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Some contributions from Graph Theory to the design and study of Large and Fault-tolerant Interconnection Networks

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Introduction

This work presents some contributions to the design and study of large and faul-tolerant interconnection networks within the framework of Graph theory.

One of the main issues to take into account when designing a multicomputer system is the choice of a suitable interconnection network. The so-called network topology affects the performance of the system to a large extent and has a decisive influence on its overall cost. It is mainly for these reasons that the design and study of *large and fault-tolerant interconnection networks* has received a great deal of attention in the last decades, especially since the advent of *very large scale integration* (VLSI) circuit technology (see [16, 19, 20, 57, 139]).

It is well known that (point-to-point) interconnection networks are usually modelled by graphs, either directed or not, in which the *vertices* represent the switching points or nodes. Communication links are depicted by *edges* if they are bidirectional or *arcs* if they are unidirectional.

When designing a multicomputer system or a large telecommunication system, some of the main requirements related to the topology of the interconnection network are:

1. A *large* number of switching points.
2. *A limited number of nodes* directly connected to a given node. Moreover, one can demand this number to have the same value d for every node.
3. *Communication* between every pair of nodes. Furthermore, the *communication delays* between nodes must be short. In other words, the minimum number of nodes that should be traversed to send a message from any switching point to any other one must be bounded.
4. Certainly, the larger a network is the proner it is to be faulty. For this reason, the network must be *fault-tolerant*. This means that if some nodes or links cease to function, it is important that the remaining nodes can still intercommunicate with reasonable efficiency. Two of the most significant issues to evaluate the reliability or fault-tolerance of a network are:
   (a) Evaluation of the minimum number k of nodes (or links) needed to disrupt the network. This means that in case of failure of less than k nodes (or

links), there is communication between every pair of still working nodes. Certainly, it is highly desirable that the value of k be as large as possible, and in this case one can approach the study of the so-called disconnecting sets of nodes (or links).

(b) In case of failure of a fixed number $s < k$ of nodes or links, the remaining network still satisfies condition 3.

Stated in terms of the (di)graph G used to model the network, these requirements are (see Chapter 1):

1. High value of the *order* $n(G)$.

2. Small value of the *maximum degree* $\Delta(G)$. Moreover, one can demand the graph G to be *d-regular.*

3. The (di)graph G is *(strongly) connected.* Furthermore, its *diameter* $D(G)$ should be rather small.

4.(a) Analysis of the connectedness of G, starting with the evaluation of the connectivity parameters $\kappa(G)$ and $\lambda(G)$. For example, a 'reliable' network should be modelled by a maximally connected (di)graph, and going one step further, by a superconnected (di)graph.

4.(b) Small *diameter vulnerability* for a certain value s lower than the *connectivity* $\kappa(G)$.

The three first requirements are, however, in conflict (see Section 1.2). Indeed, the order of any (di)graph belonging to a certain family $\mathcal{H}$ with a fixed maximum degree Δ and diameter D is limited by the so called *Moore bound* $\mathcal{M}_{\mathcal{H}}(\Delta, D)$ (see Section 1.2). On the other hand, this theoretical upper-bound is, in most cases, not attainable. For this reason, the optimization problem of finding (di)graphs of given maximum degree and diameter with an order as large as possible has deserved much attention in the literature (see Section 2.1). Some new contributions to the table of largest graphs are presented in Chapter 2.

As for the analysis of the connectedness on a certain graph or digraph family $\mathcal{H}$, the usual way of starting the study of this topic consists in finding some sufficient condition to assure any digraph $G \in \mathcal{H}$ to be maximally connected; that is to say, to be connected after the deletion of any set of vertices (or arcs) with a cardinality lower than its minimum degree $\delta(G)$. Next, a similar work can be carried out on the maximally connected (di)graphs in $\mathcal{H}$ by considering some further conditional connectivity, as for example the so-called superconnectivity (see Sections 1.7.1, 4.1 and 5.1 for more details). The two last chapters of this work are mainly devoted to the study of connectedness properties in a number of digraph families, by using the aforementioned methodology.

Apart from the connectedness, another important issue to be considered when evaluating the vertex (or arc) fault-tolerance of an interconnection work modelled by a (di)graph G with diameter D and connectivity κ is certainly the so-called diameter-vulnerability. The question to be approached (in terms of vertex removal) consists in studying how much the diameter of G can increase after the deletion of a fixed number s of vertices (or edges), s being of course less than κ (see Section 1.7.2). In Chapter 2, this study has been carried out on several graph families in the case $s = 1$.

This work is structured into five Chapters. In the first one, a review of known results, which will be referred to afterwards, is provided, where the notation and terminology used here is also introduced.

The second Chapter is devoted to the design of new large graphs of diameter six, by means of a certain variant of the so-called compounding technique, starting from the Moore bipartite graphs with this same diameter (see Sections 1.4 and 1.5.1). As a matter of fact, this specific technique is a generalization of a method introduced by Quisquater in [127].

In Chapter 3, the 1-vertex-vulnerability of the so-called *generalized compound graphs* GC is studied; it is proved there that, in most cases, it is *quasi optimal* (see Section 3.1). The GC graphs were introduced by J. Gómez in [75] by proposing several constructions, all of them inspired both in the compounding of graphs and in the design of graphs on alphabet, joining the advantages of both methods. It is mainly for this reason that they yield large graphs when the diameter is rather small. Apart from the aforementioned vulnerability study, a new and global reformulation of these class of graphs is given by means of the line digraph technique, the conjunction of digraphs and the compounding of graphs.

In Chapter 4, a connectedness study on the family of generalized p-cycles is presented. Starting from the works on this subject carried out by Balbuena, Carmona, Fàbrega and Fiol for bipartite digraphs (see [5, 9, 11, 55, 66]); that is, for generalized 2-cycles, a similar list of results for any $p \geq 3$ has been obtained.

The parameter ℓ, introduced by Fàbrega and Fiol in [54], has proved to be an excellent tool to study connectedness properties (see Sections 1.6 and 1.7.1). For this reason, a new family of digraph parameters defined by generalizing in different ways the previous one has been proposed in Chapter 5. On the other hand, with the purpose of unifying different methods used to prove constructively a number of results on connectedness involving the parameter ℓ and the diameter D, a unique framework of algorithmical or constructive proof is also put forward in this Chapter. Finally, starting from these formal contributions, a list of new results on connectedness is shown.

Chapter 1

Graph theory

1.1 Basic definitions

Graphs and digraphs

A *graph* $G = (V, E)$ consists of a nonempty set V of n elements called *vertices*, and a set of m unordered pairs of vertices called *edges*. The parameters $n = |V|$ and $m = |E|$ are named the *order* and *size* of G respectively. If $e = \{x, y\}$ is an edge of G, we say that x and y are *adjacent*, or simply $x \sim y$. It is also said that x and y are *incident* to e. A *loop* is an edge $\{x, y\}$ such that $x = y$. A *simple graph* is a graph without loops [1]. The set of vertices [edges] adjacent [incident] to a vertex x is called the *neighbourhood* [*edge-neighbourhood*] of x, denoted by $\Gamma(x)$ $[w(x)]$, and $\delta(x) = |\Gamma(x)| = |w(x)|$ is said to be the *degree* of x. The *minimum degree* of the vertices of G is denoted by $\delta(G) = \delta$ and the *maximum degree* by $\Delta(G) = \Delta$. A graph G is *k-regular* if $\delta(G) = \Delta(G) = k$.

A *digraph* $G = (V, A)$ consists of a nonempty set V of n elements called *vertices*, and a set of m ordered pairs of vertices called *arcs*. The parameters $n = |V|$ and $m = |A|$ are the *order* and *size* of G respectively. If $e = (x, y)$ is an arc of G, we say that x is *adjacent* to y, or y is *adjacent* from x, or simply $x \rightarrow y$. A *loop* is an arc (x, y) such that $x = y$. A *(simple) digraph* is a digraph without loops. The set of vertices adjacent from [to] a vertex x is called the *out-neighbourhood* [*in-neighbourhood*] of x, denoted by $\Gamma^+(x)$ $[\Gamma^-(x)]$, and similarly is defined the *out-arc-neighbourhood* [*in-arc-neighbourhood*] $w^+(x)$ $[w^-(x)]$. The integer $\delta^+(x) = |\Gamma^+(x)| = |w^+(x)|$ $[\delta^-(x) = |\Gamma^-(x)| = |w^-(x)|]$ is said to be the *out-degree* [*in-degree*] of x. The *minimum out-degree* [*minimum in-degree*] of the vertices of G is denoted by $\delta^+(G) = \delta^+$ $[\delta^-(G) = \delta^-]$ and the *maximum out-degree* [*maximum in-degree*] is denoted by $\Delta^+(G) = \Delta^+$ $[\Delta^-(G) = \Delta^-]$. The *minimum degree* [*maximum degree*] of G is $\delta = min\{\delta^-, \delta^+\}$ $[\Delta = max\{\Delta^-, \Delta^+\}]$.

Let F be a subset of vertices of a (di)graph $G = (V, A)$. The *out-neighbourhood*

[1] The simple graphs are often called *graphs*, whereas the graphs with loops are called *pseudographs* (see [39],[90]).

$\Gamma^+(F)$ and *positive boundary* $\partial^+(F)$ of F are defined as follows:

$$\Gamma^+(F) = \bigcup_{x\in F} \Gamma^+(x), \qquad \partial^+(F) = \Gamma^+(F) \setminus F.$$

The *in-neighbourhood* $\Gamma^-(F)$ and *negative boundary* $\partial^-(F)$ are defined similarly. As for the arc case, the *positive arc-boundary* $w^+(F)$ and *negative arc-boundary* $w^-(F)$ of F are defined in the following way:

$$w^+(F) = \{(x,y) \in A \ : \ x \in F \text{ and } y \in V \setminus F\},$$

$$w^-(F) = \{(x,y) \in A \ : \ x \in V \setminus F \text{ and } y \in F\}.$$

In the definition of a graph [digraph], if more than one edge [arc in the same direction] is permitted to join two vertices, the resulted structure is called a *multigraph* [*multidigraph*]. A *symmetric digraph* is a digraph such that for every pair of vertices x,y: $(x,y) \in A \Leftrightarrow (y,x) \in A$. Observe that every graph $G = (V,E)$ can be identified with a unique symmetric digraph $G = (V,A)$, and notice that $|A| = 2|E|$. In the rest of this work, we will use the following definition: A *(di)graph* $G = (V,A)$ is either a simple digraph or graph (considered, if necessary, as a symmetric digraph).

Distance in (di)graphs

Let x,y be two vertices of a (di)graph G. A *walk* of *length h* from x to y is a sequence of vertices $x = x_0x_1 \dots x_h = y$, where (x_i, x_{i+1}) is an arc. It is called a *trail* if all its arcs are distinct, and a *path*, or an $x \rightarrow y$ path, if all its internal vertices (and thus necessarily all its arcs) are different. A *circuit* is a closed walk; that is to say, a walk from a vertex x to itself. A *cycle* [2] is a closed path (if G is a graph, of length at least 3). A *digon* is a closed path of length 2 (observe that if G is a graph, an digon is not a cycle). The *distance* from x to y, denoted by $d(x,y)$, is the length of a shortest $x \rightarrow y$ path (if there are no $x \rightarrow y$ paths, then we put $d(x,y) = \infty$). Certainly, if G is a graph, then for every pair of vertices: $d(x,y) = d(y,x)$. If F is a subset of vertices of G, the distance from a vertex x to F [from F to a vertex x] is defined as follows:

$$d(x,F) = \min_{f\in F}\{d(x,f)\} \quad [d(F,x) = \min_{f\in F}\{d(f,x)\}].$$

Similarly, if F is a subset of arcs, the distance from x to F [from F to x] is defined in this way:

$$d(x,F) = \min_{(u,v)\in F}\{d(x,u)\} \quad [d(F,x) = \min_{(u,v)\in F}\{d(v,x)\}].$$

A (di)graph is called *strongly connected*, or simply *connected*, if for every pair of vertices x,y there is an $x \rightarrow y$ path. The *diameter* of a connected (di)graph is defined as follows:

$$D = D(G) = max\{d(x,y) : \ x,y \in V\}.$$

[2]Also called a *directed cycle*, if G is a digraph.

The *girth* $g = g(G)$ of a (di)graph G is the minimum length of a cycle (if there are no cycles in G, then we put $g(G) = \infty$). A (Δ, D)-*(di)graph* is a connected (di)graph with maximum degree Δ and diameter D.

Sub(di)graphs

Let $G = (V, A)$ and $G_1 = (V_1, A_1)$ two (di)graphs. The (di)graph G_1 is said to be a *sub(di)graph* of G if $V_1 \subset V$ and $A_1 \subset A$. In particular, if $V_1 = V$ $[A_1 = A \cap (V_1 \times V_1)]$, then G_1 is called a *spanning [induced] sub(di)graph* of G.

If C is a maximal connected sub(di)graph of G, then C is said to be a *(strongly) connected component* or simply a *component* of G. A component C of a non-connected digraph G is said to be a *source component*, or simply a *source*, if $w^-C = \emptyset$. Similarly, a component C of G is called a *sink* if $w^+C = \emptyset$. Otherwise, C is said to be a *transmittance component.*

For any set S of vertices of a (di)graph G, the *induced sub(di)graph* $< S >$ is the maximal sub(di)graph of G with vertex set S. Observe that two vertices of S are adjacent in $< S >$ if and only if they are adjacent in G. If $A' \subset A(G)$ then $G - A'$ denotes the (di)graph resulting from G when the edges belonging to A' are removed, i.e. $G - A' = (V, A \setminus A')$. Similarly, if $V' \subset V(G)$ then $G - V'$ is the (di)graph obtained from G by the removal of the vertices belonging to V'.

Other definitions

A set of vertices S of a (di)graph is called *stable* if no arc joins two vertices (not necessarily distinct) in S. The *complete graph* K_n^* is the graph of order n without stable sets of vertices. The *(simple) complete graph* K_n is the graph obtained from K_n^* by the deletion of all its loops. A (di)graph is *bipartite* if its vertices can be partitioned into two stable sets V_1 and V_2. The *complete bipartite graph* $K_{h,k}$ is the bipartite graph with $|V_1| = h$, $|V_2| = k$, and such that every vertex $x \in V_1$ is adjacent to all the vertices of V_2.

Two (di)graphs $G = (V, A)$ and $G' = (V', A')$ are said to be *isomorphic*, denoted by $G \cong G'$, if there exists a one-to-one correspondence ϕ between their vertex sets which preserves adjacency; that is, such that for every pair of vertices $x, y \in G$: $(x, y) \in A \Leftrightarrow (\phi(x), \phi(y)) \in A'$.

A [connected] graph without cycles is called a *forest* [*tree*]. Similarly, a non-connected digraph that contains no directed cycles is said to be an *acyclic digraph.* An *oriented tree* is an acyclic digraph whose underlying graph is a tree. An oriented tree T_z is called *out-rooted* [*in-rooted*] with root z, if for each $v \in V(T_z)$ there is a unique $z \to v$ path [$v \to z$ path]. The *level number of* a vertex v is the length of this unique $z \to v$ [$v \to z$] path, and the maximum of the level numbers of T_z is called its *height.*

The *converse digraph* $\overleftarrow{G} = (V, \overleftarrow{A})$ of a digraph $G = (V, A)$ has the same vertex set, whereas its arcs are obtained by reversing the orientation of every arc in A. To be more precise: $\forall x, y \in V,\ (x,y) \in \overleftarrow{A} \Leftrightarrow (y,x) \in A$.

The *underlying graph* $UG = (V, E)$ of $G = (V, A)$ is the graph with the same vertex set, and whose edge set is defined as follows [3]:

$$\forall x, y \in V,\ \{x, y\} \in E \Leftrightarrow (x,y) \in A \text{ or } (y,x) \in A.$$

The *condensation* G^* of a non-connected digraph G has the (strong) components as its vertices, with an arc from a component C_i to another C_j whenever there is at least one arc in G from a vertex of C_i to a vertex in C_j. Notice that, from the maximality of strong components, it follows immediately that the condensation G^* of any digraph is acyclic (see Figure 1.1).

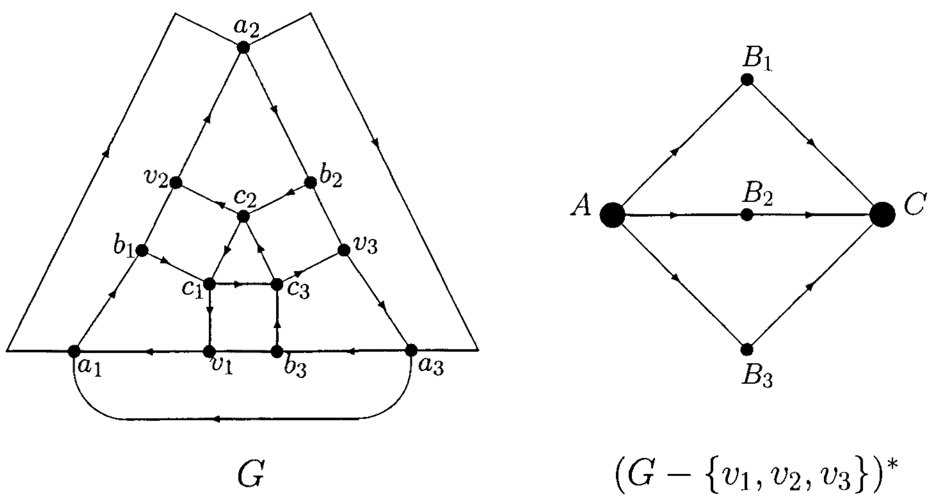

Figure 1.1: Condensation of a disconnected digraph obtained by deleting three vertices in a 2-regular digraph of diameter 3.

1.2 The (Δ, D)-problem

A question of special interest in Graph theory is the construction of connected graphs and digraphs with an order $n(\Delta, D)$ as large as possible for a given maximum degree Δ and diameter D. It is the so-called (Δ, D)*-problem*, and any of its 'solutions' is referred to as a *large (di)graph*.

Definition 1.2.1 *Let $\mathcal{H}$ be a family of connected (di)graphs. A Moore bound of $\mathcal{H}$ is a function $\mathcal{M}_{\mathcal{H}}(\Delta, D)$ such that the order of every (Δ, D)-(di)graph $G \in \mathcal{H}$ is at*

[3]The condition $x \neq y$ is sometimes added.

most $\mathcal{M}_{\mathcal{H}}(\Delta, D)$. A Moore (di)graph in $\mathcal{H}$ is a (Δ, D)-(di)graph G such that $n(G) = \mathcal{M}_{\mathcal{H}}(\Delta, D)$.

Different Moore bounds have been obtained for a wide range of families of connected graphs and digraphs. Let us show some of them:

Proposition 1.2.1 *Let $\mathcal{H}$ be a family of connected (di)graphs.*

1. $\mathcal{H}=\{G\text{: } G \text{ is a graph }\}$: $\mathcal{M}_{\mathcal{H}}(\Delta, D)= \begin{cases} 2D+1 & \textit{if } \Delta = 2 \\ \frac{\Delta(\Delta-1)^D-2}{\Delta-2} & \textit{if } \Delta > 2 \end{cases}$

2. $\mathcal{H}=\{G\text{: } G \text{ is a bipartite graph }\}$: $\mathcal{M}_{\mathcal{H}}(\Delta, D)= \begin{cases} 2D & \textit{if } \Delta = 2 \\ \frac{2(\Delta-1)^D-2}{\Delta-2} & \textit{if } \Delta > 2 \end{cases}$

3. $\mathcal{H}=\{G\text{: } G \text{ is a digraph}\}$: $\mathcal{M}_{\mathcal{H}}(\Delta, D)= \begin{cases} D+1 & \textit{if } \Delta = 1 \\ \frac{\Delta^{D+1}-1}{\Delta-1} & \textit{if } \Delta > 1 \end{cases}$

4. $\mathcal{H}=\{G\text{: } G \text{ is a bipartite digraph, } D \text{ odd }\}$: $\mathcal{M}_{\mathcal{H}}(\Delta, D)= \begin{cases} D+1 & \textit{if } \Delta = 1 \\ \frac{2\Delta^{D+1}-2}{\Delta^2-1} & \textit{if } \Delta > 1 \end{cases}$

5. $\mathcal{H}=\{G\text{: } G \text{ is a bipartite digraph, } D \text{ even }\}$: $\mathcal{M}_{\mathcal{H}}(\Delta, D)= \begin{cases} D & \textit{if } \Delta = 1 \\ \frac{2\Delta^{D+1}-2\Delta}{\Delta^2-1} & \textit{if } \Delta > 1 \end{cases}$

Proof.

1. Let v be any vertex of G and let $\Gamma_i(v)$ denote the set of vertices at distance i of v. It is clear that the set $\{\Gamma_0(v), \Gamma_1(v), \ldots, \Gamma_D(v)\}$ is a partition of the vertex set V, and $|\Gamma_i(v)| \leq \Delta(\Delta-1)^{i-1}$ for all $i \in \{1, \ldots, D\}$. Hence,
$$n = |V| = |\Gamma_0(v)| + \sum_{i=1}^{D} |\Gamma_i(v)| \leq 1 + \sum_{i=1}^{D} \Delta(\Delta-1)^{i-1},$$
and the proposed upper bound is obtained.

2. The proof of this case is similar to the previous one with the only difference that, taking into account that G is bipartite, the set $\Gamma_D(v)$ is stable and therefore, $|\Gamma_D(v)| \leq (\Delta-1)^{D-1}$.

3. Let v be any vertex of G and let $\Gamma_i^+(v)$ denote the set of vertices at distance i from v. Certainly, the set $\{\Gamma_0^+(v), \Gamma_1^+(v), \ldots, \Gamma_D^+(v)\}$ is a partition of the vertex set V, and for every $i \in \{0, 1, \ldots, D\}$: $|\Gamma_i^+(v)| \leq \Delta^i$. Hence,
$$n = |V| = \sum_{i=0}^{D} |\Gamma_i^+(v)| \leq \sum_{i=0}^{D} \Delta^i.$$

4. Let $G = (V_1 \cup V_2, E)$ a bipartite digraph with odd diameter D and suppose that $|V_1| \le |V_2|$. If v is any vertex of V_2, it is clear that the set $\{\Gamma_0^+(v), \Gamma_2^+(v), \ldots, \Gamma_{D-1}^+(v)\}$ is a partition of the stable vertex set V_2, and for any $i \in \{0, 2, \ldots, D-1\}$: $|\Gamma_i^+(v)| \le \Delta^i$. Hence,
$$n = |V| = |V_1| + |V_2| \le 2|V_2| = 2 \sum_{i=0}^{(D-1)/2} |\Gamma_{2i}^+(v)| \le 2 \sum_{i=0}^{(D-1)/2} \Delta^{2i}.$$

5. Let $G = (V_1 \cup V_2, E)$ be a bipartite digraph with even diameter D and suppose that $|V_1| \le |V_2|$. If v is any vertex of V_1, the set $\{\Gamma_1^+(v), \Gamma_3^+(v), \ldots, \Gamma_{D-1}^+(v)\}$ is crtainly a partition of the stable vertex set V_2, and for any $i \in \{1, 3, \ldots, D-1\}$: $|\Gamma_i^+(v)| \le \Delta^i$. Hence,
$$n = |V| = |V_1| + |V_2| \le 2|V_2| = 2 \sum_{i=1}^{D/2} |\Gamma_{2i-1}(v)| \le 2 \sum_{i=1}^{D/2} \Delta^{2i-1}.$$
■

The only Moore graphs are: the complete graphs $K_{\Delta+1}$, the odd cycles C_{2D+1}, the Petersen (3,2)-graph P (see Figure 2.2), the Hoffmann-Singleton (7,2)-graph, and perhaps, a $(57, 2)$-graph (see [22]). There are five families of Moore bipartite graphs: for $\Delta = 2$ the even cycles C_{2D}, for $D = 2$ the complete equi-bipartite graphs $K_{\Delta,\Delta}$, and for $D = 3, 4$ or 6 with maximum degree a prime power q plus one, the graphs denoted by P_q, Q_q and H_q respectively (see Section 1.5.1).

As for the directed case, Bridges and Toueg proved in [33] that the unique Moore digraphs are the complete graphs $K_{\Delta+1}$ and the directed cycles $\vec{C}_n$. The even directed cycles $\vec{C}_{2h}$ (that are the only connected bipartite digraphs with $\Delta = 1$) and the complete equi-bipartite graphs $K_{\Delta,\Delta}$ are Moore bipartite digraphs. In [67], Fiol and Yebra proved that for any $\Delta > 1$ and $D = 3$ or 4 there are Moore bipartite digraphs, and such digraphs do not exist for $D \ge 5$ (see Remark 1.5.1).

Due to the non-attainability, in most cases, of the Moore bounds obtained for different families of (di)graphs, the study of the (Δ, D)-problem in a certain (di)graph family $\mathcal{H}$ mainly consists in finding (Δ, D)-(di)graphs with order, on the one hand, as close as possible to the corresponding Moore bound $\mathcal{M}_{\mathcal{H}}(\Delta, D)$, and on the other hand, larger than the rest of the known (Δ, D)-(di)graphs of $\mathcal{H}$.

1.3 The line digraph technique

The line digraph technique was introduced by Harary and Norman in [92], and has proved to be very useful in the design of digraphs with 'good' properties. For instance, by means of this technique a wide range of large digraphs have been designed (see Section 1.5.2).

Let $G = (V, A)$ be a $(\delta, \Delta, D, n, m)$-digraph. In the *line digraph* of G, denoted by LG, each vertex represents an arc of G; that is to say, $V(LG) = \{xy : (x, y) \in A\}$. The arc set of LG is defined by means of the following adjacency rule:

$$uv \to xy \Leftrightarrow v = x$$

For every $h > 0$, the *h-iterated line digraph* L^hG is defined recursively by $L^hG = L(L^{h-1}G)$. Observe that the vertices of L^hG represent the walks of length h in G and can be denoted by 'words' with $h+1$ letters. Notice also that for every vertex $x_1x_2\dots x_{h+1}$,

$$\Gamma^+(x_1x_2\dots x_{h+1}) = \{x_2\dots x_{h+1}\alpha \ : \ \alpha \in \Gamma^+(x_{h+1})\}$$

As a direct consequence of the definitions the following list of properties is obtained (see Figure 1.3).

Proposition 1.3.1 *([68],[93]) Let LG be the line digraph of a digraph G. Then,*

1. $n(LG) = m$.
2. $uv \in V(LG) \Rightarrow \delta^-(uv) = \delta^-(u)$, $\delta^+(uv) = \delta^+(v)$.
3. $\delta(LG) = \delta(G)$, $\Delta(LG) = \Delta(G)$.
4. *G is k-regular $\Leftrightarrow$ LG is k-regular, and $n(LG) = kn$.*
5. *G is k-regular $\Leftrightarrow$ L^hG is k-regular, and $n(L^hG) = k^hn$.*
6. $G = \vec{C}_n \Leftrightarrow LG = \vec{C}_n$.
7. *G bipartite $\Rightarrow$ LG bipartite.*
8. *G connected: LG bipartite $\Rightarrow$ G bipartite.*
9. *G and LG have the same girth: $g(LG) = g(G)$.*
10. *For every $T \subset A(G)$: $L(G-T)$=$LG-T$.*
11. *If G is disconnected and $|A(G)| \geq 2$, then LG is disconnected.*

In addition to these ones, this technique satisfies other interesting properties (see [93]), and next we show two of them. The first one states the relation between the diameter of a connected digraph G and that of LG, and the second is the so-called Heuchenne adjacency condition, which is a characterization of the line digraphs.

Proposition 1.3.2 *([2]) Let G be a connected digraph different from a directed cycle. Then LG is a connected digraph of diameter: $D(LG) = D(G) + 1$.*

Proposition 1.3.3 *([94]) In every simple digraph G the following conditions are equivalent:*

1. *G is the line digraph of a simple multidigraph* [4].
2. *For any $u, v \in V(G)$, either $\Gamma^+(u) \cap \Gamma^+(v) = \emptyset$ or $\Gamma^+(u) = \Gamma^+(v)$.*
3. *For any $u, v \in V(G)$, either $\Gamma^-(u) \cap \Gamma^-(v) = \emptyset$ or $\Gamma^-(u) = \Gamma^-(v)$.*

[4] To guarantee that G is the line digraph of a simple digraph some more restriction must be added, see [93] for details.

1.4 Operations on (di)graphs

There are many ways of combining (simple) (di)graphs to produce new (di)graphs (see [90]). For instance, the *complement* $\overline{G}$ of a graph G has the same set of vertices $V(G)$, two vertices being adjacent in $\overline{G}$ if and only if they are not adjacent in G. In this section, we describe several binary operations on two (di)graphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, which result in a (di)graph G whose vertex set is the cartesian product $V_1 \times V_2$.

Cartesian product

In the *cartesian product* $G_1 \times G_2 = (V_1 \times V_2, A)$, a vertex (u_1, u_2) is adjacent to another vertex (v_1, v_2) if and only if either

$$u_1 = v_1 \text{ and } u_2 \to v_2$$

or

$$u_2 = v_2 \text{ and } u_1 \to v_1.$$

In the next proposition we show some properties of this operation.

Proposition 1.4.1 *([37])*

1. *$G_1 \times G_2$ has order $n_1 n_2$ and size $n_1 m_2 + n_2 m_1$.*
2. *$\delta(G_1 \times G_2) = \delta_1 + \delta_2$, $\Delta(G_1 \times G_2) = \Delta_1 + \Delta_2$.*
3. *$G_1 \times G_2 \cong G_2 \times G_1$.*
4. *If G_1 and G_2 are connected, then $G_1 \times G_2$ is also connected.*
5. *If G_1 and G_2 are bipartite, then $G_1 \times G_2$ is also bipartite.*
6. *In general, $\overline{G_1 \times G_2} \ncong \overline{G_1} \times \overline{G_2}$.*

Lexicographic product

In the *lexicographic product* $G_2(G_1) = (V_2 \times V_1, A)$, a vertex (u_2, u_1) is adjacent to another vertex (v_2, v_1) if and only if either

$$u_2 \to v_2$$

or

$$u_2 = v_2 \text{ and } u_1 \to v_1.$$

Next, we show some properties involving this operation.

Proposition 1.4.2 *([37])*

1. *$G_2(G_1)$ has order $n_1 n_2$ and size $n_2 m_1 + n_1^2 m_2$.*

2. $\delta(G_1 \times G_2) = \delta_1 + \delta_2 \cdot n_1$, $\Delta(G_1 \times G_2) = \Delta_1 + \Delta_2 \cdot n_1$

3. *In general,* $G_2(G_1) \ncong G_1(G_2)$.

4. *If* G_1 *and* G_2 *are connected, then* $G_2(G_1)$ *is also connected.*

5. *If* G_1 *and* G_2 *are bipartite, then, in general,* $G_2(G_1)$ *is not bipartite.*

6. $\overline{G_2(G_1)} \cong \overline{G_2}(\overline{G_1})$.

Both the cartesian product $G_2 \times G_1$ and the lexicographic product $G_2(G_1)$ can be defined in the following way. If G_1 and G_2 are two (di)graphs, the new (di)graph G is obtained by replacing in G_2 each vertex x by one copy G_1^x of G_1. The difference between the two constructions lies in the so-called *intercopy arcs*. So, in the cartesian product $G_2 \times G_1$ each arc (x, y) of G_2 is replaced by exactly n_1 arcs from G_1^x to G_1^y, whereas in the construction of the lexicographic product $G_2[G_1]$ each arc of G_2 gives rise to n_1^2 arcs. As expected, these operations have been generalized by several authors (see [18, 47, 48, 77]) in different ways. Most of these generalizations consist basically in replacing each vertex of a (di)graph G_2 by a (di)graph G_1 and each arc of G_2 by a certain number β of 'intercopy arcs'. When the value of β is rather small (for instance 1 or 2) the corresponding construction is called a *compounding* of (di)graphs (see section 1.5.3), whereas such a construction is called a *product* of (di)graphs if β is rather large (for instance n_1).

Conjunction

In the *conjunction* [5] $G_1 \otimes G_2 = (V_1 \times V_2, A)$, a vertex (u_1, u_2) is adjacent to another vertex (v_1, v_2) if and only if $u_1 \to v_1$ and $u_2 \to v_2$. As in the previous cases, let us see some properties involving this operation.

Proposition 1.4.3 *([37])*

1. $G_1 \otimes G_2$ *has order* $n_1 n_2$ *and size* $2m_1 m_2$.

2. $\delta(G_1 \otimes G_2) = \delta_1 \cdot \delta_2$, $\Delta(G_1 \otimes G_2) = \Delta_1 \cdot \Delta_2$

3. $G_1 \otimes G_2 \cong G_2 \otimes G_1$.

4. *If* G_1 *and* G_2 *are connected, then* $G_1 \otimes G_2$ *is connected if and only if either* G_1 *or* G_2 *is not bipartite.*

5. *If either* G_1 *or* G_2 *is bipartite, then* $G_1 \otimes G_2$ *is also bipartite.*

[5] It is also known as *tensor product* and *Kronecker product.*

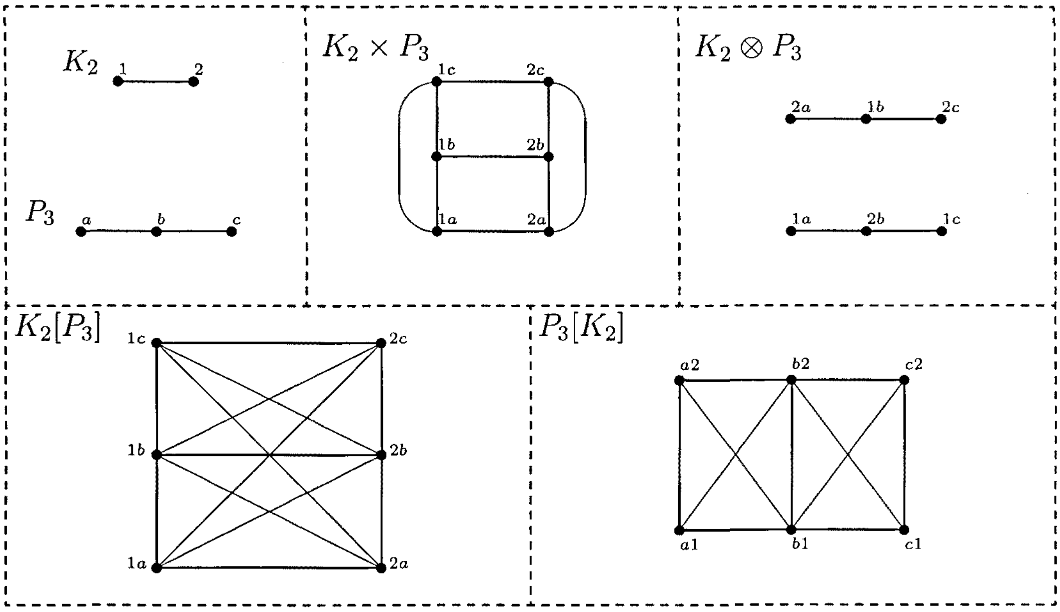

Figure 1.2: Some binary operations from the graphs K_2 and P_3.

6. *If G_1 and G_2 are connected, then $G_1 \otimes G_2$ consists of exactly two bipartite components if and only if G_1 and G_2 are both bipartite.*

7. *In general, $\overline{G_1 \otimes G_2} \not\cong \overline{G_1} \otimes \overline{G_2}$.*

The conjunction of (di)graphs satisfies another property, which reveals the excellent behaviour of the line digraph technique with respect to this operation (see [84]).

Proposition 1.4.4 *For any given two (di)graphs G_1 and G_2 it follows that*

$$L(G_1 \otimes G_2) \cong LG_1 \otimes LG_2$$

Proof. Let u_1v_1 and u_2v_2 be arcs of $A(G_1)$ and $A(G_2)$ respectively. Let us consider a one-to-one mapping ϕ from $V(L(G_1 \otimes G_2))$ onto $V(LG_1 \otimes LG_2)$; namely, $\phi((u_1, u_2)(v_1, v_2)) = (u_1v_1, u_2v_2)$. It follows that ϕ is an isomorphism, because it preserves adjacency. Indeed, $(u_1, u_2)(v_1, v_2)$ is adjacent to another vertex $(a_1, a_2)(b_1, b_2)$ of $L(G_1 \otimes G_2)$ if and only if $(v_1, v_2) = (a_1, a_2)$, which is equivalent to saying that (u_1v_1, u_2v_2) is adjacent to (a_1b_1, a_2b_2) in $LG_1 \otimes LG_2$. ■

1.5 Some families of (di)graphs

1.5.1 Moore bipartite graphs

The terms Moore graph and Moore bipartite graph introduced in Section 1.2 are also used for a certain class of graphs whose members are the so-called *cages.* A (Δ, g)-cage is a Δ-regular graph of girth g having the minimum possible order $n(\Delta, g)$. In the following result, we find the reason for this apparent contradiction.

Proposition 1.5.1 *([22])*
1. *The order of a graph with* $\Delta \geq 3$ *and odd girth* $g = 2d + 1$ *is at least*

$$n_0(\Delta, g) = \tfrac{\Delta(\Delta-1)^d-2}{\Delta-2} \text{ if } \Delta > 2$$

If there is such a graph, having exactly $n_0(\Delta, g)$ *vertices, then it has diameter* $D = d$.

2. *The order of a graph with* $\Delta \geq 3$ *and even girth* $g = 2d$ *is at least*

$$n_0(\Delta, g) = \tfrac{2(\Delta-1)^d-2}{\Delta-2} \text{ if } \Delta > 2$$

If there is such a graph, having exactly $n_0(\Delta, g)$ *vertices, then it is bipartite and has diameter* $D = d$.

Observe that these lower bounds coincide with the Moore upper bounds for graphs and bipartite graphs obtained in Proposition 1.2.1. Note also that for $\Delta = 2$, the cycles C_g are Moore $(2, \frac{g-1}{2})$-graphs if g is odd and Moore bipartite $(2, \frac{g}{2})$-graphs if g is even, and in both cases, they are $(2, g)$-cages. Therefore, taking into account that the girth of any Moore graph [Moore bipartite graph] has to be odd [even], we conclude that a graph [bipartite graph] G of maximum degree Δ and diameter D is a Moore graph [Moore bipartite graph] if and only if it is a $(\Delta, 2D + 1)$-cage [$(\Delta, 2D)$-cage] of order $n_0(\Delta, 2D + 1)$ [$n_0(\Delta, 2D)$].

Let us consider the family of Moore bipartite graphs; that is to say, the family of (Δ, g)-cages of even girth having order $n_0(\Delta, g)$. As we have already noticed, when $\Delta = 2$ the even cycles provide the complete answer to the problem. For $\Delta \geq 2$, Feit and Higman proved in [58] that Moore bipartite graphs with a girth greater than 12 do not exist. For $g = 4$, it is easy to see that the complete bipartite graph $K_{\Delta,\Delta}$ is the unique Moore bipartite graph for all $\Delta \geq 2$ (see [39]). In [14], C. T. Benson proved that, for $g = 2D \in \{6, 8, 12\}$, there exist Moore bipartite (Δ, D)-graphs if and only if $q = \Delta - 1$ is a prime power.

For $g = 2D = 6$, Singleton in [131] noted that the so-called *generalized triangles* P_q are Moore bipartite $(q+1, 3)$-graphs. In fact, P_q is the incidence graph of the projective plane $PG(2, q)$; that is to say, it is the bipartite graph whose stable sets are, on the one hand, the points of $PG(2, g)$, and on the other, its lines, being a point a and a line b adjacent if and only if $a \in b$.

In the mentioned work [14], the author also constructed the so-called *generalized quadrangles* Q_q and *generalized hexagons* H_q, and proved that they are Moore bipartite

$(q+1,D)$-graphs of girth $g = 2D = 8$ and 12 respectively. In both cases, these graphs are defined in a similar way as the generalizad triangles P_n; that is to say, as incidence graphs of certain projective geometries. For instance, the stable sets of H_q are, on the one hand, the points of the 5-dimensional projective space $PG(5,q)$, and on the other, some lines of a non-degenerate quadric surface in $PG(5,q)$.

1.5.2 De Bruijn and Kautz (di)graphs

As Fiol, Yebra and Alegre proved in [68], the line digraph technique is an excellent method to obtain large digraphs with other good properties. Two of the most important line digraph families are the so-called De Bruijn and Kautz digraphs.

Let K_d^* be the complete graph of order $d \geq 2$. The *De Bruijn digraph* $B(d,D)$ is defined as the $(D-1)$-iterated line digraph of K_d^*; that is to say, $B(d,D) = L^{D-1}K_d^*$. As a direct consequence of the properties of the line digraph technique, it is derived that $B(d,D)$ is a d-regular digraph of diameter D and order d^D. Observe that $B(d,D)$ has d loops (incident to the vertices of the form $aa\ldots a$) and $\binom{d}{2}$ digons (between each pair of vertices of the form $abab\ldots$ and $baba\ldots$).

The De Bruijn digraphs, which were introduced in [34], were originally defined as *digraphs on alphabet* in the following way. The vertices of $B(d,D)$ are the words of length D constructed from an alphabet X of cardinality d, being a vertex $x = x_1x_2\ldots x_D$ adjacent to a vertex $y = y_1y_2\ldots y_D$ if and only if the last $D-1$ letters of x are the same as the first $D-1$ letters of y. In other words,

$$\Gamma^+(x_1x_2\ldots x_D) = \{x_2\ldots x_D\alpha \;:\; \alpha \in X\}$$

Let K_{d+1} be the simple complete graph of order $d+1 \geq 3$. The *Kautz digraph* $K(d,D)$ is defined as the $(D-1)$-iterated line digraph of K_{d+1}; that is to say, $K(d,D) = L^{D-1}K_{d+1}$. As a direct consequence of the properties of the line digraph technique, it is derived that $K(d,D)$ is a d-regular digraph of diameter D and order $d^D + d^{D-1}$. Observe that $K(d,D)$ has $\binom{d+1}{2}$ digons (between each pair of vertices of the form $abab\ldots$ and $baba\ldots$), and it has no loops. For example, the digraph G of Figure 1.1 is $K(2,3)$, and it is a regular and simple digraph with 12 vertices and 3 digons.

This family of digraphs contains, in almost all cases (see Fig. 3 of [68], for a counterexample), the larger known digraphs. Indeed, the order of the Kautz digraph $K(d,D)$ is very close to the unattainable (except in the case $D=1$) Moore bound for digraphs. To be more precise,

$$n(K(d,D)) = d^D + d^{D-1} > \frac{d^2-1}{d^2}\cdot\frac{d^{D+1}-1}{d-1} = \frac{d^2-1}{d^2}\cdot\mathcal{M}(d,D)$$

The Kautz digraphs, which were introduced in [104], were originally defined in the same way as the De Bruijn digraphs, taking another vertex set. In this case, the vertices of $K(d,D)$ are the words of length D from an alphabet X of cardinal number $d+1$, in which any two consecutive letters have to be different. In other words, $K(d,D)$ is the subgraph of $B(d+1,D)$ induced by

$$V(K(d,D)) = \{x_1x_2\dots x_D \in V(B(d+1,D)) \ : \ x_i \neq x_{i+1},\ i = 1,\dots,D-1\}$$

The *De Bruijn graph* $UB(d,D)$ is defined as the simple [6] underlying graph of $B(d,D)$. Observe that, for $D \geq 2$, $UB(d,D)$ is a graph with minimum degree $\delta = 2d-2$, maximum degree $\Delta = 2d$, diameter D and order $n = (\frac{\Delta}{2})^D$. In a similar way, the *Kautz graph* $UK(d,D)$ is defined as the underlying graph of $K(d,D)$. Notice also that $UK(d,D)$ is, for $D \geq 2$, a graph with minimum degree $\delta = 2d-1$, maximum degree $\Delta = 2d$, diameter D and order $n = (\frac{\Delta}{2})^D + (\frac{\Delta}{2})^{D-1}$.

Due mainly to the fact that both the De Bruijn and Kautz graphs have even maximum degree, several authors have designed different families of dense *graphs on alphabet* (see [79] and its references). Nevertheless, in all the cases the order of these graphs is $n \approx k(\frac{\Delta}{2})^D$ with $k \in \{1,2,3\}$, that is far away from the Moore bound for graphs $\mathcal{M}(\Delta,D) \approx \Delta^D$.

Generalized De Bruijn and Kautz digraphs

The De Bruijn digraph $B(d,D)$ can also be arithmetically defined as the digraph with vertex set Z_n, $n = d^D$, whose adjacency rule is:

$$x \in V(B(d,D)) \Rightarrow \Gamma^+(x) = \{dx + t \ : \ t \in Z_d\}$$

Starting from this definition, Reddy, Pradham and Kuhl defined in [129] (see also [99]) the so-called *generalized De Bruijn digraphs* [7] $GB(d,n)$, by replacing the $n = d^D$ condition by the more general one: $2 \leq d \leq n$. Certainly, the De Bruijn digraph $B(d,D)$ coincides with $GB(d,d^D)$. It is also well known that $GB(d,n)$ is a d-regular digraph of order n, diameter $D = \lceil \log_d n \rceil$, and it has loops. Another interesting property of $GB(d,n)$ is that its line digraph is another generalized De Bruijn digraph. To be more precise: $LGB(d,n) = GB(d,dn)$.

In a similar way, the Kautz digraph $K(d,D)$ can also be arithmetically defined as the digraph with vertex set Z_n, $n = d^D + d^{D-1}$, whose adjacency rule is:

$$x \in V(B(d,D)) \Rightarrow \Gamma^+(x) = \{-dx - t \ : \ 1 \leq t \leq d\}$$

Imase and Itoh introduced in [100] the so-called *generalized Kautz digraphs* [8] $GK(d,n)$, by using the foregoing definition and replacing the $n = d^D + d^{D-1}$ condition by this other: $2 \leq d \leq n$. Obviously, $K(d,D) = GK(d, d^D + d^{D-1})$. It is also well known that $GK(d,n)$ is a d-regular digraph of order n, diameter $D \in \{\lfloor \log_d n \rfloor, \lceil \log_d n \rceil\}$, and it only has loops when n is not a multiple of $d+1$. As in De Bruijn case, this family also reveal a good behaviour with respect to the line digraph technique, since $LGK(d,n) = GK(d,dn)$.

[6] For $D \geq 2$, the loops are removed.

[7] Also known as *Reddy-Pradham-Kuhl digraphs*.

[8] Also known as *Imase-Itoh digraphs*.

1.5.3 Compound graphs

Compounding of graphs has proved to be an excellent method for obtaining large graphs when the diameter is not too large. The first compound graphs were introduced by Bermond, Delorme, and Quisquater in [18], and since then this operation has been used by several authors in order to obtain new designs of compound graphs. We have noticed that all these constructions can be unified according to one of the following definitions [9]:

Definition 1.5.1 *Let $G_2 = (V_2, E_2)$, $G_1 = (V_1, E_1)$ be two graphs. It is denoted by $G_2[G_1] = (V, E)$ any graph obtained in the following way:*

- *In G_2, each vertex $x \in V_2$ is replaced by one copy of G_1 represented by G_1^x. Hence,*

$$V = V(G_2[G_1]) = \bigcup_{x \in V_2} V(G_1^x) = \bigcup_{x \in V_2} \{(x, x') : x' \in V_1\} = V_2 \times V_1$$

- *In G_2, each edge $xy \in E_2$ is replaced by, at least, one edge that joins one vertex of G_1^x with another one of G_1^y; that is,*

$$xy \in E_2 \iff \exists x', y' \in V_1 \text{ such that } (x, x')(y, y') \in E$$

Definition 1.5.2 *Let $G_2 = (U_2 \cup V_2, E_2)$ be a bipartite graph, and let $G_1 = (V_1, E_1)$, $G_1' = (V_1', E_1')$ be two graphs. It is denoted by $G_2[G_1, G_1'] = (V, E)$ any graph obtained in the following way:*

- *In G_2, each vertex $x \in U_2$ is replaced by one copy G_1^x of G_1, and each vertex $y \in V_2$ by one copy G'^y_1 of G_1'. Hence,*

$$V = V(G_2[G_1, G_1']) = (\bigcup_{x \in U_2} V(G_1^x)) \cup (\bigcup_{y \in V_2} V(G'^y_1)) = (U_2 \times V_1) \cup (V_2 \times V_1')$$

- *In G_2, each edge $xy \in E_2$ is replaced by, at least, one intercopy edge, that is,*

$$xy \in E_2 \iff \exists x' \in V_1, y' \in V_1' \text{ such that } (x, x')(y, y') \in E,$$

in such a way that every vertex of the new graph must be the endvertex of at least one intercopy edge.

