Boundedness of the Hilbert Transform on Weighted Lorentz Spaces

Elona Agora

Universitat de Barcelona

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Elona Agora

María Jesús Carro Rossell i Francisco Javier Soria de Diego, professors del Departament de Matemàtica Aplicada i Anàlisi de la Universitat de Barcelona

CERTIFIQUEN:

Que la present memòria ha estat realitzada, sota la seva direcció, per Elona Agora i que constitueix la tesi d'aquesta per a aspirar al grau de Doctora en Matemàtiques.

María Jesús Carro Rossell   Francisco Javier Soria de Diego

Στους γονείς μου, στα αδέρφια μου Όλγα και Δημήτρη, στην josha, στον Jorge

Contents

Acknowledgements

As you set out for Ithaca
hope that your journey is a long one,
full of adventure, full of discovery.
Laistrygonians and Cyclops,
angry Poseidon, do not be afraid of them:
you'll never find things like that on your way
as long as you keep your thoughts raised high...

Hope that your journey is a long one.
May there be many summer mornings when,
with what pleasure, what joy,
you come into harbors seen for the first time...

Konstantinos Petrou Kavafis

With what pleasure, what joy, I can see the harbor now. I can smell the land. And behind me, and inside me, many stories. Stories and people. People that have made me feel in any moment that I am not alone...

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Resum

L'objectiu principal d'aquesta tesi és unificar dues teories conegudes i aparentment no relacionades entre elles, que tracten l'acotació de l'operador de Hilbert, sobre espais amb pesos, definit per

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy,$$

quan aquest límit existeix gairebé a tots els punts. Per una banda, tenim l'acotació de l'operador H sobre els espais de Lebesgue amb pesos i la teoria desenvolupada per Calderón i Zygmund. Per altra banda, hi ha la teoria de l'acotació de l'operador H desenvolupada al voltant dels espais invariants per reordenació. El marc natural per unificar aquestes teories consisteix en els espais de Lorentz amb pesos $\Lambda_u^p(w)$ i $\Lambda_u^{p,\infty}(w)$, els quals van ser definits per Lorentz a [68] i [67] de la següent manera:

$$\Lambda_u^p(w) = \left\{ f \in \mathcal{M} : \, ||f||_{\Lambda_u^p(w)} = \left(\int_0^\infty (f_u^*(t))^p w(t) dt \right)^{1/p} < \infty \right\}, \tag{1}$$

$$\Lambda_u^{p,\infty}(w) = \left\{ f \in \mathcal{M} : \, ||f||_{\Lambda_u^{p,\infty}(w)} = \sup_{t>0} W^{1/p}(t) f_u^*(t) < \infty \right\}, \tag{2}$$

on

$$f_u^*(t) = \inf\{s > 0 : \, u(\{x : |f(x)| > s\}) \leq t\} \quad \text{i} \quad W(t) = \int_0^t w(s) ds.$$

Més concretament, estudiarem l'acotació de l'operador H sobre els espais de Lorentz amb pesos:

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w), \tag{3}$$

i la seva versió de tipus dèbil

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w). \tag{4}$$

Abans de descriure els nostres resultats, presentem una breu revisió històrica de l'operador H. Aquest operador va ser introduït per Hilbert a [48] i [49]. Però, no va ser fins el 1924 quan Hardy el va anomenar "operador de Hilbert" en honor a les seves contribucions (veure [43], i [44]).

L'operador H sorgeix en molts contextos diferents, com l'estudi de valors de frontera de les parts imaginàries de funcions analítiques i la convergència de sèries de Fourier. Entre els resultats clàssics, esmentem el teorema de Riesz:

$$H : L^p \to L^p,$$

és acotat, quan $1 < p < \infty$ (veure [85], i [86]). Tot i que l'acotació en L^1 no és certa, Kolmogorov va provar a [58] l'estimació de tipus dèbil següent:

$$H : L^1 \to L^{1,\infty}. \tag{5}$$

Per a més informació en aquests temes veure [40], [94], [36], i [8].

Els resultats més rellevants que van servir per motivar aquest estudi són:

(I) Si $w = 1$, aleshores (3) i (4) corresponen a l'acotació

$$H : L^p(u) \to L^p(u), \tag{6}$$

i la seva versió dèbil

$$H : L^p(u) \to L^{p,\infty}(u), \tag{7}$$

respectivament.

Aquestes desigualtats sorgeixen naturalment quan en el teorema de Riesz, la mesura subjacent es canvia per una mesura general u. Aleshores, el problema és estudiar quines són les condicions sobre u que permeten que l'operador H sigui acotat a $L^p(u)$. Aquesta nova aproximació va donar naixement a la teoria de les desigualtats amb pesos, la qual juga un paper important en l'estudi de problemes de valor de la frontera per l'equació de Laplace en dominis Lipschitz. Altres aplicacions inclouen desigualtats vectorials, extrapolació d'operadors, i aplicacions a equacions no lineals, de derivades parcials i integrals (veure [36], [41], [56], i [57]).

L'estudi de (6) i (7) proporciona juntament amb l'acotació de l'operador maximal de Hardy-Littlewood en els mateixos espais, la teoria clàssica de pesos A_p. L'operador sublineal M, introduït per Hardy i Littlewood a [45], es defineix per

$$Mf(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)| dy,$$

on el supremum es considera en tots els intervals I de la recta real que contenen $x \in \mathbb{R}$. Per a més referències veure [38], [36], [41], [40], i [94].

Diem que $u \in A_p$ si, per a $p > 1$, tenim:

$$\sup_I \left(\frac{1}{|I|} \int_I u(x) dx \right) \left(\frac{1}{|I|} \int_I u^{-1/(p-1)}(x) dx \right)^{p-1} < \infty, \tag{8}$$

on el supremum es considera en tots els intervals I de la recta real, i $u \in A_1$ si

$$Mu(x) \approx u(x) \quad \text{a.e } x \in \mathbb{R}. \tag{9}$$

Muckenhoupt va provar a [71] que, si $p \geq 1$, la condició A_p caracteritza l'acotació

$$M : L^p(u) \longrightarrow L^{p,\infty}(u),$$

i si $p > 1$ també caracteritza

$$M : L^p(u) \longrightarrow L^p(u).$$

Hunt, Muckenhoupt i Wheeden van provar a [54] que, si $p \geq 1$, la condició A_p caracteritza (7) i si $p > 1$ la mateixa condició caracteritza també (6). Per una prova alternativa d'aquests resultats veure [26]. Per exponents $p < 1$ no hi ha cap pes u que compleixi (6) i (7).

(II) El cas $u = 1$ correspon a l'acotació de l'operador H en els espais de Lorentz clàssics, i va ser solucionat per Sawyer a [90]:

$$H : \Lambda^p(w) \longrightarrow \Lambda^p(w). \tag{10}$$

Una caracterització simplificada dels pesos pels quals l'acotació és certa, es presenta en termes de la condició $B_p \cap B_\infty^*$ introduïda per Neugebauer a [80]. Diem que $w \in B_p$ si

$$\int_r^\infty \left(\frac{r}{t}\right)^p w(t)\, dt \lesssim \int_0^r w(t)dt, \tag{11}$$

per a tot $r > 0$, i (11) caracteritza l'acotació

$$M : \Lambda^p(w) \to \Lambda^p(w),$$

provat a [5]. La condició $w \in B_\infty^*$ és

$$\int_0^r \frac{1}{t} \int_0^t w(s)ds\, dt \lesssim \int_0^r w(s)ds, \tag{12}$$

per a tot $r > 0$. Si $p > 1$ la condició $B_p \cap B_\infty^*$ caracteritza també la versió de tipus dèbil

$$H : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w), \tag{13}$$

mentre que el cas $p \leq 1$, es caracteritza per la condició $B_{p,\infty} \cap B_\infty^*$. Diem que $w \in B_{p,\infty}$ si, i només si

$$M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w) \tag{14}$$

es acotat. Precisament, tenim que:

(α) Si $p > 1$, $B_{p,\infty} = B_p$.

(β) Si $p \leq 1$, aleshores $w \in B_{p,\infty}$ si, i només si w és p quasi-concava: per a tot $0 < s \leq r < \infty$,

$$\frac{W(r)}{r^p} \lesssim \frac{W(s)}{s^p}. \tag{15}$$

(III) Recentment, Carro, Raposo i Soria van estudiar a [20] l'anàleg de la relació (3), però per a l'operador M, en comptes de l'operador H

$$M : \Lambda^p_u(w) \to \Lambda^p_u(w),$$

i la solució és la classe de pesos $B_p(u)$ definida com:

$$\frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \leq C \max_{1\leq j\leq J}\left(\frac{|I_j|}{|S_j|}\right)^{p-\varepsilon}, \tag{16}$$

per a algun $\varepsilon > 0$ i per cada família finita d'intervals disjunts, i oberts $(I_j)_{j=1}^J$, i també cada família de conjunts mesurables $(S_j)_{j=1}^J$, amb $S_j \subset I_j$, per a cada $j \in J$. Aquesta classe de pesos recupera els resultats ben coneguts en els casos clàssics; és a dir, si $w = 1$ llavors (16) és la condició A_p i si $u = 1$, llavors és la classe de pesos B_p (veure [20]). En el mateix treball es va considerar la versió de tipus dèbil del problema,

$$M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w). \tag{17}$$

Tanmateix, la caracterització geomètrica completa de l'estimació (17) no es va resoldre per a $p \geq 1$.

En aquesta tesi, caracteritzem totalment les acotacions (3) i (4), quan $p > 1$, donant una versió estesa i unificada de les teories clàssiques. També caracteritzem (17) per la condició $B_p(u)$ quan $p > 1$. Els resultats principals d'aquesta tesi proven que els enunciats següents són equivalents per a $p > 1$ (veure el Teorema 6.19):

Teorema. *Sigui $p > 1$. Els enunciats següents són equivalents:*

(i) $H : \Lambda^p_u(w) \to \Lambda^p_u(w)$ *és acotat.*

(ii) $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ *és acotat.*

(iii) $u \in A_\infty, w \in B^*_\infty$ *i* $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$ *és acotat.*

(iv) $u \in A_\infty, w \in B^*_\infty$ *i* $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ *és acotat.*

(iv) Existeix $\varepsilon > 0$, tal que per a cada família finita d'intervals disjunts, oberts $(I_j)_{j=1}^J$, i cada família de conjunts mesurables $(S_j)_{j=1}^J$, amb $S_j \subset I_j$, per a cada $j \in J$, es verifica que:

$$\min_j\left(\log\frac{|I_j|}{|S_j|}\right) \lesssim \frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \lesssim \max_j\left(\frac{|I_j|}{|S_j|}\right)^{p-\varepsilon}.$$

En particular, recuperem els casos clàssics $w = 1$, i $u = 1$. A més, reescrivim els nostres resultats en termes d'alguns índex de Boyd generalitzats. Lerner i Pérez van estendre a [66] el teorema de Lorentz-Shimogaki en espais funcionals quasi-Banach, no necessàriament invariants per reordenació. Motivats pels seus resultats, donem una extensió del teorema de Boyd, en el context dels espais de Lorentz amb pesos (veure el Teorema 6.26).

També hem solucionat el cas de tipus dèbil, $p \leq 1$ amb alguna condició addicional en w (veure el Teorema 6.20).

Els capítols són organitzats de la següent manera:

Per tal de dur a terme aquest projecte com a monografia auto-continguda, en el Capítol 2 estudiem totes les propietats bàsiques dels espais de Lorentz amb pesos. Aquest capítol també conté un resultat de densitat nou: provem que l'espai de funcions $\mathcal{C}^\infty$ amb suport compacte, $\mathcal{C}_c^\infty$, és dens en els espais de Lorentz amb pesos $\Lambda_u^p(w)$ en el cas que u i w no són integrables (veure el Teorema 2.13). Això serà important per solucionar alguns problemes tècnics de la definició de l'operador de Hilbert en $\Lambda_u^p(w)$.

El Capítol 3 recull totes les classes de pesos que apareixen en aquesta monografia. Primer estudiem les classes de pesos A_p i A_∞. A continuació, estudiem les classes de pesos B_p i $B_{p,\infty}$ que caracteritzen l'acotació de M en els espais de Lorentz clàssics. Com ja hem mencionat, la classe dels pesos B_p no és suficient per obtenir l'acotació de l'operador H sobre els espais $\Lambda_u^p(w)$, i es requereix també la condició B_∞^*. Després, investiguem la condició $B_p(u)$ i trobem algunes expressions equivalents noves estudiant el comportament asimptòtic a l'infinit d'una funció submultiplicativa (veure el Corol·lari 3.38). Finalment, definim i estudiem una classe nova de pesos AB_∞^*, que combina les classes A_∞ i B_∞^* (veure Proposició 3.46 per a més detalls).

En el Capítol 4 trobem condicions necessàries per l'acotació de tipus dèbil de l'operador H i obtenim algunes conseqüències útils. Si restringim l'acotació $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$ a funcions característiques d'intervals, tenim:

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\,ds\right)}{W\left(\int_{-b}^{b} u(s)\,ds\right)} \lesssim \left(\log \frac{1+\nu}{\nu}\right)^{-p},$$

per a cada $\nu \in (0,1]$ (veure el Teorema 4.4). En particular, això implica $u \notin L^1(\mathbb{R})$ i $w \notin L^1(\mathbb{R}^+)$ (veure la Proposició 4.5). A més a més si restringim l' acotació de tipus dèbil a funcions característiques de conjunts mesurables (veure el Teorema 4.8), obtenim

$$\frac{W(u(I))}{W(u(E))} \lesssim \left(\frac{|I|}{|E|}\right)^p,$$

i per això $W \circ u$ satisfà la condició doblant i w és p quasi-còncava. Finalment, l'acotació de tipus dèbil implica, aplicant arguments de dualitat, que es compleix:

$$||u^{-1}\chi_I||_{(\Lambda_u^p(w))'}||\chi_I||_{\Lambda_u^p(w)} \lesssim |I|,$$

per a tots els intervals I de la recta real (veure el Teorema 4.16). Estudiem aquesta condició i, en conseqüència, obtenim l'acotació de tipus dèbil de l'operador H en els espais $\Lambda^p(w)$.

En el Capítol 5 caracteritzem l'acotació de tipus dèbil en els espais de Lorentz clàssics per a $p > 0$, sota la suposició que $u \in A_1$:

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \Leftrightarrow w \in B_{p,\infty} \cap B_\infty^*,$$

(veure el Teorema 5.2). A més a més provem que si $u \in A_1$ i $p > 1$ tenim que

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w) \Leftrightarrow w \in B_p \cap B_\infty^*,$$

(veure Teorema 5.4), mentre en el cas $p \leq 1$ tenim el mateix resultat amb una condició addicional en els pesos (veure el Teorema 5.5). Per això, si $u \in A_1$, l'acotació del tipus fort (resp. del tipus dèbil) $H : \Lambda_u^p(w) \to \Lambda_u^p(w)$ (resp. $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$) coincideix amb l'acotació del mateix operador per $u = 1$.

El Capítol 6 conté la solució completa del problema quan $p > 1$; és a dir, la caracterització del tipus dèbil de l'acotació de l'operador H en els espais de Lorentz amb pesos (veure el Teorema 6.13) i també la seva versió de tipus fort (Teorema 6.18). A més, les condicions geomètriques que caracteritzen ambdós, les acotacions dels tipus dèbil i fort de l'operador H en $\Lambda_u^p(w)$ es donen per al Teorema 6.19 quan $p > 1$, i al Teorema 6.20 per l'acotació del tipus dèbil en el cas $p < 1$. Finalment, reformulem els nostres resultats en termes del teorema de Boyd (veure el Teorema 6.26).

Alguns dels resultats tècnics més significatius que hem utilitzat per provar els nostres teoremes principals són els següents:

(a) Hem caracteritzat la condició A_∞, en termes de l'operador H de la manera següent (veure el Teorema 6.3):

$$\int_I |H(u\chi_I)(x)|dx \lesssim u(I),$$

i així obtenim que (4) implica la necessitat de la condició AB_∞^*.

(b) Provem que si $p > 1$, llavors

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \Rightarrow M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w),$$

(veure el Teorema 6.8) que, en particular, proporciona una prova diferent del fet ben conegut, que correspon al cas $w = 1$:

$$H : L^p(u) \to L^{p,\infty}(u) \Rightarrow M : L^p(u) \to L^{p,\infty}(u),$$

sense fer servir explícitament la condició A_p.

(c) Solucionem completament l'acotació de (17), si $p > 1$ i la solució és la classe $B_p(u)$ (veure el Teorema 6.17). En particular, mostrem que si $p > 1$, llavors

$$M : L^p(u) \to L^{p,\infty}(u) \Rightarrow M : L^p(u) \to L^p(u),$$

sense utilitzar la desigualtat de Hölder inversa.

Les tècniques que vam fer servir per obtenir la caracterització de l'acotació

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w),$$

i la seva versió de tipus dèbil $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, quan $p > 1$ ens permeten aconseguir algunes condicions necessàries per l'acotació de tipus dèbil de l'operador H en el cas no diagonal:

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1),$$

que serà també necessari per la versió de tipus fort $H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1}_{u_1}(w_1)$. En el Capítol 7 estudiem aquestes condicions. En primer lloc, presentem una breu revisió en els casos clàssics, on, per una banda, tenim el conegut problema de dos pesos per l'operador de Hilbert,

$$H : L^p(u_0) \to L^{p,\infty}(u_1) \quad \text{i} \quad H : L^p(u_0) \to L^p(u_1),$$

que es va plantejar als anys 1970, però no s'ha resolt completament, i d'altra banda, tenim el cas no-diagonal de l'acotació de l'operador H en els espais de Lorentz clàssics.

Finalment, presentem algunes aplicacions respecte a la caracterització de l'acotació

$$H : L^{p,q}(u) \to L^{r,s}(u),$$

per alguns exponents $p, q, r, s > 0$. En particular, completem alguns dels resultats obtinguts a [25] per Chung, Hunt, i Kurtz.

Els resultats d'aquesta memoria están inclosos a [1, 2, 3].

Notations

Throughout this monograph, the following standard notations are used: The letter $\mathcal{M}$ is used for the space of measurable functions on $\mathbb{R}$, endowed with the measure $u = u(x)dx$. Moreover, u and w will denote weight functions; that is, positive, locally integrable functions defined on $\mathbb{R}$ and $\mathbb{R}^+ = [0, \infty)$, respectively. If E is a measurable set of $\mathbb{R}$, we denote $u(E) = \int_E u(x)dx$ and we write $W(r) = \int_0^r w(t)dt$, for $0 \leq r \leq \infty$. For $0 < p < \infty$, L^p denotes the usual Lebesgue space and L^p_{dec} the cone of positive, decreasing functions belonging to L^p. The limit case L^∞ is the set of bounded measurable functions defined on $\mathbb{R}$, while $L^\infty_0(u)$ refers to the space of functions belonging to L^∞, whose support has finite measure with respect to u. The letter p' denotes the conjugate of p; that is $1/p + 1/p' = 1$. In addition, $\mathcal{C}^\infty_c$ refers to the space of smooth functions defined on $\mathbb{R}$ with compact support. We denote by $\mathcal{S}$ the class of simple functions

$$\mathcal{S} = \{f \in \mathcal{M} : \operatorname{card}(f(\mathbb{R})) < \infty\}.$$

The class of simple functions with support in a set of finite measure is:

$$\mathcal{S}_0(u) = \{f \in \mathcal{S} : u(\{f \neq 0\}) < \infty\}.$$

Furthermore, we write $\mathcal{S}_c$ for the space of simple functions with compact support. The distribution function of $f \in \mathcal{M}$ is $\lambda^u_f(s) = u(\{x : |f(x)| > s\})$, the non-increasing rearrangement with respect to the measure u is

$$f^*_u(t) = \inf\{s > 0 : \lambda^u_f(s) \leq t\},$$

and $f^{**}_u(t) = \dfrac{1}{t}\displaystyle\int_0^t f^*_u(s)ds$. The rearrangement of f with respect to the Lebesgue measure is denoted as $f^*(t)$. Finally, letting A and B be two positive quantities, we say that they are equivalent ($A \approx B$) if there exists a positive constant C, which may vary even in the same theorem and is independent of essential parameters defining A and B, such that $C^{-1}A \leq B \leq CA$. The case $A \leq CB$ is denoted by $A \lesssim B$.

For any other possible definition or notation, we refer to the main reference books (e.g. [8], [36], [40], [41], [87], [94]).

Chapter 1

Introduction

The main purpose of this work is to unify two well-known and, a priori, unrelated theories dealing with weighted inequalities for the Hilbert transform, defined by

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy,$$

whenever this limit exists almost everywhere. On the one hand, we have the Calderón-Zygmund theory of the boundedness of H on weighted Lebesgue spaces. On the other hand, there is the theory developed around the boundedness of H on classical Lorentz spaces in the context of rearrangement invariant function spaces. A natural unifying framework for these two settings consists on the weighted Lorentz spaces $\Lambda^p_u(w)$ and $\Lambda^{p,\infty}_u(w)$ defined by Lorentz in [68] and [67] as follows:

$$\Lambda^p_u(w) = \left\{ f \in \mathcal{M} : \, ||f||_{\Lambda^p_u(w)} = \left(\int_0^\infty (f^*_u(t))^p w(t) dt \right)^{1/p} < \infty \right\}, \tag{1.1}$$

and

$$\Lambda^{p,\infty}_u(w) = \left\{ f \in \mathcal{M} : \, ||f||_{\Lambda^{p,\infty}_u(w)} = \sup_{t>0} W^{1/p}(t) f^*_u(t) < \infty \right\}. \tag{1.2}$$

More precisely, we will study the boundedness of H on the weighted Lorentz spaces:

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w), \tag{1.3}$$

and its weak-type version

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w). \tag{1.4}$$

Before describing our results, we present a brief historical review on the Hilbert transform. This operator was introduced by Hilbert in [48] and [49], and named "Hilbert transform" by Hardy in 1924, in honor of his contributions (see [43] and [44]). It arises in many different contexts such as the study of boundary values of the imaginary parts of analytic functions and the convergence of Fourier series. Among the classical results, we mention Riesz' theorem which states that

$$H : L^p \to L^p$$

is bounded, whenever $1 < p < \infty$ (see [85] and [86]). Although the L^1 boundedness for H fails to be true, Kolmogorov proved in [58] the following weak-type estimate:

$$H : L^1 \to L^{1,\infty}. \tag{1.5}$$

For further information on these topics see [40], [94], [36] and [8].

The following examples, involving weighted inequalities, have been historically relevant to motivate our study.

(I) If $w = 1$, then (1.3) and (1.4) correspond to the boundedness

$$H : L^p(u) \to L^p(u), \tag{1.6}$$

and its weak-type version

$$H : L^p(u) \to L^{p,\infty}(u), \tag{1.7}$$

respectively. These inequalities arise naturally when in the Riesz' theorem, the underlying measure is changed from Lebesgue measure to a general measure u. Then, the problem is to study which are the conditions over u that allow the Hilbert transform to be bounded on $L^p(u)$. This new approach gave birth to the theory of weighted inequalities, which plays a large part in the study of boundary value problems for Laplace's equation on Lipschitz domains. Other applications include vector-valued inequalities, extrapolation of operators, and applications to certain classes of nonlinear partial differential and integral equations (see [36], [41], [56], and [57]).

The study of (1.6) and (1.7) yield together with the boundedness of the Hardy-Littlewood maximal function M, on the same spaces, the classical theory of the Muckenhoupt A_p weights. The sublinear operator M, introduced by Hardy and Littlewood in [45], is defined by

$$Mf(x) = \sup_{x\in I} \frac{1}{|I|}\int_I |f(y)|dy,$$

and the supremum is considered over all intervals I of the real line containing $x \in \mathbb{R}$. For further references see [38], [36], [40], [41], and [94].

We say that $u \in A_p$ if, for $p > 1$, the following holds:

$$\sup_I \left(\frac{1}{|I|}\int_I u(x)dx\right)\left(\frac{1}{|I|}\int_I u^{-1/(p-1)}(x)dx\right)^{p-1} < \infty, \tag{1.8}$$

and the supremum is considered over all intervals of the real line, and $u \in A_1$ if

$$Mu(x) \approx u(x) \quad \text{a.e } x \in \mathbb{R}. \tag{1.9}$$

Muckenhoupt showed in [71] that, if $p \geq 1$, the A_p condition characterizes the boundedness

$$M : L^p(u) \longrightarrow L^{p,\infty}(u),$$

and if $p > 1$ it also characterizes

$$M : L^p(u) \longrightarrow L^p(u).$$

Hunt, Muckenhoupt, and Wheeden proved in [54] that, for $p \geq 1$, the A_p condition characterizes (1.7) and for $p > 1$ it also characterizes (1.6). For an alternative proof of these results see [26]. For $p < 1$ there are no weights u such that (1.6) or (1.7) hold.

(II) The case $u = 1$ corresponds to the boundedness of the Hilbert transform on the classical Lorentz spaces, solved by Sawyer in [90]. A simplified characterization of the weights for which the boundedness

$$H : \Lambda^p(w) \longrightarrow \Lambda^p(w) \tag{1.10}$$

holds, whenever $p > 0$, is given in terms of the $B_p \cap B_\infty^*$ condition, introduced by Neugebauer in [80]. We say that $w \in B_p$ if the following condition holds:

$$\int_r^\infty \left(\frac{r}{t}\right)^p w(t)\,dt \lesssim \int_0^r w(t)dt, \tag{1.11}$$

for all $r > 0$, and (1.11) characterizes the boundedness

$$M : \Lambda^p(w) \to \Lambda^p(w),$$

proved in [5]. The condition $w \in B_\infty^*$ is given by

$$\int_0^r \frac{1}{t} \int_0^t w(s)ds\ dt \lesssim \int_0^r w(s)ds, \tag{1.12}$$

for all $r > 0$. If $p > 1$ the $B_p \cap B_\infty^*$ class characterizes also the weak-type version

$$H : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w), \tag{1.13}$$

whereas the case $p \leq 1$ is characterized by the $B_{p,\infty} \cap B_\infty^*$ condition. We say that $w \in B_{p,\infty}$ if and only if

$$M : \Lambda^p(w) \longrightarrow \Lambda^{p,\infty}(w) \tag{1.14}$$

is bounded. It holds that:

(α) If $p > 1$, $B_{p,\infty} = B_p$.

(β) If $p \leq 1$, then $w \in B_{p,\infty}$ if and only if w is p quasi-concave: for every $0 < s \leq r < \infty$,

$$\frac{W(r)}{r^p} \lesssim \frac{W(s)}{s^p}. \tag{1.15}$$

(III) Recently, Carro, Raposo and Soria studied in [20] the analogous of relation (1.3), but for the Hardy-Littlewood maximal function, instead of H

$$M : \Lambda_u^p(w) \to \Lambda_u^p(w),$$

and the solution is the $B_p(u)$ class of weights, defined as follows:

$$\frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \leq C \max_{1\leq j\leq J}\left(\frac{|I_j|}{|S_j|}\right)^{p-\varepsilon}, \tag{1.16}$$

for some $\varepsilon > 0$ and for every finite family of pairwise disjoint, open intervals $(I_j)_{j=1}^J$, and also every family of measurable sets $(S_j)_{j=1}^J$, with $S_j \subset I_j$, for every j. This class of weights recovers the well-known results in the classical cases; that is, if $w = 1$ then (1.16) is the A_p condition and if $u = 1$, then it is the B_p condition (see [20]). In the same work, the weak-type version of the problem was also considered

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w). \tag{1.17}$$

However, the complete geometric characterization of the estimate (1.17) was not obtained for $p \geq 1$.

In this work, we totally solve the problem of the boundedness (1.3) and its weak-type version (1.4), whenever $p > 1$ giving a unified version of the classical theories. We also characterize (1.17) by the $B_p(u)$ condition, since it will be involved in the solution of (1.3) and (1.4). We will see that this solution is given in terms of conditions involving both underlying weights u and w in a rather intrinsic way. Summarizing, the main results of this thesis prove that the following statements are equivalent for $p > 1$ (see Theorem 6.19):

Theorem. *If $p > 1$, then the following statements are equivalent:*

(i) $H : \Lambda_u^p(w) \to \Lambda_u^p(w)$ *is bounded.*

(ii) $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$ *is bounded.*

(iii) $u \in A_\infty, w \in B_\infty^*$ *and* $M : \Lambda_u^p(w) \to \Lambda_u^p(w)$ *is bounded.*

(iv) $u \in A_\infty, w \in B_\infty^*$ *and* $M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$ *is bounded.*

(iv) There exists $\varepsilon > 0$, such that for every finite family of pairwise disjoint, open intervals $(I_j)_{j=1}^J$, and every family of measurable sets $(S_j)_{j=1}^J$, with $S_j \subset I_j$, for every $j \in J$, it holds that:

$$\min_j\left(\log\frac{|I_j|}{|S_j|}\right) \lesssim \frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \lesssim \max_j\left(\frac{|I_j|}{|S_j|}\right)^{p-\varepsilon}.$$

Furthermore, we reformulate our results in terms of some generalized upper and lower Boyd indices. Lerner and Pérez extended in [66] the Lorentz-Shimogaki theorem in quasi-Banach function spaces, not necessarily rearrangement invariant. Motivated by their results, we define the lower Boyd index and give an extension of Boyd theorem, in the context of weighted Lorentz spaces (see Theorem 6.26).

Moreover, we have solved the weak-type boundedness of H on $\Lambda^p_u(w)$ for $p \leq 1$, with some extra assumption on w (see Theorem 6.20).

The chapters are organized as follows:

In order to carry out this project as a self-contained monograph, we study in Chapter 2 all the basic properties of the weighted Lorentz spaces. This chapter also contains a new density result: we prove that the $\mathcal{C}^\infty$ functions with compact support, $\mathcal{C}^\infty_c$, is dense in weighted Lorentz spaces $\Lambda^p_u(w)$, provided u and w are not integrable (see Theorem 2.13). This will be important in order to solve technical problems, since the Hilbert transform is well-defined on $\mathcal{C}^\infty_c$.

In Chapter 3 we summarize all the classes of weights that appear throughout this work. First we study the Muckenhoupt A_p class of weights and the A_∞ condition. Then, we study the B_p and $B_{p,\infty}$ conditions that characterize the boundedness of M on classical Lorentz spaces, introducing the Hardy operator. Since, as we have already mentioned, the B_p (resp. $B_{p,\infty}$) condition is not sufficient for the strong-type (resp. weak-type) boundedness of the Hilbert transform on $\Lambda^p(w)$, we introduce and study the B^*_∞ condition. Next, we investigate the $B_p(u)$ condition, and find some new equivalent expressions studying the asymptotic behavior of some submultilplicative function at infinity (see Corollary 3.38). Finally, we define and study a new class of pairs of weights AB^*_∞, that combines the already known A_∞ and B^*_∞ classes (see Proposition 3.46 for more details). This new class of weights is involved in the study of the boundedness of the Hilbert transform on weighted Lorentz spaces (see Chapter 6).

In Chapter 4 we find necessary conditions for the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces and obtain some useful consequences. If we restrict the weak-type boundedness of the Hilbert transform, $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, to characteristic functions of intervals, we have that

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\,ds\right)}{W\left(\int_{-b}^{b} u(s)\,ds\right)} \lesssim \left(\log \frac{1+\nu}{\nu}\right)^{-p},$$

for every $\nu \in (0,1]$ (see Theorem 4.4). In particular, this implies that $u \notin L^1(\mathbb{R})$ and $w \notin L^1(\mathbb{R}^+)$ (see Proposition 4.5). We also show that, if we restrict the weak-type boundedness of H to characteristic functions of measurable sets (see Theorem 4.8), we obtain

$$\frac{W(u(I))}{W(u(E))} \lesssim \left(\frac{|I|}{|E|}\right)^p,$$

and hence $W \circ u$ satisfies the doubling condition and w is p quasi-concave. In particular $w \in \Delta_2$. Thus, in what follows after Corollary 4.9, we shall assume, without loss of generality, that

$$u \notin L^1(\mathbb{R}),\ w \notin L^1(\mathbb{R}^+),\ \text{and } w \in \Delta_2.$$

Finally, the weak-type boundedness of the Hilbert transform implies, applying duality arguments, that

$$||u^{-1}\chi_I||_{(\Lambda_u^p(w))'}||\chi_I||_{\Lambda_u^p(w)} \lesssim |I|,$$

for all intervals I of the real line (see Theorem 4.16).

In Chapter 5 we characterize the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces for $p > 0$, under the assumption that $u \in A_1$:

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \Leftrightarrow w \in B_{p,\infty} \cap B_\infty^*,$$

(see Theorem 5.2). Analogously, we prove that if $u \in A_1$ and $p > 1$ we have that

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w) \Leftrightarrow w \in B_p \cap B_\infty^*,$$

(see Theorem 5.4), while in the case $p \leq 1$ we have the same result under some extra assumption on the weights (see Theorem 5.5). Hence, if $u \in A_1$, the strong-type (resp. weak-type) boundedness of the Hilbert transform $H : \Lambda_u^p(w) \to \Lambda_u^p(w)$ (resp. $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$) coincides with the boundedness of the same operator for $u = 1$.

Chapter 6 contains the complete solution of the problem in the case $p > 1$; that is, the characterization of the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces (see Theorem 6.13) and also its strong-type version (see Theorem 6.18). Moreover, the geometric conditions that characterize both weak-type and strong-type boundedness of H on $\Lambda_u^p(w)$ are given in Theorem 6.19 for $p > 1$, and in Theorem 6.20 for the weak-type boundedness and $p < 1$. Reformulating the above results in terms of the generalized Boyd indices, we give an extension of Boyd theorem in $\Lambda_u^p(w)$ (see Theorem 6.26).

