Unstable motions in the Three Body Problem

Author
Jaime Paradela Díaz

Supervised by
Marcel Guàrdia Munárriz and Tere M-Seara Alonso

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Abstract

The 3 Body Problem is a dynamical system which models the motion of three bodies interacting via Newtonian gravitation. It is called *restricted* when one body has zero mass and the other two, the primaries, have strictly positive masses. In the region of the phase space where one body (the massless body for the restricted case) is far from the other two both models can be studied as a nearly integrable Hamiltonian system. This is the so-called *hierarchical regime*.

The present thesis deals with the existence of *unstable motions*, strongly associated to non-integrable dynamics, in the 3 Body Problem and/or its restricted versions. More concretely, we analyze the existence of topological instability, non trivial hyperbolic sets and oscillatory motions (complete orbits which are unbounded but return infinitely often to some bounded region). On one hand, the existence of (a strong form of) topological instability in the N Body Problem ($N \geq 3$) was coined by Herman to be "the oldest question in dynamical systems" [Her98]. On the other hand, oscillatory motions are the unique type of final motions for the 3 Body Problem which are not present in the integrable approximation. Their connection with the existence of non trivial hyperbolic sets has lead to the formulation of fundamental, yet unsolved, conjectures about their abundance [Ale71, GK12].

Our first main result establishes the existence of *Arnold diffusion*, a robust mechanism leading to topological instability [Arn64], in the Restricted 3 Body Problem for *any value* $m_0, m_1 > 0$ of the masses of the primaries. The transition chain leading to Arnold diffusion is built in the hierarchical region. We extend a previous result by Kaloshin, Delshams, De la Rosa and Seara [DKdlRS19], which applied to arbitrarily small mass ratio $m_1/m_0 \to 0$. Their setting, which exploits the trick, used by Arnold in his original paper, of making use of two perturbative parameters, lead to an a priori unstable model. In our setting, where the mass ratio is arbitrary, we face some of the challenges present in *a priori stable* systems.

Our second main result shows the existence of oscillatory motions in a symmetric configuration of the Restricted 3 Body Problem usually known as the Restricted Isosceles 3 Body Problem (RI3BP). This symmetry implies the existence of a conserved quantity, the angular momentum, which can be taken as a parameter of the system. For large values of the parameter, one can focus on the hierarchical region to study the existence of oscillatory motions, and therefore, make use of geometric perturbation theory. However, for non-large values of the parameter, the set of oscillatory motions is not contained in the hierarchical region. We develop new tools which blend geometric ideas with variational techniques to prove that there exist *oscillatory motions* in the RI3BP for *almost all values* of the parameter.

Our third main result proves the existence of *non trivial hyperbolic sets* and oscillatory motions in the 3 Body Problem for *all values of the masses* $m_0, m_1, m_2 > 0$. The non trivial hyperbolic set, contained in a subset of the hierarchical region where the inner bodies perform approximately circular motions, is associated to a transverse intersection between the stable and unstable manifolds of a (topological) Normally Hyperbolic Invariant Manifold. The existence of *non trivial center directions* complicates heavily both the analysis of existence of transverse intersections between these invariant manifolds and the construction of the horseshoe. The contribution of the author concerns the first of these two steps.

Our fourth main result concerns the existence of *Arnold diffusion* in the 3 Body Problem for *all values of the masses* $m_0, m_1, m_2 > 0$. The robustness of the mechanism which we use to prove the existence of Arnold Diffusion in the Restricted 3 Body Problem implies that the obtained transition chain admits a continuation in the 3 Body Problem if m_2 is sufficiently small. The substantial difference when the masses $m_0, m_1, m_2 > 0$ are fixed is that one can construct a transition chain along which there is a large exchange of momentum between the inner and outer bodies, resulting in a significant change of the eccentricity of the inner bodies. This requires considerably more work than in our construction of the transition chain in the Restricted 3 Body Problem and our construction of hyperbolic sets (contained in the nearly circular subset of the hierarchical region) for the 3 Body Problem. The first step towards establishing this result, which constitutes the subject of the last chapter of this thesis, is the analysis of the so-called Melnikov approximation associated to the aforementioned transition chain.

Contents

Chapter 1

Introduction

The 3 Body Problem is a Hamiltonian dynamical system which models the motion of three bodies interacting via Newtonian gravitation. Understanding qualitatively the global picture of its dynamics is, probably, one of the most challenging questions in dynamics, and, research in this problem, typically combines tools from many different fields, such as Hamiltonian perturbation theory, hyperbolic dynamics, calculus of variations and symplectic geometry. In this manuscript we present some modest contributions of the author to the problem.

The major complexity of the dynamics of the 3 Body Problem was already pointed out by Poincaré back in 1890 [Poi90, Poi99], when he discovered (in simplified models) the existence of transverse intersections between the stable and unstable manifolds of certain hyperbolic periodic orbits. Among the many dynamical consequences of this "realm of chaos", is the non existence of additional global and real-analytic integrals of motion apart from the already known (i.e. the Hamiltonian function and the ones associated to continuous symmetries), which implies the analytic non-integrability of the 3 Body Problem. Since the time of Poincaré many remarkable dynamical phenomena have been observed in the 3 Body Problem.

1.1 State of the art and main results

At the moment, many of the results concerning the global dynamics of the 3 Body Problem have been established in nearly integrable settings, that is, regions in the phase space, or in the space of parameters (the masses of the three bodies), where the 3 Body Problem can be studied as a small perturbation of two uncoupled 2 Body Problems (recall that the 2 Body Problem is integrable). Two examples of nearly integrable settings are the so-called *planetary* and *hierarchical* regimes. In the former one, the first body is much more heavy than the other two, so, up to first order, the light bodies do not interact between themselves and the Hamiltonian decouples into two binary systems. In the latter one, the third body is far from the other two, and therefore, up to higher order interactions, the motions of the binary system, and the motion of the third body with respect to the center of mass of the binary system, are uncoupled.

The first major result concerning the dynamics in the planetary regime, was established by Arnold in [Arn63]. Although the lack of torsion of the integrable approximation prevents the application of the classical versions of the KAM theory, he extended the KAM techniques to deal with "properly degenerate systems", and proved the existence of a positive measure set of quasiperiodic motions in the 3 Body Problem. The result was later later extended to the case of $N \geq 3$ bodies in the work of Féjoz and Herman [Fej04] (see also [Rob95, CP11]). For the existence of quasiperiodic motions in the hierarchical regime (and punctured tori, passing arbitrarily close to double collision) see [Fej01, Fej02a, Fej02b, Zha14, Zha15]. This metric stability can be seen as a remnant of the completely *elliptic dynamics* of the integrable approximations. However, although of positive measure, the KAM set is nowhere dense, leaving room for the existence of unstable motions starting arbitrarily close to it. On an effort to replace positive measure by open sets, a remarkable result by Nekhoroshev [Neh77] (see also [Nie96]), shows that the KAM set is indeed included in an open subset of the phase space where trajectories are *effectively stable*. That is, orbits starting on this set are stable for an exponentially long time (with respect to the perturbative parameter, which, in the planetary regime, is the mass ratio).

On the other hand, many striking mechanisms, leading to some sort of instability or departing from the integrable dynamics, have been found in the N Body problem with $N \geq 3$, or its restricted versions [Sit60, LS80a, Moe84, Xia92, Mos01, Bol06, Moe07, GK11, GK12, GMS16, GSMS17, CG18, DKdlRS19, SZ20, Xue20, BGG21, CGM$^+$22, BGG22, CFG22, CFG23]. Many of them (if not all) are related to the existence of (partially) *hyperbolic* invariant objects.

Complementary to the Hamiltonian perturbation theory approach, the *variational approach*, has yielded a number of remarkable results on the N Body Problem for $N \geq 3$ in *non-nearly integrable settings*, which include, for example, the existence of collisionless periodic orbits and completely parabolic and hyperbolic motions (just to cite a few results, see [CM00, FT04, TV07, MV09, MV20] and the references therein).

Despite the enormous amount of research, the global picture of the dynamics of the 3 Body Problem (and, more generally, of the N Body Problem), is still quite far to be understood. Yet, a quite remarkable result of Chazy gives a classification of the possible final motions (i.e. complete orbits) of the 3 Body Problem. To describe them, we denote by r_k the vector from the point mass m_i to the point mass m_j for $i \neq k$, $j \neq k$, $i < j$.

Theorem 1.1.1 (Chazy [Cha22], see also [AKN06])**.** *Every solution of the 3 Body Problem defined for all (future) time belongs to one of the following seven classes.*

- *Hyperbolic (H):* $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i = 0, 1, 2$, *as* $t \to \infty$.
- *Hyperbolic–Parabolic (HP_k):* $|r_i| \to \infty$, $i = 0, 1, 2$, $|\dot{r}_k| \to 0$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$, *as* $t \to \infty$.
- *Hyperbolic–Elliptic, (HE_k):* $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i = 0, 1, 2$, $i \neq k$, *as* $t \to \infty$, $\sup_{t \geq t_0} |r_k| < \infty$.
- *Parabolic-Elliptic (PE_k):* $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i = 0, 1, 2$, $i \neq k$, *as* $t \to \infty$, $\sup_{t \geq t_0} |r_k| < \infty$.
- *Parabolic (P):* $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i = 0, 1, 2$, *as* $t \to \infty$.
- *Bounded (B):* $\sup_{t \geq t_0} |r_i| < \infty$, $i = 0, 1, 2$.
- *Oscillatory (OS):* $\limsup_{t \to \infty} \sup_{i=0,1,2} |r_i| = \infty$ *and* $\liminf_{t \to \infty} \sup_{i=0,1,2} |r_i| < \infty$.

Note that this classification applies both when $t \to +\infty$ or $t \to -\infty$. To distinguish both cases we add a superindex $+$ or $-$ to each of the cases, e.g H^+ and H^-. Examples of all types of motion but the oscillatory ones were known at the time of Chazy. The first example of oscillatory motions was given by Sitnikov in [Sit60]. Since then, starting with Moser [Mos01], a number of works have shown the existence of oscillatory motions in different models in Celestial Mechanics (see Section 1.1.2 for a list of results). The next natural question is to evaluate the measure of each of these sets. It turns out that the answer is known for all sets except one, the oscillatory ones. For example, notice that, in particular, Arnold's theorem [Arn63] shows that bounded motions have positive measure. The following conjecture about the measure of the set of oscillatory motions goes back to Alexeev [Ale71] (in the English version he attributes this conjecture to Kolmogorov).

Conjecture 1.1.2 (Alexeev, Kolmogorov)**.** *The Lebesgue measure of the set of oscillatory motions is zero.*

Proving or disproving this conjecture was considered by Arnold to be the fundamental question in Celestial Mechanics. By now, it remains wide open. The only available partial results, have been obtained in [GK12].

One should also remark the importance of oscillatory motions in connection with the existence of non trivial hyperbolic sets. More concretely, the modern approach to prove the existence of oscillatory motions in Celestial Mechanics, introduced by Moser in [Mos01], consists on proving the existence of a homoclinic class which contains oscillatory motions.

Now we give a different look at the description of the qualitative behavior of solutions. In his 1998 ICM lecture, Herman asked the following question, which he considered to be the *oldest question in dynamical systems*. Let $N \geq 3$, fix the center of mass at the origin, and, on the constant energy hypersurface h, reparametrize the flow so collisions occur in infinite time.

Question 1.1.3 (Herman [Her98])**.** *Is for every h the non wandering set of the Hamiltonian flow of the N Body Problem nowhere dense on the constant energy hypersurface of energy h?*

This would imply that bounded orbits are nowhere dense and no topological stability occurs. At the moment, this conjecture is largely out of reach, and we are still looking for topologically unstable motions in negative energy levels (for positive energy levels, the identity of Jacobi-Lagrange implies that every orbit defined for all future times is wandering (see [Che98] for a proof)). What Herman believed that was not an unreasonable question to ask, "and possibly prove in a finite time with a lot of technical details" is that:

Question 1.1.4 (Herman [Her98])**.** *Consider the planetary regime, then, if the mass ratio is sufficiently small, in any neighbourhood of fixed different circular orbits around m_0 (the heavy body) moving in the same direction in a plane, there are wandering domains.*

These questions motivate, at least up to some extent, the results obtained in this thesis. In Sections 1.1.1, 1.1.2 and 1.1.3, we state the main results of our work. They concern the existence of topological instability, oscillatory motions and non trivial hyperbolic sets in the 3 Body Problem and its restricted versions. We recall that the 3 Body Problem is called *restricted* if one of the bodies has zero mass and the other two, the *primaries*, have strictly positive masses. In this limit problem, the primaries "are not affected" by the motion of the massless body and they move according to the dynamics of the 2 Body Problem. Then, the Restricted 3 Body Problem models the motion of the massless body under the gravitational potential created by the primaries. The most interesting case is when the primaries move in the region of negative energy, i.e. they perform a (bounded) circular or elliptic motion.

The common feature of the mechanisms that we present in this work is the existence of a *Normally Hyperbolic Invariant Manifold* (see [HPS77, Fen74, Fen77] for the precise definition and classical results on normal hyperbolicity). Loosely speaking, a submanifold $N \subset M$ of a Riemannian manifold M is Normally Hyperbolic for a map f if it is *f-invariant* and for each point in N there exists a *dominated splitting* of the tangent space into three subspaces: a uniformly contracting one (stable), a uniformly expanding one (unstable), and the tangent space to N, in which the dynamics is "in between" (less contracting than the stable and less expanding than the unstable). Among the remarkable properties of Normally Hyperbolic Invariant Manifolds are their persistence under perturbations and the fact that they possess stable and unstable invariant manifolds which are also robust under perturbations. As we will see, these manifolds usually act as carriers, connecting distant regions of the phase space.

We will identify certain (topological) Normally Hyperbolic Manifolds for the 3 Body Problem or its restricted versions, and introduce tools to prove that their stable and unstable manifolds intersect transversally. Then, we analyze the dynamical consequences of this phenomenon. In particular, how they imply the existence of topological instability and/or non trivial hyperbolic sets. These phenomena have been studied in the context of "general" Hamiltonian systems (see for example [Dou88, BT99, DdlLS00, MS02, BB02, Mat03, CY04, Tre04, DdlLS06, Ber08, GT08, DH09, NP12, Tre12, BKZ16, DdlLS16, Che17, GT17, KZ20, GdlLS20]), however, the application of these ideas to Celestial Mechanics is quite challenging because, in addition to many other difficulties, of the existence of many degeneracies.

Before entering into details let us do a final remark. Despite the common framework in wich we build the mechanisms leading to topological instability and chaotic dynamics, these phenomena are *quite different* in nature. On one hand, *chaotic* dynamics are associated to the existence of non trivial *hyperbolic invariant sets*, while topological *instability* is associated (in the present work) to the existence of *finite transition chains* between partially hyperbolic invariant objects. On the other hand, the hyperbolic invariant sets associated to chaotic behavior might be small and do not lead necessarily to topological instability.

1.1.1 Topological Instability in the Restricted 3 Body Problem

In accordance with the general belief that the 3 Body Problem, although strongly degenerate, displays the main features of a general Hamiltonian system, one expects (at least in nearly integrable settings) the coexistence of metric stability of quasiperiodic KAM motions with topological instability in the 3 Body Problem. This was indeed conjectured by Arnold, to be the typical situation in nearly integrable Hamiltonians (see [Arn63]). However, results concerning topological instability in Celestial Mechanics have been obtained only quite recently in [CG18, DKdlRS19] for the *Restricted* 3 Body Problem and in [CFG22, CFG23] for the spatial 4 Body Problem.

In all these results, the underlying mechanism is the so-called *Arnold diffusion mechanism*. This mechanism, proposed by Arnold in his seminal study of topological instability in nearly integrable Hamiltonian systems (see [Arn64]), is based on the existence of a *transition chain of invariant tori*, that is, a sequence of partially hyperbolic invariant tori connected by transverse heteroclinic orbits. In modern language, the Arnold diffusion mechanism relies on the existence of a Normally Hyperbolic Invariant Manifold (NHIM) whose stable and unstable manifolds intersect transversally along a homoclinic manifold. Then, if the inner dynamics on the NHIM contains "sufficient" quasiperiodic invariant tori (or other invariant objects such as Aubry-Mather sets), one can combine the outer excursions along the homoclinic manifold with quasiperiodic inner dynamics (or orbits shadowing the Aubry-Mather sets) to obtain a transition chain leading to topological instability. Another important tool in the modern (geometric) approach to Arnold diffusion is the so-called *scattering map* [DdlLS08], a suitable composition of holonomy maps along the unstable and stable foliations, which encodes the dynamics along the homoclinic manifold.

Although Arnold conjectured in [Arn64] that the mechanism of instability based on the existence of transition chains "is applicable to the general case (for example, to the problem of 3 bodies)" the implementation of these ideas is quite challenging and no such result is available so far.

The first result in this direction was obtained in [DKdlRS19], which, to the best of our knowledge, constituted the first, complete, analytic proof of Arnold diffusion in Celestial Mechanics. There, the authors considered the Restricted Elliptic 3 Body Problem, in which the primaries, of masses $m_0, m_1 > 0$, revolve around each other in Keplerian ellipses. This configuration is a a 2+1/2 degrees of freedom Hamiltonian system.

They showed that there exist a *transition chain of periodic orbits* along which the angular momentum[1] G of the massless body experiences large variations. Notice that the angular momentum is a conserved quantity in the 2 Body Problem, which can be seen as a limit problem of the Restricted 3 Body Problem when $m_1/m_0 \to 0$ [2].

The construction of the transition chain of periodic orbits goes as follows. In the Restricted 3 Body Problem there exists a 3 dimensional (topological) Normally Hyperbolic Invariant Cylinder $\mathcal{P}_\infty$ located "at infinity". It corresponds to the ω-limit set (resp. α-limit set) of the forward (resp. backwards) parabolic motions of the Restricted 3 Body Problem. We will refer to $\mathcal{P}_\infty$ as the "parabolic infinity". Since the Newtonian interaction decays with the distance, the dynamics on $\mathcal{P}_\infty$ is trivial: it is foliated by periodic orbits (fixed points of the time one map). On the other hand, although the linearized vector field vanishes on $\mathcal{P}_\infty$, it is a classical result that $\mathcal{P}_\infty$ posseses 4 dimensional stable $W^s(\mathcal{P}_\infty)$ and unstable $W^u(\mathcal{P}_\infty)$ invariant manifolds (these are indeed the set of forward and backward parabolic motions respectively). Moreover, for sufficiently large $G_* > 0$, the submanifolds $W^{u,s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ pass far from the position of the primaries. In other words, the submanifolds $W^{u,s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ are contained in the *hierarchical region*, where the Restricted 3 Body Problem can be studied as a perturbation of the 2 Body Problem. Then, by means of classical Poincaré-Melnikov theory [Mel63, DdlLS06], the authors of [DKdlRS19] established that, for $G_* \gg 1$, $W^u(\mathcal{P}_\infty \cap \{G \geq G_*\})$ and $W^s(\mathcal{P}_\infty \cap \{G \geq G_*\})$ intersect transversally along *two different homoclinic manifolds* by further asuming that $m_1/m_0 \ll 1$. Then, to overcome the fact that the inner dynamics on $\mathcal{P}_\infty$ is trivial, the authors make use of the two scattering maps associated to the two different homoclinic manifolds. They prove that these maps share no common invariant curve, which finally implies the existence of drifting orbits of the iterated function system [Moe02].

[1] In polar coordinates in the plane of motion of the massless body, the angular momentum G is symplectically conjugated to the angular coordinate.

[2] For this range of parameters, the massless body only "feels" the interacion with m_0 up to terms of order $m_1/m_0 \ll 1$.

From the a priori unstable to the a priori stable case

As already mentioned, the transition chain of periodic orbits constructed in [DKdlRS19] is contained in the hierarchical region, where the third body is far away from the primaries and the Restricted 3 Body Problem can be studied as a time periodic perturbation of the 2 Body Problem. Due to the fast decay of the Newtonian force with the distance, in the hierarchical regime, there exist *two different time scales*: the motion associated to the integrable approximation is slow compared to the evolution of the time variable. In a neighborhood of $\mathcal{P}_\infty \cap \{G \geq G_*\}$, the ratio between the two time scales is proportional to $\varepsilon = G_*^{-3}$ while the size of the perturbation is of order $\delta = G_*^{-3}(m_1/m_0) = \varepsilon(m_1/m_0)$.[3] Therefore, the effect of the perturbation along the stable and unstable manifolds of $\mathcal{P}_\infty \cap \{G \geq G_*\}$ averages out to an exponentially small remainder $\mathcal{O}(\delta \exp(-1/\varepsilon))$, which, as a matter of fact, bounds the distance between $W^s(\mathcal{P}_\infty \cap \{G \geq G_*\})$ and $W^u(\mathcal{P}_\infty \cap \{G \geq G_*\})$.

Indeed, a somehow standard averaging argument (see [Nei84]) yields a *non sharp*, exponentiall small, upper bound on the distance between these manifolds. However, in order to prove the existence of *transverse intersections* between them, one needs to obtain an asymptotic formula for the distance between these manifolds (measured along a suitable transverse section). This requires substantially more work: notice that the perturbation (in the hierarchical approximation) has size $\delta = \varepsilon(m_1/m_0)$, so (for fixed values of the masses) one has to prove that a lot of cancellations happen in order to obtain a sharp asymptotic formula with exponentially small (in $\varepsilon = G_*^{-3}$) leading term. This obstacle was removed in [DKdlRS19] by assuming that m_1/m_0, which, we recall, is proportional to the size of the perturbation, is also exponentially small in ε, measuring the ratio between time scales. With this trick, already employed by Arnold in his original paper [Arn64], although there exist different time scales, the system is also exponentially close to integrable and classical Poincaré-Melnikov theory can be used to prove the existence of transverse intersections between $W^s(\mathcal{P}_\infty \cap \{G \geq G_*\})$ and $W^u(\mathcal{P}_\infty \cap \{G \geq G_*\})$. Moreover, in this setting, the scattering maps associated to the transverse intersections between these manifolds are also exponentially close to the identity. Therefore, although the difference between the scattering maps is exponentially small, the verification of the non existence of common invariant curves can be investigated by means of classical perturbation theory.

Summing up, the transition chain in [DKdlRS19] is built in a neighbourhood of a (topological) Normally Hyperbolic Invariant Cylinder (NHIC), in which there exist different time scales. However, by the choice of the parameters in the problem, the Hamiltonian is exponentially close (with respect to the ratio between time scales) to integrable. In the Arnold diffusion literature, this setting is usually referred to the *a priori unstable case*. Indeed, for these systems, the "splitting" between the invariant manifolds of the NHIC, is of the order of the perturbation.

On the other hand, one refers to the *a priori stable case*, for perturbations of completely integrable systems. Notice that the integrable system is completely elliptic, i.e. there do not exist hyperbolic invariant objects. For sufficiently small ε-perturbations, a NHIC $\mathcal{N}_\varepsilon$ typically arises in small neighbourhoods of single resonances [Ber10]. Indeed, they survive as perturbations of the NHIC $\widetilde{\mathcal{N}}_\varepsilon$ associated to the truncated resonant normal form. Moreover, due to its integrability, in the truncated resonant normal form, the invariant manifolds of $\widetilde{\mathcal{N}}_\varepsilon$ coincide along a homoclinic manifold. However, due to their weak hyperbolicity (it is of the order of the square root of the size of the perturbation, i.e. $\sqrt{\varepsilon}$), there exist different time scales in a neighbourhood of $\mathcal{N}_\varepsilon$: the fast, non resonant, angles and the slow, resonant, angle. Studying the splitting of the homoclinic manifold of $\widetilde{\mathcal{N}}_\varepsilon$, when considering the full normal form, is now quite subtle. Although the truncated normal form is ε^α (for some $\alpha > 1$) close to the full normal form, in the real analytic setting we know that, due to the existence of different time scales, the perturbation averages out to an exponentially small remainder (in ε) which bounds the splitting of the homoclinic manifold. However, there are no extra parameters at our disposal, and, obtaining an asymptotic formula for the difference between the stable and unstable manifolds of $\mathcal{N}_\varepsilon$ is much harder than in the a priori unstable case.

Our first main contribution is the extension of the result in [DKdlRS19] to the case of arbitrary masses $m_0, m_1 > 0$, a setting in which we face some of the challenges present in the a priori stable case.

[3] The labelling of the primaries is not relevant, so without loss of generality we can suppose that $m_1 < m_0$.

Theorem 1.1.5 ([GPS23b]). *Let G be the angular momentum of the massless body and $\epsilon \in (0,1)$ be the eccentricity of the orbit of the primaries. Then, for any $m_0, m_1 > 0$, $m_1 \neq m_0$, there exists $G_* > 0$ such that, for any $\epsilon \in (0, G_*^{-3})$ and any values G_1, G_2 satisfying*

$$G_* \leq G_1 < G_2 \leq \epsilon^{-1/3},$$

there exists $T > 0$ and an orbit γ of the RPE3BP for which

$$G \circ \gamma(0) \leq G_1 \qquad \text{and} \qquad G_2 \leq G \circ \gamma(T).$$

To the best of our knowledge, this is the first result concerning Arnold diffusion for a real-analytic Hamiltonian system which displays features of the a priori stable case[4]. In particular, in a neighbourhood of $\mathcal{P}_\infty \cap \{G \geq G_*\}$, the dynamics of the RPE3BP presents two different time scales, and we study the existence of transverse intersections between its invariant manifolds without making use of extra parameters. In order to prove Theorem 1.1.5, in [GPS23b], we introduce a new approach to:

- Analyze the highly anisotropic splitting between the stable and unstable manifolds associated to pairs of partially hyperbolic fully resonant invariant tori in a singular perturbation framework, and
- Distinguish the dynamics of two exponentially close scattering maps associated to different homoclinic channels.

The first item can be seen as an extension of the formalism developed in [Sau01, LMS03], where the splitting of the stable and unstable invariant manifolds of the same quasiperiodic torus was investigated[5]. We exploit the fact that $\mathcal{P}_\infty$ is foliated by invariant (resonant) tori, whose stable and unstable invariant manifolds are Lagrangian. With this approach, we can take advantage of the symplectic features of the problem. Let us mention that, all previous works which study the existence of transverse intersections between the invariant manifolds of different invariant tori rely on an indirect approach: first one proves the existence of transverse homoclinic orbits to a given torus and then deduce the existence of heteroclinic orbits to nearby tori by direct application of the implicit function theorem. However, the directions along which the splitting is exponentially small can move as we vary the torus. Thus, to ensure that the errors in the approximation by the homoclinic connection are exponentially small, this indirect method only works when the two tori under consideration are exponentially close (in the perturbative parameter).

We, on the other hand, study the existence of transverse intersections between different pair of tori in a direct way. This enables us to establish the existence of heteroclinic connections between resonant tori separated up to a distance of the size of the perturbation.

It is key for proving that, the invariant manifolds of $\mathcal{P}_\infty \cap \{G \geq G_*\}$ intersect transversally along *two different homoclinic manifolds* which are moreover diffeomorphic to (recall that ϵ is the eccentricity of the primaries orbit)

$$\mathcal{P}_\infty(\epsilon, G_*) = \mathcal{P}_\infty \cap \{G_* < G < \epsilon^{-1/3}\}.$$

The upshot of this result is that we can prove the existence of two different scattering maps defined *globally* on $\mathcal{P}_\infty(\epsilon, G_*)$[6]. These scattering maps are polynomially close to the identity. We develop tools to obtain an asymptotic formula for their difference, which averages out to an exponentially small quantity, and then make use of an interpolation plus averaging argument to show that they do not share common invariant curves.

Another novelty of our construction is that we work directly with the generating functions of the stable and unstable manifolds of the invariant tori instead of relying on vector parametrizations. The main difficulty is the appearance of certain unbounded operator in the linearized invariant equation defining the generating functions (see [Sau01]). This obstacle was removed in all previous works by considering a different vector parametrization of the invariant manifolds. We overcome the problem, and

[4]The first example of a real-analytic a priori stable system exhibiting topological instability was recently constructed by B. Fayad in [Fay23]. The techniques are, however, different from the Arnold diffusion mechanism.

[5]The authors in [Sau01, LMS03] consider a generalized Arnold model.

[6]This is crucial in the present problem due to the degeneracy of the inner dynamics on $\mathcal{P}_\infty$

directly find the generating functions, by making use of a suitable Newton iterative scheme in a scale of Banach spaces in the spirit of the usual schemes used in KAM theory.

We believe that the main ideas developed in [GPS23b] can be of general interest for the study of Arnold diffusion in the real-analytic a priori stable setting.

1.1.2 Oscillatory motions in the Restricted 3 Body Problem

Another major question in Celestial Mechanics is the description of the final motions (i.e. defined for all future times) of the Restricted 3 Body Problem. In the restricted case, the classification in Theorem 1.1.1 reduces to four classes (as in Theorem 1.1.1, the classification also applies to $t \to -\infty$).

Theorem 1.1.6 (Chazy [Cha22]). *Every solution of the Restricted 3-body Problem defined for all (future) times belongs to one of the following classes*

- *B (bounded):* $\sup_{t\geq 0} |q(t)| < \infty$.
- *P (parabolic)* $|q(t)| \to \infty$ *and* $|\dot{q}(t)| \to 0$ *as* $t \to \infty$.
- *H (hyperbolic):* $|q(t)| \to \infty$ *and* $|\dot{q}(t)| \to c > 0$ *as* $t \to \infty$.
- *O (oscillatory)* $\limsup_{t\to\infty} |q(t)| = \infty$ *and* $\liminf_{t\to\infty} |q(t)| < \infty$.

As already mentioned in the introduction, the first example was given by Sitnikov [Sit60] in a particular symmetric configuration of the Restricted 3 Body Problem nowadays known as the Sitnikov example.

The Moser approach: a toolbox from hyperbolic dynamics

In 1973, Moser gave a new, conceptually more transparent, proof of the existence of oscillatory motions in the Sitnikov example [Mos01], making use of ideas from hyperbolic dynamics: he built a *homoclinic class* which contains oscillatory motions.

More concretely, he considered a (topologically) hyperbolic periodic orbit γ_∞ at infinity, [7] and proved that its stable and unstable invariant manifolds intersect transversally. Although γ_∞ is only topologically hyperbolic (often denoted as parabolic), Moser proved that, for a suitable (2 dimensional) return map Φ_Σ to a suitable section Σ close to $W^u(\gamma_\infty) \pitchfork W^s(\gamma_\infty)$, there exists a non trivial hyperbolic set $\mathcal{X}$. The dynamics of Φ_Σ restricted to $\mathcal{X} \subset \Sigma$ is moreover conjugated to the shift

$$\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}} \qquad\qquad (\sigma\omega)_k = \omega_{k+1},$$

acting on the space of infinite sequences. Namely, $\mathcal{X}$ is a horseshoe with "infinitely many legs" for Φ_Σ. By construction, sequences $\omega = (\cdots, \omega_{-n}, \omega_{-n+1}, \cdots, \omega_0, \cdots \omega_{n-1}, \omega_n, \cdots) \in \mathbb{N}^{\mathbb{Z}}$ for which $\limsup_{n\to\infty} \omega_n = \infty$ (resp. $\limsup_{n\to-\infty} \omega_n = \infty$) correspond to complete motions of the Sitnikov problem which are oscillatory in the future (in the past).

Moser's ideas have been very influential and have been extended to other models in Celestial Mechanics [LS80a, LS80b, Xia92, Moe07, GMS16, GSMS17, SZ20, CGM$^+$22]. The main difficulties that one faces when implementing these ideas are the following. First, proving that the stable and unstable manifolds of the parabolic infinity (a topological hyperbolic periodic orbit in symmetric configurations, and, a topological normally hyperbolic invariant cylinder in the Elliptic and in the Spatial Circular Restricted 3 Body Problem) intersect transversally. To the best of our knowledge, in all previous works, the authors consider perturbative settings in order to tackle this step. Second, to construct a non trivial hyperbolic invariant set close to the homoclinic intersection. As we will see in Section 1.1.3, this step presents major challenges for models with more than 2 degrees of freedom.

[7] Due to the existence of additional symmetries the Sitnikov model is a 1+1/2 degrees of freedom Hamiltonian system. After performing the symplectic reduction, the parabolic infinity (see Section 1.1.1 and compare with the 3 dimensional submanifold $\mathcal{P}_\infty$), reduces to a periodic orbit.

The Restricted Isosceles 3 Body Problem: a functional analytic approach to the existence of oscillatory motions

In this work we consider the so-called *Restricted Isosceles 3 Body Problem.* In this model, the primaries move periodically along a degenerate ellipse (a line) and the massless body moves on the plane perpendicular to the line in which the primaries move. The position of the primaries is symmetric with respect to this perpendicular plane, so the three bodies always form an isosceles triangle. Due to the rotational symmetry, the angular momentum G of the massless body is a conserved quantity, which can be taken as a parameter of the system. We thus obtain a one-parameter family of Hamiltonian systems H_G, with $G \in \mathbb{R}$.

Our second main result is the following.

Theorem 1.1.7 ([GPSV21, PT22]). *There eixsts a constant* $G_* > 0$ *and a subset* $\mathcal{G} \subset \mathbb{R}$ *with* $\{|G| \geq G_*\} \subset \mathcal{G}$ *and* $\mathrm{Leb}(\mathbb{R} \setminus \mathcal{G}) = 0$ *such that, for the Hamiltonian* H_G*, if* $G \in \mathcal{G}$*,*

$$X^+ \cap Y^- \neq \emptyset \qquad \textit{with} \qquad X, Y = OS, B, P, H.$$

Let us now mention a few remarks concerning Theorem 1.1.7. For all $G \in \mathbb{R}$, there exists a (topological) hyperbolic periodic orbit at infinity $\gamma_{\infty,G}$. Moreover, for $G \gg 1$, the invariant manifolds $W^{u,s}(\gamma_{\infty,G})$, are contained in the hierarchical region (recall the discussion in Section 1.1.1). Therefore, for $|G| \gg 1$, the study of transverse intersections between $W^{u,s}(\gamma_{\infty,G})$ can be studied perturbatively. This was the approach in [GPSV21]. The problem of existence of transverse intersections between $W^{u,s}(\gamma_{\infty,G})$ for $G \geq G_* \gg 1$, can indeed be considered as a simpler, lower dimensional version of our analysis of the existence of transverse intersections between the manifolds $W^{u,s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ for the Restricted Elliptic 3 Body Problem, outlined in Section 1.1.1.

However, for an arbitrary $G \in \mathbb{R}$, one cannot rely on a perturbative approach to study the existence of transverse intersections between $W^{u,s}(\gamma_{\infty,G})$. As far as we know, Theorem 1.1.7, is the first result concerning the existence of oscillatory motions relying upon a global analytical approach rather than on perturbative techniques. Some interesting related works, where the existence of oscillatory motions is obtained in a setting which is not close to integrable, are [Moe07] and [CGM+22]. While in [Moe07] the author shows the existence of oscillatory motions in the 3 Body Problem close to triple collision (perturbation from the zero angular momentum case), in [CGM+22] the authors obtain a computer assisted proof of the existence of oscillatory motions in the Restricted Circular 3 Body Problem for small values of the Jacobi constant.

As in Moser's approach, the first main step in our construction is to prove the existence of a homoclinic orbit to $\gamma_{\infty,G}$. To this end, we will adopt a global approach and deploy the powerful machinery of the theory of calculus of variations. In particular, we rephrase the problem of existence of homoclinic orbits to $\gamma_{\infty,G}$ as that of the existence of critical points of a certain action functional $\mathcal{A}_G$ defined in a suitable Hilbert space $D^{1,2}$. The existence of critical points of the action functional $\mathcal{A}_G$ is obtained by a *minmax argument* tailor made for the present problem. The use of minmax techniques to study the existence and multiplicity results for homoclinic orbits in Hamiltonian systems has already been widely exploited in the literature (see for example [Sér92, CZES90, CZR91] and [MNT99]). In the variational approach to our problem, we face two main difficulties at this step: the phase space is not compact and the vector field presents singularities (corresponding to possible collision with the massive bodies). In order to overcome the first difficulty we make use of a *renormalized action functional* defined on an appropriately chosen functional space $D^{1,2}$. In order to avoid singularities and gain compactness, we then perform a constrained deformation argument. With these techniques, together with a compactness property of the map $\mathrm{d}\mathcal{A}_G : D^{1,2} \to D^{1,2}$ and Struwe's monotonicity trick, we are able to show that, for almost all values of the angular momentum G, there exists a Palais-Smale sequence in $D^{1,2}$ which converges to a critical point of the action functional $\mathcal{A}_G$. This proves the existence of an orbit $\tilde{r}_h$ homoclinic to $\gamma_{\infty,G}$, which actually corresponds to a two sided parabolic motion of the Restricted Isosceles 3 Body Problem. It is worthwhile pointing out that half parabolic and hyperbolic motions for the N Body Problem have been obtained using variational methods in [MV09, MV20] with a different technique.

The homoclinc orbit $\tilde{r}_h$ obtained in this way is associated with an intersection between the stable and unstable manifolds of the periodic orbit $\gamma_{\infty,G}$. To proceed further, though we can not tell whether this intersection is transverse or not, we may rely on our minmax construction to deduce some topological transversality. This can be achieved by a topological degree argument based on a general result by Hofer ([Hof86]). More precisely, we exploit the mountain pass characterization of $\tilde{r}_h$ to show that for almost all values of the angular momentum G there exists a (possibly different) critical point r_h of the action functional $\mathcal{A}_G$ for which the Leray-Schauder index of the map $\nabla\mathcal{A}_G : D^{1,2} \to D^{1,2}$ at r_h is well defined and different from zero. This allows us to shadow finite segments of the homoclinic orbit r_h. The proof of Theorem 1.1.7 is then obtained by combining a suitable parabolic version of the Lambda lemma close to $\gamma_{\infty,G}$ with the outer dynamics wich shadows finite segments of r_h.

1.1.3 Hyperbolic dynamics, Oscillatory motions and Topological instability in the 3 Body Problem

Finally, we present some results concerning the existence of unstable motions in the planar 3 Body Problem (3 degrees of freedom). These motions are constructed on a region of the phase space which we denote by the *parabolic-elliptic regime* (see Theorem 1.1.1).

The parabolic-elliptic regime

On each constant, negative energy hypersurface, there exists a 3 dimensional invariant submanifold at infinity $\mathcal{E}_\infty$ for the flow of the 3 Body Problem. It corresponds to the ω-limit set (resp. α-limit set) of the points which lead to forward (resp. backwards) orbits along which the motion of one body is parabolic and the motion the other two bodies is elliptic. Since at $\mathcal{E}_\infty$ the distance between one body (the one performing the parabolic motion) and the other two (the binary, elliptic, system) is infinite, the coupling in the hierarchical approximation vanishes identically on $\mathcal{E}_\infty$. Thus, the dynamics on $\mathcal{E}_\infty$ is completely integrable. Moreover, due to the so-called super integrability of the 2 Body Problem, $\mathcal{E}_\infty$ is foliated by periodic orbits.

It is known (see [BFM20c]), that $\mathcal{E}_\infty$ posseses 4-dimensional stable and unstable invariant manifolds $W^{s,u}(\mathcal{E}_\infty)$. We focus on an invariant submanifold $\mathcal{E}_{\infty,\mathrm{circ}} \subset \mathcal{E}_\infty$ corresponding to nearly circular motion of the bodies q_0, q_1. Then, if we denote by G the angular momentum of the body q_2, and let $G_* \gg 1$, the stable and unstable manifolds $W^{u,s}(\mathcal{E}_{\infty,\mathrm{circ}} \cap \{G \geq G_*\})$ are contained in the hierarchical region, and the bodies q_0, q_1 perform nearly circular motions.

Existence of non trivial hyperbolic sets

Our third main result is the following.

Theorem 1.1.8 ([GMPS22])**.** *Consider the 3 Body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$. Then,*

$$X^+ \cap Y^- \neq \emptyset \qquad \text{with} \qquad X, Y = OS, B, PE_3, HE_3.$$

Theorem 1.1.8 is indeed a consequence of the following result, which deals with the existence of non trivial hyperbolic sets in the 3 Body Problem (see Section 1.1.2).

Theorem 1.1.9 ([GMPS22])**.** *Consider the 3 Body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$. Then, there exists a section transverse to the flow of the 3 Body Problem such that (a suitable iterate of) the induced Poincaré map possesses a non trivial hyperbolic set.*

The *contribution of the author* to the proof of Theorems 1.1.8 and 1.1.9, has been to adapt the techniques developed in [GPS23b] to prove the existence of two transverse intersections between $W^{u,s}(\mathcal{E}_{\infty,\mathrm{circ}} \cap \{G \geq G_*\})$. This corresponds to Sections 7 and 8 in [GMPS22].

Once this result is proved, close to the intersection of these invariant manifolds we build a horseshoe for certain return map associated to a transverse section. The construction of the horseshoe is rather involved. We only outline it.

We build two suitable sections Σ_1 and Σ_2 transverse to the local stable and local unstable manifolds of $\mathcal{E}_{\infty,\mathrm{circ}}$ respectively in which by using a (parabolic) Lambda lemma we are able to define a transition map.

For this four dimensional map the dynamics is hyperbolic in a pair of directions and is C^1 close to the identity in the other pair. *Two different outer maps* can be defined from certain subsets of Σ_2 to Σ_1 by following the homoclinic excursions associated to the transverse intersections between $W^{u,s}(\mathcal{E}_{\infty,\mathrm{circ}} \cap \{G \geq G_*\})$. The composition of the transition map with the outer maps yields two well defined *return maps* for the section Σ_2. Since the outer dynamics are close to the scattering map dynamics (which are close to the identity maps), the return maps are (as the transition map from Σ_1 to Σ_2) hyperbolic in a pair of directions and C^1 close to the identity in the other pair. However to build the horseshoe we need hyperbolicity in all directions. The idea to overcome this problem is to make use of the two different return maps. Indeed, we are able to prove that, for a suitable composition of these maps, there exists an *isolating block*, in which the dynamics is uniformly hyperbolic in all directions.

Topological instability in the 3 Body Problem

Due to the robustness of the mechanism, one could directly prove that the transition chain of heteroclinic orbits constructed in [DKdlRS19, GPS23b] for the R3BP can be continued to the 3BP if the mass m_2 is sufficiently small. As a consequence, one can deduce that, in the 3BP, if m_2 is sufficiently small, there exist orbits along which the angular momentum G of the third body experiences significant variations, while the eccentricity of the inner bodies remains small.

A much more challenging, and interesting question, is to prove the existence of Arnold Diffusion in the 3BP for any choice of the masses $m_0, m_1, m_2 > 0$. The substantial difference is that, due to the conservation of the total angular momentum, as the angular momentum of the third body grows, so does the eccentricity of the orbit of the binary system [8]. However, in order to construct orbits along which this transfer of angular momentum is significant, one cannot make use of the arguments developed in [DKdlRS19, GPS23b], since they strongly rely on the hypothesis that the eccentricity of the primaries orbit is small enough. Thus, new techniques have to be developed to, in particular, analyze the existence of transverse intersections between the invariant manifolds of $\mathcal{E}_\infty$.

Here, we present the first, of a series of papers, devoted to the construction of a transition chain of periodic orbits contained in $\mathcal{E}_\infty$, along which the angular momentum of the third body is transferred to the binary system, resulting in a substantial change of its eccentricity. In particular, we want to construct orbits which transition from close to circular orbits to highly eccentric ellipses (i.e., with close to collision points). This first paper is devoted to analyze the so-called Melnikov approximation of the distance between the invariant manifolds of $\mathcal{E}_\infty$. Just to motivate the result, let us recall that the invariant manifold $\mathcal{E}_\infty$ is foliated by invariant tori, whose stable and unstable manifolds are Lagrangian submanifolds of the phase space. Thus, these invariant manifolds can be parametrized by the gradient of scalar valued *generating functions*, which we denote by $S^{u,s}$. Notice that the existence of critical points of $\Delta S = S^u - S^s$ implies the existence of intersections between the invariant manifolds of $\mathcal{E}_\infty$. Thus, if one is able to prove that ΔS is approximated by certain function L in the C^2 topology, the existence of non degenerate critical points of L implies the existence of non degenerate critical points of ΔS. The function L is usually called the *Melnikov potential*.

Remark 1.1.10. *See Section 1.3.2 for a discussion on the justification of the Melnikov approximation and the actual construction of a transition chain of periodic orbits for the 3 Body Problem.*

The full result that we present in Chapter 6 (see also [GPS23a]), which concerns the asymptotic analysis of the Melnikov potential, is a rather lengthy formula which requires the introduction of a large amount of notation. Therefore, we only state here an important consequence of the full result, which is key to prove that the stable and unstable manifolds of $\mathcal{E}_\infty$ intersect transversally along two different homoclinic manifolds. Let ϵ denote the instantaneous eccentricity of ellipse formed by q_0, q_1 and let G be the angular momentum of q_2. Given $\delta \in (0, 1/2)$, and $G_* \gg 1$, we denote as

$$\mathcal{E}_\infty(\delta, G_*) = \{q \in \mathcal{E}_\infty : \epsilon \in (\delta, 1-\delta),\ G \geq G_*\}$$

Theorem 1.1.11. *Fix any $\delta \in (0, 1/2)$. Then, there exists $G_* \gg 1$ such that the Melnikov potential*

$$L(u, z) : \mathbb{R} \times \mathcal{E}_\infty(\delta, G_*) \to \mathbb{R}, \qquad u \in \mathbb{R},\ z \in \mathcal{E}_\infty(\delta, G_*),$$

[8] The change of eccentricity due to the transfer of angular momentum goes to zero with $m_2 \to 0$.

associated to the invariant manifolds of $\mathcal{E}_\infty(\delta, G_*)$ is non degenerate in the following sense. There exist two connected open sets $\Gamma_\pm \subset \mathcal{E}_\infty(\delta, G_*)$, satisfying

$$\mathrm{Leb}(\mathcal{E}_\infty(\delta, G_*) \setminus \Gamma_\pm) \lesssim \exp(-G_*^3/3),$$

such that, for $z \in \Gamma_\pm$, there exists $u_\pm = u_\pm(z)$ for which $\partial_u L(u_\pm(z), z)) = 0$ and $\partial^2_{uu} L(u_\pm(z), z) \neq 0$.

The asymptotic analysis of the Melnikov potential L in this problem is rather more complicated than in [DKdlRS19, GPS23b]. Let us explain why. As in [DKdlRS19, GPS23b], due to the existence of different time scales, the function L is given by an infinite sum of *fast oscillatory integrals*. A classical tool for analyzing fast oscillatory integrals when the integrand is a real analytic function is to change the integration contour to a *steepest descent path*. This is a complex path which visits the singularity (or singularities) of the integrand which is closest to the real axis and such that, as we move away from the singularity, the integrand decays exponentially fast.

In [DKdlRS19, GPS23b], the domain of analyticity of the integrand is close to a direct product and, moreover, its singularities are all "sufficiently close" from each other so they can be treated as one. Then, one can expand the integrand in Laurent series around this singularity to analyze the Melnikov potential asymptotically. However, in the present setting, where we consider arbitrary eccentricities, the domain of analyticity of the integrand of the oscillatory integrals defining L is not a direct product and there exist several, different singular submanifolds. Thus, one cannot rely on Laurent expansions. On the other hand, our approach makes use of standard tools from complex analysis such as the method of *analytic continuation* to first, locate all the complex singularities and, second, study the local behavior of the integrand around them.

1.2 Conclusions

The present thesis has been devoted to the study of different kinds of *unstable motions* in the 3 Body Problem and its restricted versions. The common feature in the mechanisms presented, is the existence of (topological) *hyperbolic invariant objects*. We have introduced different tools to prove the existence of transverse intersections between their invariant manifolds in a variety of contexts, including *singular perturbation frameworks* and *non perturbative settings*. Then, we have investigated the dynamical consequences of the existence of these transverse intersections. More concretely, the existence of Arnold diffusion, oscillatory motions and non trivial hyperbolic sets.

This thesis was performed from September 2019 to April 2023 by Jaime Paradela Díaz and was supervised by Professors Marcel Guàrdia Munárriz and Tere M-Seara Alonso.

The results presented here can also be found in the articles [GPSV21, GMPS22, PT22, GPS23b, GPS23a]. At the current date, the first is already published, the second, third and fourth are under revision, and the fifth one is being prepared.

1.3 Open problems

To conclude this introduction, we list three open problems, related to the main results of this work, which we find interesting and which we are currently addressing.

1.3.1 Existence of two sided completely parabolic motions in the N Body Problem

Consider a complete orbit of the N Body Problem. We say that it is *completely parabolic*, if it approaches infinity with zero asymptotic kinetic energy. It is a classical result that, for completely parabolic motions, its asymptotic shape must be a *central configuration* (see [Che98]). A number of works have proved that given an arbitrary initial configuration and an arbitrary central configuration there exists a completely parabolic orbit of the N Body Problem, passing through the given initial configuration and whose asymptotic shape is the prescribed central configuration (see [MV09, BDFT21]).

However, to the best of our knowledge, there are no results available on the existence of two sided completely parabolic orbits, i.e. orbits which are completely parabolic both in backward and forward time. By definition, the shape of this orbit is asymptotic to a central configuration both in the past and in the future. Together with S. Terracini, we are investigating the existence of such orbits. A possible roadmap is to consider orbits of the $2N$ Body Problem which are symmetric under the dihedral group. This symmetry, drastically drops the dimension of the problem to a system of two degrees of freedom [FP08, FP13]. In this setting, we plan to extend the tools developed in [PT22] to study the existence of two sided completely parabolic motions in the $2N$ Body Problem.

1.3.2 Topological Instability in the 3 Body Problem

As we already explained in Section 1.1.1, justifying the Melnikov approximation (namely, obtaining an asymptotic formula for the distance between the stable and unstable invariant manifolds), for problems involving different time scales, is a quite challenging problem. One option would be to carry on an averaging procedure with an "optimal loss of analyticity". Another approach is to follow Lazutkin ideas for the standard map [Laz87] (see also [Sau01, LMS03]), and extend the parametrizations of the invariant manifolds to a "sufficiently large" complex domain. This was the approach in [GPS23b].

Compared to [GPS23b], justifying the Melnikov approximation for the 3 Body Problem in the parabolic-elliptic regime (see Section 1.1.3), requires overcoming many major difficulties. These are indeed related to the same difficulties present in the analysis of the Melnikov potential (see the discussion at the end of Section 1.1.3). Namely, when we study the Hamiltonian of the 3 Body Problem, in a neighbourhood of the invariant manifolds of $\mathcal{E}_\infty$, its domain of analyticity is not a direct product.

Together with M. Guàrdia and T. M-Seara, we are currently extending the techniques in [GPS23b] to overcome this problem.

1.3.3 Homoclinic tangencies in Celestial Mechanics

Since the pioneering work of Newhouse [New70, New74, New79], the existence of homoclinic tangencies is widely recognized as a source of wild dynamics. For Hamiltonian systems (at least in models of 1+1/2 or 2 degrees of freedom), this includes: existence of thick hyperbolic basic sets with persistent tangencies, existence of infinitely many elliptic islands and universality (loosely speaking, this concept means that, given any diffeomorphism f of the disk, there exists a return map of the system which, after a proper renormalization, approximates the dynamics of f up to arbitrary order).

In some particular models in Celestial Mechanics, due to the existence of discrete symmetries, one can find *primary homoclinic tangencies* between the manifolds of the "parabolic infinity". Moreover, in some cases, these tangencies unfold "generically" as we move the parameters of the system. Together with J.M. Cors, M. Garrido, and P. Martín, we are studying the dynamical consequences of this phenomenon.

Disclaimer: The reader will forgive us for the text overlap between some of the chapters of the thesis, especially between their introduction sections.

Chapter 2

A degenerate Arnold diffusion mechanism in the Restricted 3 Body Problem

Abstract: A major question in dynamical systems is to understand the mechanisms driving global instability in the 3 Body Problem (3BP), which models the motion of three bodies under Newtonian gravitational interaction. The 3BP is called *restricted* if one of the bodies has zero mass and the other two, the primaries, have strictly positive masses m_0, m_1. We consider the Restricted Planar Elliptic 3 Body Problem (RPE3BP) where the primaries revolve in Keplerian ellipses. We prove that the RPE3BP exhibits topological instability: *for any values of the masses* m_0, m_1 (except $m_0 = m_1$), we build orbits along which the angular momentum of the massless body experiences an arbitrarily large variation provided the eccentricity of the orbit of the primaries is positive but small enough.

In order to prove this result we show that a *degenerate Arnold Diffusion Mechanism*, which moreover involves exponentially small phenomena, takes place in the RPE3BP. Our work extends the result obtained in [DKdlRS19] for the a priori unstable case $m_1/m_0 \ll 1$, to the case of arbitrary masses $m_0, m_1 > 0$, where the model displays features of the so-called *a priori stable* setting.

2.1 Introduction

The N Body Problem models the motion of N bodies under mutual gravitational interaction. Understanding its global dynamics for $N \geq 3$ (the system is integrable for $N = 2$) is probably one of the oldest (and more challenging) questions in dynamical systems. A major achievement in this direction was the proof of the existence of a positive measure set of quasiperiodic motions in the N Body Problem. This result was first established by Arnold in [Arn63], who gave a master application of the KAM technique to the case of 3 coplanar bodies. The proof was later extended to case of $N \geq 3$ in the work of Féjoz and Herman [Fej04] (see also [Rob95, CP11]). On the other hand, in accordance with the general belief that the N Body Problem, although strongly degenerate, displays the main features of a "typical" Hamiltonian system, in his ICM address, Herman conjectured [Her98] that the set of non wandering points for the flow of the N Body Problem is nowhere dense on every energy level for $N \geq 3$. This would imply topological instability for the N Body Problem in a very strong sense.

The existence of topological instability in Hamiltonian systems was first investigated by Arnold in [Arn64], where he constructed an example of nearly integrable Hamiltonian in which this kind of behavior occurs. To that end, Arnold proposed a mechanism giving raise to unstable motions based on the existence of a transition chain of invariant tori: a sequence of invariant irrational tori which are connected by transverse heteroclinic orbits. This mechanism is nowadays called the *Arnold mechanism*. Arnold verified that this mechanism takes place in a cleverly built model usually referred to as the Arnold model, and he conjectured that topological instability is indeed a common phenomenon in the complement of integrable Hamiltonian Systems [Arn63]. Despite the enormous amount of research (see for example [Dou88, BT99, DdlLS00, MS02, BB02, Mat03, CY04, Tre04, DdlLS06, Ber08, GT08, DH09, NP12, Tre12, BKZ16, DdlLS16, Che17, GT17, KZ20, GdlLS20]) the Arnold diffusion phenomenon, and more generally the dynamics in the complement of the KAM tori set, is still poorly understood (and even more poorly for real analytic or non-convex Hamiltonians).

In [Arn64], Arnold conjectured that the mechanism of instability based on the existence of transition chains *"is applicable to the general case (for example, to the problem of 3 bodies)"*. However, results concerning the existence of Arnold diffusion in the 3 Body Problem or related models are rather scarce (see [CG18, DKdlRS19, CFG22, CFG23] and also [DGR16, FGKR16] for numerical based results).

The 3 Body Problem is called "restricted" if one of the bodies has zero mass and the other two, the primaries, have strictly positive masses m_0, m_1. In this limit problem, the motion of the primaries is just a 2 Body Problem and the dynamics of the massless body is governed by the gravitational interaction with the primaries. In this work, we consider the case in which the primaries revolve around each other in Keplerian ellipses of eccentricity $\epsilon \in (0,1)$ and the massless body moves on the same plane as the primaries. This model, usually known in the literature as the Restricted Planar Elliptic 3 Body Problem (RPE3BP), is a $2+1/2$ degrees of freedom Hamiltonian system. For $\epsilon = 0$ (i.e. for the Restricted Planar Circular 3 Body Problem), the rotational symmetry prevents the existence of topological instability in nearly integrable settings (see Remark 2.1.4 below).

The goal of this paper is to prove that a degenerate Arnold Diffusion mechanism takes place in the RPE3BP: we show that for any value of the masses of the primaries ($m_0 \neq m_1$), there exist orbits of the RPE3BP along which the angular momentum of the massless body experiences any predetermined drift provided the eccentricity of the orbits of the primaries is positive but small enough. Notice that the angular momentum is a conserved quantity in the 2 Body Problem, which can be seen as a limit problem of the Restricted 3 Body Problem when $m_1/m_0 \to 0$.

To the best of our knowledge the first complete proof of existence of Arnold Diffusion in Celestial Mechanics was obtained in [DKdlRS19], in which the authors showed the existence of topological instability in the RPE3BP. Nevertheless, this result was established under the strong hypothesis $m_1/m_0 \ll 1$ (see Section 2.1.2 for a more precise description of the setting). Under this condition, the problem falls in the a priori unstable case for the study of Arnold diffusion and can be analyzed by means of classical perturbation theory. Our result extends the work in [DKdlRS19] to the case of *arbitrary masses* $m_0, m_1 > 0$, a setting in which the problem displays many features of the so-called *a priori stable* case.

2.1.1 Main Result

Fix a Cartesian reference system with origin at the center of mass of the primaries and choose units so that the total mass of the primaries is equal to 1. In these coordinates, the primaries, which we denote by q_0 and q_1, move along Keplerian ellipses of eccentricity $\epsilon \in (0,1)$ whose time parametrization reads

$$q_0(t) = \mu\varrho(t)(\cos f(t), \sin f(t)) \qquad\qquad q_1(t) = -(1-\mu)\varrho(t)(\cos f(t), \sin f(t)),$$

where $m_0 = 1-\mu$ and $m_1 = \mu \in (0, 1/2]$ are the masses of q_0 and q_1, $\varrho(t)$ is the distance between the primaries and is given by

$$\varrho(t) = \frac{1-\epsilon^2}{1+\epsilon\cos f(t)}$$

and the so called *true anomaly* $f(t)$ is determined implicitely by the equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{(1+\epsilon\cos f)^2}{(1-\epsilon)^{3/2}}, \qquad\qquad f(0) = 0.$$

The RPE3BP describes the motion of a massless body $q \in \mathbb{R}^2$ in the gravitational field generated by the primaries and it is governed by the second order differential equation

$$\ddot{q} = (1-\mu)\frac{q - q_0(t)}{|q-q_0(t)|^3} + \mu\frac{q-q_1(t)}{|q-q_1(t)|^3}. \tag{2.1}$$

It is a classical fact that the RPE3BP admits a Hamiltonian structure. Introducing p and E the conjugate momenta to q and t, and the gravitational potential

$$U(q,t) = \frac{1-\mu}{|q-q_0(t)|} + \frac{\mu}{|q-q_1(t)|}$$

the RPE3BP is Hamiltonian with respect to

$$\mathcal{H}(q,p,t,E) = \frac{|p|^2}{2} - U(q,t) + E$$

and the canonical symplectic structure in the extended phase space $T^*(\mathbb{R}^2 \times \mathbb{T})$. The following is our main result.

Theorem 2.1.1. *Let $G(q,p) = |q \wedge p|$ be the angular momentum of the massless body. Then, for any $\mu \in (0,1/2)$, there exists $G_* > 0$ such that, for any $\epsilon \in (0, G_*^{-3})$ and any values G_1, G_2 satisfying*

$$G_* \leq G_1 < G_2 \leq \epsilon^{-1/3},$$

there exists $T > 0$ and an orbit γ of the RPE3BP for which

$$G \circ \gamma(0) \leq G_1 \qquad \text{and} \qquad G_2 \leq G \circ \gamma(T).$$

2.1.2 Previous results: Arnold diffusion and unstable motions in Celestial Mechanics

A number of works have shown the existence of unstable motions in the 3 Body Problem or its restricted versions. For example, oscillatory orbits (orbits that leave every bounded region but return infinitely often to some fixed bounded region, see [Cha22]) and/or chaotic behavior in particular configurations of the Restricted 3 Body Problem [Sit60, LS80a, Moe84, Xia92, Bol06, Moe07, GK11, GK12, GMS16, GSMS17, Mos01, SZ20, GPSV21, CGM+22, BGG21, BGG22, GMPS22, PT22].

However, results concerning the existence of Arnold diffusion in the 3 Body Problem or related models are rather scarce. Some remarkable works are [DGR16, FGKR16, CG18, DKdlRS19, CFG22, CFG23]. In [DGR16] and [FGKR16], the authors combine numerical with analytical techniques to study the existence of diffusion orbits in the Restricted 3 Body Problem close to $L1$ and along mean motion resonances respectively. In [CG18] the authors give a computer assisted proof of the existence of Arnold diffusion in the Restricted Planar Elliptic 3 Body Problem. Moreover, some very interesting features of the random behavior, such as convergence to a stochastic process, are studied. In the recent works [CFG22, CFG23], the authors show that the Arnold Diffusion mechanism takes place in the spatial 4 Body Problem.

Of major importance, and closely related to the setting of the present work, is the paper [DKdlRS19] (see also [Xia93, MP94] for previous partial results). To the best of our knowledge it constituted the first complete analytic proof of Arnold Diffusion in Celestial Mechanics.

Theorem 2.1.2 (Theorem 1 in [DKdlRS19])**.** *There exist $G_* > 0$ and $c > 0$ such that, for any $\epsilon \in (0, cG_*^{-1})$ and any values G_1, G_2 satisfying*

$$G_* \leq G_1 < G_2 \leq c/\epsilon,$$

if the mass ratio satisfies

$$\mu \ll \exp(-G_2^3/3),$$

there exists $T > 0$ and an orbit $\gamma : [0,T] \to \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^3$ of the RPE3BP for which

$$G \circ \gamma(0) \leq G_1 \qquad G_2 \leq G \circ \gamma(T).$$

2.1.3 From the a priori unstable to the a priori stable case

The proofs of Theorem 2.1.1 and Theorem 2.1.2 rely on the existence of a rather degenerate Arnold Diffusion mechanism. In modern language, the seminal proof of existence of Arnold diffusion in [Arn64] is based on the existence of a Normally Hyperbolic Invariant Cylinder foliated by invariant tori. The stable and unstable manifolds of the cylinder intersect transversally, what allows to construct a sequence of quasiperiodic whiskered invariant tori connected by heteroclinic orbits. An important tool in the modern approach to Arnold diffusion is the so called scattering map [DdlLS08] which encodes the dynamics along these heteroclinic connections.

In [DKdlRS19], it is shown that, for the RPE3BP, there exists a 3 dimensional (topological) Normally Hyperbolic Invariant Cylinder $\mathcal{P}_\infty$ foliated by periodic orbits (see Section 2.1.4). We will see in Section 2.2.2 that, in the region of the phase space $\{G \geq G_*\}$, the RPE3BP can be studied as a fast periodic perturbation of the integrable 2BP. Theorems 2.1.1 and 2.1.2 are based on the existence of a transition chain of periodic orbits, contained in $\mathcal{P}_\infty \cap \{G \geq G_*\}$, along which the angular momentum G experiences an arbitrarily large drift.

Under the additional (and rather restrictive) hypothesis of exponentially small mass ratio $\mu \ll \exp(-G_*^3/3)$, the RPE3BP in the parabolic regime with large angular momentum $G \geq G_* \gg 1$ (see Section 2.1.4) falls in the a priori unstable setting. Indeed, one takes the parameter μ, measuring the size of the perturbation, exponentially small with respect to the one measuring the ratio between the different time scales of the problem $1/G_*^3$, as Arnold did in his original paper [Arn64]. This heavily simplifies the two main steps for the construction of the diffusion chain of heteroclinic orbits (see Section 2.1.4). On one hand, the existence of transverse intersections between the 4 dimensional stable and unstable manifolds $W^{\mathrm{s}}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ and $W^{\mathrm{u}}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ can be tackled by classical perturbative techniques (Poincaré-Melnikov method). The reason is that, although the splitting between these manifolds is exponentially small in $1/G_*$, the system is also exponentially close to integrable. To overcome the fact that the inner dynamics on $\mathcal{P}_\infty$ is trivial, the proof of Theorems 2.1.1 and 2.1.2 make use of two different scattering maps associated to two different homoclinic manifolds contained in $W^{\mathrm{s}}(\mathcal{P}_\infty \cap \{G \geq G_*\}) \pitchfork W^{\mathrm{u}}(\mathcal{P}_\infty \cap \{G \geq G_*\})$. In the doubly perturbative setting $\{G \geq G_*\}$ and $\mu \ll \exp(-G_*^3/3)$, the scattering maps are exponentially close to the identity and a (non trivial) algebraic computation shows that they share no common invariant curve. Then, the existence of drifting orbits can be deduced from classical arguments (see [Moe02]).

Theorem 2.1.1 extends Theorem 2.1.2 to the case $\mu \in (0, 1/2)$. In this setting, the problem displays many features of the so called *a priori stable* case in the real analytic category. In particular, no extra parameters are available to study the exponentially small splitting between $W^{\mathrm{u}}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ and $W^{\mathrm{s}}(\mathcal{P}_\infty \cap \{G \geq G_*\})$. To the best of our knowledge, Theorem 2.1.1 is the first result proving the existence of Arnold Diffusion for a real analytic Hamiltonian which does not fall in the a priori unstable setting [1].

We develop two main sets of tools to prove Theorem 2.1.1. In particular we introduce a new approach to:

- Analyze the highly anisotropic splitting between the stable and unstable manifolds associated to pairs of partially hyperbolic fully resonant invariant tori in a singular perturbation framework, and

- Distinguish the dynamics of two exponentially close scattering maps associated to different homoclinic channels.

We believe that the main ideas developed in this work can be of general interest for the study of Arnold diffusion in the *real analytic a priori stable setting*. We refer the interested reader to Section 2.1.5, where we introduce a degenerate version of the Arnold model to explain the main difficulties and novelties in the proof of Theorem 2.1.1.

[1] The first example of a real analytic a priori stable system exhibiting topological instability was recently constructed by B. Fayad in [Fay23]. The techniques are however different from the Arnold Diffusion mechanism.

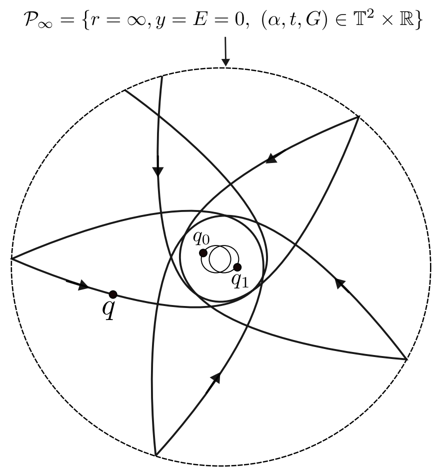

Figure 2.1: Sketch of the projection on the plane $q \in \mathbb{R}^2$ of a segment of the diffusive orbit. It shadows a finite family of parabolic orbits which are indeed heteroclinic orbits to $\mathcal{P}_\infty$. As the angular momentum grows the massless body q passes further from the primaries.

Remark 2.1.3. *We will see later (see Section 2.3.1 and, in particular, the discussion below Theorem 2.3.9) that the angle of splitting between the invariant manifolds W^{u}_I and W^{s}_I of the invariant torus $\mathcal{T}_I = \mathcal{P}_\infty \cap \{G = I\}$ is of the order $\sim \mu(1-2\mu)\epsilon\exp(-I^3/3)$. Since the inner dynamics on $\mathcal{P}_\infty$ is trivial, in the diffusion mechanism underlying the proof of Theorems 2.1.1 and 2.1.2, an estimate of the splitting angle between W^{u}_I and W^{s}_I is not enough for estimating the diffusion time, and one more ingredient comes into play: the transversality between the invariant curves of the two scattering maps associated to each transverse homoclinic intersection. Still, we will also see (see the proof of Proposition 2.3.25 below) that the angle between these invariant curves is again proportional to $\sim \mu(1-2\mu)\epsilon\exp(-I^3/3)$. Thus, the orbits obtained in Theorem 2.1.1 present significant drift only after exponentially long times.*

2.1.4 Outline of the proof of Theorem 2.1.1

We introduce the (exact symplectic) change to polar coordinates $(r, y, \alpha, G, t, E) \mapsto (q, p, t, E)$ where $q = (r\cos\alpha, r\sin\alpha)$ and (y, G) are the conjugate momenta to (r, α). In this coordinate system, the RPE3BP is a Hamiltonian system on the (extended) phase space [2]

$$(r, \alpha, t, y, G, E) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^3 \equiv M_{\mathrm{pol}} \tag{2.2}$$

with Hamiltonian function

$$H_{\mathrm{pol}}(r,\alpha,t,y,G,E) = \frac{y^2}{2} + \frac{G^2}{2r^2} - V_{\mathrm{pol}}(r,\alpha,t) + E, \qquad V_{\mathrm{pol}}(r,\alpha,t) = U(r\cos\alpha, r\sin\alpha, t). \tag{2.3}$$

The equations of motion in polar coordinates simply read

$$\begin{aligned}
\dot r &= \partial_y H_{\mathrm{pol}} = y & \dot y &= -\partial_r H_{\mathrm{pol}} = \frac{G^2}{r^3} + \partial_r V_{\mathrm{pol}}\\
\dot\alpha &= \partial_G H_{\mathrm{pol}} = \frac{G}{r^2} & \dot G &= -\partial_\alpha H_{\mathrm{pol}} = \partial_\alpha V_{\mathrm{pol}}\\
\dot t &= \partial_E H_{\mathrm{pol}} = 1 & \dot E &= -\partial_t H_{\mathrm{pol}} = \partial_t V_{\mathrm{pol}}.
\end{aligned}$$

[2] Properly one should exclude collisions. Since our analysis is performed far from collisions we abuse notation and we refer to M_{pol} as phase space.

Since $V_{\rm pol}(r,\alpha,t)\to 0$ as $r\to\infty$, we identify the invariant submanifold [3]

$$\mathcal{P}_\infty=\left\{(\infty,\varphi,t,0,I,0)\colon (\varphi,t)\in\mathbb{T}^2, I\in\mathbb{R}\right\} \tag{2.4}$$

contained in the zero energy level $\{H_{\rm pol}=0\}$.

Remark 2.1.4. *We have already pointed out in the introduction that, although the Restricted 3 Body Problem with $\mu>0,\epsilon=0$, (i.e. the RPC3BP) is non integrable (see [GMS16]), the existence of topological instability is prevented by the rotational symmetry. In particular, the conservation of the Jacobi constant $\mathtt{J}=H_{\rm pol}-G$, which is a consequence of the rotational symmetry, readily shows that, for the RPC3BP, there cannot exist heterolicinic orbits connecting periodic orbits in $\mathcal{P}_\infty$ with different values of G (see also Lemma 2.3.17).*

However, we want to remark that in the setting in Theorem 2.1.1 one cannot deduce the existence of transversal intersections between $W^{\rm u}(\mathcal{P}_\infty)$ and $W^{\rm s}(\mathcal{P}_\infty)$ from that of the invariant manifolds of the RPC3BP. Indeed, the splitting between the invariant manifolds in the case $\mu>0,\epsilon=0$ is exponentially small in $1/G_*$ and we consider eccentricities up to polynomially small values in $1/G_*$.

Despite being degenerate (the linearized vector field restricted to $\mathcal{P}_\infty$ vanishes), it is a classical result of Baldomá and Fontich [BF04b] (see also McGehee [McG73] for the circular case) that the manifold $\mathcal{P}_\infty$ posseses stable and unstable manifolds

$$\begin{aligned} W^{\rm u}(\mathcal{P}_\infty)=&\{x\in\{H_{\rm pol}=0\}\colon \exists z\in\mathcal{P}_\infty \text{ for which } \lim_{\tau\to-\infty}|\phi^\tau(x)-\phi^\tau(z)|=0\}\\ W^{s}(\mathcal{P}_\infty)=&\{x\in\{H_{\rm pol}=0\}\colon \exists z\in\mathcal{P}_\infty \text{ for which } \lim_{\tau\to\infty}|\phi^\tau(x)-\phi^\tau(z)|=0\}. \end{aligned} \tag{2.5}$$

By introducing the McGehee transformation $r=\eta_{\rm MG}(x)=2/x^2$, one can prove that the flow on a neighborhood of the invariant manifold $\tilde{\mathcal{P}}_\infty=\eta_{\rm MG}^{-1}(\mathcal{P}_\infty)$ "behaves" as the flow around a Normally Hyperbolic Invariant Cylinder. Namely, one can prove that $W^{\rm u,s}(\tilde{\mathcal{P}}_\infty)=\eta_{\rm MG}^{-1}(W^{\rm u,s}(\mathcal{P}_\infty))$ exist and are analytic submanifolds except at $x=0$, where they are C^∞, and that a parabolic version of the Lambda lemma holds. Because of this, we say that $\mathcal{P}_\infty$ is a Topological Normally Hyperbolic Invariant Cylinder (TNHIC).

In order to prove Theorem 2.1.1, we will use the invariant manifolds of $\mathcal{P}_\infty$, whose vertical direction is parametrized by the coordinate G, as a highway to obtain orbits whose angular momentum G experiences arbitrarily large variations. More concretely, we build a *transition chain of periodic orbits* in $\mathcal{P}_\infty$ along which G increases.

There are two main ingredients for the construction of the aforementioned transition chain of periodic orbits. The first one is the existence of two different transverse intersections between $W^{\rm u}(\mathcal{P}_\infty)$ and $W^{\rm s}(\mathcal{P}_\infty)$. The second one is to establish certain transversality property between the dynamics along the two different 3 dimensional homoclinic manifolds associated to the transverse intersections between $W^{\rm u}(\mathcal{P}_\infty)$ and $W^{\rm s}(\mathcal{P}_\infty)$. The application of these ideas to the RPE3BP, without assuming that μ is exponentially small with respect to $1/G_*$, is quite challenging since major difficulties are present in the verification of each of the two main ingredients for the construction of the transition chain.

Transverse homoclinic intersections between the invariant manifolds for $\mu\in(0,1/2)$: The existence of transverse intersections between the stable and unstable manifolds of a hyperbolic periodic orbit was already identified by Poincaré as a major source of dynamical complexity (see [Poi90]). The occurrence of this phenomenon, although residual in the C^r ($r\geq 1$) topology for vector fields on a compact manifold, is nevertheless rather complicated to check in a particular model and, in general, little can be said except in the case of perturbations of systems with a Normally Hyperbolic Invariant Manifold whose stable and unstable manifolds coincide along a homoclinic manifold.

Our approach to show that the invariant manifolds $W^{\rm u,s}(\mathcal{P}_\infty)$ defined in (2.5) intersect transversally is to study the RPE3BP as a small perturbation of the integrable 2BP, in which the invariant manifolds of $\mathcal{P}_\infty$ coincide along a homoclinic manifold $W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty)$ of parabolic motions. Yet, for fixed $\mu\in(0,1/2)$, the RPE3BP is far from the 2BP. We however recover a nearly integrable regime if we focus our attention

[3] The submanifold (2.4) can be described properly in McGehee variables (x,α,t,y,G,E) where $r=\eta_{\rm MG}(x)=2/x^2$.

to the region of the phase space $\{G \geq G_*\}$ with G_* sufficiently large. The reason is that, for G_* large enough, the stable and unstable manifolds $W^{u,s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ are located far away from the primaries and therefore, they can be studied as a perturbation of the homoclinic manifold $W^{h}_{2BP}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ (see Section 2.2). The (substantial) price to pay is that this regime corresponds to a singular perturbation setting. Namely, as we show in Section 2.2, for $G_* \gg 1$, the dynamics in a neighborhood of $W^{u,s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ corresponds to a fast periodic analytic perturbation coupled to the slow dynamics of the integrable 2BP. Indeed, since the Newtonian potential decays with distance, the motion of the massless body is much slower than the rotation of the primaries. The existence of these two time scales results in an *exponentially small splitting* (in $1/G_* \ll 1$) between the invariant manifolds $W^{u}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ and $W^{s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$: the effect of the perturbation along a neighborhood of the homoclinic manifold W^{h}_{2BP} averages out up to an exponentially small remainder which, as a matter of fact, bounds the distance between $W^{u}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ and $W^{s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ (see [Nei84]).

In the *a priori unstable* setting, that is, perturbations of systems with a Normally Hyperbolic Invariant Manifold whose stable and unstable manifolds coincide along a homoclinic manifold, and for which the hyperbolicity is much stronger than the size of the perturbation, there are no different time scales. Therefore, the splitting between invariant manifolds is usually tackled by means of Poincaré-Melnikov theory (see [Mel63, DdlLS06, GdlL18]), which gives an asymptotic formula for the distance between them in terms of a convergent improper integral, usually referred to in the literature as the Melnikov function. Nevertheless, establishing the validity of the Melnikov approximation in the singular perturbation framework where the splitting between $W^{u,s}$ is exponentially small (as is the case of the present problem), is a demanding problem for which no general theory is available.

One should remark that the original Arnold model, despite having two different time scales, it possesses two parameters: the one measuring the ratio between time scales ε and the one measuring the size of the perturbation μ. In this case, Melnikov theory predicts that $\mathrm{dist}(W^u, W^s) \sim M + \mathcal{O}(\mu^2)$ for a function M which "typically" has size $M \sim \mu \exp(-c/\sqrt{\varepsilon})$. Therefore, by assuming that the size of the perturbation μ is exponentially small compared to the ratio between time scales, classical Melnikov theory can be directly applied even if the splitting between $W^{u,s}$ is exponentially small in ε. This was the approach considered in [Arn64] and [DKdlRS19].

The analysis of exponentially small splitting has drawn major attention in the past decades due to its relevance for the study of instability mechanisms in real analytic Hamiltonian systems. Indeed, in the absence of extra perturbative parameters, as is the case in *a priori stable systems* (perturbations of integrable systems in action angle variables: no hyperbolicity is present in the integrable system), one has to face this phenomenon. Remarkable progress has been made in a number of works in low dimensional models (just to cite a few works, see [Laz87, DS92, Gel94, Gel97, Tre97, Gel99, BF04a, MSS11, BFGS12, Gua13]). In higher dimension, results are much more scarce. We highlight [Sau01] and [LMS03], where the exponentially small splitting between the stable and unstable manifolds of a partially hyperbolic invariant torus is investigated. When the torus under consideration is sufficiently irrational, the splitting of its invariant manifolds is exponentially small in all directions (see [DGJS97] and [Sau01]). However, if the torus is resonant, one expects that the splitting is in general highly anisotropic, involving directions in which the splitting is exponentially small and directions in which the splitting is of the order of the perturbation. This strong anisotropy complicates heavily the geometric analysis. Exponentially small splitting happens in directions close to that of the actions conjugated to the fast angles and polynomially small splitting happens in directions close to that of the actions conjugated to the resonant angles. However is not clear a priori how to locate exactly the directions of exponentially small splitting.

In [LMS03], Lochak, Marco and Sauzin developed a formalism to identify the directions of exponentially small splitting between the stable and unstable manifolds of the same partially hyperbolic invariant torus. The situation is much more intrincate when one considers the invariant manifolds associated to *two different* partially hyperbolic invariant tori. Indeed, all previous works which study the existence of transverse intersections between the invariant manifolds of different invariant tori rely on an indirect approach: first, one proves the existence of transverse homoclinic intersections between the invariant manifolds of a given torus and then deduce the existence of heteroclinic connections between nearby tori by direct application of the implicit function theorem. However, the directions along which the splitting is exponentially small can move as we vary the torus. Therefore, to ensure that all the errors in the approximation by the homoclinic connection are exponentially small, this indirect method only works when the two tori under consideration are exponentially close.

The present is, to the best of the authors knowledge, the first work in which the highly anisotropic splitting between the invariant manifolds of a pair of partially hyperbolic fully resonant invariant tori (which in the current problem foliate $\mathcal{P}_\infty$) is succesfully analyzed. Namely, by studying the problem in a direct way we establish the existence of heteroclinic connections between resonant tori separated up to a distance of the size of the perturbation. This allows us to prove the existence of two manifolds $\Gamma_\pm$ of homoclinic points to $\mathcal{P}_\infty \cap \{t = 0\}$ which moreover are diffeomorphic to

$$\mathcal{P}_\infty^* = \mathcal{P}_\infty \cap \{t = 0,\ G_* < G < \epsilon^{-1/3}\}.$$

The main idea behind our approach is to exploit the Hamilton-Jacobi formalism for, given a pair of partially hyperbolic invariant tori, building a local symplectic coordinate system, tailored made for each pair of invariant tori, in which the direction of exponentially small splitting between their associated invariant manifolds is clearly isolated from the non-exponentially small one. The coordinate system strongly depends on the pair of tori considered, what gives an idea of the subtleness of the phenomenon. The key player in this construction is the "splitting potential" ΔS, which will be defined in (2.22) as the difference between the generating function of the unstable manifold of one of the tori and the generating function of the stable manifold of the other torus [4].

Remark 2.1.5. *The splitting potential was first introduced by Eliasson in [Eli94] and later appeared in the work of Sauzin [Sau01] and Lochak, Marco and Sauzin [LMS03], to study the splitting between the invariant manifolds of a given torus. The terminology splitting potential was coined in [DG00].*

In the variational approach to Arnold Diffusion, the splitting potential also plays a major role, since it is related to the so called Peierl's barrier in Mather theory (see [Zha11]).

Another remarkable novelty of our construction is that we work directly with the generating functions associated to the stable and unstable manifolds of the invariant tori instead of relying on a vector parametrization of these invariant manifolds. The difficulty to work directly with the generating function is the appearance of certain unbounded operator in the linearized invariance equation which defines the generating functions (see [Sau01]). This obstacle was removed in all previous works by considering a different vector parametrization of the invariant manifolds. We overcome the problem, and directly find the generating functions, by making use of a suitable Newton iterative scheme in a scale of Banach spaces in the spirit of the usual schemes used in KAM theory.

Construction of a transition chain of heteroclinic orbits: The progress made in the analysis of the strongly anisotropic splitting between invariant manifolds of partially hyperbolic resonant invariant tori is of high relevance for the second step in the proposed diffusion mechanism. Indeed, we are able to prove that the *scattering maps* (see [DdlLS08]), which encode the dynamics along the homoclinic manifolds $\Gamma_\pm$, are *globally defined* on $\mathcal{P}_\infty^* \subset \mathcal{P}_\infty$. This result was already obtained in [DKdlRS19], for the a priori unstable case $\mu \ll \exp(-G_*^3/3)$ and is extended in this work to the a priori stable setting $\mu \in (0, 1/2)$. Besides the importance of this achievement for the study of Arnold Diffusion in a priori stable Hamiltonians, the existence of two globally defined scattering maps is vital for the construction of diffusive orbits in the RPE3BP. As a matter of fact, the inner dynamics on $\mathcal{P}_\infty$ is trivial and, as seen in [DKdlRS19], we can only rely on the combination of the two scattering maps.

The pair of scattering maps on $\mathcal{P}_\infty^*$ defines an iterated function system: the existence of a transition chain of heteroclinic orbits to $\mathcal{P}_\infty$, along which the drift in the angular momentum G takes place, is guaranteed after showing that the two scattering maps share no common invariant curves (see [Moe02] and also [LC07]). This is a rather challenging problem since, although both scattering maps are only polynomially close (in $1/G_*$) to the identity, the difference between them averages out up to an exponentially small quantity (in $1/G_*$) which, we show, is different from zero on an open subset of $\mathcal{P}_\infty^*$.

[4] These are Lagrangian submanifolds and, therefore, can be parametrized in terms of a generating function (see Section 2.3.1).

The key to establish this result is the construction of a generating function for each scattering map, which are exact symplectic. We moreover show that the asymptotics of these generating functions are well controlled by the asymptotics of an explicit function usually referred to as the reduced Melnikov potential. Once the dynamics of both scattering maps have been distinguished, we apply an interpolation result combined with an averaging procedure to show that the invariant curves of both maps always intersect transversally.

2.1.5 A degenerate Arnold model

Since we expect that the ideas of this work can be of interest for readers from the field of Arnold diffusion, but which might have no background in Celestial Mechanics, in this section we present a degenerate version of the Arnold model (see [Arn64]) which illustrates the two main challenges we face.

Consider the Hamiltonian system

$$H(q,p,\varphi,I,t;\varepsilon,\mu) = H_0(q,p,I;\varepsilon) + \mu H_1(q,\varphi,t;\varepsilon), \qquad (q,p)\in\mathbb{T}\times\mathbb{R},\ (\varphi,I)\in\mathbb{T}\times\mathbb{R},\ t\in\mathbb{T} \quad (2.6)$$

where

$$H_0(q,p,I;\varepsilon) = \frac{p^2}{2} + \varepsilon(\cos q - 1)\left(1+\frac{I^2}{2}\right), \qquad H_1(q,\varphi,t;\varepsilon) = \varepsilon(\cos q - 1)\left(\sin\varphi + \cos t\right).$$

We observe that, for any $\mu,\varepsilon \geq 0$,

$$\mathcal{N} = \{(q,p,\varphi,I,t)\in\mathbb{T}\times\mathbb{R}\times\mathbb{T}\times\mathbb{R}\times\mathbb{T} \colon q=p=0\}$$

is a Normally Hyperbolic Invariant Cylinder, which is foliated by periodic orbits with frequencies

$$(\omega_\varphi,\omega_t) = (0,1).$$

Due to the fact that the inner dynamics on $\mathcal{N}$ is trivial, in order to obtain orbits which present a large drift along the I component, we can only rely on the outer dynamics.

The setting $0<\varepsilon\ll 1$ and $0<\mu\ll\exp(-1/\sqrt{\varepsilon})$, which corresponds to the so called a priori unstable case, can be identified (up to major difficulties and technicalities associated to the particular form of the Hamiltonian of the RPE3BP) with the situation studied in [DKdlRS19]. Define

$$\tilde{\varepsilon} = \varepsilon\left(1+\frac{I^2}{2}\right).$$

When $\mu=0$, the system has an homoclinic manifold to $\mathcal{N}$ which can be parametrized as

$$\Gamma_{\mathrm{h}}(s,\theta,I;\tilde{\varepsilon}) = \{(q,p,\varphi,I) = (q_{\mathrm{h}}(s),\sqrt{\tilde{\varepsilon}}p_{\mathrm{h}}(s),\theta+\theta_{\mathrm{h}}(s,I;\sqrt{\tilde{\varepsilon}}),I),\ s\in\mathbb{R},\ (\theta,I)\in\mathbb{T}\times\mathbb{R}\} \quad \text{with} \quad \dot{s}=1/\sqrt{\tilde{\varepsilon}}.$$

By assuming that $\mu\ll\exp(-1/\sqrt{\tilde{\varepsilon}})$, one can use classical perturbation theory to show that, $W^{\mathrm{u}}(\mathcal{N})$ and $W^{\mathrm{s}}(\mathcal{N})$ intersect transversally along two different homoclinic manifolds $\Gamma_\pm$. Indeed, as explained in the previous section, Poincaré-Melnikov theory predicts that, when measured along the line orthogonal to the unperturbed homoclinic manifold and passing through the point $(\Gamma_{\mathrm{h}}(s,\theta,I;\tilde{\varepsilon}),t)$, the distance between the invariant manifolds is given by

$$\mathrm{dist}(W^{\mathrm{u}}(\mathcal{N}),W^{\mathrm{s}}(\mathcal{N})) \sim \mu\partial_s\mathcal{L}(\theta,I,t-s/\sqrt{\tilde{\varepsilon}};\tilde{\varepsilon}) + \mathcal{O}(\mu^2),$$

where

$$\mathcal{L}(\theta,I,\sigma;\tilde{\varepsilon}) = \frac{1}{\sqrt{\tilde{\varepsilon}}}\int_{-\infty}^{\infty} H_1(q_{\mathrm{h}}(\tau),\sqrt{\tilde{\varepsilon}}p_{\mathrm{h}}(\tau),\theta+\theta_{\mathrm{h}}(\tau,I;\sqrt{\tilde{\varepsilon}}),\sigma+\tau/\sqrt{\tilde{\varepsilon}})\mathrm{d}\tau,$$

is the so-called Melnikov potential. Using the expression for H_1, one can easily see that

$$\partial_s\mathcal{L}(\theta,I,t-s/\sqrt{\tilde{\varepsilon}};\tilde{\varepsilon}) = \frac{2\pi}{\sinh\left(\frac{\pi}{2\sqrt{\tilde{\varepsilon}}}\right)}\sin\left(t-s/\sqrt{\tilde{\varepsilon}}\right) = \mathcal{O}(\exp(-1/\sqrt{\tilde{\varepsilon}}).$$

The claim then follows from the fact that, for all $(\theta, I, t) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$, there exist two different non-degenerate zeros $s_+(t, I) = \sqrt{\tilde{\varepsilon}}t$ and $s_-(t, I) = \sqrt{\tilde{\varepsilon}}t + \pi$ of the function $s \mapsto \partial_s \mathcal{L}(\theta, I, t - s/\sqrt{\tilde{\varepsilon}}; \tilde{\varepsilon})$.

This allows to define two scattering maps $\mathbb{P}_\pm : \mathcal{N} \cap \{t = 0\} \to \mathcal{N} \cap \{t = 0\}$, associated to the homoclinic manifolds $\Gamma_\pm$. Finally, one can show (see [DdlLS08]) that the dynamics of each of the scattering maps $\mathbb{P}_\pm$ expressed in variables (θ, I), are given by

$$\mathbb{P}_\pm = \mathrm{Id} + \mu \mathcal{J} \nabla \mathcal{L}_\pm(\theta, I; \tilde{\varepsilon}) + \mathcal{O}(\mu^2),$$

where $\mathcal{J}$ is the standard complex structure in $\mathbb{R}^2$ and

$$\mathcal{L}_\pm(\theta, I; \tilde{\varepsilon}) = \mathcal{L}(\theta, I, -s_\pm(0, I)/\sqrt{\tilde{\varepsilon}}; \tilde{\varepsilon}).$$

Therefore, in the case $\mu \ll \exp(-1/\sqrt{\tilde{\varepsilon}})$, classical perturbation theory yields an asymptotic formula for the difference between $\mathbb{P}_+$ and $\mathbb{P}_-$ which can be used to verify the existence of drifting orbits.

The problem is much more intrincate if $\mu \sim \tilde{\varepsilon}^\alpha$ (for a given $\alpha > 0$). On one hand, as already explained in Section 2.1.4, one cannot make use of classical perturbation theory to study directly the existence of transverse intersections between $W^{\mathrm{u,s}}(\mathcal{N})$. The ideas developed in [Sau01] and [LMS03] could be used to prove the existence of two functions $\theta_\pm(I')$ such that for every I', $(q, p, \varphi, I, t) = (0, 0, \theta_\pm(I'), I', 0) \subset \mathcal{N}$ is a point for which there exist a homoclinic orbit to the torus $\mathcal{T}_{I'} = \mathcal{N} \cap \{I = I'\}$. Then, by application of the implicit function theorem, one can show that there exist two scattering maps $\mathbb{P}_\pm : \mathcal{N}_\pm \cap \{t = 0\} \to \mathcal{N} \cap \{t = 0\}$ where $\mathcal{N}_\pm$ are vertical strips of the form

$$\mathcal{N}_\pm = \{(q, p, \varphi, I, t) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T} \times \mathbb{R}_+ \times \mathbb{T} \colon q = p = 0,\ |\varphi - \theta_\pm(I)| \leq \exp(-1/\sqrt{\tilde{\varepsilon}})\}.$$

However, with this approach $\mathcal{N}_+ \cap \mathcal{N}_- = \emptyset$. Therefore, only one scattering map would be available on each domain and diffusion would be prevented by the existence of invariant curves of the scattering maps. The ideas we introduce in the present work allow us to prove the existence of two *globally defined* scattering maps $\mathbb{P}_\pm : \mathcal{N} \cap \{t = 0\} \to \mathcal{N} \cap \{t = 0\}$ in the case $\mu \sim \tilde{\varepsilon}^\alpha$.

Finally, in the case $\mu \sim \tilde{\varepsilon}^\alpha$, both scattering maps $\mathbb{P}_\pm$ are only $\mathcal{O}(\tilde{\varepsilon}^\beta)$ (for some $\beta > 0$) close to the identity whereas the difference between $\mathbb{P}_+$ and $\mathbb{P}_-$ averages out to up to an exponentially small $\sim \mu \exp(-1/\sqrt{\tilde{\varepsilon}})$ term (the size of the splitting). Therefore, proving that this difference is not zero, is much more demanding than in the a priori unstable case $\mu \ll \exp(-1/\sqrt{\tilde{\varepsilon}})$ (even $\mu = 1$).

We end this section with two remarks concerning the degeneracy of Hamiltonian (2.6) (and of the setting in which we build the diffusion mechanism leading to Theorem 2.1.1). The first one is that the convexity of (2.6) close to $\mathcal{N}$ is of order ε (and vanishes on $\mathcal{N}$). Therefore, the Hamiltonian (2.6) does not satisfy the assumptions of Nekhoroshev theorem and one could think that the diffusion time is polynomial in ε. This is not the case since the angle between the invariant curves of the map $\mathbb{P}_+$ and those of the map $\mathbb{P}_-$ is exponentially small with respect to ε.

The second remark is that, although in the present case all the cylinder is foliated by periodic orbits with the same frequency (which, as already discussed above, introduces certain challenges for proving the existence of diffusive orbits), we have the strong feeling that the the ideas developed in this work, specially the ones in Sections 2.3.1 and 2.3.3, can be adapted to the a priori stable case for the original Arnold model, in which there exists a Normally Hyperbolic Invariant Cylinder $\widetilde{\mathcal{N}}$ foliated by invariant tori of frequencies $(\omega_\varphi(I), \omega_t) = (I, 1)$.

2.1.6 Organization of the article

In Section 2.2 we introduce the (nearly integrable) parabolic regime with large angular momentum discussed in Section 2.1.4. We show that, in this regime, the RPE3BP can be treated as a fast time periodic perturbation of the 2BP, whose main features are also discussed in Section 2.2. Section 2.3 contains the core of the proof of Theorem 2.1.1. More concretely, Section 2.3.1 renders the main ideas behind the proof of the first main ingredient: existence of transverse intersections between the invariant manifolds of $\mathcal{P}_\infty$. The proof of this result is postponed to Section 2.4. Sections 2.3.2 to 2.3.5 are devoted to the construction of two global scattering maps on $\mathcal{P}_\infty$ and the analysis of the transversality between the invariant curves of these maps. The rather technical proofs of the results in these sections are deferred to Sections 2.5, 2.6 and Appendix 2.B. Finally, in Section 2.3.6 we state a suitable shadowing result for parabolic manifolds which completes the proof of Theorem 2.1.1. Appendix 5.C contains a detailed study of the perturbative potential and the associated Melnikov potential.

Througout the rest of the paper we fix a value $\mu \in (0, 1/2)$.

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2.2 The 2BP and a nearly integrable regime for the RPE3BP

The 2 Body Problem (2BP) in polar coordinates is the Hamiltonian system associated to

$$H_{2\mathrm{BP}}(r, y, G) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r} \tag{2.7}$$

on the phase space $(r, \alpha, y, G) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}^2$. Since the Hamiltonian $H_{2\mathrm{BP}}$ does not depend on the angle α, the angular momentum G is a first integral for the 2BP. Moreover, it is functionally independent and commutes with the energy $H_{2\mathrm{BP}}$, what makes the 2BP integrable. The dynamics of the 2BP is completely understood: positive energy levels correspond to hyperbolic motions, negative energy levels to elliptic motions and the zero energy level corresponds to parabolic motions.

2.2.1 The parabolic homoclinic manifold of the 2BP

Of special interest for us are the parabolic motions. Denote by $\mathcal{P}_\infty^{2\mathrm{BP}} = \{(\infty, \alpha, 0, G) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}^2\} = \mathcal{P}_\infty \cap \{t = E = 0\}$ the parabolic infinity in the reduced phase space (see the extended phase space in polar coordinates in Section 2.1.1), which is a 2 dimensional TNHIC. Then, the set of points leading to parabolic motions, that is, the set $\{H_{2\mathrm{BP}} = 0\}$, is a 3 dimensional submanifold $W_{2\mathrm{BP}}^{\mathrm{h}}$ homoclinic to $\mathcal{P}_\infty^{2\mathrm{BP}}$. Let $\phi_{H_{2\mathrm{BP}}}^\tau$ be the flow associated to the Hamiltonian (2.7), then [5]

$$W_{2\mathrm{BP}}^{\mathrm{h}} = \{x \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}^2 \colon \exists z \in \mathcal{P}_\infty^{2\mathrm{BP}} \quad \text{for which} \quad \lim_{\tau \pm\infty} |\phi_{H_{2\mathrm{BP}}}^\tau(x) - \phi_{H_{2\mathrm{BP}}}^\tau(z)| = 0\}. \tag{2.8}$$

The following lemma gives a parametrization of the homoclinic manifold $W_{2\mathrm{BP}}^{\mathrm{h}}$. A proof can be found in [MP94].

Lemma 2.2.1. *There exist real analytic functions $r_{\mathrm{h}}(u), \alpha_{\mathrm{h}}(u)$ and $y_{\mathrm{h}}(u)$ such that*

$$W_{2\mathrm{BP}}^{\mathrm{h}} = \{\Gamma_{2\mathrm{BP}}(u, \beta) = (G^2 r_{\mathrm{h}}(u), \beta + \alpha_{\mathrm{h}}(u), G^{-1} y_{\mathrm{h}}(u), G) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}^2 \colon u \in \mathbb{R},\ \beta \in \mathbb{T},\ G \in \mathbb{R} \setminus \{0\}\}$$

and, if we denote by $X_{2\mathrm{BP}}$ the vector field associated to the Hamiltonian (2.7),

$$X_{2\mathrm{BP}} \circ \Gamma_{2\mathrm{BP}} = D\Gamma_{2\mathrm{BP}}\ \Upsilon \qquad \textit{with} \qquad \Upsilon = (G^{-3}, 0).$$

The functions $r_{\mathrm{h}}, y_{\mathrm{h}}$ and α_{h} admit a unique analytic extension to $\mathbb{C} \setminus \{u = is \colon s \in (-\infty, -1/3] \cup [1/3, \infty)\}$ and satisfy the asymptotic behavior

$$r_{\mathrm{h}}(u) \sim u^{2/3} \qquad \exp(i\alpha_{\mathrm{h}}(u)) \sim 1 \qquad y_{\mathrm{h}}(u) \sim u^{-1/3} \qquad \textit{as} \quad u \to \pm\infty$$

[5] Note that for the r component $\pi_r \circ \phi_{H_{2\mathrm{BP}}}^\tau(x) \to \infty$ as $\tau \to \pm\infty$.

and

$$r_{\rm h}(u) \sim (u \pm i/3)^{1/2} \qquad \exp(i\alpha_{\rm h}(u)) \sim \left(\frac{u \pm i/3}{u \mp i/3}\right)^{1/2} \qquad y_{\rm h}(u) \sim (u \pm i/3)^{-1/2} \qquad \textit{as} \quad u \to \pm i/3.$$

Moreover, $y_{\rm h}(u) = 0$ *if and only if* $u = 0$ *and* $r_{\rm h}(u) \geq 1/2$ *for all* $u \in \mathbb{R}$.

2.2.2 The parabolic regime with large angular momentum for the RPE3BP

It is a fact, implied by the last item in Lemma 2.2.1, that, for $G_* \gg 1$,

$$\pi_r \left(W^{\rm h}_{\rm 2BP} \cap \{G \geq G_*\} \right) \geq G_*^2/2 \gg 1,$$

where by π_r we denote the projection into the r coordinate. Therefore, in a neighborhood of the parabolic homoclinic manifold (6.21) for the 2BP we have $r \gg 1$ (that is, the massless body is far away from the primaries) and the Hamiltonian of the RPE3BP can be studied as a perturbation of the (integrable) 2BP. Indeed, expanding the Hamiltonian (2.3) in powers of $1/r$,

$$H_{\rm pol}(r, \alpha, t, y, G, E) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r} + E + \mathcal{O}\left(\frac{1}{r^3}\right) = H_{\rm 2BP}(r, y, G) + E + \mathcal{O}\left(\frac{1}{r^3}\right),$$

where we have used that $V_{\rm pol}$ in (2.3) is given by

$$\begin{aligned} V_{\rm pol}(r, \alpha, t) = & \frac{\mu}{(r^2 + 2(1-\mu) r \varrho(t) \cos(\alpha - f(t)) + (1-\mu)^2 \varrho^2)^{1/2}} \\ & + \frac{1-\mu}{(r^2 + 2\mu r \varrho(t) \cos(\alpha - f(t)) + \mu^2 \varrho^2)^{1/2}} = \frac{1}{r} + \mathcal{O}\left(\frac{1}{r^3}\right). \end{aligned}$$

With the object of investigating this perturbative regime, we consider an arbitrarily large constant $G_* \gg 1$ and make the conformally symplectic scaling

$$(\tilde{r}, \alpha, t, \tilde{y}, \tilde{G}, \tilde{E}) \longmapsto (r, \alpha, t, y, G, E)$$

defined by

$$r = G_*^2 \tilde{r}, \qquad y = G_*^{-1} \tilde{y}, \qquad G = G_* \tilde{G}, \qquad E = G_* \tilde{E}.$$

Up to time reparametrization, the autonomous Hamiltonian in the scaled variables reads

$$\tilde{H}(\tilde{r}, \alpha, t, \tilde{y}, \tilde{G}, \tilde{E}) = \frac{\tilde{y}^2}{2} + \frac{\tilde{G}^2}{2\tilde{r}^2} - \tilde{V}(\tilde{r}, \alpha, t) + G_*^3 \tilde{E}, \qquad \tilde{V}(\tilde{r}, \alpha, t) = G_*^2 V_{\rm pol}\left(G_*^2 \tilde{r}, \alpha, t\right).$$

It is an easy computation to show that, for $G_*^2 \tilde{r} \gg 1$,

$$\tilde{V}(\tilde{r}, \alpha, t) = \frac{1}{\tilde{r}} + \mathcal{O}\left(\frac{1}{G_*^4 \tilde{r}^3}\right)$$

and, therefore

$$\tilde{H}(\tilde{r}, \alpha, t, \tilde{y}, \tilde{G}, \tilde{E}) = H_{\rm 2BP}(\tilde{r}, \tilde{y}, \tilde{G}) + G_*^3 \tilde{E} + \mathcal{O}(G_*^{-4} \tilde{r}^{-3}).$$

The nature of this perturbative regime is now clear: in the parabolic regime with large angular momentum the RPE3BP is a fast time periodic perturbation ($\dot{t} = G_*^3$) of the slow dynamics ($\dot{\tilde{r}} \sim \dot{\alpha} \sim \dot{\tilde{y}} = \mathcal{O}(1)$) of the 2BP. Since the gravitational potential $\tilde{V}$ is analytic on a complex neighborhood of the embedding of $W^{\rm h}_{\rm 2BP}$ in the extended phase space, succesive averaging steps can be performed to find a real analytic change of variables ψ defined on a complex neighborhood of $W^{\rm h}_{\rm 2BP} \cap \{\tilde{G} \geq 1\}$ in which

$$\tilde{H} \circ \psi = K + \mathcal{O}(\exp(-C G_*^3)) \tag{2.9}$$

for some $C > 0$ and where $K = K(\tilde{r}, \alpha, \tilde{y}, \tilde{G})$ is $\mathcal{O}(G_*^{-4})$ close to $\tilde{H}$ and coincides with $\tilde{H}$ at $\mathcal{P}_\infty \cap \{\tilde{G} \geq 1\}$. A simple counting dimension argument shows that, for the flow associated to the 2 degrees of freedom autonomous Hamiltonian K, the invariant manifolds associated to $\mathcal{P}_\infty \cap \{\tilde{G} \geq 1\}$ (which is also a TNHIC for K) must coincide along a homoclinic manifold. Therefore, it follows from (2.9) that the distance between $W^{\rm u,s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ is bounded by $\mathcal{O}(\exp(-CG_*^3))$.