The first definition corresponds to three known constructions. In the first one (see [18]), each edge of G_2 is replaced by exactly one intercopy edge, and the graphs so obtained are the so-called *basic compound graphs*. The second construction, proposed by Delorme in [47], yields to the so-called *bipartite compound graphs*. In it, the graph G_1 is bipartite, and each edge of G_2 is replaced by one or two edges between copies[10] (see

[9] In this section, an edge $\{x, y\}$ is denoted by xy.

[10] In fact, in [47] the author only considered the case of two intercopy edges.

Figures 3.1 and 3.2). Finally, a third design of this kind, the so-called FF compound graphs, was proposed by Fiol and Fàbrega in [64]. As in the previous case, the graph G_1 is bipartite, but now each edge of G_2 is replaced by four intercopy edges, and the resulting graph is not bipartite (see Figure 3.3).

Two types of known compound graphs which correspond to the second definition are the so-called DQ_Λ and $B_0 \nabla B_1$ graphs, introduced in [50] and [77] respectively[11]. In the design of a DQ_Λ compound graph, G_1 is bipartite, and each edge of G_2 is replaced by two intercopy edges (see Figure 3.4). The $B_0 \nabla B_1$ (bipartite) graphs are similarly obtained, but in this case both G_1 and G_1' have to be bipartite, and each edge of G_2 is replaced by two (case bipartite) or four intercopy edges (see Figures 3.5 and 3.6).

Observe that the order of any compound graph follows directly from the orders of the original graphs. For instance: $n(G_2[G_1]) = n(G_2)n(G_1)$. However, both the maximum degree Δ and the diameter D depend on the number of intercopy edges and how they are placed in each case. In any case, the following proposition provides a list of upper bounds for the diameter D of the different compound graphs that have just been presented.

Proposition 1.5.2 *Let G_2, G_1, G_1' be three graphs of diameters D_2, D_1 and D_1' respectively.*

1. *([18]) If D is the diameter of a basic compound graph: $D \leq (D_1+1)D_2 + D_1$.*
2. *([47]) If D is the diameter of a bipartite compound graph: $D \leq D_1 D_2 + D_1$.*
3. *([64]) If D is the diameter of a FF compound graph: $D \leq D_1 D_2 + D_1 - 1$.*
4. *([50][12]) If D is the diameter of a DQ_Λ compound graph and D_2 is even: $D \leq \frac{D_2(D_1+D_1'+1)}{2}$.*
5. *([77][12]) If D is the diameter of a $B_0 \nabla B_1$ compound graph with four intercopy edges and D_2 is even: $D \leq \frac{D_2(D_1+D_1')}{2}$.*
6. *([77][12]) If D is the diameter of a $B_0 \nabla B_1$ bipartite compound graph with two intercopy edges and D_2 is even: $D \leq \frac{D_2(D_1+D_1')+2}{2}$.* ■

1.5.4 Generalized p-cycles

A *generalized p-cycle* is a digraph G in which its set of vertices can be partitioned into p subsets,

$$V(G) = \bigcup_{\alpha \in Z_p} V_\alpha,$$

[11] In both definitions only the case $G_2 = K_{h,k}$ was considered.

[12] These proofs have been carried out in the particular case $G_2 = K_{h,k}$, being very similar when G_2 is an arbitrary bipartite graph.

in such a way that the vertices in the partite set V_α are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in Z_p. Observe that any digraph can be shown as a generalized p-cycle with $p = 1$, whereas the bipartite digraphs are generalized 2-cycles.

In the next proposition, proved by Gómez, Padró and Perennes in [81], we show some properties that are satisfied by this kind of digraphs. Observe that, for $p = 1$ and $p = 2$ (see [67]), all of them are known results about digraphs and bipartite digraphs.

Proposition 1.5.3 *([81])*

1. *A digraph G is a generalized p-cycle if and only if for any pair of vertices x, y, the lengths of all paths from x to y are congruent modulo p.*

2. *The diameter D of a connected generalized p-cycle is the minimum integer such that for any vertex x, all the vertices of one of the partite sets of G are at distance of at most $D - (p - 1)$ from x.*

3. *Let G be a generalized p-cycle different from a directed cycle, with minimum degree δ, maximum degree Δ and diameter D. Then, LG is a generalized p-cycle with minimum degree δ, maximum degree Δ and diameter $D + 1$.*

4. *If G is a regular generalized p-cycle of order n, then LG is a regular generalized p-cycle of order Δn.*

5. *For any digraph G, the conjunction $\vec{C}_p \otimes G$ is a generalized p-cycle.*

6. *The Moore upper-bound of a generalized p-cycle with maximum degree $\Delta \geq 2$ and diameter D is*
$$\mathcal{M}_p(\Delta, D) = \frac{p\Delta^{D+1} - p\Delta^r}{\Delta^p - 1},$$
where $r \in \{0, 1, \ldots, p-1\}$ is the residue of $(D - (p - 1))$ modulo p.

7. *If $r \in \{0, \ldots, p-2\}$, then $\mathcal{M}_p(\Delta, D + 1) = \Delta\mathcal{M}_p(\Delta, D)$*

8. *If $r = p - 1$, then $\mathcal{M}_p(\Delta, D + 1) = \Delta\mathcal{M}_p(\Delta, D) + p$*

9. *If $\Delta \geq 2$, the Moore bound is attained if and only if $p \leq D \leq 3p - 2$.*

Possibly the most important kind of generalized p-cycles designed and studied in last years, are those constructed by taking into account the fifth property of the foregoing list. Among them, let us first point out the so-called *complete generalized p-cycle of degree $d \geq 2$*, which is defined as the conjunction $\vec{C}_p \otimes K_d^*$. In [81], Gómez, Padró and Perennes generalized in different ways these digraphs. Firstly, they introduced the so-called *De Bruijn generalized p-cycles $BGC(p, d, n)$*, which were defined as the conjuntions $\vec{C}_p \otimes GB(d, n)$, where GB(d,n) is the generalized De Bruijn digraph of degree d and order n. Notice that for $n = d^D$, $BGC(p,d,n)$ is isomorphic to the $(D-1)$-iterated line digraph of $\vec{C}_p \otimes K_d^*$:

$$L^{D-1}(\vec{C}_p \otimes K_d^*) = \vec{C}_p \otimes L^{D-1}K_d^* = \vec{C}_p \otimes B(d,D) = \vec{C}_p \otimes GB(d,d^D) = BGC(p,d,d^D).$$

Secondly, the aforementioned authors, after realizing that for p even the digraphs $\vec{C}_p \otimes GB(d,n)$ and $\vec{C}_p \otimes GK(d,n)$ are isomorphic [13], defined the so-called *Kautz generalized p-cycles* $KGC(p,d,n)$ as follows (see Remark 1.5.1):

$$KGC(p,d,n) = \begin{cases} \vec{C}_p \otimes GK(d,n) & if \;\; p \;\; is \;\; odd \\ \vec{C}_p \otimes_\phi GB(d,n) & if \;\; p \;\; is \;\; even, \end{cases}$$

where the $\otimes_\phi$ symbol denotes a new binary operation on digraphs also defined in the same article by generalizing in a certain way the standard conjunction $\otimes$ (see [81] for details). Observe finally that, if p is odd and $n = d^D + d^{D-1}$, $\vec{C}_p \otimes GK(d,n)$ is isomorphic to the $(D-1)$-iterated line digraph of $\vec{C}_p \otimes K_{d+1}$.

After proving by algebraical methods the non-existence of Moore generalized p-cycles of diameter greater than $3p-2$, the authors of [81], taking into account properties 4 and 7 of the foregoing list, realized that to conclude the proof of the property 9 it suffices to design, for every $\Delta \geq 2$, two Δ-regular generalized p-cycles of orders $p\Delta$ and $p(\Delta^p + 1)$ respectively. As expected, the case $n = p\Delta$ was easily solved by considering the complete generalized p-cycle $\vec{C}_p \otimes K_\Delta^*$. The problem of finding a Moore generalized p-cycle of diameter $D = 2p-1$ and order $p(\Delta^p + 1)$ was solved for p odd in a similar way as in the previous case, by taking the generalized Kautz p-cycle $\vec{C}_p \otimes GK(\Delta, \Delta^p + 1)$. Unfortunately, for p even, the previous digraph was proven to be of diameter $D = 2p$, and therefore the problem could not be solved in this way. Nevertheless, after introducing the family of generalized Kautz p-cycles $KGC(p,\Delta,n)$, they proved that for every $\Delta \geq 2$, $KGC(p,\Delta,\Delta^p+1)$ is a generalized p-cycle of order $p(\Delta^p + 1)$ and diameter $D = 2p-1$.

Remark 1.5.1 In fact, the Kautz generalized p-cycles $KGC(p,d,n)$ are a generalization [14] of the family of *dense bipartite digraphs* $BD(d,n)$ introduced by Fiol and Yebra in [67] as follows. For any positive integers d, n, with $d \leq n$, the bipartite digraph $BD(d,n)$ has set of vertices $V = Z_2 \times Z_n = \{(\alpha, i);\ \alpha \in Z_2, i \in Z_n\}$, and for every vertex (α, i) its out-neighbourhood is:

$$\Gamma^+(\alpha, i) = \{(1-\alpha, (-1)^\alpha d(i+\alpha) + t);\ t = 0, 1, \ldots, d-1\}.$$

The digraph $BD(d,n)$ is d-regular, has order $2n$ and, when $n = d^{D-1} + d^{D-3}$, it has diameter D. Moreover, the digraphs $BD(d, d^{D-1} + d^{D-3})$ are, for $D = 3$ and $D = 4$, Moore bipartite digraphs, and when $D \geq 5$, the order of these digraphs is larger than $(d^4 - 1)/d^4$ times the Moore bound.

[13] Where $GK(d,n)$ is the generalized Kautz digraph of degree d and order n.

[14] In other words, $BD(d,n) = KGC(2,d,n)$.

1.6 The parameter ℓ

Fàbrega and Fiol introduced in [54] the so-called *parameter* ℓ of a simple connected (di)graph, which, as is pointed out in the mentioned paper, 'can be thought of as a generalization of the girth of a graph'. This parameter, whose value depends basically on the number of short paths, has proved to be an excellent tool for studying some *fault tolerance* topics in graphs and digraphs, especially their *connectedness* (see Section 1.7.1).

Definition 1.6.1 *Let $G = (V, A)$ be a simple connected (di)graph with diameter D. The parameter $\ell = \ell(G)$ of G is defined as the greatest integer belonging to $\{1, \ldots, D\}$ such that for any $x, y \in V$:*

$$0 \le d(x,y) = d \le \ell \Rightarrow \begin{cases} \text{1. There is a unique } x \to y \text{ path of length } d. \\ \text{2. If } d < \ell, \text{ there are no } x \to y \text{ paths of length } d+1. \end{cases}$$

In the following proposition, we show some properties which reveal, on the one hand, the close relation between the parameter ℓ of a graph and its girth, and on the other, the behaviour of this parameter with respect to the operations presented in Sections 1.3 and 1.4. All these results can be easily proved from the definitions involved.

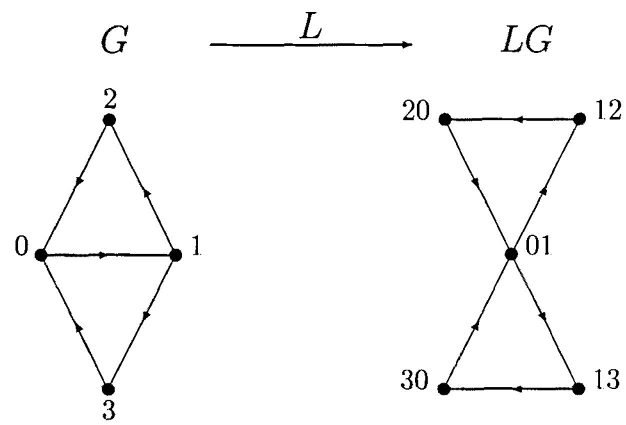

Figure 1.3: G is a digraph with $\delta = 1$, $D = 3$ and $\ell = 1$, whereas the values of these parameters for LG are: $\delta = 1$, $D = 4$ and $\ell = 4$.

Proposition 1.6.1 *Let G and H be two simple connected (di)graphs with parameters ℓ and ℓ' respectively.*

1. *If G is a generalized p-cycle with $p \ge 2$, then the second condition of the previous definition is satisfied for any d, and can therefore be omitted.*

2. *If G is a graph with girth g, then $\ell = \lfloor \frac{g-1}{2} \rfloor$.*

3. *([54])*[15] *If $\delta(G) \geq 2$, then $\ell(LG) = \ell(G) + 1$.*

4. *The parameter ℓ of the cartesian product $G \times H$ is $min\{\ell, \ell'\}$.*

5. *The parameter ℓ of the lexicographic product $G[H]$ is 1.*

6. *The parameter ℓ of the conjunction $G \otimes H$ is at least $min\{\ell, \ell'\}$.*

Examples.

1. $\ell(K_n)$=1.

2. $\ell(\vec{C}_{D+1})$=D.

3. $\ell(C_n)= \begin{cases} D \text{ if } n = 2D+1 \text{ is odd;} \\ D-1 \text{ if } 4 \leq n = 2D \text{ is even.} \end{cases}$

4. $\ell(K_{h,k})$=1.

5. $\ell(K(d, D))$=D, since $K(d, D) = L^{D-1}K_{d+1}$.

6. In any Moore graph $\ell = D$, since in all of them the girth is $g = 2D + 1$.

7. In any Moore bipartite graph $\ell = D - 1$, since in all of them the girth is $g = 2D$.

8. From the definition of a Moore bipartite digraph, and bearing in mind that in any bipartite (di)graph the second condition of the foregoing definition is always satisfied, it is easily derived that, for $\Delta \geq 2$, its parameter ℓ is $D - 1$.

9. In [81], it was implicitely proved that the diameter D of a Moore generalized p-cycle can be defined as the minimum integer such that between every pair of vertices, at most one path of length $D - (p - 1)$ exists. As a consequence [16] we conclude that the parameter ℓ in such digraphs is $D - (p - 1)$.

1.7 Fault tolerance

As it was said in the Introduction of this work, one of the most important considerations in the design and analysis of graphs and digraphs, is the so-called *fault-tolerance* or *reliability*. Two important topics in the study of the fault tolerance of a (di)graph are *connectedness* and *diameter vulnerability* (see [16], [20], [108]).

[15] See Figure 1.3.

[16] Notice that, for $p \in \{1, 2\}$, this statement is also true.

1.7.1 Connectedness

Disconnecting sets

A subset of vertices T of a (di)graph G is said to be a *vertex-cutset*, or simply a *cutset*, if $G-T$ is a disconnected (di)graph. Likewise, a subset T of arcs is an *arc-cutset* [17] of G if $G-T$ is a disconnected (di)graph. A cutset [arc-cutset] T is said to be *minimal* [18] if there are no cutsets [arc-cutsets] contained in T. A cutset [arc-cutset] T with k vertices [arcs] is said to be *minimum* if there are no cutsets [arc-cutsets] in G with cardinality lower than k.

A set of vertices T is said to be *trivial* if it contains either the out-neighbourhood or the in-neighbourhood of some vertex not belonging to T. The *trivial* sets of arcs are defined in a similar way. Certainly, every subset of a nontrivial cutset [arc-cutset] is also nontrivial. A nontrivial cutset [arc-cutset] T is said to be *minimal* if every cutset [arc-cutset] of G contained in T, if any, is trivial. Observe that the previous definition is equivalent to saying that T is a nontrivial minimal cutset. A nontrivial cutset [arc-cutset] T is said to be *minimum* if every cutset [arc-cutset] of G with cardinality lower than $|T|$, if any, is trivial. Notice that, unlike the 'minimal' case, a minimum nontrivial cutset [arc-cutset] needs not be a nontrivial minimum cutset (this issue will be dealt with in detail at the end of this section).

A *positive fragment* [*α-fragment*] of G is a subset of vertices F whose positive boundary $\partial^+(F)$ [arc-boundary $\omega^+(F)$] is a minimum cutset [arc-cutset] (the *negative fragments* [*α-fragments*] are similarly defined). Likewise, a *positive 1-fragment* [*α_1-fragment*] of G is a subset of vertices F whose positive boundary $\partial^+(F)$ [arc-boundary $\omega^+(F)$] is a minimum nontrivial cutset [arc-cutset] (in the same way, the *negative 1-fragments* [*α_1-fragments*] are defined). Observe that a vertex set F is a positive fragment [1-fragment] if and only if $\overline{F}=V\setminus(F\cup\partial^+F)$ is a negative fragment [1-fragment] and $\partial^+F=\partial^-\overline{F}$. Similarly, it is also clear that F is a positive α-fragment [α_1-fragment] if and only if $V\setminus F$ is a negative α-fragment [α_1-fragment] and $\omega^+F=\omega^-(V\setminus F)$.

Starting from the so-called *Hamidoune terminology*[19] for cutsets and arc-cutsets ('boundaries and fragments') just put forward, some more definitions were introduced in [10]. The *deepness* of a positive fragment F is $\mu(F)=\max_{x\in F} d(x,\partial^+F)$. Similarly, the deepness of a negative fragment F is $\mu(F)=\max_{x\in F} d(\partial^-F,x)$. With respect to α-fragments, the *deepness* of a positive α-fragment F is $\nu(F)=\max_{x\in F} d(x,\omega^+F)$. The *deepness* of a negative α-fragment F is defined analogously, $\nu(F)=\max_{x\in F} d(\omega^-F,x)$. If F is a positive fragment of deepness μ, then the set $\{x\in F : d(x,\partial^+F)=\mu)\}$ is called the *valley* of F. The valley of a negative fragment, a positive α-fragment and a negative α-fragment are similarly defined.

A vertex subset T is called a *positive [negative] vertex-cut* if there exists a proper

[17] The vertex-cutsets [arc-cutsets] are also called *vertex-disconnecting* [*arc-disconnecting*] sets.
[18] The minimal arc-cutsets are often called cocycles (see [40, 90]).
[19] See [86, 87].

subset of vertices $F \subset V \setminus T$ such that $\partial^+(F) = T$ $[\partial^-(F) = T]$. In a similar way, an arc subset T is called a *positive [negative] arc-cut*, or simply a *cut* [20], if there exists a proper subset of vertices F such that $w^+(F) = T$ $[w^-(F) = T]$. Certainly, every vertex-cut [arc-cut] is a vertex-cutset [arc-cutset], but the converse is, in general, not true [21] (see [40]). However, when the minimallity condition is required, both definitions coincide. This is the main result shown in the following two lemmas (the second one treats the nontrivial case).

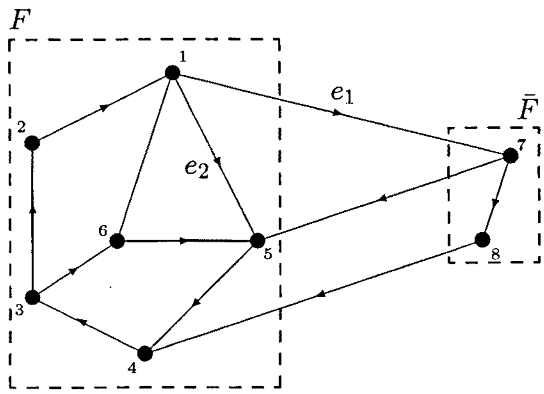

Figure 1.4: In this graph, $\{e_1, e_2\}$ is an arc-cutset, but it is not a cut.

Lemma 1.7.1 *Let T be a subset of vertices [arcs] of a (di)graph $G = (V, A)$. Then,*

1. *if T is a vertex-cutset [arc-cutset], then there exists a proper vertex subset $F \subset V \setminus T$ $[F \subset V]$ such that $\partial^+ F \subseteq T$ $[w^+ F \subseteq T]$.*
2. *T is a minimal arc-cutset if and only if it is a minimal arc-cut. Moreover,*
   - *if G is a graph, there exist exactly two components in $G - T$.*
   - *if G is a digraph, there exists a unique sink component S in $G - T$, which can be obtained as follows:*

   $$S = \{v \in V : \textit{There exists a } t \to v \textit{ path in } G - T \textit{ for some } t \in X_T\},$$

   where $X_T = \{x \in V : w^+(x) \cap T \neq \emptyset\}$.
3. *T is a minimal vertex-cutset if and only if it is a minimal vertex-cut.*
4. *T is a minimum cutset [arc-cutset] if and only if there exists a fragment [α-fragment] F such that $T = \partial^+ F$ $[T = w^+ F]$.*

[20] Also known as *coboundary*.

[21] In fact, it is well known that every cut is a disjoint union of minimal arc-cutsets.

Proof.

1. Let T and $(G-T)^*$ be a vertex-cutset and the condensation of $G-T$ respectively. Since T is a cutset, then $(G-T)^*$ has at least one source and one sink and each of which must be different. Finally, if F denotes any sink of $G-T$, then, as a consequence of the acyclicity of $(G-T)^*$, it is deduced that all the paths from F to $\overline{F}$ go through T, and hence, $\partial^+ F \subseteq T$. The arc case is proved similarly.

2. Assume that $T \subset A$ is a minimal arc-cutset of a digraph G. From the minimality of T is it deduced that the digraph $G-T$ has a unique sink component F. As a consequence, $w^+F = T$. Finally, since every arc-cut is an arc-cutset, we conclude that T is a minimal arc-cut.

   Suppose now that T is a minimal arc-cut of a digraph G. Clearly, T is an arc-cutset. On the other hand, let T' be the minimal arc-cutset contained in T. As it has just been proved, T' is a minimal arc-cut. Therefore, $T' = T$.

   To conclude, taking into account that S is connected and that all the paths from S to $V \setminus S$ go through T, the indicated characterization of S is immediately deduced.

   The undirected case is proved similarly.

3. If $T \subset V$ is a minimal vertex-cutset, then it is the positive boundary in G of every sink component S of $G-T$. The minimality of T and the converse are immediately deduced as in the arc case.

4. This point is an immediate consequence of the previous ones. ■

Lemma 1.7.2 *Let T be a nontrivial vertex [arc] subset of a digraph $G = (V, A)$. Then,*

1. *T is a minimal nontrivial vertex-cutset [arc-cutset] {vertex-cut} (arc-cut) if and only if it is a nontrivial minimal vertex-cutset [arc-cutset] {vertex-cut} (arc-cut).*

2. *T is a minimal nontrivial vertex-cutset [arc-cutset] if and only if it is a minimal nontrivial vertex-cut [arc-cut].*

3. *T is a minimum nontrivial vertex-cutset [arc-cutset] if and only if there exists a 1-fragment [α_1-fragment] F such that $T = \partial^+ F$ [$T = w^+F$].*

Proof.

1. Suppose that T is a minimal nontrivial vertex-cutset. As every subset of a nontrivial set is also nontrivial, then T does not contain any vertex-cutset, and hence it is a minimal vertex-cutset. Conversely, if T is a nontrivial minimal vertex-cutset, then it does not contain any vertex-cutset, and in particular, any nontrivial vertex-cutset. The rest of the cases are proved in the same way.

2. Assume again that T is a minimal nontrivial cutset. Then, it is a nontrivial minimal cutset, and hence, according to Lemma 1.7.1, it is a nontrivial minimal vertex-cut. Therefore, it is a minimal nontrivial vertex-cut. The converse and the arc case are proved similarly.

3. If T is a minimum nontrivial cutset, then it is a minimal nontrivial cutset; that is, it is a nontrivial minimal vertex-cut. The converse is obvious, and the arc case is proved similarly. ■

Finally, an interesting proposition with some results involving the line digraph technique is now put forward.

Proposition 1.7.1 *Let LG be the line digraph of a connected digraph $G = (V, A)$. Let $T \subset A$ be a subset of arcs of the digraph G. Then,*

1. *If $|A \setminus T| \geq 2$, then $G - T$ is a disconnected digraph if and only if $LG - T$ is a disconnected digraph.*

2. *T is a trivial subset of arcs of G if and only if T is a trivial subset of vertices of LG.*

Proof.

1. Suppose that $G - T$ is disconnected. Since $LG - T = L(G - T)$ and $|A \setminus T| \geq 2$, from Proposition 1.3.1 we conclude that $LG - T$ is also a disconnected digraph. Assume now that $G - T$ is a connected digraph. From Proposition 1.3.2 and again noticing that $LG - T = L(G - T)$, we obtain that $LG - T$ is a connected digraph too.

2. If T is a trivial subset of arcs of G, then there exists a vertex $x \in V$ such that either $w^-(x)$ or $w^+(x)$ is contained in T. Suppose, for instance, that $w^+(x) \subset T$, and consider a vertex a belonging to the in-neighbourhood of x: $a \in \Gamma^-(x)$. It is clear that ax is a vertex of LG whose out-neighbourhood is contained in T. Therefore, T is a trivial subset of vertices in LG. The converse is proved similarly. ■

Connectivity

Let G be a (strongly) connected (di)graph of minimum degree δ and diameter D. The *vertex-connectivity* index or simply the *connectivity* of G, denoted by $\kappa = \kappa(G)$, is defined as the minimum number of vertices whose removal results in a (di)graph that is either disconnected (that is, not (strongly) connected) or *trivial* (that is, an isolated vertex). To be more precise [22]:

$$\kappa(G) = \min_{F\subset V(G)} \{|F| : G - F \text{ is disconnected or trivial}\}$$

Observe that the minimum cutsets defined in Section 1.1 are precisely those having cardinality $\kappa(G)$. It is also clear that a nontrivial (di)graph has connectivity 0 if and only if it is disconnected. Furthemore, the complete graph K_n is the only connected (di)graph with no cutsets, and thus its connectivity is $\kappa(G) = n - 1$. A (di)graph G is said to be *k-connected* if $\kappa(G) \geq k$.

Similarly, the *arc-connectivity* $\lambda = \lambda(G)$ is defined as the smallest number of arcs whose removal results in a disconnected or trivial (di)graph:

$$\lambda(G) = \min_{F\subset A(G)} \{|F| : G - F \text{ is disconnected or trivial}\}$$

As in the previous case, the minimum arc-cutsets are those having cardinality $\lambda(G)$, and $\lambda(G) = 0$ if and only if G is disconnected or trivial. Likewise, a (di)graph G is said to be *k-arc-connected* if $\lambda(G) \geq k$.

Certainly, the value of both $\kappa(G)$ and $\lambda(G)$ is at most the minimum degree of G. In fact, it is not difficult to prove (see [70], [138]) that, for every (di)graph G the so-called *Whitney inequality sequence*, $\kappa(G) \leq \lambda(G) \leq \delta(G)$, is always verified. It is also well known that if LG is the line digraph of a digraph G, then the following equality holds: $\kappa(LG) = \lambda(G)$. Actually, this result is only a corollary of the first statement of Proposition 1.7.1.

A (di)graph G is said to be *maximally* connected [arc-connected] if it is δ-connected; that is, if $\kappa(G) = \delta$ [$\lambda(G) = \delta$]. One of the most important problems on connectedness approached in recent years has been to find sufficient conditions for a (di)graph to be maximally connected and maximally arc-connected. Some of the main results, for graphs and digraphs, are presented next, in chronological order.

Proposition 1.7.2 *Let G be a (δ, Δ, D, n)-graph such that $\delta \geq 3$.*

1. *([38]) If $\delta \geq \lfloor \frac{n}{2} \rfloor$, then G is maximally edge-connected.*
2. *([105]) If $\delta(u) + \delta(v) \geq n - 1$ for all pairs u, v of nonadjacent vertices of G, then G is maximally edge-connected.*
3. *([122]) If $D \leq 2$, then G is maximally edge-connected.*
4. *([135]) If $D \leq 2\ell - 1$, then G is maximally connected.*
5. *([136]) If $n > (\delta - 1)(\Delta - 1)^{D-1} + 2$, then G is maximally connected.*

[22] This formulation of $\kappa(G)$ was introduced by Harary in [89], although there are of course earlier definitions (see for instance [137, 138]).

6. ([136]) *If* $n > (\delta-1)(\frac{(\Delta-1)^{D-1}+\Delta-3}{\Delta-2})+\Delta-1$, *then* G *is maximally edge-connected.*

7. ([136]) *If* $D \leq 2\ell$, *then* G *is maximally edge-connected.*

Proposition 1.7.3 *Let* G *be a* (δ,Δ,D,n)*-digraph such that* $\delta \geq 2$.

1. ([4]) *If* $g \geq 3$ *and* $\delta \geq \lfloor \frac{n+2}{4} \rfloor$, *then* G *is maximally arc-connected.*

2. ([103]) *If* $D \leq 2$, *then* G *is maximally arc-connected.*

3. ([101]) *If* $n > \frac{(\delta-1)(\Delta^D+\Delta^2-\Delta-1)}{\Delta-1}$, *then* G *is maximally connected.*

4. ([101]) *If* $n > \frac{(\delta-1)(\Delta^{D-1}+\Delta^2-2)}{\Delta-1}$, *then* G *is maximally arc-connected.*

5. ([54]) *If* $D \leq 2\ell - 1$, *then* G *is maximally connected.*

6. ([54]) *If* $D \leq 2\ell$, *then* G *is maximally arc-connected.*

7. ([62]) *If* $n > \frac{(\delta-1)(\Delta^{D-\ell+1}+\Delta^\ell-2\Delta)}{\Delta-1} + \Delta^\ell + 1$, *then* G *is maximally connected.*

8. ([62]) *If* $n > \frac{(\delta-1)(\Delta^{D-\ell}+\Delta^\ell-2)}{\Delta-1} + \Delta^\ell$, *then* G *is maximally arc-connected.*

Observe that the different sufficient conditions that we have presented can be classified in three types: *degree*[23], *order* and *diameter* conditions. In the first case, the different results obtained so far consist always in finding a lower bound for a certain additive function which depends only on the vertex degrees of the (di)graph (see [26, 44, 45, 46, 72, 73, 141]). In the second case, the problem of finding a lower bound on the order n which guarantees maximal connectivity is equivalent to obtaining a Moore upper bound for a certain family of (di)graphs. For instance, the authors of [101] proved that the order of any digraph of maximum degree Δ, diameter D and connectivity κ is at most $\kappa(\frac{\Delta^D-1}{\Delta-1} + \Delta)$ (see also [9, 53, 60, 135]). Finally, the third type of sufficient conditions that has been studied consists in finding an upper bound on the diameter for a certain class of (di)graphs in order to assure maximal connectivity (see also [55, 125]).

Superconnectivity

In addition of the connectivity and arc-connectivity, other parameters have been defined in order to study concrete aspects of connectedness on graphs and digraphs. For instance, F. Boesch and R. Tindell introduced[24] in [24] (see also [13]) the so-called *superconnectivity* and *arc-superconnectivity* as follows. A (di)graph is said to be *superconnected* [*arc-superconnected*] [25] if and only if every minimum cutset [arc-cutset] F is trivial; that is to say, $G - F$ has at least one component with a single vertex of

[23] Also known as *Chartrand* conditions.

[24] In fact, both definitions were given for graphs, being generalized for (di)graphs by J. Fàbrega and M.A. Fiol in [54].

[25] Also known as *super-*κ [*super-*λ] (di)graph.

minimum degree 0 in $G - F$. Notice that every superconnected [arc-superconnected] (di)graph is maximally connected [arc-connected] but the converse is not true [26]. Some general results on arc-superconnectivity are shown in the following proposition (see also [11, 55, 60, 66]).

Proposition 1.7.4 *Let G be a (δ, Δ, D, n)-(di)graph such that $\delta \geq 3$. Then G is arc-superconnected if any of the following conditions hold:*

1. *([126]) $G \in \{K_n : n \geq 4\} \cup \{K_{n,m} : max\{n, m\} \geq 3\}$.*
2. *([54]) $D \leq 2\ell - 1$.*
3. *([134]) $n > \delta(\frac{\Delta^{D-1}+\Delta-2}{\Delta-1}) + \Delta^{D-1}$.*
4. *([61]) $D = 2$ and it contains no K_δ with all its vertices of out-degree δ or all its vertices of in-degree δ.*
5. *([61]) $\delta^+(u) + \delta^-(v) \geq n + 1$ for all pairs u, v of nonadjacent vertices.*
6. *([61]) $g \geq 3$ and $\delta^+ + \delta^- \geq \lfloor \frac{n}{2} \rfloor + 1$.*
7. *([63]) $n > \delta(\frac{\Delta^{D-\ell}+\Delta^{\ell}-2}{\Delta-1}) + \Delta^{D-\ell}$.*

In a similar way as in the case of the connectivity, in order to study the superconnectivity of graphs and digraphs, J. Fàbrega and M. A. Fiol introduced in [55] (see also [5]) the so-called *vertex-superconnectivity* and *arc-superconnectivity* parameters, denoted by κ_1 and λ_1 respectively, as follows:

$$\kappa_1(G) = \min_{F \subset V(G)} \{|F| : F \text{ nontrivial }, \ G - F \text{ is disconnected } \}$$

$$\lambda_1(G) = \min_{F \subset A(G)} \{|F| : F \text{ nontrivial }, \ G - F \text{ is disconnected } \}$$

Certainly, if a digraph G is not maximally connected [arc-connected], then $\kappa_1 = \kappa$ [$\lambda_1 = \lambda$]. It is also clear that G is superconnected [arc-superconnected] if and only if either every cutset [arc-cutset] in G is trivial [27] or $\kappa_1(G) > \delta(G)$ [$\lambda_1(G) > \delta(G)$]. Another interesting property, obtained as an immediate consequence of Proposition 1.7.1, is the following.

Corollary 1.7.1 *If LG is the line digraph of a connected digraph G, then $\kappa_1(LG) = \lambda_1(G)$.*

Finally, both connectivity and superconnectivity can be considered as particular cases of the so-called *conditional connectivity* introduced by Harary in [91]. This question is discussed in detail in Chapter 5.

[26] $\vec{C}_3 \times K_3$ is both maximally connected and arc-connected, but it is neither superconnected nor arc-superconnected.

[27] see Section 5.4.

1.7.2 Diameter vulnerability

Let G be a k-connected [k-arc-connected] (Δ, D)-(di)graph of order n. For any integer $s \in \{1, \ldots, k-1\}$ the *s-vertex vulnerability* [*s-edge vulnerability*] of G is defined as the maximum diameter of all the sub(di)graphs obtained from G by deleting at most s vertices [arcs]:

$$K(s; G) = max\{D(G - F) : F \subset V(G),\ |F| \leq s\}$$

$$\Lambda(s; G) = max\{D(G - F) : F \subset A(G),\ |F| \leq s\}$$

A connected (di)graph G is said to be a (Δ, D', s)-*(di)graph* if it has maximum degree Δ and s-vertex vulnerability $K(s; G) \leq D'$. Similarly, a connected (di)graph G is a (Δ, D, D', s)-*(di)graph* [28] if it is a (Δ,D)-(di)graph with s-vertex vulnerability $K(s; G) \leq D'$.

The diameter vulnerability of graphs and digraphs have been studied from different points of view: Moore bounds ([30, 80, 101, 111, 142]), lower and upper bounds of $K(s; G)$ and $\Lambda(s; G)$ ([30, 43, 59, 113, 114, 120, 132]), diameter vulnerability of known families of (di)graphs ([31, 59, 76, 80, 101, 113, 133, 142]), design of (di)graphs with vertex or arc vulnerabilities close to its diameter ([30, 74, 76, 112]), etc. One of the oldest results in this area was obtained by Plesnik in [123]. In this paper it was proved that the 1-edge vulnerability of every 2-edge-connected graph of diameter D is at most $2D$. This result was generalized by Chung and Garey in [43], where it was proved that the s-edge vulnerability of any (s+1)-edge-connected graph G of diameter D is both lower and upper bounded in terms of s and D:

$$(s + 1)(D - 2) \leq K(s; G) \leq (s + 1)D + s.$$

In the same paper [43] the case of deletion of vertices was also studied, showing that the s-vertex vulnerability $K(s; G)$ is unbounded in terms of s and D. More recently, Bond and Peirat showed in [30] that $K(s; G)$ remains small if the degree is bounded or the number of vertices is large enough. For instance, it was proved that the 1-vertex vulnerability of any 2-connected (Δ, D)-graph is at most $\Delta(2\lfloor \frac{D}{2} \rfloor - 1) + D$.

Both connectivity and diameter vulnerability can be characterized by means of *Menger-type conditions.* In 1927 ([110]), Karl Menger proved that for any pair u, v of nonadjacent vertices of a graph G, the minimum number of vertices whose deletion disconnects u and v is equal to the maximum number of internally disjoint $u - v$ paths. As a direct consequence of this theorem H. Whitney showed in [138] that a nontrivial graph G is k-connected if and only if for each pair u, v of distinct vertices there are at least k internally disjoint $u - v$ paths in G. Both Menger's Theorem and Whitney's characterization have been proved to be true for edge deletion and for digraphs (see [39], [109]). The analysis of the diameter vulnerability in graphs and digraphs is in

[28] The case of edge deletion is similarly defined.

most cases carried out in terms of the following Menger-type condition: for any pair u, v of distinct vertices such that v is not adjacent from u there are $s+1$ disjoint $u \rightarrow v$ paths of length at most D'. Certainly, if a (Δ, D)-(di)graph G satisfies this condition, then it is a (Δ, D, D', s)-(di)graph with connectivity at least $s+1$. Nevertheless, it is important to realize that the converse of this statement is not true if $D' \geq 5$ (see [16, 32, 106]).

Chapter 2

New large graphs

2.1 Introduction

A question of special interest in Graph theory is the construction of graphs with an order as large as possible for a given degree and diameter or (Δ, D)-problem. This matter deserves a lot of attention due to its implications in the design of topologies for interconnection networks and other questions such as the data alignment problem and the description of some cryptographic protocols.

The (Δ, D)-problem for undirected graphs has been approached in different ways. First of all, it has been proved that the order of any (undirected) graph G of maximum degree Δ and diameter D is at most (see Section 1.2):

$$\mathcal{M}(\Delta, D) = \begin{cases} 2D+1 & if \;\; \Delta = 2 \\ \frac{\Delta(\Delta-1)^D-2}{\Delta-2} & if \;\; \Delta > 2 \end{cases}$$

As this theoretical bound, known as Moore bound for graphs, is unattainable if G has diameter and/or maximum degree greater than 2 (see Section 1.2), most of the work deals with the construction of large graphs which, for this given diameter and degree, have a number of vertices as close as possible to the theoretical bounds (see [19] for a review).

Various techniques which depend on the way graphs are generated and how their parameters are calculated have been developed. Many large graphs correspond to Cayley graphs (see [35], [36], [51], [130]) and have been found by computer research. However, the use of computers is only efficient when both the degree and diameter are not too large. Compounding is another technique that has been proved to be useful and consists in replacing one or more vertices of a given graph by another graph or copies of a graph and rearranging the edges suitably (see, for instance, [77], [78]). Compound graphs and Cayley graphs make up most of the large (Δ, D)-graphs described in the literature for small diameter.

In this work compounding is used to construct new large (Δ, D)-graphs that improve some known results for diameter 6. This technique is a generalization of a method introduced by Quisquater in [127], based on the replacement by a complete graph of a single vertex from a Moore bipartite graph. In [42], [78] the authors used this method, although modified in order to replace several vertices. This work presents a new technique for working out a general rule for the replacement of a large number of vertices by means of a special diameter 2 graph.

In Section 1.2, it was proved that the maximum order of any (Δ, D)-bipartite graph with $\Delta > 2$ is: $\mathcal{M}(\Delta, D) = \frac{2(\Delta-1)^D-2}{\Delta-2}$. In Section 1.5.1 the different families of Moore bipartite graphs were presented; that is, the bipartite graphs whose order reaches the previous bound: the complete bipartite graphs $K_{\Delta,\Delta}$, the generalized triangles P_q, the generalized quadrangles Q_q, and finally, the generalized hexagons H_q. The construction proposed in [42] consists in expanding the generalized hexagon H_q by replacing several vertices by complete graphs and creating some new adjacencies. In it, some vertices x_{ijk} (see Figure 2.1) are replaced by copies of the complete graph K_h ($h \leq \Delta$) whose vertex set is denoted by K_{ijk}. These replacements must verify some conditions. In this work, we will use the same notation for the replaced vertices but our conditions are changed. To be more precise, these conditions have become only one, which is simpler and easier to verify.

Here we have some known properties that will be used in the rest of sections:

- If $G = (V_1 \cup V_2, E)$ is any bipartite graph of even [odd] diameter D, the distance between $x \in V_1$ and any $y \in V_2$ [$y \in V_1$] is at most $D - 1$.

- If G is a (Δ, D)-Moore bipartite graph, for any $x, y \in V(G)$ with $d(x, y) = D$, there exist Δ disjoint paths between x and y of length D.

- The girth of any Moore bipartite graph with diameter D is $g = 2D$.

The rest of this chapter consists of three sections. In the next one we introduce the so-called *[l,λ]-cliques*, and we present some examples that will be used later. In Section 2.3 we describe a general technique for the construction of large diameter 6 graphs. Finally, in Section 2.4 some new large graphs are proposed.

2.2 $[l,\lambda]$-cliques

Definition 2.2.1 *A graph G=(V,E) is an [l,λ]-clique if it is possible to partition V into λ partite sets $V_1, \cdots, V_\lambda$, whose induced subgraphs are cliques and such that $|V_1| \leq \cdots \leq |V_\lambda| = l$.*

Observe that if $\tilde{G} = (\tilde{W}, \tilde{E})$ is an [l,λ]-clique, then $|\tilde{W}| \leq l.\lambda$, and it is also a $[k, \mu]$-clique, for each $k \leq l$ and for some $\mu \geq \lambda$. Next, some examples of [l,λ]-cliques are given. Most of them will be used to construct new large graphs in Section 2.4.

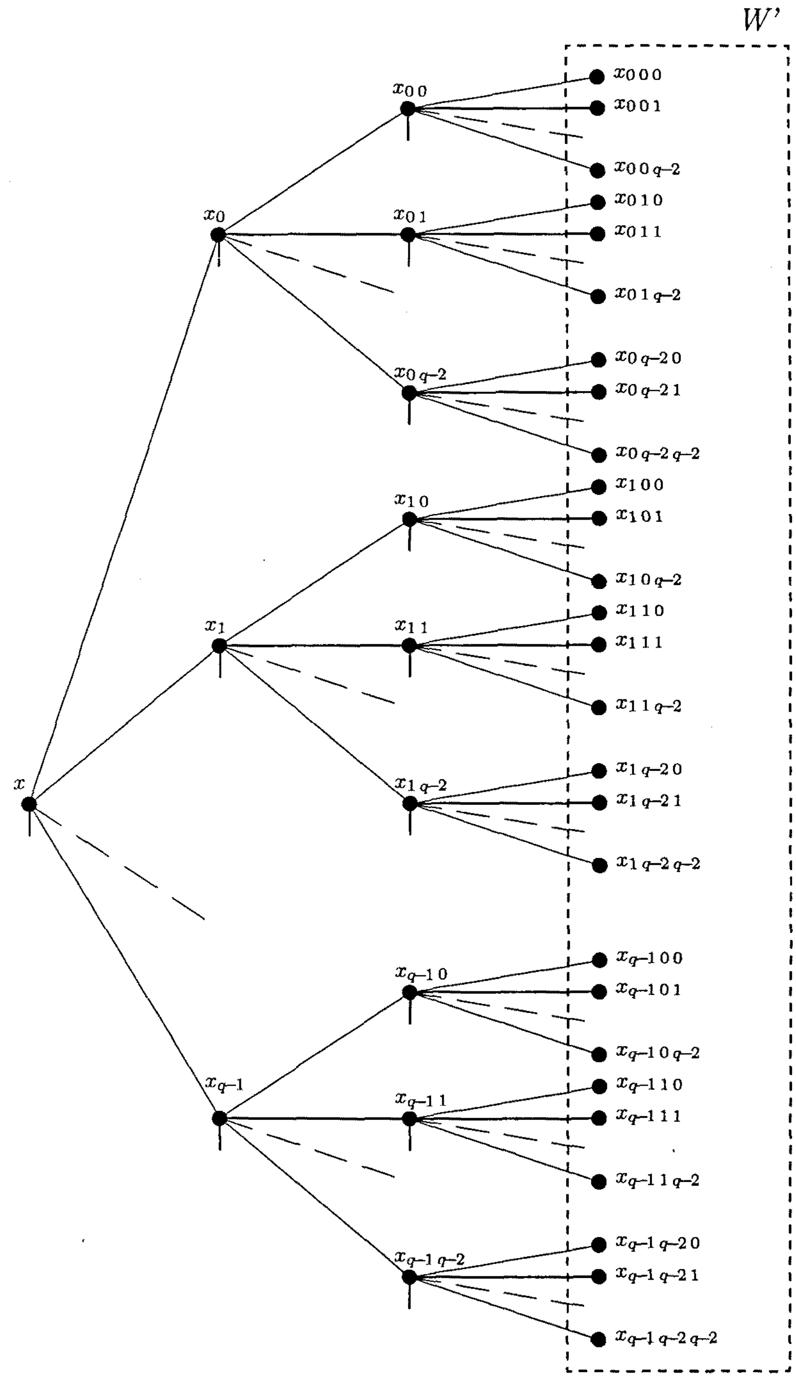

Figure 2.1: The subgraph of H_q to be modified.

Examples 2.2.1

1. Every graph is a [2,λ]-clique, but for the graphs without edges.

2. The complete graph K_h is a [h,1]-clique and it is also a [k,2]-clique for every $k \in \{\lceil \frac{h}{2} \rceil, \ldots, h-1\}$.

3. Every graph G that contains a clique K_l is an [l,λ]-clique with λ at most $n(G) - l + 1$.

4. The Petersen graph P is a [2,5]-clique. A partition is $\{\{(0,i),(1,i)\}, i = 0,1,2,3,4\}$ (see Figure 2.2).

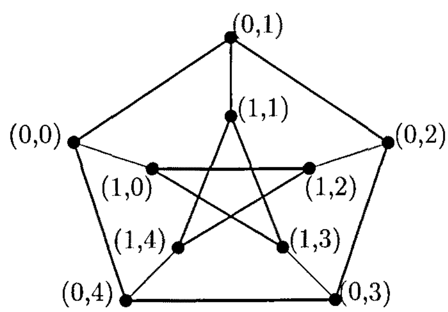

Figure 2.2: Petersen graph.

5. The largest known graph with degree 6 and diameter 2 is $K_4 * X_8$. It is constructed by means of joining 4 copies of X_8 (see[17]). Figure 2.3 shows that X_8 is a [2,4]-clique. Thus, $K_4 * X_8$ is a [2,16]-clique.

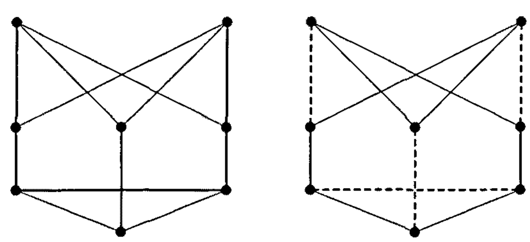

Figure 2.3: X_8 and its partition.

6. The Moore bipartite graph P_q is, for $D = 3$ and $\Delta = q + 1$, the incident graph of the projective plane $PG(2,q)$ (see Section 1.5.1). The points of $PG(2,q)$ are the 1-dimensional vector subspaces of the 3-dimensional vector space K^3 over a finite field K with q elements. This fact enables us to define the so called graph P'_q as follows: the vertex set is $PG(2,q)$ and the adjacency rule is: for each $a, b \in P(2,q)$, a and b are adjacent if and only if they are orthogonal. It is easy to see that P'_q is a regular graph with order $n = q^2 + q + 1$, degree $\Delta = q + 1$ and diameter $D = 2$. For $q = 9, 11$ and 13, the graph P'_q is the largest known graph

and, moreover, it is a [3,λ]-clique. In fact, according to Corollary 3 of Theorem 4.3.6 of [95], P'_9 can be partitioned into 7 P'_3. Besides, P'_3 is a [3,6]-clique for a partition with 1 K_3 and 5 K_2 (see Section 2.5). As a consequence, P'_9 is a [3,42]-clique with 7 K_3 and 35 K_2. The same theorem can not be applied to the graphs P'_{11} and P'_{13}. However, a study by computer shows that P'_{11} is a [3,55]-clique (see Section 2.5). The partition obtained has 32 K_3, 14 K_2 and 9 K_1. Another analogous study shows that P'_{13} is a [3,72]-clique with 44 K_3, 23 K_2 and 5 K_1 (see Section 2.5).

7. Let l be a positive integer less than or equal to q^2+q+1. Let us choose a partition into λ parts of the vertex set of P'_q so that

   - Each part has, at most, l vertices.
   - There exists a part with l vertices.

