugings, one violating weak and strong constraints, the other one satisfying both. Our notion of a locally geometric orbit of gaugings is related to the existence of at least one undoubled background giving rise to it. However, this ambiguity seems to be peculiar of gaugings containing Q flux. These can, in principle, be independently obtained by either adding a β but no \tilde{y} dependence (locally geometric choice, usually called T-fold), or by including non-trivial \tilde{y} dependence but no β (non-geometric choice) [215].

Another remarkable degeneracy occurs for the case of semi-simple gaugings, corresponding to **orbits 1 – 3** in $D = 7$. For the special case of $\alpha = 0$, we have two possible ways of generating such orbits from higher-dimensions: either a coset reduction over a sphere or analytic continuations thereof, or a duality twist involving non-geometric

coordinate dependence. Therefore d -dimensional coset reductions seem to be equivalent to $2d$ -dimensional twisted torus reductions (with the latter in fact being more general, as it leads to all values of α). Considering the complications that generally arise in proving the consistency of coset reductions, this is a remarkable reformulation that would be interesting to understand in more detail. Furthermore, when extending the notion of double field theory to type II and M-theory, this relation could also shed new light on the consistency of the notoriously difficult four-, five- and seven-sphere reductions of these theories.

Our results mainly focus on Scherk-Schwarz compactifications leading to gauged supergravities with vanishing ξ_M fluxes. In addition, we have restricted to the NSNS sector and ignored α' -effects. Also, we stress once again that relaxing the strong and weak constraints is crucial in part of our analysis. If we kept the weak constraint, typically the Jacobi identities would lead to backgrounds satisfying also the strong constraint [216]. However, from a purely (double) field theoretical analysis the weak constraint is not necessary. A sigma model analysis beyond tori would help us to clarify the relation between DFT without the weak and strong constraints and string field theory on more general backgrounds. We hope to come back to this point in the future.

Chapter 6

Studies on $N = 2$ multicenter black holes

We present a systematic study of general, stationary, multicenter black hole solutions in ungauged four dimensional Einstein-Maxwell $N = 2$ supergravity theories minimally coupled to scalars, i.e. theories with quadratic prepotentials. We show how is possible to derive in a systematic and straightforward way a fully analytic, explicit description of the multicenter black holes, the attractor mechanism and their properties making an intensive use of the stabilization matrices, their newly defined symplectic adjoints and the algebraic properties of both. The symplectic unitarity of these matrices suggests the decomposition of the $2n_V + 2$ dimensional symplectic space into a subspace generated by the center charges q_a , their associated vectors $\mathcal{S}^\dagger q_a$ and its orthogonal complement subspace. This decomposition results useful in particular the understanding of questions as entropy increasing effects in the fragmentation of a single center black holes into two or more centers, or the maximality of the solutions, in terms, for example, of simple considerations of the dimensions of the each subspaces. The results properties presented are easily extendable to general prepotentials or even theories without them.

6.1 Introduction

We are interested in this chapter in general, stationary, multicenter black hole solutions in ungauged four dimensional $N = 2$ supergravity theories coupled to an arbitrary number of $N = 2$ vector multiplets. The action of the theory can be determined, in the framework of special geometry, in terms of a holomorphic section Ω of the scalar manifold. The set of field equations and Bianchi identities associated to the action is invariant under the group of symplectic transformations $Sp(2n_v + 2)$. This group acts linearly on the section Ω , becoming this a symplectic vector, which can be written $\Omega = (X^I, F_J)$, $(I, J) = (0, n_V)$. Supergravity, $4d$, $N = 2$, black hole solutions have been extensively studied for a long term by now, see for example [61, 63, 67, 238–246] and references therein, or for multicenter black holes [247–253].

The values of the n_V scalar fields constitute the moduli space of the theory. A distinctive feature of many of these theories is that the, possibly disconnected, black hole horizon acts as an attractor for the scalar fields present in the spectrum. The values of the moduli at any of the the horizon components does not depend on their asymptotic values but only on the symplectic vector of charge assigned to that horizon component [63, 243]. The embedding of the duality group of the moduli space into the symplectic group $Sp(2n_V + 2)$ establishes in general a relation between the upper and lower components of Ω , $F_I = F_I(X^J)$. In some cases F_I is the derivative of a single function, the prepotential $F = F(X^J)$. The choice of a particular embedding determines the full lagrangian of the theory and whether a prepotential exists [254, 255].

In this chapter we focus in general quadratic prepotentials. They correspond to the simplest examples of homogeneous special Kahler manifolds, the $\mathcal{CP}^n \equiv SU(1, n)/(U(1) \times SU(n))$ case, The case $n = 1$, for example, corresponds to the Axion-Dilaton Black Hole (see for example [256, 257] or [258]). These models correspond to Maxwell-Einstein $N = 2$ supergravities minimally coupled to n_V scalars. They lead to phenomenologically interesting $N = 1$ minimal coupling supergravities [259].

Theories derived from particular examples of these quadratic prepotentials have been studied in detail. The list include the Axion-dilaton model with prepotential $F = -iX^0X^1$ and the \mathbb{CP}^n models. We define for brevity the constant matrix $\mathcal{S} \equiv \mathcal{S}_F$.

The aim of this study is the explicit, detailed study of stationary multicenter black hole solutions with any number of scalars, the study of the properties of the bosonic field solutions and their global and local properties. For this purpose we make a systematic use of, some previously well known objects of the theory, the “stabilization matrices” and some newly defined ones, their symplectic “adjoints”. These stabilization matrices, $\mathcal{S}_F, \mathcal{S}_N$, are related to the vector kinetic matrix and the matrix of second derivatives of the prepotential. They are real $Sp(2n_V + 2)$ matrices, isometries of the symplectic quadratic form, connecting the the real and imaginary parts of the special geometry sections. Their adjoints with respect to the symplectic product $\mathcal{S}_N^\dagger, \mathcal{S}_F^\dagger$ are defined and shown to lie inside the Lie algebra of the isometry group, they are such that $\mathcal{S} + \mathcal{S}^\dagger = 0$. This property together with $\mathcal{S}^2 = -1$ make these matrices “unitary”, $\mathcal{S}\mathcal{S}^\dagger = 1$, with respect the symplectic product. We show how is possible to derive or rederive again in a systematic and straightforward way a fully analytic, explicit description of the multicenter black holes and their properties (attractor mechanism, central chage, horizon areas, masses...) making an intensive use of these stabilization matrices, their adjoints and the algebraic properties of both.

As we will show, the properties of these matrices, in special their symplectic unitarity property $\mathcal{S}\mathcal{S}^\dagger = 1$, suggests, and makes useful and convenient, the separation of the $2n_V + 2$ dimensional symplectic space into a $2n_a$ dimensional subspace generated by the n_a center charges q_a and their associated vectors $\mathcal{S}^\dagger q_a$ (or $\mathcal{S}q_a$) and its orthogonal complement subspace (possibly of dimension zero depending on the number and on the linear dependency of center charge vectors). In the case of quadratic prepotentials this separation into “charge-longitudinal” and “transversal” subspaces can be made

global choosing $\mathcal{S} = \mathcal{S}_F$. A similar, but local, scalar-dependent, separation can be advantageously considered also for generic prepotentials, or even theories without them. The projection of any symplectic vector appearing in the theory, for example a subset of the charge vectors themselves or vectors characterising the black-hole ansatz at infinity, in terms of these new basis appears as a promising technique. The use of this projection allows in particular the understanding of questions as entropy increasing effects in the fragmentation of a single center black holes into two or more centers, or the maximality of the solutions, in terms, for example, of simple considerations of the dimensions of the each of the charge-longitudinal and transversal subspaces.

Although we have focused in this study in minimal coupling theories with quadratic prepotentials, the main techniques, properties and expressions presented are extendable to general prepotentials or even theories without them.

This study is organized as follows. In the first two sections we present a brief introduction of the Reissner-Nordstrom Black hole and the concept of maximality (section 6.2) and then the attractor mechanism (section 6.3). In the section 6.4 we present some well known basics of dimensional four $N = 2$ supergravity theories and their formulation in terms of special and symplectic geometry. In the following sections we first introduce the matrices $\mathcal{S}_{N,F}$, we emphasize some already known properties and derive new ones and define projection operators constructed from them and their symplectic adjoints. After the consideration of the attractor mechanism in terms of these projectors, we enter in a full explicit description multicenter black hole solutions and their horizon and asymptotic properties. We finally present a section of summary, discussion and outlook.

6.2 The Reissner-Nordström black hole. Extremality in GR

Our starting point is the Einstein-Maxwell action in 4 dimensions,

$$\mathcal{L} = \int d^4x e \left(R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) . \quad (6.1)$$

which allows charged black holes as solutions. For the sake of simplicity, we consider a static and spherically symmetric metric ansatz. The most general one satisfying these requirements is

$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} [dr^2 + r^2 d\Omega^2] , \quad (6.2)$$

where $d\Omega^2 = d\theta + \sin^2 \theta d\phi^2$ and $U(r)$ is the warp factor. Imposing the same symmetry conditions on the Maxwell field, the field strength is restricted to the form

$$F = P \sin \theta d\theta \wedge d\phi + Q dt \wedge \frac{1}{r^2} dr , \quad (6.3)$$

where the constants P and Q can be interpreted as the magnetic and electric charges, respectively. Solving the field equations derived from (6.1), we get for the *Reisner-Nordström* metric element

$$e^{2U(r)} = 1 - \frac{2M}{r} + \frac{P^2 + Q^2}{r^2} . \quad (6.4)$$

This solution enjoys a singularity at $r = 0$ hidden by the horizons which appear when the warp factor vanishes,

$$r_{\pm} = M \pm \sqrt{M^2 - (P^2 + Q^2)} . \quad (6.5)$$

Both r_{\pm} values are real when $M^2 \geq P^2 + Q^2$, the singularity becomes naked for lower values of the mass. We define a parameter c (the *extremality parameter*)

$$c = r_+ - r_- = \sqrt{M^2 - (P^2 + Q^2)} . \quad (6.6)$$

The properties of the solution depends on the values of the mass and the electric and magnetic charges. For $c > 0$ then the solution describes a *non-extremal* black hole with two, interior and exterior, horizons. Its surface gravity κ_S ¹ and are respectively given by

$$\kappa_S (\propto c) = \frac{\sqrt{M^2 - (P^2 + Q^2)}}{2M(M + \sqrt{M^2 - (P^2 + Q^2)}) - (P^2 + Q^2)}, \quad (6.7)$$

$$A = 4\pi(M + \sqrt{M^2 - (P^2 + Q^2)})^2. \quad (6.8)$$

When $c = 0$ or

$$M^2 = P^2 + Q^2$$

the two horizons coincide, the surface gravity vanishes and the horizon area is given exclusively in terms of the charge.

$$A = 4\pi(P^2 + Q^2)^2. \quad (6.9)$$

We have an *extremal black hole*. In the case $c < 0$ the event horizons disappear, the singularity at $r = 0$ is a naked singularity. The Schwarzschild black hole and the Minkowski space are special cases for respectively $M > 0, P = Q = 0$ and $M = P = Q = 0$.

In the *extremal* case, by introducing a radial coordinate $v = r - M$, the metric is

$$ds^2 = - \left(1 + \frac{M}{v}\right)^{-2} dt^2 + \left(1 + \frac{M}{v}\right)^2 [dv^2 + v^2 d\Omega^2] . \quad (6.10)$$

¹ xxx,Nota: The acceleration of a test body at the event horizon of a black hole is infinite in relativity. Because of this, a renormalized value is used that corresponds to the Newtonian value in the non-relativistic limit. The value used is generally the local proper acceleration (which diverges at the event horizon) multiplied by the gravitational redshift factor (which goes to zero at the event horizon).

The horizon is now at $v = 0$ and the near-horizon metric for $v \rightarrow 0$ is

$$ds_{NH}^2 = -\frac{v^2}{M^2}dt^2 + \frac{M^2}{v^2}dv^2 + M^2d\Omega^2 \quad . \quad (6.11)$$

Defining a new coordinate $z \equiv M^2/v$, it is rewritten as

$$ds^2 = \frac{M^2}{z^2}(-dt^2 + dz^2) + M^2d\Omega^2 \quad . \quad (6.12)$$

This line element describes the direct product of two manifolds, $\text{AdS}_2 \otimes S^2$, where the AdS scale L and the radius of the sphere r_S coincide, $L = r_S = M$. This metric is known as the *Robinson-Bertotti metric* [260, 261].

A generalization of the Reissner-Nordström solution arises from rewriting the metric ansatz (6.10) in an isotropic form,

$$ds^2 = -H^{-2}(\vec{x})dt^2 + H^2(\vec{x})dx^i dx^i \quad . \quad (6.13)$$

The equation of motion for this warp factor is expressed as

$$\Delta_3 H = 0 \quad , \quad (6.14)$$

with Δ_3 the 3-dimensional Laplacian which can be solved by generic harmonic functions

$$H \equiv e^{-U} = 1 + \sum_n \frac{M_n}{|\vec{x} - \vec{x}_n|} \quad (6.15)$$

This is the *Majumdar-Papapetrou solution* [262, 263].

6.3 Black holes in Supergravity. The Attractor mechanism

We will study now black-hole solutions in gravity theories which contain gauge and scalar fields (such it happens necessarily in supergravity). Some important mechanism appears in these theories, the *the attractor mechanism*. The attractor mechanism was originally discovered for BPS extremal black holes in $N = 2$ supergravity theories. The origin of this mechanism can be found in [61, 62, 67]. Under this mechanism, the values of the scalars at the black hole horizon are fixed, independently of their values at infinity, in terms of the electric and magnetic charges possibly carried by the black hole. The flow of the scalars towards the horizon exhibits the feature of gradient flow towards a fixed point, which is the minimum of a function related to the central charge of the algebra. Among other properties, a basic feature of the attractor mechanism is that the ADM mass² is minimized, for fixed values of the conserved charges carried by the black hole, when the scalar fields are constant, thake their attractor values through the spacetime.

² The ADM mass M is defined by $g_{tt} = 1 - M/r + \dots$

In the last years, the attractor mechanism has been investigated for extremal black holes in non-supersymmetric theories, in theories beyond GR as 5d Gauss-Bonnet gravity [264] as well as for non-BPS extremal solutions in $N \geq 1$ supersymmetric theories.

A generic lagrangian describing a $4d$, $N = 2$ supergravity coupled to scalars and n_V vector fields is of the form

$$\mathcal{S} = \int d^4x e \left(R - \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + \frac{1}{4} \mathcal{I}_{\Lambda\Sigma}(\phi) F^\Lambda \wedge F^\Sigma + \frac{1}{4} \mathcal{R}_{\Lambda\Sigma}(\phi) F^\Lambda \wedge \star F^\Sigma \right) . \quad (6.16)$$

where $g_{ij}(\phi)$ is the metric of the scalar σ -model, \mathcal{I}, \mathcal{R} describe the gauge kinetic and vector couplings, in particular \mathcal{R} is the generalization of the θ -angle terms in the presence of scalar and vector fields. We assume that there is no-scalar potential.

We are interested in finding single center, static, maximal, spherically symmetric and charged black hole solutions. In addition we assume asymptotical flatness. We introduce a metric ansatz of the form,

$$ds^2 = -e^{2U} dt^2 + e^{-2U} \left(\frac{c^4}{\sinh^4(cz)} dz^2 + \frac{c^2}{\sinh^2(cz)} d\Omega^2 \right) , \quad (6.17)$$

with the required symmetry and which cover both the maximal and non maximal cases, the constant c is an extremality parameter. The z coordinate is related to the standard radial coordinate by $z = -1/r$. It runs from $z = -\infty$ (horizon) to $z = 0$ (spatial infinity). The unknown function $U = U(z)$ is such that $\exp -2U(z \rightarrow 0) = \exp -2Mz \rightarrow 1$ (asymptotic flatness). At $z \rightarrow -\infty$ we require, in order to ensure a finite area ($c \neq 0$),

$$U(z \rightarrow -\infty) = cz, \quad (6.18)$$

$$U'(z \rightarrow -\infty) = c. \quad (6.19)$$

In the maximal limit $c \rightarrow 0$ we recover the metric

$$ds^2 = -e^{2U} dt^2 + e^{-2U} \left(\frac{1}{z^2} dz^2 + \frac{1}{z^2} d\Omega^2 \right) . \quad (6.20)$$

In this case the condition of finite area (A) at the horizon implies the boundary condition:

$$\exp(-2U) = \frac{A}{4\pi} z^2, \quad (z \rightarrow -\infty). \quad (6.21)$$

With respect to the gauge fields, a similar static, spherically symmetric ansatz can be introduced. For it electric and magnetic conserved charges can be defined in terms of the field strengths and their duals,

$$\frac{1}{4\pi} \int F^\Lambda = p^\Lambda , \quad \frac{1}{4\pi} \int G_\Lambda = q_\Lambda . \quad (6.22)$$

Due to the new couplings in (6.16) and the dependence of \mathcal{R} and \mathcal{I} on the scalars, the Bianchi identities are

$$dF^\Lambda = 0 \quad , \quad (6.23)$$

$$dG_\Lambda = d(\mathcal{R}_{\Lambda\Sigma}F^\Sigma - \mathcal{I}_{\Lambda\Sigma} \star F^\Sigma) = 0 \quad . \quad (6.24)$$

Where the second equation defines the quantity G . This set of equations remains invariant when performing a symplectic rotation of the field strengths [265]

$$\begin{pmatrix} F \\ G \end{pmatrix} \rightarrow S \begin{pmatrix} F \\ G \end{pmatrix} \quad , \quad (6.25)$$

where $S \in Sp(2n_V, \mathbb{R})$.³ We can introduce the couple of potentials (A^Σ, A_Σ) , corresponding to the symplectic vector of 2-forms (F^Σ, G_Σ) with the required symmetry.

$$A^\Lambda = \chi^\Lambda(r)dt - p^\Lambda \cos\theta d\phi \quad , \quad (6.26)$$

$$A_\Lambda = \psi_\Lambda(r)dt - q_\Lambda \cos\theta d\phi \quad . \quad (6.27)$$

The electric-magnetic duality relation imposes the constraint

$$\chi'^\Lambda = e^{2U} \mathcal{I}^{\Lambda\Sigma} (q_\Sigma - \mathcal{R}_{\Sigma\Gamma} p^\Gamma) \quad . \quad (6.28)$$

At this point, we can write the Einstein field equations for the metric (6.17) and gauge field ansatze. The equations of motion for the gauge fields may then directly solved. The equations of motion for metric and scalar fields simplify to the equations

$$U'' - e^{2U} V_{BH} = 0 \quad , \quad (6.29)$$

$$(U')^2 + \frac{1}{2} g_{ij} \phi'^i \phi'^j - e^{2U} V_{BH} - c^2 = 0 \quad , \quad (6.30)$$

$$\phi^{i''} + \Gamma_{jk}^i \phi'^j \phi'^k - e^{2U} g^{ij} \partial_j V_{BH} = 0 \quad . \quad (6.31)$$

The non-linear system of second degree differential equations 6.29 is complemented by the asymptotic boundary conditions for the metric at infinity (flatness) and at the horizon (finite area condition) and in principle two initial or boundary conditions for each of the scalars. The properties of this non-linear system, the existence of the constraint equation and the requirement of regularity everywhere will imply that of the two theoretically possible conditions for each of the scalars only one will survive. The value of the scalars and their first derivative will be fixed at the horizon, only the value at the scalars at infinity will remain as a free parameter of the theory. This is in essence the attractor mechanism.

The quantity V_{BH} is the *black hole potential*, which encodes the terms of the energy momentum tensor corresponding to the vector fields that appear in the lagrangian [67]. It can be written as

$$V_{BH} = -\frac{1}{2} Q^T \mathcal{M} Q \quad , \quad (6.32)$$

³Additional matter couplings may reduce this symmetry to $G \subset Sp(2n_V, \mathbb{R})$, being G the U-duality group of the theory

where \mathcal{M} is a scalar depending matrix

$$\mathcal{M} = \begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix} , \quad (6.33)$$

and Q is a charge symplectic vector

$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} . \quad (6.34)$$

Using the scalar matrix \mathcal{M} we can rewrite the gauge field strengths in a covariant way as

$$\begin{pmatrix} F \\ G \end{pmatrix} = e^{2U} \tilde{\Omega} \mathcal{M} \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} dt \wedge dz - \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \sin \theta d\theta \wedge d\phi , \quad (6.35)$$

where $\tilde{\Omega}$ is the symplectic metric

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\mathbb{I}_{n_V} \\ \mathbb{I}_{n_V} & 0 \end{pmatrix} . \quad (6.36)$$

At this point, let us consider the possibility of deriving the field equations above from an effective lagrangian. Let us take the following lagrangian

$$\mathcal{L} = (U')^2 + \frac{1}{2} g_{ij} \phi^{i'} \phi^{j'} + e^{2U} V_{BH}(\phi) + c^2 , \quad (6.37)$$

depending on the ‘‘fields’’ $U(z)$, $\phi(z)$. Noether’s theorem applied to this lagrangian implies that the effective ‘energy’

$$\mathcal{E} \equiv (U')^2 + \frac{1}{2} g_{ij} \phi^{i'} \phi^{j'} - e^{2U} V_{BH} - c^2 \quad (6.38)$$

is a constant, it does not depend on z . The Euler-Lagrangian equations for this system (corresponding to the fields $U(z)$, ϕ^i) agree with first and last of the equations in (6.29). The second equation in (6.29) has to be implemented by hand, as an additional constraint

$$\mathcal{E} = 0.$$

Thus, this effective action plus the constraint $\mathcal{E} = 0$ is equivalent to the system.⁴

We note that from the behaviour of the constraint equation 6.38 at spatial infinity (considering that $\exp -2U(z \rightarrow 0) = \exp -2Mz \rightarrow 1$) we get the following constraint between the black hole mass M , scalar charges⁵, and the potential at infinity

$$M^2 + \frac{1}{2} g_{ij} \Sigma^i \Sigma^{j'} - V_{BH}(\phi_\infty^i) = c^2 \quad (6.39)$$

⁴Alternatively the same lagrangian could be obtained inserting in the original action the ansatz for the metric and the gauge fields, integrating out the spatial coordinates and ensuring that the resulting motion equations were equivalent to the original ones.