Some of the most significant technical results that we have used to prove our main theorems are the following:

(a) We have characterized the A_∞ condition, in terms of the Hilbert transform (see Theorem 6.3),

$$\int_I |H(u\chi_I)(x)|dx \lesssim u(I),$$

and so we obtain that (1.4) implies the necessity of the AB_∞^* condition.

(b) We prove that if $p > 1$, then

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \Rightarrow M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w),$$

(see Theorem 6.8) which, in particular, provides a different proof of the well-known fact, that corresponds to the case $w = 1$:

$$H : L^p(u) \to L^{p,\infty}(u) \Rightarrow M : L^p(u) \to L^{p,\infty}(u),$$

without passing through the A_p condition.

(c) We completely solve the boundedness of (1.17) when $p > 1$ and the solution is the $B_p(u)$ condition (see Theorem 6.17). In particular, we show that if $p > 1$, then

$$M : L^p(u) \to L^{p,\infty}(u) \Rightarrow M : L^p(u) \to L^p(u),$$

without using the reverse Hölder inequality.

The techniques used to characterize the boundedness $H : \Lambda^p_u(w) \to \Lambda^p_u(w)$, and its weak-type version $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, whenever $p > 1$ allow us to get some necessary conditions for the weak-type boundedness of H in the non-diagonal case:

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1),$$

which will be also necessary for the strong-type version $H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1}_{u_1}(w_1)$. In Chapter 7 we study these conditions. First, we present a brief review on the classical cases: On the one hand, we have the well-known two-weighted problem for the Hilbert transform,

$$H : L^p(u_0) \to L^{p,\infty}(u_1) \quad \text{and} \quad H : L^p(u_0) \to L^p(u_1),$$

posed in the early 1970's, but still unsolved in its full generality. On the other hand, we have the non-diagonal boundedness of H on classical Lorentz spaces.

Finally, we present some applications concerning the characterization of

$$H : L^{p,q}(u) \to L^{r,s}(u)$$

for some exponents $p, q, r, s > 0$. In particular, we complete some of the results obtained in [25] by Chung, Hunt, and Kurtz.

The results of this monograph are included in [1], [2], and [3].

As far as possible, we have tried to provide precise bibliographic information about the previously known results.

Chapter 2

Review on weighted Lorentz spaces

As we have pointed out in the Introduction, the main goal of this monograph is to study the strong-type boundedness of the Hilbert transform on weighted Lorentz spaces

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w),$$

and its weak-type version $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$. For this reason, we will briefly present some basic properties of these spaces. Then, we prove a new density result: $\mathcal{C}_c^\infty$ is dense in weighted Lorentz spaces $\Lambda_u^p(w)$, under some assumptions on the weights u and w (see Theorem 2.13). This fact will be useful to solve some technical problems, since the Hilbert transform is well defined on $\mathcal{C}_c^\infty$.

2.1 Weighted Lorentz spaces

Weighted Lorentz spaces $\Lambda_u^p(w)$ (see Definition 2.1 below) are a particular class of linear function spaces of measurable functions defined on $\mathbb{R}$. These spaces were introduced and studied by Lorentz in [68], and [67] for the measure space $((0,\ell), dx)$ and $\ell < \infty$. The functional defining them depends on two measures: the non-increasing rearrangement is taken with respect to the measure u and the integral is considered with respect to w defined on $\mathbb{R}^+$. Both aspects provide measure-theoretical and functional-analytic properties, enriching the theory developed by Lorentz. We present some of these well-known properties and prove a new density result.

Definition 2.1. *If $0 < p < \infty$, the weighted Lorentz spaces are defined as*

$$\Lambda_u^p(w) = \left\{ f \in \mathcal{M} : ||f||_{\Lambda_u^p(w)} = \left(\int_0^\infty (f_u^*(t))^p w(t) dt \right)^{1/p} < \infty \right\},$$

and the weak-type weighted Lorentz spaces

$$\Lambda_u^{p,\infty}(w) = \left\{ f \in \mathcal{M} : ||f||_{\Lambda_u^{p,\infty}(w)} = \sup_{t>0} W^{1/p}(t) f_u^*(t) < \infty \right\}.$$

The weighted Lorentz spaces generalize many well-known spaces such as the weighted Lebesgue spaces $L^p(u)$ and the $L^{p,q}$ spaces.

Example 2.2. *In view of Definition 2.1, we have that*

(α) If $u = 1, w = 1$, we recover the Lebesgue spaces, $\Lambda_1^p(1) = L^p$ and $\Lambda_1^{p,\infty}(1) = L^{p,\infty}$ respectively.

(β) If $w = 1$, we obtain the weighted Lebesgue spaces $\Lambda_u^p(1) = L^p(u)$ and $\Lambda_u^{p,\infty}(1) = L^{p,\infty}(u)$ respectively.

(γ) If $u = 1$, we get the spaces $\Lambda^p(w)$ and $\Lambda^{p,\infty}(w)$ respectively, that are usually called classical Lorentz spaces.

(δ) If $u = 1$ and $w(t) = t^{(q-p)/p}$, then $\Lambda^q(t^{(q-p)/p})$ is the $L^{p,q}$ space given by

$$L^{p,q} = \left\{ f \in \mathcal{M} : ||f||_{L^{p,q}} = \left(\int_0^\infty (f^*(t))^q t^{q/p-1} dt \right)^{1/q} < \infty \right\}$$

and $\Lambda^{q,\infty}(t^{(q-p)/p})$ is

$$L^{p,\infty} = \{ f \in \mathcal{M} : ||f||_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t) < \infty \}.$$

Observe that $||f||_{\Lambda_u^p(w)} = ||f_u^*||_{L^p(w)}$. This allows us to extend the previous definition as follows (see [20]).

Definition 2.3. *For $0 < p, q \leq \infty$ set*

$$\Lambda_u^{p,q}(w) = \left\{ f \in \mathcal{M} : ||f||_{\Lambda_u^{p,q}(w)} = ||f_u^*||_{L^{p,q}(w)} = \left(\int_0^\infty ((f_u^*(t))_w^*)^q t^{q/p-1} dt \right)^{1/q} < \infty \right\}. \quad (2.1)$$

The functional defining the weighted Lorentz spaces $\Lambda_u^{p,q}(w)$ can be expressed in terms of the distribution function. In fact, it was proved in [22] that, for $q > 0$, and every decreasing function g we have that

$$\int_0^\infty g^q(s) w(s) ds = \int_0^\infty q t^{q-1} W(u(\{x \in \mathbb{R} : |g(x)| > t\})) dt. \quad (2.2)$$

Then, relation (2.2) gives several equivalent expressions for the functional $|| \cdot ||_{\Lambda_u^{p,q}(w)}$, in terms of W.

Proposition 2.4. *Let $0 < p, q < \infty$ and f measurable in $\mathbb{R}$.*

(i) $||f||_{\Lambda_u^{p,q}(w)} = \left(\int_0^\infty q t^{q-1} W^{q/p}(u(\{x \in \mathbb{R} : |f(x)| > t\}))dt\right)^{1/q}$.

(ii) $||f||_{\Lambda_u^{p}(w)} = \left(\int_0^\infty p t^{p-1} W(u(\{x \in \mathbb{R} : |f(x)| > t\}))dt\right)^{1/p}$.

(iii) $||f||_{\Lambda_u^{p,\infty}(w)} = \sup_{t>0} t W^{1/p}(u(\{x \in \mathbb{R} : |f(x)| > t\}))$.

Remark 2.5. If $w = 1$ in (2.1), then by Proposition 2.4 (i) we obtain the $L^{p,q}(u)$ spaces,

$$L^{p,q}(u) = \left\{ f \in \mathcal{M} : ||f||_{L^{p,q}(u)} = \left(\int_0^\infty (f_u^*(t))^q t^{q/p-1}\right)^{1/q} < \infty \right\}. \tag{2.3}$$

The Lorentz spaces are not necessarily Banach function spaces. Although, the study of the normability requires certain operator estimates (see Chapter 3), they are quasi-normed function spaces, provided a weak assumption on the weight w.

Definition 2.6. *We say that $w \in \Delta_2$ if $W(2r) \lesssim W(r)$, for all $r > 0$.*

Theorem 2.7. *[20] Let $0 < p < \infty$ and $0 < q \leq \infty$. Then, the following statements are equivalent:*

(i) $\Lambda_u^{p,q}(w)$ *is a quasi-normed space.*

(ii) $w \in \Delta_2$.

(iii) $W(s+t) \lesssim W(t) + W(s)$, *for all* $s, t > 0$.

Definition 2.8. *A measurable function f is said to have absolutely continuous quasi-norm in a quasi-normed space X if*

$$\lim_{n\to\infty} ||f\chi_{A_n}||_X = 0,$$

for every decreasing sequence of sets (A_n) with $\chi_{A_n} \to 0$ a.e. If every function in X has this property, we say that X has an absolutely continuous quasi-norm.

Next theorem gives an equivalent property to the dominated convergence theorem for the weighted Lorentz spaces.

Theorem 2.9. *[20] If $w \in \Delta_2$ and $f \in \Lambda_u^p(w)$ (resp. $\Lambda_u^{p,\infty}(w)$), then the following statements are equivalent:*

(i) f *has absolutely continuous quasi-norm in* $\Lambda^p_u(w)$ *(resp.* $\Lambda^{p,\infty}_u(w)$*).*

(ii) $\lim_{n\to\infty} ||g-g_n||_{\Lambda^p_u(w)} = 0$, *(resp.* $\lim_{n\to\infty} ||g-g_n||_{\Lambda^{p,\infty}_u(w)} = 0$ *) if* $|g_n| \le |f|$ *and* $\lim_{n\to\infty} g_n = g$ *a.e.*

Theorem 2.10. *[20] Let* $0 < p < \infty$ *and let* $w \in \Delta_2$*.*

(i) If $u(\mathbb{R}) < \infty$*, then* $\Lambda^p_u(w)$ *has absolutely continuous quasi-norm.*

(ii) If $u(\mathbb{R}) = \infty$*, then* $\Lambda^p_u(w)$ *has absolutely continuous quasi-norm if and only if* $w \notin L^1$*.*

Now, we prove that the space $\mathcal{C}^\infty_c$ is dense in $\Lambda^p_u(w)$, under the assumptions $u \notin L^1(\mathbb{R})$ and $w \notin L^1(\mathbb{R}^+)$. In fact, we will need this density result to define the Hilbert transform on weighted Lorentz spaces, but these assumptions are not restrictive, since we will show that they are necessary in our setting (see Proposition 4.5). First we need the following technical results.

Lemma 2.11. *[52] Let* $K \subset \mathbb{R}$ *be a compact set and* $U \subset \mathbb{R}$ *an open set, such that* $K \subset U$*. Then, there exists* $f \in \mathcal{C}^\infty(U)$ *such that* $f = 0$ *in* U^c*,* $0 \le f \le 1$ *and* $f = 1$ *in* K*.*

Lemma 2.12. *[20] Let* $w \in \Delta_2$*. Then* $\mathcal{S}_0(u)$ *is dense in* $\Lambda^p_u(w)$*.*

Theorem 2.13. *If* $u \notin L^1(\mathbb{R})$*,* $w \notin L^1(\mathbb{R}^+)$ *and* $w \in \Delta_2$*, then* $\mathcal{C}^\infty_c(\mathbb{R})$ *is dense in* $\Lambda^p_u(w)$*.*

Proof. Observe that the space of simple functions with compact support, $\mathcal{S}_c(\mathbb{R})$ is dense in $\Lambda^p_u(w)$. Indeed, by Lemma 2.12, we have that $\mathcal{S}_0(u)$ is dense in $\Lambda^p_u(w)$. On the other hand, given $f \in \mathcal{S}_0(u)$, the sequence $f_n = f\chi_{(-n,n)} \in \mathcal{S}_c(\mathbb{R})$ tends to f pointwise and hence, by Theorem 2.10 *(ii)*, it also converges to f in the quasi-norm $||\cdot||_{\Lambda^p_u(w)}$.

Now, to prove the density of $\mathcal{C}^\infty_c(\mathbb{R})$ in $\mathcal{S}_c(\mathbb{R})$ with respect to the topology induced by the quasi-norm of $\Lambda^p_u(w)$, it is enough to show that a characteristic function of a bounded measurable set can be approximated by smooth functions of compact support. Thus, let E be a bounded measurable set and let $\varepsilon > 0$. Take a compact set $K \subset \mathbb{R}$ and a bounded open set $U \subset \mathbb{R}$ such that

$$K \subset E \subset U \qquad \text{and} \qquad u(U \setminus K) \le \delta,$$

for some small δ to be chosen. Then, by Urysohn's Lemma 2.11, there exists a function $f \in \mathcal{C}^\infty_c(\mathbb{R})$ such that $f|_K = 1$, $f|_{U^c} = 0$, and $0 \le f \le 1$. Then, since $|\chi_E - f| \le \chi_{U\setminus K}$, we get

$$||\chi_E - f||^p_{\Lambda^p_u(w)} \le ||\chi_{U\setminus K}||^p_{\Lambda^p_u(w)} = \int_0^{u(U\setminus K)} w(x)\,dx \le \int_0^\delta w(x)\,dx.$$

Therefore, choosing δ small enough we obtain that $||\chi_E - f||_{\Lambda^p_u(w)} \le \varepsilon$. $\square$

2.2 Duality

The dual and associate spaces of the weighted Lorentz spaces have been studied in [20], whereas the definition in the context of Banach function spaces can be found in [8]. The authors described in [20] the associate spaces of $\Lambda^p_u(w)$ and $\Lambda^{p,\infty}_u(w)$ in terms of the so-called Lorentz spaces Γ, and identified when they are the trivial spaces. It is out of our purpose to make a complete presentation of the aforementioned subject, although we give the results that will be necessary for our work.

Definition 2.14. *Let $||\cdot||:\mathcal{M}\to[0,\infty)$ be a positively homogeneous functional and $E=\{f\in\mathcal{M}: ||f||<\infty\}$. We define the associate norm*

$$||g||_{E'} := \sup_{f\in E}\frac{\left|\int_{\mathbb{R}} f(x)g(x)u(x)\,dx\right|}{||f||}.$$

The associate space of E is then $E'=\{f\in\mathcal{M}: ||f||_{E'}<\infty\}$.

Definition 2.15. *If $0<p<\infty$ we define*

$$\Gamma^p_u(w)=\left\{f\in\mathcal{M}: ||f||_{\Gamma^p_u(w)}=\left(\int_0^\infty (f^{**}_u(t))^p w(t)dt\right)^{1/p}<\infty\right\}.$$

The weak-type version of the previous space is

$$\Gamma^{p,\infty}_u(w)=\left\{f\in\mathcal{M}: ||f||_{\Gamma^{p,\infty}_u(w)}=\sup_{t>0}W^{1/p}(t)f^{**}_u(t)<\infty\right\}.$$

Theorem 2.16. *[20] The associate spaces of the Lorentz spaces are described as follows:*

(i) If $p\le 1$, then

$$(\Lambda^p_u(w))'=\Gamma^{1,\infty}_u(\tilde{w}),$$

where $\widetilde{W}(t)=tW^{-1/p}(t), t>0$.

(ii) If $1<p<\infty$, and $f\in\mathcal{M}$, then

$$\begin{aligned}||f||_{(\Lambda^p_u(w))'} &\approx \left(\int_0^\infty\left(\frac{1}{W(t)}\int_0^t f^*_u(s)ds\right)^{p'} w(t)dt\right)^{1/p'}+\frac{\int_0^\infty f^*_u(t)dt}{W^{1/p}(\infty)}\\ &\approx \left(\int_0^\infty\left(\frac{1}{W(t)}\int_0^t f^*_u(s)ds\right)^{p'-1} f^*_u(t)dt\right)^{1/p'}.\end{aligned}$$

(iii) If $0 < p < \infty$*, then*

$$(\Lambda_u^{p,\infty}(w))' = \Lambda_u^1(W^{-1/p}).$$

A direct consequence of Theorem 2.16 is the characterization of the weights w such that $(\Lambda_u^p(w))' = \{0\}$.

Theorem 2.17.

(i) If $0 < p \leq 1$*, then* $(\Lambda_u^p(w))' \neq \{0\} \;\Leftrightarrow\; \sup\limits_{0<t<1} \dfrac{t^p}{W(t)} < \infty.$

(ii) If $1 < p < \infty$*, then* $(\Lambda_u^p(w))' \neq \{0\} \;\Leftrightarrow\; \displaystyle\int_0^1 \left(\frac{t}{W(t)}\right)^{p'-1} dt < \infty.$

(iii) If $0 < p < \infty$*, then* $(\Lambda_u^{p,\infty}(w))' \neq \{0\} \;\Leftrightarrow\; \displaystyle\int_0^1 \frac{1}{W^{1/p}(t)} dt < \infty.$

One of the most important tools in the study of the boundedness of operators is the interpolation theory. Among the results that can be found in the literature, we will be interested in the Marcinkiewicz theorem adapted to the context of weighted Lorentz spaces $\Lambda_u^{p,q}(w)$. This has been one of the subjects studied in [20], in the setting of the K functional associated to the weighted Lorentz spaces. For further information on this topic see [8], [9] and [96].

Theorem 2.18. *[20] Let* $0 < p_i, q_i, \bar{p}_i, \bar{q}_i \leq \infty$*,* $i = 0, 1$*, with* $p_0 \neq p_1$ *and* $\bar{p}_0 \neq \bar{p}_1$ *and assume that* $w, \bar{w} \in \Delta_2$*. Let* T *be a sublinear operator defined in* $\Lambda_u^{p_0,q_0}(w) + \Lambda_u^{p_1,q_1}(w)$ *satisfying*

$$T : \Lambda_u^{p_0,q_0}(w) \to \Lambda_u^{\bar{p}_0,\bar{q}_0}(\bar{w}),$$

$$T : \Lambda_u^{p_1,q_1}(w) \to \Lambda_u^{\bar{p}_1,\bar{q}_1}(\bar{w}).$$

Then, for $0 < \theta < 1$*,* $1 < r \leq \infty$*,*

$$T : \Lambda_u^{p,r}(w) \to \Lambda_u^{\bar{p},r}(\bar{w}),$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{\bar{p}} = \frac{1-\theta}{\bar{p}_0} + \frac{\theta}{\bar{p}_1}.$$

Chapter 3

Several classes of weights

This chapter will be devoted to describe the classes of weights that characterize the strong-type and weak-type boundedness of the Hardy-Littlewood maximal function and the Hilbert transform on the known cases, focusing on the properties that we will need throughout this monograph. For further information on these topics see [36], [41], [32] [38], and [94]. Moreover, we define and study a new class of weights, namely AB_∞^* that will be involved in the characterization of the boundedness of the Hilbert transform on weighted Lorentz spaces.

In the first section we present the A_p class of weights that characterizes the weak-type boundedness of both operators in weighted Lebesgue spaces:

$$M, H : L^p(u) \to L^{p,\infty}(u),$$

for $p \geq 1$ and also the strong-type boundedness for $p > 1$. We study some of the classical properties of the A_p weights that will be used in the forthcoming discussions.

In the second section we study the B_p and $B_{p,\infty}$ classes of weights that characterize the boundedness of the Hardy-Littlewood maximal function on classical Lorentz spaces

$$M : \Lambda^p(w) \to \Lambda^p(w),$$

and its weak-type version, respectively. We find equivalent conditions to B_p class, in terms of the asymptotic behavior of some submultiplicative function $\overline{W}$ at infinity. In Chapter 2, we mentioned that $\Lambda_u^p(w)$ and $\Lambda_u^{p,\infty}(w)$, are not necessarily Banach function spaces and that under the assumption $w \in \Delta_2$ they are quasi-normed. However, the B_p and $B_{p,\infty}$ classes of weights give us sufficient conditions for the normability.

The B_p condition does not characterize the boundedness of the Hilbert transform

$$H : \Lambda^p(w) \to \Lambda^p(w).$$

Another condition is, in fact, required, namely the B_∞^* condition, such that the $B_p \cap B_\infty^*$ class gives the solution to the above boundedness. For this reason, in the third section we present the well-known expressions equivalent to the B_∞^* condition, and studying the asymptotic behavior of $\overline{W}$ at 0 we obtain some new expressions.

The analogue of our problem but for the Hardy-Littlewood maximal function,

$$M : \Lambda_u^p(w) \to \Lambda_u^p(w)$$

was studied in [20], and the solution is the $B_p(u)$ class of weights. Some partial results were obtained for its weak-type version $M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$. In the fourth section we present some of these results, that will be necessary throughout our work. Besides, extending the function $\overline{W}_u$ on $[1,\infty)$, such that it involves the weight u, and studying its behavior at infinity, we obtain some equivalent expression to $B_p(u)$.

When dealing with the boundedness of the Hilbert transform in weighted Lorentz spaces

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w),$$

we note that the $B_p(u)$ condition is not sufficient, since even in the case $u = 1$, the B_∞^* condition is also required. It is therefore natural to define a new class of weights, namely AB_∞^*, that extends the B_∞^* class and, as we will see later on, it turns out to be one of the necessary and sufficient conditions for the strong-type and weak-type boundedness of the Hilbert transform on weighted Lorentz spaces. Among other equivalent expressions, we prove that $w \in AB_\infty^*$ is equivalent to $u \in A_\infty$ and $w \in B_\infty^*$.

3.1 The Muckenhoupt A_p class of weights

The characterization of the weak-type boundedness of the Hardy-Littlewood maximal function on weighted Lebesgue spaces, for $p \geq 1$

$$M : L^p(u) \to L^{p,\infty}(u),$$

led to the introduction of the Muckenhoupt A_p class of weights. If $p > 1$, it also characterizes the strong-type boundedness of M (see [71]) and gives a solution to the boundedness of the Hilbert transform on the same spaces (see [54] and [26]). Some references on these subjects are [36], [41], [38], [32] and [94].

Definition 3.1. *Let $p > 1$. We say that $u \in A_p$ if*

$$\sup_I \left(\frac{1}{|I|} \int_I u(x)dx \right) \left(\frac{1}{|I|} \int_I u^{-1/(p-1)}(x)dx \right)^{p-1} < \infty, \tag{3.1}$$

where the supremum is considered over all intervals I of the real line and, $u \in A_1$ if

$$Mu(x) \approx u(x) \quad a.e.\, x \in \mathbb{R}. \tag{3.2}$$

The A_p class can be characterized as follows:

Theorem 3.2. *[94] Let $1 < p < \infty$. Then, $u \in A_p$ if and only if there exists $\varepsilon > 0$ such that*

$$\frac{u(I)}{u(S)} \lesssim \left(\frac{|I|}{|S|}\right)^{p-\varepsilon},$$

for all intervals I and measurable sets $S \subset I$.

If $u \in A_p$, there exists $\delta \in (0,1)$ such that, given any interval I and any measurable set $S \subset I$, then

$$\left(\frac{|S|}{|I|}\right)^{p} \lesssim \frac{u(S)}{u(I)} \lesssim \left(\frac{|S|}{|I|}\right)^{\delta}. \tag{3.3}$$

Definition 3.3. *If a weight u satisfies the right hand-side inequality in (3.3), then we say that $u \in A_\infty$.*

The A_∞ condition has the property of p-independence. However, the following classical result shows its relation with the A_p classes. For further information concerning the A_∞ condition see [36], [38], [94], and [41].

Proposition 3.4. *If $u \in A_\infty$ there exists $q \geq 1$ such that $u \in A_q$.*

Proposition 3.5. *The weight $u \in A_\infty$ if and only if there exist $0 < \alpha, \beta < 1$ such that for all intervals I and all measurable sets $S \subset I$, we have*

$$|S| \leq \alpha|I| \Rightarrow u(S) \leq \beta u(I).$$

Remark 3.6. Note that if $u \in A_\infty$, then u is non-integrable. Indeed, let $S = (-1,1)$ and $I = (-n,n)$ in the right hand-side inequality of (3.3), then taking limit when n tends to infinity, we get the non-integrability of u.

One of the main results of the theory of A_p weights, is the reverse Hölder inequality proved in [26] and considered independently in [39] (for more details see also [41]). It states that if $u \in A_p$ for some $1 \leq p < \infty$, then there exists $\gamma > 0$ such that

$$\left(\frac{1}{|I|}\int_I u(t)^{1+\gamma}dt\right)^{\frac{1}{1+\gamma}} \lesssim \frac{1}{|I|}\int_I u(t)dt \tag{3.4}$$

for every interval I. Among several applications, we mention that if $u \in A_p$, then $u \in A_{p-\varepsilon}$ for some $\varepsilon > 0$, which allows to prove that the weak-type boundedness of the Hardy-Littlewood maximal function implies the strong-type one, whenever $p > 1$,

$$M : L^p(u) \to L^{p,\infty}(u) \;\; \Rightarrow \;\; M : L^p(u) \to L^p(u).$$

We will see in Chapter 6 that similar results, but without using (3.4), hold in the case of the boundedness of M on $\Lambda^p_u(w)$.

3.2 The B_p and $B_{p,\infty}$ classes of weights

The B_p and $B_{p,\infty}$ conditions characterize the boundedness of the Hardy-Littlewood maximal function on the classical Lorentz spaces, that are,

$$M : \Lambda^p(w) \to \Lambda^p(w) \quad \text{and} \quad M : \Lambda^p(w) \to \Lambda^{p,\infty}(w), \tag{3.5}$$

respectively.

It is well-known that the decreasing rearrangement of Mf, with respect to the Lebesgue measure, is pointwise equivalent (see [8]) to the Hardy operator acting on the rearrangement of f with respect to the same measure, where the Hardy operator is defined as:

$$Pf(t) = \frac{1}{t}\int_0^t f(s)\,ds,$$

for $t > 0$. Then, this relation states that,

$$(Mf)^*(t) \approx Pf^*(t), \qquad t > 0. \tag{3.6}$$

Since every decreasing and positive function in $\mathbb{R}^+$ is equal a.e. to the decreasing rearrangement of a measurable function in $\mathbb{R}$ we deduce that the boundedness (3.5) is equivalent to the boundedness of the Hardy operator on L^p_{dec},

$$P : L^p_{\text{dec}}(w) \to L^p(w) \quad \text{and} \quad P : L^p_{\text{dec}}(w) \to L^{p,\infty}(w), \tag{3.7}$$

respectively. For further information on the subject see [84], [47], [46], [8], [60], and [61].

First, we introduce the B_p condition. Then, we will define the function $\overline{W}$ on $\mathbb{R}^+$ and study its asymptotic behavior at infinity. This will give us a unified approach of the well-known equivalent conditions to B_p. Finally, we present the $B_{p,\infty}$ condition, which in fact coincides with B_p, whenever $p > 1$.

3.2.1 The Ariño-Muckenhoupt B_p class of weights

We will see that the boundedness of the Hardy operator on $L^p_{\text{dec}}(w)$, and consequently the boundedness of the Hardy-Littlewood maximal function on the classical Lorentz spaces, that follows by (3.6), is characterized by the following B_p condition introduced in [5].

Definition 3.7. *Let $0 < p < \infty$. We say that a weight $w \in B_p$, if*

$$\int_r^\infty \left(\frac{r}{t}\right)^p w(t)\,dt \lesssim \int_0^r w(t)\,dt,$$

for all $r > 0$.

Theorem 3.8. *([5], [93]) Let $0 < p < \infty$. Then, the following statements are equivalent:*

(i) $w \in B_p$.

(ii) $P : L^p_{\mathrm{dec}}(w) \to L^p(w)$.

(iii) $M : \Lambda^p(w) \to \Lambda^p(w)$.

(iv) $\displaystyle\int_0^r \frac{1}{W^{1/p}(t)}\,dt \lesssim \frac{r}{W^{1/p}(r)}$, *for every* $r > 0$.

The B_p condition can be also given in terms of the quasi-concavity property defined as follows:

Definition 3.9. *A weight is said to be p quasi-concave if for every $0 < s \leq r < \infty$,*

$$\frac{W(r)}{r^p} \lesssim \frac{W(s)}{s^p}. \tag{3.8}$$

Theorem 3.10. *([5],[79]) A weight $w \in B_p$ if and only if w is $(p - \varepsilon)$ quasi-concave for some $\varepsilon > 0$.*

We present some consequences of the B_p class and the p quasi-concavity condition in terms of the associate spaces, that will be useful to get several estimates in the next chapters.

Proposition 3.11. *For all measurable sets E, the following hold:*

(i) If $p \leq 1$ and w is p quasi-concave, then

$$||\chi_E||_{(\Lambda^p_u(w))'} \approx \frac{u(E)}{W^{1/p}(u(E))}.$$

(ii) If $p > 0$ and $w \in B_p$, then

$$||\chi_E||_{(\Lambda^{p,\infty}_u(w))'} \approx \frac{u(E)}{W^{1/p}(u(E))}.$$

Moreover, under the assumptions of (i) and (ii) we have that $(\Lambda^p_u(w))' \neq \{0\}$ and also $(\Lambda^{p,\infty}_u(w))' \neq \{0\}$, respectively.

Proof. (i) By Theorem 2.16 (i) we get

$$||\chi_E||_{(\Lambda^p_u(w))'} = ||\chi_E||_{\Gamma^{1,\infty}_u(\tilde{w})} = \sup_{t>0} \frac{\int_0^t (\chi_E)^*_u(s)ds}{W^{1/p}(t)} \leq \sup_{t>0} \sup_{A\subset E: u(A)=t} \frac{u(A)}{W^{1/p}(u(A))} \lesssim \frac{u(E)}{W^{1/p}(u(E))},$$

where $\tilde{w}$ is such that $\int_0^t \tilde{w} \approx tW^{-1/p}(t)$, for every $t > 0$, and the inequality is a consequence of the p quasi-concavity of w. The opposite inequality is clear.

(ii) On the one hand, by Theorem 2.16 (iii) we obtain

$$||\chi_E||_{(\Lambda^{p,\infty}_u(w))'} \approx ||\chi_E||_{\Lambda^1_u(W^{-1/p})} = \int_0^{u(E)} \frac{1}{W^{1/p}(s)} ds \lesssim \frac{u(E)}{W^{1/p}(u(E))},$$

where the inequality is a consequence of the condition $w \in B_p$ and Theorem 3.8 (iv). On the other hand, we have that

$$\frac{u(E)}{W^{1/p}(u(E))} \lesssim \int_0^{u(E)} \frac{1}{W^{1/p}(s)} ds,$$

since W is non-decreasing. $\square$

Now, we will define the function $\overline{W}$, which will be fundamental to prove equivalent expressions to the B_p condition.

Definition 3.12. *Define* $\overline{W} : (0,\infty) \to (0,\infty)$ *as*

$$\overline{W}(\lambda) := \sup\left\{\frac{W(t)}{W(s)} : 0 < t \leq \lambda s\right\} = \sup_{x\in[0,+\infty)} \frac{W(\lambda x)}{W(x)}.$$

Note that $\overline{W}$ is submultiplicative: $\overline{W}(\lambda\mu) \leq \overline{W}(\lambda)\overline{W}(\mu)$, for all $\lambda, \mu > 0$,

$$\frac{W(\lambda\mu x)}{W(x)} = \frac{W(\lambda\mu x)W(\mu x)}{W(\mu x)W(x)} \leq \overline{W}(\lambda)\overline{W}(\mu).$$

So, taking supremum in $x \in [0,\infty)$ we get the submultiplicativity. First, we will present some basic facts about submultiplicative functions.

Lemma 3.13. *Let* $\varphi : [1,\infty) \to [1,\infty)$ *be a non-decreasing submultiplicative function such that* $\varphi(1) = 1$. *The following statements are equivalent:*

(i) There exists $\mu \in (1,\infty)$ *such that* $\varphi(\mu) < \mu^p$.

(ii) There exists $\varepsilon > 0$ *such that* $\varphi(x) < (\mu x)^{p-\varepsilon}$, *for all* $x \in (1,\infty)$ *and some* $\mu \in (1,\infty)$.

(iii) $\lim\limits_{x\to\infty} \dfrac{\varphi(x)}{x^p} = 0.$

(iv) $\lim\limits_{x\to\infty} \dfrac{\log\varphi(x)}{\log x^p} < 1.$

Proof. Clearly, $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$. We will show that $(i) \Rightarrow (ii)$, and hence prove the equivalence between $(i), (ii)$ and (iii). If (i) holds, then there exists $\varepsilon > 0$ such that $\varphi(\mu) < \mu^{p-\varepsilon}$. Let $q = p - \varepsilon$ and define $\psi(x) = \varphi(e^{\alpha x})$ for every $x \in (0, +\infty)$, where α will be chosen later. As φ is a non-decreasing submultiplicative function, ψ is also a non-decreasing function and it satisfies:

$$\psi(x+y) \leq \psi(x)\cdot\psi(y). \tag{3.9}$$

Thus, it suffices to prove that

$$\psi(x) < \mu^q e^{\alpha x q}.$$

By equation (3.9), we obtain that $\psi(n) \leq \big(\psi(1)\big)^n$. Therefore, choosing $\alpha = \log\mu$, we get $\psi(1) = \varphi(\mu) < \mu^q = e^{\alpha q}$. Hence, for every $n \in \mathbb{N}$ we obtain $\psi(n) < e^{\alpha n q}$. So, given $x \in [1, +\infty)$, if $[x]$ denotes the integer part of x, then

$$\psi(x) \leq \psi\big([x]+1\big) < e^{\alpha q([x]+1)} \leq e^{\alpha q} e^{\alpha q x} = \mu^q e^{\alpha q x}.$$

On the other hand, for $x \in (0,1)$ we get $\psi(x) < \psi(1) = \varphi(\mu) < \mu^q = e^{\alpha q} \leq e^{\alpha q} e^{\alpha q x}$. Hence (ii) holds.

