Three Essays on Weak Condorcet Winners, Public Bads, and Contrast Campaigns

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0.1 Introduction

There are as many preferences as individuals; each of us evaluates and ranks alternatives given our own interest. A social decision might not be only over consumption, activities, or leisure; but it can be a decision over politicians, public goods, cost sharing, or any other scenario where several agents are involved in both: the decision process and in the consequences of such selection. The bridge between individual preferences and collective decision-making is not clear and has to be analyzed.

The effects about collective decisions could have a great impact for the social well-being; for this matter, the social choice functions used to aggregate preferences are required to accomplish certain properties. Two of the most important features required for a rule are: 1) non dictatorship; that is, that the rule is not the image of a certain agent; and, 2) non manipulability; that is, that no agent can become better off misrepresenting his true preferences. However, it is well known that the aggregation of preferences is not a simple task, as Gibbard (1973) and Satterthwaite (1975) proved; if the range of the rule has three or more alternatives and there is no restriction on preferences, then the rule is either dictatorial or manipulable. After this negative result, the Social Choice Theory continued exploring different solutions to solve such dilemma.

A negative result does not mean that there is nothing more to do for the subject. Along history, people have created rules to decide and try to reconcile all the individual preferences. However, when a rule has been established to decide over several alternatives, more matters can arise: either there is no a winning alternative, there is not a unique solution, or the alternatives are not well known by the agents involved in the

selection procedure. This thesis is a brief analysis over the conditions under which we can find positive results in collective decisions, and a scenario where the information available to the agent for the choice of a political candidate is not complete.

Condorcet (1785) suggested that, to aggregate preferences, the social choice function should select an alternative that beats by majority the rest of the available alternatives. If an alternative a is preferred by more than half of the agents over another alternative b, we say that a beats by majority the alternative b. Hence, a Condorcet Winner is an alternative that beats by majority to all the rest of the available alternatives; if there is a tie, we call it Weak Condorcet Winner. Several rules such as the Majority Rule, or the Copeland Rule select the Condorcet Winner. An appealing property of these rules is, for example, that if we are dealing with two alternatives, the Majority rule will be strategy-proof. However, given the individual preferences, the Condorcet Winner might not exist or might not be unique. Hence, the GS negative result can be overcame with the selection of Condorcet Winners, but we should look at the conditions on the preference's profile to ensure its existence.

In Chapter 1 we look at the preferences' profiles, that is, the group list of individual preferences involved in the selection procedure, and identify necessary and sufficient conditions of a preference profile for the existence of Weak Condorcet Winners. Sen and Pattanaik (1969) gave a sufficient condition over the preference profile to ensure the existence of Weak Condorcet Winners: the Value Restriction. This condition has been widely used to construct restrictions over preferences to obtain positive results in the pursuit of strategy-proof rules, such as single-peakness. However, we identify necessary and sufficient conditions, intending to contribute in the construction of new

preferences restrictions to develop positive results in the Social Choice Theory.

The restriction of the domain for the social choice functions is a way to have positive results in the aggregation of preferences. Single-peak preferences have been widely studied, as we are able to obtain positive results. Moulin (1980) characterized the family of strategy-proof and tops-only social choice functions under this domain. Such preferences are naturally used in setups such as in the location of public goods. Each agent reports his first best alternative along a one-dimensional space, and the farther from that alternative the location is, the least level of satisfaction does the agent receive. As it is well known, the alternatives chosen in this framework, are either one of the n voters' peak or the $(n-1)$ phantoms located along the alternative space. If we allow having an outside option, creating indifferences in the bottom, Cantala (2004) shows that the strategy-proof and efficient social choice functions select exclusively one of the n peaks of the agents.

Another important domain restriction that accomplish both Value Restriction and the necessary and sufficient conditions found in Chapter 1, are single-dipped preferences. This type of preferences assume that each agent considers one of the alternatives the least desirable, that is, each agent reports a dip or the least preferred alternative, and locations become better as they are placed farther from the dip. They are used to analyze the problem of the location of public bads, as their externalities are not desirable. Manjunath (2009) characterized the family of strategy-proof and efficient social choice functions under this domain. Barberà, Berga, and Moreno ($2011a, b$) studied the range of such functions in this domain. The result is that the range contains at most two locations, and in particular, they are the extremes of the line considered as

alternatives.

In Chapter 2, we assume that preferences are single-dipped, but we extend the preferences to allow them have indifferences on the top. It is assumed that each agent might reach his highest level of satisfaction, and then remain indifferent on the location up to the frontier of the one-dimensional space of alternatives. Hence, we include the case where an agent considers a location sufficiently far from his dip to perceive the negative externalities of the public bad; therefore, the agent considers from that location onward, the best possible locations. Under this framework, we characterize the family of strong group strategy-proof and efficient social choice functions. We want to avoid that a group of agents might profitable misrepresent their true preferences and to have an outcome which is non-dictatorial. The results show that the range of the social choice function is not necessarily the extremes of the interval of locations, but it can be a subset of alternatives in between.

The underlying assumption of these two first chapters is that there is complete information about the alternatives to choose. However, it is not always the case that agents now exactly which are the available alternatives. For instance, if the collective decision is over politicians, voters might know their names, but not the policy they would implement if running office. The incomplete information does not let the voters know which candidate to vote for, since their concerns are over the policies to be executed. Therefore, politicians send messages through campaigns; they are not exclusively over what policy they would implement, but also about the contender's policy. After the reception of the information, voters create their beliefs over the policies and vote for their favorite candidate. As Roger Ailes said to Nixon in 1968, "campaign ads

are as insulting to the viewer's intelligence as a teddy bear selling toilet tissue, and yet no candidate would dare run a campaign without them".

In Chapter 3 we design a model to capture the idea that in a campaign, a politician does not limit his messages over his own promised policy, but pronounces about his contender. The messages are costly as their reputation is negatively correlated to the distance between their promise or pronouncement and the actual policy implemented after the elections. We want to see if the messages are meaningful under competition, that is, if they reveal the true policies of each of the candidates given initial reputation indexes, and the threat to loose credibility for future elections. Polborn et. al. (2006) create a model where the messages send from the candidates can be either about themselves or about the contender, but not both. He shows that the kind of message they use is a sign of the quality of the contender, however, in political contests strong and weak candidates do talk about their contender. Aragonés et. al (2007) show that in a infinite horizon model, if there is a one-shot trigger threat where candidates might loose all their reputation, still there is a distance from the actual policy where a candidate can promise and change voter's beliefs.

To select an alternative, even if unknown, individual preferences are confronted and have to be aggregated. How to achieve a positive result? We look at properties to restrict the domain and have a Weak Condorcet Winner, characterize rules that are appealing under single-dipped preferences with indifferences, and model a signaling game to reveal the true alternatives for the voters in a campaign.

REFERENCES CITED

1. Aragonés, E.; Palfrey, T; and Postlewaite, A. (2007) "Political reputations and campaign promises", *Journal of the European Economic Association* 5 : 846 − 884

2. Barberà, S., Berga, D., and Moreno, B. (2011*a*): "Domains, Ranges and Strategy-proofness: the Case of Single-dipped Preferences". Forthcoming in *Social Choice and Welfare*

3. Barberà, S., Berga, D., and Moreno, B. (2011*b*): "Group Strategy-proof Social Choice Functions with Binary Ranges and Arbitrary Domains: Characterization Results". Forthcoming in the *International Journal of Game Theory*

4. Cantala, D. (2004): "Choosing the Level of a Public Good when Agents Have an Outside Option", *Social Choice and Welfare* 22 : 491 − 514

5. Allan G. (1973), "Manipulation of voting schemes: a general result", *Econometrica*, 41 : 587 − 601

6. M. de Condorcet. (1785): *Essai sur l'application de l'anlayse à la probabilité des décisions rendues à la pluralité des voix*

7. Manjunath, V. (2010): "Efficient and Strategy-proof Social Choice when Preferences are Single-dipped". Mimeo

8. Moulin, H. (1980): "On Strategy-Proofness and Single-Peakedness", *Public Choice* 35 : 437 − 55

9. Polborn, M. and Yi, D. (2006) "Informative positive and negative campaigning", *Quarterly Journal of Political Science* $1 : 351 - 371$

10. Mark A. S. (1975), "Strategy-proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions", *Journal of Economic Theory* $10 : 187 - 217$

11. Sen, A. and Pattanaik, P. K. (1969): "Necessary and Sufficient Conditions for Rational Choice under Majority Decision", *Journal of Economic Theory* $1 : 178 - 202$

CHAPTER 1.

ON WEAK CONDORCET WINNERS: EXISTENCE AND

UNIQUENESS

1.1 Introduction

Consider a society formed by a finite set of agents with individual weak preferences over a finite set of alternatives with cardinality bigger than three. Condorcet (1785) proposed the majority principle which states that an alternative y could not be selected if more than half of the individuals preferred x over y. However, it could be the case that there exists no alternative which beats by majority every other alternative or there might not be a unique winner. The existence is an important issue if the society agrees to use a rule which selects a Weak Condorcet Winner among the available alternatives. Rules as the Majority Voting or the Copeland Rule select the Weak Condorcet Winner when it exists. Thus, we propose a condition in preference profiles to ensure the existence of a Weak Condorcet Winner. This is the first step to achieve domain restrictions compatible with rules which select the Weak Condorcet Winner. On the other hand, the uniqueness of a Weak Condorcet Winner will be relevant given the objective of the society. If its purpose is the selection of a single alternative by the majority principle, we would like to identify conditions on preference profiles which ensure the uniqueness of a Weak Condorcet Winner. Society could have a disagreement in the tie-breaking rule and might not reach a solution if more than one Weak Condorcet Winner exists.

Black (1958) was aware that a Weak Condorcet Winner would exist; thus, the

majority decision was going to be plausible, when the aggregation of preferences yields transitive social preferences. Sen and Pattanaik (1969) improved this observation and came up with the Value Restriction as a sufficient condition to have a Weak Condorcet Winner in the preference profile. In this paper we propose necessary and sufficient conditions in the preference profile for the existence and the uniqueness of the Weak Condorcet Winner. The Upmost Condition will ensure that the set of Weak Condorcet Winners is non-empty. This condition will look only at those alternatives which are strictly preferred by a given alternative x. The Cycle Condition will guarantee that the Weak Condorcet Winner is unique looking at each triple of alternatives.

The paper is organized as follows. In section 2 we introduce the basic notation and the concept of the majority principle. Section 3 contains the results for the conditions in preferences profiles to ensure the existence and uniqueness of a Weak Condorcet Winner. We conclude with some final remarks.

1.2   Preliminaries

Let X be the non-empty set of alternatives with cardinality m. Let $N = \{1, ..., n\}$ be the set of agents. A preference profile $R = (R_1, ..., R_n)$ is an n-tuple of complete, reflexive, and transitive binary relations on the set of alternatives X. We should read $x R_i y$ as x is as least as good as y for agent i. Therefore, for all $x, y, z \in X$ and $i \in N$, either $x R_i y$ or $y R_i x$; $x R_i x$; and if $x R_i y$ and $y R_i z$ then $x R_i z$. Let P_i be the asymmetric part of the binary relation R_i and I_i denote the indifference relation of the binary relation R_i. Denote by $\mathcal{R}$ the set of individual preferences and by $\mathcal{R}^N$ the set

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of all preference profiles. A domain $\hat{D} = D_1 \times ... \times D_n \subseteq \mathcal{R}^N$ is a cartesian product subset of $\mathcal{R}^N$.

Denote, at a given profile R, the cardinality of the set of individuals who strictly prefer x to y by $N(xPy)$. That is, $N(xPy) \equiv |\{i \in N \mid xP_iy\}|$. The concerned individuals are the number of agents who are not indifferent between x and y; that is, $n^*(x, y; R) = N(xPy) + N(yPx)$.

If for all $y, x \in X$, $y \neq x$, strictly more than half of the concerned agents prefer alternative x over y, then such alternative x is called a Strong Condorcet Winner.

Definition 1..1 *A **Strong Condorcet Winner** at profile R is an alternative $x \in X$ such that $N(xPy) > N(yPx)$ for all $y \in X \backslash \{x\}$.*

If it exists, a Strong Condorcet Winner is unique and it does not tie with any alternative in pairwise majority comparison. However, there are profiles R where some alternatives defeat or tie the rest of the alternatives by pairwise majority comparison; we refer to such alternatives as the Weak Condorcet Winners.

Definition 1..2 *A **Weak Condorcet Winner** at profile R is an alternative $x \in X$ such that $N(xPy) \geq N(yPx)$ for all $y \in X \backslash \{x\}$.*

For some profiles R, we can have two type of situations when we are looking for a Weak Condorcet Winner:

a) Society is not be able to find any Weak Condorcet Winner and face the so called Condorcet Paradox. Consider three agents $N = \{1, 2, 3\}$ and three

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alternatives $X = \{a, b, c\}$. Let the profile R be such that aP_1bP_1c, bP_2cP_2a, and cP_3aP_3b. Then, by majority a beats b, b beats c, and c beats a. Thus, there exists no alternative which defeats all the rest by majority pairwise comparison.

b) Although a Weak Condorcet Winner exists for every profile $R \in \hat{D} \subseteq \mathcal{R}^N$, it might not be unique. If the objective is the selection of a single alternative by the majority principle, having more than one Weak Condorcet Winner might cause that the society does not reach an agreement.

We will characterize the profiles R where the set of Weak Condorcet Winners is non-empty and we will differentiate between profiles where this set is a singleton or has cardinality strictly bigger than one.

1.3 Results

There is not a complete characterization of profiles for which Weak Condorcet Winners exists. Sen and Pattanaik (1969) proposed the Value Restriction property on a profile R to ensure the existence of Weak Condorcet Winners.

Definition 1..3 *A profile R satisfies **Value Restriction** if for every triple $\{x, y, z\}$ $\subseteq X$ there is some alternative, say x, such that all individuals agree that x is not the worst, or agree x is not the best, or agree x is not medium. That is, one of the following three conditions hold:*

1. *for all $i \in N$, either xP_iy or xP_iz; or*

2. *for all $i \in N$, either yP_ix or zP_ix; or*

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3. for all $i \in N$, either (xP_iy and xP_iz) or (yP_ix and zP_ix).

Theorem 1..1 *(Sen and Pattanaik, 1969) A sufficient condition for a profile R to have a Weak Condorcet Winner is that R satisfies Value Restriction.*

Theorem 1 gives a sufficient condition for the existence of Weak Condorcet Winners at R. However, in Example 1 we observe that there might be cases in which not all triples satisfy Value Restriction and still have a Weak Condorcet Winner.

Example 1..1 *Let $N = \{1, 2, 3\}$ and $X = \{x, y, z, w\}$. The profile R*

P_1	P_2	P_3
w	z	y
x	x	x
y	w	z
z	y	w

does not satisfy Value Restriction since the triple $\{y, z, w\}$ has the property that no alternative is considered neither the best, nor the worst, nor the medium. Nevertheless, x is the Weak Condorcet Winner. Furthermore, as x does not tie with any other alternative, it is the Strong Condorcet Winner.

Existence of Weak Condorcet Winners

In this subsection, we will give a necessary and sufficient condition to be satisfied by a preference profile to have a non-empty set of Weak Condorcet Winners. For this purpose we will focus on those alternatives which are stricly better than a given alternatve x for each agent i at a profile R.

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Definition 1..4 *The **strict upper contour set** of a binary relation R_i at x, denoted by $U(R_i, x)$, is the set of alternatives $y \in X\backslash\{x\}$ that are strictly prefered to x; that is $U(R_i, x) = \{y \in X \mid yP_ix\}$.*

Given an alternative $x \in X$, we can look at the upper contour sets and identify the number of agents which prefer another alternative y over x.

Definition 1..5 *We say that a profile R satisfies the **Upmost Condition**, if there exists at least one alternative $x \in X$ such that for all $y \in X\backslash\{x\}$, $|\{i \in N \mid y \in U(R_i, x)\}| \leq \dfrac{n^*(x, y; R)}{2}$.*

Denote by x_{uc} a generic alternative which makes that profile R satisfies the Upmost Condition. If profile R satisfies the Upmost Condition at $x_{uc} = x$, at most half of the concerned agents have the rest of the alternatives in their upper contour sets at x.