⁵The scalar charges Σ^i of the black hole are defined by (at spatial infinity $z \rightarrow 0$) $\phi^i = \phi_\infty^i + z \Sigma^i / r + O(z^2)$.

Let us note that the equations in (6.29) can be solved by constant values $\tilde{\phi}^i$ of the scalar fields, if in addition these values represent a critical point of the effective potential, that means

$$\frac{\partial V_{BH}(\phi)}{\partial \phi^i} \Big|_{\tilde{\phi}} = 0. \quad (6.40)$$

We will construct below extremal solutions with non-constant scalars but these critical values of the effective BH potential represent possible attractor values for the moduli scalars (provided positivity of the hessian). They will have the same horizon-limit (for the value of the field and its derivative), however the asymptotic values of these scalars may be varied freely. Extremal black hole solutions in which the scalar fields take constant values are called *double-extreme* black holes.

For a constant scalar solution given by the critical point of the black hole potential

$$\partial_i V_{BH}(\tilde{\phi}_f^i, q, p) = 0 \quad . \quad (6.41)$$

The black hole charges are the only parameters that appear in (6.41), the extremal points will be solved in terms of them,

$$\tilde{\phi}^i = \tilde{\phi}^i(q, p) \quad . \quad (6.42)$$

The BH potential is a constant given by

$$\tilde{V}_{BH} = V_{BH}(\tilde{\phi}^i(q, p), q, p) \quad . \quad (6.43)$$

The first and second equations of (6.29) for the warp factor at the horizon can be directly solved giving

$$U''(z) = (U'(z))^2 \quad . \quad (6.44)$$

$$U(z) = -\log\left(\sqrt{\tilde{V}_{BH}z}\right) \quad . \quad (6.45)$$

This implies that the entropy is (taking into account that $r_H = \sqrt{\tilde{V}_{BH}}$)

$$S_{BH} = \frac{A}{4} = \pi \tilde{V}_{BH}(q, p) \quad . \quad (6.46)$$

In addition, for double extremal solution, from the equation 6.39 and taking $c = 0$, we get

$$M^2 = V_{BH}(\phi_\infty^i). \quad (6.47)$$

Next we will study general, non-constant scalar solutions. We will see how the some universal properties attractor mechanism and of the area of extremal black holes can be deduced only from the requirement of the regularity of the configuration, that the geometry as well the scalars are regular near the horizon. If the scalars and their

derivatives does not blow up near the horizon we can assume the following expression valid for $z \rightarrow \infty$

$$\phi^i = \tilde{\phi}_h^i + a^i/z + \dots \quad (6.48)$$

Inserting this series and $\exp(-2U)_h = z^2 A/4\pi$, the differential equations (6.29) become at leading orders near the horizon

$$\frac{1}{z} - \frac{4\pi}{Az^2} V_{BH,h} = 0 \quad , \quad (6.49)$$

$$\frac{1}{z^2} + \frac{a_1}{z^4} - \frac{4\pi}{Az^2} V_{BH,h} + \dots = 0 \quad , \quad (6.50)$$

$$\frac{a_2}{z^3} + \frac{a_3}{z^4} - \frac{4\pi}{Az^2} (g^{ij} \partial_j V_{BH})_h + \dots = 0 \quad . \quad (6.51)$$

or (a_1, a_2, a_3 constants)

$$1 - \frac{4\pi}{A} V_{BH,h} = 0 \quad , \quad (6.52)$$

$$a_1 + z^2 \left(1 - \frac{4\pi z^2}{A} V_{BH,h} \right) + \dots = 0 \quad , \quad (6.53)$$

$$a_2 z + a_3 - z^2 \frac{4\pi}{A} (g^{ij} \partial_j V_{BH})_h + \dots = 0 \quad . \quad (6.54)$$

If we compare coefficients order by order in $1/z$, we arrive to that, in order for solutions not to blow up at the horizon, we have the conditions (assuming that the matrix $g^{ij}(\phi)$ is invertible at the horizon) :

$$A = 4\pi V_{BH,h} \quad , \quad (6.55)$$

$$\phi^i(z \rightarrow -\infty) = 0 \quad , \quad (6.56)$$

$$\left(\frac{\partial V_{BH}}{\partial \phi^i} \right)_h = 0 \quad . \quad (6.57)$$

Thus we have shown that the area of the horizon of extreme black holes coincides with the area of the horizon of the double-extreme black holes with the same values of charges and is given by the value of the potential, as is expressed by the first equation above. Moreover, the entropy of the black hole, related to the area of the horizon, will be determined by the charges. This means that the entropy does not depend on any continuous parameter.

In addition we see that the values of the scalars at the horizon can be considered as free initial conditions, they are given by the minimization of the effective black-hole potential, as it appears in the third equation of 6.52: the horizon is an attractor point [61–64].

The gist for the existence of the attractor mechanism in extremal black holes may be consider to reside in the distance of the horizon from any observer [67]. Whilst in the extremal case the horizon results inachievable because of its infinite distance, in the non-extremal situation, the proper distance is finite. This justifies the lost of memory of the asymptotic values of the scalar fields in the former case.

Supergravity central charge and flow equations

Now we consider in more detail the special case of $N = 2$ supergravity for which the scalar manifold is a special Kahler manifold. For these $N = 2$ theories special geometry can be used and the expressions are somehow simplified. The black hole potential is given by (we will follow [65])

$$V_{BH} = |Z|^2 + 4g^{ij}\partial_i|Z|\bar{\partial}_j|Z| \quad , \quad (6.58)$$

where Z is the central charge of the $N = 2$ SUSY algebra,

$$Z = e^{-K/2}(X^\Lambda q_\Lambda - p^\Lambda F_\Lambda) = e^{-K/2}\langle\Omega, Q\rangle \quad , \quad (6.59)$$

where K is the Kahler potential and X_I, F_I are determined by the special geometry. Thus, the lagrangian (6.37) and the constraint (6.38) become

$$\mathcal{L} = (U')^2 + g_{ij}\phi^{i'}\phi^{j'} + e^{2U}(|Z|^2 + 4g^{ij}\partial_i|Z|\bar{\partial}_j|Z|) \quad , \quad (6.60)$$

$$(U')^2 + g_{ij}\phi^{i'}\phi^{j'} = e^{2U}(|Z|^2 + 4g^{ij}\partial_i|Z|\bar{\partial}_j|Z|) \quad . \quad (6.61)$$

We will reduce this system to first order differential equations in the following simplified way. The energy constraint is an equality between two different sums of squares with the same weight e^{2U} . Thus, a possible ansatz for the solution would be

$$U' = \pm e^U|Z| \quad , \quad (6.62)$$

$$\phi^{i'} = \pm 2e^U g^{i\bar{j}}\bar{\partial}_j|Z| \quad . \quad (6.63)$$

It can be checked that this ansatz is also a solution for the equations of motion (6.29) when the same signs are chosen. Thus, this is a reduction of the original second order system to a first order system governed by $|Z|$. Due to asymptotic flatness arguments, the physical sign is

$$U' = -e^U|Z| \quad , \quad (6.64)$$

$$\phi^{i'} = -2e^U g^{i\bar{j}}\bar{\partial}_j|Z| \quad . \quad (6.65)$$

The same first order equations can be obtained by analyzing the Killing spinor equations for the theory. The conditions for the gravitino and gaugino supersymmetry transformations

$$\delta\psi_\mu^A = 0 \quad , \quad (6.66)$$

$$\delta\lambda_A^i = 0 \quad . \quad (6.67)$$

are equivalent to the former and latter differential equations ⁶.

⁶Actually, the Killing spinor equation for gauginos implies a new first order equation for a phase factor. However, it is also related to the Kähler connection and once the flow equations (6.64) are fulfilled, this additional equation is automatically satisfied, showing that the phase factor is not an independent quantity.

By evaluating the equations at (6.64) at infinity and at the horizon similarly as in the previous section we infer that the central charge fully determines the solution. The fixed values of the scalars at horizon are given by the minimization condition

$$\partial_i |Z|_h = 0 \quad , \quad (6.68)$$

whose critical points are also a critical point for the black hole potential. Solutions corresponding to a critical point describes a supersymmetric extremal black hole. The central charge at the horizon is fixed in terms of the discrete charges

$$|Z|_h = |Z|_h(p, q, \phi_h^i(p, q)) \quad . \quad (6.69)$$

The Special Kähler nature of the scalar manifold guarantees that the second derivative of the central charge is such that

$$\partial_i \bar{\partial}_{\bar{j}} |Z| = g_{i\bar{j}} |Z| > 0 \quad , \quad (6.70)$$

all the critical points are minima of the central charge. No matter what is the value of the scalars at infinity, they will be driven towards the minimum of the central charge. This constitutes an attractor behaviour.

The extremal condition for the central charge was brought to an equivalent form in [266–270] under the condition that the special geometry is not singular. They are the so called “stabilization equations” of purely algebraic form and which will be used in the next sections.

6.4 $N = 2$ $D = 4$ SUGRA and Special Kähler geometry

The fields of the $N = 2$ Supergravity coupled to $N = 2$ vector multiplets theory consist of the gravitational field, n_v vectors, scalars and fermions respectively $e_\mu^a, A_\mu^I, z^\alpha, \psi_{\mu,r}, \lambda_r^\alpha$ with $\alpha = (1, n_v), I = (0, n_v)$. The theory also contain hypermultiplets which can be safely taken as constant or neglected (further details can be found in [248] whose notation and concepts we generally adopt). The bosonic $N = 2$ action can be written as, considering only gravity scalars and vectors

$$S = \int_{M(4d)} R \star 1 + \mathcal{G}_{\alpha\bar{\beta}} dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} + F^I \wedge G_I. \quad (6.71)$$

The fields F^I, G_I are not independent. The field strengths $F^I = dA^I$ and G_I are combinations of the F^I and their Hodge duals $G_I = a_{IJ} F^I + b_{IJ} \star F^I$ with scalar dependent coefficients a_{IJ}, b_{IJ} .

Abelian charges with respect the $U(1)^{n_v+1}$ local symmetry of the theory are defined by integrals of the gauge field strengths (where S_a are surfaces completely surrounding one and only one center)

$$q \equiv (p^I, q_I) \equiv \frac{1}{2\pi^2} \int_{S_a} (F^I, G_I). \quad (6.72)$$

The theory is defined, in the special geometry formalism, by the introduction of some projective scalar coordinates X^I , for example “special” projective coordinates $z^\alpha \equiv X^I/X^0$, and by the introduction of a covariantly holomorphic section of a symplectic bundle $V = V(z, \bar{z}) \equiv (V^I, V_I)$ with ${}^7 \langle V | \bar{V} \rangle \equiv V^t \omega V \equiv \bar{V}^I V_I - V^I \bar{V}_I = -i$. The scalar kinetic term metric is given by $\mathcal{G}_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}$ where the Kahler potential \mathcal{K} is defined by the relations $V = \exp(-\mathcal{K}/2)\Omega$ where the holomorphic section $\Omega \equiv (X^I, F_I)$ and

$$e^{-\mathcal{K}} = i (\bar{X}^I F_I - X^I F_I) = i \langle \Omega | \bar{\Omega} \rangle. \quad (6.73)$$

In N2 SUGRA theories the central charge central charge function can be expressed as the linear function on the charge space:

$$Z(z^\alpha, q) \equiv \langle V | q \rangle = e^{\mathcal{K}/2} (p^I F_I - q_I X^I). \quad (6.74)$$

The embedding of the isometry group of the scalar manifold metric $\mathcal{G}_{\alpha\bar{\beta}}$, into the symplectic group fixes, through the Kahler potential \mathcal{K} , a functional relation between the lower and upper parts of V and Ω , $F_I = F_I(X)$, $V_I = V_I(V^I)$ [271, 272].

There always exists an symplectic frame where the theory can be described in terms of a single holomorphic function, the prepotential $F(X)$, a second degree homogeneous function on the projective scalar coordinates X^I , such that $F_I(X) = \partial_I F(X)$. For simplicity we will assume the existence of such prepotential along this study although the results will not depend on such existence. We use the notation $F_{IJ} = \partial_I \partial_J F$.

The lower and upper components of V and are related by a field dependent matrix N , which is determined by the special geometry relations [63]

$$V_I = N_{IJ} V^J, \quad D_{\bar{i}} \bar{V}_I = N_{IJ} D_{\bar{i}} \bar{V}^J. \quad (6.75)$$

The matrix N , which fixes the vector couplings (a_{IJ}, b_{IJ}) in the action, can be related to F_{IJ} [273] by $N_{IJ} = \bar{F}_{IJ} + T_I T_K$, where T_I is proportional to the projector of the graviphoton, whose flux define the $N = 2$ central charge [273]. For our purposes it is convenient to write this relation between the N and F matrices as

$$N_{IJ} \equiv F_{IJ} + N_{IJ}^\perp \equiv F_{IJ} - 2i \Im F_{IJ} + 2i \frac{\Im F_{IK} L^K \Im F_{JQ} L^Q}{L^P \Im F_{PQ} L^Q}, \quad (6.76)$$

where we decompose the matrix N into “longitudinal” and “perpendicular” parts. The perpendicular term, N_{IJ}^\perp , acts over L^I , or any multiple of it, as

$$N_{IJ}^\perp(\alpha L^J) = 0. \quad (6.77)$$

Form here the first relation in Eq.(6.75) can be written as

$$V_I = N_{IJ} L^J = F_{IJ} L^J. \quad (6.78)$$

⁷We choose a basis where $\omega = \begin{pmatrix} 0 & -\mathbb{I}_{\bar{n}} \\ \mathbb{I}_{\bar{n}} & 0 \end{pmatrix}$.

The existence of functional dependences among the upper and lower components of the vectors V or Ω imply further relations between their respective real and imaginary parts. They are related by symplectic $Sp(2n_v + 2, R)$ matrices $\mathcal{S}_N, \mathcal{S}_F$ such that:

$$\operatorname{Re}(\Omega) = \mathcal{S}_F \operatorname{Im}(\Omega). \quad (6.79)$$

$$\operatorname{Re}(V) = \mathcal{S}_N \operatorname{Im}(V) = \mathcal{S}_F \operatorname{Im}(V), \quad (6.80)$$

where in the last equality we have used the relation (6.78). Explicit expressions for the matrices $\mathcal{S}_{N,F}$ can be found in [273].⁸ The properties of the stabilization matrices $\mathcal{S}_{N,F}$ will play an important role in what follows.

In the particular case of theories with quadratic prepotentials, of the form (or equivalently for U-duality groups of degenerate type $E7$, [259])

$$F(X) = \frac{1}{2} F_{IJ} X^I X^J, \quad (6.81)$$

with F_{IJ} a complex, constant, symmetric matrix, the matrix \mathcal{S}_F , in contrast to the matrix \mathcal{S}_N , is a field-independent, constant matrix. In fact the matrix \mathcal{S}_N is never scalar independent in $N = 2$ supergravity [259]. We can assume that $\Re F_{IJ} = 0$ and $\Im F_{IJ}$ is negative definite.

The most general stationary (time independent) metric in four dimensions compatible with supersymmetry can be written in the IWP form [274–276].

$$ds^2 = e^{2U} (dt + \omega)^2 - e^{-2U} d\vec{x}^2. \quad (6.82)$$

The 1-form ω and the function e^{-2U} are related in these theories to the Kahler potential and connection, \mathcal{K}, Q [276]. Kahler gauge fixing is accomplished if asymptotic flatness is requested $e^{-2U} \rightarrow 1$ as $\mathbf{x} \rightarrow \infty$ together with $\omega \rightarrow 0$.

Supersymmetric $N = 2$ supergravity solutions can be constructed systematically following well-established methods [248]. BPS field equation solutions for the action above, in particular expressions for the terms appearing in the metric e^{-2U}, ω , can be written in terms of real symplectic vectors \mathcal{R} and \mathcal{I} defined as:

$$\mathcal{R} = \frac{1}{\sqrt{2}} \Re \frac{V}{X}, \quad \mathcal{I} = \frac{1}{\sqrt{2}} \Im \frac{V}{X}. \quad (6.83)$$

where $X(z, \bar{z})$ is an arbitrary function such that $1/X$ is harmonic. The $2n + 2$ components of \mathcal{I}, \mathcal{R} are real harmonic functions in \mathbb{R}^3 . Due to the relations (6.79,6.80) the solutions can be written in terms only of the vector \mathcal{I} :

$$\mathcal{R} = \mathcal{S}_N \mathcal{I} = \mathcal{S}_F \mathcal{I}. \quad (6.84)$$

⁸ The matrix \mathcal{S}_N is related to the matrix appearing in the effective potential $V_{BH} = -1/2q^t \mathcal{M} q$ by $\mathcal{S}_N \omega = \mathcal{M}$.

In practise, particular solutions are determined by giving a particular ansatz for it. We are interested in general, single- or multi-center black hole-type solutions defined by an \mathcal{I} ansatz with any number of point-like singularities of the form ($a = 1, \dots, n_a$)

$$\mathcal{I} = \mathcal{I}_\infty + \sum_a \frac{q_a}{|\mathbf{x} - \mathbf{x}_a|}, \quad (6.85)$$

where $q_a = (p_a^I, q_{aI})$ and \mathcal{I}_∞ are real, constant, symplectic vectors.

With the help of these vectors we rewrite the only independent metric component as

$$e^{-2U} = e^{-\kappa} = \frac{1}{2 |X|^2} = \langle \mathcal{R} | \mathcal{I} \rangle. \quad (6.86)$$

Similarly, the time independent 3-dimensional 1-form $\omega = \omega_i dx^i$ satisfies the equation

$$d\omega = 2 \langle \mathcal{I} | \star_3 d\mathcal{I} \rangle, \quad (6.87)$$

where \star_3 is the Hodge dual on flat R^3 , together with integrability constraint

$$\langle \mathcal{I} | \Delta \mathcal{I} \rangle = 0. \quad (6.88)$$

The asymptotic flatness condition implies

$$\langle \mathcal{R}_\infty | \mathcal{I}_\infty \rangle = \langle \mathcal{S} \mathcal{I}_\infty | \mathcal{I}_\infty \rangle = 1. \quad (6.89)$$

The gauge field equations of motion and Bianchi identities can be solved directly in terms of spatially dependent harmonic functions [248]. The modulus of the central charge defined in Eq.(6.74) can be written, taking into account Eq.(6.86), as

$$|Z|^2 e^{-2U} = |\langle \mathcal{R} | q \rangle|^2 + |\langle \mathcal{I} | q \rangle|^2. \quad (6.90)$$

At infinity, assuming asymptotic flatness, we arrive to

$$|Z_\infty|^2 = |\langle \mathcal{R}_\infty | q \rangle|^2 + |\langle \mathcal{I}_\infty | q \rangle|^2. \quad (6.91)$$

The, assumed time independent, n complex scalar fields z^α solutions to the field equations, are given in this formalism by

$$z^\alpha = \frac{\Omega^\alpha}{\Omega^0} = \frac{V^\alpha}{V^0} = \frac{\mathcal{R}^\alpha + i\mathcal{I}^\alpha}{\mathcal{R}^0 + i\mathcal{I}^0}. \quad (6.92)$$

The scalars can in principle take any values (z_∞) at asymptotic infinity. But, at the Black hole horizon, according to the attractor mechanism, the moduli adjust themselves so that their values near the horizon of any center of charge vector q_a approach to the values $z(\mathbf{x}) \rightarrow z(\mathbf{x}_a) = z_{fix}^a$. The attractor equations relate these fixed values of the scalars to the charge vectors $z_{fix}^a = z_{fix}^a(q_I)$ [61, 63, 239]

$$q^a = \text{Re} \left(2i \bar{Z}(z_{fix}^a) V(z_{fix}^a) \right). \quad (6.93)$$

where it appears the prepotential, through V , and the central charge function, Eq.(6.74). The stabilization equations can be reformulated as an extremalization condition on $|Z|$ or V_{BH} evaluated at the horizons. In this way the scalar attractor values are independent of their asymptotic values and depend only on the charge of the respective black hole center.