Clearly $(iv) \Rightarrow (i)$, and we complete the proof showing that $(ii) \Rightarrow (iv)$. Indeed, if we assume that $\varphi(x) < (\mu x)^{p-\varepsilon}$, we have that

$$\frac{\log\varphi(x)}{\log\mu x} < p - \varepsilon.$$

Hence,

$$\lim_{x\to\infty} \frac{\log\varphi(x)}{\log x^p} = \frac{1}{p} \lim_{x\to\infty} \frac{\log\varphi(x)}{\log\mu x}\frac{\log\mu x}{\log x} \leq \frac{p-\varepsilon}{p} < 1.$$

□

Corollary 3.14. *The following statements are equivalent to the condition $w \in B_p$:*

(i) $\lim\limits_{x\to\infty} \dfrac{\overline{W}(x)}{x^p} = 0.$

(ii) There exists $\mu \in (1,\infty)$ such that $\overline{W}(\mu) < \mu^p$.

(iii) There exists $\varepsilon > 0$ such that w is $(p-\varepsilon)$ quasi-concave.

(iv) $\lim\limits_{x\to\infty} \dfrac{\log\overline{W}(x)}{\log x^p} < 1.$

Proof. It is a consequence of Lemma 3.13 for $\varphi = \overline{W}$. The equivalence between (iii) and the B_p condition is given by Theorem 3.10. $\square$

Remark 3.15. Condition (iv) of Corollary 3.14 can be related with the upper Boyd index and the Lorentz-Shimogaki theorem. However, we will deal with this subject in Section 6.6.

3.2.2 The $B_{p,\infty}$ class

The characterization of the weak-type boundedness $P : L^p_{\text{dec}}(w) \to L^{p,\infty}(w)$ motivates the definition of the $B_{p,\infty}$ class, introduced firstly as the W_p class in [79]. The notation $B_{p,\infty}$ appeared in [17] and [20] and this class agrees with the B_p class for $p > 1$. We will also study the case $p \leq 1$.

Definition 3.16. *Let $0 < p < \infty$. We write $w \in B_{p,\infty}$ if*

$$P : L^p_{\text{dec}}(w) \to L^{p,\infty}(w).$$

Theorem 3.17. *[79] Let $1 < p < \infty$. Then, $B_p = B_{p,\infty}$.*

The condition that characterizes the case $p \leq 1$ is expressed in terms of the p quasi-concavity property.

Theorem 3.18. *([21], [17]) Let $p \leq 1$. Then,*

$$w \in B_{p,\infty} \quad \Leftrightarrow \quad w \;\; \textit{is} \;\; p \;\; \textit{quasi-concave.}$$

We have seen that the $B_{p,\infty}$ condition coincides with B_p whenever $p > 1$. Now, we will study how far is the $B_{p,\infty}$ condition from B_p, when $p \leq 1$. In fact, we will show that in this case, the condition $w \in B_{p,\infty}$ implies that either $w \in B_p$ or $\overline{W}^{1/p}$ is equivalent to the identity.

Proposition 3.19. *Let $p \leq 1$. If $w \in B_{p,\infty}$, then one of the following statements holds:*

(i) $w \in B_p$.

(ii) $\overline{W}^{1/p}(t) \approx t$, *for all* $t > 1$.

Proof. If $w \in B_{p,\infty}$, then for all $s \leq t$

$$\frac{W^{1/p}(t)}{t} \lesssim \frac{W^{1/p}(s)}{s}.$$

Hence, we have that

$$\overline{W}^{1/p}(\mu) \lesssim \mu, \tag{3.10}$$

for all $\mu > 1$. If now for all $\mu > 1$ we have that $\overline{W}^{1/p}(\mu) \geq \mu$, then we get (ii). In opposite case there exists $\mu > 1$ such that $\overline{W}^{1/p}(\mu) < \mu$. Hence, by Corollary 3.14 we conclude that $w \in B_p$. □

As we have already mentioned, weighted Lorentz spaces $\Lambda^p_u(w)$, $\Lambda^{p,\infty}_u(w)$ are not necessarily Banach function spaces. However, if we make some assumption on w (like w to be decreasing, or to satisfy any of the B_p, $B_{p,\infty}$ conditions) we get sufficient conditions in order to obtain the normability. Lorentz characterized when the functional defining the space is a norm (see [68]), and other authors have studied this problem (see [59], [16], [42], [90], [17], [93], and [20]), summarized in the following result:

Theorem 3.20. *Let $w = w\chi_{(0,u(\mathbb{R}))}$.*

(i) If $1 \leq p < \infty$, then $|| \cdot ||_{\Lambda^p_u(w)}$ is a norm if and only if w is decreasing.

(ii) If $1 \leq p < \infty$, then $\Lambda^p_u(w)$ is normable if and only if $w \in B_{p,\infty}$.

(iii) If $0 < p < \infty$, then $\Lambda^{p,\infty}_u(w)$ is normable if and only if $w \in B_p$.

The normability of $\Lambda^p_u(w)$ and $\Lambda^{p,\infty}_u(w)$ can be characterized in terms of the associate space of $(\Lambda^p_u(w))'$ and $(\Lambda^{p,\infty}_u(w))'$, respectively, as follows (see Chapter 2 for more details):

Theorem 3.21. *[20] $\Lambda^p_u(w)$ (resp. $\Lambda^{p,\infty}_u(w)$) is normable if and only if $\Lambda^p_u(w) = (\Lambda^p_u(w))''$ (resp. $(\Lambda^{p,\infty}_u(w))''$), with equivalent norms. In particular, every normable weighted Lorentz space is a Banach function space with norm $|| \cdot ||_{(\Lambda^p_u(w))''}$ (resp. $|| \cdot ||_{(\Lambda^{p,\infty}_u(w))''}$).*

3.3 The B^*_∞ class

As we have previously pointed out, $(Mf)^*(t) \approx Pf^*(t)$, for every $t > 0$. Besides, there exists an analogue relation concerning the decreasing rearrangement of the Hilbert transform, with respect to the Lebesgue measure, and the sum of the Hardy operator and its adjoint, where the last operator is given by

$$Qf(t) = \int_t^\infty f(s)\,\frac{ds}{s},$$

for all $t > 0$. The aforementioned relation is the following:

$$(Hf)^*(t) \lesssim (P+Q)f^*(t) \lesssim (Hg)^*(t), \qquad t > 0, \tag{3.11}$$

where g is an equimeasurable function with f (see [7], [8]). Since every decreasing and positive function in $\mathbb{R}^+$ is equal a.e. to the decreasing rearrangement of a measurable function in $\mathbb{R}$ we deduce that the boundedness

$$H : \Lambda^p(w) \to \Lambda^p(w)$$

is equivalent to the boundedness

$$P, Q : L^p_{\text{dec}}(w) \to L^p(w).$$

Throughout this section we will study the boundedness of the adjoint of the Hardy operator Q, characterized by the B_∞^* condition defined below. This condition is involved in the boundedness of H on the classical Lorentz spaces solved by Sawyer [90] and Neugebauer [80].

Definition 3.22. *We say that $w \in B_\infty^*$ if*

$$\int_0^r \frac{1}{t} \int_0^t w(s)ds\, dt \lesssim \int_0^r w(s)ds, \tag{3.12}$$

for all $r > 0$.

Neugebauer in [80] and Andersen in [4] studied the weak-type and the strong-type boundedness of the adjoint of the Hardy operator on the cone of decreasing functions. Both cases are characterized by the B_∞^* condition. We present these well-known results and some new ones that will be involved in the study of the Hilbert transform.

Theorem 3.23. *([4], [80]) If for some $0 < p < \infty$, one of the following statements holds, then they are all equivalent and hold for every $0 < p < \infty$.*

(i) $w \in B_\infty^*$.

(ii) $Q : L^p_{\text{dec}}(w) \to L^p(w)$.

(iii) $Q : L^p_{\text{dec}}(w) \to L^{p,\infty}(w)$.

(iv) $\dfrac{W(t)}{W(s)} \lesssim \left(\log \dfrac{s}{t}\right)^{-p}$, *for all* $0 < t \le s < \infty$.

We characterize the following boundedness of the adjoint of the Hardy operator (see also [93]).

Proposition 3.24. *If $0 < p < \infty$, then $Q : L^{p,\infty}_{\text{dec}}(w) \to L^{p,\infty}(w)$ is bounded if and only if*

$$\int_t^\infty \frac{1}{W^{1/p}(s)} \frac{ds}{s} \lesssim \frac{1}{W^{1/p}(t)}. \tag{3.13}$$

Proof. First observe that $f(t) \lesssim ||f||_{L^{p,\infty}(w)} W^{-1/p}(t)$, for all $t > 0$ and since f is non-increasing we have that $||f||_{L^{p,\infty}(w)} = \sup_{t>0} f(t) W^{1/p}(t)$. Then, if we assume the condition (3.13), we obtain

$$Qf(t) \lesssim ||f||_{L^{p,\infty}(w)} Q(W^{-1/p}(t)) \lesssim ||f||_{L^{p,\infty}(w)} W^{-1/p}(t).$$

Hence, we get that $Qf(t)W^{1/p}(t) \lesssim ||f||_{L^{p,\infty}(w)}$ for all $t > 0$ and, taking the supremum over all $t > 0$ we get the result. On the other hand, by the boundedness of Q we get $\sup_{t>0} W^{1/p}(t) Q(W^{-1/p}(t)) \lesssim \sup_{t>0} W^{1/p}(t) W^{-1/p}(t) = 1$, since $W^{-1/p} \in L^{p,\infty}_{\text{dec}}(w)$. Therefore, we obtain (3.13). □

Remark 3.25. (i) Let $p > q$. Then, $W_p \subset W_q$, where W_p denotes condition (3.13). Indeed, let $\nu > 0$ such that $\frac{1}{p} + \nu = \frac{1}{q}$. Then, if a weight w satisfies W_p we get

$$\int_t^\infty \frac{1}{W^{1/q}(s)} \frac{ds}{s} = \int_t^\infty \frac{1}{W^{1/p+\nu}(s)} \frac{ds}{s} \leq \frac{1}{W^\nu(t)} \int_t^\infty \frac{1}{W^{1/p}(s)} \frac{ds}{s} \lesssim \frac{1}{W^{1/q}(t)},$$

since W is a non-decreasing function.

(ii) If w is p quasi-concave, then condition (3.13) is equivalent to

$$\int_0^t \frac{1}{s} \left(\int_0^s w(r) dr \right)^{1/p} ds \lesssim \left(\int_0^t w(r) dr \right)^{1/p}. \tag{3.14}$$

Indeed, if w is p quasi-concave then

$$\int_t^\infty \frac{1}{W^{1/p}(s)} \frac{ds}{s} \gtrsim \frac{1}{W^{1/p}(t)} \quad \text{and} \quad \int_0^t \frac{1}{s} \left(\int_0^s w(r) dr \right)^{1/p} ds \gtrsim \left(\int_0^t w(r) dr \right)^{1/p}.$$

Hence, we obtain the equivalence both in (3.13) and (3.14). Now, it suffices to prove that

$$\int_t^\infty \frac{1}{W^{1/p}(s)} \frac{ds}{s} \approx \frac{1}{W^{1/p}(t)} \Longleftrightarrow \int_0^t \frac{1}{s} \left(\int_0^s w(r) dr \right)^{1/p} ds \approx \left(\int_0^t w(r) dr \right)^{1/p}.$$

In fact, this is a consequence of a lemma proved by Sagher in [88]: if m is a positive function and, for all $r > 0$,

$$\int_0^r m(s) \frac{ds}{s} \approx m(r) \Longleftrightarrow \int_r^\infty \frac{1}{m(s)} \frac{ds}{s} \approx \frac{1}{m(r)}.$$

If w is p quasi-concave, then for $p=1$, the weak-type boundedness $Q : L^{p,\infty}_{\mathrm{dec}}(w) \to L^{p,\infty}(w)$ is equivalent to B_∞^* (see Remark 3.25 *(ii)*). In fact, we prove that this holds for all $p>0$, provided $w \in \Delta_2$. In order to see this, we have used some facts from the interpolation theory on the cone of positive and decreasing functions (see [24]). For further references on this topic see [9] and [8].

Theorem 3.26. *Let $0<p<\infty$ and suppose that $w \in \Delta_2$. Then, the following statements are equivalent:*

(i) $w \in B_\infty^*$.

(ii) $Q : L^{p,\infty}_{\mathrm{dec}}(w) \to L^{p,\infty}(w), \quad$ *for all* $\ 0<p<\infty$.

Proof. *(i)* $\Rightarrow$ *(ii)*. If we assume that $w \in B_\infty^*$, then $Q : L^{p_j}_{\mathrm{dec}}(w) \to L^{p_j}(w)$ hold, for $j=0,1$ and $0<p_0<p_1<\infty$, by Theorem 3.23. Then, using [24, pg. 245] we obtain that the interpolation space between $L^{p_0}_{\mathrm{dec}}(w)$ and $L^{p_1}_{\mathrm{dec}}(w)$ is $L^{p,q}_{\mathrm{dec}}$, for $p_0<p<p_1$ and $q \le \infty$, provided $w \in B_\infty^*$ and $w \in \Delta_2$. Hence, the desired result follows considering $q=\infty$.

(ii) $\Rightarrow$ *(i)*. It is an immediate consequence of the continuous inclusion $L^p(w) \subset L^{p,\infty}(w)$. In fact, we obtain that $Q : L^r_{\mathrm{dec}}(w) \to L^r(w)$ for all $0<r<\infty$. Applying Theorem 3.23 we get *(i)*. □

Now, studying the asymptotic behavior of the function $\overline{W}$ at 0, we obtain some equivalent expressions to the B_∞^* condition. First we present the following technical result.

Lemma 3.27. *Let $\varphi : (0,1] \to [0,1]$ be a non-decreasing submultiplicative function such that $\varphi(1)=1$. The following statements are equivalent:*

(i) There exists $\lambda \in (0,1)$ such that $\varphi(\lambda)<1$.

(ii) There exists $C>0$ such that $\varphi(x) \le C\Big(1+\log\dfrac{1}{x}\Big)^{-1}$, for all $x \in (0,1]$.

(iii) $\displaystyle\lim_{x\to 0}\varphi(x)=0$.

(iv) Given $p>0$, there exists $C=C(p)>0$ such that $\varphi(x) \le C\Big(1+\log\dfrac{1}{x}\Big)^{-p}$, for all $x \in (0,1]$.

(v) $\displaystyle\lim_{x\to 0}\frac{\log\varphi(x)}{\log x^p}>0$.

Proof. We will show that $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iv)$. Then, since clearly $(ii) \Rightarrow (iii) \Rightarrow (i)$, and $(iv) \Rightarrow (i)$, we get the equivalences between $(i), (ii), (iii)$ and (iv).

First we prove that $(i) \Rightarrow (ii)$. Define $\psi(x) = \varphi(e^{-\alpha x})$ for every $x \in [0, +\infty)$, where $\alpha = \log(1/\lambda)$. As φ is a non-decreasing submultiplicative function, ψ is a non-increasing function satisfying the inequality

$$\psi(x+y) \leq \psi(x) \cdot \psi(y). \tag{3.15}$$

It suffices to prove that there is a constant $C > 0$ such that

$$\psi(x) \leq \frac{C}{1+\alpha x}.$$

By equation (3.15), we obtain that $\psi(n) \leq \big(\psi(1)\big)^n$. Therefore, as $\psi(1) = \varphi(\lambda) < 1$, there exists a constant $C_0 > 0$ big enough such that, for every $n \in \mathbb{N}$

$$\psi(n) \leq \frac{C_0}{1+\alpha n}.$$

So, given $x \in [1, +\infty)$ we have that

$$\psi(x) \leq \psi\big([x]\big) \leq \frac{C_0}{1+\alpha[x]} \leq \frac{(1+\alpha)C_0}{1+\alpha x},$$

where in the last inequality we use that

$$\frac{1+\alpha x}{1+\alpha[x]} \leq \frac{1+\alpha[x]+\alpha}{1+\alpha[x]} \leq 1+\alpha.$$

On the other hand, for $x \in [0,1)$, $\psi(x) \leq \psi(0) = \varphi(1) = 1$, as φ is submultiplicative and non decreasing. So, in this case

$$\psi(x) \leq \frac{1+\alpha}{1+\alpha x}.$$

Therefore, by taking $C = (1+\alpha)\max\{1, C_0\}$ we get (ii). Applying the same arguments as before, but for the function $\tilde{\varphi} = \varphi^{1/p}$, which is also non-decreasing, submultiplicative and $\tilde{\varphi}(\lambda) < 1$ we obtain $(i) \Rightarrow (iv)$.

Clearly $(v) \Rightarrow (i)$ and it remains to prove that $(i) \Rightarrow (v)$. Note that for all $n \in \mathbb{N}$ we get that

$$\psi(n) \leq \psi(1)^n = (\varphi(\lambda))^n$$

where $\varphi(\lambda) < 1$. Hence, for all $n \in \mathbb{N}$ there exists $c > 0$ such that

$$\frac{\log\left(\frac{1}{\psi(n)}\right)}{n} \geq c.$$

If $x \in [1, \infty)$ we have

$$\log\left(\frac{1}{\psi(x)}\right) \geq \log\left(\frac{1}{\psi([x])}\right) \geq c[x] \geq Cx,$$

since ψ is non-increasing. The function $\log(\psi)$ is subadditive, then by a result of Hille and Phillips (see [50]) we have that

$$\lim_{x\to\infty} \frac{\log\left(\frac{1}{\psi(x)}\right)}{x} = \sup_{1<x<\infty} \frac{\log\left(\frac{1}{\psi(x)}\right)}{x} \geq C.$$

Taking $y = e^{-\alpha x}$, we get

$$\lim_{y\to 0^+} \frac{\log \varphi(y)}{\log y} > 0.$$

□

Corollary 3.28. *The following statements are equivalent to the condition $w \in B^*_\infty$:*

(i) $\overline{W}$ *is not identically 1.*

(ii) $\dfrac{W(t)}{W(s)} \lesssim \left(1 + \log\dfrac{s}{t}\right)^{-1}$, *for all* $0 < t \leq s$.

(iii) For every $p > 0$, $\dfrac{W(t)}{W(s)} \lesssim \left(1 + \log\dfrac{s}{t}\right)^{-p}$, *for all* $0 < t \leq s$.

(iv) $\overline{W}(0^+) = 0$.

(v) $\displaystyle\lim_{x\to\infty} \frac{\log \overline{W}(x)}{\log x^p} > 0$.

(vi) For every $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $W(t) \leq \varepsilon W(s)$, *provided* $t \leq \delta s$.

Proof. The proof is identical to that of Corollary 3.14. The equivalence between the condition *(iii)* and B^*_∞ follows by Theorem 3.23. □

The following result characterizes the strong-type boundedness of H on the classical Lorentz spaces,

$$H : \Lambda^p(w) \to \Lambda^p(w), \tag{3.16}$$

and its weak-type version $H : \Lambda^p(w) \to \Lambda^{p,\infty}(w)$. It was proved by Sawyer in [90] and Neugebauer in [80] (see also [97]). In fact, Sawyer showed a two-weighted version of (3.16). However, Neugebauer characterized (3.16) by means of the condition $w \in B_p \cap B^*_\infty$, for $p > 1$, which is simpler than the conditions of Sawyer even in the diagonal case.

Theorem 3.29. *If $p \leq 1$, then:*

(α) The boundedness $H : \Lambda^p(w) \to \Lambda^p(w)$ holds if and only if $w \in B_p \cap B_\infty^*$.

(β) The boundedness $H : \Lambda^p(w) \to \Lambda^{p,\infty}(w)$ holds if and only if $w \in B_{p,\infty} \cap B_\infty^*$.

And, if $p > 1$, the following statements are equivalent:

(i) $H : \Lambda^p(w) \to \Lambda^p(w)$.

(ii) $H : \Lambda^p(w) \to \Lambda^{p,\infty}(w)$.

(iii) $w \in B_p \cap B_\infty^*$.

Proof. The above result is a consequence of the relation (3.11). The problem reduces to the study of the boundedness of P and Q on $L^p_{\text{dec}}(w)$. ☐

Remark 3.30. Note that neither of the conditions B_p, B_∞^* is obtained from the other.
(i) If $w(t) = \chi_{(0,1)}(t)$, then $w \notin B_\infty^*$. On the other hand, it can be proved that it belongs to the B_p class for $p > 1$; that is,

$$\int_r^\infty \frac{w(x)}{x^p} dx \leq c_p \frac{1}{r^p} \int_0^r w(x)\, dx, \quad \forall r > 0.$$

Indeed, if $r > 1$ the inequality is clearly true. If $r \leq 1$,

$$\int_r^\infty \frac{w(x)}{x^p} dx = \int_r^1 \frac{1}{x^p} dx = \frac{r^{-p+1} - 1}{p-1}$$

and

$$\frac{1}{r^p} \int_0^r w(x) dx = \frac{1}{r^p} \int_0^r dx = r^{-p+1}.$$

Then, taking $c_p = \frac{1}{p-1}$ we get that $w \in B_p$.

(ii) The condition B_∞^* does not imply Δ_2 condition. If $w(t) = e^t$, then, $w \in B_\infty^*$. Indeed, by the mean value theorem there exists $\xi \in (0, t)$ such that

$$\frac{e^t - 1}{t} = e^\xi.$$

Taking into account that the exponential function is monotone, we have $e^\xi \leq e^t$. Hence, the following holds

$$\int_0^r \frac{e^t - 1}{t} dt \leq \int_0^r e^t dt = e^r - 1.$$

Besides, there is no constant such that $e^{2t} - 1 \leq c e^t$, hence $w \notin \Delta_2$. In particular, observe that this weight does not belong to B_p, whereas it belongs to B_∞^*.

3.4 The $B_p(u)$ and $B_{p,\infty}(u)$ classes of weights

The boundedness of the Hardy-Littlewood maximal function on weighted Lorentz spaces

$$M : \Lambda_u^p(w) \to \Lambda_u^p(w),$$

has been characterized in [20] and its solution is the $B_p(u)$ condition. Partial results have been obtained for its weak-type analogue $M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$. We will present some of these results, which will be necessary for our study.

Definition 3.31. *We say that $w \in B_p(u)$ if there exists $\varepsilon > 0$ such that, for every finite family of pairwise disjoint, open intervals $(I_j)_{j=1}^J$, and every family of measurable sets $(S_j)_{j=1}^J$, with $S_j \subset I_j$, for every $j \in J$, we have that*

$$\frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \lesssim \max_{1\le j\le J}\left(\frac{|I_j|}{|S_j|}\right)^{p-\varepsilon}. \tag{3.17}$$

Theorem 3.32. *[20] If $0 < p < \infty$, then*

$$M : \Lambda_u^p(w) \to \Lambda_u^p(w) \quad \textit{if and only if} \quad w \in B_p(u).$$

Remark 3.33. Theorem 3.32 recovers the well-known cases $w = 1$ and $u = 1$. Indeed, if $w = 1$, then the $B_p(u)$ condition (see (3.17)) agrees with A_p, since the last condition is equivalent by Theorem 3.2 to the existence of $\varepsilon > 0$ such that

$$\frac{u(I)}{u(S)} \lesssim \left(\frac{|I|}{|S|}\right)^{p-\varepsilon},$$

for all $S \subset I$, for all intervals I. If $u = 1$, then the $B_p(u)$ condition is equivalent to B_p by Theorem 3.10.

Now, we extend the function $\overline{W}$, such that it involves the weight u, yielding the function $\overline{W}_u$. We study the asymptotic behavior of this function at infinity and obtain equivalent expressions to the $B_p(u)$ condition.

Definition 3.34. *Define $\overline{W}_u$, for $\lambda \in [1,\infty)$ as follows:*

$$\overline{W}_u(\lambda) := \sup\left\{\frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} : \textit{such that } S_j \subseteq I_j \textit{ and } |I_j| < \lambda |S_j| \textit{ for every } j \in J\right\},$$

where I_j are pairwise disjoint, open intervals, the sets S_j are measurable and all unions are finite.

Remark 3.35. (i) Note that in the previous definition we could consider $\overline{W}_u$ as follows:

$$\overline{W_u}(\lambda) = \sup \left\{ \frac{W\left(u\left(\bigcup_{j=1}^{J} I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^{J} S_j\right)\right)} : \text{such that } S_j \subseteq I_j \text{ and } \frac{|I_j|}{|S_j|} = \lambda, \text{ for every } j \in J \right\}.$$

By the regularity of the measure u, the sets S_j can be considered as finite union of intervals.

(ii) Observe that if $u = 1$, then $\overline{W}_u$ recovers $\overline{W}$ on $[1, \infty)$. Indeed, note that since $|I_j|/|S_j| = \lambda$, for every j, we also have that

$$\frac{|\cup_j I_j|}{|\cup_j S_j|} = \frac{\sum_j (|I_j||S_j|)/|S_j|}{\sum_j |S_j|} = \lambda.$$

Hence, if $t = |\cup_j I_j|$ and $r = |\cup_j S_j|$, then by (i) we have that

$$\overline{W}_u(\lambda) = \sup \left\{ \frac{W(t)}{W(r)} : t/r = \lambda \right\},$$

which is the function $\overline{W}(\lambda)$.

We will prove the submultiplicativity of the function $\overline{W}_u$. First we need a technical result.

Lemma 3.36. *Let I be an interval and let $S = \cup_{k=1}^{N} (a_k, b_k)$ be a union of disjoint intervals such that $S \subset I$. Then, for every $t \in [1, |I|/|S|]$ there exists a collection of disjoint subintervals $\{I_n\}_{n=1}^{M}$ satisfying that $S \subset \cup_n I_n$ and for every $n \in \mathbb{N}$:*

$$t|S \cap I_n| = |I_n|. \tag{3.18}$$

Proof. Without loss of generality we can assume that $I = (0, |I|)$ and $a_1 < a_2 \cdots < a_N$. First observe that if $J = \cup I_n$ we should in particular obtain $t|S| = |J|$ applying (3.18). We use induction in N. Clearly it is true for $n = 1$. Indeed, it suffices to consider $0 \le c \le a_1 < b_1 \le d \le |I|$ such that $t(b_1 - a_1) = d - c$. Suppose that the results holds for all $k < n$. We will prove that it also holds for $n + 1$.
Case I: Let $|I| - t|S| \le a_1$. Then, it suffices to consider $I_1 = (|I| - t|S|, |I|) = J$. Hence, the problem is solved with $M = 1$.
Case II: Let $a_1 < |I| - t|S|$, and call $\bar{I} = (a_1, |I|)$. Observe that $t|S| < |\bar{I}|$ and $S \subset \bar{I}$. Hence in this case we could assume without loss of generality that $a_1 = 0$. Let now $I_1 = (0, c)$ such that $b_1 \le c \le |I|$ and $t|S \cap I_1| = c = |I_1|$. Note that $c \notin S$. In fact, suppose that there exists $S_m = (a_m, b_m)$ such that $c \in S_m$. Then, $t|S \cap [0, a_m)| > |[0, a_m)|$ which implies that $t|S \cap [0, c)| > |[0, c)| = |I_1|$; that is a contradiction. Therefore, we obtain

$$t|S \cap I_1| + t|S \cap [c, |I|)| = t|S| < |I| = |I_1| + |[c, |I|)| = t|S \cap I_1| + |[c, |I|)|,$$

and consequently $t|S \cap [c, |I|)| < |[c, |I|)|$. Then, since $[c, |I|)$ is the union of at most $n - 1$ disjoint intervals, we apply the inductive hypothesis to the intervals $[c, |I|)$ and the set $S \cap [c, |I|)$. Hence, we obtain the intervals $I_2, \ldots, I_M$ such that (3.18) is satisfied. $\square$

Theorem 3.37. *The function $\overline{W}_u$ is submultiplicative.*

Proof. Consider a finite family of pairwise disjoint intervals I_j, and measurable sets $S_j \subseteq I_j$ such that $|I_j| = \lambda\mu|S_j|$, for every j and $\lambda, \mu \in [1, \infty)$. By Remark 3.35 (i), each S_j can be considered as a finite union of intervals. Then, by Lemma 3.36 and for each j, we can get a set J_j such that it is a finite union of intervals, that we call J_{ji} :

$$S_j \subseteq J_j \subseteq I_j, \quad \lambda|S_j \cap J_{ji}| = |J_{ji}|, \quad \text{and} \quad \mu|J_j| = |I_j|.$$

So, we have that

$$\frac{W\left(u\left(\bigcup_j I_j\right)\right)}{W\left(u\left(\bigcup_j S_j\right)\right)} \leq \frac{W\left(u\left(\bigcup_j I_j\right)\right)}{W\left(u\left(\bigcup_j S_j\right)\right)} \frac{W\left(u\left(\bigcup_j J_j\right)\right)}{W\left(u\left(\bigcup_j J_j\right)\right)} \leq \overline{W}_u(\lambda)\overline{W}_u(\mu).$$

Therefore, taking supremum over all the possible choices of intervals I_j and measurable subsets S_j such that $S_j \subseteq I_j$ and $|I_j| = \lambda\mu|S_j|$, we get that $\overline{W}_u(\lambda\mu) \leq \overline{W}_u(\lambda)\overline{W}_u(\mu)$. □

Now, we will see equivalent expressions to the $B_p(u)$ condition applying the submultiplicativity of the function $\overline{W}_u(\lambda)$.

Corollary 3.38. *The following statements are equivalent:*

(i) *There exists $\mu \in (1, \infty)$ such that $\overline{W}_u(\mu) < \mu^p$.*

(ii) $w \in B_p(u)$.

(iii) $\displaystyle\lim_{x\to\infty} \frac{\overline{W}_u(x)}{x^p} = 0.$

(iv) $\displaystyle\lim_{\mu\to\infty} \frac{\log \overline{W}_u(\mu)}{\log \mu^p} < 1.$

Proof. It is a consequence of Lemma 3.13 and the fact that $\overline{W}_u$ is submultiplicative and increasing by Theorem 3.37. □

Remark 3.39. The condition (ii) has been studied in [20] by Carro, Raposo and Soria. The conditions (iii) and (iv) already appeared in a work of Lerner and Pérez in [66], and in Section 6.6 we will specially deal with (iv) in the setting of Boyd indices. Finally, the condition (i) seems to be new.

On the other hand, the study of the weak-type version

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w),$$

motivates the definition of the $B_{p,\infty}(u)$ class introduced in [20].

Definition 3.40. *We say that $w \in B_{p,\infty}(u)$ if and only if $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$.*

Theorem 3.41. *[20] If $0 < p < \infty$, then $w \in B_{p,\infty}(u)$ if and only if*

$$(Mf)^*_u(s) \lesssim \left(\frac{1}{W(s)} \int_0^s (f^*_u)^p(r) w(r) dr \right)^{1/p},$$

for every $t > 0$ and $f \in \mathcal{M}$.

Theorem 3.42. *[20] If $0 < p < \infty$, then the $B_{p,\infty}(u)$ condition implies that*

$$\frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \leq C \max_{1 \leq j \leq J} \left(\frac{|I_j|}{|S_j|} \right)^p, \tag{3.19}$$

for every finite family of pairwise disjoint, open intervals $(I_j)_{j=1}^J$, and every family of measurable sets $(S_j)_{j=1}^J$, with $S_j \subset I_j$, for every j.

Remark 3.43. It is known that if $p < 1$, then (3.19) is equivalent to $B_{p,\infty}(u)$ (see [20]), while the case $p \geq 1$ remained open. In Section 6.5 we will completely solve the problem, for $p > 1$, showing that the $B_{p,\infty}(u)$ condition is equivalent to $B_p(u)$.

3.5 The AB^*_∞ class

We have seen that the B_p condition is not sufficient for the boundedness of the Hilbert transform on the classical Lorentz spaces, since the B^*_∞ condition is required (see Theorem 3.29). When dealing with the boundedness

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w),$$

and its weak-type version, it naturally appears a new class of weights, namely AB^*_∞, which in fact is equivalent to the B^*_∞ and A_∞ conditions.

Definition 3.44. *We say that $(u, w) \in AB^*_\infty$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all intervals I and measurable sets $S \subset I$, we have*

$$|S| \leq \delta |I| \Rightarrow W(u(S)) \leq \varepsilon W(u(I)).$$

Remark 3.45. Note that if $w = 1$ in Definition 3.44, then $u \in A_\infty$ by Proposition 3.5 and if $u = 1$, then by Corollary 3.28 we have that $w \in B^*_\infty$.

Now, we will prove that the AB^*_∞ condition not only recovers the A_∞ and B^*_∞ conditions in the classical cases, as shown in Remark 3.45, but also it is equivalent to these conditions.