Proposition 1..1 *Let $n \geq 3$ and $m \geq 3$. A profile R satisfies the Upmost Condition if and only if the set of Weak Condorcet Winners is non-empty.*

Proof. To prove sufficiency, suppose R satisfies the Upmost Condition. Hence, there exists at least one alternative $x \in X$ such that for all $y \in X\backslash\{x\}$, $|\{i \in N \mid y \in U_i(R_i, x)\}| \leq \dfrac{n^*(x, y; R)}{2}$. By definition, if $y \in U_i(R_i, x)$ then yP_ix. Therefore, the number of agents who strictly prefer y to x is equal or less than half of the concerned agents; that is, $|\{i \in N \mid y \in U_i(R_i, x)\}| = N(yPx) \leq \dfrac{n^*(x, y; R)}{2}$. Thus, $2N(yPx) \leq n^*(x, y; R)$, then $N(yPx) \leq n^*(x, y; R) - N(yPx)$, which indeed is $N(xPy) \geq N(yPx)$ for all $y \in X\backslash\{x\}$. Hence, x is a Weak Condorcet Winner.

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Now, to prove necessity, assume that the set of Weak Condorcet Winners is non-empty at R. Hence, there exists $x \in X$ such that $N(xPy) \geq N(yPx)$ for all $y \in X\backslash\{x\}$. This means that $n^*(x, y; R) - N(yPx) \geq N(yPx)$, which is only possible if $\dfrac{n^*(x, y; R)}{2} \geq N(yPx)$. As $y \in U(R_i, x)$ if yP_ix then we have that $N(yPx) = |\{i \in N \mid y \in U(R_i, x)\}| \leq \dfrac{n^*(x, y; R)}{2}$ for all $y \in X\backslash\{x\}$. Thus, R satisfies the Upmost Condition. ∎

The Upmost Condition is giving us the existence of Weak Condorcet Winners at profile R. The following example illustrates a profile R satisfying the Upmost Condition.

Example 1..2 *Consider $N = \{1, 2, 3, 4\}$, $X = \{x, y, z\}$, and the profile R where*

R_1	R_2	R_3	R_4
y	x	z, y	x
x	z	x	z, y
z	y		

Take $x \in X$ and all the strict upper contour sets $U(R_i, x)$. Then,

$$|\{i \in N \mid y \in U_i(R_i, x)\}| = 2 = \frac{n^*(x, y; R)}{2} = 2, \text{ and}$$

$$|\{i \in N \mid z \in U_i(R_i, x)\}| = 1 < \frac{n^*(x, z; R)}{2} = 2.$$

Hence, R satisfies the Upmost Condition with $x_{uc} = x$. Alternative $x \in X$ beats or ties all the rest of the alternatives by pairwise majority comparison. Hence, x belongs to the set of Weak Condorcet Winners.

Eventhough a profile R satisfies the Upmost Condition, we are not excluding the possibility of having more than one Weak Condorcet Winner.

<u>Uniqueness of the Weak Condorcet Winner</u>

Having more than one Weak Condorcet Winner can be a problem to define a social choice function or to reach an agreement between individuals. However, it is not always the case in which the objective of a society is to select only one alternative under the majority principle. For example, there might be a pre-selection of alternatives from a pool of candidates previous to the final voting procedure. In either case, it is interesting to know if the Weak Condorcet Winner is unique. We next identify a condition at a profile R to determine if the set of Weak Condorcet Winners is a singleton or not. Like the Value Restriction property, we will focus on properties by each triple of alternatives in X. Hence, we will first define two configurations a triple of alternatives might have.

Definition 1..6 *Let R be a profile. A triple of alternatives $\{w, y, z\} \subseteq X$ is a **cycle** at R if:*

$N(wPy) > N(yPw),$

$N(yPz) > N(zPy),$ *and*

$N(zPw) > N(wPz).$

Definition 1..7 *Let R be a profile. A triple of alternatives $\{w, y, z\} \subseteq X$ is a **semi-cycle** at R if:*

$N(wPy) > N(yPw),$

$N(yPz) > N(zPy),$ *and*

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$N(zPw) = N(wPz)$.

In this case, we will say that w *leads* the semi-cycle, as it is the only alternative in the triple $\{w, y, z\}$ which is not beaten by pairwise majority comparison with the rest.

A set of alternatives X with cardinality m has ${}_mC_3 = \dfrac{m!}{3!(m-3)!}$ possible triples of alternatives. These triples might be cycles, semi-cycles, or neither of them. Notice that an alternative $y \in X$ is in more than one possible triple; for example, if $m = 4$, then each alternative $y \in X$ will be in 3 of the ${}_4C_3 = 4$ possible triples of alternatives. Looking at each triple, we can see if an alternative is a candidate to be a Weak Condorcet Winner and say something about the cardinality of the set.

Definition 1..8 *Let R be a profile. An alternative $y \in X$ is **beaten inside a cycle** at R if either y belongs to a cycle or y belongs to a semi-cycle and it is not the leader.*

Definition 1..9 *Let R be a profile. An alternative $y \in X$ is **beaten outside a cycle** at R if y is not beaten inside a cycle and $|\{i \in N \mid y \in U_i(R_i, a)\}| < \dfrac{n^*(y, a; R)}{2}$ for some $a \in X$.*

It is easy to see that there might be alternatives which are not beaten neither inside nor outside the cycle. If there is a unique Weak Condorcet Winner, there will only be one alternative which is not beaten in any cycle.

Definition 1..10 *We say that a profile R satisfies the **Cycle Condition** if there exists an alternative $x \in X$, such that for all $y \in X \backslash \{x\}$, y is beaten inside a cycle or y is beaten outside a cycle.*

Denote by x_{cc} a generic alternative that ensures that profile R satisfies the Cycle Condition.

Proposition 1..2 *Let $n \geq 3$ and $m \geq 3$. There exists a unique Weak Condorcet Winner $x \in X$ at R if and only if the Upmost Condition and the Cycle Condition hold with $x = x_{uc} = x_{cc}$.*

Proof. To prove necessity, suppose $x \in X$ is the unique Weak Condorcet Winner at R. By Proposition 1, the Upmost Condition holds. Moreover, as x is the unique Weak Condorcet Winner, $x = x_{uc}$ and looking at any alternative $y \in X \backslash \{x\}$, we can distinguish two cases:

Case i) $y \in X \backslash \{x\}$ belongs to a cycle or a semi-cycle without being the leader. By definition, we know that $N(wPy) > N(yPw)$ with an alternative w of the cycle or semi-cycle. As y looses by pairwise comparison within that specific triple, y is beaten inside a cycle.

Case ii) $y \in X \backslash \{x\}$ does not belong to any cycle or it is the leader of a semi-cycle; hence, y is not beaten inside a cycle. As x is the unique Weak Condorcet Winner, the rest of the alternatives $y \in X \backslash \{x\}$ should loose by pairwise majority comparison with at least one alternative in X; in other words, $N(aPy) > N(yPa)$ for some $a \in X$. Using the concerned individuals, we can rewrite it as $n^*(y, a; R) - N(yPa) > N(yPa)$; thus, $N(yPa) < \dfrac{n^*(y, a; R)}{2}$. By definition, if yP_ia, then $y \in U_i(R_i, a)$. We conclude that $N(yPa) \equiv |\{i \in N \mid y \in U_i(R_i, a)\}| < \dfrac{n^*(y, a; R)}{2}$. Hence, y is beaten outside the cycle.

Thus, all $y \in X \backslash \{x\}$ are beaten either inside or outside a cycle. Hence, R

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satisfies the Cycle Condition with $x = x_{cc}$.

To prove sufficiency suppose that R satisfies the Upmost Condition and the Cycle Condition with $x = x_{uc} = x_{cc}$. By Proposition 1, we know that if R satisfies the Upmost Condition, then the set of Weak Condorcet Winners is non-empty. In particular, there is at least one alternative $x_{uc} \in X$ which is a Weak Condorcet Winner. Hence x_{uc} beats or ties with the rest of the alternatives $y \in X \backslash \{x_{uc}\}$. Since R satisfies the Cycle Condition, there exists an alternative $x_{cc} \in X$, such that for the rest of the alternatives $y \in X \backslash \{x_{cc}\}$ one of the following two cases holds:

Case i) y is beaten inside a cycle. Hence, either (1) y belongs to a cycle $\{y, z, w\} \subseteq X$ so $N(yPz) > N(zPy)$, $N(zPw) > N(wPz)$, and $N(wPy) > N(yPw)$ or (2) y belongs to a semi-cycle $\{y, z, w\} \subseteq X$ and does not lead it; that is $N(wPy) > N(yPw)$, $N(yPz) > N(zPy)$, and $N(zPw) = N(wPz)$. Hence y is beaten in pairwise comparison by some alternative within the triple.

Case ii) y is beaten outside a cycle. Hence, either (1) y belongs to a semi-cycle and leads it or (2) y is not in a cycle or semi-cycle. Then, as y is beaten outside a cycle, there exists an alternative $a \in X \backslash \{y\}$ such that $|\{i \in N \mid y \in U_i(R_i, a)\}| < \dfrac{n^*(y, a; R)}{2}$. We can arrage the terms as $2 |\{i \in N \mid yP_ia\}| < n^*(y, a; R)$, and $N(yPa) < n^*(y, a; R) - N(yPa)$; that is, $N(aPy) > N(yPa)$ for some $a \in X \backslash \{y\}$. Therefore, y is not a Weak Condorcet Winner, as it has been beaten in pairwise comparison by a.

Thus, if the Cycle Condition holds, none of the alternatives $y \in X \backslash \{x_{cc}\}$ can be a Weak Condorcet Winner at R. Furthermore, as x_{uc} is in the set of Weak Condorcet Winners, it has neither being beaten inside nor outside a triple. Hence $x_{uc} = x_{cc} = x$ is the unique Weak Condorcet Winner at R. ∎

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Example 3 illustrates a profile R satisfying the Cycle Condition.

Example 1..3 *Consider $N = \{1, 2, 3, 4\}$, $X = \{w, x, y, z\}$, and the profile R where*

	R_1	R_2	R_3	R_4
	y	x, w	z, y	x, z
	x	z	x	y, w
	w	y	w	
	z			

Notice that

$$N(zPy) > N(yPz),$$

$$N(yPw) > N(wPy),$$

$$N(xPw) > N(wPx),$$

$$N(xPz) > N(zPx), \text{ and}$$

$$N(wPz) = N(zPw) = N(yPx) = N(xPy).$$

The possible triples of alternatives are $\{w, x, y\}$, $\{w, x, z\}$, $\{w, y, z\}$, and $\{x, y, z\}$. The Cycle Condition holds with $x = x_{cc}$ as $\{w, y, z\} \subseteq X \backslash \{x\}$ is a semi-cycle with z as the leader. Hence, w and y are beaten inside the cycle and z is beaten outside the cycle as $|\{i \in N \mid z \in U_i(R_i, x)\}| < \dfrac{n^*(z, x; R)}{2}$. Futhermore, the Upmost Condition holds with $x = x_{uc}$ since $|\{i \in N \mid y \in U_i(R_i, x)\}| \leq \dfrac{n^*(x, y; R)}{2}$ for all $y \in X \backslash \{x\}$. Hence x is the unique Weak Condorcet Winner.

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1.4 Final Remarks

The majority principle is an attractive property of any social choice election procedure. No matter if an alternative or a set of alternatives must be chosen, it is reasonable to think that those selected alternatives should not be defeated by pairwise majority comparison with any other. Nevertheless, even allowing ties, these Weak Condorcet Winners might not exist. Sen and Pattanaik (1969) gave a sufficient condition which helps to identify profiles where Weak Condorcet Winners exist. In particular, Value Restriction is a very strong condition and has been very important in Social Choice Theory, since domains such as Single-Peaked or Group Separable are derived from this property. The Upmost Condition gives a necessary and sufficient condition to have the existence of Weak Condorcet Winners.

We have given examples where the society has to choose several alternatives. As a corollary of the Proposition 1 and Proposition 2, we can state that given $n \geq 3$ and $m \geq 3$, a profile R satisfying the Upmost Condition but not the Cycle Condition has a set of Weak Condorcet Winners with cardinality strictly bigger than one. In such cases, we can be able to apply voting procedures which selects the set of Weak Condorcet Winners. Borm et. al. (2004) introduced the β and λ social choice correspondences which are consistent with the majority principle. Laffond et. al. (1995) analysed and compared several Condorcet consistent rules as the Copeland Rule, the Slater set, the Banks set, and others.

Results do not change if the preference profile R does not admit indifferences; that is, if R is asymmetric. However, in this scenario, we should always have an even

number of agents in order to allow ties by pairwise majority comparison and have Weak

Condorcet Winners.

REFERENCES CITED

1. Borm, P., Brink, R., Levínský, R., and Slikker, M. (2004): "On two new social choice correspondences", *Mathematical Social Sciences* $47 : 51 - 68$

2. Black, D. (1958): *The Theory of Committees and Elections.* Cambridge University Press, Cambridge, MA

3. Laffond, G., Laslier, J.F., and Breton, M. (1995): "Condorcet Choice Correspondences: A Set-Theoretical Comparison", *Mathematical Social Science* $30 : 23 - 35$

4. M. de Condorcet. (1785): *Essai sur l'application de l'anlayse à la probabilité des décisions rendues à la pluralité des voix*

5. Sen, A. and Pattanaik, P. K. (1969): "Necessary and Sufficient Conditions for Rational Choice under Majority Decision", *Journal of Economic Theory* $1 : 178 - 202$

CHAPTER 2.

SINGLE-DIPPED PREFERENCES WITH INDIFFERENCES: STRONG GROUP STRATEGY-PROOFNESS AND UNANIMITY

2.1 Introduction

Public bads, such as a dumpsite or a nuclear plant, cause negative externalities like pollution, noise, radioactivity, bad smells, or even illnesses. The location of a public bad is a problem which concerns the agents that will be affected by its existence. Each of them can have different preferences over the location of the public bad, making this social decision a difficult task. Where to place the public bad when the opinions about the best location differ?

Various disciplines have studied the location of a public bad and the response of the agents to such problem. Since the 1980s, sociologists observed the phenomenon of local opposition to the location of public bads and named it the NIMBY (Not In My Back Yard) phenomenon. The NIMBY literature focuses on the reasons why people refuse to have a public bad near them, even though they are facilities that provide services needed. Freudenberg and Pastor (1992) provide a useful review of the NIMBY literature and suggest that the phenomenon can be described via two distinct perspectives. The first is of NIMBY as an ignorant or irrational response by the agents. The second perspective is that it is a selfish response to the location of a public bad. However, sociologists have not studied the rules which can be useful to solve such problems.

In Economics, the Social Choice literature has studied different rules or social

choice functions which can be used for the decision of the location of a public facility. The agents' preferences have been modeled by two different views, depending on whether the future facility is a good or a bad. If the future facility is a good, each agent has an ideal point about the location of the public good, which is called the "peak", and as the location goes farther from the peak the agent's satisfaction strictly decreases. If the future facility is a bad, there exists a location which gives the least level of satisfaccction for the agent, which is called the "dip", and the agent strictly prefers any location farther from it. Furthermore, it has also been studied different social choice functions which are requiered to accomplish some features such as strategy-proofness or efficiency. These are two appealing properties for the social choice function: strategy-proofness avoids the misrepresentation of preferences, and efficiency guarantees that the well-being of all agents can not improve by changing the location of a public facility.

The misrepresentation of preferences can be done either by a single agent or by a group of agents. If no agent by himself can manipulate the rule by misrepresenting his preferences, then the rule is strategy-proof. In this paper we look at the property that no group of agents can profitable misrepresent their preferences becoming better of or remaining at least as good as with the report of their true preferences; that is, on strong group strategy-proof rules. With respect to efficiency, we use the stronger notion of unanimity. We look for rules that select an alternative considered as least as good as any other for all agents if it exists.