Single centered black hole solutions are known to exist in all regions of the moduli scalars at infinity, under very mild conditions on the charge vector. In the multicentered case, for fixed charge vectors, not all the positions in the ansatz Eq.(6.85) are allowed. The integrability condition (6.88) impose necessary conditions on the positions and on the moduli scalars at spatial infinity (through I_∞) for existence of a solution. In this framework a particular black hole solution is completely determined by a triplet of charge vectors, distances and values of the moduli at infinity $(q_a, \mathbf{x}_a, z_\infty)$.

6.5 The stabilization matrix

It can be shown by explicit computation that the real symplectic $Sp(2(n_v + 1))$ matrices $\mathcal{S}_N, \mathcal{S}_F \equiv \mathcal{S}$ given by Eqs.(6.79,6.80) satisfy the relation

$$\mathcal{S}_N^2 = \mathcal{S}_F^2 = -1. \quad (6.94)$$

We note that \mathcal{S}_F is “not” the matrix appearing in V_{BH} , however $\mathcal{S} \equiv \mathcal{S}(N)$ acting on the imaginary or real parts of the vector V or any multiple of it (for any complex function $s(z, \bar{z})$):

$$\Re(sV) = \mathcal{S}_N \Im(sV) = \mathcal{S}_F \Im(sV). \quad (6.95)$$

The matrix \mathcal{S} is an isometry of the symplectic space, we have

$$\langle \mathcal{S}A | \mathcal{S}B \rangle = \langle A | B \rangle. \quad (6.96)$$

We will find convenient to define the adjoint of the matrix \mathcal{S} , \mathcal{S}^\dagger , with respect to the symplectic bilinear product such that, for any vectors A, B :

$$\langle \mathcal{S}A | B \rangle = \langle A | \mathcal{S}^\dagger B \rangle. \quad (6.97)$$

A direct computation shows that \mathcal{S}^\dagger is given by

$$\mathcal{S}^\dagger = -\Omega \mathcal{S}^t \Omega. \quad (6.98)$$

The adjoint matrix \mathcal{S}^\dagger depends on the imaginary and real parts of F_{IJ} . It can be seen that for a symmetric F_{IJ} we have the relation

$$\mathcal{S}^\dagger = -\mathcal{S}. \quad (6.99)$$

In summary the matrix \mathcal{S} is skew-adjoint with respect to ω and its square $\mathcal{S}^2 = -1$. It fullfills an “unitarity” condition $\mathcal{S}^\dagger \mathcal{S} = 1$. We see that the matrix \mathcal{S} is an element of the symplectic Lie algebra.

It is useful to define in addition the projector operators

$$P_{\pm} = \frac{1 \pm i\mathcal{S}}{2}. \quad (6.100)$$

We have the following straightforward properties

$$P_{\pm}^2 = P_{\pm}, \quad (6.101)$$

$$\mathcal{S}P_{\pm} = \mp iP_{\pm}, \quad (6.102)$$

$$(P_{\pm})^* = P_{\mp}. \quad (6.103)$$

According to Eq.(6.102), the projectors P_{\pm} are the projectors into the eigenspaces of the matrix \mathcal{S} . With the help of the newly defined projection operators P_{\pm} we can rewrite a relation like the equation (6.95) as

$$sV = 2iP_{-}\Im(sV) = 2P_{-}\Re(sV). \quad (6.104)$$

Let us consider now the attractor equations, Eq.(6.93). We will use the properties of the stabilization matrix \mathcal{S} to solve them in a purely algebraic way and give some explicit expressions for the scalars at the fixed points. If we multiply both sides of Eq.(6.93) by $\mathcal{S}_N^f = \mathcal{S}_N(z = z_{fix})$ we arrive, using the property (6.95), to:

$$\mathcal{S}_N^f q^a = \mathcal{S}_N^f \text{Re} (2i\bar{Z}(z_{fix}^a)V(z_{fix}^a)) \quad (6.105)$$

$$= \mathcal{S}_F^f \text{Re} (2i\bar{Z}(z_{fix}^a)V(z_{fix}^a)) \quad (6.106)$$

$$= \mathcal{S}q^a. \quad (6.107)$$

We arrive to the conclusion that the attractor mechanism equation (6.93) is equivalent to any of the equations

$$\mathcal{S}_N^f q^a = \mathcal{S}_F^f q^a, \quad (6.108)$$

$$\mathcal{S}_F^f \mathcal{S}_N^f q^a = -q^a. \quad (6.109)$$

Any of the equations above can be used in practise to get the values of the scalars at the fixed points as a function of the charges. ⁹

Moreover the equation (6.93) can be written, using the expression (6.104) as

$$i\bar{Z}_{fix}V_{fix} = P_{-}q, \quad (6.110)$$

or as its conjugate equation

$$-iZ_{fix}\bar{V}_{fix} = P_{+}q. \quad (6.111)$$

⁹ Note that [273] $V_{BH} = |Z_i|^2 + |Z|^2 = -\frac{1}{2}q^t \mathcal{S}(N)\Omega q$, $|Z_i|^2 - |Z|^2 = \frac{1}{2}q^t \mathcal{S}(F)\Omega q$. At the fixed points $Z_i = 0$ then $|Z|^2 = -\frac{1}{2}q^t \mathcal{S}_N \Omega q = -\frac{1}{2}q^t \mathcal{S}_F \Omega q$. This last equation is satisfied by a solution of Eq.(6.109).

We get in addition, from eqs.(6.110,6.111), taking symplectic products,

$$|Z_{\text{fix}}|^2 \langle V_{\text{fix}} | \bar{V}_{\text{fix}} \rangle = \langle P_- q | P_+ q \rangle = \langle q | P_+ q \rangle \quad (6.112)$$

$$= -\frac{i}{2} \langle \mathcal{S}q | q \rangle. \quad (6.113)$$

If we insert the constraint $\langle V | \bar{V} \rangle = -i$, we arrive to the well known formula

$$|Z_{\text{fix}}|^2 = \frac{1}{2} \langle \mathcal{S}q | q \rangle \quad (6.114)$$

The positivity of the quadratic form $g(q) = \langle \mathcal{S}q | q \rangle$ is from here a necessary consistency condition for the existence of solutions to the attractor mechanism.

The values of the scalars at the fixed points are implicit in the equations (6.108), or, (6.109) and can be computed from them but in fact a more explicit expression can be found. The fixed values of the n_v complex scalars, z_{fix}^α , are constructed using the general formula (6.92) and Eq.(6.110). We arrive to

$$z_{\text{fix}}^\alpha = \frac{(P_- q)^\alpha}{(P_- q)^0}. \quad (6.115)$$

The fixed values of the scalars are given in terms of the projection of the charges into the eigenspaces of the matrix \mathcal{S} . For quadratic prepotentials this is a complete, explicit solution of the attractor equations.

The values of the n complex scalars at spatial infinity, $|\mathbf{x}| \rightarrow \infty$ are given by (using Eq.(6.92) and defining $\mathcal{I}_\infty = \lim_{\mathbf{x} \rightarrow \infty} \mathcal{I}$)

$$z_\infty^\alpha = \lim_{|\mathbf{x}| \rightarrow \infty} \frac{(P_- \mathcal{I})^\alpha}{(P_- \mathcal{I})^0} = \frac{(P_- \mathcal{I}_\infty)^\alpha}{(P_- \mathcal{I}_\infty)^0}. \quad (6.116)$$

The ‘moduli’ z_∞^α are simple rational functions of the $2n + 2$ real constant components of \mathcal{I}_∞ . They are thus independent of the fixed attractor values, (6.115).

Moreover, the expression (6.116) is formally identical to the expression, Eq.(6.115), giving the scalar at fix point in terms of the charges. It is suggestive to write an “effective attractor equation” at infinity where the role of the charge center is played the vector \mathcal{I}_∞

$$\mathcal{I}_\infty = \text{Re} (2i \bar{Z} V) |_\infty, \quad (6.117)$$

whose solution is given by Eq.(6.116). We can also write the following equations which allows the computation of \mathcal{I}_∞ for given values of the scalars at infinity and viceversa

$$\mathcal{S}_N^\infty \mathcal{I}_\infty = \mathcal{S}_F^\infty \mathcal{I}_\infty^a, \quad (6.118)$$

$$\mathcal{S}_F^\infty \mathcal{S}_N^\infty \mathcal{I}_\infty = -\mathcal{I}_\infty, \quad (6.119)$$

and where $\mathcal{S}_N^\infty \equiv \mathcal{S}_N(z = z_\infty)$, meanwhile for constant prepotentials $\mathcal{S}_F^\infty \equiv \mathcal{S}_F(z = z_\infty) = \mathcal{S}$.

The expressions (6.115,6.116) are written, in terms of some projective scalar coordinates, as respectively at the horizon and at infinity, as

$$\Omega_{fix} = P_{-q}, \quad (6.120)$$

$$\Omega_{\infty} = P_{-\mathcal{I}_{\infty}}. \quad (6.121)$$

Extending the arguments in [277] (and references therein) if SUSY Black Hole solutions are uniquely determined by the symplectic real vectors q_a . The also symplectic, but complex, vector $\Omega = (X^I, F_J)$ must be related to it in a some linear way, respecting symplectic covariance. The only possibility being the expressions (6.120,6.121), which at the points of maximal symmetry, the horizon and spatial infinity, are equivalent forms of the standard attractor equations at the horizon and some generalized attractor equation at infinity.

6.6 Black hole complete solutions

Let us come back to the general expression for the complex scalars and insert the ansatz (6.85) in the general expression (6.92). The values for the n_v complex scalar solutions to the field equations can be written as

$$z^{\alpha} = \frac{(P_{-\mathcal{I}})^{\alpha}}{(P_{-\mathcal{I}})^0} = \frac{(P_{-\mathcal{I}_{\infty}})^{\alpha} + \sum_a \frac{(P_{-q_a})^{\alpha}}{|\mathbf{x}-\mathbf{x}_a|}}{(P_{-\mathcal{I}_{\infty}})^0 + \sum_a \frac{(P_{-q_a})^0}{|\mathbf{x}-\mathbf{x}_a|}}. \quad (6.122)$$

The expression (6.122) is a rational interpolating expression for the value of the scalar fields in all the space:

$$z^{\alpha}(\mathbf{x}) = c_{\infty}^{\alpha}(\mathbf{x})z_{\infty}^{\alpha} + c_a^{\alpha}(\mathbf{x})z_{a,\text{fix}}^{\alpha}, \quad (6.123)$$

where $c_{\infty}^{\alpha}(\mathbf{x}), c_a^{\alpha}(\mathbf{x})$ are spatial dependent complex functions such that $c_{\infty}^{\alpha}(\infty) = 1, c_{\infty}^{\alpha}(\mathbf{x}_a) = 0, \lim_{\mathbf{x} \rightarrow \mathbf{x}_b} c_a^{\alpha}(\mathbf{x}) = \delta_{ab}$. For a single center black hole, we note that if $z_{\infty}^{\alpha} = z_{fix}^{\alpha}$ then the scalar field is constant in all the space, a double maximal solution.

The attractor mechanism is automatically fulfilled by the ansatz (6.85). The value of z^{α} at any center \mathbf{x}_a is given, taking limits in Eq.(6.122), by

$$z^{\alpha}(\mathbf{x}_a) = \frac{(P_{-q_a})^{\alpha}}{(P_{-q_a})^0} = z_{\text{fix}}^{\alpha}(q_a). \quad (6.124)$$

On the other hand, at the spatial infinity we recover spherical symmetry, we have (with $|\mathbf{x}| \equiv r$)

$$z^{\alpha}(r \rightarrow \infty) = \frac{r(P_{-\mathcal{I}_{\infty}})^{\alpha} + \sum_a (P_{-q_a})^{\alpha}}{r(P_{-\mathcal{I}_{\infty}})^0 + \sum_a (P_{-q_a})^0} \quad (6.125)$$

$$\equiv (1 - c^{\alpha}(r))z_{\infty}^{\alpha} + c^{\alpha}(r)z_{\text{fix}}^{\alpha}(Q) \quad (6.126)$$

$$= z_{\infty}^{\alpha} + \frac{\sigma^{\alpha}}{r} + O\left(\frac{1}{r^2}\right), \quad (6.127)$$

where $z_{fix}(Q)$ is the fixed point scalar value which would correspond to a total charge $Q = \sum_a q_a$, the functions $c^\alpha(r) = 1/(1 + r/r_0^\alpha)$ with $r_0^\alpha = \sum_a (P_{-q_a})^0 / \sum_a (P_{-\mathcal{I}_\infty})^0$. They are such that $c^\alpha(0) = 1, c^\alpha(\infty) = 0$. From Eq.(6.127) we read the values of the scalar charges

$$\mathcal{S}^\alpha = r_0^\alpha (z_{fix}^\alpha(Q) - z_\infty^\alpha). \quad (6.128)$$

The expression (6.128) agrees, in the special case of a single center black hole, with the well known fact that the scalar charges vanish for double maximal black holes. In the multicenter case we have a similar result, the scalar charges vanish if $z_\infty^\alpha = z_{fix}^\alpha(Q)$, but in this case this does not mean that the scalars are constant in all the space.

Near horizon and not near-horizon geometry

The metric has the form given by Eq.(6.82). with the asymptotic flatness conditions $-g_{rr} \langle \mathcal{R}_\infty | \mathcal{I}_\infty \rangle = 1$ and $\omega_\infty \rightarrow 0$. For point-like sources, as those represented by the ansatz (6.85), the compatibility equation (6.88) takes the form (see, for example, [248])

$$N \equiv \sum_a \langle \mathcal{I}_\infty | q_a \rangle = \langle \mathcal{I}_\infty | Q \rangle = 0. \quad (6.129)$$

An explicit computation of the total field strength shows that Eq.(6.129) is equivalent to the requirement of absence of NUT charges. Only after imposing the condition $N = 0$ the overall integral of the (F^I, G_J) field strengths at infinity, similarly to Eq.(6.72), is equal to $Q = \sum_a q_a$. As one can check, another consequence of the condition $N = 0$ is that the angular speeds are the same in all the componentes of the multicenter black hole horizon and equal to the infinite value.

Let us write a more explicit expression for the g_{rr} component at any space localization. We can write, using the ‘stabilization equation’ (6.84) and the ansatz (6.85), the expression

$$\begin{aligned} \langle \mathcal{R} | \mathcal{I} \rangle &= \left\langle \mathcal{S}\mathcal{I}_\infty + \sum_a \frac{\mathcal{S}q_a}{|\mathbf{x} - \mathbf{x}_a|} | \mathcal{I}_\infty + \sum_b \frac{q_b}{|\mathbf{x} - \mathbf{x}_b|} \right\rangle \\ &= 1 + \sum_b \frac{1}{|\mathbf{x} - \mathbf{x}_b|} (\langle \mathcal{S}\mathcal{I}_\infty | q_b \rangle + \langle \mathcal{S}q_b | \mathcal{I}_\infty \rangle) + \sum_{a,b} \frac{\langle \mathcal{S}q_a | q_b \rangle}{|\mathbf{x} - \mathbf{x}_a| |\mathbf{x} - \mathbf{x}_b|} \\ &= 1 + 2 \sum_b \frac{\langle \mathcal{S}\mathcal{I}_\infty | q_b \rangle}{|\mathbf{x} - \mathbf{x}_b|} + \sum_{a,b} \frac{\langle \mathcal{S}q_a | q_b \rangle}{|\mathbf{x} - \mathbf{x}_a| |\mathbf{x} - \mathbf{x}_b|}. \end{aligned} \quad (6.130)$$

where we have used the property $\mathcal{S}^\dagger = -\mathcal{S}$ and the asymptotic flatness condition $\langle \mathcal{S}\mathcal{I}_\infty | \mathcal{I}_\infty \rangle = 1$. We introduce now the quantities

$$M_a \equiv \langle \mathcal{S}\mathcal{I}_\infty | q_a \rangle, \quad (6.131)$$

$$A_{ab} \equiv \langle \mathcal{S}q_a | q_b \rangle. \quad (6.132)$$

The second one is symmetric in its indices: $A_{ab} = A_{ba}$. With these definitions, we can finally write

$$\langle \mathcal{R} | \mathcal{I} \rangle = 1 + 2 \sum_b \frac{M_b}{|\mathbf{x} - \mathbf{x}_b|} + \sum_{a,b} \frac{A_{ab}}{|\mathbf{x} - \mathbf{x}_a| |\mathbf{x} - \mathbf{x}_b|}. \quad (6.133)$$

If the metric element (6.133) describes a supersymmetric black hole then the right part is always positive, and finite for any finite $|\mathbf{x}|$.

Behaviour at fixed points and at infinity

At spatial infinity $1/|\mathbf{x} - \mathbf{x}_a| \rightarrow 1/r$, we recover spherical symmetry.

$$-g_{rr} \equiv 1 + \frac{2M_{ADM}}{r} + \frac{A_{tot}}{r^2} + o\left(\frac{1}{r}\right)^3 \quad (6.134)$$

$$= 1 + \frac{2 \sum_a M_a}{r} + \frac{\sum_{ab} A_{ab}}{r^2} + o\left(\frac{1}{r}\right)^3, \quad (6.135)$$

The first expression defines the mass M_{ADM} and the ‘‘Area’’ A_{tot} , they are ($Q = \sum_a q_a$)

$$M_{ADM} = \sum_a M_a = \langle \mathcal{S} \mathcal{I}_\infty | Q \rangle, \quad (6.136)$$

$$A_{tot} = \sum_{ab} A_{ab} = \langle \mathcal{S} Q | Q \rangle. \quad (6.137)$$

The expression for the central charge at infinity, Eq.(6.91), becomes then

$$|Z_\infty|^2 = M_{ADM}^2 + N^2 \quad (6.138)$$

where N is defined by Eq.(6.129). The condition $N = 0$ is equivalent to the BPS condition

$$|Z_\infty|^2 = M_{ADM}^2 = |\langle \mathcal{S} \mathcal{I}_\infty | Q \rangle|^2 \quad (6.139)$$

The M_{ADM} quantity, on the opposite of the A_{tot} , depends on the scalar values at infinity through the implicit dependence on them of \mathcal{I}_∞ . These can take arbitrary values there, or at least can be chosen in a continuous range. For one center, for any given charge vector one can obtain a certain particular solution by setting the scalar fields to constant values ($z_{fix}^\alpha = z_\infty^\alpha$) and this gives the minimal possible M_{ADM} mass. For multicenter solutions and generic non-trivial charge vectors, it is not possible to have constant scalar fields. But, still we can proceed to the minimization of $M_{ADM}(z_\infty^\alpha)$ with respect to the scalars at infinity for a given BH configuration. On view of the relation (6.128), if this minimum coincides with $z_\infty^\alpha = z_{fix}^\alpha(Q)$ we would have full analogy with the one center case.

For $\mathbf{x} \rightarrow \mathbf{x}_a$ the metric element given by (6.133) becomes spherically symmetric. Moreover it can be shown, that, by fixing additive integration constants, we can take $\omega_a = \omega(\mathbf{x} \rightarrow \mathbf{x}_a) = 0$ at the same time that $\omega_\infty = \omega(\mathbf{x} \rightarrow \infty) = 0$. As a consequence

the metric at any of the horizon components with charge q_a approaches an $AdS_2 \times S^2$ metric of the form

$$ds^2 = \frac{r^2}{\langle \mathcal{S}q_a | q_a \rangle} dt^2 - \frac{\langle \mathcal{S}q_a | q_a \rangle}{r^2} d\bar{x}^2 \quad (6.140)$$

The near horizon geometry is thus completely determined in terms of the horizon areas $A_{ha} = \langle \mathcal{S}q_a | q_a \rangle$.

The total horizon area A_h is the sum of the areas of its disconnected parts

$$A_h = \sum_a A_{ha} = \sum_a \langle \mathcal{S}q_a | q_a \rangle = 2 \sum_a |Z_{f,a}|^2 \quad (6.141)$$

This expression can be compared with the area corresponding to a single center black hole with charge $Q = \sum_a q_a$, in that case $A_h(q = Q) = \langle \mathcal{S}Q | Q \rangle$.