   Starting from a copy of P'_q, the so called P'_ql graph is designed by adding new edges between nonadjacent vertices of the same part so that in each part all the vertices are then adjacent among themselves. So, any graph P'_ql is composed by the union of λ complete graphs. As a consequence, the degree of P'_ql is at most $q+l$. Furthermore, it is an [1,λ]-clique for the previous partition.

2.3 $H_q(K_h)$ graphs

Let us consider the subgraph of the Moore bipartite graph $H_q = (V \cup W, E)$ showed in Figure 2.1, and consider a vertex $x \in V$. According to this figure:

$$\begin{aligned}
&\Gamma(x) = \{x_0, x_1, ...x_q\} \subset W, \\
&\Gamma(x_i) = \{x_{i0}, x_{i1}, ...x_{iq-1}\} \cup \{x\} \subset V, \qquad \forall i \in \{0, ..., q-1\} \\
&\Gamma(x_{ij}) = \{x_{ij0}, x_{ij1}, ...x_{ijq-1}\} \cup \{x_i\} \subset W, \quad \forall j \in \{0, ..., q-2\}.
\end{aligned} \tag{2.1}$$

Hence, the subset W' of W, called *set of replaceable vertices*, has the following expression:

$$W' = \bigcup_{i=0}^{q-1} \bigcup_{j=0}^{q-2} \Gamma(x_{ij}) \setminus \{x_i, x_{ijq-1}\}$$

The set of incident edges to vertex x_{ijk} is denoted by E_{ijk}.

With the notation $W_{ij} = \Gamma(x_{ij}) \setminus \{x_i, x_{ijq-1}\}$, the set $\{W_{ij},\ i = 1, 2, \ldots, q-1,\ j = 1, 2, \ldots, q-2\}$ is a partition of W' that we call *standard partition.* It consists of $q(q-1)$ parts with $q-1$ elements in each one.

Definition 2.3.1 *Let q be a prime power and let h be an integer so that $1 \leq h \leq q+1$. Let us denote $H_q(K_h)$ any graph obtained from H_q by carrying out the following steps:*

1. *Let us choose a subset $\tilde{W}$ of W'.*

2. Each vertex $x_{ijk} \in \tilde{W}$ is replaced by a complete graph K_h, whose vertex set is denoted by $K_{ijk} = \{y^1_{ijk}, ..., y^h_{ijk}\}$. The set of added vertices is called $\tilde{W}(K_h)$. Thus,

$$\tilde{W}(K_h) = \bigcup_{x_{ijk} \in \tilde{W}} K_{ijk}.$$

3. The incident edges to each $x_{ijk} \in \tilde{W}$ are joined to the vertices of K_{ijk} so that each vertex y^l_{ijk}, $l \in \{1, 2, \ldots, h\}$, is incident at least to one of them.

4. Some new edges may be added between vertices of $\tilde{W}(K_h)$ so that the constructed graph has degree $\Delta = q + 1$. The set of these additional edges is denoted by $\tilde{E}_{ijk}$.

Example 2.3.1 Consider the Moore bipartite $(6, 6)$-graph H_5 whose order is $n = 7812$. The subset chosen is $\tilde{W} = \{x_{000}\}$. This vertex is replaced by a complete graph K_6. The edges of E_{000} are now incident to vertices of K_{000} according to Figure 2.4.

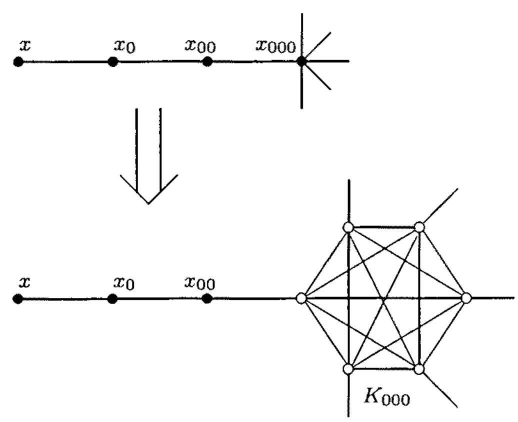

Figure 2.4: An expansion of H_5.

The graph so constructed is not bipartite anymore. However, it is still 6-regular and its order is greater than the original one (in five vertices). This example, put forward by Quisquater in [127], corresponds to a particular case of our construction. The author also showed that this graph has still diameter 6.

Observe that any $H_q(K_h)$ graph verifies the following properties:

1. After making the first three steps, each vertex of $\tilde{W}(K_h)$ has degree not greater than q+1.

2. If $h > 1$, then $H_q(K_h)$ is not bipartite.

3. $\mid \tilde{W} \mid \leq (\Delta - 1)(\Delta - 2)^2 = q(q-1)^2$.

4. $n(H_q(K_h)) = n(H_q) + \mid \tilde{W} \mid (h - 1)$.

The next proposition provides an upper bound to the diameter of any $H_q(K_h)$ graph.

Proposition 2.3.1 *The diameter of any $H_q(K_h)$ graph is at most 7.*

Proof. After the replacement of the vertices x_{ijk} of $\tilde{W}$ by complete graphs K_h, we observe the following:

1. Any path of maximum length 6 in H_q without x_{ijk} vertices is unaltered in $H_q(K_h)$.
2. The shortest path between any two vertices will increase its length by at most two (the maximum number of x_{ijk} vertices that might be contained in this path); see Figure 2.1 and remember that $g = 2D$.
3. The maximum distance between two vertices of type $y = y^l_{ijk}$ and $y' = y^o_{rst}$ is 7. In fact, since H_q is bipartite, the distance between x_{ijk} and x_{rst} has to be 0, 2, 4 or 6. If it is less than or equal to 4, then by the previous observation, the distance between y and y' is at most 6. Otherwise, since H_q contains Δ disjoint paths of length 6 between x_{ijk} and x_{rst}, at least a path of the same length runs between y and y' or between y and a vertex y^m_{rst} of K_{rst} in $H_q(K_h)$, which is adjacent to vertex y'.

Therefore, we have only to examine the following case. Let two vertices be at distance 6 in such a way that at least one of them is not of the form y^l_{ijk}. Then, as Δ disjoint paths of length 6 run between these vertices, from condition 3. of Definition 2.3.1 and Figure 2.1, it follows that there exists one path between these vertices, which is unaffected by the replacements. ■

To put forward in an easy way more complicated expansions of the H_q graphs than the ones presented by Quisquater in [127] and F. Comellas and J. Gómez in [42], we need to introduce the following definition:

Definition 2.3.2 *The Margin M of an $H_q(K_h)$ graph is the number of edges needed to add in step 4 of its construction to vertices of K_{ijk}, so that each vertex has degree $\Delta = q + 1$.*

Example 2.3.2 The vertex x_{000} is replaced by a copy of K_3 in H_3 (see Figure 2.5). It is easy to see that it is necessary to add two edges (drawn in broken line) so that each vertex has degree 4. Thus, $M = 2$ in this case.

Proposition 2.3.2 *Given an $H_q(K_h)$ graph, if $h \leq \Delta = q + 1$, then*

$$M = (\Delta - h)(h - 1). \tag{2.2}$$

Proof. After making the first three steps in order to construct the $H_q(K_h)$ graph, we have:

$$h\Delta = \Delta + h(h - 1) + M,$$

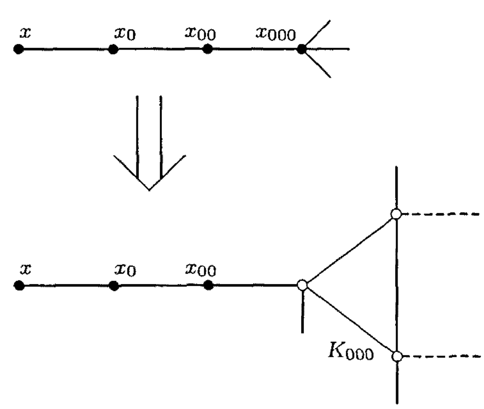

Figure 2.5: Margin of $H_3(K_3)$ is 2.

and the desired result is obtained by isolating M from this equation. ∎

When a subset $\tilde{W}$ of W' is chosen, the standard partition of W' leads to a new partition, which we call *standard partition* of $\tilde{W}$:

$$\tilde{W} = \bigcup_{i=1}^{\lambda} \tilde{W}_i, \tag{2.3}$$

where λ is the number of non-empty parts. Note that $\lambda \leq q(q-1)$ and for each $i \in \{1, 2, \ldots, \lambda\}$, $|\tilde{W}_i| \leq q - 1$.

As said above, to construct an $H_q(K_h)$ graph, each vertex $x_{ijk} \in \tilde{W}$ is replaced by a complete graph K_h. Then, the edges of E_{ijk} are joined to the vertices of K_{ijk} so that each vertex of K_{ijk} is adjacent at least to some not-replaced vertex of H_q. Finally, some new edges are added, joining vertices of the copies K_{ijk} in such a way that every new vertex has degree less than or equal to $\Delta = q + 1$. Observe that this last set $\tilde{E}_{ijk}$ of added edges has at most M elements.

As a consequence of Proposition 1, the diameter of any $H_q(K_h)$ graph is 6 or 7. As a matter of fact, it is easy to check that there are a lot of them with diameter 7. However, not all of them have this diameter (remember the construction proposed by Quisquater). To present a family of large $H_q(K_h)$ graphs with diameter 6, a new graph is required. This is denoted by $\tilde{G} = (\tilde{W}, \tilde{E})$, where $\tilde{W}$ is presented in (2.3) and $\tilde{E}$ is defined as follows:

$$(x_{ijk}, x_{rst}) \in \tilde{E} \leftrightarrow \exists \alpha, \gamma \in \{1, 2 \ldots, h\} \mid (y^{\alpha}_{ijk}, y^{\gamma}_{rst}) \in \tilde{E}_{ijk} \cap \tilde{E}_{rst} \tag{2.4}$$

The graph so constructed has degree not greater than M. Additionally, the parameters of the standard partition of $\tilde{W}$ verify that

$$l = \max_{i=1,2,\ldots,\lambda} |\tilde{W}_i|$$

$$\lambda \leq q(q-1)$$

$$l \leq q-1.$$

Theorem 2.3.1 *If the graph $\tilde{G} = (\tilde{W}, \tilde{E})$ has diameter 2 and it is an [l,λ]-clique for the standard partition of $\tilde{W}$, then $H_q(K_h)$ is a graph of (maximum) degree $\Delta = q+1$, diameter $D = 6$, and order $n(H_q(K_h)) = n(H_q) + |\tilde{W}|.(h-1)$*

Proof. According to Proposition 2.3.1, it is sufficient to consider the three following cases:

1. Let us consider two vertices in H_q, say y and z, at distance 5 joined by the path: $y, x_{ijk}, x_{ij}, x_{ijr}, t, z$. Since x_{ijk} and x_{ijr} belong to the same part of the standard partition of $\tilde{W}$, they are adjacent in $\tilde{G}$. Therefore, according to (2.4), there exist $\alpha, \gamma \in \{1, 2\ldots, h\}$ so that vertices y^{α}_{ijk} and y^{β}_{ijr} are adjacent in $H_q(K_h)$. Thus, a path of length at most 6 between y and z in $H_q(K_h)$ is: $y, y^{\beta}_{ijk}, y^{\alpha}_{ijk}, y^{\gamma}_{ijr}, y^{\delta}_{ijr}, t, z$. See Figure 2.6, where this is illustrated for $h = 4$.

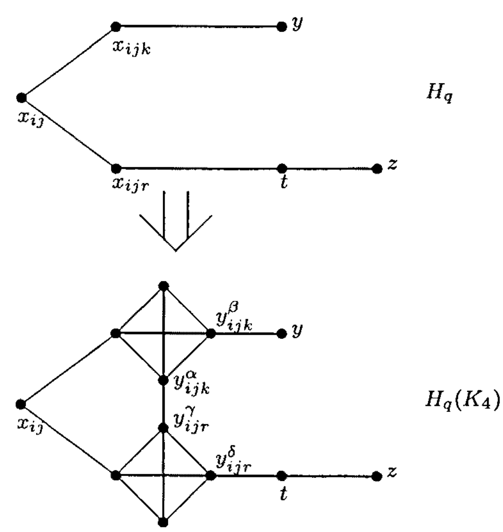

Figure 2.6: A path of length 6 between y and z in $H_q(K_4)$.

2. Let us consider these two vertices, $y = y^{\beta}_{ijk}$ and z, in $H_q(K_h)$, where x_{ijk} and z are at distance 5 in H_q and the shortest path between them is: $x_{ijk}, x_{ij}, x_i, x_{is}, x_{isr}, z$. Since x_{ijk} is at distance less than or equal to 2 of x_{isr} in $\tilde{G}$, there exist $\alpha, \gamma \in \{1, 2\ldots, h\}$ such that the vertices y^{α}_{ijk} and y^{γ}_{isr} are at most at distance 3 in $H_q(K_h)$. See Figure 2.7, where this is shown for $h = 4$. So, a path of length at most 6 between $y = y^{\beta}_{ijk}$ and z is: $y^{\beta}_{ijk}, y^{\alpha}_{ijk}, p, q, y^{\gamma}_{isr}, y^{\delta}_{isr}, z$.

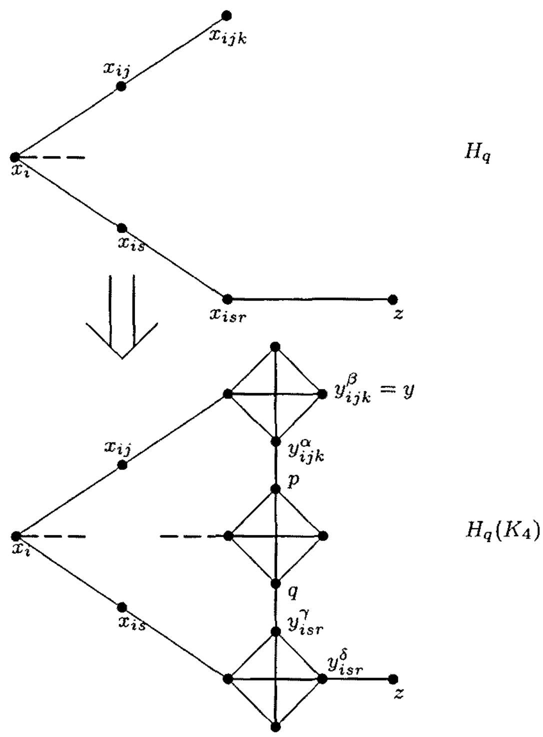

Figure 2.7: A path of length 6 between y and z.

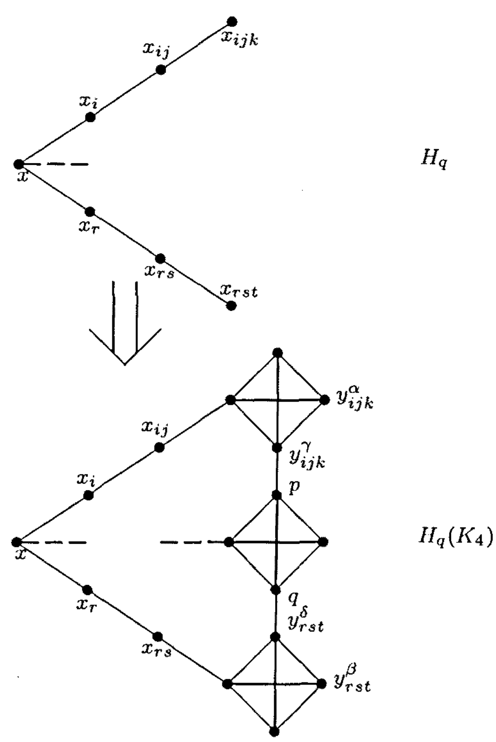

Figure 2.8: A path of length 5 between y^{α}_{ijk} and y^{β}_{rst}.

3. Let us consider these two vertices, y^{α}_{ijk} and y^{β}_{rst}, with $i \neq r$. Since x_{ijk} and x_{rst} are at distance at most 2 in $\tilde{G}$, by using the same reasoning as in the previous case, we find that the distance between y^{α}_{ijk} and y^{β}_{rst} is at most 5. (Figure 2.8 shows the path for $h = 4$.)

■

2.4 New large graphs of diameter 6

Theorem 2.3.1 provides a method to construct some $H_q(K_h)$ graphs of diameter 6. To be more precise, we take a copy of a graph H_q and we choose a complete graph K_h with $h \leq q+1$. Next, we take an [l,λ]-clique $\tilde{G} = (\tilde{W}, \tilde{E})$ with diameter $D = 2$ and degree $\Delta_{\tilde{G}}$, where

$$\lambda \leq q(q-1) \tag{2.5}$$

$$l \leq q-1 \tag{2.6}$$

$$\Delta_{\tilde{G}} \leq (q+1-h)(h-1). \tag{2.7}$$

In the following subsections, some of the examples of an [l,λ]-clique with diameter $D = 2$ are used in order to obtain new large graphs of diameter 6.

2.4.1 Large graphs from γ-graphs

All the largest known graphs with diameter 2 are called in this work *γ-graphs*. Five of them are used in this subsection in order to obtain new large graphs. To be more precise, we use the graphs: Petersen, $K_4 * X_8$, P'_9, P'_{11} and P'_{13}, which correspond to those already mentioned in Examples 2.2.1.4, 2.2.1.5 and 2.2.1.6, to obtain five new large graphs (see Table 2.1).

Δ	h	M	$\tilde{G}$	$n(\tilde{G})$	l	λ	n
5	4	3	P	10	2	5	**2760**
6	4	6	$K_4 * X_8$	32	2	16	**7908**
8	6	10	P'_9	91	3	42	**39671**
9	7	12	P'_{11}	133	3	55	**75696**
10	8	14	P'_{13}	183	3	72	**134141**

Table 2.1: New values of large graphs from γ-graphs.

Δ	h	M	$\tilde{G}$	$n(\tilde{G})$	l	λ	n
10	6	20	$P'_{17}5$	307	5	72	**134395**
12	8	28	$P'_{23}6$	553	6	110	**358183**
14	9	40	$P'_{32}7$	1057	7	151	**812924**

Table 2.2: New values of large graphs from δ-graphs.

2.4.2 Large graphs from δ-graphs

From the family presented in Example 2.2.1.7, which we call *δ-graphs*, three new large graphs are obtained (see Table 2.2). The δ-graphs used are constructed as follows:

Graph $H_9(K_6)$ is constructed by using a graph $P'_{17}5$. This is done by means of a partition of P'_{17} with $\lambda = 72$ parts (see Section 2.5). It consists of 50 C_5, 8 $K_{1,3}$, 1 K_3, 9 K_2, and 4 K_1. Observe that to obtain complete graphs from them it is enough to increase their degree by at most two units. So, $P'_{17}5$ has degree 20 which coincides with the margin in this case.

Graph $H_{11}(K_8)$ is constructed by using a graph $P'_{23}6$. This is done by means of a partition of P'_{23} with $\lambda = 110$ parts. It consists of 107 sets with order 5 each and 3 sets containing 2 K_3 each (see Section 2.5). Observe that to obtain complete graphs from them it is enough to increase their degree by at most four units. So, $P'_{23}6$ has degree 28 which again coincides with the margin in this case.

Graph $H_{13}(K_9)$ is constructed by using a graph $P'_{32}7$. This is made by means of a partition of P'_{32} with $\lambda = 151$ parts. It consists of 151 sets with 7 arbitrary vertices. In this case, to obtain 151 complete graphs K_7, the degree of each vertex is increased by at most 6 units. So, $P'_{32}7$ has degree 39 which is less than the margin in this case.

Table 2.3 shows the best values obtained in this work for degree less than or equal to 14, all of them improving the previous values (see [143]).

Δ	$x - graph$	G	$n(G)$
5	γ	$H_4(K_4)$	**2 760**
6	γ	$H_5(K_4)$	**7 908**
8	γ	$H_7(K_6)$	**39 671**
9	γ	$H_8(K_7)$	**75 696**
10	δ	$H_9(K_6)$	**134 395**
12	δ	$H_{11}(K_8)$	**358 183**
14	δ	$H_{13}(K_9)$	**812 924**

Table 2.3: New large graphs.

The method put forward in this work enables one to improve a lot of values for diameter 6 and degree Δ greater than 14, by means of the corresponding family of [l,λ]-cliques. For instance, using δ-graphs for degree $\Delta = q+1$ (with q a prime power), this method yields graphs on order about: $n = n(H_q) + \frac{\Delta^5}{2^6}$.

2.5 Appendix

In this appendix, some properties of the graphs P'_3, P'_{11}, P'_{13}, P'_{17}, P'_{23} are shown. The vertex and arc sets of the graph $P'_q = (V, E)$ can be defined as follows.

$$V = \{(0,0,1\} \cup \{(0,1,\alpha) : 0 \leq \alpha \leq q-1\} \cup \{(1,\beta,\delta) : 0 \leq \beta, \delta \leq q-1\}$$

$$\forall\ x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in V : \{x, y\} \in E\ \Leftrightarrow\ x_0y_0 + x_1y_1 + x_2y_2 = 0\ (mod\ q)$$

1. About P'_3. This graph is shown to be an $\Re_3$-graph, composed by the union of 1 K_3, 5 K_2:

   (1,0,0), (0,1,0), (0,0,1);

   (0,1,1), (0,1,2); (1,1,1), (1,2,0); (1,1,2), (1,0,1); (1,2,1), (1,0,2); (1,2,2), (1,1,0);

2. About P'_{11}. This graph is an $\Re_3$-graph, composed by the union of 32 triangles, 14 couples and 9 vertices:

   (1,0,0), (0,1,0), (0,0,1); (1,0,1), (1,1,10), (1,9,10); (1,0,2), (1,1,5), (1,7,5); (1,0,3), (1,1,7), (1,5,7); (1,0,4), (0,2,8), (1,6,8); (1,0,5), (1,1,2), (1,6,2); (1,0,6), (1,1,9), (1,6,9); (1,0,7), (1,2,3), (1,6,3); (1,0,8), (1,1,4), (1,5,4); (1,0,9), (1,1,6), (1,7,6); (1,0,10), (1,1,1), (1,9,1); (1,1,0), (1,10,1), (1,10,9); (1,2,0), (1,5,2), (1,5,9); (1,2,1), (1,2,6), (1,5,0); (1,2,2), (1,4,1), (1,9,7); (1,2,4), (1,4,6), (1,6,5); (1,2,5), (1,6,4), (0,1,4); (1,2,7), (1,4,5), (1,6,6); (1,2,9), (1,4,10), (1,9,4); (1,2,10), (1,4,9), (0,1,2); (1,3,0), (1,7,2), (1,7,8); (1,3,2), (1,3,6), (1,7,0); (1,3,3), (1,9,9), (0,1,10); (1,3,5), (1,5,10), (1,10,7); (1,3,7), (1,10,5), (0,1,9); (1,3,8), (1,9,2), (0,1,1); (1,3,9), (1,9,3), (1,10,10); (1,4,0), (1,8,2), (1,8,6); (1,4,2), (1,4,8), (1,8,0); (1,4,3), (1,6,10), (0,1,6); (1,5,8), (1,7,1), (1,10,6); (1,6,1), (1,6,7), (1,9,0);

   (1,1,3), (1,8,8); (1,1,8), (1,7,10); (1,3,1), (1,5,6); (1,3,10), (1,5,5); (1,4,4), (1,5,3); (1,4,7), (1,10,2); (1,5,1), (1,8,3); (1,6,0), (1,9,5); (1,7,3), (1,7,9); (1,7,4), (1,8,5); (1,7,7), (1,10,4); (1,8,1), (0,1,3); (1,8,10), (0,1,8); (1,10,8), (0,1,7);

   (0,1,5); (1,8,7); (1,9,8); (1,3,4); (1,8,9); (1,10,0); (1,8,4); (1,9,6); (1,10,3);

3. About P'_{13}. The following list shows that P'_{13} is an $\Re_3$-graph composed by the union of 44 triangles,.23 couples and 5 vertices:

   (1,8,5), (0,1,1), (1,4,9); (1,8,6), (0,1,3), (1,2,8); (1,8,7), (1,2,5), (0,1,10); (1,8,8), (1,2,6), (1,9,12); (0,8,9), (0,1,2), (1,9,2); (0,1,0), (0,0,1), (1,0,0); (0,1,4), (1,6,5), (1,8,11); (0,1,6), (1,2,4), (1,9,5); (1,1,9), (1,7,2), (1,5,8); (1,1,10), (1,2,1), (1,5,2); (1,1,11), (1,3,2), (1,6,10); (1,1,12), (1,2,3), (1,7,8); (1,2,0), (1,6,1), (1,6,2); (1,2,2), (1,3,3), (0,1,12); (1,2,7), (1,3,12), (1,7,9); (1,2,9), (1,2,11), (1,6,0); (1,2,10), (1,7,5), (1,1,1); (1,2,12), (1,3,7), (1,11,10); (1,3,0), (1,4,2), (1,4,11); (1,3,1), (1,5,10), (1,9,11); (1,3,5), (1,3,11), (1,4,0); (1,3,6), (1,10,10), (1,1,8); (1,3,8), (1,5,11), (1,6,9); (1,3,10), (1,7,3), (1,5,1); (1,4,1), (1,11,7), (1,5,5); (1,4,4), (1,12,4), (1,0,3); (1,4,5), (1,11,4), (0,1,7); (1,4,6), (1,7,6), (1,0,2); (1,4,7), (1,7,7), (1,0,11); (1,5,9), (1,6,11), (1,11,1); (1,5,12), (1,9,7), (1,1,6); (1,6,3), (1,12,6), (0,1,11); (1,6,4), (1,9,9), (1,12,11); (1,6,7), (1,12,10), (1,5,3); (1,6,8), (1,8,2), (0,1,9); (1,6,12), (1,7,4), (1,12,8); (1,7,0), (1,11,5), (1,11,12); (1,7,1), (1,9,1), (1,0,12); (1,7,11), (1,7,12), (1,11,0); (1,8,4), (1,12,5), (1,5,6); (1,9,0), (1,10,1), (1,10,3);

(1,9,6), (1,9,8), (1,10,0); (1,10,5), (1,11,9), (1,1,3); (1,12,2), (1,12,12), (1,1,0);

(0,1,5), (1,8,1); (1,3,4), (1,8,10); (1,3,9), (1,8,3); (1,4,3), (1,10,8); (1,6,6), (1,12,3); (1,7,10), (1,9,4); (1,8,0), (1,8,12); (1,9,3), (1,12,7); (1,9,10), (1,11,3); (1,10,2), (0,1,8); (1,10,4), (1,11,8); (1,10,6), (1,10,7); (1,10,9), (1,12,1); (1,10,11), (1,0,7); (1,10,12), (1,0,1); (1,11,2), (1,0,6); (1,11,6), (1,4,12); (1,12,0), (1,1,2); (1,12,9), (1,0,10); (1,0,5), (1,1,5); (1,0,8), (1,4,8); (1,0,9), (1,4,10); (1,5,0), (1,5,4);

(1,0,4); (1,1,4); (1,1,7); (1,5,7); (1,11,11);

4. About graph P'_{17}. The following list shows P'_{17} as composed by the union of 50 C_5, 8 $K_{1,3}$, 1 P_4, 1 K_3, 6 K_2, and 6 K_1:

(1,12,1), (1,12,8), (1,2,16), (1,0,1), (1,0,16); (1,12,2), (1,12,4), (1,2,15), (1,4,13), (1,9,5); (1,12,3), (1,1,7), (1,2,2), (1,3,5), (0,1,13); (1,12,5), (1,2,12), (1,3,15), (1,4,15), (1,1,11); (1,12,6), (1,12,7), (1,2,11), (1,2,4), (1,8,15); (1,1,4), (1,1,8), (1,0,2), (1,4,8), (1,9,6); (1,1,5), (1,2,13), (1,2,14), (1,3,8), (1,15,7); (1,1,6), (1,2,8), (1,2,10), (1,3,1), (0,1,14); (1,1,13), (1,2,5), (1,5,8), (1,9,7), (1,12,16); (1,1,14), (1,2,1), (1,3,10), (1,4,14), (1,0,6); (1,2,7), (1,2,9), (1,3,3), (1,4,7), (1,0,12); (1,2,0), (1,8,1), (1,9,12), (1,15,0), (0,0,1); (1,2,3), (1,3,14), (1,4,10), (1,3,9), (1,14,13); (1,3,2), (1,3,12), (1,4,6), (1,5,5), (1,13,14); (1,3,6), (1,4,12), (1,5,11), (1,7,6), (1,0,14); (1,3,7), (1,4,3), (1,5,10), (1,6,2), (1,6,7); (1,3,11), (1,3,13), (1,2,6), (1,5,1), (1,7,15); (1,3,16), (1,5,16), (1,6,14), (1,8,5), (1,16,15); (1,4,1), (1,5,13), (1,5,15), (1,8,12), (1,1,12); (1,4,2), (1,2,4), (1,6,1), (1,7,8), (1,11,3); (1,4,4), (1,6,15), (1,7,13), (1,8,10), (1,12,9); (1,4,5), (1,1,16), (1,8,9), (1,12,10), (1,13,3); (1,4,9), (1,5,9), (1,6,6), (1,6,8), (1,10,3); (1,4,11), (1,1,15), (1,5,3), (1,5,14), (1,14,1); (1,4,16), (1,5,4), (1,6,5), (1,6,13), (1,15,10); (1,5,2), (1,6,10), (1,7,11), (1,10,9), (1,14,7); (1,5,6), (1,5,7), (1,7,7), (1,9,3), (1,13,6); (1,5,12), (1,7,14), (1,9,2), (1,14,3), (0,1,1); (1,6,3), (1,6,16), (1,7,9), (1,10,11), (1,13,2); (1,6,4), (1,6,12), (1,7,12), (1,10,4), (1,0,4); (1,6,9), (1,6,11), (1,7,10), (1,10,15), (0,1,5); (1,7,1), (1,8,11), (1,9,15), (1,11,16), (1,15,13); (1,7,2), (1,8,14), (1,8,16), (1,11,4), (1,11,12); (1,7,3), (1,10,16), (1,11,9), (1,13,1), (1,1,3); (1,7,4), (1,9,1), (1,11,2), (1,11,7), (1,13,11); (1,7,5), (1,9,11), (1,10,11), (1,11,8), (1,14,4); (1,7,16), (1,8,6), (1,9,2), (1,9,10), (1,16,11); (1,8,3), (1,9,4), (1,12,11), (1,0,3), (1,0,11); (1,8,4), (1,12,14), (1,14,11), (0,1,8), (1,1,2); (1,8,7), (1,9,9), (1,10,5), (1,15,14), (1,16,1); (1,8,13), (1,9,14), (1,9,16), (1,13,16), (1,15,9); (1,9,8), (1,11,13), (1,12,12), (1,14,10), (1,14,16); (1,9,13), (1,10,10), (1,13,9), (1,15,16), (0,1,15); (1,10,2), (1,14,6), (1,16,5), (0,1,7), (0,1,12); (1,10,6), (1,14,2), (1,14,12), (1,15,15), (1,16,10); (1,10,8), (1,11,1), (1,14,15), (1,15,12), (0,1,3); (1,10,12), (1,15,3), (1,15,4), (1,16,12), (1,0,7); (1,11,5), (1,16,2), (1,16,16), (1,1,16), (1,1,1); (1,11,6), (1,13,10), (1,15,11), (1,16,9), (1,16,13); (1,11,10), (1,11,15), (1,13,4), (1,15,2), (1,1,9);

(1,0,0), (0,1,0), (0,1,2), (0,1,4); (1,3,0), (1,11,0), (1,11,11), (1,11,14); (1,5,0), (1,10,0), (1,10,7), (1,10,7); (1,6,0), (1,14,0), (1,14,0), (1,14,8); (1,7,0), (1,12,0), (1,12,13), (1,12,15); (1,9,0), (1,15,1), (1,15,5), (1,15,6); (1,13,0), (1,13,5), (1,13,7), (1,13,8); (1,14,9), (1,14,14), (1,1,6), (1,0,15);

(0,1,9), (1,10,14), (1,16,14), (0,1,11);

(1,16,4), (1,16,8), (1,1,0);

(1,8,8), (1,16,3); (1,13,13), (1,0,13); (1,13,15), (1,0,9); (1,15,8), (1,16,6); (1,16,0), (1,1,10); (1,0,5), (1,0,10);

(0,1,10); (1,0,8); (1,4,0); (1,8,0); (1,13,12); (1,16,7);

5. About P'_{23}. Finally, 6 K_3 are shown to be in P'_{23}.

(1,0,0), (0,1,0), (0,0,1); (1,0,1), (1,1,22), (1,21,22); (1,0,2), (1,1,11), (1,16,11); (1,0,3), (1,1,15), (1,4,15); (1,0,4), (1,1,17), (1,9,17); (1,0,5), (1,1,9), (1,10,9);

Chapter 3

1-vertex vulnerability of generalized compound graphs

3.1 Introduction

Concern over fault tolerance in the design of interconnection networks has stimulated interest in finding large graphs with maximum degree Δ and diameter D such that the subgraphs obtained by deleting any set of s vertices have diameter at most D', this value being close to D or even equal to it. This is the so-called (Δ, D, D', s)-problem (see Section 1.7.2). The study of the 1-vertex vulnerability; that is, the $(\Delta, D, D', 1)$-problem, is probably the most interesting case, and in fact, it has deserved so far the greatest attention in the literature on this subject.

A technique that has been proved to be useful for obtaining large (Δ, D)-graphs is the compounding of graphs (see Sections 1.4, 1.5.3). The 1-vertex vulnerability of different families of compound graphs have been studied by several authors (see [29, 30, 74, 121]), and the obtained result is that, in most cases, they are $(\Delta, D, D+j, 1)$-graphs with $j = 1, 2$.

Another useful method for obtaining large (Δ, D)-graphs is the design of *graphs on alphabet* (see [3, 29, 34, 49, 69, 74, 79, 104]). These graphs are constructed by labelling the vertices with words of a given alphabet, together with a rule that relates pairs of different words to define the edges. One usual procedure of obtaining graphs on alphabet consists in taking the underlying graph of a line digraph. For instance, the De Bruijn and Kautz graphs were designed in this way (see Section 1.5.2). The 1-vertex vulnerability of these two families of graphs was studied by J. Bond and C. Peyrat in [31]. They proved that both De Bruijn and Kautz graphs have *optimal* 1-vertex vulnerability; that is, they are $(\Delta, D, D, 1)$ graphs.

Generalized compound graphs, also called *GC graphs*, were introduced in [75] by J. Gómez by proposing several constructions, all of them inspired both in the compounding of graphs and in the design of graphs on alphabet. The so-called *GCG technique* joins together the advantages of these two methods. In general, compounding of graphs yields large graphs when the diameter D is rather small (see [77, 143]), but each time the procedure is iterated in order to obtain large graphs with greater diameters, the degree of the resulting graph increases. On the other hand, the design of graphs on alphabet, and particularly the line digraph technique, provides large graphs for high values of the diameter. Moreover, in these graphs the design of algorithms which provide paths between pairs of vertices is fairly simple. As was shown in [75], generalized compound graphs, which can be seen both as compound graphs and graphs on alphabet, have, in general, an order greater than those of compound graphs. In addition, they are generally larger than the known graphs on alphabet. The 1-vertex vulnerability of the GC graphs, whose study is presented in this work (see also [84]) is *quasi optimal*; that is, they are in almost all the cases $(\Delta, D, D+1, 1)$-graphs.

This chapter consists of three more sections. In the next one, three families of line digraphs are defined, being proved that the corresponding families of underlying graphs have, in most cases, optimal 1-vertex vulnerability. Section 3.3 is devoted to characterizing the GC graphs as compound graphs by using the previous families. Finally, in Section 3.4 we make use of this characterization in order to study the 1-vertex vulnerability of the GC graphs.

3.2 Three families of line digraphs

In the rest of this chapter J_m denote the set of the first m natural numbers; that is, $J_m = \{1, 2, \dots, m\}$. In this section we introduce three families of digraphs on alphabet which will be shown to be families of iterated line digraphs. The first of them, whose members are isomorphic to De Bruijn digraphs, is defined as follows.

Definition 3.2.1 *Let m, n, h be three positive integers such that $mn \geq 2$. Let us define the digraph $G^I(m,n,h)$ in the following way:*
Its vertex set is: $V = [J_m \times J_n]^h = \{(\beta_1, x_1)(\beta_2, x_2)\dots(\beta_h, x_h),\ \beta_i \in J_m,\ x_i \in J_n\}$. The adjacency rule is:

$$\Gamma^+ ((\beta_1, x_1)\dots(\beta_h, x_h)) = \{(\beta_0, x_0)(\beta_1, x_1)\dots(\beta_{h-1}, x_{h-1}),\ \beta_0 \in J_m,\ x_0 \in J_n\}$$

Corollary 3.2.1 *The digraph $G^I(m,n,h)$ satisfies the following properties:*

1. *It is isomorphic to the line digraph $L^{h-1}(K^*_{mn})$. Hence, it is a regular digraph with degree and diameter: $(\Delta, D) = (mn, h)$.*

2. *Its underlying graph has diameter h and, for $h \geq 2$, maximum degree $2mn$.*

Definition 3.2.2 *Let m, n, h be three positive integers. Let us define the digraph $G^{II}(m,n,h)$ in the following way:*
Its vertex set is: $V = \{(\beta_1, x_1)(\beta_2, x_2)\dots(\beta_h, x_h),\ \beta_i \in J_m,\ x_i \in J_{n+1}, x_i \neq x_{i+1}\}$. The adjacency rule is:

$$\Gamma^+ ((\beta_1, x_1)\dots(\beta_h, x_h)) = \{(\beta_0, x_0)(\beta_1, x_1)\dots(\beta_{h-1}, x_{h-1}),\ \beta_0 \in J_m,\ x_0 \in J_{n+1} \setminus \{x_1\}\}$$

It is easy to see that $G^{II}(1,n,h)$ is isomorphic to the Kautz digraph $K(n,h)$. Taking into account known properties of Kautz digraphs we obtain the following corollary.

Corollary 3.2.2 *The digraph $G^{II}(1,n,h)$, $n \geq 2$, satisfies the following properties:*

1. *It is isomorphic to the iterated line digraph $L^{h-1}(K_{n+1})$. Hence, it is a regular digraph with degree and diameter: $(\Delta, D) = (n,h)$.*

2. *Its underlying graph has diameter h and, for $h \geq 3$, maximum degree $2n$.*

In the case $m > 1$, the digraph $G^{II}(m,n,h)$ is also isomorphic to an iterated line digraph obtained by conjunction of a De Bruijn digraph with a Kautz digraph, as we show in the following proposition.

Proposition 3.2.1 *If $m > 1$, then the digraph $G^{II}(m,n,h)$ satisfies the following properties:*

1. *It is isomorphic to the digraph $B(m,h) \otimes K(n,h)$.*

2. *It is isomorphic to the digraph $L^{h-1}(K_m^* \otimes K_{n+1})$.*

3. *It is a regular digraph with degree and diameter: $(\Delta, D) = (mn, h+1)$.*

4. *The diameter of its underlying graph is $h+1$ and, if $h \geq 3$, then its maximum degree is $2mn$.*

Proof. The one-to-one mapping from $V[G^{II}(m,n,h)]$ onto $V[B(m,h) \otimes K(n,h)]$ defined by:

$$(\beta_1, x_1)(\beta_2, x_2)\dots(\beta_h, x_h) \longmapsto \beta_1\beta_2\dots\beta_h x_1 x_2 \dots x_h$$

induces clearly an isomorphism between both digraphs.

Point two follows immediately from Proposition 1.4.4, together with the fact that the digraphs $B(m,h)$ and $K(n,h)$ are isomorphic to $L^{h-1}(K_m^*)$ and $L^{h-1}(K_{n+1})$ respectively.

From Definition 3.2.2 it follows immediately that $\Delta = mn$. Moreover, $K_m^* \otimes K_{n+1}$ has no loops and its diameter is 2. Hence, by the previous point and Proposition 1.3.2, $D = 2 + h - 1 = h + 1$, and thus point three also holds.

Finally, from Definition 3.2.2, it immediately follows that the maximum degree of the underlying graph $UG^{II}(m,n,h)$ is $2mn$, since $h \geq 3$. It is also clear that the diameter of this graph is at most $h+1$. To see the equality it is enough to find two vertices at distance $h+1$ in $G^{II}(m,n,h)$, which remain at distance $h+1$ in $UG^{II}(m,n,h)$. Let us consider the following two vertices:

$$u = (\beta_1, x_1)(\beta_2, x_2)\dots(\beta_{h-1}, x_{h-1})(\beta_h, x_h),$$
$$v = (\gamma_1, x_h)(\gamma_2, y_2)\dots(\gamma_{h-1}, y_{h-1})(\gamma_h, x_1).$$

Notice that, if $\beta_1 \neq \gamma_h$ and $\beta_h \neq \gamma_1$, then $d(u,v) = d(v,u) = h+1$ in the original digraph and so $d(u,v) = h+1$ in the underlying graph as well. ■

Definition 3.2.3 *Let m_0, n_0, m_1, n_1, h be positive integers, h being an odd number. It is defined $G^{III}(m_0, n_0, m_1, n_1, h)$ as the bipartite digraph with partite sets:*

$$U = \{(\gamma_1, y_1)(\beta_2, x_2)\dots(\gamma_h, y_h),\ \beta_i \in J_{m_0},\ \gamma_j \in J_{m_1},\ x_i \in J_{n_0},\ y_j \in J_{n_1}\}$$

$$V = \{(\beta_1, x_1)(\gamma_2, y_2)\dots(\beta_h, x_h),\ \beta_i \in J_{m_0},\ \gamma_j \in J_{m_1},\ x_i \in J_{n_0},\ y_j \in J_{n_1}\}$$

The adjacency rules are:

$$\Gamma^+\left((\gamma_1, y_1)(\beta_2, x_2)\dots(\gamma_h, y_h)\right) = \{(\beta_0, x_0)(\gamma_1, y_1)\dots(\beta_{h-1}, x_{h-1}),\ \beta_0 \in J_{m_0},\ x_0 \in J_{n_0}\};$$

$$\Gamma^+\left((\beta_1, x_1)(\gamma_2, y_2)\dots(\beta_h, x_h)\right) = \{(\gamma_0, y_0)(\beta_1, x_1)\dots(\gamma_{h-1}, y_{h-1}),\ \gamma_0 \in J_{m_1},\ y_0 \in J_{n_1}\}.$$

As in the above two cases, this third family of bipartite digraphs is an iterated line digraph family as well, as it is shown in the following proposition.

Proposition 3.2.2 *If either $m_0n_0 \geq 2$ or $m_1n_1 \geq 2$, then $G^{III}(m_0, n_0, m_1, n_1, h)$ satisfies the following properties:*

1. *It is isomorphic to the bipartite line digraph $L^{h-1}(K_{m_0n_0, m_1n_1})$. Hence, its minimum degree is $min\{m_0n_0, m_1n_1\}$ and its maximum degree and diameter are: $(\Delta, D) = (max\{m_0n_0, m_1n_1\},\ h+1)$.*
2. *The diameter of its underlying graph is $h+1$ and, if $h \geq 3$, then its maximum degree is $m_0n_0 + m_1n_1$.*

Proof. The line digraph of a bipartite digraph is also bipartite. Furthermore, it is clear that $G^{III}(m_0, n_0, m_1, n_1, h) = L^{h-1}(K_{m_0n_0, m_1n_1})$. Hence, its maximum degree is equal to $\max\{m_0n_0, m_1n_1\}$ and by means of Proposition 1.3.2, we derive that the diameter is $h+1$. Finally, the proof of the third property is similar to that of Proposition 3.2.1. ■

Let us finalize this section by studying the extent to which the diameter of the underlying graphs of the line digraph families just presented increases when one vertex is deleted. J. Bond and C. Peyrat proved in [31] the following result.

Proposition 3.2.3 *Let G be a (strongly) connected digraph such that:*

$$\forall x, y \in V[G]: \quad |\Gamma^+(x) \cap \Gamma^+(y)| \neq 1,\ |\Gamma^-(x) \cap \Gamma^-(y)| \neq 1.$$

Then,

$$\forall v \in V[UG]: \quad D(UG - \{v\}) \leq D(G)$$

As a consequence of the Heuchenne condition (see Proposition 1.3.3) together with this result, it was proved in the same reference that, if G is a Kautz digraph or a De Bruijn digraph (with D and Δ not equal together to 2), then $D(UG - \{v\}) = D(G)$ for any vertex v of G. From these ideas we get the following result.

Theorem 3.2.1 *Let $G = LH$ be a (strongly) connected line digraph with minimum degree $\delta \geq 2$, for which $D(UG) = D(G)$. Then, UG is a $(\Delta, D, D, 1)$-graph.*

Proof. As a consequence of the Heuchenne condition, it follows that G satisfies the hypothesis of Proposition 3.2.3, since $\delta \geq 2$. Hence, for every vertex v of G, $D(UG - \{v\}) \leq D(G)$. Moreover, there exists some vertex w for which $D(UG - \{w\}) \geq D(UG)$ because $\delta \geq 2$, and since $D(G) = D(UG)$, we obtain that $D(UG - \{w\}) = D$, and hence UG is a $(\Delta, D, D, 1)$-graph. ■

As a direct consequence of the above results we have the following theorem.

Theorem 3.2.2 *Let m, n, h be positive integers with $mn \geq 2$ and $h \geq 2$. Then, $UG^I(m,n,h)$ and $UG^{II}(m,n,h)$ are $(\Delta', D, D, 1)$-graphs where $\Delta' \leq 2\Delta = 2mn$. Besides, if $h \geq 3$ is an odd number and $m_0 n_0 \geq 2$, $m_1 n_1 \geq 2$, then $UG^{III}(m_0, n_0, m_1, n_1, h)$ is a $(\Delta', D, D, 1)$-graph as well, where $\Delta' = m_0 n_0 + m_1 n_1$.*

3.3 Generalized compound graphs

The different families of generalized compound graphs were introduced by J. Gómez in [75]. In order not to increase excessively this work we refer to this paper for details about these families. In this section we present these graphs from another point of view; more specifically, we characterize any GC graph as a compound graph, either $G_2[G_1]$ or $G_2[G_0, G_1]$, G_2 being the underlying graph of one of the line digraphs introduced in the previous section. In this way, we will be able to study the 1-vertex vulnerability of all of these graphs, a subject which is presented in the next section. Since the proofs of the different propositions that are put forward can be obtained directly from the definitions involved, they are omitted.

3.3.1 GC graphs of type I

Next, we are going to define from another point of view the so-called *GC graphs of type I* put forward in [75]. They were denoted by $G_1\{m,k\}G$, and $B_1\{m,k\}G$ in the bipartite case. These constructions were inspired by De Bruijn graphs because, as we will show, they are isomorphic to compound graphs $G_2[G_1]$, where G_2 is the underlying graph of the digraph $G^I(m, n, k-1) \cong L^{k-2}K^*_{mn}$ (see Corollary 3.2.1).

Proposition 3.3.1 *Let $G_1 = (J_n, E_1)$ be a (Δ_1, D_1)-graph on n vertices, and let m, k be two positive integers, with $k \geq 2$. Then, the graph $G_1\{m,k\}G$ is isomorphic to the* *compound graph* $G_2[G_1]$ *where* $G_2 = UG^I(m,n,k-1)$, *and the adjacency rule which yields an intercopy edge is:*

$$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})x_k \sim (\beta_0,x_k)(\beta_1,x_1)\ldots(\beta_{k-2},x_{k-2})x_{k-1}$$

where $(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})$ *is any vertex of* G_2, $x_k \in J_n$ *and* $\beta_0 \in J_m$.
Its maximum degree, diameter and order are: $(\Delta, D, N) = (\Delta_1 + 2m, kD_1 + k - 1, m^{k-1}n^k)$ *whenever* $k \geq 3$, *and* $\Delta = \Delta_1 + m$ *if* $k = 2$.

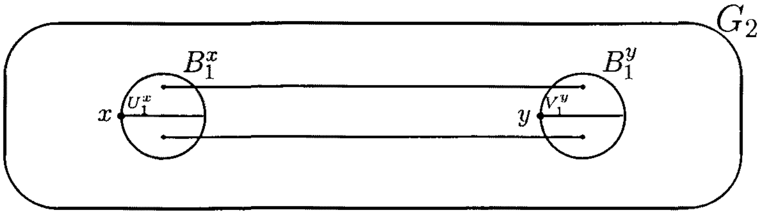

Figure 3.1: Bipartite compound graphs with two intercopy edges.