Moulin (1980) characterized the family of strategy-proof and tops-only social choice functions under *single-peaked preferences* for the location of desirable public

goods known as the "generalized median voter rules". He also characterized a particular interesting subclass, the family of strategy-proof, anonymous, and efficient social choice functions.[1] Each generalized median voter rule, in this subclass, selects the median of the n agents' peaks plus $n - 1$ phantom voters added along the alternative space. Cantala (2004) extended the concept of single-peaked preferences to allow natural and meaningful indifferences; it is assumed that agents have single-peaked preferences but each agent may be indifferent among all locations from one point onward. To illustrate such preferences, consider the following figure with 5 agents and the interval $[0, T] \subset \mathbb{R}$ as the set of alternatives.
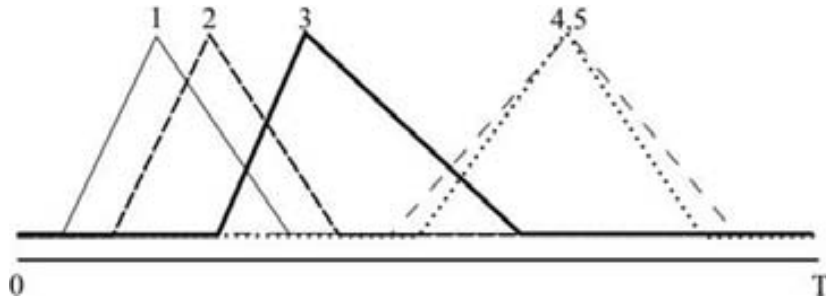

Figure 1. Single-peaked preferences with indifferences

The only strategy-proof, efficient, and anonymous social choice functions under this domain restriction are the median voter rules with the phantom voters located only at the bounds of the alternative space. Hence, the chosen alternative of the social choice function under this framework is one of the peaks of the agents. That is, under single-peaked preferences, the inclusion of indifferences reduce the set of potential chosen alternatives of any strategy-proof, efficient, and anonymous social choice function in a drastic but desirable way.

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[1]Observe that strategy-proofness and efficiency implies tops-onlyness.

When the issue is to locate a public bad, then it is natural to assume that agents' preferences are *single-dipped.* In this case, each agent considers a location as the worst possible within the set of alternatives, called the "dip", and the agent strictly prefers any location farther from it. Figure 2 illustrates the single-dipped preference R_i of agent i that strictly prefers T over 0 and has a dip $d(R_i)$.
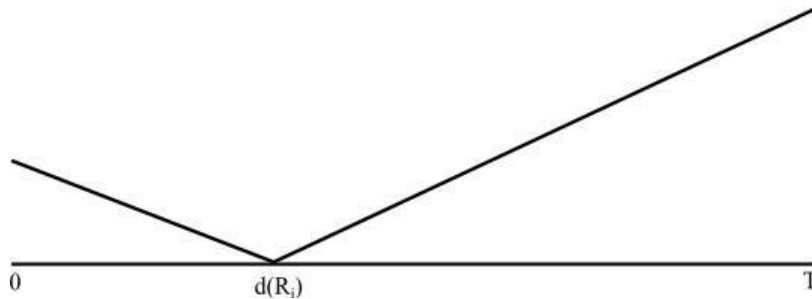

Figure 2. Single-dipped preferences

Nevertheless, single-dipped preferences exclude some interesting situations: these preferences do not allow to have a location from where the agent does not perceive the negative externalities anymore, making him reach his maximum level of satisfaction and being indifferent from that location onward. For example, an agent living near a mountain such as in Figure 3. The satisfaction of placing the dumpsite to the right hand side of his dip will increase up to a point where the mountain prevents the agent to perceive the pollution; from that location onwards the agent is indifferent about where to place the public bad. Having no reference in the literature for such type of preferences, we called them *single-dipped preferences with indifferences.*
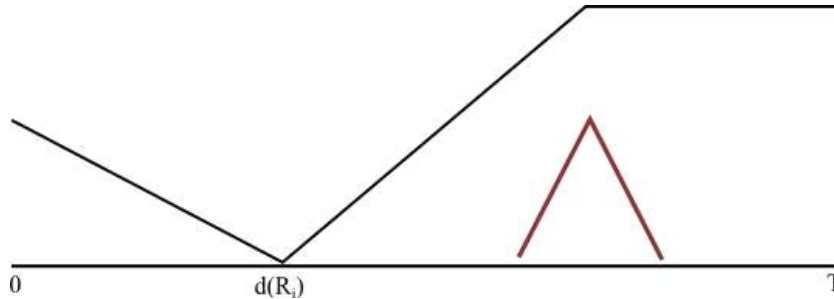
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Figure 3. An agent is indifferent in the location of a public bad when he does not

perceive the negative externalities

Manjunath (2010) studied the location of a public bad under the assumption of single-dipped preferences along a closed interval $[0, T] \subset \mathbb{R}$. He characterized the class of efficient and strategy-proof social choice functions under this preference domain. His results show that the range of an efficient and strategy-proof social choice function is the set $\{0, T\}$, the two extremes of the interval. Barberà, Berga, and Moreno (2011a) proved that the range of strategy-proof social choice functions under single-dipped preferences contains two alternatives at most.

Furthermore, Manjunath (2010) showed that under single-dipped preferences, strong group strategy-proofness is equivalent to strategy-proofness. That is, the non-profitable misrepresentation of the preferences by a group of agents or by a single agent is equivalent. Barberà, Berga, and Moreno (2011b) studied group strategy-proof social choice functions with binary ranges and proved that strong group strategy-proofness implies group strategy-proofness.[2]

In this paper we characterize the family of strong group strategy-proof and

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[2]It can be shown that the equivalence between strategy-proof and group strategy-proofness holds under our extension of single-dipped preferences with indifferences.

unanimous social choice functions under single-dipped preferences with indifferences: the class of *full agreement rules*. Given our extension of preferences, we show that the range of a strong group strategy-proof and unanimous social choice function may be larger than just two alternatives, so at some profiles the chosen alternative is not necessarily one of the extremes of the interval.

The paper is organized as follows. In section 2 we introduce the basic notation and definitions. In section 3 we prove our main characterization result. Then, in section 4 we present the proof of the main Theorem, and we finish in section 5 with some final remarks.

2.2 Notation and definitions

Let $N = \{1, ..., n\}$ be the set of agents and let $[0, T] \subseteq \mathbb{R}$ be the set of alternatives. The preference of each agent $i \in N$ is a complete, reflexive, continuous, and transitive binary relation R_i over $[0, T]$. We denote the strict part of R_i by P_i and the indifference part of R_i by I_i. Let $\mathcal{R}$ denote the class of all possible preferences on $[0, T]$. We assume that agents have to collectively choose the location of a public bad and their preferences satisfy the following condition.

Definition 2..1 *The preference $R \in \mathcal{R}$ is **single-dipped with indifferences** if there exists a unique alternative, the "dip" at R_i denoted by $d(R_i)$, and two alternatives $0 \leq l(R_i) < h(R_i) \leq T$ such that $l(R_i) \leq d(R_i) \leq h(R_i)$, and:*

1. *for all $a, b \in [l(R_i), h(R_i)]$, $[a < b \leq d(R_i)$ or $d(R_i) \leq b < a]$, implies aP_ib;*

2. *for all $a, b \in [0, l(R_i)]$, aI_ib;*

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3. for all $a, b \in [h(R_i), T]$, aI_ib; and

4. if $0 < l(R_i) < h(R_i) < T$, then $l(R_i)I_ih(R_i)$.

We denote the set of all preferences that are single-dipped with indifferences on $[0, T]$ by $\mathcal{R}_D$. Figure 4 illustrates thee preferences in $\mathcal{R}_D$.
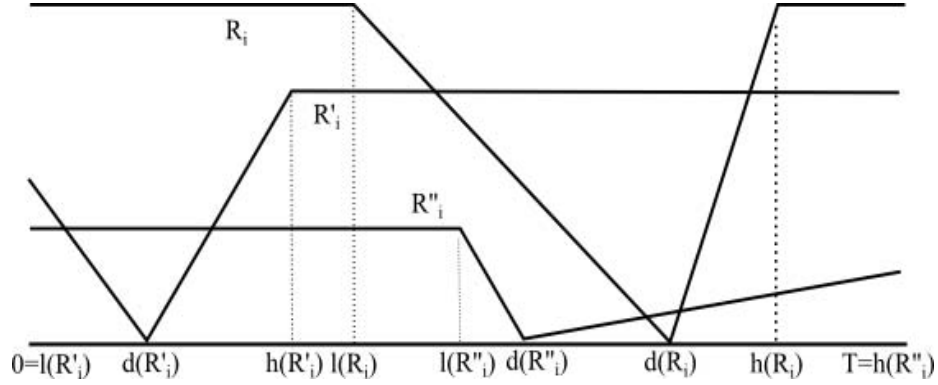

Figure 4. Single-dipped preferences with indifferences

We assume that the indifferences appear when there exist a maximum level of satisfaction or a satiation level for each agent. That is, the agent may reach an alternative where he becomes indifferent if the public bad is in that location up to the end of the set of alternatives because he does not receive the negative externality anymore. As the agent is not affected by the public bad, hence, those alternatives become the best location of the public bad for such agent.

A preference profile $R = (R_1, ..., R_n)$ is a $n-$tuple of all agents' preferences. Let $\mathcal{R}_D^n$ denote the set of all possible preference profiles where each agent's preferences is single-dipped with indifferences, i.e. $\mathcal{R}_D^n = \mathcal{R}_D \times ... \times \mathcal{R}_D$. Let $i \in N$ be an agent and R be a preference profile; denote by R_{-i} the $n-1$ tuple of all agents' preferences except

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i. For each $M \subseteq N$, R_M denotes all preferences of the agents in M, and R_{-M} denotes the preferences of all agents that are not in M.

Given $R \in \mathcal{R}_D^n$, let

$$N_0(R) = \{i \in N : 0P_iT\},$$

$$N_T(R) = \{i \in N : TP_i0\}, \text{ and}$$

$$N_{0T}(R) = \{i \in N : 0I_iT\}.$$

Given $M \subseteq N$ and $R_M \in \mathcal{R}_D^m$, let

$$l_{min}(R_M) = \min\{l(R_i) : i \in M\} \text{ and}$$

$$h_{max}(R_M) = \max\{h(R_i) : i \in M\}.$$

Notice that if $N_0(R)$ or $N_T(R)$ are non empty, then $h_{max}(R_{N_0(R)}) = T$ or $l_{min}(R_{N_T(R)}) = 0$ respectively.

Given $R_i \in \mathcal{R}_D$ and $a \in [0, T]$, the **lower contour set** set of R_i at a is $W(a; R_i) = \{b \in [0, T] : aR_ib\}$.

Given $R \in \mathcal{R}_D^n$, let

$$a_0(R) = \min_{i \in N_T(R)} [\max W(0; R_i)] \text{ and}$$

$$a_T(R) = \max_{i \in N_0(R)} [\min W(T; R_i)].$$

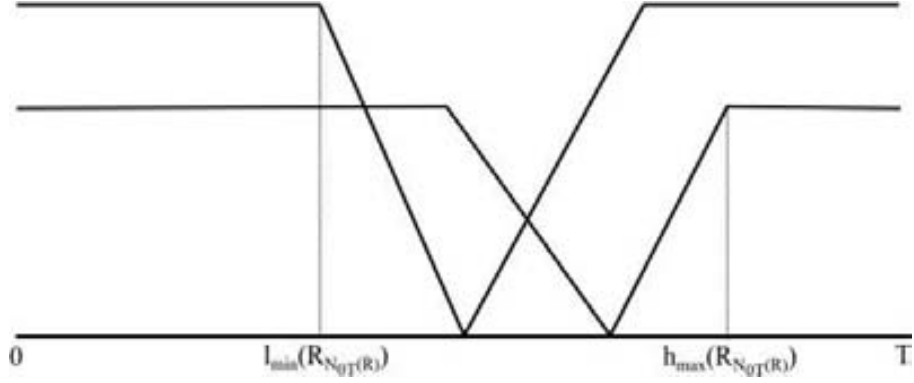Figures 5 and 6 illustrate the previous definitions.

Figure 5. Existence of $l_{min}(R_{N_{0T(R)}})$ and $h_{max}(R_{N_{0T(R)}})$
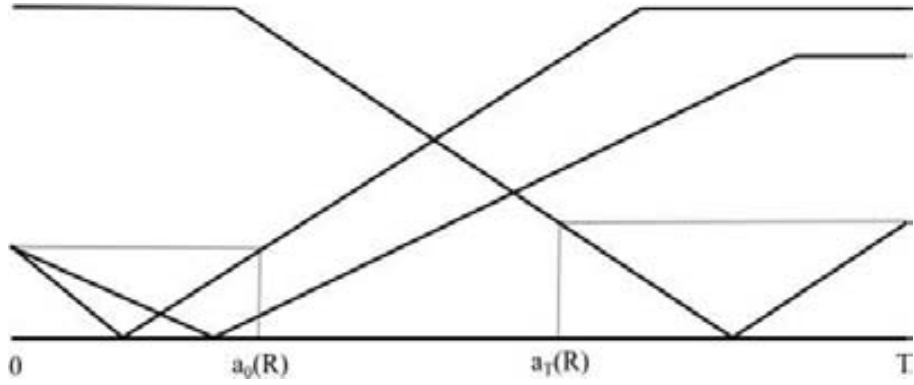

Figure 6. General setting of Single-dipped prefereces with indifferences

A social choice function f associates to each preference profile an alternative, i.e. $f : \mathcal{R}_D^n \to [0, T]$. This function can be as general as possible but we ask to achieve two requirements: to avoid strategic misrepresentations of the preferences, and to have a social outcome which takes into consideration the agents' preferences. Notice that, a trivial rule which avoids misrepresentations of preferences is the "constant rule", which chooses always the same alternative. However, this rule does not take into consideration what the preferences of the agents are. To rule out these type of rules, we will focus on a strong form of efficiency: *unanimity*. This property ensures that, whenever there

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exists a set of locations that are considered to be the best for all agents then, the chosen alternative belongs to that set.

Given $R \in \mathcal{R}_D^n$, a **unanimous interval** in R is an interval $[b, c] \subseteq [0, T]$ such that for all $a \in [b, c]$ and all $i \in N$:

$$a R_i d \text{ for each } d \in [0, T].$$

Given $R \in \mathcal{R}_D^n$, let $U(R)$ be the union of all the unanimous intervals in R. Figure 7 illustrates the set $U(R)$ obtained by the union of several unanimous invervals in R.
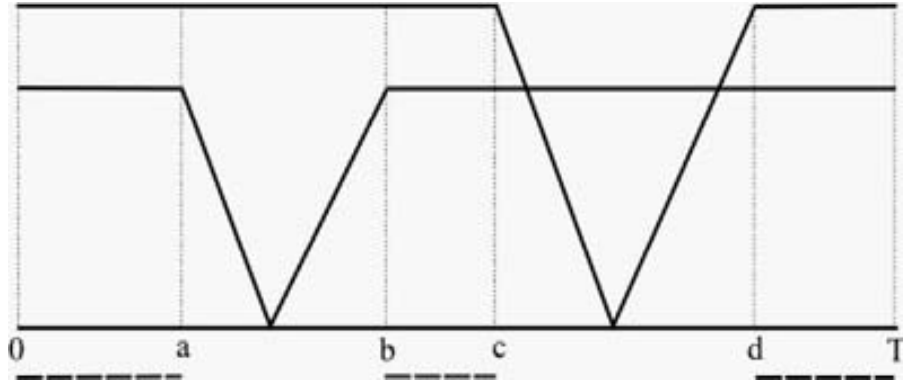

Figure 7. $U(R) = [0, a] \cup [b, c] \cup [d, T]$

Definition 2..2 *A social choice function $f : \mathcal{R}_D^n \to [0, T]$ is **unanimous** if for each $R \in \mathcal{R}_D^n$ such that $U(R) \neq \emptyset$, we have that*

$$f(R) \in U(R).$$

We may also consider a weaker concept than unanimity, that is, Pareto efficiency.