Proposition 3.46. *Let $w \in \Delta_2$. Then,*

$$(u, w) \in AB^*_\infty \quad \textit{if and only if} \quad u \in A_\infty \textit{ and } w \in B^*_\infty.$$

Proof. Assume that $(u, w) \in AB^*_\infty$. Let us prove that $u \in A_\infty$. Indeed, let $\varepsilon = 2^{1-k}$, $k \in \mathbb{N}$ and $\varepsilon' < c^{-k}$, where $c > 1$ is the Δ_2 constant. By definition, there exists $\delta = \delta'(\varepsilon')$ such that $|S| \le \delta|I|$ implies,

$$W(u(S)) \le \varepsilon' W(u(I)) < c^{-k} W(u(I)).$$

If $\dfrac{u(I)}{u(S)} \le 2^{k-1}$ we have that

$$W(u(S)) < c^{-k} W\left(\frac{u(I)}{u(S)} u(S)\right) \le c^{-1} W(u(S),$$

taking into account that $w \in \Delta_2$. Since $c > 1$, we obtain $W(u(S)) < W(u(S))$ which is a contradiction. Hence,

$$u(S) \le 2^{1-k} u(I) = \varepsilon u(I), \tag{3.20}$$

which implies that $u \in A_\infty$. Now, we prove that $w \in B^*_\infty$. For every $S \subseteq I$ there exists $\lambda \in (0, 1)$ such that

$$\frac{W(u(S))}{W(u(I))} < \frac{1}{2},$$

provided $|S| < \lambda|I|$. Since, $u \in A_\infty$, there exists $q \ge 1$ such that $u \in A_q$. Let $\delta \in (0, 1)$ such that $u(S) < \delta u(I)$. Hence

$$\frac{|S|}{|I|} \le C_u \left(\frac{u(S)}{u(I)}\right)^{1/q} = C_u \delta^{1/q}.$$

Now, choose δ such that $C_u \delta^{1/q} < \lambda$. Therefore, take $0 < t < \delta s$ and consider $S \subset I$ such that $t = u(S)$ and $s = u(I)$. Then, $|S| \le C_u \delta^{1/q} |I| < \lambda |I|$ and

$$\frac{W(t)}{W(s)} = \frac{W(u(S))}{W(u(I))} \le \overline{W}_u(\lambda) \le \frac{1}{2} < 1.$$

This implies that $\overline{W}(\delta) < 1$; that is equivalent to B^*_∞ by Corollary 3.28.

Conversely, assume that $w \in B^*_\infty$, then by Corollary 3.28, for every $\varepsilon > 0$, there exists $\beta(\varepsilon) > 0$ such that

$$t \le \beta(\varepsilon) r \Rightarrow W(t) \le \varepsilon W(r). \tag{3.21}$$

Since $u \in A_\infty$, we have that for all $\beta > 0$ and in particular for $\beta = \beta(\varepsilon)$ fixed above, we have that there exists $\delta > 0$ such that for $S \subset I$

$$|S| \leq \delta|I| \Rightarrow u(S) \leq \beta(\varepsilon)u(I). \tag{3.22}$$

Hence by (3.22) and (3.21), for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S| \leq \delta|I| \Rightarrow W(u(S)) \leq \varepsilon W(u(I)). \tag{3.23}$$

□

We will prove that the AB^*_∞ condition holds also if we substitute the set S and the interval I in Definition 3.44 by finite unions of sets S_j and intervals I_j, respectively, such that $S_j \subset I_j$. To do this we define the function $\underline{W_u}$, which is an extension of the function $\overline{W}$ on $(0,1]$. Then, we study the asymptotic behavior at 0, obtaining equivalent expressions to AB^*_∞.

Definition 3.47. *We define the function $\underline{W_u}$ in $(0,1]$ as follows:*

$$\underline{W_u}(\lambda) := \sup\left\{ \frac{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)} : \textit{such that } S_j \subseteq I_j \textit{ and } |S_j| < \lambda|I_j|, \textit{ for every } j \right\}, \tag{3.24}$$

where I_j are pairwise disjoint, open intervals, the sets S_j are measurable and all unions are finite.

Remark 3.48. Note that (3.24) is equivalent to

$$\underline{W_u}(\lambda) = \sup\left\{ \frac{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)} : \text{such that } S_j \subseteq I_j \text{ and } \frac{|S_j|}{|I_j|} = \lambda, \text{ for every } j \right\},$$

and since the measure u is regular, every S_j can be considered as a finite union of intervals. Moreover, as in Remark 3.35, we can show that $\underline{W_u}$ recovers $\overline{W}$ on $(0,1]$.

Theorem 3.49. *The function $\underline{W_u}$ is submultiplicative.*

Proof. The proof is similar to that of Theorem 3.37. Indeed, consider a finite family of pairwise disjoint, open intervals I_j, and measurable sets $S_j \subseteq I_j$, such that $|S_j| = \lambda\mu|I_j|$, where $\lambda, \mu \in (0,1]$. By Remark 3.48 we can consider S_j as a union of intervals. Then by Lemma 3.36 and for each j obtain a set J_j such that it is a union of a finite number of intervals, that we call J_{ji} :

$$S_j \subseteq J_j \subset I_j, \quad |S_j \cap J_{ji}| = \lambda|J_{ji}|, \quad \text{and} \quad |J_j| = \mu|I_j|.$$

So, we have that

$$\frac{W\left(u\left(\bigcup_j S_j\right)\right)}{W\left(u\left(\bigcup_j I_j\right)\right)} \leq \frac{W\left(u\left(\bigcup_j S_j\right)\right)}{W\left(u\left(\bigcup_j I_j\right)\right)} \frac{W\left(u\left(\bigcup_j J_j\right)\right)}{W\left(u\left(\bigcup_j J_j\right)\right)} \leq \overline{W}_u(\lambda)\overline{W}_u(\mu).$$

Therefore, taking supremum over all the possible choices of intervals I_j and measurable subsets S_j such that $S_j \subseteq I_j$ and $|S_j| = \lambda\mu|I_j|$, we get that $\underline{W_u}(\lambda\mu) \leq \underline{W_u}(\lambda)\underline{W_u}(\mu)$. ☐

The following result is the weighted version of Corollary 3.28.

Corollary 3.50. *The following statements are equivalent:*

(i) $\underline{W_u}$ *is not identically 1.*

(ii) For every finite family of pairwise disjoint, open intervals $(I_j)_{j=1}^J$*, and every family of measurable sets* $(S_j)_{j=1}^J$*, with* $S_j \subset I_j$*, for every* j *we have that*

$$\min_j \left(1 + \log \frac{|I_j|}{|S_j|}\right) \lesssim \frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)}.$$

(iii) $\underline{W_u}(0^+) = 0$.

(iv) $\lim_{\lambda\to 0} \dfrac{\log \underline{W_u}(\lambda)}{\log \lambda^p} > 0$.

Moreover, if $w \in \Delta_2$ *they are all equivalent to the* AB_∞^* *condition.*

Proof. The equivalences follow by Theorem 3.49 and Lemma 3.27. To see the last part, observe that, since clearly (iii) implies the AB_∞^* condition, it is sufficient to show that AB_∞^* implies (i). Indeed, if AB_∞^* holds, then by Proposition 3.46 it is equivalent to $w \in B_\infty^*$, and $u \in A_\infty$, since $w \in \Delta_2$. By the B_∞^* condition, in view of Corollary 3.28, there exists $\alpha \in (0, 1)$ with

$$\frac{W(t)}{W(s)} < \frac{1}{2}, \tag{3.25}$$

provided $0 < t < \alpha s$. On the other hand, if $S_j \subset I_j$ such that $|S_j| < \eta|I_j|$, with $\eta > 0$ to be chosen later on, then we have that

$$\frac{u(\bigcup S_j)}{u(\bigcup I_j)} = \frac{\sum_j u(S_j)}{\sum_j u(I_j)} \leq c_u \sum_j \frac{u(I_j)(|S_j|/|I_j|)^r}{\sum_j u(I_j)} \leq c_u \left(\max_j \frac{|S_j|}{|I_j|}\right)^r = c_u\eta^r,$$

where $r \in (0,1)$ and $c_u > 0$ are constants depending on the condition A_∞. So, choose $\eta \in (0,1)$ such that $c_u \eta^r < \alpha$. Let $t = u(\cup S_j)$ and $s = u(\cup I_j)$. Then, by (3.25)

$$\frac{W\Big(u(\bigcup S_j)\Big)}{W\Big(u(\bigcup I_j)\Big)} < \frac{1}{2}.$$

This shows that $\underline{W_u}(\eta) < 1$, which is (i). $\square$

Chapter 4

Necessary conditions for the boundedness of H on $\Lambda^p_u(w)$

Throughout this chapter we present necessary conditions on the weights u, w for the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces,

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w).$$

In the first section, we prove that if we restrict the boundedness $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ to characteristic functions of intervals, in particular we obtain

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\,ds\right)}{W\left(\int_{-b}^{b} u(s)\,ds\right)} \lesssim \left(\log \frac{1+\nu}{\nu}\right)^{-p},$$

for every $\nu \in (0,1]$ (see Theorem 4.4), which implies that $u \notin L^1(\mathbb{R})$ and $w \notin L^1(\mathbb{R}^+)$ (see Proposition 4.5). The non-integrability of the weights u and w is important since we have proved that under these assumptions, the space $\mathcal{C}^\infty_c$, where the Hilbert transform is well-defined, is dense in $\Lambda^p_u(w)$ (see Theorem 2.13).

If we restrict the boundedness $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ to characteristic functions of measurable sets (see Theorem 4.8), we obtain

$$\frac{W(u(I))}{W(u(E))} \lesssim \left(\frac{|I|}{|E|}\right)^p.$$

In particular, this implies that $W \circ u$ satisfies the doubling condition. Furthermore, w is p quasi-concave (see Corollary 4.9). These results are proved in the second section.

In the third section, applying duality arguments, we have that

$$||u^{-1}\chi_I||_{(\Lambda^p_u(w))'} ||\chi_I||_{\Lambda^p_u(w)} \lesssim |I|,$$

for all intervals I of the real line (see Theorem 4.16).

4.1 Restricted weak-type boundedness on intervals

Stein and Weiss proved in [95] that the distribution function of the Hilbert transform of the characteristic function of a measurable set depends only on the Lebesgue measure of the set. Precisely, they proved the following relation

$$|\{x \in \mathbb{R} : |H\chi_E(x)| > \lambda\}| = \frac{2|E|}{\sinh \pi\lambda}, \tag{4.1}$$

where E is a measurable set of finite Lebesgue measure and $\lambda > 0$ (for more details see [8]). An alternative proof can be found in [27], based on an already known result established in [10].

We calculate explicitly the distribution function of the Hilbert transform of a characteristic function of an interval, with respect to a weight u, generalizing the relation (4.1) when the set E is an interval.

If we consider the boundedness of the Hilbert transform on characteristic functions of intervals, we find necessary conditions for the weak-type boundedness $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$. For this reason we start by defining the restricted weak-type inequality (p, p) with respect to the pair (u, w) as given in [8].

Definition 4.1. *Let $p > 0$. We say that a sublinear operator T is of restricted weak-type (p, p) (with respect to (u, w)) if*

$$||T\chi_S||_{\Lambda^{p,\infty}_u(w)} \lesssim ||\chi_S||_{\Lambda^p_u(w)}, \tag{4.2}$$

where S is a measurable set of the real line. If S is an interval, then we say that T is of restricted weak-type (p, p) on intervals.

The following lemma gives an explicit formula for the distribution function of the Hilbert transform of the characteristic function of an interval.

Lemma 4.2. *Let $a, b \in \mathbb{R}$. For $\lambda > 0$,*

$$u\left(\left\{x \in \mathbb{R} : |H\chi_{(a,b)}(x)| > \lambda\right\}\right) = \int_{a-\psi(\lambda)}^{a+\varphi(\lambda)} u(s)\,ds + \int_{b-\varphi(\lambda)}^{b+\psi(\lambda)} u(s)\,ds, \tag{4.3}$$

where

$$\varphi(\lambda) = \frac{b-a}{1+e^{\pi\lambda}} \quad \text{and} \quad \psi(\lambda) = \frac{b-a}{e^{\pi\lambda}-1}. \tag{4.4}$$

Proof. A simple calculation shows that

$$H\chi_{(a,b)}(x) = \frac{1}{\pi}\log\frac{|x-a|}{|x-b|},$$

where $x \in (a,b)$ (for more details see [40]). Then, we obtain the following expression for the level set of $H\chi_{(a,b)}$:

$$\begin{aligned} E &= \{x\in\mathbb{R} : |H\chi_{(a,b)}(x)| > \lambda\} = \left\{x\in\mathbb{R} : \left|\frac{1}{\pi}\log\frac{|x-a|}{|x-b|}\right| > \lambda\right\} \\ &= \left\{x\in\mathbb{R} : \frac{|x-a|}{|x-b|} > e^{\pi\lambda}\right\} \cup \left\{x\in\mathbb{R} : \frac{|x-a|}{|x-b|} < e^{-\pi\lambda}\right\} = E_1 \cup E_2. \end{aligned}$$

Letting $g(x) = \dfrac{x-a}{x-b}$ and taking into account that g tends to 1, when x tends to infinity, we get

$$\begin{aligned} E_1 &= \{x\in\mathbb{R} : |g(x)| > e^{\pi\lambda}\} = \{x\in\mathbb{R} : g(x) > e^{\pi\lambda}\} \cup \{x\in\mathbb{R} : g(x) < -e^{\pi\lambda}\} \\ &= (b-\varphi(\lambda), b+\psi(\lambda)), \end{aligned}$$

where φ and ψ are given by $g(b-\varphi(\lambda)) = -e^{\pi\lambda}$ and $g(b+\psi(\lambda)) = e^{\pi\lambda}$, respectively. Following the same procedure for E_2 and letting $h = 1/g$, we obtain

$$\begin{aligned} E_2 &= \{x\in\mathbb{R} : |h(x)| > e^{\pi\lambda}\} = \{x\in\mathbb{R} : h(x) > e^{\pi\lambda}\} \cup \{x\in\mathbb{R} : h(x) < -e^{\pi\lambda}\} \\ &= (a-\psi(\lambda), a+\varphi(\lambda)). \end{aligned}$$

Then,

$$u(E) = u(E_2) + u(E_1) = \int_{a-\psi(\lambda)}^{a+\varphi(\lambda)} u(s)\,ds + \int_{b-\varphi(\lambda)}^{b+\psi(\lambda)} u(s)\,ds.$$

$\square$

Remark 4.3. If u is the Lebesgue measure on $\mathbb{R}$, we recover the result of Stein and Weiss when the set E is the interval (a,b); that is

$$|\{x\in\mathbb{R} : |H\chi_{(a,b)}(x)| > \lambda\}| = \frac{2|b-a|}{\sinh\pi\lambda},$$

with $\lambda > 0$ and $a, b \in \mathbb{R}$. Indeed, if $u = 1$ then by (4.3) and (4.4) we have that

$$|\{x\in\mathbb{R} : |H\chi_{(a,b)}(x)| > \lambda\}| = 2(\varphi(\lambda)+\psi(\lambda)) = \frac{2|b-a|}{\sinh\pi\lambda}.$$

Applying the above lemma, we find necessary conditions for the restricted weak-type (p,p) inequality on intervals with respect to the pair (u,w).

Theorem 4.4. *Let $0 < p < \infty$. If the Hilbert transform is of restricted weak-type (p,p) on intervals with respect to the pair (u,w), then necessarily*

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\,ds\right)}{W\left(\int_{-b}^{b} u(s)\,ds\right)} \lesssim \left(1+\log\frac{1}{\nu}\right)^{-p}, \tag{4.5}$$

for every $\nu \in (0,1]$, and

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{0} u(s)ds\right)}{W\left(\int_{0}^{b} u(s)ds\right)} \lesssim \left(\log\frac{1+\nu}{\nu}\right)^{-p}, \tag{4.6}$$

for every $\nu > 0$.

Proof. Let $a, b \in \mathbb{R}$. Then, by hypothesis we have that

$$\sup_{\lambda>0} W\big(\, u(\{x \in \mathbb{R} : \, |H\chi_{(a,b)}(x)| > \lambda\})\,\big)\lambda^p \lesssim W\left(\int_a^b u(s)\,ds\right),$$

which, applying (4.3), is equivalent to

$$\sup_{\lambda>0} W\left(\int_{a-\psi(\lambda)}^{a+\varphi(\lambda)} u(s)ds \; + \int_{b-\varphi(\lambda)}^{b+\psi(\lambda)} u(s)\,ds\right)\lambda^p \lesssim W\left(\int_a^b u(s)\,ds\right).$$

Let $a = 0$ and $b > 0$, then by the monotonicity of W, we necessarily obtain for every $\lambda > 0$

$$W\left(\int_{\frac{b}{1-e^{\pi\lambda}}}^{\frac{b}{1+e^{\pi\lambda}}} u(s)\,ds\right)\lambda^p \lesssim W\left(\int_0^b u(s)\,ds\right). \tag{4.7}$$

Since $\dfrac{b}{1-e^{\pi\lambda}} < \dfrac{-b}{1+e^{\pi\lambda}} < 0 < \dfrac{b}{1+e^{\pi\lambda}}$ we obtain that

$$W\left(\int_{\frac{-b}{1+e^{\pi\lambda}}}^{\frac{b}{1+e^{\pi\lambda}}} u(s)\,ds\right)\lambda^p \lesssim W\left(\int_0^b u(s)\,ds\right).$$

Writing $\nu = \dfrac{1}{1+e^{\pi\lambda}}$, we get

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\,ds\right)}{W\left(\int_{-b}^{b} u(s)\,ds\right)} \lesssim \left(\log\frac{1-\nu}{\nu}\right)^{-p}, \quad \nu \in (0,1/2). \tag{4.8}$$

Now, by the monotonicity of W, for every $\nu \in (0,1]$

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\,ds\right)}{W\left(\int_{-b}^{b} u(s)\,ds\right)} \leq 1. \tag{4.9}$$

So, (4.8) and (4.9) are equivalent to the following

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\,ds\right)}{W\left(\int_{-b}^{b} u(s)\,ds\right)} \lesssim \min\left\{1, \left(\log\frac{1-\nu}{\nu}\right)^{-p}\right\} \approx \left(1+\log\frac{1}{\nu}\right)^{-p},$$

for every $\nu \in (0,1/2)$. Moreover, by (4.9), we obtain

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\,ds\right)}{W\left(\int_{-b}^{b} u(s)\,ds\right)} \lesssim 1 \lesssim \left(1+\log\frac{1}{\nu}\right)^{-p},$$

for every $\nu \in [1/2,1]$. By the two last relations we get (4.5).

Finally, equation (4.6) is a consequence of (4.7), taking $\nu = \dfrac{1}{e^{\pi\lambda}-1}$. $\square$

If we consider the boundedness of the Hardy-Littlewood maximal function

$$M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w),$$

then u is necessarily non-integrable, whereas there are no integrability restrictions on w (see [20]). However, we will prove that the boundedness of H on $\Lambda^p_u(w)$ implies that both u and w are non-integrable. If $u = 1$, this was already proved by Sawyer in [90].

In order to avoid trivial cases, we can assume that the weights satisfy the following condition:

$$W\left(\int_{-\infty}^{+\infty} u(x)\,dx\right) > 0. \tag{4.10}$$

Proposition 4.5. *If the Hilbert transform is of restricted weak-type with respect to the pair (u,w) on intervals, then $u \notin L^1(\mathbb{R})$ and $w \notin L^1(\mathbb{R}^+)$.*

Proof. Since w is locally integrable, it is enough to prove that

$$W\left(\int_{-\infty}^{+\infty} u(x)\,dx\right) = \lim_{t\to\infty} W\left(\int_{-t}^{t} u(x)\,dx\right) = \infty. \tag{4.11}$$

Suppose that this limit is a finite number $\ell > 0$. Since, by Theorem 4.4 we have that there exists $C > 0$ such that, for all $\nu \in (0, 1]$,

$$\sup_{b>0} \frac{W\left(\int_{-b\nu}^{b\nu} u(s)\, ds\right)}{W\left(\int_{-b}^{b} u(s)\, ds\right)} \leq C \left(\log \frac{1}{\nu}\right)^{-p}, \tag{4.12}$$

taking $\nu > 0$ small enough satisfying $C\left(\log \frac{1}{\nu}\right)^{-p} < 1/2$ we obtain that

$$\lim_{b\to\infty} \frac{W\left(\int_{-\nu b}^{\nu b} u(s)\, ds\right)}{W\left(\int_{-b}^{b} u(s)\, ds\right)} \leq \sup_{b>0} \frac{W\left(\int_{-\nu b}^{\nu b} u(s)\, ds\right)}{W\left(\int_{-b}^{b} u(s)\, ds\right)} \leq \frac{1}{2}.$$

Since we also have that

$$\lim_{b\to\infty} \frac{W\left(\int_{-\nu b}^{\nu b} u(s)\, ds\right)}{W\left(\int_{-b}^{b} u(s)\, ds\right)} = \frac{\ell}{\ell} = 1,$$

we get a contradiction. Hence, (4.11) holds. □

One could think that the boundedness

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w)$$

holds if both the boundedness $H : L^p(u) \to L^p(u)$, for $p > 1$ (characterized by the A_p condition) and the boundedness $H : \Lambda^p(w) \to \Lambda^p(w)$ (characterized by $w \in B_p \cap B^*_\infty$) hold. However, the next result shows that in general these conditions ($u \in A_p$ and $w \in B_p \cap B^*_\infty$) are not sufficient for the boundedness of H on $\Lambda^p_u(w)$ for $p > 1$. In the next chapter we will see that if we assume a stronger condition; that is $u \in A_1$, then $w \in B_p \cap B^*_\infty$ characterizes the strong-type boundedness of Hilbert transform on $\Lambda^p_u(w)$, for $p > 1$ (see Theorem 5.4 below) and also prove similar results for the weak-type version (see Theorem 5.2 below).

Proposition 4.6. *If the Hilbert transform is of restricted weak-type (p, p) on intervals with respect to the pair $(|x|^k, t^l)$, then necessarily $(k+1)(l+1) \leq p$, where $k, l > -1$. In particular, there exist $u \in A_p$ and $w \in B_p \cap B^*_\infty$ such that the Hilbert transform is not bounded on $\Lambda^p_u(w)$ for $p > 1$.*

Proof. By hypothesis, it holds (4.6), which implies that

$$(k + 1)(l + 1) \leq p. \tag{4.13}$$

Indeed, since $u(x) = |x|^k$, $w(t) = t^l$ and $k, l > -1$ then by (4.6) we have that for $q = (k + 1)(l + 1)/p$

$$\nu^q \log\left(1 + \frac{1}{\nu}\right) \leq C$$

for all $\nu > 0$. Hence $q \leq 1$. Now, if we choose p and $k = l$ such that $\sqrt{p} < k+1 < p$, then $u(x) = |x|^k \in A_p$ and $w(t) = t^k \in B_p \cap B^*_\infty$. However, such $k = l > -1$ contradicts the condition (4.13) since $p < (k+1)^2$. Hence, in this case, the Hilbert transform is not bounded on $\Lambda^p_u(w)$. □

4.2 Restricted weak-type boundedness

It is well-known that the boundedness of the Hilbert transform $H : L^p(u) \to L^{p,\infty}(u)$ implies in particular that u is a doubling measure. Furthermore, if we consider $H : \Lambda^p(w) \to \Lambda^{p,\infty}(w)$ we get $w \in \Delta_2$. We will see that if we assume the boundedness

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$$

we obtain that the composition $W \circ u$ satisfies the doubling property; that is,

$$W(u(2I)) \lesssim W((u(I))), \tag{4.14}$$

for all intervals $I \subset \mathbb{R}$, where $2I$ denotes the interval with the same center than I and double size-length. In fact, this is a consequence of a stronger result which also implies that w is p quasi-concave. First we present the following known result.

Theorem 4.7. *[20] Let $0 < p < \infty$. If $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, then it implies*

$$\frac{W(u(I))}{W(u(E))} \lesssim \left(\frac{|I|}{|E|}\right)^p, \tag{4.15}$$

for every interval $I \subset \mathbb{R}$ and $S \subset I$. Moreover, (4.15) implies that w is p quasi-concave and in particular $w \in \Delta_2$.

Theorem 4.8. *Let $0 < p < \infty$. If H is of restricted weak-type (p,p) with respect to the pair (u,w), then (4.15) holds. In particular, $W \circ u$ satisfies the doubling property.*

Proof. Let E be a measurable subset of the interval I and $f = \chi_E$. Let I' be an interval of the same size touching I. For every $x \in I'$ we obtain

$$|H\chi_E(x)| = \left| \int_{\mathbb{R}} \frac{\chi_E(y)}{x-y}\, dy \right| \geq \frac{|E|}{2|I|}. \tag{4.16}$$

So, if $\lambda \leq \frac{|E|}{2|I|}$, then $I' \subseteq \{x : |H\chi_E(x)| > \lambda\}$. Therefore

$$W(u(I')) \leq W(u(\{x : |H\chi_E(x)| > \lambda\})) \lesssim \frac{1}{\lambda^p} \int_0^\infty (\chi_E)^*_u(t) w(t)\, dt \approx \frac{1}{\lambda^p} W(u(E)),$$

where the last step follows by the boundedness of H. As the above inequality holds for every $\lambda \leq \frac{|E|}{2|I|}$, we obtain that

$$\frac{W(u(I'))}{W(u(E))} \leq C \left(\frac{|I|}{|E|}\right)^p. \tag{4.17}$$

So, it only remains to prove that we can replace I' by the interval I. In fact, the quantities $W(u(I'))$ and $W(u(I))$ are comparable, since taking $E = I$ in (4.17) we get

$$W(u(I')) \lesssim W(u(I)).$$

Interchanging the roles of I and I' we get the converse inequality

$$W(u(I)) \leq C\, W(u(I')).$$

$\square$

Corollary 4.9. *Let $0 < p < \infty$. If H is of restricted weak-type (p, p) with respect to the pair (u, w), then w is p quasi-concave. In particular, w satisfies the Δ_2 condition.*

Proof. It follows by Theorems 4.8 and 4.7. $\square$

Remark 4.10. From now on, and taking into account Proposition 4.5 and Corollary 4.9, we shall always assume that $w \in \Delta_2$, $u \notin L^1$, and $w \notin L^1$.

4.3 Necessary conditions and duality

By Remark 4.10 and Theorem 2.13, $\mathcal{C}_c^\infty$ is dense in $\Lambda_u^p(w)$, and hence we can give the following definition.

Definition 4.11. *We say that the Hilbert transform $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$ if*

$$||Hf||_{\Lambda_u^{p,\infty}(w)} \lesssim ||f||_{\Lambda_u^p(w)},$$

for every $f \in \mathcal{C}_c^\infty$. Analogously, we write $H : \Lambda_u^p(w) \to \Lambda_u^p(w)$ if we have that $||Hf||_{\Lambda_u^p(w)} \lesssim ||f||_{\Lambda_u^p(w)}$ for every $f \in \mathcal{C}_c^\infty$.

Then, H can be extended to $\Lambda_u^p(w)$ in the usual way:

$$\overline{H}f = \Lambda_u^{p,\infty}(w) - \lim Hf_n \tag{4.18}$$

where $(f_n)_n \subset C_c^\infty$ and $f = \Lambda_u^p(w) - \lim f_n$.

Lemma 4.12. *Let f be bounded and with compact support. Then, $f \in \Lambda_u^p(w)$ and it holds that*

$$\overline{H}f(x) = Hf(x) := \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} dy, \qquad a.e.\ x \in \mathbb{R}.$$

Proof. The idea is to construct a sequence $(f_n)_n$ of functions in C_c^∞ such that $f = \Lambda_u^p(w) - \lim f_n$ and $f = L^1 - \lim f_n$. Then, we have that (4.18) holds and hence (see [20]) there exists a subsequence such that

$$\overline{H}f(x) = \lim_k Hf_{n_k}(x), \qquad a.e.\ x \in \mathbb{R}.$$

On the other hand, we know that $Hf = L^{1,\infty} - \lim_k Hf_{n_k}$ and that, in this case,

$$Hf(x) = \lim_j Hf_{n_{k_j}}(x), \qquad a.e.\ x \in \mathbb{R}, \tag{4.19}$$

and the result follows.

Let us see now how to construct the sequence (f_n). Let $h \in C_c^\infty$ such that $\int h(x)dx = 1$ and set $h_m(x) = mh(mx)$. Let $g_m = f * h_m$. Then $L^1 - \lim_m g_m = f$. Hence there exists a subsequence $f_n = g_{m_n}$ such that $\lim_n f_n(x) = f(x)$ for almost every x and since $|f_n| \leq C\chi_I$ for some constant C and some interval I, we can apply the dominated convergence Theorem 2.10 (for more details see [20]) to conclude that $f = \Lambda_u^p(w) - \lim_n f_n$, as we wanted to see. □

Theorem 4.13. *If* $0 < p < \infty$ *and*

$$H : \Lambda_u^p(w) \longrightarrow \Lambda_u^{p,\infty}(w)$$

then, for every $1 \leq q < \infty$ *and every* $f \in L^q \cap \Lambda_u^p(w)$, $\overline{H}f = Hf$.

Proof. We clearly have that if $f \in L^q \cap \Lambda_u^p(w)$, then the sequence

$$f_n(x) = f(x)\chi_{\{|f(x)| \leq n\}}(x)\chi_{(-n,n)}(x)$$

satisfies that $f = L^q - \lim_n f_n$ and $f = \Lambda_u^p(w) - \lim_n f_n$. Since f_n are bounded functions with compact support, $\overline{H}f_n = Hf_n$ and the result follows using the same argument as in the previous lemma. □

Remark 4.14. From now on we shall write Hf to indicate the extended operator and we shall use the previous theorem whenever it is necessary.

Assuming the boundedness of the Hilbert transform on weighted Lorentz spaces

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w),$$

we are going to obtain necessary conditions that involve the associate spaces of the weighted Lorentz spaces.

In [23] Carro and Soria studied the boundedness of the Hardy-Littlewood maximal function on weighted Lorentz spaces and obtained the following necessary condition:

Theorem 4.15. *[23] Let $0 < p < \infty$. If $M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$ then*

$$||u^{-1}\chi_I||_{(\Lambda_u^p(w))'}||\chi_I||_{\Lambda_u^p(w)} \lesssim |I|, \tag{4.20}$$

for all intervals I.

Theorem 4.16. *Let $0 < p < \infty$. If $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$, then (4.20) holds. In particular, $(\Lambda_u^p(w))' \neq \{0\}$.*

Proof. Let I and I' be as in Theorem 4.8. If $f \geq 0$ is bounded and supported in I, $f_I = \int_I f(x)dx$ and $\lambda \leq \frac{f_I}{2|I|}$, then for every $x \in I'$ we have by Lemma 4.12 that

$$|Hf(x)| = \left|\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy\right| = \left|\int_I \frac{f(y)}{x-y}\,dy\right| \geq \frac{1}{2|I|}\int_I f(y)\,dy. \tag{4.21}$$

Therefore $I' \subseteq \{x : |Hf(x)| > \lambda\}$ and so

$$W^{1/p}(u(I')) \leq W^{1/p}(u(\{x : |Hf(x)| > \lambda\})) \lesssim \frac{1}{\lambda}||f||_{\Lambda_u^p(w)},$$

where the last step follows by the boundedness of H. As the above inequality holds for every $\lambda \leq \frac{f_I}{2|I|}$, we obtain

$$\left(\frac{f_I}{||f||_{\Lambda_u^p(w)}}\right) W^{1/p}(u(I')) \lesssim |I|. \tag{4.22}$$

If f is not bounded, then we set $f_n = f\chi_{\{|f(x)|\leq n\}}$ and we can conclude, using the dominated convergence theorem in $\Lambda_u^p(w)$, that (4.22) holds for every $f \in \Lambda_u^p(w)$. Considering the supremum over all $f \in \Lambda_u^p(w)$ and taking into account that

$$f_I = \int_{\mathbb{R}} f(x)(u^{-1}(x)\chi_I(x))u(x)dx,$$

we get that

$$||u^{-1}\chi_I||_{(\Lambda_u^p(w))'}W^{1/p}(u(I')) \lesssim |I|.$$

Applying the monotonicity of W and then the doubling property for $W \circ u$, that holds by Theorem 4.8, it follows that

$$||\chi_I||^p_{\Lambda_u^p(w)} = W(u(I)) \leq W(u(3I')) \leq cW(u(I')).$$

Hence,

$$||u^{-1}\chi_I||_{(\Lambda_u^p(w))'}||\chi_I||_{\Lambda_u^p(w)} \lesssim |I|.$$

In particular, $u^{-1}\chi_I \in (\Lambda_u^p(w))'$. $\square$

Theorem 4.17. *If* $0 < p < \infty$ *and*

$$H : \Lambda^p_u(w) \longrightarrow \Lambda^{p,\infty}_u(w)$$

then, $\Lambda^p_u(w) \subset L^1_{loc}$.

Proof. Let $f \in \Lambda^p_u(w)$. Then, applying Hölder's inequality and Theorem 4.16 we obtain that

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|\chi_I(x)dx &= \int_{\mathbb{R}} |f(x)|u(x)u^{-1}(x)\chi_I(x)dx \lesssim ||f||_{\Lambda^p_u(w)}||u^{-1}\chi_I||_{(\Lambda^p_u(w))'} \\ &\lesssim ||f||_{\Lambda^p_u(w)}\frac{|I|}{W^{1/p}(u(I))} < \infty.\end{aligned}$$

□

As a consequence of (4.20), some necessary conditions on p, depending on w, were obtained in [20]. Following their approach, we see that the same results can be obtained if we assume the boundedness of the Hilbert transform on weighted Lorentz spaces. First, we need to define the index p_w:

Definition 4.18. *Let* $0 < p < \infty$. *We define*

$$p_w = \inf\left\{p > 0 : \frac{t^p}{W(t)} \in L^{p'-1}\left((0,1), \frac{dt}{t}\right)\right\},$$

where $p' = \infty$, *if* $0 < p \leq 1$.

Proposition 4.19. *Let* $0 < p < \infty$. *If* $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, *then* $p \geq p_w$. *Moreover, if* $p_w > 1$ *then* $p > p_w$. *In particular, if* $p < 1$ *there is no weight* u *for which it holds that* $H : L^p(u) \to L^{p,\infty}(u)$ *is bounded.*

Proof. See the proof of Theorem 3.4.2 and 3.4.3 in [20]. □

Since there exists an explicit description of the associate spaces of weighted Lorentz spaces by Theorem 2.16, we can provide equivalent integral expressions to (4.20). For this reason, it will be useful to associate to each weight u the family of functions $\{\phi_I\}_I$ defined as follows. For every interval I of the real line, we set

$$\phi_I(t) = \phi_{I,u}(t) = \sup\{|E| : E \subset I, u(E) = t\}, \quad t \in [0, u(I)). \tag{4.23}$$

Then, we will study the function ϕ and find some equivalent expressions depending on u, and also several concrete examples.