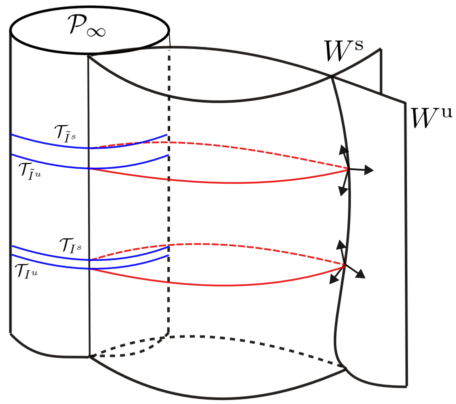

Figure 2.2: The unstable and stable manifolds $W^{\rm u,s}(\mathcal{P}_\infty)$ of the Topological Normally Hyperbolic Invariant Cylinder $\mathcal{P}_\infty$. In order to measure the splitting between $W^{\rm u}_{I^{\rm u}}$ and $W^{s}_{I^{\rm s}}$ we build a suitable symplectic coordinate system (depending non trivially on $I^{\rm u}$ and $I^{\rm s}$), in which the directions of exponentially small splitting are isolated from the non-exponentially small one.

2.3 Proof of the main theorem

The first step in the proof of Theorem 2.1.1 is to prove that the manifolds $W^{\rm u,s}(\mathcal{P}_\infty)$ (defined in (2.5)) intersect transversally. This is a rather delicate problem since we will see that the splitting angle between $W^{\rm u}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ and $W^{\rm s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ is exponentially small in $1/G_*$ and we will not study directly the existence of intersections between them. The reason is that, in order to measure this splitting, one needs to find a suitable local coordinate system which isolates the exponentially small directions. However, these directions are highly sensitive with respect to the projection along the stable and unstable foliations (see Figure 2.3) and it is not clear a priori how to locate them without exploiting the symplectic features of the problem.

To overcome this difficulty we take advantage of the fact that $\mathcal{P}_\infty$ in (2.4) is foliated by invariant tori

$$\mathcal{P}_\infty = \bigcup_{I\in\mathbb{R}} \mathcal{T}_I, \qquad \mathcal{T}_I = \left\{x = (\infty, \varphi, t, 0, I, 0),\ (\varphi, t) \in \mathbb{T}^2,\ G = I\right\}$$

and, therefore, we can express

$$W^{\rm u,s}(\mathcal{P}_\infty) = \bigcup_{I\in\mathbb{R}} W^{\rm u,s}_I$$

where $W^{\rm u,s}_I$ are the stable and unstable manifolds of the invariant torus $\mathcal{T}_I$. Since $W^{\rm u,s}_I$ are Lagrangian submanifolds, one can parametrize them (at least locally) as a graph over the configuration space and measure the splitting in the conjugate directions. Since all tori $\mathcal{T}_I \subset \mathcal{P}_\infty$ are resonant with frequencies $(\omega_\alpha, \omega_t) = (0, 1)$, we will see that the splitting between their invariant manifolds is highly anisotropic.

In the present work, we extend the formalism developed by Lochak, Marco and Sauzin in [LMS03], to analyze directly the existence of transverse intersections between the stable $W^{\rm s}_{I^{\rm s}}$ and unstable $W^{\rm u}_{I^{\rm u}}$ manifolds of two (possibly) different invariant tori $\mathcal{T}_{I^{\rm u}}$ and $\mathcal{T}_{I^{\rm s}}$.

2.3.1 The non exact Lagrangian intersection problem

In this section, we exploit the Hamilton-Jacobi formalism to reduce the problem of existence of intersections between $W^{\rm s}_{I^{\rm s}}$ and $W^{\rm u}_{I^{\rm u}}$, to the problem of existence of critical points of a certain scalar function. Before entering the details of our construction, the introduction of some notation is in order. Given a value $I_* \in \mathbb{R}_+$, for $\epsilon \in (0, I_*^{-3})$, we define the annulus

$$\Lambda(I_*, \epsilon) = \mathbb{T} \times \{I_* \leq I \leq \epsilon^{-1/3}\} \subset \mathbb{T} \times \mathbb{R} \tag{2.10}$$

and, given $\rho > 0$, we introduce the complex neighborhood

$$\Lambda_\rho(I_*, \epsilon) = \mathbb{T}_\rho \times \Lambda_{\rho,I}(I_*, \epsilon) \subset (\mathbb{C}/2\pi\mathbb{Z}) \times \mathbb{C} \tag{2.11}$$

where $\mathbb{T}_\rho$ is the strip of width ρ centered at the real torus and

$$\Lambda_{\rho,I}(I_*, \epsilon) = \{I \in \mathbb{C} \colon |\mathrm{Im} I| \leq \rho,\ I_* \leq \mathrm{Re} I \leq \epsilon^{-1/3}\}. \tag{2.12}$$

Remark 2.3.1. *In the following we will restrict our analysis to tori $\mathcal{T}_{I^{\mathrm{u,s}}}$ such that $I^{\mathrm{u}}, I^{\mathrm{s}} \in \Lambda(I_*, \epsilon)$. The introduction of the annulus $\Lambda(I_*, \epsilon) \subset \mathbb{T} \times \mathbb{R}$ is needed for the following reasons. On one hand, in order to work in the perturbative regime introduced in Section 2.2.2, one needs to consider a region of the phase space with sufficiently large angular momentum, hence the requirement $I \geq I_*$. On the other hand, the requirement $\epsilon \leq I_*^{-3}$ is of technical nature and it is related to the limitations of the method (see Appendix 5.C) that we use to compute the so-called Melnikov potential (defined in (2.24) below).*

Remark 2.3.2. *The use of complex neighborhoods of $\Lambda(I_*, \epsilon)$ is needed to make use of Cauchy estimates in Section 2.3.2. In the following, fixed a value of I_*, we will simply write Λ, Λ_ρ and $\Lambda_{\rho,I}$ and drop the dependence on I_*.*

Given $I_*, \rho > 0$ and $\epsilon \in (0, I_*^{-3})$, for any $I^{\mathrm{u}}, I^{\mathrm{s}} \in \Lambda_{\rho,I}$, we define

$$I_{\mathrm{m}} = \frac{1}{2}(I^{\mathrm{u}} + I^{\mathrm{s}}) \tag{2.13}$$

and perform the change of variables (depending on I_{m})

$$\eta_{I_{\mathrm{m}}} : (u, \beta, t, Y, J, E; I_{\mathrm{m}}) \longmapsto (r, \alpha, t, y, G, E) \tag{2.14}$$

given by

$$r = I_{\mathrm{m}}^2 r_{\mathrm{h}}(u) \qquad \alpha = \beta + \alpha_{\mathrm{h}}(u) \qquad y = I_{\mathrm{m}}^{-1} y_{\mathrm{h}}(u) + I_{\mathrm{m}}^{-2} y_{\mathrm{h}}^{-1}(u)(Y - r_{\mathrm{h}}^{-2} J) \qquad G = I_{\mathrm{m}} + J,$$

where $r_{\mathrm{h}}, y_{\mathrm{h}}$ and α_{h} are defined in Lemma 2.2.1. The change of variables $\eta_{I_{\mathrm{m}}}$ is the symplectic completion of the change in the basis given by $r = I_{\mathrm{m}}^2 r_{\mathrm{h}}(u)$, $\alpha = \beta + \alpha_{\mathrm{h}}(u)$, which is well suited to study a neighborhood of the unperturbed homoclinic orbit $W_{2\mathrm{BP}}^{\mathrm{h}} \cap \{G = I_{\mathrm{m}}\}$ (see (6.21)).

A key point in our construction, is that we use the parametrization of $W_{2\mathrm{BP}}^{\mathrm{h}} \cap \{G = I_{\mathrm{m}}\}$ in Lemma 2.2.1 as first order approximation both for the unstable manifold of $\mathcal{T}_{I^{\mathrm{u}}}$ and for the stable manifold of $\mathcal{T}_{I^{\mathrm{s}}}$.

Remark 2.3.3. *Notice that in (u, β, t, Y, J, E) coordinates, the tori $\mathcal{T}_{I^{\mathrm{u,s}}}$ are given by $\mathcal{T}_{I^{\mathrm{u}}} = \{u = -\infty, Y = E = 0,\ J = I^{\mathrm{u}} - I_{\mathrm{m}}\}$ and $\mathcal{T}_{I^{\mathrm{s}}} = \{u = \infty, Y = E = 0,\ J = I^{\mathrm{s}} - I_{\mathrm{m}}\}$.*

The proof of the following result is a straightforward computation.

Lemma 2.3.4. *Let $(M_{\mathrm{pol}}, \mathrm{d}\lambda_{\mathrm{pol}})$ be the exact symplectic manifold where M_{pol} is the phase space in polar coordinates (see (2.2)) and $\lambda_{\mathrm{pol}} = y\mathrm{d}r + G\mathrm{d}\alpha + E\mathrm{d}t$ is the canonical one form. Let $(M, \mathrm{d}\lambda)$ be the exact symplectic manifold*

$$M = \big\{(u, \beta, t, Y, J, E) \in \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^3\big\} \qquad \text{and} \qquad \lambda = Y\mathrm{d}u + J\mathrm{d}\beta + E\mathrm{d}t$$

The change of variables $\eta_{I_{\mathrm{m}}} : M \setminus \{u = 0\} \to \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^3$ defined in (2.14) satisfies

$$\eta_{I_{\mathrm{m}}}^* \lambda_{\mathrm{pol}} - \lambda = \frac{2}{I_{\mathrm{m}}^2 r_{\mathrm{h}}(u)} \mathrm{d}u + I_{\mathrm{m}} \mathrm{d}\beta.$$

In particular, $\eta_{I_{\mathrm{m}}}^*$ is a symplectic change of variables between $(M \setminus \{u = 0\}, \mathrm{d}\lambda)$ and $(M_{\mathrm{pol}}, \mathrm{d}\lambda_{\mathrm{pol}})$.

Remark 2.3.5. *The map $\eta_{I_{\mathrm{m}}}$ is not defined at $u = 0$ since $y_{\mathrm{h}}(u) = 0$ (see Lemma 2.2.1) This will introduce some technicalities at certain steps in the proof of Theorem 2.1.1.*

After time reparametrization (multiplication by $I_{\rm m}^3$), when expressed in the new coordinate system, the Hamiltonian function $H_{\rm pol}$ in (2.3) reads

$$H(u,\beta,t,Y,J,E) = Y + I_{\rm m}^3 E + \frac{\left(Y - r_{\rm h}^{-2}(u)J\right)^2}{2y_{\rm h}^2(u)I_{\rm m}} + \frac{J^2}{2r_{\rm h}^2(u)I_{\rm m}} - V(u,\beta,t;I_{\rm m},\epsilon), \tag{2.15}$$

where

$$V(u,\beta,t;I_{\rm m},\epsilon) = I_{\rm m}^3 V_{\rm pol}(I_{\rm m}^2 r_{\rm h}(u), \beta + \alpha_{\rm h}(u), t; \epsilon) - \frac{I_{\rm m}}{r_{\rm h}(u)} \tag{2.16}$$

and $V_{\rm pol}$ is the gravitational potential expressed in polar coordinates (see (2.3)).

It is easy to check that $W^{\rm u,s}_{I^{\rm u,s}}$, are Lagrangian submanifolds of M but are not exact (unless $I^{\rm s} = I^{\rm u}$) since they have defect of exactness

$$\delta^{\rm u} = (0, \frac{1}{2}(I^{\rm u} - I^{\rm s}), 0) \qquad\qquad \delta^{\rm s} = (0, \frac{1}{2}(I^{\rm s} - I^{\rm u}), 0). \tag{2.17}$$

The manifolds $W^{\rm u,s}_{I^{\rm u,s}} - (0,0,0,\delta^{\rm u,s})$ (expressing $W^{\rm u,s}_{I^{\rm u,s}}$ in (u,β,t,Y,J,E) coordinates) are, as a matter of fact, exact Lagrangian submanifolds and there exist functions (here u_0 is some positive constant)

$$T^{\rm u}(u,\beta,t;I^{\rm u},I^{\rm s},\epsilon) : (-\infty,-u_0] \times \mathbb{T}^2 \to \mathbb{R}, \qquad\qquad T^{\rm s}(u,\beta,t;I^{\rm u},I^{\rm s},\epsilon) : [u_0,\infty) \times \mathbb{T}^2 \to \mathbb{R}, \tag{2.18}$$

solutions to the Hamilton-Jacobi equation

$$H(q, \delta^{\rm u,s} + \nabla T^{\rm u,s}(q;I^{\rm u},I^{\rm s},\epsilon)) = 0, \qquad\qquad q = (u,\beta,t),$$

which, for $q = (u,\beta,t)$ belonging to $(-\infty,-u_0] \times \mathbb{T}^2 \to \mathbb{R}$ for the unstable or $[u_0,\infty) \times \mathbb{T}^2 \to \mathbb{R}$ for the stable, give parametrizations

$$\begin{aligned} \mathcal{W}^{\rm u}_{I^{\rm u}}(q;I^{\rm u},I^{\rm s},\epsilon) &= (q, \delta^{\rm u} + \nabla T^{\rm u}(q;I^{\rm u},I^{\rm s},\epsilon)) \\ \mathcal{W}^{\rm s}_{I^{\rm s}}(q;I^{\rm u},I^{\rm s},\epsilon) &= (q, \delta^{\rm u} + \nabla T^{\rm u}(q;I^{\rm u},I^{\rm s},\epsilon)) \end{aligned} \tag{2.19}$$

of (a part of) the invariant manifolds $W^{\rm u}_{I^{\rm u}}$ and $W^{\rm s}_{I^{\rm s}}$ (defined in (2.5)) in the coordinate system defined by (2.14). In the next proposition, we prove that these parametrizations can be uniquely extended to domains which intersect along an open set.

Proposition 2.3.6. *There exists $\rho_*, I_* > 0$ such that, for $\epsilon \in (0, I_*^{-3})$, and any $I^{\rm u}, I^{\rm s}$ with $I_{\rm m} = (I^{\rm u} + I^{\rm s})/2 \in \Lambda_{\rho_*,I}$, and $|I^{\rm s} - I^{\rm u}| \leq \epsilon |I_{\rm m}|^{-4}$, the functions $T^{\rm u,s}$ in* (2.18) *admit a unique analytic continuation to certain domains of the form $(u,\beta,t) \in R^{\rm u,s} \times \mathbb{T}^2$ where $R^{\rm u,s} \subset \mathbb{R}$ are such that $R^{\rm u} \cap R^{\rm s}$ is a non-empty open interval. Moreover,*

$$|\nabla T^{\rm u,s}(u,\beta,t;I^{\rm u},I^{\rm s},\epsilon)| \lesssim |I_{\rm m}|^{-4} \qquad\qquad \forall (u,\beta,t) \in R^{\rm u,s} \times \mathbb{T}^2$$

and

$$T^{\rm u}(u,\beta,t;I^{\rm u},I^{\rm s},\epsilon) = \mathcal{O}(|u|^{-1/3}) \quad \text{as } u \to -\infty \qquad \text{and} \qquad T^{\rm s}(u,\beta,t;I^{\rm u},I^{\rm s},\epsilon) = \mathcal{O}(|u|^{-1/3}) \quad \text{as } u \to +\infty.$$

Remark 2.3.7. *Ideally, one would try to extend the unstable parametrization to $R^{\rm u} = (-\infty, \tilde{u}_0]$ and the stable one to $R^{\rm s} = [-\tilde{u}_0, \infty)$ for some $\tilde{u}_0 > 0$ so $0 \in R^{\rm u} \cap R^{\rm s}$. However, we are not able to define the parametrizations* (2.19) *at $u = 0$ (see Remark 6.4.3). Yet, we can extend $\mathcal{W}^{\rm u}_{I^{\rm u}}$ to a domain $R^{\rm u}$ (which does not contain the point $u = 0$) and such that $R^{\rm u} \cap R^{\rm s}$ is a non empty open interval (see Section 2.4.3, the idea is to define a new parametrization which can be extended across $u = 0$ and then come back to the Lagrangian graph parametrization). This is crucial, since for measuring the distance between the invariant manifolds, we need their parametrizations $\mathcal{W}^{\rm u,s}_{I^{\rm u,s}}$ to be defined on an open common domain.*

Define now the *generating functions*

$$S^{\rm u,s}(q;I^{\rm u},I^{\rm s},\epsilon) = \langle \delta^{\rm u,s}, q \rangle + T^{\rm u,s}(q;I^{\rm u},I^{\rm s},\epsilon), \qquad\qquad q = (u,\beta,t), \tag{2.20}$$

where $\delta^{\mathrm{u,s}}$ are given in (2.17), which, by definition, solve the Hamilton-Jacobi equation

$$H(q, \nabla S^{\mathrm{u,s}}(q; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)) = 0,$$

and the parametrizations (2.19) can be rewritten as

$$\mathcal{W}^{\mathrm{u,s}}_{I^{\mathrm{u}}}(q; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) = (q, \nabla S^{\mathrm{u,s}}(q; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)), \qquad q \in R^{\mathrm{u,s}} \times \mathbb{T}^2. \tag{2.21}$$

In this way, we have shown that the problem of existence of transverse intersections between $W^{\mathrm{u}}_{I^{\mathrm{u}}}$ and $W^{s}_{I^{\mathrm{s}}}$ is equivalent to the existence of critical points of the *splitting potential*

$$(u, \beta, t) \mapsto \Delta S(u, \beta, t; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) = (S^{\mathrm{u}} - S^{s})(u, \beta, t; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon). \tag{2.22}$$

We point out that the functions $S^{\mathrm{u,s}}$ are no more 2π-periodic in β and must be considered as functions on the covering $\beta \in \mathbb{R}$. Indeed, now for all $k \in \mathbb{Z}$,

$$\begin{aligned} S^{\mathrm{u}}(u, \beta + 2k\pi, t; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) =& k\pi(I^{\mathrm{u}} - I^{\mathrm{s}}) + S^{\mathrm{u,s}}(u, \beta, t; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) \\ S^{s}(u, \beta + 2k\pi, t; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) =& k\pi(I^{s} - I^{\mathrm{u}}) + S^{\mathrm{u,s}}(u, \beta, t; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon). \end{aligned}$$

This fact reflects the non-exact nature of the problem and excludes the possibility (at least in a straighforward manner) of applying topological/variational methods such as Ljusternik-Schnirelman theory to prove the existence of critical points of (2.22) as is usually done when $\delta^{\mathrm{u}} = \delta^{\mathrm{s}}$ (see [Eli94]).

In Theorem 2.3.9 below we establish the existence of two manifolds of critical points for the function ΔS defined in (2.22). The main ingredient is the approximation of ΔS by the so-called *Melnikov potential* defined in (2.24).

Proposition 2.3.8. *Let ΔS be the function defined in (2.22) and let $0 < v_1 < v_2$ be two fixed real numbers. Then, there exists $\rho_*, I_* > 0$ such that, for $\epsilon \in (0, I_*^{-3})$ and any $I^{\mathrm{u}}, I^{\mathrm{s}}$ with $I_{\mathrm{m}} = (I^{\mathrm{u}} + I^{\mathrm{s}})/2 \in \Lambda_{\rho_*, I}$ and $|I^{\mathrm{s}} - I^{\mathrm{u}}| \le \epsilon |I_{\mathrm{m}}|^{-4}$, there exist an analytic (real analytic if $I^{\mathrm{u}}, I^{\mathrm{s}} \in \mathbb{R}$) close to the identity local change of variables*

$$\begin{aligned} \Phi(\cdot\,; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) : [v_1, v_2] \times \mathbb{T}^2_\rho &\longrightarrow \quad \mathbb{C} \times \mathbb{T}^2_{2\rho} \\ (v, \theta, t) &\longmapsto \quad (u, \beta, t) \end{aligned}$$

and an analytic (real analytic if $I^{\mathrm{u}}, I^{\mathrm{s}} \in \mathbb{R}$) function $\Delta\mathcal{S}(\sigma, \theta; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)$ such that

$$\Delta\mathcal{S}(t - I_{\mathrm{m}}^3 v, \theta; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) = \Delta S \circ \Phi(v, \theta, t; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon). \tag{2.23}$$

Moreover, if we define the Melnikov potential

$$L(\sigma, \theta; I_{\mathrm{m}}, \epsilon) = \int_{\mathbb{R}} V(s, \theta, \sigma + I_{\mathrm{m}}^3 s; I_{\mathrm{m}}, \epsilon) \mathrm{d}s, \tag{2.24}$$

where V is defined in (2.16), then the estimates

$$|\Delta\mathcal{S}(\sigma, \theta; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) - (I^{\mathrm{u}} - I^{\mathrm{s}})\theta - L(\sigma, \theta; I_{\mathrm{m}}, \epsilon)| \lesssim |I_{\mathrm{m}}|^{-7},$$

and (here $h^{[l]}$ denotes the l-th Fourier coefficient of a 2π-periodic function $\sigma \mapsto h(\sigma)$)

$$|\Delta\mathcal{S}^{[l]}(\theta; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) - L^{[l]}(\theta; I_{\mathrm{m}}, \epsilon)| \lesssim |C I_{\mathrm{m}}|^{-4+3|l|/2} \exp(-|l| \mathrm{Re}(I_{\mathrm{m}}^3)/3),$$

are satisfied for some $C > 0$ independent of $I^{\mathrm{u}}, I^{\mathrm{s}}$ and ϵ.

Proposition 2.3.8 is proved in Section 2.4 where we perform the analytic continuation of the stable and unstable generating functions $S^{\mathrm{u,s}}$ in 2.20 up to a common domain where we can study their difference $\Delta S = S^{\mathrm{u}} - S^{\mathrm{s}}$. The core of Proposition 2.3.8 is to give a harmonic by harmonic asymptotic approximation of $\Delta\mathcal{S}$, defined in (2.23), in terms of the Melnikov potential (2.24), whose critical points can be easily computed. Then, a direct application of the implicit function theorem yields next theorem. Again, for the sake of clarity in the ongoing discussion, its proof is deferred to Section 2.4.

Theorem 2.3.9. *There exists $\rho_*, I_* > 0$ such that, for $\epsilon \in (0, I_*^{-3})$ and all $(\theta, I^{\mathrm{u}}) \in \Lambda_{\rho_*}$, there exist two real analytic functions*

$$(\theta, I^{\mathrm{u}}) \mapsto (\sigma_\pm(\theta, I^{\mathrm{u}}), \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}))$$

such that

$$\partial_\sigma \Delta\mathcal{S}(\sigma_\pm(\theta, I^{\mathrm{u}}), \theta; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}), \epsilon) = 0 \qquad \partial_\theta \Delta\mathcal{S}(\sigma_\pm(\theta, I^{\mathrm{u}}), \theta; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}), \epsilon) = 0.$$

Moreover, the determinant of the Hessian matrix of the function $(\sigma, \theta) \mapsto \Delta\mathcal{S}(\sigma, \theta; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)$ evaluated at $(\sigma, \theta; I^{\mathrm{u}}, I^{\mathrm{s}}) = (\sigma_\pm(\theta, I^{\mathrm{u}}), \theta; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}))$ is different from zero for all $(\theta, I^{\mathrm{u}}) \in \Lambda_\rho$.

Before analyzing the consequences of Theorem 2.3.9 it is worth pointing out two remarks. The first one is that the change of coordinates Φ obtained in Proposition 2.3.8 can be completed to an exact symplectic change of coordinates

$$\tilde{\Phi}(\cdot\,; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) : (v, \theta, t, \mathcal{Y}, \mathcal{J}, \mathcal{E}) \mapsto (u, \beta, t, Y, J, E)$$

in which now the stable and unstable manifolds are locally parametrized by $\widetilde{W}^{\mathrm{u,s}}_{I^{\mathrm{u,s}}} : (v_1, v_2) \times \mathbb{T}^2 \to \mathbb{C}^6$ where

$$\widetilde{\mathcal{W}}^{\mathrm{u}}_{I^{\mathrm{u}}}(\tilde{q}; I^{\mathrm{u}}, I^{s}, \epsilon) = (\tilde{q}, \nabla(S^{\mathrm{u}} \circ \Phi)(\tilde{q}; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)) \qquad \widetilde{\mathcal{W}}^{s}_{I^{s}}(\tilde{q}; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) = (\tilde{q}, \nabla(S^{s} \circ \Phi)(\tilde{q}; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)),$$

and $\tilde{q} = (v, \theta, t)$. Therefore, as $\Delta\mathcal{S} = (S^{\mathrm{u}} - S^{\mathrm{s}}) \circ \Phi$, the existence of nondegenerate critical points of $\Delta\mathcal{S}$ found in Theorem 2.3.9 also implies the existence of transverse intersections between $W^{\mathrm{u}}_{I^{\mathrm{u}}}$ and $W^{s}_{I^{s}}$, which, in the coordinate system given by $\tilde{\Phi}$, can be parametrized as

$$\tilde{\Gamma}_\pm(\theta, t; I^{\mathrm{u}}) = \widetilde{\mathcal{W}}^{u}_{I^{\mathrm{u}}}(I_{\mathrm{m}}^{-3}(t - \sigma_\pm(\theta, I^{\mathrm{u}})), \theta, t; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}(\theta, I^{\mathrm{u}})). \tag{2.25}$$

The reason for introducing the change of coordinates $\tilde{\Phi}$ is that this coordinate system isolates the directions in which the splitting is exponentially small. Namely, we will see in the proof of Proposition 2.3.8, carried out in Section 2.4, that

$$|\partial_\sigma \Delta\mathcal{S}| \sim |I_{\mathrm{m}}|^{-1/2} \exp(-\mathrm{Re}(I_{\mathrm{m}}^3)/3) \qquad |\partial_\theta \Delta\mathcal{S}| \sim \epsilon |I_{\mathrm{m}}|^{-5}.$$

The change of coordinates $\tilde{\Phi}$ depends on the pair of tori whose splitting we are measuring, namely on I^{u} and I^{s}, a fact which reflects the subtleness of the problem (see Figure 2.3).

The second remark is that there are several different ways to look for zeros of the map

$$(\partial_\sigma \Delta\mathcal{S}, \partial_\theta \Delta\mathcal{S})(\sigma, \theta; I^{\mathrm{u}}, I^{\mathrm{s}}) : \mathbb{T}^2 \times \mathbb{R}^2 \longmapsto \mathbb{R}^2.$$

In Theorem 2.3.9 we have chosen to express σ and I^{s} in terms of θ and I^{u} since, with this approach, the functions $\sigma_\pm(\theta, I^{\mathrm{u}})$ and $\tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}})$, giving rise to critical points of $\Delta\mathcal{S}$, are globally defined on $\mathbb{T} \times \{I_* \leq I^{\mathrm{u}} \leq \epsilon^{-1/3}\}$ which is diffeomorphic to an annular region inside $\mathcal{P}_\infty \cap \{t = 0\}$. In Section 2.3.2 we exploit this construction to show the existence of two scattering maps (one for each of the manifolds $\tilde{\Gamma}_+$ and $\tilde{\Gamma}_-$) which are globally defined on $\mathcal{P}_\infty \cap \{t = 0\} \cap \{I_* \leq I \leq \epsilon^{-1/3}\}$.

In Section 2.3.3, once we have established the global existence of the scattering maps in Section 2.3.2, we describe the critical points of $\Delta\mathcal{S}$ in terms of the actions I^{u} and I^{s} labelling the tori $\mathcal{T}_{I^{\mathrm{u}}}$ and $\mathcal{T}_{I^{\mathrm{u}}}$ which are connected along the corresponding heteroclinic orbit. This, in some sense, more natural approach, sheds light on the relationship between the generating functions of the invariant manifolds and the scattering maps (see Proposition 2.3.13). As a matter of fact, we define the generating functions of each scattering map in terms of the generating functions $S^{\mathrm{u,s}}$ of the invariant manifolds.

Remark 2.3.10. *In the forthcoming sections we only write the dependence on ϵ explicitely when needed.*

2.3.2 Construction of two global scattering maps

From now on we work with the stroboscopic Poincaré map

$$P : \{t = 0\} \to \{t = 0\} \tag{2.26}$$

induced by the flow of the Hamiltonian (2.3) on the section $\{t = 0\}$. Observe that the manifold

$$\mathcal{P}_\infty^* \equiv \mathcal{P}_\infty \cap \{t = 0,\ I_* < I < \epsilon^{-1/3}\}. \tag{2.27}$$

which can be parametrized by the coordinates (α, G), is foliated by fixed points for this map.

In Theorem 2.3.9 we have found two manifolds (indexed by $\pm$) of non-degenerate critical points of the function $(\sigma, \theta) \mapsto \Delta\mathcal{S}$, each of them giving rise to a manifold $\Gamma_\pm$ consisting on heteroclinic orbits to $\mathcal{P}_\infty^*$ (see the parametrization (2.25) for $\{t = 0\}$). Our goal now is to build a map which encodes the dynamics along each of the manifolds $\Gamma_\pm$ which, following [DdlLS08], we denote as homoclinic channels . These are the so-called *scattering maps* introduced by Delshams, de la Llave and Seara in [DdlLS00, DdlLS06] (see also [DdlLS08], where the geometric properties of this object are thoroughly studied). Loosely speaking, given one of the channels $\Gamma_\pm$, at a point $(\varphi^{\rm u}, I^{\rm u}) \in \mathcal{P}_\infty^*$, its associated scattering map gives the forward asymptotic (α, G) components along the unique heteroclinic orbit through $\Gamma_\pm$ which is asymptotic in the past to $(\varphi^{\rm u}, I^{\rm u})$.

The key idea behind the proof of Theorem 2.3.9 has been the construction of a bespoke coordinate system for the analysis of each intersection problem: notice that the changes of variables $\eta_{I_{\rm m}}$ and Φ introduced for studying the intersection between the invariant manifolds $W^{\rm u}_{I^{\rm u}}$ and $W^{\rm s}_{I^{\rm s}}$ depend both on the actions $I^{\rm u,s}$. Therefore, up to now, Theorem 2.3.9 implies the existence of a bunch of heteroclinic orbits each of them described in a different coordinate system. Still, in order to build the scattering maps, we need an unified description of the asymptotic dynamics along the families of heteroclinic orbits.

The first step towards definining them is to obtain a parametrization of the homoclinic channels $\Gamma_\pm$ in the original polar coordinates (2.2). To that end, let Φ be the change of variables of Theorem 2.3.9, let $\sigma_\pm(\theta, I^{\rm u})$ and $\tilde{I}^{\rm s}_\pm(\theta, I^{\rm u})$ be the functions obtained in that theorem and for $t = 0$, define

$$\Phi_\pm(\theta, I^{\rm u}) = \Phi(-I_{\rm m}^{-3}(I^{\rm u}, \tilde{I}^{\rm s}(\theta, I^{\rm u}))\sigma_\pm(\theta, I^{\rm u}), \theta, 0; I^{\rm u}, \tilde{I}^{\rm s}_\pm(\theta, I^{\rm u})). \tag{2.28}$$

Then, the homoclinic manifolds $\Gamma_\pm \subset M_{\rm pol}$ ($M_{\rm pol}$ is the phase space in polar coordinates) can be parametrized as follows (see (2.14) and (2.21))

$$\Gamma_\pm = \left\{(r, \alpha, 0, y, G, E) = \eta_{I_{\rm m}} \circ \mathcal{W}^{\rm u}_{I^{\rm u}} \circ \Phi_\pm(\theta, I^{\rm u}) = \eta_{I_{\rm m}} \circ \mathcal{W}^{\rm s}_{I^{\rm s}} \circ \Phi_\pm(\theta, I^{\rm u}),\ (\theta, I^{\rm u}) \in \mathbb{T} \times \{I_* < I < \epsilon^{-1/3}\}\right\}. \tag{2.29}$$

Remark 2.3.11. *Eventually, we will work with the extended parametrization of the homoclinic manifolds $\Gamma_\pm$ to the complex domain $(\theta, I^{\rm u}) \in \Lambda_\rho$, which was defined in* (2.11).

Notice that the homoclinic manifolds $\Gamma_\pm$ are diffeomorphic to $\mathcal{P}_\infty^*$ in (2.27). Therefore, denoting by $\phi^\tau_{H_{\rm pol}}$ the time τ flow associated to the Hamiltonian (2.3), we can define the *backward wave map*

$$\begin{aligned} \Omega^{\rm u}_\pm : \Gamma_\pm &\to \mathcal{P}_\infty \cap \{t = 0\} \\ x &\mapsto (\varphi^{\rm u}_\pm, I^{\rm u}_\pm) = \lim_{\tau \to -\infty} (\alpha \circ \phi^\tau_{H_{\rm pol}}(x), G \circ \phi^\tau_{H_{\rm pol}}(x)) \end{aligned} \tag{2.30}$$

and the *forward wave map*

$$\begin{aligned} \Omega^{\rm s}_\pm : \Gamma_\pm &\to \mathcal{P}_\infty \cap \{t = 0\} \\ x &\mapsto (\varphi^{\rm s}_\pm, I^{\rm s}_\pm) = \lim_{\tau \to +\infty} (\alpha \circ \phi^\tau_{H_{\rm pol}}(x), G \circ \phi^\tau_{H_{\rm pol}}(x)) \end{aligned} \tag{2.31}$$

which are diffeomorphisms on their images. Notice that α and G are constants of motion in $\mathcal{P}_\infty$ and therefore, these limits are well defined. Finally, the so called *scattering maps*, which encode the dynamics along the heteroclinic excursions, are given by

$$\begin{array}{rcl} \mathbb{P}_\pm = \Omega^{\rm s}_\pm \circ \left(\Omega^{\rm u}_\pm\right)^{-1} : \mathcal{P}_\infty^* & \longrightarrow & \mathcal{P}_\infty \cap \{t = 0\} \\ (\varphi^{\rm u}, I^{\rm u}) & \longmapsto & (\varphi^{\rm s}_\pm, I^{s}_\pm). \end{array} \tag{2.32}$$

We notice at this point that our construction of the homoclinic channels gives much more information about the dynamics of the scattering map in the action component I than in the angle component φ. Namely, using the parametrization (2.29) of the homoclinic manifold $\Gamma_\pm$ and writing $x = x_\pm(\theta, I^{\mathrm{u}})$ for a point $x_\pm \in \Gamma_\pm$, the wave maps satisfy

$$\begin{aligned}\Omega^{\mathrm{u}}_\pm(x_\pm(\theta, I^{\mathrm{u}})) =& (\varphi^{\mathrm{u}}_\pm(\theta, I^{\mathrm{u}}), I^{\mathrm{u}}) = \lim_{\tau\to-\infty}(\alpha\circ\phi^\tau_{H_{\mathrm{pol}}}(x_\pm(\theta, I^{\mathrm{u}})), I^{\mathrm{u}})\\ \Omega^{\mathrm{s}}_\pm(x_\pm(\theta, I^{\mathrm{u}})) =& (\varphi^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}), I^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}})) = \lim_{\tau\to+\infty}(\alpha\circ\phi^\tau_{H_{\mathrm{pol}}}(x_\pm(\theta, I^{\mathrm{u}})), \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}))\end{aligned} \tag{2.33}$$

so, up to composing with the close to identity transformation $(\Omega^{\mathrm{u}}_\pm)^{-1}$, the projection of the scattering map in the direction of the action is given by the function $\tilde{I}^{\mathrm{s}}_\pm$ obtained in Theorem 2.3.9 and which is determined implicitely in terms of $\Delta\mathcal{S} = \Delta S\circ\Phi$ by the system of equations

$$\partial_\theta\Delta\mathcal{S}(\sigma_\pm(\theta, I^{\mathrm{u}}), \theta; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}})) = 0 \qquad \partial_\theta\Delta\mathcal{S}(\sigma_\pm(\theta, I^{\mathrm{u}}), \theta; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}})) = 0. \tag{2.34}$$

However, the existence of a direct link between the generating functions which parametrize the invariant manifolds of the tori in $\mathcal{T}^*_I = \mathcal{T}_I \cap \{t = 0\} \subset \mathcal{P}^*_\infty$, and the angular component of the wave maps, and consequently of the scattering maps, is not clear at the moment. In Section 2.3.3 we establish a relationship between the difference $\Delta S(\cdot\,; I^{\mathrm{u}}, I^{\mathrm{s}})$ defined in (2.22) between generating functions associated to the invariant manifolds of a pair of invariant tori $\mathcal{T}^*_{I^{\mathrm{u}}}, \mathcal{T}^*_{I^{\mathrm{s}}}$, and the angular dynamics along the heteroclinic orbit in $\Gamma_\pm$ which connects the tori $\mathcal{T}^*_{I^{\mathrm{u}}}, \mathcal{T}^*_{I^{\mathrm{s}}}$. This connection is crucial to obtain asymptotic formulas for the scattering maps, since the asymptotics of the difference between generating functions $\Delta S(\cdot\,; I^{\mathrm{u}}, I^{\mathrm{s}})$ of a pair of invariant tori is well controlled by the Melnikov function L defined in (2.24).

2.3.3 A generating function for the scattering maps

It is indeed quite natural to expect a direct relationship between the family of generating functions $S^{\mathrm{u}}, S^{\mathrm{s}}$ in (2.20). However, until this paper, as far as the authors know, this connection had only been established up to first order using the so called Melnikov potential (see [DdlLS08]). In Theorem 2.3.13 we show how $S^{\mathrm{u}}, S^{\mathrm{s}}$ *completely determine* the scattering maps.

To do so, we first need to look at the manifolds of critical points of the function $(\sigma, \theta) \mapsto \Delta\mathcal{S}(\sigma, \theta; I^{\mathrm{u}}, I^{\mathrm{s}})$ in a different way from that in Theorem 2.3.9. This is the content of the following proposition, which will be proved together with Theorem 2.3.9 in Section 2.4.

Proposition 2.3.12. *Let $\Delta\mathcal{S}$ be the function defined in Theorem 2.3.9. Then, there exists $I_* > 0$ such that, for any $\epsilon \in (0, I_*^{-3})$ and every pair of actions*

$$(I^{\mathrm{u}}, I^{\mathrm{s}}) \in \mathcal{R}_I \equiv \left\{(I^{\mathrm{u}}, I^{\mathrm{s}}) \in (I_*, \epsilon^{-1/3})^2 \colon |I^{\mathrm{s}} - I^{\mathrm{u}}| < \mu(1-\mu)(1-2\mu)\frac{15\pi\epsilon}{16(I^{\mathrm{u}})^5}\right\}, \tag{2.35}$$

one can find functions

$$(I^{\mathrm{u}}, I^{\mathrm{s}}) \mapsto (\hat{\sigma}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}), \hat{\theta}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}})),$$

such that

$$\partial_\sigma\Delta\mathcal{S}(\hat{\sigma}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}), \hat{\theta}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}); I^{\mathrm{u}}, I^{\mathrm{s}}) = 0 \qquad \partial_\theta\Delta\mathcal{S}(\hat{\sigma}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}), \hat{\theta}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}); I^{\mathrm{u}}, I^{\mathrm{s}}) = 0.$$

Proposition 2.3.12 provides in some sense, a more natural way to look for the critical points of the function $\Delta\mathcal{S}$ than the one in Theorem 2.3.9: We fix a sufficiently close (but not necessarily exponentially close) pair of actions $(I^{\mathrm{u}}, I^{\mathrm{s}})$ and look at the values of the angles (σ, θ) for which there exists a critical point of $(\sigma, \theta) \mapsto \Delta\mathcal{S}$. Next theorem gives the connection between the generating functions associated to the invariant manifolds and the scattering maps.

Theorem 2.3.13. *Let $(I^{\mathrm{u}}, I^{\mathrm{s}}) \in \mathcal{R}_I$ where $\mathcal{R}_I$ is the domain defined in* (2.35)*, let $\hat{\sigma}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}), \hat{\theta}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}})$ be the functions obtained in Proposition 2.3.12 and define*

$$\mathtt{S}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}) = \Delta\mathcal{S}(\hat{\sigma}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}), \hat{\theta}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}); I^{\mathrm{u}}, I^{\mathrm{s}}). \tag{2.36}$$

Then, for all $(I^{\rm u}, I^{\rm s}) \in \mathcal{R}_I$, *the angles*

$$\varphi^{\rm u}_\pm(I^{\rm u}, I^{\rm s}) = \partial_{I^{\rm s}} \mathtt{S}_\pm(I^{\rm u}, I^{\rm s}) \qquad\qquad \varphi^{\rm s}_\pm(I^{\rm u}, I^{\rm s}) = -\partial_{I^{\rm u}} \mathtt{S}_\pm(I^{\rm u}, I^{\rm s}) \tag{2.37}$$

satisfy

$$\mathbb{P}_\pm(\varphi^{\rm u}_\pm(I^{\rm u}, I^{\rm s}), I^{\rm u}) = (\varphi^{\rm s}_\pm(I^{\rm u}, I^{\rm s}), I^{\rm s}).$$

Namely, $\mathtt{S}_\pm$ *is a generating function for the scattering map* $\mathbb{P}_\pm$ *defined in* (2.32).

The rather slow decay of parabolic motions and the fact that the parametrizations (2.21) are not defined at $u = 0$ introduce certain technicalities in the proof of Theorem 2.3.13. For this reason, the proof is deferred to Section 2.5.

2.3.4 Qualitative and asymptotic properties of the scattering maps

The link established between the scattering maps $\mathbb{P}_\pm$ and the difference ΔS between the generating functions associated to the invariant manifolds of pairs of invariant tori provides very rich information about the qualitative and quantitative properties of $\mathbb{P}_\pm$. This information is split between Theorem 2.3.16 and Theorem 2.3.19 below. The former sums up their qualitative properties and states a global asymptotic formula for $\mathbb{P}_\pm$ in terms of the *reduced Melnikov potentials*

$$\mathcal{L}_\pm(\varphi^{\rm u}, I^{\rm u}; \epsilon) = \int_{\mathbb{R}} V(s, \varphi^{\rm u}, \tilde{\sigma}_\pm(\varphi^{\rm u}) + (I^{\rm u})^3 s; I^{\rm u}, \epsilon) {\rm d}s \qquad \tilde{\sigma}_+(\varphi^{\rm u}) = \varphi^{\rm u}, \qquad \tilde{\sigma}_-(\varphi^{\rm u}) = \varphi^{\rm u} + \pi, \tag{2.38}$$

where $V(u, \beta, t; I^{\rm u}, \epsilon)$ is the potential introduced in (2.16). Define also the reduced Melnikov potential associated to the circular problem

$$\mathcal{L}_{\pm,{\rm circ}}(I^{\rm u}) = \int_{\mathbb{R}} V_{\rm circ}(s, \tilde{\sigma}_\pm(\varphi^{\rm u}) - \varphi^{\rm u} + (I^{\rm u})^3 s; I^{\rm u}) {\rm d}s \tag{2.39}$$

where $V_{\rm circ}(u, t - \beta; I^{\rm u}) = V(u, \beta, t; I^{\rm u}, 0)$. Then, in Theorem 2.3.19, we establish an asymptotic formula for the difference between the scattering maps $\mathbb{P}_+$ and $\mathbb{P}_-$.

Remark 2.3.14. *In the following we identify* Λ_ρ *defined in* (2.10) *with a complex neighborhood of* $\mathcal{P}^*_\infty$.

Lemma 2.3.15. *Let* $\mathcal{L}_\pm$ *be the reduced Melnikov potentials defined in* (2.38). *Then, there exists* $\rho_*, I_* > 0$ *such that for* $\epsilon \in (0, I_*^{-3})$ *and for all* $(\varphi^{\rm u}, I^{\rm u}) \in \Lambda_{\rho_*}$ *we have*

$$\begin{aligned} \partial_{\varphi^{\rm u}} \mathcal{L}_\pm(\varphi^{\rm u}, I^{\rm u}; \epsilon) =& \mu(1-\mu)(1-2\mu)\frac{15\pi\epsilon}{8(I^{\rm u})^5} \sin\varphi^{\rm u} + \mathcal{O}(\epsilon|I^{\rm u}|^{-7}) \\ \partial_{I^{\rm u}} \mathcal{L}_\pm(\varphi^{\rm u}, I^{\rm u}; \epsilon) =& -\mu(1-\mu)\frac{3\pi}{2(I^{\rm u})^4} + \mathcal{O}(|I^{\rm u}|^{-7}, \epsilon|I^{\rm u}|^{-4}). \end{aligned}$$

Moreover, under the same assumptions

$$\begin{aligned} \partial_{I^{\rm u}}(\mathcal{L}_+ - \mathcal{L}_-)(\varphi^{\rm u}, I^{\rm u}; \epsilon) =& \mu(1-\mu)\sqrt{\frac{\pi I^3}{2}} \left((1-2\mu) + \mathcal{O}(|I^{\rm u}|^{-1}) \exp(-(I^{\rm u})^3/3)\right) \\ \partial_{\varphi^{\rm u}}(\mathcal{L}_+ - \mathcal{L}_-)(\varphi^{\rm u}, I^{\rm u}; \epsilon) =& -\mu(1-\mu)\epsilon\left(6\sqrt{2\pi I^3}\sin\varphi^{\rm u} + \mathcal{O}(|I^{\rm u}|^{1/2})\right)\exp(-(I^{\rm u})^3/3)). \end{aligned}$$

In particular, the asymptotic formula

$$\{\mathcal{L}_+, \mathcal{L}_-\}(\varphi^{\rm u}, I^{\rm u}; \epsilon) = 2\partial_{\varphi^{\rm u}}(\mathcal{L}_+ - \mathcal{L}_-)\partial_{I^{\rm u}}\mathcal{L}_+(\varphi^{\rm u}, I^{\rm u}; \epsilon) + \mathcal{O}\left(\epsilon|I^{\rm u}|^{-7/2}\exp(-(I^{\rm u})^3/3)\right), \tag{2.40}$$

which measures the transversality between the level sets of $\mathcal{L}_\pm$, *holds for all* $(\varphi^{\rm u}, I^{\rm u}) \in \Lambda$.

Lemma 2.3.15 is proven in Appendix 5.C, were we provide a detailed analysis of the asymptotic properties of the Melnikov potential L defined in (2.24). We now state the global properties satisfied by $\mathbb{P}_\pm$.

Theorem 2.3.16. *The scattering maps $\mathbb{P}_\pm : \mathcal{P}^*_\infty \to \mathcal{P}^*_\infty$ defined in* (2.32) *are exact symplectic and real analytic. Moreover, there exists $\rho_* > 0$ such that the maps $\mathbb{P}_\pm$ admit an analytic extension to Λ_{ρ_*} and for all $(\varphi^{\rm u}, I^{\rm u}) \in \Lambda_{\rho_*}$*

$$\mathbb{P}_\pm = (\mathrm{Id} + \mathcal{J}\nabla\mathcal{L}_\pm) + \left(\mathcal{O}(|I^{\rm u}|^{-7}, \mathcal{O}(\epsilon|I^{\rm u}|^{-11/2})\right)^\top, \tag{2.41}$$

where $\mathcal{L}_\pm$ has been defined in (2.38) *and $\mathcal{J}$ denotes the standard complex structure in $\mathbb{R}^2$.*

Introduce now (compare (2.27)) the domain

$$\mathcal{P}^*_{\infty,\rm circ} = \mathcal{P}_\infty \cap \{t = 0, I_* \leq I\}, \tag{2.42}$$

and denote by

$$\mathbb{P}_{\pm,\rm circ} : \mathcal{P}^*_{\infty,\rm circ} \to \mathcal{P}^*_{\infty,\rm circ} \tag{2.43}$$

the scattering map (2.32) associated to the case $\epsilon = 0$, which corresponds to the circular problem (RPC3BP). The following result is an immediate corollary of Theorem 2.3.16.

Lemma 2.3.17. *The scattering map $\mathbb{P}_{\pm,\rm circ} : \mathcal{P}^*_{\infty,\rm circ} \to \mathcal{P}^*_{\infty,\rm circ}$, associated to the circular case $\epsilon = 0$, is of the form*

$$\mathbb{P}_{\pm,\rm circ}(\varphi^{\rm u}, I^{\rm u}) = (\varphi^{\rm u} + \omega_{\rm circ}(I^{\rm u}), I^{\rm u})$$

Moreover, for all $(\varphi^{\rm u}, I^{\rm u}) \in \mathcal{P}^*_{\infty,\rm circ}$, we have

$$\omega_{\rm circ}(I^{\rm u}) = \partial_{I^{\rm u}}\mathcal{L}_{\pm,\rm circ}(I^{\rm u}) + \mathcal{O}(|I^{\rm u}|^{-7}),$$

where $\mathcal{L}_{\pm,\rm circ}$ has been defined in (2.39).

Remark 2.3.18. *The integrability of the scattering map of the circular problem ($\epsilon = 0$) is a consequence of the conservation of the Jacobi constant (see Remark 2.1.4).*

Once we have established the global existence and asymptotic behavior for the maps $\mathbb{P}_\pm$, in Theorem 2.3.19 below we provide an asymptotic formula for the difference $\mathbb{P}_+ - \mathbb{P}_-$. With the intention of clarifying the statement of Theorem 2.3.19, the recalling of some notation is in order. Let $\Phi_\pm$ be the maps defined in (2.28), let $\Omega^{\rm u}_\pm$ be the wave maps defined in (2.30), denote by $\Xi_\pm$ be the maps

$$(I^{\rm u}, I^{\rm s}) \mapsto \Xi_\pm(I^{\rm u}, I^{\rm s}) = (\hat\theta_\pm(I^{\rm u}, I^{\rm s}), I^{\rm u}) \tag{2.44}$$

obtained in Proposition 2.3.12, let $\mathtt{S}_\pm$ be the generating functions obtained in Proposition 2.3.13 and consider the function $\tilde I^{\rm s}_\pm(\theta, I^{\rm u})$ obtained in Theorem 2.3.9. Define also the vertical strip

$$\mathcal{P}^*_{\infty,\rm vert} = \mathcal{P}^*_\infty \cap \{\pi/8 \leq \varphi^{\rm u} \leq \pi/4\}. \tag{2.45}$$

Theorem 2.3.19. *The restriction $\mathbb{P}_\pm|_{\mathcal{P}^*_{\infty,\rm vert}} : \mathcal{P}^*_{\infty,\rm vert} \longrightarrow \mathcal{P}^*_\infty$ of the scattering maps $\mathbb{P}_\pm$ to $\mathcal{P}^*_{\infty,\rm vert}$ can be computed as*

$$(\varphi^{\rm u}, I^{\rm u}) \longmapsto (\varphi^{\rm u} - (\partial_{I^{\rm u}}\mathtt{S}_\pm + \partial_{I^{\rm s}}\mathtt{S}_\pm) \circ (\Omega^{\rm u}_\pm \circ \eta_{I_{\rm m}} \circ \Phi_\pm \circ \Xi_\pm)^{-1}(\varphi^{\rm u}, I^{\rm u}), \quad \tilde I^{\rm s}_\pm \circ (\Omega^{\rm u}_\pm \circ \eta_{I_{\rm m}} \circ \Phi_\pm)^{-1}(\varphi^{\rm u}, I^{\rm u})).$$

Moreover, for all $(\varphi^{\rm u}, I^{\rm u}) \in \mathcal{P}^*_{\infty,\rm vert}$,

$$\mathbb{P}_+ - \mathbb{P}_- = \mathcal{J}\nabla(\mathcal{L}_+ - \mathcal{L}_-) + \exp(-(I^{\rm u})^3/3)\left(\mathcal{O}((I^{\rm u})^{-1/2}),\ \mathcal{O}(\epsilon(I^{\rm u})^{-5/2})\right)^\top. \tag{2.46}$$

The proof of Theorems 2.3.16 and 2.3.19 is postponed until Section 2.6.

Remark 2.3.20. *Notice that to state Theorem 2.3.19, we have considered the vertical strip $\mathcal{P}^*_{\infty,\rm vert}$ instead of the whole submanifold $\mathcal{P}^*_\infty$. This is due to the fact that the maps*

$$(I^{\rm u}, I^{\rm s}) \to \Omega^{\rm u}_\pm \circ \Xi_\pm(I^{\rm u}, I^{\rm s}) = (\varphi^{\rm u}_\pm(I^{\rm u}, I^{\rm s}), I^{\rm u})$$

are not invertible everywhere on $\mathcal{P}^*_\infty$. However, it is easy to check from Lemma 2.3.15 and Theorem 2.3.16 that $\mathcal{P}^*_{\infty,\rm vert} \subset \mathrm{Dom}\ (\Omega^{\rm u}_\pm \circ \Xi_\pm)^{-1}$. This will be enough for our purposes.

Remark 2.3.21. *We point out that* (2.46) *does not mean that* $\mathbb{P}_\pm$ *are approximated by* $\mathcal{L}_\pm$ *up to an exponentially small remainder. This is a subtle point in our argument: there are non-exponentially small, i.e. polynomially small, errors in the approximation of* $\mathbb{P}_\pm$ *by* $\mathcal{L}_\pm$. *What we prove in Theorem 2.3.19 is that these errors* are the same for both approximations *of* $\mathbb{P}_+$ *and* $\mathbb{P}_-$.

Remark 2.3.22. *Throughout the rest of this section we write* (φ, I) *instead of* $(\varphi^{\mathrm{u}}, I^{\mathrm{u}})$.

2.3.5 Transversality between the scattering maps

In this section we prove that the scattering maps $\mathbb{P}_\pm$ share no common invariant curves. This transversality property will imply (see Section 2.3.6) the existence of a transition chain of heteroclinic orbits along which the angular momentum changes in any predetermined fashion.

To prove this property we first straighten the dynamics of one of the maps. Namely, we obtain a one degree of freedom Hamiltonian $\mathcal{K}_+$, defined on $\mathcal{P}_\infty^*$, such that $\mathbb{P}_+$ follows the level sets of $\mathcal{K}_+$ up to an exponentially small remainder. Then, we verify that on the vertical strip $\mathcal{P}^*_{\mathrm{vert}}$ defined in (2.45), the scalar product between the vectors $\nabla\mathcal{K}_+$ and $\mathbb{P}_- - \mathbb{P}_+$ is uniformly away from zero to guarantee the absence of common invariant curves.

We start by looking for the Hamiltonian $\mathcal{K}_+$. To this end, we first use a theorem by Kuksin and Pöschel ([KP94]) which produces a non autonomous time periodic interpolating Hamiltonian K_+ for the map $\mathbb{P}_+$. The introduction of some notation is in order. Given a domain $D \subset \mathbb{T}^n \times \mathbb{R}^n$ and $\rho > 0$ we write

$$D_\rho = \{z \in \mathbb{C}^{2n} \colon \mathrm{dist}(z, D) \leq \rho\}.$$

We write $|\cdot|_\rho$ for the sup norm for functions $f : D_\rho \to \mathbb{C}$ and use $\|\cdot\|_\rho$ for the case where f is vector valued. Also, given a domain D as before we call $\hat{D} = D \times \mathbb{T} \times \mathbb{R}$ its *extended phase space.*

Theorem 2.3.23 (Theorem 4 in [KP94])**.** *Fix* $\rho_0 > 0$ *and let* $F : D_{\rho_0} \subset \mathbb{T}^n_{\rho_0} \times \mathbb{C}^n \to \mathbb{T}^n_{\rho_0} \times \mathbb{C}^n$ *be a real analytic exact symplectic map of the form* $F = F_0 + F_1$ *where*

$$F_0(\varphi, I) = (\varphi + \partial_I h(I), I)$$

for some $h : \mathbb{R}^n \to \mathbb{R}$ *and*

$$\|F_1\|_{\rho_0} \leq \varepsilon.$$

Then, there exists $\varepsilon_0(n, \rho, |h|, |Dh|, |D^2h|) > 0$ *such that for all* $0 \leq \varepsilon \leq \varepsilon_0$*, there exists a non-autonomous time periodic real analytic Hamiltonian* $K(\varphi, \tau, I, E) : \hat{D} \subset \mathbb{T}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ *on the extended phase space and a real analytic symplectic embedding*

$$j : D \to \Sigma = \{K = 0,\ \tau = 0\}$$

such that the Poincaré map ϕ_K *for the flow of* K *on the section* Σ *is well defined and satisfies*

$$F = j^{-1} \circ \phi_K \circ j.$$

Moreover,

$$|K - h|_\rho \lesssim \varepsilon \qquad \text{and} \qquad \|j - j_0\|_\rho \lesssim \varepsilon,$$

where $j_0(\varphi, \tau, I, E) = (\varphi, 0, I, -h(I))$ *and* $\rho = \rho_0/2$.

Let $D = \mathcal{P}^*_\infty \cap \{I_* \leq I \leq I_* + 1\}$ be a horizontal strip of width 1. Then, Lemma 2.3.15, Theorem 2.3.16 and Lemma 2.3.17 imply that the map $\mathbb{P}_+$ restricted to D satisfies the hypothesis of Theorem 2.3.23 with $h(I)$ any function such that

$$\partial_I h(I) = \omega_{\mathrm{circ}}(I), \qquad\qquad \varepsilon = \epsilon I_*^{-5}$$

and some $\rho_0 > 0$ which does not depend on I_*. Thus, Theorem 2.3.23 yields a real analytic Hamiltonian function K_+ and a real analytic symplectic embedding j_+ such that

$$\mathbb{P}_+ = j_+^{-1} \circ \phi_{K_+} \circ j_+$$

and

$$|K_+ - h|_\rho \lesssim \epsilon I_*^{-5}, \qquad \|j_+ - j_0\|_\rho \lesssim \epsilon I_*^{-5},$$

where $\rho = \rho_0/2$. Writing X_{K_+} for the vector field generated by K_+ and expanding its Poincaré map ϕ_{K_+} on the section $\{\tau = 0\}$ in Taylor series we get that

$$\phi_{K_+} = \mathrm{Id} + X_{K_+} + \left(\mathcal{O}\left(|X_{K_+,\varphi}|_\rho^2 + |X_{K_+,\varphi}|_\rho |X_{K_+,I}|_\rho\right),\ \mathcal{O}\left(|X_{K_+,I}|_\rho^2 + |X_{K_+,\varphi}|_\rho |X_{K_+,I}|_\rho|\right)\right)^\top. \quad (2.47)$$

Moreover, identifying j_0 with the identity map on D and writing $j_+ = \mathrm{Id} + \Delta j_+$, for all $(\varphi, I) \in D_{\rho/2}$, a simple Taylor expansion plus Cauchy estimates show that

$$\begin{aligned}
\phi_{K_+} =& j_+ \circ \mathbb{P}_+ \circ j_+^{-1} = j_+ \circ \left(j_+^{-1} + (\mathbb{P}_+ - \mathrm{Id}) \circ j_+^{-1}\right)\\
=& (\mathrm{Id} + \Delta j_+) \circ \left(\mathrm{Id} - \Delta j_+ + (\mathbb{P}_+ - \mathrm{Id}) \circ (\mathrm{Id} + \Delta j_+)^{-1} + \mathcal{O}(\|\Delta j_+\|_\rho^2)\right)\\
=& \mathrm{Id} - \Delta j_+ + (\mathbb{P}_+ - \mathrm{Id}) + \Delta j_+ \circ \left(\mathrm{Id} - \Delta j_+ + (\mathbb{P}_+ - \mathrm{Id}) \circ (\mathrm{Id} + \Delta j_+)^{-1}\right)\\
&+ \mathcal{O}(\|D(\mathbb{P}_+ - \mathrm{Id})\Delta j_+\|_{\rho/2}) + \mathcal{O}(\|\Delta j_+\|_{\rho/2}^2)\\
=& \mathbb{P}_+ + \mathcal{O}(\|\Delta j_+\|_\rho^2) + \mathcal{O}(\|D(\mathbb{P}_+ - \mathrm{Id})\Delta j_+\|_{\rho/2})\\
=& \mathbb{P}_+ + \mathcal{O}(\epsilon^2 I_*^{-10}).
\end{aligned}$$

Therefore, using (2.47), we get that, for all $(\varphi, I) \in D_\rho$,

$$X_{K_+} - (\mathbb{P}_+ - \mathrm{Id}) = \left(\mathcal{O}(I_*^{-8} + \epsilon I_*^{-9}),\ \mathcal{O}(\epsilon I_*^{-9})\right)^\top. \quad (2.48)$$

Therefore, using Lemma 2.3.15 and Theorem 2.3.16, we observe that the Hamiltonian vector field X_{K_+} is a slow fast system on $(\varphi, \tau, I, E) \in D$ since $\dot\tau = 1$ while $\dot\varphi = \mathcal{O}(I_*^{-4})$ and $\dot I = \mathcal{O}(\epsilon I_*^{-5})$. We now obtain a Neishtadt's like normal form ([Nei84]) for the Hamiltonian function K_+ to push the τ dependence to an exponentially small remainder.

Lemma 2.3.24. *There exists a real analytic change of variables $\psi : D_{\rho/8} \to D_{\rho/2}$ with*

$$\|\mathrm{Id} - \psi\|_{\rho/8} \lesssim \epsilon I_*^{-9}$$

and a real analytic autonomous Hamiltonian function $\mathcal{K}_+ : D_{\rho/8} \to \mathbb{C}$ such that the map

$$\mathtt{P}_+ = \psi^{-1} \circ j_+ \circ \mathbb{P}_+ \circ j_+^{-1} \circ \psi$$

and the time one map $\phi_{\mathcal{K}_+}$ associated to the Hamiltonian function $\mathcal{K}_+$ satisfy

$$\|\mathtt{P}_+ - \phi_{\mathcal{K}_+}\|_{\rho/8} \lesssim \epsilon \exp\left(-c I_*^4\right) \quad (2.49)$$

for some $c = c(\rho) > 0$.

The proof follows the ideas developed in [Nei84] but in a Hamiltonian setting. We only sketch the proof in Appendix 2.B in order to keep track of the ϵ dependence of the error terms.

Let $\vartheta = j_+^{-1} \circ \psi$, which satisfies $\vartheta = \mathrm{Id} + \mathcal{O}(\epsilon I_*^{-5})$ uniformly on $D_{\rho/8}$. Then, from the previous lemma, we know that the curves $\{\mathcal{K}_+ = \mathrm{const}\}$ are almost invariant for the map

$$\mathtt{P}_+ = \vartheta^{-1} \circ \mathbb{P}_+ \circ \vartheta.$$

In the next proposition we show that this is not the case for the map $\mathtt{P}_- = \vartheta^{-1} \circ \mathbb{P}_- \circ \vartheta$. The approximation result

$$\mathbb{P}_+ - \mathbb{P}_- \sim \mathcal{J}(\mathcal{L}_+ - \mathcal{L}_-)$$

obtained in Theorem 2.3.19 and the asymptotic expression (2.40) measuring the transversality between the level sets of $\mathcal{L}_+$ and $\mathcal{L}_-$ given in Lemma 2.3.15 are the key to this result.

Proposition 2.3.25. *There exists $I_*>0$ such that, for any $\epsilon\in(0,I_*^{-3})$, the maps $\mathbb{P}_+$ and $\mathbb{P}_-$ share no common invariant curve on $\mathcal{P}^*_\infty$.*

Proof. We write $|\cdot|, \|\cdot\|$ for the scalar and vectorial sup norm on D (namely, $\rho=0$). Let $\mathcal{K}_+$ be the autonomous Hamiltonian obtained in Lemma 2.3.24. Then, for all $(\varphi,I)\in D$ the map $\mathtt{P}_+=\vartheta^{-1}\circ\mathbb{P}_+\circ\vartheta$ satisfies

$$|\mathcal{K}_+\circ\mathtt{P}_+-\mathcal{K}_+|=|\mathcal{K}_+\circ\mathtt{P}_+-\mathcal{K}_+\circ\phi_{\mathcal{K}_+}|\leq\|\nabla\mathcal{K}_+\|\|\mathtt{P}_+-\phi_{\mathcal{K}_+}\|\lesssim\epsilon\exp(-cI_*^4),$$

where $\phi_{\mathcal{K}_+}$ is the time one map of the Hamiltonian $\mathcal{K}_+$ and we have used inequality (2.49) in Lemma 2.3.24. We now claim that, for all $(\varphi,I)\in\mathcal{P}^*_{\mathrm{vert}}$, the map $\mathtt{P}_-=\vartheta^{-1}\circ\mathbb{P}_-\circ\vartheta$ satisfies

$$\mathcal{K}_+\circ\mathtt{P}_--\mathcal{K}_+=\{\mathcal{L}_+,\mathcal{L}_-\}+\mathcal{O}(\epsilon I_*^{-9/2}\exp(-I_*^3/3)),$$

from which the statement of the proposition follows using the estimates for $\{\mathcal{L}_+,\mathcal{L}_-\}$ given in Lemma 2.3.15. Indeed, these estimates prove that the maps $\mathbb{P}_\pm$ share no common invariant curve on

$$\mathcal{P}^*_\infty\cap\{I_*\leq I\leq I_*+I_*^{-2}\}$$

where the estimates are uniform, i.e. for $I\in\{I_*\leq I\leq I_*+I_*^{-2}\}$ we have

$$\exp(-I_*^3/3)\lesssim\exp(-I^3/3)\lesssim\exp(-I_*^3/3).$$

Since the choice of I_* was arbitrary this implies that the maps $\mathbb{P}_\pm$ share no common invariant curve on $\mathcal{P}^*_\infty$. To verify the claim we use the triangle inequality to write

$$|\mathcal{K}_+\circ\mathtt{P}_--\mathcal{K}_+|\geq\Big|\,|\mathcal{K}_+\circ\mathtt{P}_--\mathcal{K}_+\circ\mathtt{P}_+|-|\mathcal{K}_+\circ\mathtt{P}_+-\mathcal{K}_+|\,\Big|.$$

Now, in order to bound from below the term $|\mathcal{K}_+\circ\mathtt{P}_--\mathcal{K}_+\circ\mathtt{P}_+|$ we expand in Taylor series

$$\mathcal{K}_+\circ\mathtt{P}_--\mathcal{K}_+\circ\mathtt{P}_+=\langle\nabla\mathcal{K}_+,(\mathtt{P}_--\mathtt{P}_+)\rangle+\mathcal{O}\left(\|D^2\mathcal{K}_+(\mathtt{P}_--\mathtt{P}_+)\|\|\mathtt{P}_--\mathtt{P}_+\|\right).$$

On one hand, denoting by $\mathcal{J}$ the usual complex structure in $\mathbb{R}^2$ and using inequality (2.48), we have that

$$\begin{aligned}\nabla\mathcal{K}_+=&\mathcal{J}(\mathbb{P}_+-\mathrm{Id})+\left(\mathcal{O}\left(|X_{\mathcal{K}_+,I}-(\mathbb{P}_{+,I}-I)|\right),\mathcal{O}\left(|X_{\mathcal{K}_+,\varphi}+(\mathbb{P}_{+,I}-I)|\right)\right)^\top\\=&\mathcal{J}(\mathbb{P}_+-\mathrm{Id})+\left(\mathcal{O}\left(\epsilon I_*^{-9}\right),\ \mathcal{O}\left(I_*^{-8}\right)\right)^\top.\end{aligned}$$

On the other hand, since ϑ is a $\mathcal{O}(\epsilon I_*^{-5})$-close to identity real analytic transformation defined in a complex neighborhood of size $\rho/8\sim1$, one easily checks that $\tilde\vartheta\equiv\vartheta^{-1}-\mathrm{Id}=\mathcal{O}(\epsilon I_*^{-5})$ and

$$\begin{aligned}\mathtt{P}_--\mathtt{P}_+=&\vartheta^{-1}\circ\mathbb{P}_-\circ\vartheta-\vartheta^{-1}\circ\mathbb{P}_+\circ\vartheta=\left(\vartheta^{-1}\circ\mathbb{P}_--\vartheta^{-1}\mathbb{P}_+\right)\circ\vartheta\\=&\Big(\mathbb{P}_--\mathbb{P}_++\tilde\vartheta\circ\mathbb{P}_--\tilde\vartheta\circ\mathbb{P}_+\Big)\circ\vartheta\\=&\left((\mathbb{P}_--\mathbb{P}_+)+\left(\int_0^1D\tilde\vartheta\left(\mathbb{P}_-+s(\mathbb{P}_+-\mathbb{P}_-)\right)\mathrm{d}s\right)(\mathbb{P}_--\mathbb{P}_+)\right)\circ\vartheta\\=&\mathbb{P}_--\mathbb{P}_++\mathcal{O}\left(\epsilon I_*^{-5}\|\mathbb{P}_--\mathbb{P}_+\|\right).\end{aligned}$$

Therefore,

$$\langle\nabla\mathcal{K}_+,(\mathtt{P}_--\mathtt{P}_+)\rangle=\Big\langle\mathcal{J}(\mathbb{P}_+-\mathrm{Id})+\left(\mathcal{O}(\epsilon I_*^{-9}),\mathcal{O}(I_*^{-8})\right)^\top,\ \mathbb{P}_--\mathbb{P}_++\mathcal{O}\left(\epsilon I_*^{-5}\|\mathbb{P}_--\mathbb{P}_+\|\right)\Big\rangle.$$

Using the estimates in Theorem 2.3.16,

$$\begin{aligned}\langle\mathcal{J}(\mathbb{P}_+-\mathrm{Id}),(\mathbb{P}_--\mathbb{P}_+)\rangle=&(\partial_\varphi\mathcal{L}_++\mathcal{O}(\epsilon I_*^{-7}))(\partial_I(\mathcal{L}_+-\mathcal{L}_-)+\mathcal{O}(I_*^{1/2}\exp(-I_*^3/3)))\\&+(\partial_I\mathcal{L}_++\mathcal{O}(I_*^{-7}))(-\partial_\varphi(\mathcal{L}_+-\mathcal{L}_-)+\mathcal{O}(\epsilon I_*^{-3/2}\exp(-I_*^3/3)))\\=&\{\mathcal{L}_+,\mathcal{L}_-\}+\mathcal{O}(\epsilon I_*^{-9/2}\exp(-I_*^3/3)),\end{aligned}$$

and the proposition is proved taking into account the asymptotic expressions in Lemma 2.3.15. □

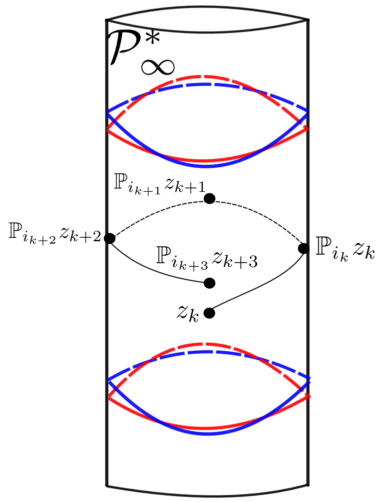

Figure 2.3: The invariant curves of the map $\mathbb{P}_+$ (in red) intersect transversally the invariant curves of the map $\mathbb{P}_-$ (in blue). We also sketch a segment of a diffusive orbit for the iterated function system generated by the maps $\mathbb{P}_\pm$.

2.3.6 Shadowing and existence of a transition chain

We now consider the iterated function system generated by the maps $\mathbb{P}_\pm$. Since the maps $\mathbb{P}_\pm$ are both twist maps and share no common invariant curve on $\mathcal{P}^*_\infty$, it is proven in [Moe02] (see also [LC07]) that the iterated function system generated by the maps $\mathbb{P}_\pm$ possesses drift orbits in $\mathcal{P}^*_\infty$.

Theorem 2.3.26. *Let $\mathbb{P}_\pm$ be the maps defined in (2.32). Then, there exists I_* such that, for any $\epsilon \in (0, I_*^{-3})$ and any pair I_1, I_2 satisfying*

$$I_* \leq I_1 \leq I_2 \leq \epsilon^{-1/3},$$

there exists $N \in \mathbb{N}$ and a sequence

$$\{(i_k, z_k)\}_{1\leq k\leq N} \subset (\{+,-\} \times \mathcal{P}^*_\infty)^N \qquad\qquad z_{k+1} = \mathbb{P}_{i_k}(z_k)$$

such that

$$\pi_I z_1 \leq I_1 \qquad\qquad \text{and} \qquad\qquad I_2 \leq \pi_I z_N.$$

Finally, the proof of Theorem 2.1.1 is completed by standard shadowing results (see Figure 2.3.6). Let $(\varphi, I) \in \mathcal{P}^*_\infty$, which is a parabolic fixed point of the Poincaré map P in (2.26) and denote by $W^{\mathrm{u,s}}_{\varphi,I}$ its stable and unstable manifolds. For a number $\delta > 0$ and a point $p \in \{t = 0\}$ and denote by $B_\delta(p) \subset \{t = 0\}$ the ball of radius δ centered at p at the Poincaré section. The following shadowing result for TNHIC, proved in [GSMS17] fits our purposes.

Proposition 2.3.27 (Proposition 2 in [GSMS17])**.** *Let $N \in \mathbb{N}\cup\{\infty\}$ and let $\{(\varphi_k, I_k)\}_{1\leq k\leq N}$ be a family of fixed points in $\mathcal{P}^*_\infty$ for the Poincaré map P such that, for all $1 \leq k \leq N$, $W^{\mathrm{u}}_{\varphi_k,I_k}$ intersects transversally $W^{\mathrm{s}}(\mathcal{P}^*_\infty)$ at a point $p_k \in W^s_{\varphi_{k+1},I_{k+1}}$. Then, for any sequence $\{\delta_k\}_{k\geq1}$ with $\delta_k > 0$ there exists a point $z \in B_{\delta_1}(\varphi_0, I_0)$ and two sequences $\{n_k\}_{1\leq k\leq N}, \{\tilde{n}_k\}_{1\leq k\leq N}, \subset \mathbb{N}$ with $n_k < \tilde{n}_k < n_{k+1} < \tilde{n}_{k+1}$ such that $P^{n_k}(z) \in B_{\delta_k}(\varphi_k, I_k)$ and $P^{\tilde{n}_k}(z) \in B_{\delta_k}(p_k)$ for all $1 \leq k \leq N$.*

Let $\{z_k\}_{1\leq k\leq N} = \{(\varphi_k, I_k)\}_{1\leq k\leq N} \subset \mathcal{P}^*_\infty$ be the sequence of fixed points for the Poincaré map P given in Theorem 2.3.26 and apply Proposition 2.3.27 with $\delta_k > 0$ small enough. The proof of Theorem 2.1.1 is complete.

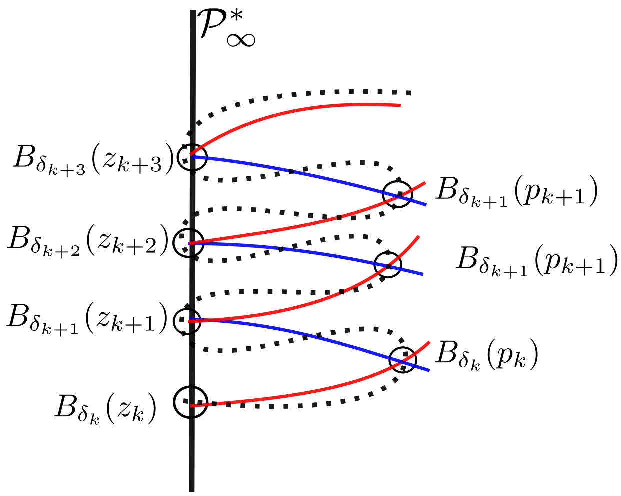

Figure 2.4: A true orbit of the RPE3BP which shadows the pseudo orbit $\{z_k\}_{1\leq k\leq N}$ obtained in Theorem 2.3.26. In blue (red) we skecth segments of the stable (unstable) manifolds associated to parabolic fixed points $\{z_k\}_{0\leq k\leq N}$ of the Poincaré map P.

2.4 The generating functions of the invariant manifolds

In this section we provide the proof of Propositions 2.3.6 and 2.3.8 and show how the latter readily implies Theorem 2.3.9. First we show the existence of real analytic solutions $T^{\mathrm{u,s}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)$ to the Hamilton-Jacobi equation associated to the Hamiltonian H in (2.15). That is,

$$H(q,\delta^{\mathrm{u,s}}+\nabla T^{\mathrm{u,s}}(q))=0 \qquad\qquad q=(u,v,t)$$

with $\delta^{\mathrm{u}}=(0,(I^{\mathrm{u}}-I^{\mathrm{s}})/2,0)$, $\delta^s=(0,(I^{\mathrm{s}}-I^{\mathrm{u}})/2,0)$ and asymptotic conditions

$$\lim_{\mathrm{Re}u\to-\infty}T^{\mathrm{u}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)=0 \qquad\qquad \lim_{\mathrm{Re}u\to\infty}T^{s}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)=0$$

on certain complex domains of the form $D^{\mathrm{u,s}}\times\mathbb{T}^2$ defined below, which satisfy

$$D^{\mathrm{u}}\cap D^{s}\neq\emptyset, \qquad \text{and} \qquad ((-\infty,-u_0]\cup[u_1,u_2])\times\mathbb{T}^2\subset D^{\mathrm{u}}\times\mathbb{T}^2, \qquad [u_0,\infty)\times\mathbb{T}^2\subset D^{s}\times\mathbb{T}^2$$

for some real values $u_0<u_1<u_2$. This is the content of Sections 2.4.2 and 2.4.3. Then, in Section 2.4.4 we study the difference

$$\Delta S(q;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)=\langle\delta^{\mathrm{u}}-\delta^{\mathrm{s}},q\rangle+(T^{\mathrm{u}}-T^{\mathrm{s}})(q;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)$$

on the complex domain $(D^{\mathrm{u}}\cap D^{\mathrm{s}})\times\mathbb{T}^2$ and show that ΔS is approximated uniformly in $(D^{\mathrm{u}}\cap D^{\mathrm{s}})\times\mathbb{T}^2$ by

$$\Delta S\sim\langle\delta^{\mathrm{u}}-\delta^{\mathrm{s}},q\rangle+\tilde{L}$$

where $\tilde{L}$ is the *Melnikov potential* (recall that $I_{\mathrm{m}}=(I^{\mathrm{u}}+I^{\mathrm{s}})/2$) defined by

$$\tilde{L}(u,\beta,t;I_{\mathrm{m}},\epsilon)=\int_{\mathbb{R}}V(s,\beta,t-I_{\mathrm{m}}^3u+I_{\mathrm{m}}^3s;I_{\mathrm{m}},\epsilon)\mathrm{d}s. \tag{2.50}$$

Remark 2.4.1. *The function $\tilde{L}$ satisfies*

$$\tilde{L}(u,\beta,t;I_{\mathrm{m}},\epsilon)=L(t-I_{\mathrm{m}}^3u,\beta;I_{\mathrm{m}},\epsilon)$$

where $L(\sigma,\beta;I_{\mathrm{m}},\epsilon)$ was defined in (2.24)*. The introduction of* (2.50) *is just a matter of convenience for the forthcoming sections.*

Finally, we prove that the existence of nondegenerate critical points of the function

$$q \mapsto \langle \delta^{\mathrm{u}} - \delta^{\mathrm{s}}, q\rangle + \tilde{L}(q; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)$$

implies the existence of nondegenerate critical points of the function $q \mapsto \Delta S(q; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)$.

2.4.1 From the circular to the elliptic problem

As pointed out in the introduction, for $\epsilon = 0$ and $\mu > 0$, which corresponds to the circular problem (RPC3BP), the system is already non integrable since there exist transverse intersections between the stable and unstable manifolds of all the tori $\mathcal{T}_I \subset \mathcal{P}_\infty$ with I sufficiently large (see [GMS16]). However, for $\epsilon = 0$, due to the conservation of the Jacobi constant, there do not exist heteroclinic connections between different tori $\mathcal{T}_I, \mathcal{T}_{I'} \subset \mathcal{P}_\infty$. In Theorem 2.3.9 we prove that for $\epsilon > 0$ there do exist heteroclinic connections between sufficiently close $\mathcal{T}_I, \mathcal{T}_{I'} \subset \mathcal{P}_\infty$. As explained at the beginning of Section 2.4 this result will be proved by approximating the difference ΔS by the Melnikov potential L. In this approximation there are errors errors coming from the circular part of the perturbation and errors exclusive of the elliptic part. For this reason, in order to obtain asymptotic formulas for the scattering maps associated to the aforementioned heteroclinic intersections, in the case $\mu, \epsilon > 0$, it is necessary to keep track of the ϵ dependent part in the generating functions $T^{\mathrm{u,s}}$. To that end, we denote by (see [GMS16])

$$T^{\mathrm{u,s}}_{\mathrm{circ}}(u, t-\beta; I_{\mathrm{m}}) = T^{\mathrm{u,s}}(u, \beta, t; I_{\mathrm{m}}, I_{\mathrm{m}}, 0), \tag{2.51}$$

the generating functions associated to the invariant manifolds of the invariant torus $\mathcal{T}_{I_{\mathrm{m}}} \subset \mathcal{P}_\infty$ for the circular problem ($\epsilon = 0$), let

$$V_{\mathrm{circ}}(u, t-\beta; I_{\mathrm{m}}) = V(u, \beta, t; I_{\mathrm{m}}, 0) \tag{2.52}$$

and introduce the Melnikov potential associated to the circular problem

$$\tilde{L}_{\mathrm{circ}}(u, t-\beta; I_{\mathrm{m}}) = \tilde{L}(u, \beta, t; I_{\mathrm{m}}, 0). \tag{2.53}$$

2.4.2 Analytic continuation of the unstable generating function

Consider the domain (see Figure 2.4.2 and Remark 2.3.7)

$$D^{\mathrm{u}}_\kappa = \{u \in \mathbb{C}\colon |\mathrm{Im} u| \leq 1/3 - \kappa |I_{\mathrm{m}}|^{-3} - \tan\beta_1 \mathrm{Re} u,\ |\mathrm{Im} u| \geq 1/6 + \kappa |I_{\mathrm{m}}|^{-3} - \tan\beta_2 \mathrm{Re} u\}, \tag{2.54}$$

where $\beta_1, \beta_2 \in (0, \pi/2)$, $\beta_1 < \beta_2$ and $\kappa > 0$ is a given constant. It is clear that for I_{m} large enough D^{u}_κ is non empty. The role of the parameter κ is to shrink the domain D^{u}_κ when, in Sections 2.4.3 and 2.4.4, we introduce close to identity changes of variables and make use of Cauchy estimates.

In this section we prove the existence of positive constants κ, ρ and σ such that for all I_* large enough and any $I_{\mathrm{m}} \in \Lambda_{\rho, I}(I_*)$, $|I^{\mathrm{u}} - I^{\mathrm{s}}| \leq \epsilon |I_{\mathrm{m}}|^{-4}$, where $\Lambda_{\rho, I}$ is introduced in (2.12), there exists a unique real analytic solution to the Hamilton-Jacobi equation

$$H(q, \delta^{\mathrm{u}} + \nabla T^{\mathrm{u}}) = (1 + A^{\mathrm{u}}(q, \delta^{\mathrm{u}} + \nabla T^{\mathrm{u}}))\partial_u T^{\mathrm{u}} + B^{\mathrm{u}}(q, \delta^{\mathrm{u}} + \nabla T^{\mathrm{u}})\partial_\beta T^{\mathrm{u}} + I^3_{\mathrm{m}} \partial_t T^{\mathrm{u}} - V(q) = 0 \tag{2.55}$$

with asymptotic condition $\lim_{\mathrm{Re} u \to -\infty} T^{\mathrm{u}} = 0$ in the complex domain $(u, \beta, t) \in D^{\mathrm{u}}_\kappa \times \mathbb{T}_\rho \times \mathbb{T}_\sigma$ and where

$$A^{\mathrm{u}} = \frac{1}{2y_{\mathrm{h}}^2 I_{\mathrm{m}}} \left(\partial_u T^{\mathrm{u}} - r_{\mathrm{h}}^{-2}(\delta^{\mathrm{u}} + \partial_\beta T^{\mathrm{u}})\right) \qquad B^{\mathrm{u}} = -\frac{1}{2y_{\mathrm{h}}^2 r_{\mathrm{h}}^2 I_{\mathrm{m}}} \left(\partial_u T^{\mathrm{u}} - r_{\mathrm{h}}^{-1}(\delta^{\mathrm{u}} + \partial_\beta S^{\mathrm{u}}\right). \tag{2.56}$$

The existence of T^{s} solving the corresponding Hamilton-Jacobi equation on $D^{\mathrm{s}} \times \mathbb{T}_\rho \times \mathbb{T}_\sigma$ with $D^s_\kappa = \{u \in \mathbb{C}\colon -u \in D^{\mathrm{u}}_\kappa\}$ is obtained by a completely analogous argument.

Remark 2.4.2. *The use of different widths for the strips in the angles β and t is only a technical issue. The solution to the Hamilton-Jacobi equation* (2.55) *will be obtained by means of a Newton method in which the size of the strip $\mathbb{T}_\rho$ for the angle β is reduced at each iteration while the size of the strip $\mathbb{T}_\sigma$ for the t variable can be kept constant. For this reason, in all the forthcoming notation, we omit the dependence on σ and only emphasize the dependence on ρ.*

The width of the strip of analyticity in the angle β is taken to be the same than the width of the complex neighborhood for the parameter I_{m}. This is an arbitrary choice to avoid introducing more notation.

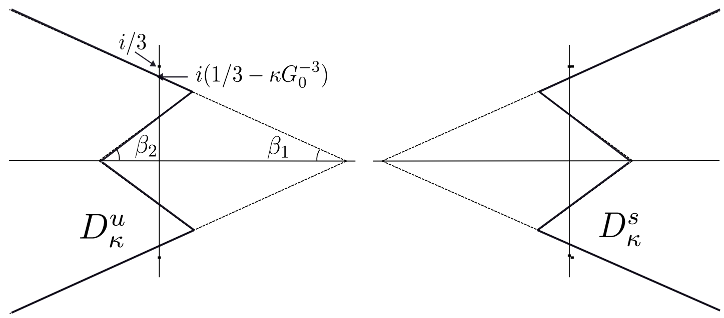

Figure 2.5: The complex domains D^{u}_κ and D^{s}_κ.

Let η,ν be positive real constants. We now introduce the family of Banach spaces of sequences of analytic functions in which we will look for solutions to (2.55)

$$\mathcal{Z}_{\eta,\nu,\rho}=\left\{h=\{h^{[l]}\}_{l\in\mathbb{Z}}:\ h^{[l]}:D^{\mathrm{u}}_\kappa\times\mathbb{T}_\rho\to\mathbb{C}\text{ is analytic for all }l\in\mathbb{Z}\text{ and }\|h\|_{\eta,\nu,\rho}<\infty\right\}\tag{2.57}$$

where $\|\cdot\|_{\eta,\nu,\rho}$ is the Fourier sup norm

$$\|h\|_{\eta,\nu,\rho}=\sum_{l\in\mathbb{Z}}\|h^{[l]}\|_{\eta,\nu,\rho,l}e^{|l|\sigma}$$

defined by

$$\begin{aligned}\|h\|_{\eta,\nu,\rho,l}=&\sup_{(u,\beta)\in(D^{\mathrm{u}}_\kappa\cap\{\mathrm{Re}(u)\le-1\})\times\mathbb{T}_\rho}\left|u^\eta h^{[l]}(u,\beta)\right|\\&+\sup_{(u,\beta)\in(D^{\mathrm{u}}_\kappa\cap\{\mathrm{Re}(u)\ge-2\})\times\mathbb{T}_\rho}\left|(u-i/3)^{\nu+l/2}(u+i/3)^{\nu-l/2}h^{[l]}(u,\beta)\right|.\end{aligned}$$

It will also be convenient for us to introduce the Banach spaces

$$\mathcal{X}_{\eta,\nu,\rho}=\left\{h=\{h^{[l]}\}_{l\in\mathbb{Z}}\ :\ h^{[l]}:D^{\mathrm{u}}_\kappa\times\mathbb{T}_\rho\to\mathbb{C}\text{ is analytic for all }l\in\mathbb{Z}\text{ and }[\![h]\!]_{\eta,\nu,\rho}<\infty\right\}.\tag{2.58}$$

where

$$[\![h]\!]_{\eta,\nu,\rho}=\|h\|_{\eta,\nu,\rho}+\|\partial_u h\|_{\eta+1,\nu+1,\rho}.\tag{2.59}$$

Remark 2.4.3. *It is straightforward to check that the elements of $\mathcal{Z}_{\eta,\nu,\rho}$ and $\mathcal{X}_{\eta,\nu,\rho}$ can be identified with Fourier series*

$$h(u,\beta,t)=\sum_{l\in\mathbb{Z}}h^{[l]}(u,\beta)e^{ilt}$$

which, for a given $u\in D^{\mathrm{u}}$, converge on the strip

$$\mathbb{T}_\sigma(u)=\left\{t\in\mathbb{C}/2\pi\mathbb{Z}:\ \left|\mathrm{Im}(t)-\frac{1}{2}\left(\ln(|u-i/3|)-\ln(|u+i/3|)\right)\right|\le\sigma\right\}.$$

That is, they yield well defined functions for $(u,\beta)\in D^{\mathrm{u}}_\kappa\times\mathbb{T}_\rho$ and $t\in\mathbb{T}_\sigma(u)$. Alternatively one can think of the elements of $\mathcal{X}_{\eta,\nu,\rho}$ as formal Fourier series on the strip $\mathbb{T}_\sigma$ (see [GMS16] and [GMPS22]).

Remark 2.4.4. *In the case $I^{\mathrm{u}},I^{\mathrm{s}}\in\mathbb{R}$, one can replace analytic by real analytic in the definition of $\mathcal{Z}_{\eta,\nu,\rho}$ and $\mathcal{X}_{\eta,\nu,\rho}$.*

In the following lemma we list some properties of the spaces $\mathcal{X}_{\eta,\nu,\rho}$ which will be useful. The proof is straightforward.

Lemma 2.4.5. *The following statements hold:*

- *(Graded algebra property) For any* $g \in \mathcal{X}_{\eta,\nu,\rho}$ *and* $f \in \mathcal{X}_{\eta',\nu',\rho}$, *their product satisfies* $gf \in \mathcal{X}_{\eta+\eta',\nu+\nu',\rho}$.
- *Let* $h \in \mathcal{X}_{\eta,\nu,\rho}$. *Then, for* $\eta' < \eta$ *and* $\nu' < \nu$ *we have* $h \in \mathcal{X}_{\eta',\nu',\rho}$ *and*

$$\|h\|_{\eta',\nu',\rho} \le |I_{\mathrm{m}}|^{3(\nu-\nu')}\|h\|_{\eta,\nu,\rho}$$

- *Let* $h \in \mathcal{X}_{\eta,\nu,\rho}$. *Then for any* $0 < \delta < \rho$ *we have that* $\partial_\beta h \in \mathcal{X}_{\eta,\nu,\rho-\delta}$ *and*

$$\|\partial_\beta h\|_{\eta,\nu,\rho-\delta} \le \delta^{-1}\|h\|_{\eta,\nu,\rho}.$$

We also state the following lemma, which will be useful to deal with compositions in the angular variable β. The proof can be found in [GMS16].

Lemma 2.4.6. *Let* $h \in \mathcal{X}_{\eta,\nu,\rho}$ *and let* $g_i \in \mathcal{X}_{0,0,\rho'}$ *with* $\rho > \rho'$, $i = 1, 2$, *and*

$$\|g_i\|_{0,0,\rho'} \le \frac{\rho-\rho'}{2}.$$

Write $h \circ (\mathrm{Id} + g_i)(u, \beta, t) = h(u, \beta + g_i(u, \beta, t), t)$. *Then,* $h \circ (\mathrm{Id} + g_i) \in \mathcal{X}_{\eta,\nu,\rho'}$ *with*

$$\|h \circ (\mathrm{Id} + g_i)\|_{\eta,\nu,\rho'} \lesssim \|h\|_{\eta,\nu,\rho}.$$

Moreover, for $f = h \circ (\mathrm{Id} + g_2) - h \circ (\mathrm{Id} + g_1)$ *we have*

$$\|f\|_{\eta,\nu,\rho'} \lesssim (\rho-\rho')^{-1}\|h\|_{\eta,\nu,\rho}\|g_2 - g_1\|_{0,0,\rho'}.$$

The choice of the functional space for solving (2.55) is motivated by the following result proved in Appendix 5.C.

Lemma 2.4.7. *Fix* $\kappa > 0$ *and* $\sigma > 0$. *Then, there exist* $\rho_0, I_* > 0$ *such that, for* $\epsilon \in (0, I_*^{-3})$, *and* $I^{\mathrm{u}}, I^{\mathrm{s}}$ *with* $I_{\mathrm{m}} \in \Lambda_{\rho_0,I}$, *the perturbative potential* $V(v, \beta, t; I_{\mathrm{m}})$ *defined in* (2.16) *satisfies* $V \in \mathcal{X}_{2,3/2,\rho_0}$. *Moreover*

$$\|V\|_{2,3/2,\rho_0} \lesssim |I_{\mathrm{m}}|^{-3}$$

and

$$\|V - V_{\mathrm{circ}}\|_{2,3/2,\rho_0} \lesssim \epsilon|I_{\mathrm{m}}|^{-3}$$

where V_{circ} *was defined in* (2.52).

We now state the main result in this section.

Theorem 2.4.8. *Let* $\kappa, \sigma > 0$ *and* $\rho_0 > 0$ *as in Lemma 2.4.7. Then, there exist* $\rho \in (0, \rho_0)$ *and* $I_* > 0$ *such that for for* $\epsilon \in (0, I_*^{-3})$, *and* $I^{\mathrm{u}}, I^{\mathrm{s}}$ *such that* $I_{\mathrm{m}} = (I^{\mathrm{u}} + I^{\mathrm{s}})/2 \in \Lambda_{\rho,I}(I_*)$, *and* $|I^{\mathrm{u}} - I^{\mathrm{s}}| \le \epsilon|I_{\mathrm{m}}|^{-4}$, *there exists* $T^{\mathrm{u}} \in \mathcal{X}_{1/3,1/2,\rho}$ *solution to the Hamilton-Jacobi equation* (2.55) *such that*

$$\|T^{\mathrm{u}}\|_{1/3,1/2,\rho} \lesssim |I_{\mathrm{m}}|^{-3} \qquad \text{and} \qquad \|T^{\mathrm{u}} - L^{\mathrm{u}}\|_{1/3,1,\rho} \lesssim |I_{\mathrm{m}}|^{-7},$$

where L^{u} *is the unstable half Melnikov potential*

$$L^{\mathrm{u}}(u, \beta, t; I_{\mathrm{m}}, \epsilon) = \int_{-\infty}^{0} V(u + s, \beta, t + I_{\mathrm{m}}^3 s; I_{\mathrm{m}}, \epsilon)\mathrm{d}s. \tag{2.60}$$

Moreover,

$$\|T^{\mathrm{u}} - T^{\mathrm{u}}_{\mathrm{circ}} - (L^{\mathrm{u}} - L^{\mathrm{u}}_{\mathrm{circ}})\|_{1/3,1,\rho} \lesssim \epsilon|I_{\mathrm{m}}|^{-7},$$

where $T^{\mathrm{u}}_{\mathrm{circ}}$ *is defined in* (2.51) *and* $L^{\mathrm{u}}_{\mathrm{circ}}(u, t - \beta; I_{\mathrm{m}}) = L^{\mathrm{u}}(u, \beta, t; I_{\mathrm{m}}, 0)$.

The proof of Theorem 2.4.8 will be accomplished by a Newton iterative scheme. That is, we obtain T^{u} as the limit of an iterative process $T^{\mathrm{u}} = \lim_{n\to\infty} T_n$ where $T_0 = 0$ and the n-th step is obtained as the solution to the linear equation

$$H(T_{n-1}) + DH(T_{n-1})[T_n - T_{n-1}] = 0 \tag{2.61}$$

where, by abuse of notation we have written (and will write in the forthcoming sections)

$$H(T_n) = H(q, \delta^{\mathrm{u}} + \nabla T_n; I^{\mathrm{u}}, I^{\mathrm{s}})$$

and H is given in (2.15). One can check that the linearized operator $DH(T)[\cdot]$ reads

$$\begin{aligned} DH(T)[\cdot] = &\left(1 + y_{\mathrm{h}}^{-2} I_{\mathrm{m}}^{-1}\left(\partial_u T - r_{\mathrm{h}}^{-2}(\delta_\beta^{\mathrm{u}} + \partial_\beta T)\right)\right)\partial_u[\cdot] \\ &- y_{\mathrm{h}}^{-2} r_{\mathrm{h}}^{-2} I_{\mathrm{m}}^{-1}\left(\partial_u T - 2r_{\mathrm{h}}^{-1}(\delta_\beta^{\mathrm{u}} + \partial_\beta T)\right)\partial_\beta[\cdot] + I_{\mathrm{m}}^3 \partial_t[\cdot], \end{aligned} \tag{2.62}$$

where we recall that

$$\delta_\beta^{\mathrm{u}} = \frac{1}{2}(I^{\mathrm{u}} - I^{\mathrm{s}}).$$

Since H is quadratic in ∇T the second differential of H is a bilinear operator and the error we accomplish at the step n is

$$H(T_n) = D^2 H\left[\Delta T_n, \Delta T_n\right] = y_{\mathrm{h}}^{-2} I_{\mathrm{m}}^{-1}\left((\partial_u \Delta T_n)^2 - 2r_{\mathrm{h}}^{-2} \partial_u \Delta T_n \partial_\beta \Delta T_n + 2r_{\mathrm{h}}^{-1}\left(\partial_\beta \Delta T_n\right)^2\right) \tag{2.63}$$

where we have introduced the notation $\Delta T_n = T_n - T_{n-1}$.

In the proof of Theorem 2.4.8 we treat $DH(T)[\cdot]$ as a small perturbation of the constant coefficients linear operator

$$\mathcal{L}[\cdot] = (\partial_u + I_{\mathrm{m}}^3 \partial_t)[\cdot]. \tag{2.64}$$

The next technical lemma, proved in [GOS10], shows the existence of a right inverse for $\mathcal{L}$ on the functional space $\mathcal{X}_{\eta,\nu,\rho}$ with $\eta > 1$.

Lemma 2.4.9. *Let $\mathcal{L}$ be the operator defined in* (2.64)*. Then, for any $\eta > 1$ there exists an operator $\mathcal{G} : \mathcal{X}_{\eta,\nu,\rho} \to \mathcal{X}_{\eta-1,\nu,\rho}$, given by*

$$\mathcal{G}(h)(u, \beta, t) = \int_{-\infty}^{0} h(u + s, \beta, t + I_{\mathrm{m}}^3 s)\mathrm{d}s, \tag{2.65}$$

such that $\mathcal{L} \circ \mathcal{G} = \mathrm{Id} : \mathcal{X}_{\eta,\nu,\rho} \to \mathcal{X}_{\eta,\nu,\rho}$. Moreover, for any $h \in \mathcal{X}_{\eta,\nu,\rho}$ with $\eta, \nu > 1$ the following estimates hold

$$\|\mathcal{G}(h)\|_{\eta-1,\nu-1,\rho} \lesssim \|h\|_{\eta,\nu,\rho} \qquad \text{and} \qquad \|\partial_u \mathcal{G}(h)\|_{\eta,\nu,\rho} \lesssim \|h\|_{\eta,\nu,\rho}.$$

Remark 2.4.10. *Lemma 2.4.9 or similar versions are usually proved for $I_{\mathrm{m}} \in \mathbb{R}$ (see [GOS10]). As for $I_{\mathrm{m}} \in \Lambda_{\rho,I}$ we have*

$$\arg(I_{\mathrm{m}}) = \tan^{-1}(\mathcal{O}(|I_{\mathrm{m}}|^{-1})) = \mathcal{O}(|I_{\mathrm{m}}|^{-1}),$$

and it is straightforward to check that the same proof applies for $I_{\mathrm{m}} \in \Lambda_{\rho,I}$.

First step of the Newton scheme

The iterative scheme proposed above defines the function T_1 as the solution to the linearized equation

$$H(0) + DH(0)[T_1] = 0. \tag{2.66}$$

Instead, it will be convenient to modify the first step of the iterative process and define T_1 as the solution to

$$\mathcal{L}T_1 = -H(0), \tag{2.67}$$

where $\mathcal{L}$ is the constant coefficients linear operator defined in (2.64). Using Lemma 2.4.9 we can rewrite (2.66) as

$$T_1 = -\mathcal{G}(H(0)). \tag{2.68}$$

The properties of the unperturbed homoclinic stated in Lemma 2.2.1, Lemma 2.4.7 for the potential V and the hypothesis $|\delta_\beta^{\mathrm{u}}| \lesssim \epsilon |I_{\mathrm{m}}|^{-4}$ imply that (here ρ_0 is the constant given in Lemma 2.4.7)

$$\|H(0)\|_{4/3,3/2,\rho_0} = \|y_{\mathrm{h}}^{-2} r_{\mathrm{h}}^{-3} I_{\mathrm{m}}^{-1} (\delta_\beta^{\mathrm{u}})^2 - V\|_{4/3,3/2,\rho_0} \lesssim |I_{\mathrm{m}}|^{-3} \equiv \varepsilon_0. \tag{2.69}$$

Therefore, it follows from Lemma 2.4.9 that $T_1 \in \mathcal{X}_{1/3,1/2,\rho_0}$ with

$$[\![T_1]\!]_{1/3,1/2,\rho_0} \lesssim \varepsilon_0. \tag{2.70}$$

The error in this first approximation is given by

$$H(T_1) = D^2 H[T_1, T_1] + (DH(T_1) - \mathcal{L})[T_1].$$

Using Lemma 2.2.1, Lemma 2.4.5, and the expressions (2.62) and (2.63), we obtain that for $0 < \delta_0 < \rho_0$

$$\begin{aligned} \|D^2 H[T_1, T_1]\|_{4/3,3/2,\rho_0-\delta_0} \lesssim |I_{\mathrm{m}}|^{1/2} \Big(&\|\partial_u T_1\|^2_{4/3,3/2,\rho_0} + \|\partial_u T_1\|_{4/3,3/2,\rho_0} \|\partial_\beta T_1\|_{1/3,1/2,\rho_0-\delta_0} \\ &+ |I_{\mathrm{m}}|^{-3/2} \|\partial_\beta T_1\|^2_{4/3,3/2,\rho_0-\delta_0} \Big) \end{aligned}$$

and

$$\|(DH(T_1) - \mathcal{L})[T_1]\|_{4/,3/2,\rho_0-\delta_0} \lesssim |I_{\mathrm{m}}|^{-1} |\delta_\beta^{\mathrm{u}}| \left(\|\partial_u T_1\|_{1/3,1/2,\rho_0-\delta_0} + \|\partial_\beta T_1\|_{1/3,1/2,\rho_0-\delta_0} \right).$$

Then, it follows from the estimate (2.70) for T_1, the hypothesis $|\delta_\beta^{\mathrm{u}}| \lesssim \epsilon |I_{\mathrm{m}}|^{-4}$ and the third and fourth items in Lemma 2.4.5, that

$$\|H(T_1)\|_{4/3,3/2,\rho_0-\delta_0} \lesssim |I_{\mathrm{m}}|^{1/2} \varepsilon_0^2 \left(1 + \delta_0^{-1} + I_{\mathrm{m}}^{-3/2} \delta_0^{-2}\right) \lesssim |I_{\mathrm{m}}|^{1/2} \varepsilon_0^2 \delta_0^{-2},$$

where ε_0 was defined in (2.69). We now take

$$\delta_0 \equiv \varepsilon_0^{1/8} \qquad \text{and define} \qquad \rho_1 = \rho_0 - \delta_0.$$

Therefore,

$$\|H(T_1)\|_{4/3,3/2,\rho_1} \lesssim |I_{\mathrm{m}}|^{1/2} \varepsilon_0^{1/4} \varepsilon_0^{3/2} \leq \varepsilon_0^{3/2} \equiv \varepsilon_1.$$

The iterative argument

Throughout this section, the symbol $a \lesssim b$ means that there exists $C > 0$ which does not depend on the step n and I_{m} such that $a \leq Cb$.