The relation between the asymptotic ‘‘Area’’ A_{tot} and the multicenter horizon area A_h is simply

$$A_{tot} = \langle \mathcal{S}Q | Q \rangle = \sum_{a,b} \langle \mathcal{S}q_a | q_b \rangle \quad (6.142)$$

$$= S + 2 \sum_{a < b} \langle \mathcal{S}q_a | q_b \rangle. \quad (6.143)$$

For one center blackhole we always have $A_{tot} = A_h$. For a multicenter black hole, the horizon area can be equal, greater or less than A_{tot} (which is also the area of the equivalent single black hole with the same total charge) depending on the last term of Eq.(6.143). For a two center black hole, for example, with charges $q_{1,2}$ the difference is

$$A_{tot} - A_h = 2 \langle \mathcal{S}q_1 | q_2 \rangle. \quad (6.144)$$

Let us study in some detail this last relation and the sign of the right term depending on the two charges. which can be either positive, negative or null. For a general analysis it is useful to define the vectors:

$$q_1 = \lambda_1 Q + \mu_1 \mathcal{S}Q + \delta_1 \hat{s}, \quad (6.145)$$

$$q_2 = \lambda_2 Q + \mu_2 \mathcal{S}Q + \delta_2 \hat{s}. \quad (6.146)$$

The two vectors $q_1 + q_2 = Q$, if we take $\lambda_1 = 1 - \lambda_2 = \lambda, \mu_1 = -\mu_2 = \mu, \delta_1 = -\delta_2 = \delta$. The vector \hat{s} is an arbitrary vector in the orthogonal complement $(Q, \mathcal{S}Q)_\perp$, that means $\langle \hat{s} | Q \rangle = \langle \hat{s} | \mathcal{S}Q \rangle = 0$. For convenience we choose the normalization $\langle \mathcal{S}\hat{s} | \hat{s} \rangle = \langle \mathcal{S}Q | Q \rangle$. With this parametrization of the two charge vectors, their crossed product is

$$\langle \mathcal{S}q_1 | q_2 \rangle = (\lambda_1 \lambda_2 + \mu_1 \mu_2 + \delta_1 \delta_2) \langle \mathcal{S}Q | Q \rangle \quad (6.147)$$

and the expression Eq.(6.144) can be written as

$$\begin{aligned} A_h \equiv A_{h1} + A_{h2} &= A_h(q_1 + q_2) - 2 \langle \mathcal{S}q_1 | q_2 \rangle = (1 - 2(\lambda_1 \lambda_2 + \mu_1 \mu_2 + \delta_1 \delta_2)) \langle \mathcal{S}Q | Q \rangle \\ &= (1 - 2\lambda(1 - \lambda) + 2\mu^2 + 2\delta^2) \langle \mathcal{S}Q | Q \rangle \end{aligned} \quad (6.148)$$

The first contribution in the right part of Eq.(6.149) can take any sign. In particular it is negative for $\lambda \in (0, 1)$, in this case $-2\lambda(1 - \lambda) \in (-1/2, 0)$. The minimum of this term is obtained for the symmetric configuration $\lambda = 1/2$. The other two contributions are always positive. We observe that $S_1 + S_2 = S_{12}$ configuration is allowed only if $\lambda \in (0, 1)$ and

$$\mu^2 + \delta^2 = \lambda(1 - \lambda) \in [0, 1/4]. \quad (6.150)$$

In the opposite side, no such configuration can exist if

$$\mu^2 + \delta^2 > (\lambda(1 - \lambda))_{max} = \frac{1}{4}. \quad (6.151)$$

The fragmentation of an initial charge Q into the longitudinal subspace (with $\lambda \in [0, 1]$) decreases the area. Meanwhile the fragmentation into the subspace generated by $(\mathcal{S}Q, \hat{s})$ is Area increasing.

It has been shown [258] that for this class of models, the single-center BPS extremal black hole area with charge $Q = q_1 + q_2$ is always larger than the corresponding two-center area $A_h(Q = q_1 + q_2) > A_{h1} + A_{h2}$. That is

$$0 < A_h(q_1 + q_2) - (A_{h1} + A_{h2}) = 2 \langle \mathcal{S}q_1 | q_2 \rangle = 2(\lambda(1 - \lambda) - \mu^2 - \delta^2) \langle \mathcal{S}Q | Q \rangle \quad (6.152)$$

We have then, for this case, assuming $\langle \mathcal{S}Q | Q \rangle$

$$\lambda(1 - \lambda) - \mu^2 - \delta^2 > 0, \quad (6.153)$$

$$\lambda(1 - \lambda) > \mu^2 + \delta^2. \quad (6.154)$$

More on the asymptotic flatness condition

In this section we will establish an explicit relation between the parameter \mathcal{I}_∞ appearing in the black hole ansatz and the values of the scalars z_∞^α at infinity. It can be shown that the vector \mathcal{I}_∞ is directly related to the charge vectors q_a and M_{ADM} . For this purpose we decompose the $2n_v + 2$ dimensional \mathcal{I}_∞ vector in a basis formed by the charge vectors q_a , $\mathcal{S}^\dagger q_a$ and possibly as many other vectors needed to complete a basis. The dimension of the space $q_{||}$ = generated by the n_a pairs $(q_a, \mathcal{S}^\dagger q_a)$ is in general $dim(q_{||}) \leq 2n_a$. The dimension of the orthogonal complement to this space, i.e. those vectors s such that $\langle q | s \rangle = \langle \mathcal{S}q | s \rangle = 0$ is $dim(q_\perp) = 2(n_v - n_a) + 2$. This dimension is zero for one scalar, one center black holes ($n_v = 0, n_a = 1$). ...We will see how black hole extremality imposes strong conditions on such extra vectors. Let us consider here the case of a single center black hole and an arbitrary number of scalars where the method is illustrated.

We assume that the vector \mathcal{I} is decomposed as (with $\langle \mathcal{S}q | q \rangle \neq 0$)

$$\mathcal{I}_\infty = \alpha q + \beta \mathcal{S}^\dagger q + \gamma s + \epsilon \mathcal{S}^\dagger s, \quad (6.155)$$

where $\alpha, \beta, \gamma, \epsilon \in \mathbb{R}$ and s is an arbitrary but fixed, normalized vector such that $s \in q_\perp$, i.e.

$$\langle s | q \rangle = \langle s | \mathcal{S}^\dagger q \rangle = 0, \quad \langle \mathcal{S} s | s \rangle = 1. \quad (6.156)$$

Such a vector s can be always determined by a modified Gram-Schmidt procedure for a given pair of vectors $(q, \mathcal{S}^\dagger q)$. By projecting the relation (6.155) over any of the individual vectors, we get

$$\langle \mathcal{I}_\infty | q \rangle = \beta \langle \mathcal{S}^\dagger q | q \rangle = -\beta \langle \mathcal{S} q | q \rangle, \quad (6.157)$$

$$\langle \mathcal{I}_\infty | \mathcal{S}^\dagger q \rangle = \langle \mathcal{S} \mathcal{I}_\infty | q \rangle = \alpha \langle q | \mathcal{S}^\dagger q \rangle = \alpha \langle \mathcal{S} q | q \rangle. \quad (6.158)$$

Using the expressions (6.129,6.136,6.137) we can rewrite these last two expressions respectively as

$$N = -\beta A_{tot}, \quad (6.159)$$

$$M_{ADM} = \alpha A_{tot} \quad (6.160)$$

from where we read the values of the α, β coefficients in terms of some other physical parameters.

Let us consider now the asymptotic flatness condition and apply the ansatz (6.155) for \mathcal{I}_∞ . We have, using the α, β values, the definition $\Delta^2 = (\gamma^2 + \epsilon^2)$ and the Eq.(6.138), the expression

$$\langle \mathcal{S} \mathcal{I}_\infty | \mathcal{I}_\infty \rangle = 1 = (\alpha^2 + \beta^2) \langle \mathcal{S} q | q \rangle + (\gamma^2 + \epsilon^2) \langle \mathcal{S} s | s \rangle \quad (6.161)$$

$$= (M_{ADM}^2 + N^2) \langle \mathcal{S} q | q \rangle / A_{tot}^2 + \Delta^2. \quad (6.162)$$

or, equivalently,

$$|Z_\infty|^2 = M_{ADM}^2 + N^2 = \langle \mathcal{S} q | q \rangle (1 - \Delta^2). \quad (6.163)$$

The BPS condition $|Z_\infty| = M_{ADM} = \langle \mathcal{S} q | q \rangle$ is only fulfilled if $N = 0$ (in concordance with Eq.(6.139)) and $\Delta = 0$.

The vanishing of these quantities can be directly seen by imposing extremality in the metric elements, by requesting extremal RN black hole type metric, or, $-g_{rr} \sim f^2$ with f an spatially harmonic function. The metric component g_{rr} at infinity is

$$-g_{rr} = 1 + \frac{2M_{ADM}}{r} + \frac{\langle \mathcal{S} q | q \rangle}{r^2}, \quad (6.164)$$

$$= 1 + \frac{2M_{ADM}}{r} + \frac{(M_{ADM}^2 + N^2)/(1 - \Delta^2)}{r^2}, \quad (6.165)$$

$$= \left(1 + \frac{M_{ADM}}{r}\right)^2 + \frac{1}{r^2} \left(\frac{M_{ADM}^2 \Delta^2}{1 - \Delta^2} + \frac{N^2}{1 - \Delta^2}\right). \quad (6.166)$$

We observe from the last line above that the conditions $N = 0, \Delta = 0$ are necessary conditions recover an extremal RN black hole type metric. In this case

$$|Z_\infty|^2 = M_{ADM}^2 \quad (6.167)$$

$$= \langle \mathcal{S} q | q \rangle. \quad (6.168)$$

The first equality, Eq.(6.167), comes by directly imposing the condition $N = 0$. The second equality, (6.168), comes however from the vanishing of the non-extremality parameter Δ ($\gamma = \epsilon = 0$ in Eq.(6.155)), i.e. from requiring that \mathcal{I}_∞ is fully contained in the subspace $q_{\parallel} = \text{span}(q, \mathcal{S}q)$.

After imposing the conditions $N = \Delta = 0$, then \mathcal{I}_∞ is simply proportional to the charge vector and the full black hole ansatz becomes

$$\mathcal{I} = \frac{q}{M_{ADM}} \left(1 + \frac{M_{ADM}}{r} \right). \quad (6.169)$$

6.7 Concluding remarks

We have presented a systematic study of general, stationary, multicenter black hole solutions in ungauged four dimensional Einstein-Maxwell $N = 2$ supergravity theories minimally coupled to scalars, i.e. theories with quadratic prepotentials. An important part of our analysis has been based on the stabilization matrices $\mathcal{S}_F, \mathcal{S}_N$ and some new matrices, their symplectic adjoints. These stabilization matrices are isometries of the symplectic quadratic form. Their adjoints with respect to the symplectic product $\mathcal{S}_N^\dagger, \mathcal{S}_F^\dagger$, which fulfill the property $\mathcal{S}^2 = -1$, are shown to lay inside the Lie algebra of the isometry group, they are such that $\mathcal{S} + \mathcal{S}^\dagger = 0$. They are “unitary”, $\mathcal{S}\mathcal{S}^\dagger = 1$, with respect to the symplectic product.

Using a generic multicenter ansatz, Eq.6.85, depending on the center charges q_a and the value at infinity \mathcal{I}_∞ , and the attractor equations we have rederived in a simple way different relations. In particular the fix point scalar values have been written in terms of the projection of the respective charges into the eigenspaces of the matrix \mathcal{S} , Eq.(6.115). The \mathcal{I}_∞ quantity, in analogy to the center charges, formally follows an effective attractor equation at infinity. The horizon attractor mechanism equations and the relation of \mathcal{I}_∞ and the scalar moduli can be written in a suggestive unified form, Eq.(6.120,6.121),

$$\Omega_{fix} = P_- q, \quad (6.170)$$

$$\Omega_\infty = P_- \mathcal{I}_\infty. \quad (6.171)$$

Explicit expressions for the spatial dependence of the scalars are written which connect the \mathcal{S} matrix, their values at infinity and the \mathcal{I}_∞ symplectic vector, Eq.(6.122). This spatial dependence interpolates between the values of the scalars at the fixed points (the horizons and the effective infinity “fixed point” previously introduced). As a consequence, the vanishing of the, non conserved, scalar charges Σ^α , is shown to be equivalent to the vanishing of the quantities $z_\infty^\alpha = z_{fix}^\alpha(Q)$ where $z_{fix}^\alpha(Q)$ are the fixed values of the scalars for a single center black hole of total charge $Q = \sum q_a$. This is in close analogy with the single center case, where the vanishing of the scalar charges is a necessary and sufficient conditions for the double extremality of the black hole [238].

The study of the near horizon and infinity geometry of the black hole lead us to the consideration of the area-like quantities $A_{ab} = \langle \mathcal{S}q_a | q_b \rangle$ and $A_{tot} = \sum A_{ab}$ in

addition to the horizon areas $A_{ha} \langle \mathcal{S}q_a | q_a \rangle$. Using different projection operators constructed from the matrix \mathcal{S} , it is discussed the relation between the horizon area of a multicenter black hole and that one of an equivalent single black hole with the same total charge. We have seen that the fragmentation of an initial charge Q into the longitudinal subspace decreases the area. Meanwhile the fragmentation into a orthogonal subspace generated would be area increasing. Finally we have studied diverse properties and given some explicit expression of the quantity \mathcal{I}_∞ by expanding it in a certain symplectic basis. The $2n_V + 2$ dimensional symplectic space is decomposed into a $2n_a$ dimensional subspace generated by the n_a center charges q_a and their associated vectors $\mathcal{S}^\dagger q_a$ (or $\mathcal{S}q_a$) and its orthogonal complement subspace (possibly of dimension zero depending on the number and on the linear dependency of center charge vectors). In the case of quadratic prepotentials this separation into “charge-longitudinal” and “transversal” subspaces is global. A similar, but local, scalar-dependent, separation can be advantageously considered also for generic prepotentials, or even theories without them.

The projection of any symplectic vector appearing in the theory, for example a subset of the charge vectors themselves or vectors characterising the black-hole ansatz at infinity, in terms of these new basis might be of general interest. The use of this projection, as it has been shown here, in particular the understanding of questions as entropy increasing effects in the fragmentation of a single center black holes into two or more centers, or the maximality of the solutions, in terms, for example, of simple considerations of the dimensions of the each of the charge-longitudinal and transversal subspaces. Preliminary results show that the use of these projection techniques together with the careful use of asymptotic unitary are sufficient to write down generalized multicenter mass formulas involving the mass of the black hole, angular momentum, horizon areas and other quantities as the “crossed central charge matrix” $Z_{ij} = \langle \mathcal{S}_\infty q_i | q_j \rangle$.

Although we have focused in this study in minimal coupling theories with quadratic prepotentials, the main techniques, properties and expressions presented are extendable to general prepotentials or even theories without them.

Chapter 7

Conclusions and prospects

This work comprises an analysis of theoretical topics of supergravity. A considerable part of it has dealt with the construction of gauged supergravities and its understanding from the string theory viewpoint. The last part treats the search of multicenter black hole solutions in $N = 2$ $D = 4$ supergravity.

The first part is about the structure of gauged supergravities and the embedding tensor formalism. We have introduced this mechanism as a covariant tool to generate gauged supergravities. Even more, this formalism scans along all the possible combinations of the global symmetry generators catching all the gaugings allowed by the global symmetry that the ungauged theory enjoys.

We have applied the embedding-tensor formalism to the study of the most general deformations (*i.e.* gaugings and massive deformations) of maximal 9-dimensional supergravity. We have used the complete global $SL(2, \mathbb{R}) \times \mathbb{R}^2$ symmetry of its equations of motion, which includes the so-called *trombone symmetry*. We have found the constraints that the deformation parameters must satisfy in order to preserve both gauge and supersymmetry invariance (the latter imposed through the closure of the local supersymmetry algebra to lowest order in fermions). We have used most of the constraints to express some components of the deformation tensors in terms of a few components of the embedding tensor which we take to be independent and which are given in Eq. (4.146). At that point we have started making contact with the results of Ref. [117], since those independent components are precisely the 8 possible deformations identified there. All of them have a higher-dimensional origin discussed in detail in Ref. [117]. The field strengths, gauge transformations and supersymmetry transformations of the deformed theory, written in terms of the independent deformation tensors, are collected in Appendix B.4.

The 8 independent deformation tensors are still subject to quadratic constraints, given in Eq. (4.148), but those constraints cannot be used to express analytically some of them in terms of the rest, and, therefore, we must keep the 8 deformation parameters and we must enforce these irreducible quadratic constraints.

In Section 4.4 we have used our knowledge of the global symmetries (and corresponding Noether 1-forms), the independent deformation tensors and the irreducible

quadratic constraints of the theory, together with the general arguments of Section 4.2.2 to determine the possible 7-, 8- and 9-forms of the theory (Table 4.4), which are dual to the Noether currents, independent deformation tensors and irreducible quadratic constraints. We have compared this spectrum of higher-rank forms with the results of Refs. [167, 168], based on E_{11} level decomposition. We have found that, in the sector unrelated to the trombone symmetry, which was excluded from that analysis, the embedding-tensor formalism predicts one doublet of 9-forms less than the E_{11} approach. However, both predictions are not contradictory: the extra doublet of 9-forms may not survive the deformations on which the embedding-tensor formalism is built: new 9-form Stückelberg shifts proportional to the deformation parameters may occur that can be used to eliminate it so only one combination of the two 9-form doubles survives. This mechanism is present in the $N = 2$ $d = 4, 5, 6$ theories [119], although the physics behind it is a bit mysterious.

The second part treats gauged supergravities and their origin from SS compactifications of higher-dimensional supergravities. Once we have a tool that provides all the possible deformations of a given supergravity, we decided to use it to extend this classification to lower dimensional theories. We performed the orbit classification of maximal and half-maximal $D = 9, 8, 7$ theories. The aim of this work is not only interesting by itself, but also is a reference to understand what orbits have a geometric origin, in the sense of arising from a SS compactification of a higher-dimensional theory. Since there is a mismatching between the existence of some gauged supergravities and the gaugings that arise from flux compactifications, several T duality constructions emerged to justify this information leak in the dimensional reduction procedure. Once again, we want to remark that the embedding tensor formalism is essential because it tells us all the possible gaugings and guarantees the existence of no more than the ones found. In other case, we would work with a set of gaugings without being sure that it is expandable and the comparison with the flux compactification gaugings could not be performed in a systematical way.

In chapter 5 we have provided a litmus test to the notion of non-geometry, by classifying the explicit orbits of consistent gaugings of different supergravity theories, and considering the possible higher-dimensional origins of these. The results turn out to be fundamentally different for the cases of U-duality orbits of maximal supergravities, and T-duality orbits of half-maximal theories.

In the former case we have managed to explicitly classify all U-duality orbits in dimensions $8 \leq D \leq 11$. This led to zero, one, four and ten discrete orbits in dimensions $D = 11, 10, 9$ and 8 , respectively, with different associated gauge groups. Remarkably, we have found that all of these orbits have a higher-dimensional origin via some geometric compactification, be it twisted reductions or compactifications on group manifolds or coset spaces. In our parlance, we have therefore found that all U-duality orbits are geometric. The structure of U-duality orbits is therefore dramatically different from the sketch of figure 1 in the introduction. Although a full classification of all orbits in lower-dimensional cases becomes increasingly cumbersome, we are not aware of any

examples that are known to be non-geometric. It could therefore hold in full generality that all U-duality orbits are necessarily geometric.

This is certainly not the case for T-duality orbits of gaugings of half-maximal supergravities. In this case, we have provided the explicit classification in dimensions $7 \leq D \leq 10$ (where in $D = 7$ we have only included three-form fluxes). The numbers of distinct families of orbits in this case are zero, one, three and eleven in dimensions $D = 10, 9, 8$ and 7 , respectively, which includes both discrete and one-parameter orbits. A number of these orbits do not have a higher-dimensional origin in terms of a geometric compactification. Such cases are **orbits 2** and **3** in $D = 8$ and **orbits 1, 2** and **3** in $D = 7$ for $\alpha \neq 0$. Indeed, these are exactly the orbits that do not admit an uplift to the maximal theory. As proven in section 5.3.2, all such orbits necessarily violate the weak and/or strong constraints, and therefore need truly doubled backgrounds. Thus, the structure of T-duality orbits is very reminiscent of figure 1 in the introduction. Given the complications that already arise in these simpler higher-dimensional variants, one can anticipate that the situation will be similar in four-dimensional half-maximal supergravity.

Fortunately, the formalism of double field theory seems tailor-made to generate additional T-duality orbits of half-maximal supergravity. Building on the recent generalisation of the definition of double field theory [216], we have demonstrated that all T-duality orbits, including the non-geometric ones in $D = 7, 8$, can be generated by a twisted reduction of double field theory. We have explicitly provided duality twists for all orbits. For locally-geometric orbits the twists only depend on the physical coordinates y , while for the non-geometric orbits these necessarily also include \tilde{y} . Again, based on our exhaustive analysis in higher-dimensions, one could conjecture that also in lower-dimensional theories, all T-duality orbits follow from this generalised notion of double field theory.

At this point we would like to stress once more that a given orbit of gaugings can be generated from different twist orbits. Therefore, there is a degeneracy in the space of twist orbits giving rise to a particular orbit of gaugings. Interestingly, as it is the case of **orbit 6** in $D = 7$ for instance, one might find two different twist orbits reproducing the same orbit of gaugings, one violating weak and strong constraints, the other one satisfying both. Our notion of a locally geometric orbit of gaugings is related to the existence of at least one undoubled background giving rise to it. However, this ambiguity seems to be peculiar of gaugings containing Q flux. These can, in principle, be independently obtained by either adding a β but no \tilde{y} dependence (locally geometric choice, usually called T-fold), or by including non-trivial \tilde{y} dependence but no β (non-geometric choice) [215].