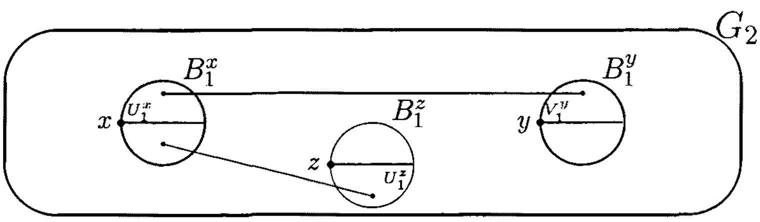

Figure 3.2: Bipartite compound graphs with one intercopy edge.

Proposition 3.3.2 *Let* B_1 *be a bipartite* (Δ_1, D_1)*-graph with partite sets* $U_1 = \{0\} \times J_n$ *and* $V_1 = \{1\} \times J_n$, *and let* m, k *be two positive integers with* $k \geq 3$. *Then,* $B_1\{m,k\}G$ *is isomorphic to the bipartite compound graph* $G_2[B_1]$, *where* $G_2 = UG^I(m,n,k-1)$ *and, according to the parity of* D_1, *the intercopy adjacency rules are:*

- *If* D_1 *is odd, then the adjacency rule produces two intercopy edges (see Figure 3.1):*

  $$(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})(\alpha,x_k) \sim (\beta_0,x_k)(\beta_1,x_1)\ldots(\beta_{k-2},x_{k-2})(\bar{\alpha},x_{k-1})$$

  where $(\beta_1,x_1)(\beta_2,x_2)\ldots(\beta_{k-1},x_{k-1})$ *is a vertex of* G_2, $\bar{\alpha} = \alpha + 1$ *in* Z_2, $\beta_0 \in J_m$ *and* $(\bar{\alpha},x_{k-1}), (\alpha,x_k) \in U_1 \cup V_1$.

  In this case the maximum degree is $\Delta = \Delta_1 + 2m$.

- *If D_1 is even, then the adjacency rule produces one intercopy edge (see Figure 3.2):*

$$(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})(0,x_k) \sim (\beta_0,x_k)(\beta_1,x_1)\dots(\beta_{k-2},x_{k-2})(1,x_{k-1})$$

$$(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})(1,x_k) \sim (\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})(\beta_k,x_k)(0,x_1)$$

where $(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})$ is a vertex of G_2, $(0,x_k) \in U_1$, $(1,x_k) \in V_1$ and $\beta_0,\beta_k \in J_m$.

In this case the maximum degree is $\Delta = \Delta_1 + m$.

In both cases the diameter and order are $(D,N) = (kD_1, 2m^{k-1}n^k)$.

3.3.2 GC graphs of type II

Generalized compound graphs of type II were denoted in [75] by $G_1(m,k)G$, $B_1(m,k)G$ and $FF(m,k)G$. These constructions were inspired by Kautz graphs, since any of these graphs is isomorphic to a certain compound graph $G_2[G_1]$, where G_2 is the underlying graph of $G^{II}(m,n,k-1) \cong L^{k-2}(K_m^* \otimes K_{n+1})$ if $m \geq 2$, and $G^{II}(1,n,k-1) \cong L^{k-2}(K_{n+1})$ (see Corollary 3.2.2 and Proposition 3.2.1). The main differences among them lie, on the one hand, in the type of graph G_1 used and, on the other, in the adjacency rules producing intercopy edges. Before showing the three characterizations of GC graphs of type II, and in order to define suitably the adjacency rules, we need firstly to put forward the following definition.

Definition 3.3.1 *Given $l \in J_{n+1}$, let f_l denote the only increasing bijection from $J_{n+1} \setminus \{l\}$ onto J_n.*

Proposition 3.3.3 *Let m,k be two positive integers, $k \geq 3$, and $G_1 = (J_n, E_1)$ a (Δ_1, D_1)-graph on n vertices. The graph $G_1(m,k)G$ is isomorphic to the compound graph $G_2[G_1]$, where $G_2 = UG^{II}(m,n,k-1)$, and the adjacency rule producing one intercopy edge is:*

$$(\beta_1,x_1)(\beta_2,x_2)\dots(\beta_{k-1},x_{k-1})x_k \sim (\beta_0,x'_k)(\beta_1,x_1)\dots(\beta_{k-2},x_{k-2})x_{k-1}$$

where $(\beta_1,x_1)\dots(\beta_{k-1},x_{k-1}) \in V(G_2)$, $x_k \in f^{-1}_{x_{k-1}}(J_n)$, $x'_k = (f^{-1}_{x_1} f_{x_{k-1}})(x_k)$, and $\beta_0 \in J_m$.
Its maximum degree, diameter and order are:

$$(\Delta, D, N) = (\Delta_1 + 2m, kD_1 + k - 1, (n+1)(mn)^{k-1}).$$

Proposition 3.3.4 *Let m,k be two positive integers with $k \geq 3$, and B_1 a bipartite (Δ_1, D_1)-graph on $2n$ vertices, with partite sets $U_1 = \{0\} \times J_n$, $V_1 = \{1\} \times J_n$. Then, $B_1(m,k)G$ is isomorphic to the bipartite compound graph $G_2[B_1]$ where $G_2 = UG^{II}(m,n,k-1)$ and, according to the parity of D_1, the intercopy adjacency rules are:*

- *If D_1 is odd, the adjacency rule produces two intercopy edges (see Figure 3.1):*

$$(\beta_1, x_1)(\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})(\alpha, x_k) \sim (\beta_0, x'_k)(\beta_1, x_1)\dots(\beta_{k-2}, x_{k-2})(\bar{\alpha}, x_{k-1})$$

where $(\beta_1, x_1)\dots(\beta_{k-1}, x_{k-1}) \in V[G_2]$, $(\alpha, x_k) \in \{0,1\} \times f^{-1}_{x_{k-1}}(J_n)$, $\bar{\alpha} = \alpha + 1$ in Z_2, $f_{x_1}(x'_k) = f_{x_{k-1}}(x_k)$, and $\beta_0 \in J_m$.

In this case the maximum degree is $\Delta = \Delta_1 + 2m$.

- *If D_1 is even, the adjacency rule yields one intercopy edge (see Figure 3.2):*

$$(\beta_1, x_1)(\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})(0, x_k) \sim (\beta_0, x'_k)(\beta_1, x_1)\dots(\beta_{k-2}, x_{k-2})(1, x_{k-1})$$

$$(\beta_1, x_1)(\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})(1, x_k) \sim (\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})(\beta_k, x_k)(0, x'_1)$$

where $(\beta_1, x_1)\dots(\beta_{k-1}, x_{k-1}) \in V[G_2]$, $x_k \in f^{-1}_{x_{k-1}}(J_n)$, $f_{x_1}(x'_k) = f_{x_{k-1}}(x_k)$, $f_{x_k}(x'_1) = f_{x_2}(x_1)$, and $\beta_0, \beta_k \in J_m$.

In this case the maximum degree is $\Delta = \Delta_1 + m$.

In both cases the diameter and order are $(D, N) = (kD_1, 2(n+1)(mn)^{k-1})$.

Proposition 3.3.5 *Let m, k be two positive integers with $k \geq 3$, and B_1 a bipartite (Δ_1, D_1)-graph on $2n$ vertices, with partite sets $U_1 = \{0\} \times J_n$, $V_1 = \{1\} \times J_n$. Then, the not bipartite graph $FF(m,k)G$ is isomorphic to $G_2[B_1]$ where $G_2 = UG^{II}(m, n, k-1)$, and the adjacency rule producing four intercopy edges (see Figure 3.3) is:*

$$(\beta_1, x_1)(\beta_2, x_2)\dots(\beta_{k-1}, x_{k-1})(\alpha, x_k) \sim (\beta_0, x'_k)(\beta_1, x_1)\dots(\beta_{k-2}, x_{k-2})(\alpha', x_{k-1})$$

where $(\beta_1, x_1)\dots(\beta_{k-1}, x_{k-1}) \in V(G_2)$, $(\alpha, x_k) \in \{0,1\} \times f^{-1}_{x_{k-1}}(J_n)$, $\beta_0 \in J_m$, $f_{x_1}(x'_k) = f_{x_{k-1}}(x_k)$, and $\alpha' \in \{0,1\}$.
Its maximum degree, diameter and order are $(\Delta, D, N) = (\Delta_1 + 4m, kD_1 - 1, 2(n+1)(mn)^{k-1})$.

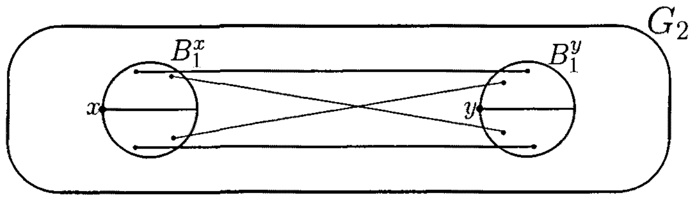

Figure 3.3: Intercopy edges of compound graphs FF.

Using the previous ideas it is possible to design a $FF(m,k)G$ graph with maximum degree $\Delta = \Delta_1 + 2m$, the same diameter and, however, an smaller order (see [75]).

3.3.3 GC graphs of type III

Both GC graphs of type I and II can be seen as compounds graphs $G_2[G_1]$, where G_2 is a certain non-bipartite graph. In other words, all of them are compound graphs constructed according to Definition 1.5.1. In a similar way, the so-called *generalized compound graphs of type III* are compound graphs $G_2[G_0, G_1]$ in which G_2 is a certain bipartite graph, and therefore, they are compound graphs designed according to definition 1.5.2. This class of graphs corresponds to the $DQ_\Lambda\{m_1, m_0, k\}G$ and $B_0\nabla B_1\{m_1, m_0, k\}G$ graphs put forward in [75]. Next, we are going to characterize the different families of designed GC graphs of type III by showing that in all of them G_2 is isomorphic to the underlying graph of $G^{III}(m_0, n_0, m_1, n_1, k-1) \cong L^{k-2}(K_{m_0n_0, m_1n_1})$ (see Proposition 3.2.2). Moreover, we see that the main differences among them lie in the type of graphs G_0 and G_1 used, and in the adjacency rules producing intercopy edges.

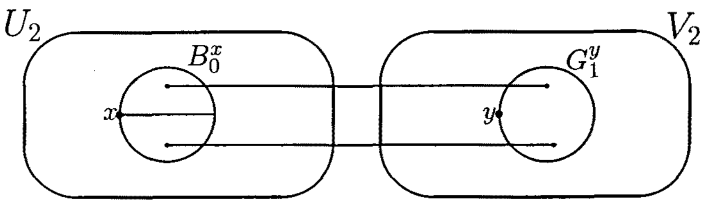

Figure 3.4: Intercopy edges of compound graphs DQ_Λ.

Proposition 3.3.6 *Let B_0 be a bipartite (Δ_0, D_0)-graph of order $2n_0$, whose partite sets are $\{0\} \times J_{n_0}$ and $\{1\} \times J_{n_0}$, and let $G_1 = (J_{n_1}, E_1)$ be a (Δ_1, D_1)-graph of order n_1. Let m_0, m_1, k be three positive integers such that $k \geq 4$ is an even number. The (non-bipartite) $DQ_\Lambda\{m_1, m_0, k\}G$ graph is isomorphic to the compound graph $G_2[B_0, G_1]$, where $G_2 = UG^{III}(m_0, n_0, m_1, n_1, k-1)$; that is, G_2 is bipartite with partite sets U_2 and V_2, and the adjacency rules which produce two intercopy edges (see Figure 3.4) are:*

$$(\gamma_1, y_1)(\beta_2, x_2)\ldots(\gamma_{k-1}, y_{k-1})(\alpha, x_k) \sim (\beta_0, x_k)(\gamma_1, y_1)\ldots(\beta_{k-2}, x_{k-2})y_{k-1}$$

for any $(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1}) \in U_2$, $(\alpha, x_k) \in \{0,1\} \times J_{n_0}$, $\beta_0 \in J_{m_0}$.

$$(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1})y_k \sim (\gamma_0, y_k)(\beta_1, x_1)\ldots(\gamma_{k-2}, y_{k-2})(\alpha, x_{k-1})$$

for any $(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1}) \in V_2$, $y_k \in J_{n_1}$, $\gamma_0 \in J_{m_1}$, $\alpha \in \{0,1\}$.
Its maximum degree, diameter and order are:

$$(\Delta, D) = (max\{\Delta_0 + 2m_0, \Delta_1 + 4m_1\}), \frac{k(D_0 + D_1 + 1)}{2}),$$

$$N = (m_0 + 2m_1)(m_0m_1)^{\frac{k}{2}-1}(n_0n_1)^{\frac{k}{2}}).$$

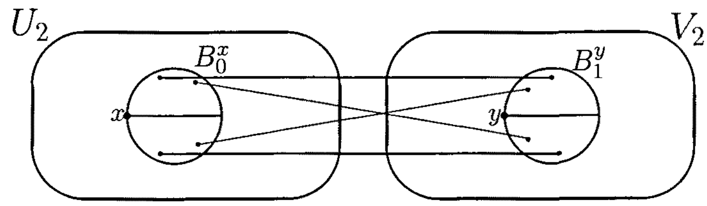

Figure 3.5: Intercopy edges of compound graphs $B_0 \nabla B_1$ (non-bipartite case).

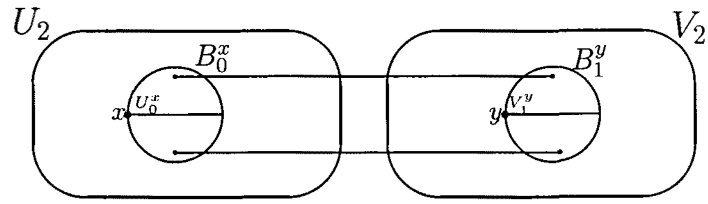

Figure 3.6: Intercopy edges of compound graphs $B_0 \nabla B_1$ (bipartite case).

Proposition 3.3.7 *Let B_i, $i \in \{0,1\}$, be two bipartite (Δ_i, D_i)-graphs of order $2n_i$ with partite sets $\{0\} \times J_{n_i}$ and $\{1\} \times J_{n_i}$. Let m_0, m_1, k be three positive integers such that $k \geq 4$ is an even number. The (non-bipartite) graph $B_0 \nabla B_1\{m_1, m_0, k\}G$ is isomorphic to the compound graph $G_2[B_0, B_1]$, where $G_2 = UG^{III}(m_0, n_0, m_1, n_1, k-1)$; that is, G_2 is bipartite with partite sets U_2 and V_2, and the adjacency rules producing four intercopy edges (see Figure 3.5) are:*

$$(\gamma_1, y_1)(\beta_2, x_2)\dots(\gamma_{k-1}, y_{k-1})(\alpha, x_k) \sim (\beta_0, x_k)(\gamma_1, y_1)\dots(\beta_{k-2}, x_{k-2})(\alpha', y_{k-1})$$

for any $(\gamma_1, y_1)(\beta_2, x_2)\dots(\gamma_{k-1}, y_{k-1}) \in U_2$, $(\alpha, x_k) \in \{0,1\} \times J_{n_0}$, $\beta_0 \in J_{m_0}$, $\alpha' \in \{0,1\}$;

$$(\beta_1, x_1)(\gamma_2, y_2)\dots(\beta_{k-1}, x_{k-1})(\alpha, y_k) \sim (\gamma_0, y_k)(\beta_1, x_1)\dots(\gamma_{k-2}, y_{k-2})(\alpha', x_{k-1})$$

for any $(\beta_1, x_1)(\gamma_2, y_2)\dots(\beta_{k-1}, x_{k-1}) \in V_2$, $(\alpha, y_k) \in \{0,1\} \times J_{n_1}$, $\gamma_0 \in J_{m_1}$, $\alpha' \in \{0,1\}$.
Its maximum degree, diameter and order are:

$$(\Delta, D) = (max\{\Delta_0 + 4m_0, \Delta_1 + 4m_1\}, \frac{k(D_0 + D_1)}{2}),$$
$$N = 2(m_0 + m_1)(m_0 m_1)^{\frac{k}{2}-1}(n_0 n_1)^{\frac{k}{2}}).$$

Proposition 3.3.8 *Let B_i, $i \in \{0,1\}$, be two bipartite (Δ_i, D_i)-graphs of order $2n_i$, with partite sets $\{0\} \times J_{n_i}$ and $\{1\} \times J_{n_i}$. Let m_0, m_1, k be three positive integers such that $k \geq 4$ is an even number. The bipartite graph $B_0 \nabla B_1\{m_1, m_0, k\}G$ is isomorphic to the compound graph $G_2[B_0, B_1]$, where $G_2 = UG^{III}(m_0, n_0, m_1, n_1, k-1)$; that is, G_2 is bipartite with partite sets U_2 and V_2, and the adjacency rules producing two intercopy edges (see Figure 3.6) are:*

$$(\gamma_1, y_1)(\beta_2, x_2)\dots(\gamma_{k-1}, y_{k-1})(\alpha, x_k) \sim (\beta_0, x_k)(\gamma_1, y_1)\dots(\beta_{k-2}, x_{k-2})(\bar{\alpha}, y_{k-1})$$

for any $(\gamma_1, y_1)(\beta_2, x_2)\ldots(\gamma_{k-1}, y_{k-1}) \in U_2$, $(\alpha, x_k) \in \{0,1\} \times J_{n_0}$, $\beta_0 \in J_{m_0}$, $\bar{\alpha} = \alpha + 1$ *in* Z_2;

$$(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1})(\alpha, y_k) \sim (\gamma_0, y_k)(\beta_1, x_1)\ldots(\gamma_{k-2}, y_{k-2})(\bar{\alpha}, x_{k-1})$$

for any $(\beta_1, x_1)(\gamma_2, y_2)\ldots(\beta_{k-1}, x_{k-1}) \in V_2$, $(\alpha, y_k) \in \{0,1\} \times J_{n_1}$, $\gamma_0 \in J_{m_1}$, $\bar{\alpha} = \alpha + 1$ *in* Z_2.
Its degree, diameter and order are:

$$(\Delta, D) = (max\{\Delta_0 + 2m_0, \Delta_1 + 2m_1\}, \frac{k(D_0 + D_1) + 2}{2}),$$
$$N = 2(m_0 + m_1)(m_0 m_1)^{\frac{k}{2}-1}(n_0 n_1)^{\frac{k}{2}}).$$

3.4 The $(\Delta, D, D', 1)$-problem in the GC graphs

In this section, the 1-vertex vulnerability of the GC graphs is studied. As it has been shown in the foregoing section, any GC graph can be regarded either as a compound graph $G_2[G_1]$ if it is of type I or II, or as a compound graph $G_2[G_0, G_1]$ if it is of type III. In both cases the graph G_2 will be referred as to be the *main graph* of the compounding, whereas the graphs G_0 and G_1 will be called *auxiliary graphs.*

Remark 3.4.1 In what follows, we will only consider, on the one hand, main graphs satisfying the hypotheses of Theorem 4.3.1, and on the other, auxiliary graphs with minimum degree δ at least two.

Lemma 3.4.1 *Let u, v, w be three vertices belonging to the same copy of a GC graph G. Let D_1 denote the diameter of any copy if G is of type I or II, and the maximum diameter of both copies if G is of type III. Then there exists a $u - v$ path avoiding w, whose length is less than or equal to:*

1. *$D_1 + 6$, if G is of type I or II [resp. III] with $m = 1$ [resp. $m_0 = 1$ or $m_1 = 1$];*
2. *$D_1 + 4$, otherwise.*

Proof. Taking into account that both in 1. and 2. all the proofs are very similar, we shall show in each case one of them.

1. Let us assume that $G \cong G_2[B_0, B_1]$ (non-bipartite case), where the main graph is $G_2 = UG^{III}(1, n_0, m_1, n_1, k-1)$ (see Proposition 3.3.7). Suppose that $D(B_1) = D_1 \geq D(B_0)$ and that u, v, w belong to a certain copy B_1^x in G; that is, $u = x(0, y_k)$, $v = x(i, y'_k)$ and $w = x(j, y''_k)$, where $i, j \in \{0,1\}$ and

$$x = (\beta, x_1)(\gamma_2, y_2)(\beta, x_3)\ldots(\gamma_{k-2}, y_{k-2})(\beta, x_{k-1}) \in G_2.$$

Since the minimum degree of any copy is at least two, we may take in B_0 a vertex $(1, \hat{x}_{k-1})$ adjacent to $(0, x_{k-1})$ such that $x_{k-1} \neq \hat{x}_{k-1}$. Besides, we may consider in B_1 a short path $\mu = (0, y_k)\ldots(0, y'_k)$. Then, we can take in G the following path joining u and v:

$$
\begin{array}{rcl}
u & = & x_1(\gamma_2, y_2)x_3 \dots (\gamma_{k-2}, y_{k-2})x_{k-1}(0, y_k) \\
& & (\epsilon_k, y_k)x_1(\gamma_2, y_2) \dots (\gamma_{k-2}, y_{k-2})(0, x_{k-1}) \\
& & (\epsilon_k, y_k)x_1(\gamma_2, y_2) \dots (\gamma_{k-2}, y_{k-2})(1, \hat{x}_{k-1}) \\
& & \left.\begin{array}{l} x_1(\gamma_2, y_2)x_3 \dots (\gamma_{k-2}, y_{k-2})\hat{x}_{k-1}(0, y_k) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_1(\gamma_2, y_2)x_3 \dots (\gamma_{k-2}, y_{k-2})\hat{x}_{k-1}(0, y'_k) \end{array}\right\} \le D_1 \\
& & (\epsilon_k, y'_k)x_1(\gamma_2, y_2) \dots (\gamma_{k-2}, y_{k-2})(1, \hat{x}_{k-1}) \\
& & (\epsilon_k, y'_k)x_1(\gamma_2, y_2) \dots (\gamma_{k-2}, y_{k-2})(0, x_{k-1}) \\
v & = & x_1(\gamma_2, y_2)x_3 \dots (\gamma_{k-2}, y_{k-2})x_{k-1}(i, y'_k)
\end{array}
$$

Since $x_{k-1} \neq \hat{x}_{k-1}$, this path avoids w and its length is at most $D_1 + 6$.

2. Suppose that $G = B_1(m,k)G \cong G_2[B_1]$ is a bipartite compound graph of type II such that $m \geq 2$ and $D_1 = D(B_1)$ is even (see Proposition 3.3.4). Since u, v, w belong to the same copy B_1^x in G, we can denote them by $u = x(0, x_k)$, $v = x(\alpha, y_k)$ and $w = x(j, z_k)$, where $\alpha, j \in \{0, 1\}$ and $x = (\beta_1, x_1)(\beta_2, x_2) \dots (\beta_{k-1}, x_{k-1})$ is a vertex of G_2. Let $\mu = (0, x_k) \dots (\bar{\alpha}, \bar{x}_k)(\alpha, y_k)$ be a short path in B_1 joining $(0, x_k)$ to (α, y_k) (notice that $d((0, x_k), (\bar{\alpha}, \bar{x}_k)) \leq D_1 - 1$). Let us consider in G the following path η joining u and v:

$$
\begin{array}{rcl}
u & = & (\beta_1, x_1)(\beta_2, x_2) \dots (\beta_{k-1}, x_{k-1})(0, x_k) \\
& & (\epsilon_k, x'_k)(\beta_1, x_1) \dots (\beta_{k-2}, x_{k-2})(1, x_{k-1}) \\
& & \left.\begin{array}{l} (\beta_1, x_1)(\beta_2, x_2) \dots (\epsilon_{k-1}, x_{k-1})(0, x_k) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (\beta_1, x_1)(\beta_2, x_2) \dots (\epsilon_{k-1}, x_{k-1})(0, \tilde{x}_k) \end{array}\right\} \le D_1 \\
& & (\epsilon_k, \tilde{x}'_k)(\beta_1, x_1) \dots (\beta_{k-2}, x_{k-2})(1, x_{k-1}) \\
& & (\beta_1, x_1)(\beta_2, x_2) \dots (\beta_{k-1}, x_{k-1})(0, \tilde{x}_k) \\
v & = & (\beta_1, x_1)(\beta_2, x_2) \dots (\beta_{k-1}, x_{k-1})(\alpha, y_k)
\end{array}
$$

where $\tilde{x}_k = y_k$ if $\alpha = 0$, and $\tilde{x}_k = \bar{x}_k$ if $\alpha = 1$. Notice that, if $\epsilon_k \neq \beta_1$ and $\epsilon_{k-1} \neq \beta_{k-1}$, then the path η avoids w and its length is at most $D_1 + 4$ since $m \geq 2$. ■

Lemma 3.4.2 *Let u, v, w be three vertices belonging to three different copies of a GG graph G with diameter D. Then, there exists a $u - v$ path avoiding w, whose length is less than or equal to:*

1. *D, if $G \notin \{B_1\{m,k\}G\} \cup \{B_1(m,k)G : D_1$ even$\}$.*
2. *$D + 1$, otherwise.*

Proof. The proof is based on the fact that the graph G is isomorphic to $G_2[G_1]$ or $G_2[G_0, G_1]$, where G_2 is isomorphic to the underlying graph of an iterated line digraph which is a $(\Delta_2, D_2, D_2, 1)$-graph (see Theorem 4.3.1).

Let us assume that $G \notin \{B_1\{m,k\}G, B_1(m,k)G : D_1$ *even*$\}$. To see point 1, we consider two cases:

(1.a) Let G be a graph of type I, type II with $m = 1$ or type III. The proof for any of these graphs is similar, and they are based on the following facts:

(i) The diameter of G attains the upper bound given in Proposition 1.5.2.

(ii) If at least one copy is a bipartite graph, then there are two or more intercopy edges placed appropriately between two copies corresponding to adjacent vertices in G_2, because D_1 is odd (see Figures 3.1, 3.4, 3.5, 3.6).

For the sake of simplicity, let us assume that $G = G_1\{m, k\}G$. Every path ρ in G_2 induces in G a path η_ρ, whose length depends on the length of ρ, on the diameter of G_1 and on the intercopy edges (see Figure 3.7). If we denote $u = xx_k$, $v = yy_k$ and $w = zz_k$, then it follows that there exists in G_2 a path ρ joining x and y which does not go through z, whose length is at most $D_2 = k - 1$. Therefore, and keeping in mind (i), η_ρ is a path between u and v avoiding w, whose length is at most D.

Notice that if G is a GG graph whose copies are bipartite graphs, then there exists an intercopy edge from each partite set. Starting from this fact, it follows that the contribution of the subpath contained in any copy of $B_1^{a_i}$ to the length of η_ρ is less than or equal to $D_1 - 1$.

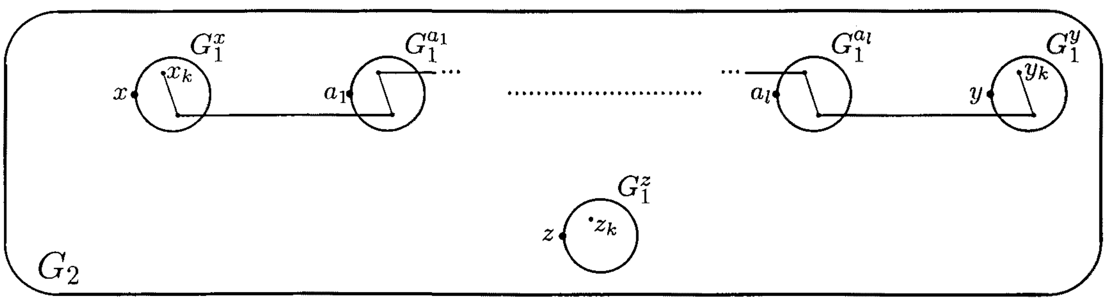

Figure 3.7: Path η_ρ induced in G by a path $\rho : xa_1 \dots a_l y$ of G_2.

(1.b) Suppose now that G is a GG graph of type II with $m \geq 2$. Notice that for such a graph the condition (ii) of the above case still holds, but not (i). For instance, the diameter of the GG graph $FF(m, k)G$ is $D = kD_1 - 1$, which is less than the upper bound obtained in Proposition 1.5.2 for this case; namely, $D_1D_2+D_1-1 = (k+1)D_1-1$. The proofs for the three families of type II are very similar, and for this reason we will only show one of them. Let G be the graph $G_1(m, k)G$ where $G_2 = UG^{II}(m, n, k-1)$. Observe that, according to Propositions 3.2.1 and 3.3.3, the graph G_2 has diameter $D_2 = k$ and G is isomorphic to the compound graph $G_2[G_1]$. Let $u = xx_k$, $v = yy_k$ and $w = zz_k$ be three vertices placed in three different copies, where:

$$\begin{aligned} x &= (\beta_1, x_1)(\beta_2, x_2) \dots (\beta_{k-1}, x_{k-1}) \\ y &= (\gamma_1, y_1)(\gamma_2, y_2) \dots (\gamma_{k-1}, y_{k-1}) \end{aligned}$$

are vertices of G_2. We distinguish four cases:

(b1) If $x_1 \neq y_{k-1}$ and $x_{k-1} \neq y_1$, then the digraph $G^{II}(m,n,k-1)$ must contain two paths of length at most $k-1$, one from x to y and the other from y to x:

$$P_{xy} : xa_1 \ldots a_r y, \qquad P_{yx} : yb_1 \ldots b_s x$$

which induce two paths in G joining u and v, of length at most $kD_1 + k - 1 = D$. Suppose that an internal vertex z lies in both P_{xy} and P_{yx}:

$$a_i = b_j = z \Rightarrow z \in \Gamma^+(a_{i-1}) \cap \Gamma^+(b_{j-1}),\ z \in \Gamma^-(a_{i+1}) \cap \Gamma^-(b_{j+1}).$$

Since $G^{II}(m,n,k-1)$ is a Δ-regular line digraph with $\Delta \geq 2$, by applying the Heuchenne condition, we obtain that $\Gamma^+(a_{i-1}) = \Gamma^+(b_{j-1})$, $\Gamma^-(a_{i+1}) = \Gamma^-(b_{j+1})$. So, we can consider $z_1 \in \Gamma^+(a_{i-1}) \setminus \{z\}$, $z_2 \in \Gamma^-(a_{i+1}) \setminus \{z\}$. Hence, G_2 has the following two paths:

$$xa_1 \ldots a_{i-1} z_1 b_{j-1} \ldots b_1 y, \quad xb_s \ldots b_{j+1} z_2 a_{i+1} \ldots a_r y$$

which do not contain the vertex z, and at least one of them is of length less than or equal to $k-1$, because $r+s \leq 2k-2$. Therefore, its corresponding induced path in G avoids w and its length is at most D.

(b2) If $x_1 = y_{k-1}$ and $x_{k-1} \neq y_1$, then $G^{II}(m,n,k-1)$ contains a path P_{yx} of length $k-1$. Moreover, if $\beta_1 = \gamma_{k-1}$, then there exists a path P_{xy} in $G^{II}(m,n,k-1)$ of length at most $k-2$ and the reasoning is the same as in (b1). If $\beta_1 \neq \gamma_{k-1}$, then we consider the following path in G:

$$\begin{array}{rcrcl} u & = & xx_k & = & (\beta_1, y_{k-1})(\beta_2, x_2) \ldots (\beta_{k-1}, x_{k-1}) x_k \\ \tilde{u} & = & \tilde{x}x'_1 & = & (\beta_2, x_2)(\beta_3, x_3) \ldots (\epsilon_k, x_k) x'_1 \\ \hat{u} & = & \hat{x}x_k & = & (\gamma_{k-1}, y_{k-1})(\beta_2, x_2) \ldots (\beta_{k-1}, x_{k-1}) x_k \end{array}$$

choosing ϵ_k in such a way that $\tilde{x} \neq z$. Therefore, we can find in $G^{II}(m,n,k-1)$ another path $P_{\hat{x}y}$ of length at most $k-2$. These two directed paths P_{yx} and $P_{\hat{x}y}$ induce two paths in the graph G between u and v, of length at most D. If the vertex z lies in both of them, then, reasoning as in case (b1), two paths in G_2 are obtained :

$$x\tilde{x}\hat{x}a_3 \ldots a_{i-1} z_1 b_{j-1} \ldots b_1 y, \quad xb_s \ldots b_{j+1} z_2 a_{i+1} \ldots a_r y$$

So, the length of one of them is at most $k-1$, which induces a path in G avoiding w of length at most D.

(b3) If $x_1 \neq y_{k-1}$ and $x_{k-1} = y_1$, then the proof is as in (b2).

(b4) Suppose that $x_1 = y_{k-1}$ and $x_{k-1} = y_1$. If $\beta_1 = \gamma_{k-1}$ or $\beta_{k-1} = \gamma_1$ then the reasoning is as in (b2). If $\beta_1 \neq \gamma_{k-1}$ and $\beta_{k-1} \neq \gamma_1$, we consider in G the following $u - \hat{u}$ and $v - \hat{v}$ paths:

$$\begin{array}{rcrcl} u & = & xx_k & = & (\beta_1, y_{k-1})(\beta_2, x_2) \ldots (\beta_{k-1}, x_{k-1}) x_k \\ \tilde{u} & = & \tilde{x}x'_1 & = & (\beta_2, x_2)(\beta_3, x_3) \ldots (\epsilon_k, x_k) x'_1 \\ \hat{u} & = & \hat{x}x_k & = & (\gamma_{k-1}, y_{k-1})(\beta_2, x_2) \ldots (\beta_{k-1}, x_{k-1}) x_k \end{array}$$

$$\begin{array}{rclcr} v & = & yy_k & = & (\gamma_1, x_{k-1})(\gamma_2, y_2)\dots(\gamma_{k-1}, y_{k-1})y_k \\ \tilde{v}' & = & \tilde{y}y'_1 & = & (\gamma_2, y_2)(\gamma_3, y_3)\dots(\epsilon_0, y_k)y'_1 \\ \hat{v} & = & \hat{y}y_k & = & (\beta_{k-1}, x_{k-1})(\gamma_2, y_2)\dots(\gamma_{k-1}, y_{k-1})y_k \end{array}$$

chossing ϵ_k and ϵ_0 in such a way that $z \notin \{\tilde{x}, \tilde{y}\}$. Thus, $G^{II}(m, n, k-1)$ must contain two paths of length at most $k-2$, one from $\hat{x}$ to y and the other from $\hat{y}$ to x:

$$P_{\hat{x}y} : \hat{x}a_3 \dots a_r y, \qquad P_{\hat{y}x} : \hat{y}b_3 \dots b_s x$$

which induce two paths in G of length at most D joining u and v. If $P_{\hat{x}y}$ and $P_{\hat{y}x}$ have a vertex z in common (see Figure 3.8), reasoning as in (b1), two paths of G_2 are obtained:

$$\rho_1 : x\tilde{x}\hat{x}a_3 \dots a_{i-1}z_1b_{j-1} \dots b_3\hat{y}\tilde{y}y, \quad \rho_2 : xb_s \dots b_{j+1}z_2a_{i+1} \dots a_r y$$

If the length of ρ_2 is at most $k-1$, then this path induces in G a path avoiding w of length at most D. Finally, if the length of ρ_2 is greater than or equal to k, then the length of the subpath ρ_1 between $\hat{x}$ and $\hat{y}$ must be less than or equal to $k-4$ and therefore, ρ_1 induces in G a path avoiding w, of length at most:

$$2 + (D_1 + 1)(k-4) + D_1 + 2 \;=\; (D_1+1)k - 3D_1 \;<\; (D_1+1)k - 1 \;=\; D.$$

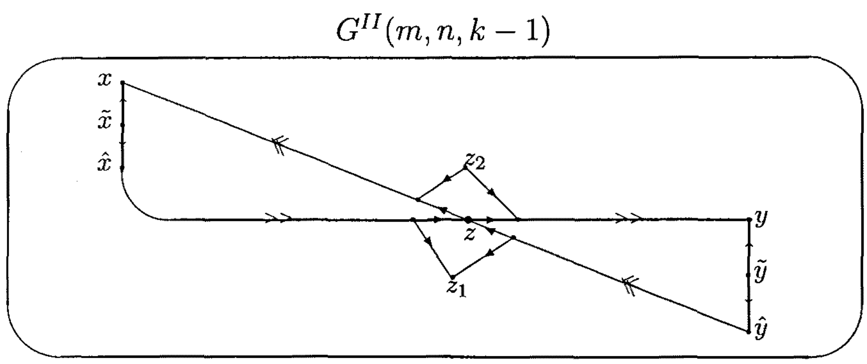

Figure 3.8: Two paths in $G^{II}(m, n, k-1)$ intersecting in z.

2. The two families of GG graphs considered in this point have an essential difference with respect to the other families studied before: there are copies corresponding to adjacent vertices in G_2 which are joined by one only edge (see Figure 3.2). Actually, the proof for $G = B_1\{m, k\}G$ is similar to that of case (a), and the proof for $G = B_1(m, k)G$ is similar to that exposed in (b). But now D_1 is even, and so it happens the following: Let $u = x(0, x_k)$, $v = y(1, y_k)$ and $w = z(i, z_k)$ be vertices of G such that $d(x, y) = k-1$ in the graph $G_2 = UG^{II}(m, n, k-1)$. Let $\rho : xx_1 \dots x_{k-2}y$ be a shortest path avoiding

z, and let us consider its induced path η_ρ in G (see Figure 3.9). Notice that the length of η_ρ is at most:

$$D_1 + 1 + D_1(k-2) + D_1 = kD_1 + 1 = D + 1$$

since the vertices of different partite sets are at distance at most $D_1 - 1$ because D_1 is even. Hence, the result of this point holds. ■

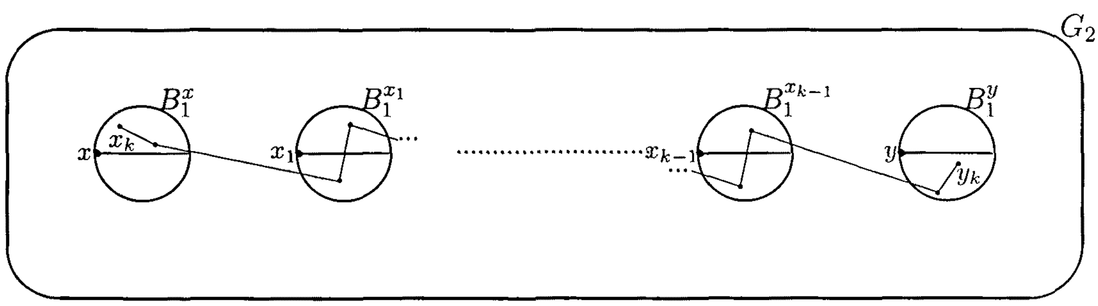

Figure 3.9: Path η_ρ between $u = x(0, x_k)$ and $v = y(1, y_k)$ induced by $\rho : xx_1 \ldots x_{k-2}y$ in G_2.

Lemma 3.4.3 *Let G_1^x and G_1^y be two copies of a GG graph G with diameter D. Let u be a vertex of G_1^x and v, w two vertices of G_1^y. Then, there exists a $u - v$ path avoiding w, whose length is less than or equal to:*

1. *$D + 1$, if $G \notin \{G_1\{1,k\}G\} \cup \{B_1\{1,k\}G\} \cup \{B_1(1,k)G,\ D_1$ even$\}$.*

2. *$D + 2$, otherwise.*

Proof. 1. To prove this point we will distinguish two cases:

(1.a) Assume that $G \notin \{B_1\{m,k\}G, B_1(m,k)G :\ D_1$ *even*$\}$. Let $\tilde{v}$ be a vertex not belonging to the copy G_1^y adjacent to v. From Lemma 3.4.2 it follows that there exists a path between u and $\tilde{v}$ avoiding w, of length less than or equal to D. Hence, by adding the edge $\tilde{v}v$, we obtain a path linking u and v of length at most $D + 1$, which does not contain w.

(1.b) Suppose now that $G \in \{B_1\{m,k\}G, B_1(m,k)G :\ D_1$ *even*, $m \geq 2\}$. Since the proof is similar for both families of graphs, we will just show one of them. Let G be the bipartite compound graph of type II $B_1(m,k)G$ where $m \geq 2$ and $D_1 = D(B_1)$ is even (see Proposition 3.3.4). Let $u = x(1, x_k)$, $v = y(i, y_k)$ and $w = y(j, z_k)$. If $i = 0$ then the proof is the same as in case (a). If $i = 1$, then we can take the following path p_{uv}^{ϵ} joining u and v:

$$\begin{array}{rcllr}
u & = & (\beta_1,x_1)(\beta_2,x_2) & \dots & (\beta_{k-1},x_{k-1})(1,x_k) \\
 & & \dots\dots\dots\dots & \dots & \dots\dots\dots\dots\dots\dots \\
 & & (\beta_1,x_1)(\beta_2,x_2) & \dots & (\beta_{k-1},x_{k-1})(0,y'_k) \\
 & & (\epsilon,y_k)(\beta_1,x_1) & \dots & (\beta_{k-2},x_{k-2})(1,x_{k-1}) \\
 & & \dots\dots\dots\dots & \dots & \dots\dots\dots\dots\dots\dots \\
 & & (\epsilon,y_k)(\beta_1,x_1) & \dots & (\beta_{k-2},x_{k-2})(0,y'_{k-1}) \\
 & & (\gamma_{k-1},y_{k-1})(\epsilon,y_k) & \dots & (\beta_{k-3},x_{k-3})(1,x_{k-2}) \\
 & & \dots\dots\dots\dots & \dots & \dots\dots\dots\dots\dots\dots \\
 & & \dots\dots\dots\dots & \dots & \dots\dots\dots\dots\dots\dots \\
 & & (\gamma_2,y_2)(\gamma_3,y_3) & \dots & (\epsilon,y_k)(1,x_1) \\
 & & \dots\dots\dots\dots & \dots & \dots\dots\dots\dots\dots\dots \\
 & & (\gamma_2,y_2)(\gamma_3,y_3) & \dots & (\epsilon,y_k)(0,y'_1) \\
v & = & (\gamma_1,y_1)(\gamma_2,y_2) & \dots & (\gamma_{k-1},y_{k-1})(1,y_k)
\end{array}$$

Notice that the length of p^{ϵ}_{uv} is at most $kD_1 = D$, because D_1 is even. If $m \geq k$, then, taking $\epsilon \notin \{\gamma_1,\dots,\gamma_{k-1}\}$, p^{ϵ}_{uv} is a $u-v$ path avoiding w. If $m < k$, then we can consider in the digraph $G^{II}(m,n,k-1)$ the path $\rho : xa_1 \dots a_{k-1}y$, which induces in G the path p^{ϵ}_{uv}. If $y \notin \{a_1,\dots,a_{k-1}\}$, then p^{ϵ}_{uv} is a $u-v$ path avoiding w. Finally, if $y \in \{a_1,\dots,a_{k-1}\}$, taking $\epsilon \neq \gamma_{k-2}$, then $y \in \{a_1,\dots,a_{k-3}\}$. Let us

$G^{II}(m,n,k-1)$

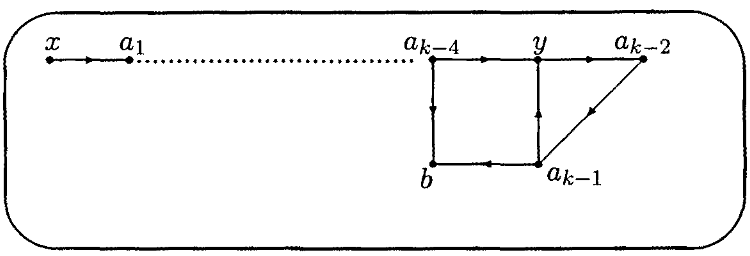

Figure 3.10: Paths ρ and μ in $G^{II}(m,n,k-1)$.

assume that $y = a_{k-3}$ (the proof in any other case is similar). Thus, it follows that $y \in \Gamma^+(a_{k-4}) \cap \Gamma^+(a_{k-1})$ and, from Heuchenne conditions, we conclude that there exists another vertex $b \in \Gamma^+(a_{k-4}) \cap \Gamma^+(a_{k-1})$ (see Figure 3.10). Let us consider the path η_μ induced by the path $\mu : xa_1 \dots a_{k-4}ba_{k-1}y$ of $G_2 = UG^{II}(m,n,k-1)$. Clearly η_μ is a $u-v$ path avoiding w, whose length is at most $k(D_1-1)+D_1+1 = D+1$.

2. Suppose first that $G = G_1\{1,k\}G$ and $v = (\beta,a)(\beta,a)\dots(\beta,a)a$. This vertex has just adjacent vertices inside its own copy G_1^y, say $\hat{v} = (\beta,a)(\beta,a)\dots(\beta,a)b$, $b \neq a$. Since the minimum degree of any copy is at least two, we can suppose that $\hat{v} \neq w$. Now we can take a vertex $\tilde{v}$ not belonging to the copy G_1^y adjacent to $\hat{v}$. From Lemma 3.4.2, it follows that there exists a path between u and $\tilde{v}$ of length less than or equal to D, avoiding w. Hence, by adding the path $\tilde{v}\hat{v}v$, we obtain a path linking u and v of length at most $D+2$, which does not contain w. Finally, if $G = B_1(1,k)G$ with D_1

even, then the reasoning is similar to that of case (1.b), but now D_1 is even and so we lose one unit. ■

The main result of this work is put forward in the following theorem, whose proof is a direct consequence of the above lemmas.

Theorem 3.4.1 *If G is a GG graph with diameter D satisfying the restrictions imposed in Remark 3.4.1, then it is a $(\Delta, D, D', 1)$-graph where:*

1. $D' = D + 1$, *if* $G \notin \{G_1\{1,k\}G\} \cup \{B_1\{1,k\}G\} \cup \{B_1(1,k)G,\ D_1 \textit{ even}\}$.
2. $D' = D + 2$, *otherwise.*

Chapter 4

Connectivity and superconnectivity of generalized p-cycles

4.1 Introduction

One of the most important topics in the study of the fault tolerance or reliability of a certain family Υ of digraphs is connectedness. One usual procedure of approaching this study is as follows. First of all, finding sufficient conditions to assure maximal connectivitity. Going one step further, the following goal is to obtain sufficient conditions for a maximally connected digraph of Υ to be superconnected. Finally, the study is completed by carrying out a similar work starting from the superconnected digraphs of Υ; that is, by searching sufficient conditions for a superconnected digraph to have minimum nontrivial cutsets large enough.

Moreover, as it was indicated in Section 1.7.1, there are three main types of sufficient conditions: degree, order and diameter conditions. This means that the aforementioned *procedure* can be implemented one or more times. For instance, the whole work can be reduced to find both diameter and order conditions to assure maximal connectivity, superconnectivity and 'good' superconnectivity respectively. Apart from the considerations indicated so far, two more questions have to be taken into account. The first of them is to choose the type of disconnecting sets which we are interested in; namely, vertex- or arc-cutsets. In fact, it is well known that both for graphs and digraphs the arc-connectivity has historically been the first and main subject of study, as is shown in Propositions 1.7.2 and 1.7.3. The second and last question to consider is the selection of suitable digraph parameters to be used. For instance, the parameter ℓ (see Section 1.6) has proved to be an excellent tool to study connectedness properties in digraphs.

The first work on connectedness in digraphs carried out by following the steps previously put forward is credited to J. Fàbrega and M. A. Fiol. As is shown in the following proposition, the study was done, on the one hand, by using the parameter ℓ, and on the other, both under diameter and order conditions.

Proposition 4.1.1 *Let G be a $(\delta, \Delta, D, n, \ell)$-digraph such that $\delta \geq 3$.*[1]

1. *([54, 62]) $\kappa = \delta$ if either $D \leq 2\ell - 1$ or $n > \frac{(\delta-1)(\Delta^{D-\ell+1}+\Delta^{\ell}-2\Delta)}{\Delta-1} + \Delta^{\ell} + 1$.*

2. *([54, 62]) $\lambda = \delta$ if either $D \leq 2\ell$ or $n > \frac{(\delta-1)(\Delta^{D-\ell}+\Delta^{\ell}-2)}{\Delta-1} + \Delta^{\ell}$.*

3. *([5, 54, 63]) $\kappa_1 > \delta$ if either $D \leq 2\ell - 2$ or $n > \frac{\delta(\Delta^{D-\ell+1}+\Delta^{\ell}-2\Delta)}{\Delta-1} + \Delta^{D-\ell+1}$.*

4. *([54, 63]) $\lambda_1 > \delta$ if either $D \leq 2\ell - 1$ or $n > \frac{\delta(\Delta^{D-\ell}+\Delta^{\ell}-2)}{\Delta-1} + \Delta^{D-\ell}$.*

Similar results were derived for bipartite digraphs by C. Balbuena, A. Carmona, J. Fàbrega and M. A. Fiol, as the next proposition states.