Through the Newton iteration scheme, at the step $n+1$, we have to solve the linearized equation

$$H(T_n) + DH(T_n)[\Delta T_{n+1}] = 0 \qquad\qquad \Delta T_{n+1} = T_{n+1} - T_n.$$

For that, we have to invert the linear operator $DH(T_n)[\cdot]$ defined in (2.62). Since this operator has a non zero coefficient multiplying ∂_β we must find a change of variables

$$(u, \beta, t) = \Psi_{n+1}(u, \varphi, t) \equiv (u, \varphi + \psi_{n+1}(u, \varphi, t), t), \tag{2.71}$$

in which the linearized operator does not involve partial derivatives with respect to the new angle φ. Define

$$A_n = \frac{1}{2 y_{\mathrm{h}}^2 I_{\mathrm{m}}} \left(\partial_u T_n - r_{\mathrm{h}}^{-2} (\delta_\beta^{\mathrm{u}} + \partial_\beta T_n) \right) \qquad\qquad B_n = -\frac{1}{2 y_{\mathrm{h}}^2 r_{\mathrm{h}}^2 I_{\mathrm{m}}} \left(\partial_u T_n - r_{\mathrm{h}}^{-1} (\delta_\beta^{\mathrm{u}} + \partial_\beta T_n) \right). \tag{2.72}$$

Then, one can check that, if one considers a change of variables (2.71) with ψ_{n+1} solving (here $\mathcal{L}$ is the operator (2.64))

$$\mathcal{L}\psi_{n+1} = B_n \circ \Psi_{n+1} - (A_n \circ \Psi_{n+1})\partial_v \psi_{n+1},$$

the following equations determining an unknown function h are equivalent

$$DH(T_n)[h] + H(T_n) = 0 \qquad \Longleftrightarrow \qquad (1 + A_n \circ \Psi_{n+1})\partial_u(h \circ \Psi_{n+1}) + I_{\mathrm{m}}^3 \partial_t (h \circ \Psi_{n+1}) + H(T_n) \circ \Psi_{n+1} = 0. \tag{2.73}$$

Indeed, if we denote by $\Delta\widetilde{T}_{n+1}$ the solution of the second equation in (2.73), then

$$\Delta T_{n+1} = \Delta\widetilde{T}_{n+1} \circ \Psi_{n+1}^{-1}$$

solves the first equation in (2.73). As a consequence, we define

$$T_{n+1} = T_n + \Delta\widetilde{T}_{n+1} \circ \Psi_{n+1}^{-1}$$

and

$$\widetilde{T}_{n+1} = T_{n+1} \circ \Psi_{n+1} = \widetilde{T}_n \circ \Psi_n^{-1} \circ \Psi_{n+1} + \Delta\tilde{T}_{n+1}.$$

In order to state the inductive hypothesis, we define now the constants

$$\varepsilon_n = \varepsilon_{n-1}^{3/2} \qquad \delta_n = \varepsilon_n^{1/8} \qquad \rho_n = \rho_{n-1} - 2\delta_{n-1}. \tag{2.74}$$

Notice that it follows (taking ε_0 small enough) from this definition that $\rho_n \geq \rho_0/2 \quad \forall n \in \mathbb{N} \cup \{\infty\}$. Suppose that:

- (H1) There exists a family of functions $\{\widetilde{T}_i\}_{1\leq i\leq n} \subset \mathcal{X}_{1/3,1/2,\rho_{i-1}}$ and a family of close to identity maps $\Psi_1 = \mathrm{Id}$ and $\{\Psi_i\}_{2\leq i\leq n}$ with $\Psi_i = \mathrm{Id} + \psi_i$, $\psi_i \in \mathcal{X}_{2/3,1/2,\rho_{i-1}}$, such that

  $$\mathcal{L}\psi_{i+1} = B_i \circ \Psi_{i+1} - (A_i \circ \Psi_{i+1})\partial_u \psi_{i+1},$$

  where now A_i, B_i are written in terms of $\tilde{T}_i$

  $$\begin{aligned} A_i =& \frac{1}{2y_{\mathrm{h}}^2 I_{\mathrm{m}}} \left(\partial_u(\widetilde{T}_i \circ \Psi_i^{-1}) - r_{\mathrm{h}}^{-2}(\delta_\beta^{\mathrm{u}} + \partial_\beta(\widetilde{T}_i \circ \Psi_i^{-1}))\right) \\ B_i =& -\frac{1}{2y_{\mathrm{h}}^2 r_{\mathrm{h}}^2 I_{\mathrm{m}}} \left(\partial_u(\widetilde{T}_i \circ \Psi_i^{-1}) - r_{\mathrm{h}}^{-1}(\delta_\beta^{\mathrm{u}} + \partial_\beta(\widetilde{T}_i \circ \Psi_i^{-1}))\right). \end{aligned}$$

- (H2) The functions ψ_i satisfy (see Remark 2.4.11 below)

  $$[\![\psi_{i+1} - \psi_i]\!]_{2/3,1/2,\rho_i} \lesssim |I_{\mathrm{m}}|^{-1}\delta_{i-1}^7.$$

- (H3) The functions $\widetilde{T}_i$ satisfy

  $$[\![\widetilde{T}_{i+1} - \widetilde{T}_i \circ \Psi_i^{-1} \circ \Psi_{i+1}]\!]_{1/3,1/2,\rho_i} \lesssim \varepsilon_i,$$

  and

  $$\|H(\widetilde{T}_{i+1} \circ \Psi_{i+1}^{-1}) \circ \Psi_{i+1}\|_{4/3,3/2,\rho_{i+1}} \lesssim \varepsilon_{i+1}.$$

Remark 2.4.11. *Hypothesis (H2) can be rephrased as*

$$[\![\Psi_i^{-1} \circ \Psi_{i+1} - \mathrm{Id}]\!]_{2/3,1/2,\rho_i} \lesssim |I_{\mathrm{m}}|^{-1}\delta_{i-1}^7.$$

We claim that, under these hypotheses, there exists a map $\Psi_{n+1} = \mathrm{Id} + \psi_{n+1}$, $\psi_{n+1} \in \mathcal{X}_{2/3,1/2,\rho_n}$ solving

$$\mathcal{L}\psi_{n+1} = B_n \circ \Psi_{n+1} - (A_n \circ \Psi_{n+1})\partial_u \psi_{n+1}$$

with

$$[\![\psi_{n+1}-\psi_n]\!]_{2/3,1/2,\rho_n} \lesssim |I_{\mathrm{m}}|^{-1}\delta_{n-1}^7$$

and $\widetilde{T}_{n+1} \in \mathcal{X}_{1/3,1/2,\rho_n}$ such that

$$[\![\widetilde{T}_{n+1}-\widetilde{T}_n\circ\Psi_n^{-1}\circ\Psi_{n+1}]\!]_{1/3,1/2,\rho_n} \lesssim \varepsilon_n$$

for which

$$\|H(\widetilde{T}_{n+1}\circ\Psi_{n+1}^{-1})\circ\Psi_{n+1}\|_{4/3,3/2,\rho_{n+1}} \lesssim \varepsilon_{n+1}.$$

The first step towards the proof of the inductive claim is to look for the change of variables Ψ_{n+1}.

Lemma 2.4.12. *Assume that (H1), (H2) and (H3) hold for all $1\le i\le n$. Then, there exists $\Psi_{n+1}=\mathrm{Id}+\psi_{n+1}$, $\psi_{n+1}\in\mathcal{X}_{2/3,1/2,\rho_n}$ such that*

$$\mathcal{L}\psi_{n+1} = B_n\circ\Psi_{n+1}-(A_n\circ\Psi_{n+1})\partial_v\psi_{n+1}$$

with $[\![\Psi_{n+1}-\Psi_n]\!]_{2/3,1/2,\rho_n}\lesssim |I_{\mathrm{m}}|^{-1}\delta_{n-1}^7$.

Proof. Throughout the proof we will use the first part of Lemma 2.4.6, which deals with compositions in the angular variable, without mentioning. We also define $\tilde{\rho}_n=\rho_{n-1}-\delta_{n-1}$ to avoid lengthy notation. Since $\mathcal{L}$ is linear, we can write

$$\begin{aligned}\mathcal{L}(\psi_{n+1}-\psi_n) =& B_n\circ\Psi_{n+1}-B_{n-1}\circ\Psi_n-(A_n\circ\Psi_{n+1})\partial_u\psi_{n+1}+(A_{n-1}\circ\Psi_n)\partial_u\psi_n\\ =& B_n\circ\Psi_{n+1}-B_n\circ\Psi_n+(B_n-B_{n-1})\circ\Psi_n-((A_n-A_{n-1})\circ\Psi_{n+1})\partial_u\psi_{n+1}\\ &-(A_{n-1}\circ\Psi_{n+1}-A_{n-1}\circ\Psi_n)\partial_u\psi_{n+1}-(A_{n-1}\circ\Psi_n)\partial_u(\psi_{n+1}-\psi_n)\end{aligned}$$

which, by the mean value theorem, can be rewritten as the fixed point equation

$$\Delta\psi_{n+1}=\mathcal{G}(F(\Delta\psi_{n+1}))$$

in a Banach space $\mathcal{X}_{\eta,\nu,\rho}$ for suitable η,ν,ρ to be chosen, where $\Delta\psi_{n+1}=\psi_{n+1}-\psi_n$, $\mathcal{G}$ is the operator introduced in Lemma 2.4.9 and

$$\begin{aligned}F(\Delta\psi_{n+1}) =& \Delta\psi_{n+1}\int_0^1\partial_\beta B_n\circ(\mathrm{Id}+s\Delta\psi_{n+1})\,\mathrm{d}s+(B_n-B_{n-1})\circ\Psi_n\\ &-((A_n-A_{n-1})\circ(\Psi_n+\Delta\psi_{n+1}))\partial_u(\psi_n+\Delta\psi_{n+1})\\ &-\Delta\psi_{n+1}\partial_u(\psi_n+\Delta\psi_{n+1})\int_0^1\partial_\beta A_n\circ(\mathrm{Id}+s\Delta\psi_{n+1})\,\mathrm{d}s-(A_{n-1}\circ\Psi_n)\partial_u\Delta\psi_{n+1}.\end{aligned}$$

We obtain $\Delta\psi_{n+1}$ by an standard application of the fixed point theorem for Banach spaces. To that end we first bound the term

$$F(0)=(B_n-B_{n-1})\circ\Psi_n-(A_n-A_{n-1})\circ\Psi_n\partial_u\psi_n.$$

We observe that

$$\begin{aligned}\partial_u\left(\widetilde{T}_n\circ\Psi_n^{-1}-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\right) =& \partial_u\left(\widetilde{T}_n-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\circ\Psi_n\right)\circ\Psi_n^{-1}\\ &+\partial_\varphi\left(\widetilde{T}_n-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\circ\Psi_n\right)\circ\Psi_n^{-1}\quad\partial_u\Psi_n^{-1}\end{aligned}$$

and

$$\partial_\beta\left(\widetilde{T}_n\circ\Psi_n^{-1}-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\right) = \partial_\varphi\left(\widetilde{T}_n-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\circ\Psi_n\right)\circ\Psi_n^{-1}\quad\partial_\beta\Psi_n^{-1}.$$

Therefore, taking into account that (for the case $n=1$ notice that $\psi_1=0$)

$$[\![\psi_n]\!]_{2/3,1/2,\rho_{n-1}}\le[\![\psi_2]\!]_{2/3,1/2,\rho_{n-1}}+\sum_{i=3}^n[\![\psi_i-\psi_{i-1}]\!]_{2/3,1/2,\rho_{n-1}}\lesssim[\![\psi_2]\!]_{2/3,1/2,\rho_{n-1}}\lesssim|I_{\mathrm{m}}|^{-1}\delta_0^7,$$

it is easy to show that the inductive hypothesis implies

$$\begin{aligned}\left\|y_{\mathrm{h}}^{-2}r_{\mathrm{h}}^{-2}I_{\mathrm{m}}^{-1}\partial_u\left(\widetilde{T}_n\circ\Psi_n^{-1}-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\right)\circ\Psi_n\right\|_{5/3,3/2,\tilde{\rho}_n}\lesssim&|I_{\mathrm{m}}|^{-1}[\![\widetilde{T}_n-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\circ\Psi_n]\!]_{1/3,1/2,\rho_{n-1}}\\ \lesssim&|I_{\mathrm{m}}|^{-1}\varepsilon_{n-1}\end{aligned}$$

and

$$\begin{aligned}\left\|y_{\mathrm{h}}^{-2}r_{\mathrm{h}}^{-3}I_{\mathrm{m}}^{-1}\partial_\beta\left(\widetilde{T}_n\circ\Psi_n^{-1}-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\right)\circ\Psi_n\right\|_{5/3,3/2,\tilde{\rho}_n}\lesssim&|I_{\mathrm{m}}|^{-1}\delta_{n-1}^{-1}\\ &\times[\![\widetilde{T}_n-\widetilde{T}_{n-1}\circ\Psi_{n-1}^{-1}\circ\Psi_n]\!]_{1/3,1/2,\rho_{n-1}}\\ \lesssim&|I_{\mathrm{m}}|^{-1}\varepsilon_{n-1}\delta_{n-1}^{-1}.\end{aligned}$$

Thus, from the definition of B_n in (2.72),

$$\|(B_n-B_{n-1})\circ\Psi_n\|_{5/3,3/2,\tilde{\rho}_n}\lesssim|I_{\mathrm{m}}|^{-1}\varepsilon_{n-1}\delta_{n-1}^{-1}=|I_{\mathrm{m}}|^{-1}\delta_{n-1}^{7}.$$

Taking into account that $\delta_n\le\delta_0=|I_{\mathrm{m}}|^{-3/8}$ and the definition of A_n in (2.72), a similar computation shows that

$$\begin{aligned}\|(A_n-A_{n-1})\circ\Psi_n\partial_u\psi_n\|_{5/3,3/2,\tilde{\rho}_n}\lesssim&\|(A_n-A_{n-1})\circ\Psi_n\|_{2/3,1/2,\tilde{\rho}_n}\|\partial_u\psi_n\|_{1,1,\rho_n}\\ \lesssim&|I_{\mathrm{m}}|^{3/2}\|(A_n-A_{n-1})\circ\Psi_n\|_{2/3,1/2,\tilde{\rho}_n}[\![\psi_n]\!]_{2/3,1/2,\rho_n}\\ \lesssim&|I_{\mathrm{m}}|^{1/2}\varepsilon_{n-1}\delta_{n-1}^{-1}[\![\psi_n]\!]_{2/3,1/2,\rho_n}\lesssim|I_{\mathrm{m}}|^{-17/8}\delta_{n-1}^{7}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|F(0)\|_{5/3,3/2,\tilde{\rho}_n}=&\|(B_n-B_{n-1})\circ\Psi_n-(A_n-A_{n-1})\circ\Psi_n\partial_u\psi_n\|_{5/3,3/2,\rho_n}\\ \lesssim&|I_{\mathrm{m}}|^{-1}\varepsilon_{n-1}\delta_{n-1}^{-1}=|I_{\mathrm{m}}|^{-1}\delta_{n-1}^{7},\end{aligned}$$

and it follows from Lemma 2.4.9 that

$$\|\mathcal{G}(F(0))\|_{2/3,1/2,\tilde{\rho}_n}\lesssim|I_{\mathrm{m}}|^{-1}\delta_{n-1}^{7}.$$

We notice that, since A_n,B_n depend linearly on T_n,

$$\begin{aligned}\|\partial_\beta A_n\|_{2/3,1/2,\tilde{\rho}_n-\delta_{n-1}}\le&\|\partial_\beta A_1\|_{2/3,1/2,\tilde{\rho}_1-\delta_0}+\sum_{i=2}^{n}\|\partial_\beta(A_i-A_{i-1})\|_{2/3,1/2,\tilde{\rho}_i-\delta_{i-1}}\\ \le&\delta_0^{-1}\|\partial_\beta A_1\|_{2/3,1/2,\tilde{\rho}_1}+\sum_{i=2}^{n}\delta_i^{-1}\|A_i-A_{i-1}\|_{2/3,1/2,\tilde{\rho}_i}\\ \lesssim&|I_{\mathrm{m}}|^{-1}\left(\varepsilon_0\delta_0^{-1}+\sum_{i=1}^{n-1}\varepsilon_i\delta_i^{-1}\right)\lesssim|I_{\mathrm{m}}|^{-1}\varepsilon_0\delta_0^{-1}=|I_{\mathrm{m}}|^{-29/8},\end{aligned}$$

and the same computation shows that

$$\|\partial_\beta B_n\|_{5/3,3/2,\tilde{\rho}_n-\delta_{n-1}}\lesssim|I_{\mathrm{m}}|^{-1}\varepsilon_0\delta_0^{-1}=|I_{\mathrm{m}}|^{-29/8}.$$

Take now any $\Delta\psi,\Delta\psi^*\in B(|I_{\mathrm{m}}|^{-1}\delta_{n-1}^{7})\subset\mathcal{X}_{2/3,1/2,\rho_n}$. From the fundamental theorem of calculus, it follows that

$$\begin{aligned}F(\Delta\psi^*)-F(\Delta\psi)=&(\Delta\psi^*-\Delta\psi)\int_0^1\partial_\beta B_n\circ(\mathrm{Id}+s(\Delta\psi^*-\Delta\psi))\,\mathrm{d}s-A_n\circ(\Psi_n+\Delta\psi)\partial_u(\Delta\psi^*-\Delta\psi)\\ &-(\Delta\psi^*-\Delta\psi)\partial_u(\psi_n+\Delta\psi^*)\int_0^1\partial_\beta A_n\circ(\mathrm{Id}+s(\Delta\psi^*-\Delta\psi))\,.\end{aligned}$$

Using the previous estimates, Lemma 2.4.5 and the second part of Lemma 2.4.6, we obtain that (recall that $\rho_n = \rho_{n-1} - 2\delta_{n-1} = \tilde{\rho}_n - \delta_{n-1}$)

$$\begin{aligned}\|(\Delta\psi^* - \Delta\psi)\int_0^1 \partial_\beta B_n \circ (\mathrm{Id} + s(\Delta\psi^* - \Delta\psi))\,\mathrm{d}s\|_{5/3,3/2,\rho_n} \lesssim & |I_{\mathrm{m}}|^{3/2}\|\partial_\beta B_n\|_{5/3,3/2,\rho_n}\|\Delta\psi - \Delta\psi^*\|_{2/3,1/2,\rho_n}\\ \lesssim & |I_{\mathrm{m}}|^{-17/8}\|\Delta\psi - \Delta\psi^*\|_{2/3,1/2,\rho_n}.\end{aligned}$$

Similar computations show that

$$\|F(\psi) - F(\psi^*)\|_{5/3,3/2,\rho_n} \lesssim |I_{\mathrm{m}}|^{-17/8}\|\psi - \psi^*\|_{2/3,1/2,\rho_n}.$$

Finally, from Lemma 2.4.9,

$$\begin{aligned}\|\mathcal{G}\left(F(\psi) - F(\psi^*)\right)\|_{2/3,1/2,\rho_n} \lesssim & \|F(\psi) - F(\psi^*)\|_{5/3,3/2,\rho_n}\\ \lesssim & |I_{\mathrm{m}}|^{-17/8}\|\psi - \psi^*\|_{2/3,1/2,\rho_n}.\end{aligned}$$

Then, the proof of the lemma follows from direct application of the fixed point theorem in the ball of radius $C|I_{\mathrm{m}}|^{-1}\delta_{n-1}^7$ (for some large enough C) centered at the origin of the Banach space $\mathcal{X}_{2/3,1/2,\rho_n}$. □

We now complete the proof of the inductive claim for $\tilde{T}_{n+1}$.

Proposition 2.4.13. *The equation*

$$(1 + A_n \circ \Psi_{n+1})\partial_u(\Delta\widetilde{T}_{n+1}) + I_{\mathrm{m}}^3\partial_t(\Delta\widetilde{T}_{n+1}) + H(T_n) \circ \Psi_{n+1} = 0 \tag{2.75}$$

admits a unique solution $\Delta\widetilde{T}_{n+1} \in \mathcal{X}_{1/3,1/2,\rho_n}$ *such that*

$$\|\Delta\widetilde{T}_{n+1}\|_{1/3,1/2,\rho_n} \lesssim \varepsilon_n.$$

Moreover, the function

$$\widetilde{T}_{n+1} = T_n \circ \Psi_n^{-1} \circ \Psi_{n+1} + \Delta\widetilde{T}_{n+1}$$

satisfies

$$\|H(\widetilde{T}_{n+1} \circ \Psi_{n+1}^{-1}) \circ \Psi_{n+1}\|_{1/3,1/2,\rho_{n+1}} \lesssim \varepsilon_{n+1}.$$

Proof. Again, throughout the proof we will use the first part of Lemma 2.4.6, which deals with compositions in the angular variable, without mentioning. We rewrite (2.75) as the affine fixed point equation for $\Delta\widetilde{T}_{n+1}$

$$\Delta\widetilde{T}_{n+1} = -\mathcal{G}\left(H(\widetilde{T}_n \circ \Psi_n^{-1}) \circ \Psi_{n+1} - (A_n \circ \Psi_{n+1})\partial_u(\Delta\widetilde{T}_{n+1})\right)$$

where $\mathcal{G}$ is the operator introduced in Lemma 2.4.9. The existence of a fixed point $\Delta\widetilde{T}_{n+1} \in \mathcal{X}_{1/3,1/2,\rho_n}$ with

$$[\![\Delta\widetilde{T}_{n+1}]\!]_{1/3,1/2,\rho_n} \lesssim \varepsilon_n$$

is easily completed using the properties of $\mathcal{G}$ in Lemma 2.4.9 and the estimates

$$\|(H(\widetilde{T}_n \circ \Psi_n^{-1})) \circ \Psi_{n+1}\|_{4/3,3/2,\rho_n} \lesssim \varepsilon_n \qquad \|A_n \circ \Psi_{n+1}\|_{0,0,\rho_n} \lesssim I_{\mathrm{m}}^{3/2}\|A_n \circ \Psi_{n+1}\|_{2/3,1/2,\rho_n} \lesssim |I_{\mathrm{m}}|^{-17/8},$$

which are obtained from the inductive hypothesis after writing

$$(H(\widetilde{T}_n \circ \Psi_n^{-1})) \circ \Psi_{n+1} = (H(\widetilde{T}_n \circ \Psi_n^{-1})) \circ \Psi_n + \left((H(\widetilde{T}_n \circ \Psi_n^{-1})) \circ \Psi_{n+1} - (H(\widetilde{T}_n \circ \Psi_n^{-1})) \circ \Psi_n\right),$$

and using the estimate for $[\![\psi_{n+1} - \psi_n]\!]_{1/3,1/2,\rho_n}$ given in Lemma 2.4.12. In order to prove the estimate for the error, it follows from our construction that

$$H(\widetilde{T}_{n+1} \circ \Psi_{n+1}^{-1}) = D^2H\left[\Delta\widetilde{T}_{n+1} \circ \Psi_{n+1}^{-1}, \Delta\widetilde{T}_{n+1} \circ \Psi_{n+1}^{-1}\right].$$

The proof is completed in a straightforward manner from expression (2.63), the estimate for $[\![\Delta\tilde{T}_{n+1}]\!]_{1/3,1/2,\rho_n}$ and the estimate for $[\![\psi_{n+1} - \psi_n]\!]_{1/3,1/2,\rho_n}$ given in Lemma 2.4.12. □

We can now conclude the proof of Theorem 2.4.8.

Proof of Theorem 2.4.8. Notice that the function T_1 obtained in (2.68) and the map $\Psi_1 = \mathrm{Id}$ satisfy the inductive hypothesis assumed at the beginning of Section 2.4.2. Therefore, for all $n \in \mathbb{N}$, we can find maps $\Psi_n = \mathrm{Id} + \psi_n$ with $\psi_n \in \mathcal{X}_{1/3,1/2,\rho_{n-1}}$ satisfying

$$[\![\psi_{n+1} - \psi_n]\!]_{1/3,1/2,\rho_n} \lesssim |I_{\mathrm{m}}|^{-1}\delta_{n-1}^7$$

and functions $\widetilde{T}_n \in \mathcal{X}_{1/3,1/2,\rho_{n-1}}$ such that

$$\|\widetilde{T}_{n+1} - \widetilde{T}_n \circ \Psi_n^{-1} \circ \Psi_{n+1}\|_{1/3,1/2,\rho_n} \lesssim \varepsilon_n \qquad \left\|(H(\widetilde{T}_n \circ \Psi_n^{-1})) \circ \Psi_n\right\|_{4/3,3/2,\rho_n} \lesssim \varepsilon_{n+1} = \varepsilon_0^{(3/2)^{n+1}}.$$

Then, Ψ_n converges uniformly on $\mathcal{X}_{1/3,1/2,\rho_0/4}$ to an analytic change of coordinates

$$\Psi_\infty = \mathrm{Id} + \psi_\infty \qquad [\![\psi_\infty]\!]_{1/3,1/2,\rho_n} \lesssim |I_{\mathrm{m}}|^{-1}\delta_0^7 = |I_{\mathrm{m}}|^{-29/8} \tag{2.76}$$

and the sequence $\{T_n\}_{n\in\mathbb{N}}$ defined by

$$T_n = \widetilde{T}_n \circ \Psi_n^{-1}$$

converges uniformly to an analytic function $T \in \mathcal{X}_{1/3,1/2,\rho_0/4}$ such that

$$\|T\|_{1/3,1/2,\rho_0/4} \lesssim \|T_1\|_{1/3,1/2,\rho/4} + \sum_{n=1}^{\infty} \|T_{n+1} - T_n\|_{1/3,1/2,\rho/4} \lesssim \varepsilon_0$$

and

$$\|H(T)\|_{4/3,3/2,\rho_0/4} = \lim_{n\to\infty} \|H(T_n)\|_{4/3,3/2,\rho_0/4} = 0.$$

This proves the existence of a solution $T \in \mathcal{X}_{1/3,1/2,\rho_0/4}$ to the Hamilton-Jacobi equation (2.55). Moreover, recalling the definition of the half Melnikov potential L^{u} in (2.60), we have

$$T^{\mathrm{u}} - L^{\mathrm{u}} = \mathcal{G}\left(\frac{1}{2y_{\mathrm{h}}^2 I_{\mathrm{m}}}(\partial_u T^{\mathrm{u}} - r_{\mathrm{h}}^{-2}(\delta_\beta^{\mathrm{u}} + \partial_\beta T^{\mathrm{u}}))^2 + \frac{1}{2y_{\mathrm{h}}^2 r_{\mathrm{h}}^2 I_{\mathrm{m}}}(\delta_\beta^{\mathrm{u}} + \partial_\beta T^{\mathrm{u}})^2\right)$$

Therefore, using that $|\delta_\beta^{\mathrm{u}}| \lesssim \epsilon |I_{\mathrm{m}}|^{-4}$ and $\|T^{\mathrm{u}}\|_{1/3,1/2,\rho_0/4}$, one easily obtains that

$$\|T^{\mathrm{u}} - L^{\mathrm{u}}\|_{1/3,1,\rho_0/8} \lesssim |I_{\mathrm{m}}|^{-7}.$$

We set $\rho = \rho_0/8$.

Now we prove the estimate for the difference $T^{\mathrm{u}} - T^{\mathrm{u}}_{\mathrm{circ}}$. The function $T^{\mathrm{u}}_{\mathrm{circ}}$ satisfies (compare (2.55))

$$(1 + A^{\mathrm{u}}_{\mathrm{circ}})\partial_u T^{\mathrm{u}}_{\mathrm{circ}} + B^{\mathrm{u}}_{\mathrm{circ}}\partial_\beta T^{\mathrm{u}}_{\mathrm{circ}} + I_{\mathrm{m}}^3 \partial_t T^{\mathrm{u}}_{\mathrm{circ}} - V_{\mathrm{circ}} = 0,$$

with

$$A^{\mathrm{u}}_{\mathrm{circ}} = \frac{1}{2y_{\mathrm{h}}^2 I_{\mathrm{m}}}\left(\partial_u T^{\mathrm{u}}_{\mathrm{circ}} - r_{\mathrm{h}}^{-2}\partial_\beta T^{\mathrm{u}}_{\mathrm{circ}}\right) \qquad B^{\mathrm{u}}_{\mathrm{circ}} = -\frac{1}{2y_{\mathrm{h}}^2 r_{\mathrm{h}}^2 I_{\mathrm{m}}}\left(\partial_u T^{\mathrm{u}}_{\mathrm{circ}} - r_{\mathrm{h}}^{-1}\partial_\beta T^{\mathrm{u}}_{\mathrm{circ}}\right). \tag{2.77}$$

Using that (see Lemma 2.4.7) $\|V - V_{\mathrm{circ}}\|_{2,3/2,\rho_0} \lesssim \epsilon |I_{\mathrm{m}}|^{-3}$ and, by hypothesis, $|\delta_\beta^{\mathrm{u}}| \lesssim \epsilon |I_{\mathrm{m}}|^{-4}$, one easily obtains that

$$\|T^{\mathrm{u}} - T^{\mathrm{u}}_{\mathrm{circ}}\|_{1/3,1/2,\rho} \lesssim \epsilon |I_{\mathrm{m}}|^{-3}.$$

This estimate implies that

$$\|A^{\mathrm{u}} - A^{\mathrm{u}}_{\mathrm{circ}}\|_{2/3,1/2,\rho} \lesssim \epsilon |I_{\mathrm{m}}|^{-4} \qquad \|B^{\mathrm{u}} - B^{\mathrm{u}}_{\mathrm{circ}}\|_{2,3/2,\rho} \lesssim \epsilon |I_{\mathrm{m}}|^{-4},$$

and we obtain that

$$\|T^{\mathrm{u}} - T^{\mathrm{u}}_{\mathrm{circ}} - (L^{\mathrm{u}} - L^{\mathrm{u}}_{\mathrm{circ}})\|_{1/3,1,\rho} \lesssim \epsilon |I_{\mathrm{m}}|^{-7},$$

as was to be shown. ∎

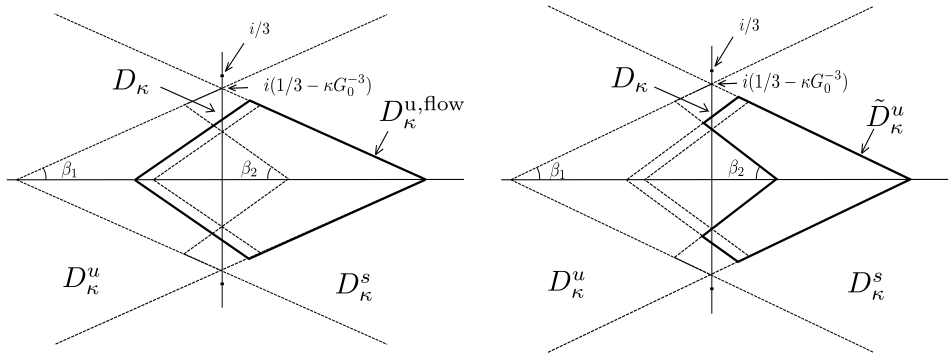

Figure 2.6: The domains $D_{\kappa_1}^{\mathrm{u,flow}}$ and $\tilde{D}_{\kappa_2}^{\mathrm{u}}$ defined in (2.81) and (2.82).

2.4.3 Extension of the parametrization by the flow

Theorem 2.4.8 provides the existence of a Lagrangian graph parametrization $\mathcal{W}^{\mathrm{u}}$ of the form (2.19) of the unstable manifold of the invariant torus $\mathcal{T}_{I^{\mathrm{u}}}$, on the domain $(u,\beta,t) \in D_\kappa^{\mathrm{u}} \times \mathbb{T}_\rho \times \mathbb{T}_\sigma$. As already discussed in Remark 2.3.7 (see also Section 2.4.4 below), to study the difference between $W_{I^{\mathrm{u}}}^{\mathrm{u}}$ and $W_{I^{\mathrm{s}}}^{\mathrm{s}}$ we need to extend their parametrizations to a common domain containing a subset of the real line. This is (at least in a direct manner) not possible using the parametrizations (2.19) since $y_{\mathrm{h}}(0) = 0$.

We sketch the simple solution to this technical issue. The details can be found in [GMS16]: since in polar coordinates (r,α,t,y,G,E) the vector field associated to the Hamiltonian (2.3) is not singular (except at $r = 0$), we look for a different parametrization $\mathtt{W}^{\mathrm{u}}(\tilde{u},\tilde{\beta},t)$ of the unstable manifold in polar coordinates

$$\begin{aligned}\mathtt{W}^{u}(\tilde{u},\tilde{\beta},t) = \big\{(r,\alpha,t,y,G,E) =& (I_{\mathrm{m}}^2 r_{\mathrm{h}}(\tilde{u}) + R_{\mathrm{flow}}(\tilde{u},\tilde{\beta},t), \tilde{\beta} + \alpha_{\mathrm{h}}(\tilde{u}) + \Omega_{\mathrm{flow}}(\tilde{u},\tilde{\beta},t), t, I_{\mathrm{m}}^{-1} \\ & y_{\mathrm{h}}(\tilde{u}) + Y_{\mathrm{flow}}(\tilde{u},\tilde{\beta},t, I_{\mathrm{m}} + J_{\mathrm{flow}}(\tilde{u},\tilde{\beta},t), E_{\mathrm{flow}}(\tilde{u},\tilde{\beta},t))\big\}\end{aligned} \tag{2.78}$$

such that

$$\phi^s(\mathtt{W}^{\mathrm{u}}(\tilde{u},\tilde{\beta},t)) = \mathtt{W}^{\mathrm{u}}(\tilde{u}+s,\tilde{\beta},t+I_{\mathrm{m}}^3 s) \tag{2.79}$$

where $\phi_{\mathrm{pol}}^{\mathrm{s}}$ is the time s flow generated by the Hamiltonian (2.3). Notice that this extension is a rather standard procedure since we will consider domains which are at distance order ~ 1 from the singularities $u = \pm i/3$.

Let $\eta_{I_{\mathrm{m}}}$ be the change of coordinates defined in (2.14) and let $\mathcal{W}^{\mathrm{u}}$ be the Lagrangian graph parametrization associated to the generating function T^{u} obtained in Theorem 2.4.8. The first step is to perform a change of variables h of the form

$$(u,\beta,t) = h(\tilde{u},\tilde{\beta},t) = (\tilde{u} + h_u(\tilde{u},\tilde{\beta},t), \tilde{\beta} + h_\beta(\tilde{u},\tilde{\beta},t), t) \tag{2.80}$$

such that the parametrization $\eta_{I_{\mathrm{m}}} \circ \mathcal{W}^{\mathrm{u}} \circ h$ is of the form (2.78) and satisfies (2.79). This is the content of Lemma 2.4.14 below. Second, we use the flow $\phi_{\mathrm{pol}}^{\mathrm{s}}$ to extend this parametrization to a domain $(\tilde{u},\tilde{\beta},t) \in D_{\kappa_1}^{u,\mathrm{flow}} \times \mathbb{T}_{\rho_1} \times \mathbb{T}_{\sigma_1}$ (for suitable $\kappa_1 > \kappa, \rho_1 < \rho, \sigma_1 < \sigma$) where

$$D_{\kappa_1}^{\mathrm{u,flow}} = \{\tilde{u} \in \mathbb{C} \colon |\mathrm{Im}\tilde{u}| \le -\tan\beta_1 \mathrm{Re} u + 1/3 - \kappa_1 |I_{\mathrm{m}}|^{-3},\ |\mathrm{Im}\tilde{u}| \le \tan\beta_2 \mathrm{Re}\tilde{u} + 1/6 + \kappa_1 |I_{\mathrm{m}}|^{-3}\}. \tag{2.81}$$

This domain contains $\tilde{u} = 0$, is at distance $\sim \mathcal{O}(1)$ from $u = \pm i/3$, and satisfies $D_{\kappa_1}^{\mathrm{u,flow}} \cap D_\kappa^{\mathrm{u}} \neq \emptyset$, $D_{\kappa_1}^{\mathrm{u,flow}} \cap D^{\mathrm{s}} \cap \mathbb{R} \neq \emptyset$ (see Figure 2.4.3).

Lemma 2.4.14. *Let $\|\cdot\|_{0,0,\rho}$ be as in* (2.58) *but referred to the domain $D_{\kappa_1}^{\mathrm{u,flow}} \cap D_\kappa^{\mathrm{u}}$. Then, on the overlapping domain $(D_{\kappa_1}^{\mathrm{u,flow}} \cap D_\kappa^{\mathrm{u}}) \times \mathbb{T}_{\rho_1} \times \mathbb{T}_{\sigma_2}$, there exists an analytic change of coordinates h of the form* (2.80) *such that*

$$\|h_u\|_{0,0,\rho} \lesssim |I_{\mathrm{m}}|^{-4} \qquad \|h_\beta\|_{0,0,\rho} \lesssim |I_{\mathrm{m}}|^{-5/2},$$

and for which the parametrization $\eta_{I_{\mathrm{m}}} \circ \mathcal{W}^{\mathrm{u}} \circ h : (D_{\kappa_1}^{\mathrm{u,flow}} \cap D_\kappa^{\mathrm{u}}) \times \mathbb{T}_{\rho_1} \times \mathbb{T}_{\sigma_2} \to \mathbb{C}^6$ is of the form (2.78) *and satisfies* (2.79).

The proof of this lemma follows the same lines as the proof of Theorem 5.16 in [GMS16]. As commented above, we now extend the parametrization obtained in Lemma 2.4.14 to the domain $D_{\kappa_1}^{u,\mathrm{flow}}$. Notice that this parametrization will be well defined at $\tilde{u}=0$ since the vector field associated to the Hamiltonian (2.3) is not singular at $r=I_{\mathrm{m}}^2 r_{\mathrm{h}}(0)\neq 0$.

Lemma 2.4.15. *The parametrization $\eta_{I_{\mathrm{m}}}\circ\mathcal{W}^{\mathrm{u}}\circ h:(D_{\kappa_1}^{\mathrm{u,flow}}\cap D_{\kappa}^{\mathrm{u}})\times\mathbb{T}_{\rho_1}\times\mathbb{T}_{\sigma_2}\to\mathbb{C}^6$ obtained in Lemma 2.4.14 can be extended analitically to a parametrization $\mathtt{W}^{\mathrm{u}}:D_{\kappa_1}^{u,\mathrm{flow}}\times\mathbb{T}_{\rho_1}\times\mathbb{T}_{\sigma_1}\to\mathbb{C}^6$ of the form* (2.78) *which satisfies* (2.79) *and such that*

$$|I_{\mathrm{m}}|^2(\ln(|I_{\mathrm{m}}|)^{-1}\|R_{\mathrm{flow}}\|_{0,0,\rho_1},\ |I_{\mathrm{m}}|^{-1/2}\ \|\Omega_{\mathrm{flow}}\|_{0,0,\rho_1},\ |I_{\mathrm{m}}|^{-1}\ \|Y_{\mathrm{flow}}\|_{0,0,\rho_1},\ |I_{\mathrm{m}}|^{-3/2}\|J_{\mathrm{flow}}\|_{0,0,\rho_1}\lesssim|I_{\mathrm{m}}|^{-3},$$

where the norm $\|\cdot\|_{0,0,\rho}$ is as in (2.58) *but referred to the domain $D_{\kappa_1}^{\mathrm{u,flow}}$.*

The proof of this lemma follows the same lines as the proof of Proposition 5.20 in [GMS16]. Finally, we come back to the graph parametrization. To that end, for suitable $\kappa_2>\kappa_1,\rho_2<\rho_1,\sigma_2<\sigma_1$, we define the domain

$$\begin{aligned}\tilde{D}_{\kappa_2}^{\mathrm{u}}=\{u\in\mathbb{C}\colon &|\mathrm{Im}u|\leq-\tan\beta_1\mathrm{Re}u+1/3-\kappa_2|I_{\mathrm{m}}|^{-3},\ |\mathrm{Im}u|\leq\tan\beta_2\mathrm{Re}u+1/6-\kappa_2|I_{\mathrm{m}}|^{-3},\\ &|\mathrm{Im}u|\geq-\tan\beta_2\mathrm{Re}u+1/6+\kappa_2|I_{\mathrm{m}}|^{-3}\}\end{aligned}\tag{2.82}$$

which is at distance $\sim\mathcal{O}(1)$ from $u=0$ and verifies $\tilde{D}_{\kappa_2}^{\mathrm{u}}\subset D_{\kappa_1}^{\mathrm{u,flow}}$ (see Figure 2.4.3).

Lemma 2.4.16. *Let $\mathtt{W}^{\mathrm{u}}$ be the parametrization obtained in Lemma 2.4.15, which is of the form* (2.78)*. Let $\|\cdot\|_{0,0,\rho}$ be as in* (2.58) *but referred to the domain $\tilde{D}_{\kappa_2}^{\mathrm{u}}$. Then, there exists an analytic change of coordinates $g=(\tilde{u}+g_u(\tilde{u},\tilde{\beta},t),\tilde{\beta}+g_\beta(\tilde{u},\tilde{\beta},t),t)$ such that*

$$\|g_u\|_{0,0,\rho}\lesssim|I_{\mathrm{m}}|^{-4}\qquad\qquad \|g_\beta\|_{0,0,\rho}\lesssim|I_{\mathrm{m}}|^{-5/2},$$

and such that $\eta_{I_{\mathrm{m}}}^{-1}\circ\mathtt{W}^{\mathrm{u}}\circ g$ constitutes the unique analytic extension, to the domain $(u,\beta,t)\in\tilde{D}_{\kappa_2}^{\mathrm{u}}\times\mathbb{T}_{\rho_2}\times\mathbb{T}_{\sigma}$, of the Lagrangian graph parametrization $\mathcal{W}^{\mathrm{u}}$ associated to the function T^{u} obtained in Theorem 2.4.8.

The proof of this lemma follows the same lines as the proof of Proposition 5.21 in [GMS16]. In conclusion, we have proven the existence of the analytic continuation of the unstable generating function T^{u} to the domain (see Figure 2.4.3)

$$\begin{aligned}D_{\kappa_2}=\{u\in\mathbb{C}\colon &|\mathrm{Im}u|\leq-\tan\beta_1\mathrm{Re}u+1/3-\kappa_2|I_{\mathrm{m}}|^{-3},\ |\mathrm{Im}u|\leq\tan\beta_1\mathrm{Re}u+1/3-\kappa_2|I_{\mathrm{m}}|^{-3},\\ &|\mathrm{Im}u|\geq-\tan\beta_2\mathrm{Re}u+1/6+\kappa_2|I_{\mathrm{m}}|^{-3}\}.\end{aligned}\tag{2.83}$$

Indeed, introducing the Banach spaces

$$\mathcal{Y}_{\nu,\rho}=\left\{h=\{h^{[l]}\}_{l\in\mathbb{Z}}\colon h^{[l]}:D_{\kappa_2}\times\mathbb{T}_\rho\to\mathbb{C}\text{ is analytic for all }l\in\mathbb{Z}\text{ and }\|h\|_{\nu,\rho}<\infty\right\},\tag{2.84}$$

where $\|\cdot\|_{\nu,\rho}$ is the Fourier sup norm

$$\|h\|_{\nu,\rho}=\sum_{l\in\mathbb{Z}}\|h^{[l]}\|_{\nu,\rho,l}e^{-|l|\sigma}\qquad \|h^{[l]}\|_{\nu,\rho,l}=\sup_{(u,\beta)\in D_{\kappa_2}\times\mathbb{T}_\rho}\left|(u-i/3)^{\nu+l/2}(u+i/3)^{\nu-l/2}h^{[l]}(u,\beta)\right|\tag{2.85}$$

(notice that the weight u^η becomes now meaningless since D_{κ_2} is bounded), the following proposition, which extends the domain of definition of the function element T^{u} in Theorem 2.4.8, holds.

Proposition 2.4.17. *There exist $\kappa_2,\sigma_2,\rho_2>0$ and $I_*>0$ such that for $\epsilon\in(0,I_*^{-3})$ and $I^{\mathrm{u}},I^{\mathrm{s}}$ with $I_{\mathrm{m}}\in\Lambda_{\rho_2,I}$, and $|I^{\mathrm{u}}-I^{\mathrm{s}}|\leq\epsilon|I_{\mathrm{m}}|^{-4}$, there exists $T^{\mathrm{u}}\in\mathcal{Y}_{1/2,\rho_2}$ which constitutes the unique analytic continuation to $(u,\beta,t)\in D_{\kappa_2}\times\mathbb{T}_{\rho_2}\times\mathbb{T}_{\sigma_2}$ of the function obtained in Theorem 2.4.8. Moreover, in this domain*

$$\|T^{\mathrm{u}}\|_{1/2,\rho_2}\lesssim|I_{\mathrm{m}}|^{-3}\qquad\text{and}\qquad\|T^{\mathrm{u}}-L^{\mathrm{u}}\|_{1,\rho_2}\lesssim|I_{\mathrm{m}}|^{-7},$$

where L^{u} is the unstable half Melnikov potential defined in (2.60)*. In addition, we have that*

$$\|T^{\mathrm{u}}-T^{\mathrm{u}}_{\mathrm{circ}}-(L^{\mathrm{u}}-L^{\mathrm{u}}_{\mathrm{circ}})\|_{1,\rho_2}\lesssim\epsilon|I_{\mathrm{m}}|^{-7}$$

where $T^{\mathrm{u}}_{\mathrm{circ}}$ is defined in (2.51) *and $L^{\mathrm{u}}_{\mathrm{circ}}(u,t-\beta;I_{\mathrm{m}})=L^{\mathrm{u}}(u,\beta,t;I_{\mathrm{m}},0)$.*

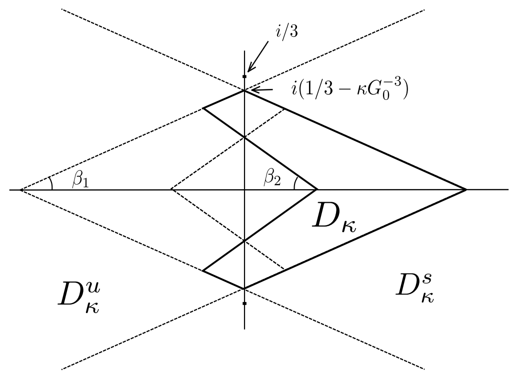

Figure 2.7: The domain D_κ defined in (2.83).

2.4.4 The difference ΔS between the generating functions of the invariant manifolds

In Theorem 2.4.8 we have proved that, for suitable $\kappa, \sigma > 0$ and $\rho > 0$, the formal Fourier series $T^{\rm u}$ (see Remark 2.4.3) in the parametrization (2.19) of the unstable manifold of the torus $\mathcal{T}_{I^{\rm u}}$ is uniformly $\mathcal{O}(|I_{\rm m}|^{-7})$ approximated in $\mathcal{X}_{1/3,1/2,\rho}$, by the half Melnikov potential $L^{\rm u}$ introduced in (2.60). Moreover, in Proposition 2.4.17 we have shown that $T^{\rm u}$ admits a unique analytic continuation to the domain $(u,\beta,t) \in D_{\kappa_2} \times \mathbb{T}_{\rho_2} \times \mathbb{T}_{\sigma_2}$ for suitable $\kappa_2 > \kappa, \rho_2 < \rho$ and $\sigma_2 < \sigma$.

The very same argument in the proof of Theorem 2.4.8 shows the stable counterpart for the formal Fourier series $T^{\rm s}$ on the domain $(u,\beta,t) \in D^{\rm s}_\kappa \times \mathbb{T}_\rho \times \mathbb{T}_\sigma$ where $D^{\rm s}_\kappa = \{u \in \mathbb{C}\colon\ -u \in D^{\rm u}_\kappa\}$. Moreover, denoting by $\mathcal{X}^{\rm s}_{1/3,1/2,\rho}$ the associated Banach space for formal Fourier series defined on $D^{\rm s}_\kappa \times \mathbb{T}_\rho \times \mathbb{T}_\sigma$, $T^{\rm s}$ is uniformly $\mathcal{O}(|I_{\rm m}|^{-7})$ approximated in $\mathcal{X}^{\rm s}_{1/3,1/2,\rho}$ by the stable half Melnikov potential

$$L^{\rm s}(u,\beta,t;I_{\rm m},\epsilon) = \int_{+\infty}^{0} V(u+s,\beta,t+I_{\rm m}^3 s;I_{\rm m},\epsilon){\rm d}s. \tag{2.86}$$

Since $D_{\kappa_2} \subset D^{\rm s}_\kappa$, we can now analyze the difference between the generating functions of the stable and unstable manifolds (see equation (2.20) and the discussion below it)

$$\Delta S = S^{\rm u} - S^{\rm s} = \langle \delta^{\rm u} - \delta^{\rm s}, q\rangle + T^{\rm u} - T^{\rm s} \tag{2.87}$$

on the common domain $q = (u,\beta,t) \in D_{\kappa_2} \times \mathbb{T}_{\rho_2} \times \mathbb{T}_{\sigma_2}$. For the sake of clarity in the forthcoming arguments, we summarize in Theorem 2.4.18 the previous discussion. We denote by

$$\Delta S_{\rm circ}(u,t-\beta;I_{\rm m}) \equiv T^{\rm u}_{\rm circ}(u,t-\beta;I_{\rm m}) - T^{\rm s}_{\rm circ}(u,t-\beta;I_{\rm m}). \tag{2.88}$$

Theorem 2.4.18. *There, there exist $\kappa_2, \sigma_2, \rho_2 > 0$ and $I_* > 0$ such that for $\epsilon \in (0, I_*^{-3})$ and $I^{\rm u}, I^{\rm s}$ with $I_{\rm m} \in \Lambda_{\rho_2,I}$ and $|I^{\rm u} - I^{\rm s}| \le \epsilon |I_{\rm m}|^{-4}$, the difference $\Delta S = S^{\rm u} - S^{\rm s}$ defined in* (2.20) *satisfies $\Delta S \in \mathcal{Y}_{1/2,\rho_2}$ and*

$$\|\Delta S - \langle \delta^{\rm u} - \delta^{\rm s}, q\rangle - \tilde{L}\|_{1/2,\rho_2} \lesssim |I_{\rm m}|^{-7},$$

where the norm $\|\cdot\|_{\nu,\rho}$ is defined in (2.85) *and the Melnikov potential $\tilde{L}$ is defined in* (2.50). *Moreover, we have*

$$\|\Delta S - \Delta S_{\rm circ} - (\langle \delta^{\rm u} - \delta^{\rm s}, q\rangle + \tilde{L} - \tilde{L}_{\rm circ})\|_{1,\rho_2} \lesssim \epsilon |I_{\rm m}|^{-7},$$

where $\Delta S_{\rm circ}$ is defined in (2.88) *and $\tilde{L}_{\rm circ}$ is defined in* (2.53).

We now recall that the aim of Section 2.4 is to show that the existence of nondegenerate critical points of the function $\langle(\delta^{\rm u} - \delta^{\rm s}), q\rangle + \tilde{L}$ implies the existence of critical points of the function $q \to \Delta S$. Namely, our goal is to prove Theorem 2.3.9. As first step, we provide a proof of Proposition 2.3.8. With that objective we state the following lemma, whose proof is given in Appendix 5.C.

Lemma 2.4.19. *Let $\rho_0 > 0$ be given in Lemma 2.4.7. Then, there exists $I_* > 0$ such that for $\epsilon \in (0, I_*^{-3})$ and $I_\mathrm{m} \in \Lambda_{\rho_0, I}$, the Melnikov potential $\tilde{L}$ defined in (2.50) is an analytic function of all its arguments (real analytic if $I_\mathrm{m} \in \mathbb{R}$) and can be expressed as the absolutely convergent series*

$$\tilde{L}(u, \beta, t; I^\mathrm{u}, I^\mathrm{s}, \epsilon) = \sum_{l \in \mathbb{N}} \mathcal{L}_l(t - I_\mathrm{m}^3 u, \beta; I_\mathrm{m}, \epsilon),$$

where, writing $\sigma = t - I_\mathrm{m}^3 u$,

- $\mathcal{L}_0(\beta; I_\mathrm{m}) = \mu(1-\mu)\big(L_{0,0}(I_\mathrm{m}, \epsilon) + L_{0,1}(I_\mathrm{m}, \epsilon)\cos\beta + E_0(\beta; I_\mathrm{m}, \epsilon)\big)$ *with*

$$\begin{aligned} L_{0,0}(I_\mathrm{m}, \epsilon) =& \frac{\pi}{2I_\mathrm{m}^3}\left(1 + \mathcal{O}(|I_\mathrm{m}|^{-4}, \epsilon^{-2})\right) \\ L_{0,1}(I_\mathrm{m}, \epsilon) =& -(1-2\mu)\frac{15\pi\epsilon}{8I_\mathrm{m}^3}\left(1 + \mathcal{O}(|I_\mathrm{m}|^{-4}, \epsilon^{-2})\right) \\ |E_0(\beta; I_\mathrm{m}, \epsilon)| \lesssim& \epsilon^2 |I_\mathrm{m}|^{-7}, \end{aligned}$$

- $\mathcal{L}_1(\sigma, \beta; I_\mathrm{m}, \epsilon) = \mu(1-\mu)\big(2L_{1,1}(I_\mathrm{m}, \epsilon)\cos(\sigma - \beta) + 2L_{1,2}(I_\mathrm{m}, \epsilon)\cos(\sigma - 2\beta) + E_1(\sigma, \beta; I_\mathrm{m}, \epsilon)\big)$ *with*

$$\begin{aligned} 2L_{1,1}(I_\mathrm{m}, \epsilon) =& (1-2\mu)\sqrt{\frac{\pi}{8I_\mathrm{m}}}\left(1 + \mathcal{O}(|I_\mathrm{m}|^{-1}, \epsilon^{-2})\right)\exp(-I_\mathrm{m}^3/3) \\ 2L_{1,2}(I_\mathrm{m}, \epsilon) =& -3\epsilon\sqrt{2\pi I_\mathrm{m}^3}\left(1 + \mathcal{O}(|I_\mathrm{m}|^{-1}, \epsilon^{-1})\right)\exp(-I_\mathrm{m}^3/3) \\ |E_1(\sigma, \beta; I_\mathrm{m}, \epsilon)| \lesssim& \epsilon(|I_\mathrm{m}|^{-3/2} + \epsilon|I_\mathrm{m}|^{5/2})\exp(-I_\mathrm{m}^3/3), \end{aligned} \tag{2.89}$$

- *The sum of the higher coefficients*

$$\mathcal{L}_{\geq 2}(u, \beta, t; I_\mathrm{m}) = \sum_{l \geq 2} \mathcal{L}_l(\sigma, \beta; I_\mathrm{m}, \epsilon)$$

  satisfies the estimate

$$|\mathcal{L}_{\geq 2}| \lesssim |I_\mathrm{m}|^{3/2}\exp(-2\mathrm{Re}(I_\mathrm{m}^3)/3).$$

Notice that the estimates in Theorem 2.4.8 only imply

$$|\partial_u \Delta(S - \tilde{L})| \lesssim |I_\mathrm{m}|^{-7}$$

while

$$|\partial_u \tilde{L}| \sim |I_\mathrm{m}|^{3/2}\exp(-\mathrm{Re}(I_\mathrm{m}^3)/3).$$

The existence of critical points of ΔS as a consequence of the existence of nondegenerate critical points of the function $\langle(\delta^\mathrm{u} - \delta^\mathrm{s}), q\rangle + \tilde{L}$ is therefore not clear at the moment. This "mismatch" is caused by not looking at the problem in the right set of coordinates. In Lemma 2.4.20 below, we prove the existence of a change of variables $(u, \beta, t) = \Phi(v, \theta, t)$ such that $\Delta\mathcal{S} = \Delta S \circ \Phi$ only depends on v and t through the difference $\sigma = t - I_\mathrm{m}^3 v$. This fact is equivalent to $\Delta\mathcal{S} \in \mathrm{Ker}\mathcal{L}$ where $\mathcal{L}$ is the linear operator

$$\mathcal{L} = \partial_v + I_\mathrm{m}^3 \partial_t.$$

Then, in Lemma 2.4.21 it is shown that functions in $\mathcal{Y}_{\nu,\rho} \cap \mathrm{Ker}\mathcal{L}$ (see (2.84)), present an exponential decay in the size of their Fourier coefficients. Finally, this last property, together with the approximation of $\Delta\mathcal{S}$ by $\langle(\delta^\mathrm{u} - \delta^\mathrm{s}), q\rangle + \tilde{L}$ in the norm (2.85), given in Theorem 2.4.18, are used to complete the proof of Proposition 2.3.8.

Lemma 2.4.20. *There exists $\rho_3, I_* > 0$ such that for $\epsilon \in (0, I_*^{-3})$ and $I^{\rm u}, I^{\rm s}$ with $I_{\rm m} \in \Lambda_{\rho_3,I}$ and $|I^{\rm u} - I^{\rm s}| \le \epsilon |I_{\rm m}|^{-4}$, there exists an analytic change of variables of the form*

$$(u, \beta, t) = \Phi(v, \theta, t) = (v + \phi_v(v, \theta, t), \theta + \phi_\theta(v, \theta, t), t)$$

with $\phi_v \in \mathcal{Y}_{0,\rho_3}$, $\phi_\theta \in \mathcal{Y}_{1/2,\rho_3}$ and $\|\phi_v\|_{0,\rho_3} \lesssim |I_{\rm m}|^{-4}$, $\|\phi_\theta\|_{1/2,\rho_3} \lesssim |I_{\rm m}|^{-4}$, such that $\Delta\mathcal{S} = \Delta S \circ \Phi$ satisfies

$$\mathcal{L}\Delta\mathcal{S} = (\partial_v + I_{\rm m}^3 \partial_t)\Delta\mathcal{S} = 0. \tag{2.90}$$

Moreover, under the same hypotheses, there exists an analytic change of variables

$$(u, t - \beta) = \Phi_{\rm circ}(v, t - \theta) = (v + \phi_{v,{\rm circ}}(v, t - \theta), \theta + \phi_{\theta,{\rm circ}}(v, t - \theta))$$

with $\phi_{v,{\rm circ}} \in \mathcal{Y}_{0,\rho_3}$, $\phi_{\theta,{\rm circ}} \in \mathcal{Y}_{1/2,\rho_3}$ and $\|\phi_v - \phi_{v,{\rm circ}}\|_{0,\rho_3} \lesssim \epsilon|I_{\rm m}|^{-4}$, $\|\phi_\theta - \phi_{\theta,{\rm circ}}\|_{1/2,\rho_3} \lesssim \epsilon|I_{\rm m}|^{-4}$, such that $\Delta\mathcal{S}_{\rm circ} = \Delta S_{\rm circ} \circ \Phi_{\rm circ}$ satisfies

$$\Delta\mathcal{S}_{\rm circ}(v, t - \theta) = \Delta\widehat{\mathcal{S}}_{\rm circ}(t - \theta - I_{\rm m}^3 v; I_{\rm m}), \tag{2.91}$$

for some periodic function $\Delta\widehat{\mathcal{S}}_{\rm circ}(t - \theta - I_{\rm m}^3 v; I_{\rm m})$.

Proof. Using that both $S^{\rm u,s}$ satisfy the same Hamilton-Jacobi equation $H(q, \nabla S^{\rm u,s}) = 0$ it is an straightforward computation to check that ΔS is a solution to $\widetilde{\mathcal{L}}\Delta S = 0$ with

$$\widetilde{\mathcal{L}} = (1 + (A^{\rm s} + A^{\rm u}))\,\partial_u + (B^{\rm s} + B^{\rm u})\partial_\beta + I_{\rm m}^3\partial_t \tag{2.92}$$

and where $A^{\rm u,s}, B^{\rm u,s}$ are defined as in (2.56). One can now check that $\Delta\mathcal{S} \in {\rm Ker}\mathcal{L}$, if and only if, Φ satisfies

$$\mathcal{L}\phi_v = (A^{\rm s} + A^{\rm u}) \circ \Phi \qquad \text{and} \qquad \mathcal{L}\phi_\beta = (B^{\rm s} + B^{\rm u}) \circ \Phi. \tag{2.93}$$

In order to rewrite (2.93) as a fixed point equation for Φ we introduce the left inverse operator $\mathcal{G}$ for $\mathcal{L}$ defined by the expression (here v_+ and v_- are the top and bottom points and v_0 is any real point in the domain D_{κ_2} defined in (2.83)),

$$\mathcal{G}(h) = \sum_{l\in\mathbb{Z}} \mathcal{G}^{[l]}(h) \tag{2.94}$$

with

$$\begin{aligned}
\mathcal{G}^{[l]}(h) &= \int_{v_+-v}^{0} h^{[l]}(v+s,\theta)e^{ilI_{\rm m}^3 s}{\rm d}s && \text{for} \quad l > 0\\
\mathcal{G}^{[0]}(h) &= \int_{v_0-v}^{0} h^{[0]}(v+s,\theta){\rm d}s && \text{for} \quad l = 0\\
\mathcal{G}^{[l]}(h) &= \int_{v_--v}^{0} h^{[l]}(v+s,\theta)e^{ilI_{\rm m}^3 s}{\rm d}s && \text{for} \quad l < 0.
\end{aligned}$$

Therefore, it is enough to look for Φ satisfying

$$\phi_v = \mathcal{G}((A^{\rm s} + A^{\rm u}) \circ \Phi) \qquad \text{and} \qquad \phi_\theta = \mathcal{G}((B^{\rm s} + B^{\rm u}) \circ \Phi).$$

The proof of the first part of the lemma now follows from a standard fixed point argument along the lines (but considerably simpler) of the proof of Lemma 2.4.12 (see also Theorem 6.3 in [GMS16]). In particular, the proof is easily completed using the estimates

$$\|A^{\rm u,s}\|_{1/2,\rho} \lesssim |I_{\rm m}|^{-4} \qquad \|B^{\rm u,s}\|_{3/2,\rho} \lesssim |I_{\rm m}|^{-4},$$

which are obtained in an straightforward manner from Proposition 2.4.17 and the discussion at the beginning of Section 2.4.4 by taking, for example, $\rho_3 \le \rho_2/2$. To deal with compositions, we make use of a natural extension of Lemma 2.4.6 which allows to treat also changes of variables in v (details can be found in [GMS16]).

We now prove the second part of the lemma. Introduce the angle $\xi = t - \beta$, and write $\Delta S_{\rm circ}(u,\xi;I_{\rm m})$. Therefore, $\Delta S_{\rm circ}$ is a solution to $\widetilde{\mathcal{L}}_{\rm circ}\Delta S_{\rm circ} = 0$ where

$$\widetilde{\mathcal{L}}_{\rm circ} = (1 + (A^{\rm s}_{\rm circ} + A^{\rm u}_{\rm circ}))\,\partial_u + (I_{\rm m}^3 - B^{\rm s}_{\rm circ} - B^{\rm u}_{\rm circ})\partial_\xi \tag{2.95}$$

and $A^{\rm u,s}_{\rm circ}, B^{\rm u,s}_{\rm circ}$ are defined in (2.77). Thus, $\Delta\mathcal{S}_{\rm circ} \in {\rm Ker}\mathcal{L}$, if and only if, $\tilde\Phi_{\rm circ}(v,\tilde\xi) \equiv (v + \phi_{v,{\rm circ}}(v,\tilde\xi), \tilde\xi - \phi_{\theta,{\rm circ}}(v,\tilde\xi))$ satisfies

$$\mathcal{L}\phi_{v,{\rm circ}} = (A^{\rm s}_{\rm circ} + A^{\rm u}_{\rm circ}) \circ \tilde\Phi_{\rm circ} \qquad \text{and} \qquad \mathcal{L}\phi_{\theta,{\rm circ}} = (B^{\rm s}_{\rm circ} + B^{\rm u}_{\rm circ}) \circ \tilde\Phi_{\rm circ}.$$

The lemma follows using the estimates (see the proof of Theorem 2.4.8)

$$\|A^{\rm u,s} - A^{\rm u,s}_{\rm circ}\|_{1/2,\rho} \lesssim \epsilon |I_{\rm m}|^{-4} \qquad \|B^{\rm u,s} - B^{\rm u,s}_{\rm circ}\|_{3/2,\rho} \lesssim \epsilon |I_{\rm m}|^{-4}.$$

□

The following lemma gives the exponential decay of the Fourier coefficients for functions in $\mathcal{Y}_{\nu,\rho} \cap {\rm Ker}\mathcal{L}$ (see also Lemma 6.7 in [GMS16])

Lemma 2.4.21. *Fix $\nu, \rho \geq 0$ and let $h \in \mathcal{Y}_{\nu,\rho}$ be such that $h \in {\rm Ker}\mathcal{L}$. Then h can be written as*

$$h(v,\theta,t) = \sum_{l\in\mathbb{Z}} \Lambda^{[l]}(\theta) e^{il(t - I_{\rm m}^3 v)}$$

and, for some $C > 0$ independent of $\|h\|_{\nu,\rho}$ and $I_{\rm m}$,

$$\sup_{\theta\in\mathbb{T}_\rho} |\Lambda^{[l]}(\theta)| \lesssim \|h\|_{\nu,\rho}\ (C|I_{\rm m}|)^{3(\nu + |l|/2)} \exp(-|l|{\rm Re}(I_{\rm m}^3)/3).$$

Proof. Write

$$h(v,\theta,t) = \sum_{l\in\mathbb{Z}} h^{[l]}(v,\theta) e^{ilt}.$$

Since $h \in {\rm Ker}\mathcal{L}$

$$h(v,\theta,t) = \sum_{l\in\mathbb{Z}} \Lambda^{[l]}(\theta) e^{il(t - I_{\rm m}^3 v)},$$

where $\Lambda^{[l]}(\theta) = h^{[l]}(v,\theta)e^{ilI_{\rm m}^3 v}$ is independent of v. For $l > 0$, we evaluate at $v_+ = i(1/3 - \kappa|I_{\rm m}|^{-3})$ and use that

$$\|h^{[l]}\|_{\nu,\rho} \leq |I_{\rm m}|^{3|l|/2}\|h\|_{\nu,\rho}$$

to obtain that

$$\begin{aligned} |\Lambda^{[l]}| \leq& |I_{\rm m}|^{3\nu}\|h^{[l]}\|_{\nu,\rho}\ \exp(-|l|{\rm Re}(I_{\rm m}^3)(1/3 - \kappa|I_{\rm m}|^{-3})) \\ \leq& \|h\|_{\nu,\rho}\ (C|I_{\rm m}|)^{3(\nu+|l|/2)} \exp(-|l|{\rm Re}(I_{\rm m}^3)/3) \end{aligned}$$

for some $C > 0$. The result for $l < 0$ is obtained analogously evaluating at $v_- = -i(1/3 - \kappa|I_{\rm m}|^{-3})$. □

We now have all the ingredients to complete the proof of Proposition 2.3.8.

Remark 2.4.22. *In the following, we rename as ρ the constant $\rho_3 > 0$ which was obtained in Lemma 2.4.20.*

Proof of Proposition 2.3.8. Recall that $\tilde L$, which was defined in (2.50), satisfies

$$\tilde L(u,\beta,t;I_{\rm m},\epsilon) = L(t - I_{\rm m}^3 u, \beta; I_{\rm m}, \epsilon),$$

where $L(\sigma,\beta;I_{\rm m},\epsilon)$ was defined in (2.24). Let $\tilde q = (v,\theta,t)$. Since $\mathcal{E} = \Delta\mathcal{S} - \langle(\delta^{\rm u} - \delta^{\rm s}), \tilde q\rangle - \tilde L \in {\rm Ker}\mathcal{L}$, and $\mathcal{E} \in \mathcal{Y}_{1/2,\rho}$, it is enough to estimate $\|\mathcal{E}\|_{1/2,\rho}$ and apply Lemma 2.4.21. To that end, we write

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$$

with $\mathcal{E}_1 = \Delta S - \langle \delta^{\rm u} - \delta^{\rm s}, \tilde{q}\rangle - \tilde{L}$, and $\mathcal{E}_2 = \Delta\mathcal{S} - \Delta S$. Using that $|\delta^{\rm u,s}| \lesssim \epsilon |I_{\rm m}|^{-4}$, the estimate for $\|\phi_\theta\|_{0,\rho}$ in Lemma 2.4.20 and the estimate for $\Delta S - \langle \delta^{\rm u} - \delta^{\rm s}, q\rangle - \tilde{L}$ in Theorem 2.4.18 we obtain

$$\|\mathcal{E}_1\|_{1/2,\rho} \lesssim |I_{\rm m}|^{-7}.$$

In order to bound $\mathcal{E}_2$, it follows from the mean value theorem, the estimates for $\|S^{\rm u,s}\|_{3/2,\rho}$, which can be deduced from Proposition 2.4.17 and the analogous version for $T^{\rm s}$ (see the discussion at Section 2.4.4), and the estimates for$\|\phi_v\|_{0,\rho}, \|\phi_\theta\|_{1/2,\rho}$ in Lemma 2.4.20, that

$$\|\mathcal{E}_2\|_{1/2,\rho} \lesssim |I_{\rm m}|^{-11/2}.$$

Applying Lemma 2.4.21, we obtain that

$$\Delta\mathcal{S} - \langle \delta^{\rm u} - \delta^{\rm s}, \tilde{q}\rangle - \tilde{L} = \sum_{l\in\mathbb{Z}} \mathcal{E}^{[l]}(\theta) e^{il(t - I_{\rm m}^3 v)} \tag{2.96}$$

where, there exists some $C > 0$, such that for $l \neq 0$,

$$\begin{aligned}\sup_{\theta\in\mathbb{T}_\rho} |\mathcal{E}^{[l]}(\theta)| \lesssim& (C|I_{\rm m}|)^{3(1+|l|)/2} \|\mathcal{E}\|_{1/2,\rho} \exp(-|l|\mathrm{Re}(I_{\rm m}^3)/3)\\ \lesssim& (C|I_{\rm m}|)^{-4+3|l|/2} \exp(-|l|\mathrm{Re}(I_{\rm m}^3)/3)\end{aligned}$$

as was to be shown. □

Finally, we also state the following lemma which will prove useful in the proof, in Section 2.4.5, of Theorem 2.3.9.

Lemma 2.4.23. *Define the function $\mathcal{E}_{\rm circ}(\theta, \sigma; I^{\rm u}, I^{\rm s})$ given by*

$$\mathcal{E}_{\rm circ} = \Delta\mathcal{S} - \Delta\mathcal{S}_{\rm circ} - ((I^{\rm u} - I^{\rm s})\theta + L - L_{\rm circ}) \tag{2.97}$$

where $\Delta\mathcal{S}$ and $\Delta\mathcal{S}_{\rm circ}$ are defined in Lemma 2.4.20, L is defined in (2.24) *and $L_{\rm circ}(\sigma - \theta; I_{\rm m}) = L(\sigma, \theta; I_{\rm m}, 0)$. Then, we have that*

$$\|\mathcal{E}_{\rm circ}\|_{1/2,\rho} \lesssim \epsilon |I_{\rm m}|^{-11/2}.$$

Proof. We write $\mathcal{E}_{\rm circ} = \mathcal{E}_{\rm circ,1} + \mathcal{E}_{\rm circ,2} + \mathcal{E}_{\rm circ,3} + \mathcal{E}_{\rm circ,4}$ with

$$\begin{aligned}\mathcal{E}_{\rm circ,1} =& \Big(\Delta S - \Delta S_{\rm circ} - (I^{\rm u} - I^{\rm s})\beta - (\tilde{L} - \tilde{L}_{\rm circ})\Big) \circ \Phi\\ \mathcal{E}_{\rm circ,2} =& \Delta S_{\rm circ} \circ \Phi - \Delta\mathcal{S}_{\rm circ}\\ \mathcal{E}_{\rm circ,3} =& -(I^{\rm u} - I^{\rm s})\phi_\theta\\ \mathcal{E}_{\rm circ,4} =& -(\tilde{L} - \tilde{L}_{\rm circ}) + (\tilde{L} - \tilde{L}_{\rm circ}) \circ \Phi\end{aligned}$$

On one hand, Theorem 2.4.18 implies that $\|\mathcal{E}_{\rm circ,1}\|_{1,\rho} \lesssim \epsilon |I_{\rm m}|^{-7}$. On the other hand, the estimates

$$\|\partial_v \Delta S_{\rm circ}\|_{3/2,\rho}, \|\partial_\beta \Delta S_{\rm circ}\|_{1/2,\rho} \lesssim |I_{\rm m}|^{-3},$$

which can be deduced from Theorem 2.4.18, and the estimates $\|\phi_v - \phi_{v,\rm circ}\|_{0,\rho}, \|\phi_\theta - \phi_{\theta,\rm circ}\|_{1/2,\rho} \lesssim \epsilon |I_{\rm m}|^{-4}$ obtained in Lemma 2.4.20 imply that $\|\mathcal{E}_{\rm circ,2}\|_{1/2,\rho} \lesssim \epsilon |I_{\rm m}|^{-11/2}$. For the third term, since $|I^{\rm u} - I^{\rm s}| \lesssim \epsilon |I_{\rm m}|^{-4}$, the estimate $\|\phi_v\|_{0,\rho} \lesssim |I_{\rm m}|^{-4}$ in Lemma 2.4.20 shows that $\|\mathcal{E}_{\rm circ,3}\|_{1/2,\rho} \lesssim \epsilon |I_{\rm m}|^{-8}$. Finally, since $\|\partial_v(\tilde{L} - \tilde{L}_{\rm circ})\|_{3/2,\rho}, \|\partial_\beta(\tilde{L} - \tilde{L}_{\rm circ})\|_{1/2,\rho} \lesssim \epsilon |I_{\rm m}|^{-3}$, which can be deduced from Theorem 2.4.18, we obtain that $\|\mathcal{E}_{\rm circ,4}\|_{1/2,\rho} \lesssim \epsilon |I_{\rm m}|^{-11/2}$. □

2.4.5 The critical points of the function $\Delta\mathcal{S}$

In this section we use Lemma 2.4.19 and Proposition 2.3.8 to provide a proof of Theorem 2.3.9 and Proposition 2.3.12.

Remark 2.4.24. *Since we always assume that $|I^{\mathrm{u}} - I^{\mathrm{s}}| \leq \epsilon |I^{\mathrm{u}}|^{-4}$, all the errors, which a priori depend on both $I^{\mathrm{u}}, I^{\mathrm{s}}$ can be estimated in terms of the value of I^{u} alone. This is an arbitrary choice motivated by the fact that, in the following, we take I^{u} as independent variable.*

Proof of Theorem 2.3.9. Throughout the proof we will use the following notation. Let

$$K = \{(\theta, \sigma, I^{\mathrm{u}}, I^{\mathrm{s}}) \in \mathbb{T}^2_\rho \times \Lambda^2_{\rho,I} \colon |I^{\mathrm{u}} - I^{\mathrm{s}}| \leq \epsilon |I^{\mathrm{u}}|^{-4}\}.$$

We look for zeros of the function

$$F(\theta, \sigma, I^{\mathrm{u}}, I^{\mathrm{s}}) \equiv (\partial_\sigma \Delta\mathcal{S}, \partial_\theta \Delta\mathcal{S})(\theta, \sigma; I^{\mathrm{u}}, I^{\mathrm{s}}), \tag{2.98}$$

which are of the form $(\theta, \sigma, I^{\mathrm{u}}, I^{\mathrm{s}}) = (\theta, \sigma_\pm(\theta, I^{\mathrm{u}}), I^{\mathrm{u}}, I^{\mathrm{s}}(\theta, I^{\mathrm{u}}))$.

In order to obtain asymptotic formulas for the critical points, we divide the proof in two steps. First we study the existence of critical points $\sigma_{\pm,\mathrm{circ}}(\theta, I^{\mathrm{u}})$ of the function $\Delta\mathcal{S}_{\mathrm{circ}}(\sigma, \theta, I^{\mathrm{u}}) = \Delta\widehat{\mathcal{S}}_{\mathrm{circ}}(\sigma - \theta; I^{\mathrm{u}})$, and then prove the existence of critical points of the function F which are ϵ close to $(\sigma, I^{\mathrm{s}}) = (\sigma_{\pm,\mathrm{circ}}(\theta, I^{\mathrm{u}}), I^{\mathrm{u}})$. Since for all $(\theta, \sigma, I^{\mathrm{u}}, I^{\mathrm{s}}) \in K$ (see (2.89))

$$\partial_\sigma L_{\mathrm{circ}}(\sigma - \theta; I^{\mathrm{u}}) = \mu(1-\mu)(1-2\mu)\sqrt{\frac{\pi}{2I^{\mathrm{u}}}} \exp(-(I^{\mathrm{u}})^3/3) \sin(\sigma - \theta) + \mathcal{O}(|I^{\mathrm{u}}|^{-3/2} \exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3))$$

and

$$|\partial_\sigma \Delta S_{\mathrm{circ}}(\sigma, \theta; I^{\mathrm{u}}) - L_{\mathrm{circ}}(\sigma - \theta; I^{\mathrm{u}})| \lesssim |I^{\mathrm{u}}|^{-5/2} \exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3),$$

a direct application of the implicit function theorem shows that there exist non degenerate critical points

$$\sigma_{+,\mathrm{circ}}(\theta, I^{\mathrm{u}}) = \theta + \mathcal{O}(|I^{\mathrm{u}}|^{-1}) \qquad \sigma_{-,\mathrm{circ}}(\theta, I^{\mathrm{u}}) = \theta + \pi + \mathcal{O}(|I^{\mathrm{u}}|^{-1}) \tag{2.99}$$

of the function $\partial_\sigma \Delta\mathcal{S}_{\mathrm{circ}}(\sigma, \theta, I^u)$.

Therefore, to analyze the zeros of F, we write

$$\begin{aligned}
\partial_\sigma \Delta\mathcal{S} =& \partial_\sigma \Delta\mathcal{S}_{\mathrm{circ}} + \mathcal{E}_\sigma \\
\partial_\theta \Delta\mathcal{S} =& (I^{\mathrm{u}} - I^{\mathrm{s}}) + \partial_\theta (L - L_{\mathrm{circ}}) + \mathcal{E}_{\theta,1} + \mathcal{E}_{\theta,2}
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{E}_\sigma =& \partial_\sigma (\Delta\mathcal{S} - \Delta\mathcal{S}_{\mathrm{circ}}) \\
\mathcal{E}_{\theta,1} =& \partial_\theta (\Delta\mathcal{S} - \Delta\mathcal{S}_{\mathrm{circ}} - ((I^{\mathrm{u}} - I^{\mathrm{s}})\theta + L - L_{\mathrm{circ}})) \\
\mathcal{E}_{\theta,2} =& \partial_\theta \Delta\mathcal{S}_{\mathrm{circ}}
\end{aligned} \tag{2.100}$$

The existence of nondegenerate zeros of the function $F(\theta, \sigma, I^{\mathrm{u}}, I^{\mathrm{s}})$ will be a direct consequence of the asymptotic formulas in Lemma 2.4.19, the estimates in Lemma 2.4.23 and the implicit function theorem. The first step is to estimate the error terms $\mathcal{E}_\sigma, \mathcal{E}_{\theta,1}$ and $\mathcal{E}_{\theta,2}$. We write $\mathcal{E}_\sigma = \partial_\sigma (L - L_{\mathrm{circ}}) + \mathcal{E}_{\mathrm{circ}}$ where $\mathcal{E}_{\mathrm{circ}}$ has been defined in (2.97). Therefore, the asymptotic formulas in Lemma 2.4.19, the fact that $\mathcal{E}_{\mathrm{circ}} \in \mathrm{Ker}\mathcal{L}$ and the estimates in Lemma 2.4.23 imply that

$$|\mathcal{E}_\sigma| \lesssim \epsilon |I^{\mathrm{u}}|^{3/2} \exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3).$$

The estimate in Lemma 2.4.23 implies that

$$|\mathcal{E}_{\theta,1}| \lesssim \epsilon |I^{\mathrm{u}}|^{-11/2}.$$

Moreover, since

$$|\partial^2_{\sigma\theta} \Delta\mathcal{S}_{\mathrm{circ}}| \lesssim |I^{\mathrm{u}}|^{-1/2} \exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3),$$

if we define (for a sufficiently large, but fixed, $C>0$)

$$K_{\pm}=\{(\theta,\sigma,I^{\mathrm{u}},I^{\mathrm{s}})\in K\colon |\sigma-\sigma_{\pm,\mathrm{circ}}(\theta,I^{\mathrm{u}})|\leq C\epsilon|I^{\mathrm{u}}|^2\}$$

we obtain that

$$\sup_{(\theta,\sigma)\in K_{\pm}}|\mathcal{E}_{\theta,2}|\lesssim \epsilon|I^{\mathrm{u}}|^{3/2}\exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3).$$

Therefore, in view of the asymptotic expression in Lemma 2.4.19

$$\partial_\theta(L-L_{\mathrm{circ}})(\theta,I^{\mathrm{u}})=\mu(1-\mu)(1-2\mu)\frac{15\pi\epsilon}{8(I^{\mathrm{u}})^5}\sin\theta+\mathcal{O}(\epsilon|I^{\mathrm{u}}|^{-11/2}),\tag{2.101}$$

we take

$$\tilde{\sigma}_{\pm}(\theta,I^{\mathrm{u}})=\sigma_{\pm,\mathrm{circ}}(\theta,I^{\mathrm{u}})\qquad\qquad \hat{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}})=I^{\mathrm{u}}+\partial_\theta(L-L_{\mathrm{circ}})(\theta,\tilde{\sigma}_{\pm}(\theta,I^{\mathrm{u}}),I^{\mathrm{u}}),\tag{2.102}$$

where $\sigma_{\pm,\mathrm{circ}}(\theta,I^{\mathrm{u}})$ are defined in (2.99), as approximate solutions. Indeed, taking into account the estimates for $\mathcal{E}_\sigma,\mathcal{E}_{\theta,1}$ and $\mathcal{E}_{\theta,2}$ defined in(2.100), for all $(\theta,\tilde{\sigma}_{\pm},I^{\mathrm{u}},\hat{I}^{\mathrm{s}}_{\pm})\in K$

$$F(\theta,\tilde{\sigma}_{\pm},I^{\mathrm{u}},\tilde{I}^{\mathrm{s}})=\left(\mathcal{O}(\epsilon\exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3)),\mathcal{O}(\epsilon|I^{\mathrm{u}}|^{-11/2})\right),\tag{2.103}$$

and these estimates extend to $(\theta,\sigma,I^{\mathrm{u}},I^{\mathrm{s}})\in\tilde{K}_{\pm}\equiv\{(\theta,\sigma,I^{\mathrm{u}},I^{\mathrm{s}})\in K_{\pm}\colon |I^{\mathrm{s}}-\hat{I}^{\mathrm{s}}_{\pm}|\leq\epsilon|I^{\mathrm{u}}|^{-5}\}$. Denote by $A\pm$ the differential of the map $(\sigma,I^{\mathrm{s}})\mapsto F(\theta,\sigma,I^{\mathrm{u}},I^{\mathrm{s}})$ evaluated at $(\theta,\tilde{\sigma}_{\pm},I^{\mathrm{u}},\hat{I}^{\mathrm{s}}_{\pm})$. It is an straightforward but tedious computation to check that the asymptotic expression in Lemma 2.4.19 and the estimates in 2.4.23 imply

$$\begin{aligned}A_{\pm}=&\begin{pmatrix}\pm2\mu(1-\mu)L_{1,1} & 0\\ 0 & -1\end{pmatrix}\\ &+\begin{pmatrix}\mathcal{O}((|I^{\mathrm{u}}|^{-5/2}+\epsilon|I^{\mathrm{u}}|^{3/2})\exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3)) & \mathcal{O}(|I^{\mathrm{u}}|^{3/2}\exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3))\\ \mathcal{O}(|I^{\mathrm{u}}|^{-1/2}\exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3)) & \mathcal{O}(|I^{\mathrm{u}}|^{-7})\end{pmatrix}.\end{aligned}$$

Therefore, a direct application of the Implicit function theorem, together with the fact that (see (2.89))

$$|L_{1,1}|\sim|I_{\mathrm{m}}|^{-1/2}\exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3)),$$

yields the existence of $I_*\gg1$ and

$$\sigma_{\pm}(\theta,I^{\mathrm{u}})=\tilde{\sigma}_{\pm}(\theta,I^{\mathrm{u}})+\mathcal{O}(\epsilon|I^{\mathrm{u}}|^2)\qquad\qquad \tilde{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}})=\hat{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}})+\mathcal{O}(\epsilon|I_{\mathrm{m}}|^{-11/2})\tag{2.104}$$

such that, for all $(\theta,I^{\mathrm{u}})\in\Lambda_\rho=\mathbb{T}_\rho\times\Lambda_{\rho,I}$, we have

$$F(\theta,\sigma_{\pm}(\theta,I^{\mathrm{u}}),I^{\mathrm{u}},\tilde{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}}))=0.$$

□

It will be convenient for the proofs of Theorems 2.3.16 and 2.3.19, which will be given in Section 2.6, to state now the following more technical version of Theorem 2.3.9, which includes the asymptotic formulas for the functions $\sigma_{\pm}(\theta,I^{\mathrm{u}}),\tilde{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}})$ obtained in the proof of Theorem 2.3.9 above.

Lemma 2.4.25. *Let $(\theta,I^{\mathrm{u}})\in\Lambda_\rho$ and let*

$$(\theta,I^{\mathrm{u}})\mapsto(\sigma_{\pm}(\theta,I^{\mathrm{u}}),\tilde{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}}))$$

be the real analytic functions satisfying

$$\partial_\sigma\Delta\mathcal{S}(\sigma_{\pm}(\theta,I^{\mathrm{u}}),\theta;I^{\mathrm{u}},\tilde{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}}))=0\qquad\qquad \partial_\theta\Delta\mathcal{S}(\sigma_{\pm}(\theta,I^{\mathrm{u}}),\theta;I^{\mathrm{u}},\tilde{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}}))=0,$$

which were obtained in Theorem 2.3.9. Then, for all $(\theta,I^{\mathrm{u}})\in\Lambda_\rho$ we have

$$\sigma_+=\theta+\mathcal{O}(|I^{\mathrm{u}}|^{-1}),\qquad \sigma_-=\theta+\pi+\mathcal{O}(|I^{\mathrm{u}}|^{-1})$$

and

$$\tilde{I}^{\mathrm{s}}_{\pm}(\theta,I^{\mathrm{u}})=I^{\mathrm{u}}+\partial_\theta\mathcal{L}_{\pm}(\theta,I^{\mathrm{u}})+\mathcal{O}(\epsilon|I^{\mathrm{u}}|^{-11/2}),$$

where $\mathcal{L}_{\pm}(\theta,I^{\mathrm{u}})$ is the Melnikov potential defined in (2.38).

Proof. The result for $\sigma_\pm$ is deduced from (2.99) and (2.104). We only have to prove the result for $\tilde{I}^{\rm s}_\pm$. In the proof of Theorem 2.3.9 above, we have obtained that (see (2.102) and (2.104))

$$\tilde{I}^{\rm s}_\pm(\theta, I^{\rm u}) = I^{\rm u} + \partial_\theta (L - L_{\rm circ})(\theta, \sigma_\pm(\theta, I^{\rm u}); I^{\rm u}) + \mathcal{O}(\epsilon |I^{\rm u}|^{-11/2}).$$

Let $\tilde{\sigma}_+(\theta) = \theta$ and $\tilde{\sigma}_-(\theta) = \theta + \pi$. On one hand, since

$$|\sigma_\pm - \tilde{\sigma}_\pm| \lesssim |I^{\rm u}|^{-1},$$

from the asymptotic expression in Lemma 2.4.19, we obtain that, for all $(\theta, I^{\rm u}) \in \Lambda_\rho$,

$$|\partial_\theta (L - L_{\rm circ})(\theta, \tilde{\sigma}_\pm(\theta); I^{\rm u}) - \partial_\theta (L - L_{\rm circ})(\theta, \sigma_\pm(\theta, I^{\rm u}); I^{\rm u})| \lesssim \epsilon |I^{\rm u}|^{3/2} \exp(-{\rm Re}((I^{\rm u})^3/3)$$

and,

$$|\partial_\sigma (L - L_{\rm circ}) \partial_\theta \tilde{\sigma}_\pm(\theta, \tilde{\sigma}_\pm(\theta); I^{\rm u})| \lesssim \epsilon |I^{\rm u}|^{3/2} \exp(-{\rm Re}((I^{\rm u})^3/3),$$

and the lemma follows from the fact that $L_{\rm circ}(\tilde{\sigma}_\pm(\theta) - \theta; I^{\rm u})$ does not depend on θ. □

We now finish this section with the proof of Proposition 2.3.12.

Proof of Proposition 2.3.12. For $I^{\rm u}, I^{\rm s}$ such that

$$|I^{\rm s} - I^{\rm u}| \leq \frac{\mu(1-\mu)(1-2\mu)15\pi\epsilon}{16|I^{\rm u}|^5},$$

we define

$$\tilde{\theta}(I^{\rm s}, I^{\rm u}) = \sin^{-1}\left(\frac{8(I^{\rm u})^5 (I^{\rm s} - I^{\rm u})}{\mu(1-\mu)(1-2\mu)15\pi\epsilon}\right) \qquad \tilde{\sigma}_\pm(I^{\rm u}, I^{\rm s}) = \sigma_{\pm,{\rm circ}}(\tilde{\theta}(I^{\rm s}, I^{\rm u}), I^{\rm u}).$$

where $\sigma_{\pm,{\rm circ}}$ are defined in (2.99). Denoting by $\widetilde{A}_\pm$ the differential of the map $(\sigma, \theta) \mapsto F(\theta, \sigma, I^{\rm u}, I^{\rm s})$, where F is defined in (2.98), evaluated at $(\tilde{\theta}(I^{\rm u}, I^{\rm s}), \tilde{\sigma}_\pm(I^{\rm u}, I^{\rm s}), I^{\rm u}, I^{\rm s})$, we obtain that

$$\begin{aligned}\widetilde{A}_\pm &= \begin{pmatrix} \pm\mu(1-\mu)L_{1,1} & 0 \\ 0 & \mu(1-\mu)(1-2\mu)\frac{15\pi\epsilon}{8I^5}\cos\tilde{\theta} \end{pmatrix} \\ &+ \begin{pmatrix} \mathcal{O}((|I^{\rm u}|^{-5/2} + \epsilon|I^{\rm u}|^{3/2})\exp(-{\rm Re}((I^{\rm u})^3)/3)) & \mathcal{O}(|I^{\rm u}|^{-1/2}\exp(-{\rm Re}((I^{\rm u})^3)/3)) \\ \mathcal{O}(|I^{\rm u}|^{-1/2}\exp(-{\rm Re}((I^{\rm u})^3)/3)) & \mathcal{O}(\epsilon|I^{\rm u}|^{-11/2}) \end{pmatrix},\end{aligned}$$

and again, it follows from direct application of the Implicit function theorem the existence of a value I_* (which might be different from the one obtained in the proof of Theorem 2.3.9) and functions

$$\hat{\sigma}_\pm(I^{\rm u}, I^{\rm s}) = \tilde{\sigma}_\pm(\theta, I^{\rm u}) = +\mathcal{O}(\epsilon|I^{\rm u}|^2) \qquad \hat{\theta}_\pm(I^{\rm u}, I^{\rm s}) = \tilde{\theta}(I^{\rm u}, I^{\rm s}) + \mathcal{O}(\epsilon|I^{\rm u}|^{-11/2})$$

such that

$$F(\theta_\pm(I^{\rm u}, I^{\rm s}), \hat{\sigma}_\pm(I^{\rm u}, I^{\rm s}), I^{\rm u}, I^{\rm s}) = 0$$

for all

$$(I^{\rm u}, I^{\rm s}) \in \left\{(I^{\rm u}, I^{\rm s}) \in \mathbb{C}^2 \colon I^{\rm u} \in \Lambda_{\rho,I},\ |I^{\rm s} - I^{\rm u}| \leq \frac{\mu(1-\mu)(1-2\mu)15\pi\epsilon}{16|I^{\rm u}|^5}\right\}.$$

□

2.5 The generating functions of the scattering maps. Proof of Theorem 2.3.13

As explained in Section 2.3.2, Theorem 2.3.9 implies the existence of two scattering maps $\mathbb{P}_\pm : \mathcal{P}^*_\infty \to \mathcal{P}^*_\infty$ (see (2.32)). In this section, we provide the rather technical proof of Theorem 2.3.13, in which we prove the existence (and obtain an explicit expression) of a generating function for each of the scattering maps $\mathbb{P}_\pm$.