Another remarkable degeneracy occurs for the case of semi-simple gaugings, corresponding to **orbits 1 – 3** in $D = 7$. For the special case of $\alpha = 0$, we have two possible ways of generating such orbits from higher-dimensions: either a coset reduction over a sphere or analytic continuations thereof, or a duality twist involving non-geometric coordinate dependence. Therefore d -dimensional coset reductions seem to be equiv-

alent to $2d$ -dimensional twisted torus reductions (with the latter in fact being more general, as it leads to all values of α). Considering the complications that generally arise in proving the consistency of coset reductions, this is a remarkable reformulation that would be interesting to understand in more detail. Furthermore, when extending the notion of double field theory to type II and M-theory, this relation could also shed new light on the consistency of the notoriously difficult four-, five- and seven-sphere reductions of these theories.

Our results mainly focus on Scherk-Schwarz compactifications leading to gauged supergravities with vanishing ξ_M fluxes. In addition, we have restricted to the NSNS sector and ignored α' -effects. Also, we stress once again that relaxing the strong and weak constraints is crucial in part of our analysis. If we kept the weak constraint, typically the Jacobi identities would lead to backgrounds satisfying also the strong constraint [216]. However, from a purely (double) field theoretical analysis the weak constraint is not necessary. A sigma model analysis beyond tori would help us to clarify the relation between DFT without the weak and strong constraints and string field theory on more general backgrounds. We hope to come back to this point in the future.

At this point, we wonder whether we could generalize this study to lower dimensions. Unfortunately, this is a considerable more complicated goal, due to how the global symmetry groups quickly grow. This means that the classification of the orbits is extraordinarily difficult. However, some questions based on some insights of our results could be set out. What is the relation between geometric orbits and maximal supergravities? That is, is there any underlying reason why the maximal theories analyzed only host geometric orbits? On the other hand, we wonder whether all the gaugings of half-maximal theories have a description in terms of DFT. What about 1/4-BPS states? Is DFT powerful enough to reproduce those solutions? Do these states violate even the relaxed version of the strong constraint? What about the supersymmetric completion of DFT? This is an issue that has already been studied. Finally, a sizzling problem is the generalization of DFT towards the M theory goal. Some hot off the press attempts have been constructed [278–280].

The third part of the manuscript treats the multicenter black hole solutions in $N = 2$ theories. Despite of an exact solution was worked out in [281], it is not trivial to find a set of parameters that satisfy the physical constraints of these solutions. We provide a general mechanism for theories with a quadratic prepotentials that, in some way, rearranges the fashion in that black hole solutions are found. That is, we are able to implement the restrictions over the physical parameters (mass, areas, NUT charges, etc) at the very beginning, guaranteeing the validity of our solutions. The stubborn (in a friendly way) attractor mechanism is continuously telling us that the final configuration, among other things, only depends on the electromagnetic charges of the BH, and we use this advice to formulate the problem in terms of a particular basis of the symplectic vector space.

In chapter 6 we have presented a systematic study of general, stationary, multicenter black hole solutions in ungauged four dimensional Einstein-Maxwell $N = 2$ supergrav-

ity theories minimally coupled to scalars, i.e. theories with quadratic prepotentials. An important part of our analysis has been based on the stabilization matrices $\mathcal{S}_F, \mathcal{S}_N$ and some new matrices, their symplectic adjoints. These stabilization matrices are isometries of the symplectic quadratic form. Their adjoints with respect to the symplectic product $\mathcal{S}_N^\dagger, \mathcal{S}_F^\dagger$, which fulfill the property $\mathcal{S}^2 = -1$, are shown to lay inside the Lie algebra of the isometry group, they are such that $\mathcal{S} + \mathcal{S}^\dagger = 0$. They are “unitary”, $\mathcal{S}\mathcal{S}^\dagger = 1$, with respect the symplectic product.

Using a generic multicenter ansatz, Eq.6.85, depending on the center charges q_a and the value at infinity \mathcal{I}_∞ , and the attractor equations we have rederived in a simple way different relations. In particular the fix point scalar values have been written in terms of the projection of the respective charges into the eigenspaces of the matrix \mathcal{S} , Eq.(6.115). The \mathcal{I}_∞ quantity, in analogy to the center charges, formally follows an effective attractor equation at infinity. The horizon attractor mechanism equations and the relation of \mathcal{I}_∞ and the scalar moduli can be written in a suggestive unified form, Eq.(6.120,6.121),

$$\Omega_{fix} = P_- q, \quad (7.1)$$

$$\Omega_\infty = P_- \mathcal{I}_\infty. \quad (7.2)$$

Explicit expressions for the spatial dependence of the scalars are written which connect the \mathcal{S} matrix, their values at infinity and the \mathcal{I}_∞ symplectic vector, Eq.(6.122). This spatial dependence interpolates between the values of the scalars at the fixed points (the horizons and the effective infinity “fixed point” previously introduced). As a consequence, the vanishing of the, non conserved, scalar charges Σ^α , is shown to be equivalent to the vanishing of the quantities $z_\infty^\alpha = z_{fix}^\alpha(Q)$ where $z_{fix}^\alpha(Q)$ are the fixed values of the scalars for a single center black hole of total charge $Q = \sum q_a$. This is in close analogy with the single center case, where the vanishing of the scalar charges is a necessary and sufficient conditions for the double extremality of the black hole [238].

The study of the near horizon and infinity geometry of the black hole lead us to the consideration of the area-like quantities $A_{ab} = \langle \mathcal{S}q_a | q_b \rangle$ and $A_{tot} = \sum A_{ab}$ in addition to the horizon areas $A_{ha} \langle \mathcal{S}q_a | q_a \rangle$. Using different projection operators constructed from the matrix \mathcal{S} , it is discussed the relation between the horizon area of a multicenter black hole and that one of an equivalent single black hole with the same total charge. We have seen that the fragmentation of an initial charge Q into the longitudinal subspace decreases the area. Meanwhile the fragmentation into a orthogonal subspace generated would be area increasing. Finally we have studied diverse properties and given some explicit expression of the quantity \mathcal{I}_∞ by expanding it in a certain symplectic basis. The $2n_V + 2$ dimensional symplectic space is decomposed into a $2n_a$ dimensional subspace generated by the n_a center charges q_a and their associated vectors $\mathcal{S}^\dagger q_a$ (or $\mathcal{S}q_a$) and its orthogonal complement subspace (possibly of dimension zero depending on the number and on the linear dependency of center charge vectors). In the case of quadratic prepotentials this separation into “charge-longitudinal” and “transversal” subspaces is global. A similar, but local, scalar-dependent, separa-

tion can be advantageously considered also for generic prepotentials, or even theories without them.

The projection of any symplectic vector appearing in the theory, for example a subset of the charge vectors themselves or vectors characterising the black-hole ansatz at infinity, in terms of these new basis might be of general interest. The use of this projection, as it has been shown here, in particular the understanding of questions as entropy increasing effects in the fragmentation of a single center black holes into two or more centers, or the maximality of the solutions, in terms, for example, of simple considerations of the dimensions of the each of the charge-longitudinal and transversal subspaces. Preliminary results show that the use of these projection techniques together with the careful use of asymptotic unitary are sufficient to write down generalized multicenter mass formulas involving the mass of the black hole, angular momentum, horizon areas and other quantities as the “crossed central charge matrix” $Z_{ij} = \langle \mathcal{S}_\infty q_i | q_j \rangle$.

Although we have focused in this study in minimal coupling theories with quadratic prepotentials, the main techniques, properties and expressions presented are extendable to general prepotentials or even theories without them.

Appendix A

Dimensional reduction and general properties

Dimensional reduction is a mechanism that allows to convert a theory defined in a given dimension into a lower-dimensional theory. Despite of this is essential to justify the plausible existence of theories that, by construction, necessarily live in $D > 4$, its origin was the opposite: Kaluza and Klein tried to unify electromagnetism and gravity formulating a 5-dimensional theory and applying a KK dimensional reduction on a circle. When more than one dimensions have to be compactified, a wide variety of manifolds can be used and, depending of their topological characteristics, the properties of the effective theory show the consequences of the choice.

The relation between dimensional reduction and SUGRA becomes crucial, especially in the work that we are going to perform in the following chapters. After applying this mechanism to higher-dimensional SUGRAs, we will motivate the emergence of a new field theory, Double Field Theory.

A.1 Kaluza-Klein reduction

We will briefly show a Kaluza-Klein (KK) reduction on a scalar manifold from $\hat{D} = D + 1$ to D dimensions on a circle. Hatted indices and fields are defined on \hat{D} dimensions whereas the unhatted ones correspond to D dimensions. We will refer as z the compactified coordinate.

Let us consider a massless scalar field $\hat{\phi}(x^{\hat{\mu}})$, where $x^{\hat{\mu}} = (x^\mu, z)$. This field satisfies the Klein-Gordon equation

$$(\hat{\square} - m^2)\hat{\phi} = \left(\square + \frac{\partial^2}{\partial z^2} - m^2\right)\hat{\phi} = 0. \quad (\text{A.1})$$

This equation admits a Fourier decomposition of the field as solution,

$$\hat{\phi}(x^\mu, z) = \sum_{k=-\infty}^{+\infty} e^{ikz/L} \phi_k(x^\mu), \quad (\text{A.2})$$

such that any spacetime function associated to the k th Fourier mode, $\phi_k(x^\mu)$, satisfies

$$\left[\square - \left(\frac{k}{L} \right)^2 - m^2 \right] \phi_k = 0. \quad (\text{A.3})$$

Thus, the spectrum of the theory contains an infinite set of massive scalar fields, even having started with $m = 0$. These are the so-called KK modes; only one of them becomes massless, the case $k = 0$.

The equations of motion (A.3) are decoupled, so that we can deal with all of the states independently. This is useful because when performing the reduction, one keeps only a finite set of modes, which must be a *consistent truncation* of the full set.

In fact, a consistent truncation in Kaluza-Klein reduction [282] is one such that all the gauge bosons of the isometry group G of the compact manifold are retained in a truncation keeping only a finite number of lower-dimensional fields, with the additional condition that setting the truncated fields to zero is consistent with the eoms of the higher-dimensional fields. That is, the reduction ansatz is consistent if all the higher-dimensional equations of motion are satisfied as a consequence of the equations of motion for the surviving fields in the lower-dimensional theory.

It is common to truncate to the massless sector because, as we see in (A.3), the masses are inversely proportional to the characteristic size of the internal manifold. Thus, since we, effectively, live in $D = 4$, any internal direction is very small and therefore the mass of states with non-zero momentum blows up. Then, we can discard these modes because they are too massive and not physically interesting. However, the size of the internal manifold is not essential to argue the truncation to lightest modes. From the consistency viewpoint, the solution $\square\phi_0 = 0$ and $\phi_n = 0, \forall n \neq 0$ is also a solution for $\hat{\square}\hat{\phi} = 0$ and, as we have mentioned, this is enough for this to be a consistent truncation.

Thus, we conclude that dimensional reduction is an expansion of the fields over the internal manifold and a truncation to the lightest modes. However, in practice we make a reduction ansatz, relating higher-dimensional fields to a set of lower-dimensional ones, so that the latter are taken to be the lightest sector of the expansion. This reduction ansatz can be formulated by using internal coordinates, so that the equations of motion (or Lagrangian) could explicitly show this dependence. Nevertheless, it is strictly necessary that the effective theory does not exhibit this dependence.

Let us overview some reduction schemes in the following sections to understand the different properties that the lower-dimensional theory enjoys.

A.2 Toroidal reduction

In this section, we will consider toroidal reductions of pure gravity,

$$\hat{\mathcal{L}} = \sqrt{|\hat{g}|} \hat{R}. \quad (\text{A.4})$$

Firstly, let us consider a reduction from \hat{D} to $D = \hat{D} - 1$ dimensions in a circle. Due to its periodicity, the metric admits a Fourier decomposition,

$$\hat{g}_{\mu\nu}(x^\mu, z) = \sum_n g_{\mu\nu}^{(n)}(x^\mu) e^{inz/R} \quad , \quad (\text{A.5})$$

where R is the compactification radius. Since all the non-zero modes have a mass proportional to $1/R$, we restrict ourselves to the massless sector, which is indeed a consistent truncation. Our higher-dimensional metric can be decomposed as

$$\hat{d}s^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi} (dz + A_\mu dx^\mu)^2 \quad . \quad (\text{A.6})$$

Since our ansatz does not depend on the internal coordinates, consistency is guaranteed. The ansatz gives rise to the following lower-dimensional lagrangian

$$\mathcal{L} = \sqrt{|g|} \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{2(\beta-\alpha)\phi} F^2 \right) \quad , \quad (\text{A.7})$$

where we impose α and β to be

$$\alpha^2 = \frac{1}{2(D-2)(D-1)} \quad , \quad \beta = -(D-2)\alpha \quad , \quad (\text{A.8})$$

in order to have a Ricci scalar without dilaton coupling and a suitable dilaton kinetic term. This lagrangian led to Kaluza [26] and Klein [27] to think about the possibility of unifying gravity and electromagnetism when they performed this reduction for $D = 5$. The corresponding equations of motion are

$$G_{\mu\nu} = \frac{1}{2} \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}(\partial\phi)^2 g_{\mu\nu} \right) + \frac{1}{2} e^{-2(D-1)\alpha\phi} \left(F_\mu{}^\rho F_\nu{}^\rho - \frac{1}{4} F^2 g_{\mu\nu} \right) \quad , \quad (\text{A.9})$$

$$\nabla_\mu (e^{-2(D-1)\alpha\phi} F^{\mu\nu}) = 0 \quad , \quad (\text{A.10})$$

$$\square\phi = -\frac{1}{2}(D-1)\alpha e^{-2(D-1)\alpha\phi} F^2 \quad . \quad (\text{A.11})$$

From (A.11), we notice that ϕ cannot be set to constant in order to get electromagnetism; in fact it has to be a dynamical field.

From the lagrangian, we infer that the symmetry of the higher-dimensional space, diffeomorphisms in \hat{D} dimensions, has splitted into diffeomorphisms in D dimensions plus an internal (gauge) symmetry. In more detail, we can see that the general coordinate transformation under which the lagrangian (A.4) is invariant,

$$\delta x^{\hat{\mu}} = -\hat{\xi}^{\hat{\mu}} \quad , \quad \delta \hat{g}_{\hat{\mu}\hat{\nu}} = \hat{\xi}^{\hat{\rho}} \partial_{\hat{\rho}} \hat{g}_{\hat{\mu}\hat{\nu}} + \hat{g}_{\hat{\rho}\hat{\nu}} \partial_{\hat{\mu}} \hat{\xi}^{\hat{\rho}} + \hat{g}_{\hat{\mu}\hat{\rho}} \partial_{\hat{\nu}} \hat{\xi}^{\hat{\rho}} \quad , \quad (\text{A.12})$$

has transformed into

$$\hat{\xi}^{\hat{\mu}} = \xi^\mu \quad , \quad \hat{\xi}^z = \lambda(x) + cz \quad . \quad (\text{A.13})$$

That is, $(D + 1)$ -dimensional diffeomorphisms have transformed into D -dimensional diffeomorphisms, a $U(1)$ gauge transformation and a global scale symmetry.

What we have done up to now can be generalized to a torus $T^n = S^1 \times \dots \times S^1$ when the reduction is from \hat{D} to $D = \hat{D} - n$ dimensions. In this case, a valid ansatz is

$$\hat{d}s^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi} M_{mn} (dz^m + A_\mu^m dx^\mu) (dz^n + A_\nu^n dx^\nu) \quad , \quad (\text{A.14})$$

where M_{mn} is a scalar matrix parameterized by $SL(n)/SO(n)$. We find that the Einstein-Hilbert lagrangian converts into

$$\mathcal{L} = \sqrt{|g|} \left(R - \frac{1}{2}(\partial\phi)^2 + \frac{1}{4}\text{Tr}(\partial^\mu M_{mn}\partial_\mu M^{mn}) - \frac{1}{4}e^{2(\beta-\alpha)\phi} M_{mn} F^m F^n \right) \quad , \quad (\text{A.15})$$

where

$$\alpha^2 = \frac{n}{2(D+n-2)(D-2)} \quad , \quad \beta = -\frac{(D-2)\alpha}{n} \quad . \quad (\text{A.16})$$

It is natural then to wonder what symmetries induce the \hat{D} -dimensional diffeomorphisms in D dimensions. The answer is:

$$\hat{\xi}^\mu = \xi^\mu \quad , \quad \hat{\xi}^m = \lambda^m(x) + \Lambda^m_n z^n \quad . \quad (\text{A.17})$$

So, we have a D -dimensional general coordinate transformations, $U(1)^n$ gauge transformations and a global $GL(n, \mathbb{R})$ symmetry.¹

For completeness arguments, let us consider, at least schematically, the reduction of a gauge potential p -form over a circle. Let us assume a naive lagrangian made out by a p -form $\hat{C}^{(p)}$ coupled to a dilaton $\hat{\varphi}$,

$$\hat{\mathcal{L}} = \sqrt{|\hat{g}|} \left(-\frac{1}{2}(\partial\hat{\varphi})^2 - \frac{1}{2}e^{a\hat{\varphi}} \hat{G}^{(p+1)} \cdot \hat{G}^{(p+1)} \right) \quad , \quad (\text{A.18})$$

where $\hat{G}^{(p+1)} = d\hat{C}^{(p)}$. Additionally to the gravity ansatz, we can take

$$\hat{C}^{(p)} = C^{(p)} + (dz + A) \wedge C^{(p-1)} \quad , \quad \hat{\varphi} = \varphi \quad . \quad (\text{A.19})$$

Substituting in the lagrangian, we have

$$\mathcal{L} = \sqrt{|g|} \left(-\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}e^{a\varphi-2p\alpha\phi} G^{(p+1)} \cdot G^{(p+1)} - \frac{1}{2}e^{a\varphi+2(D-p-1)\alpha\phi} G^{(p)} \cdot G^{(p)} \right) \quad , \quad (\text{A.20})$$

where the field strengths are $G^{(p+1)} = dC^{(p)} + F \wedge dC^{(p-1)}$ and $G^{(p)} = dC^{(p-1)}$. Thus, in general a suitable reduction ansatz of a p -form over a n -torus implies

$$\binom{n}{D-\tilde{p}} \quad (\text{A.21})$$

¹Mention about the $GL(n, \mathbb{R}) = SL(n, \mathbb{R}) \times \mathbb{R}^+$ splitting and the trombone symmetry linear combination to get the right trombone symmetry in $D = \hat{D} - n$ dimensions.

forms of rank \tilde{p} . In this case, the gauge symmetry of the p -form

$$\delta\hat{C}^{(p)} = d\hat{\lambda}^{(p-1)} \quad , \quad (\text{A.22})$$

splits up into different lower-dimension gauge transformations

$$\hat{\lambda}^{(p-1)} = \lambda^{(p-1)} + (dz + A) \wedge d\lambda^{(p-2)} \quad , \quad (\text{A.23})$$

where the gauge parameters $\lambda^{(d-1)}$, $\lambda^{(d-2)}$ correspond to the $C^{(d)}$, $C^{(d-1)}$ potentials, respectively. Moreover, these potentials also transform properly under (A.17), because the lagrangian (A.20) also satisfied the higher-dimensional general coordinate transformations (A.12).

A.3 Twisted reduction

We can go beyond toroidal reduction by including a new ingredient in our reduction scheme: new periodic conditions in which the field is not exactly the same one once we have surrounded the circle, but it is a field transformed by an element of the global symmetry group of the theory. That is, if we have a theory whose global symmetry group is G and the scalar field transforms as $\phi \rightarrow g\phi$, $g \in G$, we can impose the following twisted boundary condition;

$$\hat{\phi}(x^\mu, 2\pi R) \rightarrow M(g)\hat{\phi}(x^\mu, 0) \quad . \quad (\text{A.24})$$

This is known as a *monodromy transformation*. If we assume the following ansatz,

$$\hat{\phi}(x^\mu, z) = g(z)\phi(x^\mu) \quad , \quad (\text{A.25})$$

then we have a condition on M , $M(g) = g(z = 2\pi R)g^{-1}(z = 0)$. On the other hand, in order to avoid an explicit z -dependence, the element $g(z)$ has to satisfy that

$$C \equiv g^{-1}(z)\partial_z g(z) \quad (\text{A.26})$$

is a constant. This can be solved as

$$g(z) = e^{Cz} \quad , \quad M(g) = e^{2\pi RC} \quad . \quad (\text{A.27})$$

If G is not a symmetry of the theory, the reduction ansatz will not be consistent because we would not be able to cancel the coordinate dependence in the eoms. Moreover, it would not be possible to perform this reduction if the G group was a local symmetry.

A.4 Reduction over a group manifold

This kind of reductions can be performed by making use of the global symmetries of the internal manifold. This means we can apply this method only to theories where a dynamic metric is involved.