Proposition 4.1.2 *Let G be a $(\delta, \Delta, D, n, \ell)$-bipartite digraph such that $\delta \geq 3$.*[2]

1. *([9, 55, 65]) $\kappa = \delta$ if either $D \leq 2\ell$ or $n > \frac{(\delta-1)(\Delta^{D-\ell}+\Delta^{\ell+1}-2\Delta)}{\Delta-1} + 2$.*

2. *([9, 55, 65]) $\lambda = \delta$ if either $D \leq 2\ell + 1$ or $n > \frac{(\delta-1)(\Delta^{D-\ell-1}+\Delta^{\ell+1}-2)}{\Delta-1}$.*

3. *([11, 55]) $\kappa_1 > \delta$ if either $D \leq 2\ell - 1$ or $n > \frac{\delta(\Delta^{D-\ell}+\Delta^{\ell}-2\Delta)}{\Delta-1} + \Delta^{D-\ell} + \Delta^{\ell}$.*

4. *([11, 55]) $\lambda_1 > \delta$ if either $D \leq 2\ell$ or $n > \frac{\delta(\Delta^{D-\ell-1}+\Delta^{\ell}-2)}{\Delta-1} + \Delta^{D-\ell-1} + \Delta^{\ell}$.*

As it was indicated in Section 1.5.4, every digraph can be seen as a generalized p-cycle with $p = 1$, whereas the bipartite digraphs coincide with the generalized 2-cycles. This chapter is devoted to generalizing similar results to the previous ones for the family of generalized p-cycles (see Section 1.5.4) for $p \geq 3$, following the aforementioned procedure. As for the terminology and notation employed, we have made use of the Hamidoune terminology presented in Section 1.7.1 (see also [10, 61]).

4.2 Maximal connectivity

In this section the first two points of Propositions 4.1.1 and 4.1.2 are generalized. To be more precise, we obtain both diameter and order sufficient conditions for a generalized p-cycle to be maximally connected [arc-connected].

[1] Points *1.* and *2.* are also verified for $\delta = 2$. In point *3.* [*4.*], if $D \leq 2\ell - 2$ [$D \leq 2\ell - 1$], then $\kappa_1 \geq 2\delta - 2$ [$\lambda_1 \geq 2\delta - 2$].

[2] Points *1.* and *2.* are also verified for $\delta = 2$. In point *3.* [*4.*], if $D \leq 2\ell - 1$ [$D \leq 2\ell$], then $\kappa_1 \geq 2\delta - 2$ [$\lambda_1 \geq 2\delta - 2$].

4.2.1 Diameter conditions

In [54], it was implicitly shown that the parameter ℓ is related to the deepness of any fragment or α-fragment. For the sake of convenience we repeat the proof of this useful fact in the following lemma.

Lemma 4.2.1 *Let G be a digraph with parameter ℓ, minimum degree $\delta \geq 2$, diameter D and connectivities κ and λ. Let F denote a positive fragment or α-fragment of G. Then,*

(a) if $\kappa < \delta$, then $\mu(F) \geq \ell$ and $\mu(\overline{F}) \geq \ell$;

(b) if $\lambda < \delta$, then $\nu(F) \geq \ell$ and $\nu(V \setminus F) \geq \ell$.

Proof. Let F be a positive fragment; that is, $|\partial^+ F| = \kappa \leq \delta - 1$, and assume that $\mu(F) \leq \ell - 1$. Let x be a vertex of F such that $d(x, \partial^+ F) = \mu(F)$ and consider δ of its out-neighbours, $x_1, x_2, \ldots, x_\delta$. For each x_i, let f_i be a vertex in $\partial^+(F)$ at minimum distance from x_i. Hence, $f_i = f_j$ for some $i \neq j$, and then there would be two different $x \to f_i$ paths of length $\ell - 1$ or ℓ, which contradicts the definition of parameter ℓ. Considering the converse digraph of G and recalling that $\overline{F} = V \setminus (F \cup \partial^+ F)$, the assertion $\mu(\overline{F}) \geq \ell$ is immediately proved.

(b) The arc case is similarly proved if $\nu(F) \geq 1$. So, let us see that the assumption $\lambda = |\omega^+ F| < \delta$ implies $\nu(F) > 0$. To this end, observe firstly that certainly $|F| > 1$. Moreover, if we suppose that $\nu(F) = 0$, then $|F| < \delta$ and the number β of arcs which have their initial and final vertices in F satisfies $|F|(|F| - 1) \geq \beta = \sum_{x \in F} \delta^+(x) - |\omega^+ F| \geq |F|\delta - \delta$. But this means that $|F| \geq \delta$, which is a contradiction. ■

As a consequence of the foregoing result and going one step further, it is easy to prove that in every part of a not maximally connected generalized p-cycle there are vertices at distance at least ℓ. This fact is shown in the following lemma.

Lemma 4.2.2 *Let $G = (\cup_{i=0}^{p-1} V_i, A)$ be a connected generalized p-cycle with parameter ℓ, minimum degree $\delta \geq 2$ and connectivity $\kappa < \delta$. If F is a positive [negative] fragment of G, then, for every $i \in \{0, \ldots, p-1\}$, there exists a vertex $x \in V_i$ such that $d(x, \partial^+ F) \geq \ell$ [$d(\partial^- F, x) \geq \ell$].*

Proof. Let us suppose that F is a positive fragment of G (the negative fragment case is proved similarly), and consider the following subset of vertices:

$$F(\ell) = \{x \in F, d(x, \partial^+ F) \geq \ell\},$$

which, according to the previous lemma, is non-empty. Therefore, we can take, for some α, a vertex $x_0 \in F(\ell) \cap V_\alpha$ satisfying $d(x_0, \partial^+ F) = \ell$. Then, $\Gamma^+(x_0) \cap F(\ell) \neq \emptyset$, since otherwise there would be two distinct paths from x_0 to some vertex of $\partial^+ F$ of length ℓ, which contradicts the definition of this parameter. Hence, we can consider a vertex $x_1 \in F(\ell) \cap V_{\alpha+1}$ so that $d(x_1, \partial^+ F) = d(x_1, x_r) + d(x_r, \partial^+ F) = d(x_1, x_r) + \ell$, where certainly $x_r \in V_{\alpha+r}$ for some $1 \leq r$ and so $F(\ell) \cap V_{\alpha+j} \neq \emptyset$ for each $0 \leq j \leq r$. Since $d(x_r, \partial^+ F) = \ell$ we find again that $\Gamma^+(x_r) \cap F(\ell) \neq \emptyset$, and thus $F(\ell) \cap V_{\alpha+r+1} \neq \emptyset$. This shows that in $F(\ell)$ vertices of every partite set must exist. ∎

The main result of this section, which guarantees a generalized p-cycle with diameter small enough to be maximally connected, is shown below.

Theorem 4.2.1 *Let G be a connected generalized p-cycle with parameter ℓ, minimum degree $\delta \geq 2$, diameter D and connectivities κ and λ. Then,*

(a) $\kappa = \delta$ if $D \leq 2\ell + p - 2$;

(b) $\lambda = \delta$ if $D \leq 2\ell + p - 1$.

Proof. Observe that, for p=1, this theorem is a corollary of Lemma 4.2.1. Let us thus suppose that $p \geq 2$.[3]

(a) Assume that $\kappa < \delta$ and let F be a positive fragment (see Lemma 1.7.1). As every path from F to $\overline{F} = V \setminus (F \cup \partial^+ F)$ goes through $\partial^+ F$, we can consider a vertex $x \in F$ and a vertex $y \in \overline{F}$ so that $d(x, y) \geq d(x, \partial^+ F) + d(\partial^+ F, y) = \mu(F) + \mu(\overline{F})$, where $\mu(F)$ and $\mu(\overline{F})$ are the deepness of the positive and negative fragment F and $\overline{F}$ respectively. As a consequence, we conclude that $D \geq \mu(F) + \mu(\overline{F})$. Furthermore, from Lemma 4.2.1, it follows that $\ell \leq \mu(F)$ and $\ell \leq \mu(\overline{F})$, and so we can consider the non-empty sets:

$$F(\ell) = \{x \in F, d(x, \partial^+ F) \geq \ell\}, \quad \overline{F}(\ell) = \{y \in \overline{F}, d(\partial^+ F, y) \geq \ell\}.$$

As G is a p-cycle, its set of vertices can be partitioned into p parts, $V = \bigcup_{\alpha \in Z_p} V_\alpha$, in such a way that the vertices in the partite set V_α are only adjacent to vertices in $V_{\alpha+1}$, where the sum is in Z_p. From Lemma 4.2.2 we can ensure that for each $0 \leq \alpha \leq p-1$, $F(\ell) \cap V_\alpha \neq \emptyset$ and $\overline{F}(\ell) \cap V_\alpha \neq \emptyset$.

Now, let us consider the integer r, $0 \leq r \leq p-1$, so that $D + 1 \equiv r (\text{mod } p)$. If $x \in F(\ell) \cap V_\alpha$, then, for every vertex $y \in \overline{F}(\ell) \cap V_{\alpha+r}$, there exists an integer $h \geq 1$ such that $2\ell \leq d(x, y) = D + 1 - hp \leq D - p + 1$, because the length of every path from V_α to $V_{\alpha+r}$ is congruent with r modulo p. But this means that $D \geq 2\ell + p - 1$, which contradicts the hypothesis.

(b) Suppose that there is a generalized p-cycle with $\delta > 1$, parameter ℓ, arc-connectivity $\lambda < \delta$ and $D \leq 2\ell + p - 1$. Thus, according to Propositions 1.5.3 and 1.6.1, its line digraph LG would be a generalized p-cycle with the same minimum degree, parameter $\ell' = \ell + 1$, vertex-connectivity $\kappa' = \lambda < \delta$ and diameter $D' = D + 1 \leq 2\ell + p = 2\ell' + p - 2$, contradicting (a). ∎

[3] In fact, this theorem was proved, for $p = 1$ and $p = 2$, in [54] and [55] respectively.

As a consequence of the above proof, it is easy to see that the deepness of the fragments increases by at least one unit if the minimum degree is small enough[4]. This fact is shown in the following lemma.

Lemma 4.2.3 *Let G be a connected generalized p-cycle with parameter ℓ, minimum degree δ satisfying $p \geq \delta \geq 2$, diameter D and connectivities κ and λ. Let F be a positive fragment or α-fragment. Then,*

(a) if $\kappa < \delta$, then $\mu(F) \geq \ell + 1$ and $\mu(\overline{F}) \geq \ell + 1$;

(b) if $\lambda < \delta$, then $\nu(F) \geq \ell + 1$ and $\nu(V \setminus F) \geq \ell + 1$.

Proof. (a) Assume that $\kappa < \delta$ and let F be a positive fragment. Suppose that $\mu(F) = \ell$ and let $F(\ell)$, $\overline{F}(\ell)$ be as in the proof of Theorem 4.2.1. We know that, for each $0 \leq \alpha \leq p-1$, $F(\ell) \cap V_\alpha \neq \emptyset$ and $\overline{F}(\ell) \cap V_\alpha \neq \emptyset$. As $\kappa < \delta$ and $p \geq \delta$, there exists some α so that $V_\alpha \cap \partial^+ F = \emptyset$. Let us now consider a vertex $x \in F(\ell) \cap V_{\alpha-\ell}$. The vertices of $\partial^+ F$ at distance ℓ from x must belong to V_α, and then $V_\alpha \cap \partial^+ F \neq \emptyset$, which is a contradiction, and hence, $\ell + 1 \leq \mu(F)$. Considering the converse digraph of G, we can also prove $\mu(\overline{F}) \geq \ell + 1$.

(b) Let us assume that $\lambda < \delta$ and let F be a positive α-fragment. From Lemma 4.2.1, it follows that $\ell \leq \nu(F)$. Let us consider the non-empty sets $F_0 = \{x \in F : (x,y) \in \omega^+ F\}$ and $\overline{F}_0 = \{y \in V \setminus F : (x,y) \in \omega^+ F\}$ and define the following sets:

$F(\ell) = \{x \in F : d(x, F_0) \geq \ell\}$, $\overline{F}(\ell) = \{y \in V \setminus F : d(\overline{F}_0, y) \geq \ell\}$.

As in the proof of Theorem 4.2.1, we find that, for each $0 \leq \alpha \leq p-1$, $F(\ell) \cap V_\alpha \neq \emptyset$ and $\overline{F}(\ell) \cap V_\alpha \neq \emptyset$. Hence, reasoning as in case (a), we arrive at the desired result. ∎

As a direct consequence of the previous lemma, the lower bounds for the diameter obtained in Theorem 4.2.1 can be slightly improved, as is shown next.

Theorem 4.2.2 *Let G be a connected generalized p-cycle with parameter ℓ, minimum degree δ satisfying $p \geq \delta \geq 2$, diameter D and connectivities κ and λ. Then,*

(a) $\kappa = \delta$ if $D \leq 2\ell + p - 1$;

(b) $\lambda = \delta$ if $D \leq 2\ell + p$.

Proof. (a) Assume that $\kappa < \delta$ and let F be a positive fragment. By Corollary 4.2.3, we can take a vertex $x \in F$ so that $d(x, \partial^+ F) = \mu(F) \geq \ell + 1$. Suppose that $x \in V_\alpha$. Then, we can consider a vertex $y \in \overline{F}(\ell) \cap V_{\alpha+D+1}$ in such a way that $2\ell + 1 \leq d(x,y) \leq D - p + 1$. This leads to $D \geq 2\ell + p$, which contradicts the hypothesis.

The proof of (b) is identical as that of previous theorem. ∎

Finally, from the previous results and taking into account known properties of generalized p-cycles (see Propositions 1.5.3 and 1.6.1), the following sufficient condition for a k-iterated line digraph to be maximally connected is obtained.

[4]This lemma was implicitly proved, for bipartite digraphs with minimum degree $\delta = 2$, in [9].

Corollary 4.2.1 *Let G be a connected generalized p-cycle, $p \geq 2$, with parameter ℓ, minimum degree $\delta \geq 2$, diameter D and connectivities κ and λ. Then, for every integer $k \geq 1$, L^kG is a generalized p-cycle with parameter $\ell + k$, minimum degree δ and diameter $D + k$, whose connectivities verify:*

(a) $\kappa(L^kG) = \delta$ *if* $k \geq D - 2\ell - p + 2$;
$\lambda(L^kG) = \delta$ *if* $k \geq D - 2\ell - p + 1$.

(b) If $p \geq \delta$,
$\kappa(L^kG) = \delta$ *if* $k \geq D - 2\ell - p + 1$;
$\lambda(L^kG) = \delta$ *if* $k \geq D - 2\ell - p$.

4.2.2 Order conditions

This section is devoted to obtaining sufficient conditions which guarantee that a generalized p-cycle with a given maximum degree Δ and diameter D is maximally connected if its order is large enough. To begin with, let us consider the case of vertex-connectivity. First, we compute the minimum and maximum deepness of any positive or negative fragment.

Lemma 4.2.4 *Let G be a generalized p-cycle with minimum degree $\delta \geq 2$, connectivity $\kappa < \delta$, diameter D and parameter ℓ. If F is a positive or negative fragment of G, then*

(i) $\mu(F) \geq \ell$ *and* $\mu(\overline{F}) \leq D - \ell - p + 1$;

(ii) If $p \geq \delta$, *then* $\mu(F) \geq \ell + 1$ *and* $\mu(\overline{F}) \leq D - \ell - p$.

Proof. To prove (i), let F be a positive fragment of G; that is, a subset of vertices such that its positive boundary is a cutset of cardinal $|\partial^+ F| = \kappa$. From Theorem 4.2.1 and Lemma 4.2.1, it is clear that $D \geq 2\ell + p - 1$ and $\mu, \mu' \geq \ell$ respectively, where $\mu = \mu(F)$ and $\mu' = \mu(\overline{F})$. Let us further consider the partition of F [$\overline{F}$] induced by the distance of each one of its vertices to [from] the minimum cutset $\partial^+ F$:

$$\partial^+ F \cup F = \bigcup_{i=0}^{\mu} F_i = \bigcup_{i=0}^{\mu} \{x \in F : d(x, \partial^+ F) = i\}$$

$$[\partial^- \overline{F} \cup \overline{F} = \bigcup_{j=0}^{\mu'} \overline{F}_j = \bigcup_{j=0}^{\mu'} \{x \in \overline{F} : d(x, \partial^+ F) = j\}],$$

where $F_0 = \partial^+ F = \partial^- \overline{F} = \overline{F}_0$. As every path from F to $\overline{F}$ goes through $\partial^+ F$, the distance from any vertex in F_μ to $\overline{F}_{\mu'}$ is at least $\mu + \mu' \leq D$. From Lemma 4.2.2, we have that, for every $\alpha \in \{1, \ldots, p-1\}$, $F(\ell) \cap V_\alpha \neq \emptyset$. Therefore, we can consider a set of vertices $\{x_\alpha\}_{\alpha=0}^{p-1}$ so that $x_\alpha \in V_\alpha \cap F(\ell)$ and $d(x_\alpha, x_{\alpha+1}) = 1$. Suppose that $\mu' \geq D - \ell - p + 2$ and consider a vertex $y \in \overline{F}_{D-\ell-p+2}$. Hence, $d(x_\alpha, y) \geq$

$\ell+D-\ell-p+2 = D-p+2$, for $0 \le \alpha \le p-1$. In particular, $d(x_{p-1}, y) = D-p+2+k$ for some $0 \le k \le p-2$, since $d(x_{p-1}, y) \le D$. Hence, we have two paths from x_k to y; namely, a shortest path of length $d(x_k, y)$, and the path $x_k, x_{k+1}, \ldots, x_{p-1} \to y$ of length $d(x_{p-1}, y)+p-1-k = D+1$. Since the length of these two paths are congruent modulo p and $d(x_k, y) \le D$, we find that $d(x_k, y) = D+1-hp$ for some positive integer h. But this contradicts the fact that $d(x_k, y) \ge D-p+2$. Therefore, $\mu' \le D-\ell-p+1$.

To prove (ii) notice that, from Lemma 4.2.3, it follows that $\mu = \mu(F) \ge \ell+1$, since $p \ge \delta$. Reasoning again as in case (i), we find that $\mu' = \mu(\overline{F}) \le D-\ell-p$. ∎

The next two theorems provide a Moore bound for a certain family of generalized p-cycles, and a sufficient order condition for any digraph of this family to have maximal connectivity respectively. As a matter of fact, the second theorem is just a restatement of the first one.

Theorem 4.2.3 *Let G be a $(\delta, \Delta, D, n, \kappa, \ell)$-generalized p-cycle with $p \ge 3$ and $\delta \ge 2$. Then* [5],

(i) $\kappa < \delta \Rightarrow n \le \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2.$

(ii) If $p \ge \delta$, $\kappa < \delta \Rightarrow n \le \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1 - \Delta^{\ell+2}\}.$

Proof. Let T be a minimum cutset and F a positive fragment of G such that $\partial^+ F = T$. Observe that $|F_i| \le \Delta|F_{i-1}|$, $1 \le i \le \mu$, and $|\overline{F}_j| \le \Delta|\overline{F}_{j-1}|$, $1 \le j \le \mu'$ (see [6]). Notice also that, without loss of generality, we can suppose that $\mu \le \mu'$. Starting from Lemma 4.2.4, we can distinguish two cases:

(a) $\mu' < D-\ell-p+1$ (notice that this is the only case in which $p \ge \delta$.)

(a.1) If $\ell \le \mu \le \ell+p-2$, then:

$$\begin{aligned} n &= \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{\mu'} |\overline{F}_j| - |\partial^+ F| \le \kappa\{n(\Delta, \mu) + n(\Delta, D-\ell-p) - 1\} \\ &\le \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - \kappa\Delta^{D-\ell-p+1} \\ &\le \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2, \end{aligned}$$

because in this case $\ell \le \mu \le \mu' \le D-\ell-p$, hence $D \ge 2\ell+p$, and then $\kappa\Delta^{D-\ell-p+1} \ge \Delta^{\ell+1} \ge \Delta^2 \ge 2(\delta-1)$.

Observe that if $p \ge \delta$, from Lemma 4.2.4 (ii), it follows that $\ell+1 \le \mu \le \mu' \le D-\ell-p$. Hence, $\kappa\Delta^{D-\ell-p+1} \ge \kappa\Delta^{\ell+2}$, and the point (ii) of this theorem is proved.

(a.2) If $\mu \ge \ell+p-1$, then:

$$\begin{aligned} n &= \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{\mu'} |\overline{F}_j| - |\partial^+ F| \le \kappa\{n(\Delta, \mu) + n(\Delta, D-\ell-p) - 1\} \\ &= \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} + \kappa\{\sum_{i=\ell+p-1}^{\mu} \Delta^i - \Delta^{D-\ell-p+1}\} \end{aligned}$$

[5]From now on the expression $\sum_{i=0}^{s} d^i$ is denoted by $n(d, s)$.

[6]See the proof of Lemma 4.2.4 for notation questions.

$$\leq \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - \kappa\Delta^{\ell+2}$$

$$\leq \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2,$$

since $\ell + p - 1 \leq \mu \leq \mu' \leq D - \ell - p$. Hence,

$$\kappa\left(\frac{\Delta^{\mu+1}-\Delta^{\ell+p-1}}{\Delta-1} - \Delta^{D-\ell-p+1}\right) \leq \kappa\left(\frac{\Delta^{D-\ell-p+1}-\Delta^{\ell+p-1}}{\Delta-1} - \Delta^{D-\ell-p+1}\right) =$$
$$\frac{\kappa}{\Delta-1}\left((2-\Delta)\Delta^{D-\ell-p+1} - \Delta^{\ell+p-1}\right) \leq \frac{\kappa}{\Delta-1}(1-\Delta)\Delta^{\ell+p-1} \leq -\kappa\Delta^{\ell+1} \leq -2\delta+2.$$

(b) $\mu' = D - \ell - p + 1$. From Lemma 4.2.4, it follows that $p < \delta$, and so $\delta \geq 3$. On the other hand, since every path from $x \in F_\mu$ to $y \in \overline{F}_{\mu'}$ goes through ∂^+F, then $D \geq d(x,y) \geq d(x, \partial^+F) + d(\partial^+F, y) = \mu + \mu' = \mu + D - \ell - p + 1$. Therefore, $\ell \leq \mu \leq \ell + p - 1$.

(b.1) Suppose that $\mu = \ell + p - 1$ and consider two vertices $x \in F_\mu$ and $y \in \overline{F}_{\mu'}$. Certainly, $d(x,y) \geq \mu + \mu' = \ell + p - 1 + D - \ell - p + 1 = D$. Therefore, if there exists a vertex $x' \in \Gamma^+(x) \cap F_{\ell+p-1}$, then we would have two different paths from x to y, one of length D and the other, $xx' \to y$, of length $D+1$, which is impossible in a generalized p-cycle with $p \geq 2$. As a consequence, for all $x \in F_{\ell+p-1}$, $\Gamma^+(x) \subset F_{\ell+p-2}$, which implies that $|F_{\ell+p-1}| \leq \frac{\Delta}{\delta}|F_{\ell+p-2}|$. In a similar way, it is proved that for any vertex $y \in \overline{F}_{D-\ell-p+1}$, $\Gamma^-(y) \subset \overline{F}_{D-\ell-p}$ and therefore, $|\overline{F}_{D-\ell-p+1}| \leq \frac{\Delta}{\delta}|\overline{F}_{D-\ell-p}|$. In this way we obtain that

$$n = \sum_{i=0}^{\ell+p-2}|F_i| + \sum_{j=0}^{D-\ell-p}|\overline{F}_j| - |\partial^+F| + |F_{\ell+p-1}| + |\overline{F}_{D-\ell-p+1}|$$
$$\leq \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p) - 1\} + \tfrac{\kappa}{\delta}\{\Delta^{\ell+p-1} + \Delta^{D-\ell-p+1}\}$$
$$= \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} + \tfrac{\kappa}{\delta}\{\Delta^{\ell+p-1} + \Delta^{D-\ell-p+1}\} -$$
$$\kappa\Delta^{D-\ell-p+1} \leq \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2,$$

since $\kappa(\frac{1}{\delta}\{\Delta^\mu + \Delta^{\mu'}\} - \Delta^{\mu'}) \leq \kappa(\frac{2}{\delta} - 1)\Delta^{\mu'} \leq -2\delta + 2$, because $2 \leq \ell + 1 \leq \ell + p - 1 = \mu \leq \mu'$ and $\delta \geq 3$.

(b.2) Suppose that $\mu = \ell + p - 2$, and consider two vertices $x \in F_\mu$ and $y \in \overline{F}_{\mu'}$. As every path from $x \in F_\mu$ to y goes through ∂^+F, it must be that $d(x,y) \geq d(x, \partial^+F) + d(\partial^+F, y) = \mu + \mu' = \ell + p - 2 + D - \ell - p + 1 = D - 1$.

In addition, either for all $x \in F_{\ell+p-2}$, $\Gamma^+(x) \subset F_{\ell+p-3}$ or there exists a vertex $x \in F_{\ell+p-2}$ with some out-neighbour $x' \in \Gamma^+(x) \cap F_{\ell+p-2}$. In the first case, we would find that $|F_{\ell+p-2}| \leq \frac{\Delta}{\delta}|F_{\ell+p-3}|$, whereas in the second one, we obtain that $d(x', y) \geq D - 1$. Notice that in this last case, if $d(x,y) = D - 1$, then the length of the path $xx' \to y$ must be $1 + d(x', y) = D - 1 + hp$ for some integer $h \geq 0$. From $d(x', y) \geq D - 1$, it follows that $h \geq 1$, and so $d(x', y) \geq D - 2 + p \geq D + 1$ because $p \geq 3$, which is a contradiction. Hence, the only possibility is that $d(x,y) = D$, $d(x', y) = D - 1$ and $\Gamma^+(x') \subset F_{\ell+p-3}$. We thus have proved that $|F_{\ell+p-2}| \leq \Delta|F_{\ell+p-3}| - (\delta - 1)$. Similarly, it is proved that $|F_{D-\ell-p+1}| \leq \Delta|F_{D-\ell-p}| - (\delta - 1)$. Therefore, the order n of G can be upper-bounded as follows:

$$n = \sum_{i=0}^{\ell+p-3} |F_i| + \sum_{j=0}^{D-\ell-p} |\overline{F}_j| - |\partial^+ F| + |F_{\ell+p-2}| + |\overline{F}_{D-\ell-p+1}|$$
$$\leq \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2.$$

(b.3) Finally, if $\ell \leq \mu \leq \ell + p - 3$, then

$$n = \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{D-\ell-p+1} |\overline{F}_j| - |\partial^+ F|$$
$$\leq \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - \kappa\Delta^{\ell+p-2}$$
$$\leq \kappa\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2.$$

■

Theorem 4.2.4 *Let G be a $(\delta, \Delta, D, n, \kappa, \ell)$-generalized p-cycle with $p \geq 3$. Then, $\kappa = \delta$ if*

(i) $n > (\delta - 1)\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 3\}$.

(ii) $p \geq \delta$ *and* $n > (\delta - 1)\{n(\Delta, \ell+p-2) + n(\Delta, D-\ell-p+1) - 1 - \Delta^{\ell+2}\}$.

Recalling that a digraph is a generalized p-cycle if and only if its line digraph is, we can apply the line digraph technique to the digraphs of Theorem 4.2.3. So, we obtain a sufficient condition on the size m for a generalized p-cycle to have maximum arc-connectivity.

Corollary 4.2.2 *Let G be a $(\delta, \Delta, D, m, \kappa, \lambda, \ell)$-generalized p-cycle with $p \geq 3$. Then,*

(i) $\lambda < \delta \Rightarrow m \leq \lambda\{n(\Delta, \ell+p-1) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2$.

(ii) If $p \geq \delta$, $\lambda < \delta \Rightarrow m \leq \lambda\{n(\Delta, \ell+p-1) + n(\Delta, D-\ell-p+1) - 1 - \Delta^{\ell+3}\}$.

Proof. Suppose that the result is not true. Then, there would be a generalized p-cycle G with $p \geq 3$, m arcs, parameters δ, Δ, ℓ, D and arc-connectivity $\lambda < \delta$ such that

$$m > \lambda\{n(\Delta, \ell+p-1) + n(\Delta, D-\ell-p+1) - 1\} - 2\delta + 2.$$

Therefore, its line digraph LG would have $n' = m$ vertices, minimum and maximum degree δ and Δ, diameter $D' = D+1$, parameter $\ell' = \ell+1$, and connectivity $\kappa' = \lambda < \delta$, satisfying

$$n' > \kappa'\{n(\Delta, \ell'+p-2) + n(\Delta, D'-\ell'-p+1) - 1\} - 2\delta + 2,$$

which contradicts Theorem 4.2.3. The case $p \geq \delta$ is similarly proved. ■

When the digraph G is d-regular, it has $m = dn$ arcs and we get the following corollary.

Corollary 4.2.3 *If G is a d-regular $(D, n, \kappa, \lambda, \ell)$-generalized p-cycle with $p \geq 3$, then*

(i) $\kappa = d$ *if* $\begin{cases} n > d^{\ell+p-1} + d^{D-\ell-p+2} - 3d + 1; \\ n > d^{\ell+p-1} + d^{D-\ell-p+2} - (d^{\ell+2} + 1)(d-1) - 2 \text{ and } p \geq d. \end{cases}$

(ii) $\lambda = d$ *if* $\begin{cases} n > d^{\ell+p-1} + d^{D-\ell-p+1} - 3; \\ n > d^{\ell+p-1} + d^{D-\ell-p+1} - d^{\ell+2}(d-1) - 1 \text{ and } p \geq d. \end{cases}$ ∎

From the above results we can deduce the following sufficient condition for a k-iterated line d-regular generalized p-cycle to be maximally connected.

Corollary 4.2.4 *Let G be a d-regular $(D, n, \kappa, \lambda, \ell)$-generalized p-cycle with $p \geq 3$. Then,*

(i) $\kappa(L^k G) = d$ *if* $k > \log_d \dfrac{d^{D-\ell-p+2} - 3d + 1}{n - d^{\ell+p-1}}$.

(ii) $\lambda(L^k G) = d$ *if* $k > \log_d \dfrac{d^{D-\ell-p+1} - 3}{n - d^{\ell+p-1}}$. ∎

Finally, from the foregoing corollary and Corollary 4.2.1 the following result is directly derived.

Corollary 4.2.5 *Let G be a d-regular $(D, n, \kappa, \lambda, \ell)$-generalized p-cycle with $p \geq 3$. Then,*

(i) $\kappa(L^k G) = d$ *if* $k > \min\{D - \ell - p + 2 - \log_d(n - d^{\ell+p-1}), D - 2\ell - p + 2\}$.

(ii) $\lambda(L^k G) = d$ *if* $k > \min\{D - \ell - p + 1 - \log_d(n - d^{\ell+p-1}), D - 2\ell - p + 1\}$.

An upper bound on the number of vertices for any generalized p-cycle with $p \geq 3$, $\delta \geq 2$ and arc-connectivity $\lambda < \delta$, similar to that of Theorem 4.2.3, can be obtained by carrying out the same type of proof. As in the vertex case, we need firstly to bound the deepness of α-fragments.

Lemma 4.2.5 *Let G be a generalized p-cycle with minimum degree $\delta \geq 2$, arc-connectivity $\lambda < \delta$, diameter D and parameter ℓ. Then, for every positive or negative α-fragment F,*

(i) $\nu(F) \geq \ell$ *and* $\nu(V \setminus F) \leq D - \ell - p$;

(ii) $\nu(F) \geq \ell + 1$ *and* $\nu(V \setminus F) \leq D - \ell - p - 1$ *if* $p \geq \delta$.

Proof. To prove (i), let F be a positive α-fragment of G. Then, $|\omega^+ F| = \lambda$, $D \geq 2\ell + p$ and $\nu, \nu' \geq \ell$ where $\nu = \nu(F)$ and $\nu' = \nu(V \setminus F)$. In addition, since every path from F to $V \setminus F$ goes through an arc of $\omega^+ F$, it follows that $\nu + 1 + \nu' \leq D$. It is also clear that the two non-empty disjoint sets $F_0 = \{x \in F : (x, y) \in \omega^+ F\}$ and $\overline{F}_0 = \{y \in V \setminus F : (x, y) \in \omega^+ F\}$ have cardinal at most $|\omega^+ F|$. On the other hand, let us consider the sets of vertices $F_i = \{x \in F;\ d(x, F_0) = i\}$, $0 \leq i \leq \nu$ and $\overline{F}_j = \{y \in V \setminus F : d(\overline{F}_0, y) = j\}$, $0 \leq j \leq \nu'$. Finally, reasoning in the same way as in the proof of Lemma 4.2.4, we conclude that $\nu' \leq D - \ell - p$. Point (*ii*) is analogously proved. ∎

Theorem 4.2.5 *Let G be a $(\delta, \Delta, D, n, \lambda, \ell)$-generalized p-cycle with $p \geq 3$ and $\delta \geq 2$. Then,*

(i) $\lambda < \delta \Rightarrow n \leq \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\}$.

(ii) If $p \geq \delta$, $\lambda < \delta \Rightarrow n \leq \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p) - \Delta^{\ell+2}\}$.

Proof. Although this proof is very similar to that of Theorem 4.2.3, we present it briefly for the sake of completeness. We make use of the same notation as in Lemma 4.2.5 and assume that $\nu \leq \nu'$. We distinguish two cases:

(a) $\nu' < D - \ell - p$ (this is the only case in which $p \geq \delta$).

(a.1) If $\ell \leq \nu \leq \ell + p - 2$, then

$$\begin{aligned} n &= \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{\nu'} |\overline{F}_j| \leq \lambda\{n(\Delta, \nu) + n(\Delta, D - \ell - p - 1)\} \\ &= \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\} - \lambda\Delta^{D-\ell-p}. \end{aligned}$$

If $p \geq \delta$, $\ell + 1 \leq \nu \leq \nu' \leq D - \ell - p - 1$. As a consequence, $\lambda\Delta^{D-\ell-p} \geq \lambda\Delta^{\ell+2}$, and the point (ii) is proved.

(a.2) If $\nu \geq \ell + p - 1$, then

$$\begin{aligned} n &= \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{\nu'} |\overline{F}_j| \leq \lambda\{n(\Delta, \nu) + n(\Delta, D - \ell - p - 1)\} \\ &= \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\} + \lambda\{\sum_{i=\ell+p-1}^{\nu} \Delta^i - \Delta^{D-\ell-p}\} \\ &\leq \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\} - \lambda\Delta^{\ell+2} \end{aligned}$$

(b) $\nu' = D - \ell - p$. Since all the paths from $x \in F_\nu$ to $y \in \overline{F}_{\nu'}$ go through ω^+F, then $D \geq d(x, y) \geq d(x, \omega^+F) + 1 + d(\omega^+F, y) \geq \nu + 1 + \nu' = \nu + 1 + D - \ell - p$. Therefore, $\nu \leq \ell + p - 1$.

(b.1) If $\nu = \ell + p - 1$ and we consider two vertices $x \in F_\nu$ and $y \in \overline{F}_{\nu'}$, then $d(x, y) \geq \nu + 1 + \nu' = \ell + p - 1 + 1 + D - \ell - p = D$. Therefore, if there exists a vertex $x' \in \Gamma^+(x) \cap F_{\ell+p-1}$, then we would have two different paths from x to y, one of length D and the other, $xx' \to y$, of length $D + 1$, which is impossible in a generalized p-cycle with $p \geq 2$. As a consequence, for all $x \in F_{\ell+p-1}$, $\Gamma^+(x) \subset F_{\ell+p-2}$, which implies that $|F_{\ell+p-1}| \leq \frac{\Delta}{\delta}|F_{\ell+p-2}|$. In a similar way it is proved that for any vertex $y \in \overline{F}_{D-\ell-p}$, $\Gamma^-(y) \subset \overline{F}_{D-\ell-p-1}$, and therefore $|\overline{F}_{D-\ell-p}| \leq \frac{\Delta}{\delta}|\overline{F}_{D-\ell-p-1}|$. In this way we obtain that

$$\begin{aligned} n &= \sum_{i=0}^{\ell+p-2} |F_i| + \sum_{j=0}^{D-\ell-p-1} |\overline{F}_j| + |F_{\ell+p-1}| + |\overline{F}_{D-\ell-p}| \\ &\leq \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p - 1)\} + \tfrac{\lambda}{\delta}\{\Delta^{\ell+p-1} + \Delta^{D-\ell-p}\} \\ &= \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\} + \tfrac{\lambda}{\delta}\{\Delta^{\ell+p-1} + \Delta^{D-\ell-p}\} - \lambda\Delta^{D-\ell-p} \\ &\leq \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\}, \end{aligned}$$

(b.2) Finally, if $\ell \leq \nu \leq \ell + p - 2$, then

	p	$D \le$	$n >$
$\kappa = d$	1	$2\ell - 1$ [54]	$d^D + d^2 - d - 1$ [101] $d^{D-\ell+1} + 2d^{\ell} - 2d + 1$ [62]
	2	2ℓ [55], [*]	$2(d^{D-1} - 1)$ [1] $d^{D-\ell} + d^{\ell+1} - 2d + 2$ [9]
	≥ 3	$2\ell + p - 2$ [*]	$d^{D-\ell-p+2} + d^{\ell+p-1} - 3d + 1$ [*]
$\lambda = d$	1	2ℓ [54]	$d^{D-1} + d^2 - 2$ [101] $d^{D-\ell} + 2d^{\ell} - 2$ [62]
	2	$2\ell + 1$ [55], [*]	$2d^{D-2}$ [1] $d^{D-\ell-1} + d^{\ell+1} - 2$ [9]
	≥ 3	$2\ell + p - 1$ [*]	$d^{D-\ell-p+1} + d^{\ell+p-1} - 3$ [*]

Table 4.1: Sufficient conditions for a d-regular generalized p-cycle to have maximum connectivities. The * symbol indicates this work.

$$n = \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{D-\ell-p} |\overline{F}_j| \le \lambda\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\}. \qquad \blacksquare$$

Finally, the following corollary gives a sufficient condition on the number of vertices for any generalized p-cycle to have maximum arc-connectivity.

Theorem 4.2.6 *Let G be a $(\delta, \Delta, D, n, \lambda, \ell)$-generalized p-cycle with $p \ge 3$. Then,*

(i) $\lambda = \delta$ if $n > (\delta - 1)\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p)\}$.

(ii) If $p \ge \delta$, $\lambda = \delta$ if $n > (\delta - 1)\{n(\Delta, \ell + p - 2) + n(\Delta, D - \ell - p) - \Delta^{\ell+2}\}$.

In Table 4.1, we put forward some of the main sufficient conditions that have been obtained for a d-regular generalized p-cycle to be maximally connected and/or arc-connected.

4.3 Superconnectivity

This section is devoted to generalizing the two last points of Propositions 4.1.1 and 4.1.2. In other words, the aim is to obtain both diameter and order sufficient conditions for a generalized p-cycle to be superconnected.

4.3.1 Diameter conditions

The first aim of this section is to obtain a diameter sufficient condition that guarantees any generalized p-cycle to be superconnected. Bearing this in mind, we need firstly to study the 1-fragments more deeply (see Section 1.7.1). To be more precise, we are going to stablish some properties relating the parameter ℓ to the deepness of any 1-fragment when its order is rather small.

Let x be a vertex belonging to a positive [negative] 1-fragment F of a connected digraph G. In what follows, it will be denoted by $S^+(x)$ [$S^-(x)$] the subset of ∂^+F [∂^-F] whose vertices are at minimum distance from [to] x. To be more precise:

$$S^+(x) = \{f \in \partial^+F \,:\, d(x,f) = d(x,\partial^+F)\}$$

$$[S^-(x) = \{f \in \partial^-F \,:\, d(f,x) = d(\partial^-F,x)\}]$$

Lemma 4.3.1 *Let F be a positive 1-fragment [7] of a connected digraph with parameter ℓ.*

1. *If $x \in F$ and $x_i, x_j \in \Gamma^+(x)$, then $S^+(x_i) \cap S^+(x_j) = \emptyset$ if one of the following conditions is satisfied:*
   (a) *$\mu(F) \leq \ell - 1$, and x belongs to the valley of F.*
   (b) *$d(x_i,\partial^+F) = d(x_j,\partial^+F) = d(x,\partial^+F) - 1 \leq \ell - 1$.*
2. *If $y \in \overline{F}$ and $y_i, y_j \in \Gamma^-(y)$, then $S^-(y_i) \cap S^-(y_j) = \emptyset$ if one of the following conditions is satisfied:*
   (a) *$\mu(\overline{F}) \leq \ell - 1$, and y belongs to the valley of $\overline{F}$.*
   (b) *$d(x_i,\partial^-\overline{F}) = d(x_j,\partial^-\overline{F}) = d(x,\partial^-\overline{F}) - 1 \leq \ell - 1$.*

Proof. To prove (1)(a), let us first assume that $\mu(F) \geq 2$. If $f \in S^+(x_i) \cap S^+(x_j)$, then there are two distinct short paths from the vertex x to f; namely, $xx_i \to f$ and $xx_j \to f$, the lengths of which are $\mu(F)$ or $\mu(F)+1$, contradicting the definition of parameter ℓ, since $\mu(F) \leq d(x,f) \leq \mu(F)+1 \leq \ell$. Suppose now that $\mu(F) = 1$ and $x_i, x_j \in \partial^+F$. In this case the result is obvious, since $S^+(x_i) = x_i$ and $S^+(x_j) = x_j$. Finally, if $\mu(F) = 1$ and either $x_i \in F$ or $x_j \in F$, then the results follow directly from the fact that $2 = \mu(F) + 1 \leq \ell$. Points (1)(b) and (2) are similarly proved. ■

Next, we present several results on 1-fragments in generalized p-cycles, which will allow us to prove the first theorem of this section.

Lemma 4.3.2 *Let F be a positive 1-fragment of a generalized p-cycle, $p \geq 3$, with parameter ℓ, minimum degree $\delta \geq 2$ and vertex-superconnectivity $\kappa_1 \leq 2\delta - 2$. Let $x \in F$, $y \in \overline{F}$ be two vertices belonging to the valley of F and valley of $\overline{F}$ respectively.*

(a) If $\mu(F) \leq \ell - 1$, then there exists a vertex $x_i \in \Gamma^+(x)$ into the valley of F;

(b) If $\mu(\overline{F}) \leq \ell - 1$, then there exists a vertex $y_i \in \Gamma^-(y)$ into the valley of $\overline{F}$.

[7] Notice that this lemma can be generalized for any proper vertex subset F.

Proof. To prove (a), notice that if $\mu = \mu(F) = 1$, then the valley of F is F. Moreover, $\partial^+ F$ is a nontrivial set because F is a positive 1-fragment, and thus for every $x \in F$ there exists some vertex $y \in \Gamma^+(x) \cap F$. So, let us assume that $\mu \geq 2$ and for each $x_i \in \Gamma^+(x)$, $d(x_i, \partial^+ F) = \mu - 1$. As a consequence of Lemma 4.3.1, it follows that $|S^+(x)| = \sum_{x_i \in \Gamma^+(x)} |S^+(x_i)| \geq \delta$. Furthermore, $\Gamma^+(x) \subset F$ because $\mu \geq 2$. Thus, there exists some $y \in \Gamma^+(x)$ for which $|S^+(y)| = 1$, since otherwise $\kappa_1 = |\partial^+ F| \geq |S^+(x)| \geq 2\delta$, contradicting the hypotheses. Then, there exist $\delta - 1$ out-neighbours of y, $\{y_1, \dots, y_{\delta-1}\}$, for which $\mu - 1 \leq d(y_j, \partial^+ F) \leq \mu$. Let f_i be a vertex of $\partial^+ F$ at minimum distance from y_i. The set $B = \{f_1, \dots, f_{\delta-1}\}$ must have cardinality $\delta - 1$; on the contrary, there would exist two distinct paths; namely, $yy_i \rightarrow f_i$ and $yy_j \rightarrow f_i$, of lengths μ or $\mu + 1$, contradicting the definition of parameter ℓ, since $\mu \leq \ell - 1$. Therefore, $B \cup S^+(x) \subset \partial^+ F$. As $\kappa_1 = |\partial^+ F| \leq 2\delta - 2$, we conclude that some vertex f_j must belong to $S^+(x)$. So, we have two distinct paths from x to f_j, on the one hand, the short path $x \rightarrow f_j$ of length μ, and on the other, a path $xyy_j \rightarrow f_j$, whose length is $\mu + 1$ or $\mu + 2$. But this fact is impossible because G is a generalized p-cycle with $p \geq 3$. Hence, there exists some $x_i \in \Gamma^+(x)$ such that $d(x_i, \partial^+ F) = \mu$.

The proof of (b) is similar. ∎

Lemma 4.3.3 *Let G be a generalized p-cycle, $p \geq 3$, with parameter ℓ and minimum degree $\delta \geq 2$. Let F denote a positive 1-fragment of G. If $\kappa_1 \leq 2\delta - 2$, then $\mu(F) \geq \ell$ and $\mu(\overline{F}) \geq \ell$.*

Proof. Suppose, for example, that $\mu = \mu(F) \leq \ell - 1$. If x is a vertex belonging to the valley of F, then, according to Lemma 4.3.2, there exists a vertex $z \in \Gamma^+(x)$ belonging to the valley of F. Notice that $\Gamma^+(x) \cap \Gamma^+(z) = \emptyset$ because the digraph is a p-cycle with $p \geq 3$. Moreover, we know that each pair of vertices $x_i, x_j \in \Gamma^+(x)$ satisfies $S^+(x_i) \cap S^+(x_j) = \emptyset$ (see Lemma 4.3.1). For the same reason, every pair of vertices $z_i, z_j \in \Gamma^+(z)$ satisfies $S^+(z_i) \cap S^+(z_j) = \emptyset$. Starting from the fact that $p \geq 3$, it follows that $x \notin \Gamma^+(z)$ and hence,

$$\sum_{x_i \in \Gamma^+(x) - \{z\}} |S^+(x_i)| + \sum_{z_j \in \Gamma^+(z)} |S^+(z_j)| \geq 2\delta - 1.$$

But $\kappa_1 = |\partial^+ F| \leq 2\delta - 2$, which means that there exist at least two vertices, $x_r \in \Gamma^+(x) \setminus \{z\}$ and $z_s \in \Gamma^+(z)$, such that $S^+(x_r) \cap S^+(z_s) \neq \emptyset$. This fact implies that there are two different paths from x to a certain vertex $f \in S^+(x_r) \cap S^+(z_s)$; namely, a $xx_r \rightarrow f$ path of length μ or $\mu + 1$, and a $xzz_s \rightarrow f$ path of length $\mu + 1$ or $\mu + 2$. But G is a generalized p-cycle, and hence the length of these two paths must be congruent modulo $p \geq 3$. Therefore, both paths must have length $\mu + 1$, which contradicts the definition of parameter ℓ, since $\mu + 1 \leq \ell$. Hence, $\mu = \mu(F) \geq \ell$. ∎

Lemma 4.3.4 *Let G be a generalized p-cycle, $p \geq 3$, with parameter ℓ, minimum degree $\delta \geq 2$ and vertex-superconnectivity $\kappa_1 \leq 2\delta - 2$. If F is a positive 1-fragment, then*

(a) for each vertex $x \in F$ such that $d(x, \partial^+ F) \geq \ell$, there exists a vertex $\hat{y} \in F$ such that: $d(x, \hat{y}) \leq 2$ and $d(\hat{y}, \partial^+ F) \geq \ell$;

(b) for each vertex $x \in F$ such that $d(x, \partial^+ F) \geq \ell$, there exists a vertex $\hat{y} \in F$ such that: $d(\hat{y}, x) \leq 2$ and $d(\partial^+ F, \hat{y}) \geq \ell$.