Proof of Theorem 2.3.13. Consider the time T map $\phi^T_{H_{\rm pol}}$ of the Hamiltonian $H_{\rm pol}$ introduced in (2.3). The transformation $\phi^T_{H_{\rm pol}}$ is exact symplectic and therefore there exists a function $P^T : M_{\rm pol} \to \mathbb{R}$ such that $\mathrm{d}P^T = (\phi^T_{H_{\rm pol}})^*\lambda - \lambda_{\rm pol}$. The function (it is defined modulo constants) P^T is known as the *primitive function* associated to the exact symplectic map $\phi^T_{H_{\rm pol}}$. It is a standard computation (see the proof of Theorem 13 in [DdlLS08]) that (up to a constant)

$$P^T = \int_0^T (i_{H_{\rm pol}}\lambda_{\rm pol} + H_{\rm pol}) \circ \phi^\tau_{H_{\rm pol}} \mathrm{d}\tau,$$

where $i_{H_{\rm pol}}\lambda_{\rm pol}$ denotes the contraction of the one form $\lambda_{\rm pol} = y\mathrm{d}r + G\mathrm{d}\alpha + E\mathrm{d}t$ with the vector field associated to the Hamiltonian $H_{\rm pol}$. Now we obtain an expression for the primitive function associated to the (exact symplectic) scattering maps $\mathbb{P}_\pm$. The natural candidate to consider as primitive function of the scattering map $\mathbb{P}_\pm$ defined in (2.32) would be (see Theorem 13 in [DdlLS08]) to consider the function P^T restricted to $\Gamma_\pm$, which is given by

$$\tilde{P}_\pm(\varphi^{\rm u}, I^{\rm u}) = \lim_{T\to\infty} \int_{-T}^{T} i_{H_{\rm pol}}\lambda_{\rm pol} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u})\ \mathrm{d}\tau, \tag{2.105}$$

where we have already taken into account that $H_{\rm pol} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm = 0$ and that the dynamics in $\mathcal{P}_\infty$ is trivial. However, the improper integral (2.105) is not convergent (Theorem 13 in [DdlLS08] is proved for scattering maps associated to Normally Hyperbolic Invariant Manifolds, however, in the present case the rate of contraction/expansion along the stable/unstable leaves of $\mathcal{P}_\infty$ is only polynomial). Indeed, for $\tau \to \pm\infty$ (see Lemma 2.2.1 and Proposition 2.3.6)

$$\begin{aligned} i_{H_{\rm pol}}\lambda_{\rm pol} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}) &= \left(y^2 + \frac{G^2}{r^2}\right) \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}) \\ &\sim I_{\rm m}^{-2}\left(y_{\rm h}^2(\tau) + \frac{1}{r_{\rm h}^2(\tau)}\right) = \frac{2}{I_{\rm m}^2 r_{\rm h}(\tau)} \sim \tau^{-2/3}. \end{aligned}$$

Therefore, we consider instead the renormalized primitive function $P_\pm : \mathcal{P}^*_\infty \to \mathbb{R}$, defined as

$$P_\pm(\varphi^{\rm u}, I^{\rm u}) = \int_{\mathbb{R}} \left(i_{H_{\rm pol}}\lambda_{\rm pol} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}) - Q'(\tau)\right) \mathrm{d}\tau, \tag{2.106}$$

where $Q(u)$ is any function satisfying $Q'(u) = 2/I_{\rm m}^2 r_{\rm h}(u)$. We now want to express the integrand in (2.106) in terms of the parametrizations (2.21). To that end we notice that

$$i_{H_{\rm pol}}\lambda_{\rm pol} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm = i_{H_{\rm pol}}\lambda_{\rm pol} \circ \phi^\tau_{H_{\rm pol}} \circ \eta_{I_{\rm m}} \circ (\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm = i_{H_{\rm pol}}\lambda_{\rm pol} \circ \eta_{I_{\rm m}} \circ \phi^\tau_{\rm h} \circ (\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm.$$

Then, using Lemma 6.4.2 and the definition of Q,

$$i_{H_{\rm pol}}\lambda_{\rm pol} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm = i_H(\lambda + \mathrm{d}Q + I_{\rm m}\mathrm{d}\beta) \circ \phi^\tau_{\rm h} \circ (\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm$$

where $\lambda = Y\mathrm{d}u + G\mathrm{d}\beta + E\mathrm{d}t$. Yet, the parametrization (2.21) is not defined at $u = 0$ so $\phi^\tau_{\rm h} \circ (\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm$ might not be defined for all $\tau \in \mathbb{R}$. The rather simple solution to this annoyance goes as follows. By Cauchy's initial value theorem, the function $\phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm$ can be extended analitically to a real analytic function, which, by abuse of notation, we denote as $\tau \mapsto \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm$, defined in a complex neighborhood of $\mathbb{R}$ and such that

$$\frac{\rm d}{{\rm d}\tau}\left(\phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm\right) = X_{H_{\rm pol}} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm,$$

where $X_{H_{\rm pol}}$ is the vector field associated to the Hamiltonian (2.3). Therefore, we can change the integration path in the definition of $P_\pm$ to a complex path $\gamma \subset \mathbb{C}$ on the domain of analyticity of the function $\tau \mapsto \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm$ and such that $0 \notin \gamma$. Moreover, we can choose γ to also satisfy that (π_u denotes the projection onto the u component)

$$u(\tau;\varphi^{\rm u}, I^{\rm u}) \equiv \pi_u\left(\eta^{-1}_{I_{\rm m}} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u})\right) \neq 0 \qquad \forall \tau \in \gamma. \tag{2.107}$$

This is possible since (see the expresion (2.15) of the Hamiltonian H and Proposition 2.3.6) away from $u \neq 0$

$$\frac{\rm d}{{\rm d}\tau}\pi_u \circ \phi^\tau_{\rm h}(u,\beta,t) = \partial_Y H \circ \phi^\tau_H(u,\beta,t) = 1 + \mathcal{O}(|I_{\rm m}|^{-5}),$$

so by taking γ which does not enter a $\mathcal{O}(|I_{\rm m}|^{-5})$ neighborhood of $\tau = 0$ we can guarantee that (2.107) holds. Then,

$$\begin{aligned} P_\pm(\varphi^{\rm u}, I^{\rm u}) &= \int_\gamma i_{H_{\rm pol}}\lambda_{\rm pol} \circ \phi^\tau_{H_{\rm pol}} \circ (\Omega^{\rm u})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}) - Q'(\tau)\ {\rm d}\tau \\ &= \int_\gamma i_H(\lambda + {\rm d}Q + I_{\rm m}{\rm d}\beta) \circ \phi^\tau_H \circ (\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}) - Q'(\tau){\rm d}\tau \end{aligned} \tag{2.108}$$

is well defined. Moreover,

$$\begin{aligned} &\lim_{T\to\infty}\int_{-T}^{T} i_{\rm h}{\rm d}Q\ \circ \phi^\tau_H \circ (\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}) - Q'(\tau)\ {\rm d}\tau \\ &\qquad = \lim_{T\to\infty}\int_{-T}^{T}\frac{\rm d}{{\rm d}\tau}(Q \circ \phi^\tau_H \circ (\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm)(\varphi^{\rm u}, I^{\rm u}) - Q'(\tau)\ {\rm d}\tau \\ &\qquad = \lim_{T\to\infty}(Q(u(T;\varphi^{\rm u}, I^{\rm u})) - Q(T)) + (Q(-T) - Q(u(-T;\varphi^{\rm u}, I^{\rm u}))). \end{aligned}$$

We claim that this limit is zero. Indeed, from the expresion (2.15) of the Hamiltonian H and Proposition 2.3.6 we observe, that for large values of u,

$$\begin{aligned} \frac{\rm d}{{\rm d}\tau}u(\tau;\varphi^{\rm u}, I^{\rm u}) =& \frac{\rm d}{{\rm d}\tau}\pi_u \circ \phi^\tau_H \circ (\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}) = \partial_Y H \circ \phi^\tau_H(\Omega^{\rm u} \circ \eta_{I_{\rm m}})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}) \\ =& 1 + \mathcal{O}(|u(\tau;\varphi^{\rm u}, I^{\rm u})|^{-2/3}). \end{aligned}$$

So for large T we have

$$|u(\pm T;\varphi^{\rm u}, I^{\rm u}) \mp T)| = \mathcal{O}(T^{1/3}).$$

Moreover, $Q'(\pm T) \sim T^{-2/3}$ for $T \to \infty$ so, by application of the mean value theorem

$$|Q(u(\pm T;\varphi^{\rm u}, I^{\rm u})) - Q(\pm T)| \lesssim Q'(\pm T)|u(\pm T;\varphi^{\rm u}, I^{\rm u}) \mp T| \leq \mathcal{O}(T^{-1/3}).$$

Therefore, expression (2.108) reduces to

$$P_\pm(\varphi^{\rm u}, I^{\rm u}) = \int_\gamma (i_H\lambda + I_{\rm m}{\rm d}\beta) \circ \phi^\tau_H \circ (\Omega^{\rm u})^{-1}_\pm(\varphi^{\rm u}, I^{\rm u}){\rm d}\tau. \tag{2.109}$$

Let now $\gamma^{\rm u} = \gamma|_{\tau \leq 0}$, $\gamma^{\rm s} = \gamma|_{\tau \geq 0}$ and introduce the functions

$$\begin{aligned} P^{\rm u}(u,\beta,t; I^{\rm u}, I^{\rm s}) &= \int_{\gamma^{\rm u}} i_H(\lambda + I_{\rm m}{\rm d}\beta) \circ \phi^\tau_{\rm h} \circ \mathcal{W}^{\rm u}(u,\beta,t; I^{\rm u}, I^{\rm s}){\rm d}\tau \\ P^{\rm s}(u,\beta,t; I^{\rm u}, I^{\rm s}) &= \int_{\gamma^{\rm s}} i_H(\lambda + I_{\rm m}{\rm d}\beta) \circ \phi^\tau_{\rm h} \circ \mathcal{W}^{\rm s}(u,\beta,t; I^{\rm u}, I^{\rm s}){\rm d}\tau, \end{aligned}$$

where $\mathcal{W}^{\mathrm{u,s}}$ are the parametrizations of the invariant manifolds introduced in (2.21). Therefore

$$\begin{aligned} P^{\mathrm{u}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)=&\int_{\gamma^{\mathrm{u}}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left((S^{\mathrm{u}}+I_{\mathrm{m}}\beta)\circ\phi_{\mathrm{h}}^{\tau}\circ\mathcal{W}^{\mathrm{u}}\right)(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)\mathrm{d}\tau\\ =&S^{\mathrm{u}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)+I_{\mathrm{m}}\beta-I^{\mathrm{u}}\varphi^{\mathrm{u}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)\\ P^{s}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)=&\int_{\gamma^{\mathrm{s}}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left((S^{\mathrm{s}}+I_{\mathrm{m}}\beta)\circ\phi_{\mathrm{h}}^{\tau}\circ\mathcal{W}^{\mathrm{s}}\right)(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)\mathrm{d}\tau\\ =&-S^{s}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)-I_{\mathrm{m}}\beta+I^{\mathrm{s}}\varphi^{\mathrm{s}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon), \end{aligned} \tag{2.110}$$

where $\varphi^{\mathrm{u,s}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)$ denotes the asymptotic value of the α coordinate along the unstable or unstable leave of a point in W^{u} or W^{s} given by the parametrization (2.21). Notice that, in particular,

$$(P^{\mathrm{u}}+P^{\mathrm{s}})(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)=\Delta S(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)+I^{\mathrm{s}}\varphi^{\mathrm{s}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)-I^{\mathrm{u}}\varphi^{\mathrm{u}}(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon).$$

Let now $(I^{\mathrm{u}},I^{\mathrm{s}})\in\mathcal{R}_I$ (see (2.35)) and denote by $\varphi_{\pm}^{\mathrm{u,s}}(I^{\mathrm{u}},I^{\mathrm{s}})$ the backwards and forward asymptotic value of the β component along the heteroclinic orbit which passes through the heteroclinic point $x_{\pm}=(u,\beta,t,Y,J,E)$ given by

$$\begin{aligned} x_{\pm}(I^{\mathrm{u}},I^{\mathrm{s}})=&\mathcal{W}^{\mathrm{u}}\circ\Phi(-I_{\mathrm{m}}^{3}(I^{\mathrm{u}},I^{\mathrm{s}})\hat{\sigma}_{\pm}(I^{\mathrm{u}},I^{\mathrm{s}}),\hat{\theta}_{\pm}(I^{\mathrm{u}},I^{\mathrm{s}}),0;I^{\mathrm{u}},I^{\mathrm{s}})\\ =&\mathcal{W}^{\mathrm{s}}\circ\Phi(-I_{\mathrm{m}}^{3}(I^{\mathrm{u}},I^{\mathrm{s}})\hat{\sigma}_{\pm}(I^{\mathrm{u}},I^{\mathrm{s}}),\hat{\theta}_{\pm}(I^{\mathrm{u}},I^{\mathrm{s}}),0;I^{\mathrm{u}},I^{\mathrm{s}}), \end{aligned}$$

where $\Phi(v,\theta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)$ is the change of variables obtained in Proposition 2.3.8 and $\hat{\sigma}_{\pm},\hat{\theta}_{\pm}$ were obtained in Proposition 2.3.12. That is,

$$\mathbb{P}_{\pm}(\varphi_{\pm}^{\mathrm{u}}(I^{\mathrm{u}},I^{\mathrm{s}}),I^{\mathrm{u}})=(\varphi_{\pm}^{\mathrm{s}}(I^{\mathrm{u}},I^{\mathrm{s}}),I^{\mathrm{s}}).$$

Then, using expression (2.110), we obtain that the primitive function in (2.109) can be expressed as

$$\begin{aligned} P_{\pm}(\varphi_{\pm}^{\mathrm{u}}(I^{\mathrm{u}},I^{\mathrm{s}}),I^{\mathrm{u}})=&(P^{\mathrm{u}}+P^{\mathrm{s}})\circ\Phi_{\pm}(\hat{\theta}_{\pm}(I^{\mathrm{u}},I^{\mathrm{s}});I^{\mathrm{u}})\\ =&\mathtt{S}_{\pm}(I^{\mathrm{u}},I^{\mathrm{s}})+I^{\mathrm{s}}\varphi_{\pm}^{\mathrm{s}}(I^{\mathrm{u}},I^{\mathrm{s}})-I^{\mathrm{u}}\varphi_{\pm}^{\mathrm{u}}(I^{\mathrm{u}},I^{\mathrm{s}}), \end{aligned}$$

where $\mathtt{S}_{\pm}$ is the function defined in (2.36). The proposition plainly follows from the definition of primitive function of an exact symplectic map. Indeed

$$\mathrm{d}\mathtt{S}_{\pm}=\mathrm{d}P_{\pm}-I^{\mathrm{s}}\mathrm{d}\varphi_{\pm}^{\mathrm{s}}-\varphi_{\pm}^{\mathrm{s}}\mathrm{d}I^{\mathrm{s}}+I^{\mathrm{u}}\mathrm{d}\varphi_{\pm}^{\mathrm{u}}+\varphi_{\pm}^{\mathrm{u}}\mathrm{d}I^{\mathrm{u}}=\varphi_{\pm}^{\mathrm{u}}\mathrm{d}I^{\mathrm{u}}-\varphi_{\pm}^{\mathrm{s}}\mathrm{d}I^{\mathrm{s}}.$$

□

2.6 Asymptotic analysis of the scattering maps

In this section we prove Theorems 2.3.16 and 2.3.19. Namely, we establish an asymptotic formula for the scattering maps defined in (2.32) and for their difference in terms of the reduced Melnikov potentials $\mathcal{L}_{\pm}$ defined in (2.38). Let $\eta_{I_{\mathrm{m}}}$ be the change of variables defined in (2.14), consider the function $\tilde{I}_{\pm}^{\mathrm{s}}(\theta,I^{\mathrm{u}})$ obtained in Theorem 2.3.9 (see also Lemma 2.4.25), let $\Phi_{\pm}$ be the map defined in (2.28) and let $\Omega_{\pm}^{\mathrm{u}}$ be the wave maps introduced in (2.30). By the expressions (2.33) for the wave maps, it follows that for all $(\varphi^{\mathrm{u}},I^{\mathrm{u}})\in\mathcal{P}_{\infty}^{*}$, the G coordinate of the scattering map $\mathbb{P}_{\pm}$ is given by

$$I_{\pm}^{\mathrm{s}}(\varphi^{\mathrm{u}},I^{\mathrm{u}})=\tilde{I}_{\pm}^{\mathrm{s}}\circ(\Omega_{\pm}^{\mathrm{u}}\circ\eta_{I_{\mathrm{m}}}\circ\Phi_{\pm})^{-1}(\varphi^{\mathrm{u}},I^{\mathrm{u}}).$$

2.6.1 The wave maps and their difference

Let

$$\varphi^{\mathrm{u}}=\Theta(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)\equiv\beta+\vartheta(u,\beta,t;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon) \tag{2.111}$$

be the map which to $q = (u, \beta, t)$ associates the backward asymptotic β component along the leave of the unstable foliation which passes through the point $\mathcal{W}^{\mathrm{u}}(q; I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon)$. The map (2.111) is indeed the inverse of the map $\Psi_\infty = \mathrm{Id} + \psi_\infty$ in (2.76) and therefore (the norm $[\![\cdot]\!]_{\eta,\nu,\rho}$ is defined in (2.59))

$$[\![\vartheta]\!]_{1/3,1/2,\rho} \lesssim |I^{\mathrm{u}}|^{-29/8}. \tag{2.112}$$

Remark 2.6.1. *A more refined analysis of the first step in the iterative process carried out in Sections 2.4.2 and 2.4.2 shows that the map $\Psi_\infty = \mathrm{Id} + \psi_\infty$ in* (2.76) *satisfies indeed $[\![\psi_\infty]\!]_{1/3,1/2,\rho_n} \lesssim |I^{\mathrm{u}}|^{-4}$ and consequently $[\![\vartheta]\!]_{1/3,1/2,\rho} \lesssim |I^{\mathrm{u}}|^{-4}$. Performing this extra step would complicate unnecessarily the iterative process in Sections 2.4.2 and 2.4.2. Therefore we continue our discussion making use of the rougher estimate* (2.112) *which is sufficient for our purposes.*

Let now $\Phi_\pm$ be the change of coordinates defined in (2.28) and $\sigma_\pm, \tilde{I}^{\mathrm{s}}_\pm$ be the functions obtained in Theorem 2.3.9. Define now the maps

$$\widetilde{\Omega}^{\mathrm{u}}_\pm = \Omega^{\mathrm{u}}_\pm \circ \eta_{I_{\mathrm{m}}} \circ \Phi_\pm \tag{2.113}$$

where $\Omega^{\mathrm{u}}_\pm$ are the backward wave maps introduced in (2.30). By construction

$$\widetilde{\Omega}^{\mathrm{u}}_\pm(\theta, I^{\mathrm{u}}) = \Big(\theta + (\vartheta \circ \Phi_\pm)(\theta, I^{\mathrm{u}}) + \phi_\theta(-I^3_{\mathrm{m}}(I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}))\sigma_\pm(\theta, I^{\mathrm{u}})), \theta, 0; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}})), \quad I^{\mathrm{u}}\Big) \tag{2.114}$$

where $\Phi = (v + \phi_v, \theta + \phi_\theta, t)$ was defined in Lemma 2.4.20. In this section we show that $\widetilde{\Omega}^{\mathrm{u}}_\pm$ is a $\mathcal{O}(|I_{\mathrm{m}}|^{-29/4})$-close to identity map and show that the difference between the map $\widetilde{\Omega}^{\mathrm{u}}_+$ and $\widetilde{\Omega}^{\mathrm{u}}_-$ is exponentially small. To do so we will show that the function

$$\Upsilon = \vartheta \circ \Phi + \phi_\theta \tag{2.115}$$

is the sum of a function $\Upsilon_{\mathrm{hom}} \in \mathrm{Ker}\mathcal{L}$ and a function which vanishes when evaluated at $(\theta, \sigma_\pm(\theta, I^{\mathrm{u}}), I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}))$.

By construction, if we denote by X_H the vector field associated to the Hamiltonian (2.15) and write

$$X^{\mathrm{u}}_H = (X^{\mathrm{u}}_{H,u}, X^{\mathrm{u}}_{H,\beta}, X^{\mathrm{u}}_{H,t}) = (X_{H,u} \circ \mathcal{W}^{\mathrm{u}}, X_{H,\beta} \circ \mathcal{W}^{\mathrm{u}}, I^3_{\mathrm{m}}),$$

then Θ, defined in (2.111), conjugates the vector field

$$\dot{u} = X^{\mathrm{u}}_{H,u} \circ \Theta^{-1} \qquad \dot{\varphi}^{\mathrm{u}} = 0 \qquad \dot{t} = I^3_{\mathrm{m}}$$

to the vector field $X^{\mathrm{u}}_{\mathrm{h}}$. That is, Θ^{-1} straightens the dynamics in the φ^{u} component. It is straightforward to check that this conjugacy is equivalent to the fact that, ϑ defined in (2.111), solves

$$\mathcal{L}^{\mathrm{u}}\vartheta = -X^{\mathrm{u}}_{H,\beta} \tag{2.116}$$

with

$$\mathcal{L}^{\mathrm{u}} = X^{\mathrm{u}}_{H,u}\partial_u + X^{\mathrm{u}}_{H,\beta}\partial_\beta + I^3_{\mathrm{m}}\partial_t$$

Notice now that

$$X^{\mathrm{u}}_{H,u} = 1 + 2A^{\mathrm{u}} \qquad X^{\mathrm{u}}_{H,\beta} = 2B^{\mathrm{u}},$$

where A^{u} and B^{u} are the functions defined in (2.56). Therefore, denoting by $\widetilde{\mathcal{L}}$ the differential operator defined in (2.92) one can rewrite (2.116) as

$$\widetilde{\mathcal{L}}\vartheta = -2B^{\mathrm{u}} + (A^{\mathrm{s}} - A^{\mathrm{u}})\,\partial_u\vartheta + (B^{\mathrm{s}} - B^{\mathrm{u}})\,\partial_\beta\vartheta.$$

It now follows from the definition of Φ in Lemma 2.4.20 that Υ, defined in (2.115), solves

$$\mathcal{L}\Upsilon = \Big((A^{\mathrm{u}} - A^{\mathrm{s}})\partial_u\vartheta + (B^{\mathrm{s}} - B^{\mathrm{u}})\,(1 + \partial_\beta\vartheta)\Big) \circ \Phi. \tag{2.117}$$

Write $\Phi^{-1} = (u + \tilde{\phi}_u, \beta + \tilde{\phi}_\beta, t)$. Thus, from the definition of $A^{\mathrm{u,s}}$ and $B^{\mathrm{u,s}}$, expression (2.117) can be rewritten as

$$\mathcal{L}\Upsilon = F\partial_v\Delta\mathcal{S} + G\partial_\theta\Delta\mathcal{S}$$

where $\Delta\mathcal{S}$ is defined in (2.23),

$$F = f(1+\partial_v\tilde{\phi}_u) + g\partial_v\tilde{\phi}_\beta \qquad\qquad G = g(1+\partial_\beta\tilde{\phi}_\beta) + f\partial_\beta\tilde{\phi}_v$$

and

$$f = -\frac{1}{2y_{\mathrm{h}}^2 r_{\mathrm{h}}^2 I_{\mathrm{m}}}\left(1+\partial_\beta\vartheta - r_{\mathrm{h}}^2\partial_u\vartheta\right) \qquad\qquad g = \frac{1}{y_{\mathrm{h}}^2 r_{\mathrm{h}}^3 I_{\mathrm{m}}}\left(1+\partial_\beta\vartheta - r_{\mathrm{h}}^{-1}\partial_u\vartheta\right)$$

Let $\mathcal{G}$ be the left inverse operator for $\mathcal{L}$ defined in (2.94). Since $\Delta\mathcal{S} \in \mathrm{Ker}\mathcal{L}$ (and $\partial_v\Delta\mathcal{S}, \partial_\theta\Delta\mathcal{S}$ too),

$$\mathcal{L}\left(\mathcal{G}(F)\partial_v\Delta\mathcal{S} + \mathcal{G}(G)\partial_\theta\Delta\mathcal{S}\right) = F\partial_v\Delta\mathcal{S} + G\partial_\theta\Delta\mathcal{S}$$

and hence,

$$\Upsilon = \Upsilon_{\mathrm{hom}} + \mathcal{G}(F)\partial_v\Delta\mathcal{S} + \mathcal{G}(G)\partial_\theta\Delta\mathcal{S}$$

for some function $\Upsilon_{\mathrm{hom}} \in \mathrm{Ker}\mathcal{L}$. Define now

$$\Upsilon_\pm(\theta, I^{\mathrm{u}}) = \Upsilon(-I_{\mathrm{m}}^{-3}(I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}))\sigma_\pm(\theta, I^{\mathrm{u}}), \theta, 0; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}})) \tag{2.118}$$

where $\sigma_\pm(\theta, I^{\mathrm{u}}), \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}})$ are the functions obtained in Theorem 2.3.9. Then, the functions $\widetilde{\Omega}_\pm$ defined in (2.113) satisfy

$$\widetilde{\Omega}_\pm(\theta, I^{\mathrm{u}}) = (\theta + \Upsilon_\pm(\theta, I^{\mathrm{u}}), I^{\mathrm{u}}). \tag{2.119}$$

Lemma 2.6.2. *For all* $(\theta, I^{\mathrm{u}}) \in \Lambda_\rho$,

$$|\Upsilon_\pm| \lesssim |I^{\mathrm{u}}|^{-29/8} \qquad \text{and} \qquad |\Upsilon_+ - \Upsilon_-| \lesssim |I^{\mathrm{u}}|^{-5/8}\exp(-\mathrm{Re}((I^{\mathrm{u}})^3))/3).$$

Proof. Taking into account the estimate for ϕ_θ in Lemma 2.4.20 and the estimate (2.112) (the norm $\|\cdot\|_{\nu,\rho}$ is defined in (2.85))

$$\|\Upsilon\|_{1/2,\rho} \le \|\vartheta\|_{1/2,\rho} + \|\phi_\theta\|_{1/2,\rho} \lesssim |I^{\mathrm{u}}|^{-29/8},$$

which implies the first estimate. In order to prove the result for the difference we only need to estimate

$$\|\Upsilon_{\mathrm{hom}}\|_{1/2,\rho} = \|\Upsilon - \mathcal{G}(F)\partial_v\Delta\mathcal{S} - \mathcal{G}(G)\partial_\theta\Delta\mathcal{S}\|_{1/2,\rho}.$$

Indeed, since $\partial_v\Delta\mathcal{S}(\sigma_\pm(\theta, I^{\mathrm{u}}), \theta; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}), \epsilon) = \partial_\theta\Delta\mathcal{S}(\sigma_\pm(\theta, I^{\mathrm{u}}), \theta; I^{\mathrm{u}}, \tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}), \epsilon) = 0$, it follows from Lemma 2.4.21

$$|\Upsilon_+ - \Upsilon_-| \le 2\,|(\mathrm{Id} - \pi_0)\Upsilon_{\mathrm{hom}}| \lesssim |I^{\mathrm{u}}|^3\,\|\Upsilon_{\mathrm{hom}}\|_{1/2,\rho}\exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3).$$

To estimate $\|\Upsilon_{\mathrm{hom}}\|_{1/2,\rho}$, one can check from the definition of F, G and the estimates for $\|\Delta S\|_{1/2,\rho}$ which can be deduced from Theorem 2.4.18 that

$$\begin{aligned}\|\mathcal{G}(F)\partial_v\Delta\mathcal{S}\|_{1/2,\rho} &\lesssim |I^{\mathrm{u}}|^{-1}\|\partial_v\Delta\mathcal{S}\|_{3/2,\rho} \lesssim |I^{\mathrm{u}}|^{-4}\\ \|\mathcal{G}(G)\partial_\theta\Delta\mathcal{S}\|_{1/2,\rho} &\lesssim |I^{\mathrm{u}}|^{-1}\|\partial_\theta\Delta\mathcal{S}\|_{1,\rho} \lesssim |I^{\mathrm{u}}|^{-4}\end{aligned}$$

and the proof is completed. □

Remark 2.6.3. *From now on we will decrease the value of* $\rho > 0$ *without mentioning.*

2.6.2 Proof of Theorem 2.3.16

From the definition of the scattering maps $\mathbb{P}_\pm(\varphi^{\mathrm{u}}, I^{\mathrm{u}}) = (\varphi^{\mathrm{s}}_\pm, I^{\mathrm{s}}_\pm)$ in (2.32), the expressions (2.33) for the wave maps $\Omega^{\mathrm{u}}_\pm$ and $\Omega^{\mathrm{s}}_\pm$, and the definition of $\widetilde{\Omega}_\pm$ in (2.113)

$$I^{\mathrm{s}}_\pm = \tilde{I}^{\mathrm{s}}_\pm \circ (\widetilde{\Omega}_\pm)^{-1}.$$

We then write

$$I^{\mathrm{s}}_\pm = I^{\mathrm{u}} + \partial_\theta \mathcal{L}_\pm + (\tilde{I}^{\mathrm{s}}_\pm - (I^{\mathrm{u}} + \partial_\theta \mathcal{L}_\pm)) \circ (\widetilde{\Omega}_\pm)^{-1} + ((I^{\mathrm{u}} + \partial_\theta \mathcal{L}_\pm) \circ (\widetilde{\Omega}_\pm)^{-1} - (I^{\mathrm{u}} + \partial_\theta \mathcal{L}_\pm)).$$

Therefore, from Lemma 2.4.25, we obtain that for all $(\varphi^{\mathrm{u}}, I^{\mathrm{u}}) \in \Lambda_\rho$

$$|(\tilde{I}^{\mathrm{s}}_\pm - (I^{\mathrm{u}} + \partial_\theta \mathcal{L}_\pm)) \circ (\widetilde{\Omega}_\pm)^{-1}| \lesssim \epsilon |I^{\mathrm{u}}|^{-11/2}.$$

Also, from the estimates of the Melnikov potential L given in Lemma 2.4.19, the expression (2.119) for $\widetilde{\Omega}_\pm$ and the estimate for $\Upsilon_\pm$ in Lemma 2.6.2, we deduce that, for all $(\varphi^{\mathrm{u}}, I^{\mathrm{u}}) \in \Lambda_\rho$,

$$|(I^{\mathrm{u}} + \partial_\theta \mathcal{L}_\pm) \circ (\widetilde{\Omega}_\pm)^{-1} - (I^{\mathrm{u}} + \partial_\theta \mathcal{L}_\pm)| \lesssim \epsilon |I^{\mathrm{u}}|^{-69/8}.$$

Combining both estimates

$$|I^{\mathrm{s}}_\pm - (I^{\mathrm{u}} + \partial_\theta \mathcal{L}_\pm)| \lesssim \epsilon |I^{\mathrm{u}}|^{-11/2}.$$

The result

$$|\varphi^{\mathrm{s}}_\pm - (\varphi^{\mathrm{u}} - \partial_{I^{\mathrm{u}}} \mathcal{L}_\pm)| \lesssim |I^{\mathrm{u}}|^{-7}$$

has already been proved in [GSMS17] (see also the proof of Proposition 2.6.5).

Remark 2.6.4. *In [GSMS17] the authors consider the case* $0 \leq \epsilon \leq \exp(-\mathrm{Re}(I^{\mathrm{u}})^3/3))$*, however, since both the main term in the asymptotic expansion and the error come from the circular part, the result holds for* $0 \leq \epsilon < 1$*).*

2.6.3 Proof of Theorem 2.3.19

We now derive asymptotic formulas for the difference between the components of each of the maps $\mathbb{P}_+$ and $\mathbb{P}_-$ defined in (2.32), thus completing the proof of Theorem 2.3.19.

Let $\tilde{I}^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}})$ be the functions obtained in Theorem 2.3.9, let $\hat{\sigma}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}), \hat{\theta}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}})$ be the functions obtained in Proposition 2.3.12, denote by $\Xi_\pm$ be the maps

$$(I^{\mathrm{u}}, I^{\mathrm{s}}) \mapsto \Xi_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}) = (\hat{\theta}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}), I^{\mathrm{u}}) \tag{2.120}$$

and define the function (see Proposition 2.3.13)

$$\mathtt{S}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}) = \Delta \mathcal{S}(\hat{\sigma}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}), \hat{\theta}_\pm(I^{\mathrm{u}}, I^{\mathrm{s}}); I^{\mathrm{u}}, I^{\mathrm{s}}).$$

Then, it follows from Proposition 2.3.13 that, for $(\varphi^{\mathrm{u}}, I^{\mathrm{u}}) \in \mathcal{P}^*_{\mathrm{vert}} = \mathcal{P}^*_\infty \cap \{\pi/8 \leq \varphi^{\mathrm{u}} \leq \pi/4\}$ (see Remark 2.3.21) the scattering maps $\mathbb{P}_\pm : (\varphi^{\mathrm{u}}, I^{\mathrm{u}}) \mapsto (\varphi^{\mathrm{s}}_\pm, I^{\mathrm{s}}_\pm)$ are given by the implicit expression

$$(\varphi^{\mathrm{u}}, I^{\mathrm{u}}) \mapsto (\varphi^{\mathrm{u}} + (\partial_{I^{\mathrm{u}}} \mathtt{S}_\pm + \partial_{I^{\mathrm{s}}} \mathtt{S}_\pm) \circ (\Omega^{\mathrm{u}}_\pm \circ \eta_{I_{\mathrm{m}}} \circ \Phi_\pm \circ \Xi_\pm)^{-1},\ \tilde{I}^{\mathrm{s}}_\pm \circ (\Omega^{\mathrm{u}}_\pm \circ \eta_{I_{\mathrm{m}}} \circ \Phi_\pm)^{-1}). \tag{2.121}$$

Proposition 2.6.5. *Let* $\mathcal{L}_\pm(\theta, I^{\mathrm{u}})$ *be the reduced Melnikov potentials introduced in* (2.38)*. Then, there exists* $I_* > 0$ *such that*

- *For all* $(\theta, I^{\mathrm{u}}) \in \Lambda$ *(see* (2.10)*),*

$$|(\tilde{I}^{\mathrm{s}}_+ - \tilde{I}^{\mathrm{s}}_-)(\theta, I^{\mathrm{u}}) - \partial_\theta(\mathcal{L}_+ - \mathcal{L}_-)(\theta, I^{\mathrm{u}})| \lesssim \epsilon (I^{\mathrm{u}})^{-5/2} \exp(-(I^{\mathrm{u}})^3)/3) \tag{2.122}$$

- *For all* $(\theta, I^{\mathrm{u}}) \in [\pi/8, \pi/4] \times \{I^{\mathrm{u}} \geq I_*\} \subset \mathrm{Dom}(\Xi_\pm^{-1})$*, we have*

$$\begin{aligned}|(\partial_{I^{\mathrm{u}}}\mathtt{S}_+ + \partial_{I^{\mathrm{s}}}\mathtt{S}_+ - \partial_{I^{\mathrm{u}}}\mathcal{L}_+)\circ\Xi_+^{-1} - (\partial_{I^{\mathrm{u}}}\mathtt{S}_- + \partial_{I^{\mathrm{s}}}\mathtt{S}_- - \partial_{I^{\mathrm{u}}}\mathcal{L}_-)\circ\Xi_-^{-1}| \\ \lesssim (I^{\mathrm{u}})^{-1/2}\exp(-(I^{\mathrm{u}})^3)/3).\end{aligned} \tag{2.123}$$

Proof. We first check (2.122). To do so, we write

$$\begin{aligned}\partial_\sigma \Delta\mathcal{S} =& \partial_\sigma \Delta\mathcal{S}_{\mathrm{circ}} + \partial_\sigma(L - L_{\mathrm{circ}}) + \mathcal{E}_\sigma \\ \partial_\theta \Delta\mathcal{S} =& \partial_\theta \Delta\mathcal{S}^{[0]} + \partial_\theta\left((\mathrm{Id} - \pi_0)(L - L_{\mathrm{circ}})\right) + \partial_\theta\left((\mathrm{Id} - \pi_0)\Delta\mathcal{S}_{\mathrm{circ}}\right) + \mathcal{E}_\theta,\end{aligned}$$

where $\mathcal{E}_\sigma = \partial_\sigma \mathcal{E}_{\mathrm{circ}}$, $\mathcal{E}_\theta = \partial_\theta\left((\mathrm{Id} - \pi_0)\mathcal{E}_{\mathrm{circ}}\right)$ and $\mathcal{E}_{\mathrm{circ}}$ has been defined in (2.97). Let now $\hat\sigma_\pm(\theta, I^{\mathrm{u}}, I^{\mathrm{s}})$ and $\hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}})$ be such that

$$\partial_\sigma \Delta\mathcal{S}(\theta, \tilde\sigma_\pm(\theta, I^{\mathrm{u}}, I^{\mathrm{s}}), I^{\mathrm{u}}, I^{\mathrm{s}}) = 0 \qquad\qquad (\partial_\theta \Delta\mathcal{S})^{[0]}(\theta, I^{\mathrm{u}}, \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}})) = 0.$$

One expects that the solution $(\sigma, I^{\mathrm{s}}) = (\sigma_\pm(\theta, I^{\mathrm{u}}), \tilde I^{\mathrm{s}}_\pm(\theta, I^{\mathrm{u}}))$ to $(\partial_\sigma\Delta\mathcal{S}, \partial_\theta\Delta\mathcal{S}) = 0$ is close to $(\sigma, I^{\mathrm{s}}) = (\hat\sigma_\pm(\theta, I^{\mathrm{u}}, \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}})), \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}}))$. The main term in the correction of the solution to the second equation of the system $(\partial_\sigma\Delta\mathcal{S}, \partial_\theta\Delta\mathcal{S}) = 0$ is given by the term

$$\partial_\theta\left((\mathrm{Id} - \pi_0)(L - L_{\mathrm{circ}})\right)(\theta, \hat\sigma_\pm(\theta, I^{\mathrm{u}}, I^{\mathrm{s}}); I^{\mathrm{u}}, \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}}), \epsilon) + \partial_\theta\left((\mathrm{Id} - \pi_0)\Delta\mathcal{S}_{\mathrm{circ}}\right)(\theta, \hat\sigma_\pm(\theta, I^{\mathrm{u}}, \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}})); I^{\mathrm{u}}, I^{\mathrm{s}}). \tag{2.124}$$

Therefore, using the fact that $\Delta\mathcal{S}_{\mathrm{circ}}(\theta, \sigma; I^{\mathrm{u}}, I^{\mathrm{s}}) = \Delta\widehat{\mathcal{S}}_{\mathrm{circ}}(\sigma - \theta; I^{\mathrm{u}}, I^{\mathrm{s}})$ and the definition of $\hat\sigma_\pm(\theta, I^{\mathrm{u}}, I^{\mathrm{s}})$, the term (2.124) can be expressed as

$$\begin{aligned}\partial_\theta\left((\mathrm{Id} - \pi_0)(L - L_{\mathrm{circ}})\right)(\theta, \hat\sigma_\pm(\theta, I^{\mathrm{u}}, I^{\mathrm{s}}); I^{\mathrm{u}}, \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}}), \epsilon) + \partial_\sigma(L - L_{\mathrm{circ}})(\theta, \hat\sigma_\pm(\theta, I^{\mathrm{u}}, \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}})); I^{\mathrm{u}}, I^{\mathrm{s}}, \epsilon) \\ + \mathcal{E}_\sigma(\theta, \hat\sigma_\pm(\theta, I^{\mathrm{u}}, I^{\mathrm{s}}); I^{\mathrm{u}}, \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}}), \epsilon).\end{aligned}$$

It follows from the fact that $L_{\mathrm{circ}} = L_{\mathrm{circ}}(\sigma - \theta, I^{\mathrm{u}})$ and the definition of $\mathcal{L}_\pm(\theta, I^{\mathrm{u}}; \epsilon)$ that

$$\partial_\theta \mathcal{L}_\pm(\theta, I^{\mathrm{u}}; \epsilon) = \partial_\theta\left((\mathrm{Id} - \pi_0)(L - L_{\mathrm{circ}})\right)(\theta, \tilde\sigma_\pm(\theta); I^{\mathrm{u}}, \epsilon) + \partial_\sigma(L - L_{\mathrm{circ}})(\theta, \tilde\sigma_\pm(\theta); I^{\mathrm{u}}, \epsilon),$$

where

$$\tilde\sigma_+(\theta) = \theta \qquad \tilde\sigma_+(\theta) = \theta + \pi.$$

Then, the asymptotic formula (2.122) follows from the estimates

$$|\mathcal{E}_\sigma|, |\mathcal{E}_\theta| \lesssim \epsilon |I^{\mathrm{u}}|^{-5/2}\exp(-\mathrm{Re}((I^{\mathrm{u}})^3/3))$$

the fact that

$$|\tilde\sigma(\theta) - \hat\sigma(\theta, I^{\mathrm{u}}, \hat I^{\mathrm{s}}(\theta, I^{\mathrm{u}}))| \lesssim |I^{\mathrm{u}}|^{-1},$$

and Lemma 2.4.19.

We now prove the asymptotic formula (2.123). Let $\Phi_\pm$ be defined in (2.28) and $\Xi_\pm$ be defined in (2.44). Then, using that

$$(\partial_\sigma\Delta\mathcal{S})\circ\Phi_\pm\circ\Xi_\pm = (\partial_\theta\Delta\mathcal{S})\circ\Phi_\pm\circ\Xi_\pm = 0$$

we have

$$\begin{aligned}\partial_{I^{\mathrm{u}}}\mathtt{S}_\pm + \partial_{I^s}\mathtt{S}_\pm =& \partial_{I^{\mathrm{u}}}(\Delta\mathcal{S}\circ\Phi_\pm\circ\Xi_\pm) + \partial_{I^s}(\Delta\mathcal{S}\circ\Phi_\pm\circ\Xi_\pm) = (\partial_{I^{\mathrm{u}}}\Delta S + \partial_{I^s}\Delta S)\circ\Phi_\pm\circ\Xi_\pm \\ =& (\partial_{I^{\mathrm{u}}}\tilde L + \partial_{I^s}\tilde L)\circ\Phi_\pm\circ\Xi_\pm + \mathcal{E}_I\end{aligned}$$

where $\tilde L$ is the Melnikov potential defined in (2.50), and

$$\mathcal{E}_I = (\partial_{I^{\mathrm{u}}}(\Delta S - \tilde L) + \partial_{I^{\mathrm{s}}}(\Delta S - \tilde L))\circ\Phi_\pm\circ\Xi_\pm.$$

It follows from the estimate

$$|(\mathrm{Id}-\pi_0)(\Delta S-\tilde{L})(\theta,\sigma;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)|\lesssim (I^{\mathrm{u}})^{-5/2}\exp(-(I^{\mathrm{u}})^3/3)$$

in Proposition 2.3.8, that for all $(\theta,I^{\mathrm{u}})\in\Lambda$

$$|(\mathrm{Id}-\pi_0)\partial_{I^{\mathrm{u,s}}}(\Delta S-\tilde{L})(\theta,\sigma;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)|\lesssim (I^{\mathrm{u}})^{-1/2}\exp(-(I^{\mathrm{u}})^3/3).$$

so

$$|\mathcal{E}_I(\theta,\sigma;I^{\mathrm{u}},I^{\mathrm{s}},\epsilon)|\lesssim (I^{\mathrm{u}})^{-1/2}\exp(-(I^{\mathrm{u}})^3/3).$$

Therefore, it follows from the definition of $\Phi_\pm(\theta,I^{\mathrm{u}})$ that

$$|\mathcal{E}_I\circ\Phi_+-\mathcal{E}_I\circ\Phi_-|\lesssim (I^{\mathrm{u}})^{-1/2}\exp(-(I^{\mathrm{u}})^3/3),$$

for all $(\theta,I^{\mathrm{u}})\in\Lambda$ and the asymptotic formula (2.123) is inmediate. □

Finally, we complete the proof of Theorem 2.3.19.

Proof of Theorem 2.3.19. We write

$$I^s_+-I^{\mathrm{s}}_-=(\tilde{I}^{\mathrm{s}}_+-\tilde{I}^{\mathrm{s}}_-)\circ(\widetilde{\Omega}_+)^{-1}+\mathcal{E}_1\qquad\text{with}\qquad\mathcal{E}_1=\tilde{I}^{\mathrm{s}}_-\circ(\widetilde{\Omega}_+)^{-1}-\tilde{I}^{\mathrm{s}}_-\circ(\widetilde{\Omega}_-)^{-1}$$

and the result for the G component follows using that $\widetilde{\Omega}_\pm(\theta,I^{\mathrm{u}})=(\theta+\Upsilon_\pm(\theta,I^{\mathrm{u}}),I^{\mathrm{u}})$ and the estimate $|\Upsilon_+-\Upsilon_-|\lesssim|I^{\mathrm{u}}|^{-5/8}\exp(-\mathrm{Re}((I^{\mathrm{u}})^3)/3)$ given in Lemma 2.6.2. Indeed by the mean value theorem

$$|\mathcal{E}_1|=|\tilde{I}^{\mathrm{s}}_-\circ(\widetilde{\Omega}_+)^{-1}-\tilde{I}^{\mathrm{s}}_-\circ(\widetilde{\Omega}_-)^{-1}|\lesssim\sup_{\theta\in\mathbb{T}_\rho}|\partial_\theta\tilde{I}^{\mathrm{s}}_-||\Upsilon_+-\Upsilon_-|\lesssim\epsilon(I^{\mathrm{u}})^{-45/8}\exp(-(I^{\mathrm{u}})^3/3),$$

where we have used that for all $(\theta,I^{\mathrm{u}})\in\Lambda_\rho$

$$|\partial_\theta\tilde{I}^{\mathrm{s}}_-|\lesssim|\partial^2_{\theta\theta}\mathcal{L}_-|\lesssim\epsilon(I^{\mathrm{u}})^{-5}.$$

We now study the angular component, which for $(\varphi^{\mathrm{u}},I^{\mathrm{u}})\in\Lambda_{\mathrm{vert}}$, is given by

$$\varphi^{\mathrm{s}}_+-\varphi^{\mathrm{s}}_-=((\partial_{I^{\mathrm{u}}}\mathtt{S}_++\partial_{I^{\mathrm{s}}}\mathtt{S}_+)\circ\Xi_+^{-1}-(\partial_{I^{\mathrm{u}}}\mathtt{S}_-+\partial_{I^{\mathrm{s}}}\mathtt{S}_-)\circ\Xi_-^{-1})\circ\widetilde{\Omega}_+^{-1}+\mathcal{E}_2,$$

where

$$\mathcal{E}_2=(\partial_{I^{\mathrm{u}}}\mathtt{S}_-+\partial_{I^{\mathrm{s}}}\mathtt{S}_-)\circ(\widetilde{\Omega}_+\circ\Xi_-)^{-1}-(\partial_{I^{\mathrm{u}}}\mathtt{S}_-+\partial_{I^{\mathrm{s}}}\mathtt{S}_-)\circ(\widetilde{\Omega}_-\circ\Xi_-)^{-1}.$$

The asymptotic formulas for the Melnikov potential given in Lemma 2.4.19 and the uniform estimates in Proposition 2.6.5 imply that

$$|\partial_\theta((\partial_{I^{\mathrm{u}}}\mathtt{S}_-+\partial_{I^{\mathrm{s}}}\mathtt{S}_-)\circ\Xi_-^{-1})|\lesssim\epsilon(I^{\mathrm{u}})^{-6}.$$

Since

$$|\Upsilon_+-\Upsilon_-|\lesssim(I^{\mathrm{u}})^{-5/8}\exp(-(I^{\mathrm{u}})^3)/3),$$

we obtain that, for all $(\theta,I^{\mathrm{u}})\in\Lambda_{\mathrm{vert}}$,

$$|\mathcal{E}_2|\lesssim\epsilon(I^{\mathrm{u}})^{-53/8}\exp(-(I^{\mathrm{u}})^3/3).$$

Theorem 2.3.19 now follows combining these estimates with the ones given in Proposition 2.6.5.

□

2.A The perturbative potential V and the Melnikov potential L

In this appendix we provide we provide the proofs of Lemma 2.4.7, which describes the behavior of the perturbative potential V defined in (2.16), Lemma 2.4.19 which states the main properties of the Melnikov potential L defined in (2.50) and Lemma 2.3.15 concerning the reduced Melnikov potentials $\mathcal{L}_\pm$ introduced in (2.38). We start by recalling the following well known result, a proof of which can be found in [MP94].

Lemma 2.A.1. *Let $r_\mathrm{h}(u)$ and $\alpha_\mathrm{h}(u)$ the functions defined in Lemma 2.2.1. Then, under the real analytic change of variables $u=(\tau+\tau^3/3)/2$, and using the same notation $r_\mathrm{h}(\tau)$ and $\alpha_\mathrm{h}(\tau)$, we have that*

$$r_\mathrm{h}(\tau)=\frac{\tau^2+1}{2}\qquad\qquad e^{i\alpha_\mathrm{h}(\tau)}=\frac{\tau-i}{\tau+i}.$$

2.A.1 Proof of Lemma 2.4.7

From the definition of $V(u,\beta,t;I_\mathrm{m})$ in (2.16) and straightforward manipulations we obtain that

$$\begin{aligned}U(\tau,\beta,t;I_\mathrm{m})=&V(u(\tau),\beta,t;I_\mathrm{m})\\=&\frac{\mu I_\mathrm{m}}{r_\mathrm{h}(\tau)\left(1+\frac{2(1-\mu)\varrho(t)}{I_\mathrm{m}^2r_\mathrm{h}(\tau)}e^{i(\beta+\alpha_\mathrm{h}(\tau)-f(t))}\right)^{1/2}\left(1+\frac{2(1-\mu)\varrho(t))}{I_\mathrm{m}^2r_\mathrm{h}(\tau)}e^{-i(\beta+\alpha_\mathrm{h}(\tau)-f(t))}\right)^{1/2}}\\&+\frac{(1-\mu)I_\mathrm{m}}{r_\mathrm{h}(\tau)\left(1-\frac{2\mu\varrho(t)}{I_\mathrm{m}^2r_\mathrm{h}(\tau)}e^{i(\beta+\alpha_\mathrm{h}(\tau)-f(t))}\right)^{1/2}\left(1-\frac{2\mu\varrho(t)}{I_\mathrm{m}^2r_\mathrm{h}(\tau)}e^{-i(\beta+\alpha_\mathrm{h}(\tau)-f(t))}\right)^{1/2}}-\frac{I_\mathrm{m}}{r_\mathrm{h}(\tau)}.\end{aligned}\tag{2.125}$$

As we need to bound the Fourier coefficients of $V(u,\beta,t;I_\mathrm{m})$ for $u\in D^\mathrm{u}_\kappa$, we will use the transformation in Lemma 2.A.1 and bound the potential in these variables, where we have the explicit expressions of r_h and α_h. Important in the sequel is that when $u\in D^\mathrm{u}_\kappa$ we know that $|\tau^2+1|\geq\kappa|I_\mathrm{m}|^{-3/2}$. We now define the Fourier coefficients of $t\to U(\tau,\beta,t;I_\mathrm{m})$ as the integral expression

$$U^{[l]}(\tau,\beta;I_\mathrm{m})=\frac{1}{2\pi}\int_0^{2\pi}U(\tau,\beta,t;I_\mathrm{m})e^{-ilt}\mathrm{d}t.\tag{2.126}$$

In this proof we will perform several changes of variables in this integral but we will keep the same notation for the functions ϱ and f. In order to analyze this integral, we change the integration variable to the eccentric anomaly ξ by means of Kepler equation $t=\xi-\epsilon\sin\xi$ so that (2.126) reads

$$U^{[l]}(\tau,\beta;I_\mathrm{m})=\frac{1}{2\pi}\int_0^{2\pi}(1-\epsilon\cos\xi)U(\tau,\beta,\xi-\epsilon\sin\xi;I_\mathrm{m})e^{-il(\xi-\epsilon\sin\xi)}\mathrm{d}t.\tag{2.127}$$

In this way, we have the explicit formulas

$$\varrho(\xi)=1-\epsilon\cos\xi\qquad\qquad\varrho(\xi)e^{if(\xi)}=a^2e^{i\xi}-\epsilon+\frac{\epsilon^2}{4a^2}e^{-i\xi},\tag{2.128}$$

where $a=(\sqrt{1+\epsilon}+\sqrt{1-\epsilon})/2$. Changing the integration contour in (2.127) to the line $\{\xi\in\mathbb{C}/2\pi\mathbb{Z}\colon\xi=\alpha_\mathrm{h}(\tau)+s,\ s\in[0,2\pi]\}$ we obtain that,

$$U^{[l]}(\tau,\beta;I_\mathrm{m})=\frac{e^{-il\alpha_\mathrm{h}(\tau)}}{2\pi}\int_0^{2\pi}(1-\epsilon\cos(\alpha_\mathrm{h}(\tau)+s))U(\tau,\beta,\alpha_\mathrm{h}(\tau)+s-\epsilon\sin(\alpha_\mathrm{h}(\tau)+s);I_\mathrm{m})e^{-il(s-\epsilon\sin(\alpha_\mathrm{h}(\tau)+s))}\mathrm{d}s,$$

and

$$\varrho(s)=(1-\epsilon\cos(\alpha_\mathrm{h}(\tau)+s))\qquad\varrho(s)e^{if(s)}=e^{i\alpha_\mathrm{h}(\tau)}\left(a^2e^{is}-\epsilon e^{-i\alpha_\mathrm{h}(\tau)}+\frac{\epsilon^2}{4a^2}e^{-i(2\alpha_\mathrm{h}(\tau)+s))}\right)\tag{2.129}$$

Now, the main observation is that, using the assumption $\epsilon \lesssim I_{\rm m}^{-2}$, for fixed $\kappa, \sigma > 0$ and all $(\tau, \beta) \in \{|\tau^2+1| \geq \kappa |I_{\rm m}|^{-3/2}\} \times \mathbb{T}_\sigma$ one can easily see that

$$\left|\epsilon e^{\pm i\alpha_{\rm h}(\tau)}\right| \lesssim I_{\rm m}^{-\frac{1}{2}}$$

and, therefore,

$$|\varrho(s)| \lesssim 1 \qquad \left|\varrho(s)\, e^{\pm i(\beta+\alpha_{\rm h}(\tau)-f(t))}\right| \lesssim 1.$$

Using these inequalities, as well as the fact that

$$\left|\frac{1}{I_{\rm m}^2 r_{\rm h}(\tau)}\right| \lesssim I_{\rm m}^{-\frac{1}{2}},$$

we obtain,

$$\left|\frac{2(1-\mu)\,\varrho(t)}{I_{\rm m}^2 r_{\rm h}(\tau)} e^{\pm i(\beta+\alpha_{\rm h}(\tau)-f(t))}\right| \lesssim \frac{1}{I_{\rm m}^2 r_{\rm h}(\tau)} \lesssim I_{\rm m}^{-\frac{1}{2}} \qquad \left|\frac{2\mu\varrho(t)}{I_{\rm m}^2 r_{\rm h}(\tau)} e^{\pm i(\beta+\alpha_{\rm h}(\tau)-f(t))}\right| \lesssim \frac{1}{I_{\rm m}^2 r_{\rm h}(\tau)} \lesssim I_{\rm m}^{-\frac{1}{2}}.$$

This justifies that we can use the Taylor formula

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \mathcal{O}(x^2),$$

to bound the Fourier coefficients of the potential. Using the cancellations of the order 0 and 1 terms we get, for a certain $\rho_0, \tilde{\sigma} > 0$ small enough but independent of $|I_{\rm m}|$, and for $(\tau, \beta) \in \{|\tau^2+1| \geq \kappa |I_{\rm m}|^{-3/2}\} \times \mathbb{T}_{\rho_0}$

$$|U^{[l]}(\tau, \beta)| \lesssim |I_{\rm m}|^{-3} |r_{\rm h}(\tau)|^{-3} |e^{-il\alpha_{\rm h}(\tau)}| e^{-|l|\tilde{\sigma}}.$$

Equivalenty, for $(u, \beta) \in D_\kappa^{\rm u} \times \mathbb{T}_{\rho_0}$

$$|V^{[l]}(u, \beta)| \lesssim |I_{\rm m}|^{-3} |r_{\rm h}(u)|^{-3} |e^{-il\alpha_{\rm h}(u)}| e^{-|l|\tilde{\sigma}}, \tag{2.130}$$

that taking into account Lemma 2.2.1 gives the desired bound for the norm of $V^{[l]}$ and V and completes the proof of the first estimate in Lemma 2.4.7. The estimate for the difference $V - V_{\rm circ}$ is obtained from the fact that V depends analytically on ϵ and a straightforward application of Schwarz's lemma.

2.A.2 Proof of Lemmas 2.3.15 and 2.4.19

The estimates (2.130) are enough to bound the associated Fourier coefficients $L^{[l]}(\beta; I_{\rm m})$ of the Melnikov potential $\tilde{L}(u, \beta, t; I_{\rm m})$ defined in (2.50). In fact

$$\tilde{L}(u, \beta, t; I_{\rm m}, \epsilon) = \sum e^{il(t-I_{\rm m}^3 u)} L^{[l]}(\beta; I_{\rm m}, \epsilon), \quad L^{[l]}(\beta; I_{\rm m}) = \int_{-\infty}^{\infty} V^{[l]}(s, \beta; I_{\rm m}, \epsilon) e^{ilI_{\rm m}^3 s} ds, \tag{2.131}$$

so we can write

$$\tilde{L}(u, \beta, t; I^{\rm u}, I^{\rm s}, \epsilon) = \sum_{l \in \mathbb{N}} \mathcal{L}_l(t - I_{\rm m}^3 u, \beta; I_{\rm m}, \epsilon)$$

where

$$\mathcal{L}_l(t - I_{\rm m}^3 u, \beta; I_{\rm m}, \epsilon) = e^{il(t-I_{\rm m}^3 u)} L^{[l]}(\beta; I_{\rm m}, \epsilon) + e^{-il(t-I_{\rm m}^3 u)} L^{[-l]}(\beta; I_{\rm m}, \epsilon).$$

Then, for $l \geq 1$, it is enough to change the path of integration to $\operatorname{Im} u = \frac{1}{3} - |I_{\rm m}|^3$ to bound $|L^{[l]}|$, use the bounds (2.130), use that

$$\left|e^{\pm i\alpha_{\rm h}(u)}\right| \lesssim |I_{\rm m}|^{\frac{3}{2}}$$

and the fact that $L^{[-l]} = \overline{L^{[l]}}$, to obtain, writing $\sigma = t - I_{\rm m}^3 u$,

$$|\mathcal{L}_l(\sigma, \beta; I_{\rm m})| \leq |I_{\rm m}|^{\frac{3l}{2}+\frac{3}{2}} \exp(-l\operatorname{Re}(I_{\rm m}^3)/3)$$

and, therefore, for the sum,

$$|\mathcal{L}_{\geq 2}(\sigma,\beta;I_{\rm m})|\leq |I_{\rm m}|^{\frac{9}{2}}\exp(-2{\rm Re}(I_{\rm m}^3)/3).$$

The coefficients $L^{[0]}$ and $L^{[1]}$ can be computed expanding the potential U up to order four in powers of $1/r_{\rm h}(\tau)$ and bounding the remainder in an analogous way. We do not do the computations here because they can be found in Lemmas 31 and 36 in [DKdlRS19]. Define the coefficients

$$c_l^{k,n}(\mu)=\frac{1}{2\pi}((1-\mu)^{k-1}-(-\mu)^{k-1})\int_0^{2\pi}\varrho^k(t)e^{-inf(t))}e^{-ilt}{\rm d}t.$$

Then, one has

$$\mathcal{L}_0(\beta;I_{\rm m})=\mu(1-\mu)\left(c_0^{2,0}(0)\frac{\pi}{2I_{\rm m}^3}+(1-2\mu)c_0^{3,1}(0)\frac{3\pi}{4I_{\rm m}^5}\cos\beta+\mathcal{O}(\epsilon^2|I_{\rm m}|^{-7})\right),$$

and

$$\mathcal{L}_1(\sigma,\beta;I_{\rm m},\epsilon)=\mu(1-\mu)\left(2L_{1,1}(I_{\rm m},\epsilon)\cos(\sigma-\beta)+2L_{1,2}(I_{\rm m},\epsilon)\cos(\sigma-2\beta)+\mathcal{O}(\epsilon|I_{\rm m}|^{-3/2},|c_1^{3,3}I_{\rm m}^4|)\right),$$

with

$$\begin{aligned}L_{1,1}(I_{\rm m},\epsilon)=&(1-2\mu)\left(c_1^{3,1}(0)\sqrt{\frac{\pi}{8I_{\rm m}}}+\mathcal{O}(|I_{\rm m}|^{-2})\right)\exp(-I_{\rm m}^3/3)\\ L_{1,2}(I_{\rm m},\epsilon)=&\left(c_1^{2,2}(0)\sqrt{\frac{\pi I_{\rm m}^3}{2}}+\mathcal{O}(\epsilon)\right)\exp(-I_{\rm m}^3/3).\end{aligned}$$

The proof of Lemma 2.4.19 is now completed by making use of Lemma 28 in [DKdlRS19] where the coefficients $c_l^{k,n}(0)$ are computed. An analogous computation is done in [GMPS22].
Finally, the proof of Lemma 2.3.15 is straightforward after noticing that

$$\mathcal{L}_+(\beta;I_{\rm m})=\sum_{l\in\mathbb{Z}}e^{il\beta}L^{[l]}(\beta;I_{\rm m})\qquad\qquad \mathcal{L}_-(\beta;I_{\rm m})=\sum_{l\in\mathbb{Z}}(-1)^le^{il\beta}L^{[l]}(\beta;I_{\rm m}).$$

2.B Proof of Lemma 2.3.24

We look for a symplectic change of variables as the time one map of the Hamiltonian flow ϕ_{F_1} induced by a function F_1 to be determined. We write $K_+=K_0+R_0$ where K_0 does not depend on time and the average $\langle R_0\rangle=0$. Notice that

$$\|X_{K_0}\|_{\rho/2}\equiv\varepsilon\lesssim I_*^{-4}\qquad\qquad \|X_{R_0}\|_{\rho/2}\equiv\tilde{\varepsilon}\lesssim\epsilon I_*^{-9}$$

By Taylor's formula with integral remainder we find that

$$K_+\circ\phi_{F_1}=K_0+\partial_\tau F_1+R_0+P_0$$

where

$$P_0=\{K_0,F_1\}+\int_0^1\{R_0+(1-s)\{K_0,F_1\},F_1\}\circ\phi_{F_1}^{\rm s}{\rm d}s.$$

Since $\langle R_0\rangle=0$ we can choose F_1 periodic and satisfying $F_1=-\int_0^\tau R_0{\rm d}s$ so

$$\|X_{F_1}\|_{\rho/2}\leq\tilde{\varepsilon}.$$

Now we write $K_+\circ\phi_{F_1}=K_1+R_1$ where $K_1=K_0+\langle P_0\rangle$ and $R_1=P_0-\langle P_0\rangle$. Write $\tilde{\rho}=\rho/2$, then, the estimates

$$\|X_{F_1}\|_{\tilde{\rho}}\lesssim\tilde{\varepsilon},\qquad\qquad |R_1|_{\tilde{\rho}-\delta}\lesssim\tilde{\varepsilon}\varepsilon,\qquad\qquad \|X_{K_1}-X_{K_0}\|_{\tilde{\rho}-2\delta}\lesssim\tilde{\varepsilon}\delta^{-1}\varepsilon \tag{2.132}$$

for any $0 < \varepsilon < \delta < \tilde{\rho}$ are straightforward. Indeed

$$|\{K_0, F_1\}|_{\tilde{\rho}} \lesssim \|X_{K_+}\|_{\tilde{\rho}} \|X_{F_1}\|_{\tilde{\rho}} \lesssim \varepsilon\tilde{\varepsilon} \qquad |\{R_0, F_1\}|_{\tilde{\rho}} \lesssim \|X_{R_0}\|_{\tilde{\rho}} \|X_{F_1}\|_{\tilde{\rho}} \lesssim \tilde{\varepsilon}^2$$

and

$$|\{\{K_0, F_1\}, F_1\}|_{\tilde{\rho}-\delta} \lesssim \delta^{-2} \|X_{K_+}\|_{\tilde{\rho}} \|X_{F_1}\|_{\tilde{\rho}} \|X_{F_1}\|_{\tilde{\rho}-\delta} \lesssim \varepsilon\tilde{\varepsilon}^2\delta^{-2} \leq \varepsilon\delta^{-1}\tilde{\varepsilon}$$

from where the second and third inequalities (2.132) plainly follow. Assume now that we are able to carry on the process iteratively and find n functions $F_i,\ i = 1, \ldots, n$ such that

$$K \circ \phi_{F_1} \circ \cdots \circ \phi_{F_n} = K_n + R_n$$

with

$$\|X_{F_n}\|_{\tilde{\rho}-2(n-1)\delta} \lesssim \tilde{\varepsilon}\delta^{-n+1}\varepsilon^{n-1}, \qquad |R_n|_{\tilde{\rho}-(2n-1)\delta} \lesssim \tilde{\varepsilon}\delta^{-n+1}\varepsilon^n, \qquad \left\|X_{K_n} - X_{K_{n-1}}\right\|_{\tilde{\rho}-2n\delta} \lesssim \tilde{\varepsilon}\delta^{-n}\varepsilon^n$$

where the symbol $a \lesssim b$ means that there exists $C > 0$ which does not depend on $n, \varepsilon, \tilde{\varepsilon}$ and δ such that $a \leq Cb$.

Then, if $\delta^{-1}\varepsilon < 1$ and $\tilde{\rho} - 2(n+1)\delta > 0$ is an easy computation to show that we can perform one averaging step more to obtain a new function F_{n+1} such that

$$K \circ \phi_{F_1} \circ \cdots \circ \phi_{F_{n+1}} = K_{n+1} + R_{n+1}$$

with

$$\left\|X_{F_{n+1}}\right\|_{\tilde{\rho}-2n\delta} \lesssim \tilde{\varepsilon}\delta^{-n}\varepsilon^n, \qquad |R_{n+1}|_{\tilde{\rho}-(2n+1)\delta} \lesssim \tilde{\varepsilon}\delta^{-n}\varepsilon^{n+1},$$

and

$$\left\|X_{K_{n+1}} - X_{K_n}\right\|_{\tilde{\rho}-2(n+1)\delta} \lesssim \tilde{\varepsilon}\delta^{-(n+1)}\varepsilon^{n+1}.$$

Therefore, taking $\delta = 2\varepsilon$, after a number $N = \left[\tilde{\rho}\delta^{-1}\right]/4$ of averaging steps we get that $\tilde{\rho} - 2n\delta \geq \tilde{\rho}/2 = \rho/8$ and the reminder has size

$$|R_N|_{\rho/8} \lesssim \tilde{\epsilon}\delta\,(\epsilon/\delta)^N = 2\tilde{\varepsilon}\varepsilon 2^{-N} = 2\tilde{\varepsilon}\varepsilon \exp\left(\frac{-[\tilde{\rho}\delta^{-1}]\ln 2}{4}\right)$$

from where the estimate (2.49) follows using the definition of δ. On the other hand, by construction

$$\|X_{K_N} - X_{K_0}\|_{\rho/8} \lesssim \sum_{n=1}^{N} \left\|X_{K_n} - X_{K_{n-1}}\right\|_{\rho/8} \lesssim \tilde{\varepsilon} \sum_{n=1}^{N} \delta^{-n}\varepsilon^n \leq \tilde{\varepsilon}$$

and we have shown that $\|X_{K_N} - (\mathbb{P}_+ - \mathrm{Id})\|_{\rho/8} \lesssim I_*^{-8}$. Finally, for $\psi = \phi_{F_1} \circ \cdots \circ \phi_{F_N}$ we have

$$\|\mathrm{Id} - \psi\|_{\rho/8} \lesssim \|X_{F_1}\|_{\rho/2} \lesssim \epsilon I_*^{-9}.$$

Chapter 3

Symbolic dynamics in the Restricted Elliptic Isosceles 3 Body Problem

Abstract: The restricted elliptic isosceles three body problem (REI3BP) models the motion of a massless body under the influence of the Newtonian gravitational force caused by two other bodies called the primaries. The primaries of masses $m_1 = m_2$ move along a degenerate Keplerian elliptic collision orbit (on a line) under their gravitational attraction, whereas the third, massless particle, moves on the plane perpendicular to their line of motion and passing through the center of mass of the primaries. By symmetry, the component of the angular momentum G of the massless particle along the direction of the line of the primaries is conserved.

We show the existence of symbolic dynamics in the REI3BP for large G by building a Smale horseshoe on a certain subset of the phase space. As a consequence we deduce that the REI3BP possesses oscillatory motions, namely orbits which leave every bounded region but return infinitely often to some fixed bounded region. The proof relies on the existence of transversal homoclinic connections associated to an invariant manifold at infinity. Since the distance between the stable and unstable manifolds of infinity is exponentially small, Melnikov theory does not apply.

3.1 Introduction

The restricted three body problem studies the motion of three bodies, one of them massless, under Newtonian gravitational force. The massless body does not exert any force on the other two, the primaries, and move therefore according to Kepler laws. As a particular case, in the restricted elliptic isosceles three body problem (REI3BP), the primaries move along a degenerate ellipse and the third (massless) body moves on the perpendicular plane to their line of motion passing through their center of mass, which is invariant. In this configuration the primaries collide, but since it is a Keplerian motion its collisions can be regularized. In a coordinate system with origin at the center of mass of the primaries, the position of the primaries is given by

$$q_1(t) = \frac{\rho(t)}{2}(0,0,1) \qquad q_2(t) = \frac{\rho(t)}{2}(0,0,-1), \tag{3.1}$$

where

$$\rho(t) = 1 - \cos E(t) \tag{3.2}$$

and the eccentric anomaly $E(t)$ satisfies

$$t = E - \sin E. \tag{3.3}$$

Introducing polar coordinates (r, y, α, G) in the plane of motion of the third body, where (y, G) denote the conjugated momenta to (r, α) the REI3BP is Hamiltonian with respect to

$$H(r, y, G, t) = \frac{y^2}{2} + \frac{G^2}{r^2} - \frac{1}{\sqrt{r^2 + \frac{\rho^2(t)}{4}}}. \tag{3.4}$$

It is inmediate to check that G is a conserved quantity so the REI3BP is a system of $1 + 1/2$ degrees of freedom. We fix $G \neq 0$ in order to avoid triple collisions.

In [BDV08] the authors the study the existence of symmetric periodic solutions of the Hamiltonian system associated to (3.4). In the present paper we prove the existence of chaotic dynamics in the REI3BP for large values of the angular momentum G, by building a Smale horseshoe with infinitely many symbols on a certain subset of the phase space. To build this horseshoe we first prove that the stable and unstable manifold associated to a certain invariant manifold intersect transversally, giving rise to homoclinic connections to the invariant manifold.

As a consequence, from the way the horseshoe is built, we deduce the existence of different types of orbits of the REI3BP according to their behavior as $t \to \pm\infty$. In particular, the existence of infinitely many periodic orbits of arbitrary large period is obtained. A complete classification of the orbits of the three body problem according to their final motion was already established by Chazy in 1922 [Cha22] (see also [AKN06]). For the restricted three body problem (either planar or spatial, circular or elliptic) the possibilities reduce to four:

- $H^{\pm}$(hyperbolic) : $\|r(t)\| \to \infty$ and $\|\dot{r}(t)\| \to c > 0$ as $t \to \pm\infty$.
- $P^{\pm}$(parabolic) : $\|r(t)\| \to \infty$ and $\|\dot{r}(t)\| \to 0$ as $t \to \pm\infty$.
- $B^{\pm}$(bounded) : $\limsup_{t \to \pm\infty} \|r(t)\| < \infty$.
- $OS^{\pm}$(oscillatory) : $\limsup_{t \to \pm\infty} \|r(t)\| = \infty$ and $\liminf_{t \to \pm\infty} \|r(t)\| < \infty$.

Examples of hyperbolic, parabolic and bounded motions were already known by Chazy (in particular they are present in the two body problem). However, no examples of oscillatory motions were known until Sitnikov [Sit60] proved their existence on a certain symmetric configuration of the spatial restricted three body problem, now called *the Sitnikov example*. We shall prove that any past-future combination of the four possible final motions exists in the REI3BP.

The connection between chaotic dynamics and the existence of different types of final motions was first devised by Moser [Mos01], who gave a new proof of the existence of oscillatory motions in the Sitnikov model. Moser's approach relying on the connection between final motions, transversal homoclinic points and symbolic dynamics has been successfully extended to provide more examples of these motions [LS80a, LS80b, Moe84, Moe07, GK12, GMS16]. When dealing with perturbations of integrable systems the classical strategy for showing the existence of transversal intersections between the invariant manifolds is to find non-degenerate zeros of the Melnikov function, which gives an asymptotic expression for the distance between them. However, when considering fast non-autonomous perturbations, the Melnikov function is exponentially small with respect to the perturbative parameter and the validity of Melnikov theory is not justified. This difficulty can be solved when the system in consideration has two perturbative parameters and an exponentially smallness condition between them is assumed. This was the approach in [LS80a], where the existence of oscillatory motions in the restricted planar circular three body problem (RPC3BP) was shown for values of the mass ratio exponentially small compared to the value of the inverse of the Jacobi constant.

The study of the existence of intersections between invariant manifolds for fast non-autonomous perturbations without assuming smallness conditions on extra parameters requires showing that the distance between invariant manifolds is indeed exponentially small. This problem, now known as *exponentially small splitting of separatrices*, has drawn remarkable attention in the past decades, but, due to its difficulty most of the available results concern concrete models [HMS88, DS92, Gel00, GOS10, GaG11] or in general systems under very restrictive hypothesis to be applicable to problems in Celestial Mechanics

[DGJS97, BF04a, BF04b, Gua12, BFGS12, Ga12]. Following these ideas, [GMS16] proves the transversality of certain invariant manifolds of the RPC3BP for any mass ratio and large Jacobi constants, extending the result in [LS80a] of existence of oscillatory motions to any mass ratio.

Following the same approach in [GMS16], the present paper proves the exponentially small splitting of separatrices in a real problem arising from Celestial Mechanics, the aforementioned REI3BP, under the only assumption of large angular momentum G. It is worth pointing out that the Hamiltonian (3.4) is, in general, far from being integrable. However, we will see in Section 3.2 that for orbits with large angular momentum G, the Hamiltonian (3.4) can be considered as a fast non-autonomous perturbation of the two body problem, which is integrable.

From our result we deduce the existence of transverse homoclinic connections and we are able to build a Smale horseshoe on a certain subset which is close to the homoclinc points. This result completes the previous work [BDDV17], where the existence of symbolic dynamics in the EIR3BP was investigated for large values of G using numerical techniques for analyzing the exponentially small splitting of separatrices.

The main result of the present paper, which gives the existence of chaotic dynamics in the REI3BP, is the following.

Theorem 3.1.1. *Denote by ψ the Poincaré map induced by the flow of the Hamiltonian* (3.4) *on the section $\Sigma_+ = \{(r, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} : y = 0,\ \dot{y} > 0\}$. Then, there exists $0 < G^* < \infty$ such that for $G > G^*$ there exists an invariant set $S \subset \Sigma_+$ such that the dynamics of $\psi : S \to S$ is topologically conjugated to the shift*

$$\begin{aligned} \sigma : \mathbb{N}^{\mathbb{Z}} &\to \mathbb{N}^{\mathbb{Z}} \\ \{a_n\}_{n\in\mathbb{Z}} &\mapsto \{a_{n-1}\}_{n\in\mathbb{Z}} \end{aligned}$$

Namely ψ has a Smale horseshoe of infinite symbols.

An immediate consequence of Theorem 3.1.1 is the existence of infinitely many periodic orbits in the system associated to Hamiltonian (3.4). Moreover, from the way the Smale horseshoe of Theorem 3.1.1 is built, we obtain the second main result (see Section 3.2 for a detailed exposition of this connection).

Theorem 3.1.2. *Denote by X^+ (respectively Y^-) either H^+, P^+, B^+ or OS^+ (respectively H^-, P^-, B^- or OS^-). Then, there exists $G^* < \infty$ such that if $G > G^*$ we have*

$$X^+ \cap Y^- \neq \emptyset$$

for all possible combinations of X^+ and Y^-. In particular, the Hamiltonian system (3.4) *posses oscillatory orbits, that is, orbits such that*

$$\limsup_{t\to\pm\infty} |r(t)| = \infty \quad \text{and} \quad \liminf_{t\to\pm\infty} |r(t)| < \infty.$$

As commented above, the proof of Theorem 3.1.1 relies on two main ingredients: establishing the existence of transversal intersections between the invariant manifolds $\mathcal{W}_\infty^{u,s}$ associated to a periodic orbit at infinity and showing the existence of a Smale horseshoe on a certain subset close to the homoclinc points. The latter follows from the arguments presented in [Mos01] without significant modifications. These arguments are sketched in Section 3.2 for the sake of self-completeness.

For the analysis of the splitting of the invariant manifolds, we use the fact that $\mathcal{W}_\infty^{u,s}$ are Lagrangian submanifolds so they can be parametrized as graphs which satisfy the Hamilton-Jacobi equation associated to H. Then, we study solutions to this equation in a suitable complex domain to get exponentially small asymptotics for the distance between $\mathcal{W}_\infty^{s}$ and $\mathcal{W}_\infty^{u}$. In order to obtain the appropiate exponent these parameterizations must be analyzed in a neighbourhood $\mathcal{O}\left(G^{-3}\right)$ of the singularities of the unperturbed homoclinic $(G \to \infty)$.

The document is organized as follows. In Section 3.2 we introduce the invariant manifolds at infinity and discuss the proofs and connection between Theorem 3.1.1 and Theorem 3.1.2. In particular, from Theorem 3.2.1, which claims the existence of transverse intersections of the infinity manifolds, we build a Smale horseshoe that is then used to show the existence of any past-future combination of final motions.

The rest of the paper is devoted to the proof of Theorem 3.2.1. We discuss the integrable system ($G \to \infty$) and its homoclinic manifold in Section 3.3.1. Section 3.3.2 is devoted to rewrite the problem of existence of the infinity manifolds as a fixed point equation. We solve this equation and bound the solution in a suitable complex domain in Section 3.4. In Section 3.5 we show that the distance between the invariant manifolds is given, up to first order, by the Melnikov function and then we compute its asymptotic expansion for large G in Section 3.A.

3.2 Description of the proof of theorems 3.1.1 and 3.1.2

We notice from the Hamiltonian (3.4) that the angular momentum G is a conserved quantity. Therefore, we apply the conformally symplectic change of variables

$$r = G^2\tilde{r}, \quad y = G^{-1}\tilde{y}, \quad t = G^3 s,$$

to the equations of motion associated to the Hamiltonan (3.4) to obtain a new system which is also Hamiltonian with respect to the scaled Hamiltonian.

$$\begin{aligned}\tilde{H}\left(\tilde{r},\tilde{y},s;G\right) &= G^2 H\left(G^2\tilde{r}, G^{-1}\tilde{y}, G^3 s\right)\\ &= \frac{\tilde{y}^2}{2} + \frac{1}{\tilde{r}^2} - \frac{1}{\tilde{r}} + U\left(\tilde{r}, G^3 s\right)\end{aligned} \tag{3.5}$$

where

$$U\left(\tilde{r}, G^3 s\right) = \frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2 + \rho^2\left(G^3 s\right)/4G^4}} = \frac{\rho^2\left(G^3 s\right)}{8G^4\tilde{r}^3}\left(1 + \mathcal{O}\left(\frac{1}{\tilde{r}^2 G^4}\right)\right). \tag{3.6}$$

Observe that, for G large, the system associated to the Hamiltonian (3.5) can be studied as a fast and small non-autonomous perturbation of the Kepler two-body problem. Adding time t as a phase variable, which we now denote by ξ, we see from the equations of motion associated to the Hamiltonian (3.5)

$$\begin{aligned}\frac{\mathrm{d}\tilde{r}}{\mathrm{d}s} &= \tilde{y}\\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}s} &= \frac{1}{\tilde{r}^3} - \frac{1}{\tilde{r}^2} - \partial_{\tilde{r}} U\\ \frac{\mathrm{d}\xi}{\mathrm{d}s} &= G^3,\end{aligned} \tag{3.7}$$

that $\Lambda = \{(\tilde{r},\tilde{y},\xi) = (\infty, 0, \xi) : \xi \in \mathbb{T}\}$ is a parabolic periodic orbit, which we will call infinity.

Denoting by $\phi_s = (\phi_s^{\tilde{r}}, \phi_s^{\tilde{y}}, \phi_s^{\xi})$ the flow of the system (3.7), we define the stable and unstable manifolds of infinity as

$$\begin{aligned}\mathcal{W}_\infty^s &= \left\{(\tilde{r},\tilde{y},\xi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} : \lim_{s\to+\infty} \phi_s^{\tilde{r}}(\tilde{r},\tilde{y},\xi) = \infty, \ \lim_{s\to+\infty} \phi_s^{\tilde{y}}(\tilde{r},\tilde{y},\xi) = 0\right\}\\ \mathcal{W}_\infty^u &= \left\{(\tilde{r},\tilde{y},\xi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} : \lim_{s\to-\infty} \phi_s^{\tilde{r}}(\tilde{r},\tilde{y},\xi) = \infty, \ \lim_{s\to-\infty} \phi_s^{\tilde{y}}(\tilde{r},\tilde{y},\xi) = 0\right\}.\end{aligned} \tag{3.8}$$

The usual way to study the dynamics near infinity is to use McGehee coordinates $r = 2x^{-2}$ which map neighbourhoods of infinity into bounded domains containing the origin. In particular, the periodic orbit Λ corresponds to the periodic orbit $\{(x, y, \xi) = (0, 0, \xi) : \xi \in \mathbb{T}\}$ in McGehee coordinates. This transformation was used in [McG73] to show that $\mathcal{W}_\infty^{u,s}$ exist and are analytic submanifolds except at infinity, where only C^∞ regularity is proven (see [BFM20c] for more general results). However, in the present work we prefer to stick to the original variables since the symplectic form is non canonical in McGehee coordinates.

For $G \to \infty$ the system is integrable since $U \to 0$ and therefore $\mathcal{W}_\infty^s$ and $\mathcal{W}_\infty^u$ coincide along a two dimensional homoclinic manifold which is foliated by Keplerian parabolic orbits. Hence, it can be

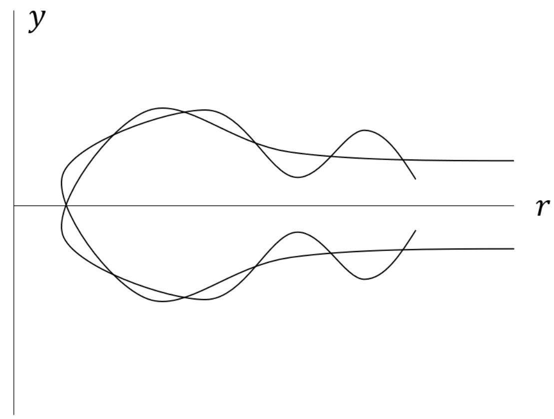

Figure 3.1: Stable and unstable invariant manifolds of infinity for the Poincaré map $\mathcal{P}_{\xi_0}$ in (3.10).

parametrized by the time section ξ and a suitable time parametrization $(\tilde{r}_h(v), \tilde{y}_h(v))$ of the parabolic orbit. We denote the parametrization of this invariant manifold as

$$\tilde{z}_h(v,\xi) = (\tilde{r}_h(v), \tilde{y}_h(v), \xi) \qquad \text{where} \qquad (v,\xi) \in \mathbb{R}\times\mathbb{T}, \tag{3.9}$$

and fix the origin of v such that $\tilde{y}_h(0) = 0$, which makes the homoclinic orbit symmetric under the map $v \to -v$. Some properties of this parametrization are discussed in Section 3.3.1.

We will prove that in the full problem (3.5), this two dimensional homoclinic manifold breaks down for $1 \ll G < \infty$, and $\mathcal{W}^s_\infty, \mathcal{W}^u_\infty$ do not longer coincide. In order to measure the distance between the invariant manifolds we introduce the Poincaré stroboscopic map

$$\begin{array}{rcl} \mathcal{P}_{\xi_0} : \{\xi = \xi_0\} & \to & \{\xi = \xi_0 + 2\pi\} \\ (\tilde{r}, \tilde{y}) & \mapsto & \mathcal{P}_{\xi_0}(\tilde{r}, \tilde{y}) \end{array} \tag{3.10}$$

so $\mathcal{W}^{s,u}_\infty \cap \{\xi = \xi_0\}$ become invariant curves $\gamma^{s,u}$ (see Figure 3.1).

Then, for $y > 0$, considering a parametrization of $\gamma^{s,u}$ of the form

$$\begin{array}{rl} \tilde{r} & = \tilde{r}_h(v) \\ \tilde{y} & = Y^{s,u}_{\xi_0}(v) \end{array} \tag{3.11}$$

where $\tilde{r}_h(v)$ is the parametrization of the unperturbed homoclinic (3.9), we observe that to measure the distance between the invariant manifolds along a suitable section $v = v^*$ it suffices to measure the difference between the functions $Y^{s,u}_{\xi_0}$. The following theorem is one of the two main ingredients needed for the proof of Theorem 3.1.1.

Theorem 3.2.1. *Let $\mathcal{W}^s_\infty$ and $\mathcal{W}^u_\infty$ be the infinity manifolds associated to the periodic orbit Λ and $\gamma^{s,u}$ the corresponding curves of the map $\mathcal{P}_{\xi_0}$. Then, for G large enough,*

(i) The curves $\gamma^{s,u}$ exist and have a parametrization of the form (3.11),

(ii) If we fix a section $\tilde{r} = \tilde{r}(v^*)$ the distance d between these curves along this section is given by

$$d = \frac{J_1(1)\sqrt{2\pi}}{\tilde{y}_h(v^*)} G^{1/2} e^{-\frac{G^3}{3}} \sin\left(\xi_0 - G^3 v^*\right) + E, \qquad |E| \le C G^{-1/2} e^{-\frac{G^3}{3}}, \tag{3.12}$$

where J_1 is the first Bessel function of first kind and $\tilde{y}_h$ correspond to the $\tilde{y}$ component of the unperturbed homoclinic and $C > 0$ is a constant independent of G.

(iii) There exist (at least) two transverse homoclinic connections to the periodic orbit Λ.

Item (iii) is a direct consequence of Item (ii). Indeed, since

$$J_1(1) \sim 0.44051 \neq 0$$

we observe that formula (3.12) in Theorem 3.2.1, implies that the zeros of the distance are given, up to first order, by the zeros of the function $\sin\left(\xi_0 - G^3 v^*\right)$. Therefore, transversal intersections of the invariant curves $\gamma^{s,u}$ will occur for values of $\xi_0 - G^3 v^*$ located in a neighbourhood $\mathcal{O}\left(G^{-1}\right)$ of the points $\xi_0 - G^3 v^* = 0, \pi$. These transversal intersections give rise to two homoclinic connections to the invariant manifold Λ as stated in the third item of Theorem 3.2.1.

Observe that the distance between the invariant manifolds is exponentially small with respect to G. As usually happens in exponentially small splitting of separatrices phenomena, the smaller the period of the fast perturbation (in our case $2\pi/G^3$), the smaller the distance between the manifolds (see [Nei84]).

3.2.1 Symbolic dynamics and oscillatory orbits

Once Theorem 3.2.1 is proven, the existence of chaotic dynamics is obtained following the techniques introduced in [Mos01]. For that we define the section

$$\Sigma_+ = \left\{(\tilde{r}, \tilde{y}, \xi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} : \tilde{y} = 0, \dot{\tilde{y}} > 0\right\} \tag{3.13}$$

and use coordinates $(\tilde{r}_0, \xi_0)$ for this section. Then, we define the Poincaré map

$$\begin{aligned} \psi : \Sigma_+ &\to \Sigma_+ \\ (\tilde{r}_0, \xi_0) &\mapsto (\tilde{r}_1, \xi_1) \end{aligned} \tag{3.14}$$

where $\xi_1 = \xi_0 + G^3 s$, and $s > 0$ is the first time in which $\phi_s(\tilde{r}_0, 0, \xi_0)$ intersects Σ_+ again and $\tilde{r}_1$ is such that $\phi_s(\tilde{r}_0, 0, \xi_0) = (\tilde{r}_1, 0, \xi_1)$. We set $\xi_1 = \infty$ for points $(\tilde{r}_0, \xi_0)$ which do not intersect Σ_+ anymore in the future and define $D_0 \subset \Sigma_+$ as the set of points for which $\xi_1 < \infty$. In the unperturbed problem $(G \to \infty)$ one easily deduces, using the conservation of energy, that Σ_+ is divided in two open sets, corresponding to initial conditions leading to hyperbolic and elliptic motions, whose common boundary is the curve in which the homoclinic manifold (3.9) intersects Σ_+. In this case, D_0 corresponds to the set of initial conditions leading to elliptic motions.

In order to characterize the set D_0 in the full problem (3.5) we make use of the following proposition, already proven in [BDDV17], which describes the intersection $\mathcal{W}^{s,u} \cap \Sigma_+$.

Proposition 3.2.2. *The stable manifold $\mathcal{W}^s$ intersects Σ_+ backwards for the first time in a simple curve*

$$\tilde{\gamma}^s = \left\{\left(\tilde{r}_0^s(\xi_0), \xi_0\right) \in \Sigma_+ : \tilde{r}_0^s(\xi_0 + 2\pi) = \tilde{r}_0^s(\xi_0)\right\}. \tag{3.15}$$

Analogously, the unstable manifold $\mathcal{W}^u$ intersects Σ_+ forward for the first time in a simple curve

$$\tilde{\gamma}^u = \left\{\left(\tilde{r}_0^u(\xi_0), \xi_0\right) \in \Sigma_+ : \tilde{r}_0^u(\xi_0 + 2\pi) = \tilde{r}_0^u(\xi_0)\right\}. \tag{3.16}$$

Remark 3.2.3. *From Theorem 3.2.1 we deduce that the curves $\tilde{\gamma}^{s,u}$ described in Proposition 3.2.2 intersect transversally, a fact which is crucial for the proof of Theorem 3.2.4.*

The curve $\tilde{\gamma}^s$ divides Σ_+ in two connected components. One of these components correspond to D_0 and the other component consists of initial conditions leading to orbits which do not intersect Σ_+ again and which escape to infinity with positive asymptotic radial velocity. We also define the set $D_1 \subset \Sigma_+$ of initial conditions $(\tilde{r}_0, \xi_0)$, in which the map ψ^{-1} is well defined. A similar argument to the one above using $\tilde{\gamma}^u$ instead of $\tilde{\gamma}^s$ can be used to identify this set.

Once we have identified D_0 and D_1, given a point $(\tilde{r}_0, \xi_0) \in D_0 \cap D_1$ we consider the sequence of consecutive times ξ_n given by $\psi^n(\tilde{r}_0, \xi_0) = (\tilde{r}_n, \xi_n)$ for $n \in \mathbb{Z}$ (whenever they exist) to define the sequence of integers

$$a_n = \left[\frac{\xi_n - \xi_{n-1}}{2\pi}\right],$$

where $[\cdot]$ defines the integer part. Thus, $a_n \in \mathbb{N}$ measures the number of binary collisions of the primaries between consecutive approaches of the third body. We introduce some technical concepts needed for stating the theorem that establishes the existence of symbolic dynamics on a subset of the closure $D_0 \cap D_1$ by conjugating ψ with the shift acting on a space of doubly infinite sequences.

Let A denote the set of all doubly infinite sequences

$$a = (\dots a_{-2}, a_{-1}, a_0; a_1, a_2 \dots)$$

of elements $a_n \in \mathbb{N}$. Equipping A with the product topology, the shift $\sigma : A \to A$ given by

$$\sigma\left(\{a_n\}_{n\in\mathbb{Z}}\right) = \{a_{n-1}\}_{n\in\mathbb{Z}} \tag{3.17}$$

is a homeomorphism.

We can define the compactification $\bar{A}$ of A admitting elements of the following type: For α, β integers satysfying $\alpha \leq 0$, $\beta \geq 1$, let

$$a = (\infty, a_{\alpha+1}, \dots, a_{\beta-1}, \infty) \quad a_n \in \mathbb{N}.$$

We also admit half infinite sequences with $\alpha = -\infty$, $\beta < \infty$ or $\alpha > -\infty$, $\beta = \infty$. It is possible to extend the topology defined on A to $\bar{A}$ in a way such that the shift (3.17) is a homeomorphism when restricted to

$$\bar{A}_0 = \left\{a \in \bar{A} \colon a_0 \neq \infty\right\}$$

(see [Mos01] for details).

The proof of the following theorem, from which Theorems 3.1.1 and 3.1.2 are deduced, follows from direct adaptation of the ideas presented in [Mos01] for the Sitnikov problem. The main ingredients are the transversal intersection of the curves $\gamma^{s,u}$ and a C^1 Lambda-Lemma for the parabolic invariant manifold Λ. This Lambda-Lemma follows from a careful analysis of the dynamics near Λ using McGehee coordinates which map neighbourhoods of infinity into bounded neighborhoods of the origin.

Theorem 3.2.4. *There exists a set $S \subset (D_0 \cap D_1)$ which is invariant under the Poincaré map ψ defined in* (3.14) *and such that its restriction to S, is conjugated to the shift σ defined in* (3.17). *That is, there exists an homeomorphism $\chi : A \to S$ such that*

$$\psi\chi = \chi\sigma.$$

Moreover, χ can be extended to $\bar{\chi} : \bar{A} \to \bar{S}$ such that

$$\psi\bar{\chi} = \bar{\chi}\sigma$$

if both sides are restricted to $\bar{A}_0$.

In other words, to each point $p = (r_0, \xi_0) \in S$ we associate a sequence $a(p) \in A$ which codifies the time between successive intersections of the flow $\phi_s(r_0, 0, \xi_0)$ with Σ_+. In this setting, the connection between Theorem 3.1.1 and Theorem 3.1.2 becomes clear. The first part of Theorem 3.2.4 corresponds to sequences

- $a(p) = (\dots a_{-2}, a_{-1}, a_0, a_1, \dots)$ with $a_n \in \mathbb{N}$ for all $n \in \mathbb{Z}$. These represent orbits which perform an infinite number of "close" approaches to the line where the primaries move both in the past and in the future. From this result we deduce the existence of any past-future combination of bounded ($\sup_{n\in\mathbb{Z}} a_n < \infty$) and oscillatory ($\limsup_{n\to\pm\infty} a_n = \infty$) motions.

The second part of the theorem, concerns sequences of the following type

- $a(p) = (\infty, a_{-k}, a_{-k+1}, \dots)$ with $a_n \in \mathbb{N}$ for all $n > -k$, which represent capture orbits, i.e., orbits where the third body comes from infinity at $t \to -\infty$ and remains revolving around the line of primaries for all future times. In particular, we obtain orbits which are hyperbolic or parabolic in the past and bounded or oscillatory in the future.

- $a(p) = (\dots a_{l-1}, a_l, \infty)$ with $a_n \in \mathbb{N}$ for all $n < l$. In this case the third body performed an infinite number of oscillations in the past but escapes to infinity as $t \to \infty$. These sequences correspond to orbtis which are bounded or oscillatory in the past and parabolic or hyperbolic in the future.

- $a(p) = (\infty, a_k, \dots, a_l, \infty)$ with $a_n \in \mathbb{N}$ for all $n \in \mathbb{Z}$ which corresponds to orbits coming from infinity, revolving around the primaries a finite number of times and escaping again to infinity as $t \to \infty$. They correspond to past-future combinations of parabolic and hyperbolic motions.

Finally, we point out that the existence of infinitely many periodic orbits in the REI3BP is deduced from Theorem 3.1.1 since fixed points for the shift correspond to periodic orbits of the Hamiltonian (3.5).

3.3 The invariant Manifolds as graphs

3.3.1 The unperturbed homoclinic solution

For the unperturbed problem, $G \to \infty$ in (3.5), the equations of motion reduce to

$$\begin{aligned} \frac{\mathrm{d}\tilde{r}}{\mathrm{d}v} &= \tilde{y} \\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}v} &= \frac{1}{\tilde{r}^3} - \frac{1}{\tilde{r}^2}. \end{aligned} \tag{3.18}$$

In this case the infinity manifolds $\mathcal{W}^{s,u}_{\infty}$ associated to Λ coincide along the two dimensional homocinic manifold $\tilde{z}_h$ introduced in (3.9). The (complex) singularities of $\tilde{z}_h(v,\xi)$ will be crucial for studying the existence of the invariant manifolds of the perturbed problem in certain complex domains. Thus, we state the following results, which were already obtained in [MP94].

1. The homoclinic solution (3.9) behaves as

$$\tilde{r}_h(v) \sim 3v^{2/3}, \qquad \tilde{y}_h(v) \sim 2v^{-1/3} \qquad \text{as} \qquad |v| \to \infty.$$

2. The homoclinic solution (3.9) is a real analytic function of v with singularities at $v = \pm i/3$.

3. Close to its singularities, the homoclinic solution (3.9) behaves as

$$\tilde{r}_h(v) \sim C\left(v \mp \frac{i}{3}\right)^{1/2}, \qquad \tilde{y}_h(v) \sim \frac{C}{2}\left(v \mp \frac{i}{3}\right)^{1/2}, \qquad \text{where} \qquad C^2 = \pm 2i.$$

3.3.2 The perturbed invariant manifolds and their difference

In this section we look for parametrizations of the infinity manifolds $\mathcal{W}^{u,s}_{\infty}$ in certain complex domains defined below. More concretely we look for graph parametrizations of $\mathcal{W}^{u,s}_{\infty}$ as solutions to a PDE. To do this we observe that the canonical form $\lambda = \tilde{r}\mathrm{d}\tilde{y} - \tilde{H}\mathrm{d}s$ is closed on the infinity manifolds (since the infinity manifolds are invariant by the flow it is enough to check that $\mathrm{d}\lambda$ is null on Λ). Then, one can see λ as the differential of a function $S(\tilde{r},\xi)$ such that

$$\partial_{\tilde{r}} S = \tilde{y} \qquad G^3 \partial_\xi S = -\tilde{H}$$

or, putting this together, as a solution of the Hamilton-Jacobi equation

$$G^3 \partial_\xi S + H(\tilde{r}, \partial_{\tilde{r}} S, \xi) = 0.$$

We write $S = S_0 + S_1$ where S_0 is the solution to the unperturbed problem

$$G^3 \partial_\xi S_0 + \frac{(\partial_{\tilde{r}} S_0)^2}{2} + \frac{1}{2\tilde{r}^2} - \frac{1}{\tilde{r}} = 0$$

and perform the change of variables

$$(\tilde{r}, \xi) \mapsto (\tilde{r}_h(v), \xi). \tag{3.19}$$

Then, the equation for $T_1(v, \xi) = S_1(\tilde{r}_h(v), \xi)$ becomes

$$\partial_v T_1 + \frac{1}{2\tilde{y}_h{}^2}(\partial_v T_1)^2 + G^3 \partial_\xi T_1 + V(v, \xi) = 0, \tag{3.20}$$

where

$$V(v, \xi) = U(\tilde{r}_h(v), \xi). \tag{3.21}$$

Note that the change of variables (3.19) implies that we are looking for parametrizations of the stable and unstable manifolds of the form

$$\begin{aligned} \tilde{r} &= \tilde{r}_h(v) \\ \tilde{y} &= \tilde{y}_h(v) + \frac{1}{\tilde{y}_h(v)} \partial_v T_1^{u,s} \end{aligned} \tag{3.22}$$

where $\tilde{r}_h(v), \tilde{y}_h(v)$ correspond to the unperturbed homoclinic (3.9) and $T_1^{u,s}(v, \xi)$ are solutions of equation (3.20) with asymptotic boundary condition for the unstable manifold

$$\lim_{v \to -\infty} \frac{1}{\tilde{y}_h(v)} \partial_v T_1^u = 0 \tag{3.23}$$

and the analogous one for the stable manifold. Once we show the existence of the unstable manifold, the existence of the stable one is guaranteed by symmetry. Indeed, if $T_1(v, \xi)$ is a solution of (3.20), $-T_1(-v, -\xi)$ is also a solution satisfying the opposite boundary condition.

Before going into the analysis of the existence of the generating functions $T_1^{u,s}$ we recall that our goal is to have a first asymptotic approximation of the distance between the infinity manifolds which now boils down to obtain an asymptotic formula for $\partial_v(T_1^u - T_1^s)$. To this end, we introduce the Melnikov potential

$$L(v, \xi; G) = \int_{-\infty}^{\infty} V\left(\tilde{r}_h(v+s), \xi + G^3 s\right) \mathrm{d}s, \tag{3.24}$$

which, as we state in Theorem 3.3.2 below approximates to first order the difference $\Delta = T_1^s - T_1^u$.

We point out that the parametrization (3.22) becomes undefined at $v = 0$ since we have fixed v such that $\tilde{y}_h(0) = 0$. Since in order to measure $\partial_v(T_1^u - T_1^s)$ we need both functions to be defined in a common domain we will introduce a different parametrization to extend the unstable manifold across $v = 0$. This is discussed in full detail in Section 3.4.

The next proposition gives the first asymptotic term of the Melnikov potential and will be proved in Section 3.A.

Proposition 3.3.1. *The function function $L(v, \xi; G)$ defined in (3.24) satisfies*

$$L(v, \xi; G) = L^{[0]}(G) + 2\sum_{l=1}^{\infty} L^{[l]}(G) \cos\left(l\left(\xi - G^3 v\right)\right),$$

where

$$\begin{aligned} L^{[1]}(G) &= -J_1(1)\sqrt{2\pi} G^{-5/2} e^{\frac{-G^3}{3}} \left(1 + \mathcal{O}\left(G^{-3/2}\right)\right) \\ \left|L^{[l]}(G)\right| &\leq K G^{-5/2} e^{l-1/2} e^{\frac{-|l|G^3}{3}}, \quad \text{for } l > 1, \end{aligned}$$

with J_1 the first Bessel function of the first kind and $K > 0$ a constant independent of G.

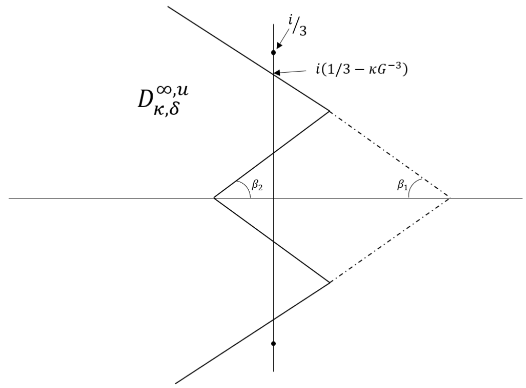

Figure 3.2: The domain $D^{\infty,u}_{\kappa,\delta}$ defined in (3.25).

Theorem 3.3.2. *Choose any* $0 < v_- < v_+ < \infty$. *Then, there exists* $K > 0$ *such that for any* $v \in [v_-, v_+]$ *and for any* G *large enough, the generating functions* $T_1^{u,s}(v,\xi)$ *satisfy*

$$|T_1^s(v,\xi) - T_1^u(v,\xi) - L(v,\xi) - E| \leq KG^{-7/2}e^{\frac{-G^3}{3}},$$

where $E \in \mathbb{R}$ *is a constant and*

$$|\partial_v (T_1^s(v,\xi) - T_1^u(v,\xi)) - \partial_v L(v,\xi)| \leq KG^{-1/2}e^{\frac{-G^3}{3}}.$$

From Proposition 3.3.1, Theorem 3.3.2 and Equation (3.22) we deduce Theorem 3.2.1. We devote Sections 3.4 and 3.5 to the proof of Theorem 3.3.2.

3.4 The invariant manifolds in complex domains

The classical procedure when studying exponentially small splitting of separatrices is to look for the functions T_1^u and T_1^s in a complex common domain $D \times \mathbb{T}$ where $D \subset \mathbb{C}$ is a connected domain which reaches a neighborhood of size $\mathcal{O}(G^{-3})$ (recall that the period of the perturbation (3.6) is $2\pi/G^3$) of the singularities of the unperturbed separatrix, i.e., $v = \pm i/3$ (see Section 3.3.1). The idea behind this approach is that for $v \in \mathbb{R}$ we will get exponentially small bounds on the distance $d(v,\xi)$ between the invariant manifolds if we show that d is a quasiperiodic function in some suitable coordinates and we manage to bound $|d|$ in a connected domain D which contains a subset of the real line and gets close to the singularities $v = \pm i/3$.

Since boundary conditions are imposed at infinity, we need to solve the equation (3.20) for T_1^u (resp. T_1^s) in a complex unbounded domain reaching $v \to -\infty$ (resp. $v \to \infty$). On the other hand, in order to measure their difference we need them to be defined in a common domain, we need to extend one of them across $v = 0$. However, the equation (3.20) becomes singular at $v = 0$ since $\tilde{y}_h(0) = 0$. To overcome this problem we divide the process of extension of the invariant manifolds into three steps.

We first solve equation (3.20) together with the boundary condition (3.23) in the domain

$$D^{\infty,u}_{\kappa,\delta} = \left\{v \in \mathbb{C}\colon\ |\mathrm{Im}(v)| < -\tan\beta_1 \mathrm{Re}(v) + 1/3 - \kappa G^{-3},\ |\mathrm{Im}(v)| > \tan\beta_2 \mathrm{Re}(v) + 1/6 - \delta\right\}, \qquad (3.25)$$

which does not contain $v = 0$ and where κ, δ and $\beta_1, \beta_2 \in (0, \pi/2)$ are fixed independently of G (see Figure 3.2). One can check that for $\delta \in (0, 1/12)$, $\kappa \sim \mathcal{O}(1)$, we can always find G big enough such that this domain is non empty. Once the existence of T_1^u in the domain $D^{\infty,u}_{\kappa,\delta}$ is proven, we exploit the

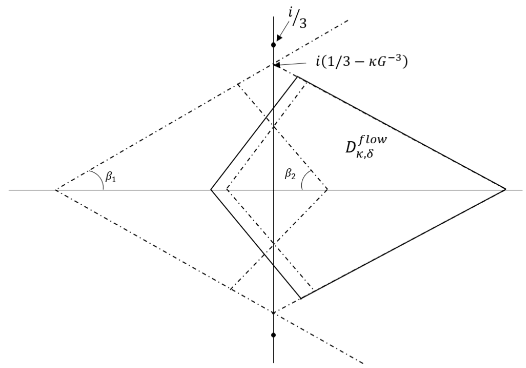

Figure 3.3: The domain $D^{\text{flow}}_{\kappa,\delta}$ defined in (3.28).

symmetry of equation (3.20) under the map $(v,\xi)\to(-v,-\xi)$ to atutomatically deduce the existence of T_1^s in the domain

$$D^{\infty,s}_{\kappa,\delta}=\left\{v\in\mathbb{C}\colon\ |\mathrm{Im}(v)|<\tan\beta_1\mathrm{Re}(v)+1/3-\kappa G^{-3},\ |\mathrm{Im}(v)|>-\tan\beta_2\mathrm{Re}(v)+1/6-\delta\right\}. \tag{3.26}$$

The next step is to perform the analytical continuation of T_1^u accross the imaginary axis. Thus, we would have both invariant manifolds defined on a common domain (this domain will be contained in $D^{\infty,s}_{\kappa,\delta}$ where T_1^s is already defined). Since $y_h(0)=0$, the equation (3.20) becomes singular at $v=0$ so we change to a parametrization invariant by the flow in the bounded domain

$$D_{\rho,\kappa,\delta}=D^{\infty,u}_{\kappa,\delta}\cap(\mathrm{Re}(v)>-\rho) \tag{3.27}$$

for some finite $\rho>0$. Then, we use the flow ϕ_s associated to the system (3.7) to extend the unstable manifold T_1^u to the domain

$$D^{\text{flow}}_{\kappa,\delta}=\left\{v\in\mathbb{C}\colon\ |\mathrm{Im}(v)|<-\tan\beta_1\mathrm{Re}(v)+1/3-\kappa G^{-3},\ |\mathrm{Im}v|<\tan\beta_2\mathrm{Re}(v)+1/6+\delta\right\} \tag{3.28}$$

which contains $v=0$ (see Figure 3.3). Then we go back to the original parametrization in a "boomerang domain"

$$\begin{aligned}D_{\kappa,\delta}=\big\{v\in\mathbb{C}:|\mathrm{Im}(v)|<-\tan\beta_1\mathrm{Re}(v)+1/3-\kappa G^{-3},|\mathrm{Im}(v)|<\tan\beta_1\mathrm{Re}(v)+1/3-\kappa G^{-3},\\ |\mathrm{Im}(v)|>-\tan\beta_2\mathrm{Re}(v)+1/6-\delta\big\},\end{aligned} \tag{3.29}$$

(which does not contain $v=0$) in order to measure the distance between the stable and unstable manifold.

3.4.1 Existence of the invariant manifolds close to infinity

In order to prove existence of the invariant manifolds we rewrite equation (3.20) as a fixed point equation in a suitable Banach space. We start by defining the linear operator

$$\mathcal{L}=\partial_v+G^3\partial_\xi \tag{3.30}$$

so equation (3.20) reads

$$\mathcal{L}\left(T_1^{u,s}\right)=\mathcal{F}\left(T_1^{u,s}\right)\qquad\text{where}\qquad\mathcal{F}\left(T_1^{u,s}\right)=-\frac{1}{2\tilde{y}_h^2}\left(\partial_vT_1^{u,s}\right)^2-V\left(v,\xi\right). \tag{3.31}$$

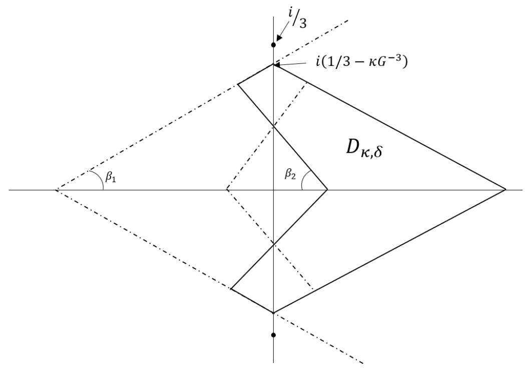

Figure 3.4: The domain $D_{\kappa,\delta}$ defined in (3.29)
.