Proposition 4.20. *(i) If $p \le 1$, the condition (4.20) is equivalent to the following:*

$$\frac{W^{1/p}(u(I))}{|I|} \lesssim \frac{W^{1/p}(u(E))}{|E|}, \quad E \subset I. \tag{4.24}$$

(ii) If $p > 1$, the condition (4.20) is equivalent to the following expression:

$$\left(\int_0^{u(I)} \left(\frac{\phi_I(t)}{W(t)} \right)^{p'} w(t) dt \right)^{1/p'} \lesssim \frac{|I|}{W^{1/p}(u(I))}. \tag{4.25}$$

Proof. (i) If $p \le 1$, the condition (4.20) is equivalent to

$$||u^{-1}\chi_I||_{\Gamma_u^{1,\infty}(\tilde{w})} ||\chi_I||_{\Lambda_u^p(w)} \lesssim |I|,$$

which, by Theorem 2.16 gives

$$\sup_{t>0} (u^{-1}\chi_I)_u^{**}(t) \widetilde{W}(t) W^{1/p}(u(I)) \lesssim |I|,$$

where $\widetilde{W}(t) = tW^{-1/p}(t)$; that is equivalent to

$$\sup_{t>0} \frac{\int_0^t (u^{-1}\chi_I)_u^*(s) ds}{W^{1/p}(t)} \lesssim \frac{|I|}{W^{1/p}(u(I))}. \tag{4.26}$$

Taking into account that

$$\int_0^\infty f_u^*(s) g_u^*(s) ds = \sup_{h_u^* = g_u^*} \int f(x) h(x) u(x) dx$$

(see [8]) for every measurable functions f, g we obtain

$$\int_0^t (u^{-1}\chi_I)_u^*(s) ds = \sup_{E \subset I, u(E) = t} |E| = \phi_I(t), \tag{4.27}$$

where $t \le u(I)$. Hence

$$\sup_{t>0} \frac{\int_0^t (u^{-1}\chi_I)_u^*(s) ds}{W^{1/p}(t)} = \sup_{t>0} \frac{\sup_{E \subset I : u(E) = t} |E|}{W^{1/p}(t)} = \sup_{E \subset I} \frac{|E|}{W^{1/p}(u(E))}.$$

Therefore, (4.26) can be rewritten as

$$\frac{|E|}{W^{1/p}(u(E))} \lesssim \frac{|I|}{W^{1/p}(u(I))},$$

for all $E \subset I$.

(ii) On the other hand, for $p>1$ we obtain by Theorem 2.16 and the fact that $w \notin L^1$, which holds by Proposition 4.5, that the condition (4.20) is equivalent to

$$\left[\int_0^\infty \left(\frac{1}{W(t)}\int_0^t (u^{-1}\chi_I)^*_u(s)ds\right)^{p'} w(t)dt\right]^{1/p'} \lesssim \frac{|I|}{W^{1/p}(u(I))}. \tag{4.28}$$

Applying the relation (4.27) we have that the condition (4.28) is equivalent to (4.25) whenever $t \le u(I)$. If $t > u(I)$ then $\phi_I(t) = |I|$ and in this case we get that

$$\left[\int_{u(I)}^\infty \left(\frac{1}{W(t)}\int_0^t (u^{-1}\chi_I)^*_u(s)ds\right)^{p'} w(t)dt\right]^{1/p'} = |I|\left[\int_{u(I)}^\infty \frac{w(t)}{W(t)^{p'}}dt\right]^{1/p'} \approx \frac{|I|}{W^{1/p}(u(I))}.$$

□

In order to study the function ϕ_I, we fix the interval $I=[a,b]$, where $a<b$ and associate to this interval the functions

$$U(s) = \int_a^{a+s} u(x)dx, \quad V(s) = \int_{b-s}^b u(x)dx, \tag{4.29}$$

for $s \in [0,|I|]$. We will see that if u is increasing (resp. decreasing) the function ϕ_I can be expressed as the inverse function of U (resp. V). Then, we analyze the function ϕ_I for a general weight u and prove that it can be expressed in terms of the inverse function of

$$\psi_I(t) = \sup\{u(F) : F \subset I \text{ and } |F| = t\}, \quad t \in [0,|I|). \tag{4.30}$$

Proposition 4.21. *Let $t \in [0,|I|]$.*

(i) If u is an increasing function, then

$$\phi_I(t) = U^{-1}(t),$$

where U^{-1} is the inverse function of U defined in (4.29).

(ii) If u is a decreasing function, then

$$\phi_I(t) = V^{-1}(t),$$

where V^{-1} is the inverse function of V defined in (4.29).

(iii) In general,

$$\phi_I(t) = |I| - \psi_I^{-1}(u(I)-t),$$

where ψ_I^{-1} is the inverse function of ψ_I defined in (4.30).

Proof. (i) Note that the supremum defining the function ϕ_I is attained in an interval of the form $[a, a+s]$ since u is increasing, for some $s \in [0, |I|]$ such that $U(s) = t$. Hence $s = U^{-1}(t)$ which implies $\phi_I(t) = U^{-1}(t)$.

(ii) In this case, the supremum defining the function ϕ_I is attained in an interval $[b-s, s]$, since u is decreasing, and $s \in [0, |I|]$ is such that $V(s) = t$. Hence $\phi_I(t) = V^{-1}(t)$.

(iii) First observe that

$$\phi_I(t) = \sup\{|F| : F \subset [0, b-a] \text{ and } (u\chi_I)^*(F) = t\}.$$

Taking into account that $(u\chi_I)^*$ is decreasing, we get by (ii)

$$\phi_I(t) = \mathcal{V}^{-1}(t),$$

where

$$\mathcal{V}(s) = \int_{|I|-s}^{|I|} (u\chi_I)^*(r)dr = u(I) - \psi_I(|I| - s) = t.$$

Hence $\phi_I(t) = |I| - \psi_I^{-1}(u(I) - t)$. □

The condition (4.25) recovers the classical results when $u = 1$ and $w = 1$ (see [20]). Besides, we show some new consequences: under some assumptions on the weight w (for example let w be a power weight) the condition (4.25) implies that u belongs necessary to the A_p class.

Proposition 4.22. *[20] Let $p > 1$.*

(i) If $u = 1$, (4.25) is equivalent to the condition $w \in B_{p,\infty}$.

(ii) If $w = 1$, (4.25) is equivalent to the condition $u \in A_p$.

Proposition 4.23. *Assume that $w(t) = t^\alpha$, $\alpha > 0$. Then the condition (4.25) implies that $u \in A_p$, for $p > 1$. In particular, if $H : L^{r,p}(u) \to L^{r,\infty}(u)$ and $r < p$, then $u \in A_p$.*

Proof. If $w(t) = t^\alpha$, then the condition (4.25) is

$$\left(\int_0^{u(I)} \left(\frac{\phi_I(t)}{t^{\alpha+1}} \right)^{p'} t^\alpha dt \right)^{1/p'} \lesssim \frac{|I|}{u(I)^{(\alpha+1)/p}}. \tag{4.31}$$

Let $\gamma = \alpha(p' - 1)$. Since $t \le u(I)$ and $\gamma > 0$, we obtain by (4.31)

$$\begin{aligned} \left(\int_0^{u(I)} \left(\frac{\phi_I(t)}{t} \right)^{p'} dt \right)^{1/p'} &\lesssim \left(\int_0^{u(I)} \left(\frac{\phi_I(t)}{t} \right)^{p'} \left(\frac{u(I)}{t} \right)^{\gamma} dt \right)^{1/p'} \\ &= u(I)^{\gamma/p'} \left(\int_0^{u(I)} \left(\frac{\phi_I(t)}{t^{\alpha+1}} \right)^{p'} t^\alpha dt \right)^{1/p'} \lesssim \frac{|I|}{u^{1/p}(I)}. \end{aligned}$$

It remains to show that the condition

$$\left(\int_0^{u(I)}\left(\frac{\phi_I(t)}{t}\right)^{p'}dt\right)^{1/p'}\lesssim\frac{|I|}{u(I)^{1/p}},$$

implies A_p. In fact, since

$$(u^{-1}\chi_I)^*_u(t)\leq\frac{1}{t}\int_0^t(u^{-1}\chi_I)^*_u(s)ds=\frac{\phi_I(t)}{t},$$

then

$$\left(\int_0^{u(I)}\left((u^{-1}\chi_I)^*_u(t)\right)^{p'}dt\right)^{1/p'}\lesssim\frac{|I|}{u(I)^{1/p}},$$

which is equivalent to the A_p condition. Indeed, (see [94])

$$\begin{aligned}u\in A_p&\Leftrightarrow\left(\frac{1}{|I|}\int_I u^{-p/p'}(x)dx\right)^{p/p'}\lesssim\frac{|I|}{u(I)}\\&\Leftrightarrow\left(\int_I(u^{-1}\chi_I)^{p'}(x)u(x)dx\right)^{1/p'}\lesssim\frac{|I|}{u^{1/p}(I)}\\&\Leftrightarrow\left(\int_0^{u(I)}((u^{-1}\chi_I)^*_u(t))^{p'}dt\right)^{1/p'}\lesssim\frac{|I|}{u^{1/p}(I)}.\end{aligned}$$

Finally, observe that if $w(t)=t^\alpha$ and $\alpha>0$, then $\Lambda^p_u(w)=L^{r,p}(u)$ and $\Lambda^{p,\infty}_u(w)=L^{r,\infty}(u)$ for $r<p$. Hence, by the previous argument we get that $u\in A_p$. □

We extend the previous result, considering more general weights.

Proposition 4.24. *Let $p>1$. If $W(t)/t$ is increasing, then (4.25) implies that $u\in A_p$.*

Proof. Note that if $W(t)/t$ is increasing then $w\in B^*_\infty$ since

$$\int_0^r\frac{W(t)}{t}dt\lesssim W(r),\quad\text{for every } r>0.$$

If $w\in B^*_\infty$ then, for every f decreasing we have, by Theorem 3.23, that

$$||Qf||_{L^1(w)}\lesssim||f||_{L^1(w)},$$

which is equivalent to

$$\int_0^\infty f(s)\frac{W(s)}{s}ds\lesssim\int_0^\infty f(s)w(s)ds.$$

Hence

$$\begin{aligned}\left(\int_0^{u(I)}\left(\frac{\phi_I(t)}{t}\right)^{p'}dt\right)^{1/p'} &= \left(\int_0^{u(I)}\left(\frac{\phi_I(t)}{t}\right)^{p'}\frac{t}{W(t)}\frac{W(t)}{t}dt\right)^{1/p'}\\ &\lesssim \left(\int_0^{u(I)}\left(\frac{\phi_I(t)}{t}\right)^{p'}\frac{t}{W(t)}w(t)dt\right)^{1/p'}\\ &= \left(\int_0^{u(I)}\left(\frac{\phi_I(t)}{W(t)}\right)^{p'}\left(\frac{W(t)}{t}\right)^{p'-1}w(t)dt\right)^{1/p'}\\ &\lesssim \left(\frac{W(u(I))}{u(I)}\right)^{1/p}\frac{|I|}{W(u(I))^{1/p}} = \frac{|I|}{u(I)^{1/p}},\end{aligned}$$

from which the result follows. □

If the Hardy-Littlewood maximal function is bounded on weighted Lorentz spaces

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$$

then it is bounded on the same spaces with $u = 1$ (see [20]). The following theorem establishes a similar result, but for H instead of M.

Theorem 4.25. *Let* $0 < p < \infty$*. If* $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$*, then* $w \in B_{p,\infty}$*.*

Proof. Since by Theorem 4.16, the equation (4.20) holds, we can follow the same arguments used in [20, Proposition 3.4.4 and Theorem 3.4.8]. □

Chapter 5

The case $u \in A_1$

In the previous chapter we showed that the boundedness

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w)$$

does not necessary hold even if we assume the following conditions (see Proposition 4.6):

(i) $H : L^p(u) \to L^p(u)$, $p > 1$, characterized by the condition $u \in A_p$;

(ii) $H : \Lambda^p(w) \to \Lambda^p(w)$ characterized by the condition $w \in B_p \cap B_\infty^*$.

However, in the first section we will prove that the situation is different if we assume the condition $u \in A_1$. In fact, we will show that under this assumption, for $p > 1$

$$H : \Lambda_u^p(w) \to \Lambda_u^p(w) \Leftrightarrow w \in B_p \cap B_\infty^*.$$

Analogously, if $u \in A_1$ and $0 < p < \infty$ we have that

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \Leftrightarrow w \in B_{p,\infty} \cap B_\infty^*.$$

Hence, in this case, the boundedness of the Hilbert transform on weighted Lorentz spaces $\Lambda_u^p(w)$ coincides with the boundedness of the same operator for the weight $u = 1$.

The results of this chapter and part of Chapter 4 are included in [2].

It is known that under the assumption $u \in A_1$, the boundedness of the Hardy-Littlewood maximal function on $\Lambda_u^p(w)$ is equivalent to the boundedness of the same operator for $u = 1$ (see [20]).

Theorem 5.1. *[20] If $u \in A_1$, and $0 < p < \infty$, then*

$$M : \Lambda_u^p(w) \to \Lambda_u^p(w) \Leftrightarrow M : \Lambda^p(w) \to \Lambda^p(w),$$

and

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \Leftrightarrow M : \Lambda^p(w) \to \Lambda^{p,\infty}(w).$$

If we rearrange the Hilbert transform with respect to a weight $u \in A_1$, we obtain a generalization of (3.11); that is, if $f \in \mathcal{C}_c^\infty$ (see [6]), then

$$(Hf)_u^*(t) \le Pf_u^*(t) + Qf_u^*(t). \tag{5.1}$$

Applying this relation, we characterize the boundedness of the Hilbert transform on weighted Lorentz spaces.

Theorem 5.2. *Let $u \in A_1$ and let $0 < p < \infty$. Then*

$$H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w) \Longleftrightarrow w \in B_{p,\infty} \cap B_\infty^*.$$

Proof. First we prove the necessary condition. By Theorem 4.25 we obtain that $w \in B_{p,\infty}$. Let us see now that it is also in B_∞^*. Let $0 < t \le s < \infty$. Then, since $u \notin L^1(\mathbb{R})$ by Proposition 4.5, there exists $\nu \in (0, 1]$ and $b > 0$ such that

$$t = \int_{-b\nu}^{b\nu} u(r)\,dr \le \int_{-b}^{b} u(r)\,dr = s.$$

By Theorem 4.4 we obtain (4.5) and hence

$$\frac{W(t)}{W(s)} \lesssim \left(1 + \log \frac{1}{\nu}\right)^{-p}.$$

Let $S = (-b\nu, b\nu)$ and $I = (-b, b)$. Since $u \in A_1$, we obtain that

$$\nu = \frac{|S|}{|I|} \lesssim \frac{u(S)}{u(I)} = \frac{t}{s}$$

and therefore

$$\frac{W(t)}{W(s)} \lesssim \left(1 + \log \frac{s}{t}\right)^{-p},$$

which is equivalent to the condition $w \in B_\infty^*$ by Corollary 3.28.

To prove the converse, we just have to use that if $u \in A_1$ and $f \in \mathcal{C}_c^\infty$ then we have (5.1). Now, since $w \in B_{p,\infty}$, we have that

$$\sup_{t>0} Pf_u^*(t) W(t)^{1/p} \lesssim ||f_u^*||_{L^p(w)} = ||f||_{\Lambda_u^p(w)},$$

and the condition $w \in B_\infty^*$ implies the same inequality for the operator Q; that is (see Section 3.3)

$$\sup_{t>0} Qf_u^*(t) W(t)^{1/p} \lesssim ||f||_{\Lambda_u^p(w)},$$

and the result follows. □

As a direct application we get the following known result (see [25]).

Corollary 5.3. *Let $q > 0$ and $p \geq 1$. If $u \in A_1$,*

$$H : L^{p,q}(u) \to L^{p,\infty}(u). \tag{5.2}$$

Proof. The boundedness $H : L^{p,q}(u) \to L^{p,\infty}(u)$ can be rewritten as $H : \Lambda^q_u(t^{q/p-1}) \to \Lambda^{q,\infty}_u(t^{q/p-1})$. Observe that the weight $t^{q/p-1}$ is in B^*_∞ and if $p \geq 1$, then $t^{q/p-1} \in B_{q,\infty}$. Indeed, if $q \leq 1$, then by Theorem 3.18 we have that $t^{q/p-1} \in B_{q,\infty}$ since it is q quasi-concave. On the other hand, if $q > 1$, $t^{q/p-1} \in B_{q,\infty}$ in view of Theorems 3.8 and 3.17. Finally, applying Theorem 5.2 we obtain the result. □

With a completely similar proof and using the properties of the B_p class, we obtain the following:

Theorem 5.4. *Let $u \in A_1$ and let $1 < p < \infty$. Then*

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w) \iff w \in B_p \cap B^*_\infty.$$

Proof. By Theorem 4.25 we obtain that $w \in B_{p,\infty}$, which by Theorem 3.17 is equivalent to the B_p condition for $p > 1$. The necessity of the B^*_∞ condition is identical to the proof of Theorem 5.2.

As for the converse, again $u \in A_1$ and $f \in \mathcal{C}^\infty_c$ imply (5.1). Then, since $w \in B_p$, we have that

$$||Pf^*_u||_{\Lambda^p_u(w)} \lesssim ||f^*_u||_{L^p(w)} = ||f||_{\Lambda^p_u(w)},$$

and the condition $w \in B^*_\infty$ implies the same inequality for the operator Q; that is (see Section 3.3)

$$||Qf^*_u||_{\Lambda^p_u(w)} \lesssim ||f||_{\Lambda^p_u(w)},$$

and the result follows. □

The strong-type boundedness of the Hilbert transform on $\Lambda^p_u(w)$ for $p \leq 1$ presents some extra difficulties. Even though we show that the $B_p \cap B^*_\infty$ condition is sufficient, provided $u \in A_1$, we prove the necessity, under some extra assumption on the function $\overline{W}$.

Theorem 5.5. *Let $u \in A_1$ and let $0 < p \leq 1$.*

(i) If $w \in B_p \cap B^*_\infty$, then $H : \Lambda^p_u(w) \to \Lambda^p_u(w)$ is bounded.

(ii) The boundedness $H : \Lambda^p_u(w) \to \Lambda^p_u(w)$ implies that $w \in B^*_\infty$ and, under the assumption $\overline{W}^{1/p}(t) \not\approx t,\ t > 1$, we get that $w \in B_p$.

Proof. The sufficiency of the conditions B_p and B_∞^* in (i), and the necessity of the condition B_∞^* in (ii) are identical to the proof of Theorem 5.4. The necessity of the condition B_p is a consequence of Theorem 4.25 and Proposition 3.19. □

Corollary 5.6. *Let $q > 0$ and $p > 1$. If $u \in A_1$,*

$$H : L^{p,q}(u) \to L^{p,q}(u).$$

Proof. As in Corollary 5.3, $H : L^{p,q}(u) \to L^{p,q}(u)$ can be rewritten as

$$H : \Lambda_u^q(t^{q/p-1}) \to \Lambda_u^q(t^{q/p-1}).$$

The weight $t^{q/p-1}$ is in B_∞^* and if $p > 1$, then $t^{q/p-1} \in B_q$. The boundedness $H : L^{p,q}(u) \to L^{p,q}(u)$ follows by Theorem 5.4 if $q > 1$ and, if $q \leq 1$ it follows by Theorem 5.5. □

Chapter 6

Complete characterization of the boundedness of H on $\Lambda^p_u(w)$

In the previous chapter we studied the boundedness of the Hilbert transform on weighted Lorentz spaces whenever $u \in A_1$, which, in general is not a necessary condition.

Throughout this chapter we completely characterize the weak-type boundedness of H, as follows:

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \quad \Leftrightarrow \quad M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \quad \text{and} \quad (u, w) \in AB^*_\infty,$$

for $p > 1$, whereas the case $p \le 1$ is partially solved. We also characterize the boundedness

$$M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w),$$

which was open for $p \ge 1$. In fact, we show that the solution for $p > 1$ is the $B_p(u)$ class of weights, which also characterizes the strong-type version $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$. Hence as in the classical cases, both the weak-type and strong-type boundedness of M on $\Lambda^p_u(w)$ coincide. The solution, which extends and unify the classical results for $u = 1$ and $w = 1$, can be reformulated in the context of generalized Boyd indices, providing an extension of Boyd theorem in the setting of weighted Lorentz spaces that are not necessarily rearrangement invariant. Our main results are summarized in Theorem 6.19 for $p > 1$, and in Theorem 6.20 for $p < 1$. The results of this chapter are included in [3] and [1].

The sections are organized as follows:

In the first section we prove that the weak-type boundedness $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ implies that $u \in A_\infty$ (see Theorem 6.4). Moreover, we show a different characterization of the A_∞ condition, in terms of the following expression involving H (see Theorem 6.3),

$$\int_I |H(u\chi_I)(x)|dx \lesssim u(I).$$

In the second section we prove that the weak-type boundedness $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ implies that $(u, w) \in AB^*_\infty$ (see Theorem 6.6).

In the third section we prove that the weak-type boundedness of H implies that of M on the same spaces:

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \quad \Rightarrow \quad M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w),$$

if $p > 1$ (see Theorem 6.8). In particular, we recover the following well-known result

$$H : L^p(u) \to L^{p,\infty}(u) \quad \Rightarrow \quad M : L^p(u) \to L^{p,\infty}(u),$$

without passing through the A_p condition.

In the fourth section we prove the sufficiency of the conditions; that is:

$$M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \quad \text{and} \quad (u, w) \in AB^*_\infty \quad \Rightarrow \quad H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w),$$

and

$$M : \Lambda^p_u(w) \to \Lambda^p_u(w) \quad \text{and} \quad (u, w) \in AB^*_\infty \quad \Rightarrow \quad H : \Lambda^p_u(w) \to \Lambda^p_u(w),$$

for all $p > 0$ (see Theorem 6.10, and Corollary 6.11).

The fifth section contains our main results that are Theorem 6.19 for $p > 1$ and Theorem 6.20 for $p < 1$. However, we first solve the boundedness of $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ when $p > 1$ and the solution is the $B_p(u)$ condition (see Theorem 6.17). In particular, we show that if $p > 1$, then

$$M : L^p(u) \to L^{p,\infty}(u) \quad \Rightarrow \quad M : L^p(u) \to L^p(u)$$

without using the reverse Hölder inequality.

In the sixth section we give an extension of Boyd theorem, reformulating our results in terms of some generalized Boyd indices.

6.1 Necessary conditions involving the A_∞ condition

In general, the $B_p(u)$ condition, which characterizes the strong-type boundedness of the Hardy-Littlewood maximal function on weighted Lorentz spaces does not imply that $u \in A_\infty$. Indeed, if $u(x) = e^{|x|}$, $x \in \mathbb{R}$ and $w = \chi_{(0,1)}$, it was proved in [20] that

$$M : \Lambda^p_u(w) \to \Lambda^p_u(w),$$

is bounded, for $p > 1$, whereas u is a non-doubling measure.

However, the situation is different when we consider the weak-type (and consequently the strong-type) boundedness of the Hilbert transform on the weighted Lorentz spaces

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w), \tag{6.1}$$

since we will prove that it necessarily implies that $u \in A_\infty$. In fact, we find a necessary condition involving the operator itself,

$$\int_I |H(u\chi_I)(x)|dx \lesssim u(I). \tag{6.2}$$

Although our aim is to prove the equivalence between (6.2) and the A_∞ condition, for technical reasons, we first show that (6.1) implies the strong-type boundedness of the same operator, for all $r > p$.

Theorem 6.1. *Let $p > 0$. If $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, then $H : \Lambda^{2p,p}_u(w) \to \Lambda^{2p,\infty}_u(w)$. Moreover, we have that $H : \Lambda^r_u(w) \to \Lambda^r_u(w)$, for all $r > p$.*

Proof. If $f \in \mathcal{C}^\infty_c$, then

$$(Hf)^2 = f^2 + 2H(fHf), \tag{6.3}$$

(see [40]), and taking into account that $w \in \Delta_2$, we have that

$$\begin{aligned} ||Hf||_{\Lambda^{2p,\infty}_u(w)} &= ||(Hf)^2||^{1/2}_{\Lambda^{p,\infty}_u(w)} = ||f^2 + 2H(fHf)||^{1/2}_{\Lambda^{p,\infty}_u(w)} \\ &\leq C(||f^2||_{\Lambda^{p,\infty}_u(w)} + 2||H(fHf)||_{\Lambda^{p,\infty}_u(w)})^{1/2} \\ &\leq (C||f||^2_{\Lambda^{2p,\infty}_u(w)} + 2C_p||fHf||_{\Lambda^p_u(w)})^{1/2}, \end{aligned}$$

where the last estimate is a consequence of the hypothesis.

Now, we will see that

$$(fHf)^*_u(t) \leq f^*_u(t_1)(Hf)^*_u(t_2), \tag{6.4}$$

for all $t = t_1 + t_2$. Indeed, let $G = fHf$ and $\mu_1 = f^*_u(t_1)$, $\mu_2 = (Hf)^*_u(t_2)$. Then, applying properties of the decreasing rearrangements and distribution functions (see [40], and [8]) we have that

$$\begin{aligned} G^*_u(t) &= G^*_u(t_1 + t_2) \leq G^*_u(\lambda^u_f(\mu_1) + \lambda^u_{(Hf)}(\mu_2)) \\ &= G^*_u(\lambda^u_G(\mu_1\mu_2)) \leq \mu_1\mu_2 \leq f^*_u(t_1)(Hf)^*_u(t_2). \end{aligned}$$

Let $t_1 = t_2 = 1/2$ in (6.4). Then, since $w \in \Delta_2$ we obtain that

$$\begin{aligned} ||fHf||_{\Lambda^p_u(w)} &\lesssim \left(\int_0^\infty (f^*_u(t))^p((Hf)^*_u(t))^p w(t)dt\right)^{1/p} \\ &= \left(\int_0^\infty \frac{(f^*_u(t))^p}{W^{1/2}(t)}(W^{1/2p}(t)(Hf)^*_u(t))^p w(t)dt\right)^{1/p} \\ &\leq \left(\int_0^\infty \frac{(f^*_u(t))^p}{W^{1/2}(t)}\left(\sup_{t>0} W^{1/2p}(t)(Hf)^*_u(t)\right)^p w(t)dt\right)^{1/p} \\ &= ||Hf||_{\Lambda^{2p,\infty}_u(w)}||f||_{\Lambda^{2p,p}_u(w)}. \end{aligned}$$

Therefore, we have that

$$||Hf||_{\Lambda_u^{2p,\infty}(w)} \leq (C||f||^2_{\Lambda_u^{2p,\infty}(w)} + 2C_p||f||_{\Lambda_u^{2p,p}(w)}||Hf||_{\Lambda_u^{2p,\infty}(w)})^{1/2}$$

and consequently

$$\frac{||Hf||^2_{\Lambda_u^{2p,\infty}(w)}}{||f||^2_{\Lambda_u^{2p,p}(w)}} \leq C\frac{||f||^2_{\Lambda_u^{2p,\infty}(w)}}{||f||^2_{\Lambda_u^{2p,p}(w)}} + C_p\frac{||Hf||_{\Lambda_u^{2p,\infty}(w)}}{||f||_{\Lambda_u^{2p,p}(w)}}.$$

Using that $\Lambda_u^{2p,p}(w) \hookrightarrow \Lambda_u^{2p,\infty}(w)$ (see [20, pg. 31]), then

$$\frac{||Hf||^2_{\Lambda_u^{2p,\infty}(w)}}{||f||^2_{\Lambda_u^{2p,p}(w)}} \leq C + C_p\frac{||Hf||_{\Lambda_u^{2p,\infty}(w)}}{||f||_{\Lambda_u^{2p,p}(w)}},$$

Thus, studying the quadratic equation we have that

$$||Hf||_{\Lambda_u^{2p,\infty}(w)} \leq \frac{C_p + (C_p^2 + 4C)^{1/2}}{2}||f||_{\Lambda_u^{2p,p}(w)}. \tag{6.5}$$

for every $f \in \Lambda_u^p(w)$ (see Remark 4.14). Taking into account the hypothesis and (6.5), we obtain by Theorem 2.18, that

$$H : \Lambda_u^r(w) \to \Lambda_u^r(w) \quad \text{for} \quad p < r < 2p.$$

Iterating this result we have that

$$H : \Lambda_u^r(w) \to \Lambda_u^r(w),$$

for all $r > p$. □

Theorem 6.2. *Let $p > 0$. If $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$, then*

$$\int_I |H(u\chi_I)(x)|dx \lesssim u(I), \tag{6.6}$$

for all intervals I of the real line.

Proof. First note that the hypothesis implies the $B_{p,\infty}$ condition by Theorem 4.25. If $p > 1$, then $B_{p,\infty} = B_p$ by Theorem 3.17, and by Proposition 3.11 (ii) we have that

$$||\chi_E||_{(\Lambda_u^{p,\infty}(w))'} \approx \frac{u(E)}{W^{1/p}(u(E))}, \tag{6.7}$$

for every measurable set E. Let now $I_n = \{x \in I : u(x) \leq n\}$. By duality arguments we have that

$$\begin{aligned} ||u^{-1}H(u\chi_{I_n})||_{(\Lambda^p_u(w))'} &= \sup_{f\in\mathcal{C}^\infty_c} \frac{|\int_{\mathbb{R}} u^{-1}(x)H(u\chi_{I_n})(x)f(x)u(x)dx|}{||f||_{\Lambda^p_u(w)}} \\ &= \sup_{f\in\mathcal{C}^\infty_c} \frac{|\int_{\mathbb{R}} H(u\chi_{I_n})(x)f(x)dx|}{||f||_{\Lambda^p_u(w)}} = \sup_{f\in\mathcal{C}^\infty_c} \frac{|\int_{\mathbb{R}} u(x)\chi_{I_n}(x)Hf(x)dx|}{||f||_{\Lambda^p_u(w)}} \\ &\lesssim \sup_{f\in\mathcal{C}^\infty_c} \frac{||\chi_{I_n}||_{(\Lambda^{p,\infty}_u(w))'}||Hf||_{\Lambda^{p,\infty}_u(w)}}{||f||_{\Lambda^p_u(w)}} \lesssim ||\chi_{I_n}||_{(\Lambda^{p,\infty}_u(w))'}, \end{aligned}$$

where we have used the hypothesis. Hence,

$$||u^{-1}H(u\chi_{I_n})||_{(\Lambda^p_u(w))'} \lesssim ||\chi_{I_n}||_{(\Lambda^{p,\infty}_u(w))'}. \tag{6.8}$$

Then, applying Hölder's inequality and taking into account (6.8) and (6.7) we obtain

$$\begin{aligned} \int_{I_n} |H(u\chi_{I_n})(x)|dx &= \int \chi_{I_n}(x)u(x)u^{-1}(x)|H(u\chi_{I_n})(x)|dx \\ &\lesssim ||\chi_{I_n}||_{\Lambda^p_u(w)}||u^{-1}H(u\chi_{I_n})||_{(\Lambda^p_u(w))'} \\ &\lesssim W^{1/p}(u(I_n))||\chi_{I_n}||_{(\Lambda^{p,\infty}_u(w))'} \lesssim u(I_n) \leq u(I). \end{aligned}$$

Since $h_n = \chi_{I_n}|H(u\chi_{I_n})|$ converges to $h = \chi_I|H(u\chi_I)|$ in $L^{1,\infty}$, there exists a subsequence h_{n_k} that converges almost everywhere to h, and so by Fatou's lemma we have that

$$\int_I |H(u\chi_I)(x)|dx \leq \liminf_{k\to\infty} \int_{I_{n_k}} |H(u\chi_{I_{n_k}})(x)|dx \lesssim u(I).$$

If $p \leq 1$, then we apply the hypothesis and by Theorem 6.1 we obtain that $H : \Lambda^r_u(w) \to \Lambda^r_u(w)$ for all $r > p$ and so, in particular it holds for exponents bigger than 1. Hence, the problem is reduced to the previous case. □

It is known that the following condition, involving the Hardy-Littlewood maximal

$$\int_I M(u\chi_I)(x)dx \lesssim u(I) \tag{6.9}$$

is equivalent to the A_∞ condition (for more details see [55], [98], [99], [100], and [65]). We prove the following similar result involving H.

Theorem 6.3. *The condition (6.6) is equivalent to the condition $u \in A_\infty$.*

Proof. If $u \in A_\infty$ then condition (6.6) is satisfied. Indeed, if $u \in A_\infty$ there exists $q \geq 1$ such that $u \in A_q$. If $q > 1$, by Hölder's inequality,

$$\begin{aligned}
\int_I |H(u\chi_I)(x)|dx &= \int u(x)u^{-1}(x)\chi_I(x)|H(u\chi_I)(x)|dx \lesssim ||u^{-1}H(u\chi_I)||_{L^{q'}(u)}||\chi_I||_{L^q(u)} \\
&= \left(\int |H(u\chi_I)(x)|^{q'}u^{1-q'}(x)dx\right)^{1/q'} u(I)^{1/q} \\
&\lesssim \left(\int (u\chi_I)^{q'}(x)u^{1-q'}(x)dx\right)^{1/q'} u(I)^{1/q} = u(I),
\end{aligned}$$

taking into account that $u \in A_q$ if and only if $u^{1-q'} \in A_{q'}$, which characterizes the boundedness of the Hilbert transform on weighted Lebesgue spaces $L^{q'}(u^{1-q'})$. If $q = 1$, then $u \in A_1$ implies that $u \in A_r$, for all $r > 1$, hence this case is reduced to the previous one.