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The **Pareto efficient set** of $R \in \mathcal{R}_D^n$ is the set $P(R) = \{a \in [0,T] : \nexists\, b \in [0,T]$

such that $bR_i a$ for all $i \in N$ and $bP_j a$ for some $j \in N\}$.

Definition 2..3 *A social choice function* $f : \mathcal{R}_D^n \to [0,T]$ *is* **Pareto efficient** *if for*

each $R \in \mathcal{R}_D^n$ *we have that*

$$f(R) \in P(R).$$

Definition 2..4 *Given* $R \in \mathcal{R}_D^n$ *and* $A \subseteq [0,T]$ *the* **Pareto improvement set** *at* A

is $E(A; R) = \{a \in P(R) :$ *for all* $b \in A \setminus \{a\}$, $aR_i b$ *for all* $i \in N$ *and* $aP_j b$ *for some*

$j \in N\}$.

Notice that $E(A; R)$ may be empty. Obviously, $E(A; R) \subseteq P(R)$. To illustrate

the previous concepts, consider the following example.

Example 2..1 *Let* $n = 2$ *and let* $R \in \mathcal{R}_D^2$ *be such that* $0P_1T$ *and* TP_20 *as in figure*

8. Notice that $P(R) = \{0\} \cup (a_0(R), h(R_2)]$. *Moreover,* $E(\{T\}; R) = \{h(R_2)\}$ *and*

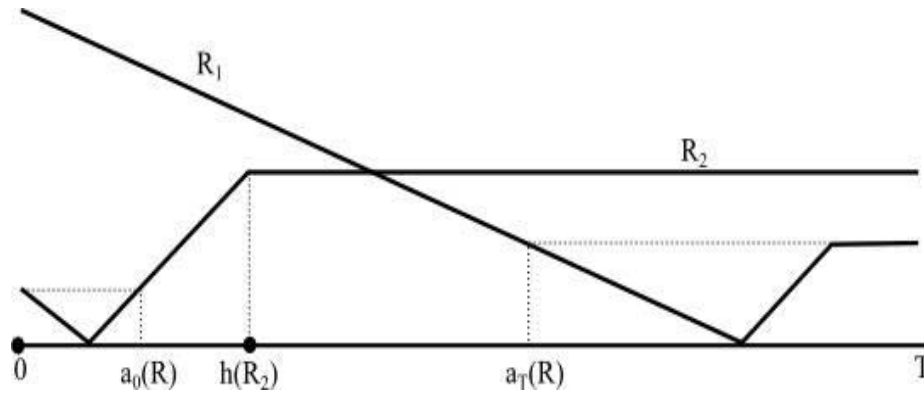$E([0, a_0(R)]; R) = \{0\}$.

Figure 8. $E(\{T\}; R) = \{h(R_2)\}$ and $E([0, a_0(R)]; R) = \{0\}$

We can say more about the Pareto improvement set and its cardinality. The following lemma shows that, at profiles with an empty union of unanimous intervals, if the Pareto improvement set with respect to the extremes of the set of alternatives is non-empty then, it contains only one element. This will be useful for the subsequent results, and will write $E(\{0\}; R)$ and $E(\{T\}; R)$ to denote both the set and its unique element, indistinctly.

Lemma 2..1 *Let $R \in \mathcal{R}_D^n$ be such that $U(R) = \emptyset$. If $E(\{0\}; R) \neq \emptyset$, then $\#E(\{0\}; R) = 1$. If $E(\{T\}; R) \neq \emptyset$, then $\#E(\{T\}; R) = 1$.*

Proof. Assume that $R \in \mathcal{R}_D^n$ is such that $U(R) = \emptyset$ and $E(\{0\}; R) \neq \emptyset$. To obtain a contradiction, suppose that $\#E(\{0\}; R) \neq 1$. Since $E(\{0\}; R) \neq \emptyset$ and $\#E(\{0\}; R) \neq 1$, there exist at least two alternatives $a, b \in E(\{0\}; R)$ such that aR_i0, bR_i0 for all $i \in N$ and aP_j0, $bP_{j'}0$ for some $j, j' \in N$; and $a, b \in P(R)$. First, aI_i0 and bI_i0 for all $i \in N_0(R) \cup N_{0T}(R)$. Hence, aI_ib for all $i \in N_0(R) \cup N_{0T}(R)$. Second, aP_i0 and bP_i0 for all $i \in N_T(R)$. To establish the binary relation between a and b according to P_i for all $i \in N_T(R)$, without loss of generality assume that $a > b$. Then, by the definition of single-dipped preferences with indifferences, it must be that aP_ib for all $i \in N_T(R)$. Since $U(R) = \emptyset$ and $N_T(R) \neq \emptyset$. Then there is no $c \in [0, T]$ such that for all $i \in N$, cR_id for all $d \in [0, T]$. Furthermore, as $a, b \in P(R)$, then there is no $c \in [0, T]$ such that cR_ia and cR_ib for all $i \in N$ and cP_ja and $cP_{j'}b$ for some $j, j' \in N$. However, aP_ib for all $i \in N_T(R)$. Thus, $b \notin P(R)$. Hence $b \notin E(\{0\}; R)$, which is a contradiction.

A symmetric argument can be used for the case where $U(R) = \emptyset$ and $E(\{T\}; R) \neq \emptyset$. ■

We have mentioned another appealing property related to the strategic incentives that a social choice function gives to agents when reporting their preferences: strategy-proofness. A social choice function is strategy-proof if no agent gain by misrepresenting her preference.

Definition 2..5 *A social choice function $f : \mathcal{R}_D^n \to [0, T]$ is **strategy-proof** if for each $R \in \mathcal{R}_D^n$ and each $i \in N$, there is not $R_i' \in \mathcal{R}$ such that*

$$f(R_i', R_{-i}) P_i f(R).$$

Not only do we care about the misrepresentation of the preferences done by a single agent, but also by a subset of them. There are different definitions of group manipulation of a social choice function. The difference between group strategy-proofness and strong group strategy-proofness is that, in the first one, the misrepresentation of the preferences done by a subset of agents has to lead to an outcome which is strictly preferred than the alternative selected under the true preference profile for every agent in the deviating subset. The latter condition asks that all the agents involved in the misrepresentation of the preferences remain as least as good with the new alternative as with the one chosen under the true preference profile, and at least one of them is striclty better off.

Definition 2..6 *A social choice function $f : \mathcal{R}_D^n \to [0, T]$ is **group strategy-proof** if for each $R \in \mathcal{R}_D^n$ and each $M \subseteq N$, there is not $R_M' \in \mathcal{R}_D^m$, such that*

$$f(R_M', R_{-M}) P_i f(R)$$

for all $i \in M$.

Definition 2..7 *A social choice function $f : \mathcal{R}_D^n \to [0, T]$ is **strong group strategy-proof** if for each $R \in \mathcal{R}_D^n$ and each $M \subseteq N$, there is not $R'_M \in \mathcal{R}_D^m$, such that*

$$f(R'_M, R_{-M}) R_i f(R)$$

for all $i \in M$, and $f(R'_M, R_{-M}) P_j f(R)$ for some $j \in M$.

In general, strong group strategy-proofness implies group strategy-proofness, and the latter implies strategy-proofness. It is possible to show that under single-dipped preferences with indifferences, strategy-proofness implies group strategy-proofness; however, strategy-proofness does not imply strong group strategy-proofness. To illustrate this case, consider the family of the serial dictatorship rules in the following example.

Example 2..2 *Let $\prec$ be a linear order on N represented by $1^{\prec} \prec 2^{\prec} \prec ... \prec n^{\prec}$, where $\sigma(j)$ is the player that is in the jth position according to $\prec$. Consider any mapping $g : \mathcal{R}_D^n \to \{0, T\}$. The **serial dictatorship rule** given the ordering $\prec$ and the mapping g is a social choice function $\hat{f} : \mathcal{R}_D^n \to [0, T]$ such that for each $R \in \mathcal{R}_D^n$ we have that*

$$\hat{f}(R) = \begin{cases} 0 & \text{if } 1^{\prec} \in N_0(R) \\ T & \text{if } 1^{\prec} \in N_T(R) \\ 0 & \text{if } 1^{\prec} \in N_{0T}(R) \text{ and } 2^{\prec} \in N_0(R) \\ T & \text{if } 1^{\prec} \in N_{0T}(R) \text{ and } 2^{\prec} \in N_T(R) \\ \vdots \\ g(R) & \text{if } N_{0T}(R) = N. \end{cases}$$

The serial dictatorship rule is strategy-proof but it is not strong group strategy-proof. Consider the ordering $\prec$ and a profile $R \in \mathcal{R}_D^n$ such that $1^{\prec} \in N_{0T}(R)$, $2^{\prec} \in$

$N_0(R)$, and $3^{\prec} \in N_T(R)$. Then, $\hat{f}(R) = 0$. The subset of agents $M = \{1^{\prec}, 3^{\prec}\}$ misrepresents their preferences by reporting any $R'_M \in \mathcal{R}_D^2$ such that $1^{\prec} \in N_T(R'_M, R_{-M})$ and $3^{\prec} \in N_T(R'_M, R_{-M})$. Then, $\hat{f}(R'_M, R_{-M}) = T$. Hence $\hat{f}(R)$ is not strong group strategy-proof as $TI_{1^{\prec}}0$ and $TP_{3^{\prec}}0$.

Strong group strategy-proofness and unanimity are two independent properties that a social choice function can satisfy, as we can see in Example 3.

Example 2..3 *Consider any mapping $g : \mathcal{R}_D^n \to \{0, T\}$. The **$i$ and j winners for 0**rule $\widetilde{f}^{i,j} : \mathcal{R}_D^n \to [0, T]$ is a social choice function such that for each $R \in \mathcal{R}_D^n$ we have that*

$$\widetilde{f}^{i,j}(R) = \begin{cases} \min(U(R)) & \text{if } U(R) \neq \emptyset \\ \\ 0 & \text{if } U(R) = \emptyset, \ 0R_iT, \ \text{and } 0R_jT \\ \\ T & \text{if } U(R) = \emptyset, \ TP_i0, \ \text{and } TP_j0 \\ \\ g(R) & \text{otherwise} \end{cases}$$

where $\min(A)$ stands for the minimum number from the set $A \subset [0, T]$.

This rule is unanimous but not strong group strategy-proof. Suppose that $U(R) = \emptyset$ and $i, j \in N$ are such that TP_i0, and TP_j0. Then, $\widetilde{f}^{i,j}(R) = T$. Suppose that $a = E(\{T\}, R)$ is such that aI_iT, aI_jT, and aP_kT for all $k \in N \setminus \{i, j\}$. Then, there exist a profile R' such that $U(R') = a$, hence N manipulates $\widetilde{f}^{i,j}$ via R' since $\widetilde{f}^{i,j}(R') = aR_iT = \widetilde{f}^{i,j}(R)$ for all $i \in N$ and aP_jT for some $j \in N$.

Obviously, any constant rule is strong group strategy-proof but not unanimous.

2.3 Characterization

A mapping $t : \mathcal{R}_D^n \to [0, T] \cup \{\emptyset\}$ is a tie-breaker if for each $R \in \mathcal{R}_D^n$, $t(R) \in U(R)$ if $U(R) \neq \emptyset$ and $t(R) = \emptyset$ if $U(R) = \emptyset$; namely, it selects an alternative within the union of unanimous intervals, whenever this set is non-empty.

Consider the following family of social choice functions named *full agreement rules*.

Definition 2..8 *A **full agreement rule** $f^{(b,t)} : \mathcal{R}_D^n \to [0, T]$ with bias $b \in \{0, T\}$ and tie-breaker t is a social choice function such that for each $R \in \mathcal{R}_D^n$:*

$$
f^{(b,t)}(R) = \begin{cases}
t(R) & \text{if } U(R) \neq \emptyset \\[2mm]
E(\{0\}; R) & \text{if } U(R) = \emptyset, \ b = 0, \ \text{and } E(\{0\}; R) \neq \emptyset \\[2mm]
0 & \text{if } U(R) = \emptyset, \ b = 0 \ \text{and } E(\{0\}; R) = \emptyset \\[2mm]
E(\{T\}; R) & \text{if } U(R) = \emptyset, \ b = T \ \text{and } E(\{T\}; R) \neq \emptyset \\[2mm]
T & \text{if } U(R) = \emptyset, \ b = T \ \text{and } E(\{T\}; R) = \emptyset.
\end{cases}
$$

The following observation will be useful in the sequel. Let $f^{(b,t)}$ be a full agreement rule. Consider a profile $R \in \mathcal{R}_D^n$ such that $E(\{0\}; R) = \emptyset$. If $b = 0$ then T can only chosen by $f^{(b,t)}$ only if $T \in U(R)$.

In case there are only two agents, the class of strong group strategy-proof and unanimous social choice functions is wider than the class of all full agreement rules.

Example 2..4 *Let $n = 2$ and $f^1 : \mathcal{R}_D^n \to [0, T]$ be a social choice function such that*

for each $R \in \mathcal{R}_D^n$:

$$f^1(R) = \begin{cases} 0 & \text{if } 0 P_1 T \\ T & \text{otherwise} \end{cases}$$

Then, f^1 is a strong group strategy-proof and unanimous social choice function, however, it does not belong to the family of full agreement rules. If f^1 is to be a full agreement rule, then the bias b should depend on the name of the individual. However, as we defined $f^{(b,t)}$, the bias b is just an exogenous alternative $b \in \{0, T\}$.

Theorem 2..1 *When $n > 2$, a social choice function $f : \mathcal{R}_D^n \to [0, T]$ is strong group strategy-proof and unanimous if and only if there exist a bias $b \in \{0, T\}$ and a tie-breaker t such that $f = f^{(b,t)}$.*

We leave the case where $n = 2$ for future research.

2.4   Proof of Theorem 1

It is straightforward to show that any full agreement rule is a strong group strategy-proof and unanimous social choice function.

To prove the converse, we first describe the range of any strong group strategy-proof and unanimous social choice function under single-dipped preferences with indifferences.

Lemma 2..2 *Let $f : \mathcal{R}_D^n \to [0, T]$ be a strong group strategy-proof and unanimous*

social choice function. Then, for all $R \in \mathcal{R}_D^n$ such that $N_T(R) \neq \emptyset$ and $N_0(R) \neq \emptyset$,

$$f(R) \notin (0, a_0(R)) \text{ and}$$

$$f(R) \notin (a_T(R), T).^3$$

Proof. Let $f : \mathcal{R}_D^n \to [0, T]$ be strong group strategy-proof and unanimous. Suppose that there exists $R \in \mathcal{R}_D^n$ such that $f(R) \in (0, a_0(R))$.

Notice that $f(R) \in W(0, R_i)$ for each $i \in N_T(R)$. Moreover, for each $i \in N_T(R)$ we have that $0 P_i f(R)$.

For each $i \in N_T(R)$, construct $R_i' \in \mathcal{R}_D$ such that $0 P_i' T$ and $l_{min}(R_{N_T(R)}') = 0$. Notice, $N_T(R_{N_T(R)}', R_{-N_T(R)}) = \emptyset$. Thus, $U(R_{N_T(R)}', R_{-N_T(R)}) = \{0\}$, and by unanimity $f(R') = 0$. This violates strong group strategy-proofness.