We introduce the left inverse operators

$$\begin{aligned}\mathcal{G}^u(f)(v,\xi) &= \int_{-\infty}^{0} f\left(v+s,\xi+G^3 s\right)\mathrm{d}s\\ \mathcal{G}^s(f)(v,\xi) &= \int_{+\infty}^{0} f\left(v+s,\xi+G^3 s\right)\mathrm{d}s,\end{aligned}\tag{3.32}$$

so we can rewrite equation (3.31) as the fixed point equation

$$T_1^{u,s} = \mathcal{G}^{u,s}\circ\mathcal{F}\left(T_1^{u,s}\right).\tag{3.33}$$

Remark 3.4.1. *Throughout this section we will only work with the unstable manifold so we will omit the superindex u and write $D^{\infty}_{\kappa,\delta}$, T_1 and $\mathcal{G}$ instead of $D^{\infty,u}_{\kappa,\delta}$, T_1^u and $\mathcal{G}^u$ if there is no possible confusion.*

We look for solutions of this equation in the Banach spaces

$$\begin{aligned}\mathcal{Z}^{\infty}_{\nu,\mu} = \Big\{ & h(v,\xi) : D^{\infty}_{\kappa,\delta}\times\mathbb{T}\to\mathbb{C}\colon v\mapsto h(v,\xi) \text{ is real analytic, } \xi\mapsto h(v,\xi) \text{ is continuous,}\\ & \text{and } \|h\|_{\nu,\mu}<\infty\Big\},\end{aligned}\tag{3.34}$$

where

$$\|h\|_{\nu,\mu} = \sum_{l\in\mathbb{Z}}\left\|h^{[l]}\right\|_{\nu,\mu}$$

and

$$\left\|h^{[l]}\right\|_{\nu,\mu} = \sup_{v\in D^{\infty}_{\kappa,\delta}\setminus D_{\rho,\kappa,\delta}}\left|v^{\nu}h^{[l]}(v)\right| + \sup_{v\in D_{\rho,\kappa,\delta}}\left|\left(v^2+1/9\right)^{\mu}h^{[l]}(v)\right|.$$

Notice that the first term takes account of the behaviour at infinity and the second one of the behaviour near the singularities since $v^2+1/9=(v-i/3)(v+i/3)$. As we see from (3.33) we will also need to take control on the derivatives so we introduce

$$\begin{aligned}\tilde{\mathcal{Z}}^{\infty}_{\nu,\mu} = \Big\{ & h(v,\xi) : D^{\infty}_{\kappa,\delta}\times\mathbb{T}\to\mathbb{C}\colon v\mapsto h(v,\xi) \text{ is real analytic, } \xi\mapsto h(v,\xi) \text{ is continuous,}\\ & \text{and } [\![h]\!]_{\nu,\mu}<\infty\Big\},\end{aligned}\tag{3.35}$$

where

$$[\![h]\!]_{\nu,\mu} = \|h\|_{\nu,\mu} + \|\partial_v h\|_{\nu+1,\mu+1}.$$

The following lemma provides estimates for the norm of the perturbative potential.

Lemma 3.4.2. *Let V be the perturbative potential defined in* (3.21). *Then, for G large enough we have that*

$$\|V\|_{2,3/2} \leq KG^{-4}$$

for a constant $K > 0$ independent of G.

Proof. Since the domain $D^{\infty}_{\kappa,\delta}$ reaches a neighbourhood of order $\mathcal{O}\left(G^{-3}\right)$ of $v = \pm i/3$ we have that for G sufficiently large

$$\left|\frac{1}{G^4 \tilde{r}_h^2(v)}\right| \leq KG^{-1},$$

for $K > 0$ independent of G. Therefore, from (3.6) we deduce that for all $(v, \xi) \in D^{\infty}_{\kappa,\delta} \times \mathbb{T}$

$$|V(v,\xi)| \leq \frac{K}{G^4 \left|\tilde{r}_h(v)\right|^3}.$$

The conclusion follows now using the asymptotic expressions for $\tilde{r}_h(v)$ obtained in Section 3.3.1. □

We also state algebra-like properties for these spaces, which are straightforward from their definition and will be useful when dealing with the fixed point equation.

Lemma 3.4.3. *Let $\mathcal{Z}^{\infty}_{\nu,\mu}$ be the Banach spaces defined in* (3.34). *Then*

i) If $h \in \mathcal{Z}^{\infty}_{\nu,\mu}$ and $g \in \mathcal{Z}^{\infty}_{\nu',\mu'}$ then $hg \in \mathcal{Z}^{\infty}_{\nu+\nu',\mu+\mu'}$ with

$$\|hg\|_{\nu+\nu',\mu+\mu'} \leq \|h\|_{\nu,\mu} \|g\|_{\nu',\mu'}.$$

ii) If $h \in \mathcal{Z}^{\infty}_{\nu,\mu}$, then $h \in \mathcal{Z}^{\infty}_{\nu-\alpha}$ for $\alpha > 0$ with

$$\|h\|_{\nu-\alpha,\mu} \leq K \|h\|_{\nu,\mu}.$$

iii) If $h \in \mathcal{Z}^{\infty}_{\nu,\mu}$ then, for $\alpha > 0$ we have that $h \in \mathcal{Z}^{\infty}_{\nu,\mu-\alpha}$ with

$$\|h\|_{\nu,\mu-\alpha} \leq KG^{3\alpha} \|h\|_{\nu,\mu}.$$

iv) If $h \in \mathcal{Z}^{\infty}_{\nu,\mu}$ then, for $\alpha > 0$ we have that $h \in \mathcal{Z}^{\infty}_{\nu,\mu+\alpha}$ with

$$\|h\|_{\nu,\mu+\alpha} \leq K \|h\|_{\nu,\mu}.$$

The following lemma provide estimates for the inverse operator. The proof follows the exact same lines as in Lemma 5.5. in [GOS10] (see also [BFGS12]).

Lemma 3.4.4. *The operator $\mathcal{G}$ defined on* (3.32) *satisfies the following properties*

i) For any $\nu > 1$, $\mu\, 1$, $\mathcal{G} : \mathcal{Z}_{\nu,\mu} \to \mathcal{Z}_{\nu-1,\mu-1}$ is well defined, linear and satisfies $\mathcal{L} \circ \mathcal{G} = \mathrm{Id}$.

ii)If $h \in \mathcal{Z}_{\nu,\mu}$ for some $\nu > 1$, $\mu > 1$, then

$$\|\mathcal{G}(h)\|_{\nu-1,\mu-1} \leq K \|h\|_{\nu,\mu}. \tag{3.36}$$

iii) If $h \in \mathcal{Z}_{\nu,\mu}$ for some $\nu \geq 1$, $\mu \geq 1$, then

$$\|\partial_v \mathcal{G}(h)\|_{\nu,\mu} \leq K \|h\|_{\nu,\mu}. \tag{3.37}$$

Now we are ready to solve the fixed point equation.

Theorem 3.4.5. *Fix $\kappa > 0$ and$\delta > 0$. Then, for G large enough the fixed point equation* (3.33) *has a unique solution T_1^u on $D^{\infty}_{\kappa,\delta} \times \mathbb{T}$ which satisfies*

$$[\![T_1^u]\!]_{1,1/2} \leq b_0 G^{-4}$$

with $b_0 > 0$ independent of G. Moreover, if we define the function

$$L_1^u(v,\xi) = \mathcal{G}^u(V)(v,\xi)$$

we have

$$\|T_1^u - L_1^u\|_{1,1/2} \leq KG^{-13/2} \tag{3.38}$$

where $K > 0$ is independent of G.

Proof. We show that T_1 is the unique solution the fixed point equation (3.33). For that we first check that the operator $\mathcal{G}\circ\mathcal{F}$ is well defined from $\tilde{\mathcal{Z}}_{1,1/2}$ to itself. Indeed, from Lemma 3.4.2 we have that

$$\|V\|_{2,3/2} \leq KG^{-4}.$$

Then, the result follows from direct application of the properties of the homoclinic solution stated in Section 3.3.1, the algebra properties of the norm stated in Lemma 3.4.3 and Lemma 3.4.4 since we obtain that for $h \in \tilde{\mathcal{Z}}_{1,1/2}$

$$[\![\mathcal{G}\circ\mathcal{F}(h)]\!]_{1,1/2} \leq K\min\left([\![h]\!]_{1,1/2}, G^{-4}\right) \tag{3.39}$$

for some $K>0$ independent of G. In particular we deduce that there exists $b_0>0$ independent of G such that

$$[\![\mathcal{G}\circ\mathcal{F}(0)]\!]_{1,1/2} \leq \frac{b_0}{2}G^{-4}.$$

Then in order to show existence and uniqueness of solutions it is enough to show that the map $\mathcal{G}\circ\mathcal{F}$ is contractive on the ball $B\left(b_0G^{-4}\right)\subset\tilde{\mathcal{Z}}_{1,1/2}$ centered at 0. For that purpose we write

$$\mathcal{F}(h_2)-\mathcal{F}(h_1) = \frac{1}{2y_h^2}\left(\partial_v h_1+\partial_v h_2\right)\left(\partial_v h_1-\partial_v h_2\right)$$

so using that $h_1,h_2\in B\left(b_0G^{-4}\right)\subset\tilde{\mathcal{Z}}_{1,1/2,\kappa,\delta}$ we have

$$\begin{aligned}
\|\mathcal{F}(h_2)-\mathcal{F}(h_1)\|_{2,3/2} &\leq \left\|\frac{1}{2y_h^2}\left(\partial_v h_1+\partial_v h_2\right)\right\|_{0,0}\|\partial_v h_1-\partial_v h_2\|_{2,3/2}\\
&\leq KG^{3/2}\left\|\frac{1}{2y_h^2}\left(\partial_v h_1+\partial_v h_2\right)\right\|_{0,1/2}[\![h_1-h_2]\!]_{1,1/2}\\
&\leq KG^{-5/2}[\![h_1-h_2]\!]_{1,1/2},
\end{aligned}$$

and contractivity follows from Lemma 3.4.4 (enlarging G if necessary).

To obtain (3.38) we notice that

$$\begin{aligned}
\|T_1-L_1^u\|_{1,1/2} &= \|\mathcal{G}\circ\left(\mathcal{F}(T_1)-\mathcal{F}(0)\right)\|_{1,1/2}\\
&\leq [\![\mathcal{G}\circ\left(\mathcal{F}(T_1)-\mathcal{F}(0)\right)]\!]_{1,1/2}\\
&\leq KG^{-5/2}[\![T_1]\!]_{1,1/2}\leq KG^{-13/2}.
\end{aligned}$$

□

Since the parametrization (3.22) becomes singular at $v=0$, in the next section we look for a new parametrization of the unstable manifold which is regular at $v=0$ and therefore allows us to extend it across $v=0$.

3.4.2 Analytic continuation of the solution to the domain $D^{\mathbf{flow}}_{\kappa,\delta}$

In order to measure the distance between the stable and unstable manifolds we need them to be defined in a common domain. However, a parametrization of the form

$$\Gamma(v,\xi) = \begin{pmatrix}\tilde{r}(v,\xi)\\ \tilde{y}(v,\xi)\end{pmatrix} = \begin{pmatrix}\tilde{r}_h(v)\\ \frac{1}{\tilde{y}_h(v)}\partial_v T^u\end{pmatrix}$$

becomes undefined at $v=0$. To avoid this difficulty we look for a different parametrization of the unstable manifold in the domain $D_{\rho,\kappa,\delta}$ (3.27) which does not contain $v=0$ and then extend it by the flow. In order to proceed, we introduce the Banach spaces

$$\mathcal{Y}_{\mu,\rho,\kappa,\delta} = \Big\{h(v,\xi): D^{\infty}_{\kappa,\delta}\times\mathbb{T}\to\mathbb{C}\colon v\mapsto h(v,\xi) \text{ is real analytic, } \xi\mapsto h(u,\xi) \text{ is continuous,} \\ \text{and } \|h\|_{\mu}<\infty\Big\}, \tag{3.40}$$

where

$$\|h\|_\mu = \sum_{l\in\mathbb{Z}} \left\|h^{[l]}\right\|_\mu \tag{3.41}$$

and

$$\left\|h^{[l]}\right\|_\mu = \sup_{v\in D_{\rho,\kappa,\delta}} \left|\left(v^2+1/9\right)^\mu h^{[l]}(v)\right| \tag{3.42}$$

and the analogues of (3.35)

$$\tilde{\mathcal{Y}}_{\mu,\rho,\kappa,\delta} = \Big\{ h(v,\xi) : D^\infty_{\rho,\kappa,\delta}\times\mathbb{T}\to\mathbb{C}\colon v\mapsto h(v,\xi) \text{ is real analytic, } \xi\mapsto h(v,\xi) \text{ is continuous, and } [\![h]\!]_\mu<\infty \Big\},$$

with

$$[\![h]\!]_\mu = \|h\|_\mu + \|\partial_v h\|_{\mu+1}.$$

Remark 3.4.6. *Throughout this section we will work on different domains $D_{\rho,\kappa,\delta}$, $D^{flow}_{\kappa,\delta}$ and $\tilde{D}_{\kappa,\delta}$ (the latter is defined in* (3.56)*). We will denote by $\mathcal{Y}_{\mu,\kappa,\delta}$ the analogue to the Banach spaces* (3.40) *associated to the domain $\tilde{D}_{\kappa,\delta}$, and by $\mathcal{Y}^{flow}_{\mu,\kappa,\delta}$ the analogues for domain $D^{flow}_{\kappa,\delta}$* (3.28) *(in this case for vectorial functions since we will work with vector fields on the plane).*

From Hamilton-Jacobi parametrizations to parametrizations invariant by the flow

We look for a change of variables of the form $\mathrm{Id}+g:(v,\xi)\mapsto(v+g(v,\xi),\xi)$ such that

$$\hat{\Gamma}(v,\xi) = \Gamma\circ(\mathrm{Id}+g)(v,\xi) \tag{3.43}$$

satisfies

$$\phi_s\left(\hat{\Gamma}(v,\xi)\right) = \hat{\Gamma}\left(v+s,\xi+G^3 s\right).$$

Denoting by X the vector field generated by the Hamiltonian (3.5), this equation is equivalent to

$$X\circ\hat{\Gamma} = \mathcal{L}\left(\hat{\Gamma}\right), \tag{3.44}$$

which we can rewrite as

$$\mathcal{L}(g)(v,\xi) = \mathcal{F}\circ(\mathrm{Id}+g)(v,\xi) \qquad \text{where} \qquad \mathcal{F} = \frac{1}{y_h^2}\partial_v T_1 \tag{3.45}$$

and $\mathcal{L}$ stands for the differential operator (3.30). As before we transform (3.45) into a fixed point equation. Thus, we introduce the inverse operator

$$\tilde{\mathcal{G}}(h) = \sum_{l\in\mathbb{Z}} \tilde{\mathcal{G}}(h)^{[l]} e^{il\xi}$$

where

$$\begin{aligned} \tilde{\mathcal{G}}(h)^{[l]} &= \int_{v_1}^{v} e^{ilG^3(t-v)} h^{[l]}(t)\,\mathrm{d}t \\ \tilde{\mathcal{G}}(h)^{[0]} &= \int_{-\rho}^{v} h^{[l]}(t)\,\mathrm{d}t \\ \tilde{\mathcal{G}}(h)^{[l]} &= \int_{\bar{v}_1}^{v} e^{ilG^3(t-v)} h^{[l]}(t)\,\mathrm{d}t. \end{aligned} \tag{3.46}$$

and $v_1, \bar{v}_1$ are the top and bottom points of the domain $D_{\rho,\kappa,\delta}$ defined in equation (3.27). The following lemma is proved as Lemma 5.5 in [GOS10].

Lemma 3.4.7. *The operator $\tilde{\mathcal{G}}$ defined on 3.46 satisfies the following properties.*

i) For any $\mu \geq 0$, $\tilde{\mathcal{G}} : \mathcal{Y}_{\mu,\rho,\kappa,\delta} \to \mathcal{Y}_{\mu,\rho,\kappa,\delta}$ is well defined, linear and satisfies $\mathcal{L} \circ \tilde{\mathcal{G}} = Id$.

ii) If $h \in \mathcal{Y}_{\mu,\rho,\kappa,\delta}$ for some $\mu > 1$, then

$$\left\| \tilde{\mathcal{G}}(h) \right\|_{\mu-1} \leq K \|h\|_{\mu}. \tag{3.47}$$

iii) If $h \in \mathcal{Y}_{\mu,\rho,\kappa,\delta}$ for some $\mu \geq 1$, then

$$\left\| \partial_v \tilde{\mathcal{G}}(h) \right\|_{\mu} \leq K \|h\|_{\mu}. \tag{3.48}$$

Therefore, solutions of (3.45) are also fixed points of

$$g = \tilde{\mathcal{G}} \circ \mathcal{F} \circ (\mathrm{Id} + g). \tag{3.49}$$

We state two technical lemmas which will be useful for dealing with compositions of functions and are deduced from the proofs of Lemmas 5.14 and 5.15 in [GMS16].

Lemma 3.4.8. *Fix constants $\delta' < \delta$, $\rho' < \rho$, $\kappa' > \kappa$ and take $h \in \mathcal{Y}_{\mu,\rho,\kappa,\delta}$. Then, $\partial_v h \in \mathcal{Y}_{\mu,\rho',\kappa',\delta'}$ and satisfy*

$$\|\partial_v h\|_{\mu} \leq \frac{G^3}{(\kappa' - \kappa)} \left(\frac{\kappa'}{\kappa} \right)^{\mu} \|h\|_{\mu}.$$

Lemma 3.4.9. *Fix constants $\rho' < \rho$, $\delta' < \delta$ and $\kappa' > \kappa + 1$. Then,*

i) If $h \in \mathcal{Y}_{\mu,\rho,\kappa,\delta}$ and $g \in B\left(G^{-3}\right) \subset \mathcal{Y}_{\mu,\rho',\kappa',\delta'}$ we have that $\tilde{h} = h \circ (\mathrm{Id} + g) \in \mathcal{Y}_{\mu,\rho',\kappa',\delta'}$ and

$$\left\| \tilde{h} \right\|_{\mu} \leq \left(\frac{\kappa'}{\kappa} \right)^{\mu} \|h\|_{\mu}.$$

ii) Moreover if $g_1, g_2 \in B\left(G^{-3}\right) \subset \mathcal{Y}_{\mu,\rho',\kappa',\delta'}$, then $f = h \circ (\mathrm{Id} + g_1) - h \circ (\mathrm{Id} + g_2)$ satisfies

$$\|f\|_{\mu} \leq \frac{G^3}{(\kappa' - \kappa)} \left(\frac{\kappa'}{\kappa} \right)^{\mu} \|h\|_{\mu} \|g_1 - g_2\|_{0,0}.$$

Theorem 3.4.10. *Let δ and κ be the constants given by Theorem 3.4.5. Let $\rho_1 < \rho$, $\delta_1 < \delta$, and $\kappa_1 > \kappa$. Then, for G big enough, there exist a function $g \in \mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1}$ satisfying*

$$\|g\|_0 \leq b_1 G^{-7/2}$$

for $b_1 > 0$ independent of G and such that

$$\hat{\Gamma} = \Gamma \circ (\mathrm{Id} + g)$$

satisfies (3.44).

Proof. To find g we solve the fixed point equation (3.49). For that, we take $g \in B\left(KG^{-7/2}\right) \subset \mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1}$, with K a constant independent of G. Then by Lemma 3.4.9 and using the estimate for $\partial_v T_1$ obtained in Theorem 3.4.5 we have

$$\begin{aligned} \|\mathcal{F} \circ (\mathrm{Id} + g)\|_{1/2} &\leq \left(\frac{\kappa_1}{\kappa} \right)^{1/2} \|\mathcal{F}\|_{1/2} \\ &\leq \left(\frac{\kappa_1}{\kappa} \right)^{1/2} KG^{-4} \\ &\leq KG^{-4} \end{aligned}$$

where K is a constant depending only on the reduction of the domain. From here it is clear using Lemma 3.4.7 that the map $\tilde{\mathcal{G}} \circ \mathcal{F} \circ (\mathrm{Id} + g) : B\left(KG^{-7/2}\right) \subset \mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1} \to \mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1}$, is well defined. Moreover, we obtain that

$$\begin{aligned} \left\| \tilde{\mathcal{G}} \circ \mathcal{F} \circ (\mathrm{Id} + g)_{|g=0} \right\|_0 &\leq KG^{1/2} \left\| \tilde{\mathcal{G}} \circ \mathcal{F} \circ (\mathrm{Id} + g)_{|g=0} \right\|_{1/6} \\ &\leq KG^{1/2} \left\| \mathcal{F} \circ (\mathrm{Id} + g)_{|g=0} \right\|_{7/6} \\ &\leq KG^{1/2} \left\| \mathcal{F} \circ (\mathrm{Id} + g)_{|g=0} \right\|_{1/2} \\ &\leq b_1 G^{-7/2} \end{aligned} \tag{3.50}$$

for some b_1 independent of G. It only remains to show that the map $\tilde{\mathcal{G}} \circ \mathcal{F} \circ (\mathrm{Id} + g)$ is contractive in a neighbourhood of the origin. Take $g_1, g_2 \in B\left(b_1 G^{-7/2}\right) \subset \mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1}$, using again Lemma 3.4.9 we have that

$$\left\| \mathcal{F} \circ (\mathrm{Id} + g_1) - \mathcal{F} \circ (\mathrm{Id} + g_2) \right\|_{1/2} \leq \tilde{K} G^{-1/2} \left\| g_1 - g_2 \right\|_0 .$$

Direct application of Lemma 3.4.4 yields

$$\left\| \tilde{\mathcal{G}} \left(\mathcal{F} \circ (\mathrm{Id} + g_1) - \mathcal{F} \circ (\mathrm{Id} + g_2) \right) \right\|_0 \leq \tilde{K} G^{-1/2} \left\| g_1 - g_2 \right\|_0 ,$$

so for G big enough the map $g \mapsto \tilde{\mathcal{G}} \circ \mathcal{F} \circ (\mathrm{Id} + g)$ is contractive on $B\left(b_1 G^{-7/2}\right) \subset \mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1}$ and the proof is completed. □

Analytic extension of the unstable manifold by the flow parametrization

Now we perform the analytic continuation of the parametrization (3.43) given by Theorem 3.4.5 to the domain $D^{\mathrm{flow}}_{\kappa,\delta}$ defined in (3.28) using the flow of the Hamiltonian (3.5). Notice that since the domain $D^{\mathrm{flow}}_{\kappa,\delta}$ is bounded and at distance of order $\mathcal{O}(1)$ with respect to the singularities all norms $\|h\|_\mu$ are equivalent, therefore it will suffice to get estimates on the norm $\|h\|_0$.

Write $\hat{\Gamma} = \hat{\Gamma}_0 + \hat{\Gamma}_1$, where

$$\hat{\Gamma}_0(v,\xi) = \Gamma_0 \circ (\mathrm{Id} + g)(v,\xi) \qquad \Gamma_0(v) = \left(\tilde{r}_h(v), \tilde{y}_h(v)\right). \tag{3.51}$$

Then, the equation (3.44) that defines this extension is rewritten as

$$\hat{\mathcal{L}}\left(\hat{\Gamma}_1\right) = \hat{\mathcal{F}}\left(\hat{\Gamma}_1\right) \tag{3.52}$$

where

$$\begin{aligned} \hat{\mathcal{L}}(h) =& \mathcal{L}(h) - DX_0\left(\hat{\Gamma}_0\right) h \\ \hat{\mathcal{F}}(h) =& X_0\left(\hat{\Gamma}_0 + h\right) - X_0\left(\hat{\Gamma}_0\right) - DX_0\left(\hat{\Gamma}_0\right) h + X_1\left(\hat{\Gamma}_0 + h\right). \end{aligned}$$

Denote by $\Psi(v)$ the fundamental matrix of the linear system

$$\dot{z}(v) = DX_0\left(\Gamma_0(v,\xi)\right) z(v), \qquad v \in D^{\mathrm{flow}}_{\kappa,\delta}.$$

Then, equation (3.52), together with a suitable initial condition $\hat{\Gamma}_h$, can be reformulated as the fixed point equation

$$\hat{\Gamma}_1 = \hat{\Gamma}_h + \hat{\mathcal{G}} \circ \hat{\mathcal{F}}\left(\hat{\Gamma}_1\right), \tag{3.53}$$

where

$$\begin{aligned} \hat{\Gamma}_h \quad = \quad & \sum_{l>0} \Psi(v) \Psi^{-1}(v_1) \hat{\Gamma}_1^{[l]}(v_1) e^{ilG^3(v_1 - v)} e^{il\xi} \\ & + \sum_{l<0} \Psi(v) \Psi^{-1}(\bar{v}_1) \hat{\Gamma}_1^{[l]}(\bar{v}_1) e^{ilG^3(\bar{v}_1 - v)} e^{il\xi} \\ & + \Psi(v) \Psi^{-1}(-\rho_1) \hat{\Gamma}_1^{[0]}(-\rho_1) \end{aligned}$$

is the solution of the homogeneous equation $\hat{\mathcal{L}}(h)=0$ (observe that since $v_1, \bar{v}_1, -\rho_1$ are contained in $D_{\rho,\kappa,\delta}$, the terms $\hat{\Gamma}_1^{[l]}(v_1), \hat{\Gamma}_1^{[l]}(\bar{v}_1)$ and $\hat{\Gamma}_1^{[l]}(-\rho_1)$ are already defined) and

$$\hat{\mathcal{G}}(h)=\Psi\tilde{\mathcal{G}}\left(\Psi^{-1}h\right)$$

is a right inverse operator. Notice that since $DX\left(\hat{\Gamma}_0(v,\xi)\right)$ is continuous and $D^{\text{flow}}_{\kappa,\delta}$ is a compact domain at distance $\mathcal{O}(1)$ from the singularities, we have that there exists $K>0$ such that

$$\sup_{v\in D^{\text{flow}}_{\kappa,\delta}} \max\left\{\|\Psi\|_0, \left\|\Psi^{-1}\right\|_0\right\}\leq K, \tag{3.54}$$

in the matrix norm associated to the usual vector norm in $\mathbb{C}^2$.

Lemma 3.4.11. *Assume $h,\tilde{h}\in B\left(KG^{-4}\right)\subset\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1}$ for some $K>0$. Then there exists $K^{'}>0$ such that*

i) Defining $Y(h)=X_0\left(\hat{\Gamma}_0+h\right)-X_0\left(\hat{\Gamma}_0\right)-DX_0\left(\hat{\Gamma}_0\right)h$ we have that $Y(h)\in\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1}$. and

$$\|Y(h)\|_0\leq K^{'}G^{-4},$$

ii) $X_1\left(\hat{\Gamma}_0+h\right)\in\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1}$. with $\left\|X_1\left(\hat{\Gamma}_0+h\right)\right\|_0\leq K^{'}G^{-4}$,

iii) $\left\|Y(h)-Y\left(\tilde{h}\right)\right\|_0\leq K^{'}G^{-4}\left\|h-\tilde{h}\right\|_0$,

iv) $\left\|X_1\left(\hat{\Gamma}_0+h\right)-X_1\left(\hat{\Gamma}_0+\tilde{h}\right)\right\|_0\leq K^{'}G^{-4}\left\|h-\tilde{h}\right\|_0$.

Proof. The proof follows from the mean value theorem together with the straightforward bounds

$$\left\|DX_0\left(\hat{\Gamma}_0\right)\right\|_0\leq K^{'}\quad \left\|X_1\left(\hat{\Gamma}_0\right)\right\|_0\leq K^{'}G^{-4}\quad \left\|DX_1\left(\hat{\Gamma}_0\right)\right\|_0\leq K^{'}G^{-4}.$$

□

Proposition 3.4.12. *Let κ_1 and δ_1 be the constants considered in Theorem 3.4.10. Then, there exists $b_2>0$ such that if G is large enough, the fixed point equation* (3.53) *has a unique solution $\hat{\Gamma}_1\in B\left(b_2G^{-4}\right)\subset\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1}$.*

Proof. As $v_1,\bar{v}_1,\rho_1\in D_{\rho_1,\kappa_1,\delta_1}$ we have that $\hat{\Gamma}_h\in\mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1}$ with

$$\left\|\hat{\Gamma}_h\right\|_0\leq KG^{-4}.$$

We claim using Lemma 3.4.11 that the map $\hat{\mathcal{K}}:h\mapsto\Gamma_h+\hat{\mathcal{G}}\circ\hat{\mathcal{F}}(h)$ is well defined from $B\left(KG^{-4}\right)\subset\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1}$ to $\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1}$.. Moreover, we see from the estimate (3.54) for the fundamental matrix $\Psi(v)$ that there exists b_2 such that

$$\left\|\hat{\mathcal{K}}(0)\right\|_0=\left\|\Gamma_h+\hat{\mathcal{G}}(X_1\circ\Gamma_0)\right\|_0\leq\frac{b_2}{2}G^{-4}.$$

Finally, from Lemma 3.4.11, we conclude that for G big enough $\hat{\mathcal{K}}$ is Lipschitz in $B\left(b_2G^{-4}\right)\subset\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1}$ with Lipschitz constant KG^{-4}. □

From flow parametrization to Hamilton-Jacobi parametrization

Now that we have extended the parametrization (3.43) across $v=0$, the next step is to come back to the Hamilton-Jacobi parametrization (3.22) so we have both stable and unstable manifolds parametrized as graphs of the rofm $(\tilde{r}_h(v),\tilde{y}^{u,s}(v,\xi))$ and we can easily measure the distance between them.

We look for a change of variables of the form $\text{Id}+f$ such that

$$\pi_1\circ\hat{\Gamma}\circ(\text{Id}+f)(v,\xi)=\tilde{r}_h(v) \tag{3.55}$$

in the domain

$$\tilde{D}_{\kappa_1,\delta_1} = D^{flow}_{\kappa_1,\delta_1} \cap D_{\kappa_1,\delta_1}, \tag{3.56}$$

where $D^{flow}_{\kappa_1,\delta_1}, D_{\kappa_1,\delta_1}$ are the domains defined in (3.28) and (3.29). Therefore, in $D^u_{\rho_1,\kappa_1,\delta_1} \cap \tilde{D}_{\kappa_1,\delta_1}$ the change $\mathrm{Id} + f$ is the inverse of the change $\mathrm{Id} + g$ obtained in Theorem 3.4.10. We will see that this change of variables is unique under certain conditions, therefore, once we have f, the second component of the unstable manifold is given by

$$\pi_2 \circ \hat{\Gamma}_1 \circ (\mathrm{Id} + f)(v,\xi) = \frac{1}{y_h(v)} \partial_v T_1. \tag{3.57}$$

Using the properties of the unperturbed solution, i.e. $\pi_1 \circ \Gamma_0 (v,\xi) = \tilde{r}_h(v)$, we can write equation (3.55) as

$$f = \mathcal{P}(f)$$

where

$$\mathcal{P}(f) = \frac{-1}{y_h(v)} \left(\tilde{r}_h (v + f(v,\xi)) - \tilde{r}_h(v) - \tilde{y}_h(v) f(v,\xi) - \pi_1 \circ \Gamma_1 \circ (\mathrm{Id} + f)(v,\xi) \right).$$

Proposition 3.4.13. *Consider the constants κ_1 and δ_1 given by Proposition 3.4.12 and any $\kappa_2 > \kappa_1$, $\delta_2 < \delta_1$. Then,*

i) There exists $b_3 > 0$ such that for G large enough, the operator $\mathcal{P}$ has a unique fixed point $f \in \mathcal{Y}_{0,\kappa_2,\delta_2}$ with

$$\|f\|_0 \leq b_3 G^{-4}.$$

ii) Equation (3.57) defines the graph of the unstable manifold wich can be written as $T^u = T_0 + T_1^u$ where T_1^u satisfies

$$\|\partial_v T_1^u\|_0 \leq K G^{-4}.$$

Proof. For the first part we observe that, for $f_2, f_1 \in B\left(KG^{-4}\right) \subset \mathcal{Y}_{0,\kappa_2,\delta_2}$,

$$\begin{aligned} \left|\tilde{r}_h(v + f_2) - \tilde{r}_h(v + f_1) - \tilde{y}_h (f_2 - f_1)\right| &\leq K \left| f_2^2 - f_1^2 \right| \\ &\leq K G^{-4} \left| f_2 - f_1 \right|. \end{aligned}$$

Then, from Lemma 3.4.9 and the fact and $\left\| \hat{\Gamma}_1^u \right\|_0 \leq KG^{-4}$ we deduce that

$$\left| \mathcal{P}(f_2) - \mathcal{P}(f_1) \right| \leq KG^{-4} \left| f_2 - f_1 \right|,$$

i.e. $\mathcal{P}(f)$ is a contractive mapping on $B\left(b_3 G^{-4}\right) \subset \mathcal{Y}_{0,\kappa_2,\delta_2}$ for some $b_3 > 0$ so there exist a unique $f \in B\left(b_3 G^{-4}\right) \subset \mathcal{Y}_{0,\kappa_2,\delta_2}$ solving $f = \mathcal{P}(f)$.

For the second part we have from equation (3.57) that

$$\pi_2 \circ \hat{\Gamma}_1 \circ (\mathrm{Id} + f)(v,\xi) = \frac{1}{y_h(v)} \partial_v T_1.$$

Therefore,

$$\begin{aligned} \|\partial_v T_1\|_{0,0} &\leq K \left\| \frac{1}{y_h(v)} \partial_v T_1 \right\|_0 \\ &= K \left\| \pi_2 \circ \hat{\Gamma}_1 \circ (\mathrm{Id} + f) \right\|_0 \\ &\leq K \left\| \hat{\Gamma}_1 \circ (\mathrm{Id} + f) \right\|_0 \\ &\leq K \left\| \hat{\Gamma}_1 \right\|_0 \leq KG^{-4}, \end{aligned}$$

where we have used Lemma 3.4.9 and the estimate for $\left\| \hat{\Gamma}_1 \right\|_0$ obtained in Proposition 3.4.12. □

We sum up the results obtained in this section in the following theorem.

Theorem 3.4.14. *Let κ_2 and δ_2 be the constants given by Proposition 3.4.13. Then, for G big enough there exist real analytic functions $T_1^{u,s}$ defined in D_{κ_2,δ_2} which are solutions of equation* (3.20) *and satisfy*

$$\|\partial_v T_1^{u,s}\|_{3/2} \leq b_4 G^{-4}$$

for a certain $b_4 > 0$ independent of G.

Proof. For the stable manifold, the result was obtained in Theorem 3.4.5 since $D_{\kappa_2,\delta_2} \subset D^{\infty,s}_{\kappa,\delta}$. For the unstable manifold, using that $D_{\kappa_2,\delta_2} \subset D^{\infty,u}_{\kappa,\delta} \cup \tilde{D}_{\kappa_2,\delta_2}$ the result follows from the combination of Theorem 3.4.5 and Proposition 3.4.13. □

3.5 The difference between the manifolds

Once we have obtained the parametrization of the invariant manifolds in the common domain $D_{\kappa,\delta}$ defined in (3.4), the next step is to study their difference. To this end we define

$$\tilde{\Delta}(v,\xi) = T^s(v,\xi) - T^u(v,\xi). \tag{3.58}$$

Substracting equation (3.20) for T_1^s and T_1^u one obtains that

$$\tilde{\Delta} \in \mathrm{Ker}\tilde{\mathcal{L}}$$

where $\tilde{\mathcal{L}}$ is the differential operator

$$\tilde{\mathcal{L}} = (1 + A(v,\xi))\,\partial_v - G^3 \partial_\xi$$

with

$$A(v,\xi) = \frac{1}{2\tilde{y}_h^2}\left(\partial_v T_1^s - \partial_v T_1^u\right). \tag{3.59}$$

To obtain exponentially small bounds on the difference between the invariant manifolds we will look for a close to identity change of variables $(v,\xi) = (w + C(w,\xi),\xi)$ such that the function

$$\Delta(w,\xi) = \tilde{\Delta}(w + C(w,\xi),\xi) \qquad (w,\xi) \in D_{\kappa,\delta} \times \mathbb{T}, \tag{3.60}$$

satisfies

$$\Delta \in \mathrm{Ker}\mathcal{L}$$

where $\mathcal{L}$ is the differential operator defined in (3.30). The condition $\Delta \in \mathrm{Ker}\mathcal{L}$ implies that $\Delta = f\left(\xi - G^3 w\right)$. Therefore, since Δ is periodic in ξ it must be periodic in w. Since Δ is real analytic and bounded in a strip that reaches up to points $\mathcal{O}\left(G^{-3}\right)$ close to the singularities the exponentially small bound for $|\Delta(w,\xi)|$ where $w \in \mathbb{R}$ comes straightforward by a classical argument (see Lemma 3.5.2 below). We devote the rest of the section to make this rigorous.

3.5.1 Straightening the operator $\tilde{\mathcal{L}}$

As we did in the previous sections we introduce the Banach spaces

$$\mathcal{Q}_{\mu,\rho,\kappa,\delta} = \Big\{ h(w,\xi) : D_{\kappa,\delta} \times \mathbb{T} \to \mathbb{C} \colon w \mapsto h(w,\xi) \text{ is real analytic, } \xi \mapsto h(w,\xi) \text{ is continuous} \\ \text{and } \|h\|_\mu < \infty \Big\}$$

where

$$\|h\|_\mu = \sum_{l \in \mathbb{Z}} \left\| h^{[l]} \right\|_\mu, \qquad \left\| h^{[l]} \right\|_\mu = \sup_{w \in D_{\kappa,\delta}} \left| \left(w^2 + 1/9\right)^\mu h(v) \right|.$$

Theorem 3.5.1. *Let κ_2 and δ_2 the constants defined in Theorem 3.4.14. Let $\kappa_3>\kappa_2$and $\delta_3<\delta_2$ be fixed. Then, for G big enough, there exists $C\in\mathcal{Q}_{0,\kappa_3,\delta_3}$ such that the function*

$$\Delta(w,\xi)=\tilde{\Delta}(w+C(w,\xi),\xi)$$

satisfies that $\Delta\in\mathrm{Ker}\mathcal{L}$. Moreover, we have that

$$\|C\|_0\leq b_5G^{-7/2}$$

for a certain $b_5>0$ independent of G.

Proof. Using the chain rule we obtain that the implication $\Delta\in\mathrm{Ker}\mathcal{L}$ if and only if $\Delta\in\mathrm{Ker}\tilde{\mathcal{L}}$, is equivalent to finding C satisfying

$$\begin{aligned}\mathcal{L}(C)&=A_{|v=w+C(w)}\\&=A\circ(\mathrm{Id}+C),\end{aligned}$$

where $A(v,\xi)$ was defined in (3.59). We can rewrite this equation as a fixed point equation

$$C=\tilde{\mathcal{G}}(A\circ(\mathrm{Id}+C)),$$

where $\tilde{\mathcal{G}}$ is the inverse operator defined in (3.46). Using the bounds for $\partial_vT_1^{u,s}$ in Theorem 3.4.14, the properties of the homoclinic orbit stated in Section 3.3.1 , and Lemma 3.4.9 for the composition, we obtain that, for $C\in B\left(KG^{-4}\right)\subset\mathcal{Q}_{0,\kappa_3,\delta_3}$,

$$\|A\circ(\mathrm{Id}+C)\|_{1/2}\leq K^{'}G^{-4}$$

for some $K^{'}>0$ independent of G. Hence, from Lemma 3.4.9 we observe that the map $C\mapsto\tilde{\mathcal{G}}(A\circ(\mathrm{Id}+C))$ is well defined from $C\in B\left(KG^{-7/2}\right)\subset Q_{0,\kappa_3,\delta_3}\to\mathcal{Q}_{0,\kappa_3,\delta_3}$. Moreover, we also get

$$\left\|\tilde{\mathcal{G}}\left(A\circ(\mathrm{Id}+C)_{|C=0}\right)\right\|_0\leq\frac{b_5}{2}G^{-7/2},$$

for some b_5 independent of G. Hence, it only remains to prove that the map $C\mapsto\tilde{\mathcal{G}}(A\circ(\mathrm{Id}+C))$ is contractive on the ball $B\left(b_5G^{-7/2}\right)\subset\mathcal{Q}_{0,\kappa_3,\delta_{3,}}$. Again by Lemma 3.4.9 we have that if $C_1,C_2\in B\left(b_5G^{-7/2}\right)\subset\mathcal{Q}_{0,\kappa_3,\delta_3}$, then

$$\begin{aligned}\|A\circ(\mathrm{Id}+C_2)-A\circ(\mathrm{Id}+C_1)\|_{1/2}&\leq KG^3\|A\|_{1/2}\|C_2-C_1\|_0\\&\leq KG^{-1}\|C_2-C_1\|_0,\end{aligned}$$

and contractivity follows from Lemma 3.4.7 for G big enough. □

3.5.2 Estimates for the difference between the invariant manifolds

Now we exploit the fact that the function $\Delta(w,\xi)$ defined in (3.60) satisfies

$$\Delta\in\mathrm{Ker}\mathcal{L}$$

to get exponentially small bounds on the real line.

Lemma 3.5.2. *Let $h:D_{\kappa,\delta}\times\mathbb{T}\to\mathbb{C}$ be a real-analytic function such that $h\in\mathcal{Q}_{0,\kappa,\delta}$ and $h\in\mathrm{Ker}\mathcal{L}$. Then,*

i) h is of the form

$$h(w,\xi)=\sum_{l\in\mathbb{Z}}h^{[l]}(w)e^{il\xi}=\sum_{l\in\mathbb{Z}}\beta^{[l]}e^{il\left(\xi-G^3w\right)}.$$

ii) the coefficients $\beta^{[l]}$ satisfy the bounds

$$\left|\beta^{[l]}\right|\leq\|h\|_0K^{|l|}e^{\frac{-|l|G^3}{3}}.$$

Proof. Since $h \in \mathrm{Ker}\mathcal{L}$ and is periodic in ξ, we have that each Fourier coefficient $h^{[l]}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}w}h^{[l]} + ilG^3h^{[l]} = 0$$

so it has to be

$$h^{[l]}(w) = \beta^{[l]}e^{-ilG^3w}$$

for certain constants $\beta^{[l]}$. Moreover, evaluating this equality at the top vertex $w_2 = i\left(1/3 - \kappa G^{-3}\right)$ of the domain $D_{\kappa,\delta}$ for $l < 0$ and at the bottom vertex $\bar{w}_2 = i\left(1/3 - \kappa G^{-3}\right)$ for $l > 0$ we obtain that

$$\begin{aligned}\left|\beta^{[l]}\right| &\leq \max\left\{h^{[l]}(w_2), h^{[l]}(\bar{w}_2)\right\} e^{\frac{-|l|G^3}{3}} e^{|l|\kappa_3} \\ &\leq \|h\|_0 e^{|l|\kappa_3} e^{\frac{-|l|G^3}{3}} \\ &\leq \|h\|_0 K^{|l|} e^{\frac{-|l|G^3}{3}},\end{aligned}$$

for a constant K independent of G and l. Therefore, for $u \in \mathbb{R} \cap D_{\kappa,\delta}$

$$\left|h^{[l]}(u)\right| = \left|\beta^{[l]}\right| \leq \|h\|_0 K^{|l|} e^{\frac{-|l|G^3}{3}}.$$

□

Using this lemma we already have exponentially small bounds for $\Delta(w,\xi)$. Nevertheless, our goal is to prove that the function L defined in (3.24) is the main term in Δ. Thus we study the function

$$\mathcal{E}(w,\xi) = \Delta(w,\xi) - L(w,\xi).$$

Lemma 3.5.3. *Consider the constants κ_3 and δ_3 defined in Theorem 3.5.1. Then, for $(w,\xi) \in (D_{\kappa_3,\delta_3} \cap \mathbb{R}) \times \mathbb{T}$ we get*

$$|\mathcal{E}(w,\xi) - E| \leq KG^{-7/2}e^{\frac{-G^3}{3}}.$$

where E is a constant and

$$|\partial_w \mathcal{E}| \leq KG^{-1/2}e^{\frac{-G^3}{3}}.$$

Proof. Notice that $L = L^s - L^u$ where $L^* = \mathcal{G}^*(V)$, with $\mathcal{G}^{u,s}$ are the left inverse operators introduced in (3.32). Then, it is clear that $\mathcal{L}(L) = 0$ and we have that $\mathcal{E} \in \mathrm{Ker}\mathcal{L}$. We bound $\mathcal{E}$ in the domain $D_{\kappa,\delta}$ so that we can apply Lemma 3.5.2. We decompose $\mathcal{E} = \mathcal{E}_1^s - \mathcal{E}_1^u + \mathcal{E}_2$ where

$$\begin{aligned}\mathcal{E}_1^* &= T_1^* - L^* \\ \mathcal{E}_2 &= \Delta - \tilde{\Delta}.\end{aligned}$$

From Lemma 3.4.3 and equation (3.38) we have

$$\|\mathcal{E}_1^*\|_0 = \|T_1^* - L^*\|_0 \leq KG^{3/2}\|T_1^* - L^*\|_{1/2} \leq KG^{-5}.$$

For the second term we use Lemmas 3.4.3, 3.4.9 and the bounds for $\tilde{\Delta}$ and C from Theorems 3.4.14 and 3.5.1 to obtain

$$\begin{aligned}\|\mathcal{E}_2\|_0 = \left\|\tilde{\Delta} \circ (\mathrm{Id} + C) - \tilde{\Delta}\right\|_0 &\leq KG^3\left\|\tilde{\Delta}\right\|_0 \|C\|_0 \\ &\leq KG^{9/2}\left\|\tilde{\Delta}\right\|_{1/2}\|C\|_0 \leq KG^{-7/2},\end{aligned}$$

Combining these results

$$\|\mathcal{E}\|_0 \leq KG^{-7/2}.$$

Hence, by direct application of Lemma 3.5.2 we obtain that for $u \in D_{\kappa_3,\delta_3} \cap \mathbb{R}$

$$\left|\mathcal{E}^{[l]}(w)\right| \leq G^{-7/2} K^{|l|} e^{\frac{-|l|G^3}{3}}.$$

Now, defining $E = \mathcal{E}^{[0]}$ (notice that by Lemma 3.5.2, $\mathcal{E}^0$ is constant) we have that for $(w,\xi) \in (D_{\kappa_3,\delta_3} \cap \mathbb{R}) \times \mathbb{T}$

$$\begin{aligned}
|\mathcal{E} - E| &\leq \sum_{|l|>1} \left|\mathcal{E}^{[l]}(w)\right| \\
&\leq G^{-7/2} e^{\frac{-G^3}{3}} \sum_{|l|>2} \left(K e^{\frac{-G^3}{3}}\right)^{|l|} \\
&\leq K G^{-7/2} e^{\frac{-G^3}{3}}.
\end{aligned}$$

Finally, it is a straightforward computation to check that

$$\left|\frac{\mathrm{d}}{\mathrm{d}w} \mathcal{E}^{[l]}(w)\right| \leq G^{-1/2} K^{|l|} e^{\frac{-|l|G^3}{3}}$$

so we conclude that

$$|\partial_w \mathcal{E}| \leq K G^{-1/2} e^{\frac{-G^3}{3}}.$$

□

There is only one step left for achieving our goal, going back to the original variables (v,ξ). This is done in the next lemma.

Lemma 3.5.4. *Consider the function*

$$\tilde{\mathcal{E}}(v,\xi) = \tilde{\Delta}(v,\xi) - L(v,\xi)$$

where $\tilde{\Delta}(v,\xi)$ *is defined in* (3.58) *and* $L(v,\xi)$ *is defined in* (3.24). *Fix* $\kappa_4 > \kappa_3$ *and* $\delta_4 < \delta_3$. *Then, for* $(v,\xi) \in (D_{\kappa_4,\delta_4} \cap \mathbb{R}) \times \mathbb{T}$,

$$\left|\tilde{\mathcal{E}}(v,\xi) - E\right| \leq K G^{-7/2} e^{\frac{-G^3}{3}} \tag{3.61}$$

where E *is a constant and*

$$\left|\partial_v \tilde{\mathcal{E}}(v,\xi)\right| \leq K G^{-1/2} e^{\frac{-G^3}{3}}. \tag{3.62}$$

Proof. We look for a function $\varphi(v,\xi)$ such that $(\mathrm{Id} + C) \circ (\mathrm{Id} + \varphi)(v,\xi) = (v,\xi)$, i.e., φ must satisfy

$$v = v + \varphi(v,\xi) + C(v + \varphi(v,\xi), \xi)$$

or what is the same

$$\varphi(v,\xi) = -C(v + \varphi(v,\xi), \xi). \tag{3.63}$$

In order to solve this fixed point equation we first use Lemma 3.4.9 to obtain that for $\varphi \in B\left(KG^{-4}\right) \subset \mathcal{Y}_{0,\kappa_4,\delta_4}$

$$\|C \circ (\mathrm{Id} + \varphi)\|_0 \leq \|C\|_0 \leq KG^{-4}$$

so the map $\varphi \mapsto C \circ (\mathrm{Id} + \varphi)$ is well defined from $B\left(KG^{-4}\right) \subset \mathcal{Y}_{0,\kappa_4,\delta_4} \to \mathcal{Y}_{0,\kappa_4,\delta_4}$. Moreover we get that there exists b_6 such that

$$\left\|C \circ (\mathrm{Id} + \varphi)_{|\varphi=0}\right\|_0 \leq \frac{b_6}{2} G^{-4}.$$

Since for $\varphi_1, \varphi_2 \in B\left(KG^{-4}\right) \subset \mathcal{Y}_{0,\kappa_4,\delta_4}$ we have

$$\|C \circ (\mathrm{Id} + \varphi_2) - C \circ (\mathrm{Id} + \varphi_1)\|_0 \leq KG^{-4} \|\varphi_2 - \varphi_1\|_0$$

we have shown the existence of a unique $\varphi \in B\left(b_6 G^{-4}\right) \subset \mathcal{Y}_{0,\kappa_4,\delta_4}$ solving (3.63).

Now that we have obtained the inverse change of variables, the bounds (3.61) and (3.62) follow from direct application of Lemma 3.4.9 if we notice that

$$\begin{aligned}\mathcal{E}\left(w\left(v,\xi\right),\xi\right) &= \left(\Delta - L\right)\circ\left(\mathrm{Id}+\varphi\right)\left(v,\xi\right)\\ &= \left(\tilde{\Delta}\circ\left(\mathrm{Id}+C\right) - L\right)\circ\left(\mathrm{Id}+\varphi\right)\left(v,\xi\right)\\ &= \tilde{\Delta}\left(v,\xi\right) - L\circ\left(\mathrm{Id}+\varphi\right)\left(v,\xi\right)\end{aligned}$$

so

$$\tilde{\mathcal{E}}\left(v,\xi\right) = \mathcal{E}\left(v,\xi\right) + L\circ\left(\mathrm{Id}+\varphi\right)\left(v,\xi\right) - L\left(v,\xi\right).$$

Then, the result follows from Lemma 3.4.9 and the estimates on Proposition 3.3.1. □

3.A Computation of the melnikov potential

We devote this section to the computation of the Melnikov potential $L\left(v,\xi\right)$ whose partial derivative with respect to v gives us the first order term of the distance between the infinity manifolds. From its definition (3.24) we have

$$\begin{aligned}L\left(v,\xi\right) &= \int_{-\infty}^{\infty} V\left(\tilde{r}_h\left(v+s\right),\xi+G^3 s\right)\mathrm{d}s\\ &= \int_{-\infty}^{\infty} V\left(\tilde{r}_h\left(s\right),\xi+G^3\left(s-v\right)\right)\mathrm{d}s.\end{aligned}$$

Expanding in Taylor series the square root in (3.6) we obtain that

$$V\left(\tilde{r}_h\left(s\right),\xi+G^3\left(s-v\right)\right) = -\sum_{k=1}^{\infty}\begin{pmatrix}\frac{-1}{2}\\ k\end{pmatrix}\left(4G^4\right)^{-k}\int_{-\infty}^{\infty}\frac{\rho^{2k}\left(\xi+G^3\left(s-v\right)\right)\mathrm{d}s}{\tilde{r}_h^{2k+1}\left(s\right)}.$$

Hence, expanding now the terms ρ^{2k} in Fourier series we get

$$L\left(v,\xi\right) = -\sum_{l\in\mathbb{Z}} e^{il\left(\xi-G^3 v\right)}\sum_{k=1}^{\infty}\begin{pmatrix}\frac{-1}{2}\\ k\end{pmatrix} a_{l,k}\left(4G^4\right)^{-k}\int_{-\infty}^{\infty}\frac{e^{ilG^3 s}\mathrm{d}s}{\tilde{r}_h^{2k+1}\left(s\right)},$$

where

$$a_{l,k} = \frac{1}{2\pi}\int_0^{2\pi}\rho^{2k}\left(\sigma\right)e^{-il\sigma}\mathrm{d}\sigma.$$

Since for all $\sigma\in[0,2\pi]$ we have $|\rho|<2$ we easily bound

$$|a_{l,k}| \leq 4^k. \tag{3.64}$$

Moreover, changing the integration variable to the eccentric anomaly E defined by $t = E - \sin E$

$$\rho\left(E\right) = 1 - \cos E,$$

we obtain that

$$a_{1,1} = -2J_1\left(1\right) \neq 0 \tag{3.65}$$

where J_1 is the Bessel function of first kind.

Under the time reparametrization

$$s = \frac{1}{2}\left(\tau + \frac{\tau^3}{3}\right),$$

we can write

$$
\begin{aligned}
L(v,\xi) &= -2\sum_{l\in\mathbb{Z}} e^{il\left(\xi-G^3v\right)}\sum_{k=1}^{\infty}\begin{pmatrix}\frac{-1}{2}\\ k\end{pmatrix} a_{l,k}G^{4k}\int_{-\infty}^{\infty}\frac{e^{ilG^3\left(\tau+\frac{\tau^3}{3}\right)/2}\mathrm{d}\tau}{(\tau-i)^{2k}(\tau+i)^{2k}}\\
&= -2\sum_{l\in\mathbb{Z}} e^{il\left(\xi-G^3v\right)}\sum_{k=1}^{\infty}\begin{pmatrix}\frac{-1}{2}\\ k\end{pmatrix} a_{l,k}G^{4k}I(l,k)\\
&= \sum_{l\in\mathbb{Z}} L^{[l]}e^{il\left(\xi-G^3v\right)}.
\end{aligned}
\tag{3.66}
$$

The harmonic with $l=0$ is readily bounded using that

$$
I(0,k)=\sqrt{\pi}\frac{\Gamma(2k-1/2)}{\Gamma(2k)},
$$

where Γ stands for the Gamma function.

A standard computation shows that $L^{[l]}=L^{[-l]}$ so we focus only on the case $l>0$. The next proposition, which can be deduced from Propositions 19 and 22 in [DKdlRS19] gives estimates for $|I(l,k)|$ and the asymptotic first order term for $I(1,1)$ which we use to identify the main term in $L^{[1]}(v,\xi)$.

Proposition 3.A.1. *Let G be large enough, then the estimate*

$$
|I(l,k)|\le 8e^{l}G^{3k-3/2}e^{\frac{-lG^3}{3}},
$$

holds for $l\ge 1,\ k\ge 1$. Moreover we have that

$$
I(1,1)=\sqrt{\pi}\left(\frac{G}{2}\right)^{3/2}e^{\frac{-G^3}{3}}\left(1+\mathcal{O}\left(G^{-3/2}\right)\right).
$$

For $l=1$ we have

$$
L^{[1]}=-2\left(-\frac{1}{2}a_{1,1}G^{-4}I_{1,1}+\sum_{k=2}^{\infty}\begin{pmatrix}\frac{-1}{2}\\ k\end{pmatrix} a_{1,k}G^{4k}I(1,k)\right).
$$

Using Proposition 3.A.1 and the estimate in (3.64) we have that

$$
\begin{aligned}
\left|\sum_{k=2}^{\infty}\begin{pmatrix}\frac{-1}{2}\\ k\end{pmatrix} a_{1,k}G^{4k}I(1,k)\right| &\le 8e^{1/2}e^{\frac{-G^3}{3}}G^{-3/2}\sum_{k=2}^{\infty}G^{-k}\\
&\le 16e^{1/2}e^{\frac{-G^3}{3}}G^{-7/2}.
\end{aligned}
$$

Therefore

$$
L^{[1]}=a_{1,1}\sqrt{\pi}2^{-3/2}G^{-5/2}e^{\frac{-G^3}{3}}\left(1+\mathcal{O}\left(G^{-1}\right)\right).
$$

For $l\ge 2$ we have

$$
L^{[l]}=-2\sum_{k=1}^{\infty}\begin{pmatrix}\frac{-1}{2}\\ k\end{pmatrix} a_{l,k}G^{-4k}I_{l,k}
$$

and again from Proposition 3.A.1 and the estimate in (3.64) we obtain

$$
\left|L^{[l]}\right|\le 32e^{l-1/2}G^{-5/2}e^{\frac{-lG^3}{3}}.
$$

From the estimates we have obtained for $\left|L^{[l]}\right|$ the double series is absolutely convergent, which justify the expansions in Taylor and Fouier series and the proof of Proposition 3.3.1 is completed.

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Chapter 4

Oscillatory motions in the Restricted 3 Body Problem: a functional-analytic approach

Abstract: In this paper we introduce a functional-analytic approach to the existence of parabolic and oscillatory motions for the Restricted Isosceles 3-body Problem (RI3BP) for almost all values of the angular momentum. According to the classification given by Chazy back in 1922, we name oscillatory an entire motion of the massless body q which is unbounded but returns infinitely often inside some bounded region:

$$\limsup_{t \to \pm\infty} |q(t)| = \infty \qquad \text{and} \qquad \liminf_{t \to \pm\infty} |q(t)| < \infty.$$

In contrast with the other possible final motions in Chazy's classification, oscillatory motions do not occur in the 2-body Problem, while they do for larger numbers of bodies. A further point of interest is their appearance in connection with the existence of chaotic dynamics.

In this paper we introduce new tools to study the existence of oscillatory motions and prove that oscillatory motions exist in a particular configuration known as the Restricted Isosceles 3-body Problem (RI3BP) for almost all values of the angular momentum. Our method, which is global in nature and not limited to nearly integrable settings, extends the previous results [GPSV21] by blending variational and geometric techniques with tools from nonlinear analysis such as the mountain pass theorem and the topological degree theory. To the best of our knowledge, the present work constitutes the first complete analytic proof of existence of oscillatory motions in a non perturbative regime.

4.1 Introduction

One of the oldest questions in Dynamical Systems is to understand the mechanisms driving the global dynamics of the 3 Body Problem, which models the motion of three bodies interacting through Newtonian gravitational force. The 3 Body Problem is called *restricted* if one of the bodies has mass zero and the other two have strictly positive masses. In this limit problem, the massless body is affected by, but does not affect, the motion of the massive bodies. A fundamental question concerning the global dynamics of the Restricted 3 Body Problem is the study of its possible final motions, that is, the qualitative description of its complete (defined for all time) orbits as time goes to infinity. In 1922 Chazy gave a complete classification of the possible final motions of the Restricted 3 Body Problem [Cha22]. To describe them we denote by q the position of the massless body in a Cartesian reference frame with origin at the center of mass of the primaries.

Theorem 4.1.1 ([Cha22])**.** *Every solution of the Restricted 3 Body Problem defined for all (future) times belongs to one of the following classes*

- *B (bounded):* $\sup_{t\geq 0}|q(t)|<\infty$.
- *P (parabolic)* $|q(t)|\to\infty$ *and* $|\dot{q}(t)|\to 0$ *as* $t\to\infty$.
- *H (hyperbolic):* $|q(t)|\to\infty$ *and* $|\dot{q}(t)|\to c>0$ *as* $t\to\infty$.
- *O (oscillatory)* $\limsup_{t\to\infty}|q(t)|=\infty$ *and* $\liminf_{t\to\infty}|q(t)|<\infty$.

Notice that this classification also applies for $t\to-\infty$. We distingish both cases adding a superindex $+$ or $-$ to each of the cases, e.g. H^+ and H^-.

Unlike oscillatory, bounded, parabolic and hyperbolic motions also exist in the 2 Body Problem and examples of each of these classes of motion in the Restricted 3 Body Problem were already known by Chazy. However, the existence of oscillatory motions in the Restricted 3 Body Problem was an open question for a long time. Their existence was first established by Sitnikov in a particular configuration of the Restricted 3 Body Problem nowadays known as the Sitnikov problem.

4.1.1 The Moser approach to the existence of oscillatory motions: literature

After Sitnikov's work, Moser gave a new proof of the existence of oscillatory motions in the Sitnikov problem [Mos01]. His approach makes use of tools from the geometric theory of dynamical systems, in particular, hyperbolic dynamics. More concretely, Moser considered an invariant periodic orbit "at infinity" (see Section 4.1.2) which is degenerate (the linearized vector field vanishes) but posseses stable and unstable invariant manifolds. Then, he proved that its stable and unstable manifolds intersect transversally. Close to this intersection, he built a section Σ transverse to the flow and established the existence of a non trivial hyperbolic set $\mathcal{X}$ for the Poincaré map Φ_Σ induced on Σ. The dynamics of Φ_Σ restricted to $\mathcal{X}\subset\Sigma$ is moreover conjugated to the shift

$$\sigma:\mathbb{N}^{\mathbb{Z}}\to\mathbb{N}^{\mathbb{Z}}\qquad\qquad (\sigma\omega)_k=\omega_{k+1}$$

acting on the space of infinite sequences. Namely, $\mathcal{X}$ is a horseshoe with "infinitely many legs" for the map Φ_Σ. By construction, sequences $\omega=(\ldots,\omega_{-n},\omega_{-n+1}\ldots\omega_0,\ldots\omega_{n-1}\ldots\omega_n\ldots)\in\mathbb{N}^{\mathbb{Z}}$ for which $\limsup_{n\to\infty}\omega_n$ (respectively $\limsup_{n\to-\infty}\omega_n$) correspond to complete motions of the Sitnikov problem which are oscillatory in the future (in the past).

Moser's ideas have been very influential. In [LS80a] Simó and Llibre implemented Moser's approach in the Restricted Circular 3 Body Problem (RC3BP) in the region of the phase space with large Jacobi constant provided the values of the ratio between the masses of the massive bodies is small enough. Their result was later extended by thm:chazyintro [Xia92] and closed by Guardia, Martín and Seara in [GMS16] where oscillatory motions for the RC3BP for all mass ratios are constructed in the region of the phase space with large Jacobi constant. The same result is obtained in [CGM$^+$22] for low values of the Jacobi constant relying on a computer assisted proof. In [GSMS17] and [SZ20], the Moser approach is applied to the Restricted Elliptic 3 Body Problem and the Restricted 4 Body Problem respectively. For the 3 Body Problem, results in certain symmetric configurations (which reduce the dimension of the phase space) were obtained in [Ale69] and [LS80b]. Another interesting result, which however holds for non generic choices of the 3 masses, is obtained in [Moe07]. In the recent preprint [GMPS22], the first author together with Guardia, Martín and Seara, has proved the existence of oscillatory motions in the planar 3 Body Problem (5 dimensional phase space after symplectic reductions) for all choices of the masses (except all equal) and large total angular momentum.

The first main ingredient in Moser's strategy is the detection of a transversal intersection between the invariant manifolds of the periodic orbit at infinity. Yet, checking the occurrence of this phenomenon in a physical model is rather problematic, and in general little can be said except for perturbations of integrable systems with a hyperbolic fixed point whose stable and unstable manifolds coincide along a homoclinic manifold. As far as the authors know, all the previous works concerning the existence of oscillatory motions in the 3 Body Problem (restricted or not) adopt a perturbative approach to prove the existence of transversal intersections between the stable and unstable manifolds of infinity. In some cases the perturbative regime is obtained by assuming that certain parameter related to the motion of the massive bodies (in general the ratio between the masses of the massive bodies or the eccentriticy of their orbit) is small, and others by working in a region of the phase space where the massless body is located far away from the primaries. The latter situation falls in what is usually called singular perturbation theory and (in general) needs a much more involved analysis than the former one, usually referred to as regular perturbation theory.

The second key ingredient is the construction of a horseshoe close to the transversal intersections of the invariant manifolds. For the Sitnikov and Isosceles Restricted 3 Body Problem (which is introduced in Section 4.1.2) are non autonomous Hamiltonian systems with $1 + 1/2$ degrees of freedom (3 dimensional phase space), its dynamics can be reduced to the study of a two dimensional area preserving map in which the periodic orbit at infinity becomes a fixed point which, despite being degenerate, behaves as a hyperbolic fixed point. The same happens in the RC3BP after reducing by rotational symmetry and in certain symmetric configurations of the 3BP. In all of these problems Moser's ideas for constructing a horseshoe close to the transverse intersections between the invariant manifolds of the parabolic fixed point can be implemented directly. In the planar 3 Body Problem, the dynamics can be reduced to a 4 dimensional symplectic map and the parabolic fixed point becomes a 2 dimensional (degenerate) normally hyperbolic invariant manifold. Due to the existence of central directions the construction of the horseshoe in [GMPS22] becomes much more involved. In [Moe07], the author analyzes orbits which pass close to triple collision. In this setting, the close encounters with triple collision, produce stretching also in the central directions.

An approach different in nature from Moser's is developed by Galante and Kaloshin in [GK11]. By making use of Aubry-Mather theory and semi-infinite regions of instability, the authors prove the existence of oscillatory orbits for the RC3BP with a realistic value of the mass ratio.

It is worthwhile mentioning that another fundamental issue in Celestial Mechanics, besides that of existence of oscillatory motions, is about their abundance. In the conference in honor of the 70th anniversary of Alexeev, Arnol'd posed the question whether the Lebesgue measure of the set of oscillatory motions is positive (cfr [GK12]). This question was considered by Arnol'd to be the fundamental issue of Celestial Mechanics. It has been conjectured by Alexeev that the Lebesgue measure is zero. Neverthless, this conjecture remains wide open. The only partial results in this direction are due to Gorodetski and Kaloshin [GK12]. They consider the RC3BP and the Sitnikov problem and prove that for both problems and a Baire generic subset of an open set of parameters (eccentricity in the Sitnikov problem and mass ratio in the RC3BP), the Hausdorff dimension of the set of oscillatory motions is maximal.

4.1.2 The Isosceles configuration of the Restricted 3 Body Problem: main results

In the present work we consider a particular configuration of the Restricted 3 Body Problem known as the Restricted Isosceles 3 Body Problem. In this configuration, the two primaries have equal masses $m_0 = m_1 = 1/2$ and move periodically on a degenerate ellipse of eccentricity one (a line), according to the Kepler laws for the motion of the 2 Body Problem. The massless particle moves on the plane perpendicular to the line along which the primaries move (see Figure 4.1.2).

In the plane of motion of the massless body we fix a Cartesian reference frame with origin at the point where the line along which the primaries move intersects the plane. Then, in Cartesian coordinates $(q, p, t) \in \mathbb{R}^4 \times \mathbb{T} \setminus \{q = 0\}$, the motion of the massless body is given by the Hamiltonian system

$$H(q, p, t) = \frac{|p|^2}{2} - V_{\text{cart}}(q, t) \qquad V_{\text{cart}}(q, t) = \frac{1}{\sqrt{|q|^2 + \rho^2(t)}}.$$

where $\rho(t) : \mathbb{T} \to [0, 1/2]$ is a half of the distance between the primaries.

Remark 4.1.2. *One can obtain an explicit expression of the function $\rho(t)$ after introducing the change of variables $t = u - \sin u$, commonly known as the Kepler equation. When expressed in terms of the new variable u (which is the eccentric anomaly) we have $\rho(t(u)) = (1 - \cos u)/2$. Yet, our analysis does not require to have an explicit expression of the function $\rho(t)$, so we work directly with the original variable t.*

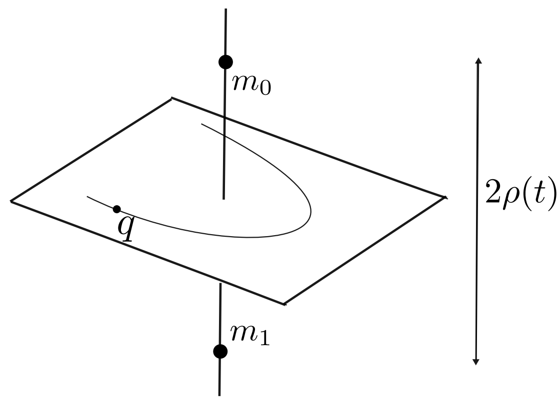

Figure 4.1: Sketch of the motion in the Restricted Isosceles 3 Body Problem.

It will be convenient for our analysis to introduce polar coordinates $(r, \alpha, t, y, G) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^2$ where $q = (r\cos\alpha, r\sin\alpha)$ and (y, G) denote the conjugate momenta to (r, α). In polar coordinates, the Hamiltonian of the Restricted Isosceles 3 Body Problem reads

$$H(r,t,y,G) = \frac{y^2}{2} + \frac{G^2}{2r^2} - V(r,t) \qquad\qquad V(r,t) = \frac{1}{\sqrt{r^2+\rho^2(t)}}. \tag{4.1}$$

We inmediately notice that G is a conserved quantity for the flow of (4.1). It is therefore natural to consider the one-parameter family of Hamiltonian systems

$$H_G(r,t,y) = H(r,t,y,G) \qquad\qquad (r,t,y) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}. \tag{4.2}$$

Since $\lim_{r\to\infty} V(r,t) = 0$, for all $G \in \mathbb{R}$ the Hamiltonian (4.2) posses a periodic orbit at infinity

$$\gamma_\infty = \{r = \infty,\ y = 0\} \subset \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}. \tag{4.3}$$

Homoclinic and heteroclinic to such a periodic orbit at infinity are entire parabolic motions. In [GPSV21], the first author together with M. Guardia, T. Seara and C.Vidal, proved the following result.

Theorem 4.1.3 ([GPSV21])**.** *Consider the Hamiltonian system H_G defined in* (4.2)*. Denote by X^+ (respectively Y^-) either H^+, P^+, B^+ or OS^+ (respectively H^-, P^-, B^- or OS^-) according to Chazy's classification in Theorem 4.1.1. Then, there exists $G_* \gg 1$ such that for all $G \in \mathbb{R}$ such that $|G| \geq G_*$, the Hamiltonian system H_G satisfies*

$$X^+ \cap Y^- \neq \emptyset$$

for all possible combinations of X^+ and Y^-.

Theorem 4.1.3 is proved by exploiting the fact that for G large enough, in a suitable region of the phase space, the Hamiltonian H_G can be studied as a perturbation of the (integrable) 2 Body Problem. This allowed the authors to prove that the periodic orbit γ_∞ posses global stable and unstable invariant manifolds which intersect transversally (see Theorem 4.4.1). As a corollary of this result, a rather straightforward implementation of Moser's ideas shows the truth of Theorem 4.1.3.

The following is the first main result of the present work.

Theorem 4.1.4. *Consider the Hamiltonian system H_G defined in* (4.2)*. Denote by X^+ (respectively Y^-) either H^+, P^+, B^+ or OS^+ (respectively H^-, P^-, B^- or OS^-) according to Chazy's classification in Theorem 4.1.1. Then, for almost all $G \in \mathbb{R}$ the Hamiltonian system H_G satisfies*

$$X^+ \cap Y^- \neq \emptyset$$

for all possible combinations of X^+ and Y^-.

To the best of our knowledge, Theorem 4.1.4 is the first complete analytic proof of the existence of oscillatory motions relying upon a global analytical approach rather than on perturbative techniques. Some interesting related works, where the existence of oscillatory motions is obtained in a setting which is not close to integrable, are [Moe07] and [CGM+22]. While in [Moe07] the author shows the existence of oscillatory motions in the 3 Body Problem close to triple collision (small values of the total angular momentum), in [CGM+22] the authors obtain a computer assisted proof of the existence of oscillatory motions in the Restricted Circular 3 Body Problem for small values of the Jacobi constant.

Theorem 4.1.4 is indeed obtained as a consequence of the following result.

Theorem 4.1.5 (Symbolic Dynamics). *Let $\{l_j\} \subset \mathbb{Z}$ be an increasing sequence and define the time intervals $I_j = [(l_j - l_{j-1})/2,\ (l_{j+1} - l_j)/2]$. Then, for almost all $G \in \mathbb{R}$, all $\varepsilon > 0$ and all R sufficiently large, there exists an orbit $r_{\rm h}(s) : \mathbb{R} \to \mathbb{R}_+$ of (4.2) homoclinic to γ_∞ and a constant $L > 0$ such that if the sequence $\{l_j\} \subset \mathbb{Z}$ satisfies $l_{j+1} - l_j \geq L$, then, for any sequence $\sigma = \{\sigma_j\} \subset \{0,1\}^{\mathbb{Z}}$ there exists an orbit $r_\sigma(s) : \mathbb{R} \to \mathbb{R}_+$ of (4.1) such that , if $\sigma_j = 0$*

$$|r_\sigma|_{C^1(I_j)} \geq R$$

and if $\sigma_j = 1$

$$|r_\sigma - r_{\rm h}|_{C^1(I_j)} \leq \varepsilon,$$

Moreover, if σ has only a finite number of non zero entries, then r_σ is a homoclinic solution.

Theorem 4.1.5 can be read as follows. For almost all $G \in \mathbb{R}$ there exist an orbit r_h of (4.2) homoclinic to γ_∞ such that the following holds. Let $z_* = (r, y, t) = (r_h(0), \dot{r}_h(0), 0) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$, let $z_\infty = (r, y, t) = (\infty, 0, 0) = \gamma_\infty \cap \{t = 0\} \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$ and denote by Φ the Poincaré map induced on the section $\{t = 0\}$ by the flow to the Hamiltonian (4.2). Then, for any $\delta > 0$ and any sequence $\{z_k\}_{k\in\mathbb{Z}} \subset \{z_\infty, z_*\}^{\mathbb{Z}}$ there exists a point $z \in B_\delta(z_0)$ and a sequence $\{n_k\}_{k\in\mathbb{Z}} \in \mathbb{N}^{\mathbb{Z}}$ such that $\Phi^{n_k}(z_0) \in B_\delta(z_k)$ [1]. The statement in Theorem 4.1.5 is indeed stronger since it also provides control on the orbit in all the intervals $[(n_k - n_{k-1})/2, (n_k + n_{k+1})/2]$.

The following corollary of Theorem 4.1.5 can obtained by nowadays well known arguments (see for example [MNT99] and [Koz83]).

Corollary 4.1.6. *For almost all $G \in \mathbb{R}$ the Restricted Isosceles 3 Body Problem is not C^ω integrable and has positive topological entropy.*

4.1.3 Outline of the proof: new tools for the study of oscillatory motions

As in Moser's approach, the first main step in our construction is to prove the existence of a homoclinic orbit to γ_∞. Yet, in the setting of Theorem 4.1.5, geometric perturbation theory is not available since the Hamiltonian system H_G in (4.2) is not nearly integrable. Instead, we will adopt a global approach and deploy the powerful machinery of the theory of calculus of variations. In particular, we rephrase the problem of existence of homoclinic orbits to γ_∞ as that of the existence of critical points of a certain action functional $\mathcal{A}_G$ (cfr 4.11) defined in a suitable Hilbert space $D^{1,2}$ (cfr (4.10)). The existence of critical points of the action functional $\mathcal{A}_G$ is obtained by a minmax argument tailored made for the present problem. The use of minmax techniques to study the existence and multiplicity results for homoclinic orbits in Hamiltonian systems has already been widely exploited in the literature (see for example [Sér92, CZES90, CZR91] and [MNT99]). In the variational approach to our problem, we face two main difficulties at this step: the phase space is not compact and the vector field presents singularities (corresponding to possible collision with the massive bodies). In order to overcome the first difficulty we make use of a *renormalized action functional* (see Remark 4.4.2) defined on a appropriately chosen functional space $D^{1,2}$. In order to avoid singularities and gain compactness we then perform a constrained deformation argument. With these techniques, together with a compactness property of the map $\mathrm{d}\mathcal{A}_G : D^{1,2} \to D^{1,2}$ (Struwe's monotonicity trick), we are able to show that, for almost all values of

[1] By $B_\delta(z_\infty)$ we mean the set $\{|y| \leq \delta,\ |r|^{-1} \leq \delta\}$.

the angular momentum G [2], there exists a Palais-Smale sequence in $D^{1,2}$ which converges to a critical point of the action functional $\mathcal{A}_G$. This proves the existence of an orbit $\tilde{r}_h$ homoclinic to γ_∞, which actually correspond to a doubly parabolic motion of our problem. It is worthwhile pointing out that half parabolic and hyperbolic motions for the n Body Problemem have been obtained using variational methods in [MV09, MV20] with a different technique.

The homoclinc orbit $\tilde{r}_h$ obtained in this way is associated with an intersection between the stable and unstable manifolds of the periodic orbit γ_∞. To proceed further, though we can not tell whether this intersection is transversal or not, we may rely on our minmax construction to deduce some topological transversality. This can be achieved by a topological degree argument based on a general result by Hofer ([Hof86]). More precisely, we exploit the mountain pass characterization of $\tilde{r}_h$ to show that for almost all values of the angular momentum G (except possibly a finite set of values) there exists a (possibly different) critical point r_h of the action functional $\mathcal{A}_G$ for which the Leray-Schauder index of the map $\nabla\mathcal{A}_G : D^{1,2} \to D^{1,2}$ at r_h is well defined and different from zero [3]. This allows us to shadow finite segments of the homoclinic orbit r_h. The proof of Theorem 4.1.5 is then obtained by combining a suitable parabolic version of the Lambda lemma close to γ_∞ with the outer dynamics wich shadows finite segments of r_h.

4.1.4 Organization of the paper

In Section 4.2 we recall some well known facts about the 2 Body Problem. Then, in Section 4.3 we analyze the dynamics around the periodic orbit γ_∞. In particular, the existence of stable and unstable manifolds $W^\pm(\gamma_\infty; G)$ and a parabolic version of the lambda lemma close to γ_∞. In Section 4.4 we introduce the variational formulation and prove the existence of a homoclinic orbit to γ_∞ by means of a minmax argument. Then, in Section 4.5 we obtain a (possibly different) homoclinic orbit associated with a topologically transverse intersection between $W^\pm(\gamma_\infty; G)$. Finally in Section 4.6 we combine the parabolic Lambda lemma of Section 4.3 together with the robustness of the topological degree under perturbations to construct "multibump" homoclinics and finish the proof of Theorem 4.1.5.

4.2 The 2 Body Problem

In this section we recall some well known facts about the 2 Body Problem (2BP) which will be used in the following. In polar coordinates, the Hamiltonian of the 2BP reads (compare (4.1))

$$H_{\mathrm{2BP}}(r,\alpha,y,G) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r}. \tag{4.4}$$

As for (4.1), the rotational symmetry implies that G is a conserved quantity, so we look at (4.4) as a one-parameter family of Hamiltonian functions $H_{\mathrm{2BP},G}(r,y)$. For each $G \in \mathbb{R}$ the Hamiltonian $H_{\mathrm{2BP},G}(r,y)$ is integrable and the motion can be classified in terms of the value of the energy: negative values correspond to elliptic motions, positive energies correspond to hyperbolic motions and for zero energy the motion is parabolic.

It is also straightforward to check that for all $G \in \mathbb{R}$

$$z_\infty = \{r = \infty,\ y = 0\} \subset \mathbb{R}_+ \times \mathbb{R}.$$

is a fixed point for the flow of (4.4)[4]. Moreover, for all $G \in \mathbb{R}$ the fixed point z_∞ posses stable and unstable manifolds which coincide along a one dimensional homoclinic manifold $W^h_{2BP}(z_\infty, G)$. The homoclinic orbit $W^h(z_\infty, G)$ is indeed the parabolic orbit of the 2BP with angular momentum G.

[2] See the discussion at the beginning of Section 4.4.2.

[3] In Proposition 4.5.12 show that the topological degree being non zero implies that the intersection between the invariant manifolds of γ_∞ at r_h is topologically transverse.

[4] To analyze this fixed point properly one should work in McGehee coordinates, which are introduced in Section 4.3.2.

Lemma 4.2.1. *There exist real analytic functions $r_0(u;G)$ and $y_0(u;G)$, defined for all $u \in \mathbb{R}$, such that*

$$W^h_{\mathrm{2BP}}(z_\infty;G) = \{r = r_0(u;G),\ y = y_0(u;G),\ u \in \mathbb{R}\}. \tag{4.5}$$

Moroever, $r_0(u;G) \geq G^2/2$ for all $u \in \mathbb{R}$ and

$$r_0(u;G) \sim u^{2/3} \qquad y_0(u;G) \sim u^{-1/3} \qquad as \qquad u \to \pm\infty.$$

In addition, for any $G, G_* \in \mathbb{R}$ we have

$$|r_0(u;G) - r_0(u;G_*)| \lesssim |G^2 - G_0^2| \qquad as \qquad u \to \pm\infty.$$

Remark 4.2.2. *In the last item of Lemma 4.2.1 we compare solutions associated with different values of the angular momentum G. The fact that we need that kind of information in our argument is due to a technical step (Struwe's monotonicity trick) in Section 4.4 (see Remark 4.4.2 and Lemma 4.4.10).*

Proof. A proof of the first two items can be found in [MP94], where the authors also show that

$$r_h(u;G) = \frac{G^2(\tau^2(u)+1)}{2} \qquad \text{for} \qquad u = \frac{G^3}{2}\left(\tau(u) + \frac{\tau^3(u)}{3}\right).$$

One can check that for $\tau \in \mathbb{R}$ the second equality admits the unique inverse

$$\tau(u) = \left(3G^{-3}u + \sqrt{9G^{-6}u^2 - 1}\right)^{1/3} - \left(3G^{-3}u + \sqrt{9G^{-6}u^2 - 1}\right)^{-1/3}$$

which for large u yields that

$$\tau(u) = (6G^{-3}u)^{1/3}\left(1 + \mathcal{O}(u^{-1})\right).$$

Therefore, as $u \to \pm\infty$

$$r_h(u;G) = \frac{G^2}{2} + \frac{(6u)^{2/3}}{2}\left(1 + \mathcal{O}(u^{-1})\right)$$

and the conclusion follows. □

Define the local stable and unstable manifolds[5]

$$\begin{aligned} W^+_{2BP,loc}(z_\infty;G) =& W^h_{2BP}(z_\infty;G) \cap \{y > 0\} \\ W^-_{2BP,loc}(z_\infty;G) =& W^h_{2BP}(z_\infty;G) \cap \{y < 0\}. \end{aligned}$$

It is a standard fact that $W^\pm_{2BP,loc}(z_\infty;G)$ are exact Lagrangian submanifolds so they can therefore be parametrized in terms of a generating function.

Lemma 4.2.3. *There exists $S_0(r;G) : (G^2/2, \infty) \to \mathbb{R}_+$, which satisfies*

$$H_{2BP;G}(r, \partial_r S_0(r;G)) = 0$$

and such that

$$W^\pm_{2BP,loc}(z_\infty;G) = \{(r, \pm\partial_r S_0(r;G)) \in \mathbb{R}_+ \times \mathbb{R} \colon r > G^2/2\}.$$

4.3 The dynamics close to γ_∞

In this section we study the dynamics in a neighbourhood of the periodic orbit at infinity defined in (4.3). Despite being degenerate (the linearized vector field vanishes at γ_∞) the flow close to the periodic orbit γ_∞ behaves in a similar way to the flow on a neighbourhood of a hyperbolic periodic orbit.

[5] One can prove that orbits starting at points in $W^+_{2BP,loc}(z_\infty;G)$ (respectively $W^-_{2BP,loc}(z_\infty;G)$) are confined in the region $\{r > G^2/2, y \geq 0\}$ for all positive times (respectively in the region $\{r > G^2/2, y \leq 0\}$ for all negative times).

4.3.1 The local invariant manifolds

Let ϕ_G^s be the time s flow associated with the Hamiltonian H_G defined in (4.2). It is a classical result by McGehee [McG73] (see also [BF04b]) that γ_∞ posses local stable and unstable invariant manifolds (by π_r, π_y we denote the projection on the r and y coordinates of a point $(r, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T}$)

$$\begin{aligned} W^+_{\mathrm{loc},R}(\gamma_\infty; G) =& \{x \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} \colon \pi_r \phi_G^s(x) \geq R,\ \pi_y \phi^s(x) \leq 1/R,\ \forall s \geq 0\} \\ W^-_{\mathrm{loc},R}(\gamma_\infty; G) =& \{x \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} \colon \pi_r \phi_G^s(x) \geq R,\ \pi_y \phi^s(x) \leq 1/R,\ \forall s \leq 0\} \end{aligned} \tag{4.6}$$

It is also a standard fact that $W^\pm_{\mathrm{loc},R}(\gamma_\infty; G)$ are exact Lagrangian submanifolds so they can be parametrized in terms of a generating function. The following result follows directly from the arguments in the proof of Theorem 4.4. in [GPSV21] (see Remark 4.3.2).

Proposition 4.3.1 ([GPSV21])**.** *Let H_G be the one parameter family of Hamiltonians defined in* (4.2) *and fix any $G_* > 0$. Then, there exist $R > 0$ such that for all $G \in [-G_*, G_*]$ there exist two functions $S^\pm(r, t; G) : [R, \infty) \times \mathbb{T} \to \mathbb{R}$, real analytic on r and G, solutions to the Hamilton-Jacobi equation*

$$H_G(r, t, \partial_r S^\pm(r, t; G)) + \partial_t S^\pm(r, t; G) = 0$$

and such that

$$W^\pm_{\mathrm{loc},R}(\gamma_\infty; G) = \{(r, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \colon r \in [R, \infty),\ y = \partial_r S^\pm(r, t; G)\}.$$

Moreover, if we let $S_0(r; G)$ be the function defined in Lemma 4.2.3, we have that

$$S^\pm(r; G) - S_0(r; G) \sim r^{-3/2} \qquad \text{as} \qquad r \to \infty.$$

Remark 4.3.2. *In Theorem 4.4. in [GPSV21] the authors only show the existence of the generating functions $S^\pm(r, t; G)$ for large values of G. The reason is that, under the hypothesis of large G, they can extend the generating functions to a common domain where they can measure their diference. However, if we are only concerned with the existence and behaviors of the generating functions close to infinity, the problem is already perturbative, and the very same arguments apply to obtain the conlcusion in Proposition 4.3.1.*

Define the global stable and unstable invariant manifolds

$$W^+(\gamma_\infty; G) = \bigcup_{s \leq 0} \phi_G^s(W^+_{\mathrm{loc},R}(\gamma_\infty; G)) \qquad W^-(\gamma_\infty; G) = \bigcup_{s \geq 0} \phi_G^s(W^-_{\mathrm{loc},R}(\gamma_\infty; G)). \tag{4.7}$$

The analytic dependence of the functions $S^\pm(r, t; G)$ on r and G will be key to prove that transversal intersections (whenever they exist) between the global stable and unstable invariant manifolds (4.7) are topologically transverse except for (possibly) a finite subset of values of G. This is key for the multibump construction. On the other hand, the estimate $S^\pm - S^0 \sim r^{-3/2}$ as $r \to \infty$ will be needed in the proof of certain technical steps in Lemma 4.5.7 (see Appendix 4.A).

4.3.2 The parabolic Lambda Lemma

We now analyze the topology of the flow lines close to the periodic orbit γ_∞. For that, it is convenient to introduce the McGehee transformation $r = 2/x^2$ in which the equations of motion associated with the Hamiltonian system H_G in (4.2) read

$$\dot{x} = -\frac{x^3}{4}\frac{\partial H_G}{\partial y} = -\frac{x^3 y}{4} \qquad \dot{y} = \frac{x^3}{4}\frac{\partial H_G}{\partial x} = -\frac{x^4}{4}\frac{1}{(1 + \frac{x^4 \rho^2(t)}{4})^{3/2}} + \frac{x^6 G^2}{8}.$$

In this variables, the periodic orbit at infinity (4.3) now corresponds to the periodic orbit $\hat{\gamma}_\infty = \{x = y = 0,\ t \in \mathbb{T}\}$. Following Moser [Mos01], we now straighten the stable and unstable directions associated with this periodic orbit. To that end, we introduce the change of variables

$$\tilde{q} = \frac{x - y}{2} \qquad \tilde{p} = \frac{x + y}{2}.$$

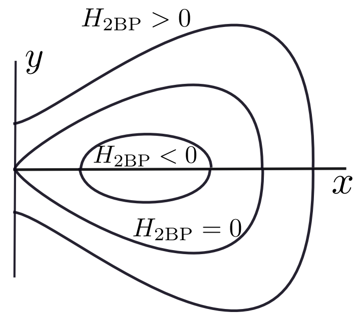

Figure 4.2: Phase portrait of the 2BP in McGehee coordinates. The fixed point z_∞ corresponds in McGehee coordinates to the origin in the $(x, y) \in \mathbb{R}^2$ plane.

In these coordinates

$$\dot{\tilde{q}} = \frac{1}{4}(\tilde{q} + \tilde{p})^3(q + \mathcal{O}_3(\tilde{q}, \tilde{p})) \qquad\qquad \dot{\tilde{p}} = -\frac{1}{4}(\tilde{q} + \tilde{p})^3(\tilde{p} + \mathcal{O}_3(\tilde{q}, \tilde{p})) \tag{4.8}$$

so it is clear that the local stable and unstable invariant manifolds associated with the periodic orbit $\tilde{\gamma}_\infty = \{\tilde{q} = \tilde{p} = 0, t \in \mathbb{T}\}$, which, by the work of McGehee [McG73] (see also Proposition 4.3.1) we already know exist, are close (for small $|\tilde{q}|, |\tilde{p}|$) to $\{\tilde{q} = 0\}$ and $\{\tilde{p} = 0\}$ respectively. Let now, for sufficiently small $\delta > 0$, define the set

$$Q_\delta = \{(\tilde{q}, \tilde{p}, t) \in \mathbb{R}^2 \times \mathbb{T} \colon |\tilde{q}| \leq \delta,\ |\tilde{p}| \leq \delta\}.$$

and, let $(0, \tilde{p}, t) \in Q_\delta \to (\tilde{p}, \gamma^s(\tilde{p}, t), t) \subset Q_\delta$ and $(\tilde{q}, 0, t) \in Q_\delta \to (\tilde{q}, \gamma^u(\tilde{q}, t), t) \subset Q_\delta$ be graph parametrizations of these local invariant manifolds. Introduce new variables on Q_δ given by

$$q = \tilde{q} - \gamma^s(\tilde{p}, t) \qquad\qquad p = \tilde{p} - \gamma^u(\tilde{q}, t).$$

From the invariance equation satisfied by $\gamma^{u,s}$ one can deduce their Taylor expansion around $\tilde{q} = \tilde{p} = 0$. Then, an easy computation, shows that

$$\dot{q} = -\frac{q}{4}\left((q + p)^3 + \mathcal{O}_4(q, p)\right) \qquad\qquad \dot{p} = \frac{p}{4}\left((q + p)^3 + \mathcal{O}_4(q, p)\right) \tag{4.9}$$

so in coordinates $(q, p, t) \subset Q_\delta$ the local stable and unstable manifolds are the sets $\{p = 0\} \cap Q_\delta$ and $\{q = 0\} \cap Q_\delta$ respectively. Define now, for $a < \delta$ the sections (see Figure 4.3.2)

$$\Sigma_a^+ = \{(q, p, t) \in Q_{2\delta} \colon p = \delta,\ 0 < q \leq a\} \qquad\qquad \Sigma_a^- = \{(q, p, t) \in Q_{2\delta} \times \mathbb{T} \colon q = \delta,\ 0 < p \leq a\}$$

and the associated Poincaré map $\Phi_{\mathrm{loc}} : \Sigma_a^+ \to \Sigma_{a'}^-$, associated with the flow (4.8), whenever is well defined. Lemma 4.3.3 shows that a parabolic version of the Lambda Lemma holds for the degenerate periodic orbit $\{p = q = 0\}$. In order to build orbits whose final motions are hyperbolic, we also introduce the outer sections

$$\Sigma_{a,\mathrm{hyp}}^+ = \{(q, p, t) \in Q_{2\delta} \colon p = \delta,\ -a \leq q < 0\} \qquad\qquad \Sigma_{a,\mathrm{hyp}}^- = \{(q, p, t) \in Q_{2\delta} \colon q = \delta,\ -a \leq p < 0\}.$$

The proof of the following proposition follows plainly from the arguments in Chapter IV of [Mos01], where an analogous result is proved for the Sitnikov problem. See also Theorem 5.4. in [GMPS22].

Lemma 4.3.3. *Fix any $G_* > 0$. Then, there exists $C > 0$ sufficiently large and $\delta > 0$ sufficiently small such that for any $G \in [-G_*, G_*]$ and any $a \in (0, \delta/2)$ the Poincaré map*

$$\Phi_{\mathrm{loc}} : \Sigma_a^+ \longrightarrow \Sigma_{a^{1-C\delta}}^-$$

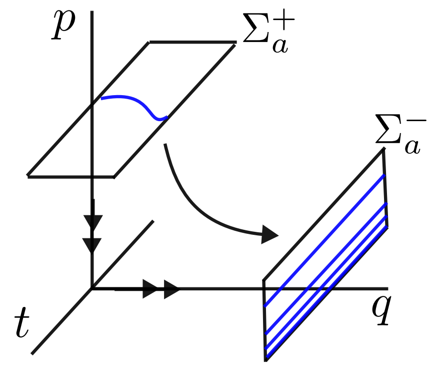

Figure 4.3: The sections $\Sigma_a^{\pm}$. The Poincaré map $\Phi_{\rm loc}:\Sigma_a^+\to\Sigma_{a'}^-$ sends the blue line in the section Σ_a^+ into the blue line in the section Σ_a^-, which accumulates to $\{p=0\}$.

is well defined. Moreover, for any t_1 sufficiently large there exist unique q and p_1, which satisfy

$$q^{1+C\delta}\le p_1\le q^{1-C\delta}\qquad\qquad q^{-3(1-C\delta)/2}\lesssim t_1\lesssim q^{-3(1+C\delta)/2},$$

for which $\Phi_{\rm loc}(q,a,0)=(a,p_1,t_1)$.

In addition, for any $(q,a,0)\in\Sigma^+_{a,\rm hyp}$ (respectively $(a,p,0)\in\Sigma^-_{a,\rm hyp}$), the orbit $(q_{\rm hyp}(s),p_{\rm hyp}(s),s)$ of (4.9) with initial condition $(q,a,0)$ (respectively $(a,p,0)$) is defined for all forward (respectively backward) times and satisfies

$$\lim_{s\to\infty}y(q_{\rm hyp}(s),(p_{\rm hyp}(s))>0\qquad\qquad (\textit{respectively}\ \lim_{s\to-\infty}y(q_{\rm hyp}(s),(p_{\rm hyp}(s))<0).$$

The first item in Lemma 4.3.3 shows that the iterates of curves which are transversal to the local stable manifold accumulate along the unstable manifold (see also Figure 4.3.2). The second item ensures that orbits with initial conditions on $\Sigma^+_{a,\rm hyp}$ (respectively $\Sigma^-_{a,\rm hyp}$) have forward (respectively backward) hyperbolic final motions. We now translate these results to the original coordinates. To that end we introduce the sections

$$\begin{aligned}\Lambda^+_{R,\delta}=&\{(r,y,t)\colon r=R,\ 0<\partial_rS^+(R,t;G)-y\le\delta,\ t\in\mathbb{T}\}\\ \Lambda^-_{R,\delta}=&\{(r,y,t)\colon r=R,\ 0<y-\partial_rS^-(R,t;G)\le\delta,\ t\in\mathbb{T}\}\end{aligned}$$

and the map $\Phi_{{\rm loc},R_1,R_2}:\Lambda^+_{R_1,\delta}\to\Lambda^-_{R_2,\delta'}$ whenever is well defined. We also define the sections leading to hyperbolic final motions

$$\begin{aligned}\Lambda^+_{R,\delta}=&\{(r,y,t)\colon r=R,\ -\delta\le\partial_rS^+(R,t;G)-y<0,\ t\in\mathbb{T}\}\\ \Lambda^-_{R,\delta}=&\{(r,y,t)\colon r=R,\ -\delta\le y-\partial_rS^-(R,t;G)<0,\ t\in\mathbb{T}\}.\end{aligned}$$

Lemma 4.3.4. *Fix any $G_*>0$. Then, there exist $R>0$ sufficiently large such that for any $R_1,R_2\ge R$ there exists $\delta_0(R_1,R_2)$ such that for all $G\in[-G_*,G_*]$ the Poincaré map*

$$\Phi_{{\rm loc},R_1,R_2}:\Lambda^+_{R_1,\delta}\to\Lambda^-_{R_2,\delta'}$$

is well defined for $\delta\le\delta_0$ and some $\delta'(R_1,R_2,\delta)>0$. There exists T_* such that for any $T\ge T_*$ there exist unique y_0,y_1 such that $\Phi_{{\rm loc},R_1,R_2}(R_1,y_0,0)=(R_2,y_1,T)$. Moreover, for any $\varepsilon>0$ there exists T_{**} such that, if $T\ge T_{**}$ and $\Phi_{{\rm loc},R_1,R_2}(R_1,y_0,0)=(R_2,y_1,T)$, then

$$\partial_rS^+(R_1,0;G)-y_0\le\varepsilon\qquad\qquad y_1-\partial_rS^+(R_2,T;G)\le\varepsilon.$$

In addition, the orbit $(r_{\rm hyp}(s),y_{\rm hyp}(s),s)$ of (4.2) with initial condition $(R_1,y,0)\in\Lambda^+_{R_1,\delta,\rm hyp}$ (respectively $(R_2,y,0)\in\Lambda^-_{R_2,\delta,\rm hyp}$), is defined for all forward (respectively backward) times and satisfies

$$\lim_{s\to\infty}y_{\rm hyp}(s)>0\qquad\qquad(\textit{respectively}\ \lim_{s\to-\infty}y_{\rm hyp}(s)<0).$$

4.4 Existence of homoclinic orbits to γ_∞

In this section we establish the existence of orbits of the Hamiltonian (4.1), which are homoclinic to γ_∞. For $|G| \gg 1$, the Hamiltonian (4.1) can be considered as a perturbation of the integrable 2BP, in which there exists a homoclinic manifold to γ_∞ (see Lemma 4.2.1). Therefore, for $|G| \gg 1$, one can use geometric perturbation theory to prove that the global invariant manifolds $W^+(\gamma_\infty; G)$ and $W^-(\gamma_\infty; G)$ defined in (4.7) intersect transverally. This was the approach used in [GPSV21] where the following result was proved.

Theorem 4.4.1 ([GPSV21])**.** *There exists $G_* < \infty$ such that for all G such that $|G| \geq G_*$ the global stable and unstable manifolds $W^+(\gamma_\infty; G)$ and $W^-(\gamma_\infty; G)$ defined in* (4.7)*, intersect transversally.*

Yet, for a fixed $G \in \mathbb{R}$, the Hamiltonian (4.1) is not close to the 2BP. Therefore, geometric perturbation theory cannot help to study the existence of transversal intersections between $W^+(\gamma_\infty; G)$ and $W^-(\gamma_\infty; G)$. We however exploit the variational formulation of the problem, in which the powerful techniques from nonlinear functional analysis are available.

More concretely, in Section 4.4.1 we introduce a suitable action functional, defined on a suitable Hilbert space, whose critical points are indeed orbits of (4.1) which are homoclinic to γ_∞. Then, in Section 4.4.2 we establish the existence of a critical point of the aforementioned action functional using a minmax argument. The minmax characterization of the critical point obtained is crucial for the construction in Section 4.6.

4.4.1 The Variational Formulation

We introduce the vector space of real valued functions

$$D^{1,2} = \{\varphi \in C(\mathbb{R})\colon \exists v_\varphi \in L^2(\mathbb{R}) \text{ such that } \varphi(s) = \varphi(0) + \int_0^s v_\varphi(t)\mathrm{d}t \quad \forall s \in \mathbb{R}\}. \tag{4.10}$$

In the following, we will write $\dot{\varphi} = v_\varphi$ (i.e. v_φ is the weak derivative of φ). It is easy to chek that

$$\langle \varphi, \psi \rangle_{D^{1,2}} = |\varphi(0)\psi(0)| + \langle \dot{\varphi}, \dot{\psi} \rangle_{L^2}$$

defines an inner product on $D^{1,2}$ for which the functional space $D^{1,2}$ equiped with this inner product is a Hilbert space. We write

$$\|\varphi\|_{D^{1,2}} = \left(\langle \varphi, \varphi \rangle_{D^{1,2}}\right)^{1/2}.$$

for the induced norm. Notice that for all $\varphi \in D^{1,2}$ and all $s \in \mathbb{R}$

$$|\varphi(s)| \leq |\varphi(0)| + \|\dot{\varphi}\|_{L^2}\sqrt{|s|}.$$

After the introduction of the functional space $D^{1,2}$ it is an easy computation to show that the existence of orbits of (4.2) homoclinic to the periodic orbit at infinity $\gamma_\infty = \{r = \infty, y = 0, t \in \mathbb{T}\}$ is equivalent to the existence of critical points of the action functional $\mathcal{A}_G : D^{1/2} \to \mathbb{R}$ given by

$$\mathcal{A}_G(\varphi; G_0) = \int_{\mathbb{R}} \mathcal{L}_{\mathrm{ren}}(\varphi, \dot{\varphi}, s; G, G_0)\mathrm{d}s, \tag{4.11}$$

where

$$\mathcal{L}_{\mathrm{ren}}(\varphi, \dot{\varphi}, s; G, G_0) = \frac{\dot{\varphi}^2}{2} + V_G(r_0 + \varphi) - V_0(r_0) - \ddot{r}_0\varphi, \tag{4.12}$$

V_G stands for the effective potential

$$V_G(r, t) = \frac{G^2}{2r^2} - \frac{1}{\sqrt{r^2 + \rho^2(t)}}, \tag{4.13}$$

and $V_0(r_0) = \frac{G_0^2}{2r_0^2} - \frac{1}{r_0}$ with r_0 being the parabolic orbit of the 2BP with angular momentum $G_0 \in \mathbb{R}$ (see Remark 4.4.3).

Remark 4.4.2. *$\mathcal{L}_{\rm ren}(\varphi,\dot\varphi,s;G)$ is indeed a renormalized Lagrangian, that is, we have substracted the term $V_0(r_0)$ in the integrand of what would be the "natural" action functional. The reason behind the definition of (4.11) is that the action of a parabolic orbit is infinite. Indeed, the Lagrangian of the 2BP reads*

$$\mathcal{L}_0(r_0,\dot r_0)=\frac{\dot r_0^2}{2}-V_0(r_0)$$

and for a parabolic orbit $r_0(s)\sim s^{2/3}$ for $s\to\pm\infty$.

Remark 4.4.3. *It might seem surprising that when defining the renormalized Lagrangian $\mathcal{L}_{\rm ren}$, we let G_0 be an independent parameter instead of taking $G=G_0$. The reason is that in this way, for a fixed $G_0\in\mathbb{R}$ and fixed $\varphi\in D^{1,2}$ the function $G\to\mathcal{A}_G(\varphi)$ is monotonely decreasing. This will allow us to use a monotonicity trick due to Struwe which is key to obtain uniform bounds for certain (Palais-Smale) sequences $\{\varphi_n\}_{n\in\mathbb{N}}\subset D^{1,2}$ for which $\mathrm{d}\mathcal{A}_G(\varphi_n)\to 0$ (see Section 4.4.2 and, in particular, 4.4.10). On the other hand, the asymptotic behavior of parabolic solutions as $s\to\pm\infty$ becomes independent of the value of the angular momentum G (see Lemma 4.2.1) so the definition of the renormalized Lagrangian $\mathcal{L}_{\rm ren}$ makes sense for $G\neq G_0$.*

Remark 4.4.4. *Throughout the rest of the paper the value $G_0\in\mathbb{R}_+$ will be fixed. Thus, we omit the dependence of all quantities on G_0. Having fixed $G_0\in\mathbb{R}_+$, we state results for $G\in[-G_0,G_0]$ (or full measure subsets of this set). This choice is completely arbitrary: the results proved below are certainly true if we replace $[-G_0,G_0]$ by any other bounded subset. However, since we have always the freedom to choose G_0 as large as we want it is enough to state results for $G\in[-G_0,G_0]$.*

The following observation will play an important role in our construction.