Let us define a group manifold G with coordinates $\{z^m\}_{m=1,\dots,\dim G}$ as the group element $g = g(z^m) \in G$. Then points on the manifold correspond to elements of the group. Group multiplication $g \rightarrow \Lambda_L g$, $g \rightarrow g \Lambda_R$, corresponds to a coordinate transformation. These coordinate transformations are not necessarily isometries of the metric. To choose left multiplication as isometries, we make

$$ds_G^2 = g_{mn} \sigma^m \sigma^n \quad , \quad T_m \sigma^m = g^{-1} dg \quad , \quad (\text{A.28})$$

where g_{mn} is arbitrary, T_m are the generators of G . The σ^m are called *Maurer-Cartan 1-forms* and can be written as

$$\sigma^m = U^m_n dz^n \quad , \quad (\text{A.29})$$

where $U^m_n = U^m_n(z)$ are functions of G . The isometries are generated by the Killing vectors L_m which, by definition, satisfy the Maurer-Cartan equations,

$$[L_m, L_n] = f_{mn}^p L_p \quad . \quad (\text{A.30})$$

These f_{mn}^p are the *structure constants* of the group G and can be written as

$$f_{mn}^p = -2(U^{-1})^r_m (U^{-1})^s_n \partial_{[r} U^p_{s]} \quad . \quad (\text{A.31})$$

Lie's second theorem ensures the z -independence of f_{mn}^p .

Let us see now how gravity can be dimensionally reduced using this technique. We start out with the toroidal reduction ansatz (A.14). If we apply U^m_n transformation on all the fields that 'feel' this symmetry, we have

$$\hat{ds}^2 = e^{2\alpha\phi} ds^2 + e^{2\beta\phi} M_{mn} (\sigma^m + A_\mu^m dx^\mu) (\sigma^n + A_\nu^n dx^\nu) \quad , \quad (\text{A.32})$$

where σ^m is defined in (A.29). Substituting this metric into the Einstein-Hilbert higher-dimensional lagrangian, we get

$$\mathcal{L} = \sqrt{|g|} \left(R - \frac{1}{2} (\partial\phi)^2 + \frac{1}{4} \text{Tr} (D^\mu M_{mn} D_\mu M^{mn}) - \frac{1}{4} e^{2(\beta-\alpha)\phi} M_{mn} F^m F^n - V \right) \quad , \quad (\text{A.33})$$

where

$$F_{\mu\nu}^m = 2\partial_{[\mu} A_{\nu]}^m + f_{mn}^p A_\mu^m A_\nu^n \quad , \quad D_\mu M_{mn} = \partial_\mu M_{mn} + 2f_{q(m}^p M_{n)p} A_\mu^q \quad , \quad (\text{A.34})$$

and the scalar potential is

$$V = \frac{1}{4} e^{2(\beta-\alpha)\phi} (2M^{nq} f_{mn}^p f_{pq}^m + M^{mq} M^{nr} M_{ps} f_{mn}^p f_{qr}^s) \quad . \quad (\text{A.35})$$

The consistency of the truncation is guaranteed by two aspects: the dependence on the internal manifold relies on the structure constants and the KK tower of fields is truncated to fields that are singlet under the left-acting isotropy group. Finally, let us study the symmetries inferred by this kind of reduction. The general coordinate transformation acts as

$$\hat{\xi}^\mu = \xi^\mu \quad , \quad \hat{\xi}^\mu = U^m{}_n \lambda^n(x) \quad . \quad (\text{A.36})$$

They are a general coordinate and a non-abelian gauge transformations with parameter $\lambda^n(x)$. The latter acts on the fields

$$\delta A_\mu{}^m = \partial_\mu \lambda^m + f_{np}{}^m \lambda^n A_\mu{}^p \quad , \quad \delta M_{mn} = f_{mp}{}^q \lambda^p M_{qn} + f_{np}{}^q \lambda^p M_{mq} \quad , \quad (\text{A.37})$$

leaving the metric invariant.

Due to the internal coordinate transformations

$$\hat{\xi}^m = \Lambda^m{}_n z^n \quad , \quad (\text{A.38})$$

the internal group manifold enjoyed a global symmetry $GL(n, \mathbb{R})$ that gets broken when the reduction is done. In the gauged theory, only the part of $GL(n, \mathbb{R})$ that preserves the structure constant covariance,

$$f_{mn}{}^p = \Lambda_m{}^q \Lambda_n{}^r (\Lambda^{-1})_s{}^p f_{qr}{}^s \quad , \quad (\text{A.39})$$

remains unbroken. This part includes the emerging gauge group, which is realized in the global symmetry group as

$$\Lambda_n{}^m = e^{\lambda^k f_{kn}{}^m} \quad , \quad (\text{A.40})$$

where λ^k are the local parameters.

We can infer here that gauge groups with non-trivial adjoint representations lead to gaugings of a part of the global symmetry. However, for the case of a reduction on a torus T^n , the $U(1)^n$ adjoint is trivial and its abelian character implies no gaugings.

A.5 The scalar sector in SUGRA theories

In general, scalar span a non-linear σ -model and their kinetic terms appear in the lagrangian as

$$G_{ab} \partial_\mu \phi^a \partial^\mu \phi^b \quad , \quad (\text{A.41})$$

where G_{ab} is the metric of the σ -model and a, b are scalar indices. The scalar manifold can have isometries generated by the Killing vectors. If the manifold is a *homogeneous space*, *i.e.*, if any point of the manifold can be reached from any other point by a symmetry operation, then the manifold is characterized by the isometry group G . This

means that there are the same number of linearly independent Killing vectors as the dimension of the manifold.

The subgroup H of the isometry group G that leaves a point invariant is called the *isotropy group* and, in case that H is non-trivial, the manifold is identified with the coset G/H . If we consider the algebras $\mathfrak{g} = \text{Lie } G$ and $\mathfrak{h} = \text{Lie } H$, we can define a complementary space \mathfrak{l} to \mathfrak{h} , such that any $g \in \mathfrak{g}$ can be decomposed into

$$g = h + k, \quad h \in \mathfrak{h}, k \in \mathfrak{l}, \quad (\text{A.42})$$

and for any $h_1, h_2 \in \mathfrak{h}$, $k_1, k_2 \in \mathfrak{l}$, we have

$$[h_1, h_2] \in \mathfrak{h}, \quad [h_1, k_1] \in \mathfrak{l}, \quad [k_1, k_2] \in \mathfrak{h}. \quad (\text{A.43})$$

The scalar fields span the adjoint representation of G_0 , the global symmetry group. However, not all of the compact generators are physical, i.e., they can always be rotated away. This means that the scalar degrees of freedom span a coset G_0/H , where H is the maximal compact subgroup of G_0 .

This decomposition can be done by means of the Cartan-Killing metric in \mathfrak{g} . Thus, any simply connected homogeneous space for which the isometry and isotropy algebras satisfy these conditions is called *symmetric space*. Scalar manifolds of SUGRA theories with more than 8 real supercharges are symmetric spaces G/H , where G is non-compact and H is its maximal compact subgroup. In fact, H is the R symmetry group for pure SUGRA and has the R symmetry group as a factor when matter multiplets are considered.

A convenient formulation of this σ -model is by means of a matrix V , which plays the role of a vielbein. It transforms under global G transformations from the left and local H transformations from the right, $V \rightarrow \Lambda V k(x)$, or equivalently,

$$\delta V = \Lambda V - V k(x), \quad (\text{A.44})$$

where $\Lambda \in G$ and $k \in H$.

This vielbein becomes crucial in the description of the fermionic sector, especially with the fermionic fields that transform in linear representations of H . In the couplings between fermions and bosons, V acts as a link of both due to its transformation rule.

In order to keep the local H freedom, it is convenient to formulate the theory in terms of manifestly H invariant objects. For example, we can define

$$M = V \Delta V, \quad (\text{A.45})$$

where Δ is a constant H -invariant positive definite matrix. M transforms under G as

$$\delta M = \Lambda M + M \Lambda^T, \quad (\text{A.46})$$

and the lagrangian takes the form

$$\mathcal{L}_{\text{scalar}} = \frac{1}{8} \text{Tr} (\partial_\mu M \partial^\mu M^{-1}). \quad (\text{A.47})$$

For instance, for the coset $SL(2)/SO(2)$, M can be expressed in terms of a complex scalar field $\tau = \chi + ie^{2\phi}$, giving rise to

$$M = \frac{1}{\text{Im}(\tau)} \begin{pmatrix} |\tau|^2 & \text{Re}(\tau) \\ \text{Re}(\tau) & 1 \end{pmatrix} . \quad (\text{A.48})$$

A.6 Central charges in supergravity

The superalgebra of the maximal supergravities that we have studied can be generalized in many ways. One of them is by the explicit inclusion of the gauge transformation associated with the antisymmetric gauge fields that appear in the theories. That is, supersymmetry transformations generate general coordinate transformations, local SUSY transformations and gauge transformations. However, the last ones do not appear in the superalgebra (1.15). This happens because at perturbative level, the theory does not contain charged fields, so the additional charges simply vanish on physical states. However, if we consider non-perturbative states, these solutions usually feel the gauge fields and carry their associated charges. Black holes or branes configurations can always be associated to these charges.

The 'gauge charges' usually appear in the anticommutator of two supercharges and may be regarded as disguised *central charges* $Z_{\mu_1, \dots}^{ij}$, of the form

$$\{Q_{\alpha}^i, Q_{\beta}^j\} = (\gamma_{\mu} C^{-1})_{\alpha\beta} P_{\mu} \delta^{ij} + \sum_k (\Gamma^{\mu_1 \dots \mu_k} C)_{\alpha\beta} Z_{\mu_1 \dots \mu_p}^{ij} . \quad (\text{A.49})$$

These Z 's are no longer Lorentz scalars in general, and thus do not commute with the Lorentz generators. They are therefore not strictly central in the group-theoretical meaning of the word.

The possible combinations of central extensions will depend on the dimension of the theory. For example, for $D = 11$, we have [99]

$$\{Q_{\alpha}, Q_{\beta}\} = (\gamma_{\mu} C^{-1})_{\alpha\beta} P_{\mu} \delta^{ij} + \sum_n (\Gamma^{\mu_1 \mu_2} C)_{\alpha\beta} Z_{\mu_1 \mu_2} + \sum_n (\Gamma^{\mu_1 \dots \mu_5} C)_{\alpha\beta} Z_{\mu_1 \dots \mu_5} . \quad (\text{A.50})$$

It is remarkable the coincidence of these two central charge with the existence of such solitonic M2- and M5-branes in this theory, as is show in Table 2.5.

Appendix B

Gaugings in $N = 2$ $D = 9$ supergravity

B.1 Conventions

We follow the conventions of Ref. [117]. In particular, we use mostly plus signature $(-, +, \dots, +)$ and the gamma matrices satisfy

$$\gamma_a^* = -\gamma_a, \quad \gamma_a = \eta_{aa} \gamma_a^\dagger. \quad (\text{B.1})$$

The Dirac conjugate of a spinor ϵ is defined by

$$\bar{\epsilon} \equiv \epsilon^\dagger \gamma_0. \quad (\text{B.2})$$

Then, we have

$$\begin{aligned} (\bar{\epsilon} \gamma^{(n)} \lambda)^* &= a_n \bar{\epsilon}^* \gamma^{(n)} \lambda^*, \\ (\bar{\epsilon} \gamma^{(n)} \lambda)^* &= b_n \bar{\lambda} \gamma^{(n)} \epsilon, \end{aligned} \quad (\text{B.3})$$

where the signs a_n and b_n are given in Table B.1

B.1.1 Spinor bilinears

We define the following real bilinears of the supersymmetry parameters ϵ_1 and ϵ_2 :

n	0	1	2	3	4	5	6	7	8	9
a_n	-	+	-	+	-	+	-	+	-	+
b_n	+	-	-	+	+	-	-	+	+	-

Table B.1: Values of the coefficients a_n and b_n defined in Eqs. (B.3).

$$\bar{\epsilon}_2 \epsilon_1 \equiv a + ib, \quad (\text{B.4})$$

$$\bar{\epsilon}_2 \epsilon_1^* \equiv c + id, \quad (\text{B.5})$$

$$\bar{\epsilon}_2 \gamma_{\mu_1 \dots \mu_n} \epsilon_1 \equiv \xi_{\mu_1 \dots \mu_n} + i \zeta_{\mu_1 \dots \mu_n}, \quad (\text{B.6})$$

$$\bar{\epsilon}_2 \gamma_{\mu_1 \dots \mu_n} \epsilon_1^* \equiv \sigma_{\mu_1 \dots \mu_n} + i \rho_{\mu_1 \dots \mu_n}, \quad (\text{B.7})$$

B.2 Relation with other conventions

The electric fields used in this paper are related to those used in Ref. [116] (which uses a mostly minus signature) as follows:

$$K = e^{\frac{\sqrt{7}}{3}\varphi}, \quad (\text{B.8})$$

$$\lambda \equiv C^{(0)} + ie^{-\varphi} = \tau \equiv \chi + ie^{-\phi}, \quad (\text{B.9})$$

$$A_{(1)} = A^0, \quad (\text{B.10})$$

$$\mathbf{A}_{(1)} = A^{\mathbf{i}}, \quad (\text{B.11})$$

$$\mathbf{A}_{(2)} = B^i + \frac{1}{2} A^{0\mathbf{i}}, \quad (\text{B.12})$$

$$A_{(3)} = -C + \frac{1}{2} \varepsilon_{\mathbf{ij}} A^{\mathbf{i}} \wedge B^j - \frac{1}{12} \varepsilon_{\mathbf{ij}} A^{0\mathbf{ij}}, \quad (\text{B.13})$$

$$A_{(4)} = -\tilde{C} + C \wedge A^0 - \frac{1}{4} \varepsilon_{\mathbf{ij}} B^i \wedge A^{0\mathbf{j}}. \quad (\text{B.14})$$

The field strengths are related by

$$F_{(2)} = F^0, \quad (\text{B.15})$$

$$\mathbf{F}_{(2)} = F^i, \quad (\text{B.16})$$

$$\mathbf{F}_{(3)} = H^i, \quad (\text{B.17})$$

$$F_{(4)} = -G, \quad (\text{B.18})$$

$$F_{(5)} = -\tilde{G}. \quad (\text{B.19})$$

The relation with the fields used in Ref. [117] (which also uses mostly plus signature) is given by (our fields are in the r.h.s. of these equations)

$$B^i = -(B^i + \frac{1}{2}A^{0i}), \quad (\text{B.20})$$

$$C = -(C - \frac{1}{6}\varepsilon_{\mathbf{ij}}A^{0\mathbf{ij}}), \quad (\text{B.21})$$

while the field strengths are related by

$$H^i = -H^i, \quad (\text{B.22})$$

$$G = -G. \quad (\text{B.23})$$

The rest of the fields are identical.

B.3 Noether currents

The Noether 1-form currents of the undeformed theory j_A are given by

$$\begin{aligned} \star j_m = & \star d\mathcal{M}_{ij} (\mathcal{M}^{-1})_{jk} T_{mi}{}^k + e^{\frac{4}{\sqrt{7}}\varphi} (\mathcal{M}_{\mathbf{ij}}^{-1}) T_{m\mathbf{k}}{}^{\mathbf{i}} A^{\mathbf{k}} \wedge \star F^{\mathbf{j}} \\ & + T_{mk}{}^i \left[e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} (B^k - \frac{1}{2}A^{0k}) \wedge \star H^j + \frac{1}{2}\varepsilon_{ij} \left(-2e^{\frac{2}{\sqrt{7}}\varphi} A^{\mathbf{j}} \wedge B^k \wedge \star G \right. \right. \\ & \left. \left. + (B^j - A^{0j}) \wedge B^k \wedge G + \varepsilon_{ln} A^l \wedge B^{jk} \wedge (H^n - \frac{1}{2}A^n \wedge F^0) \right. \right. \\ & \left. \left. + \frac{1}{4}\varepsilon_{ln} A^{0ln} \wedge B^k \wedge H^j \right) \right], \quad (\text{B.24}) \end{aligned}$$

$$\begin{aligned}
\star j_4 = & \frac{6}{\sqrt{7}} \star d\varphi + 3 \left[e^{\frac{4}{\sqrt{7}}\varphi} A^0 \wedge \star F^0 + e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} (B^i + \frac{1}{2}A^{0i}) \wedge \star H^j + e^{\frac{2}{\sqrt{7}}\varphi} (C - \frac{1}{6}\varepsilon_{ij}A^{0ij}) \wedge \star G \right. \\
& \left. + A^0 \wedge (C + \varepsilon_{ij}A^i \wedge B^j) \wedge G \right] + \frac{3}{2}\varepsilon_{ij} \left[(-C + \varepsilon_{kl}A^k \wedge B^l - \frac{7}{12}\varepsilon_{kl}A^{0kl}) \wedge B^i \wedge H^j \right. \\
& \left. - \frac{3}{2}A^{0i} \wedge C \wedge H^j + (A^i \wedge B^j - \frac{1}{2}A^{0ij}) \wedge F^0 \wedge C \right], \tag{B.25}
\end{aligned}$$

$$\begin{aligned}
\star j_5 = & \frac{\sqrt{7}}{4} \star d\varphi - \frac{3}{8} \star \frac{\tau d\bar{\tau} + \text{c.c.}}{(\Im m\tau)^2} + e^{\frac{4}{\sqrt{7}}\varphi} T_{50}^0 A^0 \wedge \star F^0 + e^{\frac{3}{\sqrt{7}}\varphi} T_{5\mathbf{k}}^{\mathbf{i}} \mathcal{M}_{ij}^{-1} A^{\mathbf{k}} \wedge \star F^{\mathbf{j}} \\
& + e^{-\frac{1}{\sqrt{7}}\varphi} \mathcal{M}_{ij}^{-1} \left[T_{5k}^i (B^k - \frac{1}{2}A^{0k}) + \frac{1}{4}A^{0i} \right] \wedge \star H^j \\
& + e^{\frac{2}{\sqrt{7}}\varphi} (T_5 C - \frac{1}{12}\varepsilon_{ij}A^{0ij} - T_{5k}^i \varepsilon_{ij} (A^k \wedge B^j - \frac{1}{6}A^{0kj})) \wedge \star G \\
& + \frac{1}{4}\varepsilon_{ij} \left[T_{5k}^i (-2B^{jk} + 3A^{0j} \wedge B^k - 5A^{0k} \wedge B^j) - \frac{1}{2}A^{0i} \wedge B^j \right] \wedge G \\
& + \frac{1}{4}\varepsilon_{ij} \left[T_{5k}^i (+2\varepsilon_{ln}A^l \wedge B^{nk} - \varepsilon_{ln}A^{0ln} \wedge B^k) - T_5 (6A^{0i} + B^i) \wedge C - \frac{1}{12}\varepsilon_{kl}A^{0kl} \wedge B^i \right] \wedge H^j \\
& + \varepsilon_{ij}\varepsilon_{ln} T_{5k}^i \left[\frac{5}{6}A^{0jk} \wedge B^l - A^{0lj} \wedge B^k + \frac{1}{2}A^k \wedge B^{jl} \right] \wedge H^n \\
& + T_5 \left[A^0 \wedge C \wedge G + \frac{1}{2}\varepsilon_{ij} (B^j + \frac{1}{2}A^{0j}) \wedge A^i \wedge F^0 \wedge C \right] \tag{B.26}
\end{aligned}$$

B.4 Final results

In this Appendix we give the final form of the deformed covariant field strengths, covariant derivatives, gauge and supersymmetry transformations in terms of the independent deformation parameters given in Eq. 4.146. We must bear in mind that they are assumed to satisfy the irreducible quadratic constraints given in Eq. (4.148) and only then the field strengths etc. have the right transformation properties.