The proof is symmetric for the case where there exists a profile $R \in \mathcal{R}_D^n$ such that $f(R) \in (a_T(R), T)$. ∎
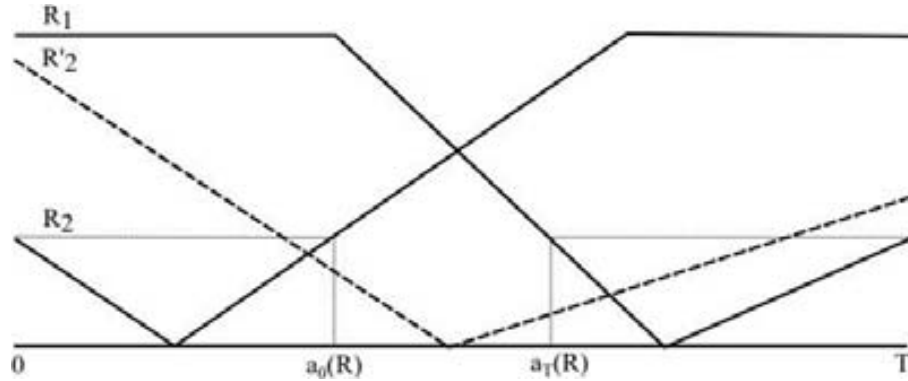

Figure 9. Let $N = \{1, 2\}$ such that $0 P_1 T$ and $T P_2 0$. If $f(R) \in (0, a_0(R))$, then agent 2 reports R_2' such that $0 P_2 T$ and by unanimity $f(R') = 0$. Hence,

$$f(R, R_2') P_2 f(R) \text{ and } f(R, R_2') I_1 f(R)$$

Lemma 2 implies the following Corollaries illustrated in Figure 10.

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[3] Figure 9 illustrates the statement of Lemma 2.

Corollary 2..1 *Let $f : \mathcal{R}_D^n \to [0, T]$ be a strong group strategy-proof and unanimous social choice function. Then, for all $R \in \mathcal{R}_D^n$ such that $a_0(R) > a_T(R)$,*

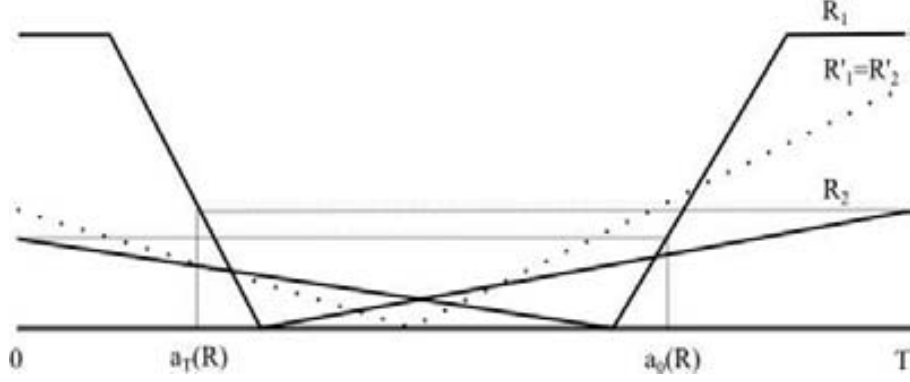$$f(R) \notin (a_T(R), a_0(R)).$$

Figure 10. If $a_0(R) > a_T(R)$, then $f(R) \notin (a_T(R), a_0(R))$.

Corollary 2..2 *Let $f : \mathcal{R}_D^n \to [0, T]$ be a strong group strategy-proof and unanimous social choice function. Then, for all $R \in \mathcal{R}_D^n$ such that $a_0(R) > a_T(R)$,*

$$f(R) \in \{0, T\}.$$

Lemma 2..3 *Let $f : \mathcal{R}_D^n \to [0, T]$ be a strong group strategy-proof and unanimous social choice function. Then, for all $R, R' \in \mathcal{R}_D^n$ such that $N_0(R) = N_0(R')$, $N_T(R) = N_T(R')$, $R_{N_{0T}(R)} = R'_{N_{0T}(R)} \in \mathcal{R}_D^{\#N_{0T}(R)}$, $a_0(R) > a_T(R)$, and $a_0(R') > a_T(R')$,*

$$f(R) = 0 \text{ and } f(R') = 0, \text{ or}$$
$$f(R) = T \text{ and } f(R') = T.$$

Proof. Let f be a strong group strategy-proof and unanimous social choice function. Let $R, R' \in \mathcal{R}_D^n$ be such that $N_0(R) = N_0(R')$, $N_T(R) = N_T(R')$, $R_{N_{0T}(R)} = R'_{N_{0T}(R)} \in \mathcal{R}_D^{\#N_{0T}(R)}$, $a_0(R) > a_T(R)$, and $a_0(R') > a_T(R')$. By Corollary 2, $f(R) \in$

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$\{0, T\}$ and $f(R') \in \{0, T\}$. Suppose that $f(R) = 0$, (a symmetric argument applies if $f(R) = T$). Notice that for all $i \in N_0(R)$, $0 P_i a$ for all $a \in (0, T]$. Hence, no agent in $N_0(R)$ has incentives to misrepresent his preferences. Construct a profile $(R'_{N_T(R)}, R_{-N_T(R)})$ such that $N_0(R) = N_0(R'_{N_T(R)}, R_{-N_T(R)})$, $N_T(R) = N_T(R'_{N_T(R)}, R_{-N_T(R)})$, $R_{N_{0T}(R)} = R'_{N_{0T}(R'_{N_T(R)}, R_{-N_T(R)})} \in \mathcal{R}_D^{\#N_{0T}(R)}$, and $a_0(R'_{N_T(R)}, R_{-N_T(R)}) > a_T(R'_{N_T(R)}, R_{-N_T(R)})$. Suppose that $f(R'_{N_T(R)}, R_{-N_T(R)}) = T$; hence, $f(R'_{N_T(R)}, R_{-N_T(R)}) P_i f(R)$ for all $i \in N_T(R)$. Thus, the subset $N_T(R) \subseteq N$ manipulates f via R'. This contradicts that f is strong group strategy-proof. ∎

Now, let us consider the case where $a_0(R) \leq a_T(R)$.

Lemma 2..4 *Let $f : \mathcal{R}_D^n \to [0, T]$ be a strong group strategy-proof and unanimous social choice function. Assume that $R \in \mathcal{R}_D^n$ is such that $a_0(R) \leq a_T(R)$, $a_0(R) \neq 0$, and $a_T(R) \neq T$. If $f(R) \in [a_0(R), a_T(R)]$ then,*

$$f(R) \in U(R) \cup E(\{0\}; R) \cup E(\{T\}; R).^4$$

Proof. Let f be a strong group strategy-proof and unanimous social choice function. Let $R \in \mathcal{R}_D^n$ be such that $0 \neq a_0(R) \leq a_T(R) \neq T$, $f(R) \notin U(R)$, $f(R) \notin E([0]; R)$, $f(R) \notin E([T]; R)$ and $f(R) \in [a_0(R), a_T(R)]$.

Construct a profile $R' \in \mathcal{R}_D^n$ such that $N_0(R) = N_0(R')$, $N_T(R) = N_T(R')$, $R_{N_{0T}(R)} = R'_{N_{0T}(R)}$, $a_0(R') > a_T(R')$. By Corollary 2, $f(R') \in \{0, T\}$. Assume first that $f(R') = 0$.

---

[4]Figures 11 and 12 illustrate the case where $f(R) \in [a_0(R), a_T(R)]$. In Figure 11 the union of unanimous intervals is non-empty, while in Figure 12, $U(R) = \emptyset$.

For all $i \in N_0(R)$, let $R''_i \in \mathcal{R}_D$ be such that $N_0(R''_{N_0(R)}, R_{-N_0(R)}) = N_0(R')$, $N_T(R''_{N_0(R)}, R_{-N_0(R)}) = N_T(R')$, $R''_{N_{0T}(R)} = R'_{N_{0T}(R')}$, and $a_0(R''_{N_0(R)}, R_{-N_0(R)}) > a_T(R''_{N_0(R)}, R_{-N_0(R)})$. Then by Lemma 3, $f(R''_{N_0(R)}, R_{-N_0(R)}) = 0$. Notice that, $f(R''_{N_0(R)}, R_{-N_0(R)}) R_i f(R)$ for all $i \in N_0(R)$.

As $f(R) \notin U(R)$ and $f(R) \notin E([0]; R)$, there are two cases:

(i) There exists $i \in N_0(R)$ such that $f(R''_{N_0(R)}, R_{-N_0(R)}) P_i f(R)$. Hence, $N_0(R)$ manipulates f via R''. This violates strong group strategy-proofness.

(ii) If there is no $i \in N_0(R)$ such that $f(R''_{N_0(R)}, R_{-N_0(R)}) P_i f(R)$, then it must be that $f(R) \in [a_0(R), l_{min}(R_{N_0(R)})]$. Then there exists $i^* \in N_T(R)$ such that $f(R) P_{i^*} f(R''_{N_0(R)}, R_{-N_0(R)})$. Then the group $\{i^*\} \cup N_0(R)$ manipulates f via R; which violates strong group strategy-proofness.

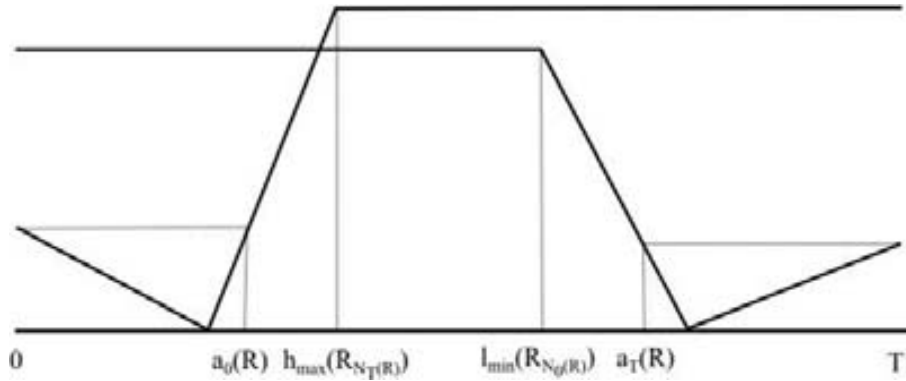The symmetric argument is used for the case where $f(R') = T$. ■

Figure 11. $U(R) \subseteq (a_0(R), a_T(R))$, then $f(R) \in [h_{max}(R_{N_T(R)}), l_{min}(R_{N_0(R)})] = U(R)$
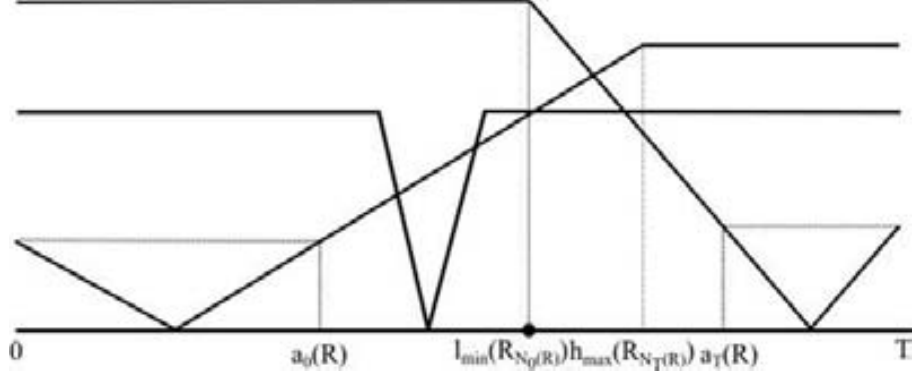
Figure 12. $E(\{0\}; R) \neq \emptyset$, then $f(R) = l_{min}(R_{N_0(R)})$

Lemma 2..5 *Let* $f : \mathcal{R}_D^n \rightarrow [0, T]$ *be a strong group strategy-proof and unanimous social choice function. Assume that* $R \in \mathcal{R}_D^n$ *is such that* $a_0(R) \leq a_T(R)$ *and either* $a_0(R) = 0$ *or* $a_T(R) = T$. *If* $f(R) \in [a_0(R), a_T(R)]$ *then,*

$$f(R) \in U(R) \cup E(\{0\}; R) \cup E(\{T\}; R) \cup \{0\} \cup \{T\}.$$

Proof. Let f be a strong group strategy-proof and unanimous social choice function. Let $R \in \mathcal{R}_D^n$ be such that $f(R) \notin U(R)$, $f(R) \notin E([0]; R)$, $f(R) \notin E([T]; R)$, $f(R) \neq 0$, $f(R) \neq T$, and $f(R) \in [a_0(R), a_T(R)]$. Assume that $0 = a_0(R) \leq a_T(R)$. Consider the following two cases:

(i) $a_T(R) \neq T$.

For all $i \in N_T(R)$, let $R'_i \in \mathcal{R}_D$ be such that $a_0(R'_{N_T(R)}, R_{-N_T(R)}) \neq 0$, $N_0(R) = N_0(R'_{N_T(R)}, R_{-N_T(R)})$, and $f(R'_{N_T(R)}, R_{-N_T(R)}) = T$. Notice that $R'_{N_T(R)}$ always exists since by Corollaries 2 and 3, $f(R'_{N_T(R)}, R_{-N_T(R)}) \in \{0, T\}$. If $f(R'_{N_T(R)}, R_{-N_T(R)}) = 0$ consider $R''_i \in \mathcal{R}_D$ for all $i \in N_T(R)$ such that $f(R)P_i f(R''_{N_T(R)}, R_{-N_T(R)}) = 0$. Then the group $N_T(R)$ manipulates f at $(R''_{N_T(R)}, R_{-N_T(R)})$ via R. Hence $f(R'_{N_T(R)}, R_{-N_T(R)}) = T$.

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Since $f(R) \notin U(R)$ and $f(R) \notin E(\{T\}; R)$, there are two cases.

(a) There exists $i \in N_T(R)$ such that $f(R'_{N_T(R)}, R_{-N_T(R)}) P_i f(R)$. Hence, $N_T(R)$ manipulates f via R'. This violates strong group strategy-proofness.

(b) If there is no $i \in N_T(R)$ such that $f(R'_{N_T(R)}, R_{-N_T(R)}) P_i f(R)$, it must be that $f(R) \in [h_{max}(R_{N_T(R)}), a_T(R)]$. Then there exists $i^* \in N_0(R) \cup N_{0T}(R)$ such that $f(R) P_{i^*} f(R'_{N_T(R)}, R_{-N_T(R)})$. Then the group $\{i^*\} \cup M$ manipulates f, violating strong group strategy-proofness.

(ii) $a_T(R) = T$.

Case (i) implies that there is $R''' \in \mathcal{R}_D^n$ such that $a_0(R''') \leq a_T(R''')$, $[a_0(R''') \neq 0$ or $a_T(R''') \neq T]$, and $f(R''') \in \{0, T\}$. Then a similar argument than the one used in case (i) can be used here in order to obtain a contradiction.

The symmetric argument is used for the case $0 < a_0(R) \leq a_T(R) = T$. ∎

Lemmas 2 to 5 imply the following Corollary.

Corollary 2..3 *Let $f : \mathcal{R}_D^n \to [0, T]$ be a strong group strategy-proof and unanimous social choice function. Then, $f(R) \in U(R)$ if $U(R) \neq \emptyset$ and $f(R) \in \{0\} \cup \{T\} \cup E(\{0\}; R) \cup E(\{T\}; R)$ if $U(R) = \emptyset$.*

Corollary 3 describes, for each preference profile, the set of possible selected alternatives by a strong group strategy-proof and unanimous social choice functions under single-dipped preferences with indifferences. It has to be shown that there exist a t and a b such that the strong group strategy-proof and unanimous social choice function f can be described as a full agreement rule.

Lemma 2..6 *Let $f : \mathcal{R}_D^n \to [0, T]$ be a strong group strategy-proof and unanimous social choice function. If $E(\{0\}; R) \neq \emptyset$, then $f(R) \neq 0$; and if $E(\{T\}; R) \neq \emptyset$, then $f(R) \neq T$.*

Proof. Let f be a strong group strategy-proof and unanimous social choice function.

Suppose $E(\{0\}; R) \neq \emptyset$ and that $f(R) = 0$. By Lemma 1, we know that $\#E(\{0\}; R) = 1$. Furthermore $E(\{0\}; R)R_i 0$ for all $i \in N$ and $E(\{0\}; R)P_j 0$ for some $j \in N$. Consider a profile R' where $E(\{0\}; R)P_i'0$ for all $i \in N$. By unanimity, $f(R') = E(\{0\}; R)$. Hence, N manipulate f via R'. Contradiction.