Lemma 4.4.5. *Let $\tau\in\mathbb{Z}$ and define the translation operator*

$$T_\tau(\varphi)(s)=\varphi(s+\tau)+r_0(s+\tau)-r_0(s).$$

Then, for all $\tau\in\mathbb{Z}$

$$\mathcal{A}_G(T_\tau(\varphi))=\mathcal{A}_G(\varphi).$$

We now state a technical lemma which will prove useful in later compactness arguments.

Lemma 4.4.6. *Let $\gamma\geq 0$ and let L^2_γ be the weighted L^2 space with norm given by*

$$\|\varphi\|_{L^2_\gamma}=\left(\int_{\mathbb{R}}\frac{|\varphi|^2}{r_0^{3+\gamma}}\right)^{1/2}.$$

Then, $D^{1,2}$ is continuously embedded in L^2_γ for $\gamma\geq 0$ and compactly embedded in L^2_γ for $\gamma>0$.

Proof. The proof of the continuous embedding for $\gamma\geq 0$ is obtained by the very same argument used in the proof of Proposition 3.2. in [BDFT21] taking into account that $r_0(s)\sim s^{2/3}$ for $s\to\pm\infty$ and $r_0(s)\geq G_0^2/2$ $\forall s\in\mathbb{R}$. We now prove that the embedding for $\gamma>0$ is compact. Take any bounded sequence $\{\varphi_n\}_{n\in\mathbb{N}}\subset D^{1,2}$ such that $\varphi_n\to 0$ weakly in $D^{1,2}$. In particular $\varphi_n(s)\to 0$ pointwise for all $s\in\mathbb{R}$. Since, for any $\varphi\in D^{1,2}$ and any $s\in\mathbb{R}$ we have

$$|\varphi(s)|\leq|\varphi(0)|+\|\dot\varphi\|_{L^2}\sqrt{|s|}$$

we obtain that for all $s\in\mathbb{R}$

$$\frac{|\varphi_n(s)|^2}{r_0^{3+\gamma}(s)}\lesssim\frac{\|\varphi_n\|^2_{D^{1,2}}}{1+|s|^{1+\gamma}}.$$

Therefore, a direct application of the dominated convergence theorem shows that

$$\lim_{n\to\infty}\|\varphi_n\|^2_{L^2_\gamma}=\lim_{n\to\infty}\int_{\mathbb{R}}\frac{|\varphi_n|^2}{r_0^{3+\gamma}(s)}=0.$$

□

We now show that $\mathcal{A}_G$ is continuous and has a continuous differential on a suitable subset $Q \subset D^{1,2}$.

Lemma 4.4.7. *Let $K > 0$ and $\underline{m} > 0$ be two fixed constants and define*

$$Q = \{\varphi \in D^{1,2} \colon \|\varphi\|_{D^{1,2}} \leq K,\ \min_{s\in\mathbb{R}} r_0(s) + \varphi(s) \geq \underline{m}\}$$

Then, for any $G \in [-G_0, G_0]$ we have $\mathcal{A}_G \in C^1(\mathrm{int}(Q), \mathbb{R})$.

Proof. Let $\varphi, \psi \in Q$ and make use of the mean value theorem to write

$$\mathcal{A}_G(\varphi) - \mathcal{A}_G(\psi) = \int_{\mathbb{R}} \frac{1}{2}(\dot\varphi + \dot\psi)(\dot\varphi - \dot\psi) + \partial_r V_G(r_0 + \xi)(\varphi - \psi) - \ddot r_0(\varphi - \psi) \tag{4.14}$$

where $\xi = \lambda\varphi + (1-\lambda)\psi$ for some $\lambda(s) \in [0,1]$. On one hand,

$$\left| \int_{\mathbb{R}} (\dot\varphi + \dot\psi)(\dot\varphi - \dot\psi) \right| \leq \left(\int_{\mathbb{R}} |\dot\varphi + \dot\psi|^2 \right)^{1/2} \left(\int_{\mathbb{R}} |\dot\varphi - \dot\psi|^2 \right)^{1/2} \to 0$$

as $\|\varphi - \psi\|_{D^{1,2}} \to 0$. On the other hand, since for $\varphi, \psi \in D^{1,2}$

$$\begin{aligned}\min_{s\in\mathbb{R}}\ r_0(s) + \xi(s) &= \min_{s\in\mathbb{R}}\ r_0(s) + \lambda\varphi(s) + (1-\lambda)\psi(s) = \min_{s\in\mathbb{R}}\ \lambda(r_0(s) + \varphi(s)) + (1-\lambda)(r_0(s) + \psi(s)) \\ &\geq \min_{s\in\mathbb{R}}\ \lambda\underline{m} + (1-\lambda)\underline{m} = \underline{m} > 0\end{aligned}$$

and convergence in $D^{1,2}$ implies uniform convergence in compact intervals, we have, taking into account the expression of V_G (4.13), that

$$\left(\partial_r V_G(r_0 + \xi) - \ddot r_0\right)(\varphi - \psi) \to 0$$

pointwise as $\|\varphi - \psi\|_{D^{1,2}} \to 0$. Moreover, for $s \to \pm\infty$

$$r_0(s) + \varphi(s) \geq r_0(s) - \left(|\varphi(0)| + \|\dot\varphi\|_{L^2} \sqrt{|s|} \right) \sim s^{2/3}$$

so, from the definition of V_G in (4.13), a straightforward computation shows that for $r_0 \to \infty$

$$\partial_r V_G(r_0) - \ddot r_0 \sim r_0^{-3}.$$

Thus, using again that $\min_{s\in\mathbb{R}}\ r_0(s) + \xi(s) \geq \underline{m} > 0$, we obtain the existence of $C > 0$ depending only on K and $\underline{m}$ such that for all $s \in \mathbb{R}$

$$|\partial_r V_G(r_0(s) + \xi(s)) - \ddot r_0(s)| \leq C r_0^{-3}(s)$$

Therefore,

$$\begin{aligned}\left| \int_{\mathbb{R}} \left(\partial_r V_G(r_0 + \xi) - \ddot r_0\right)(\varphi - \psi) \right| &\leq \left(\int_{\mathbb{R}} |\left(\partial_r V_G(r_0 + \xi) - \ddot r_0\right)| \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{R}} |\left(\partial_r V_G(r_0 + \xi) - \ddot r_0\right)| |\varphi - \psi|^2 \right)^{1/2} \\ &\leq C \int_{\mathbb{R}} \frac{|\varphi - \psi|^2}{r_0^3} = C\|\varphi - \psi\|_{L_0^2},\end{aligned}$$

and the continuity of the map $\mathcal{A}_G : Q \subset D^{1,2} \to \mathbb{R}$ is implied by Lemma 4.4.6. The proof that $\mathrm{d}\mathcal{A}_G : Q \subset D^{1,2} \to D^{1,2}$ is a continuous map follows from similar arguments. □

Lemma 4.4.8. *Let $K > 0$ and $\underline{m} > 0$ be two fixed constants and let $Q \subset D^{1,2}$ be the subset defined in Lemma 4.4.7. Then, for any for any $G \in [-G_0, G_0]$, $\mathrm{d}\mathcal{A}_G : \mathrm{int}(Q) \to D^{1,2}$ is a compact perturbation of the identity. In partiuclar, this implies that for any compact set $F \subset D^{1,2}$ the set $Q \cap (\mathrm{d}\mathcal{A}_G)^{-1}(F)$ is compact.*

Proof. We write

$$
\begin{aligned}
\mathrm{d}\mathcal{A}_G(\varphi)[\psi] =& \langle \dot\varphi, \dot\psi\rangle_{L^2} - \int_{\mathbb{R}} \left(\frac{r_0+\varphi}{((r_0+\varphi)^2+\rho^2)^{3/2}} - \frac{1}{r_0^2}\right)\psi + \int_{\mathbb{R}} \left(\frac{G^2}{(r_0+\varphi)^3} - \frac{G_0}{r_0^3}\right)\psi \\
=& \langle \dot\varphi, \dot\psi\rangle_{L^2} + 2\int_{\mathbb{R}} \frac{\varphi\psi}{r_0^3} - \int_{\mathbb{R}} \left(\frac{r_0+\varphi}{((r_0+\varphi)^2+\rho^2)^{3/2}} - \frac{1}{r_0^2} + \frac{2\varphi}{r_0^3}\right)\psi \\
&+ \int_{\mathbb{R}} \left(\frac{G^2}{(r_0+\varphi)^3} - \frac{G_0}{r_0^3}\right)\psi \\
=& \langle \dot\varphi, \dot\psi\rangle_{L^2} + 2\int_{\mathbb{R}} \frac{\varphi\psi}{r_0^3} + P(\varphi)[\psi]
\end{aligned}
\tag{4.15}
$$

where we have introduced the functional

$$
P(\varphi)[\psi] = \int_{\mathbb{R}} \left(\frac{G^2}{(r_0+\varphi)^3} - \frac{G_0}{r_0^3}\right)\psi - \int_{\mathbb{R}} \left(\frac{r_0+\varphi}{((r_0+\varphi)^2+\rho^2)^{3/2}} - \frac{1}{r_0^2} + \frac{2\varphi}{r_0^3}\right)\psi
$$

Thanks to Lemma 4.4.6 we can take

$$
\langle\langle \varphi, \psi\rangle\rangle_{D^{1,2}} = 2\int_{\mathbb{R}} \frac{\varphi\psi}{r_0^3} + \langle \dot\varphi, \dot\psi\rangle_{L^2}
$$

as an equivalent inner product in $D^{1,2}$. It follows from Lemma 4.4.7 that for all $\varphi \in Q$, $\mathrm{d}\mathcal{A}_G(\varphi) : D^{1,2} \to \mathbb{R}$ and $P(\varphi) : D^{1,2} \to \mathbb{R}$ are continuous linear functionals and thanks to Riesz representation theorem, for every $\varphi \in Q \subset D^{1,2}$ there exist unique $\eta_A(\varphi), \eta_P(\varphi) \in D^{1,2}$ such that

$$
\langle\langle \eta_A(\varphi), \psi\rangle\rangle_{D^{1,2}} = \mathrm{d}\mathcal{A}_G(\varphi)[\psi] \qquad \langle\langle \eta_P(\varphi), \psi\rangle\rangle_{D^{1,2}} = P(\varphi)[\psi].
$$

and $\eta_A = \mathrm{Id} + \eta_P$. After writing

$$
\langle\langle \eta_P(\varphi_*) - \eta_P(\varphi), \eta_P(\varphi_*) - \eta_P(\varphi)\rangle\rangle_{D^{1,2}} = P(\varphi_*)[\eta_P(\varphi_*) - \eta_P(\varphi)] - P(\varphi)[\eta_P(\varphi_*) - \eta_P\varphi)],
$$

a tedious but straightforward computation shows that for any $\varphi_*, \varphi \in Q$

$$
\|\eta_P(\varphi_*) - \eta_P(\varphi)\|^2_{D^{1,2}} \leq \|\varphi_* - \varphi\|_{L^2_{1/4}} \|\eta_P(\varphi^*) - \eta_P(\varphi)\|_{L^2_{1/4}} \leq \|\varphi_* - \varphi\|_{L^2_{1/4}} \|\eta_P(\varphi^*) - \eta_P(\varphi)\|_{D^{1,2}} \tag{4.16}
$$

what implies that $\eta_P : Q \to D^{1,2}$ is a compact operator (recall that the embedding of $D^{1,2}$ in $L^2_{1/4}$ is compact). The second item in the lemma plainly follows after writing

$$
\eta_A(\varphi) = \varphi + \eta_P(\varphi).
$$

Indeed, for a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset Q \subset D^{1,2}$ whose image under $\mathrm{d}\mathcal{A}_G$ is contained in a compact subset $F \subset D^{1,2}$ there exists a subsequence (which we do not relabel) for which $\{\eta_A(\varphi_n)\}_{n\in\mathbb{N}}$ is convergent in $D^{1,2}$. Then, the proof is finished since η_P being a compact operator and implies that (up to passing to a further subsequence) $\{\eta_P(\varphi_n)\}_{n\in\mathbb{N}}$ is also convergent in $D^{1,2}$. □

From now on we will omit the subscript in the inner product and norm defined in $D^{1,2}$.

4.4.2 Existence of critical points of the action functional

In this section we prove the existence of critical points of the action functional $\mathcal{A}_G$ defined in (4.11) using a minmax argument. In particular, we will employ a constrained version of the celebrated mountain pass theorem of Ambrosetti and Rabinowitz [?]. The first step is to verify that the level sets of $\mathcal{A}_G$ have a mountain pass geometry. This is the content of the following proposition.

Proposition 4.4.9. *Take any constant $M > 0$. Then, for all $G \in [-G_0, G_0]\backslash\{0\}$ there exist $\psi_1, \psi_2 \in D^{1,2}$ such that*

$$\mathcal{A}_G(\psi_i) \leq -M \qquad i = 1, 2.$$

Moreover, there exists $M^* > 0$ such that if we take $M \geq M^*$, then for any curve $\gamma \in C([0,1], D^{1,2})$ joining ψ_1 and ψ_2 there exist a point ψ_γ for which

$$\mathcal{A}_G(\psi_\gamma) \geq -M/2.$$

Proof. Let $\mu > 0$ so

$$\begin{aligned}\mathcal{A}_G(\mu) &= \int_{\mathbb{R}} V_G(r_0 + \mu) - V_0(r_0) = \int_{\mathbb{R}} \frac{1}{((r_0+\mu)^2 + \rho^2)^{1/2}} - \frac{1}{r_0} - \frac{G^2}{2(r_0+\mu)^2} + \frac{G_0^2}{2r_0^2} \\ &\leq \int_{\mu} \frac{1}{r_0+\mu} - \frac{1}{r_0}.\end{aligned}$$

It follows from Fatou's lemma that

$$\limsup_{\mu\to\infty} \mathcal{A}_G(\mu) = \limsup_{\mu\to\infty} \int_{\mathbb{R}} \frac{1}{r_0+\mu} - \frac{1}{r_0} \leq -\int_{\mathbb{R}} \frac{1}{r_0} = -\infty.$$

On the other hand, take $\eta \in (0, 1/2)$. Then, for some finite (and uniform for $\eta \in (0, 1/2)$) $C > 0$ we have

$$\begin{aligned}\mathcal{A}_G(\eta) &= \int_{\mathbb{R}} V_G(r_0 + \eta) - V_0(r_0) = \int_{\mathbb{R}} \frac{1}{((r_0+\eta)^2 + \rho^2)^{1/2}} - \frac{1}{r_0} + \frac{G_0^2}{2r_0^2} - \frac{G^2}{2(r_0+\eta)^2} \\ &\leq C + \int_0^1 \frac{1}{r_0+\eta} - \frac{G^2}{2(r_0+\eta)^2}.\end{aligned}$$

Using that $r_0(s) = 1/2 + s^2 + \mathcal{O}(s^3)$ for $s \to 0$ (this can be deduced from the proof of Lemma 4.2.1) one can easily check that

$$\limsup_{\eta\to 1/2} \mathcal{A}_G(\eta) = -\infty.$$

The first part of the lemma is proven by taking $\psi_1 = \mu$ with μ large enough and $\psi_2 = \eta$ with $\eta \to 1/2$. In order to prove the second item of the lemma we let $R > 0$ be such that

$$\partial^2_{rr} V_G(r) \geq 0 \qquad \forall r \geq R$$

and denote by T the value of s for which $r_0(s) \geq R$ for all s such that $|s| \geq T$. Notice that R exists because of the convexity of $V_G(r)$ for large values of r, which can be checked explicitely from the expression of V_G in (4.13). We now take $\varphi \in D^{1,2}$ such that $\min_{s\in\mathbb{R}} r_0(s) + \varphi(s) = R$. We claim that $\mathcal{A}_G(\varphi) \geq -M/2$ so the lemma follows since, by continuity, for all $\gamma \in C([0,1], D^{1,2})$ joining ψ_1 and ψ_2 there exist a point $\varphi \in \gamma$ for which

$$\min_{s\in\mathbb{R}} r_0(s) + \varphi(s) = R.$$

We now prove the claim. Lemma 4.4.5 implies that, withouth lost of generality, we can suppose that the minimum is attained at the interval $s \in [0,1]$. We express

$$\mathcal{A}_G(\varphi) = \frac{\|\dot{\varphi}\|^2_{L^2}}{2} + J_{\leq}(\varphi) + J_{\geq}(\varphi) + E(\varphi)$$

where

$$\begin{aligned}J_{\geq}(\varphi) &= \int_{|s|\geq T} \frac{1}{((r_0+\varphi)^2 + \rho^2)^{1/2}} - \frac{1}{(r_0^2 + \rho^2)^{1/2}} + \frac{r_0\varphi}{(r_0^2+\rho^2)^{3/2}} \\ &\quad - G^2 \int_{|s|\leq T} \frac{1}{2(r_0+\varphi)^2} - \frac{1}{2r_0^2} + \frac{\varphi}{r_0^3} \\ J_{\leq}(\varphi) &= \int_{|s|\leq T} \frac{1}{((r_0+\varphi)^2 + \rho^2)^{1/2}} - \frac{1}{(r_0^2 + \rho^2)^{1/2}} + \frac{r_0\varphi}{(r_0^2+\rho^2)^{3/2}} \\ &\quad - G^2 \int_{|s|\geq T} \frac{1}{2(r_0+\varphi)^2} - \frac{1}{2r_0^2} + \frac{\varphi}{r_0^3}\end{aligned}$$

and

$$E(\varphi) = \int_{s\in\mathbb{R}} \frac{1}{(r_0^2+\rho^2)^{1/2}} - \frac{r_0\varphi}{(r_0^2+\rho^2)^{3/2}} - \frac{1}{r_0} + \frac{\varphi}{r_0^2} + (G_0^2 - G^2)\left(\frac{1}{2r_0^2} - \frac{\varphi}{r_0^3}\right)$$

For the first term, after applying the mean value theorem twice, we obtain that

$$J_{\geq}(\varphi) = \int_{|s|\geq T} \partial_{rr}^2 V_G(r_0+\xi)\eta\varphi$$

with $\eta = \sigma\varphi$, $0 \leq \sigma \leq 1$ and $\xi = \lambda\eta$, $0 \leq \lambda \leq 1$. Since

$$\min(r_0+\xi) \geq \min(r_0, r_0+\varphi) \geq R$$

we have $J_{\geq}(\varphi) \geq 0$ by the definition of R. For the second term we use that $\min_{s\in\mathbb{R}} r_0(s) + \varphi(s) \geq R > 0$ and that for all $s \in \mathbb{R}$ we have $|\varphi(s)| \leq |\varphi(0)| + \|\dot{\varphi}\|_{L^2}\sqrt{|s|}$ so we obtain

$$\begin{aligned} J_{\leq}(\varphi) \geq & -C + \int_{|s|\leq T} \frac{r_0\varphi}{(r_0^2+\rho^2)^{3/2}} - G^2 \int_{|s|\leq T} \frac{1}{2R^2} - \frac{1}{2r_0^2} - \frac{\varphi}{r_0^3} \\ \geq & -C + \int_{|s|\leq T} \left(\frac{r_0}{(r_0^2+\rho^2)^{3/2}} - \frac{G_0^2}{r_0^3}\right)\varphi \geq -C(1+\|\dot{\varphi}\|_{L^2}) \end{aligned}$$

for some $C > 0$ which depends only on R. An analogous computation shows that for the third term we have

$$E(\varphi) \geq \int_{s\in\mathbb{R}} \frac{1}{(r_0^2+\rho^2)^{1/2}} - \frac{1}{r_0} + \left(\frac{1}{r_0^2} - \frac{(r_0+\rho)}{(r_0^2+\rho^2)^{3/2}} + \frac{(G_0^2-G^2)}{r_0^3}\right)\varphi \geq -C(1+\|\dot{\varphi}\|_{L^2})$$

for some $C > 0$ which depends only on R. Therefore

$$\mathcal{A}_G(\varphi) \geq \frac{\|\dot{\varphi}\|_{L^2}^2}{2} - C(1+\|\dot{\varphi}\|_{L^2})$$

for C depending only on R and the result follows after enlarging M (if necessary) while keeping R fixed. □

We now have established the existence of the mountain pass geometry for the level sets of the functional $\mathcal{A}_G$. The next natural step would be to apply the classical deformation lemma to obtain a Palais-Smale (PS) sequence for the functional $\mathcal{A}_G$. There are however two difficulties. The first one is that, a priori, a suboptimal path, might contain points $\varphi \in D^{1,2}$ for which $\min_{s\in\mathbb{R}}(r_0+\varphi)(s) = 0$, at which the functional $\varphi \mapsto \mathcal{A}_G$ is not continuous. The second difficulty is that, even if we can guarantee that $\min_{s\in\mathbb{R}}(r_0+\varphi)(s) > 0$ for all φ in the region where we carry the deformation argument, without further constraints we are not able to show that the PS sequence obtained is precompact. For that reason, we take $\overline{m} > 0$ large enough and we carry the deformation argument in the region

$$\mathcal{F}_{\overline{m}} = \left\{\varphi \in D^{1,2} \colon \min_{s\in\mathbb{R}}(r_0+\varphi)(s) \leq \overline{m}\right\}. \tag{4.17}$$

In Lemma 4.4.15 we show that on a suitable subset $\mathcal{F}_{\overline{m},\delta,b} \subset \mathcal{F}_{\overline{m}}$, the functional $\mathcal{A}_G(\varphi)$ is bounded and coercive, from where we deduce a uniform bound for $\|\varphi\|$ when $\varphi \in \mathcal{F}_{\overline{m},\delta,b}$. This will be crucial to obtain uniformly bounded PS sequences.

The deformation argument

We now introduce the set of curves

$$\mathit{\Gamma} = \left\{\gamma \in C([0,1], D^{1,2}) \colon \gamma(0) = \psi_1,\ \gamma(1) = \psi_2\right\} \tag{4.18}$$

and for $\overline{m} > 0$ large enough the candidate to critical value

$$c_G = \inf_{\gamma\in\Gamma} \max\{\mathcal{A}_G(\gamma(t))\colon \gamma(t) \in \mathcal{F}_{\overline{m}},\ t \in [0,1]\} \tag{4.19}$$

The first step in the deformation argument is to prove that there exists a positive δ such that for all bounded $\varphi \in \{\varphi \in D^{1,2}\colon |\mathcal{A}_G - \varphi| \geq \delta\}$, we have $\min_{s\in\mathbb{R}}(r_0 + \varphi)(s) > 0$. To that end we notice that

$$\mathcal{A}_G(\varphi) = A(\varphi) - G^2\ B(\varphi) \tag{4.20}$$

with

$$\begin{aligned} A(\varphi) &= \int_{\mathbb{R}} \frac{\dot\varphi^2}{2} + \frac{1}{((r_0+\varphi)^2+\rho^2)^{1/2}} - \frac{1}{r_0} + \frac{\varphi}{r_0^2} + G_0^2\left(\frac{1}{2r_0^2} - \frac{\varphi}{r_0^3}\right) \\ B(\varphi) &= \int_{\mathbb{R}} (r_0+\varphi)^{-2} \end{aligned} \tag{4.21}$$

and apply a monotonicity trick due to Struwe (see [?] and [?]) to show that for almost every G, the functional $B(\varphi)$ is bounded if $|\mathcal{A}_G(\varphi) - c_G|$ is small enough (see Remark 4.4.12). The following version of the monotonicity trick was proved in [?]. We provide the proof for the sake of self completeness.

Lemma 4.4.10. *There exists a full measure subset $J \subset [-G_0, G_0]$ such that for all $G \in J$ there exists constants $\delta > 0$ and $C > 0$ for which if $|\mathcal{A}_G(\varphi) - c_G| \leq \delta$ then $B(\varphi) \leq C$.*

Proof. Since $B(\varphi) \geq 0$ it follows from expression (4.20) and the definition of c_G in (4.19) that $G \mapsto c_G$ is a monotone decreasing function. Therefore, it is differentiable on a subset $J \subset \mathbb{R}$ whose complement has zero measure. Let $G^* \in J$, $\delta > 0$ and take φ such that $|I_{G^*}(\varphi) - c_{G^*}| \leq \delta$. Take now $G < G^*$, then, by decreasing (if necessary) the value of δ we can assume that

$$\mathcal{A}_G(\varphi) \geq c_{G^*} - (G^* - G) \qquad\qquad \mathcal{A}_{G^*}(\varphi) \leq c_{G^*} + (G^* - G)$$

Then

$$B(\varphi) = \frac{\mathcal{A}_{G^*}(\varphi) - \mathcal{A}_G(\varphi)}{G^* - G} \leq \frac{c_G + (G^*-G) - c_G^* + (G^*-G)}{G^*-G}$$

By the hypothesis on G^* there exists an open neighbourhood around G^* for which

$$-c'_{G^*} - 1 \leq \frac{c_G - c_{G^*}}{G^* - G} \leq -c'_{G^*} + 1$$

and the lemma is proven. □

Boundedness of the functional $B(\varphi)$ allows us to obtain an a priori estimate for $\min_{s\in\mathbb{R}}(r_0 + \varphi)(s)$ if $\varphi \in D^{1,2}$ is bounded.

Lemma 4.4.11. *Let $\varphi \in D^{1,2}$ be such that $B(\varphi) \leq C$. Then, there exists a constant $\underline{m} > 0$, depending only on $\|\dot\varphi\|_{L^2}$ such that*

$$r_0(s) + \varphi(s) \geq \underline{m} \qquad\qquad \forall s \in \mathbb{R}.$$

Proof. Suppose there exists $s_* \in \mathbb{R}$ such that $\lim_{s\to s_*} r_0(s) + \varphi(s) = 0$. Since $\varphi \in D^{1,2}$ it holds that $|s_*| < \infty$ and we can assume without loss of generalitiy that $r_0(s) + \varphi(s) > 0$ for all $s < s_*$. Take now $s_0 = s_* - 1$ and write $r(s) = r_0(s) + \varphi(s)$. Then, by the fundamental theorem of calculus, for any $s \in [s_0, s_*)$

$$\ln(r(s)) - \ln(r(s_0)) = \int_{r(s_0)}^{r(s)} r^{-1}\mathrm{d}r = \int_{s_0}^{s} r^{-1}(t)\dot r(t)\mathrm{d}t$$

and Hölder's inequality implies

$$|\ln(r(s)) - \ln(r(s_0))| \leq B(\varphi)\left(\int_{s_0}^{s}(\dot r_0 + \dot\varphi)^2\right)^{1/2} \leq C\left(1 + \|\dot\varphi\|_{L^2}\right).$$

□

Remark 4.4.12. *In Lemma 4.5.2, we show that for all $G \in \mathbb{R} \setminus \{0\}$, we have $\min_{s\in\mathbb{R}}(r_0+\varphi)(s) \geq G^2/2$ for orbits of (4.2) which are homoclinic to γ_∞. However, that argument does not allow us to conclude that there exist $\delta > 0$ such that for all bounded $\varphi \in \{\varphi \in D^{1,2} \colon |\mathcal{A}_G - \varphi| \geq \delta\}$ we have $\min_{s\in\mathbb{R}}(r_0+\varphi)(s) > 0$. Therefore, is not clear how to incorporate the a priori estimate in Lemma 4.5.2 to obtain a minmax critical point.*

Assume now that φ is such that $|\mathcal{A}_G(\varphi) - c_G| \leq \delta$. Therefore, thanks to Lemmas 4.4.10 and 4.4.11 it is possible to obtain an inequality of the form

$$\mathcal{A}_G(\varphi) \geq \frac{\|\dot{\varphi}\|_{L^2}}{2} - C\|\varphi\| \tag{4.22}$$

for some $C > 0$. Thus, if we moreover assume that $\varphi \in \mathcal{F}_{\overline{m}}$ and that $\inf_{s\in\mathbb{R}}(r_0+\varphi)(s)$ happens for $s \in [0,1]$ we can obtain a uniform bound for the $D^{1,2}$ norm of φ. In general, in problems in which the action functional is invariant under integer time translations, the latter assumption introduces no loss of generality and this argument can be employed to obtain uniformly bounded PS sequences.

However, in the present problem, the translation operator $T_\tau(\varphi) = \varphi(s+\tau) + r_0(s+\tau) - r_0(s)$, for which we have $\mathcal{A}_G(T_\tau(\varphi)) = \mathcal{A}_G(\varphi)$, is not an isometry in $D^{1,2}$. This introduces certain technicalities in the deformation argument. In order to overcome this technical annoyance we introduce the following definition.

Definition 4.4.13. *Given $\varphi \in D^{1,2}$ we define its barycenter as the functional* $\mathrm{Bar} : D^{1,2} \to \mathbb{R}$ *given by*

$$\mathrm{Bar}(\varphi) = \left(\int_{\mathbb{R}} (1+(r_0+\varphi)^2)^{-2} \mathrm{d}s\right)^{-1} \int_{\mathbb{R}} s(1+(r_0+\varphi)^2)^{-2}\mathrm{d}s.$$

The following properties of the barycenter functional will be crucial for the deformation argument.

Lemma 4.4.14. *Let $\mathrm{Bar}(\varphi)$ be the functional introduced in Definition 4.4.13. The following statements hold:*

- *Behaviour under translations: For any $\tau \in \mathbb{Z}$*

  $$B(T_\tau \varphi) = B(\varphi) - \tau.$$

  where the translation operator T_τ was introduced in Lemma 4.4.5.

- *Local Lipschitzianity: For any $K > 0$ there exists $L_{\mathrm{Bar}} > 0$ such that*

  $$\sup_{\|\varphi\|\leq K, \|\varphi'\|\leq K} \frac{|\mathrm{Bar}(\varphi) - \mathrm{Bar}(\varphi')|}{\|\varphi - \varphi'\|} \leq L_{\mathrm{Bar}}.$$

Proof. The proof of the first part is a trivial computation. For the second one we express

$$B(\varphi) = B_2(\varphi)/B_1(\varphi)$$

with

$$B_1(\varphi) = \int_{\mathbb{R}} (1+(r_0+\varphi)^2)^{-2}\mathrm{d}s \qquad B_2(\varphi) = \int_{\mathbb{R}} s(1+(r_0+\varphi)^2)^{-2}\mathrm{d}s.$$

First we notice that there exists $C > 0$ such that for all $\|\varphi\| \leq K$ we have $B_1(\varphi) \geq C > 0$ and $|B_2(\varphi)| \leq C$. Indeed, for all $s \in \mathbb{R}$

$$|\varphi(s)| \leq |\varphi(0)| + \|\dot{\varphi}\|_{L^2}\sqrt{|s|} \leq (1+\sqrt{|s|})\|\varphi\| \leq C(1+\sqrt{|s|})$$

so there exists $T > 0$, depending only on $\|\varphi\|$, such that

$$r_0(s) + \varphi(s) \geq r_0(s)(1 - \mathcal{O}(s^{-1/6}))$$

for all $|s| \geq T$. Therefore, for C depending only on T,

$$|B_2(\varphi)| \leq C + \int_{|s|\geq T} |s|(1+(r_0+\varphi)^2)^{-2}\mathrm{d}s \leq C\left(1+\int_{|s|\geq T}|s|r_0^{-4}(s)\mathrm{d}s\right) \leq C.$$

The uniform bound from below for $B_1(\varphi)$ follows since there exists C and T, depending only on $\|\varphi\|$, such that

$$r_0(s)+\varphi(s) \leq C(1+\sqrt{|s|})$$

for all $|s| \leq T$. Take now $\varphi, \varphi^* \in D^{1,2}$ and write

$$B(\varphi^*)-B(\varphi) = (B_2(\varphi^*)-B_2(\varphi))/B_1(\varphi) + (B_1(\varphi^*)-B_1(\varphi))B_2(\varphi)/B_1(\varphi^*)B_1(\varphi)$$

Let $g(\varphi) = (1+(r_0+\varphi)^2)^{-2}$. Then for $\varphi, \varphi^* \in D^{1,2}$ we can write

$$B_2(\varphi^*)-B_2(\varphi) = \int_{\mathbb{R}} s(\varphi^*-\varphi)\int_0^1 g'(\lambda(\varphi^*-\varphi))\mathrm{d}\lambda\mathrm{d}s$$

On one hand for all $s \in \mathbb{R}$

$$|\varphi^*(s)-\varphi(s)| \leq |\varphi^*(0)-\varphi(0)| + \|\dot{\varphi}^*-\dot{\varphi}\|_{L^2}\sqrt{|s|} \leq (1+\sqrt{|s|})\|\varphi^*-\varphi\|$$

and it follows that

$$\begin{aligned}|B_2(\varphi^*)-B_2(\varphi)| \leq & C\|\varphi^*-\varphi\|\int_{\mathbb{R}}(1+|s|^{3/2})\int_0^1 |r_0+\lambda(\varphi^*-\varphi)|(1+|r_0+\lambda\varphi^*-\varphi|^2)^{-3}\mathrm{d}\lambda\mathrm{d}s \\ \leq & C\|\varphi^*-\varphi\|\int_{\mathbb{R}}(1+|s|^{3/2})r_0^{-5} \leq C\|\varphi^*-\varphi\|.\end{aligned}$$

The same computation shows that there exists C such that

$$|B_1(\varphi^*)-B_1(\varphi)| \leq C\|\varphi^*-\varphi\|$$

and the lemma is proven. □

Together, Lemma 4.4.10 and Lemma 4.4.14 imply the following result, which is key in our constrained deformation argument.

Lemma 4.4.15. *Let $J \subset [-G_0, G_0] \subset \mathbb{R}$ be the subset obtained in Lemma 4.4.10. Let $G \in J$, let $\delta > 0$ be the constant in Lemma 4.4.10, let $b > 0$ and define*

$$\mathcal{F}_{\overline{m},\delta,b} = \mathcal{F}_{\overline{m}} \cap \left\{\varphi \in D^{1,2} \colon |\mathcal{A}_G(\varphi)-c_G| \leq \delta, \quad |\mathrm{Bar}(\varphi)| \leq b\right\} \tag{4.23}$$

Then, there exists $K > 0$ such that

$$\sup_{\varphi\in\mathcal{F}_{\overline{m},\delta,b}} \|\varphi\| \leq K$$

Proof. Let

$$\mathcal{S} = \{\bar{s} \in \mathbb{R} \colon \min_{s\in\mathbb{R}}(r_0+\varphi)(s) = (r_0+\varphi)(\bar{s})\}$$

We claim that under the hypothesis of the lemma there exists $C > 0$ such that

$$|\mathrm{Bar}(\varphi)-\bar{s}| \leq C \qquad \forall \bar{s} \in \mathcal{S}$$

Suppose not, then, by continuity, there exist $\bar{s} \in \mathcal{S}$ and sequences $\{\varphi_n\}$, $\{\bar{s}_n\}$ such that $\varphi_n \to \varphi$ in $D^{1,2}$, $\bar{s}_n \to \bar{s}$ and

$$\varphi_n \in \mathcal{F}_{\overline{m}}, \qquad |\mathcal{A}_G(\varphi_n)-c_G| \leq \delta, \qquad \text{and} \qquad |\mathrm{Bar}(\varphi_n)-\bar{s}_n| \to \infty.$$

By invariance of the action functional under translation $T_\tau(\varphi)(s) = \varphi(s+\tau) + r_0(s+\tau) - r_0(s)$ (see Lemma 4.4.5) and the fact that

$$B(T_\tau(\varphi)) = B(\varphi) - \tau$$

the sequence $\tilde{\varphi}_n = T_{[\bar{s}_n]}\varphi_n$, satisfies

$$\tilde{\varphi}_n \in \mathcal{F}_{\overline{m}}, \qquad |\mathcal{A}_G(\tilde{\varphi}_n) - c_G| \le \delta \qquad \text{and} \qquad |\mathrm{Bar}(\tilde{\varphi}_n)| \to \infty.$$

However, the first two properties imply the existence of K such $\|\tilde{\varphi}_n\| \le K$ for all $n \in \mathbb{N}$. Indeed, by the construction of $\tilde{\varphi}_n$, for all $\tilde{\varphi}_n$ we have

$$|\varphi(s)| \le \overline{m} + \|\dot{\tilde{\varphi}}_n\|_{L^2}\sqrt{|s+1|}$$

Moreover, since $G \in J$ and $|\mathcal{A}_G(\tilde{\varphi}_n) - c_G| \le \delta$, Lemma 4.4.10 implies that there exists $C > 0$ such that $B(\tilde{\varphi}_n) \le C$ for all $n \in \mathbb{N}$. Therefore, it is easy to check that there exists $C > 0$ such that

$$\mathcal{A}_G(\tilde{\varphi}_n) \ge \frac{\|\dot{\tilde{\varphi}}_n\|_{L^2}^2}{2} - C(1 + \|\dot{\tilde{\varphi}}_n\|_{L^2})$$

for all $n \in \mathbb{N}$. Thus, since $|\mathcal{A}_G(\tilde{\varphi}_n) - c_G| \le \delta$ and $|c_G|$ is bounded, the sequence $\{\tilde{\varphi}_n\}$ must be uniformly bounded. It is easy to check that the existence of K such that $\|\tilde{\varphi}_n\| \le K$ is in contradition with $\mathrm{Bar}(\tilde{\varphi}_n)$ being unbounded.

Once we know that the claim $|\mathrm{Bar}(\varphi) - \bar{s}| \le C$ holds, we obtain that $\bar{s} \le C + b$ for all $\bar{s} \in \mathcal{S}$. Therefore, since $\varphi \in \mathcal{F}_{\overline{m}}$, we have that

$$|\varphi(0)| \le |\varphi(\bar{s})| + \|\dot{\varphi}\|_{L^2}\sqrt{|\bar{s}|} \le C(1 + \|\dot{\varphi}\|_{L^2})$$

for some $C > 0$ depending only on $\overline{m}, \delta$ and b. The result follows since now we can show that for all φ_n we have

$$\mathcal{A}_G(\varphi_n) \ge \frac{\|\dot{\varphi}_n\|_{L^2}^2}{2} - C(1 + \|\dot{\varphi}_n\|_{L^2})$$

for some C uniform on n. □

In Proposition 4.4.17 we show the existence of a PS sequence contained in $\mathcal{F}_{\overline{m},\delta,b}$ for large enough values of $\overline{m}$ and b. Notice that in particular, thanks to Lemma 4.4.15 this sequence will be uniformly bounded.

We split the proof of Proposition 4.4.17 in two parts. First, we assume by contradiction that there exists no critical point of the action functional $\mathcal{A}_G$ in $\mathcal{F}_{\overline{m},\delta,b}$. Under this assumption, we build a pseudo gradient vector field for $\mathcal{A}_G$, this is the content of Proposition 4.4.16. Then, in Proposition 4.4.17 we use this pseudo gradient vector field to build a localized deformation which yields points $\varphi \in \mathcal{F}_{\overline{m},\delta,b}$ for which $\mathcal{A}_G(\varphi) < c_G$, a contradiction.

Before stating Proposition 4.4.16 some definitions are in order. Let $b > 0$ and $0 < \varepsilon < \delta/2$ where $\delta > 0$ is the constant in Lemma 4.4.10. For we want the flow along the pseudogradient vector field to leave $D^{1,2} \setminus \mathcal{F}_{\overline{m}}$ positively invariant, we express it as the convex combination of two localized vector fields: a gradient-like vector field supported on

$$P = \{\varphi \in D^{1,2} \colon |\mathcal{A}_G(\varphi) - c_G| \le \varepsilon,\ \min_{s\in\mathbb{R}} r_0(s) + \varphi(s) \le \overline{m},\ \mathrm{Bar}(\varphi) \le 2b\} \tag{4.24}$$

and a vector field supported on

$$Y = \{\varphi \in D^{1,2} \colon |\mathcal{A}_G(\varphi) - c_G| \le \delta,\ |\min_{s\in\mathbb{R}} r_0(s) + \varphi(s) - \overline{m}| \le \varepsilon,\ \mathrm{Bar}(\varphi) \le 2b\}. \tag{4.25}$$

for which $D^{1,2} \setminus \mathcal{F}_{\overline{m}}$ is positively invariant. This construction is made explicit in the following proposition.

Proposition 4.4.16. *Let $J \subset [G_0, G_0]\mathbb{R}$ be the subset obtained in Lemma 4.4.10. Let $\delta > 0$ be the constant in Lemma 4.4.10 and take $\overline{m} > 0$ large enough. Assume that for all $b > 0$ there exists $\alpha > 0$ for which*

$$\inf\{\|\nabla\mathcal{A}_G(\varphi)\| \colon \varphi \in \mathcal{F}_{\overline{m},\delta,2b}\} \geq \alpha$$

Then, there exists b_0 such that for all $b \geq b_0$ there exists a Lipschitz pseudogradient vector field $W : D^{1,2} \to D^{1,2}$ such that

- *$\|W\| \leq 1$,*
- *There exists a constant $\beta > 0$ such that*

$$\mathrm{d}\mathcal{A}_G(\varphi)\,[W(\varphi)] \leq -\beta \qquad \forall \varphi \in P \cup Y,$$

- *The region $D^{1,2} \setminus \mathcal{F}_{\overline{m}}$ is positively invariant under the flow of W.*

Proof. Let $\varepsilon < \delta/2$, define the sets

$$\begin{aligned}
Y =& \{\varphi \in D^{1,2} \colon |\mathcal{A}_G(\varphi) - c_G| \leq \delta,\ |\min_{s\in\mathbb{R}} r_0(s) + \varphi(s) - \overline{m}| \leq \varepsilon,\ \mathrm{Bar}(\varphi) \leq 2b\} \\
Z =& \{\varphi \in D^{1,2} \colon |\mathcal{A}_G(\varphi) - c_G| \leq \delta,\ |\min_{s\in\mathbb{R}} r_0(s) + \varphi(s) - \overline{m}| \geq 2\varepsilon,\ \mathrm{Bar}(\varphi) \leq 2b\}
\end{aligned}$$

and the function

$$\Psi = \frac{\mathrm{dist}Z}{\mathrm{dist}Y + \mathrm{dist}Z},$$

which satisfies $\Psi = 0$ on Z and $\Psi = 1$ on Y. We also introduce

$$\begin{aligned}
P =& \{\varphi \in D^{1,2} \colon |\mathcal{A}_G(\varphi) - c_G| \leq \varepsilon,\ \min_{s\in\mathbb{R}} r_0(s) + \varphi(s) \leq \overline{m},\ \mathrm{Bar}(\varphi) \leq 2b\} \\
Q =& \{\varphi \in D^{1,2} \colon |\mathcal{A}_G(\varphi) - c_G| \geq \delta,\ \min_{s\in\mathbb{R}} r_0(s) + \varphi(s) \leq \overline{m},\ \mathrm{Bar}(\varphi) \leq 2b\}
\end{aligned}$$

and define the function

$$\Phi = \frac{\mathrm{dist}Q}{\mathrm{dist}P + \mathrm{dist}Q}.$$

which satisfies $\Phi = 0$ on Q and $\Phi = 1$ on P. Take now a sufficiently small open neighbourhood $U \subset D^{1,2}$, $\mathrm{supp}(\Phi) \subset D^{1,2}$. Notice that by Lemma 4.4.15, there exists $K > 0$ such that

$$\sup\{\|\varphi\| \colon \varphi \in U\} \leq K.$$

Then, since $G \in \mathcal{J}$, Lemmas 4.4.10 and 4.4.11, impy that there exists $\underline{m} > 0$ such that

$$\inf\{\min_{s\in\mathbb{R}} r_0(s) + \varphi(s) \colon \varphi \in U\} \geq \underline{m}.$$

Therefore, by Lemma 4.4.7 we have that $\mathrm{d}\mathcal{A}_G \in C^1(U, D^{1,2})$, what implies the existence of a constant $C > 0$ such that $\mathrm{dist}P + \mathrm{dist}Q > C$. We introduce now the pseudogradient vector field

$$W = \frac{1}{\sqrt{2}}\left(-(1-\Psi)\Phi\frac{\nabla\mathcal{A}_G}{\|\nabla\mathcal{A}_G\|} + \Psi v\right) \tag{4.26}$$

where v is the constant vector field given by the constant $v = 1 \in D^{1,2}$. Notice that for a large enough fixed $\overline{m}$, and for all $b > 0$ there exists $\tilde{\alpha} > 0$ such that

$$\sup\{\mathrm{d}\mathcal{A}_G(\varphi)[v] \colon \varphi \in \mathrm{supp}\Psi\} \leq -\tilde{\alpha} \tag{4.27}$$

Indeed

$$\mathrm{d}\mathcal{A}_G(\varphi)[v] = \int_{\mathbb{R}} \left(\frac{G^2}{(r_0+\varphi)^3} - \frac{r_0+\varphi}{((r_0+\varphi)^2+\rho^2)^{3/2}}\right) v$$

and the claim follows since for large enough $\overline{m}$ the integrand is non positive and moreover it is strictly negative on a positive measure subset of the real line since (thanks to Lemma 4.4.15) there exists $K > 0$ such that $\sup_{\varphi\in\mathrm{supp}\Psi}\|\varphi\| \leq K$. It is straightforward to chek that the pseudogradient vector field W introduced in (4.26) satisfies the properties listed in the statement in the lemma with $\beta = \min\{\alpha, \tilde{\alpha}\} > 0$. □

Proposition 4.4.17. *Let $J \subset [-G_0, G_0] \subset \mathbb{R}$ be the subset obtained in Lemma 4.4.10. Let $\delta > 0$ be the constant in Lemma 4.4.10 and take $\overline{m} > 0$ large enough. Then, for $b > 0$ large enough there exists a Palais-Smale sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_{\overline{m},\delta,2b}$ for $\mathcal{A}_G$ at the level c_G.*

Proof. Let $\Gamma \subset C([0,1], D^{1,2})$ be the set defined in (4.18). Let $\varepsilon < \delta/2$ and take a suboptimal path $\gamma_\varepsilon \subset \Gamma$ for which $\mathcal{A}_G(\gamma_\varepsilon(t)) \leq c_G + \varepsilon$ for all $t \in [0,1]$ such that $\min_{s\in\mathbb{R}} r_0(s) + (\gamma_\varepsilon(t))(s) \leq \overline{m}$. For all $\gamma \in \Gamma$ we define the set

$$\mathcal{B}_\gamma \equiv \left\{ \mathrm{Bar}(\gamma_\varepsilon(t)) \colon |\mathcal{A}_G(\gamma(t)) - c_G| \geq \varepsilon, \quad \min_{s\in\mathbb{R}} r_0(s) + (\gamma(t))(s) \leq \overline{m}, \quad t \in [0,1] \right\}.$$

Notice that for each $\gamma \in \Gamma$, the set $\mathcal{B}_\gamma$ is a compact subset of $\mathbb{R}$. Denote by

$$b_{\min} = \min \mathcal{B}_{\gamma_\varepsilon} \qquad\qquad b_{\max} = \max \mathcal{B}_{\gamma_\varepsilon}$$

and consider the translated path $\gamma_\varepsilon^1 = T_{b_0}\gamma_\varepsilon$ for $b_0 = [b_{\min}]$. It satisfies that

$$\mathcal{B}_{\gamma_\varepsilon^1} \subset [0, b_{\max} - b_0 + 1]$$

Let $W : D^{1,2} \to D^{1,2}$ be the pseudogradient vector field built in Proposition 4.4.16 and denote by η_τ its time τ flow. Notice that since W is Lipschitz the flow η_τ is well defined at least for sufficiently small τ. Let $\beta > 0$ be the constant in Proposition 4.4.16. We claim that the deformed path $\gamma_\varepsilon^1 \circ \eta_{\tau^*}$ with $\tau_* = 2\varepsilon/\beta$ satisfies

$$\begin{aligned}\max\{&\mathcal{A}_G(\eta_{\tau_*}(\gamma_\varepsilon^1)(t)) \colon \min_{s\in\mathbb{R}} r_0(s) + \left(\eta_{\tau_*}(\gamma_\varepsilon^1)(t)\right)(s) \leq \overline{m},\ |\mathrm{Bar}(\eta_{\tau_*}(\gamma_\varepsilon^1)(t))| \leq b,\ t \in [0,1]\} \\ &\leq c_G - \varepsilon.\end{aligned}$$

To verify the claim we first notice that the maximal displacement is bounded by

$$\|\eta_{\tau^*}(\varphi) - \varphi\| \leq \tau^*\|W\| \leq \tau^* = 2\varepsilon/\beta.$$

Therefore, applying Lemma 4.4.14, we obtain that for any $\varphi \in \gamma_\varepsilon$ (taking b sufficiently large)

$$|\mathrm{Bar}(\eta_{\tau^*}(\varphi)) - \mathrm{Bar}(\varphi)| \leq L_{\mathrm{Bar}} 2\varepsilon/\beta \leq b/4.$$

Thus, since the region $\{\min_{s\in\mathbb{R}} r_0(s) + \varphi(s) \geq \overline{m}\}$ is forward invariant by the flow η_τ and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{A}_G \circ \eta) \leq 0,$$

in order to verify the claim, it is enough to check that there does not exist

$$\varphi \in \left\{ \min_{s\in\mathbb{R}} r_0(s) + \left(\gamma_\varepsilon^1(t)\right)(s) \leq \overline{m}, \quad \mathrm{Bar}(\gamma_\varepsilon^1(t)) \leq 5b/4, \quad t \in [0,1] \right\}$$

for which $\eta_\tau(\gamma_\varepsilon^1) \in P \ \ \forall \tau \in [0, \tau^*]$ where $P \subset D^{1,2}$ is the set defined in (4.24). This is clearly not possible since for $\varphi \in P$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\mathcal{A}_G \circ \eta) \leq -\beta$$

so

$$\mathcal{A}_G(\eta_{\tau^*}(\gamma_\varepsilon^1)) \leq \mathcal{A}_G(\gamma_\varepsilon^1) - \tau^*\beta \leq c_G - \varepsilon$$

a contradiction. Now that the claim is verified consider the path

$$\gamma_\varepsilon^2 = T_{-b_0}(\gamma_\varepsilon^1).$$

It satisfies that

$$\mathcal{B}_{\gamma_\varepsilon^2} \subset [b_0 + b, b_{\max} + 1]$$

and

$$\max\{\mathcal{A}_G(\gamma_\varepsilon^2(t))\colon \min_{s\in\mathbb{R}} r_0(s) + \left(\gamma_\varepsilon^2(t)\right)(s) \le \overline{m}, \quad \mathrm{Bar}(\gamma_\varepsilon^2(t)) \le b_0 + b, \quad t \in [0,1]\} \le c_G - \varepsilon$$

If $b_{\max} - b_0 + 1 \le b$ the proposition is proved. In the case $b_{\max} - b_0 + 1 \ge b$ we repeat the argument above with the path γ_ε^2 to obtain a path γ_ε^3 satisfying

$$\mathcal{B}_{\gamma_\varepsilon^3} \subset [b_0 + 2b, b_{\max} + 1]$$

and

$$\max\{\mathcal{A}_G(\gamma_\varepsilon^3(t))\colon \min_{s\in\mathbb{R}} r_0(s) + \left(\gamma_\varepsilon^3(t)\right)(s) \le \overline{m}, \quad \mathrm{Bar}(\gamma_\varepsilon^3(t)) \le b_0 + 2b, \quad t \in [0,1]\} \le c_G - \varepsilon$$

The result follows after repeating the construction no more than $[(b_{\max} - b_0 + 1)/b]$ steps. □

Finally, we obtain the existence of a critical point of the functional $\mathcal{A}_G$ at a level c_G .

Theorem 4.4.18. *Let $J \subset [-G_0, G_0] \subset \mathbb{R}$ be the subset obtained in Lemma 4.4.10. Then, for all $G \in J$ there exists a critical point of the action functional $\mathcal{A}_G$ at the level c_G.*

Proof. By Proposition 4.4.17, for sufficiently large $\overline{m}$ and b there exists a Palais-Smale sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_{\overline{m},\delta,2b}$ so it follows from Lemma 4.4.15 that it is bounded. The theorem is then proved since the Palais Smale sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfies the hypothesis for the set Q of the compactness Lemma 4.4.8. □

4.5 Topological transversality between the stable and unstable manifolds

For the choice of $G_0 > 0$ was arbitrary, Theorem 4.4.18 implies that for any compact subset $[-G_0, G_0]$ of the real line, there exists a full measure subset $J \subset [-G_0, G_0]$ such that for all $G \in J$ there exists an orbit of (4.1) which is homoclinic to γ_∞. Another way of rephrasing Theorem 4.4.18 is that the invariant manifolds manifolds $W^\pm(\gamma_\infty, G)$ defined in (4.7) intersect for almost all values of G in $[-G_0, G_0]$. However, Theorem 4.4.18 contains no information about the geometry of the intersection, in particular wether it is transversal or not.

Theorem 4.4.1, proved in [GPSV21], shows that the intersection between $W^\pm(\gamma_\infty, G)$ is transverse for all G sufficiently large. Moreover, the local stable manifolds $W^\pm_{\mathrm{loc}}(\gamma_\infty; G)$ (see (4.6)) depend analytically on r and G (see Proposition 4.3.1). We want to exploit this facts to deduce that the intersection of the manifolds $W^\pm(\gamma_\infty, G)$ (which, by Theorem 4.4.18 we already know that exists for almost all values of $G \in [-G_0, G_0]$) must be topologically transverse for almost all values of G in $[-G_0, G_0]$.

Remark 4.5.1. *In the following we fix a sufficiently large value of G_0 and work with $G \in J \subset [-G_0, G_0]$.*

The first step is to obtain an a priori estimate from below for $\min_{s\in\mathbb{R}} r_h(s)$.

Lemma 4.5.2. *Let $G \in \mathbb{R}$ and let $r_h(s; G) : \mathbb{R} \to \mathbb{R}$ be an orbit of of the Hamiltonian H_G in* (4.1) *which is homoclinic to γ_∞. Then, for all $s \in \mathbb{R}$ we have*

$$r_h(s) \ge \frac{G^2}{2}.$$

Proof. Since $r_h(s; G) : \mathbb{R} \to \mathbb{R}$ is an orbit of of the Hamiltonian H_G we have that

$$\ddot{r}_h = \frac{G^2}{r_h^3} - \frac{r_h}{(r_h^2 + \rho^2)^{3/2}} \ge \frac{G^2}{r_h^3} - \frac{1}{r_h^2}. \tag{4.28}$$

Let now $I \subset \mathbb{R}$ be an interval in which $\dot{r}_h(s) \le 0$ for all $s \in I$. Then, multiplying (4.28) by $\dot{r}_h$, for all $s \in I$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\dot{r}_h^2}{2} + \frac{G^2}{2r_h^2} - \frac{1}{r_h}\right) \le 0,$$

that is, the energy

$$E(s) = \frac{\dot r_h^2(s)}{2} + \frac{G^2}{2r_h^2(s)} - \frac{1}{r_h(s)}$$

is non increasing on the interval I. Let I be a maximal interval in which $r_h(s)$ is decreasing: we distinguish between two alternatives, either

- $I = (-\infty, s_1]$, and $\lim_{s\to-\infty} r_h(s) = \infty$, $\lim_{s\to-\infty} \dot r_h(s) = 0$ and $\dot r_h(s_1) = 0$, or
- $I = [s_0, s_1]$ and $\dot r_h(s_0) = 0$, $\ddot r_h(s_0) \le 0$ and $\dot r_h(s_1) = 0$

for some $-\infty < s_0 < s_1 < \infty$. In the first case we have

$$\lim_{s\to-\infty} E(s) = 0.$$

In the second case, using that $\dot r_h(s_0) = 0$, $\ddot r_h(s_0) \le 0$ and the inequality (4.28), we obtain

$$E(s_0) = \frac{\dot r_h^2(s_0)}{2} + \frac{G^2}{2r_h^2(s_0)} - \frac{1}{r_h(s_0)} = \frac{G^2}{2r_h^2(s_0)} - \frac{1}{r_h(s_0)} \le \frac{G^2}{r_h^2(s_0)} - \frac{1}{r_h(s_0)} \le 0.$$

Therefore, in both cases, for all $s \in I$ we have $E(s) \le 0$, which implies that

$$r(s) \ge r(s_1) \ge \frac{G^2}{2}.$$

□

Lemma 4.5.2 implies that for $G \neq 0$, homoclinic orbits do not intersect the section $\{r = 0\}$. This fact allows us to exploit the analytic dependence of the Hamiltonian H_G in the parameter G to prove the following result.

Lemma 4.5.3. *The set of values of $G \in \mathbb{R} \setminus \{0\}$ for which $W^-(\gamma_\infty, G) = W^+(\gamma_\infty, G)$, is finite.*

Proof. Fix any $\delta > 0$ and let G_* be the constant in Theorem 4.4.1 and let $1 \ll R_1 < R_2$ be such that for all $G \in [-2G_*, 2G_*]$ the generating function $S^+(r, t; G)$ associated with the local stable manifold (see 4.3.1) is well defined for all $(r, t) \in [R_1, R_2] \times \mathbb{T}$. Define the set

$$Q = \{(r, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} \colon r \in (R_1, R_2),\ y > 0,\ t = 0\}.$$

Whenever it exists, denote by $\gamma_G^- \subset Q \cap \mathcal{W}^u(\gamma_\infty; G)$ the connected component of $Q \cap \mathcal{W}^u(\gamma_\infty; G)$ associated with the first backwards intersection of $\mathcal{W}^u(\gamma_\infty; G)$ with Q (see Figure 4.5). Define now the set

$$\widetilde{\mathcal{G}} = \{G \in \mathbb{R} \colon \delta \le |G| \le 2G_*,\ \gamma_G^- \neq \emptyset \text{ and } \exists\ \varphi_G^- \in C^\omega([R_1, R_2], \mathbb{R}) \text{ such that } \gamma_G^- = \operatorname{graph}(\varphi_G^-)\}.$$

Clearly, $\mathcal{G} \subset \widetilde{\mathcal{G}}$ where

$$\mathcal{G} = \{G \in \mathbb{R} \colon \delta \le |G| \le 2G_*,\ W^+(\gamma_\infty; G) = W^-(\gamma_\infty; G)\}.$$

In view of Lemma 4.5.2, for all $G \in \mathcal{G}$

$$\operatorname{dist}(W^\pm(\gamma_\infty, G), \{r = 0\}) \ge G^2/2. \tag{4.29}$$

so $\mathcal{G}$ is a closed set. Moreover, since the Hamiltonian (4.2) depends analytically on G, and, by (4.29), for all $G \in \mathcal{G}$ the vector field associated with (4.2) is analytic on a neighbourhood of $W^\pm(\gamma_\infty, G)$, there exists an open subset $\mathcal{O} \subset \widetilde{\mathcal{G}}$ such that $\mathcal{G} \subset \mathcal{O}$ and in which $\varphi_G^- \in C^\omega([R_1, R_2] \times \mathcal{O})$. Define now the function $\Delta(r, G) : [R_1, R_2] \times \widetilde{\mathcal{G}} \to \mathbb{R}$ given by

$$\Delta(r, G) = \varphi_G^-(r) - \partial_r S^+(r, 0; G)$$

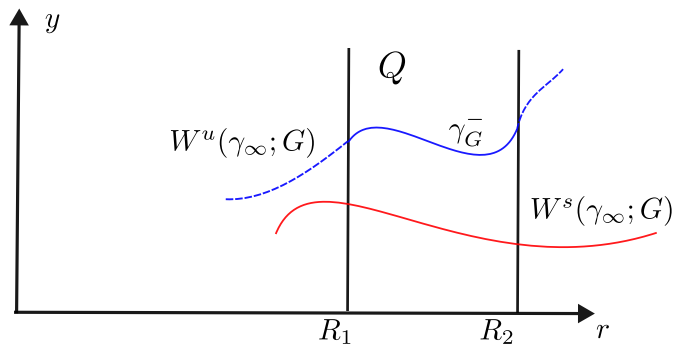

Figure 4.4: The domain Q and a sketch of the intersection of the stable manifolds $W^{\pm}(\gamma_\infty; G)$ with Q for a value of $G \in \widetilde{\mathcal{G}}$.

which satisfies $\Delta(r, G) = 0$ for all $G \in \widetilde{\mathcal{G}}$ and $\Delta \in C^\omega([R_1, R_2] \times \mathcal{O})$. Suppose now that $\mathcal{G} = \{\delta \leq |G| \leq 2G_*\}$. Then, $\Delta(r, G) = 0$ for all $(r, G) \in [R_1, R_2] \times \{\delta \leq |G| \leq 2G_*\}$ and we obtain a contradiction with the fact that for $|G| \geq G_*$ the manifolds $W^u(\gamma_\infty, G)$, $W^s(\gamma_\infty, G)$ intersect transversally (see Theorem 4.4.1). Therefore, $\mathcal{G} \subsetneq \{\delta \leq |G| \leq 2G_*\}$. We now show that, moreover, $\mathcal{G}$ cannot contain any accumulation point. To see this suppose that there exists $G_{max} \in \mathcal{G}$ such that

$$G_{max} = \max\{G \in \mathcal{G} \colon G \text{ is an accumulation point of } \mathcal{G}\}.$$

Since $G_{max} \in \mathcal{G}$ there exists an open interval $\mathcal{V} \subset \mathcal{O}$ such that $G \in \mathcal{V}$. Then, the fact that $\mathcal{V} \subset \mathcal{O}$ implies that$\Delta(r, G) \in C^\omega([R_1, R_2] \times \mathcal{V})$ and since G_{max} is an acuumulation point of $\mathcal{G}$ we conclude that $\Delta(r, G) = 0$ on $[R_1, R_2] \times \mathcal{V}$. Then $\mathcal{V} \subset \mathcal{G}$, so there exists $\tilde{G} \in \mathcal{V} \subset \mathcal{G}$ such that $\tilde{G} > G_{max}$, a contradiction with the definition of G_{max}. □

Denote now by $\mathcal{J} \subset \mathbb{R}$ the set

$$\mathcal{J} = \{G \in J \colon G \neq 0,\ W^+(\gamma_\infty; G) \neq W^-(\gamma_\infty; G)\} \tag{4.30}$$

where J was defined in Lemma 4.4.10 (see also Theorem 4.4.18).

Lemma 4.5.4. *For all $G \in \mathcal{J}$ the set* $\mathrm{Crit}(\mathcal{A}_G) = \{\varphi \in D^{1,2} \colon \mathrm{d}\mathcal{A}_G(\varphi) = 0\}$ *is isolated in* $D^{1,2}$.

Proof. Following [MNT99], we define the map $T_R : \mathrm{Crit}(\mathcal{A}) \subset D^{1,2} \to \mathbb{R}$ given by

$$T_R = \sup\{s \in \mathbb{R} \colon r_0(s) + \varphi(s) = R,\ \varphi \in \mathrm{Crit}(\mathcal{A}) \subset D^{1,2}\}.$$

We now show that the set $T_R(\mathrm{Crit}(\mathcal{A}))$ is isolated in $\mathbb{R}$. Suppose on the contrary that there exists an accumulation point $T_* \in T_R(\mathrm{Crit}(\mathcal{A}))$, then, there exist $\{(\varphi_n, t_n)\}_{n\in\mathbb{N}} \subset \mathrm{Crit}(\mathcal{A}) \times \mathbb{R}$ and $R \in \mathbb{R}_+$ such that $t_n \to T_R$, $(r_0 + \varphi_n)(t_n) = R$ and

$$((r_0 + \varphi_n)(t_n), (\dot{r}_0 + \dot{\varphi}_n)(t_n)) \in W^+_{\mathrm{loc}}(\gamma_\infty; G).$$

Thus, there exist infinitely many different homoclinic points contained in the piece of the local stable manifold $\gamma_+ = \{y = \partial_r S^+(r, t),\ t = T_*,\ r \in [R, R_1]\}$ for any $R_1 < R$. This would imply the existence of $T_{**} < T_*$, $R_2 < R_3$ such that $\gamma_+ \cap \phi^{T_* - T_{**}}(\gamma_-)$ intersect at infinitely many points, where $\gamma_- = \{y = \partial_r S^-(R, t),\ t = T_{**},\ r \in [R_2, R_3]\}$.However, γ_+ and γ_- are compact analytic curves, and since $G \in \mathcal{J}$ they cannot intersect at infinitely many points.

By Lemma 3.3. in [MNT99], the function $T_R : \mathrm{Crit}(\mathcal{A}) \subset D^{1,2} \to \mathbb{R}$ is continuous, so the lemma is proven, for if it were to be false there would exist an accumulation point $T_* \in T_R(\mathrm{Crit}(\mathcal{A}))$. □

The fact that the critical points are isolated implies the following non-degeneracy property at, at least, one of the critical points of $\mathcal{A}_G$ at the level c_G. We say that $\varphi_* \in \mathrm{Crit}(\mathcal{A}_G)$ has a *local mountain pass structure* if, for all neighbourhood $U \subset D^{1,2}$ of φ_*, the set $\{\varphi \in U \colon \mathcal{A}_G(\varphi) < \mathcal{A}_G(\varphi_*)\}$ is not path connected. The following result is a direct consequence of Lemma 4.5.4 and Theorem 1 in [Hof86].

Proposition 4.5.5. *For all $G \in \mathcal{J}$ there exists $\varphi_* \in \mathrm{Crit}(\mathcal{A}_G)$ such that $\mathcal{A}_G(\varphi_*) = c_G$ which has a local mountain pass structure.*

Remark 4.5.6. *In all the forthcoming sections we fix $G \in \mathcal{J}$ where $\mathcal{J}$ is the set defined in* (4.30) *and omit the dependence on G.*

4.5.1 The reduced action functional

For $n \in \mathbb{N} \setminus \{0\}$ we denote by $H^1([-n, n])$ the usual Sobolev space consisting of functions defined on the interval $[-n, n] \subset \mathbb{R}$ with one weak derivative in $L^2([-n, n])$ and introduce the restriction operator

$$\begin{aligned} j : D^{1,2} &\longrightarrow H^1([-n, n]) \\ \varphi &\longmapsto j(\varphi) = \varphi|_{[-n,n]} \end{aligned} \tag{4.31}$$

Then, for a sufficiently small neighbourhood $\tilde{U} \subset H^1([-n, n])$ of a point $\tilde{\varphi}_* = j(\varphi_*)$ where $\varphi_* \in D^{1,2}$ and a sufficiently large $n \in \mathbb{N}$ (depending on φ_*) we define the reduced action functional $\widetilde{\mathcal{A}} : \tilde{U} \subset H^1([-n, n]) \to \mathbb{R}$ given by

$$\widetilde{\mathcal{A}}(\tilde{\varphi}) = \int_{-n}^{n} \mathcal{L}_{\mathrm{ren}}(\tilde{\varphi}, \dot{\tilde{\varphi}}, s)\mathrm{d}s - S^{+}((r_0 + \tilde{\varphi})(n)) + S^{-}((r_0 + \tilde{\varphi})(-n)) + \dot{r}_0(n)(\tilde{\varphi}(n) - \tilde{\varphi}(-n)),$$

where the renormalized Lagrangian $\mathcal{L}_{\mathrm{ren}}$ is defined in (4.12) and $S^{\pm}$ are the generating functions of the local stable and unstable manifolds which were obtained in Proposition 4.3.1. Notice that for n sufficiently large (depending on φ_*) and $\tilde{\varphi}$ sufficiently close to $j(\varphi_*)$ the values $(r_0+\tilde{\varphi})(\pm n)$ are contained in $\mathrm{Dom}(S^{\pm})$.

We now want to translate the results we have obtained for the functional $\mathcal{A}$, in particular Proposition 4.5.5, in results for the functional $\widetilde{\mathcal{A}}$. To that end, given any constant $c \in \mathbb{R}$ and $n \in \mathbb{N}$ we introduce the functional spaces

$$D^{1,2}_{+}(c, n) = \{\varphi \in C([n, \infty)) \colon \exists v_\varphi \in L^2([n, \infty)) \text{ such that } \varphi(s) = c + \int_n^s v_\varphi(t)\mathrm{d}t,\ \forall s \in [n, \infty)\}$$

and

$$D^{1,2}_{-}(c, n) = \{\varphi \in C((-\infty, -n]) \colon \exists v_\varphi \in L^2((-\infty, -n]) \text{ such that } \varphi(s) = c - \int_s^{-n} v_\varphi(t)\mathrm{d}t,\ \forall s \in (-\infty, -n]\}$$

Define also the weakly closed subsets

$$\begin{aligned} \tilde{D}^{1,2}_{+}(c, n) =& \{\varphi \in D^{1,2}_{+}(c, n) \colon r_0(s) + \varphi(s) \geq r_0(n) + c,\ \forall s \in [n, \infty)\} \\ \tilde{D}^{1,2}_{-}(c, n) =& \{\varphi \in D^{1,2}_{-}(c, n) \colon r_0(s) + \varphi(s) \geq r_0(-n) + c,\ \forall s \in (-\infty, -n]\} \end{aligned}$$

Then, we define the asymptotic actions

$$\mathcal{A}^{\pm}(\varphi) = \pm \int_{\pm n}^{\pm\infty} \mathcal{L}_{\mathrm{ren}}(\varphi, \dot{\varphi}, s)\mathrm{d}s \tag{4.32}$$

Lemma 4.5.7. *For all $c \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exists a unique $\varphi_{\pm} \in \tilde{D}^{1,2}_{\pm}(c, n)$ such that*

$$\mathcal{A}^{\pm}(\varphi_{\pm}) = \min\{\mathcal{A}^{\pm}(\psi) \colon \psi \in \tilde{D}^{1,2}_{\pm}(c, n))\}.$$

Moreover,

$$\mathcal{A}^{\pm}(\varphi_{\pm}) = \mp S^{\pm}((r_0(\pm n) + c) \pm S^{0}((r_0)(\pm n)) \mp \dot{r}_0(\pm n)c.$$

Proof. A simple computation shows that

$$\partial^2_{rr} V(r,t) = -\frac{3G^2}{r^4} + \frac{3r^2}{(r^2+\rho^2(t))^{5/2}} - \frac{1}{(r^2+\rho^2(t))^{3/2}}$$

from where we easily deduce that there exists $R>0$ such that, if

$$r_0(n) + c \geq R$$

then, the functional $\varphi \mapsto \mathcal{A}^+(\varphi)$ in (4.32) is strictly convex on the strictly convex set $\tilde{D}^{1,2}_+(c,n)$. Therefore, there exists a unique minimizer $\varphi_+ \in \tilde{D}^{1,2}_+(c,n)$ for which

$$\mathcal{A}^+(\varphi_+) = \min\{\mathcal{A}^+(\psi) \colon \psi \in \tilde{D}^{1,2}_+(\tilde{\varphi}(n))\}.$$

Moreover, is easy to check that φ_+ is a critical point of the functional $\mathcal{A}^+(\varphi)$. Consequently, $r(s) = r_0(s) + \varphi_+(s)$ is an orbit of (4.1) asymptotic in the future to γ_∞.

Let now $S^+(r,s)$ be the generating function of the local stable manifold introduced in Proposition 4.3.1. By uniqueness of the local stable manifold, the function $\varphi_+(s)$ satisfies that

$$(\dot{r}_0 + \dot{\varphi}_+)(s) = \partial_r S^+(r_0(s) + \varphi_+(s), s)$$

for all $s \in [n,\infty)$. In particular, since moreover $\varphi_+(s) \in D^{1,2}_+$ Lemma 4.A.2 in Appendix 4.A implies that $|\dot{\varphi}_+(s)| \lesssim s^{1/6}$ as $s \to \infty$ and since $\dot{r}_0(s) \sim s^{-1/3}$ as $s \to \infty$, we can integrate by parts to obtain

$$\begin{aligned}\int_n^\infty \mathcal{L}_{\mathrm{ren}}(\varphi_+, \dot{\varphi}_+, s) &= \int_n^\infty \frac{\dot{\varphi}_+^2}{2} + V(r_0+\varphi_+) - V_0(r_0) - \ddot{r}_0\varphi_+ \\ &= -\,\dot{r}_0(n)c + \int_n^\infty \frac{\dot{\varphi}_+^2}{2} + \dot{r}_0\dot{\varphi} + V(r_0+\varphi_+) - V_0(r_0).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_n^\infty \frac{\dot{\varphi}_+^2}{2} &+ \dot{r}_0\dot{\varphi} + V(r_0+\varphi_+) - V_0(r_0) \\ &= \int_n^\infty \big((\dot{r}_0+\dot{\varphi}_+)\partial_r S^+(r_0+\varphi_+) - H(r_0+\varphi_+, \partial_r S^+(r_0+\varphi_+), s) \\ &\quad -\dot{r}_0\ \partial_r S^0(r_0) - H_0(r_0, \partial_r S^0(r_0))\big) \\ &= \int_n^\infty \frac{\mathrm{d}}{\mathrm{d}s} S^+((r_0+\varphi_+)(s)) - \frac{\mathrm{d}}{\mathrm{d}s} S^0(r_0(s)) \\ &= -\,S^+((r_0(n)+c) + S^0(r_0(n))\end{aligned}$$

where we have used that $H(r_0+\varphi_+, \partial_r S^+(r_0+\varphi_+), s) + \partial_t S^+(r_0+\varphi_+, s) = 0$ and the fact that

$$\lim_{s\to\infty} S^+((r_0+\varphi_+)(s)) - S^0(r_0(s)) = 0,$$

which is also proved in Lemma 4.A.2. □

Introduce now the extension operator $E : \tilde{U} \subset H^1([-n,n]) \to D^{1,2}$

$$E(\tilde{\varphi}) = \begin{cases} E_-(\tilde{\varphi}) & \text{for} \quad s \in (-\infty,-n) \\ \tilde{\varphi} & \text{for} \quad s \in [-n,n] \\ E_+(\tilde{\varphi}) & \text{for} \quad s \in (n,\infty) \end{cases} \tag{4.33}$$

where

$$E_\pm(\tilde{\varphi}) = \{\varphi \in \tilde{D}^{1,2}_\pm(\tilde{\varphi}(\pm n)) \colon \mathcal{A}^\pm(\varphi) \leq \mathcal{A}^\pm(\psi),\ \forall \psi \in \tilde{D}^{1,2}_\pm(\tilde{\varphi}(\pm n))\}.$$

From the proof of Lemma 4.32 we deduce the following.

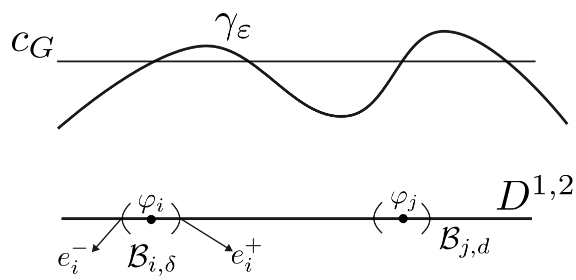

Figure 4.5: Sketch of the suboptimal path γ_ε.

Lemma 4.5.8. *Let $\varphi \in D^{1,2}$, let $n \in \mathbb{N}$ sufficiently large, let $\tilde{\varphi} = j(\varphi)$ and let $\widetilde{U} \subset H^1([-n,n])$ a sufficiently small neighbourhood of $\tilde{\varphi}$. Then, the extension operator* (4.33) *is well defined on $\widetilde{U}$.*

The proof of the following Lemma is an straightforward consequence of the definition of the extension operator E.

Lemma 4.5.9. *Let $\varphi_* \in D^{1,2}$. Then, for $n \in \mathbb{N}$ sufficiently large and all φ contained in a sufficiently small neigubourhood $U \subset D^{1,2}$ of φ_**

$$\widetilde{\mathcal{A}}(j(\varphi)) \leq \mathcal{A}(\varphi).$$

Also, for all $\tilde{\varphi}$ in a sufficiently small neighbourhood $\tilde{U} \subset H^1([-n,n])$ of $j(\varphi_*)$

$$\widetilde{\mathcal{A}}(\tilde{\varphi}) = \mathcal{A}(E(\tilde{\varphi})).$$

Moreover, for $\varphi_* \in D^{1,2}$ such that $\mathrm{d}\mathcal{A}(\varphi_*) = 0$ we have $\mathrm{d}\widetilde{\mathcal{A}}(j(\varphi_*)) = 0$.

We can now translate the result for $\mathcal{A}$ stated in Proposition 4.5.4 in an analogous result for $\tilde{\mathcal{A}}$.

Proposition 4.5.10. *There exists $n \in \mathbb{N}$ and $\tilde{\varphi}_* \subset H^1([-n,n])$ which is a critical point of $\widetilde{\mathcal{A}}$ and has a local mountain pass structure.*

Proof. The proof is a simple combination of the proof of Theorem 1 in [Hof86] together with the relationship between the functionals $\mathcal{A}$ and $\tilde{\mathcal{A}}$ which was obtained in Lemma 4.5.9. We sketch here the details for the sake of completeness.

Denote by $\mathrm{Crit}(\mathcal{A}, c_G) = \{\varphi \in \mathrm{Crit}(\mathcal{A}) \subset D^{1,2} \colon \mathcal{A}(\varphi) = c_G\}$ where c_G is the critical value defined in (4.19). Lemma 4.5.4 implies, in particular, that $\mathrm{Crit}(\mathcal{A}, c_G)$ is an isolated subset in $D^{1,2}$. Moreover, fixing $\overline{m}$ sufficiently large $\mathrm{Crit}(\mathcal{A}, c_G) \subset \mathcal{F}_{\overline{m}}$ where $\mathcal{F}_{\overline{m}}$ was defined in (4.17). Let now $\varepsilon > 0$ and $\gamma_\varepsilon \subset \mathit{\Gamma} \subset D^{1,2}$ be a suboptimal path at level c_G. Then, γ_ε intersects a finite number of elements in $\mathrm{Crit}(\mathcal{A}, c_G)$, which we denote by $\{\varphi_1, \ldots, \varphi_k\}$ for some finite k. Let now $\delta > 0$ sufficiently small and denote by $\mathcal{B}_{i,\delta} \subset D^{1,2}$ the ball of radius δ around φ_i. Without loss of generality we can assume that γ_ε intersects each $\mathcal{B}_{i,\delta}$ only once so we can define (see Figure 4.5.1)

$$t_i^- = \inf\{t \in [0,1] \colon \gamma(t) \in \mathcal{B}_{i,\delta}\} \qquad t_i^+ = \sup\{t \in [0,1] \colon \gamma(t) \in \mathcal{B}_{i,\delta}\}.$$

and $e_i^- = \gamma(t_i^-)$ and $e_i^+ = \gamma(t_i^+)$. Let now $n \in \mathbb{N}$ large enough and δ small enough so the restricition operator j in(4.31) is well defined on $\cup_{1 \leq i \leq k} \mathcal{B}_{i,\delta}$. Let $\tilde{\mathcal{B}}_{i,\delta} = j(\mathcal{B}_{i,\delta})$. Again, without loss of generality, we can assume that for all $i = 1, \ldots, k$, $e_i^\pm \in D^{1,2}$ have minimizing tails, that is, $e_i^\pm|_{[n,\infty)} \subset \tilde{D}^{1,2}(e_i^\pm(n), n)$ is the unique minimizer of $\mathcal{A}^+$ on $\tilde{D}^{1,2}(e_i^\pm(n), n)$ and $e_i^\pm|_{(-\infty,-n]} \subset \tilde{D}^{1,2}(e_i^\pm(-n), n)$ is the unique minimizer of $\mathcal{A}^-$ on $\tilde{D}^{1,2}(e_i^\pm(-n), n)$. Now, define the paths

$$\tilde{\gamma}_i = j(\gamma|_{[t_i^- t_i^+]}) \subset H^1([-n,n])$$

for $i = 1, \ldots, k$, and the points $\tilde{\varphi}_i = j(\varphi_i) \in H^1([-n,n])$, which are indeed critical points of the reduced action functional $\tilde{\mathcal{A}}$. Suppose the point $\tilde{\varphi}_i$ does not have a local mountain pass structure. Then, we can build (see Lemma 1 in [Hof86]) a continuous deformation $\eta : [0,1] \times \tilde{\mathcal{B}}_{i,\delta} \to H^1([-n,n])$ such that

$$\eta\left(\{1\} \times (\{\tilde{\varphi} \in \tilde{\mathcal{B}}_{i,\delta} \colon \tilde{\mathcal{A}}(\varphi) \leq c_G + \varepsilon\} \setminus \tilde{\mathcal{B}}_{i,\delta/2})\right) \subset \{\tilde{\varphi} \in \tilde{\mathcal{B}}_{i,\delta} \colon \tilde{\mathcal{A}}(\varphi) \leq c_G - \varepsilon\}$$

$$\eta\left([0,1]\times \mathrm{Cl}(\tilde{\mathcal{B}}_{i,\delta/2})\right)\subset \tilde{\mathcal{B}}_{i,\delta}$$
$$\eta(z,\varphi)=\varphi \qquad \forall (z,\varphi)\in[0,1]\times\{\tilde\varphi\in\tilde{\mathcal{B}}_{i,\delta}\colon |\tilde{\mathcal{A}}(\varphi)-c_G|\geq\varepsilon\}\ .$$

where by $\mathrm{Cl}(\tilde{\mathcal{B}}_{i,\delta/2})$ we denote the closure of the ball $\tilde{\mathcal{B}}_{i,\delta/2}$. Write $\eta(\tilde\gamma_i)=\eta(\{1\}\times\tilde\gamma_i)$, which satisfies

$$\tilde{\mathcal{A}}(\eta(\tilde\gamma_i))\leq c_G-\varepsilon$$

and

$$\eta(\tilde\gamma_i)(t_i^-)=j(e_i^-) \qquad\qquad \eta(\tilde\gamma_i)(t_i^+)=j(e_i^+).$$

Now, for the extension operator E is well defined on $\tilde{\mathcal{B}}_{i,\delta}$ (shrinking δ if necessary) and $\eta(\tilde\gamma_i)\subset\tilde{\mathcal{B}}_{i,\delta}$, we can define the path $E(\eta(\tilde\gamma_i))\subset D^{1,2}$ which, by construction, satisfies

$$\mathcal{A}(E(\eta(\tilde\gamma_i)))\leq c_G-\varepsilon$$

and

$$\eta(\tilde\gamma_i)(t_i^-)=e_i^- \qquad\qquad \eta(\tilde\gamma_i)(t_i^+)=e_i^+.$$

The proposition is therefore proved for, if none of the points $\tilde\varphi_i$ posses a local mountain pass structure, the continuous path $\gamma\subset D^{1,2}$ defined by gluing (in the obvious way) the segments $\gamma_\varepsilon\setminus\bigcup_{1\leq i\leq k}\gamma_\varepsilon|_{[t_i^-,t_i^+]}$ with the segments $E(\eta_i(\tilde\gamma_i))$ satisfies $\mathcal{A}(\gamma)\leq c-\varepsilon$, a contradiction. □

Proposition 4.5.10 entails a non degeneracy condition for the intersection of the invariant manifolds $W^+(\gamma_\infty)$ and $W^-(\gamma_\infty)$ at the homoclinic orbit associated with $\tilde\varphi_*$. We now make use of topological degree theory to exploit this non degeneracy condition. Let $\tilde\varphi_*\in H^1([-n,n])$ be the critical point obtained in Proposition 4.5.10 and consider a sufficiently small neighbourhood $\widetilde{U}\in H^1([-n,n])$ such that $\tilde\varphi_*\in\widetilde{U}$. By definition of the functional $\widetilde{\mathcal{A}}$, and the fact that

$$\min_{s\in[-n,n]} r_0(s)+\tilde\varphi_*(s)>0$$

the differential $\mathrm{d}\widetilde{\mathcal{A}}(\tilde\varphi):\widetilde{U}\to H^1([-n,n])$ is a continuous linear functional and, for any $\tilde\varphi\in\widetilde{U}$ and any $\psi\in H^1([-n,n])$, we can express

$$\mathrm{d}\widetilde{\mathcal{A}}(\varphi)[\psi]=\langle\dot\varphi,\dot\psi\rangle_{L^2([-n,n])}+2\int_{-n}^{n}\frac{\varphi\psi}{r_0^3}+P(\varphi)[\psi], \tag{4.34}$$

where we have introduced the functional (compare expression (4.15) in the proof of Lemma 4.4.8)

$$\begin{aligned}\widetilde{P}(\varphi)[\psi]=&\int_{-n}^{n}\left(\frac{G^2}{(r_0+\varphi)^3}-\frac{G_0}{r_0^3}\right)\psi-\int_{-n}^{n}\left(\frac{r_0+\varphi}{((r_0+\varphi)^2+\rho^2)^{3/2}}-\frac{1}{r_0^2}+\frac{2\varphi}{r_0^3}\right)\psi\\&-\left(\partial_r S^+((r_0+\tilde\varphi)(n))-\dot r_0(n)\right)\psi(n)+\left(\partial_r S^-((r_0+\tilde\varphi)(-n))-\dot r_0(-n)\right)\psi(-n).\end{aligned}$$

Since $r_0(s)>0$ and the interval $[-n,n]$ is compact, the expression

$$\langle\langle\varphi,\psi\rangle\rangle=\langle\dot\varphi,\dot\psi\rangle_{L^2([-n,n])}+2\int_{-n}^{n}\frac{\varphi\psi}{r_0^3},$$

defines an equivalent inner product on $H^1([-n,n])$. For $\tilde\varphi\in\widetilde{U}$, denote by $\nabla\widetilde{\mathcal{A}}(\tilde\varphi)$ the unique element of $H^1([-n,n])$ such that for all $\psi\in H^1([-n,n])$

$$\langle\langle\nabla\widetilde{\mathcal{A}}(\tilde\varphi),\psi\rangle\rangle=\mathrm{d}\widetilde{\mathcal{A}}(\varphi)[\psi]. \tag{4.35}$$

From (4.34) one easily deduces that the map $\nabla\widetilde{\mathcal{A}}:\widetilde{U}\to H^1([-n,n])$ is a compact perturbation of the identity. Therefore, for any subset $\widetilde{V}\subset\widetilde{U}\in H^1([-n,n])$ and any point $\tilde z\in H^1([-n,n])$ such that $\tilde z\notin\nabla\widetilde{\mathcal{A}}(\partial\widetilde{V})$ the Leray-Schauder degree [6] associated with the triple $(\nabla\widetilde{\mathcal{A}},\widetilde{V},\tilde z)$, which we denote by

$$\deg(\nabla\widetilde{\mathcal{A}},\widetilde{V},\tilde z),$$

is well defined. Proposition 4.5.10, together with Theorem 2 in [Hof86], imply the following result.

[6] The Leray-Schauder degree is a generalization of the Brouwer degree to maps between infinite dimensional spaces wich are of the form identity+compact. Details about its definition and properties can be found in [**?**].

Proposition 4.5.11. *Let $\tilde{\varphi}_* \in H^1([-n,n])$ be the critical point of $\widetilde{\mathcal{A}}$ which was obtained in Proposition 4.5.10 and, for $\varepsilon > 0$, denote by $B_\varepsilon(\tilde{\varphi}_*) \subset H^1([-n,n])$ the ball of radius ε centered at $\tilde{\varphi}_*$. Then, there exists ε_0 such that for all $0 \leq \varepsilon \leq \varepsilon_0$,*

$$\deg(\nabla \widetilde{A}, B_\varepsilon(\tilde{\varphi}_*), 0) = -1.$$

As a consequence of Proposition 4.5.11 we can prove that the manifolds $W^+(\gamma_\infty; G)$ and $W^-(\gamma_\infty; G)$ intersect transversally for $G \in \mathcal{G}$. First, we introduce some notation which will be useful in the proof of Proposition 4.5.12 and in Section 4.6. Let $s \in [-n,n]$, then, we denote by $\mathrm{ev}_s : H^1([-n,n]) \to \mathbb{R}$ the evaluation operator given by

$$\mathrm{ev}_s \tilde{\varphi} = \tilde{\varphi}(s).$$

In addition we denote by $\mathtt{ev}_s \in H^1([-n,n])$ the unique element such that for all $\psi \in H^1([-n,n])$

$$\langle\langle \mathtt{ev}_s, \psi \rangle\rangle = \mathrm{ev}_s(\psi).$$

Proposition 4.5.12. *For all $G \in \mathcal{J}$ there exists a topologically transverse intersection between $W^+(\gamma_\infty; G)$ and $W^-(\gamma_\infty; G)$.*

Proof. Let $\tilde{\varphi}_* \subset H^1([-n,n])$ be the critical point obtained in Proposition 4.5.10. Then, there exists $\varepsilon_0 > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$

$$\nabla \tilde{\mathcal{A}}(\tilde{\varphi}_*) = 0 \qquad \text{and} \qquad \nabla \tilde{\mathcal{A}}(\tilde{\varphi}) \neq 0 \qquad \forall \varphi \in B_\varepsilon(\varphi_*) \setminus \{\varphi_*\}.$$

In particular, there exists $\delta_0 > 0$ such that

$$\sup_{\varphi \in \partial B_\varepsilon(\varphi_*)} \|\nabla \tilde{\mathcal{A}}(\tilde{\varphi})\| \geq \delta_0.$$

Define now, for $\delta \in \mathbb{R}$, the one parameter family of maps $F_\delta : H^1([-n,n]) \to H^1([-n,n])$ given by

$$F_\delta(\varphi) = \nabla \tilde{\mathcal{A}}(\tilde{\varphi}_*) + \delta \mathtt{ev}_n = \nabla \left(\int_{-n}^{n} \mathcal{L}_{\mathrm{ren}}(\varphi, \dot{\varphi}, s) \right) + \partial_r S^-((r_0+\varphi)(-n))\mathtt{ev}_{-n} - (\partial_r S^+((r_0+\varphi)(n)) + \delta)\mathtt{ev}_n$$

Then, it is possible to find $\delta_1 > 0$ such that $F_\delta(\varphi)$ is an admissible homotopy for $\delta \in [-\delta_1, \delta_1]$ so by invariance of the degree under admissible homotopies

$$\deg(F_\delta, B_\varepsilon(\varphi_*), 0) = -1 \qquad \forall \delta \in [-\delta_1, \delta_1].$$

We now show how this implies the desired conclusion. Let $Q = \{\varphi \in B_\varepsilon \colon F_\delta(\varphi) = 0,\ \delta \in [-\delta_1, \delta_1]\}$. Then, denoting by π_r, π_y the projections onto the r, y coordinates of a point $(r, y, t) \in \mathbb{R}^2 \times \mathbb{T}$, and by ϕ^s the flow at time s associated to Hamiltonian (4.2) we have that

$$[-\delta_1, \delta_1] \subset \{\pi_y \circ \phi_H^{2n}(r, \partial_r S^-(r, -n), -n) - \partial_r S^+(\pi_r \circ \phi_H^{2n}(r, \partial_r S^-(r, -n), -n) \colon r \in R_{\delta_1}\}$$

for $R_{\delta_1}\{r = (r_0 + \varphi)(-n) \colon \varphi \in Q\}$. This completes the proof. □

4.6 Construction of multibump solutions

We now show how Proposition 4.5.11 together with the parabolic lambda Lemma 4.3.3 can be used to the deduce the existence of homoclinic orbits to γ_∞ which perform any arbitrary number of "bumps". We start by stating the following lemma, which is nothing but a reformulation of the parabolic lambda Lemma 4.3.4.

Lemma 4.6.1. *There exists R large enough such that for $R_0, R_1 \geq R$ there exists T_* such that for all $T \geq T_*$ there exists a unique orbit $\hat{r}(t; T, R_0, R_1)$ of (4.1) for which $\hat{r}(0) = R_0$ and $\hat{r}(T) = R_1$. Moreover, for all $\varepsilon > 0$ there exists $T_{**}(\varepsilon)$ such that for all $T \geq T_{**}$ the unique solution $\hat{r}(t; T, R_0, R_1)$ satisfies*

$$\partial_r S^+(R_0) - \dot{\hat{r}}(0) \leq \varepsilon \qquad \dot{\hat{r}}(T) - \partial_r S^-(R_1) \leq \varepsilon.$$

Given $R_0, R_1 \geq R$ and $T \geq T_*$ we denote by

$$v^+(T, R_0, R_1) = \dot{\hat{r}}(0; T, R_0, R_1) \qquad v^-(T, R_0, R_1) = \dot{\hat{r}}(T; T, R_0, R_1).$$

where $\hat{r}(t; T, R_0, R_1)$ is the orbit segment found in Lemma 4.6.1.

4.6.1 Proof of Theorem 4.1.5

We are now ready to build the multibump solutions. By proposition 4.5.11 we know that there exists a critical point $\tilde{\varphi}_* \in H^1([-n,n])$ of $\widetilde{\mathcal{A}}$ and $\varepsilon_0 \geq 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_0$,

$$\deg(\nabla\widetilde{A}, B_\varepsilon(\tilde{\varphi}_*), 0) = -1,$$

where $B_\varepsilon(\tilde{\varphi}_*) \subset H^1([-n,n])$ stands for the ball of radius ε centered at $\tilde{\varphi}_*$. For any $L \in \mathbb{N}$, introduce now the map

$$\begin{aligned} F : (B_\varepsilon(\varphi_*))^{L+1} \times (\{l \in \mathbb{N} \colon l \geq T_{**}\})^L &\longrightarrow (H^1([-n,n]))^{L+1} \\ (\varphi_1, \dots, \varphi_{L+1}, l_1, \dots, l_L) &\longmapsto (F_1, \dots, F_{L+1}) \end{aligned} \tag{4.36}$$

where the maps F_j, $1 \leq j \leq L+1$ are given by

$$\begin{aligned} F_1 =& \nabla\widetilde{\mathcal{A}}_G + \big(\partial_r S^+(\varphi_1(n), n) - v^+(l_1, \varphi_1(n), \varphi_2(-n))\big)\, \mathtt{ev}_n \\ F_{L+1} =& \nabla\widetilde{\mathcal{A}}_G + \big(v^-(l_L, \varphi_L(n), \varphi_{L+1}(-n)) - \partial_r S^-(\varphi_{L+1}(-n), -n)\big)\, \mathtt{ev}_n \end{aligned}$$

and for $2 \leq j \leq L$ (this set is empty for $L = 1$)

$$\begin{aligned} F_j =& \nabla\widetilde{\mathcal{A}}_G + \big(\partial_r S^+(\varphi_j(n), n) - v^+(l_j, \varphi_j(n), \varphi_{j+1}(-n))\big)\, \mathtt{ev}_n \\ &+ \big(v^-(l_{j-1}, \varphi_{j-1}(n), \varphi_j(-n)) - \partial_r S^-((\varphi_j)(-n), -n)\big)\, \mathtt{ev}_{-n}. \end{aligned}$$

The proof of the following result follows inmediately after from Proposition 4.5.11 and Lemma 4.6.1.

Theorem 4.6.2. *There exists $\tilde{\varphi}_* \in H^1([-n,n])$, $T > 0$ and $\varepsilon > 0$ such that for all $L \in \mathbb{N}$*

$$\deg(F, (B_\varepsilon(\varphi_*))^{L+1} \times (\{l \in \mathbb{N} \colon l \geq T\}^L, 0) = (-1)^L$$

In particular, for any sequence $\mathbf{l} = \{l_j\}_{1 \leq j \leq L} \subset (\{l \in \mathbb{N} \colon l \geq T\}^L$ there exists $\boldsymbol{\varphi}(\mathbf{l}) = \{\varphi_j(\mathbf{l})\}_{1 \leq j \leq L+1} \subset (H^1([-n,n]))^{L+1}$ such that

$$F(\boldsymbol{\varphi}(\mathbf{l}), \mathbf{l}) = 0.$$

Theorem 4.6.2 shows the truth of the first item in Theorem 4.1.5 for sequences $\sigma \in \{0,1\}^{\mathbb{Z}}$ with finite, but arbitrarily large number of nonzero entries. For the time interval T_{**} in the definition of (4.36) does not depend on L, the existence of solutions r_σ such that σ has infinitely many non-zero entries follows by a standard diagonal argument in the C^1_{loc} topology. In order to deduce the second item, namely the existence of infinitely many periodic orbits, we define the functional

$$\begin{aligned} F_{\mathrm{per}} : (B_\varepsilon(\varphi_*))^{L} \times (\{l \in \mathbb{N} \colon l \geq T_{**}\})^L &\longrightarrow (H^1([-n,n]))^{L} \\ (\varphi_1, \dots, \varphi_{L+1}, l_1, \dots, l_L) &\longmapsto (F_1, \dots, F_{L}) \end{aligned} \tag{4.37}$$

with periodic boundary conditions

$$\begin{aligned} F_1 =& \nabla\widetilde{\mathcal{A}}_G + \big(\partial_r S^+(\varphi_1(n), n) - v^+(l_1, \varphi_1(n), \varphi_2(-n))\big)\, \mathtt{ev}_n \\ &+ \big(v^-(l_L, \varphi_L(n), \varphi_1(-n)) - \partial_r S^-((\varphi_1)(-n), -n)\big)\, \mathtt{ev}_{-n} \\ F_L =& \nabla\widetilde{\mathcal{A}}_G + \big(v^-(l_L, \varphi_L(n), \varphi_{L+1}(-n)) - \partial_r S^-(\varphi_{L+1}(-n), -n)\big)\, \mathtt{ev}_n \\ &+ \big(v^+(L, \varphi_L(n), \varphi_1(-n)) - \partial_r S^+((\varphi_L)(-n), -n)\big)\, \mathtt{ev}_{-n} \end{aligned}$$

and such that for $2 \leq j \leq L$ (this set is empty for $L = 1$) F_j has the same expression as in in the non periodic case. The proof of Theorem 4.1.5 is complete.

4.6.2 Proof of Theorem 4.1.4

From the proof of Theorem 4.1.5 it follows that

$$X^+ \cap Y^- \neq \emptyset$$

for any possible combination of $X^+ \in \{P^+, B^+, OS^+\}$ and $Y^- \in \{P^-, B^-, OS^-\}$. We now show that $H^+ \cap P^- \neq \emptyset$ (the proof for the other combinations being similar). The following result is implied by the second part of Lemma 4.3.4.

Lemma 4.6.3. *There exists R large enough and ε_0 such that for all $R \geq R_0$ and all $0 \leq \varepsilon \leq \varepsilon_0$ there exists a unique orbit $\hat{r}(s;, R_0, \delta)$ of (4.1) for which*

$$\hat{r}(0) = R_0, \qquad\qquad \dot{\hat{r}}(0) = \partial_r S^+(R_0, 0) + \varepsilon.$$

Moreover, $\hat{r}(s;, R_0, \delta)$ is defined for all $s \geq 0$ and satisfies

$$\lim_{s\to\infty} \hat{r}(s; R_0, \varepsilon) = \infty \qquad\qquad \lim_{s\to\infty} \dot{\hat{r}}(s; R_0, \varepsilon) > 0$$

Given $R_0 \geq R$ and $\varepsilon > 0$ we denote by

$$v^+_{\mathrm{hyp}}(R_0, \varepsilon) = \dot{\hat{r}}(0; R_0, \varepsilon)$$

where $\hat{r}(s; R_0, \varepsilon)$ is the orbit segment found in Lemma 4.6.3. Fix $0 \leq \varepsilon \leq \varepsilon_0$. The fact that $H^+ \cap P^- \neq \emptyset$ follows from the fact that the functional

$$\begin{aligned} F_{\mathrm{hyp}} : (B_\varepsilon(\varphi_*))^{L+1} \times (\{l \in \mathbb{N} \colon l \geq T_{**}\})^L &\longrightarrow (H^1([-n, n]))^{L+1} \\ (\varphi_1, \ldots, \varphi_{L+1}, l_1, \ldots, l_L) &\longmapsto (F_{1,\mathrm{hyp}}, \ldots, F_{L+1,\mathrm{hyp}}) \end{aligned}$$

where the maps $F_{j,\mathrm{hyp}}$, $1 \leq j \leq L+1$ are given by

$$\begin{aligned} F_{1,\mathrm{hyp}} =& \nabla\widetilde{\mathcal{A}}_G + \left(\partial_r S^+(\varphi_1(n), n) - v^+(l_1, \varphi_1(n), \varphi_2(-n))\right) \mathtt{ev}_n \\ F_{L+1,\mathrm{hyp}} =& \nabla\widetilde{\mathcal{A}}_G + \left(v^-(l_L, \varphi_L(n), \varphi_{L+1}(-n)) - \partial_r S^-(\varphi_{L+1}(-n), -n)\right) \mathtt{ev}_n \\ &+ \Big(\partial_r S^+(\varphi_{L+1}(n), n) - v^+_{\mathrm{hyp}}(\varphi_{L+1}(n), \varepsilon)\Big) \mathtt{ev}_n \end{aligned}$$

and for $2 \leq j \leq L$ (this set is empty for $L = 1$)

$$\begin{aligned} F_{j,\mathrm{hyp}} =& \nabla\widetilde{\mathcal{A}}_G i + \left(\partial_r S^+(\varphi_j(n), n) - v^+(l_j, \varphi_j(n), \varphi_{j+1}(-n))\right) \mathtt{ev}_n \\ &+ \left(v^-(l_{j-1}, \varphi_{j-1}(n), \varphi_j(-n)) - \partial_r S^-((\varphi_j)(-n), -n)\right) \mathtt{ev}_{-n}. \end{aligned}$$

In order to prove that $P^+ \cap OS^- \neq \emptyset$ we take $L \to \infty$ and argue as in the proof of Theorem 4.1.5. In order to show that $H^+ \cap B^-$ we impose periodic boundary conditions. The proof of Theorem 4.1.4 is complete.

4.A Proof of of the technical claims in Lemma 4.5.7

We first prove the following result, which will be needed for the proof of Lemma 4.A.2.

Lemma 4.A.1. *Let $S_0(r; G)$ be the generating function defined in Lemma 4.2.3. Then, for any $G, G_* \in \mathbb{R}$ we have that*

$$|S_0(r; G) - S_0(r; G_*)| \lesssim \frac{|G^2 - G_*^2|}{r^{1/2}}.$$

Proof. Denote by $\tilde{u}(r)$ the unique function such that, for all $y<0$, the y component of the parametrization (4.5) is given by $y_h(\tilde{u}(r))$. Writing $\Delta S(r;G,G_*) = S_0(r;G) - S_0(r;G_*)$ we observe that $\Delta S(r;G,G_*)$ satisfies

$$y_h(\tilde{u}(r))\partial_r \Delta S + \frac{G^2 - G_*^2}{2r^2} + \frac{(\partial_r \Delta S)^2}{2} = 0$$

Using now that $y_h(\tilde{u}(r)) \sim r^{-1/2}$ for large $r \gg 1$ we obtain that

$$\partial_r \Delta S \sim \frac{G^2 - G_*^2}{r^{3/2}} + \mathcal{O}(r^{1/2}(\partial_r \Delta S)^2)$$

so

$$|\Delta S(r;G,G_*))| \lesssim \frac{|G^2 - G_*^2|}{r^{1/2}}$$

as was to be shown. □

The claims in Lemma 4.5.7 follow from the following result.