The covariant derivatives of the scalar fields are given by

$$\mathfrak{D}\varphi = -\frac{137}{24\sqrt{7}}\vartheta_0^5 A^0 + \left(-\frac{\sqrt{7}}{4}\vartheta_{\mathbf{i}}^4 + \frac{6}{\sqrt{7}}\vartheta_{\mathbf{i}}^5 \right) A^{\mathbf{i}}, \tag{B.27}$$

$$\mathfrak{D}\tau = \vartheta_0^m k_m^\tau A^{\mathbf{1}} - \frac{3}{4}\vartheta_0^5 \tau A^0 + \frac{3}{4} (\vartheta_{\mathbf{1}}^5 \tau + \vartheta_{\mathbf{2}}^5) (A^{\mathbf{1}} - \tau A^{\mathbf{2}}), \tag{B.28}$$

and their gauge transformations are explicitly given by

$$\delta_\Lambda \varphi = -\frac{137}{24\sqrt{7}}\vartheta_0^5 \Lambda^0 + \left(-\frac{\sqrt{7}}{4}\vartheta_1^4 + \frac{6}{\sqrt{7}}\vartheta_1^5\right) \Lambda^{\mathbf{i}}, \quad (\text{B.29})$$

$$\delta_\Lambda \tau = \vartheta_0^m k_m^\tau \Lambda^0 - \frac{3}{4}\vartheta_0^5 \tau \Lambda^0 + \frac{3}{4}(\vartheta_1^5 \tau + \vartheta_2^5)(\Lambda^{\mathbf{1}} - \tau \Lambda^{\mathbf{2}}). \quad (\text{B.30})$$

The deformed p -form field strengths are given by

$$F^0 = dA^0 - \frac{1}{2}(3\vartheta_1^4 + \frac{1}{2}\vartheta_1^5) A^{0\mathbf{i}} + (3\vartheta_1^4 + \frac{1}{2}\vartheta_1^5) B^{\mathbf{i}}, \quad (\text{B.31})$$

$$\begin{aligned} F^{\mathbf{i}} &= dA^{\mathbf{i}} + \frac{1}{2}(\vartheta_0^m (T_m^{(3)})_{\mathbf{j}}^{\mathbf{i}} A^{0\mathbf{j}} - \frac{3}{4}\delta_1^{\mathbf{i}} \vartheta_0^5 A^{0\mathbf{1}} + \frac{3}{2}\varepsilon^{\mathbf{ij}} \vartheta_1^5 A^{\mathbf{12}}) \\ &\quad + \vartheta_0^m (T_m^{(3)})_{\mathbf{j}}^{\mathbf{i}} B^{\mathbf{j}} - \frac{3}{4}\delta_1^{\mathbf{i}} \vartheta_0^5 B^{\mathbf{1}}, \end{aligned} \quad (\text{B.32})$$

$$\begin{aligned} H^i &= \mathfrak{D}B^i + \frac{1}{2}(A^0 \wedge dA^{\mathbf{i}} + A^{\mathbf{i}} \wedge dA^0) + \frac{1}{6}\varepsilon^{\mathbf{ij}}(3\vartheta_{\mathbf{j}}^4 + \frac{1}{2}\vartheta_{\mathbf{j}}^5) A^{0\mathbf{12}} \\ &\quad + \varepsilon^{\mathbf{ij}}(3\vartheta_{\mathbf{j}}^4 - \frac{1}{4}\vartheta_{\mathbf{j}}^5) C, \end{aligned} \quad (\text{B.33})$$

$$\begin{aligned} G &= \mathfrak{D}C - \varepsilon_{\mathbf{ij}}[F^{\mathbf{i}} \wedge B^{\mathbf{j}} - \frac{1}{2}\delta^{\mathbf{j}}_{\mathbf{i}}(A^{\mathbf{i}} \wedge dA^{\mathbf{j}} - \frac{1}{3}d(A^{0\mathbf{ij}}))] \\ &\quad + \frac{1}{2}(\varepsilon_{\mathbf{ij}}\vartheta_0^m (T_m^{(2)})_{\mathbf{k}}^{\mathbf{i}} B^{\mathbf{j}\mathbf{k}} - \frac{3}{4}\vartheta_0^5 B^{\mathbf{12}}) + Z\tilde{C}, \end{aligned} \quad (\text{B.34})$$

where the covariant derivatives acting on the different fields are given by

$$\begin{aligned} \mathfrak{D}B^i &= dB^i + \vartheta_0^m (T_m^{(2)})_{\mathbf{j}}^{\mathbf{i}} A^0 \wedge B^{\mathbf{j}} - \frac{3}{4}\delta_1^{\mathbf{i}} \vartheta_0^5 A^0 \wedge B^{\mathbf{1}} \\ &\quad + (3\vartheta_{\mathbf{k}}^4 - \frac{1}{4}\vartheta_{\mathbf{k}}^5) A^{\mathbf{k}} \wedge B^i + \frac{3}{4}\delta_{\mathbf{j}}^{\mathbf{i}} \vartheta_{\mathbf{k}}^5 A^{\mathbf{j}} \wedge B^{\mathbf{k}}, \end{aligned} \quad (\text{B.35})$$

$$\mathfrak{D}C = dC - \frac{3}{4}\vartheta_0^5 A^0 \wedge C + (3\vartheta_1^4 - \frac{1}{4}\vartheta_1^5) A^{\mathbf{i}} \wedge C. \quad (\text{B.36})$$

The field strengths transform covariantly under the gauge transformations

$$\delta_\Lambda A^0 = -\mathfrak{D}\Lambda^0 + (3\vartheta_{\mathbf{i}}^4 + \frac{1}{2}\vartheta_{\mathbf{i}}^5) \Lambda^{\mathbf{i}}, \quad (\text{B.37})$$

$$\delta_\Lambda A^{\mathbf{i}} = -\mathfrak{D}\Lambda^{\mathbf{i}} + \vartheta_0^m (T_m^{(3)})_{\mathbf{j}}^{\mathbf{i}} \Lambda^{\mathbf{j}} - \frac{3}{4} \delta_{\mathbf{i}}^{\mathbf{j}} \vartheta_0^5 \Lambda^{\mathbf{j}}, \quad (\text{B.38})$$

$$\begin{aligned} \delta_\Lambda B^i &= -\mathfrak{D}\Lambda^i + F^0 \wedge \Lambda^{\mathbf{i}} + F^{\mathbf{i}} \Lambda^0 + \frac{1}{2} (A^0 \wedge \delta_\Lambda A^{\mathbf{i}} + A^{\mathbf{i}} \wedge \delta_\Lambda A^0) \\ &\quad + \varepsilon^{\mathbf{ij}} (3\vartheta_{\mathbf{j}}^4 - \frac{1}{4}\vartheta_{\mathbf{j}}^5) \Lambda, \end{aligned} \quad (\text{B.39})$$

$$\begin{aligned} \delta_\Lambda (C - \frac{1}{6} \varepsilon_{\mathbf{ij}} A^{0\mathbf{ij}}) &= -\mathfrak{D}\Lambda - \varepsilon_{\mathbf{ij}} (\Lambda^{\mathbf{i}} H^{\mathbf{j}} + F^{\mathbf{i}} \wedge \Lambda^{\mathbf{j}} - \delta_\Lambda A^{\mathbf{i}} \wedge B^{\mathbf{j}}) \\ &\quad - \frac{1}{2} \varepsilon_{\mathbf{ij}} A^{0\mathbf{i}} \delta_\Lambda A^{\mathbf{j}} + Z\tilde{\Lambda}, \end{aligned} \quad (\text{B.40})$$

where the covariant derivatives of the different gauge parameters are given by

$$\mathfrak{D}\Lambda^0 = d\Lambda^0 + (3\vartheta_{\mathbf{i}}^4 + \frac{1}{2}\vartheta_{\mathbf{i}}^5) A^{\mathbf{i}} \Lambda^0, \quad (\text{B.41})$$

$$\mathfrak{D}\Lambda^{\mathbf{i}} = d\Lambda^{\mathbf{i}} + \vartheta_0^m (T_m^{(3)})_{\mathbf{j}}^{\mathbf{i}} A^0 \Lambda^{\mathbf{j}} - \frac{3}{4} \delta_{\mathbf{i}}^{\mathbf{j}} \vartheta_0^5 A^0 \Lambda^{\mathbf{j}} + \frac{3}{4} \varepsilon^{\mathbf{ij}} \varepsilon_{\mathbf{kl}} \vartheta_{\mathbf{j}}^5 A^{\mathbf{k}} \Lambda^{\mathbf{l}}, \quad (\text{B.42})$$

$$\begin{aligned} \mathfrak{D}\Lambda^i &= d\Lambda^i + \vartheta_0^m (T_m^{(2)})_j^i A^0 \wedge \Lambda^j + (3\vartheta_{\mathbf{k}}^4 - \frac{1}{4}\vartheta_{\mathbf{k}}^5) A^{\mathbf{k}} \wedge \Lambda^i \\ &\quad + \frac{3}{4} \delta_{\mathbf{j}}^{\mathbf{i}} \vartheta_{\mathbf{k}}^5 A^{\mathbf{j}} \wedge \Lambda^{\mathbf{k}}, \end{aligned} \quad (\text{B.43})$$

$$\mathfrak{D}\Lambda = d\Lambda - \frac{3}{4} \vartheta_0^5 A^0 \wedge \Lambda + (3\vartheta_{\mathbf{i}}^4 - \frac{1}{4}\vartheta_{\mathbf{i}}^5) A^{\mathbf{i}} \wedge \Lambda. \quad (\text{B.44})$$

The supersymmetry transformation rules of the fermion fields are given by

$$\begin{aligned} \delta_\epsilon \psi_\mu &= \mathfrak{D}_\mu \epsilon + f \gamma_\mu \epsilon + k \gamma_\mu \epsilon^* + \frac{i}{8 \cdot 2!} e^{-\frac{2}{\sqrt{7}} \varphi} \left(\frac{5}{7} \gamma_\mu \gamma^{(2)} - \gamma^{(2)} \gamma_\mu \right) F^0 \epsilon \\ &\quad - \frac{1}{8 \cdot 2!} e^{\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} \left(\frac{5}{7} \gamma_\mu \gamma^{(2)} - \gamma^{(2)} \gamma_\mu \right) (F^1 - \tau F^2) \epsilon^* \\ &\quad - \frac{i}{8 \cdot 3!} e^{-\frac{1}{2\sqrt{7}} \varphi} \left(\frac{3}{7} \gamma_\mu \gamma^{(3)} + \gamma^{(3)} \gamma_\mu \right) (H^1 - \tau H^2) \epsilon^* \\ &\quad - \frac{1}{8 \cdot 4!} e^{\frac{1}{\sqrt{7}} \varphi} \left(\frac{1}{7} \gamma_\mu \gamma^{(4)} - \gamma^{(4)} \gamma_\mu \right) G \epsilon, \end{aligned} \quad (\text{B.45})$$

$$\begin{aligned}
\delta_\epsilon \tilde{\lambda} &= i \mathcal{D} \varphi \epsilon^* + \tilde{g} \epsilon + \tilde{h} \epsilon^* - \frac{1}{\sqrt{7}} e^{-\frac{2}{\sqrt{7}} \varphi} \mathcal{F}^0 \epsilon^* - \frac{3i}{2 \cdot 2! \sqrt{7}} e^{\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} (\mathcal{F}^1 - \tau^* \mathcal{F}^2) \epsilon \\
&\quad - \frac{1}{2 \cdot 3! \sqrt{7}} e^{-\frac{1}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} (\mathcal{H}^1 - \tau^* \mathcal{H}^2) \epsilon - \frac{i}{4! \sqrt{7}} e^{\frac{1}{\sqrt{7}} \varphi} \mathcal{G} \epsilon^* ,
\end{aligned} \tag{B.46}$$

$$\begin{aligned}
\delta_\epsilon \lambda &= -e^\phi \mathcal{D} \tau \epsilon^* + g \epsilon + h \epsilon^* - \frac{i}{2 \cdot 2!} e^{\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} (\mathcal{F}^1 - \tau \mathcal{F}^2) \epsilon \\
&\quad + \frac{1}{2 \cdot 3!} e^{-\frac{1}{2\sqrt{7}} \varphi + \frac{1}{2} \phi} (\mathcal{H}^1 - \tau \mathcal{H}^2) \epsilon ,
\end{aligned} \tag{B.47}$$

where

$$\mathfrak{D}_\mu \epsilon = \left\{ \nabla_\mu + \frac{i}{2} \left[\frac{1}{2} e^\phi \mathfrak{D}_\mu^5 \chi + A^I{}_\mu \vartheta_I{}^m \mathcal{P}_m \right] + \frac{9}{14} \gamma_\mu A^I \vartheta_I{}^4 \right\} \epsilon , \tag{B.48}$$

$$\mathfrak{D}_\mu^5 \chi = \partial_\mu \chi - \frac{3}{4} A^I{}_\mu \vartheta_I{}^5 \chi , \tag{B.49}$$

and where the fermion shifts are given by

$$f = \frac{1}{14} e^{\frac{2}{\sqrt{7}} \varphi} (\vartheta_0{}^m \mathcal{P}_m + \frac{3i}{2} \vartheta_0{}^5) , \tag{B.50}$$

$$k = -\frac{9i}{14} e^{-\frac{3\varphi}{2\sqrt{7}} + \frac{\phi}{2}} (\vartheta_1{}^4 \tau + \vartheta_2{}^4) , \tag{B.51}$$

$$\tilde{g} = e^{-\frac{3\varphi}{2\sqrt{7}} + \frac{\phi}{2}} \left[\frac{6}{\sqrt{7}} (\vartheta_1{}^4 \tau^* + \vartheta_2{}^4) + \frac{\sqrt{7}}{4} (\vartheta_1{}^5 \tau^* + \vartheta_2{}^5) \right] , \tag{B.52}$$

$$\tilde{h} = \frac{4}{\sqrt{7}} e^{\frac{2}{\sqrt{7}} \varphi} \left(\frac{3}{16} \vartheta_0{}^5 + \vartheta_0{}^m \mathcal{P}_m \right) , \tag{B.53}$$

$$g = \frac{3}{4} e^{-\frac{3\varphi}{2\sqrt{7}} + \frac{\phi}{2}} (\vartheta_1{}^5 \tau + \vartheta_2{}^5) , \tag{B.54}$$

$$h = i e^{\frac{2\varphi}{\sqrt{7}} + \phi} (\vartheta_0{}^m k_m{}^\tau - \frac{3}{4} \vartheta_0{}^5 \tau) . \tag{B.55}$$

The supersymmetry transformations of the bosonic fields are

$$\delta_\epsilon \varphi = -\frac{i}{4} \tilde{\epsilon} \tilde{\lambda}^* + \text{h.c.} , \tag{B.56}$$

$$\delta_\epsilon \tau = -\frac{1}{2} e^{-\phi} \tilde{\epsilon}^* \lambda , \tag{B.57}$$

$$\delta_\epsilon A^0{}_\mu = \frac{i}{2} e^{\frac{2}{\sqrt{7}}\varphi} \bar{\epsilon} \left(\psi_\mu - \frac{i}{\sqrt{7}} \gamma_\mu \tilde{\lambda}^* \right) + \text{h.c.}, \quad (\text{B.58})$$

$$\delta_\epsilon A^1{}_\mu = \frac{i}{2} \tau^* e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left(\bar{\epsilon}^* \psi_\mu - \frac{i}{4} \bar{\epsilon} \gamma_\mu \lambda + \frac{3i}{4\sqrt{7}} \bar{\epsilon}^* \gamma_\mu \tilde{\lambda}^* \right) + \text{h.c.}, \quad (\text{B.59})$$

$$\delta_\epsilon A^2{}_\mu = \frac{i}{2} e^{-\frac{3}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left(\bar{\epsilon}^* \psi_\mu - \frac{i}{4} \bar{\epsilon} \gamma_\mu \lambda + \frac{3i}{4\sqrt{7}} \bar{\epsilon}^* \gamma_\mu \tilde{\lambda}^* \right) + \text{h.c.} \quad (\text{B.60})$$

$$\begin{aligned} \delta_\epsilon B^1 &= \tau^* e^{\frac{1}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left[\bar{\epsilon}^* \gamma_{[\mu} \psi_{\nu]} - \frac{i}{8} \bar{\epsilon} \gamma_{\mu\nu} \lambda - \frac{i}{8\sqrt{7}} \bar{\epsilon}^* \gamma_{\mu\nu} \tilde{\lambda}^* \right] + \text{h.c.} \\ &\quad - \delta^1{}_{\mathbf{i}} \left(A^0{}_{[\mu} \delta_\epsilon A^{\mathbf{i}}{}_{\nu]} + A^{\mathbf{i}}{}_{[\mu} \delta_\epsilon A^0{}_{\nu]} \right), \end{aligned} \quad (\text{B.61})$$

$$\begin{aligned} \delta_\epsilon B^2 &= e^{\frac{1}{2\sqrt{7}}\varphi + \frac{1}{2}\phi} \left[\bar{\epsilon}^* \gamma_{[\mu} \psi_{\nu]} - \frac{i}{8} \bar{\epsilon} \gamma_{\mu\nu} \lambda - \frac{i}{8\sqrt{7}} \bar{\epsilon}^* \gamma_{\mu\nu} \tilde{\lambda}^* \right] + \text{h.c.} \\ &\quad - \delta^2{}_{\mathbf{i}} \left(A^0{}_{[\mu} \delta_\epsilon A^{\mathbf{i}}{}_{\nu]} + A^{\mathbf{i}}{}_{[\mu} \delta_\epsilon A^0{}_{\nu]} \right), \end{aligned} \quad (\text{B.62})$$

$$\begin{aligned} \delta_\epsilon C_{\mu\nu\rho} &= -\frac{3}{2} e^{-\frac{1}{\sqrt{7}}\varphi} \bar{\epsilon} \gamma_{[\mu\nu} \left(\psi_{\rho]} + \frac{i}{6\sqrt{7}} \tilde{\lambda}^* \right) + \text{h.c.} \\ &\quad + 3\delta_\epsilon A^I{}_{[\mu} \left(g_{Ii} B^i{}_{\nu\rho]} + \frac{2}{3} h_{IJ}{}^i g_{Ki} A^{JK}{}_{\nu\rho]} \right). \end{aligned} \quad (\text{B.63})$$

Appendix C

Duality orbits of non-geometric fluxes

C.1 Different solvable and nilpotent gaugings

In section 5.3.4 we have studied the T-duality orbits of gaugings in half-maximal $D = 7$ supergravity and for each of them, we identified the gauge algebra and presented the results in table 5.6. Since there is no exhaustive classification of non-semisimple algebras of dimension 6, we would like to explicitly give the form of the algebras appearing in table 5.6.

Solvable algebras

The $\text{CSO}(2, 0, 2)$ and $\text{CSO}(1, 1, 2)$ algebras

The details about these algebras can be found in ref. [283]; we summarise here some relevant facts.

The six generators are labelled as $\{t_0, t_i, s_i, z\}_{i=1,2}$, where t_0 generates $\text{SO}(2)$ ($\text{SO}(1, 1)$), under which $\{t_i\}$ and $\{s_i\}$ transform as doublets

$$[t_0, t_i] = \epsilon_i^j t_j \quad , \quad [t_0, s_i] = \epsilon_i^j s_j \quad , \quad (\text{C.1})$$

where the Levi-Civita symbol ϵ_i^j has one index lowered with the metric $\eta_{ij} = \text{diag}(\pm 1, 1)$ depending on the two different signatures. z is a central charge appearing in the following commutators

$$[t_i, s_j] = \delta_{ij} z \quad . \quad (\text{C.2})$$

The Cartan-Killing metric is $\text{diag}(\mp 1, \underbrace{0, \dots, 0}_{6 \text{ times}})$, where the \mp is again related to the two different signatures.

The \mathfrak{f}_1 and \mathfrak{f}_2 algebras

These are of the form $\text{Solv}_4 \times \text{U}(1)^2$. The 4 generators of Solv_4 are labeled by $\{t_0, t_i, z\}_{i=1,2}$, where t_0 generates $\text{SO}(2)$ ($\text{SO}(1,1)$), under which $\{t_i\}$ transform as a doublet

$$[t_0, t_i] = \epsilon_i^j t_j, \quad (\text{C.3})$$

$$[t_i, t_j] = \epsilon_{ij} z. \quad (\text{C.4})$$

The Cartan-Killing metric is $\text{diag}(\mp 1, \underbrace{0, \dots, 0}_{6 \text{ times}})$.

The \mathfrak{h}_1 and \mathfrak{h}_2 algebras

The 6 generators are $\{t_0, t_i, s_i, z\}_{i=1,2}$ and they satisfy the following commutation relations

$$\begin{aligned} [t_0, t_i] &= \epsilon_i^j t_j, \quad [t_0, s_i] = \epsilon_i^j s_j + t_i, \\ [t_i, s_j] &= \delta_{ij} z, \quad [s_i, s_j] = \epsilon_{ij} z. \end{aligned} \quad (\text{C.5})$$

The Cartan-Killing metric is $\text{diag}(\mp 1, \underbrace{0, \dots, 0}_{6 \text{ times}})$.

The \mathfrak{g}_0 algebra

The 6 generators are $\{t_0, t_I, z\}_{I=1, \dots, 4}$, where t_0 transforms cyclically the $\{t_I\}$ amongst themselves such that

$$[[[t_I, t_0], t_0], t_0] = t_I, \quad (\text{C.6})$$

and

$$[t_1, t_3] = [t_2, t_4] = z. \quad (\text{C.7})$$

Note that this algebra is solvable and not nilpotent even though its Cartan-Killing metric is *completely zero*.

Nilpotent algebras

The $\text{CSO}(1, 0, 3)$ algebra

The details about this algebra can be again found in ref. [283]; briefly summarizing, the 6 generators are given by $\{t_m, z^m\}_{m=1,2,3}$ and they satisfy the following commutation relations

$$[t_m, t_n] = \epsilon_{mnp} z^p, \quad (\text{C.8})$$

with all the other brackets being vanishing. The order of nilpotency of this algebra is 2.

The \mathfrak{l} algebra

The 6 generators $\{t_1, \dots, t_6\}$ satisfy the following commutation relations

$$[t_1, t_2] = t_4 \quad , \quad [t_1, t_4] = t_5 \quad , \quad [t_2, t_4] = t_6 \quad . \quad (\text{C.9})$$

The corresponding central series reads

$$\{t_1, t_2, t_3, t_4, t_5, t_6\} \supset \{t_4, t_5, t_6\} \supset \{t_5, t_6\} \supset \{0\} \quad , \quad (\text{C.10})$$

from which we can immediately conclude that its nilpotency order is 3.