The symmetric argument works if $E(\{T\}; R) \neq \emptyset$. ∎

Now, we are ready to prove that any strong group strategy-proof and unanimous social choice function is a full agreement rule; namely,

Lemma 2..7 *Let $f : \mathcal{R}_D^n \to [0, T]$ be a strong group strategy-proof and unanimous social choice function. Then, there exists a tie- breaking rule $t : \mathcal{R}_D^n \to [0, T] \cup \{\emptyset\}$ and a bias $b \in \{0, T\}$ such that for each $R \in \mathcal{R}_D^n$, $f(R) = f^{(b,t)}(R)$.*

Proof. Let f be a strong group strategy-proof and unanimous social choice function. Let $R \in \mathcal{R}_D^n$ be arbitrary. Two general cases are possible.

(i) $U(R) \neq \emptyset$. Then, by Corollary 3, $f(R) \in U(R)$. Then, set $t(R) = f(R)$.

(ii) $U(R) = \emptyset$. Then, set $t(R) = \emptyset$. By Corollary 3 we know that $f(R) \in E(\{0\}; R) \cup E(\{T\}; R) \cup \{0\} \cup \{T\}$. We distinguish among for different cases.

(a) $E(\{0\}; R) = \emptyset$ and $E(\{T\}; R) = \emptyset$. Then $f(R) \in \{0, T\}$. Set $b = 0$ if $f(R) = 0$ and $b = T$ if $f(R) = T$.

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(b) $E(\{0\}; R) \neq \emptyset$ and $E(\{T\}; R) \neq \emptyset$. Then, by unanimity and strong group strategy-proofness, $f(R) \in \{E(\{0\}; R), E(\{T\}; R)\}$. Set $b = 0$ if $f(R) = E(\{0\}; R)$ and $b = T$ if $f(R) = E(\{T\}; R)$.

(c) $E(\{0\}; R) \neq \emptyset$ and $E(\{T\}; R) = \emptyset$. Then, by unanimity and strong group strategy-proofness, $f(R) \in \{E(\{0\}; R), T\}$. Set $b = 0$ if $f(R) = E(\{0\}; R)$ and $b = T$ if $f(R) = T$.

(d) $E(\{0\}; R) = \emptyset$ and $E(\{T\}; R) \neq \emptyset$. Then, by unanimity and strong group strategy-proofness, $f(R) \in \{0, E(\{T\}; R)\}$. Set $b = 0$ if $f(R) = 0$ and $b = T$ if $f(R) = E(\{T\}; R)$. ∎

2.5   Final Remarks

In this paper we extend the family of single-dipped preferences to allow indifferences on the top. Under single-dipped preferences with indifferences, we take into consideration the cases where agents might not perceive the negative externalities caused by the public bad, reaching their highest level of satisfaction, and becoming indifferent from that location onward.

Under this framework we have shown that the range of the strong group strategy-proof and unanimous social choice function may expand to the whole interval $[0, T] \in \mathbb{R}$. This is remarkable, as under the case of single-dipped preferences, previous results have shown that the range is restricted to the two extremes of the interval. Furthermore, even it is not shown in this paper, the results hold if we allow to have indifferences at both sides of the dip with different level of welfare for each. The alternative selected by

a unanimous and strong group strategy-proof social choice function under single-dipped preferences with indifferences are Pareto efficient.

It must be noticed that opposite to single-peak preferences where the rule is tops-only, in single dipped preferences with indifferences, not only does the best alternatives are relevant for the social choice function. In particular, it is important to know if there is a Pareto improvement with respect to any of the extremes of the alternative's space.

We leave for future research to characterize the class of all strategy-proof social choice functions under single-dipped preferences with indifferences. Moreover, we also leave for future research to extend the problem to a two-dimensional alternative space such as Ehlers (2002 and 2003) or Bogomolnaia and Nicolò (2005) did under single-peaked preferences. In this way, we could answer the question of how to locate two public bads, as a nuclear plant and its dumpsite.

REFERENCES CITED

1. Barberà, S. (2010): "Strategy-proof Social Choice", in K. J. Arrow, A. K. Sen and K. Suzumura (eds.), Handbook of Social Choice and Welfare, Volume 2, Chapter 25. North-Holland:Amsterdam

2. Barberà, S., Berga, D., and Moreno, B. (2010): "Individual versus Group Strategy-Proofness: When do They Coincide?", *Journal of Economic Theory* 145 : 1648−1674

3. Barberà, S., Berga, D., and Moreno, B. (2011*a*): "Domains, Ranges and Strategy-proofness: the Case of Single-dipped Preferences". Forthcoming in *Social Choice and Welfare*

4. Barberà, S., Berga, D., and Moreno, B. (2011*b*): "Group Strategy-proof Social Choice Functions with Binary Ranges and Arbitrary Domains: Characterization Results". Forthcoming in the *International Journal of Game Theory*

5. Bogomolnaia, A., and Nicolò, A. (2005): "Stable Assignment of Public Facilities under Congestion", *Journal of Public Economic Theory* 7 : 65 − 91

6. Cantala, D. (2004): "Choosing the Level of a Public Good when Agents Have an Outside Option", *Social Choice and Welfare* 22 : 491 − 514

7. Ehlers, L. (2002): "Multiple Public Goods and Lexicographic Preferences: Replacement Principle", *Journal of Mathematical Economics* 37 : 1 − 15

8. Ehlers, L. (2003): "Mutiple Public Goods, Lexicographic Preferences and Single-Plateaued Preferences Rules", *Games and Economic Behavior* $43 : 1 - 27$

9. Freudenberg, W., and Pastor, S. (1992): "NIMBYs and LULUs: stalking the syndromes", *Journal of Social Issues* $48 : 36 - 61$

10. Manjunath, V. (2010): "Efficient and Strategy-proof Social Choice when Preferences are Single-dipped". Mimeo

11. Moulin, H. (1980): "On Strategy-Proofness and Single-Peakedness", *Public Choice* $35 : 437 - 55$

CHAPTER 3.

CAMPAIGNS: PROMISES AND PRONOUNCEMENTS

3.1 Introduction

A common feature in recent political campaigns is the increasing number of spots where the message is not about the propositions of the candidates but it is related to the opponent's characteristics. We call negative campaigning to the advertising in which, rather to expose the virtues of a candidate, it highlights the drawbacks of the contender. It might be that the negative campaigning focuses on the lack of personal abilities, results obtained in a sphere different from the issues discussed in the campaign, or on the real position of the policies in debate on the current campaign. The negative messages are directed to harm the desirability and credibility of the candidates.

Elections and voting procedures are surrounded by positive and negative publicity about the candidates. It is a fact that not only in the U.S. the expenditure in campaigns has increased in the last years, but it is a worldwide phenomenon. In the 1996 presidential elections, 6% of the expenditure in Clinton's campaign was for negative advertising versus a 70% of negative campaign conducted by Bob Dole. UNDP has run surveys in different Latin American countries to evaluate the level of negative campaigning and the impact of mass media in voters. It was shown that, in Mexico, 11% of the spots where conducting a negative message for the opponent candidate, and the messages increased as the campaign got closer to an end. Hence, it is a fact that candidates not only expose their political position to the voters, but they also talk

about the opposition. This information is transmitted to voters, so they can create their expectations about the true position of the candidates.

There exist a vast literature in negative campaigning in Psychology, Political Science, and Economics. Psychology studies give three arguments for the existence of negative campaigning: 1) it stimulates attention to and awareness of the campaign; 2) campaigns may arouse anxiety which stimulates interest, and; 3) negative campaign might be a sign of a close race, which is directly related with the marginal utility of going to vote in some cases. People are very aware of negative information, they attend to it more, think about it more, remember it better, and it is more powerful in shaping our impressions of things (Hodges 1974). Skowronsky et.al. (1989) showed that given equal amounts of positive and negative descriptions of a person, the overall impression formed is skewed towards the negative, and Richey et.al. (1967) argued that negative data are more persistent over time.

Candidates have generally spent resources building positive images of themselves among voters. The longer a good reputation of a party is preserved in the mind of voters, the more influence they will have over them. A candidate must provide evidence or claim that the positive images voters have of their contenders are inaccurate. It is generally not enough for a candidate to simply present a positive image of him or herself. There is fragmented evidence suggesting that negative campaigning is effective to a degree, but no evidence that it is superior to a positive campaign.

There is no consensus about the measure and definition of a negative campaign. However, there is an emerging type of negative campaign called *contrasting campaign.*

This is a subclass of negative campaigns, whose principal strategy is to reveal or say something contrasting what a candidate has said of himself and what it has been really doing. This type of campaign is having a great impact in countries where explicit defamation and injury is forbidden in the campaign, such as in Mexico.[1] In the last Mexican presidential elections (2006) the winner candidate Felipe Calderón was 8% behind López Obrador one month before elections. Calderón started a campaign[2] where he affirmed that Obrador was a "danger" to Mexico, as he promised to reduce expenses and create economic growth, but instead had duplicated the debt in Mexico City, where he was the governor. Another spot first shows how Obrador is asking for tolerance; afterwards he appears in another image contemptuously calling the current president. Felipe Calderón pronounced about the "true" policies that Obrador would implement if running into office versus what he was promising to achieve. Even though Calderon did not give any further proves about his statements, he achieved to win the presidential election.

The Economic literature has focused mainly in the signaling of the position of the candidates' ideal policies. Polborn et.al. (2006) develop a model in which candidates can either send a positive or a negative message to the voters. The candidates have unknown qualities that will be signaled by the decision between doing positive

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[1] The electoral law (Código federal de instituciones y procedimientos electorales) forbids any kind of negative campaign. Its Article 38 says: "Abstain from any expression involving diatribe, libel, slander, libel, defamation, or demeaning to the citizens, public institutions or other political parties and their candidates, particularly during election campaigns and political propaganda that is used in an electoral period."

[2] Calderón posted 60 hours of such TV spots in the last phase of the campaign and sent 40 million e-mails with messages contrasting the promises of his contender. A deep analysis about such election processes is due to the political scientist Sergio Aguayo (see Aguayo (2010)).

or negative campaigning, however they cannot do both. Soubeyran (2009) provides sufficient conditions for the existence and uniqueness of a symmetric Nash equilibrium where candidates have as strategies: to attack and to defend. He considers that each candidate has a transformation function which allows them to overcome the attacks and turn them into a positive effect in the probability of winning. Aragonés et.al. (2007) analyze the conditions under which candidates' reputations may affect voters' beliefs over what policy will be implemented by the winning candidate of an election. They use a dynamic game where candidates can promise a policy different from their ideal point eventhough the true ideal policies are observable. Rational voters will believe the promises which will be implemented in the future as long as the reputation has value for the candidates. Callander et.al. (2007) provide a model where candidates are willing to lie about their policy's intentions creating an effect on other candidates' behavior, changing the nature of political campaigns.

In this paper we develop a theoretical model to justify the existence of negative campaigning, in the subclass of contrasting campaign, via a two-period repeated game. In particular, we know that in a one-period game with incomplete information about the candidates' policies and with the unique strategy available to be promises, the result is for both candidates to promise the median voter's policy; hence, promises are non-informative. Therefore, a natural question arises: does the existence of both, promises and pronouncements, and the risk to lose credibility for future elections help to reveal the true policies of the candidates?

We do not limit the candidates on giving either a promise or a pronouncement,

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but they use both. The threat of loosing reputation, hence, credibility for the next period, can help to elicit the true policy's positions of the candidates. Candidate's promises are their real policies if the opportunity cost is high enough. Therefore, the existence of pronouncements can be a tool to prevent the candidates to promise the implementation of the median voter's ideal policy. Furthermore, the results show that extreme pronouncements are always present, regardless the credibility with which candidates start the campaign.

In this setup it is hard to predict the behavior of the weak candidates, that is, those who have less reputation. There are cases where it does not matter which promises and pronouncements they make as the strong candidate, which has a higher reputation, will be able to win both periods without caring about the contender. Therefore, the weak candidate might present ambiguity in his pronouncement or promise to be placed where the median voter is without winning the election. The strong candidate is more interested in preserving his reputation and will be more accurate in revealing their position of policies via promises. Nevertheless, they will also tend to exaggerate in their pronouncement as there will no be any cost of doing so.

The paper is organized as follows: in section 2 we describe the general model. In section 3 we analyze the possible strategies and describe in detail what happens in equilibrium. We conclude with some final remarks.

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3.2   The Model

The aim of the model is to describe an electoral competition where candidates give information to the uninformed voters via promises and pronouncements. Does contrast campaign helps to achieve victory? Do the trade off between winning the election in the first period and preserving reputation to compete in the future helps to elicit the true policy positions of the candidates? To achieve our purpose, we are going to model a two-period repeated game. We first describe the elements of the game and how a one-period game would work.

Elements of the Game

- *Players and payoffs:*

Two candidates $C = \{L, R\}$ compete to run office and implement their policies in a one-dimensional space $[-1, 1] \in \mathbb{R}$. At the beginning of each period, the candidates have already chosen a policy position $x_L \in [-1, 0]$ and $x_R \in [0, 1]$. We do not model the way they have previously decided which policy to implement if going into office.[3] Both candidates know their own policy position as well as their contender's true policy. Thus, we can think of each candidate as already having solved for their optimal behavior in office conditional on being the winner. Hence, candidates have as objective to win the

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[3]Banks (1990) also assumes that prior to the election each candidates have already solved the problem of a policy selection. The underlying assumption is that candidates have full commitment to implement their political position and in the game presented, their only concern is about winning the election.

election. For any candidate $i \in C$, the payoff in each period is

$$U_i(p; r) = \begin{cases} 1 & \text{if } i \text{ wins} \\ \dfrac{1}{2} & \text{if } i \text{ and } j \text{ tie} \\ 0 & \text{if } i \text{ looses,} \end{cases}$$

where r stands for the vector of reputation indexes and p is the vector of promises and pronouncements described further on. The second period's payoff U' is discounted by a factor $\beta \in (0, 1)$ and will be determined by the second-period reputation indexes r' and the promises and pronouncements of the second campaign, p'. Thus, the total payoff of the two-period game for each candidate $i \in C$ is

$$\mathbf{U}_i(p, p'; r, r') = U_i(p; r) + \beta U'_i(p'; r').$$

We assume there is a mass of voters which have single-peaked preferences over the policy space $[-1, 1] \in \mathbb{R}$. It is common knowledge that they are uniformly distributed, hence the median voter's peak is $x_m = 0$.

- *Promises and pronouncements:*

Each candidate has two policies to decide to announce: a *promise* and a *pronouncement*. A promise is defined as the announcement of the policy to perform if going into office and a pronouncement is a statement of a policy about the contender. That is, candidate L announces about himself a policy position $p_{LL} \in [-1, 0]$ and pronounces about the contender's policy $p_{LR} \in [0, 1]$, while candidate R promises a policy $p_{RR} \in [0, 1]$ and pronounces about his contender $p_{RL} \in [-1, 0]$.

We assume that a positive campaign refers to the action of talking about their own policy position to execute if winning the election, while contrasting campaignin is the action of declaring a position about the contender. As both candidates know the true policies, they try to advise or warn voters about the possible false promises of the contender.

- *Reputation index:*

Reputation stands for the estimation in which a person is held, especially by the community or the public generally. In our context, the reputation index, r, measures how accurate a candidate was in his promises and pronouncements. The reputation will be known to all the players of the game: voters and candidates. We assume that each candidate $i, j \in \{L, R\}$ has two different reputation indexes: one for his promises (r_{ii}) and another for his pronouncements (r_{ij}). That is, we consider the possibility where a candidate might be good at stating his own policy platforms but not credible when talking about his contender's policy and vice versa. The further the promise or pronounce is from the winners true policy, the worst reputation does the candidate achieve.