On the other hand, assume that condition (6.6) holds. It is well-known that if $0 \leq f, \tilde{f} \in L^1[-\pi,\pi]$ then

$$||Mf||_{L^1[-\pi,\pi]} \lesssim ||f||_{L^1[-\pi,\pi]} + ||\tilde{f}||_{L^1[-\pi,\pi]}, \tag{6.10}$$

where $\tilde{f}(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta-\varphi)\cot\left(\frac{\varphi}{2}\right)d\varphi$ is the conjugate Hilbert transform (for more details see [8]).

Now, we will show that if $f, Hf \in L^1[-\pi,\pi]$, then

$$||\tilde{f}||_{L^1[-\pi,\pi]} \lesssim ||Hf||_{L^1[-\pi,\pi]} + ||f||_{L^1[-\pi,\pi]}. \tag{6.11}$$

Indeed, let

$$k(s) = \frac{1}{s} - \frac{1}{2}\cot\left(\frac{s}{2}\right),$$

whenever $0 < |s| < \pi$ and 0 elsewhere. The function k is continuous and increasing on $(-\pi,\pi)$, hence bounded on the real line by $1/\pi$. Now, if $0 < \varepsilon < \pi$ and $|\theta| \leq \pi$ we obtain

$$\begin{aligned}
|\tilde{f}_\varepsilon(\theta)| &= \frac{1}{2\pi}\left|\int_{\varepsilon<|\varphi|\leq\pi} f(\theta-\varphi)\cot\left(\frac{\varphi}{2}\right)d\varphi\right| \\
&= \frac{1}{\pi}\left|\int_{\varepsilon<|\varphi|\leq\pi} f(\theta-\varphi)\left[\frac{1}{\varphi}-k(\varphi)\right]d\varphi\right| \\
&\lesssim \frac{1}{\pi}\left|\int_{\varepsilon<|\varphi|\leq\pi} f(\theta-\varphi)\frac{1}{\varphi}d\varphi\right| + \frac{1}{\pi}\int_{|\varphi|\leq\pi}|f(\theta-\varphi)||k(\varphi)|d\varphi \\
&\lesssim \frac{1}{\pi}\left|\int_{\varepsilon<|\varphi|\leq\pi} f(\theta-\varphi)\frac{1}{\varphi}d\varphi + \int_{|\varphi|\geq\pi} f(\theta-\varphi)\frac{1}{\varphi}d\varphi - \int_{|\varphi|\geq\pi} f(\theta-\varphi)\frac{1}{\varphi}d\varphi\right| \\
&\quad + \frac{1}{\pi}(|f| * |k|)(\theta) \\
&\lesssim \frac{1}{\pi}\left(|H_\varepsilon f(\theta)| + (|f| * |g|)(\theta) + (|f| * |k|)(\theta)\right),
\end{aligned}$$

where $g(\varphi)=\frac{1}{\varphi}\chi_{\{|\varphi|\geq\pi\}}$. We take the limit when ε tends to 0 and obtain

$$|\tilde{f}(\theta)|\lesssim\frac{1}{\pi}\left(|Hf(\theta)|+(|f|*|g|)(\theta)+(|f|*|k|)(\theta)\right).$$

Hence,

$$||\tilde{f}||_{L^1[-\pi,\pi]}\lesssim||Hf||_{L^1[-\pi,\pi]}+||g||_{L^\infty}||f||_{L^1[-\pi,\pi]}+||k||_{L^\infty}||f||_{L^1[-\pi,\pi]}.$$

Therefore, if $0\leq f, Hf\in L^1[-\pi,\pi]$, by (6.10) and (6.11)

$$||Mf||_{L^1[-\pi,\pi]}\lesssim||f||_{L^1[-\pi,\pi]}+||Hf||_{L^1[-\pi,\pi]}. \tag{6.12}$$

Now, we will show that

$$||M(u\chi_I)||_{L^1(I)}\lesssim||u\chi_I||_{L^1(I)}+||H(u\chi_I)||_{L^1(I)}. \tag{6.13}$$

Let $D_ag(x)=g(ax)$ and $T_cg(x)=g(c+x)$ be the dilation and the translation operators respectively, where $a>0$, $c\in\mathbb{R}$. It suffices to prove (6.13) for all dilations and translations of $[-\pi,\pi]$, since every interval I can be seen as composition of dilations and translations of $[-\pi,\pi]$. First, let $I=[-b,b]$, and $a>0$ such that $a\pi=b$. Since M and H are dilation invariant operators, we have that

$$\int_I M(u\chi_I)(x)dx=\int_{-b}^{b}D_{1/a}MD_a(u\chi_I)(x)dx=a\int_{-\pi}^{\pi}M[(D_au)\chi_{[-\pi,\pi]}](x)dx, \tag{6.14}$$

and

$$\int_I|H(u\chi_I)(x)|dx=\int_{-b}^{b}D_{1/a}|HD_a(u\chi_I)(x)|dx=a\int_{-\pi}^{\pi}|H[(D_au)\chi_{[-\pi,\pi]}](x)|dx. \tag{6.15}$$

If we take $f=(D_au)\chi_{[-\pi,\pi]}$ then by (6.12), (6.14) and (6.15) we get

$$\int_I M(u\chi_I)(x)dx\lesssim\int_I u(x)dx+\int_I|H(u\chi_I)(x)|dx.$$

Let $I=[c-\pi,c+\pi]$. Since M and H are translation invariant operators, we obtain

$$\int_I M(u\chi_I)(x)dx=\int_{c-\pi}^{c+\pi}T_{-c}MT_c(u\chi_I)(x)dx=\int_{-\pi}^{\pi}M[(T_cu)\chi_{[-\pi,\pi]}](x)dx, \tag{6.16}$$

and

$$\int_I|H(u\chi_I)(x)|dx=\int_{c-\pi}^{c+\pi}T_{-c}|HT_c(u\chi_I)(x)|dx=\int_{-\pi}^{\pi}|H[(T_cu)\chi_{[-\pi,\pi]}](x)|dx. \tag{6.17}$$

Now if $f=(T_cu)\chi_{[-\pi,\pi]}$, then by (6.12), (6.16) and (6.17) we have that

$$\int_I M(u\chi_I)(x)dx\lesssim\int_I u+\int_I|H(u\chi_I)(x)|dx.$$

Finally, applying the condition (6.6) we get

$$\int_I M(u\chi_I)(x)dx\lesssim u(I) \tag{6.18}$$

which implies the A_∞ condition. □

As a consequence of the previous theorem, we obtain that the weak-type, and consequently the strong-type boundedness of the Hilbert transform on weighted Lorentz spaces implies that $u \in A_\infty$, for all $p > 0$.

Theorem 6.4. *Let $0 < p < \infty$. If $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, then $u \in A_\infty$.*

Proof. By Theorem 6.2, the hypothesis implies relation (6.6) which is equivalent to the A_∞ condition by Theorem 6.3. □

6.2 Necessity of the B^*_∞ condition

We will prove that, if $p > 0$, the weak-type boundedness of the Hilbert transform,

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w),$$

implies that $w \in B^*_\infty$. We start by showing the following consequence of the restricted weak-type boundedness of the Hilbert transform.

Proposition 6.5. *Let $u \in A_\infty$. If the Hilbert transform is of restricted weak-type (p, p) on intervals with respect to the pair (u, w), then $w \in B^*_\infty$.*

Proof. Let $0 < t \le s < \infty$. Then, since $u \notin L^1$, there exists $\nu \in (0, 1]$ such that

$$t = \int_0^{b\nu} u(r)\, dr \le \int_0^b u(r)\, dr = s, \qquad \text{for some} \qquad b > 0.$$

By the hypothesis we obtain (4.5). So,

$$\frac{W(t)}{W(s)} \le C_0 \left(\log \frac{1}{\nu}\right)^{-p}. \tag{6.19}$$

Let $S = (-b\nu, b\nu)$ and $I = (-b, b)$. Since $u \in A_\infty$, we obtain that

$$\nu = \frac{|S|}{|I|} \le c \left(\frac{u(S)}{u(I)}\right)^{1/q} = c \left(\frac{t}{s}\right)^{1/q},$$

for some $q \ge 1$. Fix $r > q$, and let $\alpha = 1/q - 1/r > 0$. Then

$$\nu \le c \left(\frac{t}{s}\right)^{\alpha} \left(\frac{t}{s}\right)^{1/r}.$$

We consider the following two cases in order to estimate (6.19):

If $\left(\frac{t}{s}\right)^{\alpha} \leq \frac{1}{c}$, then

$$\frac{W(t)}{W(s)} \leq C_0 \left(\log \frac{1}{\nu}\right)^{-p} \leq C_0 \left(\frac{1}{r} \log \frac{s}{t}\right)^{-p} = C_0 r^p \left(\log \frac{s}{t}\right)^{-p}.$$

If $\frac{t}{s} > \left(\frac{1}{c}\right)^{1/\alpha} = k > 0$, then we choose C_1 such that

$$\frac{W(t)}{W(s)} \leq 1 \leq C_1 \left(\log \frac{1}{k}\right)^{-p} \leq C_1 \left(\log \frac{s}{t}\right)^{-p}.$$

Therefore, taking $C = \max\{C_0 r^p, C_1\}$ we get, for every $0 < t \leq s$,

$$\frac{W(t)}{W(s)} \leq C \left(\log \frac{s}{t}\right)^{-p},$$

that is equivalent to B^*_∞. □

Theorem 6.6. *Let $p > 0$. If $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, then $(u, w) \in AB^*_\infty$.*

Proof. The hypothesis implies that $u \in A_\infty$ by Theorem 6.4. Then, the B^*_∞ condition is a consequence of Proposition 6.5. Finally, since $w \in \Delta_2$, applying Proposition 3.46 we have that $(u, w) \in AB^*_\infty$. □

6.3 Necessity of the weak-type boundedness of M

The main result of this section is to show that the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$$

implies that of the Hardy-Littlewood maximal function on the same spaces, whenever $p > 1$, while for the case $p \leq 1$ an extra assumption is required. In particular, we give a different proof of the following classical result:

$$H : L^p(u) \to L^{p,\infty}(u) \quad \Rightarrow \quad M : L^p(u) \to L^{p,\infty}(u),$$

for $p > 1$, that does not pass through the A_p condition.

First, we present a slightly modified version of the Vitali covering lemma (see [8], [40]).

Notation: The letter I will denote an open interval of the real line, and αI the interval concentric with I but with side length $\alpha > 0$ times as large.

Lemma 6.7. *Let K be a compact set of the real line. Let $\mathcal{F}$ be a collection of open intervals that covers K. Then, there exist finitely many intervals, let say $I_1, \dots, I_n$ from $\mathcal{F}$ such that $101I_i$ are disjoint and*

$$K \subset \bigcup_{i=1}^{n} 303 I_i.$$

Proof. By the compactness of K, there exists a finite subcover of open intervals of $\mathcal{F}$. Hence, we can assume that $\mathcal{F}$ is finite. Consider the collection $\widetilde{\mathcal{F}}$ of the dilations $101I_j$ of $I_j \in \mathcal{F}$, and form the following subcollection $\widetilde{\mathcal{F}}_{\text{sub}}$: let $101I_1$ be the largest interval, of $\widetilde{\mathcal{F}}$. Let $101I_2$ be the largest disjoint than $101I_1$ open interval, let $101I_3$ be the largest disjoint than $101I_1$ and $101I_2$, open interval and so on. Since $\mathcal{F}$ is finite, and so is $\widetilde{\mathcal{F}}$, the process will end after let say n steps, yielding a collection of disjoint intervals $\widetilde{\mathcal{F}}_{\text{sub}} = \{101I_i\}_{i=1,\dots,n}$. Now, we will see that

$$K \subset \bigcup_{i=1}^{n} 303 I_i.$$

Assume that some interval, let say $101I_l$, has not been selected for the subcollection $\widetilde{\mathcal{F}}_{\text{sub}}$; that is, there exists $101I_m \in \widetilde{\mathcal{F}}_{\text{sub}}$ such that $101I_l \cap 101I_m \neq \varnothing$. By the construction of $\widetilde{\mathcal{F}}_{\text{sub}}$, $101I_l$ should be smaller than $101I_m$, and hence it will be contained in $303I_m$. Similarly we can show that the union of the non-selected intervals of $\widetilde{\mathcal{F}}$ is contained in the union of the triples of the selected ones. □

Theorem 6.8. *If any of the following conditions holds:*

(i) $p > 1$,

(ii) $p \le 1$ *and assume that* $\overline{W}^{1/p}(t) \not\approx t, \ t > 1$,

and if $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, *then* $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$.

Proof. (i) Assume that the Hilbert transform is bounded $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$, $p > 1$. Fix $\lambda > 0$ and consider $f \in \mathcal{C}^\infty_c$ non-negative. Let K be a compact set of $E_\lambda = \{x \in \mathbb{R} : Mf(x) > \lambda\}$. By the Calderón-Zygmund decomposition (see [36]) there exists a collection of open intervals $\{I_i\}$ such that their union covers K and

$$\lambda < \frac{1}{|I_i|}\int_{I_i} f \le 2\lambda. \tag{6.20}$$

By Lemma 6.7 there exist finitely many disjoint open intervals I_i from this collection, such that they are far away from each other, concretely $\{101I_i\}_i$ are pairwise disjoint and

$$K \subset \bigcup_{i}^{n} 303 I_i. \tag{6.21}$$

Fix $i \in \{1, \dots, n\}$. If $j_i \in \{-50, 50\}$, then I_{i,j_i} denotes the interval with the same Lebesgue measure as I_i, $|I_{i,j_i}| = |I_i|$, situated on the left hand-side of I_i, if $j_i \in \{-50, -1\}$ and on the right hand-side of I_i, if $j_i \in \{1, 50\}$ and such that $\mathrm{d}(I_i, I_{i,j_i}) = (|j_i| - 1)|I_i|$. For $j_i = 0$, both I_i and I_{i,j_i} coincide.

Claim: for each I_i there exists an interval I_{i,j_i} such that, if $x \in \cup_i I_{i,j_i}$ we get

$$|H(f\chi_{\cup_i I_i})(x)| \geq \frac{1}{5}\lambda. \tag{6.22}$$

Assume that the claim holds. Let $E = \cup_i I_{i,j_i}$, then by (6.22) and applying Hölder's inequality we get

$$\begin{aligned}
\lambda W^{1/p}(u(E)) &\lesssim W^{1/p}(u(E)) \frac{1}{u(E)} \int_E |H(f\chi_{\cup_i I_i})(x)| u(x) dx \\
&\lesssim W^{1/p}(u(E)) \frac{1}{u(E)} ||H(f\chi_{\cup_i I_i})||_{\Lambda^{p,\infty}_u(w)} ||\chi_E||_{(\Lambda^{p,\infty}_u(w))'} \\
&\lesssim ||f\chi_{\cup_i I_i}||_{\Lambda^p_u(w)} \leq ||f||_{\Lambda^p_u(w)},
\end{aligned}$$

where in the third inequality we have used the hypothesis of the boundedness of the Hilbert transform, and the fact that $w \in B_p$ (see Theorems 4.25 and 3.17), which, by Proposition 3.11 *(ii)* implies

$$||\chi_E||_{(\Lambda^{p,\infty}_u(w))'} \lesssim \frac{u(E)}{W^{1/p}(u(E))}. \tag{6.23}$$

Now, by Theorem 6.4, the boundedness of the Hilbert transform implies that $u \in A_\infty$, hence u is a doubling measure and then $u(I_i) \lesssim u(I_{i,j_i})$ for every $i = 1, 2, \dots, n$. Then, by (6.21) we have that

$$u(K) \lesssim \sum_i^n u(303 I_i) \lesssim \sum_{i=1}^n u(I_{i,j_i}).$$

Thus, since $w \in \Delta_2$ we obtain

$$W^{1/p}(u(K)) \lesssim W^{1/p}(u(E)).$$

Hence,

$$\lambda W^{1/p}(u(K)) \lesssim ||f||_{\Lambda^p_u(w)}.$$

Since this holds for all compact sets of E_λ, by Fatou's lemma we obtain that

$$\lambda W^{1/p}(u(E_\lambda)) \lesssim ||f||_{\Lambda^p_u(w)}.$$

Proof of the claim: Fix $1 \leq k \leq n$ and define

$$C_k(x) = \sum_{i=1}^{k-1} \int_{I_i} \frac{f(y)}{x-y} dy + \sum_{i=k+1}^{n} \int_{I_i} \frac{f(y)}{x-y} dy.$$

Let $A_k = \bigcup_{j_k=-50}^{0} I_{k,j_k}$, hence $A_k \subset 101 I_k$ and note that A_k and I_i are disjoint for all $i \in \{1,\dots,k-1,k+1,n\}$, since by construction, $\{101 I_i\}_{i=1,\dots,n}$ is a family of pairwise disjoint, open intervals.

We will prove that C_k is decreasing in A_k. Let $x_1, x_2 \in A_k$, such that $x_1 \leq x_2$. If $1 \leq i \leq k-1$ each interval I_i is situated on the left hand-side of A_k, and so $0 < x_1 - y \leq x_2 - y$, where $y \in I_i$. Hence,

$$\int_{I_i} \frac{f(y)}{x_1 - y} dy \geq \int_{I_i} \frac{f(y)}{x_2 - y} dy. \tag{6.24}$$

If now $k+1 \leq i \leq n$, each interval I_i is situated on the right hand-side of A_k, and so $y - x_1 \geq y - x_2 > 0$, where $y \in I_i$. Hence

$$\int_{I_i} \frac{f(y)}{y - x_1} dy \leq \int_{I_i} \frac{f(y)}{y - x_2} dy. \tag{6.25}$$

Therefore by (6.24) and (6.25) we obtain

$$\begin{aligned} C_k(x_1) &= \sum_{i=1}^{k-1} \int_{I_i} \frac{f(y)}{x_1 - y} dy + \sum_{i=k+1}^{n} \int_{I_i} \frac{f(y)}{x_1 - y} dy \\ &= \sum_{i=1}^{k-1} \int_{I_i} \frac{f(y)}{x_1 - y} dy - \sum_{i=k+1}^{n} \int_{I_i} \frac{f(y)}{y - x_1} dy \\ &\geq \sum_{i=1}^{k-1} \int_{I_i} \frac{f(y)}{x_2 - y} dy - \sum_{i=k+1}^{n} \int_{I_i} \frac{f(y)}{y - x_2} dy = C_k(x_2) \end{aligned}$$

If $k = 1$ we choose the left hand-side interval of I_1; that is $I_{1,-1}$. Hence, for $x \in I_{1,-1}$ we obtain

$$|H(f\chi_{\cup_i I_i})(x)| = \left| \sum_{i=1}^{n} \int_{I_i} \frac{f(y)}{x - y} dy \right| = \sum_{i=1}^{n} \int_{I_i} \frac{f(y)}{y - x} dy \geq \frac{1}{2|I_1|} \int_{I_1} f(y) dy \geq \frac{\lambda}{2}.$$

If $k = n$ we choose the right hand-side interval of I_n; that is $I_{n,1}$. Then, for $x \in I_{n,1}$ we get

$$|H(f\chi_{\cup_i I_i})(x)| = \sum_{i=1}^{n} \int_{I_i} \frac{f(y)}{x - y} dy \geq \frac{1}{2|I_n|} \int_{I_n} f(y) dy \geq \frac{\lambda}{2}.$$

If $1 < k < n$ fix $\alpha \in I_{k,-25}$. Then, we consider the following two cases and the election of I_{k,j_k} will vary depending on the value of $C_k(\alpha)$.

Case 1: If $C_k(\alpha) \leq \dfrac{\lambda}{4}$, then we choose $j_k = -1$, that corresponds to the interval $I_{k,-1} \subset A_k$, situated on the right hand-side of $I_{k,-25}$. So, for $x \in I_{k,-1}$ we get $C_k(x) < C_k(\alpha) \leq \frac{\lambda}{4}$ since C_k is decreasing. Moreover, if $x \in I_{k,-1}$ and $y \in I_k$ we obtain $0 < y - x \leq \frac{1}{2|I_k|}$, and hence by (6.20)

$$D_k(x) = \int_{I_k} \frac{f(y)}{y - x} dy \geq \frac{1}{2|I_k|} \int_{I_k} f(y) dy \geq \frac{\lambda}{2}.$$

Then, since $D_k(x) \geq C_k(x)$ we obtain

$$|H(f\chi_{\cup_i I_i})(x)| = |C_k(x) - D_k(x)| = D_k(x) - C_k(x) \geq \frac{\lambda}{2} - \frac{\lambda}{4} = \frac{\lambda}{4}.$$

Case 2: If $C_k(\alpha) > \dfrac{\lambda}{4}$, then we choose $j_k = -50$, that corresponds to the interval $I_{k,-50} \subset A_k$, situated on the left hand-side of $I_{k,-25}$. So, for $x \in I_{k,-50}$ we obtain $C_k(x) > C_k(\alpha) > \frac{\lambda}{4}$ since C_k is decreasing. In addition, if $x \in I_{k,-50}$ and $y \in I_k$ we obtain $0 < y - x \leq \frac{1}{49|I_k|}$, hence by (6.20)

$$D_k(x) = \int_{I_k} \frac{f(y)}{y-x} dy \leq \frac{1}{49|I_k|} \int_{I_k} f(y)dy \leq \frac{2\lambda}{49} < \frac{\lambda}{20}.$$

Then, since $D_k(x) \leq C_k(x)$

$$|H(f\chi_{\cup_i I_i})(x)| = |C_k(x) - D_k(x)| = C_k(x) - D_k(x) \geq \frac{\lambda}{4} - \frac{\lambda}{20} = \frac{\lambda}{5}.$$

Therefore, if $x \in \cup_i^n I_{i,j_i}$, with the intervals I_{i,j_i} chosen as before, we have proved that

$$|H(f\chi_{\cup_i I_i})(x)| \geq \frac{\lambda}{5}.$$

(ii) The proof is similar to that of (i). The only difference is the way we obtain the relation (6.23). In this case, $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ implies that $w \in B_{p,\infty}$, which taking into account the assumption and Proposition 3.19, it also implies the condition $w \in B_p$. Hence, applying Proposition 3.11 we obtain (6.23). □

6.4 Sufficient conditions

In this section, we will show that the boundedness of the Hardy-Littlewood maximal function on weighted Lorentz spaces, together with the AB^*_∞ condition, are sufficient for the boundedness of the Hilbert transform on the same spaces. In fact, we shall prove something stronger since those conditions will imply the boundedness of the Hilbert maximal operator

$$H^*f(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \frac{f(x-y)}{y} dy \right|.$$

In 1974, Coifman and Fefferman proved in [26] the so-called good-λ inequality, that relates the Hardy-Littlewood maximal function and H^* in the following way:

$$u(\{x \in \mathbb{R} : H^*f(x) > 2\lambda \text{ and } Mf(x) \leq \gamma\lambda\}) \leq C(\gamma)u(\{x \in \mathbb{R} : H^*f(x) > \lambda\}), \tag{6.26}$$

provided $u \in A_\infty$ (see also [41]). In the classical theory of weighted Lebesgue spaces, the good-λ inequality has been used to prove that the boundedness of the Hardy-Littlewood maximal function implies that of H^* on the same spaces. Although, we could apply (6.26) to obtain sufficient conditions for H^* to be bounded on $\Lambda^p_u(w)$, with some extra condition on w, we will use a somehow different approach proved in [6] by Bagby and Kurtz. In fact, they replaced (6.26) by the following rearrangement inequality: if $u \in A_\infty$, then for every $t > 0$, we have that

$$(H^*f)^*_u(t) \leq C(Mf)^*_u(t/2) + (H^*f)^*_u(2t). \tag{6.27}$$

Iterating (6.27) the following result, involving the adjoint of the Hardy operator, is obtained. For a non-weighted version of this inequality see also [7].

Theorem 6.9. *Let $u \in A_\infty$. Then,*

$$(H^*f)^*_u(t) \lesssim \Big(Q\,(Mf)^*_u\Big)(t/4), \tag{6.28}$$

for all $t > 0$, whenever the right hand side is finite.

As a consequence we obtain the following result:

Theorem 6.10. *Let $0 < p < \infty$.*

(i) If $(u, w) \in AB^*_\infty$ and $w \in B_{p,\infty}(u)$ then, $H^* : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$.

(ii) If $(u, w) \in AB^*_\infty$ and $w \in B_p(u)$ then, $H^* : \Lambda^p_u(w) \to \Lambda^p_u(w)$.

Proof. (i) If $w \in B_{p,\infty}(u)$ we have, by Theorem 3.41, that

$$(Mf)^*_u(s) \lesssim \left(\frac{1}{W(s)}\int_0^s (f^*_u(r))^p w(r)dr\right)^{1/p},$$

(see [20] for further details). Therefore, we obtain that

$$\begin{aligned}\int_t^\infty (Mf)^*_u(s)\frac{ds}{s} &\lesssim \int_t^\infty \left(\frac{1}{W(s)}\int_0^s (f^*_u(r))^p w(r)dr\right)^{1/p}\frac{ds}{s}\\ &\lesssim ||f||_{\Lambda^p_u(w)}\int_t^\infty \frac{1}{W^{1/p}(s)}\frac{ds}{s} \lesssim ||f||_{\Lambda^p_u(w)}\frac{1}{W^{1/p}(t)} < \infty,\end{aligned}$$

where the last inequality is a consequence of Theorems 3.24 and 3.26.

Then, by Theorem 6.9,

$$\begin{aligned}\sup_{t>0} W^{1/p}(t)(H^*f)^*_u(t) &\lesssim \sup_{t>0} W^{1/p}(t)\left(Q(Mf)^*_u\right)(t/4) \lesssim \sup_{t>0} W^{1/p}(t)\left(Q(Mf)^*_u\right)(t)\\ &\lesssim \sup_{t>0} W^{1/p}(t)(Mf)^*_u(t) \lesssim \left(\int_0^\infty (f^*_u(s))^p w(s)\,ds\right)^{1/p},\end{aligned}$$

where the third inequality follows by the B^*_∞ condition, in view of Theorem 3.26, and the last step is a consequence of $B_{p,\infty}(u)$.

(ii) With a similar proof we obtain that, in this case,

$$\begin{aligned}||H^*f||^p_{\Lambda^p_u(w)} &\lesssim \int_0^\infty \left(Q(Mf)^*_u\right)^p(t/4)w(t)\,dt \lesssim \int_0^\infty \left(Q(Mf)^*_u\right)^p(t)w(t)\,dt\\ &\lesssim \int_0^\infty \left((Mf)^*_u\right)^p(t)w(t)\,dt \lesssim \int_0^\infty (f^*_u(t))^p w(t)\,dt,\end{aligned}$$

where in the third inequality we have used the B^*_∞ condition, taking into account Theorem 3.23, and the last step follows by the strong-type boundedness of the Hardy-Littlewood maximal function on weighted Lorentz spaces, that is characterized by $B_p(u)$ in Theorem 3.32. □

Using the standard techniques we obtain the following result:

Corollary 6.11. *Under the hypotheses of Theorem 6.10, it holds that, for every $f \in \Lambda^p_u(w)$, there exists*

$$\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy, \qquad a.e.\ x\in\mathbb{R}.$$

Moreover,

(i) if $(u,w)\in AB^*_\infty$ and $w\in B_{p,\infty}(u)$ then, $H:\Lambda^p_u(w)\to\Lambda^{p,\infty}_u(w)$;

(ii) if $(u,w)\in AB^*_\infty$ and $w\in B_p(u)$ then, $H:\Lambda^p_u(w)\to\Lambda^p_u(w)$.

Remark 6.12. Under the hypothesis (i) of Theorem 6.10 we obtain $H:\Lambda^p_u(w)\to\Lambda^{p,\infty}_u(w)$. Recall that H is well-defined in $\mathcal{C}^\infty_c$, and can be extended to $\overline{H}$ on $\Lambda^p_u(w)$, by continuity (see Section 4.3 for more details). We will see that for every $f\in\Lambda^p_u(w)$

$$\overline{H}f(x) = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy, \qquad a.e.\ x\in\mathbb{R},$$

where the limit in the right-hand side exists by Corollary 6.11. Indeed, we have that for every $f\in\Lambda^p_u(w)$ there exists $f_n\in\mathcal{C}^\infty_c$ such that

$$\lim_{n\to\infty}||\overline{H}f - Hf_n||_{\Lambda^{p,\infty}_u(w)} = 0,$$

and so there is a partial f_{n_k} such that

$$\lim_{k\to\infty}|\overline{H}f(x) - Hf_{n_k}(x)| = 0, \qquad a.e.\ x\in\mathbb{R}. \tag{6.29}$$

On the other hand, we have that

$$||H^*(f-f_{n_k})||_{\Lambda^{p,\infty}_u(w)} \lesssim ||f-f_{n_k}||_{\Lambda^p_u(w)},$$

and so there exists a partial, which is denoted again by f_{n_k}, such that

$$\lim_{k\to\infty} H^*(f - f_{n_k})(x) = 0, \qquad a.e.\ x \in \mathbb{R}. \tag{6.30}$$

Fix $x \in \mathbb{R}$, satisfying (6.29) and (6.30). Hence, we have that for every $\eta > 0$ there exists $k > 0$ such that

$$|\overline{H}f(x) - Hf_{n_k}(x)| < \frac{\eta}{3} \quad \text{and} \quad H^*(f - f_{n_k})(x) < \frac{\eta}{3}.$$

For this k, since $f_{n_k} \in \mathcal{C}_c^\infty$ we have that

$$\lim_{\varepsilon\to 0} |Hf_{n_k}(x) - H_\varepsilon f_{n_k}(x)| = 0. \tag{6.31}$$

Therefore, for every $\eta > 0$ there exists $\delta > 0$ such that for every $\varepsilon \in (0, \delta)$

$$|Hf_{n_k}(x) - H_\varepsilon f_{n_k}(x)| < \frac{\eta}{3}.$$

Hence, for those $x \in \mathbb{R}$ satisfying (6.29) and (6.30), we have that for every $\eta > 0$, there exists $\delta > 0$ such that for every $\varepsilon \in (0, \delta)$

$$\begin{aligned}|\overline{H}f(x) - H_\varepsilon f(x)| &\le |\overline{H}f(x) - Hf_{n_k}(x)| + |Hf_{n_k}(x) - H_\varepsilon f_{n_k}(x)| + |H_\varepsilon f_{n_k}(x) - H_\varepsilon f(x)| \\ &= \frac{\eta}{3} + \frac{\eta}{3} + |H_\varepsilon(f_{n_k} - f)(x)| \le \frac{2\eta}{3} + H^*(f_{n_k} - f)(x) \le \frac{2\eta}{3} + \frac{\eta}{3} = \eta,\end{aligned}$$

and so

$$\lim_{\varepsilon\to 0} H_\varepsilon f(x) = \overline{H}f(x).$$

6.5 Complete characterization

In this section, we will present our main results, namely, the complete characterization of the weak-type and strong-type boundedness of the Hilbert transform on weighted Lorentz spaces, whenever $p > 1$, whereas the case $p \le 1$ is solved under an extra assumption.

Although the following result characterizes totally the weak-type boundedness of the Hilbert transform on weighted Lorentz spaces, one of the required conditions, namely, the boundedness of the Hardy-Littlewood maximal on weighted Lorentz spaces

$$M : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w), \tag{6.32}$$

remains open for $p \ge 1$. For this reason, in the next subsection, we characterize (6.32) by the $B_p(u)$ condition, whenever $p > 1$. Then, since this is also a solution to the corresponding strong-type version,

$$M : \Lambda_u^p(w) \to \Lambda_u^p(w), \tag{6.33}$$

we give the solution to the strong-type boundedness of H on $\Lambda_u^p(w)$.

Theorem 6.13. *If any of the following conditions holds:*

(i) $p > 1$,

(ii) $p \le 1$ *and assume that* $\overline{W}^{1/p}(t) \not\approx t,\ t > 1$,

then, $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ *if and only if* $(u, w) \in AB^*_\infty$ *and* $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$.

Proof. The sufficiency is given by Corollary 6.11. On the other hand, if $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ is bounded then by Theorem 6.6 we have that $(u, w) \in AB^*_\infty$. Finally, the necessity of the boundedness of the Hardy-Littlewood maximal function is given by Theorem 6.8. □

Remark 6.14. Theorem 6.13 asserts that in particular if $p \le 1$, then under the condition $\overline{W}^{1/p}(t) \not\approx t$ we have that

$$H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \quad \Leftrightarrow \quad (u, w) \in AB^*_\infty \ \text{ and } \ M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w).$$

However, the assumption on the weight w is not necessary in general. For more details see Remark 7.25 (ii) and Theorem 7.24 of Chapter 7.

6.5.1 Geometric conditions

We prove that the weak-type boundedness of the Hardy-Littlewood maximal function

$$M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w),$$

implies the strong-type boundedness of the same operator, whenever $p > 1$ and hence, we get the equivalence between the $B_{p,\infty}(u)$ and $B_p(u)$ conditions. In particular, we recover the following classical result,

$$M : L^p(u) \to L^{p,\infty}(u) \quad \text{implies that} \quad M : L^p(u) \to L^p(u),$$

for all $p > 1$, without using the reverse Hölder's inequality (see [36], [41]).

Furthermore, the equivalence between $B_p(u)$ and $B_{p,\infty}(u)$ allows us to complete the characterization of the strong-type boundedness of the Hilbert transform on weighted Lorentz spaces. We also give some partial results for the case $p \le 1$.