Lemma 4.A.2. *Suppose that for $G \in [-G_*, G_*]$ there exists an orbit $r(s;G): \mathbb{R} \to \mathbb{R}_+$ of the Hamiltonian H_G in(4.2) which is homoclinic to γ_∞ and, for some $G_0 \in [-G_*, G_*]$ satisfies*

$$|r(s;G) - r_0(s,G_0)| \lesssim s^{1/2} \qquad \text{as} \qquad s \to \pm\infty.$$

Then,

$$|\partial_r S^+(r(s;G),G) - \partial_r S_0(r_0(s;G_0),G_0)| \lesssim s^{-5/6} \qquad \text{as} \qquad s \to \pm\infty,$$

and, in particular

$$\lim_{s\to\pm\infty} S^\pm(r(s;G),s;G) - S^0(r_0(s;G_0),G_0) = 0.$$

Proof. We write

$$\begin{aligned}
\partial_r S^+(r(s;G),G) - \partial_r S_0(r_0(s;G_0),G_0) =& \big(\partial_r S^+(r(s;G),G) - \partial_r S_0(r(s;G),G)\big) \\
&+ \big(\partial_r S^0(r(s;G),G) - \partial_r S_0(r_0(s;G_0),G)\big) \\
&+ \big(\partial_r S^0(r_0(s;G_0),G) - \partial_r S_0(r_0(s;G_0),G_0)\big) \\
=& E_1 + E_2 + E_3.
\end{aligned}$$

On one hand, it follows from the last item in Lemma 4.3.1 that as $s \to \pm\infty$

$$|E_1| \lesssim r^{-5/2}(s;G) \lesssim r_0^{-5/2}(s;G) \lesssim s^{-5/3}.$$

On the other hand, it follows from the mean value theorem, the definition of $S^0(r;G)$ and the hypothesis in the statement of the lemma that as $s \to \pm\infty$

$$|E_2| \lesssim \sup_{r\in I} |\partial^2_{rr} S^0(r;G)||r(s;G) - r_0(s,G_0)| \lesssim r_0^{-2}(s;G_0)|r(s;G) - r_0(s,G_0)| \lesssim s^{-5/6}$$

for $I = \{r \in \mathbb{R}_+ : r = \lambda r_0(s;G_0) + (1-\lambda)r(s;G),\ \lambda \in [0,1]\}$. Also, from Lemma 4.2.3 we deduce that

$$|E_3| \lesssim r^{-3/2} \lesssim s^{-1}.$$

The proof of the first item follows combining the estimates for E_1, E_2, E_3 and integrating. The second part follows from the obtained estimate and straightforward computations. □

Chapter 5

Hyperbolic dynamics and Oscillatory motions in the 3 Body Problem

Abstract: Consider the planar 3 Body Problem with masses $m_0, m_1, m_2 > 0$. In this paper we address two fundamental questions: the existence of oscillatory motions and chaotic hyperbolic sets.

In 1922, Chazy classified the possible final motions of the three bodies, that is, the behaviors the bodies may have when time tends to infinity. One of the possible behaviors are oscillatory motions: solutions of the 3 Body Problem such that the positions of the bodies q_0, q_1, q_2 satisfy

$$\liminf_{t\to\pm\infty} \sup_{i,j=0,1,2,i\neq j} \|q_i - q_j\| < +\infty \quad \text{and} \quad \limsup_{t\to\pm\infty} \sup_{i,j=0,1,2,i\neq j} \|q_i - q_j\| = +\infty.$$

Assume that all three masses $m_0, m_1, m_2 > 0$ are not equal. Then, we prove that such motions exist. We also prove that one can construct solutions of the 3 Body Problem whose forward and backward final motions are of different type.

This result relies on constructing invariant sets whose dynamics is conjugated to the (infinite symbols) Bernouilli shift. These sets are hyperbolic for the symplectically reduced planar 3 Body Problem. As a consequence, we obtain the existence of chaotic motions, an infinite number of periodic orbits and positive topological entropy for the 3 Body Problem.

For the sake of completeness, we reproduce here the full article [GMPS22], although as already mentioned in the introduction, the contribution of the author reduces to Sections 7, 8 and 9.1 in [GMPS22].

5.1 Introduction

The 3 Body Problem models the motion of three punctual bodies q_0, q_1, q_2 of masses $m_0, m_1, m_2 > 0$ under the Newtonian gravitational force. In suitable units, it is given by the equations

$$\begin{aligned} \ddot{q}_0 &= m_1 \frac{q_1 - q_0}{\|q_1 - q_0\|^3} + m_2 \frac{q_2 - q_0}{\|q_2 - q_0\|^3} \\ \ddot{q}_1 &= m_0 \frac{q_0 - q_1}{\|q_0 - q_1\|^3} + m_2 \frac{q_2 - q_1}{\|q_2 - q_1\|^3} \\ \ddot{q}_2 &= m_0 \frac{q_0 - q_2}{\|q_0 - q_2\|^3} + m_1 \frac{q_1 - q_2}{\|q_1 - q_2\|^3}. \end{aligned} \tag{5.1}$$

In this paper we want to address two fundamental questions for this classical model: The analysis of the possible *Final Motions* and the existence of *chaotic motions (symbolic dynamics)*. These questions go back to the first half of the XX century.

Final motions: We call final motions to the possible qualitative behaviors that the complete (i.e. defined for all time) trajectories of the 3 Body Problem may possess as time tends to infinity (forward or backward). The analysis of final motions was proposed by Chazy [Cha22], who proved that the final motions of the 3 Body Problem should fall into one of the following categories. To describe them, we denote by r_k the vector from the point mass m_i to the point mass m_j for $i \neq k$, $j \neq k$, $i < j$.

Theorem 5.1.1 (Chazy, 1922, see also [AKN06])**.** *Every solution of the 3 Body Problem defined for all (future) time belongs to one of the following seven classes.*

- *Hyperbolic (H):* $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i = 0, 1, 2$, *as* $t \to \infty$.
- *Hyperbolic–Parabolic (HP_k):* $|r_i| \to \infty$, $i = 0, 1, 2$, $|\dot{r}_k| \to 0$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$, *as* $t \to \infty$.
- *Hyperbolic–Elliptic, (HE_k):* $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i = 0, 1, 2$, $i \neq k$, *as* $t \to \infty$, $\sup_{t \geq t_0} |r_k| < \infty$.
- *Parabolic-Elliptic (PE_k):* $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i = 0, 1, 2$, $i \neq k$, *as* $t \to \infty$, $\sup_{t \geq t_0} |r_k| < \infty$.
- *Parabolic (P):* $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i = 0, 1, 2$, *as* $t \to \infty$.
- *Bounded (B):* $\sup_{t \geq t_0} |r_i| < \infty$, $i = 0, 1, 2$.
- *Oscillatory (OS):* $\limsup_{t \to \infty} \sup_{i=0,1,2} |r_i| = \infty$ *and* $\liminf_{t \to \infty} \sup_{i=0,1,2} |r_i| < \infty$.

Note that this classification applies both when $t \to +\infty$ or $t \to -\infty$. To distinguish both cases we add a superindex $+$ or $-$ to each of the cases, e.g H^+ and H^-.

At the time of Chazy all types of motions were known to exist except the oscillatory motions [1]. Their existence was proven later by Sitnikov [Sit60] for the Restricted 3 Body Problem and by Alekseev [Ale69] for the (full) 3 Body Problem for some choices of the masses. After these seminal works, the study of oscillatory motions have drawn considerable attention (see Section 5.1.2 below) but all results apply under non-generic assumptions on the masses.

Another question posed by Chazy was whether the future and past final motion of any trajectory must be of the same type. This was disproved by Sitnikov and Alekseev, who showed that there exist trajectories with all possible combinations of future and past final motions (among those permitted at an energy level).

The first result of this paper is to construct oscillatory motions for the 3 Body Problem provided $m_0 \neq m_1$ and to show that all possible past and future final motions at negative energy can be combined.

Besides the question of existence of such motions, there is the question about their abundance. As is pointed out in [GK12], V. Arnol'd, in the conference in honor of the 70th anniversary of Alexeev, posed the following question.

Question 5.1.2. *Is the Lebesgue measure of the set of oscillatory motions positive?*

Arnol'd considered it the fundamental question in Celestial Mechanics. Alexeev conjectured in [Ale71] that the Lebesgue measure is zero (in the English version [Ale81] he attributes this conjecture to Kolmogorov). This conjecture remains wide open.

Symbolic dynamics: The question on existence of chaotic motions in the 3 Body Problem can be traced back to Poincaré and Birkhoff. It has been a subject of deep research during the second half of the XX century. The second goal of this paper is to construct hyperbolic invariant sets for (a suitable Poincaré map and after symplectically reducing for the classical first integrals of) the 3 Body Problem whose dynamics is conjugated to that of the usual shift

$$\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}, \qquad (\sigma\omega)_k = \omega_{k+1}, \tag{5.2}$$

[1] Indeed, note that in the limit $m_1, m_2 \to 0$, where the model becomes two uncoupled Kepler problems, all final motions are possible except $OS^{\pm}$.

in the space of sequences, one of the paradigmatic models of chaotic dynamics. Note that these dynamics implies positive topological entropy and an infinite number of periodic orbits.

The known results on symbolic dynamics in Celestial Mechanics require rather restrictive hypotheses on the parameters of the model (in particular, the values of the masses). Moreover, all the proofs of existence of hyperbolic sets with symbolic dynamics in Celestial Mechanics deal with very symmetric configurations which allow to reduce the 3 Body Problem to a two dimensional Poincaré map (see the references in Section 5.1.2 below).

5.1.1 Main results

The two main results of this paper are the following. The first theorem deals with the existence of different final motions and, in particular, of oscillatory motions.

Theorem 5.1.3. *Consider the 3 Body Problem with masses* $m_0, m_1, m_2 > 0$ *such that* $m_0 \neq m_1$*. Then,*

$$X^- \cap Y^+ \neq \emptyset \qquad \textit{with} \qquad X, Y = OS, B, PE_2, HE_2.$$

Note that this theorem gives the existence of orbits which are oscillatory in the past and in the future. It also gives different combinations of past and future final motions. Indeed,

- The bodies of masses m_0 and m_1 perform approximately circular motions. That is, $|q_0 - q_1|$ is aproximately constant.
- The third body may have radically different behaviors: oscillatory, bounded, hyperbolic or parabolic.

The motions given by Theorem 5.1.3 have negative energy. In such energy levels, only *OS*, *B*, PE_k, HE_k are possible and therefore we can combine all types of past and future final motions.

The second main result of this paper deals with the existence of chaotic dynamics for the 3 Body Problem.

Theorem 5.1.4. *Consider the 3 Body Problem with masses* $m_0, m_1, m_2 > 0$ *such that* $m_0 \neq m_1$*. Fix the center of mass at the origin and denote by* Φ_t *the corresponding flow. Then, there exists a section* Π *transverse to* Φ_t *such that the induced Poincaré map*

$$\mathcal{P} : \mathcal{U} = \mathring{\mathcal{U}} \subset \Pi \to \Pi$$

satisfies the following. There exists $M \in \mathbb{N}$ *such that the map* $\mathcal{P}^M$ *has an invariant set* $\mathcal{X}$ *which is homeomorphic to* $\mathbb{N}^{\mathbb{Z}} \times \mathbb{T}$*. Moreover, the map* $\mathcal{P}^M : \mathcal{X} \to \mathcal{X}$ *is topologically conjugated to*

$$\begin{aligned} \mathbb{N}^{\mathbb{Z}} \times \mathbb{T} &\to \mathbb{N}^{\mathbb{Z}} \times \mathbb{T} \\ (\omega, \theta) &\mapsto (\sigma\omega, \theta + f(\omega)) \end{aligned}$$

where σ *is the shift introduced in* (5.2) *and* $f : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}$ *is a continuous function.*

The set $\mathcal{X}$ is a hyperbolic set once the 3 Body Problem is reduced by its classical first integrals. The obtained conjugation implies positive topological entropy and an infinite number of periodic orbits for the 3 Body Problem for any values of the masses (except all equal). The oscillatory motions given by Theorem 5.1.3 also belong to this invariant set $\mathcal{X}$. In fact, Theorem 5.1.3 will be a consequence of Theorem 5.1.4.

5.1.2 Literature

Oscillatory motions: The first proof of oscillatory motions was achieved by Sitnikov in [Sit60] for what is called nowadays the *Sitnikov problem*. It is the Restricted 3 Body Problem when the two primaries have equal mass (the mass ratio is $\mu = 1/2$), and perform elliptic motion whereas the third body (of zero mass) is confined to the line perpendicular to the plane defined by the two primaries and passing through their center of mass. This configuration implies that this model can be reduced to a one and a half degrees of freedom Hamiltonian system, i.e. the phase space is three dimensional.

Later, Moser [Mos01] gave a new proof. His approach to prove the existence of such motions was to consider the invariant manifolds of infinity and to prove that they intersect transversally. Then, he established the existence of symbolic dynamics close to these invariant manifolds, which lead to the existence of oscillatory motions. His ideas have been very influential and are the base of the present work. In Section 5.1.3 we explain this approach as well as the challenges to apply it to the 3 Body Problem.

Since the results by Moser, there has been quite a number of works dealing with the Restricted 3 Body Problem. In the planar setting, the first one was by Simó and Llibre [LS80a]. Following the same approach as in [Mos01], they proved the existence of oscillatory motions for the RPC3BP for small enough values of the mass ratio μ between the two primaries. One of the main ingredients of their proof, as in [Mos01], was the study of the transversality of the intersection of the invariant manifolds of infinity. They were able to prove this transversality provided μ was exponentially small with respect to the Jacobi constant, which was taken large enough. Their result was extended by Xia [Xia92] using the real-analyticity of the invariant manifolds of infinity. The problem of existence of oscillatory motions for the RPC3BP was closed by the authors of the present paper in [GMS16], which proved the existence of oscillatory motions for any value of the mass ratio $\mu \in (0, 1/2]$. These oscillatory motions possess large Jacobi constant. The authors with M. Capinski and P. Zgliczyński showed the existence of oscillatory motions with "low" Jacobi constant relying on computer assisted proofs in [CGM$^+$22]. A different approach using Aubry-Mather theory and semi-infinite regions of instability was developed in [GK11, GK10b, GK10a]. In [GPSV21], the Moser approach is applied to the Restricted Isosceles 3 Body Problem. The existence of oscillatory motions has also been proven for the (full) 3 Body Problem by [Ale69] and [LS80b]. The first paper deals with the Sitnikov problem with a third positive small mass and the second one with the collinear 3 Body Problem.

A fundamental feature of the mentioned models is that they can be reduced to two dimensional area preserving maps. In particular, one can implement the Moser approach [Mos01], that is, they relate the oscillatory motions to transversal homoclinic points to infinity and symbolic dynamics. The works by Galante and Kaloshin do not rely on the Moser approach but still rely on two dimensional area preserving maps tools. Moreover, most of these works require rather strong assumptions on the masses of the bodies.

Results on Celestial Mechanics models of larger dimension such as the 3 Body Problem or the Restricted Planar Elliptic 3 Body Problem are much more scarce.

The authors with L. Sabbagh (see [GSMS17]) proved the existence of oscillatory motions for the Restricted Planar Elliptic 3 Body Problem for any mass ratio and small eccentricity of the primaries. This work relies on "soft techniques" which allow to prove that $OS^{\pm} \neq \emptyset$ but unfortunately do not imply that $OS^{-} \cap OS^{+} \neq \emptyset$ (this stronger result could be proven with the tools developed in the present paper). The same result, that is $OS^{\pm} \neq \emptyset$, is obtained for a Restricted Four Body Problem in [SZ20].

In [Moe07], R. Moeckel proves the existence of oscillatory motions for the (non-restricted) 3 Body Problem relying on passage close to triple collision, and therefore for arbitrarily small total angular momentum. This result applies to a "big" set of mass choices (however its complement also contains an open set).

The present paper is the first one which "implements" the ideas developed by Moser to the planar 3 body problem (see Sections 5.1.3 and 5.2 below). Conditional results had been previously obtained in [Rob84, Rob15], where C. Robinson proved the existence of oscillatory motions under the assumption that the so-called scattering map has a hyperbolic fixed point. As far as the authors know, such assumption has not been proven yet. We follow a different approach (see Section 5.2).

The mentioned works deal with the problem of existence of oscillatory motions in different models of Celestial Mechanics. As far as the authors know, there is only one result dealing with their abundance [GK12] (recall the fundamental Question 5.1.2). In this paper, Gorodetksi and Kaloshin analyze the Hausdorff dimension of the set of oscillatory motions for the Sitnikov example and the RPC3BP. They prove that for both problems and a Baire generic subset of an open set of parameters (the eccentricity of the primaries in the Sitnikov example and the mass ratio and the Jacobi constant in the RPC3BP) the Hausdorff dimension is maximal.

A dynamics strongly related to oscillatory motions is the Arnold diffusion behavior attached to the parabolic invariant manifolds of infinity. Such unstable behavior leads to growth in angular momentum. This is proven in [DKdlRS19] for the Restricted Planar Elliptic 3 Body Problem for mass ratio and eccentricity small enough (some formal computations on the associated Melnikov function had been done previously in [MP94]).

Symbolic dynamics and hyperbolic sets for the 3 Body Problem Starting from the 90's, there is a wide literature proving the conjugacy or semi-conjugacy of the dynamics of N-Body Problem models with the shift (5.2). These results give the existence of symbolic dynamics for such models. Results proving the existence of hyperbolic sets with symbolic dynamics are much more scarce and, as far as the authors know, all of them are in models which can be reduced to 2 dimensional maps. Namely, until the present paper no hyperbolic sets with symbolic dynamics had been proven to exist for the (symplectically reduced) planar 3 Body Problem. Note also that all the previous results dealing with symbolic dynamics in Celestial Mechanics must impose non-generic conditions on the masses.

Concerning the Restricted 3 Body Problem, there are several papers proving the existence of hyperbolic invariant sets with symbolic dynamics. On the one hand there are the results mentioned above which construct oscillatory motions relying on the invariant manifolds of infinity. There is also a wide literature constructing symbolic dynamics (providing semiconjugacy with the shift) by means of orbits passing very close to binary collision [BM00, BM06, Bol06].

For models which can be reduced to a two dimensional Poincaré map (such as the Restricted Circular Planar 3 Body Problem), there are also results which rely on Computer Assisted Proofs to show the existence of transverse homoclinic points and therefore symbolic dynamics (see for instance [Ari02, WZ03, Cap12, GZ19]).

On the full 3 Body Problem, as far as the authors know, the only results up to now proving symbolic dynamics rely on dynamics close to triple collision [Moe89, Moe07]. These great results give semiconjugacy between the 3 Body Problem and the shift (5.2) and apply to a "large" open set (but not generic) of masses (see also [RS83]). However, they do not lead to the existence of hyperbolic sets with symbolic dynamics.

The results on symbolic dynamics for the N-Body Problem with $N \geq 4$ are very scarce (see [KMJ19] for chaotic motions in a Restricted 4 Body Problem). See also [BN03, ST12] for the N center problem.

5.1.3 The Moser approach

The proof of Theorems 5.1.3 and 5.1.4 rely on the ideas developed by J. Moser [Mos01] to prove the existence of symbolic dynamics and oscillatory motions for the Sitnikov problem. Let us explain here these ideas. Later, in Section 5.2, we explain the challenges we have to face to apply these ideas to the 3 Body Problem.

The Sitnikov problem models two particles of equal mass ($m_0 = m_1 = 1/2$) performing elliptic orbits with eccentricity ε and a third body of mass 0 which is confined along the line perpendicular to the ellipses plane and passing through the center of mass of the two other bodies. This is a Hamiltonian system of one and a half degrees of freedom defined by

$$H(p, q, t) = \frac{p^2}{2} - \frac{1}{\sqrt{q^2 + R(t)}} \tag{5.3}$$

where $R(t)$ is the distance between each of the primaries to the center of mass and satisfies

$$R(t) = \frac{1}{2} + \frac{\varepsilon}{2} \cos t + \mathcal{O}(\varepsilon^2).$$

For this model, J. Moser proposed the following steps to construct oscillatory motions:

1 One can consider $P = (q, p, t) = (+\infty, 0, t)$, $t \in \mathbb{T}$, as a periodic orbit at infinity. This periodic orbit is degenerate (the linearitzation of the vector field at it is the zero matrix). Nevertheless, one can prove that it has stable and unstable invariant manifolds [McG73]. Note that these manifolds correspond to the parabolic-elliptic motions (see Theorem 5.1.1).

2 One can prove that these invariant manifolds intersect transversally, leading to transverse homoclinic orbits "to infinity". Indeed, when $\varepsilon = 0$ the Hamiltonian (5.3) has one degree of freedom and is therefore integrable. Then, the invariant manifolds coincide. For $0 < \varepsilon \ll 1$ one can apply Poincaré-Melnikov Theory [Mel63] to prove their splitting.

If P would be a hyperbolic periodic orbit, one could apply the classical Smale Theorem [Sma65] to construct invariant sets with symbolic dynamics and, inside them, oscillatory motions. However, since P is degenerate one needs a more delicate analysis than rather just applying the Smale Theorem. In particular, one needs the further steps:

3 Analyze the local behavior of (5.3) close to the infinity periodic orbit P. In hyperbolic points/periodic orbits this is encoded in the classical Lambda lemma (see for instance [PdM82]). In this step one needs to prove a suitable version of the Lambda lemma for degenerate (parabolic) periodic orbits.

4 From Steps 2 and 3 one can construct a 2–dimensional return map close to the invariant manifolds of the periodic orbit P. The final step is to construct a sequence of "well aligned strips" for this return map plus cone conditions. This leads to the existence to a hyperbolic set whose dynamics is conjugated to that of the shift (a Smale horseshoe with an "infinite number of legs").

Acknowledgements

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5.2 Outline of the proof

To apply the Moser ideas to the 3 Body Problem is quite challenging, even more if one wants to give results for a wide choice of masses. Note here the main difficulties:

- After reducing by the first integrals, the Sitnikov model, the Alekseev model and the Restricted Planar Circular 3BP are 3 dimensional flows whereas the planar 3 Body Problem is a 5 dimensional flow. This is by no means a minor change. In particular infinity goes from a periodic orbit to a two dimensional family of periodic orbits with the same period. This adds "degenerate" dimensions which makes considerably more difficult to build hyperbolic sets.

- We do not assume any smallness condition on the masses. This means that one cannot apply classical Melnikov Theory to prove the transversality between the invariant manifolds of infinity. We consider a radically different nearly integrable regime: we take the third body far away from the other two (usually such regime is referred to as *hierarchical*). This adds multiple time scales to the problem which leads to a highly anisotropic transversality between the invariant manifolds: in some directions the transversality is exponentially small whereas in the others is power-like.

These issues make each of the steps detailed in Section 5.1.3 considerably difficult to be implemented in the 3 Body Problem. In the forthcoming sections we detail the main challenges and the novelties of our approach.

We believe that the ideas developed for each of these steps have interest beyond the results of the present paper and could be used in other physical models (certainly in Celestial Mechanics) to construct all sorts of unstable motions such as chaotic dynamics or Arnold diffusion.

5.2.1 Outline of Step 0: A good choice of coordinates

Before implementing the Moser approach in the Steps 1, 2, 3, and 4 below, one has to consider first a preliminary step: to choose a good system of coordinates. This is quite usual in Celestial Mechanics where typically cartesian coordinates do not capture "well" the dynamics of the model.

In this case, keeping in mind that we want to construct hyperbolic sets, it is crucial that

- We symplectically reduce the planar 3 Body Problem by the classical first integrals.
- We consider coordinates which capture the near integrability of the model in such a way that the first two bodies perform close to circular motion whereas the third one performs close to parabolic motion (see Figure 5.1).

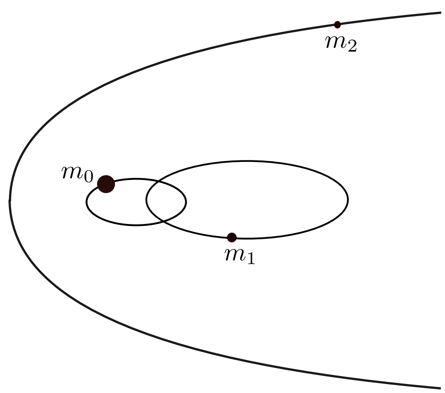


Figure 5.1: We consider two bodies performing approximately circular motions while the third body is close to a parabola, which is arbitrarily large and far from the other two bodies.

To this end we first consider the classical Jacobi coordinates (Q_1, Q_2) as seen in Figure 5.7 and conjugate momenta (P_1, P_2). This reduces the model to a 4 degrees of freedom Hamiltonian system.

Then, for the first pair (Q_1, P_1), we consider the classical Poincaré variables (λ, L, η, ξ) and for the second one (Q_2, P_2) we consider polar coordinates (r, y, α, Γ) where y is the radial momentum and Γ is the angular momentum. Finally, we "eliminate" the pair (α, Γ) by reducing the system by rotations.

Therefore, we finally have a three degrees of freedom Hamiltonian system defined in the coordinates $(\lambda, L, \eta, \xi, r, y)$ which depends on the total angular momentum, which can be treated as a parameter and which we take large enough. In Section 5.3 we perform these changes of coordinates in full detail and give expressions for the resulting Hamiltonian.

We fix the total energy to a negative value. Following the Moser approach explained in Section 5.1.3, we consider the "parabolic infinity" manifold, which is now defined by

$$\mathcal{E}_\infty = \{r = \infty,\ y = 0\},$$

and therefore can be parameterized by the coordinates (λ, L, η, ξ) (we actually eliminate the variable L by means of the energy conservation). More properly speaking, we consider McGehee coordinates

$$r = \frac{2}{x^2},$$

so that "infinity" becomes $(x, y) = (0, 0)$.

The dynamics at infinity is foliated by the periodic orbits $(\eta, \xi) =$ constant of the same period. The first step in our proof is to analyze the invariant manifolds of these periodic orbits and their intersections.

5.2.2 Outline of Step 1: Transverse homoclinic orbits to infinity

In suitably modified McGehee coordinates, the infinity manifold becomes $\mathcal{E}_\infty = \{(x, y) = (0, 0),\, z \in U \subset \mathbb{R}^2,\, t \in \mathbb{T}\}$. The dynamics in a neighborhood of infinity is given by

$$\begin{aligned} \dot{x} &= -x^3 y(1 + \mathcal{O}_2(x, y)), & \dot{z} &= \mathcal{O}_6(x, y), \\ \dot{y} &= -x^4(1 + \mathcal{O}_2(x, y)), & \dot{t} &= 1, \end{aligned}$$

for $x + y > 0$, $z \in U \subset \mathbb{R}^2$, and $t \in \mathbb{T}$. Note that infinity is foliated by the periodic orbits $z =$ constant. Thanks to [BFM20a, BFM20b], these periodic orbits have local stable and unstable invariant manifolds, which are analytic (away from infinity) and smooth with respect to parameters and to the base periodic orbit. The union of these invariant manifolds form the stable and unstable invariant manifolds of infinity, $W^s(\mathcal{E}_\infty)$ and $W^u(\mathcal{E}_\infty)$, which are four dimensional in a five dimensional energy level.

In order to control the globalization of these invariant manifolds, we consider a hierarchical regime in our system. We consider a configuration such that the first two bodies perform approximately circular motions whereas the third body performs approximately parabolic motion along a parabola which is taken arbitrarily large compared with the circle of the two first bodies (see Figure 5.1).

In other words, we choose the fixed value of the energy to be negative and of order 1, and take the total angular momentum Θ large. This choice has two consequences. On the one hand, the motion of the third body takes place far from the first two. This implies that the system becomes close to integrable, since, being far from the first bodies, the third one sees them almost as a single one and hence its motion is governed at a first order by a Kepler problem with zero energy — since its motion is close to parabolic — while the motion of the first two bodies is given at first order by another Kepler problem with negative energy. On the other hand, in this regime the system has two time scales, since the motion of the third body is $\mathcal{O}(\Theta^{-3})$ slower than that of the first ones. This implies that the coupling term between the two Kepler problems is a fast and small perturbation.

In the framework of averaging theory, the fact that the perturbation is fast implies that the difference between the stable and unstable invariant manifolds of infinity is typically exponentially small in Θ^{-3}, which precludes the application of the standard Poincaré-Melnikov theory to compute the difference of these invariant manifolds. Indeed, the perturbation can be averaged out up to any power of Θ^{-3}, making the distance between the manifolds a *beyond all orders* quantity. We need to resort to more delicate techniques to obtain a formula of this distance which is exponentially small in Θ^{-3}, proven in Theorem 5.4.3 below. From this formula we are able to deduce that the invariant manifolds of infinity do intersect transversally along two distinct intersections. These intersections are usually called homoclinic channels, which we denote by Γ^1 and Γ^2 (see Figure 5.3).

The fact that the perturbed invariant manifolds are exponentially close is usually referred to as *exponentially small splitting of separatrices*. This phenomenon was discovered by Poincaré [Poi90, Poi99]. It was not until the 80's, with the pioneering work by Lazutkin for the *standard map* (see [Laz84, Laz03]) that analytic tools to analyze this phenomenon were developed. Nowadays, there is quite a number of works proving the existence of transverse homoclinic orbits following the seminal ideas by Lazutkin, see for instance [DS92, Gel94, DS97, DGJS97, Gel97, DR98, Gel99, Gel00, Lom00, GS01, BF04b, GOS10, GaG11, Gua12, MSS11, BCS13, BCS18a, BCS18b]. Note, however, that most of these results deal with rather low dimensional models (typically area preserving two dimensional maps or three dimensional flows), whereas the model considered in the present paper has higher dimension (see also [GGSZ21], which deals with an exponentially small splitting problem in infinite dimensions). The high dimension makes the analysis in the present paper considerably more intrincate. Of special importance for the present paper are the works by Lochak, Marco and Sauzin (see [Sau01, LMS03]) who analyze such phenomenon considering certain graph parameterizations of the invariant manifolds. Other methods to deal with exponentially small splitting of separatrices are Treschev's *continuous averaging* (see [Tre97]) or "direct" series methods (see [GGM99]).

As far as the authors know, the first paper to prove an exponentially small splitting of separatrices in a Celestial Mechanics problem is [GMS16] (see also [GPSV21, BGG22, BGG21]).

The results in the aforementioned Theorem 5.4.3 allows us to define and control two different return maps from a suitable section transverse to the unstable manifold of infinity. The section, four dimensional, is close to $\mathcal{E}_\infty$. Each of these return maps will be, in turn, the composition of a *local map*, that describes the passage close to infinity, and a *global map*, following the dynamics along the global invariant manifolds. These are the subject of study of Steps 2 and 3 below. Finally, a suitable combination of the two return maps will give rise to chaotic dynamics as it is explained in Step 4.

5.2.3 Outline of Step 2: The parabolic Lambda lemma and the Local map

To analyze the local behavior close to infinity, we develop a *parabolic Lambda lemma*. The classical Lambda lemma applies to (partially) hyperbolic invariant objects and is no longer true in the parabolic setting. The statement has to be adapted and the proof we provide has to face considerable subtleties.

The first step in proving a Lambda lemma is to perform a normal form procedure which straightens the invariant manifolds and the associated stable and unstable foliations. In the present paper, thus, we need to set up a *parabolic normal form*. Indeed, for any fixed $N \geq 3$, we construct local coordinates in a neighborhood of infinity in which the (symplectically reduced) 3BP is written as

$$\begin{aligned} \dot{q} &= q((q+p)^3 + \mathcal{O}_4(q,p)), & \dot{z} &= q^N p^N \mathcal{O}_4(q,p), \\ \dot{p} &= -p((q+p)^3 + \mathcal{O}_4(q,p)), & \dot{t} &= 1, \end{aligned} \tag{5.4}$$

where $p = q = 0$ corresponds to the parabolic infinity, $\mathcal{E}_\infty$. Note that in these coordinates the (local) unstable manifold of infinity is given by $p = 0$ and the (local) stable manifold is $q = 0$. The key point, however, is that the dynamics on the "center" variables z is extremely slow in a neighborhood of infinity. This normal form is obtained in Theorem 5.5.2.

The *parabolic* Lambda Lemma is proven in these normal form variables. However, since the statement fails at the infinity manifold, first we consider two 4-dimensional sections at a fixed but small distance of $\mathcal{E}_\infty$: Σ_1 transverse to $W^s(\mathcal{E}_\infty)$ and Σ_2 transverse to $W^u(\mathcal{E}_\infty)$ (see Figure 5.2). We call *local map* to the induced map by the flow between the sections Σ_1 and Σ_2.

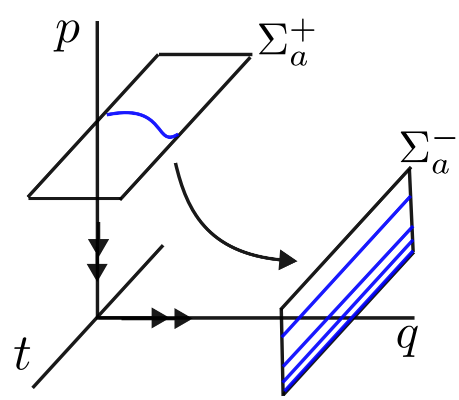


Figure 5.2: Behavior of the local map from the section Σ_1 to the section Σ_2. We are omitting the dynamics of the z-components, which are "very close" to the identity.

The parabolic Lambda lemma given in Theorem 5.5.4 below implies that the intersection of manifolds transverse to $W^s(\mathcal{E}_\infty)$ within Σ_1 gets mapped by the local map to an immersed manifold which accumulates in a $\mathcal{C}^1$ way to $W^u(\mathcal{E}_\infty) \cap \Sigma_2$. Furthermore, in the z-variables, the local map is close to the identity at the $\mathcal{C}^1$ level. As a consequence, the local map and its inverse have one and only one expanding direction.

When combining the local map with a global map along a homoclinic chanel, this construction provides a map with a single expanding direction and a single contracting direction. This was enough for Moser since in the Sitnikov problems one deals with 2-dimensional sections. However, in order to obtain a true hyperbolic object, we need hyperbolicity in *all four directions*. That is, we need to "gain" hyperbolicity in the z-directions, whose dynamics is mainly given by the behavior of the global map. We will achieve hyperbolicity by combining two different global maps, related to the two different homoclinic channels obtained in Step 1 (see Theorem 5.4.3). The Lambda Lemma ensures that the dynamics in the z-variables induced by the travel along the homoclinic manifold is essentially preserved by the local passage.

5.2.4 Outline of Step 3: The Scattering map and the Global maps

A crucial tool to understand the dynamics close to the invariant manifolds of infinity is the so-called *Scattering map*. The Scattering map was introduced by Delshams, de la Llave and Seara [DdlLS00, DdlLS06, DdlLS08] to analyze the heteroclinic connections to a normally hyperbolic invariant manifold. However, as shown in [DKdlRS19] (see Section 5.4.3), the theory in [DdlLS08] can be adapted to the parabolic setting of the present paper.

From Theorem 5.4.3 we obtain that the transversal intersection of the invariant manifolds $W^s(\mathcal{E}_\infty)$ and $W^u(\mathcal{E}_\infty)$ contains at least two homoclinic channels, $\Gamma^j \subset W^s(\mathcal{E}_\infty) \cap W^u(\mathcal{E}_\infty)$ $j = 1, 2$ (see Figure 5.3). Then, associated to each homoclinic channel, one can define the scattering map S^j as follows. We say that $x_+ = S^j(x_-)$ if there exists a heteroclinic point in Γ^j whose trajectory is asymptotic to the trajectory of x_+ in the future and asymptotic to the trajectory of x_- in the past. Such points $x_\pm$ are well defined even if $\mathcal{E}_\infty$ is not a normally hyperbolic manifold. Once Γ^j is fixed, thanks to the transversality between the invariant manifolds, the associated scattering map is locally unique and inherits the regularity of the invariant manifolds.

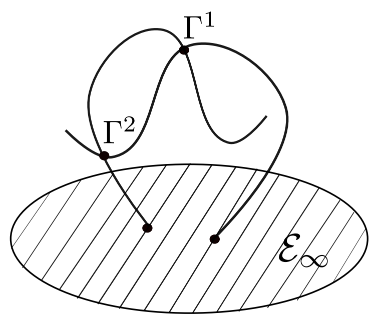


Figure 5.3: Transverse intersection of the invariant manifolds of $\mathcal{E}_\infty$ along two homoclinic channels Γ_1 and Γ_2.

The construction of scattering maps in the parabolic setting was already done in [DKdlRS19]. Note, however, that in the present paper the transversality between the invariant manifolds is highly anisotropic (exponentially small in some directions and polynomially small in the others). This complicates considerably the construction of the scattering maps, which is done in Section 5.4.3 (see Section 5.9 for the proofs). Moreover, we show that the domains of the two scattering maps S^1 and S^2, associated to the two different channels, overlap.

The scattering maps are crucial to analyze the global maps which have been introduced in Step 2 and are defined from the section Σ_2 to the section Σ_1. Indeed, we show that the dynamics of the z-variables in the two global maps are given (at first order) by the corresponding variables of the associated scattering maps. The additional hyperbolicity in the z-directions we need will come from a suitable high iterate of a combination of the two scattering maps $\widehat{S} = (S^1)^M \circ S^2$ (for a suitable large M). To prove the existence of this hyperbolicity, we construct an isolating block for this combination.

By isolating block we mean the following: There exists a small rectangle in the z-variables, in the common domain of the scattering maps, whose image under $\widehat{S}$ is another rectangle "correctly aligned" with the original one, as seen in Figure 5.4, that is, the horizontal and vertical boundaries are mapped into horizontal and vertical boundaries, respectively, it is stretched along the horizontal direction, shrunk in the vertical direction, and the left and right vertical boundaries are mapped to the left and right of the vertical boundaries, respectively, while the top and bottom horizontal boundaries are mapped below

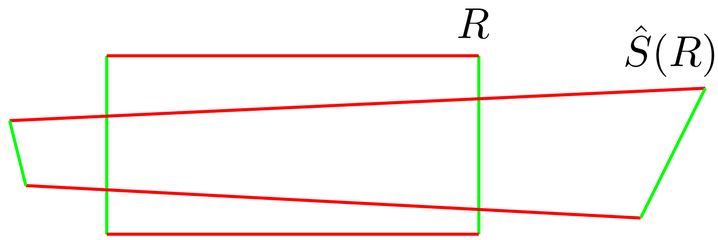


Figure 5.4: The isolating block R of the iterate of the scattering map $\widehat{S}$

and above, respectively, of the top and bottom horizontal boundaries.

To construct such isolating block we proceed as follows. Each of the scattering maps is a nearly integrable twist map around an elliptic fixed point (see Figure 5.5). The two fixed points are different but exponentially close to each other with respect Θ^{-3}. Combining the two rotations around the distinct elliptic points, we use a transversality-torsion argument (in the spirit of [Cre03]) to build the isolating block.

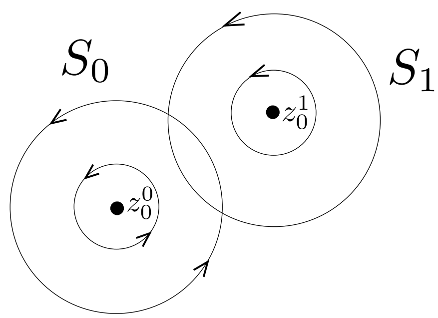


Figure 5.5: The dynamics of the two scattering maps S^1 and S^2

5.2.5 Outline of Step 4: The isolating block for the return map

The last step of the proof combines Steps 2 and 3. We consider the return map Ψ given by M iterates of the return map along the first homoclinic chanel and 1 iterate along the second homoclinic chanel. Each of the maps has two hyperbolic directions given by the passage close to the infinity manifolds as we have seen in Section 5.2.3. The projection onto the z-variables of each of the maps is close to the corresponding projection of the scattering maps. The same happens to the projection onto the z-variables of the whole composition Ψ. Hence, the map Ψ, possesses two "stable" and two "unstable" directions in some small domain. Even if the two stretching rates in the two expanding directions are drastically different, we are able to check that the restriction of Ψ to this small domain satisfies the standard hypotheses that ensure that Ψ is conjugated to the Bernouilli shift with infinite symbols. In particular, we prove cone conditions for the return map Ψ.

In conclusion, we obtain a product-like structure as seen in Figure 5.6. In the left part of the figure, one obtains the usual structure of infinite horizontal and vertical strips as obtained by Moser in [Mos01] whereas the right part of the figure corresponds to the isolating block construction in the z directions. This structure leads to the existence of a hyperbolic set whose dynamics is conjugated to that of the usual shift (5.2). Since the strips accumulate to the invariant manifolds of infinity, one can check that there exists oscillatory orbits inside the hyperbolic invariant set.

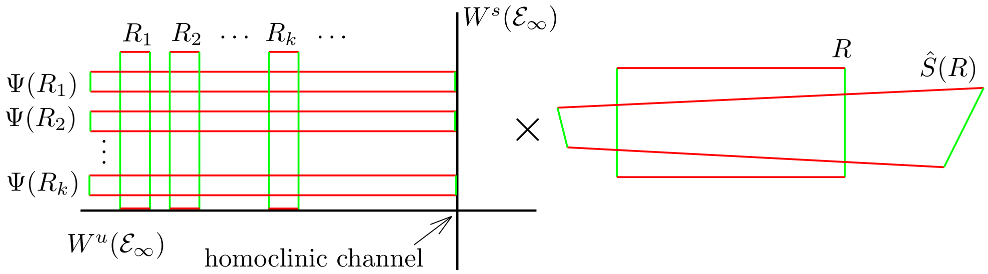
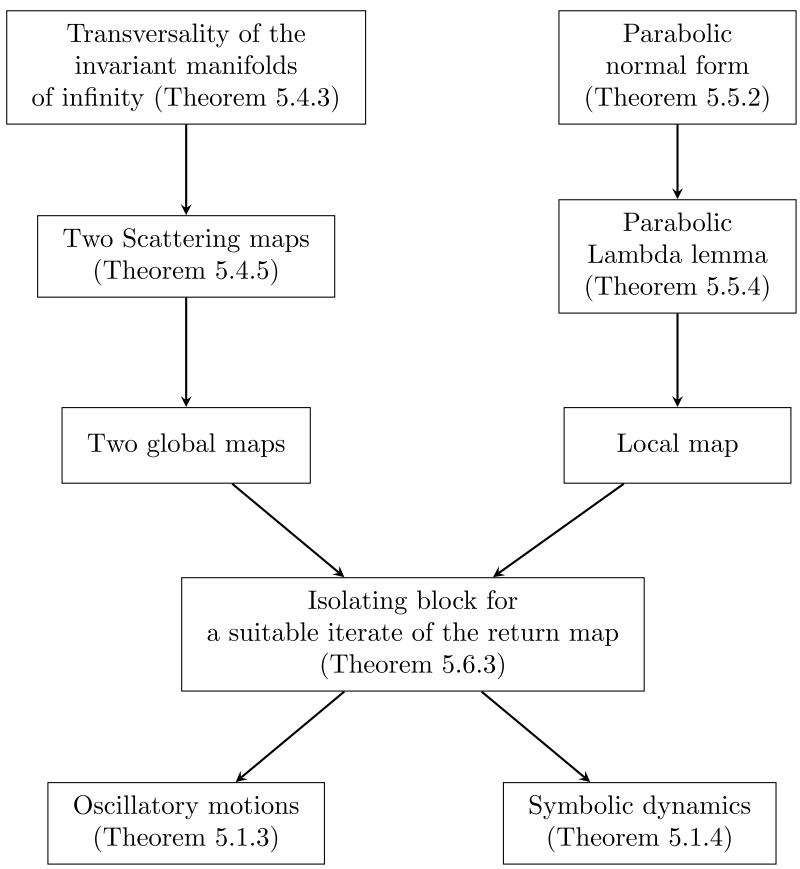


Figure 5.6: The horizontal and vertical four dimensional strips which lead to the conjugation with the Bernouilli shift of infinite symbols.

5.2.6 Summary of the outline and structure of the paper

To summarize, we present here in a diagram the main steps in the proof of Theorems 5.1.3 and 5.1.4.



5.3 A good system of coordinates for the 3 Body Problem

To analyze the planar 3 Body Problem (6.1), the first step is to choose a good system of coordinates which, on the one hand, reduces symplectically the classical first integrals of the model and, on the other hand, makes apparent the nearly integrable setting explained in Section 5.2. That is, we consider a good system of coordinates so that we obtain, at first order, that the two first bodies, $q_0, q_1 \in \mathbb{R}^2$, move on ellipses, whereas the third body, $q_2 \in \mathbb{R}^2$, moves on a coplanar parabola which is far away from the ellipses.

5.3.1 Symplectic reduction of the planar 3 Body Problem

Introducing the momenta $p_i = m_i\dot{q}_i$, $i = 0, 1, 2$, equation (6.1) defines a six degrees of freedom Hamiltonian system. We start by reducing it by translations with the classical Jacobi coordinates to obtain a four degrees of freedom Hamiltonian system. That is, we define the symplectic transformation

$$\begin{aligned} Q_0 &= q_0 & P_0 &= p_0 + p_1 + p_2 \\ Q_1 &= q_1 - q_0 & P_1 &= p_1 + \frac{m_1}{m_0 + m_1} p_2 \\ Q_2 &= q_2 - \frac{m_0 q_0 + m_1 q_1}{m_0 + m_1} & P_2 &= p_2. \end{aligned}$$

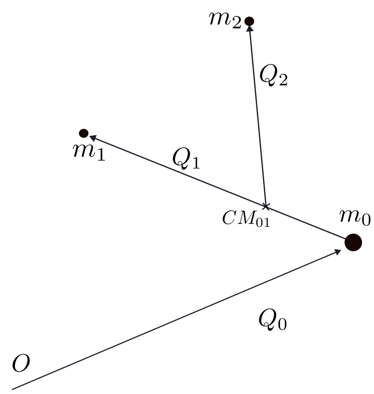


Figure 5.7: The Jacobi coordinates. CM_{01} stands for the center of mass of the bodies q_0 and q_1.

These coordinates allow to reduce by the total linear momentum since now P_0 is a first integral. Assuming $P_0 = 0$, the Hamiltonian of the 3 Body Problem becomes

$$\widetilde{H}(Q_1, P_1, Q_2, P_2) = \sum_{j=1}^{2} \frac{|P_j|^2}{2\mu_j} - \widetilde{U}(Q_1, Q_2)$$

where

$$\frac{1}{\mu_1} = \frac{1}{m_0} + \frac{1}{m_1}, \qquad \frac{1}{\mu_2} = \frac{1}{m_0 + m_1} + \frac{1}{m_2}$$

and

$$\widetilde{U}(Q_1, Q_2) = \frac{m_0 m_1}{\|Q_1\|} + \frac{m_0 m_2}{\|Q_2 + \sigma_0 Q_1\|} + \frac{m_1 m_2}{\|Q_2 - \sigma_1 Q_1\|}$$

with

$$\sigma_0 = \frac{m_1}{m_0 + m_1}, \qquad \sigma_1 = \frac{m_0}{m_0 + m_1} = \frac{1}{1 + \sigma_0}. \tag{5.5}$$

Next step is to express the Hamiltonian $\widetilde{H}$ in polar coordinates. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we consider the symplectic transformation

$$Q_1 = \rho e^{i\theta}, \qquad Q_2 = r e^{i\alpha}, \qquad P_1 = z e^{i\theta} + i\frac{\Gamma}{\rho} e^{i\theta}, \qquad P_2 = y e^{i\alpha} + i\frac{G}{r} e^{i\alpha}$$

which leads to the Hamiltonian

$$H^*(\rho, z, \theta, \Gamma, r, y, \alpha, G) = \frac{1}{\mu_1}\left(\frac{z^2}{2} + \frac{\Gamma^2}{2\rho^2}\right) + \frac{1}{\mu_2}\left(\frac{y^2}{2} + \frac{G^2}{2r^2}\right) - \widetilde{U}\left(\rho e^{i\theta}, r e^{i\alpha}\right).$$

where

$$\widetilde{U}\left(\rho e^{i\theta}, re^{i\alpha}\right) = \frac{m_0 m_1}{\rho} + \frac{m_0 m_2}{|re^{i\alpha} + \sigma_0 \rho e^{i\theta}|} + \frac{m_1 m_2}{|re^{i\alpha} - \sigma_1 \rho e^{i\theta}|}$$
$$= \frac{m_0 m_1}{\rho} + \frac{1}{r}\left(\frac{m_0 m_2}{|1 + \sigma_0 \frac{\rho}{r} e^{i(\theta-\alpha)}|} + \frac{m_1 m_2}{|1 - \sigma_1 \frac{\rho}{r} e^{i(\theta-\alpha)}|}\right).$$

We study the regime where the third body is far away from the other two and its angular momentum is very large. That is,

$$r \gg \rho \quad \text{and} \quad G \gg \Gamma.$$

Then, we have

$$H^*(\rho, z, \theta, \Gamma, r, y, \alpha, G) = \frac{1}{\mu_1}\left(\frac{z^2}{2} + \frac{\Gamma^2}{2\rho^2}\right) + \frac{1}{\mu_2}\left(\frac{y^2}{2} + \frac{G^2}{2r^2}\right) - \frac{m_0 m_1}{\rho} - \frac{m_2(m_0 + m_1)}{r} + \mathcal{O}\left(\frac{\rho^2}{r^3}\right).$$

Thus, *at first order* we have two uncoupled Hamiltonians, one for $(\rho, z, \theta, \Gamma)$ and the other for (r, y, α, G),

$$\begin{aligned} H_{\mathrm{El}}(\rho, z, \theta, \Gamma) &= \frac{1}{\mu_1}\left(\frac{z^2}{2} + \frac{\Gamma^2}{2\rho^2}\right) - m_0 m_1 \frac{1}{\rho} \\ H_{\mathrm{Par}}(r, y, \alpha, G) &= \frac{1}{\mu_2}\left(\frac{y^2}{2} + \frac{G^2}{2r^2}\right) - m_2(m_0 + m_1)\frac{1}{r}. \end{aligned} \tag{5.6}$$

To have the first order Hamiltonians H_{El} and H_{Par} independent of the masses, we make the following scaling to the variables, which is symplectic,

$$\rho = \frac{1}{\mu_1 m_0 m_1}\widetilde{\rho}, \quad z = \mu_1 m_0 m_1 \widetilde{z}, \quad r = \frac{1}{\mu_2 m_2 (m_0 + m_1)}\widetilde{r} \quad \text{and} \quad y = \mu_2 m_2 (m_0 + m_1)\widetilde{y}.$$

We also rescale time as

$$t = \frac{\tau}{\mu_2 m_2^2 (m_0 + m_1)^2}.$$

Then, we obtain the Hamiltonian

$$\tilde{H}^*(\widetilde{\rho}, \widetilde{z}, \theta, \Gamma, \widetilde{r}, \widetilde{y}, \alpha, G) = \nu\left(\frac{\widetilde{z}^2}{2} + \frac{\Gamma^2}{2\widetilde{\rho}^2} - \frac{1}{\widetilde{\rho}}\right) + \left(\frac{\widetilde{y}^2}{2} + \frac{G^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}}\right) - W(\widetilde{\rho}, \widetilde{r}, \theta - \alpha).$$

with

$$W(\widetilde{\rho}, \widetilde{r}, \theta - \alpha) = \frac{\widetilde{\nu}}{\widetilde{r}}\left(\frac{m_0}{|1 + \widetilde{\sigma}_0 \frac{\widetilde{\rho}}{\widetilde{r}} e^{i(\theta-\alpha)}|} + \frac{m_1}{|1 - \widetilde{\sigma}_1 \frac{\widetilde{\rho}}{\widetilde{r}} e^{i(\theta-\alpha)}|} - (m_0 + m_1)\right), \tag{5.7}$$

and

$$\nu = \frac{\mu_1 m_0 m_1}{\mu_2 m_2 (m_0 + m_1)}, \qquad \widetilde{\nu} = (m_0 + m_1) m_2^2 \qquad \text{and} \qquad \widetilde{\sigma}_i = \frac{\mu_2 m_2 (m_0 + m_1)}{\mu_1 m_0 m_1}\sigma_i. \tag{5.8}$$

Note that the potential W only depends on the angles through $\theta - \alpha$ due to the rotational symmetry of the system.

Now, we change the polar variables $(\widetilde{\rho}, \widetilde{z}, \theta, \Gamma)$ to the classical Delaunay coordinates (see, for instance, [Sze67])

$$(\widetilde{\rho}, \widetilde{z}, \theta, \Gamma) \mapsto (\ell, L, g, \Gamma). \tag{5.9}$$

This change is symplectic. As usual, from the Delaunay actions, which are the square of the semimajor axis L and the angular momentum Γ, one can compute the eccentricity

$$e_c(L, \Gamma) = \sqrt{1 - \frac{\Gamma^2}{L^2}}. \tag{5.10}$$

The position variables $(\widetilde{\rho}, \theta)$ can be expressed in terms of Delaunay variables as

$$\widetilde{\rho} = \widetilde{\rho}(\ell, L, \Gamma) = L^2(1 - e_c \cos E) \qquad \text{and} \qquad \theta = \theta(\ell, L, g, \Gamma) = v(\ell, L, \Gamma) + g, \tag{5.11}$$

where the angles true anomaly v and eccentric anomaly E are defined in terms of the mean anomaly ℓ and eccentricity e_c as

$$\ell = E - e_c \sin E \qquad \text{and} \qquad \tan\frac{v}{2} = \sqrt{\frac{1+e_c}{1-e_c}}\tan\frac{E}{2}. \tag{5.12}$$

One could also write an expression for $\tilde{z}$, but it is not necessary to obtain the new Hamiltonian

$$\mathcal{H}(\ell, L, g, \Gamma, \widetilde{r}, \widetilde{y}, \alpha, G) = -\frac{\nu}{2L^2} + \left(\frac{\widetilde{y}^2}{2} + \frac{G^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}}\right) + W(\widetilde{\rho}(\ell, L, \Gamma), \widetilde{r}, v(\ell, L, \Gamma) + g - \alpha),$$

where W is the potential introduced in (6.4). Now, by (6.8), the distance condition corresponds to $\widetilde{r} \gg L^2$ and the *first order* uncoupled Hamiltonians are

$$\mathcal{H}_{\mathrm{El}}(\ell, L, g, \Gamma) = -\frac{\nu}{2L^2} \qquad \text{and} \qquad \mathcal{H}_{\mathrm{Par}}(\widetilde{r}, \widetilde{y}, \alpha, G) = \frac{\widetilde{y}^2}{2} + \frac{G^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}}$$

whereas $W = \mathcal{O}(\frac{\widetilde{\rho}^2}{\widetilde{r}^3}) = \mathcal{O}(\frac{L^4}{\widetilde{r}^3})$.

Now, we make the last reduction which uses the rotational symmetry. We define the new angle $\phi = g - \alpha$. To have a symplectic change of coordinates, we consider the transformation

$$(\ell, L, \phi, \Gamma, \widetilde{r}, \widetilde{y}, \alpha, \Theta) = (\ell, L, g - \alpha, \Gamma, \widetilde{r}, \widetilde{y}, \alpha, G + \Gamma). \tag{5.13}$$

Then, we obtain the following Hamiltonian, which is independent of α,

$$\begin{aligned} \widetilde{\mathcal{H}}(\ell, L, \phi, \Gamma, \widetilde{r}, \widetilde{y}; \Theta) &= \mathcal{H}(\ell, L, \phi + \alpha, \Gamma, \widetilde{r}, \widetilde{y}, \alpha, \Theta - \Gamma) \\ &= -\frac{\nu}{2L^2} + \left(\frac{\widetilde{y}^2}{2} + \frac{(\Theta - \Gamma)^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}}\right) + W(\widetilde{\rho}(\ell, L, \Gamma), \widetilde{r}, v(\ell, L, \Gamma) + \phi). \end{aligned} \tag{5.14}$$

Since this Hamiltonian is independent of α, the total angular momentum Θ is a conserved quantity which can be taken as a parameter of the system. We assume $\Theta \gg 1$.

5.3.2 The Poincaré variables

We consider nearly circular motions for the first two bodies. Since Delaunay variables are singular at the circular motions $\Gamma \simeq L$ (equivalently by (6.7), $e_c \simeq 0$), we introduce Poincaré variables

$$(\ell, L, \phi, \Gamma, \widetilde{r}, \widetilde{y}) \mapsto (\lambda, L, \eta, \xi, \widetilde{r}, \widetilde{y}),$$

defined by

$$\begin{aligned} &\lambda = \ell + \phi, && L = L, \\ &\eta = \sqrt{L - \Gamma}e^{i\phi}, && \xi = \sqrt{L - \Gamma}e^{-i\phi}, \end{aligned} \tag{5.15}$$

which are symplectic in the sense that the form $d\ell \wedge dL + d\phi \wedge d\Gamma$ is mapped into $d\lambda \wedge dL + i d\eta \wedge d\xi$. These coordinates make the Hamiltonian $\widetilde{\mathcal{H}}$ well defined at circular motions (i.e. at $\eta = \xi = 0$). The transformed Hamiltonian can be written as

$$\widetilde{\mathcal{K}}(\lambda, L, \eta, \xi, \widetilde{r}, \widetilde{y}; \Theta) = -\frac{\nu}{2L^2} + \frac{\widetilde{y}^2}{2} + \frac{(\Theta - L + \eta\xi)^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}} + \widetilde{W}(\lambda, L, \eta, \xi, \widetilde{r}) \tag{5.16}$$

where, using that

$$e^{i\phi} = \frac{\eta}{\sqrt{\eta\xi}}, \tag{5.17}$$

the potential becomes

$$\begin{aligned} \widetilde{W}(\lambda, L, \eta, \xi, \widetilde{r}) &= W(\widetilde{\rho}(\ell, L, \Gamma), \widetilde{r}, v(\ell, L, \Gamma) + \phi) \\ &= \frac{\widetilde{\nu}}{\widetilde{r}}\left(\frac{m_0}{|1 + \widetilde{\sigma}_0 \frac{\widetilde{\rho}}{\widetilde{r}} \frac{\eta}{\sqrt{\eta\xi}} e^{iv}|} + \frac{m_1}{|1 - \widetilde{\sigma}_1 \frac{\widetilde{\rho}}{\widetilde{r}} \frac{\eta}{\sqrt{\eta\xi}} e^{iv}|} - (m_0 + m_1)\right), \end{aligned} \tag{5.18}$$

where the functions v and $\tilde{\rho}$ are evaluated at

$$(\ell, L, \Gamma) = \left(\lambda + \frac{i}{2}\log\frac{\eta}{\xi}, L, L - \eta\xi\right). \tag{5.19}$$

In particular, by (6.7) and (5.15), the eccentricity is given by

$$e_c = \frac{1}{L}\sqrt{\eta\xi}\sqrt{2L - \eta\xi}.$$

The associated equations are

$$\begin{aligned} \lambda' &= \frac{\nu}{L^3} - \frac{\Theta - L + \eta\xi}{\widetilde{r}^2} + \partial_L \widetilde{W}, & L' &= -\partial_\lambda \widetilde{W} \\ \eta' &= -i\frac{\Theta - L + \eta\xi}{\widetilde{r}^2}\eta - i\partial_\xi \widetilde{W}, & \xi' &= i\frac{\Theta - L + \eta\xi}{\widetilde{r}^2}\xi + i\partial_\eta \widetilde{W} \\ \widetilde{r}' &= \widetilde{y}, & \widetilde{y}' &= \frac{(\Theta - L + \eta\xi)^2}{\widetilde{r}^3} - \frac{1}{\widetilde{r}^2} - \partial_r \widetilde{W}. \end{aligned} \tag{5.20}$$

Remark 5.3.1. *Notice that the Hamiltonian* (6.12) *was not analytic at a neighborhood of circular motions for the two first bodies, that is* $L = \Gamma$*. Nevertheless, it is well known that once this Hamiltonian is expressed in Poincaré variables, that is Hamiltonian* (5.16)*, the system becomes* analytic *for* (η, ξ) *in a neighborhood of* $(0, 0)$*. See, for instance, [Féj13].*

5.4 The manifold at infinity and the associated invariant manifolds

The Hamiltonian $\widetilde{\mathcal{K}}$ in (5.16) has an invariant manifold at infinity. Indeed, the potential $\widetilde{W}$ in (5.18) satisfies $\widetilde{W} = \mathcal{O}(L^4/\widetilde{r}^3)$. Therefore, the manifold

$$\mathcal{P}_\infty = \{(\lambda, L, \eta, \xi, \widetilde{r}, \widetilde{y}) : \ \widetilde{r} = +\infty, \ \widetilde{y} = 0\}$$

is invariant[2].

Note that, at $\mathcal{P}_\infty$, the Hamiltonian $\widetilde{\mathcal{K}}$ satisfies

$$\widetilde{\mathcal{K}}_{|\mathcal{P}_\infty} = -\frac{\nu}{2L^2}$$

and $\dot{L}_{|\mathcal{P}_\infty} = 0$. Therefore, we can fix $L = L_0$ and restrict to an energy level $\widetilde{\mathcal{K}} = -\frac{\nu}{2L_0^2}$. We consider the restricted infinity manifold

$$\mathcal{E}_\infty = \mathcal{P}_\infty \cap \widetilde{\mathcal{K}}^{-1}\left(-\frac{\nu}{2L_0^2}\right) = \{(\lambda, L, \eta, \xi, \widetilde{r}, \widetilde{y}) : L = L_0, \ \widetilde{r} = +\infty, \ \widetilde{y} = 0, \ (\eta, \xi) \in \mathcal{U}, \ \lambda \in \mathbb{T}\}, \tag{5.21}$$

where $\mathcal{U} \subset \mathbb{R}^2$ is an open set containing the origin[3] which is specified below. By the particular form of the Hamiltonian $\widetilde{\mathcal{K}}$ in (5.16), it is clear that the manifold $\mathcal{E}_\infty$ is foliated by periodic orbits as

$$\mathcal{E}_\infty = \bigcup_{(\eta_0, \xi_0) \in \mathcal{U}} P_{\eta_0, \xi_0}$$

with

$$P_{\eta_0, \xi_0} = \{(\lambda, L, \eta, \xi, \widetilde{r}, \widetilde{y}) : \quad \eta = \eta_0, \ \xi = \xi_0, \ L = L_0, \ \widetilde{r} = +\infty, \ \widetilde{y} = 0, \ \lambda \in \mathbb{T}\},$$

[2]To analyze this manifold properly, one should consider McGehee coordinates $\widetilde{r} = 2/x^2$. This is done in Section 5.5.
[3]Observe that $(\eta, \xi) \in \mathbb{C}^2$ but they satisfy $\xi = \overline{\eta}$.

whose dynamics is given by

$$\lambda(t) = \lambda_0 + \frac{\nu}{L_0^3} t.$$

These periodic orbits are parabolic, in the sense that its linearization (in McGehee coordinates) is degenerate. Nonetheless, we will see Theorem 5.4.3 that they have stable and unstable invariant manifolds whose union form the invariant manifolds of the infinity manifold $\mathcal{E}_\infty$.

The goal of this section is to analyze the stable and unstable invariant manifolds of $\mathcal{E}_\infty$ and show that, restricting to suitable open domains of $\mathcal{E}_\infty$, they intersect transversally along two homoclinic channels Γ^1 and Γ^2 (see Figure 5.3). This will allow us to define two different scattering maps on suitable domains of $\mathcal{E}_\infty$.

5.4.1 The unperturbed Hamiltonian system

Since we are considering the regime $\widetilde{r} \gg L^2$ and $\widetilde{W}$ satisfies $\widetilde{W} = \mathcal{O}(L^4/\widetilde{r}^3)$, we first analyze the Hamiltonian $\widetilde{\mathcal{K}}$ in (5.16) with $\widetilde{W} = 0$. We consider this as *the unperturbed Hamiltonian*. In fact, when $\widetilde{W} = 0$, $\widetilde{\mathcal{K}}$ becomes integrable and therefore the invariant manifolds of the periodic orbits P_{η_0,ξ_0} coincide.

Indeed, it is easy to check that L and $\eta\xi$ (and the Hamiltonian) are functionally independent first integrals. Therefore, if we restrict to the energy level $\widetilde{\mathcal{K}} = -\frac{\nu}{2L_0}$ and we define

$$G_0 = \Theta - L_0 + \eta_0\xi_0,$$

the invariant manifolds of any periodic orbit P_{η_0,ξ_0} should satisfy

$$\Theta - L + \eta\xi = G_0$$

and therefore they must be a solution of the equations

$$\begin{aligned} \lambda' &= \frac{\nu}{L_0^3} - \frac{G_0}{\widetilde{r}^2}, \\ \eta' &= -i\frac{G_0}{\widetilde{r}^2}\eta, \qquad \xi' = i\frac{G_0}{\widetilde{r}^2}\xi \\ \widetilde{r}' &= \widetilde{y}, \qquad\qquad \widetilde{y}' = \frac{G_0^2}{\widetilde{r}^3} - \frac{1}{\widetilde{r}^2}. \end{aligned} \tag{5.22}$$

The invariant manifolds of the periodic orbit P_{η_0,ξ_0} associated to equation (5.22) are analyzed in the next lemma.

Lemma 5.4.1. *The invariant manifolds of the periodic orbit P_{η_0,ξ_0} associated to equation* (5.22) *coincide along a homoclinic manifold which can be parameterized as*

$$\begin{aligned} \lambda &= \gamma + \phi_{\mathrm{h}}(u) \quad & L &= L_0 \\ \eta &= \eta_0 e^{i\phi_{\mathrm{h}}(u)} & \xi &= \xi_0 e^{-i\phi_{\mathrm{h}}(u)} \\ \widetilde{r} &= G_0^2 \widehat{r}_{\mathrm{h}}(u) & \widetilde{y} &= G_0^{-1}\widehat{y}_{\mathrm{h}}(u), \end{aligned} \tag{5.23}$$

where $(\widehat{r}_{\mathrm{h}}(u), \widehat{y}_{\mathrm{h}}(u), \phi_{\mathrm{h}}(u))$ *are defined as*

$$\begin{aligned} \widehat{r}_{\mathrm{h}}(u) &= r_0(\tau(u)), \qquad & r_0(\tau) &= \frac{1}{2}(\tau^2+1), \\ \widehat{y}_{\mathrm{h}}(u) &= y_0(\tau(u)), & y_0(\tau) &= \frac{2\tau}{(\tau^2+1)}, \\ \phi_{\mathrm{h}}(u) &= \phi_0(\tau(u)), & \phi_0(\tau) &= i\log\left(\frac{\tau - i}{\tau + i}\right), \end{aligned} \tag{5.24}$$

where $\tau(u)$ *is obtained through*

$$u = \frac{1}{2}\left(\frac{1}{3}\tau^3 + \tau\right).$$

In particular

$$e^{i\phi_0(\tau)} = \frac{\tau+i}{\tau-i}$$

and ϕ_h *satisfies*

$$\lim_{u\to\pm\infty}\phi_\mathrm{h}(u) = 0 \ (mod\ 2\pi) \qquad and \qquad \phi_\mathrm{h}(0)=\pi. \tag{5.25}$$

Moreover, the dynamics in the homoclinic manifold (5.23) *is given by*

$$u' = G_0^{-3}, \quad \gamma' = \frac{\nu}{L_0^3}.$$

Note that the dynamics in this homoclinic manifold makes apparent the slow-fast dynamics. Indeed the motion on the $(\widetilde{r},\widetilde{y})$ variables is much slower than the rotation dynamics in the λ variable.

Proof of Lemma 5.4.1. To prove this lemma it is convenient to scale the variables and time as

$$\widetilde{r} = G_0^2\widehat{r}, \ \ \widetilde{y} = G_0^{-1}\widehat{y}, \quad \text{and} \quad t = G_0^3 s \tag{5.26}$$

in equation (5.22) to obtain

$$\begin{aligned}
\frac{d\lambda}{ds} &= \frac{\nu G_0^3}{L^3} - \frac{1}{\widehat{r}^2}, \\
\frac{d\eta}{ds} &= -\frac{i}{\widehat{r}^2}\eta, \qquad & \frac{d\xi}{ds} &= \frac{i}{\widehat{r}^2}\xi \\
\frac{d\widehat{r}}{ds} &= \widehat{y}, & \frac{d\widehat{y}}{ds} &= \frac{1}{\widehat{r}^3} - \frac{1}{\widehat{r}^2}.
\end{aligned} \tag{5.27}$$

The last two equations are Hamiltonian with respect to

$$h(\widehat{r},\widehat{y}) = \frac{\widehat{y}^2}{2} + \frac{1}{2\widehat{r}^2} - \frac{1}{\widehat{r}} \tag{5.28}$$

and, following [LS80a], they have a solution $(\widehat{r}_\mathrm{h}(s),\widehat{y}_\mathrm{h}(s))$ as given in (5.24) which satisfies

$$\lim_{s\to\pm\infty}(\widehat{r}_\mathrm{h}(s),\widehat{y}_\mathrm{h}(s)) = (\infty,0) \quad \text{and} \quad \widehat{y}_\mathrm{h}(0)=0. \tag{5.29}$$

Moreover, for the (η,ξ) components, it is enough to define the function $\phi_\mathrm{h}(s)$ which satisfies

$$\frac{d\phi}{ds} = -\frac{1}{\widehat{r}_h^2} \qquad \text{and} \qquad \phi_\mathrm{h}(0)=\pi.$$

Following again [LS80a], it is given in (5.24) and satisfies the asymptotic conditions in (5.25).

To complete the proof of the lemma it is enough to integrate the rest of equations in (5.27) and undo the scaling (5.26). □

Observe that the union of the homoclinic manifolds of the periodic orbits P_{η_0,ξ_0} form the homoclinic manifold of the infinity manifold (restricted to the energy level) $\mathcal{E}_\infty$, which is four dimensional.

5.4.2 The invariant manifolds for the perturbed Hamiltonian

In this section we analyze the invariant manifolds of the infinity manifold $\mathcal{E}_\infty$ (see (5.21)) and their intersections for the full Hamiltonian $\widetilde{\mathcal{K}}$ in (5.16) (that is, incorporating the potential $\widetilde{W}$ in (5.18)). Given a periodic orbit $P_{\eta_0,\xi_0}\in\mathcal{E}_\infty$, we want to study its 2 dimensional unstable manifold and its possible intersections with the 2 dimensional stable manifold of nearby periodic orbits

$$P_{\eta_0+\delta\eta,\xi_0+\delta\xi}\in\mathcal{E}_\infty \qquad \text{for some} \qquad |\delta\eta|,|\delta\xi|\ll 1.$$

This will lead to heteroclinic connections and, therefore, to the definition of scattering maps. To this end, we consider parameterizations of a rather particular form. The reason, as it is explained in Section 5.7.1

below, is to keep track of the symplectic properties of these parameterizations. Using the unperturbed parameterization introduced in Lemma 5.4.1 and the constant

$$G_0 = \Theta - L_0 + \eta_0\xi_0,$$

we define parameterizations of the following form, where $*$ stands for $* = u, s$,

$$\begin{aligned}
\lambda &= \gamma + \phi_{\rm h}(u)\\
L^*(u,\gamma) &= L_0 + \Lambda^*(u,\gamma)\\
\eta^*(u,\gamma) &= e^{i\phi_{\rm h}(u)}(\eta_0 + \alpha^*(u,\gamma))\\
\xi^*(u,\gamma) &= e^{-i\phi_{\rm h}(u)}(\xi_0 + \beta^*(u,\gamma))\\
\widetilde{r} &= G_0^2\widehat{r}_{\rm h}(u)\\
\widetilde{y}^*(u,\gamma) &= \frac{\widehat{y}_{\rm h}(u)}{G_0} + \frac{Y^*(u,\gamma)}{G_0^2\widehat{y}_{\rm h}(u)} + \frac{\Lambda^*(u,\gamma) - (\eta_0 + \alpha^*(u,\gamma))(\xi_0 + \beta^*(u,\gamma)) + \eta_0\xi_0}{G_0^2\widehat{y}_{\rm h}(u)(\widehat{r}_{\rm h}(u))^2}
\end{aligned} \tag{5.30}$$

where the functions $\Lambda^*, \alpha^*, \beta^*, Y^*$ satisfy

$$(\Lambda^u(u,\gamma), \alpha^u(u,\gamma), \beta^u(u,\gamma), (\widehat{y}_{\rm h}(u))^{-1}Y^u(u,\gamma)) \to (0,0,0,0), \quad \text{as} \quad u \to -\infty$$
$$(\Lambda^s(u,\gamma), \alpha^s(u,\gamma), \beta^s(u,\gamma), (\widehat{y}_{\rm h}(u))^{-1}Y^s(u,\gamma)) \to (0,\delta\eta,\delta\xi,0), \quad \text{as} \quad u \to +\infty.$$

The rather peculiar form of these parameterizations relies on the fact that one can interpret them through the change of coordinates given by

$$(\lambda, L, \eta, \xi, \widetilde{r}, \widetilde{y}) \to (\gamma, \Lambda, \alpha, \beta, u, Y).$$

Then, one can keep track of the symplectic properties of the invariant manifolds since this change is symplectic in the sense that it sends the canonical form into $d\gamma \wedge d\Lambda + i d\alpha \wedge d\beta + du \wedge dY$. This is explained in full detail in Section 5.7.1 (see the symplectic transformation (5.71)).

If the functions $(Y^*, \Lambda^*, \alpha^*, \beta^*)$ are small, as stated in Theorem 5.4.3 below, these parameterizations are close to those of the unperturbed problem, given in Lemma 5.4.1. Furthermore, note that, to analyze the difference between the invariant manifolds, it is enough to measure the differences

$$(Y^s - Y^u, \Lambda^s - \Lambda^u, \alpha^s - \alpha^u, \beta^s - \beta^u) \tag{5.31}$$

for u in a suitable interval and $\gamma \in \mathbb{T}$. The zeros of this difference will lead to homoclinic connections to P_{η_0,ξ_0}, if one chooses $\delta\eta = \delta\xi = 0$, and to heteroclinic connections between P_{η_0,ξ_0} and $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$, otherwise.

The analysis of the difference (5.31) is done in Proposition 5.4.2 and Theorem 5.4.3 below. First, in Proposition 5.4.2, we define a Melnikov potential, which provides the first order of the difference between the invariant manifolds through the difference (5.31). Then, Theorem 5.4.3 gives the existence of parameterizations of the form (5.30) for the unstable manifold of P_{η_0,ξ_0} and the stable manifold of $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$ and shows that, indeed, the derivatives of the Melnikov potential given in Proposition 5.4.2 plus an additional explicit term depending on $(\delta\eta, \delta\xi)$ gives the first order of their difference when the parameter Θ is large enough.

We then introduce a Melnikov potential

$$\mathcal{L}(\sigma, \eta_0, \xi_0) = G_0^3 \int_{-\infty}^{+\infty} \widetilde{W}\left(\sigma + \omega s + \phi_{\rm h}(s), L_0, e^{i\phi_{\rm h}(s)}\eta_0, e^{-i\phi_{\rm h}(s)}\xi_0, G_0^2\widehat{r}_{\rm h}(s)\right) ds, \tag{5.32}$$

where $\widetilde{W}$ is given in (5.18), $(\widehat{r}_{\rm h}(u), \phi_{\rm h}(u))$ are introduced in Lemma 5.4.1 and

$$\omega = \frac{\nu G_0^3}{L_0^3}, \quad \text{with} \quad G_0 = \Theta - L_0 + \eta_0\xi_0. \tag{5.33}$$

Note that, as usual, it is just the integral of the perturbing potential $\widetilde{W}$ evaluated at the unperturbed homoclinic manifold (5.23).

To provide asymptotic formulas for the Melnikov potential $\mathcal{L}$ we use the parameter

$$\tilde{\Theta} = \Theta - L_0. \tag{5.34}$$

Proposition 5.4.2. *Fix $L_0 \in [1/2, 2]$. Then, there exists $\Theta^* \gg 1$ and $0 < \varrho^* \ll 1$ such that for $\Theta \geq \Theta^*$ and (η_0, ξ_0) satisfying $\xi_0 = \overline{\eta_0}$ and $|\eta_0|\Theta^{3/2} \leq \varrho^*$, the Melnikov potential introduced in* (5.32) *is 2π-periodic in σ and can be written as*

$$\mathcal{L}(\sigma, \eta_0, \xi_0) = \mathcal{L}^{[0]}(\eta_0, \xi_0) + \mathcal{L}^{[1]}(\eta_0, \xi_0)e^{i\sigma} + \mathcal{L}^{[-1]}(\eta_0, \xi_0)e^{-i\sigma} + \mathcal{L}^{\geq}(\sigma, \eta_0, \xi_0),$$

and the Fourier coefficients satisfy $\mathcal{L}^{[q]}(\eta_0, \xi_0) = \overline{\mathcal{L}^{[-q]}}(\xi_0, \eta_0)$ and

$$\mathcal{L}^{[0]}(\eta_0, \xi_0) = \tilde{\nu}\pi L_0^4(\tilde{\Theta} + \eta_0\xi_0)^{-3}\left[\frac{N_2}{8}\left(1 + 3\frac{\eta_0\xi_0}{L_0} - \frac{3}{2}\frac{\eta_0^2\xi_0^2}{L_0^2}\right) - N_3\frac{15}{64}\frac{L_0^2}{\sqrt{2L_0}}\tilde{\Theta}^{-2}(\eta_0 + \xi_0) + \mathcal{R}_0(\eta_0, \xi_0)\right]$$

$$\mathcal{L}^{[1]}(\eta_0, \xi_0) = \tilde{\nu}\mathrm{e}^{-\frac{\tilde{\nu}(\tilde{\Theta}+\eta_0\xi_0)^3}{3L_0^3}}\left[\frac{N_3}{32}\sqrt{\frac{\pi}{2}}L_0^6\tilde{\Theta}^{-\frac{1}{2}} - 3\frac{N_2}{4}\sqrt{\pi}L_0^{\frac{7}{2}}\tilde{\Theta}^{\frac{3}{2}}\eta_0 + \mathcal{R}_1(\eta_0, \xi_0)\right]$$

where $\tilde{\nu}$ is the constant introduced in (5.8) *and*

$$N_2 = \frac{m_2^4(m_0 + m_1)^5}{m_0^3 m_1^3}, \qquad N_3 = \frac{m_2^6(m_0 + m_1)^7}{m_0^5 m_1^5}(m_1 - m_0), \tag{5.35}$$

and

$$\mathcal{R}_0(\eta_0, \xi_0) = \mathcal{O}\left(\Theta^{-4}\right) + \mathcal{O}\left(\Theta^{-2}|\eta_0|^3\right), \qquad \mathcal{R}_1(\eta_0, \xi_0) = \mathcal{O}\left(\Theta^{-1}, |\eta_0|, |\eta_0|^2\Theta^{5/2}\right)$$

and, for $i, j \geq 1$,

$$\left|\partial_{\eta_0}^i\partial_{\xi_0}^j\mathcal{R}_0(\eta_0, \xi_0)\right| \leq C(i, j)\Theta^{-2}, \qquad \left|\partial_{\eta_0}^i\partial_{\xi_0}^j\mathcal{R}_1(\eta_0, \xi_0)\right| \leq C(i, j)\Theta^{(-1+3i+3j)/2},$$

for some constants $C(i, j)$ independent of Θ.

Moreover, for $i, j \geq 0$, $k \geq 1$,

$$|\partial_{\eta_0}^i\partial_{\xi_0}^j\partial_\sigma^k\mathcal{L}^{\geq}| \leq C(i, j, k)\Theta^{7/2+3(i+j)/2}\mathrm{e}^{-\frac{2\tilde{\nu}\tilde{\Theta}^3}{3L_0^3}}.$$

where $C(i, j, k)$ is a constant independent of Θ.

This proposition is proven in Appendix 5.C.

The next theorem gives an asymptotic formula for the diference between the unstable manifold of the periodic orbit P_{η_0,ξ_0} and the stable manifold of the periodic orbit $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$, which is measured by (5.31).

Theorem 5.4.3. *Fix $L_0 \in [1/2, 2]$ and u_1, u_2 such that $u_1 > u_2 > 0$. Then, there exists $\Theta^* \gg 1$ and $0 < \varrho^* \ll 1$ such that for $\Theta \geq \Theta^*$, (η_0, ξ_0) satisfying $\xi_0 = \overline{\eta_0}$ and $|\eta_0|\Theta^{3/2} \leq \varrho^*$ and $(\delta\eta, \delta\xi)$ satisfying $\delta\xi = \overline{\delta\eta}$ and $|\delta\eta|\Theta^3 \leq \varrho^*$, the unstable manifold of P_{η_0,ξ_0} and the stable manifold of $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$ can be parameterized as graphs with respect to $(u, \gamma) \in (u_1, u_2) \times \mathbb{T}$ as in* (5.30) *for some functions $(Y^*, \Lambda^*, \alpha^*, \beta^*)$, $* = u, s$ which satisfy*

$$|Y^*| \leq \Theta^{-3}, \quad |\Lambda^*| \leq C\Theta^{-6}, \quad |\alpha^*| \leq \Theta^{-3}, \quad |\beta^*| \leq C\Theta^{-3}. \tag{5.36}$$

Moreover, its difference satisfies

$$\begin{pmatrix} Y^u(u,\gamma,z_0) - Y^s(u,\gamma,z_0,\delta z) \\ \Lambda^u(u,\gamma,z_0) - \Lambda^s(u,\gamma,z_0,\delta z) \\ \alpha^u(u,\gamma,z_0) - \alpha^s(u,\gamma,z_0,\delta z) \\ \beta^u(u,\gamma,z_0) - \beta^s(u,\gamma,z_0,\delta z) \end{pmatrix} = \mathcal{N}(u,\gamma,z_0,\delta z)\begin{pmatrix} \mathcal{M}_Y(u,\gamma,z_0,\delta z) \\ \mathcal{M}_\Lambda(u,\gamma,z_0,\delta z) \\ \delta\eta + \mathcal{M}_\alpha(u,\gamma,z_0,\delta z) \\ \delta\xi + \mathcal{M}_\beta(u,\gamma,z_0,\delta z) \end{pmatrix} \tag{5.37}$$

where $z_0 = (\eta_0, \xi_0)$, $\delta z = (\delta\eta, \delta\xi)$, $\mathcal{N}$ *is a matrix which satisfies*

$$\mathcal{N} = \mathrm{Id} + \mathcal{O}(\Theta^{-3}) \tag{5.38}$$

and the vector $\mathcal{M}$ *is of the form*

$$\begin{pmatrix} \mathcal{M}_Y(u,\gamma,z_0,\delta z) \\ \mathcal{M}_\Lambda(u,\gamma,z_0,\delta z) \\ \mathcal{M}_\alpha(u,\gamma,z_0,\delta z) \\ \mathcal{M}_\beta(u,\gamma,z_0,\delta z) \end{pmatrix} = \begin{pmatrix} \omega\partial_\sigma \mathcal{L}(\gamma - \omega u, z_0) + \mathcal{O}\left(\Theta^{-6}\ln^2\Theta\right) \\ -\partial_\sigma \mathcal{L}(\gamma - \omega u, z_0) + \mathcal{O}\left(\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}}\Theta^{-5/2}\ln^2\Theta\right) \\ -i\partial_{\xi_0}\mathcal{L}(\gamma - \omega u, z_0) + \mathcal{O}\left(\Theta^{-6}\right) \\ i\partial_{\eta_0}\mathcal{L}(\gamma - \omega u, z_0) + \mathcal{O}\left(\Theta^{-6}\right), \end{pmatrix} \tag{5.39}$$

where $\mathcal{L}$ *is the Melnikov potential introduced in* (5.32) *and* ω *is given in* (5.33).

Moreover, the function $\mathcal{M}$ *satisfies the following estimates*

$$\begin{aligned} |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^k_\gamma(\mathcal{M}_Y + \omega\partial_\sigma\mathcal{L})| &\leq C(i,j,k)\Theta^{-6} \\ |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^k_\gamma(\mathcal{M}_\alpha + i\partial_{\xi_0}\mathcal{L})|, |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^k_\gamma(\mathcal{M}_\beta - i\partial_{\eta_0}\mathcal{L})| &\leq C(i,j,k)\Theta^{-6}. \end{aligned} \tag{5.40}$$

Furthermore, both for the derivatives of the component $\mathcal{M}_\Lambda$ *and the* γ*-derivatives of the other components one has the following exponentially small estimates. For any* $i, j, k \geq 0$

$$\begin{aligned} |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^k_\gamma(\mathcal{M}_\Lambda + \partial_\sigma\mathcal{L})| &\leq C(i,j,k)\Theta^{-5/2+3(i+j)/2}\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}}\log^2\Theta \\ |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^{k+1}_\gamma(\mathcal{M}_Y - \omega\partial_\sigma\mathcal{L})| &\leq C(i,j,k)\Theta^{2-3(i+j)/2}\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}} \\ |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^{k+1}_\gamma(\mathcal{M}_\alpha + i\partial_{\eta_0}\mathcal{L})|, |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^{k+1}_\gamma(\mathcal{M}_\beta - i\partial_{\xi_0}\mathcal{L})| &\leq C(i,j,k)\Theta^{1/2-3(i+j)/2}\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}}. \end{aligned} \tag{5.41}$$

Note that for $\mathcal{M}_Y$ the error in (5.39) and (5.40) is bigger than the first order given by the Melnikov potential. Modifying slightly the matrix $\mathcal{N}$ this error could be made smaller. However, this is not needed. The reason is that, by conservation of energy, one does not need to take care of the distance between the invariant manifolds on the Y component.

Remark 5.4.4. *The estimates in Proposition 5.4.2 and the bounds* (5.40) *imply the following estimates that are needed for analyzing the scattering maps in Section 5.4.3,*

$$\begin{aligned} &|D^N_{\eta_0,\xi_0}\mathcal{M}_Y| \leq C(N)\Theta^{-6} \\ &|D^2_{\eta_0,\xi_0}\mathcal{M}_x| \lesssim \Theta^{-5} + \Theta^{-3}(|\eta_0| + |\xi_0|) \quad \text{for} \quad x = \alpha, \beta \\ &|D^N_{\eta_0,\xi_0}\mathcal{M}_x| \leq C(N)\Theta^{-3} \quad \text{for} \quad x = \alpha, \beta \ \text{and} \ N \geq 1,\ N \neq 2, \end{aligned}$$

where $C(N)$ *is a constant only depending on* N.

Analogously Proposition 5.4.2 and the bounds in (5.41) *imply*

$$|\partial^k_\gamma\mathcal{M}_\Lambda| \leq C(k)(\Theta^{-1/2} + \Theta^{-3/2}|\eta_0|)\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}}, \qquad |\partial_{\eta_0}\mathcal{M}_\Lambda| \lesssim \Theta^{3/2}\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}}, \qquad |\partial_{\xi_0}\mathcal{M}_\Lambda| \lesssim \Theta^{3/2}\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}}$$

and, when $i + j \geq 1$,

$$|\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^k_\gamma\mathcal{M}_\Lambda| \leq C(i,j,k)\Theta^{3(i+j)/2}\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}},$$

where $C(k)$ *and* $C(i,j,k)$ *are independent of* Θ.

Finally, $i, j, k \geq 0$,

$$\begin{aligned} |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^{k+1}_\gamma\mathcal{M}_Y| &\leq C(i,j,k)\Theta^{3-3(i+j)/2}\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}} \\ |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^{k+1}_\gamma\mathcal{M}_\alpha|, |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^{k+1}_\gamma\mathcal{M}_\beta| &\leq C(i,j,k)\Theta^{3/2-3(i+j)/2}\mathrm{e}^{-\frac{\tilde\nu\tilde\Theta^3}{3L_0^3}}. \end{aligned}$$

Theorem 5.4.3 is proved in several steps. First, in Section 5.7, we prove the existence of parameterizations of the form (5.30) for the invariant manifolds. These parameterizations are analyzed in complex domains. They fail to exist at $u = 0$ since at this point, when written in the original coordinates $(\lambda, L, \eta, \xi, \widetilde{r}, \widetilde{y})$, the unperturbed invariant manifold is not a graph over the variables $\widetilde{r}, \lambda$. Thus, in Section 5.7.5, we extend the unstable invariant manifold using a different parameterization. This allows us to have at the end a common domain (intersecting the real line) where both manifolds have graph parameterizations as (5.30). Finally in Section 5.8 we analyze the difference between the invariant manifolds and complete the proof of Theorem 5.4.3.

5.4.3 The scattering maps associated to the invariant manifolds of infinity

Once we have the asymptotic formulas (5.37) for the difference of the stable and unstable manifolds for nearby periodic orbits in $\mathcal{E}_\infty$, next step is to look for their intersections to find heteroclinic connections between different periodic orbits. This will allow us to define scattering maps in suitable domains of $\mathcal{E}_\infty$. Now we provide two homoclinic channels and the two associated scattering maps whose domains inside $\mathcal{E}_\infty$ overlap. The construction of the homoclinic channels relies on certain non-degeneracies of the Melnikov potential analyzed in Proposition 5.4.2. In particular, we need non-trivial dependence on the angle γ. If one analyzes the first γ-Fourier coefficient of the potential $\mathcal{L}(\gamma - \omega u, \eta_0, \xi_0)$ given by (5.32), one can easily see that it vanishes at a point of the form

$$\overline{\xi_{\mathrm{bad}}} = \eta_{\mathrm{bad}} = \frac{N_3}{24\sqrt{2}N_2} L_0^{5/2}\tilde{\Theta}^{-2} + \mathcal{O}\left(\tilde{\Theta}^{-5/2}\right).$$

Therefore, we will be able to define scattering maps for $|\eta_0| \ll \Theta^{-3/2}$ and $\xi_0 = \overline{\eta_0}$ (that is the domain considered in Theorem 5.4.3 minus a small ball around the point $(\eta_{\mathrm{bad}}, \overline{\eta_{\mathrm{bad}}})$.

The main idea behind Theorem 5.4.5 is the following: We fix a section $u = u^*$ and, for (η_0, ξ_0) in the good domain $\widetilde{\mathbb{D}}$ introduced below (see (5.44)), we analyze the zeros of equations (5.37), which lead to two solutions

$$\gamma^j = \gamma^j(u^*, \eta_0, \xi_0),\ \delta\xi^j = \delta\xi^j(u^*, \eta_0, \xi_0),\ \delta\eta^j = \delta\eta^j(u^*, \eta_0, \xi_0),\ j = 1, 2.$$

These solutions provide two heteroclinic points through the parameterization (5.30) as

$$\begin{aligned} z^j_{\mathrm{het}} &= z^j_{\mathrm{het}}(u^*, \eta_0, \xi_0) = (\lambda_{\mathrm{het}}, L_{\mathrm{het}}, \eta_{\mathrm{het}}, \xi_{\mathrm{het}}, \widetilde{r}_{\mathrm{het}}, \tilde{y}_{\mathrm{het}}) \\ &= (\lambda, L^u, \eta^u, \xi^u, \widetilde{r}, \tilde{y}^u)\,(u^*, \gamma^j(u^*, \eta_0, \xi_0)) \in W^u(P_{\eta_0,\xi_0}) \cap W^s(P_{\eta_0+\delta\eta^j, \xi_0+\delta\xi^j}). \end{aligned}$$

Varying (η_0, ξ_0) and u, one has two 3–dimensional homoclinic channels which define homoclinic manifolds to infinity. These channels are defined by

$$\Gamma^j = \left\{ z^j_{\mathrm{het}}(u, \eta_0, \xi_0) :\ u \in (u_1, u_2), (\eta_0, \xi_0) \in \widetilde{\mathbb{D}} \right\} \tag{5.42}$$

and associated to these homoclinic channels one can define scattering maps which are analyzed in the next theorem.

To define the domain $\widetilde{\mathbb{D}}$ of the scattering maps we introduce the notation

$$\mathbb{D}_\rho(\eta_0, \xi_0) = \left\{ w \in \mathbb{R}^2 : |(\eta, \xi) - (\eta_0, \xi_0)| < \rho \right\}. \tag{5.43}$$

Theorem 5.4.5. *Assume that $m_0 \neq m_1$. Fix $L_0 \in [1/2, 2]$ and $0 < \varrho \ll \varrho^*$ where ϱ^* is the constant introduced in Theorem 5.4.3. Then, there exists $\Theta^* \gg 1$ such that, if $\Theta \geq \Theta^*$, one can define scattering maps*

$$\widetilde{\mathcal{S}}^j : \mathbb{T} \times \left[\frac{1}{2}, 1\right] \times \widetilde{\mathbb{D}} \to \mathbb{T} \times \left[\frac{1}{2}, 1\right] \times \mathbb{C}, \quad j = 1, 2$$

where (see Figure 5.8)

$$\widetilde{\mathbb{D}} = \mathbb{D}_{\varrho\Theta^{-3/2}}(0, 0) \setminus \mathbb{D}_{\varrho\Theta^{-2}}(\eta_{\mathrm{bad}}, \overline{\eta_{\mathrm{bad}}}), \tag{5.44}$$

associated to the homoclinic channels Γ^j *introduced in* (5.42). *These scattering maps are of the form*

$$\widetilde{\mathcal{S}}^j(\lambda, L_0, \eta_0, \xi_0) = \begin{pmatrix} \lambda + \Delta_j(\eta_0, \xi_0) \\ L_0 \\ \mathcal{S}^j(\eta_0, \xi_0) \end{pmatrix}$$

where $\mathcal{S}^j$ *is independent of* λ *but may depend*[4] *on* L_0 *and is given by*

$$\mathcal{S}^j(\eta_0, \xi_0) = \begin{pmatrix} \eta_0 - i\widetilde{\nu}\pi L_0^4(\tilde{\Theta} + \eta_0\xi_0)^{-3}\left[A_1\eta_0 + 2A_2\eta_0^2\xi_0 + A_3\tilde{\Theta}^{-2}\right] \\ \xi_0 + i\widetilde{\nu}\pi L_0^4(\tilde{\Theta} + \eta_0\xi_0)^{-3}\left[A_1\xi_0 + 2A_2\eta_0\xi_0^2 + A_3\tilde{\Theta}^{-2}\right] \end{pmatrix} + \mathcal{R}^j(\eta_0, \xi_0) \tag{5.45}$$

where $\tilde{\Theta} = \Theta - L_0$,

$$A_1 = \frac{3N_2}{8L_0}, \quad A_2 = -\frac{3N_2}{16L_0^2}, \quad A_3 = -N_3\frac{15}{\sqrt{2}64}L_0^{\frac{3}{2}}. \tag{5.46}$$

(see (5.35)*) and* $\mathcal{R}^j$ *satisfies*

$$\mathcal{R}^j(\eta_0, \xi_0) = \mathcal{O}\left(\Theta^{-5}, \Theta^{-4}|\eta_0|\right)$$

Moreover,

- $\mathcal{S}^j$ *is symplectic in the sense that it preserves the symplectic form* $d\eta_0 \wedge d\xi_0$.
- *Fix* $N \geq 3$. *Then, the derivatives of* $\mathcal{R}^j$ *satisfy*

  $$|D^k\mathcal{R}^j(z)| \leq C(k)\Theta^{-5}, \ k = 1 \dots N$$

  for $z \in \widetilde{\mathbb{D}}$, *where* $C(k)$ *is a constant which may depend on* k *but is independent of* Θ.
- *There exists points* $z_0^j = (\eta_0^j, \xi_0^j)$, $j = 1, 2$, *of the form*

  $$\xi_0^j = \overline{\eta_0^j} \qquad \text{and} \qquad \eta_0^j = \frac{5N_3}{8N_2}\frac{L_0^3}{\sqrt{2L_0}}\tilde{\Theta}^{-2} + \mathcal{O}\left(\Theta^{-3}\log^2\Theta\right) \tag{5.47}$$

  where N_2 *and* N_3 *are the constants introduced in* (5.35), *such that* $\mathcal{S}^j(\eta_0^j, \xi_0^j) = (\eta_0^j, \xi_0^j)$. *Furthermore, the distance between these two fixed points is exponentially small as*

  $$\eta_0^2 - \eta_0^1 = \overline{\xi_0^2} - \overline{\xi_0^1} = -\frac{4}{\sqrt{\pi}}L_0^{1/2}\tilde{\Theta}^{9/2}\mathrm{e}^{-\frac{\tilde{\nu}\tilde{\Theta}^3}{3L_0^3}}\left(1 + \mathcal{O}\left(\Theta^{-1}\ln^2\Theta\right)\right). \tag{5.48}$$

This theorem is proven in Section 5.9.1.

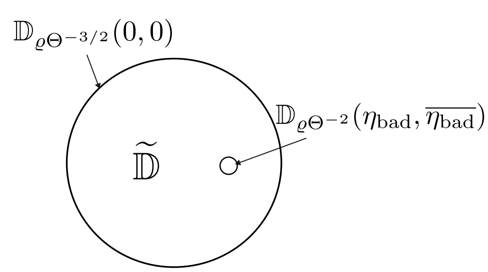


Figure 5.8: The domain $\widetilde{\mathbb{D}}$ in (5.44) of the scattering maps (see Theorem 5.4.5).

To analyze the return map from a neighborhood of infinity to itself along the homoclinic channels it is convenient to reduce the dimension of the model. To this end, we apply the classical Poincaré-Cartan

[4] To simplify the notation we omit the dependence of $\mathcal{S}^j$ on L_0. In fact, from now one we will restrict the scattering map to a level of energy. Since the energy determines L_0 it can be treated as a fixed constant

reduction. We fix the energy level $\widetilde{\mathcal{K}} = -\frac{\nu}{2L_0^2}$. Then, by the Implicit Function Theorem, we know that there exists a function $\mathtt{L}(\lambda, \eta, \xi, \widetilde{r}, \widetilde{y}; \Theta)$ satisfying

$$\widetilde{\mathcal{K}}\left(\lambda, -\mathtt{L}(\lambda, \eta, \xi, \widetilde{r}, \widetilde{y}), \eta, \xi, \widetilde{r}, \widetilde{y}\right) = -\frac{\nu}{2L_0^2}.$$

The function $\mathtt{L}$ depends also on Θ and L_0 (which can be treated now as a parameter). We omit this dependence to simplify the notation.

The function $\mathtt{L}$ can be seen as a Hamiltonian system of two and a half degrees of freedom with λ as time. Then, it is well known that the trajectories of $\mathtt{L}$ coincide with the trajectories of $\widetilde{\mathcal{K}}$ at the energy level $\widetilde{\mathcal{K}} = -\frac{\nu}{2L_0^2}$ (up to time reparameterization). From now on and, in particular, to analyze the return map from a neighborhood of infinity to itself (see Section 5.2), we consider the flow given by the Hamiltonian $\mathtt{L}$.

Recall that our goal is to construct a hyperbolic invariant set with symbolic dynamics for this return map by means of the usual isolating block construction (see Figure 5.4). To capture the hyperbolicity in the (η, ξ)-directions we must rely on the scattering maps. Indeed, in these directions, the dynamics close to infinity is close to the identity up to higher order (see Theorem 5.5.4 and the heuristics in Section 5.2.3) and, therefore, hyperbolicity can only be created through the dynamics along the invariant manifolds, which is encoded in the scattering maps. Thus, as a first step we construct an isolating block for a suitable (large) iterate of the scattering maps associated to $\mathtt{L}$.

Therefore, we need to compute them from the scattering maps $\widetilde{\mathcal{S}}_j$ obtained in Theorem 5.4.5 (restricted to the energy level $\widetilde{\mathcal{K}} = -\frac{\nu}{2L_0^2}$). Indeed, if we denote by

$$\check{\mathcal{S}}^j : \mathbb{T} \times \widetilde{\mathbb{D}} \subset \mathbb{T} \times \mathbb{C} \to \mathbb{T} \times \mathbb{C}, \quad j = 1, 2$$

the scattering maps associated to $\mathtt{L}$, they are of the form

$$\check{\mathcal{S}}^j(\lambda, \eta_0, \xi_0) = \begin{pmatrix} \lambda \\ \mathcal{S}^j(\eta_0, \xi_0) \end{pmatrix}.$$

where $\mathcal{S}^j$ are the functions introduced in (5.45). Note that the fact that λ is now time, implies that the corresponding component in the scattering map is the identity. Nevertheless, as the (η, ξ) coordinates of the periodic orbit in $\mathcal{E}_\infty$ do not evolve in time, the associated components of the scattering maps for the non-autonomous Hamiltonian $\mathtt{L}$ and the original one $\widetilde{\mathcal{K}}$ coincide.

Using Theorem 5.4.5 and Taylor-expanding the scattering maps around their fixed points, one can prove the following proposition.

Proposition 5.4.6. *Assume that $m_0 \neq m_1$, fix $N \geq 3$ and take $\Theta \gg 1$ large enough. Then, for $j = 1, 2$, the expansion of the scattering map $\mathcal{S}^j$ introduced in* (5.45) *around its fixed point z_0^j, obtained in Theorem 5.4.5, is of the form*

$$\mathcal{S}^j(z) = z_0^j + \mu_j(z - z_0^j) + \sum_{k=2}^{N} P_k\left(z - z_0^j\right) + \mathcal{O}\left(z - z_0^j\right)^{N+1}$$

where $z = (\eta, \xi)$ and

$$\mu_j = e^{i\omega_j} \qquad \text{with} \qquad \omega_j = \widetilde{\nu}\pi L_0^4 A_1 \tilde{\Theta}^{-3} + \mathcal{O}\left(\Theta^{-4}\right) \tag{5.49}$$

and P_k are homogeneous polynomials in $\eta - \eta_0^j$ and $\xi - \xi_0^j$ of degree k. Moreover they satisfy

$$P_2(z) = \sum_{i+j=2} b_{ij}^2 \xi^i \eta^j \qquad \text{with} \qquad b_{ij}^2 = \mathcal{O}\left(\Theta^{-5}\right)$$
$$P_3(z) = \mathcal{T}\tilde{\Theta}^{-3}|z|^2 z + \Theta^{-5}\mathcal{O}\left(z^3\right)$$

where

$$\mathcal{T} = -4i\widetilde{\nu}\pi L_0^4 A_2 + \mathcal{O}\left(\Theta^{-1}\right) \tag{5.50}$$

(see (5.46) *and* (5.35)*) satisfies $\mathcal{T} \neq 0$ and the coefficients of P_k for $k \geq 4$ are of order $\mathcal{O}\left(\Theta^{-3}\right)$.*

This proposition shows that the scattering maps are close to a rotation around an elliptic fixed point (with a very small frequency). For this reason it is rather hard to obtain hyperbolicity for any one of them. Instead, we obtain it for a suitable high iterate of a combination of the scattering maps. This is stated in the next theorem, whose prove is deferred to Section 5.9.2.

Theorem 5.4.7. *Assume that $m_0 \neq m_1$, fix $L_0 \in [1/2, 2]$ and take $\Theta \gg 1$ large enough. Then, there exists $0 < \widetilde{\kappa}_0 \ll 1$ and a change of coordinates*

$$\Upsilon : (-\widetilde{\kappa}_0, \widetilde{\kappa}_0)^2 \to \widetilde{\mathbb{D}}, \quad (\eta_0, \xi_0) = \Upsilon(\varphi, J)$$

where $\widetilde{\mathbb{D}}$ is the domain introduced in (5.44), such that the scattering maps

$$\widehat{\mathtt{S}}^j = \Upsilon^{-1} \circ \mathcal{S}^j \circ \Upsilon,$$

where $\mathcal{S}^j$ have been introduced in Theorem 5.4.5, satisfy the following statements.