In this paper we are going to use a linear function as the functional form for the reputation; in particular we have that $r_{ij} = 1 - |x_j - p_{ij}|$, as shown in Figure 1. The maximum level of reputation, $r_{ij} = 1$, will be held only if the promised or pronounced policy by candidate i is the same as the winner j implemented policy $p_{ij} = x_j$.
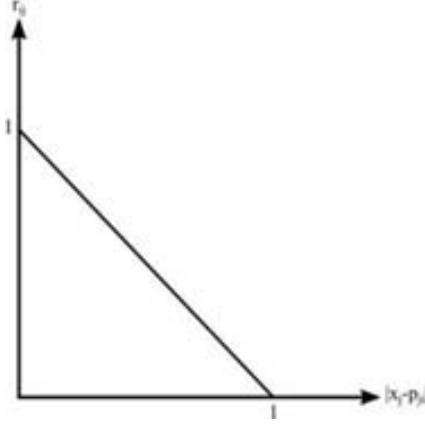
Figure 1. Reputation index

If candidate i promises the policy x_i but looses the election, then there is no way to know which was the true policy candidate i was going to implement in office. Hence, if there is no policy to contrast the promise or the pronouncement done by the candidates, then candidate i remains with the same reputation index in promises, and candidate j holds his initial reputation index in pronouncements, inherited previous to the campaign. Hence, the reputation index update for the second period is as follows:

1. Reputation index at period 2 for promises:

$$r'_{ii} = \begin{cases} 1 - |x_i - p_{ii}| & \text{if } i \text{ wins} \\ \\ r_{ii} & \text{if } i \text{ looses} \end{cases}$$

2. Reputation index at period 2 for pronouncements (i.e., $i \neq j$):

$$r'_{ij} = \begin{cases} r_{ij} & \text{if } i \text{ wins} \\ \\ 1 - |x_j - p_{ij}| & \text{if } i \text{ looses.} \end{cases}$$

We could also consider other functional forms to update the reputation index. For example, it could be weighted by the reputation they had in the previous period or

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could penalize differently if the promise is closer to the median voter with respect to the implemented policy or further, to the extreme of the policy space. The key feature is the penalization in the credibility of a candidate if his promise or pronouncement was far from what actually was the implemented policy.

The credibility of each candidate is build by the reputation indexes. Voters can compare which is the reputation index of candidate i of promising policies versus the reputation index of candidate j. Let θ_{ij} be the credibility share i has over promises or pronouncements of candidate j. We assume that $\theta_{ij}(r_{ij}, r_{jj}) \in [0, 1]$ is an increasing function in r_{ij}; that is, the more reputation candidate i has over his promise or pronouncement of candidate j, the greater the weight of the reputation of candidate i. We also assume that $\theta_{ij}(r_{ij}, r_{jj})$ is decreasing in r_{jj}, hence, if j has a better reputation than i, the share of credibility for i should be lower. The functional form we use for the credibility share is:

$$\theta_{ij}(r_{ij}, r_{jj}) = \frac{r_{ij}}{r_{jj} + r_{ij}}.^{4}$$

Therefore, $\theta_{ij}(r_{ij}, r_{jj}) + \theta_{jj}(r_{ij}, r_{jj}) = 1$.

Definition 3..1 *A one-period game, $G_{t=1}$, is a five-tuple $(C, P, x, r, \mathcal{U})$, where C are the candidates, P is the set of all vectors of promises and pronouncements $p = (p_{LL}, p_{LR}, p_{RR}, p_{RL})$, x are the true policy's positions $x = (x_L, x_R)$, r is the vector of initial reputations indexes $r = (r_{LL}, r_{LR}, r_{RR}, r_{RL})$, and $\mathcal{U} = (\mathbf{U}_L, \mathbf{U}_R)$ are the payoffs.*

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[4]Grofman, et.al. (1995) use a similar function to describe the share of votes attracted by positive and negative campaigns.

The strategy of the candidate specifies the action they will take in each stage for each possible history of play through the previous stage.

Definition 3..2 *Given the reputation indexes r, a strategy for candidate $i \in C$ is a promise p_{ii} and a pronouncement p_{ij} for each realization of his policy x_i and the policy x_j of his contender.*

Definition 3..3 *A Nash Equilibrium (NE) in pure strategies of the game $G_{t=1}$ is a pair of strategies $p_L = (p_{LL}, p_{LR})$ and $p_R = (p_{RR}, p_{RL})$ such that*

$$U_L(p_L, p_R; r) \geq U_L(\hat{p}_L, p_R; r) \text{ for every } \hat{p}_L \in [-1, 0] \times [0, 1] \text{ and}$$

$$U_R(p_L, p_R; r) \geq U_R(p_L, \hat{p}_R; r) \text{ for every } \hat{p}_R \in [-1, 0] \times [0, 1].$$

One-period Game

Each period begins with the realization of a policy for each candidate x_L and x_R and with an inherited reputation index r. Candidates have complete information, but voters only observe r. The game consists of three stages in the period: campaigning stage, voting stage, and office stage.

- *Campaigning Stage:*

During the campaigning stage, each candidate promises a policy to implement if going into office, and pronounces about the policy intended by his contender. That is, candidate L promises about himself to have as intended policy $p_{LL} \in [-1, 0]$, as well as R announces the promise of his true policy, $p_{RR} \in [0, 1]$. Each candidate also announce a policy about their contender; that is, candidate L announces a policy about of R,

$p_{LR} \in [0, 1]$, and candidate R makes a pronouncement about candidate L, $p_{RL} \in [-1, 0]$.

Candidates make their promise and pronouncement taking into account the trade-off between ensuring victory today and loosing reputation for tomorrow, which will be increasing with respect to the distance between the promises and pronouncements made and the implemented policy of the winner.

- *Voting Stage:*

Voters know the reputation indexes of each candidate, $r = (r_{LL}, r_{LR}, r_{RR}, r_{RL})$. They observe the promises and pronouncements $p = (p_{LL}, p_{LR}, p_{RR}, p_{RL})$ done by the candidates, and evaluate them by a weighted sum of promises and pronouncements according to the shares of credibility $\theta = (\theta_{LL}, \theta_{LR}, \theta_{RR}, \theta_{RL})$. That is, given r and p, voters evaluate the promises and pronouncements about the true policy of the candidate $j \in C$ and express that $j's$ implemented policy will be

$$\widetilde{x}_j = \theta_{jj}(r_{ij}, r_{jj})p_{jj} + \theta_{ij}(r_{ij}, r_{jj})p_{ij}.$$

As it is common knowledge that $x_m = 0$, then, if $-\tilde{x}_L < \tilde{x}_R$, then candidate $L \in C$ wins the election. In the case where $-\tilde{x}_L = \tilde{x}_R$, then each candidate wins with probability $\frac{1}{2}$. Figure 2 illustrates the region where each of the candidates win, with respect to the evaluation of the voters.
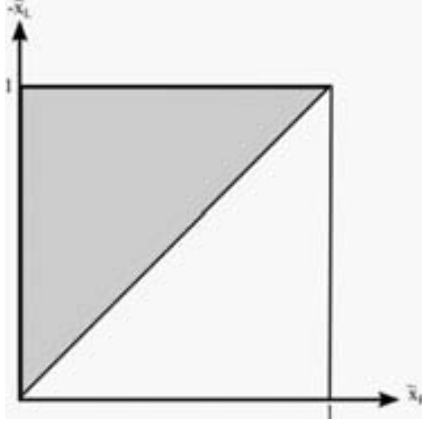
Figure 2. The shadowed space corresponds to the area of R winning the election

- *Office Stage:*

The winner candidate $i \in C$ goes into office and implements his policy x_i, which will be compared to the promise he made in the campaigning stage p_{ii} and to the pronouncement candidate j made about him, p_{ji}, to create a reputation index r' for the following period and for each candidate. This will be done as described before, where the reputation index was introduced.

What is the equilibrium in a one-period game? The best response for any candidate $i \in C$ to the promise and pronouncement of his contender $j \in C$, $i \neq j$, is to promise $p_{ii} = 0$ and pronounce $|p_{ij}| = 1$. Notice that if the game finishes in one period, even though the reputation index is updated when the winner implements his policy, any promise and pronouncement are costless as the new reputation index is of no use. The best chance to win is to proomise the implementation of the median voter policy. If it is credible or not by the voters will be determined by their inherited reputation indexes. If they pronounce something different from the furthest policy with respect to

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the median voter, then it is not a best response as the candidate pronouncing will be giving advantage to his contender. Hence, both promise $p_{LL} = p_{RR} = 0$ and pronounce $p_{LR} = -p_{RL} = 1$. The winner of the game will be decided by the inherited reputation indexes.

3.3   Promises and Pronouncements

In this section, we are going to analize a two-period repeated game. At the beginning of the first period, both candidates have inherited reputation indexes about their accurency of promising and pronouncing policies. Initially at each period, the candidates have already decided which policy to implement if going into office, $(x_L,$ $x_R)$ in the first period, and (x'_L, x'_R) in the second period. The game described in the previous section is played twice.

We begin solving the game by backwards induction at the second period. As we have seen, if there is no future and there exist no trade-off, then the campaigning becomes completeley extremist. The promises will be to implement the median voter's ideal policy, and the pronouncement will be the worst policy of the contender. That is, $p' = (0, 1, 0, -1)$ for any (x'_L, x'_R). Any strategy different from this is weakly dominated, and despite the existence of contrast campaign, the result from the positive campaigning or promises is the classical result of Downs (1957). The optimal strategies for the second period do not mean that the candidates will tie. In fact, the reputation they had achieved from the first period will determine their victory or failure. In particular,

the evaluation from the voters for the second period policies, given p', are:

$$\tilde{x}'_L = -\frac{r'_{RL}}{r'_{LL} + r'_{RL}} = -\theta'_{RL}(r'_{LL}, r'_{RL}), \text{ and}$$

$$\tilde{x}'_R = \frac{r'_{LR}}{r'_{RR} + r'_{LR}} = \theta'_{LR}(r'_{RR}, r'_{LR}).$$

Notice that these values depend on who won the first period. Therefore, we are going to focus on the promises and pronouncements done in first period as there is a threat of loosing credibility for the second period. It is easy to see that the second-period outcome will depend on the update of the reputation indexes via the first-period implemented policies versus the promises and pronouncements done. That is, the optimal promises and pronouncements in the first period should take into consideration that they will be not only used to compete in the current period, but will construct the reputation indexes for the next period. Is it possible to win in both periods? Would this threat be enough to elicit the true policies of each candidate? The candidates act with respect to how much they value the present and the future looking at the distance there exist between the median voter, their own position, and the policy of the contender knowing the reputation indexes.

As we can see, the promises and pronouncements of the first period are going to lead all the results of the game. Hence, the two-period game can be reduced to a one-period game where candidates' actions are the promises and pronouncements in the first period. The cost of the campaign will be the reputation held for the competition in the second period. Hence, given the initial reputation vector (r) and the realization of the first-period policy's positions (x_L, x_R), each candidate maximizes the first period utility subject to the minimum reputation required for tomorrow. We can rewrite the

utility of each candidate $i \in C$ as:

$$\mathbf{U}_i(p; r) = U_i(p; r) + \beta U_i'(p; r)$$

Hence, the relevant decisions of each candidate will be the promises and pronoucements done in the first period. These will ensure them running office in the first period and keep enough reputation to beat the contender in the second period. Therefore, the realization of the policy to be implemented in the second period (x_L', x_R') is not relevant, as the last reputation indexes will not be used. We reduce the game to a one-period game with the threat of loosing credibility share, hence, the equilibrium strategies that we will show correspond only to the first-period game.

In the following subsections, we are going to analyze different cases about the initial reputation indexes. That is, the results of the game depend on the initial credibility shares, and we are going to focus on some interesting situations. What happens when initial reputations are the same? What if a candidate has a better reputation in promising but not in pronouncing? This differences in the initial reputation indexes will determine how much does a candidate can promise away from the median voter and still be able to win the election and preserve enough reputation for the second period.

Candidates with equal initial reputation

As a first case, we consider the possibility that the initial credibility shares are the same for each candidate in either making promises and pronouncing policies. That is, the intial reputation indexes are such that $r_{LL} = r_{RL}$ and $r_{RR} = r_{LR}$. As no candidate

has an initial advantage, then the campaign becomes no-informative.

Lemma 3..1 *Let the initial credibility shares be $\theta_{LL} = \theta_{LR} = \theta_{RR} = \theta_{RL}$. Then, a NE in pure strategies is $p_{RR} = p_{LR} = -p_{LL} = -p_{RL}$ for all x_L and x_R.*

Proof. Let $r_{LL} = r_{RL}$ and $r_{RR} = r_{LR}$.

First period:

The evaluation rule when the initial credibility shares are equal is

$$-p_{LL} - p_{RL} \underset{>}{\overset{<}{\lessgtr}} p_{RR} + p_{LR}.$$

For candidate L, the best he can do is

$$-p_{LL} - p_{LR} \leq p_{RR} + p_{RL},$$

and for candidate R

$$-p_{LL} - p_{LR} \geq p_{RR} + p_{RL}.$$

Hence,

$$-p_{LL} - p_{LR} = p_{RR} + p_{RL}. \tag{1}$$

Second period:

Case 1) Candidate L won in the first period, and x_L is established. Then, candidate L wins the second period is if and only if,

$$\frac{1 - |x_L - p_{RL}|}{2 - |x_L - p_{RL}| - |x_L - p_{LL}|} \leq \theta_{LR} = \frac{1}{2}$$

Using this condition, the best response for candidate L to win is:

$$|x_L - p_{LL}| \leq |x_L - p_{RL}|,$$

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and for candidate R:

$$|x_L - p_{LL}| \geq |x_L - p_{RL}|.$$

Therefore,

$$p_{LL} = p_{RL}. \tag{2}$$

Case 2) Candidate R won in the first period, and x_R is established. Then, candidate R wins the second period is if and only if,

$$\frac{1 - |x_R - p_{LR}|}{2 - |x_R - p_{LR}| - |x_R - p_{RR}|} \leq \theta_{RL} = \frac{1}{2}$$

Hence, the best response for candidate L to win is

$$|x_R - p_{RR}| \geq |x_R - p_{LR}|,$$

and for candidate R

$$|x_R - p_{RR}| \leq |x_R - p_{LR}|.$$

Therefore,

$$p_{RR} = p_{LR}. \tag{3}$$

Adding up $(1), (2)$, and (3) we conclude that $p_{RR} = p_{LR} = -p_{LL} = -p_{RL}$. ∎

We say that the campaing is no-informative, as no relevant information will be disclosed. In this case, two extreme cases can be possible: one can be that all promises and pronouncements are the worst policies possible with respect to the median voter $p = (-1, 1, 1, -1)$ and the other is to promise and pronounce the median voter's ideal policy $p = (0, 0, 0, 0)$.

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Example 3..1 *Let $x_R = 0.8$, $x_L = -0.7$, $r_{LL} = r_{RL} = 0.7$ and $r_{RR} = r_{LR} = 0.3$. Therefore, $\theta_{LL} = \theta_{LR} = \theta_{RR} = \theta_{RL} = \dfrac{1}{2}$.*

If they promise and pronounce the symmetric policies, for example, $p = (0,0,0,0)$, then they both tie in any situation. Hence,

$$\boldsymbol{U}_L(p;r) = \boldsymbol{U}_R(p;r) = \frac{1}{2} + \frac{1}{2}\beta.$$

Suppose that candidate L thinks that becoming aggresive in his pronouncement will harm so much his contender R, that he will have a better outcome; that is, $\hat{p} = (0,1,0,0)$. Is it an equilibrium?

In the first period, as $-\hat{p}_{LL} - p_{RL} = p_{RR} + \hat{p}_{LR}$, then there is a tie and with probability $\dfrac{1}{2}$ each party wins the election.

Second period:

(i) Candidate L won the first period, hence $x_L = -0.7$ is revealed. Candidates L and R tie in the second period:

$$\frac{1 - |x_L - p_{RL}|}{2 - |x_L - p_{RL}| - |x_L - \hat{p}_{LL}|} = \frac{0.3}{2 - 0.7 - 0.7} = \theta_{LR} = \frac{1}{2}.$$

(ii) Candidate R won the first period, hence $x_R = 0.8$ is revealed. Candidate R wins the second period:

$$\frac{1 - |x_R - \hat{p}_{LR}|}{2 - |x_R - \hat{p}_{LR}| - |x_R - p_{RR}|} = \frac{0.8}{2 - 0.2 - 0.8} > \theta_{RL} = \frac{1}{2}.$$

The payoffs for candidate L who became aggresive in his pronouncement is:

$$\boldsymbol{U}_L(\hat{p}_L, p_R; r) = \frac{1}{2} + \frac{1}{4}\beta.$$

Clearly, $\boldsymbol{U}_L(\hat{p}_L, p_R; r) < \boldsymbol{U}_L(p;r)$, so $\hat{p}_L$ is not a profitable deviation.