We first need the following technical results:

Lemma 6.15. *Let E be a subset of an interval I, which is a union of pairwise disjoint, open intervals $E = \cup_{k=1}^N E_k$. Then, there exists a function $F_{I,E}$ supported on I, with values in $[0, 1]$ such that for every $\lambda \in [|E|/|I|, 1]$ the set*

$$J_\lambda = \{x : F_{I,E}(x) > \lambda\},$$

can be expressed as union of pairwise disjoint, open intervals $J_{\lambda,k}$,

$$J_\lambda = \bigcup_{k=1}^{N'} J_{\lambda,k}, \quad N' \le N,$$

and

$$|E \cap J_{k,\lambda}| = \lambda |J_{k,\lambda}|, \quad \textit{for every } k. \tag{6.34}$$

Proof. Let $I = (a,b)$, $E = (c,d)$ such that $E \subset I$ and $a \le c \le d \le b$. Let $e \in \mathbb{R}$ be such that $\frac{b-e}{d-e} = \frac{e-a}{e-c} = \frac{|I|}{|S|}$. Define the function $f_{I,E}$ as follows

$$f_{E,I}(x) = \begin{cases} \dfrac{e-x}{e-c}, & \text{if } x \in (a,c], \\ 1, & \text{if } x \in [c,d], \\ \dfrac{x-e}{d-e}, & \text{if } x \in [d,b). \end{cases}$$

Then by construction and Thalis' theorem we get that for every $t \in [1, |I|/|E|]$ we have that

$$|\{x \in \mathbb{R} : f_{I,E}(x) < t\}| = t|E|. \tag{6.35}$$

Let $F_{I,E}(x) = 1/f_{E,I}(x)$ for every $x \in I$ and 0 elsewhere. Then, by (6.35) we have that for every $\lambda \in [|E|/|I|, 1]$

$$\lambda |\{x \in \mathbb{R} : F_{I,E}(x) > \lambda\}| = |E|,$$

and so the property (6.34) holds for $N = 1$.

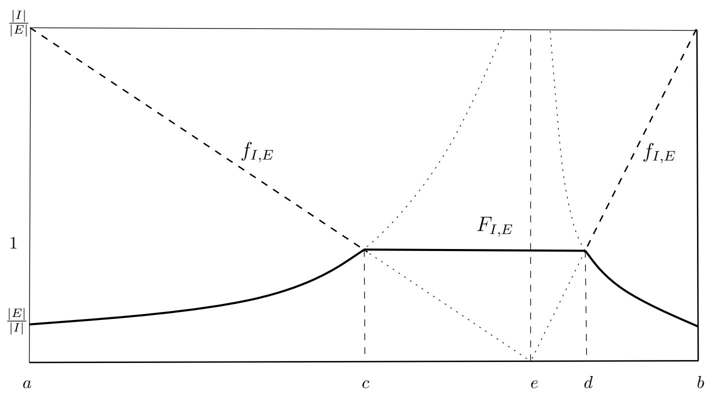

Figure 6.1: Function $F_{I,E}$

We will use induction to prove the general result. Assume that there exists a function $G_{I,S}$ which satisfies (6.34) for $S = \bigcup_{k=1}^{N} S_k$ and $N \le n$, where S_k are pairwise disjoint, open intervals. We will prove that if a set E has $n+1$ intervals; that is $E = \bigcup_{k=1}^{n+1} E_k$, there exists a function satisfying (6.34). For all $k \leq n+1$, let T_k be open intervals such that

$$E_k \subset T_k \subset I \quad \text{and} \quad \frac{|E_k|}{|T_k|} = \frac{|E|}{|I|}.$$

Case I: If T_j are pairwise disjoint then

$$F_{I,E}(x) = \max\left\{\frac{|E|}{|I|}, F_{T_1,E_1}(x), \dots, F_{T_{n+1},E_{n+1}}(x)\right\}, \tag{6.36}$$

which implies that

$$\{x \in I : F_{I,E}(x) > \lambda\} = \bigcup_{k=1}^{n+1} \{x \in I : F_{T_k,E_k}(x) > \lambda\} = \bigcup_{k=1}^{n+1} J_{\lambda,k} = J_\lambda,$$

and the property (6.34) holds for each $J_{\lambda,k}$ as in the case of one interval.

Case II: If at least two consecutive T_j's are not disjoint, let say T_j and T_{j+1}, then the corresponding functions F_{T_j,E_j} and $F_{T_{j+1},E_{j+1}}$ intersect at some point. Let λ_0 be the supremum of such points and let

$$E_{\lambda_0} = \{x : F_{I,E}(x) > \lambda_0\} = \bigcup_{k=1}^{N'} E_{\lambda_0,k},$$

where $N' \leq n$. Then the function satisfying (6.34) is:

$$\overline{F}_{I,E}(x) = \begin{cases} F_{I,E}(x) & \text{if } x \in E_{\lambda_0} \\ \lambda_0 G_{I,E_{\lambda_0}}(x) & \text{if } x \notin E_{\lambda_0} \\ 0 & \text{elsewhere,} \end{cases}$$

where $G_{I,E_{\lambda_0}}$ is obtained by the inductive hypothesis.

To see (6.34), observe that on the one hand, for the values between λ_0 and 1, the functions F_{T_j,E_j} do not intersect for any j, since λ_0 is the supremum of the intersecting values. Hence $\overline{F}$ satisfies (6.34).

On the other hand, if $\lambda \in [|E|/|I|, \lambda_0]$, we have that

$$J_\lambda = \{x : \overline{F}_{I,E}(x) > \lambda\} = \{x : \lambda_0 G_{I,E_{\lambda_0}}(x) > \lambda\} = J'_\lambda.$$

Indeed, if $x \notin E_{\lambda_0}$ then by definition of $\bar{F}_{I,E}$ we have that $x \in J_\lambda$ if and only if $x \in J'_\lambda$. Clearly, we have that $x \in E_{\lambda_0} \cap J_\lambda = E_{\lambda_0} \subset J'_\lambda$, and $x \in E_{\lambda_0} \cap J'_\lambda = E_{\lambda_0} \subset J_\lambda$ as well. Now, since $\lambda_0|E_{\lambda_0}| = |E|$ we have that $\lambda/\lambda_0 \in [|E_{\lambda_0}|/|I|, 1]$, which by the inductive hypothesis implies that

$$\left\{x : G_{I,E_{\lambda_0}}(x) > \frac{\lambda}{\lambda_0}\right\} = J_\lambda = \bigcup_{k=1}^{K} J_{\lambda,k} \quad \text{and} \quad \frac{|J_{\lambda,k} \cap E_{\lambda_0}|}{|J_{\lambda,k}|} = \frac{\lambda}{\lambda_0}, \tag{6.37}$$

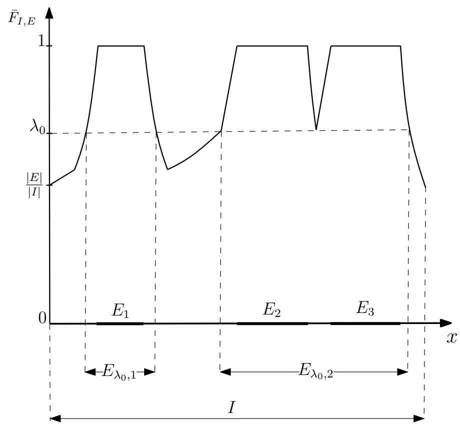

Figure 6.2: Function $\bar{F}_{I,E}$

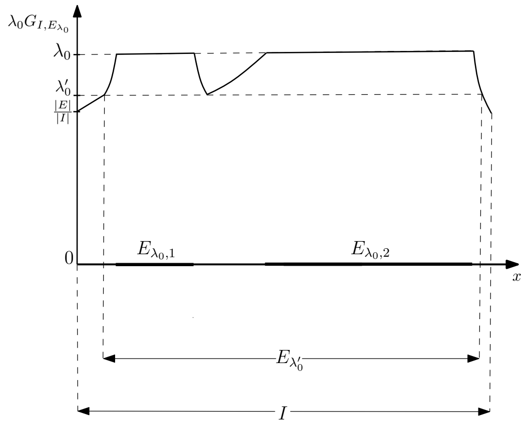

Figure 6.3: Function $\lambda_0 G_{I,E_{\lambda_0}}$

with $K \leq N'$ and $N' \leq n$. Now we will show that for every $\lambda \in [|E|/|I|, 1]$

$$\frac{|J_{\lambda,k} \cap E|}{|J_{\lambda,k}|} = \lambda, \qquad \text{for every} \qquad k \leq K.$$

Observe that the set E_{λ_0} can be expressed as a union of pairwise disjoint open intervals

$$E_{\lambda_0} = \bigcup_{i=1}^{M} E_{\lambda_0,i}, \qquad \text{for} \qquad M \leq N',$$

and we also have that

$$\lambda_0 |E_{\lambda_0,i}| = |E \cap E_{\lambda_0,i}| \tag{6.38}$$

for every $i \in \Lambda = \{1, 2, \ldots, M\}$. Fix $k \leq K$. Then the set $J_{\lambda,k}$, which will be also union of intervals and will contain some of the intervals $E_{\lambda,i}$, let's say $\bigcup_{t\in\Lambda_k} E_{\lambda,t}$, where $\Lambda_k \subset \Lambda$. Then,

$$|J_{\lambda,k} \cap E_{\lambda_0}| = |\bigcup_{t\in\Lambda_k} E_{\lambda_0,t}|.$$

Since $E \subset E_{\lambda_0}$ we have that $J_{\lambda,k} \cap E = (J_{\lambda,k} \cap E_{\lambda_0}) \cap E = (\cup_{t\in\Lambda_k} E_{\lambda_0,t}) \cap E$. Hence we get by (6.38)

$$\frac{|J_{\lambda,k} \cap E|}{|J_{\lambda,k} \cap E_{\lambda_0}|} = \frac{|(\cup_{t\in\Lambda_k} E_{\lambda_0,t}) \cap E|}{|\cup_{t\in\Lambda_k} E_{\lambda,t}|} = \lambda_0.$$

Finally, by (6.37) we have that

$$\frac{|J_{\lambda,k} \cap E|}{|J_{\lambda,k}|} = \frac{|J_{\lambda,k} \cap E|}{|J_{\lambda,k}|} \frac{|J_{\lambda,k} \cap E_{\lambda_0}|}{|J_{\lambda,k} \cap E_{\lambda_0}|} = \lambda.$$

□

Lemma 6.16. *Let E and I be as in Lemma 6.15. If $s=\dfrac{|I|}{|E|}$, then*

$$\frac{1}{|I|}\int_I F_{I,E}(x)dx=\frac{1+\log s}{s}.$$

Proof. Note that

$$|\{x: F_{I,E}(x)>\lambda\}|=\begin{cases}|I|, & \text{if } \lambda\in(0,1/s),\\ |E|/\lambda, & \text{if } \lambda\in[1/s,1),\\ 0, & \text{if } \lambda\geq 1.\end{cases}$$

Then,

$$\frac{1}{|I|}\int_I F_{I,E}(x)dx=\frac{1}{|I|}\int_0^\infty |\{x: F_{I,E}(x)>\lambda\}|d\lambda=\frac{1+\log s}{s}.$$

□

The following proof is inspired by the fact that $B_{p,\infty}$ implies B_p, for $p>1$, proved by Neugebauer in [79].

Theorem 6.17. *If $p>1$, then $M:\Lambda^p_u(w)\to\Lambda^{p,\infty}_u(w)$ implies that $M:\Lambda^p_u(w)\to\Lambda^p_u(w)$. In particular, $B_p(u)=B_{p,\infty}(u)$.*

Proof. Let $(I_j)_{j=1}^J$ be a finite family of pairwise disjoint, open intervals, and let $(E_j)_{j=1}^J$ be such that $E_j\subseteq I_j$, E_j is a finite union of disjoint, closed intervals and $\dfrac{|I_j|}{|E_j|}=s$ for every j. Let $f:\mathbb{R}\to[0,1]$ be as follows

$$f(x)=\sum_{j=1}^J F_{I_j,E_j}(x). \tag{6.39}$$

By the weak-type boundedness of M we get for all $t>0$

$$W\big(u(\{x\in\mathbb{R}:\ Mf(x)>t\})\big)\lesssim\frac{1}{t^p}||f||^p_{\Lambda^p_u(w)}. \tag{6.40}$$

On the one hand

$$\begin{aligned}||f||^p_{\Lambda^p_u(w)}&=\int_0^\infty p\lambda^{p-1}W\big(u(\{x:\ f(x)>\lambda\})\big)d\lambda\\&\leq\int_0^{1/s}p\lambda^{p-1}W\big(u(\{x:\ f(x)\geq\lambda\})\big)d\lambda+\int_{1/s}^1 p\lambda^{p-1}W\big(u(\{x:\ f(x)\geq\lambda\})\big)d\lambda\\&=(I)+(II).\end{aligned}$$

Using that

$$\frac{W\left(u\left(\bigcup_{j=1}^{J} I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^{J} E_j\right)\right)} \lesssim \max_{1\le j\le J}\left(\frac{|I_j|}{|E_j|}\right)^p \approx s^p,$$

we obtain

$$(I) \lesssim \int_0^{1/s} p\lambda^{p-1}s^p W\big(u(\cup_{j=1}^J E_j)\big)d\lambda = W\big(u(\cup_{j=1}^J E_j)\big).$$

Now we estimate (II). By Lemma 6.15, if $\lambda \in (1/s,1)$ and $E = \cup_j E_j$, then the set $J_\lambda = \{x : \ f(x) \ge \lambda\}$ is the union of disjoint intervals $J_{\lambda,k}$ such that

$$\frac{|J_{\lambda,k}|}{|E\cap J_{\lambda,k}|} = \frac{1}{\lambda} \quad \forall k,$$

and $E \subseteq J_\lambda$. Therefore

$$\frac{W\left(u\left(\bigcup_k J_{\lambda,k}\right)\right)}{W\left(u\left(E\right)\right)} = \frac{W\left(u\left(\bigcup_k J_{\lambda,k}\right)\right)}{W\left(u\left(\bigcup_k E\cap J_{\lambda,k}\right)\right)} \lesssim \max_k\left(\frac{|J_{\lambda,k}|}{|E\cap J_{\lambda,k}|}\right)^p = \lambda^{-p}.$$

Hence

$$\begin{aligned}(II) &\lesssim \int_{1/s}^1 p\lambda^{p-1}\lambda^{-p} W\left(u\left(E\right)\right)d\lambda = p(1+\log s)W\left(u\left(E\right)\right)\\ &= p(1+\log s)W\left(u\left(\cup_{j=1}^J E_j\right)\right).\end{aligned}$$

So, we have that

$$||f||^p_{\Lambda^p_u(w)} \lesssim (1+\log s)W(u\left(\cup_{j=1}^J E_j\right)). \tag{6.41}$$

On the other hand, for every j

$$I_j \subseteq \left\{x\in\mathbb{R} : \ Mf(x) > \frac{1}{2|I_j|}\int_{I_j} f(y)dy\right\},$$

and by Lemma 6.16

$$\frac{1}{|I_j|}\int_{I_j} f(x)dx = \frac{1}{|I_j|}\int_{I_j} F_{I_j,E_j}(x)dx = \frac{1+\log s}{s},$$

for every j. Hence,

$$W\left(u\left(\cup_{j=1}^J I_j\right)\right) \le W\left(u\left(\{x\in\mathbb{R} : \ Mf(x) > (1+\log s)/2s\}\right)\right). \tag{6.42}$$

Finally, if we fix $t = (1+\log s)/2s$ in (6.40), and combine (6.41) and (6.42) we obtain

$$\frac{W\left(u\left(\bigcup_{j=1}^{J} I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^{J} E_j\right)\right)} \lesssim (1+\log s)^{1-p}s^p.$$

Then, taking supremum, by Remark 3.35, we get

$$\overline{W_u}(s) \lesssim (1+\log s)^{1-p} s^p.$$

If we choose s big enough, we have that $w \in B_p(u)$ by (i) of Corollary 3.38, and hence $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$.

On the other hand, $B_p(u)$ implies obviously $B_{p,\infty}(u)$ and so we have the equality between the two classes of weights. $\square$

Now, we deduce the strong-type characterization of the Hilbert transform on weighted Lorentz spaces.

Theorem 6.18. *Let $p > 1$. Then,*

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w) \quad \textit{if and only if} \quad (u,w) \in AB^*_\infty \textit{ and } M : \Lambda^p_u(w) \to \Lambda^p_u(w).$$

Proof. The sufficiency follows by Corollary 6.11. The necessity of the AB^*_∞ condition and the strong-type boundedness of the Hardy-Littlewood maximal function is a consequence of Theorem 6.13, taking into account Theorem 6.17. $\square$

In the following theorem, we summarize our main results, giving the complete characterization of the strong-type and weak-type boundedness of the Hilbert transform on weighted Lorentz spaces, for $p > 1$. In particular, it recovers the classical cases $w = 1$ and $u = 1$.

Theorem 6.19. *The following statements are equivalent for $p > 1$:*

(i) $H : \Lambda^p_u(w) \to \Lambda^p_u(w)$ *is bounded.*

(ii) $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ *is bounded.*

(iii) $(u,w) \in AB^*_\infty$ *and* $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$ *is bounded.*

(iv) $(u,w) \in AB^*_\infty$ *and* $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ *is bounded.*

(v) *There exists $\varepsilon > 0$, such that for every finite family of pairwise disjoint open intervals $(I_j)_{j=1}^J$, and every family of measurable sets $(S_j)_{j=1}^J$, with $S_j \subset I_j$, for every $j \in J$ it holds that:*

$$\min_j \left(\log \frac{|I_j|}{|S_j|} \right) \lesssim \frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \lesssim \max_j \left(\frac{|I_j|}{|S_j|}\right)^{p-\varepsilon}.$$

Proof. The equivalences $(i) \Leftrightarrow (iii)$ and $(ii) \Leftrightarrow (iv)$ are Theorems 6.18 and 6.13, respectively. The equivalence $(iii) \Leftrightarrow (iv)$ follows by Theorem 6.17. Finally, we have that $(iii) \Leftrightarrow (v)$, since the left hand-side of (v) is equivalent to the AB^*_∞ condition by Corollary 3.50, and the right hand-side of (v) is just the $B_p(u)$ condition that characterizes both the boundedness $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$ by Theorem 3.32 and also $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ by Theorem 6.17. $\square$

If we consider the boundedness

$$H : L^p(u) \to L^{p,\infty}(u),$$

then as we have already pointed out in the Introduction, there are no weights such that the above boundedness holds for $p \leq 1$, whereas the boundedness

$$H : \Lambda^p(w) \to \Lambda^{p,\infty}(w),$$

is characterized by the $B_p \cap B^*_\infty$ class of weights. Next theorem summarizes the partial results obtained for the weak-type boundedness of H on the weighted Lorentz spaces and in the following remark we discuss the strong-type boundedness of H on the weighted Lorentz spaces for $p \leq 1$.

Theorem 6.20. *Assume that $\overline{W}^{1/p}(t) \not\approx t$ for all $t > 1$. Then, the following statements are equivalent for all $p < 1$:*

(i) $H : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ *is bounded.*

(ii) $(u, w) \in AB^*_\infty$ *and* $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$.

(iii) For every finite family of pairwise disjoint, open intervals $(I_j)_{j=1}^J$, every family of measurable sets $(S_j)_{j=1}^J$, with $S_j \subset I_j$ and for every $j \in J$ it holds that:

$$\min_j \left(\log \frac{|I_j|}{|S_j|} \right) \lesssim \frac{W\left(u\left(\bigcup_{j=1}^J I_j\right)\right)}{W\left(u\left(\bigcup_{j=1}^J S_j\right)\right)} \lesssim \max_j \left(\frac{|I_j|}{|S_j|} \right)^p.$$

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is given in Theorem 6.13. The left hand-side inequality in (iii) is equivalent to the AB^*_∞ condition by Corollary 3.50 and the left hand-side estimate in (iii) characterizes the boundedness $M : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w)$ (see Remark 3.43). Hence, we get the equivalence $(ii) \Leftrightarrow (iii)$. $\square$

Remark 6.21. (i) The first two statements of Theorem 6.20 hold also for $p = 1$.

(ii) We know that the conditions $(u,w) \in AB^*_\infty$ and $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$ are sufficient for the strong-type boundedness

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w)$$

by Theorem 6.10 and also the condition AB^*_∞ is necessary by Theorem 6.6. Nevertheless, we do not know if the boundedness $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$ is also necessary for $p \leq 1$.

(iii) Observe that Theorem 6.19 also holds for H^*. Namely,

$$H^* : \Lambda^p_u(w) \to \Lambda^p_u(w) \iff (u,w) \in AB^*_\infty \text{ and } w \in B_p(u), \;\; p > 1,$$

where the sufficiency follows by Theorem 6.10 and the necessity by Fatou's lemma and Theorem 6.19. Similarly, we obtain that

$$H^* : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \iff (u,w) \in AB^*_\infty \text{ and } w \in B_{p,\infty}(u), \;\; p > 1,$$

and finally, under the assumption $\overline{W}^{1/p}(t) \not\approx t, \; t > 1$, and taking into account Theorem 6.20 we have that

$$H^* : \Lambda^p_u(w) \to \Lambda^{p,\infty}_u(w) \iff (u,w) \in AB^*_\infty \text{ and } w \in B_p(u), \;\; p \leq 1.$$

6.6 Remarks on the Lorentz-Shimogaki and Boyd theorems

It is well-known that the boundedness of the Hardy-Littlewood maximal function and the Hilbert transform on rearrangement invariant function spaces have been characterized in terms of the so-called Boyd indices, leading to Lorentz-Shimogaki and Boyd theorems (see [69], [91], and [12]), that will be presented later on. The aim of this section is to present a reformulation of our results in the context of the Boyd indices. Although, we have already characterized the strong-type boundedness of H on $\Lambda^p_u(w)$ in the previous sections, whenever $p > 1$, this new approach provides an extension of Boyd theorem for weighted Lorentz spaces, that are not necessarily rearrangement-invariant.

Given any function $f \in \mathcal{M}(\mathbb{R}^+)$, the dilation operator is defined by

$$E_t f(s) = f(st) \;\; 0 < s < \infty.$$

Let X be a rearrangement invariant Banach space and let $h_X(t)$ denote the operator norm of E_t from $\bar{X}$ to $\bar{X}$,

$$h_X(t) = \sup_{||f||_{\bar{X}} \leq 1} ||E_t f||_{\bar{X}}, \;\; t > 0,$$

where $\bar{X}$ is the corresponding rearrangement Banach invariant space over $(0,\infty)$ such that $||f||_X = ||f^*||_{\bar{X}}$, in view of the Luxemburg representation theorem (for more details see [8]).

The upper (resp. lower) Boyd indices introduced by Boyd in a series of papers [11], [12], [13], [14], and [15], are given by:

$$\alpha_X := \inf_{0<t<1} \frac{\log h_X(t)}{\log 1/t} = \lim_{t\to 0^+} \frac{\log h_X(t)}{\log 1/t}, \tag{6.43}$$

and

$$\beta_X := \sup_{1<t<\infty} \frac{\log h_X(t)}{\log 1/t} = \lim_{t\to \infty} \frac{\log h_X(t)}{\log 1/t}, \tag{6.44}$$

respectively. The equality is proved by Hille and Phillips (see [50]). If X denotes a Banach rearrangement invariant space, then Lorentz and Shimogaki proved independently the following result:

Theorem 6.22. *([69], [91]) It holds that*

$$M : X \to X \quad \Leftrightarrow \quad \alpha_X < 1. \tag{6.45}$$

The boundedness of the Hilbert transform on X requires one more condition in terms of the lower index and the characterization is given by the Boyd theorem, which states the following:

Theorem 6.23. *[12] It holds that*

$$H : X \to X \quad \Leftrightarrow \quad 0 < \beta_X \le \alpha_X < 1. \tag{6.46}$$

There exists a generalization of this result for quasi-Banach function spaces (see [70]). Now, if $X = \Lambda^p(w)$ and w is a decreasing function, Boyd proved in [15] that

$$h_{\Lambda^p(w)}(t) = \left(\sup_{r>0} \frac{W(r)}{W(rt)}\right)^{1/p}, \quad t > 0. \tag{6.47}$$

Hence, in this case

$$\alpha_{\Lambda^p(w)} = \lim_{\mu\to\infty} \frac{\log \overline{W}(\mu)}{\log \mu^p}, \tag{6.48}$$

and

$$\beta_{\Lambda^p(w)} = \lim_{\lambda\to 0} \frac{\log \overline{W}(\lambda)}{\log \lambda^p}, \tag{6.49}$$

for $p \ge 1$.

Recently, Lerner and Pérez generalized in [66] the Lorentz-Shimogaki theorem for every quasi-Banach function space, not necessarily rearrangement invariant. For this reason, they defined a generalized Boyd index, which in the particular case of the weighted Lorentz spaces is given in terms of the expression $\alpha_{\Lambda^p_u(w)}$ defined below. Analogously to the index $\alpha_{\Lambda^p_u(w)}$, we define the generalized lower Boyd index $\beta_{\Lambda^p_u(w)}$ in the context of the weighted Lorentz spaces $\Lambda^p_u(w)$, and for $u = 1$ both coincide with relations (6.48) and (6.49) respectively.

Definition 6.24. *We define the generalized upper (resp. lower) Boyd index associated to* $\Lambda^p_u(w)$ *as*

$$\alpha_{\Lambda^p_u(w)} = \lim_{\mu\to\infty} \frac{\log \overline{W}_u(\mu)}{\log \mu^p},$$

and

$$\beta_{\Lambda^p_u(w)} = \lim_{\lambda\to 0} \frac{\log \underline{W}_u(\lambda)}{\log \lambda^p},$$

respectively.

We can state Theorems 6.22 and 6.23 in terms of these new indices. In Chapter 3, we have proved equivalent expressions to the $B_p(u)$ condition, that characterizes

$$M : \Lambda^p_u(w) \to \Lambda^p_u(w),$$

studying the asymptotic behavior of the function $\overline{W}_u$ at infinity (see Corollary 3.38). Hence, we can reformulate this result in terms the generalized upper Boyd index, reproving in a different way the extended Lorentz-Shimogaki theorem for $\Lambda^p_u(w)$ appeared in [66].

Theorem 6.25. *If* $p > 0$, *then*

$$M : \Lambda^p_u(w) \to \Lambda^p_u(w) \quad \Leftrightarrow \quad \alpha_{\Lambda^p_u(w)} < 1. \tag{6.50}$$

Proof. The boundedness $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$ is characterized by the $B_p(u)$ condition, in Theorem 3.32. The equivalence of the $B_p(u)$ condition and $\alpha_{\Lambda^p_u(w)} < 1$ is a consequence of Corollary 3.38 (iv). □

We have characterized the strong-type boundedness of the Hilbert transform on weighted Lorentz spaces. We can reformulate this result in the context of Boyd theorem as follows:

Theorem 6.26. *If* $p > 1$, *then*

$$H : \Lambda^p_u(w) \to \Lambda^p_u(w) \quad \Leftrightarrow \quad 0 < \beta_{\Lambda^p_u(w)} \le \alpha_{\Lambda^p_u(w)} < 1. \tag{6.51}$$

Proof. By Theorem 6.18 we have that the boundedness $H : \Lambda^p_u(w) \to \Lambda^p_u(w)$ is characterized by the boundedness $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$ and the AB^*_∞ condition, whenever $p > 1$. On the one hand, $M : \Lambda^p_u(w) \to \Lambda^p_u(w)$ is characterized by $\alpha_{\Lambda^p_u(w)} < 1$, applying Theorem 6.25. On the other hand, the AB^*_∞ condition is characterized by the condition $\beta_{\Lambda^p_u(w)} > 0$, applying Corollary 3.50 (iv). □

Chapter 7

Further results and applications on $L^{p,q}(u)$ spaces

In the previous chapter we characterized the strong-type diagonal boundedness of the Hilbert transform on weighted Lorentz spaces $H : \Lambda_u^p(w) \to \Lambda_u^p(w)$, and its weak-type version $H : \Lambda_u^p(w) \to \Lambda_u^{p,\infty}(w)$, whenever $p > 1$ and we partially solved the case $p \leq 1$. The techniques used in order to obtain the solution allow us to get some necessary conditions for the weak-type boundedness of H in the non-diagonal case:

$$H : \Lambda_{u_0}^{p_0}(w_0) \to \Lambda_{u_1}^{p_1,\infty}(w_1), \tag{7.1}$$

which will be also necessary for the strong-type version $H : \Lambda_{u_0}^{p_0}(w_0) \to \Lambda_{u_1}^{p_1}(w_1)$. It is known that the case $u_0 = u_1 = 1$ can be derived by the boundedness of the Hardy operator and its adjoint. Nonetheless, this problem is still open when $w_0 = w_1 = 1$ and $p_0 = p_1 = p \geq 1$ for the weak-type inequality, and $p_0 = p_1 = p > 1$ for the strong-type inequality; that is

$$H : L^p(u_0) \to L^{p,\infty}(u_1) \quad \text{and} \quad H : L^p(u_0) \to L^p(u_1),$$

respectively. This is the well-known two-weighted problem for the Hilbert transform, posed in the early 1970's, but still unsolved completely. In the first section we present a brief survey on the efforts done towards the solution of the aforementioned problems and finally we give some necessary conditions for (7.1).

The second section is devoted to the characterization of the boundedness of

$$H : L^{p,q}(u) \to L^{r,s}(u)$$

for some exponents $p, q, r, s > 0$. In particular, we complete some results obtained in [25] by Chung, Hunt, and Kurtz.

7.1 Non-diagonal problem

Throughout this section, we present a brief historical review on the strong and weak-type boundedness of the Hilbert transform on weighted Lebesgue spaces

$$H : L^p(u_0) \to L^p(u_1) \quad \text{and} \quad H : L^p(u_0) \to L^{p,\infty}(u_1),$$

respectively (mostly based in [32], [36], and [38]). We also discuss the boundedness of the Hilbert transform on the classical Lorentz spaces (see [90])

$$H : \Lambda^{p_0}(w_0) \to \Lambda^{p_1}(w_1) \quad \text{and} \quad H : \Lambda^{p_0}(w_0) \to \Lambda^{p_1,\infty}(w_1).$$

Then, we present the following results: If

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1), \tag{7.2}$$

is bounded, we have that

$$\sup_{b>0} \frac{W_1^{1/p_1}\left(\int_{-b\nu}^{b\nu} u_1(s)\,ds\right)}{W_0^{1/p_0}\left(\int_{-b}^{b} u_0(s)\,ds\right)} \lesssim \left(\log \frac{1-\nu}{\nu}\right)^{-1}, \tag{7.3}$$

for every $\nu \in (0, 1/2]$ (see Theorem 7.9). In particular, we obtain that the weights u_0, w_0 are non-integrable, whereas u_1, w_1 could be integrable. As we have already mentioned, the techniques are similar to the diagonal case, and in some cases we have to assume that the composition of the weights $W_1 \circ u_1$ satisfies the doubling property. In this case, (7.2) implies (7.3), for $\nu \in (0, 1]$. Furthermore, under the doubling property, we have that (7.2) implies:

$$\frac{W_1^{1/p_1}(u_1(I))}{W_0^{1/p_0}(u_0(E))} \lesssim \frac{|I|}{|E|}, \tag{7.4}$$

for all measurable sets $E \subset I$, and all intervals I (see Theorem 7.12) and,

$$||u_0^{-1}\chi_I||_{(\Lambda^{p_0}_{u_0}(w_0))'} ||\chi_I||_{\Lambda^{p_1}_{u_1}(w_1)} \lesssim |I|, \tag{7.5}$$

for all intervals I (see Theorem 7.16). In particular, we reduce the range of indices p_0 for which (7.2) holds.

Finally we prove that if $w_1 \in B_{p_1}$ and u_1 is a doubling measure, then,

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1) \Rightarrow M : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1),$$

(see Theorem 7.20). In particular, if $p_1 > 1$ and u_1 is a doubling measure we get that

$$H : L^{p_0}(u_0) \to L^{p_1,\infty}(u_1) \Rightarrow M : L^{p_0}(u_0) \to L^{p_1,\infty}(u_1).$$

7.1.1 Background of the problem in the non-diagonal case

In [37], Fefferman and Stein proved the following estimate for M

$$\sup_{\lambda>0} \lambda u(\{x \in \mathbb{R} : |Mf(x)| > \lambda\}) \lesssim \int_{\mathbb{R}} |f(x)| Mu(x) dx, \tag{7.6}$$

which can be seen as a precursor of the two-weighted problem for an operator, let's say T,

$$T : L^p(u_0) \to L^{p,\infty}(u_1),$$

that consists on characterizing the pair of weights (u_0, u_1) such that the above holds.

The weighted-norm inequalities for the one-weight problem ($u_0 = u_1$) for the Hardy-Littlewood maximal function and the Hilbert transform, have been solved in the setting of the A_p theory. Thus, an immediate candidate for the solution of the two-weighted problem for M and H was the two-weighted version of the A_p condition (see [32]). More precisely, we say that the pair $(u_0, u_1) \in A_p$, for $p > 1$, if

$$\sup_I \left(\frac{1}{|I|} \int_I u_1(x) dx \right) \left(\frac{1}{|I|} \int_I u_0(x)^{1-p'} dx \right)^{p-1} < \infty, \tag{7.7}$$

where the supremum is considered over all intervals I of the real line. We say that $(u_0, u_1) \in A_1$ if

$$Mu_1(x) \leq C u_0(x), \quad \text{a.e.} \quad x \in \mathbb{R}. \tag{7.8}$$

Although the A_p condition characterizes the two-weighted weak-type boundedness of M, in 1976 Muckenhoupt and Wheeden proved that this is not a sufficient condition for the Hilbert transform to be bounded (see [72]). In general, the two-weighted problem for H

$$H : L^p(u_0) \to L^p(u_1), \tag{7.9}$$

and its weak-type version

$$H : L^p(u_0) \to L^{p,\infty}(u_1) \tag{7.10}$$

remains still unsolved completely. Furthermore, Muckenhoupt and Whedeen conjectured (see [32]) that we could consider H, instead of M on the left hand-side estimate of (7.6), but recently it has been disproved by Reguera and Thiele in [83] (for further information see also the references therein).