Proof. (a) If $\ell = 1$, then the lemma is obvious, since $\partial^+ F$ is a nontrivial vertex set. Thus assume that $\ell \geq 2$. In this case, this lemma is also clear if for each $x \in F$ such that $d(x, \partial^+ F) = \ell$, there exists some $y \in \Gamma^+(x)$ for which $d(y, \partial^+ F) \geq \ell$. Suppose finally that there exists a vertex $x \in F$ such that $d(x, \partial^+ F) = \ell$ and for all $y \in \Gamma^+(x)$, $d(y, \partial^+ F) = \ell - 1$. As a consequence of Lemma 4.3.1, it follows that $|S^+(x)| = \sum_{x_i \in \Gamma^+(x)} |S^+(x_i)| \geq \delta$. Therefore, there exists a vertex $y \in \Gamma^+(x)$ for which $|S^+(y)| = 1$, since otherwise $\kappa_1 = |\partial^+ F| \geq |S^+(x)| \geq 2\delta$. Let $y_1, y_2, \ldots, y_{\delta-1}$ be $\delta - 1$ out-neighbours of y for which $\ell - 1 \leq d(y_j, \partial^+ F)$. Let $f_i \in \partial^+ F$ be a vertex at minimum distance from y_i. As $\kappa_1 = |\partial^+ F| \leq 2\delta - 2$, we conclude that there exists a vertex f_h such that either $f_h \in S^+(x)$ or $f_h = f_j$ for some $j \neq h$. In the first case, there would exist two distinct paths from x to f_h, the shortest $x \to f_h$ path and the path $xyy_h \to f_h$, of lengths ℓ and $2 + d(y_h, f_h) \geq 2 + (\ell - 1) = \ell + 1$ respectively. Since both lengths have to be congruent modulo p, we conclude that $d(y_h, f_h) > \ell$. In the second case, there would exist two distinct paths from y to f_h of lengths at least ℓ. This means that either y_h or y_j is at distance at least ℓ to $\partial^+ F$.

(b) To prove this point, it is enough to apply (a) to the converse digraph of G. ∎

As a consequence of this lemma the following corollary is obtained.

Corollary 4.3.1 *Let G be a generalized p-cycle, $p \geq 3$, with parameter ℓ, minimum degree $\delta \geq 2$ and vertex-superconnectivity $\kappa_1 \leq 2\delta - 2$. If F is a positive 1-fragment, then*

(a) for each vertex $x \in F$ such that $d(x, \partial^+ F) \geq \ell$, there exists a $\rho : x \to z$ path in F of length $p - 1$ such that for any vertex y of ρ, $d(y, \partial^+ F) \geq \ell - 1$;

(b) for each vertex $x \in \overline{F}$ such that $d(\partial^+ F, x) \geq \ell$, there exists a $\rho : z \to x$ path in $\overline{F}$ of length $p - 1$ such that for any vertex y of ρ, $d(\partial^+ F, y) \geq \ell - 1$.

We are now ready to prove a first theorem which provides a diameter sufficient condition for a generalized p-cycle to be vertex-superconnected.

Theorem 4.3.1 *Let G be a generalized p-cycle, $p \geq 3$, with parameter ℓ and minimum degree $\delta \geq 2$. Then, $\kappa_1 \geq 2\delta - 1$ and thus G is super-κ, if $D \leq 2\ell + p - 3$.*

Proof. Assume that $\kappa_1 \leq 2\delta - 2$ and let F be a positive 1-fragment such that $|\partial^+ F| \leq 2\delta - 2$. Without loss of generality we can suppose that $\mu(F) \leq \mu(\overline{F})$ (if not, use the converse digraph of G). Therefore, according to Lemma 4.3.3, we obtain that $\ell \leq \mu(F) \leq \mu(\overline{F})$ and hence, we can consider the nonempty sets:

$$F(\ell - 1) = \{x \in F, d(x, \partial^+ F) \geq \ell - 1\}$$

$$\overline{F}(\ell - 1) = \{y \in \overline{F}, d(\partial^+ F, y) \geq \ell - 1\}.$$

Notice that, as was proved in Corollary 4.3.1, we can find a path of length $p - 1$ into $\overline{F}(\ell - 1)$. As G is a generalized p-cycle, we conclude that in $\overline{F}(\ell - 1)$ vertices of every partite set V_α exist. Now, let us consider the integer r, $0 \leq r \leq p - 1$, so that $D + 1 \equiv r (\text{mod } p)$. Let $x \in V_\alpha$ be such that $d(x, \partial^+ F) = \mu(F) \geq \ell$, and consider a vertex $y \in V_{\alpha+r} \cap \overline{F}(\ell - 1)$. Then, $D - (p-1) \geq d(x, y) \geq d(x, \partial^+ F) + d(\partial^+ F, y) \geq 2\ell - 1$, because the length of every path from V_α to $V_{\alpha+r}$ is congruent with r modulo p. This means that $D \geq 2\ell + p - 2$, which contradicts the hypothesis. Therefore, $\kappa_1 \geq 2\delta - 1 > \delta$ and so the digraph is super-κ. ∎

For the arc case, a similar result to the previous one is derived.

Theorem 4.3.2 *Let G be a generalized p-cycle, $p \geq 3$, with parameter ℓ and minimum degree $\delta \geq 2$. Then, $\lambda_1 \geq 2\delta - 1$ and thus G is super-λ, if $D \leq 2\ell + p - 2$.*

Proof. Assume that the result does not hold. Then, there would be a generalized p-cycle G, $p \geq 3$, with parameter ℓ, minimum degree $\delta \geq 2$, $\lambda_1 \leq 2\delta - 2$ and $D \leq 2\ell + p - 2$. Thus, according to Propositions 1.5.3(3), 1.6.1(3) and Corollary 1.7.1, its line digraph LG would also be a generalized p-cycle having the same minimum degree δ, parameter $\ell' = \ell'(LG) = \ell + 1$, $\kappa_1(LG) = \lambda_1(G) \leq 2\delta - 2$ and diameter $D' = D + 1 \leq 2\ell' + p - 3$, contradicting Theorem 4.3.1. ∎

Finally, from Propositions 1.5.3(3), 1.6.1(3) and Corollary 1.7.1 together with Theorems 4.3.1 and 4.3.2, we obtain the following sufficient condition for the k-iterated line digraph of any generalized p-cycle to be superconnected.

Corollary 4.3.2 *Let G be a connected generalized p-cycle, $p \geq 3$, with parameter ℓ, minimum degree $\delta \geq 2$ and diameter D. Then,*

(a) $\kappa_1(L^k G) \geq 2\delta - 1$ *if* $k \geq D - 2\ell - p + 3$;

(b) $\lambda_1(L^k G) \geq 2\delta - 1$ *if* $k \geq D - 2\ell - p + 2$.

4.3.2 Order conditions

Lemma 4.3.5 *Let G be a generalized p-cycle, $p \geq 3$, with minimum degree $\delta \geq 2$, vertex-superconnectivity $\kappa_1 \leq 2\delta - 2$, diameter D and parameter ℓ. Then, for every positive [negative] 1-fragment F, $\mu(F) \geq \ell$ and $\mu(\overline{F}) \leq D - \ell - p + 2$.*

Proof. If F is a positive 1-fragment of a generalized p-cycle G, then $|\partial^+ F| = \kappa_1$. By Lemma 4.3.3, $\mu, \mu' \geq \ell$, where $\mu = \mu(F)$ and $\mu' = \mu(\overline{F})$; and from Corollary 4.3.1, it follows that F contains a path $P: x_0, x_1, \ldots, x_{p-1}$ such that $d(x_\alpha, \partial^+ F) \geq \ell - 1$, $\alpha = 0, 1, \ldots, p-1$ (notice that $x_\alpha \in V_\alpha$). Suppose that $\mu' \geq D - \ell - p + 3$ and consider a vertex y into the valley of $\bar{F}$. Hence, as every path from F to $\overline{F}$ goes through $\partial^+ F$, we have that $d(x_\alpha, y) \geq d(x_\alpha, \partial^+ F) + d(\partial^+ F, y) \geq \ell - 1 + \mu' \geq D - p + 2$ for all $x_\alpha \in P$. In particular, $d(x_{p-1}, y) = D - p + 2 + k$, for some $0 \leq k \leq p-2$, since $d(x_{p-1}, y) \leq D$. Hence, we have two paths from x_k to y; namely, a shortest path of length $d(x_k, y)$ and the path $x_k, x_{k+1}, \ldots, x_{p-1} \to y$ of length $d(x_{p-1}, y) + p - 1 - k = D + 1$. Since the lengths of these two paths are congruent modulo p and $d(x_k, y) \leq D$, we find that $d(x_k, y) = D + 1 - hp$ for some positive integer h. But this contradicts the fact that $d(x_k, y) \geq D - p + 2$. Therefore, $\mu' \leq D - \ell - p + 2$. ■

Theorem 4.3.3 *Let G be a generalized p-cycle with $p \geq 3$, order n, maximum and minimum degrees Δ and $\delta \geq 3$ respectively, diameter D and parameter ℓ. If $\kappa_1 = \delta$, then*

$$n \leq \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 2) - 2\} + \Delta^{\ell + p - 3} + 1.$$

Proof. Let T be a minimum cutset and F a positive fragment of G such that $\partial^+ F = T$. Observe that $|F_i| \leq \Delta |F_{i-1}|$, $1 \leq i \leq \mu$, and $|\overline{F}_j| \leq \Delta |\overline{F}_{j-1}|$, $1 \leq j \leq \mu'$ (see [8]). Notice also that, without loss of generality, we can suppose that $\mu \leq \mu'$. Starting from Lemma 4.2.4, we can distinguish two cases:

(a) $\mu' < D - \ell - p + 2$. Then, $\ell \leq \mu \leq \mu' \leq D - \ell - p + 1$; that is, $D \geq 2\ell + p - 1$.

(a.1) If $\ell \leq \mu \leq \ell + p - 4$, then the order of G must satisfy that

$$\begin{aligned} n = \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{\mu'} |\overline{F}_j| - |\partial^+ F| &\leq \delta\{n(\Delta, \mu) + n(\Delta, D - \ell - p + 1) - 1\} \\ &\leq \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 2) - 1\} - \delta \Delta^{D - \ell - p + 2} \\ &\leq \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 2) - 2\}. \end{aligned}$$

(a.2) If $\mu \geq \ell + p - 3$, then the order of G must satisfy that

$$\begin{aligned} n = \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{\mu'} |\overline{F}_j| - |\partial^+ F| &\leq \delta\{n(\Delta, \mu) + n(\Delta, D - \ell - p + 1) - 1\} \\ &= \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 2) - 1\} + \delta\{\sum_{i=\ell+p-3}^{\mu} \Delta^i - \Delta^{D - \ell - p + 2}\} \end{aligned}$$

[8] See the proof of Lemma 4.2.4 for notation questions.

$$\leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\} - \delta\Delta^{\ell+p-3}$$
$$\leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 2\},$$

since $\frac{\Delta^{\mu+1}-\Delta^{\ell+p-3}}{\Delta-1} - \Delta^{D-\ell-p+2} \leq \frac{\Delta^{D-\ell-p+2}-\Delta^{\ell+p-3}}{\Delta-1} - \Delta^{D-\ell-p+2} = \frac{2-\Delta}{\Delta-1}\Delta^{D-\ell-p+2} - \frac{\Delta^{\ell+p-3}}{\Delta-1} \leq \frac{1-\Delta}{\Delta-1}\Delta^{\ell+p-3} = -\Delta^{\ell+p-3}$, because $\ell+p-3 \leq \mu \leq D-\ell-p+1$.

(b) $\mu' = D-\ell-p+2$. Then, $\ell \leq \mu \leq \ell+p-2$. Indeed, since all the paths from $x \in F_\mu$ to $y \in \overline{F}_{\mu'}$ go through ∂^+F, it must be that $D \geq d(x,y) \geq d(x,\partial^+F) + d(\partial^+F, y) = \mu + \mu' = \mu + D - \ell - p + 2$. Therefore, $\mu \leq \ell + p - 2$.

(b.1) Suppose that $\mu = \ell + p - 2$ and consider two vertices $x \in F_\mu$ and $y \in \overline{F}_{\mu'}$. Certainly, $d(x,y) \geq \mu + \mu' = \ell + p - 2 + D - \ell - p + 2 = D$. Therefore, if there exists a vertex $x' \in \Gamma^+(x) \cap F_{\ell+p-2}$, then we would have two different paths from x to y, one of length D and the other, $xx' \to y$, of length $D+1$, which is impossible in a generalized p-cycle with $p \geq 3$. As a consequence, for all $x \in F_{\ell+p-2}$, $\Gamma^+(x) \subset F_{\ell+p-3}$, which implies that $|F_{\ell+p-2}| \leq \frac{\Delta}{\delta}|F_{\ell+p-3}| \leq \Delta^{\ell+p-2}$. In a similar way, we prove that for any vertex $y \in \overline{F}_{D-\ell-p+2}$, $\Gamma^-(y) \subset \overline{F}_{D-\ell-p+1}$, and therefore $|\overline{F}_{D-\ell-p+2}| \leq \frac{\Delta}{\delta}|\overline{F}_{D-\ell-p+1}| \leq \Delta^{D-\ell-p+2}$. In this way we obtain that

$$\begin{aligned}
n &= \sum_{i=0}^{\ell+p-3} |F_i| + \sum_{j=0}^{D-\ell-p+1} |\overline{F}_j| - |\partial^+F| + |F_{\ell+p-2}| + |\overline{F}_{D-\ell-p+2}| \\
&\leq \delta\{n(\Delta, \ell+p-3) + n(\Delta, D-\ell-p+1) - 1\} + \Delta^{\ell+p-2} + \Delta^{D-\ell-p+2} \\
&= \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\} + (\delta+\Delta)\Delta^{\ell+p-3} + (1-\delta)\Delta^{D-\ell-p+2} \\
&\leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\} + (\delta + (2-\delta)\Delta)\Delta^{\ell+p-3} \\
&\leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 2\} + \delta,
\end{aligned}$$

since $(\delta+\Delta)\Delta^{\mu-1} + (1-\delta)\Delta^{\mu'} \leq (\delta+\Delta)\Delta^{\mu-1} + (\Delta-\delta\Delta)\Delta^{\mu-1} = (\delta+(2-\delta)\Delta)\Delta^{\mu-1} \leq (\delta-\Delta)\Delta^{\mu-1} \leq 0$, because $\mu \leq \mu'$ and $\delta \geq 3$.

(b.2) Suppose that $\mu = \ell + p - 3$, and consider two vertices $x \in F_\mu$ and $y \in \overline{F}_{\mu'}$. As every path from x to y goes through ∂^+F, it must be that $d(x,y) \geq d(x,\partial^+F) + d(\partial^+F, y) = \mu + \mu' = \ell + p - 3 + D - \ell - p + 2 = D - 1$. Let us see that $|F_{\ell+p-3}| + |F_{D-\ell-p+2}| \leq \Delta^{\ell+p-3} + \delta\Delta^{D-\ell+p+2} - (\delta - 1)$. This result is clear when $\Gamma^+(x) \subset F_{\mu-1}$ and $\Gamma^-(y) \subset \overline{F}_{\mu'-1}$ for each $x \in F_\mu$ and $y \in \overline{F}_{\mu'}$. So, suppose that there exists a vertex $y \in \overline{F}_{\mu'}$ with some in-neighbour $y' \in \Gamma^-(y) \cap \overline{F}_{\mu'}$. As a consequence, we conclude that $d(x,y') \geq D-1$. Notice that if $d(x,y) = D-1$, then the length of the path $x \to y'y$ must be congruent with $D-1$ modulo p; that is, $1 + d(x,y') = D - 1 + hp$ for some integer $h \geq 0$. As $d(x,y') \geq D-1$, it follows that $h \geq 1$ and therefore, $d(x,y') \geq D - 2 + p \geq D + 1$ because $p \geq 3$, which is a contradiction. Hence, the only possibility is that $d(x,y) = D$, $d(x,y') = D-1$ and $\Gamma^-(y') \subset \overline{F}_{\mu'-1}$. Let us further see that $\Gamma^+(x) \subset F_{\mu-1}$. Indeed, if there exists a vertex $x' \in \Gamma^+(x) \cap F_\mu$, then $D - 1 \leq d(x',y') \leq D$. Hence, the length of the path $xx' \to y'$ is D or $D+1$, which is a contradiction with the fact that $d(x,y') = D-1$ and G is a p-cycle with $p \geq 3$. Therefore, we can conclude that $|F_{\ell+p-3}| \leq \frac{\Delta}{\delta}|F_{\ell+p-4}| \leq \Delta^{\ell+p-3}$, and $|F_{D-\ell-p+2}| \leq \delta\Delta^{D-\ell+p+2} - (\delta-1)$. As a consequence, the order of G must satisfy that

$$n = \sum_{i=0}^{\ell+p-4} |F_i| + \sum_{j=0}^{D-\ell-p+1} |\overline{F}_j| - |\partial^+ F| + |F_{\ell+p-3}| + |\overline{F}_{D-\ell-p+2}|$$
$$\leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 2\} + \Delta^{\ell+p-3} + 1.$$

(b.3) Finally, if $\mu \leq \ell + p - 4$, then

$$n = \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{D-\ell-p+2} |\overline{F}_j| - |\partial^+ F|$$
$$\leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 2\} + \delta. \qquad \blacksquare$$

Theorem 4.3.4 *Let G be a generalized p-cycle with $p \geq 3$, order n, maximum and minimum degrees Δ and $\delta \geq 3$ respectively, diameter D and parameter ℓ. If $\kappa_1 \leq 2\delta - 2$, then*

$$n \leq \kappa_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\} + 2\Delta^{\ell+p-3} - \delta - 1.$$

Proof. Let T be a minimum cutset and F a positive fragment of G such that $\partial^+ F = T$ and thus satisfying $|\partial^+ F| = \kappa_1$. As in the above theorem we suppose $\mu \leq \mu'$ and study the following cases:

(a) $\mu' < D - \ell - p + 2$. Then, $\ell \leq \mu \leq \mu' \leq D - \ell - p + 1$; that is, $D \geq 2\ell + p - 1$.

(a.1) If $\ell \leq \mu \leq \ell + p - 4$, then the order of G must satisfy that

$$n = \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{\mu'} |\overline{F}_j| - |\partial^+ F| \leq \kappa_1\{n(\Delta, \mu) + n(\Delta, D-\ell-p+1) - 1\}$$
$$\leq \kappa_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\} - \kappa_1 \Delta^{D-\ell-p+2}.$$

(a.2) If $\mu \geq \ell + p - 3$, then the order of G must satisfy that

$$n = \sum_{i=0}^{\mu} |F_i| + \sum_{j=0}^{\mu'} |\overline{F}_j| - |\partial^+ F| \leq \kappa_1\{n(\Delta, \mu) + n(\Delta, D-\ell-p+1) - 1\}$$
$$= \kappa_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\} + \kappa_1\{\sum_{i=\ell+p-3}^{\mu} \Delta^i - \Delta^{D-\ell-p+2}\}$$
$$\leq \kappa_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\} - \kappa_1 \Delta^{\ell+p-3}$$

since $\frac{\Delta^{\mu+1} - \Delta^{\ell+p-3}}{\Delta - 1} - \Delta^{D-\ell-p+2} \leq \frac{\Delta^{D-\ell-p+2} - \Delta^{\ell+p-3}}{\Delta - 1} - \Delta^{D-\ell-p+2} = \frac{2-\Delta}{\Delta-1}\Delta^{D-\ell-p+2} - \frac{\Delta^{\ell+p-3}}{\Delta-1} \leq \frac{1-\Delta}{\Delta-1}\Delta^{\ell+p-3} = -\Delta^{\ell+p-3}$, because $\ell + p - 3 \leq \mu \leq D - \ell - p + 1$.

(b) $\mu' = D - \ell - p + 2$. As in case (b) of Theorem 4.3.3 we get $\ell \leq \mu \leq \ell + p - 2$.

(b.1) If $\mu = \ell + p - 2$, then, reasoning as in case (b.1) of Theorem 4.3.3 we have that $|F_{\ell+p-2}| \leq \frac{\kappa_1}{\delta}\Delta^{\ell+p-2}$, and $|\overline{F}_{D-\ell-p+2}| \leq \frac{\kappa_1}{\delta}\Delta^{D-\ell-p+2}$. In this way, we obtain that

$$n = \sum_{i=0}^{\ell+p-3} |F_i| + \sum_{j=0}^{D-\ell-p+1} |\overline{F}_j| - |\partial^+ F| + |F_{\ell+p-2}| + |\overline{F}_{D-\ell-p+2}|$$
$$\leq \kappa_1\{n(\Delta, \ell+p-3) + n(\Delta, D-\ell-p+1) - 1\} + \frac{\kappa_1}{\delta}\{\Delta^{\ell+p-2} + \Delta^{D-\ell-p+2}\}$$
$$= \kappa_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\} + \frac{\kappa_1}{\delta}\{\Delta^{\ell+p-2} + \Delta^{D-\ell-p+2}\} +$$
$$+ \kappa_1(\Delta^{\ell+p-3} - \Delta^{D-\ell-p+2}) \leq \kappa_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+2) - 1\},$$

since $\frac{1}{\delta}\{\Delta^{\mu}+\Delta^{\mu'}\}-\Delta^{\mu'}+\Delta^{\mu-1}\leq(\frac{2}{\delta}-1)\Delta^{\mu'}+\Delta^{\mu-1}\leq(\frac{2}{\delta}-1)\Delta^{\mu}+\Delta^{\mu-1}=(\frac{2\Delta}{\delta}-\Delta+1)\Delta^{\mu-1}\leq 0$, because $\ell+p-2=\mu\leq\mu'$ and $\delta\geq 3$.

(b.2) Suppose that $\mu=\ell+p-3$, and consider two vertices $x\in F_{\mu}$ and $y\in\overline{F}_{\mu'}$. As every path from $x\in F_{\mu}$ to y goes through $\partial^{+}F$, it must be that $d(x,y)\geq d(x,\partial^{+}F)+d(\partial^{+}F,y)=\mu+\mu'=\ell+p-3+D-\ell-p+2=D-1$. As in case (b.2) of Theorem 4.3.3, we obtain that $|F_{\ell+p-3}|+|F_{D-\ell-p+2}|\leq\frac{\kappa_1}{\delta}\Delta^{\ell+p-3}+\kappa_1\Delta^{D-\ell+p+2}-(\delta-1)$. Therefore, the order of G must satisfy that

$$\begin{aligned}
n &= \sum_{i=0}^{\ell+p-4}|F_i|+\sum_{j=0}^{D-\ell-p+1}|\overline{F}_j|-|\partial^{+}F|+|F_{\ell+p-3}|+|\overline{F}_{D-\ell-p+2}| \\
&\leq \kappa_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+2)-1\}+\tfrac{\kappa_1}{\delta}\Delta^{\ell+p-3}-\delta+1 \\
&\leq \kappa_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+2)-1\}+2\Delta^{\ell+p-3}-\delta-1,
\end{aligned}$$

because $\frac{\kappa_1}{\delta}\Delta^{\ell+p-3}-\delta+1\leq 2\Delta^{\ell+p-3}-\frac{2}{\delta}\Delta^{\ell+p-3}-\delta+1\leq 2\Delta^{\ell+p-3}-\delta-1$, since $\kappa_1\leq 2\delta-2$, $p\geq 3$, $\ell\geq 1$ and $\Delta\geq\delta$.

(b.3) Finally, if $\mu\leq\ell+p-4$, then

$$n=\sum_{i=0}^{\mu}|F_i|+\sum_{j=0}^{D-\ell-p+2}|\overline{F}_j|-|\partial^{+}F|\leq\kappa_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+2)-1\}.$$

■

The following corollary, which is just a restatement of the above theorem, gives a sufficient condition on the number of vertices for any generalized p-cycle with $p\geq 3$ to have $\kappa_1\geq\delta+1$ and $\kappa_1\geq 2\delta-1$ respectively.

Corollary 4.3.3 *Let G be a generalized p-cycle with $p\geq 3$, order n, maximum and minimum degrees Δ and $\delta\geq 3$ respectively, diameter D and parameter ℓ. Then,*

(a) $\kappa_1\geq\delta+1$ *if* $n>\delta\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+2)-2\}+\Delta^{\ell+p-3}+1$.

(b) $\kappa_1\geq 2\delta-1$ *if* $n>(2\delta-2)\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+2)-1\}+2\Delta^{\ell+p-3}-\delta-1$.

■

Recalling that a digraph is a generalized p-cycle if and only if its line digraph is, we can apply the line digraph technique to the digraphs of Theorem 4.3.3 and Theorem 4.3.4. So, we obtain a sufficient condition on the number of arcs for G to have $\lambda_1\geq\delta+1$ and $\lambda_1\geq 2\delta-1$ respectively.

Corollary 4.3.4 *Let G be a generalized p-cycle with $p\geq 3$, arc-superconnectivity λ_1, size m, maximum and minimum degrees Δ and $\delta\geq 3$ respectively, diameter D and parameter ℓ. Then,*

(a) $\lambda_1\geq\delta+1$ *if* $m>\delta\{n(\Delta,\ell+p-3)+n(\Delta,D-\ell-p+2)-2\}+\Delta^{\ell+p-2}+1$.

(b) $\lambda_1\geq 2\delta-1$ *if* $m>(2\delta-2)\{n(\Delta,\ell+p-3)+n(\Delta,D-\ell-p+2)-1\}+2\Delta^{\ell+p-2}-\delta-1$.

Proof. To prove case (b), suppose that this statement is not true. Then, there would be a generalized p-cycle G with $p \geq 3$, m arcs, parameters $\delta \geq 3$, Δ, ℓ, D and $\lambda_1 \leq 2\delta - 2$ so that $m > (2\delta-2)\{n(\Delta, \ell+p-3)+n(\Delta, D-\ell-p+2)-1\}+2\Delta^{\ell+p-2}-\delta-1$. Then, its line digraph LG would have $n' = m$ vertices, minimum and maximum degree δ and Δ respectively, diameter $D' = D+1$, parameter $\ell' = \ell+1$ and superconnectivities $\kappa_1(LG) = \lambda_1(G) \leq 2\delta - 2$ (see Corollary 1.7.1) satisfying
$n' > \kappa_1(LG)\{n(\Delta, \ell'+p-4)+n(\Delta, D'-\ell'-p+2)-1\}+2\Delta^{\ell'+p-3}-\delta-1$,
which contradicts Theorem 4.3.4. The proof of case (a) is similar. ∎

Upper bounds on the number of vertices for any generalized p-cycle to have $\lambda_1 \geq \delta+1$ and $\lambda_1 \geq 2\delta-1$ respectively, for $p \geq 3$, can be obtained by using a direct reasoning. With this aim, we first need to bound the deepness of the α_1-fragments for which we present the following two lemmas.

Lemma 4.3.6 *Let F be a positive α_1-fragment of a digraph G. Then, the subset of vertices of the line digraph LG; namely,*

$$C = \{uv \in V(LG) : u, v \in F\},$$

is a positive 1-fragment of LG. Moreover, $|\omega^+F| = |\partial^+C|$.

Proof. Let us consider the set of arcs, denoted by $E(F)$, whose initial and terminal vertices belong to F. Since F is a positive α_1-fragment, $E(F)$ and $E(V \setminus F)$ are nonempty arc sets because ω^+F is nontrivial. Therefore, $C = E(F)$ is a nonempty subset of vertices of LG and we have that $V(LG) = C \cup \partial^+C \cup \overline{C}$, where $\overline{C} \neq \emptyset$ because it contains the nonempty set $E(V \setminus F)$. In order to prove that ∂^+C is nontrivial, let us see that $\partial^+C \subset \omega^+F$. To this end, consider a vertex $uv \in \partial^+C$. Certainly, there must exist a vertex $wu \in C$ adjacent to uv. But this means that $u \in F$ and $v \notin F$ because $uv \notin C$, and so we can conclude that $uv \in \omega^+F$. Hence, $\lambda_1(G) = |\omega^+F| \geq |\partial^+C| \geq \kappa_1(LG)$. Finally, according to Corollary 1.7.1, we know that $\lambda_1(G) = \kappa_1(LG)$. Therefore, $|\partial^+C| = \kappa_1(LG)$ and so the result holds. ∎

Lemma 4.3.7 *Let G be a generalized p-cycle, $p \geq 3$, with minimum degree $\delta \geq 2$, arc-superconnectivity $\lambda_1 \leq 2\delta - 2$, diameter D and parameter ℓ. Then, for every positive [negative] α_1-fragment F, $\nu(F) \geq \ell$ and $\nu(V \setminus F) \leq D - \ell - p + 1$.*

Proof. Let F be a positive α_1-fragment of G. By Lemma 4.3.6, $C = \{uv \in V(LG),\ u, v \in F\}$ is an 1-fragment of LG. Thus, $\lambda_1 = |\omega^+F| = |\partial^+C| = \kappa_1(LG) \leq 2\delta - 2$. This fact implies that C must be different from a cycle, for if not, $|\partial^+C| \geq p(\delta-1) \geq 3(\delta-1)$. Furthemore, by Lemma 4.3.3, we have that $\mu(C), \mu(\bar{C}) \geq \ell(LG) = \ell + 1$. Now, let $uv \in C$ and $ab \in \partial^+C$ such that $d(uv, \partial^+C) = d(uv, ab) = d(v, a) + 1 \leq \nu(F) + 1$. Then, $\mu(C) \leq \nu(F) + 1$ which implies $\nu(F) \geq \ell$. In a similar way, we prove that

$\nu' = \nu(V \setminus F) \geq \ell$. Following the same lines of reasoning as in Corollary 4.3.1, we can prove that F contains a path $P : x_0, x_1, \ldots, x_{p-1}$ such that $d(x_\alpha, \omega^+ F) \geq \ell - 1$, $\alpha = 0, 1, \ldots, p-1$ (notice that $x_\alpha \in V_\alpha$). Suppose that $\nu' \geq D - \ell - p + 2$ and consider a vertex y into the valley of $V \setminus F$. Hence, as every path from F to $V \setminus F$ goes through an arc of $\omega^+ F$, it follows that $d(x_\alpha, y) \geq \ell - 1 + 1 + \nu' \geq \ell + D - \ell - p + 2 = D - p + 2$ for every $x_\alpha \in P$. Reasoning in the same way as in the proof of Lemma 4.3.5, we obtain that $\nu' \leq D - \ell - p + 1$. ■

Theorem 4.3.5 *Let G be a generalized p-cycle with $p \geq 3$, order n, maximum and minimum degrees Δ and $\delta \geq 3$ respectively, diameter D and parameter ℓ. If $\lambda_1 = \delta$, then*

$$n \leq \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 1) - 1\} + \Delta^{\ell+p-3} + 1.$$

Proof. Let us consider the two non-empty disjoint sets $F_0 = \{x \in F : (x, y) \in \omega^+ F\}$ and $\overline{F}_0 = \{y \in V \setminus F : (x, y) \in \omega^+ F\}$. It is clear that $|F_0| \leq |\omega^+ F| = \lambda_1$ and $|\overline{F}_0| \leq |\omega^+ F| = \lambda_1$. Let us now consider the sets $F_i = \{x \in F; d(x, F_0) = i\}$, $0 \leq i \leq \nu$ and $\overline{F}_j = \{y \in V \setminus F : d(\overline{F}_0, y) = j\}$, $0 \leq j \leq \nu'$, and assume that $\nu \leq \nu'$. Starting from Lemma 4.3.7, we can distinguish the following cases:

(a) $\nu' < D - \ell - p + 1$. Then, $\ell \leq \nu \leq \nu' \leq D - \ell - p$; that is, $D \geq 2\ell + p$.

(a.1) If $\ell \leq \nu \leq \ell + p - 4$, then the order of G must satisfy that

$$\begin{aligned} n = \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{\nu'} |\overline{F}_j| &\leq \delta\{n(\Delta, \nu) + n(\Delta, D - \ell - p)\} \\ &\leq \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 1)\} - \delta\Delta^{D-\ell-p+1}. \\ &\leq \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 1)\}. \end{aligned}$$

(a.2) If $\nu \geq \ell + p - 3$, then the order of G must satisfy that

$$\begin{aligned} n = \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{\nu'} |\overline{F}_j| &\leq \delta\{n(\Delta, \nu) + n(\Delta, D - \ell - p)\} \\ &= \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 1)\} + \delta\{\sum_{i=\ell+p-3}^{\nu} \Delta^i - \Delta^{D-\ell-p+1}\} \\ &\leq \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 1)\} - \delta\Delta^{\ell+p-3} \\ &\leq \delta\{n(\Delta, \ell + p - 4) + n(\Delta, D - \ell - p + 1)\}, \end{aligned}$$

since $\frac{\Delta^{\nu+1} - \Delta^{\ell+p-3}}{\Delta - 1} - \Delta^{D-\ell-p+1} \leq \frac{\Delta^{D-\ell-p+1} - \Delta^{\ell+p-3}}{\Delta - 1} - \Delta^{D-\ell-p+1} = \frac{2-\Delta}{\Delta-1}\Delta^{D-\ell-p+1} - \frac{\Delta^{\ell+p-3}}{\Delta-1} \leq \frac{1-\Delta}{\Delta-1}\Delta^{\ell+p-3} \leq -\Delta^{\ell+p-3}$, because $\ell + p - 3 \leq \nu \leq \nu' \leq D - \ell - p$ and $\delta \geq 2$.

(b) $\nu' = D - \ell - p + 1$. Since every path from $x \in F_\nu$ to $y \in \overline{F}_{\nu'}$ goes through $\omega^+ F$, it must be that $D \geq d(x, y) \geq d(x, \omega^+ F) + 1 + d(\omega^+ F, y) \geq \nu + 1 + \nu' = \nu + 1 + D - \ell - p + 1$. Thus, $\ell \leq \nu \leq \ell + p - 2$.

(b.1) If $\nu = \ell + p - 2$ and we consider two vertices $x \in F_\nu$ and $y \in \overline{F}_{\nu'}$, then $d(x, y) \geq \nu + 1 + \nu' = \ell + p - 2 + 1 + D - \ell - p + 1 = D$. As in case (b.1) of Theorem

4.3.3, we conclude that $|F_{\ell+p-2}| \leq \frac{\Delta}{\delta}|F_{\ell+p-1}| \leq \Delta^{\ell+p-2}$. Similarly, we prove that $|\overline{F}_{D-\ell-p+1}| \leq \frac{\Delta}{\delta}|\overline{F}_{D-\ell-p}| \leq \Delta^{D-\ell-p+1}$. In this way, we obtain that

$$\begin{aligned} n &= \sum_{i=0}^{\ell+p-3} |F_i| + \sum_{j=0}^{D-\ell-p} |\overline{F}_j| + |F_{\ell+p-2}| + |\overline{F}_{D-\ell-p+1}| \\ &\leq \delta\{n(\Delta, \ell+p-3) + n(\Delta, D-\ell-p)\} + \Delta^{\ell+p-2} + \Delta^{D-\ell-p+1} \\ &= \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\} + \Delta^{\ell+p-3}(\delta+\Delta) + \Delta^{D-\ell-p+1}(1-\delta) \\ &\leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\}, \end{aligned}$$

since $(\delta+\Delta)\Delta^{\nu-1}+(1-\delta)\Delta^{\nu'} \leq (\delta+\Delta)\Delta^{\nu-1}+(\Delta-\delta\Delta)\Delta^{\nu-1} = (\delta+(2-\delta)\Delta)\Delta^{\nu-1} \leq (\delta-\Delta)\Delta^{\nu-1} \leq 0$, because $\ell+p-2 = \nu \leq \nu' = D-\ell-p+1$ and $\delta \geq 3$.

(b.2) Suppose that $\nu = \ell+p-3$, and consider two vertices $x \in F_\nu$ and $y \in \overline{F}_{\nu'}$. As every path from $x \in F_\nu$ to y goes through ω^+F, it must be that $d(x,y) \geq d(x,\omega^+F) + 1 + d(\omega^+F, y) = \nu+1+\nu' = \ell+p-3+1+D-\ell-p+1 = D-1$. As in case (b.2) of Theorem 4.3.3, we obtain that $|F_{\ell+p-3}| + |F_{D-\ell-p+1}| \leq \Delta^{\ell+p-3} + \delta\Delta^{D-\ell+p+1} - (\delta-1)$.
Therefore, the order of G must satisfy that

$$\begin{aligned} n &= \sum_{i=0}^{\ell+p-4} |F_i| + \sum_{j=0}^{D-\ell-p} |\overline{F}_j| + |F_{\ell+p-3}| + |\overline{F}_{D-\ell-p+1}| \\ &\leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1) - 1\} + \Delta^{\ell+p-3} + 1. \end{aligned}$$

(b.3) Finally, if $\ell \leq \nu \leq \ell+p-4$, then

$$n = \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{D-\ell-p+1} |\overline{F}_j| \leq \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\}. \qquad \blacksquare$$

Theorem 4.3.6 *Let G be a generalized p-cycle with $p \geq 3$, order n, maximum and minimum degrees Δ and $\delta \geq 3$ respectively, diameter D and parameter ℓ. If $\lambda_1 \leq 2\delta-2$, then*

$$n \leq \lambda_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\} + 2\Delta^{\ell+p-3} - \delta - 1.$$

Proof. As in Theorem 4.3.5, we can consider the following cases:
(a) $\nu' < D-\ell-p+1$. Then, $\ell \leq \nu \leq \nu' \leq D-\ell-p$.
(a.1) If $\ell \leq \nu \leq \ell+p-4$, then the order of G must satisfy that

$$\begin{aligned} n &= \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{\nu'} |\overline{F}_j| \leq \lambda_1\{n(\Delta,\nu) + n(\Delta, D-\ell-p)\} \\ &\leq \lambda_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\} - \lambda_1\Delta^{D-\ell-p+1}. \\ &\leq \lambda_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\}. \end{aligned}$$

(a.2) If $\nu \geq \ell+p-3$, then the order of G must satisfy that

$$\begin{aligned} n &= \sum_{i=0}^{\nu} |F_i| + \sum_{j=0}^{\nu'} |\overline{F}_j| \leq \lambda_1\{n(\Delta,\nu) + n(\Delta, D-\ell-p)\} \\ &= \lambda_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\} + \lambda_1\{\sum_{i=\ell+p-3}^{\nu} \Delta^i - \Delta^{D-\ell-p+1}\} \\ &\leq \lambda_1\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\} - \lambda_1\Delta^{\ell+p-3} \end{aligned}$$

$$\le \lambda_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+1)\},$$

since $\frac{\Delta^{\nu+1}-\Delta^{\ell+p-3}}{\Delta-1} - \Delta^{D-\ell-p+1} \le \frac{\Delta^{D-\ell-p+1}-\Delta^{\ell+p-3}}{\Delta-1} - \Delta^{D-\ell-p+1} = \frac{2-\Delta}{\Delta-1}\Delta^{D-\ell-p+1} - \frac{\Delta^{\ell+p-3}}{\Delta-1} \le \frac{1-\Delta}{\Delta-1}\Delta^{\ell+p-3} \le -\Delta^{\ell+p-3}$, because $\ell+p-3 \le \nu \le \nu' \le D-\ell-p$ and $\delta \ge 2$.

(b) $\nu' = D-\ell-p+1$. Since all the paths from $x \in F_\nu$ to $y \in \overline{F}_{\nu'}$ go through ω^+F, it must be that $D \ge d(x,y) \ge d(x,\omega^+F)+1+d(\omega^+F,y) \ge \nu+1+\nu' = \nu+1+D-\ell-p+1$. Thus, $\ell \le \nu \le \ell+p-2$.

(b.1) If $\nu = \ell+p-2$ and we consider two vertices $x \in F_\nu$ and $y \in \overline{F}_{\nu'}$, then $d(x,y) \ge \nu+1+\nu' = \ell+p-2+1+D-\ell-p+1 = D$. As in case (b.1) of Theorem 4.3.3, we conclude that $|F_{\ell+p-2}| \le \frac{\Delta}{\delta}|F_{\ell+p-1}| \le \frac{\lambda_1}{\delta}\Delta^{\ell+p-2}$. Similarly, we prove that $|\overline{F}_{D-\ell-p+1}| \le \frac{\Delta}{\delta}|\overline{F}_{D-\ell-p}| \le \frac{\lambda_1}{\delta}\Delta^{D-\ell-p+1}$. In this way, we obtain that

$$\begin{aligned} n &= \sum_{i=0}^{\ell+p-3}|F_i| + \sum_{j=0}^{D-\ell-p}|\overline{F}_j| + |F_{\ell+p-2}| + |\overline{F}_{D-\ell-p+1}| \\ &\le \lambda_1\{n(\Delta,\ell+p-3)+n(\Delta,D-\ell-p)\} + \tfrac{\lambda_1}{\delta}\{\Delta^{\ell+p-2}+\Delta^{D-\ell-p+1}\} \\ &= \lambda_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+1)\} + \lambda_1\left(\Delta^{\ell+p-3}(1+\tfrac{\Delta}{\delta}) + \Delta^{D-\ell-p+1}(\tfrac{1}{\delta}-1)\right) \\ &\le \lambda_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+1)\}, \end{aligned}$$

since $\Delta^{\nu-1}(1+\frac{\Delta}{\delta})+\Delta^{\nu'}(\frac{1}{\delta}-1) \le \Delta^{\nu-1}(1+\frac{\Delta}{\delta})+\Delta^{\nu-1}(\frac{\Delta}{\delta}-\Delta) \le \Delta^{\nu-1}(1+(\frac{2}{\delta}-1)\Delta) \le 0$, because $\ell+p-2 = \nu \le \nu' = D-\ell-p+1$ and $\delta \ge 3$.

(b.2) Suppose that $\nu = \ell+p-3$, and consider two vertices $x \in F_\nu$ and $y \in \overline{F}_{\nu'}$. As all the paths from $x \in F_\nu$ to y go through ω^+F, it must be that $d(x,y) \ge d(x,\omega^+F)+1+d(\omega^+F,y) = \nu+1+\nu' = \ell+p-3+1+D-\ell-p+1 = D-1$. As in case (b.2) of Theorem 4.3.3, we obtain that $|F_{\ell+p-3}| + |F_{D-\ell-p+1}| \le \frac{\lambda_1}{\delta}\Delta^{\ell+p-3} + \lambda_1\Delta^{D-\ell+p+1} - (\delta-1)$.

Therefore, the order of G must satisfy that

$$\begin{aligned} n &= \sum_{i=0}^{\ell+p-4}|F_i| + \sum_{j=0}^{D-\ell-p}|\overline{F}_j| + |F_{\ell+p-3}| + |\overline{F}_{D-\ell-p+1}| \\ &\le \lambda_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+1)\} + \tfrac{\lambda_1}{\delta}\Delta^{\ell+p-3} - \delta + 1 \\ &\le \lambda_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+1)\} + 2\Delta^{\ell+p-3} - \delta - 1, \end{aligned}$$

because $\frac{\lambda_1}{\delta}\Delta^{\ell+p-3} - \delta + 1 \le 2\Delta^{\ell+p-3} - \frac{2}{\delta}\Delta^{\ell+p-3} - \delta + 1 \le 2\Delta^{\ell+p-3} - \delta - 1$, since $\lambda_1 \le 2\delta-2$, $p \ge 3$, $\ell \ge 1$ and $\Delta \ge \delta$.

(b.3) Finally, if $\ell \le \nu \le \ell+p-4$, then

$$n = \sum_{i=0}^{\nu}|F_i| + \sum_{j=0}^{D-\ell-p+1}|\overline{F}_j| \le \lambda_1\{n(\Delta,\ell+p-4)+n(\Delta,D-\ell-p+1)\}. \qquad \blacksquare$$

It is interesting to note that, since $n \ge m/\Delta$, the above theorems also imply the result of Corollary 4.3.4. The following corollary gives a sufficient condition on the number of vertices for any generalized p-cycle to have $\lambda_1 > \delta$ and $\lambda_1 \ge 2\delta - 1$ respectively.

	p	$D \leq$	$n >$
$\kappa_1 > \delta$	1	$2\ell - 2$ [54]	$\delta\{n(\Delta, D-\ell) + n(\Delta, \ell-1) - 2\} + \Delta^{D-\ell+1}$ [5]
	2	$2\ell - 1$ [55]	$\delta\{n(\Delta, D-\ell-1) + n(\Delta, \ell-1) - 2\} + \Delta^{D-\ell} + \Delta^{\ell}$ [11]
	≥ 3	$2\ell + p - 3$ [*]	$\delta\{n(\Delta, D-\ell-p+2) + n(\Delta, \ell+p-4) - 2\} + \Delta^{\ell+p-3} + 1$ [*]
$\lambda_1 > \delta$	1	$2\ell - 1$ [54]	$\delta\{n(\Delta, D-2) + 1\} + \Delta^{D-1}$ [134] $\delta\{n(\Delta, D-\ell-1) + n(\Delta, \ell-1)\} + \Delta^{D-\ell}$ [63]
	2	2ℓ [55]	$\delta\{n(\Delta, D-\ell-2) + n(\Delta, \ell-1)\} + \Delta^{D-\ell-1} + \Delta^{\ell}$ [11]
	≥ 3	$2\ell + p - 2$ [*]	$\delta\{n(\Delta, D-\ell-p+1) + n(\Delta, \ell+p-4) - 1\} + \Delta^{\ell+p-3} + 1$ [*]

Table 4.2: Sufficient conditions for a generalized p-cycle to be superconnected. The * symbol indicates this work.

Corollary 4.3.5 *Let G be a generalized p-cycle with $p \geq 3$, order n, maximum and minimum degrees Δ and $\delta \geq 3$ respectively, diameter D and parameter ℓ.*

(a) $\lambda_1 > \delta$ *if* $n > \delta\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1) - 1\} + \Delta^{\ell+p-3} + 1$.

(b) $\lambda_1 \geq 2\delta - 1$ *if* $n > (2\delta-2)\{n(\Delta, \ell+p-4) + n(\Delta, D-\ell-p+1)\} + 2\Delta^{\ell+p-3} - \delta - 1$.

■

In Table 4.2, we put forward some of the main sufficient conditions that have been obtained for a generalized p-cycle to be superconnected and/or arc-superconnected.

4.4 Good superconnected generalized p-cycles

A generalized p-cycle G is said to be *good superconnected* if $\kappa_1(G) \geq p(\delta - 1)$. In a similar way, G is said to be *good arc-superconnected* if $\lambda_1(G) \geq p(\delta - 1)$. A reason for introducing these definitions can be found in the following list of properties of generalized p-cycles.

Lemma 4.4.1 *Let G be a connected generalized p-cycle, $p \geq 2$, with minimum degree $\delta \geq 2$. If G contains a δ-regular directed cycle $\vec{C}_p$ of order p, then*

1. $|\partial^+\vec{C}_p| = |w^+\vec{C}_p| = p(\delta - 1)$.
2. $\partial^+\vec{C}_p$ *[$w^+\vec{C}_p$] is a nontrivial vertex set [arc set].*
3. $\lambda_1 \leq p(\delta - 1)$.
4. *If* $\vec{C}_p \cup \partial^+\vec{C}_p \neq V$, *then* $\kappa_1 \leq p(\delta - 1)$.

Proof.

1. It is clear that $|w^+\vec{C}| = p(\delta - 1)$, because $\vec{C}_p$ has p vertices, each of them being of out-degree $\delta - 1$ in $G - \vec{C}_p$. Furthemore, if x, y are two vertices belonging to $\vec{C}_p$, then they are in two different stable parts of $V(G)$, and hence the same happens with their respective out-neighbourhoods $\Gamma^+(x)$ and $\Gamma^+(y)$. In particular, we can assure that $\Gamma^+(x) \cap \Gamma^+(y) = \emptyset$. Therefore, $|\partial^+\vec{C}_p| = p(\delta - 1)$.

2. Certainly, $w^+\vec{C}_p$ is a nontrivial arc set. As for the nontriviality of $\partial^+\vec{C}_p$, it directly follows from the fact that any subset of $\partial^+\vec{C}_p$ of cardinality δ contains vertices belonging to at least two partite sets.

   Points 3. and 4. are immediately deduced from the previous ones.

■

The rest of this section is devoted to finding diameter conditions for any generalized p-cycle to be good superconnected. Actually, we have restricted the study to those generalized p-cycles satisfying $\ell \geq p$. Next, let us show some properties on generalized p-cycles which will turn out to be useful for proving the main result.