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Candidiates with initial reputation advantage

The previous results do not give a positive answer to our question. The existence of pronouncements is of no use if initial reputation indexes coincide. However, when a candidate has an advantage, interesing results arise. In some cases, the strong candidate will promise his true policy or the weak candidate will pronounce the true policy of his contender. In the following lemmas, we present interesting results when initial reputation indexes differ from candidates and from promises and pronouncements.

Whenever one candidate has an advantage in both promises and pronouncements at the beginning of the first period, he takes advangate of it, and he is able to promise a policy distinct to the median voter. Assume that candidate L is in advantage as $r_{LL} > r_{RL}$ and $r_{LR} > r_{RR}$. Therefore the credibility shares are such that $\theta_{LL} > \theta_{LR} > \theta_{RR} > \theta_{RL}$. We will see that even if the strong candidate's true policy is further from the median voter with respect to the policy of the weak candidate, he is able to win in both periods.

Lemma 3..2 *Let the initial credibility shares be $\theta_{LL} > \theta_{LR} > \theta_{RR} > \theta_{RL}$. Then, a NE in pure strategies is $p_{LL} \in (\dfrac{\theta_{RR} - \theta_{LL}}{\theta_{LL}}, \dfrac{1 - 2\theta_{RR}}{1 - \theta_{RR}} + x_L)$, $p_{LR} = 1$, $p_{RR} \in [0, 1]$, and $p_{RL} \in [-1, 0]$, for all x_R provided that $x_L \geq \dfrac{2\theta_{RR} - 1}{1 - \theta_{RR}} + \dfrac{\theta_{RR}}{\theta_{LL}} - 1$.*

Proof. Let the initial credibility shares be such that $\theta_{LL} > \theta_{LR} > \theta_{RR} > \theta_{RL}$. Therefore, candidate L has advantage in both promising and pronouncing.

First period:

For any value of promises and pronouncements of R, the promise of the strong candidate L should accomplish that $-p_{LL} < \dfrac{1}{\theta_{LL}}(p_{RR} + (1 - \theta_{RR})p_{LR} + (1 - \theta_{LL})p_{RL})$

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to win in the first period. Considering the worst scenario for L, in which R becomes aggresive in the first period with the strategy $p_{RR} = 0$ and $p_{RL} = -1$, the condition to win for L becomes $-p_{LL} < \frac{1}{\theta_{LL}}((1 - \theta_{RR})p_{LR} + \theta_{LL} - 1)$.

Candidate L knows that he has an advantage over R in both credibility shares. If he manages to win in the first period, the policy of R will never be disclosed. This means that he knows that even if L exagerates over the policy of R, there will not be any repercusion in the reputation index for the second period. Therefore, he does not care about what he is going to pronounce about his contender, that is, $p_{LR} = 1$.

Hence, candidate L will promise

$$-p_{LL} < \frac{1}{\theta_{LL}}(\theta_{LL} - \theta_{RR}). \tag{4}$$

Second period:

Given that candidate L won the first period, x_L will be revealed and the reputation indexes will be updated as: $r'_{LL} = 1 - |x_L - p_{LL}|$, $r'_{RL} = 1 - |x_L - p_{RL}|$, $r'_{RR} = r_{RR}$, and $r'_{LR} = r_{LR}$.

As we know, candidate L wins in the second period, if and only if,

$$\frac{r'_{RL}}{r'_{RL} + r'_{LL}} = \frac{1 - |x_L - p_{RL}|}{2 - |x_L - p_{RL}| - |x_L - p_{LL}|} < 1 - \theta_{RR}.$$

This means that the distance between the implemented policy x_L and the promise made by L should be sufficiently small as: $|x_L - p_{LL}| < \frac{1 - \theta_{RR}(2 - |x_L - p_{RL}|)}{1 - \theta_{RR}}.$

Notice that the left hand-side will always be positive, therefore if the right-side of the condition is negative the condition will never hold. Hence, to ensure victory for candidate L it should be that $1 - \theta_{RR}(2 - |x_L - p_{RL}|) \geq 0$. This condition can be writen

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as $|x_L - p_{RL}| \geq \dfrac{2\theta_{RR} - 1}{\theta_{RR}}$, which means that for any distance between x_L and p_{RL} if $\theta_{RR} \leq \dfrac{1}{2}$ then the condition will hold, which is true as $\theta_{LR} > \theta_{RR}$ and we know that $\theta_{LR} + \theta_{RR} = 1$.

Case 1) $x_L - p_{LL} < 0$. Then,

$$-p_{LL} > \frac{\theta_{RR}(2 - |x_L - p_{RL}|) - 1}{1 - \theta_{RR}} - x_L.$$

Once again, the distance $|x_L - p_{RL}|$ appears in the condition. This distance takes values from zero to one, where zero is when party R made the exact statement of policy x_L, that is, when party R obtains the greatest reputation level $r'_{RL} = 1$. This means that the worst case for L is when R revealed his true type, in which case $|x_L - p_{RL}| = 0$. Hence,

$$-p_{LL} > \frac{2\theta_{RR} - 1}{1 - \theta_{RR}} - x_L. \tag{5}$$

Case 2) $x_L - p_{LL} > 0$. Then,

$$-p_{LL} < \frac{1 - \theta_{RR}(2 - |x_L - p_{RL}|)}{1 - \theta_{RR}} - x_L.$$

Taking into account the worst case for L, when candidate R has the best reputation, then the promise of L should be as small as

$$-p_{LL} < \frac{1 - 2\theta_{RR}}{1 - \theta_{RR}} - x_L. \tag{6}$$

Adding up conditions (4), (5), and (6) we can conclude that for L to be able to win in both periods for $p_{RR} \in [0, 1]$, $p_{RL} \in [-1, 0]$, and any x_R,

$$p_{LL} \in \left(\frac{\theta_{RR} - \theta_{LL}}{\theta_{LL}}, \frac{1 - 2\theta_{RR}}{1 - \theta_{RR}} + x_L \right), \text{ and} \tag{7}$$

$$p_{LR} = 1.$$

From the last result (7), $\dfrac{\theta_{RR} - \theta_{LL}}{\theta_{LL}} \leq \dfrac{1 - 2\theta_{RR}}{1 - \theta_{RR}} + x_L$ holds. Finally, notice that it

holds for any $p_{RR} \in [0, 1]$ and $p_{RL} \in [-1, 0]$. ∎

Example 3..2 *Let the initial reputation indexes be $r_{LL} = 0.6$, $r_{LR} = 0.3$, $r_{RR} = 0.6$,*

and $r_{RL} = 0.15$. Then, the initial credibility shares are $\theta_{RR} = 0.4$ and $\theta_{LL} = 0.8$. If

the true policy of the strong candidate L in the first period is $x_L = -0.7$, then, he

promises a policy $p_{LL} \in (-\frac{1}{2}, -0.366)$ and pronounces $p_{LR} = 1$. For any promise and

pronouncement of the weak candidate R, $p_{RR} \in [0, 1]$ and $p_{RL} \in [-1, 0]$, and any x_R,

candidate L is able to win in both periods with a payoff $\mathbf{U}_L(p; r) = 1 + \beta$ and candidate

R looses both periods obtaining as a final payoff $\mathbf{U}_R(p; r) = 0$.

This shows that the initial reputation indexes are crucial for the results and,

in this model, the distance between $-x_L$ and x_R does not matter at all. However,

the initial reputation indexes and the distance between the strong candidate and the

median voter will determine the outcome.

When $\theta_{LL} = \theta_{LR} > \theta_{RR} = \theta_{RL}$, then, candidate L still has an advantage in

the initial credibility share than his contender R in both promising and pronouncing.

However, the credibility shares about the promises and pronouncements stated by each

candidates are the same. As $\theta_{LL} = \theta_{LR} > \dfrac{1}{2}$, then the strenght of candidate L is

sufficiently high to promise his true policy, x_L, if it is close enough to the median

voter's ideal policy $x_m = 0$.

Lemma 3..3 *Let the initial credibility shares be $\theta_{LL} = \theta_{LR} > \theta_{RR} = \theta_{RL}$. Then, a NE*

in pure strategies is $p_{LL} = x_L$ and $p_{LR} = 1$, $p_{RR} \in [0,1]$, and $p_{RL} \in [-1,0]$, for any x_R provided that $-x_L < \dfrac{2\theta_{LL} - 1}{\theta_{LL}}$.

Proof. Let $\theta_{LL} = \theta_{LR} > \theta_{RR} = \theta_{RL}$. As candidate L has advantage in both promising and pronouncing in the first period, it is easier for him to win.

First period:

If candidate L runs office in the first period, it will not be harmful to pronounce against R, then he states $p_{LR} = 1$. Assume that R has an extremist strategy, that is $p_{RR} = 0$ and $p_{RL} = -1$. If L is able to win even with the extreme promises and pronouncements of his contender, then he will win with any value of p_{RR} and p_{RL}.

Therefore, the promise of L must be sufficiently close to the median voter to win in the first period,

$$-p_{LL} < \frac{1}{\theta_{LL}}(2\theta_{LL} - 1). \tag{8}$$

Second period:

Given that L won on the first period, the credibility shares that will change are θ'_{LL} and θ'_{RL} as the reputation indexes are updated, $r'_{LL} = 1 - |x_L - p_{LL}|$ and $r'_{RL} = 1 - |x_L - p_{RL}|$.

Therefore, for candidate L to be able to win in the second period, it must be that:

$$\frac{1 - |x_L - p_{RL}|}{2 - |x_L - p_{RL}| - |x_L - p_{LL}|} < \theta_{LR} = \theta_{LL}.$$

Hence, the distance between the promised policy and the actual implemented policy should be sufficiently small; hence,

$$|x_L - p_{LL}| < \frac{2\theta_{LL} + (1 - \theta_{LL})|x_L - p_{RL}| - 1}{\theta_{LL}}.$$

The worst scenario for candidate L is that the distance between the pronounce-ment made by his contender, p_{RL} and the policy implemented x_L is zero. Hence, if $|x_L - p_{LL}| < \dfrac{2\theta_{LL} - 1}{\theta_{LL}}$, then candidate L wins at the second period. If $p_{LL} \geq x_L$, then the condition becomes $-p_{LL} > \dfrac{-x_L\theta_{LL} - 2\theta_{LL} + 1}{\theta_{LL}}$. Notice that the left hand side of the expression is a positive term, and it will always be strictly greater than the right hand side if the numerator is negative, then for sure the condition will be accomplished as the whole number will be negative. Hence, it must be that

$$-x_L < \frac{1}{\theta_{LL}}(2\theta_{LL} - 1). \tag{9}$$

Candidate L is able to win both periods and accomplish both (8) and (9) in par-ticular when $p_{LL} = x_L$, $p_{LR} = 1$, $p_{RR} \in [0,1]$, $p_{RL} \in [-1,0]$ when $-x_L < \dfrac{1}{\theta_{LL}}(2\theta_{LL} - 1)$ and for any x_R. Furthermore, it has to be that $\theta_{LL} \geq \frac{1}{2}$, that along with the assumption that $\theta_{LL} = \theta_{LR}$ it ensures that $\theta_{LL} > \theta_{RR}$. ∎

Example 3..3 *Let the initial reputation indexes be $r_{LL} = 0.6$, $r_{LR} = 0.9$, $r_{RR} = 0.225$, and $r_{RL} = 0.15$. Then, the initial credibility shares are $\theta_{LL} = \theta_{LR} = 0.8$. Let the first-period true policies be $x_L = -0.7$ and $x_R = 0.2$. Then, if $p_{LL} = -0.7$ and $p_{LR} = 1$ the strong candidate L wins both periods no matter which promise or pronouncement does R. The payoffs of the candidates are $\boldsymbol{U}_L(p; r) = 1 + \beta$ and $\boldsymbol{U}_R(p; r) = 0$.*

3.4   Final Remarks

Campaigns have as an objective to reveal unknown information to the voters. Voters receive the propositions of the policies to be implemented and vote according to their own preferences and believes about the policies that will be implemented by the

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candidates. However, candidates do not limit their messages to their own policies, but do manifest or warn the voters about their contender's policies.

In a one-period game, making promises about self policies or pronouncements about the contender would be meaningless, as there is no cost on saying anything. However, if the promises and pronouncements represent a reputational cost for the candidate in a second round election, there is a trade-off between running office in the first stage, and preserving enough credibility to repeat office on the second period.

This is a first attempt to catch the glance of the importance of contrasting campaigning in an electoral competition. In a two-period game, the results rely importantly in the initial reputations indexes that are given, that is, the image inherited from previous events, jobs, or public image created before the campaign is necessary to win or loose an election. We could relax the importance of the initial reputation in our results by means of an infinite horizon game. However, interesting results are present in this simple model.

We can see that no matter the initial credibility shares with which candidates start the campaign, pronouncements are always present. We are not limiting the candidates on giving promises or pronouncements, but they combine both. The threat of being attacked can help to elicit the true positions of the contenders. That is, the promises of the candidates become the real policy they are going to implement if going into office, when the cost of opportunity is high enough.

If the candidates have no advantage, that is, their credibility shares coincide

within each candidate, then, the promises and pronouncements are non-informative, as they all promise and pronounce the same policy. This means that candidates can both promise and pronounce the median voter's ideal policy. The candidates will tie in both periods and will run office with probability 0.5. However, interesting results arise when credibility shares differ.

When there is a strong candidate, with a greater reputation index in both promising and pronouncing at the beginning of the first period, he can use the advantage to win both periods even if he reveals his true policy intentions. It is not necessarily true in this case, that the strong candidate promises the median voter's ideal policy. However, the strong candidate i will still pronounce the worst policy of his contender j, $i \neq j$; that is, $|p_{ij}| = 1$.

Contrasting campaigning is a tool to prevent the candidates to pronounce themselves as the median voter, as if they were in a one shot game. The strong candidate that has greater credibility shares is more interested in preserving his reputation and will be more accurate in revealing his position of policies via promises. Nevertheless, they will also tend to exaggerate in his pronouncements, as there is no cost of doing so.

It is hard to predict in this setup the behavior of the weak candidates that have a lower initial credibility share. There are cases where it does not matter which promises and pronouncements he does, as the strong candidate will be able to win both periods without caring about the contender. Therefore, the weak candidate might present ambiguity in his pronouncement or promise to be placed where the median voter is.

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The role of the pronouncements seems to be too harsh in our model. The fact that the reputation indexes are not correlated and that the true policy of the loosing candidate is never revealed, leads these results. However, if we incorporate a cost of defaming the contender, then the pronouncement will not be so far from the contender's true policy but will continue be present as it is an important tool to discredit the contender.

REFERENCES CITED

1. Aguayo, S. *La transición en México: 1910-2010*, 1^a edición, México, Fondo de Cultura Económica, 2010

2. Aragonés, E.; Palfrey, T; and Postlewaite, A. (2007) "Political reputations and campaign promises", *Journal of the European Economic Association* 5 : 846 − 884

3. Banks, S. J. (1990) "A model of electoral competition with incomplete information", *Journal of Economic Theory* 50 : 309 − 325

4. Callander, S. and Wilkie, S. (2007) "Lies, damned lies, and political campaigns", *Games and Economic Behavior* 60 : 262 − 286

5. Downs, A. (1957) "An Economic Theory of Political Action in a Democracy", *The Journal of Political Economy* 65 : 135 − 150

6. Grofman, B. and Skaperdas, S. (1995) "Modeling negative campaigning", *American Political Science Review* 89 : 49 − 61

7. Polborn, M. and Yi, D. (2006) "Informative positive and negative campaigning", *Quarterly Journal of Political Science* 1 : 351 − 371

8. Soubeyran, R. (2009) "Contest with attack and defense: does negative campaigning increase or decrease voter turnout?", *Social Choice and Welfare* 32 : 337 − 353