C.2 $SO(2, 2)$ and $SO(3, 3)$ 't Hooft symbols

In section 5.3.2 we discuss the origin of a given flux configuration from DFT backgrounds specified by twist matrices U . The deformations of half-maximal supergravity in $D = 10 - d$ which can be interpreted as the gauging of a subgroup of the T-duality group $O(d, d)$ can be described by a 3-form of $O(d, d)$ f_{ABC} which represents a certain (non-)geometric flux configuration.

In $D = 8$ and $D = 7$, the T-duality group happens to be isomorphic to $SL(2) \times SL(2)$ and $SL(4)$ respectively. As a consequence, in order to explicitly relate flux configurations and embedding tensor orbits, we need to construct the mapping between T-duality irrep's and irrep's of $SL(2) \times SL(2)$ and $SL(4)$ respectively.

From the $(2, 2)$ of $SL(2) \times SL(2)$ to the 4 of $SO(2, 2)$

The 't Hooft symbols $(G_A)^{\alpha i}$ are invariant tensors which map the fundamental representation of $SO(2, 2)$ (here denoted by A), into the $(2, 2)$ of $SL(2) \times SL(2)$

$$v^{\alpha i} = (G_A)^{\alpha i} v^A \quad , \quad (\text{C.11})$$

where v^A denotes a vector of $SO(2, 2)$ and the indices α and i are raised and lowered by means of $\epsilon_{\alpha\beta}$ and ϵ_{ij} respectively. $(G_A)^{\alpha i}$ and $(G_A)_{\alpha i}$ satisfy the following identities

$$(G_A)_{\alpha i} (G_B)^{\alpha i} = \eta_{AB} \quad , \quad (\text{C.12})$$

$$(G_A)^{\alpha i} (G^A)^{\beta j} = \epsilon^{\alpha\beta} \epsilon^{ij} \quad , \quad (\text{C.13})$$

where η_{AB} is the $SO(2, 2)$ metric.

After choosing light-cone coordinates for $SO(2, 2)$, our choice for the tensors $(G_A)^{\alpha i}$ is the following

$$(G_1)^{\alpha i} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad (G_2)^{\alpha i} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad (\text{C.14})$$

$$(G_{\bar{1}})^{\alpha i} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad (G_{\bar{2}})^{\alpha i} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \quad . \quad (\text{C.15})$$

By making use of the mapping (C.11), we can rewrite the structure constants $(X_{\alpha i})_{\beta j}{}^{\gamma k}$ as a 3-form of $\text{SO}(2, 2)$ as follows:

$$f_{ABC} = (X_{\alpha i})_{\beta j}{}^{\gamma k} (G_A)^{\alpha i} (G_B)^{\beta j} (G_C)_{\gamma k} . \quad (\text{C.16})$$

From the **6** of $\text{SL}(4)$ to the **6** of $\text{SO}(3, 3)$

The 't Hooft symbols $(G_A)^{mn}$ are invariant tensors which map the fundamental representation of $\text{SO}(3, 3)$, *i.e.* the **6** into the anti-symmetric two-form of $\text{SL}(4)$

$$v^{mn} = (G_A)^{mn} v^A , \quad (\text{C.17})$$

where v^A denotes a vector of $\text{SO}(3, 3)$. The two-form irrep of $\text{SL}(4)$ is real due to the role of the Levi-Civita tensor relating v^{mn} to v_{mn}

$$v_{mn} = \frac{1}{2} \epsilon_{mnpq} v^{pq} . \quad (\text{C.18})$$

The 't Hooft symbols with lower $\text{SL}(4)$ indices $(G_A)_{mn}$ carry out the inverse mapping of the one given in (C.17). The tensors $(G_A)^{mn}$ and $(G_A)_{mn} = \frac{1}{2} \epsilon_{mnpq} (G_A)^{pq}$ satisfy the following identities

$$(G_A)_{mn} (G_B)^{mn} = 2 \eta_{AB} , \quad (\text{C.19})$$

$$(G_A)_{mp} (G_B)^{pn} + (G_B)_{mp} (G_A)^{pn} = -\delta_m^n \eta_{AB} , \quad (\text{C.20})$$

$$(G_A)_{mp} (G_B)^{pq} (G_C)_{qr} (G_D)^{rs} (G_E)_{st} (G_F)^{tn} = \delta_m^n \epsilon_{ABCDEF} , \quad (\text{C.21})$$

where η_{AB} and ϵ_{ABCDEF} are the $\text{SO}(3, 3)$ metric and Levi-Civita tensor respectively.

After choosing light-cone coordinates for $\text{SO}(3, 3)$ vectors, our choice of the 't Hooft symbols is

$$(G_1)^{mn} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad (G_2)^{mn} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad (\text{C.22})$$

$$(G_3)^{mn} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \quad (G_{\bar{1}})^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad (\text{C.23})$$

$$(G_{\bar{2}})^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad (G_{\bar{3}})^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \quad (\text{C.24})$$

Thus, we can rewrite the structure constants in the **6**, $(X_{mn})_{pq}{}^{rs}$, arising from (5.119) as a 3-form of $\text{SO}(3, 3)$ as follows:

$$f_{ABC} = (X_{mn})_{pq}{}^{rs} (G_A)^{mn} (G_B)^{pq} (G_C)_{rs} . \quad (\text{C.25})$$

Abstract/Resumen

Resumen español

La presente tesis doctoral trata del estudio de soluciones en teorías de supergravedad y gravedad masiva. Estas teorías pueden considerarse extensiones de la teoría de la Relatividad General (RG) de Einstein, la cual describe la dinámica de una partícula sin masa de $\text{spin} = 2$: el gravitón. Supergravedad es una extensión supersimétrica de la RG que incorpora nuevas partículas sin masa (escalares, bosones de gauge, dilatinos, gravitinos, ...), mientras que gravedad masiva explora la dinámica de un gravitón con masa.

Las teorías de supergravedad aparecen al tomar el límite de baja energía de la teoría de cuerdas o de la teoría M. Una de las señas de identidad de estas supergravedades es que requieren un espacio-tiempo con 10 o incluso 11 dimensiones (10D/11D), en lugar de las cuatro dimensiones (4D) que experimentamos en la vida cotidiana. En consecuencia las dimensiones extras han de ser compactas y muy pequeñas para que resulten “invisibles” a los experimentos. Sin embargo, su forma y tamaño afectan a la gravitación que percibimos en 4D ya que el espacio-tiempo sería verdaderamente 10D/11D. Esto da lugar a desviaciones con respecto a la RG de Einstein que abren la posibilidad de testar las nuevas dimensiones.

Tras elegir una geometría determinada para un número d de dimensiones extras, es posible derivar la extensión de la RG que resulta en $D = 10 - d$ o $D = 11 - d$ dimensiones. Sin embargo, el proceso contrario no siempre es directo: empezando con una extensión consistente de la RG, ¿se puede inferir la geometría de las dimensiones extras de las que proviene? ¿Y la configuración de la teoría de cuerdas subyacente? Estas preguntas ocupan actualmente un lugar central en investigación en teoría de cuerdas y han sido abordadas en la presente tesis. Mas concretamente, el caso de las desviaciones de RG ocasionadas por ciertas deformaciones (gaugings) en supergravedad que preservan $\mathcal{N} = 4, 8$ supersimetrías. Estas deformaciones han sido estudiadas y clasificadas exhaustivamente en dimensiones $D = 9, 8y7$ utilizando el marco teórico del tensor de embedding, dando lugar a un análisis completo de su origen en 10D/11D. Uno de los resultados más relevantes de la tesis es la clasificación de los gaugings en órbitas cada una de las cuales produce una dinámica inequivalente en D dimensiones. Como resultado de esta clasificación, se observa que todos los gaugings analizados se pueden obtener como reducciones de una teoría de campos doble (Double Field Theory \equiv

DFT) en espacios toroidales con torsion (twisted tori), aun cuando un origen en teoría de cuerdas (10D) o teoría M (11D) puede no ser posible. Algunos de estos gaugings se corresponden con compactificaciones no-geométricas de cuerdas, demostrando que DFT tiene acceso a configuraciones genuinas de cuerdas para las que la geometría Riemanniana deja de ser válida. Estas no-geometrías están asociadas a configuraciones de cuerdas que no son compatibles con descartar los modos de enrollamiento y tomar el límite habitual de teoría de campos.

En la segunda parte de la tesis, se presenta una extensión de RG en $D = 4$ que describe la dinámica de un gravitón con masa. La primera consecuencia directa de tener un gravitón con masa es que la RG se ve modificada a grandes escalas. Esto afecta directamente a lo que se conoce como energía oscura del Universo suavizando el problema de la constante cosmológica - el Universo podría estar auto-acelerándose de acuerdo a una modificación a grandes escalas de la RG sin necesidad de invocar a la energía oscura -. La segunda consecuencia es que la deflexión de la luz es tan solo $3/4$ del valor que se obtiene con la RG (tras normalizar la interacción al caso de partículas test masivas). Esto se debe a que un gravitón con masa propaga 5 grados de libertad frente a los 2 de un gravitón sin masa.

En resumen, la RG es la única teoría que describe la dinámica de una partícula de $\text{spin} = 2$ sin masa. Cualquier modificación de la RG para dotar de masa al gravitón describirá un número distinto de grados de libertad con sus correspondientes consecuencias. La principal de ellas es que al tomar el límite de gravitón sin masa ($m_{grav} \rightarrow 0$) no se obtiene la RG sino una modificación de ésta que incluye un escalar. El límite de masa nula de un gravitón con masa no es pues un gravitón sin masa, sino un gravitón sin masa acoplado a un escalar (discontinuidad vDVZ).

El modelo de gravedad masiva en $D = 4$ que se presenta en la tesis proporciona un nuevo enfoque basado en resultados anteriores para el caso de $D = 3$. La idea que se presenta es incluir interacciones con derivadas de orden superior las cuales, a nivel linearizado, no introducen “fantasmas” debido a una formulación dual “conexión/métrica” del gravitón que describe los mismos grados de libertad (el campo del gravitón está en una representación exótica del grupo de Lorentz). También se presenta una nueva manera de tomar el límite de gravitón sin masa que respeta el número total de simetrías gauge, eliminando cualquier discontinuidad en el número de grados de libertad propagados. Como resultado, hay dos grados de libertad (partícula de $\text{spin} = 1$ sin masa) que son absorbidos por el gravitón para adquirir masa en lo que representa una versión adaptada del mecanismo de Higgs para partículas de $\text{spin} = 2$. En una posible versión completa del modelo (más allá del régimen linearizado), la RG sería reemplazada por una teoría de gravitón con masa a grandes escalas que se aproxima a una teoría escalar-tensor con un campo de Maxwell a escalas pequeñas. El modelo de gravedad masiva descrito en la tesis supone un gran avance hacia la descripción de gravitones masivos en 4D.

En la parte final de la tesis, se presenta un estudio de soluciones tipo agujero negro con multicentros en el contexto de supergravedad en $D = 4$ con $\mathcal{N} = 2$ super-

simetrías. Estas soluciones corresponden a configuraciones no-triviales de los campos del multiplete de supergravedad acoplado a un número n_v de multipletes vectoriales. El contenido de campos de la teoría se reduce pues a la métrica, escalares y vectores así como los compañeros supersimétricos de éstos (fermiones). Junto con el gravifotón, los vectores se transforman bajo rotaciones electromagnéticas de acuerdo con el grupo de simetría $Sp(2n_v + 2)$. Dado el vector de cargas electromagnéticas $Q = (p^\Lambda, q_\Lambda)$, con $\Lambda = 1, \dots, n_v + 1$, asociado a una configuración de agujero negro, los perfiles de los campos escalares fluyen desde un valor genérico en el infinito hasta un valor fijo en el horizonte del agujero negro que viene determinado por el vector de cargas Q (mecanismo atractor). Este mecanismo simplifica el cálculo de los perfiles para los escalares los cuales están codificados en una función conocida como potencial de agujero negro V_{BH} que depende de los campos escalares y las cargas electromagnéticas Q . A partir de estos perfiles, se pueden obtener las cantidades que caracterizan al agujero negro subyacente: masa, entropía, etc.

En la tesis se estudia de manera completa y detallada las configuraciones estáticas de tipo multicentro con los escalares mínimamente acoplados mediante un prepotencial cuadrático. Para llevar a cabo un análisis detallado, se introducen unas matrices de estabilización (junto con sus adjuntas), las cuales se emplean para tomar proyecciones de las cargas. Tras hacer un ansatz genérico de solución tipo multicentro, los perfiles de los escalares se calculan de forma explícita. Como resultado, se observa que una fragmentación de agujero negro a lo largo del subespacio longitudinal de las cargas reduce el área, mientras que en el caso de ser a lo largo del subespacio ortogonal lo aumenta. El uso de estas proyecciones resulta ser bastante eficiente a la hora de abordar cuestiones relacionadas con la entropía de los agujeros negros así como con su masa. Los métodos expuestos en esta parte de la tesis son extensibles a casos con prepotenciales más complejos e incluso modelos en los que no existe prepotencial, lo que los convierte en una herramienta de cálculo muy potente.

Abstract

This doctoral thesis considers the study of solutions in supergravity and massive gravity theories. These theories can be understood as extensions of Einstein's theory of General Relativity (GR), which describes the dynamics of a massless spin-2 particle: the graviton. Supergravity is a supersymmetric extension of GR that entails new massless particles (scalar fields, gauge bosons, dilatons, gravitinos, . . .), whereas massive gravity explores the dynamics of a massive graviton.

Supergravity theories arise as the low energy limit of string theories or M theory. One of their main features is the requirement of 10 or 11 dimension (10D/11D) instead of the four dimensions (4D) that we are used to experience in our daily life. This implies that the supplementary dimensions have to be compact and considerably small in such a way that they result almost imperceptible by the experiments. However, their

shape and size have direct effects on the gravity that we feel in 4D. In particular, the fact of having a truly 10D/11D spacetime gives rise to predictions that differ from the GR ones and open new scenarios where these extra dimensions can be tested.

Once we choose a certain geometry for the d extra dimensions, it is possible to obtain the effective theory in $D = 10 - d$ or $D = 11 - d$ dimensions that results an extension of General Relativity. Nevertheless, the inverse procedure is not always straightforward: namely, given a particular extension of GR, can be infer the underlying geometry of the extra dimensions that gives rise to this theory? Is the string theory configuration that generates this modification accessible? These questions get in on the main research topics in string theory and are addressed in this thesis. In particular, the GR discrepancies caused by a kind of deformations (gaugings) in supergravity theories that preserve $\mathcal{N} = 4, 8$ supersymmetries. These deformations have been exhaustively studied and classified for dimensions $D = 9, 8, 7$ by means of the theoretical approach of the *embedding tensor mechanism*. One of the most relevant results of this thesis is the classification of the gaugings in orbits that produce that give rise to inequivalent dynamics in D dimensions. As a consequence of this indexing, it is found that all of these gaugings can be obtained as a dimensional reduction of the so-called *Double Field Theory* (DFT) over twisted toroidal spaces. Some of these gaugings correspond to non-geometric compactifications of string theories, showing that DFT warrants the access to genuine string theory configurations that are not accessible by means of Riemannian geometry.

The second part of the thesis discusses a GR extension in $D = 4$ that describes the dynamics of a massive spin-2 particle. The first consequence of having a massive graviton is that GR results modified at large scales. This is immediately related to the dark energy of the Universe - the Universe could be self-accelerating according to UV modification of GR without a dispensable existence of dark energy -. The second consequences is the effect on light deflection: it implies only $3/4$ of the deflection angle that GR predicts. This can be understood as a consequence that the massive graviton propagates 5 degrees of freedom, in comparison with the 2 ones that a massless graviton does.

Summarizing, GR is the only theory that describes the dynamics of a spin-2 particle. Any GR modification that assumes a massive spin-2 graviton describes a different number of degrees of freedom with respect to the massless case and this have remarkable consequences. One of them is the so-called *vDVZ discontinuity*. When taking the massless limit of the massive gravity, we do not obtain GR, but we have a theory of a massless spin-2 particle coupled to a scalar field. Namely, the massless limit of massive gravity theory is not a massless gravity theory, but a massless gravity theory coupled to a scalar field. The fact of not having a continuity in the mass parameters originates the name of *discontinuity*.

This massive gravity model that is shown in this thesis provides a new perspective for $D = 4$ theories based on the existing ones for $D = 3$. The main idea is the addition of new interactions by means of new higher-derivative terms which, at the

linearized level, do not introduced “ghosts”. Due to a smart choice of a dual formulation “connection-like” of the graviton (the graviton field is realized as a exotic representation of the Lorentz group), this problem is circumvented and the same degrees of freedom remain propagated. In addition, the Stückelberg mechanism is used to gauge away the corresponding degrees of freedom of the graviton and take the massless limit so that the discontinuity in the number of degrees of freedom disappears. As a result, there are two degrees of freedom (a massless spin-1 particle) that are absorbed by the graviton so that it becomes massive. This represents an adapted version of the Higgs mechanism for spin-2 particles. In a possible complete version of the theory (beyond the linearized level), GR would be substituted by a massive graviton theory at UV regime that behaves as tensor-scalar field plus a Maxwell field theory at low energy. This massive gravity model provides a new contribution in the description of massive gravitons in 4D.

The last part of the thesis deals with multicenter-type black hole solutions in a $D = 4$ supergravity scenario with $\mathcal{N} = 2$ supergravities. These solutions correspond to extremal configurations (the only ones that allow the superposition of solutions as a solution of GR equations) of the supergravity multiplet coupled to n_v vector multiplets. Together with the graviphoton, the vector fields of the vector multiplets transform under the symmetry group $Sp(2n_v + 2)$. Assuming a vector of electromagnetic charges $Q = (p^\Lambda, q_\Lambda)$, where $\Lambda = 1, \dots, n_v + 1$, the scalar fields flow from a certain value at the infinity towards a fixed value at the black hole horizon, which is determined by the electromagnetic charges that span Q (attractor mechanism). This mechanism simplifies the calculation of the behaviour of the scalar fields, which is encoded in the so-called *black hole potential* (V_{BH}). This potential is a function that depends on the scalar fields and the electromagnetic charges of the black holes, and provides information about the quantities that characterize the black hole: mass, entropy, etc.

This thesis treats the study of black hole multicenter configurations with minimally coupled scalar fields. These scalar fields parameterize a special Kähler manifold and in this work quadratic prepotentials are analyzed. In order to carry out a detailed investigation, stabilization matrices are introduced (together with their adjoint matrices). These stabilization matrices are isometries of the symplectic quadratic form. Their adjoints with respect to the symplectic product $\mathcal{S}_N^\dagger, \mathcal{S}_F^\dagger$, which fullfills the property $\mathcal{S}^2 = -1$, are shown to lay inside the Lie algebra of the isometry group, they are such that $\mathcal{S} + \mathcal{S}^\dagger = 0$. They are “unitary”, $\mathcal{S}\mathcal{S}^\dagger = 1$, with respect the symplectic product. Using a generic multicenter ansatz, depending on the center charges Q_a and the value at infinity \mathcal{I}_∞ , and the attractor equations we have rederived in a simple way different relations. In particular the fix point scalar values have been written in terms of the projection of the respective charges into the eigenspaces of the matrix \mathcal{S} . Although we have focused in this study in minimal coupling theories with quadratic prepotentials, the main techniques, properties and expressions presented are extendable to general prepotentials or even theories without them.

Publications

List of publications that have given rise to the research done during the PhD period.

1. J. Fernandez-Melgarejo, T. Ortin, and E. Torrente-Lujan, “Maximal Nine Dimensional Supergravity, General gaugings and the Embedding Tensor,” *Fortsch.Phys.* **60** (2012) 1012–1018, [arXiv:1209.3774 \[hep-th\]](#).
2. G. Dibitetto, J. Fernandez-Melgarejo, D. Marques, and D. Roest, “Duality orbits of non-geometric fluxes,” *Fortsch.Phys.* **60** (2012) 1123–1149, [arXiv:1203.6562 \[hep-th\]](#).
3. E. A. Bergshoeff, J. Fernandez-Melgarejo, J. Rosseel, and P. K. Townsend, “On ‘New Massive’ 4D Gravity,” *JHEP* **1204** (2012) 070, [arXiv:1202.1501 \[hep-th\]](#).
4. L. Granda, E. Torrente-Lujan, and J. Fernandez-Melgarejo, “Non-minimal kinetic coupling and Chaplygin gas cosmology,” *Eur.Phys.J.* **C71** (2011) 1704, [arXiv:1106.5482 \[hep-th\]](#).
5. J. Fernandez-Melgarejo, T. Ortin, and E. Torrente-Lujan, “The general gaugings of maximal d=9 supergravity,” *JHEP* **1110** (2011) 068, [arXiv:1106.1760 \[hep-th\]](#).
6. M. Picariello, B. Chauhan, C. Das, Fernandez-Melgarejo, D. Montanino, *et al.*, “Neutrino Dipole Moments and Solar Experiments,” [arXiv:0907.0637 \[hep-ph\]](#).

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