Lemma 4.4.2 *Let G be a generalized p-cycle, $p \geq 2$, with parameter $\ell \geq p - 1$. If $x \in V(G)$ and $j \leq p - 1$, then there are at least δ^j vertices at distance j from [to] x.*

Proof. Assume, for instance, that there are less than δ^j vertices at distance j from x. This means that there must exist two distinct paths from x to some vertex y satisfying $d(x, y) \leq j$, the length of which must be congruent modulo p and at most j. Since $j \leq p - 1$, the length of these two paths coincide. But this contradicts the definition of parameter ℓ, since by hypothesis $\ell \geq p - 1$. ■

Lemma 4.4.3 *Let G be a generalized p-cycle, $p \geq 3$, with parameter $\ell \geq p$, minimum degree $\delta \geq 2$ and $\kappa_1 \leq p(\delta - 1) - 1$. If F is a positive 1-fragment of G, then*

$$\mu(F), \mu(\overline{F}) \geq \ell - \lceil \log_\delta p(\delta - 1) \rceil + 1$$

Proof. Let us firstly see that $\mu(F) \geq \ell - \lceil (p-1)/2 \rceil$. For it, assume that $1 \leq \mu(F) \leq \ell - \lceil (p - 1)/2 \rceil - 1$. As F is a 1-fragment, $\partial^+ F$ is nontrivial, and thus for every vertex of F we can consider a path P of length $p - 1$ in F, starting from this vertex. Notice that $|\partial^+ P| \geq p(\delta - 1)$ because G is a generalized p-cycle. As a consequence, there must exist a vertex u in P with out-degree in F at least two, because $|\partial^+ F| = \kappa_1 \leq p(\delta - 1) - 1$. Therefore, we can consider in F two paths starting from a vertex u, each one of length $\lceil (p - 1)/2 \rceil$, and such that $d(u, \partial^+ F) = \mu(F)$. Since G is a generalized p-cycle, these two paths coincide only in vertex u; that is, they form an out-rooted tree T with root u and height $\lceil (p-1)/2 \rceil$, and clearly, $|\partial^+ T| \geq p(\delta-1)+1$. In consequence, there must exist two distinct vertices $\omega_1, \omega_2 \in \partial^+ T$ and a vertex $f \in \partial^+ F$, satisfying $d(\omega_i, \partial^+ F) = d(\omega_i, f)$, $i = 1, 2$. Therefore, we have two different paths from u to f; namely, $u \to \omega_i \to f$, $i = 1, 2$, whose lengths belong to the set of numbers $\{\mu(F), \mu(F)+1, \ldots, \mu(F)+\lceil (p-1)/2 \rceil +1\}$. Since G is a generalized p-cycle and $p \geq 3$, we conclude that these two paths must have the same length, fact which contradicts the definition of parameter ℓ, because we are supposing that $\mu(F) \leq \ell - \lceil (p-1)/2 \rceil - 1$. Therefore, $\mu(F) \geq \ell - \lceil (p-1)/2 \rceil$.

To end the proof, let us suppose that $\lceil \log_\delta p(\delta-1) \rceil \leq \lceil (p-1)/2 \rceil$ and $\mu(F) \leq \ell - \lceil \log_\delta p(\delta-1) \rceil$. Since $h = \lceil \log_\delta p(\delta-1) \rceil - 1 < p - 1$, the result obtained in Lemma 4.4.2 allows us to consider an out-rooted tree T_z of height h, with root in a vertex z belonging to the valley of F; that is, verifying $d(z, \partial^+ F) = \mu(F)$, and such that all their internal vertices have out-degree at least δ in T_z. Moreover, the tree T_z is contained in F, since by hypothesis $\ell \geq p$ and hence, $h = \lceil \log_\delta p(\delta-1) \rceil - 1 \leq \lceil (p-1)/2 \rceil - 1 \leq \ell - \lceil (p-1)/2 \rceil - 1 < \mu(F)$. Furthermore, as $h + 1 \leq p - 1$, we conclude, again by Lemma 4.4.2, that $|\partial^+ T_z| \geq \delta^{h+1} = \delta^{\lceil \log_\delta p(\delta-1) \rceil} \geq p(\delta-1)$. In consequence, there must exist two distinct vertices $\omega_1, \omega_2 \in \partial^+ T_z$ and a vertex $f \in \partial^+ F$, satisfying $d(\omega_i, \partial^+ F) = d(\omega_i, f)$, $i = 1, 2$. Reasoning as before, we have two different paths from the root z to f whose lengths belong to the set of numbers $\{\mu(F), \mu(F)+1, \ldots, \mu(F)+h+1\}$. Since G is a p-cycle, these two paths must be congruent modulo p, and hence both paths have the same length, fact which contradicts the definition of parameter ℓ because we are supposing that $\mu(F) \leq \ell - h - 1$. Therefore, $\mu(F) \geq \ell - \lceil \log_\delta p(\delta-1) \rceil + 1$.

The same lower bound is obtained for $\mu(\overline{F})$ by considering the converse digraph of G. ■

Lemma 4.4.4 *Let G be a generalized p-cycle, $p \geq 3$, with parameter $\ell \geq p$, minimum degree $\delta \geq 2$ and $\kappa_1 \leq p(\delta-1) - 1$. If F is a positive 1-fragment of G, then*

(a) for each vertex $x \in F$ such that $d(x, \partial^+ F) = \ell - \lceil \log_\delta p(\delta-1) \rceil + 1$, there exists a $\rho : x \to z$ path in F of length $p-1$ such that for any vertex y of ρ, $d(y, \partial^+ F) \geq \ell - 2\lceil \log_\delta p(\delta-1) \rceil + 2$;

(b) for each vertex $x \in \overline{F}$ such that $d(\partial^+ F, x) = \ell - \lceil \log_\delta p(\delta-1) \rceil + 1$, there exists a $\rho : z \to x$ path in F of length $p-1$ such that for any vertex y of ρ, $d(\partial^+ F, y) \geq \ell - 2\lceil \log_\delta p(\delta-1) \rceil + 2$.

Proof. (a) Let us consider the integer $h = \lceil \log_\delta p(\delta-1) \rceil - 1$. It is easy to see that for every $\delta \geq 2$ and $p \geq 3$, the inequality $2h + 1 \leq p$ is always satisfied [9]. Since by

[9]Observe that it is enough to prove that $p^2(\delta-1)^2 \leq \delta^{p+1}$ $[p^2(\delta-1)^2 \leq \delta^p]$ if p is odd [even].

Lemma 4.4.3 we know that $\mu(F) \geq \ell - \lceil \log_\delta p(\delta-1) \rceil + 1$, and by hypothesis we suppose that $\ell \geq p$, we conclude that $h \leq \mu(F) - 1$. This fact allows us to consider in F an out-rooted tree T_x of height h with root in a vertex verifying $d(x, \partial^+ F) = \ell - h$, and such that all their internal vertices have out-degree at least δ in T_x. As $h+1 \leq p-1$, by Lemma 4.4.2, we conclude that $|\partial^+ T_x| \geq \delta^{h+1} = \delta^{\lceil \log_\delta p(\delta-1) \rceil} \geq p(\delta-1)$. Consequently, there must exist two distinct vertices $\omega_1, \omega_2 \in \partial^+ T_z$ and a vertex $f \in \partial^+ F$, satisfying $d(\omega_i, \partial^+ F) = d(\omega_i, f)$, $i = 1, 2$. As a result, we have two different paths from x to f; namely, $\rho_i : x \to w_i \to f$, i=1,2. Observe that, on the one hand, for every $y \in T_x$ $d(y, \partial^+ F) \geq \ell - 2h$, and on the other, $|\rho_i| \geq \ell - h$, $i = 1, 2$. Notice also that to end the proof it is enough to see that either w_1 or w_2 is at a distance of at least $\ell - h$ to $\partial^+ F$. Indeed, if we suppose that $d(w_1, f) < \ell - h$, then necessarily $d(w_2, f) \geq \ell - h$, because otherwise there would exist two paths from x to f, the lengths of which belonging to the set $\{\ell - h, \ell - h + 1, \ldots, \ell\}$, which is impossible because G is a generalized p-cycle of parameter ℓ.

Case (b) is proved in the same way. ∎

The main theorem of this section is put forward next.

Theorem 4.4.1 *Let G be a generalized p-cycle with parameter $\ell \geq p$ and minimum degree $\delta \geq 2$. Then,*

(a) $\kappa_1 \geq p(\delta - 1)$ if $D \leq 2\ell + p + 1 - 3\lceil \log_\delta p(\delta - 1) \rceil$;

(b) $\lambda_1 \geq p(\delta - 1)$ if $D \leq 2\ell + p + 2 - 3\lceil \log_\delta p(\delta - 1) \rceil$.

Proof. Notice that, for $p \in \{1, 2\}$, the theorem holds as a consequence of the results given in Propositions 4.1.1 and 4.1.2. To prove (a), assume thus that $p \geq 3$, $\kappa_1 \leq p(\delta - 1) - 1$ and let us consider a positive 1-fragment F. Without loss of generality, we can suppose that $\mu(F) \leq \mu(\overline{F})$ (if not, use the converse digraph of G). Therefore, as was proved in Lemma 4.4.3, we can assure that $\ell - \lceil \log_\delta p(\delta - 1) \rceil + 1 \leq \mu(F) \leq \mu(\overline{F})$, and hence, we can consider the nomempty sets:

$$A = \{x \in F, d(x, \partial^+ F) \geq \ell - 2\lceil \log_\delta p(\delta - 1) \rceil + 2\}$$

$$B = \{y \in \overline{F}, d(\partial^+ F, y) \geq \ell - 2\lceil \log_\delta p(\delta - 1) \rceil + 2\}.$$

As a consequence of the results obtained in Lemma 4.4.4, we conclude that we can find both in A and B a path of length $p-1$. As G is a p-cycle, this shows that both in A and B vertices of every partite set V_α exist. Now, let us consider the integer r, $0 \leq r \leq p-1$, so that $D+1 \equiv r (\text{mod } p)$. Let $x \in V_\alpha$ be such that $d(x, \partial^+ F) = \mu(F)$ and consider a vertex $y \in V_{\alpha+r} \cap B$. Since the length of every path from V_α to $V_{\alpha+r}$ must be congruent with r modulo p, then $D - (p-1) \geq d(x, y) \geq d(x, \partial^+ F) + d(\partial^+ F, y) \geq 2\ell - 3\lceil \log_\delta p(\delta - 1) \rceil + 3$. This means that $D \geq 2\ell + p + 2 - 3\lceil \log_\delta p(\delta - 1) \rceil$, which contradicts the hypothesis. Therefore, $\kappa_1 \geq p(\delta - 1)$.

Point (b) is easily proved from the previous one, by using the line digraph technique in the same way as it was carried out in Theorem 4.3.2, and for this reason we omit it. ∎

As in the above section, we can obtain the following corollary, which shows that if the iteration is large enough, the iterated line digraph of a generalized p-cycle is good superconnected.

Corollary 4.4.1 *Let G be a generalized p-cycle with parameter $\ell \geq p$ and minimum degree $\delta \geq 2$. Then,*

(a) $\kappa_1(L^kG) \geq p(\delta - 1)$ *if* $k \geq D - 2\ell - p - 1 + 3\lceil \log_\delta p(\delta - 1)\rceil$;

(b) $\lambda_1(L^kG) \geq p(\delta - 1)$ *if* $k \geq D - 2\ell - p - 2 + 3\lceil \log_\delta p(\delta - 1)\rceil$.

Finally, let us apply this new results to the Moore generalized p-cycles (see Section 1.5.4).

Examples 4.4.1

1. The complete generalized p-cycle $\vec{C}_p \otimes K_d^*$ is d-regular, has diameter p and parameter $\ell = 1$. Therefore, it satisfies: $\kappa = \lambda = d$ and $\lambda_1 \geq 2d - 1$.

2. The De Bruijn generalized p-cycle $BGC(p, d, d^k)$ is isomorphic to the $(k-1)$-iterated line digraph of $\vec{C}_p \otimes K_d^*$. This means that this digraph is d-regular, has diameter $p + k - 1$ and parameter $\ell = k$. As a consequence, we obtain that for every $k \geq 2$, $BGC(p, d, d^k)$ verifies: $\kappa = \lambda = d$, $\lambda_1 \geq 2d - 1$ and $\kappa_1 \geq 2d - 1$. Moreover, since $BGC(p, d, d^p)$ is a d-regular digraph with diameter $D = 2p - 1$ and parameter $\ell = p$, it is good superconnected when $3 \log_d p(d - 1) \leq p + 2$.

3. The Kautz generalized p-cycle $KGC(p, d, d^p + 1)$ is d-regular, has diameter $D = 2p - 1$ and parameter $\ell = p$. Therefore, it satisfies: $\kappa = \lambda = d$, $\lambda_1 \geq 2d - 1$ and $\kappa_1 \geq 2d - 1$. Furthermore, the digraph $KGC(p, d, d^{p+k} + d^k)$ is isomorphic to $L^k(KGC(p, d, d^p + 1))$, and this fact allows us to conclude that it is good superconnected for $k \geq 3\lceil \log_d p(d - 1)\rceil - p - 2$.

Chapter 5

Superconnectivity and extraconnectivity of digraphs

5.1 Introduction

Graph and digraph connectedness has always been one of the main topics of interest in graph theory, with a wide range of relevant applications in many different areas. For instance, in the design of reliable communication or interconnection networks, where the connectivity properties of the topology used to link the nodes is one of the main features to be taken into account.

In the design of a good topology for an interconnection network, the first connectedness property of interest is the avoidance of small cutsets. If δ is the minimum number of links incident from any node, it is convenient that the network should still connect the surviving elements after the failure of at most $\delta - 1$ nodes or links. This means that the graph or digraph used to model the network must be maximally connected. Proceeding one step further, among the maximally connected networks, the aim is to maximize the size of a minimum cutset, apart from the 'trivial' cutsets constituted by the set of nodes or edges incident to some given node v (since the failure of the elements of such cutset isolates v from the rest of the network). This leads to the study of superconnected graphs and digraphs. A logical generalization of these ideas can be to look for interconnection networks having η-*nontrivial* cutsets with as large as possible cardinality, where η-*nontrivial* means that the considered cutsets contain neither the out-neighbourhood nor the in-neighbourhood of a vertex set formed by a given number η of nodes. These are the so-called η-*extraconnected* digraphs. The following section of this chapter is devoted to the theoretical analysis of all these connectedness parameters, starting from the so-called *conditional connectivity* introduced by Harary.

As has been said several times along this work, the parameter ℓ has proved to be a suitable tool for studying both connectivity and superconnectivity, especially under diameter conditions (see Sections 1.6, 1.7.1 and 4.1). Furthermore, after studying in detail the different constructive proofs of the results on connectedness under diameter sufficient conditions involving these parameters (see Propositions 4.1.1 and 4.1.2), we realized that all of them were very similar in a certain sense. Section 5.3 is devoted to presenting in a structured and simpler way the model of constructive proof which has been repeatedly used.

In Section 5.4, the so-called *η-nontrivial disconnecting sets* are introduced and they are also shown some interesting properties. These results will enable to approach the study of the extraconnected digraphs under diameter conditions using the aforementioned model of algorithmical proof. The following section is devoted to generalizing the definition of the parameter ℓ. As a consequence, we introduce the so-called FF-parameters $\ell(\alpha, \eta, \pi)$ which will prove to be very useful in some particular cases.

The last three sections show a list of new results on connectedness under diameter sufficient conditions, all of them proved by using the so-called PW-algorithm proposed in Section 5.3. As for the first of them, it is, in our opinion, particularly interesting the result stated in Theorem 5.6.2, because it shows a fact which had passed unnoticed so far, and it can be, not only the key point to re-prove in a simpler way the classical results (this issue is also shown in this section), but also the starting point to obtain new connectedness results of different kinds, for example, under degree conditions. In Section 5.7, a new theorem involving the FF-parameter $\ell^{\pi} = \ell(1, 1, \pi)$ is presented, wich improves a result by Fàbrega and Fiol (see [54]). Moreover, this improvement is significant because it enables, unlike the classical theorem, to approach the study of the superconnectivity in maximally connected digraphs. Finally, Section 5.8 is devoted to the study of η-extraconnected digraphs with large girth under diameter conditions, by using the FF-parameter $\ell_{\eta} = \ell(1, \eta, 0)$. The section and chapter finish by applying the obtained results to some families of large iterated line digraphs.

And now, before starting the following section, let us comment some issues related to the notation and terminology employed.

Remark 5.1.1 Let F be a set of vertices of a connected digraph $G = (V, A)$, and x a vertex belonging to $V \setminus F$ such that $d(x, F) = d$. In the rest of this chapter, we make use of the following vertex sets related to x:

- For every $\hat{d} \geq d$, the set of vertices in F at distance at most $\hat{d}$ from x:

$$F^{+}_{\hat{d}}(x) = \{f \in F \mid d(x, f) \leq \hat{d}\}. \tag{5.1}$$

  When $d(F, x) = d \leq \hat{d}$, the set $F^{-}_{\hat{d}}(x)$ is similarly defined.

- The set of vertices adjacent from x which are on the shortest paths from x to the vertices of F:

$$\nu(x \to F) = \{y \in \Gamma^{+}(x) \mid \exists f \in F \ s.t. \ d(y, f) = d(x, f) - 1\}. \tag{5.2}$$

- When there is a unique path from x to a vertex $f \in F$, it is denoted by $x \mapsto f$. As expected, the vertex in this path belonging to $\Gamma^+(x)$ is denoted by $\nu(x \mapsto f)$.
- The set of vertices adjacent from x which are not on the shortest paths from x to the vertices of $F^+_{d+1}(x)$:

$$\Theta^+(x) = \Gamma^+(x) \setminus \nu(x \to F^+_{d+1}(x)). \tag{5.3}$$

5.2 Conditional connectivities

Given a (di)graph G and a (di)graph-theoretic property $\mathcal{P}$, Harary defined in [91] the *(universal) conditional connectivity* $\kappa(G;\mathcal{P})$ [*arc-connectivity* $\lambda(G;\mathcal{P})$] as the minimum cardinality of a set of vertices [arcs], if any, whose deletion disconnects G and every remaining component has property $\mathcal{P}$. Observe that if ϕ represents the empty set of properties, then of course $\kappa(G;\phi) = \kappa(G)$ when G is not complete, and $\lambda(G;\phi) = \lambda(G)$ if $G \neq K_1$. In the aforementioned article, Harary points out several ways in which this definition can be meaningfully modified. For example, one can relax the requirement of every component having property $\mathcal{P}$ by demanding this property be satisfied by at least two of the components, while the remaining need not be so [1]. Bearing these ideas in mind, we propose a new kind of conditional connectivity, which we will use in the rest of this section.

Definition 5.2.1 *Let G be a (di)graph and $\mathcal{P}$ a (di)graph-theoretic property. The S-conditional connectivity $\kappa_s(G;\mathcal{P})$ [$\lambda_s(G;\mathcal{P})$] is defined as the minimum cardinality of a set of vertices [arcs], if any, whose deletion disconnects G and every remaining non-transmittance component has property $\mathcal{P}$.*

Certainly, the previous definition can be alternatively stated by saying that both source and sink components must satisfy property $\mathcal{P}$, while the transmittance components need not do so. Notice also that every component of a disconnected graph is both a sink and a source, and hence $\kappa_s(G;\mathcal{P}) = \kappa(G;\mathcal{P})$, $\lambda_s(G;\mathcal{P}) = \lambda(G;\mathcal{P})$.

In a similar way as we have seen that the connectivity [2] can be considered as a particular case of universal conditional connectivity, it is also easy to verify that the superconnectivity $\kappa_1(G)$ of a certain (di)graph G coincides with its S-conditional connectivity $\kappa_s(G;\mathcal{P}_1)$, where $\mathcal{P}_1$ is the property of having more than one vertex.

Fàbrega and Fiol introduced in [56] the so-called *η-extraconnectivity* $\kappa_\eta(G)$ of a graph G as the universal conditional connectivity $\kappa(G;\mathcal{P}_\eta)$, $\mathcal{P}_\eta$ being the property of having more than η vertices. Observe that the 1-extraconnectivity of a graph coincides with its superconnectivity. With the aim of generalizing this definition to the case of digraphs, we present the following definition.

[1] In [27], Bollobás implicitly defined the *cyclic edge-connectivity* of a graph G as the smallest cardinality of a disconnecting set S of edges, if any, for which at least two components of $G - S$ contain a cycle (see p. 113).

[2] Also called *unconditional connectivity* by Harary in [91].

Definition 5.2.2 *Let G be a (di)graph and $\eta \geq 0$ an integer. The η-extraconnectivity $\kappa_\eta(G)$ [η-arc-extraconnectivity $\lambda_\eta(G)$] is defined as the S-conditional connectivity $\kappa_s(G;\mathcal{P}_\eta)$ [$\lambda_s(G;\mathcal{P}_\eta)$].*

Certainly, the 0-extraconnectivity $\kappa_0(G) = \kappa_s(G;\mathcal{P}_0) = \kappa(G;\mathcal{P}_0)$ and 1-extraconnectivity $\kappa_1(G) = \kappa_s(G;\mathcal{P}_1)$ of a (di)graph G coincide with its connectivity $\kappa(G)$ and superconnectivity [3] $\kappa_1(G)$ respectively (the same happens in the arc case). Notice also that if G is a digraph with girth g, then, for every $\eta \leq g-1$, the η-extraconnectivity $\kappa_\eta(G)$ [$\lambda_\eta(G)$] can be seen as the minimum cardinality of a set of vertices [arcs], if any, whose deletion disconnects G, and every remaining component has either at least g vertices or a single vertex, this last case being a transmittance component.

A digraph G is said to be *η-extraconnected* if $\kappa_\eta(G) \geq \tau_\eta = (\eta+1)(\delta-1)$. In a similar way, G is said to be *η-arc-extraconnected* if $\lambda_\eta(G) \geq \tau_\eta$. A good reason for introducing these definitions can be found in the following lemma, whose proof is similar to that of Lemma 4.4.1.

Lemma 5.2.1 *Let G be a connected digraph with minimum degree $\delta \geq 2$ and girth $g < \delta$. If G contains a δ-regular directed cycle $\vec{C}_g$ of order g, then*

1. $|\partial^+\vec{C}_g| \leq |w^+\vec{C}_g| = g(\delta-1)$.
2. $w^+\vec{C}_g$ *is a nontrivial arc set.*
3. $\lambda_1 \leq g(\delta-1)$.
4. *If $\partial^+\vec{C}_g$ is a nontrivial vertex set and $\vec{C}_g \cup \partial^+\vec{C}_g \neq V$, then $\kappa_1 \leq g(\delta-1)$.*

Observe that for $\delta \geq 3$, every 1-extraconnected digraph is superconnected, and it is possibly for this reason that they are often called *optimally superconnected* (see [5]). Notice also that, for $\eta \geq 1$, an $(\eta+1)$-extraconnected digraph needs not be η-extraconnected because, for instance, it could verify the following inequality sequence: $\delta \leq \kappa_1 \leq 2\delta-2 < 3\delta-3 \leq \kappa_3$ (see Section 5.8 for more details).

Finally, another possibility of modifying the original definition of conditional connectivity given by Harary consists in demanding that a certain property $\mathcal{P}$ be satisfied by the cutsets [arc-cutsets], instead of the remaining components. This idea leads us to put forward the following new kind of conditional connectivity.

Definition 5.2.3 *Let G be a (di)graph and $\mathcal{P}$ a (di)graph-theoretic property. The C-conditional connectivity $\kappa_c(G;\mathcal{P})$ [$\lambda_c(G;\mathcal{P})$] is defined as the minimum cardinality of a set of vertices [arcs] satisfying property $\mathcal{P}$, if any, whose deletion disconnects G.*

Although the study of certain C-conditional connectivities may be interesting in itself, we are mainly concerned in this work with some S-conditional connectivities;

[3] For this reason, we have used the same notation.

namely, the connectivity, superconnectivity and extraconnectivity, both in graphs and digraphs. For this reason, given a (di)graph-theoretic property $\mathcal{P}$, a suitable way of studying the S-conditional connectivity $\kappa_s(G;\mathcal{P})$ for a certain family of (di)graphs is frequently to look for another property $\hat{\mathcal{P}}$ in such a way that $\kappa_c(G;\hat{\mathcal{P}}) = \kappa_s(G;\mathcal{P})$. Indeed, whenever this goal is achieved, what remains to be done afterwards, for the aforementioned (di)graph family, is to study the $\kappa_c(G;\hat{\mathcal{P}})$ parameter.

As pointed out in Section 1.7.1, one of the most important connectedness problems addressed in recent years has been to find sufficient conditions for a (di)graph to be maximally connected, superconnected, etc. Among the different sufficient conditions which have been considered (see Propositions 1.7.2, 1.7.3, and 1.7.4), in this work we are solely concerned with the so-called diameter conditions. As stated in the aforementioned section, these kinds of sufficient conditions involve finding an upper bound on the diameter for a certain class of (di)graphs in order to assure maximal connectivity, superconnectivity, etc.

In the following section we propose an algorithm; that is, a particular type of constructive proof, designed for the study of C-conditional connectivities, starting from a certain diameter sufficient condition imposed on a particular family of (di)graphs.

5.3 Progressive withdrawal algorithm

Suppose we are interested in proving a theorem such as the following.

Theorem 5.3.1 *Let Υ be a family of connected (di)graphs and $G \in \Upsilon$. Let $\hat{\mathcal{P}}$ be a (di)graph-theoretic property and $\hat{\mu}, \hat{\rho} \geq 2$ two fixed integers. Then*

$$D(G) \leq 2\hat{\mu} - 1 \;\Rightarrow\; \kappa_c(G;\hat{\mathcal{P}}) \geq \hat{\rho}$$

First of all, it is clear that the previous result can be seen as a corollary of the following proposition.

Proposition 5.3.1 *Let F be a vertex subset of G satisfying property $\hat{\mathcal{P}}$, with cardinality $|F| < \hat{\rho}$. Then,*

$$D(G) \leq 2\hat{\mu} - 1 \;\Rightarrow\; G - F \text{ is connected.}$$

Secondly, this proposition is certainly a corollary of the following lemma (see Figure 5.1).

Lemma 5.3.1 *Let F be a vertex subset of G satisfying property $\hat{\mathcal{P}}$, with cardinality $|F| < \hat{\rho}$. Then, for every vertex $u \in V \setminus F$*

1. *there exists a $u \to v$ path in $V \setminus F$ such that: $d(v, F) \geq \hat{\mu}$.*
2. *there exists a $w \to u$ path in $V \setminus F$ such that: $d(F, w) \geq \hat{\mu}$.*

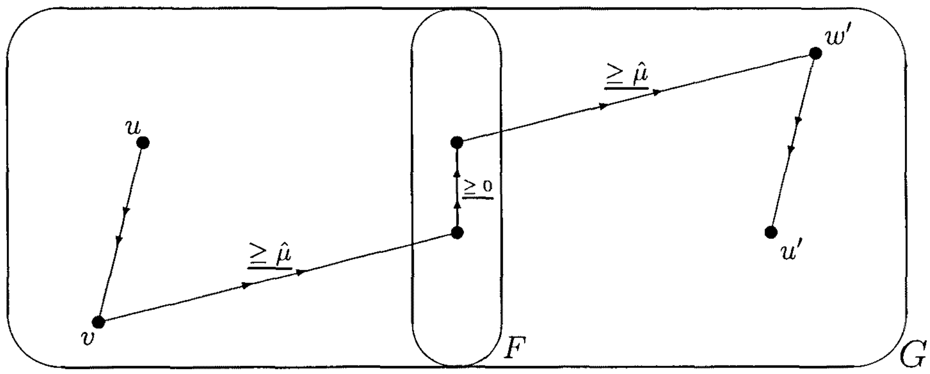

Figure 5.1: If $G - F$ is disconnected, then necessarily $D \geq 2\hat{\mu}$.

Finally, the proof of this lemma can be algorithmically approached by carrying out the so-called *progressive withdrawal algorithm*, or simply *PW-algorithm*, which is put forward next.

Theorem 5.3.2 *Let Υ be a family of connected (di)graphs and $G = (V, A) \in \Upsilon$. Let $\hat{\mathcal{P}}$ be a (di)graph-theoretic property and $\hat{\mu}, \hat{\rho} \geq 2$ two integers. Let F be a vertex subset of G satisfying property $\hat{\mathcal{P}}$, with cardinality $|F| < \hat{\rho}$. If $u \in V \setminus F$ such that $d(u, F) = d \leq \hat{\mu} - 1$, then there exists a $u \to v$ path in $V \setminus F$ satisfying $d(v, F) \geq d$ and:*

[PW1] $F \setminus F_d^+(v) \neq \emptyset$.

[PW2] $2 \leq |F_d^+(u)| < |F| \Rightarrow |F_d^+(v)| < |F_d^+(u)|$.

[PW3] $|F_d^+(u)| = 1 \Rightarrow d(v, F) \geq d + 1$.

Remarks 5.3.1

1. The arc case; that is, the study of a certain C-conditional arc-connectivity $\lambda_c(G; \hat{\mathcal{P}})$, can be similarly approached by considering the corresponding arc version of the foregoing theorem.

2. Sometimes, it can be more suitable to prove the first two steps of the algorithm at the same time. This means that $[PW1]$ and $[PW2]$ can be replaced by this single statement:
$$[PW1 + PW2] : \quad |F_d^+(v)| \leq 1 \tag{5.4}$$

3. It is also possible to prove all three steps at once. In other words, it is enough to prove this single assertion:

   [PW1+PW2+PW3]: If $u \in V \setminus F$ such that $d(u, F) = d \leq \hat{\mu} - 1$, then there exists a $u \to v$ path in $V \setminus F$ satisfying $d(v, F) \geq d$ and $|F_d^+(v)| < |F_d^+(u)|$.

4. Furthermore, it can be interesting to prove a result similar to those stated in Theorem 5.3.1, but with the new hypothesis: $D(G) \leq 2\hat{\mu}$. In this case, a variant of this algorithm can be used, which consists of the following four steps:

   (a) If $u \in V \setminus F$ such that $d(u, F) = d \leq \hat{\mu}$, then there exists a $u \to v$ path in $V \setminus F$ satisfying $d(v, F) \geq d$ and:

   [PW1'] $F \setminus F_d^+(v) \neq \emptyset$.

   [PW2'] $2 \leq |F_d^+(u)| < |F| \Rightarrow |F_d^+(v)| < |F_d^+(u)|$.

   [PW3'] $d \leq \hat{\mu} - 1$, $|F_d^+(u)| = 1 \Rightarrow d(v, F) \geq d + 1$.

   (b) If $u \in V \setminus F$ such that $d(u, F) = d \geq \hat{\mu}$ and $\hat{f} \in F$, then there exists a $u \to v$ path in $V \setminus F$ satisfying $d(v, F) \geq d$ and:

   [PW4'] $|F_{\hat{\mu}}^+(v)| \leq 1$ and $\hat{f} \notin F_{\hat{\mu}}^+(v)$.

5. In some cases, an extra difficulty may appear when trying to prove the different steps for $d = 1$. On such occasions, a solution to this problem should be added as a first step:

$$[PW0]: \quad |F_d^+(v)| \leq 2 \tag{5.5}$$

6. In the last sections of this chapter, different versions of the PW-algorithm are used to prove either known results in a more efficient way (see Section 5.6) or several new results (see Sections 5.7 and 5.8).

5.4 η-nontrivial disconnecting sets

Let $G = (V, A)$ be a (di)graph and $F \subset V$ [$F \subset A$] a vertex [arc] set. In Section 1.7.1, F was said to be trivial if it contained either the in-neighbourhood [in-arc-neighbourhood] or the out-neighbourhood [out-arc-neighbourhood] of some vertex not belonging to F. Generalizing this definition, we say that a subset of vertices F is *η-trivial*, where $\eta \geq 1$ is a fixed integer, if there exists a vertex set $S \subset V \setminus F$, with $1 \leq |S| \leq \eta$, such that F contains either $\partial^+ S$ or $\partial^- S$. That is, the deletion of the vertices of F isolates a subdigraph of G with at most η vertices. Analogously, an arc set $F \subset A$ is called *η-trivial* if there exists a vertex set $S \subset V$, with $1 \leq |S| \leq \eta$, such that F contains either its positive arc-boundary $\omega^+ S$ or its negative arc-boundary $\omega^- S$. A subset of vertices or arcs that is not η-trivial for a certain $\eta \geq 1$ is said to be *η-nontrivial*. Furthermore, every vertex or arc set will we supposed to be *0-nontrivial*. As an immediate consequence of this definition, we obtain the following list of properties.

Proposition 5.4.1 *Let $G = (V, A)$ be a (di)graph, $\eta \geq 0$ an integer, and $F \subset V$ [$F \subset A$] a vertex [arc] set. If F is a vertex [arc] set of G, then*

1. *F is nontrivial if and only if it is 1-nontrivial.*

2. *If F is $(\eta+1)$-nontrivial then it is η-nontrivial.*

3. *F is an η-nontrivial cutset [arc-cutset] if and only if every non-transmittance component of $G-F$ has at least $\eta+1$ vertices.*

4. *If $\hat{\mathcal{P}}_\eta$ denotes the property of being η-nontrivial, then $\kappa_\eta(G) = \kappa_c(G; \hat{\mathcal{P}}_\eta)$ [$\lambda_\eta(G) = \lambda_c(G; \hat{\mathcal{P}}_\eta)$].*

5. $\kappa_0 \leq \kappa_1 \leq \cdots \leq \kappa_\eta$, [$\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_\eta$]

Observe that as a consequence of the fourth property from the previous list, we can conclude that the study of the η-extraconnectivity $\kappa_\eta(G) = \kappa_s(G; \mathcal{P}_\eta)$ in a certain (di)graph G can be approached by means of the PW-algorithm described in Section 5.3.

There exist some superconnected digraphs in which every disconnecting vertex set is trivial; that is to say, without nontrivial disconnecting sets. For instance, the complete symmetric digraphs K_n and the complete symmetric bipartite digraphs $K_{n,m}$ are of this type, see [91]. Certainly, in a certain sense, such digraphs can be considered as optimally superconnected, since they cannot possibly be disconnected unless one vertex is isolated. Similarly, the complete symmetric digraph K_3 is a superconnected digraph without nontrivial arc-disconnecting sets. From now on, we will only deal with digraphs containing both nontrivial vertex- and arc-disconnecting sets.

5.5 FF-parameters $\ell(\alpha, \eta, \pi)$

As was pointed out in Section 1.6, the parameter ℓ has proved to be an excellent tool for studying some fault tolerance topics in graphs and digraphs, especially their connectedness. This section is devoted to the introduction of the so-called *FF-parameters* $\ell(\alpha, \eta, \pi)$, by generalizing the definition of parameter ℓ (see Definition 1.6.1) as follows.

Definition 5.5.1 *Let $G = (V, A)$ be a simple connected (di)graph with diameter D, and $\alpha \geq 1, \eta \geq 1, \pi \geq 0$ three fixed integers. The FF-parameter $\ell(\alpha, \eta, \pi)$ of G is defined as the greatest integer belonging to $\{1, \ldots, D\}$ such that for any $x, y \in V$, if $0 \leq d(x, y) = d \leq \ell(\alpha, \eta, \pi)$, then:*

1. *There exist, at most, α paths from x to y of length d.*

2. *If $d < \ell(\alpha, \eta, \pi)$: There exist, at most, π paths from x to y of length between $d+1$ and $min\{d+\eta, \ell(\alpha, \eta, \pi)\}$.*

Certainly, when $\alpha = 1$, $\eta = 1$ and $\pi = 0$ we obtain the definiton of parameter ℓ; that is, $\ell = \ell(1, 1, 0)$. Apart from this case, we have considered two more particular cases, which, for the sake of simplicity, are denoted as follows [4]:

$$\ell(1, 1, \pi) = \ell^\pi, \quad \ell(1, \eta, 0) = \ell_\eta.$$

[4]As a matter of fact, the FF-parameter ℓ^π was introduced by Fàbrega and Fiol in [54].

Notice that, with this alternative notation, $\ell = \ell^0 = \ell_1$. Observe also that, from the previous definition, this inequality sequence immediately follows :

$$\ell_\eta \le \cdots \le \ell_2 \le \ell_1 = \ell = \ell^0 \le \ell^1 \le \cdots \le \ell^\pi$$

Finally, a proposition showing the good behaviour of the FF-parameters ℓ^π and ℓ_η is now put forward (see Proposition 1.6).

Proposition 5.5.1 *Let G be a simple connected digraph with minimum degree $\delta \ge 2$, diameter D and girth g. If LG denotes the line digraph of G, then*

1. *([54]) $\ell^\pi(LG) = \ell^\pi(G) + 1$.*
2. *$1 \le \eta \le g - 1 \Rightarrow \ell_\eta(LG) = \ell_\eta(G) + 1$.*

Proof.

1. Observe first of all that LG is a simple connected digraph whose diameter is $D' = D(LG) = D + 1$. Notice also that if $x = x_0x_1$ and $y = y_0y_1$ are two vertices belonging to LG such that $1 \le d(x, y) = d \le \ell^\pi(G) + 1 \le D + 1 = D'$, then $0 \le d(x_1, y_0) = d - 1 \le \ell^\pi(G)$.

   Suppose that $d \ge 2$ and there are two $x \to y$ paths of length d in LG. This means that there are two $x_1 \to y_0$ paths of length $d - 1$ in G, which contradict [5] the hypothesis $d \le \ell^\pi(G) + 1$. Assume now that $d < \ell^\pi(G) + 1$ and there exist $\pi + 1$ paths from x to y of length $d + 1$ in LG. This means that there would be $\pi + 1$ paths from x_1 to y_0 of length d in G, which contradict [6] the aforementioned hypothesis $d < \ell^\pi(G) + 1$. Hence, we have proved that $\ell^\pi(G) + 1 \le \ell^\pi(LG)$.

   Notice that if $\ell^\pi(G) = D$, then the obtained inequality $\ell^\pi(G) + 1 \le \ell^\pi(G)$ must necessarily be an equality. Thus assume that $\ell^\pi(G) < D$ and $\ell^\pi(LG) \ge \ell^\pi(G) + 2$. If $\ell^\pi(G) = \ell^\pi < \ell^\pi + 1$, then there exist two vertices $a, b \in V$ such that either

   1. $d(a, b) = \ell^\pi$ and there exist $\pi + 1$ paths from a to b of length $\ell^\pi + 1$, or
   2. $d(a, b) = \ell^\pi + 1$ and there exist two $a \to b$ paths of length $\ell^\pi + 1$.

   As $\delta \ge 2$, we can consider a vertex $c \in \Gamma^-(a)\setminus\{b\}$ and a vertex $d \in \Gamma^+(b)\setminus\{a\}$. As a consequence, we conclude that if case 1. is verified, then $d(ca, bd) = \ell^\pi + 1$, and there would be $\pi + 1$ paths from ca to bd of length $\ell^\pi + 2$, which contradicts the hypothesis $\ell^\pi(LG) \ge \ell^\pi(G) + 2$. Analogously, if case 2. occurs, then $d(ca, bd) = \ell^\pi + 2$, and there would be two $ca \to bd$ paths of length $\ell^\pi + 2$, again contradicting the same hypothesis.

[5]Certainly, for $d = 2$, this situation is not possible.
[6]Certainly, for $d = 1$, this situation is not possible.

2. As in the previous case, let us see that for the integer $\ell_\eta + 1$ the conditions of Definition 5.5.1 hold. To this end, let us consider, first of all, a vertex uu' of LG. Any $uu' \to uu'$ path in LG different from the trivial one (of length zero) has length at least equal to the girth g. Therefore, the shortest trivial $uu' \to uu'$ path in LG is unique and there are no paths of length d', with $1 \le d' \le \min\{\eta, \ell_\eta+1\} \le g-1$. Suppose now that uu' and vv' are two different vertices of LG such that $1 \le d_{LG}(uu', vv') \le \ell_\eta + 1$. This means that $0 \le d_G(u', v) \le \ell_\eta$, and hence, the shortest $u' \to v$ path in G is unique and there are no $u' \to v$ paths of length d', with $d+1 \le d' \le \min\{d+\eta, \ell_\eta\}$, where $d = d_G(u', v)$. As a consequence, the shortest $uu' \to vv'$ path in LG is unique and there are no $uu' \to vv'$ paths of length d', with $d+2 \le d' \le \min\{d+1+\eta, \ell_\eta+1\}$. Hence, we have proved that $\ell_\eta(LG) \ge \ell_\eta + 1$. Moreover, since the digraph G has parameter ℓ_η, there must exist two different vertices u' and v, with $d_G(u', v) = d \le \ell_\eta + 1$, and such that either the shortest $u' \to v$ path is not unique or there exists at least another $u' \to v$ path with length d' such that $d+1 \le d' \le \min\{d+\eta, \ell_\eta+1\}$. But, since $\delta \ge 2$, this implies that there exist different vertices uu' and vv' of LG, with $d_{LG}(uu', vv') = d+1 \le \ell_\eta + 2$, and such that either the shortest $uu' \to vv'$ path is not unique or there exists at least another $uu' \to vv'$ path with length d' such that $d+2 \le d' \le \min\{d+1+\eta, \ell_\eta+2\}$. Thus, the parameter $\ell_\eta(LG)$ can not be greater than $\ell_\eta + 1$. ∎

In the rest of this chapter, a digraph G with FF-parameter $\ell(\alpha, \eta, \pi)$ will be referred as to be an *$\ell(\alpha, \eta, \pi)$-digraph.*

5.6 Superconnected ℓ-digraphs

This section is mainly devoted to presenting a new result involving ℓ-digraphs, which has been obtained thanks to the PW-algorithm introduced in Section 5.3. As a matter of fact, the first objective was to prove the 'classical' result shown in the next theorem by means of the aforementioned algorithm.

Theorem 5.6.1 *([66]) Let G be a digraph with minimum degree $\delta \ge 3$, diameter D and parameter $\ell \ge 2$. Then, $\kappa_1 \ge 2\delta - 2$ [$\lambda_1 \ge 2\delta - 2$] if $D(G) \le 2\ell - 2$ [$D(G) \le 2\ell - 1$].*

When trying to prove the vertex case of the above statement with the PW-algorithm, we realized that it would not be possible, unless the following result was true.

Theorem 5.6.2 *Let $G = (V, A)$ be a (δ, ℓ)-digraph such that $\delta \ge 3$ and $\ell \ge 2$. Let F be a nontrivial vertex set with cardinality $|F| \le 2\delta - 3$, $x \in V \setminus F$ and $\hat{f} \in F$. Then,*

1. *there exists an $x \to w$ path in $V \setminus F$ such that: $d(w, F) \ge \ell - 1$, $|F^+_{\ell-1}(w)| \le 1$ and $\hat{f} \notin F^+_{\ell-1}(w)$.*

2. *there exists a $w \to x$ path in $V \setminus F$ such that: $d(F, w) \geq \ell - 1$, $|F^-_{\ell-1}(w)| \leq 1$ and $\hat{f} \notin F^-_{\ell-1}(w)$.*

Fortunately, we succeeded in proving this statement, and the rest of this section is devoted to presenting the proof of the different steps of the *PW*-algorithm, carried out under the hypotheses of this theorem. First of all, however, let us consider two previous lemmas (see Remark 5.1.1).

Lemma 5.6.1 *Let $F \subset V$ and let x be a vertex in $V \setminus F$ such that $d(x, F) = d \leq \ell - 1$. Then,*

1. *$d(y, F) \geq d + 1$, for any $y \in \Gamma^+(x) \setminus \nu(x \to F)$.*
2. *$F^+_d(y) \cap F^+_d(y') = \emptyset$ for any two different vertices $y, y' \in \Gamma^+(x)$.*
3. *$F^+_d(y) \cap F^+_d(x) = \emptyset$ for any $y \in \Gamma^+(x) \setminus \nu(x \to F^+_d(x))$.*

Proof.

1. Let f be a vertex in F. If $d(x, f) < \ell$, then the $xy \to f$ path must have length at least $d(x, f) + 2$ because the shortest $x \mapsto f$ path is unique and there are no $x \to f$ paths of length $d(x, f) + 1$. Hence, $d(y, f) \geq d(x, f) + 1$. If $d(x, f) = \ell$, then $d(y, f) \geq \ell$, for otherwise the definition of the parameter ℓ is again contradicted. Finally, if $d(x, f) > \ell$, then $d(y, f) \geq \ell$ by the triangular inequality. Since $d(x, F) = \min\{d(x, f) : f \in F\}$, the lemma holds.

2. We have $d(y, F) \geq d - 1$ for any $y \in \Gamma^+(x)$. If either $d(y, F)$ or $d(y', F)$ is greater than d, then the result trivially holds. Hence, assume that $d(y, F)$ and $d(y', F)$ are equal to d or $d - 1$, and suppose that $f \in F^+_d(y) \cap F^+_d(y')$. In this case, we have two $x \to f$ paths; namely, $x, y \to f$ and $x, y' \to f$, whose lengths are $d + 1$ or d according to the value d or $d - 1$ of $d(y, f)$ and $d(y', f)$. But this fact contradicts the definition of the parameter ℓ because the shortest $x \mapsto f$ path has length $d(x, f) \geq d(x, F) = d$. Therefore, $F^+_d(y) \cap F^+_d(y') = \emptyset$.

3. If $y \in \Gamma^+(x) \setminus \nu(x \to F^+_d(x))$, then $d(y, F^+_d(x)) \geq d + 1$. Hence, $F^+_d(y) \subset F \setminus F^+_d(x)$ and the result follows. ∎

Lemma 5.6.2 *Let $\delta \geq 3$, $F \subset V$, $|F| \leq 2\delta - 3$ and $x \in V \setminus F$. If $d(x, F) = d \leq \ell - 1$ and $2 \leq |F^+_d(x)| \leq \delta^+(x) - 1$, then there exists a vertex $y \in \Gamma^+(x)$ satisfying $d(y, F) \geq d$ and $|F^+_d(y)| < |F^+_d(x)|$.*

Proof. Let $r = |F^+_d(x)| \leq \delta^+(x) - 1$ and let Y be a set of $\delta^+(x) - r$ vertices adjacent from x that are not in the shortest $x \to F^+_d(x)$ paths. By Lemma 5.6.1, we have $F^+_d(y) \cap F^+_d(y') = \emptyset$ for any two different vertices y and y' of Y and $F^+_d(y) \subset F \setminus F^+_d(x)$

for any $y \in Y$. We are going to show that at least one $y \in Y$ satisfies the lemma; that is, $|F_d^+(y)| < |F_d^+(x)| = r$. On the contrary, if $|F_d^+(y)| \geq r$ for any $y \in Y$, we would have

$$(\delta^+(x) - r)r \leq \sum_{y \in Y} |F_d^+(y)| \leq |F \setminus F_d^+(x)| \leq 2\delta - 3 - r \leq 2\delta^+(x) - 3 - r,$$

which would imply that $(r-2)\delta^+(x) \leq r^2 - r - 3$. So, by taking into account that $\delta^+(x) \geq r + 1$ and $r^2 - r - 3 = (r-2)(r+1) - 1$, we get a contradiction if $r \geq 2$. ■

We are now ready to show the proof of Theorem 5.6.2.

Proof of Theorem 5.6.2 As expected, the proofs of both points are identical, and thus we only show the first. To this end, we make use of the variant of the *PW*-algorithm put forward in Remarks 5.3.1.4, but also taking into account the second point on this list of remarks. To be more precise, we intend to prove the following assertions:

Let $F \subset V$ be nontrivial and $x \in V \setminus F$ such that $|F| \leq 2\delta - 3$ and $1 \leq d(x, F) = d \leq \ell - 1$ respectively. Then,

(a) there exists an $x \to x'$ path in $V \setminus F$ satisfying $d(x', F) \geq d$ and:

[PW1'+PW2'] $|F_d^+(x')| \leq 1$.

[PW3'] $d \leq \ell - 2$, $|F_d^+(x)| = 1 \Rightarrow d(x', F) \geq d + 1$.

(b) if $d = \ell - 1$ and $\hat{f} \in F$, there exists an $x \to x'$ path in $V \setminus F$ satisfying $d(x', F) \geq \ell - 1$ and:

[PW4'] $|F_{\ell-1}^+(x')| \leq 1$ and $\hat{f} \notin F_{\ell-1}^+(x')$.