In what follows, we present the characterization of the weak-type boundedness of M in terms of the two-weighted A_p condition. This condition which is not sufficient for the strong-type and weak-type boundedness of H, neither works for the strong-type boundedness of M. In fact, the last one was solved in 1982, by Sawyer in terms of conditions involving M (see [89]).

Theorem 7.1. *[71] If $1 \leq p < \infty$, then*

$$M : L^p(u_0) \to L^{p,\infty}(u_1) \Leftrightarrow (u_0, u_1) \in A_p.$$

Theorem 7.2. *[89] Given $1 < p < \infty$, the following are equivalent:*

(i) $M : L^p(u_0) \to L^p(u_1)$.

(ii) The pair (u_0, u_1) satisfies:

$$\int_I M(\chi_I \sigma)^p(x) u_0(x) dx \lesssim \sigma(I), \tag{7.11}$$

for all intervals I of the real line and $\sigma = {u_1}^{1-p'}$.

Definition 7.3. *The pair (u_0, u_1) satisfies the S_p condition if and only if (7.11) holds.*

Corollary 7.4. *Let $u_1 \in A_\infty$.*

(i) Let $1 \leq p < \infty$. If $(u_0, u_1) \in A_p$, then $H : L^p(u_0) \to L^{p,\infty}(u_1)$.

(ii) Let $1 < p < \infty$. If $(u_0, u_1) \in S_p$, then $H : L^p(u_0) \to L^p(u_1)$.

Proof. Conditions (i) and (ii) are consequences of Theorems 7.1 and 7.2 respectively, taking into account the following result proved by Coifman and Fefferman in [26]: $||Hf||_{L^{p,\infty}(u_1)} \lesssim ||Mf||_{L^{p,\infty}(u_1)}$ and $||Hf||_{L^p(u_1)} \lesssim ||Mf||_{L^p(u_1)}$, provided $u_1 \in A_\infty$. □

Currently, there are two relevant approaches of two-weighted norm inequalities, an area of active research nowadays. On the one hand, we have the theory of the so-called "Ap bump conditions", related to the following result of Neugebauer who, in 1983 found a new sufficient condition for the Hilbert transform to be bounded in terms of two-weighted A_p condition (see [78]). On the other hand, we have the theory of "testing conditions" in the context of Sawyer's S_p conditions. For further information on these topics see [32].

Theorem 7.5. *Let $1 < p < \infty$. If $(u_0^r, u_1^r) \in A_p$ for some $r > 1$, then $H : L^p(u_0) \to L^p(u_1)$.*

Note that $(u_0^r, u_1^r) \in A_p$ can be rewritten as

$$\sup_I ||u_1^{1/p}||_{rp,I} ||u_0^{-1/p}||_{rp',I} < \infty$$

(see [32]). Hence, in view of Theorem 7.5, if we replace the normalized $L^p, L^{p'}$ norms in the A_p condition by larger norms $L^{rp}, L^{rp'}$, that are called "power bumps", we can get sufficient conditions.

In 2007, Cruz-Uribe, Martell and Pérez considered in [31] the question whether one could replace the "power bumps" by other function space norms, larger than L^p, but smaller than the "power bumps", in order to get sufficient conditions for the Hilbert transform to be bounded. Pérez first posed the same question for the Hardy-Littlewood maximal function, for potential and maximal fractional type operators (see [81] and [82]). Other developments towards this direction can be found in [31], [78], [34], [30], and [64], and for the weak-type boundedness of the Hilbert transform see [33]. Another approach can be found in [73]. For historical references and further information see [32].

On the other hand, Nazarov, Treil and Volberg characterized the two-weighted problem for the Hilbert transform for $p = 2$ in [77], under some assumption on the weights. The solution is given in the context of Sawyer's S_2 conditions and T_1 conditions of David-Journé (see [35]). Their approach involves techniques developed in [74], [75], and [76]. Later on, Lacey, Sawyer and Uriarte-Tuero proved in [62] that the extra condition, assumed in [77], is not necessary and applying similar techniques they provide the characterization with a new weaker assumption.

It should be mentioned that Cotlar and Sadosky have given a necessary and sufficient condition for the two-weighted boundedness of the Hilbert transform for $p = 2$ (see [28], and [29]). However, their condition, related to Helson-Szegö theorem, is difficult to check as it was observed in [89] and [31].

The boundedness of the Hilbert transform on the classical Lorentz spaces can be derived from the study of two operators, the Hardy operator and its adjoint as proved by Sawyer in [90].

Theorem 7.6. *If $p_0, p_1 > 0$, then*

$$H : \Lambda^{p_0}(w_0) \to \Lambda^{p_1}(w_1) \quad \textit{if and only if} \quad P, Q : L^{p_0}_{\mathrm{dec}}(w_0) \to L^{p_1}(w_1),$$

and

$$H : \Lambda^{p_0}(w_0) \to \Lambda^{p_1,\infty}(w_1) \quad \textit{if and only if} \quad P, Q : L^{p_0}_{\mathrm{dec}}(w_0) \to L^{p_1,\infty}(w_1).$$

Proof. The proof is based on the equivalence (3.11). Hence the study is reduced to the characterization of operators P, Q as in the diagonal case. □

Remark 7.7. The characterization of the boundedness of the operators P, Q in the weak-type case

$$P, Q : L^{p_0}_{\mathrm{dec}}(w_0) \to L^{p_1,\infty}(w_1)$$

is given in [4]. The strong-type boundedness of the Hardy operator $P : L^{p_0}_{\mathrm{dec}}(w_0) \to L^{p_1}(w_1)$ is characterized by different authors:

(i) The cases $1 < p_0 \leq p_1 < \infty$ and $1 < p_1 < p_0 < \infty$ have been characterized in [90].

(ii) The case $0 < p_1 < 1 < p_0 < \infty$ has been solved in [97].

(iii) The case $0 < p_0 \leq p_1$ has been characterized in [22], and for $0 < p_0 \leq p_1$, $0 < p_0 < 1$ can be also found in [97].

(iv) The case $p_1 = 1 < p_0 < \infty$ has been characterized in [19].

(v) The case $0 < p_1 < 1 = p_0$ has been characterized in [92].

(vi) The case $0 < p_1 < p_0 \leq 1$ has been solved in [18].

The strong-type boundedness of the adjoint of the Hardy operator $Q : L^{p_0}_{\mathrm{dec}}(w_0) \to L^{p_1}(w_1)$ has been studied in [21] and in [22].

7.1.2 Basic necessary conditions in the non-diagonal case

Now, we study necessary conditions for the weak-type boundedness of H,

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1).$$

In particular, we obtain that the weights u_0, w_0 are non-integrable, whereas u_1, w_1 could be integrable. The techniques are similar to the diagonal case, with some extra difficulties that are solved assuming the doubling property on the composition of the weights $W_1 \circ u_1$ (see (4.14)).

In what follows we assume that $w_0, w_1 \in \Delta_2$.

Definition 7.8. *Let $p > 0$. We say that an operator T is of restricted weak-type (p_0, p_1) (with respect to (u_0, u_1, w_0, w_1)) if*

$$\|T\chi_S\|_{\Lambda^{p_1,\infty}_{u_1}(w_1)} \lesssim \|\chi_S\|_{\Lambda^{p_0}_{u_0}(w_0)}, \tag{7.12}$$

for all measurable sets S of the real line. If S is an interval, then we say that T is of restricted weak-type (p_0, p_1) on intervals (with respect to (u_0, u_1, w_0, w_1)).

Theorem 7.9. *Let $0 < p_0, p_1 < \infty$. If the Hilbert transform is of restricted weak-type (p_0, p_1) on intervals with respect to (u_0, u_1, w_0, w_1) then*

$$\sup_{b>0} \frac{W_1^{1/p_1}\left(\int_{-b\nu}^{b\nu} u_1(s)\,ds\right)}{W_0^{1/p_0}\left(\int_{-b}^{b} u_0(s)\,ds\right)} \lesssim \left(\log \frac{1-\nu}{\nu}\right)^{-1}, \tag{7.13}$$

for every $\nu \in (0, 1/2]$.

Proof. The proof is similar to that of Theorem 4.4. □

Proposition 7.10. *Let $0 < p_0, p_1 < \infty$. If the Hilbert transform is of restricted weak-type (p_0, p_1) on intervals with respect to (u_0, u_1, w_0, w_1), then $u_0 \notin L^1(\mathbb{R})$ and $w_0 \notin L^1(\mathbb{R}^+)$.*

Proof. By Theorem 7.9 we get the relation (7.13). Let $c > 0$ such that $W_1^{1/p_1}(u_1(-c, c)) > 0$. Then fix $\nu = c/b$. Therefore we obtain that

$$\frac{W_0^{1/p_0}(u_0(-b, b))}{W_1^{1/p_1}(u_1(-c, c))} \gtrsim \log(b - 1), \tag{7.14}$$

for all $b \in (2c, \infty)$. Since $u_1 \in L^1_{\text{loc}}$ and $w_1 \in L^1_{\text{loc}}$ we get that $W_1^{1/p_1}(u_1(-c, c)) = C$. If we take the limit when b tends to infinity, then

$$W_0^{1/p_0}(u_0(-\infty, \infty)) \gtrsim \infty. \tag{7.15}$$

Thus, we obtain the result. □

Remark 7.11. (i) As in the diagonal case, since $u_0 \notin L^1$ and $w_0 \notin L^1$, and $\mathcal{C}_c^\infty$ is dense in $\Lambda^{p_0}_{u_0}(w_0)$ by Theorem 2.13, we say that $H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1)$, if

$$||Hf||_{\Lambda^{p_1,\infty}_{u_1}(w_1)} \lesssim ||f||_{\Lambda^{p_0}_{u_0}(w_0)},$$

for every $f \in \mathcal{C}_c^\infty$. Then, H can be extended to $\Lambda^{p_0}_{u_0}(w_0)$ as a bounded linear operator $\overline{H}$, which by Theorem 4.13 coincides with the Hilbert transform, for every function belonging to $f \in L^q \cap \Lambda^{p_0}_{u_0}(w_0)$ and $q \geq 1$. For further details see Section 4.3.

(ii) If the Hilbert transform is bounded

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1),$$

then it also satisfies that

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1'}(w_1'),$$

where $u_1' = u_1\chi_{B(0,r)}$, for some $r > 0$ such that u_1' is not identically 0, and $w_0' = w_0\chi_{(0,t)}$, for some $t > 0$ and w_0' is not identically 0. Therefore, we see that w_1', u_1' are not necessary integrable.

Theorem 7.12. *Let $0 < p_0, p_1 < \infty$ and assume that $W_1 \circ u_1$ satisfies the doubling property. If the Hilbert transform is of restricted weak-type (p_0, p_1) with respect to (u_0, u_1, w_0, w_1), then:*

(i) For all measurable subsets $E \subset I$, it holds

$$\frac{W_1^{1/p_1}(u_1(I))}{W_0^{1/p_0}(u_0(E))} \lesssim \frac{|I|}{|E|}. \tag{7.16}$$

(ii) For all $\nu \in (0,1]$, it holds

$$\sup_{b>0} \frac{W_1^{1/p_1}\left(\int_{-b\nu}^{b\nu} u_1(s)\,ds\right)}{W_0^{1/p_0}\left(\int_{-b}^{b} u_0(s)\,ds\right)} \lesssim \left(\log \frac{1+\nu}{\nu}\right)^{-1}. \tag{7.17}$$

Proof. (i) As in Theorem 4.8 we obtain

$$\frac{W_1^{1/p_1}(u_1(I'))}{W_0^{1/p_0}(u_0(E))} \leq C\frac{|I|}{|E|}.$$

Applying the monotonicity of W_1 and then the doubling property, we have that $W_1(u_1(I)) \leq W_1(u_1(3I')) \leq cW_1(u_1(I'))$. Hence,

$$\frac{W_1^{1/p_1}(u_1(I))}{W_0^{1/p_0}(u_0(E))} \leq C\frac{|I|}{|E|}.$$

(ii) By Theorem 7.9 we get

$$\sup_{b>0} \frac{W_1^{1/p_1}\left(\int_{-b\nu}^{b\nu} u_1(s)\,ds\right)}{W_0^{1/p_0}\left(\int_{-b}^{b} u_0(s)\,ds\right)} \lesssim \left(\log \frac{1-\nu}{\nu}\right)^{-1}, \tag{7.18}$$

for $\nu \in (0,1/2]$. Besides, by relation (7.16) and the monotonicity of W_1, we obtain that

$$\sup_{b>0} \frac{W_1^{1/p_1}\left(\int_{-b\nu}^{b\nu} u_1(s)\,ds\right)}{W_0^{1/p_0}\left(\int_{-b}^{b} u_0(s)\,ds\right)} \lesssim \sup_{b>0} \frac{W_1^{1/p_1}\left(\int_{-b}^{b} u_1(s)\,ds\right)}{W_0^{1/p_0}\left(\int_{-b}^{b} u_0(s)\,ds\right)} \lesssim 1, \tag{7.19}$$

for all $\nu \in (0,1]$. Then by (7.18) and (7.19) we obtain (7.17) for all $\nu \in (0,1]$ (for more details see the proof of Theorem 4.4). □

In [51], Hörmander proved that if a translation invariant, linear operator is bounded from L^p to L^q, then necessarily $p \leq q$. We prove that if $H : \Lambda_u^p(w) \to \Lambda_u^{q,\infty}(w)$, then $p = q$, if $W \circ u$ satisfies the doubling property.

Proposition 7.13. *Let $H : \Lambda_u^p(w) \to \Lambda_u^{q,\infty}(w)$, and $W \circ u$ satisfy the doubling condition. Then, $p = q$.*

Proof. By Theorem 7.12 we obtain that

$$\frac{W^{1/q}(u(I))}{W^{1/p}(u(E))} \lesssim \frac{|I|}{|E|}. \tag{7.20}$$

Letting $E = I$ we obtain that $W^{1/q-1/p}(u(I)) \le C$ for all interval I. Since $u \notin L^1$ we get $W^{1/q-1/p}(r) \le C$ for all $r > 0$. As $\lim_{t\to 0} W(t) = 0$ we get that $q \le p$. On the other hand, as $w \notin L^1$, we have that $\lim_{t\to\infty} W(t) = \infty$ and the inequality $W^{1/q-1/p}(r) \le C$ holds only if $p = q$. □

In Section 4.3 we studied a necessary condition for the boundedness of the Hilbert transform in terms of the associate Lorentz spaces. Now we will prove that an analogue condition holds in the non-diagonal case

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1),$$

under the assumption that $W_1 \circ u_1$ satisfies the doubling property. First, we present the non-diagonal version of the boundedness of the Hardy-Littlewood maximal function studied by Carro and Soria in [23].

Theorem 7.14. *[23] Let $0 < p_0, p_1 < \infty$. If $M : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1)$, then*

$$||u_0^{-1}\chi_I||_{(\Lambda^{p_0}_{u_0}(w_0))'}||\chi_I||_{\Lambda^{p_1}_{u_1}(w_1)} \lesssim |I|, \tag{7.21}$$

for all intervals I.

Under some additional conditions on the weights w_0, w_1, condition (7.21) is also sufficient for the boundedness of the Hardy-Littlewood maximal function (for more details see [23]).

Theorem 7.15. *[23] Let $0 < p_0, p_1 < \infty$. If there exists $\alpha > 0$ such that $\alpha p_1/p_0 \ge 1$ and for every sequence $\{t_j\}_j$ we have that*

$$W_1^{\alpha}\left(\sum_j t_j\right) \lesssim \sum_j W_1^{\alpha}(t_j) \tag{7.22}$$

and

$$\sum_j W_0^{\alpha p_1/p_0}(t_j) \lesssim W_0^{\alpha p_1/p_0}\left(\sum_j t_j\right), \tag{7.23}$$

then $M : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1)$ if and only if condition (7.21) holds.

We will show that if $W_1 \circ u_1$ satisfies the doubling property, then the boundedness of H,

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1)$$

implies (7.21).

Theorem 7.16. *Let $0 < p_0, p_1 < \infty$ and assume that $W_1 \circ u_1$ satisfies the doubling property. If the Hilbert transform*

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1)$$

is bounded, then condition (7.21) holds.

Proof. The proof follows the ideas of Theorem 4.8. Indeed, we have that

$$||u_0^{-1}\chi_I||_{(\Lambda^{p_0}_{u_0}(w_0))'} W_1^{1/p_1}(u_1(I')) \lesssim |I|.$$

Now, by the doubling property it follows that

$$||\chi_I||^{1/p_1}_{\Lambda^{p_1}_{u_1}(w_1)} = W_1(u_1(I)) \leq W_1(u_1(3I')) \leq cW_1(u_1(I')).$$

Hence,

$$||u_0^{-1}\chi_I||_{(\Lambda^{p_0}_{u_0}(w_0))'} ||\chi_I||_{\Lambda^{p_1}_{u_1}(w_1)} \lesssim |I|.$$

□

Corollary 7.17. *Let $0 < p_0, p_1 < \infty$. Assume that $W_1 \circ u_1$ satisfies the doubling property and the weights w_0, w_1 satisfy the conditions (7.23) and (7.22) respectively. If*

$$H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1),$$

then

$$M : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1).$$

The necessary condition (7.21) implies some restrictions depending on w_0 that reduce the range of indices p_0 for which the boundedness $H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1)$ holds. We follow the same approach as in [20].

Proposition 7.18.

(i) Let $0 < p_1 < \infty$ and assume that $W_1 \circ u_1$ satisfies the doubling property. If $H : \Lambda^{p_0}_{u_0}(w_0) \to \Lambda^{p_1,\infty}_{u_1}(w_1)$, then $p_0 \geq p_{w_0}$. If $p_{w_0} > 1$, then $p_0 > p_{w_0}$.

(ii) Let $p_0 < 1$ and assume that u_1 is a doubling measure. Then, there are no weights u_0, u_1 such that $H : L^{p_0}(u_0) \to L^{p_1,\infty}(u_1)$ is bounded, for $0 < p_1 < \infty$.

Proof. We follow the same ideas as in [20, Theorem 3.4.2 and 3.4.3]. □

It is known that the boundedness of the Hardy-Littlewood maximal function

$$M : \Lambda_u^{p_0}(w_0) \to \Lambda_u^{p_1,\infty}(w_1)$$

implies the boundedness of the same operator on the same spaces with $u = 1$ (see [20]). We will see that if $W_1 \circ u$, satisfies the doubling property and the Hilbert transform satisfies the weak-type boundedness

$$H : \Lambda_u^{p_0}(w_0) \to \Lambda_u^{p_1,\infty}(w_1),$$

then we obtain the boundedness of the Hardy-Littlewood maximal function on the classical Lorentz spaces,

$$M : \Lambda^{p_0}(w_0) \to \Lambda^{p_1,\infty}(w_1).$$

Theorem 7.19. *Let $0 < p_0, p_1 < \infty$. Assume that $W_1 \circ u$ satisfies the doubling property. If $H : \Lambda_u^{p_0}(w_0) \to \Lambda_u^{p_1,\infty}(w_1)$ then*

$$M : \Lambda^{p_0}(w_0) \to \Lambda^{p_1,\infty}(w_1).$$

Proof. By Theorem 7.16 we obtain the relation (7.21), taking into account the doubling property. Then, we can follow the same arguments as in [20, Proposition 3.4.4 and Theorem 3.4.8]. □

7.1.3 Necessity of the weak-type boundedness of M

In this section we prove that the boundedness of the Hilbert transform on weighted Lorentz spaces implies the boundedness of the Hardy-Littlewood maximal function on the same spaces, in the non-diagonal case.

Theorem 7.20. *Let $p_1 > 1$, $w_1 \in B_{p_1}$ and let u_1 be a doubling measure. Then*

$$H : \Lambda_{u_0}^{p_0}(w_0) \to \Lambda_{u_1}^{p_1,\infty}(w_1) \Rightarrow M : \Lambda_{u_0}^{p_0}(w_0) \to \Lambda_{u_1}^{p_1,\infty}(w_1).$$

Proof. The proof is identical to the proof of Theorem 6.8 in the diagonal case. Let E, E_λ and K be as in the aforementioned theorem. Note that the condition $w_1 \in B_{p_1}$ implies by Proposition 3.11

$$||\chi_E||_{(\Lambda_{u_1}^{p_1,\infty}(w_1))'} \lesssim \frac{u_1(E)}{W_1^{1/p_1}(u_1(E))}. \tag{7.24}$$

Then, applying relation (6.22), Hölder's inequality and the hypothesis, we have that

$$\begin{aligned}\lambda W_1^{1/p_1}(u_1(E)) &\lesssim W_1^{1/p_1}(u_1(E))\frac{1}{u_1(E)}\int_E |H(f\chi_{\cup_i I_i})(x)|u_1(x)dx \\ &\lesssim W_1^{1/p_1}(u_1(E))\frac{1}{u_1(E)}||H(f\chi_{\cup_i I_i})||_{\Lambda_{u_1}^{p_1,\infty}(w_1)}||\chi_E||_{(\Lambda_{u_1}^{p_1,\infty}(w_1))'} \\ &\lesssim ||f\chi_{\cup_i I_i}||_{\Lambda_{u_0}^{p_0}(w_0)} \leq ||f||_{\Lambda_{u_0}^{p_0}(w_0)}.\end{aligned}$$

Now, since u_1 is a doubling measure and $w_1 \in \Delta_2$, we have that

$$W_1^{1/p_1}(u_1(K)) \lesssim W_1^{1/p_1}(u_1(E)).$$

Hence,

$$\lambda W^{1/p}(u(K)) \lesssim ||f||_{\Lambda_u^p(w)}.$$

Since this holds for all compact sets of E_λ, by Fatou's lemma we obtain that

$$\lambda W^{1/p}(u(E_\lambda)) \lesssim ||f||_{\Lambda_u^p(w)}.$$

□

Corollary 7.21. *Assume that u_1 is a doubling measure. Then, if $p_1 > 1$*

$$H : L^{p_0}(u_0) \to L^{p_1,\infty}(u_1) \Rightarrow M : L^{p_0}(u_0) \to L^{p_1,\infty}(u_1).$$

Remark 7.22. Corollary 7.21 has been already proved for $p_0 = p_1 = p > 1$, without assuming the doubling property (see [72]).

7.2 Applications on $L^{p,q}(u)$ spaces

It is known that the following condition

$$\frac{u(I)}{|I|^p} \lesssim \frac{u(E)}{|E|^p}, \quad E \subset I, \tag{7.25}$$

characterizes the boundedness of M

$$M : L^{p,1}(u) \to L^{p,\infty}(u),$$

for $1 \leq p < \infty$ (see for example [20]). In [25], Chung, Hunt, and Kurtz proved that condition (7.25) is also sufficient for the boundedness of the Hilbert transform

$$H : L^{p,1}(u) \to L^{p,\infty}(u), \tag{7.26}$$

for the same exponent $1 \leq p < \infty$. Throughout this section we study the boundedness of the Hilbert transform on the Lorentz spaces $L^{p,q}(u)$ and among other results, we show that condition (7.25) characterizes also (7.26).

First we present the collection of the known results concerning the boundedness of the Hardy-Littlewood maximal function

$$M : L^{p,q}(u) \to L^{r,s}(u),$$

such as it appears in [20], and then we study the boundedness of Hilbert transform on the same spaces.

Theorem 7.23. *([71], [25], [20], [53], [63]) Let $p, r \in (0, \infty)$, $q, s \in (0, \infty]$.*

(i) If $p < 1, p \neq r$ or $s < q$, there are no weights u such that $M : L^{p,q}(u) \to L^{r,s}(u)$.

(ii) The boundedness

$$M : L^{1,q}(u) \to L^{1,s}(u)$$

holds if and only if $q \leq 1$, $s = \infty$ and in this case a necessary and sufficient condition is $u \in A_1$.

(iii) If $p > 1$ and $0 < q \leq s \leq \infty$ then the boundedness

$$M : L^{p,q}(u) \to L^{p,s}(u)$$

holds if and only if

(a) Case $q \leq 1$, $s = \infty$: $\dfrac{u(I)}{|I|^p} \lesssim \dfrac{u(E)}{|E|^p}, \quad E \subset I$.

(b) Case $q > 1$ or $s < \infty$: $u \in A_p$.

Theorem 7.24. *Let $p, r \in (0, \infty)$ and $q, s \in (0, \infty]$.*

(α) Let $p = 1$ and $s = \infty$.

(i) If $q \leq 1$, the boundedness

$$H : L^{1,q}(u) \to L^{1,\infty}(u)$$

holds if and only if $u \in A_1$.

(ii) If $1 < q \leq \infty$, the boundedness

$$H : L^{1,q}(u) \to L^{1,\infty}(u)$$

does not hold for any u.

(β) Let $p > 1$ and $s = \infty$.

(i) If $q \leq 1$, then the boundedness

$$H : L^{p,q}(u) \to L^{p,\infty}(u)$$

holds if and only if $\dfrac{u(I)}{|I|^p} \lesssim \dfrac{u(E)}{|E|^p}$, *for all $E \subset I$.*

(ii) If $1 < q \leq \infty$, then the boundedness

$$H : L^{p,q}(u) \to L^{p,\infty}(u)$$

holds if and only if $u \in A_p$.

(γ) If $p > 1$ and $s = q > 1$, then the boundedness

$$H : L^{p,q}(u) \to L^{p,q}(u)$$

holds if and only if $u \in A_p$.

(δ) If $p < 1$ or $p \neq r$, there are no doubling weights u such that

$$H : L^{p,q}(u) \to L^{r,s}(u)$$

is bounded.

Proof. Some of the proofs follow the same ideas of [20, Theorem 3.5.1]:

Case α: (i) The boundedness $H : L^{1,q}(u) \to L^{1,\infty}(u)$ can be rewritten as $H : \Lambda_u^q(t^{q-1}) \to \Lambda_u^{q,\infty}(t^{q-1})$, and by (4.15) we get

$$\frac{u(I)}{|I|} \lesssim \frac{u(E)}{|E|}, \quad E \subset I,$$

which is equivalent to the A_1 condition. On the other hand if $u \in A_1$ then

$$H : L^{1,q}(u) \to L^{1,\infty}(u),$$

by Corollary 5.3.

(ii) If $H : L^{1,q}(u) \to L^{1,\infty}(u)$ is bounded, we also have the boundedness of $H : L^{1,r}(u) \to L^{1,\infty}(u)$, for $r < 1$ and thus, by (i), we have that $u \in A_1$. But in this case, Theorem 5.4 shows that w must be in B_q, while $t^{q-1} \notin B_q$, if $q > 1$.

Case β: (i) First we prove the necessity: if $q \leq 1$ and $s = \infty$, the weak-type boundedness of the Hilbert transform

$$H : L^{p,q}(u) \to L^{p,\infty}(u)$$

can be rewritten as $H : \Lambda_u^q(t^{q/p-1}) \to \Lambda_u^{q,\infty}(t^{q/p-1})$, which by (4.15) implies $\dfrac{u(I)}{|I|^p} \lesssim \dfrac{u(E)}{|E|^p}$ for all $E \subset I$ as we wanted to see. Conversely, if

$$\frac{u(I)}{|I|^p} \lesssim \frac{u(E)}{|E|^p}, \quad E \subset I,$$

we have that $u \in A_\infty$ and by Theorem 7.23 it implies the weak-type boundedness of the Hardy-Littlewood maximal function $M : L^{p,q}(u) \to L^{p,\infty}(u)$. Rewriting the last estimate as $M : \Lambda_u^q(t^{q/p-1}) \to \Lambda_u^{q,\infty}(t^{q/p-1})$ and taking into account that $w(t) = t^{q/p-1} \in B_\infty^*$ we have, by Theorem 6.10, that $H : L^{p,q}(u) \to L^{p,\infty}(u)$.

(*ii*) Let $q > 1$. The boundedness $H : L^{p,q}(u) \to L^{p,\infty}(u)$ can be equivalently expressed by

$$H : \Lambda_u^q(t^{q/p-1}) \to \Lambda_u^{q,\infty}(t^{q/p-1}). \tag{7.27}$$

Then, since $t^{q/p-1} \in B_\infty^*$, by Theorem 6.13 we have that (7.27) is equivalent to the boundedness of M on the same spaces, which by Theorem 7.23 (*iii*) is characterized by the A_p condition.

Case γ: Since $t^{q/p-1} \in B_\infty^*$, we have by Theorem 6.13 that

$$H : \Lambda_u^q(t^{q/p-1}) \to \Lambda_u^q(t^{q/r-1})$$

is bounded if and only if M is bounded on the same spaces, characterized by the A_p condition in view of Theorem 7.23.

Case δ: Since $L^{r,s}(u) \subset L^{r,\infty}(u)$, if $H : L^{p,q}(u) \to L^{r,s}(u)$ we would have that $H : L^{p,q}(u) \to L^{r,\infty}(u)$ is bounded, which is equivalent to having that $H : \Lambda_u^q(t^{q/p-1}) \to \Lambda_u^{q,\infty}(t^{q/r-1})$ is bounded. By Theorem 7.16, taking into account that u is non-doubling, we have that

$$\frac{u^{1/r}(I)}{|I|} \lesssim \frac{u^{1/p}(E)}{|E|}, \quad E \subset I. \tag{7.28}$$

Then, by the Lebesgue differentiation theorem we get first that $p \geq 1$. On the other hand, if we take $E = I$, then (7.28) implies $u^{1/r-1/p}(I) \lesssim 1$ and hence $p = r$, since $u \notin L^1$ by Proposition 4.5. $\square$

Remark 7.25. (i) In [25] Chung, Hunt, and Kurtz proved the sufficiency of the case $\beta, (i)$ of Theorem 7.24 for the exponent $q = 1$. The necessity of the A_p condition in γ of Theorem 7.24 can be obtained directly, applying β of the same theorem and the continuous inclusion $L^{p,q} \hookrightarrow L^{p,\infty}$.

(ii) In Chapter 6 we characterized the boundedness of the Hilbert transform on weighted Lorentz spaces $\Lambda_u^p(w)$, and for the case $p \leq 1$, we solved the problem under the assumption $\overline{W}^{1/p}(t) \not\approx t$, for all $t > 1$ (see Theorem 6.13). However, we will see that this assumption is not necessary in general. Indeed, consider the weight $w(t) = t^{q-1}$, $q \leq 1$. Although, it holds that $\overline{W}^{1/q}(t) = t$, we have by α, (i) of Theorem 7.24 and Theorem 7.23 that

$$H : \Lambda_u^q(t^{p-1}) \to \Lambda_u^{q,\infty}(t^{q-1}) \;\Leftrightarrow\; M : \Lambda_u^q(t^{p-1}) \to \Lambda_u^{q,\infty}(t^{q-1}) \;\Leftrightarrow\; u \in A_1,$$

and we also have by β, (i) of Theorem 7.24 and Theorem 7.23 that

$$H : \Lambda_u^q(t^{p-1}) \to \Lambda_u^{q,\infty}(t^{q-1}) \;\Leftrightarrow\; M : \Lambda_u^q(t^{p-1}) \to \Lambda_u^{q,\infty}(t^{q-1}) \;\Leftrightarrow\; \frac{u(I)}{|I|^p} \lesssim \frac{u(E)}{|E|^p},\; E \subset I.$$

We obtain the characterization of the boundedness

$$H : L^{p_0,q_0}(u_0) \to L^{q_1,\infty}(u_1)$$

under the additional hypothesis that $u_1 \in A_\infty$. In fact, we will prove that it is equivalent to the boundedness of the Hardy-Littlewood maximal function on the same spaces, which follows as a special case of Theorem 7.15 (for more details see [23]).

Theorem 7.26. *Assume that $u_1 \in A_\infty$ and there exists $\alpha \geq p_0/p_1$, with $q_0/p_1 \leq \alpha \leq q_1/p_1$ and $\max(p_0, q_0) \leq q_1$. Then,*

$$H : L^{q_0,p_0}(u_0) \to L^{q_1,\infty}(u_1)$$

if and only if

$$||u_0^{-1}\chi_I||_{L^{q_0',p_0'}(u_0)} ||\chi_I||_{L^{q_1,p_1}(u_1)} \lesssim |I|, \tag{7.29}$$

for all intervals I of the real line.

Proof. Assume that the operator $H : L^{q_0,p_0}(u_0) \to L^{q_1,\infty}(u_1)$ is bounded, which can be rewritten as $H : \Lambda_{u_0}^{p_0}(t^{p_0/q_0-1}) \to \Lambda_{u_1}^{p_1,\infty}(t^{p_1/q_1-1})$. Then, since the weights $w_0(t) = t^{p_0/q_0-1}$ and $w_1(t) = t^{p_1/q_1-1}$ satisfy conditions (7.23) and (7.22) respectively, we obtain, by Corollary 7.17, the boundedness of the Hardy-Littlewood maximal function $M : L^{q_0,p_0}(u_0) \to L^{q_1,\infty}(u_1)$ which, by Theorem 7.15, is characterized by relation (7.29).

On the other hand, condition (7.29) implies the boundedness $H : L^{q_0,p_0}(u_0) \to L^{q_1,\infty}(u_1)$ provided $w_1(t) = t^{q_1/p_1-1} \in B_\infty^*$ and $u_1 \in A_\infty$ (similar to the proof of Theorem 6.10). $\square$

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