• *Proof of [PW1'+PW2']*: Let us distinguish the following two cases: i. $d = 1$ and ii. $2 \geq d \leq \ell - 1$.

i. Suppose $|F_1^+(x)| \geq 2$. Since F is nontrivial we have $|F_1^+(x)| \leq \delta^+(x) - 1$. Moreover, as $d(x, F) = 1 \leq \ell - 1$, we can apply Lemma 5.6.2. It follows that there is a vertex $y \in \Gamma^+(x)$ such that $d(y, F) \geq 1$ and $|F_1^+(y)| < |F_1^+(x)|$. By iterating this reasoning at most $\delta^+(x) - 2$ times, the claimed $x \to x'$ path is obtained.

ii. If there exists a vertex $y \in \Gamma^+(x)$ such that $d(y, F) \geq d + 1$, then the result trivially holds because $F_d^+(y) = \emptyset$. Hence, assume that $d - 1 \leq d(y, F) \leq d$ for each $y \in \Gamma^+(x)$. From Lemma 5.6.1(2.), we have $F_d^+(y) \cap F_d^+(y') = \emptyset$ for any two different vertices $y, y' \in \Gamma^+(x)$. It follows that there exists a vertex $y^\star \in \Gamma^+(x)$ such that $|F_d^+(y^\star)| = 1$, for otherwise we would have $|F| \geq \sum_{y \in \Gamma^+(x)} |F_d^+(y)| \geq 2\delta$, contradicting $|F| \leq 2\delta - 3$. Hence, let $F_d^+(y^\star) = \{f^\star\}$. If $d(y^\star, f^\star) = d$, then we are finished. Therefore, let us assume that $d(y^\star, f^\star) = d - 1$. Consider the set $Z = \Gamma^+(y^\star) \setminus \nu(y^\star \to \{f^\star\})$ that contains at least $\delta - 1$ vertices. For any $z \in Z$, we have $d(z, f^\star) \geq d$ (see Lemma 5.6.1(1.)) and

$d(z, F \setminus \{f^\star\}) \geq d$ (by the triangular inequality because $d(y^\star, F \setminus \{f^\star\}) \geq d+1$). Let us see that there exists some vertex $z^\star \in Z$ such that $|F_d^+(z^\star)| \leq 2$. On the contrary, suppose that $|F_d^+(z)| \geq 3$ for any $z \in Z$. By applying Lemma 5.6.1(2.) to $F \setminus \{f^\star\}$, we conclude that $F_d^+(z) \cap F_d^+(z') \subset \{f^\star\}$. That is, we are assuming that $|F_d^+(z) \setminus \{f^\star\}| \geq 2$ for any $z \in Z$. Hence, $|F| \geq \sum_{z \in Z} |F_d^+(z) \setminus \{f^\star\}| \geq 2(\delta - 1)$, arriving at a contradiction. Finally, if $|F_d^+(z^\star)| \leq 1$, then we are finished, and if $|F_d^+(z^\star)| = 2 \leq \delta - 1$, then the result follows from Lemma 5.6.2.

• *Proof of [PW3']*: If $|F| < \delta$, take $x' \in \Gamma^+(x) \setminus \nu(x \to F)$ and by Lemma 5.6.1(1.), the result holds. Hence, let us suppose that $\delta \leq |F| \leq 2\delta - 3$. Let $F_d^+(x) = \{f^\star\}$ and partition $F - f^\star$ into two nonempty disjoint subsets F' and F'' in such a way that $|F'| = \delta - 2$ and $1 \leq |F''| \leq \delta - 2$. Let $w \in \Gamma^+(x) \setminus \nu(x \to \{f^\star\} \cup F')$. Since $d + 2 \leq \ell$, it follows from Lemma 5.6.1(1.) that $d(w, f^\star) \geq d+1$ and $d(w, F') \geq d+2$. Moreover, by the triangular inequality, $d(w, F'') \geq d(x, F'') - 1 \geq d$. Now, if $x' \in \Gamma^+(w) \setminus \nu(w \to \{f^\star\} \cup F'')$, we conclude (again by Lemma 5.6.1(1.) and the triangular inequality) that $d(x', f^\star) \geq d+2$, $d(x', F'') \geq d+1$ and $d(w, F') \geq d+1$. Hence, the path xwx' satisfies the statement of the lemma.

• *Proof of [PW4']*: As a consequence of the previous steps, we conclude that there is an $x \to x''$ path such that $d(x'', F) \geq \ell - 1$ and $F_{\ell-1}^+(x'') \leq 1$. Let us suppose that $F_{\ell-1}^+(x'') = \{\hat{f}\}$ (otherwise, the proof is finished by taking x'=x"). Consider the set of vertices: $Y = \Gamma^+(x'') \setminus \nu(x \to \hat{f})$, whose cardinality is certainly at least $\delta - 1$. It is also clear that, for every $y \in Y$, $d(y, F) \geq \ell - 1$ and $d(y, \hat{f}) \geq \ell$ (see Lemma 5.6.1(3.)). To end the prove, it is enough to see that there exists a vertex $y \in Y$ satisfying $|F_{\ell-1}^+(y)| \leq 1$. To this end, suppose that, for every $y \in Y$, $|F_{\ell-1}^+(y)| \geq 2$. Then, taking into account Lemma 5.6.1(2.), we obtain the following contradiction:

$$2\delta - 2 \leq \sum_{y \in Y} |F_{\ell-1}^+(y)| = |\bigcup_{y \in Y} F_{\ell-1}^+(y)| \leq |F \setminus \{\hat{f}\}| \leq 2\delta - 4.$$ ■

Finally, as an immediate consequence of the above statement, Theorem 5.6.1 is proved.

Proof of Theorem 5.6.1 Let us consider a nontrivial subset of vertices F with $|F| \leq 2\delta - 3$. Let x, y be two different vertices not belonging to F. From Theorem 5.6.2 there exist in $V \setminus F$ two paths $x \to x'$, $y' \to y$, such that: $d(x', F) \geq \ell - 1$, $|F_{\ell-1}^+(x')| \leq 1$, $d(F, y') \geq \ell - 1$ and $|F_{\ell-1}^-(y')| \leq 1$, in such a way that $F_{\ell-1}^+(x') \cap F_{\ell-1}^-(y') = \emptyset$. Therefore, every $x' \to y'$ path through F must be of length at least $2\ell - 1$. Since $d(x', y') \leq D \leq 2\ell - 2$, there exists a path from x' to y' which avoids F. In this way, we find a path from x to y ($x \to x' \to y' \to y$) which does not go through F. Therefore $\kappa_1 \geq |F| + 1 \geq 2\delta - 2$. . ■

5.7 Superconnected ℓ^1-digraphs

The first generalization of the parameter ℓ was carried out by Fàbrega and Fiol in [54]. To be more precise, they introduced the FF-parameter $\ell^\pi = \ell(1,1,\pi)$, whose definition is presented next (see also Definition 5.5.1):

Definition 5.7.1 *Let $G = (V, A)$ be a simple connected digraph with diameter D, and $\pi \geq 0$ a fixed integer. The FF-parameter ℓ^π of G is defined as the greatest integer belonging to $\{1, \ldots, D\}$ such that for any $x, y \in V$, if $0 \leq d(x,y) = d \leq \ell^\pi$, then:*

1. *There is a unique $x \to y$ path of length d.*
2. *If $d < \ell$, there are at most π $x \to y$ paths of length $d+1$.*

In the same paper [54], a first result involving this parameter was shown, which is put forward next together with a new proof based on the PW-algorithm. But firstly, a lemma is shown which will be useful for proving the mentioned result in a easier way [7].

Lemma 5.7.1 *Let $G = (V, A)$ an ℓ^π-digraph, $F \subset V$ and $x \in V \setminus F$ such that $1 \leq d(x, F) = d \leq \ell^\pi - 1$. Then, for every $y \in \Theta^+(x)$:*

1. *$f \in F_d^+(x) \Rightarrow d(y, f) \geq d$.*
2. *$f \in F \setminus F_d^+(x) \Rightarrow d(y, f) \geq d+1$.*

Proof.

1. Suppose that $d(y, f) < d$. If $d(y, f) < d-1$, then $d(x, f) \leq d(x, y) + d(y, f) < 1 + d - 1 = d = d(x, f)$, which is certainly a contradiction. If $d(y, f) = d-1$, then necessarily $y = \nu(x \mapsto f)$, because of condition 1. of Definition 5.7.1. Thus in this case we also obtain a contradiction.
2. Suppose firstly that $f \in F_{d+1}^+(x) \setminus F_d^+(x)$; that is, $d(x, f) = d+1$. This means that $y \neq \nu(x \mapsto f)$, and thus necessarily $d(y, f) \geq d+1$.

   Assume lastly that $f \in F \setminus F_{d+1}^+(x)$; that is, $d(x, f) > d+1$. Therefore, $d(y, f) \geq d(x, f) - 1 \geq d + 2 - 1 = d + 1$. ∎

Theorem 5.7.1 *([54]) Let $G = (V, A)$ be an ℓ^π-digraph with minimum degree $\delta \geq 2$ and diameter D. If $0 \leq \pi \leq \delta - 2$, then*

1. *$D \leq 2\ell^\pi - 1 \Rightarrow \kappa(G) \geq \delta - \pi$*
2. *$D \leq 2\ell^\pi \Rightarrow \lambda(G) \geq \delta - \pi$*

[7] See Remark 5.1.1 for notation questions.

Proof. Since both proofs are very similar, we are going to show only the first of them. To this end, we make use of the PW-algorithm introduced in Section 5.3. Besides, in this particular case we can reduce the three steps of this algorithm to only one (see Remark 5.3.1.3):

[$PW1 + PW2 + PW3$]: Let F be a vertex set with cardinality $|F| < \delta - \pi$, and x a vertex belonging to $V \setminus F$ such that $d(x, F) = d \leq \ell^\pi - 1$. Then, there exists a vertex $w \in \Gamma^+(x)$ such that $d(w, F) \geq d$ and $|F_d^+(w)| < |F_d^+(x)|$.

To prove this, consider the set of vertices $\Theta^+(x)$. As was proved in Lemma 5.7.1, we know that for every $y \in \Theta^+(x)$: $f \in F \setminus F_d^+(x) \Rightarrow d(y, f) \geq d + 1$. Furthermore, it is clear that $\Theta^+(x)$ is a subset of $\Gamma^+(x)$ with cardinality $|\Theta^+(x)| \geq \pi + 1$. As a consequence, if $f \in F_d^+(x)$, then there must exist a vertex $w \in \Theta^+(x)$ such that $d(y, f) \geq d + 1$. ∎

This result provides, for $\pi = 1$, a diameter sufficient condition for any connected digraph to have a connectivity of at least $\delta - 1$. Besides, for $\pi = 0$, we obtain again the classical result by Fàbrega and Fiol (see Proposition 1.7.3,(4.)). Next, we present a new result that, in general, improves the previous one.

Theorem 5.7.2 *Let G be an ℓ^π-digraph with minimum degree $\delta \geq 4$, diameter D and parameter $\ell \geq 2$. If $1 \leq \pi \leq \delta - 3$, then*

1. $D \leq 2\ell^\pi - 2 \Rightarrow \kappa(G) \geq \delta - \pi + 1$
2. $D \leq 2\ell^\pi - 1 \Rightarrow \lambda(G) \geq \delta - \pi + 1$

Proof.

1. As in the previous theorem, we make use again of the PW-algorithm, but in this case the variant pointed out in Remarks 5.3.1.4, together with the fifth point of this same list. To be more precise, we are going to prove the following statements:

   Let $F \subset V$ and $x \in V \setminus F$ such that $|F| \leq \delta - \pi$ and $1 \leq d(x, F) = d \leq \ell^\pi - 1$ respectively. Then,

   (a) there exists an $x \to w$ path in $V \setminus F$ satisfying $d(w, F) \geq d$ and:

   [PW0] $d(w, F) \geq 2$.
   [PW1'] $2 \leq d = d(x, F) \leq \ell^\pi - 1 \Rightarrow F \setminus F_d^+(w) \neq \emptyset$.
   [PW2'] $2 \leq |F_d^+(x)| < |F| \Rightarrow |F_d^+(w)| < |F_d^+(x)|$.
   [PW3'] $d \leq \ell^\pi - 2$, $|F_d^+(x)| = 1 \Rightarrow d(w, F) \geq d + 1$.

   (b) if $d = \ell^\pi - 1$ and $\hat{f} \in F$, there exists an $x \to w$ path in $V \setminus F$ satisfying $d(w, F) \geq \ell^\pi - 1$ and:

   [PW4'] [8] $F_{\ell^\pi - 1}^+(w) \subseteq \{\hat{f}\}$.

[8] Observe that this assertion is 'stronger' than that originally stated (see Remarks 5.3.1.4).

First of all, observe that the set $\Theta^+(x)$ has cardinality at least π, and so we can distinguish the following two cases: (I)$|\Theta^+(x)| \geq \pi + 1$, and (II)$|\Theta^+(x)| = \pi$.

(I) In this case, the proof is the same as that of Theorem 5.7.2; that is to say, it can be reduced to show the step $[PW1+PW2+PW3]$ exhibited in the aforementioned proof.

(II) Suppose that $|\Theta^+(x)| = \pi$. Observe that in this case $F^+_{d+1}(x) = F$. Notice also that the following property is verified:

$$\forall f_1, f_2 \in F,\ f_1 \neq f_2 \ \Rightarrow\ \nu(x \mapsto f_1) \neq \nu(x \mapsto f_2) \tag{5.6}$$

• *Proof of [PW0']*: Let us consider a vertex w belonging to the set $\Theta^+(x)$ and $f \in F$. If $d(x,f) \geq 2$, then $d(w,f) \geq 2$ (see Lemma 5.7.1). Finally, if $d(x,f) = 1$, then f must necessarily be at a distance of at least 2 from w, since by hypothesis $\ell \geq 2$.

• *Proof of [PW1']*: Suppose that $F^+_d(x) = F$. Observe that by Lemma 5.7.1, we know that if $y \in \Theta^+(x)$ and $f \in F$, then $d(y,f) \geq d$. Let $\hat{f}$ be a vertex belonging to F, and consider the vertex $z = \nu(x \mapsto \hat{f})$. Certainly, $d(z,\hat{f}) \geq d - 1 \geq 1$. Furthermore, if we suppose that for every $y \in \Theta^+(x)$ and for every $f \in F$ $d(y,f) = d$, then, taking into account (5.6), we conclude that for every $f \in F \setminus \{\hat{f}\}$, $d(z,f) \geq d+1$.

Finally, consider a vertex w belonging to $\Theta^+(z)$. Certainly, for every $f \in F \setminus \{\hat{f}\}$, $d(w,f) \geq d+1$. Therefore, to conclude the proof it is enough to prove that $d(w,\hat{f}) \geq d$. But this inequality must be verified; otherwise there would be $\pi + 1$ paths from x to $\hat{f}$ of length $d+1$, contradicting the second condition of Definition 5.7.1.

• *Proof of [PW2']*: Taking into account Lemma 5.7.1, if $d(y,f) \geq d+1$ for some pair of vertices $y \in \Theta^+(x)$ and $f \in F^+_d(x)$, then the proof is concluded. Thus assume that:

$$\forall y \in \Theta^+(x),\ \forall f \in F^+_d(x),\ d(y,f) = d. \tag{5.7}$$

Let $\hat{f}$ be a vertex belonging to $F^+_{d+1}(x) \setminus F^+_d(x)$, and consider the vertex $w = \nu(x \mapsto \hat{f})$. As a consequence of (5.6) and (5.7), we can conclude that for every $f \in F^+_d(x)$, $d(w,f) \geq d+1$. On the other hand, let f be a vertex belonging to $F^+_{d+1}(x)$ different from $\hat{f}$. Again taking into account (5.6), we know that $w \neq \nu(x \mapsto f)$ and therefore, from the first condition of Definition 5.7.1, we obtain that $d(w,f) \geq d+1$. Hence, we have seen that $|F^+_d(w)| = 1$, and the proof of this step is concluded.

• *Proof of [PW3']*: If $F^+_d(x) = \{f_1\}$, let us suppose that, for every $y \in \Theta^+(x)$, $d(y,f_1) = d$ (otherwise, the proof is finished). By Lemma 5.7.1, we know that, for every vertex $y \in \Theta^+(x)$ and for every vertex $f \in F \setminus \{f_1\}$, $d(y,f) \geq d+1$. Consequently, we can distinguish these two cases:

(a) There exists a vertex $\hat{y} \in \Theta^+(x)$ s.t. $d(\hat{y}, \hat{f}) > d+1$ for some $\hat{f} \in F \setminus \{f_1\}$.
(b) For every $y \in \Theta^+(x)$ and for every $f \in F \setminus \{f_1\}$, $d(y, f) = d+1$.

(a) We know that the vertex $\hat{y}$ satisfies: $d(\hat{y}, F) = d$ and $F_d^+(\hat{y}) = \{f_1\}$. Furthermore, from the fact that $F \setminus F_{d+1}^+(\hat{y}) \neq \emptyset$, we derive that $|\Theta^+(\hat{y})| \geq \pi+1$. Therefore, we have seen that $\hat{y}$ satisfies the hypotheses of the previous case (I).
(b) Let $f_2 \in F$ be a vertex different from f_1, and consider the vertex $z = \nu(x \mapsto f_2)$. Firstly, it is clear that $d(z, f_2) = d$. Secondly, we intend to prove that $d(z, f_3) \geq d+2$ for any vertex f_3 belonging to the nonempty set $F \setminus \{f_1, f_2\}$. Indeed, if $d(z, f_3) = d$, then it would mean that $z = \nu(x \mapsto f_3)$, contradicting (5.6). Furthermore, if the hypothesis $d(z, f_3) = d+1$ is assumed, then there would be $\pi+1$ paths from x to f_3 of length $d+2 \leq \ell^\pi$, which is also a contradiction, because $d(x, f_3) = d+1$. And lastly, it is clear that $d(z, f_1) = d+1$, because if we suppose that $d(z, f_1) = d$, then there would be $\pi+1$ paths from x to f_1 of length $d+1 \leq \ell^\pi - 1$, again contradicting the second condition of Definition 5.7.1. Therefore, we have proved that the vertex z satisfies: $|\Theta^+(z)| \geq \pi+1$, and this means that this vertex verifies the hypotheses of case (I).

- *Proof of [PW4']*: We can assume that $d(x, F) = \ell^\pi - 1$ and $F_{\ell^\pi-1}^+(x) = \{f_1\} \neq \{\hat{f}\}$ (if not, either the proof is finished, or we can apply step [PW2']). Certainly, if $d(y, f_1) > d$ for some vertex $y \in \Theta^+(x)$, the proof is finished. Thus suppose that for every $y \in \Theta^+(x)$, $d(y, f_1) = d$. Finally, as a consequence of all these hypotheses, it is easy to see that the vertex $w = \nu(x \mapsto \hat{f})$ satisfies, on the one hand, $d(w, \hat{f}) = d$, and on the other, $d(w, f) \geq d+1$ for every $f \in F \setminus \{\hat{f}\}$ (see Figure 5.2).

2. This case is easily proved by means of the line digraph technique (see proof of Theorem 4.2.1). ∎

It is important to notice that, for $\pi = 1$, this theorem provides a new diameter condition for any digraph with minimum degree $\delta \geq 4$ and parameter $\ell \geq 2$ to be maximally connected.

Corollary 5.7.1 *If G is an ℓ^1-digraph with minimum degree $\delta \geq 4$, parameter $\ell \geq 2$ and diameter $D \leq 2\ell^1 - 2$ [$D \leq 2\ell^1 - 1$], then it is a maximally connected [arc-connected] digraph.*

From this last result, we have attempted an approach to the study of superconnected ℓ^1-digraphs. In fact, we have already obtained a first result, but the proof is too long and 'complicated'; thus it must be revised. For this reason we have decided to present it in this work as a conjecture.

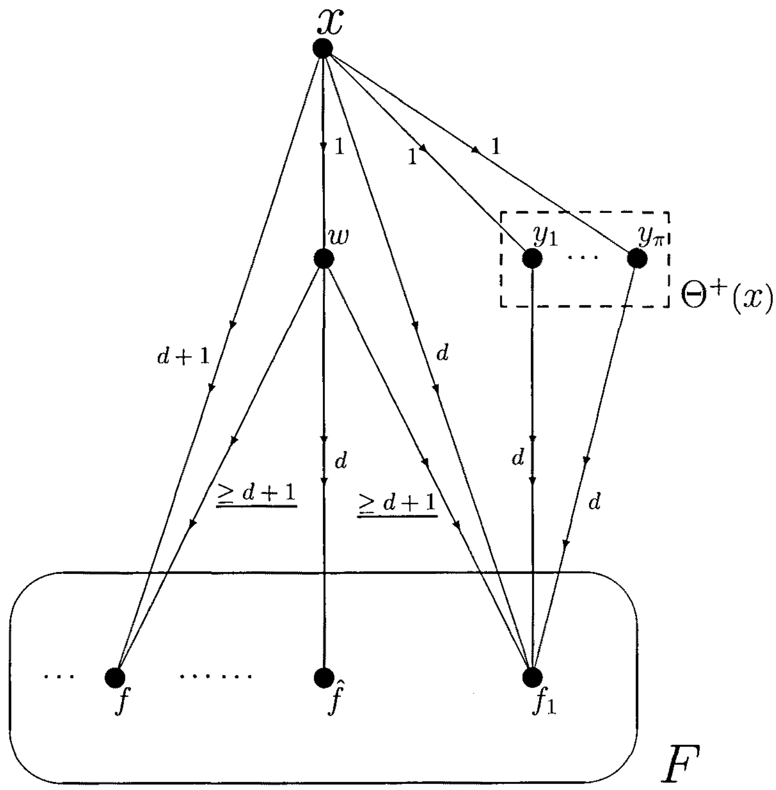

Figure 5.2: In this picture: $d(x, F) = d = \ell^\pi - 1$, $|F| = \delta - \pi$ and $|\Theta^+(x)| = \pi$.

Conjecture 5.7.1 *Let G be a $(\delta, D, \ell, \ell^1)$-digraph such that : $\delta \geq 4$, $\ell \geq 2$ and $\ell^1 \geq 3$. Then,*

1. $D \leq 2\ell^1 - 3 \Rightarrow \kappa_1 \geq 2\delta - 2$
2. $D \leq 2\ell^1 - 2 \Rightarrow \lambda_1 \geq 2\delta - 2$

5.8 η−extraconnected ℓ_η-digraphs

This section is devoted to obtaining diameter sufficient conditions for a certain class of ℓ_η-digraphs to be η−extraconnected. As a matter of fact, this is the first work on connectedness that has been carried out by using the FF-parameter ℓ_η. Maybe for this reason, we have restricted ourselves to digraphs having a 'large' girth g, since extentions to other cases, such as for example digraphs with digons, pose some additional difficulties which, however, we have the intention of approaching shortly. To begin with, let us recall the definition of the FF-parameter $\ell_\eta = \ell(1, \eta, 0)$.

Definition 5.8.1 *Let $G = (V, A)$ be a simple connected (di)graph with diameter D, and $\eta \geq 1$ a fixed integer. The FF-parameter ℓ_η of G is defined as the greatest integer belonging to $\{1, \dots, D\}$ such that for any $x, y \in V$, if $0 \leq d(x, y) = d \leq \ell_\eta$, then:*

1. *There is a unique path from x to y of length d.*
2. *If $d < \ell_\eta$: There are no paths from x to y of length between $d+1$ and $\min\{d+\eta, \ell_\eta\}$.*

Remark 5.8.1 For instance, a class of digraphs with a nontrivial parameter ℓ_η is the family of s-geodetic digraphs (see [6, 10, 124]). A digraph G is said to be *s-geodetic*, $1 \leq s \leq D$, if there exists at most one $u \to v$ path of length less than or equal to s, for any $u, v \in V(G)$. Notice that the girth of an s-geodetic digraph must be at least $s + 1$. Moreover, it is easily checked that $\ell_\eta \geq s$ for any $\eta \geq 1$.

As we have just stated, we are going to approach the problem of finding some diameter condition to assure a certain ℓ_η-digraph with a girth g large enough to be η-extraconnected. To be more precise, we will only consider those digraphs satisfying the inequality $\ell_\eta \geq \eta + 1$ which means, among other facts, that they have a girth greater than η.

Lemma 5.8.1 *Let $\eta \geq 1$ be a fixed integer, and G an ℓ_η-digraph with girth g. Then,*

$$\ell_\eta \geq \eta \quad \Rightarrow \quad g \geq \eta + 1 \tag{5.8}$$

Proof. Let u be any vertex belonging to a directed cycle $\vec{C_g}$ of length g in G. Certainly, the shortest path from u to itself has length 0. It is also clear that $\vec{C_g}$ is a $u \to u$ path of length g. Therefore, $\ell_g \leq g - 1$. Finally, if we suppose that $g \leq \eta$, then $\ell_\eta \leq \ell_g \leq g - 1 \leq \eta - 1$, contradicting the main hypothesis. ∎

Next, let us see a second lemma showing some properties involving out- and in-rooted oriented trees, wich will be used later on [9].

[9] For the sake of clearness, only the out-rooted case is stated and proved. Certainly, the in-rooted case is an immediate consequence of this one by considering the converse digraph.

Lemma 5.8.2 *Let $\eta \geq 1$ be a fixed integer, and G a digraph with minimum degree $\delta \geq 2$ and FF-parameter $\ell_\eta \geq \eta + 1$. Let $F \subset V(G)$ and T an out-rooted tree with root z and height $h \leq \eta - 1$ in G. Then,*

(a) $|\partial^+ T| \geq (\delta - 1)|T| + 1$.

(b) If $|F| < |\partial^+ T|$, then either $d(z, F) \geq \ell_\eta - h$ or there exists some vertex $t \in T \cup \partial^+ T$ such that $d(t, F) > d(z, F)$.

(c) If T is contained in $G - F$ and $|F| < |\partial^+ T|$, then there exists a $z \to z'$ path in $G - F$ such that $d(z', F) \geq \ell_\eta - h$. Moreover, the length of this path is at most $(\ell_\eta - h - 1)(h + 1)$.

Proof. (a) Suppose that T is an out-rooted tree with root z and height $h \leq \eta - 1$, and assume that, for two distinct vertices $t_1, t_2 \in T$, there exists a vertex $w \in \Gamma^+(t_1) \cap \Gamma^+(t_2)$. Then, there exist two different paths from the root z to vertex w; namely, $z \to t_1 w$ and $z \to t_2 w$, whose lengths belong to the set of numbers $L = \{1, \ldots, h, h+1\}$. However, this contradicts the definition of parameter ℓ_η since $h+1 \leq \eta < \ell_\eta$, and hence, $h + 1 \leq \min\{d(z, w) + \eta, \ell_\eta\}$. Thus, apart from the shortest $z \to w$ path in G, there cannot exist another $z \to w$ path with length belonging to L. Therefore, for any two different vertices $t_1, t_2 \in T$, $\Gamma^+(t_1) \cap \Gamma^+(t_2) = \emptyset$. Moreover, by Lemma 5.8.1, we have $g \geq \eta + 1$, and thus $z \notin \Gamma^+(t)$ for any $t \in T$.

Hence, $|\partial^+ T| = |\Gamma^+(T) \backslash T| \geq \sum_{t \in T} \left(\delta - \delta_T^+(t)\right) = (\delta - 1)|T| + 1$, since $\sum_{t \in T} \delta_T^+(t) = |T| - 1$ because T is a tree ($\delta_T^+(t)$ denotes the out-degree in T of vertex t). This proof allows us to assure that the subdigraph of G induced by the arcs of T and those from T to $\partial^+ T$ is a new out-rooted tree with root z and height $h + 1$.

(b) Suppose that $h + 1 \leq \eta$ and $d(t, F) \leq d(z, F) \leq \ell_\eta - h - 1$ for each vertex $t \in T \cup \partial^+ T$. Since $|\partial^+ T| > |F|$, there must exist two distinct vertices $w_1, w_2 \in \partial^+ T$, which are both of them at minimum distance to the same vertex $f \in F$; that is, $d(w_i, F) = d(w_i, f) \leq d(z, F)$, $i = 1, 2$ (it may happen that $w_1 \in F$ or $w_2 \in F$; in this case, $w_1 = f$ or $w_2 = f$ respectively). Furthermore, the root z of T satisfies $d(z, F) \leq d(z, f) \leq d(z, w_i) + d(w_i, f) \leq h + 1 + d(z, F)$. Therefore, we have two different paths from z to f; namely, $z \to w_i \to f$, $i = 1, 2$, whose lengths belong to the set of numbers $L = \{d(z, F), d(z, F) + 1, \ldots, d(z, F) + h + 1\}$. This fact contradicts again the definition of parameter ℓ_η because $d(z, F) + h + 1 \leq \ell_\eta$ and $h + 1 \leq \eta$ imply $d(z, F) + h + 1 \leq \min\{d(z, f) + \eta, \ell_\eta\}$. Therefore, either $d(z, F) \geq \ell_\eta - h$ or there exists some vertex $t \in T \cup \partial^+ T$ satisfying $d(t, F) > d(z, F)$.

(c) [10] From case (b), it follows that condition $|F| < |\partial^+ T|$ implies that either $d(z, F) \geq \ell_\eta - h$ or there exists some vertex $z_1 \in T \cup \partial^+ T$ such that $d(z_1, F) > d(z, F)$. In the first case, the lemma holds by taking $z' = z$. In the second one, the lemma also holds if $d(z_1, F) \geq \ell_\eta - h$ by taking $z' = z_1$. If $d(z_1, F) < \ell_\eta - h$, let us consider another

[10]See Figure 5.3.

out-rooted tree isomorphic to T, say T_{z_1}, with root z_1 and the same height h, which is contained in $G-F$. Hence, $|\partial^+ T_{z_1}| = |\partial^+ T| > |F|$, and, again by applying case (b), we obtain that there exists some vertex $z_2 \in T_{z_1} \cup \partial^+ T_{z_1}$ such that $d(z_2, F) > d(z_1, F)$. By repeating this procedure, we obtain a $z \to z'$ path in $G-F$ such that $d(z', F) \geq \ell_\eta - h$. Moreover, this iterative procedure assures the existence in $G-F$ of a $z \to z_1 \to \cdots \to z_{u_p} = z'$ path whose length is at most $p(h+1) \leq (\ell_\eta - h - 1)(h+1)$, since the number p of steps needed to attain the vertex z' satisfying the lemma is at most $(\ell_\eta - h - 1)$. ■

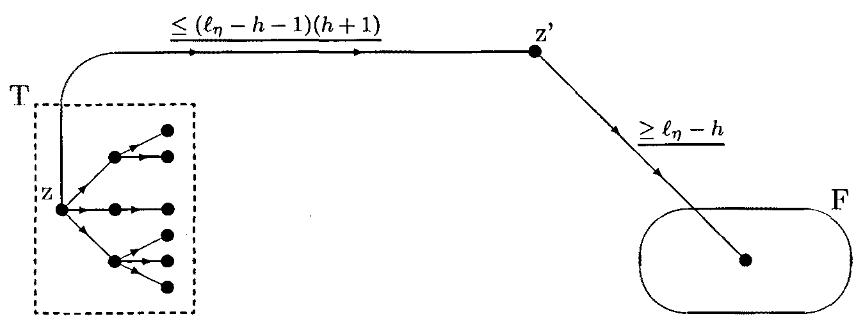

Figure 5.3: A $z \to z'$ path of length at most $(\ell_\eta - h - 1)(h+1)$, with its terminal vertex z' at a distance of at least $\ell_\eta - h$ to F.

And now we are ready to prove the main result of this section by means of the *PW* algorithm (see Section 5.3). In this particular case we are going to use the standard version stated in Theorem 5.3.2 together with the third point of Remarks 5.3.1.

Lemma 5.8.3 *Let $\eta \geq 2$ be a fixed integer and G a digraph with minimum degree $\delta \geq 2$ and FF-parameter $\ell_\eta \geq \eta + 1$. Let F be a nontrivial subset of vertices [arcs] such that $|F| < \tau_\eta = (\eta+1)(\delta-1)$. Then, for every vertex $v \in V \setminus F$, there exists*

(a) a $v \to v'$ path in $G-F$ such that $d(v', F) \geq \ell_\eta - \lceil \log_\delta \tau_\eta \rceil + 1$;

(b) a $w' \to v$ path in $G-F$ such that $d(F, w') \geq \ell_\eta - \lceil \log_\delta \tau_\eta \rceil + 1$.

Moreover, the length of these paths is at most $\eta + (\lceil \eta/2 \rceil + 1)(\ell_\eta - \lceil \eta/2 \rceil - 1 + \lceil \log_\delta \tau_\eta \rceil) - (\lceil \log_\delta \tau_\eta \rceil)^2$.

Proof.

(a) Let us prove the vertex case. By Lemma 5.8.1, the girth of G satisfies $g \geq \eta + 1$. Therefore, since F is a nontrivial set, for any vertex $v \in V \setminus F$, we can consider in $G-F$ a path $P = v \to u_\eta$ of length η starting in v. Notice that $|\partial^+ P| \geq (\delta - 1)(\eta + 1) = \tau_\eta$.

Let us see that there must exist a vertex u in P with out-degree in $G - F$ at least two. Suppose, on the contrary, that any vertex of P has in $G - F$ out-degree at most one. If the root v is adjacent from the vertex u_η, then $F \supset \partial^+ P$, and hence $|F| \geq |\partial^+ P| \geq \tau_\eta$, contradicting the hypothesis $|F| < \tau_\eta$. Furthermore, if v is not adjacent from u_η, then $|\partial^+ P| \geq \tau_\eta + 1$ and $|\partial^+ P \cap (V \setminus F)| \leq 1$. So, in this case, $|F| \geq |\partial^+ P| - 1 \geq (\eta + 1)(\delta - 1) = \tau_\eta$, and we arrive at the same contradiction. Therefore, a vertex u with out-degree in $G - F$ at least two exists.

Now, let us see that there exists in $G-F$ a $u \to z$ path such that $d(z, F) \geq \ell_\eta - \lceil \eta/2 \rceil$. To this end, let us consider in $G - F$ two paths starting in vertex u, each one of length $\lceil \eta/2 \rceil$. Since $\ell_\eta \geq \eta + 1$, these two paths coincide only in the vertex u, and hence they form an out-rooted tree T' with root u. The height of T' is $h = \lceil \eta/2 \rceil$, and hence, $h + 1 \leq \eta$ because $\eta \geq 2$. By Lemma 5.8.2(a), we have $|\partial^+ T'| \geq (\delta - 1)|T'| + 1 \geq (\delta - 1)(\eta + 1) + 1 > |F|$, and thus we can apply Lemma 5.8.2(c). Then, we can conclude that there exists in $G - F$ a $u \to z$ path such that $d(z, F) \geq \ell_\eta - \lceil \eta/2 \rceil$.

To end the proof, let us see that there exists in $G - F$ a $z \to v'$ path such that $d(v', F) \geq \ell_\eta - \lceil \log_\delta \tau_\eta \rceil + 1$. Let $h' = \lceil \log_\delta \tau_\eta \rceil - 1$. If $\lceil \eta/2 \rceil \leq h'$, then the result holds for $v' = z$. Hence, assume that $h' < \lceil \eta/2 \rceil$. Since $h' + \lceil \eta/2 \rceil < 2\lceil \eta/2 \rceil \leq \eta + 1 \leq \ell_\eta$, and hence, $h' < \ell_\eta - \lceil \eta/2 \rceil \leq d(z, F)$, we can consider in $G - F$ an out-rooted tree T_z with root z and height h', such that their vertices with level number less than h' have all of them out-degree at least δ. Then, we have $|\partial^+ T_z| \geq \delta^{h'+1} = \delta^{\lceil \log_\delta \tau_\eta \rceil} \geq \tau_\eta$. Finally, we can apply Lemma 5.8.2 (c) and the proof of part (a) concludes.

Certainly, point (b) is an immediate consequence of the previous one by considering the converse digraph. As for the arc case, the proof is similar and for this reason we omit it.

Finally, it can be easily checked that the length of the $v \to u \to z \to v'$ path is at most $\eta + (\ell_\eta - \lceil \eta/2 \rceil - 1)(\lceil \eta/2 \rceil + 1) + \lceil \log_\delta \tau_\eta \rceil (\lceil \eta/2 \rceil + 1 - \lceil \log_\delta \tau_\eta \rceil)$. ∎

And finally, as a consequence of this lemma we obtain the following results.

Theorem 5.8.1 *Let $\eta \geq 1$ be a fixed integer, and G a digraph with minimum degree $\delta \geq 2$, FF-parameter $\ell_\eta \geq \eta + 1$ and diameter D. Then*

(a) $D \leq 2\ell_\eta + 1 - 2\lceil \log_\delta \tau_\eta \rceil \Rightarrow \kappa_1 \geq \tau_\eta$

(b) $D \leq 2\ell_\eta + 2 - 2\lceil \log_\delta \tau_\eta \rceil \Rightarrow \lambda_1 \geq \tau_\eta$.

Proof. First of all, notice that, for $\eta = 1$, the theorem holds as a consequence of the results given in Theorem 5.6.1. So, assume that $\eta \geq 2$ is a fixed integer, and let us prove the arc version of the theorem, the proof of the vertex case being analogous [11].

Let us consider a nontrivial set of arcs A such that $|A| < \tau_\eta$. Take two different vertices u, v of the digraph G. By Lemma 5.8.3, there exist $u \to u'$ and $v' \to v$

[11] Alternatively, one can prove the vertex case, and then the arc case is immediately obtained by using the line digraph technique (see proof of Theorem 4.3.2).

paths in $G - A$ such that $d(u', A), d(A, v') \geq \ell_\eta - \lceil \log_\delta \tau_\eta \rceil + 1$. Hence, any $u' \to v'$ path in G containing an arc of A has length at least $2\ell_\eta - 2\lceil \log_\delta \tau_\eta \rceil + 3$. Since $D \leq 2\ell_\eta + 2 - 2\lceil \log_\delta \tau_\eta \rceil$, a shortest $u' \to v'$ path does not contain any arc of A. Hence, $G - A$ is still connected and we have $\lambda_1 \geq \tau_\eta$. ■

Taking into account the result given in Lemma 5.8.3 on the length of the $u \to u'$ and $v' \to v$ paths, we have the following corollary concerning the diameter vulnerability of G. We only state the result in the vertex case.

Corollary 5.8.1 *Let G be a digraph with minimum degree $\delta \geq 2$ and FF-parameter $\ell_\eta \geq \eta + 1$, $\eta \geq 2$, and let F be a nontrivial subset of vertices such that $|F| < \tau_\eta = (\eta+1)(\delta-1)$. If $D \leq 2\ell_\eta + 1 - 2\lceil \log_\delta \tau_\eta \rceil$, then $G' = G - F$ is connected and has diameter $D' \leq D + 2\rho$, where $\rho = \eta + (\lceil \eta/2 \rceil + 1)(\ell_\eta - \lceil \eta/2 \rceil - 1 + \lceil \log_\delta \tau_\eta \rceil) - (\lceil \log_\delta \tau_\eta \rceil)^2$.* ■

Another consequence of considering ℓ_η-digraphs having a girth $g \geq \eta + 1$, is that in this kind of digraphs the results stated in the previous theorem can be used to obtain a similar sufficient diameter condition to guarantee η-extraconnectivity for $\eta \in \{1, \ldots, g-1\}$. As a matter of fact, this new result is a corollary of both Theorem 5.8.1 and the following proposition.

Proposition 5.8.1 *Let η be a fixed integer such that $1 \leq \eta \leq g - 1$, G a digraph with girth $g \geq \eta + 1$, and F subset of subset of vertices [arcs]. Then*

1. *F is η-trivial if and only if it is 1-trivial.*

2. *$\kappa_1 = \kappa_2 = \ldots = \kappa_{g-1} \leq \kappa_g \leq \ldots$ [$\lambda_1 = \lambda_2 = \ldots = \lambda_{g-1} \leq \lambda_g \leq \ldots$].*

3. *F is a nontrivial vertex-cutset [arc-cutset] if and only if every component of $G - F$ either has at least g vertices or it is a transmittance component with a single vertex.*

Proof.

1. Certainly, if F is trivial, then it is η-trivial for any $\eta \geq 1$ (see Proposition 5.4.1). Reciprocally, suppose that $F \subset V$ is η-trivial. So, there exists $S \subset V \setminus F$, with $1 \leq |S| \leq \eta$, such that F contains $\partial^+ S$ or $\partial^- S$. Without loss of generality, we can assume that $F \supset \partial^+ S$; otherwise we can reason on the converse digraph obtained from G by reversing the direction of every arc. If $\Gamma^+(s) \not\subset F$ for every $s \in S$, then S contains at least one cycle. But this is impossible since $g \geq \eta + 1$. This contradiction proves the existence in S of a vertex v such that $\partial^+\{v\} \subset F$, and hence, F is trivial. The proof for the arc case is analogous.

2. These sequences are an immediate consequence of the previous point.

3. This result is immediately obtained by considering the third point of Proposition 5.4.1 and the first of this one. ■

And now, as a consequence of the results given in Proposition 5.4.1 we can reformulate Theorem 5.8.1 in the following way:

Theorem 5.8.2 *Let $\eta \geq 1$ be a fixed integer, and G a digraph with minimum degree $\delta \geq 2$, FF-parameter $\ell_\eta \geq \eta + 1$ and diameter D. Then*

(a) $D \leq 2\ell_\eta + 1 - 2\lceil \log_\delta \tau_\eta \rceil \Rightarrow \kappa_1 = \kappa_2 = \ldots = \kappa_\eta = \ldots = \kappa_{g-1} \geq \tau_\eta$

(b) $D \leq 2\ell_\eta + 2 - 2\lceil \log_\delta \tau_\eta \rceil \Rightarrow \lambda_1 = \lambda_2 = \ldots = \lambda_\eta = \ldots = \lambda_{g-1} \geq \tau_\eta.$ ■

The excellent behaviour of the FF-parameter ℓ_η with respect to the line digraph technique (see Proposition 5.5.1) allows us to obtain the following result as a corollary of the previous theorem.

Corollary 5.8.2 *Let G be a connected digraph with minimum degree $\delta \geq 2$, diameter D, girth g, and parameter ℓ_η, where $1 \leq \eta \leq g-1$. If L^kG is the k-iterated line digraph of G and $k \geq \eta + 1 - \ell_\eta$, then*

(a) $k \geq D - 2\ell_\eta - 1 + 2\lceil \log_\delta \tau_\eta \rceil \Rightarrow \kappa_1(L^kG) = \ldots = \kappa_\eta(L^kG) = \ldots = \kappa_{g-1}(L^kG) \geq \tau_\eta;$

(b) $k \geq D - 2\ell_\eta - 2 + 2\lceil \log_\delta \tau_\eta \rceil \Rightarrow \lambda_1(L^kG) = \ldots = \lambda_\eta(L^kG) = \ldots = \lambda_{g-1}(L^kG) \geq \tau_\eta.$

Notice that this result means that if the iteration order is large enough, any iterated line digraph is η-extraconnected. Finally, let us apply these results to some families of large iterated line digraphs which have been considered as good models of interconnection networks, such as the dense bipartite digraphs, the De Bruijn and Kautz generalized p-cycles.

Examples 5.8.1

1. The Moore bipartite digraph $G = BD(d, d^2 + 1)$ (see Remark 1.5.1) is d-regular, has diameter $D = 3$, girth $g = 4$ and parameter $\ell = 2$. Therefore, $\ell = \ell_1 = \ell_2 = \ell_3 = 2$. Since the bipartite digraph $BD(d, d^{D-1} + d^{D-3})$ is the $(D-3)$-iterated line digraph of G, we derive that $\ell_2(L^{D-3}G) \geq 3$ for $D \geq 4$ and $\ell_3(L^{D-3}G) \geq 4$ for $D \geq 5$. Applying Theorem 5.8.2 to $G = BD(d, d^{D-1} + d^{D-3})$ we obtain that this digraph has $\lambda_1 = \lambda_2 = \lambda_3 \geq 3(d-1)$ and $\kappa_1 = \kappa_2 = \kappa_3 \geq 3(d-1)$ for $D \geq 4$, and $\lambda_1 = \lambda_2 = \lambda_3 \geq 4(d-1)$ and $\kappa_1 = \kappa_2 = \kappa_3 \geq 4(d-1)$ for $D \geq 5$.

2. The De Bruijn generalized p-cycle $BGC(p, d, d^p)$ is isomorphic to the $(p-1)$-iterated line digraph of $\vec{C}_p \otimes K_d^*$. This means that this digraph is d-regular, has diameter $2p-1$, girth $g = p$ and FF-parameters $\ell = \ldots = \ell_{p-1} = p$. If we apply the previous theorem to this graph we obtain that if the iteration order satisfies $k \geq 2\lceil \log_d p(d-1) \rceil - 2$ $[k \geq 2\lceil \log_d p(d-1) \rceil - 3]$, then $BGC(p, d, d^{p+k})$ has $\kappa_1 = \kappa_2 = \ldots = \kappa_{p-1} \geq p(d-1)$ $[\lambda_1 = \lambda_2 = \ldots = \lambda_{p-1} \geq p(d-1)]$.

3. The Kautz generalized p-cycle $KGC(p, d, d^p + 1)$ is d-regular, has diameter $2p-1$, girth $g = p$ and FF-parameters $\ell = \ldots = \ell_{p-1} = p$. As a consequence, we can proceed in the same way as in the previous case, and thus we obtain the same results for $KGC(p, d, d^{p+k} + d^k)$.

Conclusions and open problems

This work has dealt with some issues related to the design and study of large and reliable graphs and digraphs. To be more precise, we have carried out four specific contributions:

- (Δ, D)-problem: Design of new large graphs (Chapter 2).
- Vulnerability analysis: 1-vertex vulnerability of GC graphs (Chapter 3).
- Specific connectedness study: Connectivity and superconnectivity of generalized p-cycles both under diameter and order sufficient conditions (Chapter 4).
- General connectedness study: Connectivity, superconnectivity and extraconnectivity of digraphs under diameter conditions, by using several FF-parameters and a unique kind of constructive proof (Chapter 5).

In Chapter 2, a list of new solutions to the (Δ, D)-problem is obtained for diameter $D = 6$ (see Table 2.3). Each of these graphs has been contructed by compounding a Moore bipartite graph of diameter six with a family of complete graphs. The main difficulty has been to find for each case a large enough $[l, \lambda]$-clique of diameter 2 (see Section 2.2). We have presented in this work the new large graphs obtained with a maximum degree $\Delta \leq 14$, although the method theoretically allows to yield large graphs with the sole restriction that $\Delta - 1$ must be a prime power. The results of this work were presented in *Combinatorix'98 (Palermo, Italy)* and they can be found in [83, 143].

In Chapter 3, a new reformulation of the generalized compound graphs GC has been proposed, by using the conjunction of digraphs, the line digraph technique and the compounding of graphs (see Section 3.3). Starting from this simpler characterization, the $(\Delta, D, D', 1)$-problem for these graph families has been studied. The result obtained is that, except for a few cases, every generalized compound graph has a quasi optimal 1-vertex vulnerability (see Theorem 4.4.1). This work was presented in *British'99 (Canterbury, Great Britain)* and it can be found in [84].

The two last Chapters of this work contain a number of new results on digraph connectedness. In Chapter 4, both the connectivity and superconnectivity of generalized

p-cycles has been studied. The methodology used in this work has been inspired in those carried out by C. Balbuena, A. Carmona, J. Fàbrega and M. A. Fiol on this subject for digraphs and bipartite digraphs (see Section 4.1). Starting from the parameter ℓ and using the so-called Hamidoune terminology (see Section 1.7.1), we have obtained both diameter and order conditions to assure maximal connectivity and superconnectivity (see Tables 4.1 and 4.2). Furthermore, we have defined the good superconnected generalized p-cycles and studied this property under diameter conditions (see Section 4.4). Finally, these results have been applied to the Moore generalized p-cycles. The main results of this work were shown in *Combinatorix'98 (Palermo, Italy)* and they can be found in [116, 12].

In Chapter 5, a number of new results involving connectivity, superconnectivity and extraconnectivity have been presented (see Sections 5.6, 5.7 and 5.8). It is particularly interesting the result stated in Theorem 5.6.2, because it shows a fact which had passed unnoticed so far, and it can be, not only the key point to re-prove in a simpler way the classical results (this issue is also shown in Section 5.6), but also the starting point for obtaining new connectedness results of different kinds, for example, under degree conditions. In Section 5.7, a new theorem involving the FF-parameter $\ell^\pi = \ell(1, 1, \pi)$ is presented, wich improves a result by Fàbrega and Fiol (see [54]). Moreover, this improvement is significant because it enables, unlike the classical theorem, to approach the study of the superconnectivity in maximally connected digraphs. Finally, in Section 5.8 the study of the η-extraconnected digraphs has been approached. Certainly, the main result of this section is Theorem 5.8.2, which provides a diameter sufficient condition for any digraph with girth large enough to be η-extraconnected.

Open problems

As it was to be expected, several new unsolved problems, directly related to those approached in this work, have appeared. In the following list we briefly present some of them.

- Design of new large graphs with diameter different from six by using a similar methodology to that presented in Chapter 2.
- Design of new large graphs constructed as GC graphs.
- Study of connectedness properties of the GC graphs, especially its connectivity. Study of other cases of diameter-vulnerability.
- Study of connectedness properties on the generalized p-cycles with the FF-parameters and the PW-algorithm. Improvement of the order conditions presented in this work.
- Behaviour of the FF-parameters with respect to the line digraph technique.

- Analysis of the existence problem related to the η-nontrivial cutsets and arc-cutsets. Whitney-like inequality sequence for $\eta \geq 1$.
- Computational problems of the FF-parameters.
- Generalizing the diameter-vulnerability: supervulnerability, extravulnerability, etc.
- Generalizing the cages for the directed case, by using the parameter ℓ or other suitable FF-parameters.
- Study of the η-extraconnectivity under Chartrand and order conditions. Study of the η-extraconnectivity for digraphs with small girth.

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