1. *They are of the form*

$$\widehat{\mathtt{S}}^1(\varphi, J) = \begin{pmatrix} \varphi + \widetilde{B}(J) + \mathcal{O}\left(J^2\right) \\ J + \mathcal{O}\left(J^3\right) \end{pmatrix} \qquad \text{and} \qquad \widehat{\mathtt{S}}^2(\varphi, J) = \begin{pmatrix} \widehat{\mathtt{S}}^2_\varphi(\varphi, J) \\ \widehat{\mathtt{S}}^2_J(\varphi, J) \end{pmatrix}$$

   which satisfies

$$\mathtt{b} = \partial_\varphi \widehat{\mathtt{S}}^2_J(0,0) \neq 0 \qquad \text{and} \qquad \widehat{\mathtt{S}}^2_J(0, J) = 0 \quad \text{for} \quad J \in (-\widetilde{\kappa}_0, \widetilde{\kappa}_0).$$

2. *For any $0 < \widetilde{\kappa} \ll \widetilde{\kappa}_0$, there exists $M = M(\widetilde{\kappa})$ such that the rectangle*

$$\mathcal{R} = \left\{(\varphi, J) :\, 0 \leq \varphi \leq 2\mathtt{b}^{-1}\tilde{\kappa},\, 0 \leq J \leq \tilde{\kappa}\right\} \tag{5.51}$$

   is an isolating block for $\widehat{\mathtt{S}} = (\widehat{\mathtt{S}}^1)^M \circ \widehat{\mathtt{S}}^2$. Namely, if one considers a $\mathcal{C}^1$ curve $J = \gamma(\varphi)$ with $\gamma : [0, 2\mathtt{b}^{-1}\tilde{\kappa}] \to \mathbb{R}$ with $\gamma(\varphi) \in [0, \tilde{\kappa}]$, then, its image $(\varphi_1(\varphi), J_1(\varphi)) = \widehat{\mathtt{S}}(\varphi, \gamma(\varphi))$ is a graph over its horizontal component and satisfies that

$$J_1(\varphi) \in (0, \tilde{\kappa}), \quad \varphi_1(0) < 0 \quad \text{and} \quad \varphi_1(2\mathtt{b}^{-1}\tilde{\kappa}) > 2\mathtt{b}^{-1}\tilde{\kappa}.$$

3. *For $z = (\varphi, J) \in \mathcal{R}$, the matrix $D\widehat{\mathtt{S}}(z)$ is hyperbolic with eigenvalues $\lambda_{\widehat{\mathtt{S}}}(z), \lambda_{\widehat{\mathtt{S}}}(z)^{-1} \in \mathbb{R}$ with*

$$\lambda_{\widehat{\mathtt{S}}}(z) \gtrsim \tilde{\kappa}^{-1}$$

   Furthermore, there exist two vectors fields $V_j : \mathcal{R} \to T\mathcal{R}$ of the form

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} V_{21}(z) \\ 1 \end{pmatrix} \quad \text{with} \quad |V_{21}(z)| \lesssim \tilde{\kappa},$$

   which satisfy, for $z \in \mathcal{R}$,

$$D\widehat{\mathtt{S}}(z)V_1 = \lambda_{\widehat{\mathtt{S}}}(z)\left(V_1 + \widehat{V}_1(z)\right) \qquad \text{with} \qquad |\widehat{V}_1(z)| \lesssim \tilde{\kappa}$$

$$D\widehat{\mathtt{S}}(z)V_2(z) = \lambda_{\widehat{\mathtt{S}}}(z)^{-1}\left(V_2(\widehat{\mathtt{S}}(z)) + \widehat{V}_2(z)\right) \qquad \text{with} \qquad |\widehat{V}_2(z)| \lesssim \tilde{\kappa}.$$

5.5 Local behavior close to infinity and a parabolic Lambda Lemma

5.5.1 McGehee coordinates

To study the behavior of Hamiltonian $\widetilde{\mathcal{K}}$ in (5.16) close to the infinity manifold $\mathcal{E}_\infty$, we introduce the classical McGehee coordinates $\widetilde{r} = 2/x^2$. To simplify the notation, in this section we drop the tilde of $\widetilde{y}$. The Hamiltonian $\widetilde{\mathcal{K}}$ becomes

$$\mathcal{J}(\lambda, L, \eta, \xi, x, y; \Theta) = -\frac{\nu}{2L^2} + \frac{y^2}{2} + \frac{(\Theta - L + \eta\xi)^2}{2}\frac{x^4}{4} - \frac{x^2}{2} + V(\lambda, L, \eta, \xi, x)$$

where

$$V(\lambda, L, \eta, \xi, x) = \widetilde{W}\left(\lambda, L, \eta, \xi, \frac{2}{x^2}\right) = \mathcal{O}(x^6),$$

and $\widetilde{W}$ is the potential in (5.18), while the canonical symplectic form $d\lambda \wedge dL + id\eta \wedge d\xi + dr \wedge dy$ is transformed into

$$d\lambda \wedge dL + id\eta \wedge d\xi - \frac{4}{x^3} dx \wedge dy.$$

Hence, the equations of motion associated to $\mathcal{J}$ are

$$\begin{aligned}
\lambda' &= \partial_L \mathcal{J} = \frac{\nu}{L^3} + \mathcal{O}(x^4), & L' &= -\partial_\lambda \mathcal{J} = \mathcal{O}(x^6),\\
\eta' &= -i\partial_\xi \mathcal{J} = -\frac{i}{4}\left(\Theta - L + \eta\xi\right) x^4 \eta + \mathcal{O}(x^6), & \xi' &= i\partial_\eta \mathcal{J} = \frac{i}{4}\left(\Theta - L + \eta\xi\right) x^4 \xi + \mathcal{O}(x^6),\\
x' &= -\frac{x^3}{4}\partial_y \mathcal{J} = -\frac{1}{4}x^3 y, & y' &= -\frac{x^3}{4}\left(-\partial_x \mathcal{J}\right) = -\frac{1}{4}x^4 + \frac{(\Theta - L + \eta\xi)^2}{8} x^6 + \mathcal{O}(x^8).
\end{aligned}$$

In the new variables, the periodic orbits P_{η_0,ξ_0} in the energy level $\widetilde{\mathcal{K}} = -\frac{\nu}{2L_0^2}$ become

$$P_{\eta_0,\xi_0} = \{\lambda \in \mathbb{T},\ L = L_0,\ \eta = \eta_0,\ \xi = \xi_0,\ x = y = 0\}$$

for any $\eta_0, \xi_0 \in \mathbb{C}$ with $|\eta_0|, |\xi_0| \leq L_0^{1/2}$ (see (5.15)). To study the local behavior around $\mathcal{E}_\infty$, we consider the new variables

$$a = \eta\, e^{-i(\Theta - L + \eta\xi)y} \qquad \text{and} \qquad b = \xi\, e^{i(\Theta - L + \eta\xi)y}. \tag{5.52}$$

The equations of motion become

$$\begin{aligned}
\lambda' &= \partial_L \mathcal{J} = \frac{\nu}{L^3} + \mathcal{O}(x^4), & L' &= -\partial_\lambda \mathcal{J} = \mathcal{O}(x^6),\\
a' &= \mathcal{O}(x^6), & b' &= \mathcal{O}(x^6),\\
x' &= -\frac{1}{4}x^3 y, & y' &= -\frac{1}{4}x^4 + \frac{(\Theta - L + ab)^2}{8} x^6 + \mathcal{O}(x^8).
\end{aligned}$$

Remark 5.5.1. *Note that the change of coordinates* (5.52) *is the identity on* $\mathcal{E}_\infty$. *Therefore, Theorem 5.4.5 is still valid in these coordinates.*

As we have done in Section 5.4, we restrict to the energy level $\mathcal{J} = -\frac{\nu}{2L_0^2}$ and express L in terms of the rest of the variables in a neighborhood of $x = y = 0$. An immediate computation shows that L is an even function of x and y and

$$L = L_0 + \frac{1}{2\nu}(x^2 - y^2) + \mathcal{O}_2(x^2, y^2).$$

Taking λ as the new time and denoting the derivative with respect to this new time by a dot, we obtain the 2π-periodic equation

$$\begin{aligned}
\dot{x} &= -K_0 x^3 y \left(1 + B(x^2 - y^2) + \mathcal{O}_2(x^2, y^2)\right),\\
\dot{y} &= -K_0 x^4 \left(1 - (4A(z) + B)x^2 - By^2 + \mathcal{O}_2(x^2, y^2)\right),\\
\dot{z} &= \mathcal{O}(x^6),\\
\dot{t} &= 1,
\end{aligned}$$

where, abusing notation, we denote again by t the new time λ, $z = (a, b)$ belongs to a compact set and

$$\begin{aligned}
A(z) &= \frac{1}{8}\left(\Theta - L_0 + ab\right)^2 = \frac{\Theta^2}{8}\left(1 - \frac{1}{\Theta}\left(L_0 - ab\right)\right)^2,\\
B &= \frac{3}{2}\frac{1}{L_0 \nu}, \qquad K_0 = \frac{L_0^3}{4\nu}.
\end{aligned} \tag{5.53}$$

Observe that, since (a, b) belong to a compact set, taking Θ large enough, we can assume that

$$A = \frac{\Theta^2}{8}\left(1 - \frac{1}{\Theta}\left(L_0 - ab\right)\right)^2 > \frac{1}{16}\Theta^2 > 0.$$

Scaling x and y by $K_0^{1/3}$, one obtains

$$\begin{aligned} \dot{x} &= -x^3 y(1 + Bx^2 - By^2 + R_1(x, y, z, t)), \\ \dot{y} &= -x^4(1 + (B - 4A)x^2 - By^2 + R_2(x, y, z, t)), \\ \dot{z} &= R_3(x, y, z, t), \\ \dot{t} &= 1, \end{aligned} \tag{5.54}$$

where we have kept the notation x, y for the scaled variables and A,B for the scaled constants. Moreover,

1. the functions R_i, $i = 1, 2, 3$, are even in x,
2. $R_3(x, y, z, t) = \mathcal{O}_3(x^2)$ and $R_i(x, y, z, t) = \mathcal{O}_2(x^2, y^2)$, $\quad i = 1, 2$.

The periodic orbit P_{η_0,ξ_0} becomes $P_{\eta_0,\xi_0} = \{x = y = 0,\ a = \eta_0,\ b = \xi_0,\ t \in \mathbb{T}\}$.

We now apply the change of variables $(x, y, z) = (x, y, \Upsilon(\varphi, J))$, where Υ is given in Theorem 5.4.7. The transformed equation has the same form as in (5.54) with statements 1 and 2 above. From now on, we will assume $z = (\varphi, J)$. In particular, the scattering maps associated to the infinity manifold $\{x = y = 0\}$ will satisfy the properties of Theorem 5.4.7.

5.5.2 C^1 behavior close to infinity

To study the local behavior of system (5.54) close to $\mathcal{E}_\infty$ we start by finding a suitable set of coordinates, provided by the next theorem, whose proof is deferred to Section 5.10.

Theorem 5.5.2. *Let $K \subset \mathbb{R}^2$ be a compact set. For any $N \geq 1$, there exists a neighborhood U of the subset $\mathcal{E}_\infty^K = \cup_{z_0 \in K} P_{z_0} \subset \mathcal{E}_\infty$, in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{T}$ and a $\mathcal{C}^N$ change of variables*

$$\Phi : (x, y, z, t) \in U \mapsto (q, p, \tilde{z}, t) = \left(\frac{x - y}{2}, \frac{x + y}{2}, z, t\right) + \mathcal{O}_2(x, y)$$

that transforms system (5.54) *into*

$$\begin{aligned} \dot{q} &= q\left((q + p)^3 + \mathcal{O}_4(q, p)\right), \qquad \dot{\tilde{z}} = q^N p^N \mathcal{O}_4(q, p), \\ \dot{p} &= -p\left((q + p)^3 + \mathcal{O}_4(q, p)\right), \quad \dot{t} = 1. \end{aligned} \tag{5.55}$$

Remark 5.5.3. *It is worth to remark that the change of variables in Theorem 5.5.2 is analytic in some complex sectorial domain of $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{T}$ with $\mathcal{E}_\infty^K$ in its vertex. This claim is made precise in the proof of the Theorem 5.5.2. To prove this fact, it is important to control the terms of degree 6 of the equations for x and y in* (5.54)*, in particular, the sign of A in* (5.53) *(see Section 5.10).*

To simplify the notation, we drop the tilde from the new z variable. Let $N > 10$ be fixed. We are interested in the behavior of system (5.55) in the region $\Phi(U) \cap \{q + p \geq 0\}$. The stable and unstable invariant manifolds of P_{z_0} in this region are, respectively, $W^s(P_{z_0}) = \{q = 0,\ p > 0,\ z = z_0,\ t \in \mathbb{T}\}$ and $W^u(P_{z_0}) = \{p = 0,\ q > 0,\ z = z_0,\ t \in \mathbb{T}\}$. Even if the invariant manifold $\mathcal{E}_\infty^K$ is not normally hyperbolic, it behaves as such and possesses smooth stable and unstable invariant manifolds (see [BFM20a, BFM20b]) defined by

$$W^*(\mathcal{E}_\infty^K) = \bigcup_{z_0 \in K} W^*(P_{z_0}), \quad * = u, s.$$

Moreover, the invariant manifolds $W^*(\mathcal{E}_\infty^K)$ are foliated by stable and unstable leaves $W^*_{w_0}$, $* = u, s$, which are defined as follows. Denote by $\varphi_\tau(w)$ the flow associated to equation (5.55). Then, $w \in W^s_{w_0}$ if and only if

$$|\varphi_\tau(w) - \varphi_\tau(w_0)| \to 0 \qquad \text{as} \qquad \tau \to +\infty$$

and analogously for the unstable leaves with backward time.

This allows us to define the classical wave maps associated to the stable and unstable foliations, which we denote by Ω^s and Ω^u, as

$$\Omega^*(w) = w_0 \quad \text{if and only if} \quad w \in W^*_{w_0} \qquad \text{for } * = u, s. \tag{5.56}$$

Observe that in the local coordinates given by Theorem 5.5.2, one has that, locally, $W^s(\mathcal{E}^K_\infty) = \{q = 0\}$ and $W^u(\mathcal{E}^K_\infty) = \{p = 0\}$. Moreover,

$$\Omega^s(0, p, z, t) = (0, 0, z, t) \qquad \text{and} \qquad \Omega^s(q, 0, z, t) = (0, 0, z, t).$$

The next step is to prove a Lambda Lemma that will describe the local dynamics close to $\mathcal{E}^K_\infty$, in the coordinates given by Theorem 5.5.2. Note that the particular form and invariance of $W^*(\mathcal{E}^K_\infty), * = u, s$, implies that the solution $\varphi_\tau(w_0)$ through any point $w_0 = (q_0, p_0, z_0, t_0) \in \Phi(U) \cap \{q > 0,\ p > 0\}$ satisfies $\varphi_\tau(w_0) \in \Phi(U) \cap \{q > 0,\ p > 0\}$ for all τ such that $\varphi_\tau(w_0) \in \Phi(U)$. We define, then,

$$V_\rho = \{(q, p) \mid |q|, |p| < \rho,\ q > 0,\ p > 0\}.$$

Let $W \subset \mathbb{R}^2$ be an bounded open set. Given $0 < \delta < a \le \rho$ and $\widetilde{W} \subset W$, we define the sections

$$\begin{aligned} \Lambda^-_{a,\delta}(\widetilde{W}) &= \{(q, p, z, t) \in V_\rho \times \widetilde{W} \times \mathbb{T} \mid p = a,\ 0 < q < \delta\}, \\ \Lambda^+_{a,\delta}(\widetilde{W}) &= \{(q, p, z, t) \in V_\rho \times \widetilde{W} \times \mathbb{T} \mid q = a,\ 0 < p < \delta\}. \end{aligned} \tag{5.57}$$

and the associated Poincaré map

$$\Psi_{\mathrm{loc}} : \Lambda^-_{a,\delta}(\widetilde{W}) \longrightarrow \Lambda^+_{a,\delta}(W) \tag{5.58}$$

induced by the flow of (5.55), wherever it is well defined.

Theorem 5.5.4. *Assume $N > 10$ in system* (5.55). *Let $K \subset W$ be a compact set. There exists $0 < \rho < 1$ and $C > 0$, satisfying $C\rho < 3/5$, such that, for any $0 < a \le \rho$ and any $\delta \in (0, a/2)$, the Poincaré map $\Psi_{\mathrm{loc}} : \Lambda^-_{a,\delta}(K) \to \Lambda^+_{a,\delta^{1-Ca}}(W)$ associated to system* (5.55) *is well defined. Moreover, Ψ_{loc} satisfies the following.*

1. *There exist $\widetilde{C}_1, \widetilde{C}_2 > 0$ such that, for any $(q, a, z_0, t_0) \in \Lambda^-_{a,\delta}(K)$, $\Psi_{\mathrm{loc}}(q, a, z_0, t_0) = (a, p_1, z_1, t_1)$ satisfies*

$$\begin{aligned} q^{1+Ca} \le p_1 &\le q^{1-Ca}, \\ |z_1 - z_0| &\le \frac{1}{2N} a^{N(1+Ca)} q^{N(1-Ca)}, \\ \widetilde{C}_1 q^{-3(1-Ca)/2} \le t_1 - t_0 &\le \widetilde{C}_2 q^{-3(1+Ca)/2}. \end{aligned}$$

2. *Fix any $M > 0$. Then, there exists δ_0 and $\widetilde{C}_3 > 0$, such that for any $\delta \in (0, \delta_0)$ and $\gamma : [0, \delta) \to \overline{V_\rho} \times W \times \mathbb{T}$, a $\mathcal{C}^1$ curve with $\gamma((0, \delta)) \subset \Lambda^-_{a,\delta}(K)$ of the form $\gamma(q) = (q, a, z_0(q), t_0(q))$ and satisfying $\|\gamma\|_{\mathcal{C}^1} \le M$ the following is true. Its image $\Psi_{\mathrm{loc}}(\gamma(q)) = (a, p_1(q), z_1(q), t_1(q))$ satisfies*

$$|p_1'(q)| \le \widetilde{C}_3, \qquad \left|\frac{p_1'(q)}{t_1'(q)}\right| \le \widetilde{C}_3 q^{1-Ca}, \qquad |z_1'(q)| \le \widetilde{C}_3, \qquad |t_1'(q)| \ge \widetilde{C}_3 \frac{1}{q^{3/5-Ca}}.$$

3. *There exists $\widetilde{C}_4 > 0$ such that, if $\gamma : [0, 1] \to \Lambda^-_{a,\delta}(K)$ is a $\mathcal{C}^1$ curve of the form $\gamma(u) = (q_0(u), a, z_0(u), t_0(u))$, then $\Psi_{\mathrm{loc}}(\gamma(u)) = (a, p_1(u), z_1(u), t_1(u))$ satisfies, for all $u \in [0, 1]$,*

$$|z_1'(u) - z_0'(u)| \le \widetilde{C}_4 \|\gamma'(u)\| q_0(u)^{N-10}.$$

4. Fix any $M>0$. Then, there exists δ_0 such that for any $\delta\in(0,\delta_0)$, any $\mathcal{C}^1$ curve $(z_0(u),t_0(u))\in K\times\mathbb{T}$, $u\in[0,1]$, satisfying $\|(z_0(u),t_0(u))\|_{\mathcal{C}^1}\leq M$ and any $\tilde{q}_0\in(0,\delta)$, there exists a function $q_0:[0,1]\to(0,\delta)$ with $q_0(0)=\tilde{q}_0$, $|q_0'(0)|<\tilde{q}_0^{1/5}$, such that $\Psi_{\mathrm{loc}}(q_0(u),a,z_0(u),t_0(u))=(a,p_1(u),z_1(u),t_1(u))$ satisfies

$$
\begin{aligned}
|p_1'(0)| &\leq \widetilde{C}_5\tilde{q}_0^{\frac{3}{5}-Ca}\|(q_0'(0),z_0'(0),t_0'(0))\|,\\
|z_1'(0)-z_0'(0)| &\leq \widetilde{C}_5\tilde{q}_0\|(q_0'(0),z_0'(0),t_0'(0))\|,\\
t_1'(0) &= t_0'(0),
\end{aligned}
$$

for some $\widetilde{C}_5>0$ independent of the curve and δ.

The proof of Theorem 5.5.4 is deferred to Section 5.11.

5.6 Construction of the hyperbolic set

The final step in constructing the Smale horseshoe for the 3 Body Problem given by the Hamiltonian (5.16) is to combine the dynamics in the vicinity of the disk $\mathcal{E}_\infty^K$ at infinity (see Theorem 5.5.2) with the dynamics along their invariant manifolds analyzed in Theorem 5.4.3.

5.6.1 The return map

We construct a return map in a suitable section transverse to the invariant manifolds. This map is built as a composition of the local map (close to infinity) studied in Theorem 5.5.4 (see also Figure 5.2), and a global map (close to the invariant manifolds), which we analyze now. To build the hyperbolic set, we will have to consider a suitable high iterate of the return map. To be more precise, we consider different return maps associated to two different homoclinic channels (and therefore, associated to the two different scattering maps obtained in Theorem 5.4.5).

To define these return maps, we consider the sections $\Lambda^\pm_{a,\delta}$ given in (5.57), which are transverse to the stable/unstable invariant manifolds respectively, and we call

$$
\begin{aligned}
\Sigma_1 &\equiv \Lambda^-_{a,\delta}(K)=\{p=a, 0<q<\delta, t\in\mathbb{T}, z\in K\},\\
\Sigma_2 &\equiv \Lambda^+_{a,\delta^{1-Ca}}(K)=\{q=a, 0<p<\delta^{1-Ca}, t\in\mathbb{T}, z\in K\},
\end{aligned}
\tag{5.59}
$$

where (q,p,z,t) are the coordinates defined by Theorem 5.5.2, $K\subset\mathbb{R}^2$ is a compact set and we take

$$
\delta<\frac{a}{2}\qquad\text{and}\qquad a\leq\rho.
$$

Theorem 5.5.4) ensures that there exists $C>0$ such that the local map

$$
\Psi_{\mathrm{loc}}:\Sigma_1\to\Sigma_2
$$

(see (5.58)) is well defined.

The global maps will be defined from suitable open sets in Σ_2 to Σ_1. They are defined as the maps induced by Hamiltonian (5.16) expressed in the coordinates given by Proposition 5.5.2. In fact, to construct them, we use slightly different coordinates which are defined on suitable neighborhoods of the homoclinic channels at Σ_2, i.e. $\Gamma^j\cap\Sigma_2$, $j=1,2$ (see (5.42)).

These coordinates are constructed as follows.

1. In the coordinates[5] (p,t,z) in Σ_2, $\Sigma_2\cap W^u=\{p=0\}$. Since we are in a perturbative setting (when Θ is large enough), we have that $W^s\cap\Sigma_2=\{p=w^s(t,z)\}$. Moreover, $\Gamma^j\cap\Sigma_2$, $j=1,2$, where Γ^j are

[5] Note that we have reordered the variables. The reason is that, in the section Σ_2, the variable t will play a similar role as the variable p whereas the variable z is treated as a center variable.

the homoclinic channels given by Theorem 5.4.5, can be parametrized as $\{(p,t,z)=(0,t_j^2(z),z)\}$. In particular, the functions t_j^2 satisfy

$$w^s(t_j^2(z),z)=0, \qquad \partial_t w^s(t_j^2(z),z)\neq 0.$$

Hence, the equation $p=w^s(t,z)$ defines in the neighborhood of each homoclinic channel two functions $\tilde{w}_j^s(p,z)$, satisfying $\tilde{w}_j^s(0,z)=0$, such that

$$p=w^s(t,z) \qquad \Longleftrightarrow \qquad t=t_j^2(z)+\tilde{w}_j^s(p,z),$$

in a neighborhood of $\Gamma^j\cap\Sigma_2$, $j=1,2$. That is, $(p,t_j^2(z)+\tilde{w}_j^s(p,z),z)$ parametrizes $W^s\cap\Sigma_2$ in a neighborhood of $\Gamma^j\cap\Sigma_2$. We define two new sets of coordinates in Σ_2, defined in a neighborhood of $\Gamma^j\cap\Sigma_2$,

$$(p,\tau,z)=A_j(p,t,z)=(p,t-t_j^2(z)-\tilde{w}_j^s(p,z),z), \qquad j=1,2. \tag{5.60}$$

In these coordinates, $W^s\cap\Sigma_2$ in each of the neighborhoods of Γ^j is given by $\tau=0$.

2. We proceed analogously in Σ_1. In the coordinates (q,γ,z) in Σ_1, $\Sigma_1\cap W^s=\{q=0\}$, $W^u\cap\Sigma_1=\{q=w^u(t,z)\}$ for some function w^u and the intersection of the homoclinic channels Γ^j, $j=1,2$, with Σ_1 are given by $\{(q,t,z)=(0,t_j^1(z),z)\}$ for some functions t_j^1. In particular,

$$w^u(t_j^1(z),z)=0, \qquad \partial_t w^u(t_j^1(z),z)\neq 0.$$

Hence, the equation $q=w^u(t,z)$ can be inverted in the neighborhood of $\Gamma^j\cap\Sigma_1$, $j=1,2$, by defining two functions $\tilde{w}_j^u(q,z)$ satisfying $\tilde{w}_j^u(0,z)=0$, such that

$$q=w^u(t,z) \qquad \Longleftrightarrow \qquad t=t_j^1(z)+\tilde{w}_j^u(q,z).$$

That is, $(q,t_j^1(z)+\tilde{w}_j^u(q,z),z)$ parametrizes $W^u\cap\Sigma_1$ in a neighborhood of $\Gamma^j\cap\Sigma_1$. We define two new sets of coordinates in Σ_1, defined in a neighborhood of $\Gamma^j\cap\Sigma_1$,

$$(q,\sigma,z)=B_j(q,t,z)=(q,t-t_j^1(z)-\tilde{w}_j^u(q,z),z), \qquad j=1,2. \tag{5.61}$$

In these coordinates, $W^u\cap\Sigma_1$ in each of the neighborhoods of Γ^j is given by $\sigma=0$.

Let $\Psi_{\mathrm{glob},j}$, $j=1,2$, be the two *global* maps from a neighborhood of $\Gamma^j\cap\Sigma_2$ in Σ_2, which we denote by U_j^2, to a neighborhood of $\Gamma^j\cap\Sigma_1$ in Σ_1, which we denote by U_j^1, defined by the flow. Choosing the coordinates (p,τ,z) in U_j^2 and (q,σ,z) in U_j^1, given by A_j and B_j respectively (see (5.60), (5.61)), we can define

$$\widetilde{\Psi}_{\mathrm{glob},j}(p,\tau,z)=B_j\circ\Psi_{\mathrm{glob},j}\circ A_j^{-1}(p,\tau,z).$$

Then, for points $(p,\tau,z)=(0,0,z)\in\Gamma^j\cap\Sigma_2$, the global map $\widetilde{\Psi}_{\mathrm{glob},j}$ map is given by:

- Compute $(0,0,\hat{t},z)=\Omega^u(a,0,t_j^2(z),z)\in\mathcal{E}_\infty^K$, where Ω^u is the wave map introduced in (5.56).
- Compute $(0,0,\hat{t},\widehat{\mathtt{S}}^j(z))\in\mathcal{E}_\infty^K$, where $\widehat{\mathtt{S}}^j$ is the scattering map analyzed in Theorem 5.4.7.
- Compute $(\Omega^s)^{-1}(0,0,\hat{t},\widehat{\mathtt{S}}^j(z))=(0,a,\tilde{t},\widehat{\mathtt{S}}^j(z))\in\Gamma^j\cap\Sigma_1$.
- Finally in coordinates (q,σ,z) this last point becomes $(0,0,\widehat{\mathtt{S}}^j(z))$
- In conclusion, $\widetilde{\Psi}_{\mathrm{glob},j}(0,0,z)=(0,0,\widehat{\mathtt{S}}^j(z))$

Using this fact and the changes of coordinates in (5.60), (5.61) we have that

$$\begin{pmatrix} q_1\\ \sigma_1\\ z_1\end{pmatrix}=\widetilde{\Psi}_{\mathrm{glob},j}(p,\tau,z)=B_j\circ\Psi_{\mathrm{glob},j}\circ A_j^{-1}(p,\tau,z)=\begin{pmatrix}\tau\nu_1^j(z)(1+\mathcal{O}_1(p,\tau))\\ p\nu_2^j(z)(1+\mathcal{O}_1(p,\tau))\\ \widehat{\mathtt{S}}^j(z)+\mathcal{O}_1(p,\tau)\end{pmatrix}, \tag{5.62}$$

where $\nu_1^j(z)\nu_2^j(z) \neq 0$. Indeed, the claim follows from the fact that

$$\begin{aligned}\widetilde{\Psi}_{\mathrm{glob},j}(\{p=0\}\cap U_j^2) &= \widetilde{\Psi}_{\mathrm{glob},j}(W^u\cap\Sigma_2\cap U_j^2) = W^u\cap\Sigma_1\cap U_j^1 = \{\sigma=0\}\cap U_j^1\\ \widetilde{\Psi}_{\mathrm{glob},j}(\{\tau=0\}\cap U_j^2) &= \widetilde{\Psi}_{\mathrm{glob},j}(W^s\cap\Sigma_2\cap U_j^2) = W^s\cap\Sigma_1\cap U_j^1 = \{q=0\}\cap U_j^1\end{aligned}$$

and expanding around $(0,0,z)$. The fact that $\nu_1^j(z)\nu_2^j(z) \neq 0$ follows immediately from the fact that $\widetilde{\Psi}_{\mathrm{glob},j}$ are diffeomorphisms. It is then immediate that $\widetilde{\Psi}^{-1}_{\mathrm{glob},j} : U_j^1 \to U_j^2$ is of the form

$$\widetilde{\Psi}^{-1}_{\mathrm{glob},j}(q,\sigma,z) = A_j \circ \Psi^{-1}_{\mathrm{glob},j} \circ B_j^{-1}(q,\sigma,z) = \begin{pmatrix} \sigma\mu_1^j(z)(1+\mathcal{O}_1(q,\sigma))\\ q\mu_2^j(z)(1+\mathcal{O}_1(q,\sigma))\\ (\widehat{\mathtt{S}}^j)^{-1}(z)+\mathcal{O}_1(q,\sigma)\end{pmatrix}, \tag{5.63}$$

where $\mu_1^j(z)\mu_2^j(z) \neq 0$.

Now we deal with the local map. Notice that Theorem 5.5.4 implies that, for $1 \le i,j \le 2$, $\Psi_{\mathrm{loc}}(U_i^1)\cap U_j^2 \neq \emptyset$. We will denote by $\Psi_{\mathrm{loc},i,j} = \Psi_{\mathrm{loc}|\Psi^{-1}_{\mathrm{loc}}(U_j^2)\cap U_i^1} : U_i^1 \to U_j^2$ and its expression in coordinates

$$\widetilde{\Psi}_{\mathrm{loc},i,j} = A_j \circ \Psi_{\mathrm{loc},i,j} \circ B_i^{-1}.$$

Observe that the map $\Psi_{\mathrm{loc},i,j}$ does not depend on i and j and the dependence of $\widetilde{\Psi}_{\mathrm{loc},i,j}$ on i and j is only through the systems of coordinates A_j and B_i.

The combination of the global maps along the homoclinic channels and the local map allows to define four different maps $\Psi_{ij} : U_i^2 \to U_j^2$ by setting $\Psi_{i,j} = \Psi_{\mathrm{loc},i,j} \circ \Psi_{\mathrm{glob},i}$. We will denote its expression in coordinates as

$$\widetilde{\Psi}_{i,j} = \widetilde{\Psi}_{\mathrm{loc},i,j} \circ \widetilde{\Psi}_{\mathrm{glob},i}. \tag{5.64}$$

Let us specify the domains we will consider. Given $\delta \in (0, a/2)$, let $\mathcal{Q}_\delta^i \subset \Sigma_2$ be the set

$$\mathcal{Q}_\delta^i = \{0<p<\delta,\ 0<\tau<\delta,\ z\in\mathcal{R}\} \subset U_i^2 \tag{5.65}$$

where $\mathcal{R}$ has been introduced in (5.51). We remark that the "sides" $\{p=0\}$ and $\{\tau=0\}$ of $\mathcal{Q}_\delta^i$ are $W^u\cap\Sigma_2$ and $W^s\cap\Sigma_2$, respectively, and the "edge" $\{p=\tau=0\}$ is $\Gamma^i\cap\Sigma_2$.

Let Ψ be the map defined as

$$\Psi = \Psi_{1,2}\circ\Psi_{1,1}^{M-1}\circ\Psi_{2,1}, \tag{5.66}$$

where M is given by (5.173). We will denote by $\widetilde{\Psi}$ its expression in the A_2 coordinate system, that is,

$$\widetilde{\Psi} = \widetilde{\Psi}_{1,2}\circ\widetilde{\Psi}_{1,1}^{M-1}\circ\widetilde{\Psi}_{2,1} : \mathcal{Q}_\delta^2 \longrightarrow \Sigma_2. \tag{5.67}$$

5.6.2 Symbolic dynamics: conjugation with the shift

We consider in $\mathcal{Q}_\delta^2$, defined in (5.65), the set of coordinates (p,τ,φ,J) given by A_2 and Theorem 5.4.7. The coordinates have been chosen in such a way that (τ,φ) variables are "expanding" by $\widetilde{\Psi}$, while the (p,J) variables are "contracting". To formalize this idea, we introduce the classical concepts of *vertical* and *horizontal rectangles* in our setting (see Figure 5.6) as well as *cone fields* (see [Sma65, Mos01, Wig90, KH95]).

We will say that $H \subset \mathcal{Q}_\delta^2$ is a *horizontal rectangle* if

$$H = \{(p,\tau,\varphi,J)\in\mathcal{Q}_\delta^2;\ h_1^-(\tau,\varphi)\le p\le h_1^+(\tau,\varphi),\ h_2^-(\tau,\varphi)\le J\le h_2^+(\tau,\varphi)\}, \tag{5.68}$$

where $h_i^\pm : (0,\delta)\times(0,\tilde\kappa) \to (0,\delta)\times(0,\tilde\kappa)$, $i=1,2$, are ℓ_h-Lipschitz. Analogously, $V \subset \mathcal{Q}_\delta^2$ is a *vertical rectangle* if

$$V = \{(p,\tau,\varphi,J)\in\mathcal{Q}_\delta^2;\ v_1^-(p,J)\le\tau\le v_1^+(p,J),\ v_2^-(p,J)\le\varphi\le v_2^+(p,J)\}, \tag{5.69}$$

with ℓ_v-Lipschitz functions $v_i^\pm : (0,\delta)\times(0,\tilde\kappa) \to (0,\delta)\times(0,\tilde\kappa)$, $i=1,2$.

If H is the horizontal rectangle (5.68), we split $\partial H = \partial_h H \cup \partial_v H$ as

$$\partial_h H = \{\omega \in \mathcal{Q}^2_\delta;\ (p,J) = (h_1^-, h_2^-)(\tau,\varphi) \text{ or } (p,J) = (h_1^+, h_2^+)(\tau,\varphi)\},$$
$$\partial_v H = \{\omega \in \mathcal{Q}^2_\delta;\ (\tau,\varphi) = (0,0) \text{ or } (\tau,\varphi) = (\delta,\tilde{\kappa})\}$$

and, analogously, if V is the vertical rectangle (5.69), we split $\partial V = \partial_h V \cup \partial_v V$ as

$$\partial_h V = \{\omega \in \mathcal{Q}^2_\delta;\ (p,J) = (0,0) \text{ or } (p,J) = (\delta,\tilde{\kappa})\},$$
$$\partial_v V = \{\omega \in \mathcal{Q}^2_\delta;\ (\tau,\varphi) = (v_1^-, v_2^-)(p,J) \text{ or } (\tau,\varphi) = (v_1^+, v_2^+)(p,J)\}.$$

Additionally, we define the *stable and unstable cone fields* in the following way. For $\omega \in \mathcal{Q}^2_\delta$, we consider in $T_\omega \mathcal{Q}^2_\delta$ the basis given by the coordinates (p,τ,φ,J) and write $x \in T_\omega \mathcal{Q}^2_\delta$ as $x = (x_u, x_s)$ meaning $x = x_{s,p}\frac{\partial}{\partial p} + x_{u,\tau}\frac{\partial}{\partial \tau} + x_{u,\varphi}\frac{\partial}{\partial \varphi} + x_{s,J}\frac{\partial}{\partial J}$. We define $\|x_u\| = \max\{|x_{u,\tau}|, |x_{u,\varphi}|\}$ and $\|x_s\| = \max\{|x_{s,p}|, |x_{s,J}|\}$. Then, a κ_s*-stable cone* at $\omega \in \mathcal{Q}^2_\delta$ is

$$S^s_{\omega,\kappa_s} = \{x \in T_\omega \mathcal{Q}^2_\delta;\ \|x_u\| \le \kappa_s \|x_s\|\}$$

and a κ_u*-unstable cone* at $\omega \in \mathcal{Q}^2_\delta$,

$$S^u_{\omega,\kappa_u} = \{x \in T_\omega \mathcal{Q}^2_\delta;\ \|x_s\| \le \kappa_u \|x_u\|\}. \tag{5.70}$$

Having in mind [Wig90] (see also [Mos01]), we introduce the following hypotheses. Let $F : \mathcal{Q}^2_\delta \to \mathbb{R}^4$ be a C^1 diffeomorphism onto its image.

H1 There exists two families $\{H_n\}_{n\in\mathbb{N}}$, $\{V_n\}_{n\in\mathbb{N}}$ of horizontal and vertical rectangles in $\mathcal{Q}^2_\delta$, with $\ell_h \ell_v < 1$, such that $H_n \cap H_{n'} = \emptyset$, $V_n \cap V_{n'} = \emptyset$, $n \neq n'$, $H_n \to \{p = 0\}$, $V_n \to \{\tau = 0\}$, when $n \to \infty$, in the sense of the Hausdorff distance, $F(V_n) = H_n$, homeomorphically, $F^{-1}(\partial_v V_n) \subset \partial_v H_n$, $n \in \mathbb{N}$.

H2 There exist $\kappa_u, \kappa_s, \mu > 0$ satisfying $0 < \mu < 1 - \kappa_u \kappa_s$ such that if $\omega \in \cup_n V_n$, then $DF(\omega) S^u_{\omega,\kappa_u} \subset S^u_{F(\omega),\kappa_u}$, whereas if $\omega \in \cup_n H_n$, then $DF^{-1}(\omega) S^s_{\omega,\kappa_s} \subset S^s_{F^{-1}(\omega),\kappa_s}$. Moreover, denoting $x^+ = DF(\omega)x$ and $x^- = DF^{-1}(\omega)x$, if $x \in S^u_{\omega,\kappa_u}$, then $|x_u^+| \ge \mu^{-1}|x_u|$ whereas, if $x \in S^s_{\omega,\kappa_s}$, then $|x_s^-| \ge \mu^{-1}|x_s|$.

Finally, we introduce symbolic dynamics in our context (see [Mos01] for a complete discussion). Consider the *space of sequences* $\Sigma = \mathbb{N}^{\mathbb{Z}}$, with the topology[6] induced by the neighborhood basis of $s^* = (\dots, s^*_{-1}, s^*_0, s^*_1, \dots)$

$$J_j = \{s \in \Sigma;\ s_k = s^*_k,\ |k| < j\}, \qquad J_{j+1} \subset J_j$$

and the *shift map* $\sigma : \Sigma \to \Sigma$ defined by $\sigma(s)_j = s_{j+1}$. The map σ is a homeomorphism.

We have the following theorem, which is a direct consequence of Theorems 2.3.3 and 2.3.5 of [Wig90].

Theorem 5.6.1. *Assume that $F : \mathcal{Q}^2_\delta \to \mathbb{R}^4$, a C^1 diffeomorphism onto its image, satisfies* **H1** *and* **H2**. *Then there exists a subset $\mathcal{X} \subset \mathcal{Q}^2_\delta$ and a homeomorphism $h : \mathcal{X} \to \Sigma$ such that $h \circ F_{|\mathcal{X}} = \sigma \circ h$.*

Remark 5.6.2. *Hypothesis* **H2** *implies that the set $\mathcal{X}$ given by Theorem 5.6.1 is hyperbolic.*

Theorem 5.6.3. *If $\tilde{\kappa}$ and δ are small enough, $\widetilde{\Psi}$ satisfies* **H1** *and* **H2**.

Theorem 5.6.3 is an immediate consequence of the following two propositions. Proposition 5.6.4 implies that $\widetilde{\Psi}$ indeed satisfies **H1** and Proposition 5.6.5 implies that $\widetilde{\Psi}$ satisfies **H2**.

[6] This topology can be also defined by the distance $d(s,r) = \sum_{k\in\mathbb{Z}} 4^{-|k|}\delta(s_k, r_k)$ where $\delta(n,m) = 1$ if $n = m$ and $\delta(n,m) = 0$ if $n \neq m$.

Proposition 5.6.4. *If δ is small enough, $\widetilde{\Psi}(\mathcal{Q}_\delta^2) \cap \mathcal{Q}_\delta^2$ has an infinite number of connected components. More concretely, there exists $0 < \tau_1 < \tau_2 < \delta$ such that the set*

$$H_{\tau_1,\tau_2} = \{(p, \tau, \varphi, J) \in \mathcal{Q}_\delta^2 \mid \tau_1 \leq \tau \leq \tau_2\}$$

satisfies the following. There exists $\{H_n\}_{n\in\mathbb{N}}$, a family of horizontal rectangles,

$$H_n = \{(p, \tau, \varphi, J) \in \mathcal{Q}_\delta^2 \mid h_{1,n}^-(\tau, \varphi) \leq p \leq h_{1,n}^+(\tau, \varphi),\ h_{2,n}^-(\tau, \varphi) \leq J \leq h_{2,n}^+(\tau, \varphi)\},$$

with $H_n \cap H_{n'} = \emptyset$, if $n \neq n'$, such that $h_{1,n}^-, h_{1,n}^+ \to 0$ uniformly when $n \to \infty$, and

$$\sup_n \operatorname{Lip} h_{2,n}^-, \sup_n \operatorname{Lip} h_{2,n}^+ \lesssim \mathcal{O}(\tilde{\kappa}) + \mathcal{O}(\delta),$$

with $\widetilde{\Psi}(H_{\tau_1,\tau_2}) \cap \mathcal{Q}_\delta^2 = \cup_{n\in\mathbb{N}} H_n$.

The analogous claim holds for vertical rectangles and $\widetilde{\Psi}^{-1}$, that is, there exist $0 < p_1 < p_2 < \delta$ such that the set

$$V_{p_1,p_2} = \{(p, \tau, \varphi, J) \in \mathcal{Q}_\delta^2 \mid p_1 \leq p \leq p_2\}$$

satisfies the following. There exists $\{V_n\}_{n\in\mathbb{N}}$, a family of vertical rectangles,

$$V_n = \{(p, \tau, \varphi, J) \in \mathcal{Q}_\delta^2 \mid v_{1,n}^-(p, J) \leq \tau \leq v_{1,n}^+(p, J),\ v_{2,n}^-(p, J) \leq \varphi \leq v_{2,n}^+(p, J)\},$$

with $V_n \cap V_{n'} = \emptyset$, if $n \neq n'$, such that $v_{1,n}^-, v_{1,n}^+ \to 0$ uniformly when $n \to \infty$, and

$$\sup_n \operatorname{Lip} v_{2,n}^-, \sup_n \operatorname{Lip} v_{2,n}^+ \lesssim \mathcal{O}(1),$$

with $\widetilde{\Psi}^{-1}(V_{p_1,p_2}) \cap \mathcal{Q}_\delta^2 = \cup_{n\in\mathbb{N}} V_n$.

In particular, $\widetilde{\Psi}$ satisfies **H1**.

The proof of this proposition is placed in Section 5.12.2.

Proposition 5.6.5. *$\widetilde{\Psi}$ satisfies* **H2** *with $\kappa_u = \mathcal{O}(\delta) + \mathcal{O}(\tilde{\kappa})$, $\kappa_s = \mathcal{O}(1)$ and $\mu = \mathcal{O}(\tilde{\kappa})$.*

The proof of this proposition is placed in Section 5.12.4.

Propositions 5.6.4 and 5.6.5 imply that the map $\widetilde{\Psi}$ satisfies hypotheses **H1** and **H2**. Therefore, one can apply Theorem 5.6.1 to $\widetilde{\Psi}$ to obtain that $\widetilde{\Psi}$ has a a hyperbolic invariant set whose dynamics is conjugated to the shift of infinite symbols.

To complete the proof of Theorem 5.1.4 we need to "undo" the symplectic reduction by rotations (see (6.11)). This adds one degree of freedom to the system: now one has to take into account Θ (which is a first integral) and its conjugate variable $\alpha \in \mathbb{T}$. Since α is a cyclic variable, its dynamics is just a rotation determined by the other variables.

Then, one can consider as Poincaré section Π just the section Σ_2 introduced in (5.59) expressed in the original coordinates, and the invariant set given by Theorem 5.6.3 becomes a set $\mathcal{X}$ which is homeomorphic to $\mathbb{N}^\mathbb{Z} \times \mathbb{T}$. Note that in Theorem 5.1.4 we are fixing the center of mass at the origin and therefore we do not need to pay attention to the reduction by translation; indeed, the variables used for Theorem 5.6.3 are based on Jacobi coordinates which also reduce by translations.

Theorem 5.1.3 is also a direct consequence of Theorem 5.6.3. Indeed, note that, the symbols in $\mathbb{N}$ keep track of the closeness of the corresponding strip of each point in $\mathcal{X}$ to the invariant manifolds of infinity. That is, the larger the symbol, the closer the strip is to the invariant manifolds. This implies that these points get closer to infinity. For this reason, by construction, if one considers a bounded sequence in Σ, the corresponding orbit in $\mathcal{X}$ is bounded. If one considers a sequence $\{\omega_k\}_{k\in\mathbb{Z}}$ which is unbounded both as $k \to \pm\infty$, the corresponding orbit belongs to $OS^- \cap OS^+$. Indeed, the orbit keeps visiting for all forward and backward times a fixed neighborhood of the homoclinic channel (which is "uniformly far" from infinity) but at the same time keeps getting closer and closer to infinity because the sequence is unbounded. By considering sequences which are bounded at one side and unbounded at the other, one can construct trajectories which belong to $B^-\cap OS^+$ and $OS^-\cap B^+$. The trajectories which are (in the future or in the past) parabolic-elliptic or hyperbolic-elliptic do not belong to $\mathcal{X}$ but they can be built analogously. Indeed, as is done by Moser [Mos01], one can consider sequences of the form

$$\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots, \omega_{M-1}, \infty).$$

That is, points whose $M-1$ forward iterates come back to the section Σ_2 and then the trajectory goes to infinity. By the construction of the horizontal strips, one can built orbits which have these behavior since the strips get streched and therefore its image hit the invariant manifolds of infinity (which correspond to the motions PE_2) and hit points "at the other side" of the invariant manifolds, which correspond to hyperbolic motions HE_2 (see Figure 5.6). The same can be achieved for the inverse return map $\widetilde{\Psi}^{-1}$ and the vertical strips. For this reason one can combine future/past PE_2 and HE_2 with any other types of motion.

5.7 Proof of Theorem 5.4.3: Parameterization of the invariant manifolds of infinity

Theorem 5.4.3 gives an asymptotic formula for the distance between the unstable manifold of the periodic orbit P_{η_0,ξ_0} and the stable manifold of the periodic orbit $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$. In this section we carry out the first step of its proof. We consider suitable graph parameterizations of the invariant manifolds and we analyze their analytic extensions to certain complex domains. Later, in Section 5.8, we use these analytic extensions to obtain asymptotic formulas for the difference between the parameterizations for real values of the parameters.

This section is structured as follows. First, in Section 5.7.1 we consider symplectic coordinates which are adapted to have graph paramerizations of the invariant manifolds, which are constructed in Section 5.7.2. Then, in Section 5.7.3 we analyze the analytic extension of this graph paramerizations to certain complex domains. Such analysis is performed in Section 5.7.4 by means of a fixed point argument in suitable Banach spaces of formal Fourier series. These graphs paramerizations are singular at a certain point (where the invariant manifolds cease to be graphs). To overcome this problem, in Section 5.7.5, we consider a different type of parameterizations.

5.7.1 An adapted system of coordinates

To study $W^u(P_{\eta_0,\xi_0})$ and $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi})$ with $|\delta\eta|, |\delta\xi| \ll 1$, we perform a change of variables to the coordinates introduced in (5.15). This transformation

$$(\widetilde{r}, \widetilde{y}, \lambda, L, \eta, \xi) \to (u, Y, \gamma, \Lambda, \alpha, \beta)$$

relies on the parameterization of the unperturbed separatrix associated to the periodic orbit P_{η_0,ξ_0} given by Lemma 5.4.1 and is defined as

$$\begin{aligned}
\widetilde{r} &= G_0^2 \widehat{r}_{\rm h}(u), & \widetilde{y} &= \frac{\widehat{y}_{\rm h}(u)}{G_0} + \frac{Y}{G_0^2 \widehat{y}_{\rm h}(u)} + \frac{\Lambda - (\eta_0+\alpha)(\xi_0+\beta) + \eta_0\xi_0}{G_0^2 \widehat{y}_{\rm h}(u)(\widehat{r}_{\rm h}(u))^2} \\
\lambda &= \gamma + \phi_{\rm h}(u), & L &= L_0 + \Lambda \\
\eta &= e^{i\phi_{\rm h}(u)}(\eta_0+\alpha), & \xi &= e^{-i\phi_{\rm h}(u)}(\xi_0+\beta)
\end{aligned} \tag{5.71}$$

where

$$G_0 = \Theta - L_0 + \eta_0\xi_0. \tag{5.72}$$

This change of coordinates is consistent with the particular form of the parameterization of the perturbed invariant manifolds given in (5.30). Indeed, we look for parameterizations of the unstable manifold of the periodic orbit P_{η_0,ξ_0} and the stable manifold of the periodic orbit $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$ as graphs in (u,γ) as

$$(u,\gamma) \mapsto (Y, \Lambda, \alpha, \beta) = Z^*(u,\gamma), \quad * = u, s.$$

It can be easily checked, using Lemma 5.4.1, that the change of coordinates (5.71) is symplectic in the sense that the pull back of the canonical symplectic form is just $\omega = du\wedge dY + d\gamma\wedge d\Lambda + i d\alpha\wedge d\beta$. This fact will be strongly used later in Section 5.8.5.

To analyze the dynamics it is enough to express the Hamiltonian $\widetilde{\mathcal{K}}$ (5.16) in terms of the new variables. We also scale time as

$$t = G_0^3 s \tag{5.73}$$

to have parabolic motion of speed one coupled with a fast rotation. Then, the Hamiltonian becomes

$$\begin{aligned}\mathcal{P}(u,Y,\gamma,\Lambda,\alpha,\beta) =& -\frac{G_0^3\nu}{2(L_0+\Lambda)^2} + \frac{G_0}{2}\left(\widehat{y}_{\mathrm{h}}(u) + \frac{Y}{G_0\widehat{y}_{\mathrm{h}}(u)} + \frac{\Lambda-(\eta_0+\alpha)(\xi_0+\beta)+\eta_0\xi_0}{G_0\widehat{y}_{\mathrm{h}}(u)(\widehat{r}_{\mathrm{h}}(u))^2}\right)^2\\ &+\frac{(\Theta-L_0-\Lambda+(\eta_0+\alpha)(\xi_0+\beta))^2}{2G_0\widehat{r}_{\mathrm{h}}(u)^2} - \frac{G_0}{\widehat{r}_{\mathrm{h}}(u)}\\ &+G_0^3\widetilde{W}\left(\gamma+\phi_{\mathrm{h}}(u), L_0+\Lambda, e^{i\phi_{\mathrm{h}}(u)}(\eta_0+\alpha), e^{-i\phi_{\mathrm{h}}(u)}(\xi_0+\beta), G_0^2\widehat{r}_{\mathrm{h}}(u)\right).\end{aligned} \tag{5.74}$$

Observe that we do not write the dependence of $\mathcal{P}$ on the parameters L_0, η_0, ξ_0, nor on G_0. In a natural way we can write $\mathcal{P} = \mathcal{P}_0(u,Y,\gamma,\Lambda,\alpha,\beta) + \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta)$ where, using (5.72),

$$\begin{aligned}\mathcal{P}_0(u,Y,\gamma,\Lambda,\alpha,\beta) =& -\frac{G_0^3\nu}{2(L_0+\Lambda)^2} + \frac{G_0}{2}\left(\widehat{y}_{\mathrm{h}}(u) + \frac{Y}{G_0\widehat{y}_{\mathrm{h}}(u)} + \frac{\Lambda-(\eta_0+\alpha)(\xi_0+\beta)+\eta_0\xi_0}{G_0\widehat{y}_{\mathrm{h}}(u)(\widehat{r}_{\mathrm{h}}(u))^2}\right)^2\\ &+\frac{(\Theta-L_0-\Lambda+(\eta_0+\alpha)(\xi_0+\beta))^2}{2G_0\widehat{r}_{\mathrm{h}}(u)^2} - \frac{G_0}{\widehat{r}_{\mathrm{h}}(u)}\\ =& -\frac{G_0^3\nu}{2(L_0+\Lambda)^2} + Q_0(u,Y,\Lambda-(\eta_0+\alpha)(\xi_0+\beta)+\eta_0\xi_0)\end{aligned} \tag{5.75}$$

where, taking into account (5.28), Q_0 can be written as

$$\begin{aligned}Q_0(u,Y,q) =& \frac{G_0}{2}\left(\widehat{y}_{\mathrm{h}}(u) + \frac{Y}{G_0\widehat{y}_{\mathrm{h}}(u)} + \frac{q}{G_0\widehat{y}_{\mathrm{h}}(u)(\widehat{r}_{\mathrm{h}}(u))^2}\right)^2 + \frac{(G_0-q)^2}{2G_0\widehat{r}_{\mathrm{h}}(u)^2} - \frac{G_0}{\widehat{r}_{\mathrm{h}}(u)}\\ =& Y + \frac{Y^2}{2G_0\widehat{y}_{\mathrm{h}}^2(u)} + f_1(u)Yq + f_2(u)\frac{q^2}{2}\end{aligned} \tag{5.76}$$

with

$$f_1(u) = \frac{1}{G_0 y_h^2(u) r_h^2(u)}, \qquad f_2(u) = \frac{2}{G_0 r_h^3(u) y_h^2(u)} \tag{5.77}$$

and

$$\mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) = G_0^3\widetilde{W}\left(\gamma+\phi_{\mathrm{h}}(u), L_0+\Lambda, e^{i\phi_{\mathrm{h}}(u)}(\eta_0+\alpha), e^{-i\phi_{\mathrm{h}}(u)}(\xi_0+\beta), G_0^2\widehat{r}_{\mathrm{h}}(u)\right). \tag{5.78}$$

The periodic orbits at infinity P_{η_0,ξ_0} and $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$ are now given by

$$\begin{aligned}P_{\eta_0,\xi_0} =& \{(u,Y,\gamma,\Lambda,\alpha,\beta) = (\pm\infty,0,\gamma,0,0,0), \gamma\in\mathbb{T}\}\\ P_{\eta_0+\delta\eta,\xi_0+\delta\xi} =& \{(u,Y,\gamma,\Lambda,\alpha,\beta) = (\pm\infty,0,\gamma,0,\delta\eta,\delta\xi), \gamma\in\mathbb{T}\}\end{aligned}$$

The equations for the integrable system, which corresponds to $\mathcal{P}_1 = 0$, are

$$\begin{aligned}\dot{u} &= \partial_Y\mathcal{P}_0 = \partial_Y Q_0 = 1 + \frac{Y}{G_0\widehat{y}_h^2} + f_1(u)q\\ \dot{Y} &= -\partial_u\mathcal{P}_0 = -\partial_u Q_0 = \frac{\widehat{y}_{\mathrm{h}}'(u)Y^2}{\widehat{y}_{\mathrm{h}}^3(u)G_0} + f_1'(u)Yq + f_2'(u)\frac{q^2}{2}\\ \dot{\gamma} &= \partial_\Lambda\mathcal{P}_0 = \frac{G_0^3\nu}{(L_0+\Lambda)^3} + \partial_q Q_0 = \frac{G_0^3\nu}{(L_0+\Lambda)^3} + f_1(u)Y + f_2(u)q\\ \dot{\Lambda} &= -\partial_\gamma\mathcal{P}_0 = -\partial_\gamma Q_0 = 0\\ \dot{\alpha} &= -i\partial_\beta\mathcal{P}_0 = i\alpha\partial_q Q_0 = i\alpha\left(f_1(u)Y + f_2(u)q\right)\\ \dot{\beta} &= i\partial_\alpha\mathcal{P}_0 = -i\beta\partial_q Q_0 = -i\beta\left(f_1(u)Y + f_2(u)q\right)\end{aligned}$$

where $q = \Lambda - (\eta_0 + \alpha)(\xi_0 + \beta) + \eta_0\xi_0$ and f_1 and f_2 are given in (5.77).

This system has a 2-dimensional homoclinic manifold to the periodic orbit P_{η_0,ξ_0} given by

$$\{u_{\mathrm{h}} = u,\, Y_{\mathrm{h}} = 0,\, \gamma_{\mathrm{h}} = \gamma,\, \Lambda_{\mathrm{h}} = 0,\, \alpha_{\mathrm{h}} = 0,\, \beta_{\mathrm{h}} = 0,\, u \in \mathbb{R}, \gamma \in \mathbb{T}\} \tag{5.79}$$

whose dynamics is given by

$$(\dot u, \dot Y, \dot\gamma, \dot\Lambda, \dot\alpha, \dot\beta) = \left(1, 0, \frac{G_0^3\nu}{L_0^3}, 0, 0, 0\right).$$

(recall that we have scaled time as (5.73)).

5.7.2 Graph parameterizations of the perturbed invariant manifolds

We look for parameterizations of $W^u(P_{\eta_0,\xi_0})$ and $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi})$ as perturbations of the same homoclinic manifold (5.79) as

$$(u,\gamma) \mapsto (Y,\Lambda,\alpha,\beta) = Z^*(u,\gamma) \qquad \text{where} \qquad Z^*(u,\gamma) = \begin{pmatrix} Y^*(u,\gamma) \\ \Lambda^*(u,\gamma) \\ \alpha^*(u,\gamma) \\ \beta^*(u,\gamma) \end{pmatrix}, \quad * = u, s. \tag{5.80}$$

Note that in the unperturbed case, $Z = 0$ is a manifold homoclinic to P_{η_0,ξ_0}.

The graph parameterizations (5.80) are not defined in a neighborhood of $u = 0$ since the symplectic transformation (5.71) is not well defined at $u = 0$. For this reason, we shall use different parameterizations depending on the domain.

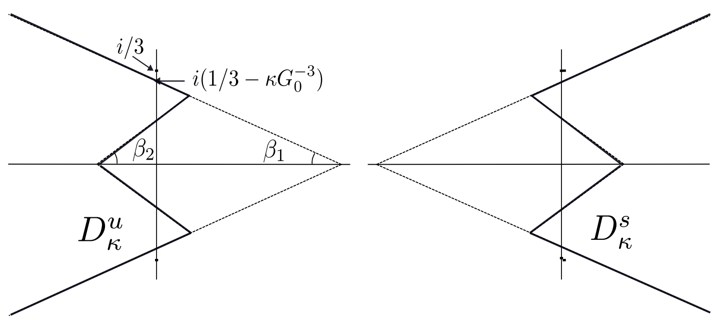


Figure 5.9: The domains $D^u_{\kappa,\delta}$ and $D^s_{\kappa,\delta}$ defined in (5.81).

First in Sections 5.7.3 and 5.7.4, we obtain graph parameterizations (5.80) in the domains $D^{u,s}_{\kappa,\delta} \times \mathbb{T}$, where

$$\begin{aligned} D^s_{\kappa,\delta} &= \left\{u \in \mathbb{C}; |\mathrm{Im}\, u| < \tan\beta_1 \mathrm{Re}\, u + 1/3 - \kappa G_0^{-3}, |\mathrm{Im}\, u| > -\tan\beta_2 \mathrm{Re}\, u + 1/6 - \delta\right\} \\ D^u_{\kappa,\delta} &= \{u \in \mathbb{C}; -u \in D^s_{\kappa,\delta}\}, \end{aligned} \tag{5.81}$$

which do not contain $u = 0$ (see Figure 5.9). These are the same domains that were used in [GMS16]. However, the intersection domain $D^s_{\kappa,\delta} \cap D^u_{\kappa,\delta}$ has empty intersection for real values of u and therefore, to compare both manifolds one needs to extend the stable manifold to a domain which overlaps with $D^u_{\kappa,\delta} \cap \mathbb{R}$. This is done in Section 5.7.5.

5.7.3 The invariance equation for the graph parameterizations

The graph parameterizations introduced in (5.80) satisfy the invariance equation

$$(\partial_u Z, \partial_\gamma Z)(u,\gamma) \cdot \begin{pmatrix} \partial_Y \mathcal{P} \\ \partial_\Lambda \mathcal{P} \end{pmatrix} (u,\gamma,Z(u,\gamma)) = \begin{pmatrix} -\partial_u P \\ -\partial_\gamma \mathcal{P} \\ -i\partial_\beta \mathcal{P} \\ i\partial_\alpha \mathcal{P} \end{pmatrix} (u,\gamma,Z(u,\gamma)) \tag{5.82}$$

Using vector notation $Z = Z(x)$ is invariant if

$$DZ(x)X_x(x, Z(x)) = X_Z(x, Z(x)) \tag{5.83}$$

where $X_x = (\partial_Y \mathcal{P}, \partial_\Lambda \mathcal{P})^\top$ and $X_Z = (-\partial_u \mathcal{P}, -\partial_\gamma \mathcal{P}, -i\partial_\beta \mathcal{P}, i\partial_\alpha \mathcal{P})^\top$.

Observe that $X = X^0 + X^1$ where X^i are the Hamiltonian vector fields associated to $\mathcal{P}_i$. Of course, when $\mathcal{P}_1 = 0$, $Z = 0$ satisfies the invariance equation (5.82). In fact,

$$\begin{aligned} X_x^0(x,0) &= (\partial_Y \mathcal{P}_0, \partial_\Lambda \mathcal{P}_0)^\top (x,0) = \left(1, \nu \frac{G_0^3}{L_0^3}\right)^\top \\ X_Z^0(x,0) &= (-\partial_u \mathcal{P}_0, -\partial_\gamma \mathcal{P}_0, -i\partial_\beta \mathcal{P}_0, i\partial_\alpha \mathcal{P}_0)^\top (x,0) = 0. \end{aligned}$$

Proposition 5.7.1. *The invariance equation* (5.82) *can be rewritten as*

$$\mathcal{L}Z = AZ + F(Z) \qquad \text{with} \qquad F(Z) = -\mathcal{G}_1(Z)\partial_u Z - \mathcal{G}_2(Z)\partial_\gamma Z + \mathcal{Q}(Z) \tag{5.84}$$

where

- $\mathcal{L}$ *is the operator*

$$\mathcal{L}(Z) = \partial_u Z + \nu \frac{G_0^3}{L_0^3} \partial_\gamma Z. \tag{5.85}$$

- *The functions* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are defined as*

$$\begin{aligned} \mathcal{G}_1(u,\gamma,Y,\Lambda,\alpha,\beta) &= \frac{Y}{G_0 \widehat{y}_h^2(u)} + f_1(u)q \\ \mathcal{G}_2(u,\gamma,Y,\Lambda,\alpha,\beta) &= \frac{G_0^3\nu}{(L_0+\Lambda)^3} - \frac{G_0^3\nu}{L_0^3} + f_1(u)Y + f_2(u)q + \partial_\Lambda \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) \end{aligned} \tag{5.86}$$

  where $q = \Lambda - \eta_0\beta - \xi_0\alpha - \alpha\beta$

- *The matrix* A *is*

$$A(u) = \begin{pmatrix} 0 & 0 \\ \mathcal{A}(u) & \mathcal{B}(u) \end{pmatrix} \tag{5.87}$$

  with

$$\mathcal{A}(u) = i \begin{pmatrix} f_1(u)\eta_0 & f_2(u)\eta_0 \\ -f_1(u)\xi_0 & -f_2(u)\xi_0 \end{pmatrix}, \qquad \mathcal{B}(u) = i f_2(u) \begin{pmatrix} -\eta_0\xi_0 & -\eta_0^2 \\ \xi_0^2 & \eta_0\xi_0 \end{pmatrix} \tag{5.88}$$

  where f_1 *and* f_2 *are defined in* (5.77).

- *The function* $\mathcal{Q}$ *is*

$$\begin{aligned} \mathcal{Q}_1(u,\gamma,Y,\Lambda,\alpha,\beta) &= \frac{\widehat{y}_h'(u)}{G_0\widehat{y}_h^3(u)} Y^2 - f_1'(u)Yq - f_2'(u)\frac{q^2}{2} - \frac{\partial \mathcal{P}_1}{\partial u}(u,\gamma,\Lambda,\alpha,\beta) \\ \mathcal{Q}_2(u,\gamma,Y,\Lambda,\alpha,\beta) &= -\frac{\partial \mathcal{P}_1}{\partial \gamma}(u,\gamma,\Lambda,\alpha,\beta) \\ \mathcal{Q}_3(u,\gamma,Y,\Lambda,\alpha,\beta) &= i\alpha\left[f_1(u)Y + f_2(u)\left(\Lambda - 2\eta_0\beta - \alpha\xi_0 - \alpha\beta\right)\right] - i\partial_\beta \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) \\ \mathcal{Q}_4(u,\gamma,Y,\Lambda,\alpha,\beta) &= -i\beta\left[f_1(u)Y + f_2(u)\left(\Lambda - \eta_0\beta - 2\alpha\xi_0 - \alpha\beta\right)\right] + i\partial_\alpha \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta). \end{aligned} \tag{5.89}$$

The proof of this proposition is done in Appendix 5.A.

To solve the invariance equation, we first integrate the linear system $\mathcal{L}Z = A(u)Z$, which, writing $Z = (Z_{Y\Lambda}, Z_{\alpha\beta})$, reads

$$\begin{aligned} \mathcal{L}Z_{Y\Lambda} &= 0 \\ \mathcal{L}Z_{\alpha\beta} &= \mathcal{A}(u)Z_{Y\Lambda} + \mathcal{B}(u)Z_{\alpha\beta} \end{aligned} \tag{5.90}$$

where $\mathcal{A}(u)$ and $\mathcal{B}(u)$ are given in (5.88). The proof of the following lemma is straighforward.

Lemma 5.7.2. *A fundamental matrix of the linear system* (5.90) *is*

$$\Phi_A^*(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \eta_0 g_1(u) & \eta_0 g_2^*(u) & 1-\eta_0\xi_0 g_2^*(u) & -\eta_0^2 g_2^*(u) \\ -\xi_0 g_1(u) & -\xi_0 g_2^*(u) & \xi_0^2 g_2^*(u) & 1+\eta_0\xi_0 g_2^*(u) \end{pmatrix}, \qquad * = u,s \tag{5.91}$$

where $g_1'(u) = if_1(u)$ *and* $(g_2^*)'(u) = if_2(u)$, $* = u,s$. *In particular we can choose the functions* g_2^* *satisfying that* $\lim_{\mathrm{Re}\, u\to+\infty} g_2^s(u) = 0$ *and* $\lim_{\mathrm{Re}\, u\to-\infty} g_2^u(u) = 0$.

We define two different inverse operators of $\mathcal{L}$,

$$\begin{aligned} \mathcal{G}^u(h)(u,\gamma) &= \int_{-\infty}^0 h(u+s,\gamma+\nu G_0^3 L_0^{-3} s)ds \\ \mathcal{G}^s(h)(u,\gamma) &= \int_{+\infty}^0 h(u+s,\gamma+\nu G_0^3 L_0^{-3} s)ds. \end{aligned} \tag{5.92}$$

We use them to prove the existence of the stable and unstable invariant manifolds. Here we only deal with the stable manifold of $P_{\eta_0+\delta\eta,\xi_0+\delta_\xi}$ and we take $\mathcal{G} = \mathcal{G}^s$ and $\Phi_A = \Phi_A^s$ (the unstable manifold of P_{η_0,ξ_0} is obtained analogously from the particular case $\delta\eta = \delta\xi = 0$).

We use this operator and the fundamental matrix Φ_A to derive an integral equation equivalent to the invariance equation (5.84).

Lemma 5.7.3. *The parameterization* Z^s *of* $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta_\xi})$ *is a fixed point of the operator*

$$\mathcal{F}(Z) = \Phi_A\delta z + \Phi_A\mathcal{G}\left(\Phi_A^{-1}F(Z)\right) = \Phi_A\delta z + \mathcal{G}_A\circ F(Z) \tag{5.93}$$

where $\delta z = (0,0,\delta\eta,\delta\xi)^\top$ *and*

$$\mathcal{G}_A(h) = \begin{pmatrix} \mathcal{G}(h_1) \\ \mathcal{G}(h_2) \\ \mathcal{G}(h_3) + \eta_0\mathcal{G}\left(f_1\mathcal{G}(h_1)\right) + \eta_0\mathcal{G}\left(f_2\mathcal{G}(h_2)\right) - \eta_0\mathcal{G}\left(f_2\mathcal{G}\left(\xi_0 h_3 + \eta_0 h_4\right)\right) \\ \mathcal{G}(h_4) - \xi_0\mathcal{G}\left(f_1\mathcal{G}(h_1)\right) - \xi_0\mathcal{G}\left(f_2\mathcal{G}(h_2)\right) + \xi_0\mathcal{G}\left(f_2\mathcal{G}\left(\xi_0 h_3 + \eta_0 h_4\right)\right). \end{pmatrix} \tag{5.94}$$

Proof. Using the fundamental matrix $\Phi_A(u)$ and the variation of constants formula give the first equality in (5.93). We point out that $\lim_{u\to+\infty}\Phi_A(u)\delta z = \delta z$. For the second one we write $\Phi_A\mathcal{G}\left(\Phi_A^{-1}F(Z)\right)$ in components as

$$\Phi_A\mathcal{G}\left(\Phi_A^{-1}F(Z_1)\right) = \begin{pmatrix} \mathcal{G}(F_1(Z_1)) \\ \mathcal{G}(F_2(Z_1)) \\ \mathcal{G}(F_3(Z_1)) + \eta_0 g_1\mathcal{G}(F_1(Z_1)) + \eta_0 g_2\mathcal{G}(F_2(Z_1)) \\ \quad +\eta_0\mathcal{G}\left(-g_1F_1(Z_1) - g_2F_2(Z_1) + \xi_0 g_2 F_3(Z_1) + \eta_0 g_2 F_4(Z_1)\right) \\ \quad -\eta_0 g_2\mathcal{G}\left(\xi_0 F_3(Z_1) + \eta_0 F_4(Z_1)\right) \\ \mathcal{G}(F_4(Z_1)) - \xi_0 g_1\mathcal{G}(F_1(Z_1)) - \xi_0 g_2\mathcal{G}(F_2(Z_1)) \\ \quad -\xi_0\mathcal{G}\left(-g_1F_1(Z_1) - g_2F_2(Z_1) - \xi_0 g_2F_3(Z_1) - \eta_0 g_2 F_4(Z_1)\right) \\ \quad +\xi_0 g_2\mathcal{G}\left(\xi_0 F_3(Z_1) + \eta_0 F_4(Z_1)\right) \end{pmatrix}.$$

Then, it only suffices to note that the terms of the form $g_i\mathcal{G}(F_j) - \mathcal{G}(g_iF_j)$ can be rewritten as

$$g_i\mathcal{G}(F_j) - \mathcal{G}(g_iF_j) = \mathcal{G}(\partial_u g_i\mathcal{G}(F_j)) = \mathcal{G}(f_i\mathcal{G}(F_j)).$$

Indeed, it is enough to apply the operator $\mathcal{L}$ to both sides. ☐

5.7.4 A fixed point of the operator $\mathcal{F}$

To obtain a fixed point of the operator $\mathcal{F}$ in (5.93), for $(u,\gamma)\in D^s_{\kappa,\delta}\times\mathbb{T}_\sigma$, we introduce the functional setting we work with. We consider functions of the form $Z=(Y,L,\alpha,\beta)^\top$. Take h any of these components and define its Fourier series

$$h(u,\gamma)=\sum_{q\in\mathbb{Z}}h^{[q]}(u)e^{iq\gamma}$$

Denote by f any of the Fourier coefficients, which are only functions of u, and take $\rho>0$. We consider the following norm, which captures the behavior as $\operatorname{Re}u\to\infty$ and also the behavior "close" to the singularities of the unperturbed homoclinic (see Lemma 5.7.8),

$$\|f\|_{n,m,q}=\sup_{D^s_{\kappa,\delta}\cap\{\operatorname{Re}u\geq\rho\}}|u^nf(u)|+\sup_{D^s_{\kappa,\delta}\cap\{\operatorname{Re}u\leq\rho\}}\left|\left(u-\frac{i}{3}\right)^m\left(u+\frac{i}{3}\right)^m e^{-iq\phi_{\rm h}(u)}f(u)\right|.$$

Now, for a fixed $\sigma>0$, we define the norm for h as

$$\|h\|_{n,m}=\sum_{k\in\mathbb{Z}}\|h^{[q]}\|_{n,m,q}e^{|q|\sigma}.$$

We denote the corresponding Banach space by $\mathcal{Y}_{n,m}$. Note that such norms do not define necessarily functions in $D^s_{\kappa,\delta}\times\mathbb{T}_\sigma$. Indeed, the Fourier series may be divergent for complex u due to the term $e^{-iq\phi_{\rm h}(u)}$ which grows exponentially as $|q|\to\infty$. Still, since $|e^{-iq\phi_{\rm h}(u)}|=1$ for real values of u, the Fourier series define actual functions for real values of u.

To prove the existence of the invariant manifolds, we need to keep control of the first derivatives for sequences of Fourier coefficients $h\in\mathcal{Y}_{n,m}$. The derivatives of sequences are defined in the natural way

$$\partial_u h(u,\gamma)=\sum_{q\in\mathbb{Z}}\partial_u h^{[q]}(u)e^{iq\gamma},\qquad \partial_\gamma h(u,\gamma)=\sum_{q\in\mathbb{Z}}(i\ell)h^{[q]}(u)e^{iq\gamma}. \tag{5.95}$$

Then, we also consider the norm

$$\llbracket h\rrbracket_{n,m}=\|h\|_{n,m}+\|\partial_u h\|_{n+1,m+1}+G_0^3\|\partial_\gamma h\|_{n+1,m+1}$$

and denote by $\mathcal{X}_{n,m}$ the associated Banach space. Since each component of $Z=(Y,L,\alpha,\beta)$ has different behavior, we define the weighted norms

$$\begin{aligned}\|Z\|_{n,m,\text{vec}}&=\|Y\|_{n+1,m+1}+\|\Lambda\|_{n,m}+\|\alpha e^{i\phi_{\rm h}(u)}\|_{n,m}+\|\beta e^{-i\phi_{\rm h}(u)}\|_{n,m}\\ \llbracket Z\rrbracket_{n,m,\text{vec}}&=\|Z\|_{n,m,\text{vec}}+\|\partial_u Z\|_{n+1,m+1,\text{vec}}+G_0^3\|\partial_\gamma Z\|_{n+1,m+1,\text{vec}}.\end{aligned} \tag{5.96}$$

We denote by $\mathcal{Y}_{n,m,\text{vec}}$ and $\mathcal{X}_{n,m,\text{vec}}$ the associated Banach spaces.

Since the Banach space $\mathcal{X}_{n,m,\text{vec}}$ is a space of formal Fourier series, the terms $\partial_z\mathcal{P}_1(u,\gamma,Z)$, $z=u,\gamma,\Lambda,\alpha,\beta$, which appear in Proposition 5.7.1 for $Z=(Y,\alpha,\Lambda,\beta)\in\mathcal{X}_{n,n,\text{vec}}$ are understood formally by the formal Taylor expansion[7]

$$\partial_z\mathcal{P}_1(u,\gamma,Z)=\sum_{\ell\geq0}\frac{1}{\ell!}\sum_{i_1=0}^{\ell}\sum_{i_2=0}^{\ell-i_1}\frac{\ell!}{i_1!i_2!(\ell-i_1-i_1)!}\frac{\partial^\ell(\partial_z\mathcal{P}_1)}{\partial^{i_1}\alpha\partial^{i_2}\beta\partial^{\ell-i_1-i_2}\Lambda}(u,\gamma,0,0,0)\alpha^{i_1}\beta^{i_2}\Lambda^{\ell-i_1-i_2}. \tag{5.97}$$

where $z=u,\gamma,\Lambda,\alpha,\beta$. In Lemma 5.7.7 below we give conditions on Z which make this formal composition meaningful.

Finally, we define

$$\widetilde{Z}=Z-\delta z\qquad\text{where}\qquad\delta z=(0,0,\delta\eta,\delta\xi)^\top$$

and introduce the operator

$$\widetilde{\mathcal{F}}(\widetilde{Z})=\mathcal{F}(\widetilde{Z}+\delta z)-\delta z \tag{5.98}$$

where $\mathcal{F}$ is defined in (5.93). It is clear that Z is a fixed point of $\mathcal{F}$ if and only if $\widetilde{Z}$ is a fixed point of $\widetilde{\mathcal{F}}$.

[7]Note that the function $\mathcal{P}_1$ in (5.84) does not depend on Y

Theorem 5.7.4. *Let $\delta z = (0, 0, \delta\eta, \delta\xi)^\top$ and denote by B_ρ the ball of radius ρ in $\mathcal{X}_{1/3,1/2,\text{vec}}$. There exists $b_0 > 0$ such that if $G_0 \gg 1$, $|\delta\eta|, |\delta\xi| \lesssim G_0^{-3}$ and*

$$|\eta_0| G_0^{3/2} \ll 1, \tag{5.99}$$

then, the operator $\widetilde{\mathcal{F}}$ defined in (5.98) has the following properties.

1. *$\widetilde{\mathcal{F}} : B_{b_0 G_0^{-3} \ln G_0} \to B_{b_0 G_0^{-3} \ln G_0}$,*
2. *It is Lipschitz in $B_{b_0 G_0^{-3} \ln G_0}$ with Lipschitz constant $\text{Lip}\,(\widetilde{\mathcal{F}}) \lesssim G_0^{-3/2} \ln^2 G_0$.*

Therefore, $\widetilde{\mathcal{F}}$ has a fixed point $\widetilde{Z}^s$ in $B_{b_0 G_0^{-3} \ln G_0}$. Denoting by $Z^s = \delta z + \widetilde{Z}^s$ and by $\mathcal{F}$ the operator (5.93) we have that $Z^s - \mathcal{F}(0) \in B_{b_0 G_0^{-3} \ln G_0}$ and

$$\llbracket Z^s - \mathcal{F}(0) \rrbracket_{1/3,1/2,\text{vec}} \lesssim G_0^{-9/2} \ln^3 G_0.$$

Moreover, $\widetilde{\Lambda}^s$ satisfies

$$\llbracket \widetilde{\Lambda}^s \rrbracket_{1/3,1} \lesssim G_0^{-9/2}. \tag{5.100}$$

Next proposition gives estimates for the derivatives of the invariant manifolds parameterizations for real values of (u, γ).

Proposition 5.7.5. *The parameterization $Z^s = \delta z + \widetilde{Z}^s$ obtained in Theorem 5.7.4 can be extended analytically to the domain*

$$u \in D^s_{\kappa,\delta} \cap \mathbb{R}, \gamma \in \mathbb{T}, |\eta_0| \le \frac{1}{2}, |\xi_0| \le \frac{1}{2}. \tag{5.101}$$

Moreover, in this domain the function $Z^s = (Y^s, \Lambda^s, \alpha^s, \beta^s)$ satisfies that

$$|Y^s| \le G_0^{-3}, \quad |\Lambda^s| \le C G_0^{-6}, \quad |\alpha^s| \le G_0^{-3}, \quad |\beta^s| \le C G_0^{-3}$$

and, for $N \ge 0$, its derivatives satisfy

$$|D^N (Z^s - \mathcal{F}(0))| \le C(N) G_0^{-6}, \tag{5.102}$$

where D^N denotes the differential of order N with respect to the variables $(u, \gamma, \eta_0, \xi_0)$ and $C(N)$ is a constant which may depend on N but independent of G_0.

Note that the condition (5.99) is not required in Proposition 5.7.5. Indeed this condition is needed to extend the Fourier coefficients of Z^s into points of $D^s_{\kappa,\delta}$ wich are G_0^{-3}-close to the singularities $u = \pm i/3$. The extension to the disk $|\eta_0| \le \frac{1}{2}, |\xi_0| \le \frac{1}{2}$ is needed to apply Cauchy estimates to obtain (5.102), which is needed (jointly with the analogous estimate for the parameterization of the unstable manifold) to obtain the estimates for the difference between the invariant manifolds given in (5.40).

We devote the rest of this section to proof Theorem 5.7.4. First in Section 5.7.4, we state several lemmas which give properties of the norms and the functional setting. Then, in Section 5.7.4 we give the fix point argument which proves Theorem 5.7.4. Finally, in Section 5.7.4, we explain how to adapt the proof of Theorem 5.7.4 to prove Proposition 5.7.5.

Technical lemmas

We devote this section to state several lemmas which are needed to prove Theorem 5.7.4. The first one, whose prove is straighforward, gives properties of the Banach spaces $\mathcal{Y}_{n,m}$.

Lemma 5.7.6. *The spaces $\mathcal{Y}_{n,m}$ satisfy the following properties:*

- *If $h \in \mathcal{Y}_{n,m}$ and $g \in \mathcal{Y}_{n',m'}$, then the formal product of Fourier series hg defined as usual by*

$$(hg)^{[\ell]}(v) = \sum_{k\in\mathbb{Z}} h^{[k]} g^{[\ell-k]}$$

satisfies that $hg \in \mathcal{Y}_{n+n',m+m'}$ and $\|hg\|_{n+n',m+m'} \leq \|h\|_{n,m}\|g\|_{n',m'}$.

- *If $h \in \mathcal{Y}_{n,m}$, then $h \in \mathcal{Y}_{n-\eta,m}$ with $\eta > 0$ and $\|h\|_{n-\eta,m} \leq K\|h\|_{n,m}$.*

- *If $h \in \mathcal{Y}_{n,m}$, then $h \in \mathcal{Y}_{n,m+\eta}$ with $\eta > 0$ and $\|h\|_{n,m+\eta} \leq K\|h\|_{n,m}$.*

- *If $h \in \mathcal{Y}_{n,m}$, then $h \in \mathcal{Y}_{n,m-\eta}$ with $\eta > 0$ and $\|h\|_{n,m-\eta} \leq KG_0^{3\eta}\|h\|_{n,m}$.*

We are going to find a fixed point of the operator (5.93) in a suitable space $\mathcal{X}_{n,m,\mathrm{vec}}$ of *formal* Fourier series. The previous lemma ensures that $\mathcal{X}_{n,m,\mathrm{vec}}$ is an algebra with respect to the usual product, but we need to ensure that the composition (5.97) is also well defined.

Lemma 5.7.7. *Consider $Z \in \mathcal{X}_{1,1/2,\mathrm{vec}}$ satisfying $\|Z\|_{1,1/2,\mathrm{vec}} \ll G_0^{-3/2}$. Then, the formal compositions $\partial_z\mathcal{P}_1(u,\gamma,Z(u,\gamma))$, $z = u,\gamma,\Lambda,\alpha,\beta$, defined in* (5.97) *satisfy*

$$(\partial_u\mathcal{P}_1(\cdot,\cdot,Z), \partial_\gamma\mathcal{P}_1(\cdot,\cdot,Z), \partial_\beta\mathcal{P}_1(\cdot,\cdot,Z), \partial_\alpha\mathcal{P}_1(\cdot,\cdot,Z)) \in \mathcal{X}_{2,3/2,\mathrm{vec}}$$

and

$$\|(\partial_u\mathcal{P}_1(\cdot,\cdot,Z), \partial_\gamma\mathcal{P}_1(\cdot,\cdot,Z), \partial_\beta\mathcal{P}_1(\cdot,\cdot,Z), \partial_\alpha\mathcal{P}_1(\cdot,\cdot,Z))\|_{2,3/2,\mathrm{vec}} \lesssim G_0^{-3}.$$

Moreover, if one defines

$$\begin{aligned}\Delta\mathcal{P}_1(Z,Z') =&(\partial_u\mathcal{P}_1(\cdot,\cdot,Z), \partial_\gamma\mathcal{P}_1(\cdot,\cdot,Z), \partial_\beta\mathcal{P}_1(\cdot,\cdot,Z), \partial_\alpha\mathcal{P}_1(\cdot,\cdot,Z)) \\ &- (\partial_u\mathcal{P}_1(\cdot,\cdot,Z'), \partial_\gamma\mathcal{P}_1(\cdot,\cdot,Z'), \partial_\beta\mathcal{P}_1(\cdot,\cdot,Z'), \partial_\alpha\mathcal{P}_1(\cdot,\cdot,Z')),\end{aligned}$$

then, for $Z, Z' \in \mathcal{X}_{1,1/2,\mathrm{vec}}$,

$$\|\Delta\mathcal{P}_1(Z,Z')\|_{3,2,\mathrm{vec}} \lesssim G_0^{-3}\|Z - Z'\|_{1,1/2,\mathrm{vec}}.$$

The proof of this lemma is a straighforward computation.

We also need a precise knowledge of the behavior of the paramerization of the unperturbed homoclinic introduced in Lemma 5.4.1 as $u \to \pm\infty$ and close to its complex singularities. They are given in the next two lemmas.

Lemma 5.7.8. *The homoclinic* (5.24) *with initial conditions* (5.29) *behaves as follows:*

- *As $|u| \to +\infty$,*

$$\widehat{r}_{\mathrm{h}}(u) \sim u^{2/3}, \quad \widehat{y}_{\mathrm{h}}(u) \sim u^{-1/3} \quad \text{and} \quad \phi_{\mathrm{h}}(u) - \pi \sim u^{-1/3} \ (\text{mod } 2\pi).$$

- *As $u \longrightarrow \pm i/3$,*

$$\widehat{r}_{\mathrm{h}}(u) \sim \left(u \mp \frac{i}{3}\right)^{1/2}, \quad \widehat{y}_{\mathrm{h}}(u) \sim \left(u \mp \frac{i}{3}\right)^{-1/2}, \quad e^{i\phi_{\mathrm{h}}(u)} \sim \left(\frac{u + \frac{i}{3}}{u - \frac{i}{3}}\right)^{1/2}.$$

The proof of this lemma is given in [GMS16]. From Lemma 5.7.8, one can derive properties for the functions f_1 and f_2 introduced in (5.77).

Lemma 5.7.9. *The functions f_1 and f_2 introduced in* (5.77) *satisfy $f_1 \in \mathcal{X}_{2/3,0}$ and $f_2 \in \mathcal{X}_{4/3,1/2}$. Moreover,*

$$[\![f_1]\!]_{2/3,0} \lesssim G_0^{-1} \quad \text{and} \quad [\![f_2]\!]_{4/3,1/2} \lesssim G_0^{-1}.$$

Finally we give properties of the operators introduced in (5.92) and (5.94).

Lemma 5.7.10. *The operator $\mathcal{G} = \mathcal{G}^s$ in (5.92), when considered acting on the spaces $\mathcal{X}_{n,m}$ and $\mathcal{Y}_{n,m}$ has the following properties.*

1. *For any $n > 1$ and $m \geq 1$, $\mathcal{G} : \mathcal{Y}_{n,m} \longrightarrow \mathcal{Y}_{n-1,m-1}$ is well defined and linear continuous. Moreover $\mathcal{L} \circ \mathcal{G} = \mathrm{Id}$.*

2. *If $h \in \mathcal{Y}_{n,m}$ for some $n > 1$ and $m > 1$, $\mathcal{G}(h) \in \mathcal{Y}_{n-1,m-1}$ and*
$$\|\mathcal{G}(h)\|_{n-1,m-1} \lesssim \|h\|_{n,m}.$$

3. *If $h \in \mathcal{Y}_{n,1}$ for some $n > 1$, $\mathcal{G}(h) \in \mathcal{Y}_{n-1,0}$ and*
$$\|\mathcal{G}(h)\|_{n-1,0} \lesssim \ln G_0 \|h\|_{n,1}$$

4. *If $h \in \mathcal{Y}_{n,m}$ for some $n \geq 1$ and $m \geq 1$ satisfies $\langle h \rangle_\gamma = 0$, $\mathcal{G}(h) \in \mathcal{Y}_{n,m}$ and*
$$\|\mathcal{G}(h)\|_{n,m} \lesssim G_0^{-3} \|h\|_{n,m}.$$

5. *If $h \in \mathcal{Y}_{n,m}$ for some $n \geq 1$ and $m \geq 1$, $\partial_u \mathcal{G}(h), \partial_\gamma \mathcal{G}(h) \in \mathcal{Y}_{n,m}$ and*
$$\|\partial_u \mathcal{G}(h)\|_{n,m} \lesssim \|h\|_{n,m}$$
$$\|\partial_\gamma \mathcal{G}(h)\|_{n,m} \lesssim G_0^{-3} \|h\|_{n,m}.$$

6. *From the previous statements, one can conclude that if $h \in \mathcal{Y}_{n,m}$ for some $n > 1$ and $m \geq 1$, then $\mathcal{G}(h) \in \mathcal{X}_{n-1,m-1}$ and*
$$\llbracket \mathcal{G}(h) \rrbracket_{n-1,m-1} \lesssim \|h\|_{n,m} \qquad \text{if } m > 1$$
$$\llbracket \mathcal{G}(h) \rrbracket_{n-1,m-1} \lesssim \ln G_0 \|h\|_{n,m} \qquad \text{if } m = 1.$$

Additionally,

(7) if $h \in \mathcal{Y}_{n,m}$ for $n > 1$, $m > 3/2$ then
$$\|e^{\pm i\phi_h} \mathcal{G}(e^{\mp i\phi_h} h)\|_{n-1,m-1} \lesssim \|h\|_{n,m},$$

(8) if $h \in \mathcal{Y}_{n,3/2}$ for $n > 1$, then
$$\|e^{\pm i\phi_h} \mathcal{G}(e^{\mp i\phi_h} h)\|_{n-1,1/2} \lesssim \ln G_0 \|h\|_{n,3/2}.$$

Claims 1 to 6 in this lemma are proved for $m > 1$ in [GMS16]. The case $m = 1$ can be proven analogously. Claims 7 and 8 can be deduced analogously taking into account the expression of $e^{\mp i\phi_h}$ given in Lemma 5.7.8

From this result we can deduce the following lemma, which is a direct consequence of Lemmas 5.7.9 and 5.7.10.

Lemma 5.7.11. *Consider $h \in \mathcal{Y}_{n,m,\mathrm{vec}}$ for $n > 1$ and $m \geq 3/2$. Then, the operator $\mathcal{G}_A$ introduced in (5.94) satisfies the following:*

- *If $m > 3/2$, $\mathcal{G}_A(h) \in \mathcal{X}_{n-1,m-1,\mathrm{vec}}$ and $\llbracket \mathcal{G}_A(h) \rrbracket_{n-1,m-1,\mathrm{vec}} \lesssim \|h\|_{n,m,\mathrm{vec}}$,*
- *If $m = 3/2$, $\mathcal{G}_A(h) \in \mathcal{X}_{n-1,1/2,\mathrm{vec}}$ and $\llbracket \mathcal{G}_A(h) \rrbracket_{n-1,1/2,\mathrm{vec}} \lesssim \ln G_0 \|h\|_{n,3/2,\mathrm{vec}}$.*

The fixed point argument: Proof of Theorem 5.7.4

To prove the existence of a fixed point of the operator $\widetilde{\mathcal{F}}$ defined in (5.98), we start by analyzing $\widetilde{\mathcal{F}}(0) = (\Phi_A - \mathrm{Id})\delta z + \mathcal{G}_A \circ F(\delta z)$ (see (5.84) and (5.94)) where $\delta z = (0, 0, \delta_\eta, \delta_\xi)^\top$.

Since F has several terms, we split $F(Z) = -\mathcal{G}_1(Z)\partial_u Z - \mathcal{G}_2(Z)\partial_\gamma Z + \mathcal{Q}(Z)$ (see (5.84)) as $F = F^1 + F^2 + F^3$ where

$$F^1(Z) = -\mathcal{G}_1(Z)\partial_u Z - \mathcal{G}_2(Z)\partial_\gamma Z \tag{5.103}$$

$$F^2(Z) = \begin{pmatrix} \frac{\widehat{y}_\mathrm{h}'(u)}{G_0 \widehat{y}_\mathrm{h}^3(u)} Y^2 - f_1'(u) Y q - f_2'(u) \frac{q^2}{2} \\ 0 \\ i\alpha \left(f_1(u)Y + f_2(u)\Lambda - 2 f_2(u)\eta_0\beta - f_2(u)\alpha\xi_0 - f_2(u)\alpha\beta \right) \\ -i\beta \left(f_1(u)Y + f_2(u)\Lambda - f_2(u)\eta_0\beta - 2 f_2(u)\alpha\xi_0 - f_2(u)\alpha\beta \right) \end{pmatrix} \tag{5.104}$$

$$F^3(Z) = \begin{pmatrix} -\partial_u \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) \\ -\partial_\gamma \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) \\ -i\partial_\beta \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) \\ i\partial_\alpha \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) \end{pmatrix} \tag{5.105}$$

and $q = \Lambda - \eta_0\beta - \xi_0\alpha - \alpha\beta$.

First we notice that $F^1(\delta z) = 0$. Denoting by $|\delta z| = |\delta\eta| + |\delta\xi|$ it is straightforward to check, using Lemma 5.7.9, that

$$\left\| F^2(\delta z) \right\|_{4/3,1/2} \lesssim G_0^{-1} |\delta z|^2.$$

On the other hand, by the bounds of $\mathcal{P}_1$ in Lemma 5.B.1 (see the estimates (5.273)) and the estimates for $\widehat{r}_\mathrm{h}$ and $\widehat{y}_\mathrm{h}$ in Lemma 5.7.8, one has that $F^3(\delta z) \in \mathcal{Y}_{2,3/2,\mathrm{vec}}$ and

$$\left\| F^3(\delta z) \right\|_{2,3/2,\mathrm{vec}} \lesssim G_0^{-3}.$$

Then, applying Lemma 5.7.11 one obtains $\widetilde{\mathcal{F}}(0) \in \mathcal{X}_{1/3,1/2,\mathrm{vec}}$ and that there exists $b_0 > 0$ such that

$$\llbracket \widetilde{\mathcal{F}}(0) \rrbracket_{1/3,1/2,\mathrm{vec}} \leq \frac{b_0}{4} (G_0^{-2} + |\delta z| + |\delta z|^2) G_0^{-1} \ln G_0 \leq \frac{b_0}{2} G_0^{-3} \ln G_0. \tag{5.106}$$

where we have used that

$$\|(\Phi_A - \mathrm{Id})\delta z\|_{1/3,1/2,\mathrm{vec}} \lesssim G_0^{-1} |\delta z|$$

and the hypothesis $|\delta z| \lesssim G_0^{-3}$.

Next step is to prove that $\widetilde{\mathcal{F}}$ is contractive in the ball $B(b_0 G_0^{-3} \ln G_0) \subset \mathcal{X}_{1/3,1/2,\mathrm{vec}}$. For that we compute separately the Lipschitz constant of each of the terms $F^i(Z)$ for $i = 1, 2, 3$.

Notation 5.7.12. *In the statements of the forthcoming lemmas, given an element $\widetilde{Z} \in \mathcal{X}_{1/3,1/2}$ we write $Z = \widetilde{Z} + \delta z$.*

We also assume without mentioning that $|\delta\eta|, |\delta\xi| \lesssim G_0^{-3}$ and $\eta_0 G_0^{3/2} \ll 1$.

Lemma 5.7.13. *Consider $\widetilde{Z}, \widetilde{Z}' \in \mathcal{X}_{1/3,1/2,\mathrm{vec}}$ with $\llbracket \widetilde{Z} \rrbracket_{1/3,1/2,\mathrm{vec}}, \llbracket \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}} \lesssim G_0^{-3} \ln G_0$. Then, the functions $\mathcal{G}_1$ and $\mathcal{G}_2$ introduced in* (5.86) *satisfy that*

$$\|\mathcal{G}_1(Z)\|_{2/3,1/2}, \|\mathcal{G}_1(Z')\|_{2/3,1/2} \lesssim G_0^{-4} \ln G_0$$
$$\|\mathcal{G}_2(Z)\|_{1/3,1/2}, \|\mathcal{G}_2(Z')\|_{1/3,1/2} \lesssim \ln G_0$$

and

$$\|\mathcal{G}_1(Z) - \mathcal{G}_1(Z')\|_{2/3,1/2} \lesssim G_0^{-1} \llbracket \widetilde{Z} - \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}}$$
$$\|\mathcal{G}_2(Z) - \mathcal{G}_2(Z')\|_{1/3,1/2} \lesssim G_0^{3} \llbracket \widetilde{Z} - \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}}.$$

This lemma is a direct consequence of the definition of $\mathcal{G}_1$ and $\mathcal{G}_2$ in (5.86) and Lemmas 5.7.8 and 5.7.6. We use this lemma to compute the Lipschitz constant of F^1.

Lemma 5.7.14. *Consider $\widetilde{Z}, \widetilde{Z}' \in \mathcal{X}_{1/3,1/2,\mathrm{vec}}$ with $\llbracket \widetilde{Z} \rrbracket_{1/3,1/2,\mathrm{vec}}, \llbracket \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}} \leq b_0 G_0^{-3} \ln G_0$. Then, the function F^1 introduced in* (5.103) *satisfies*

$$\|F^1(Z)\|_{5/3,2,\mathrm{vec}} \lesssim G_0^{-6} \ln^2 G_0$$
$$\|F^1(Z) - F^1(Z')\|_{5/3,2,\mathrm{vec}} \lesssim G_0^{-3} \ln G_0 \llbracket \widetilde{Z} - \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}}.$$

Proof. The first part plainly follows from the definition of F^1 and the estimates in Lemma 5.7.13. We now obtain the result for the difference. The second component can be written as

$$\begin{aligned} F_2^1(Z) - F_2^1(Z') = & \left(\partial_u \Lambda - \partial_u \Lambda'\right) \mathcal{G}_1(Z) + \partial_u \Lambda' \left(\mathcal{G}_1(Z) - \mathcal{G}_1(Z')\right) \\ & \left(\partial_\gamma \Lambda - \partial_\gamma \Lambda'\right) \mathcal{G}_2(Z) + \partial_\gamma \Lambda' \left(\mathcal{G}_2(Z) - \mathcal{G}_2(Z')\right). \end{aligned}$$

Then, the estimate for the second component is a consequence of Lemmas 5.7.6 and 5.7.13 and the fact that $\llbracket \widetilde{Z} \rrbracket_{1/3,1/2,\mathrm{vec}}, \llbracket \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}} \leq b_0 G_0^{-3} \ln G_0$. The other components can be estimated analogously. □

Lemma 5.7.15. *Consider $\widetilde{Z}, \widetilde{Z}' \in \mathcal{X}_{1/3,1/2,\mathrm{vec}}$ with $\llbracket \widetilde{Z} \rrbracket_{1/3,1/2,\mathrm{vec}}, \llbracket \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}} \leq b_0 G_0^{-3} \ln G_0$. Then, the function F^2 introduced in* (5.104) *satisfies*

$$\|F^2(Z)\|_{4/3,2,\mathrm{vec}} \lesssim G_0^{-7} \ln^2 G_0$$
$$\|F^2(Z) - F^2(Z')\|_{4/3,2,\mathrm{vec}} \lesssim G_0^{-4} \ln G_0 \llbracket \widetilde{Z} - \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}}.$$

Proof. We recall that F^2 was defined in (5.104). For the first component F_1^2 we obtain

$$\|F_1^2(Z)\|_{7/3,3} \lesssim G_0^{-1} \left(\|Y\|_{4/3,3/2}^2 + \|Y\|_{2/3,5/2} \|q\|_{0,1/2} + \|q\|_{0,1/2}^2 \right) \lesssim G_0^{-7} \ln^2 G_0,$$

where we have used that $\|q\|_{0,1/2} \lesssim (|\delta z| + \|\tilde{Z}\|_{0,1/2}) \lesssim G_0^{-3} \ln G_0$. On the other hand, for the difference

$$\left\| F_1^2(Z) - F_1^2(Z') \right\|_{7/3,3} \lesssim G_0^{-4} \ln G_0 \|\widetilde{Z} - \widetilde{Z}'\|_{1/3,1/2,\mathrm{vec}}.$$

Similar computations lead to the following estimate for the third component

$$\left\| F_3^2(Z) \right\|_{4/3,2} \lesssim G_0^{-7} \ln G_0$$

and to the bound for the difference

$$\left\| F_3^2(Z) - F_3^2(Z') \right\|_{4/3,2} \lesssim G_0^{-4} \ln G_0 \|\widetilde{Z} - \widetilde{Z}'\|_{1/3,1/2,\mathrm{vec}}.$$

Proceeding analogously one obtains the same estimate for F_4^2. Since $F_2^2 = 0$, the claim follows. □

Lemma 5.7.16. *Consider $\widetilde{Z}, \widetilde{Z}' \in \mathcal{X}_{1/3,1/2,\mathrm{vec}}$ with $\llbracket \widetilde{Z} \rrbracket_{1/3,1/2,\mathrm{vec}}, \llbracket \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}} \leq b_0 G_0^{-3} \ln G_0$. Then, the function F^3 introduced in* (5.105) *satisfies*

$$\|F^3(Z) - F^3(0)\|_{2,2,\mathrm{vec}} \lesssim G_0^{-6} \ln^2 G_0$$
$$\|F^3(Z) - F^3(Z')\|_{2,2,\mathrm{vec}} \lesssim G_0^{-3} \llbracket \widetilde{Z} - \widetilde{Z}' \rrbracket_{1/3,1/2,\mathrm{vec}}.$$

Proof. To prove the first statement, one can write

$$F^3(Z) - F^3(0) = (F^3(\delta z + \widetilde{Z}) - F^3(\delta z)) + (F^3(\delta z) - F^3(0))$$

For the first term in the right hand side, it is enough to apply Lemma 5.7.7. To estimate the second term, one can use the mean value theorem and the estimates for the derivatives of $\mathcal{P}_1$ given in Lemma 5.B.1 (and Cauchy estimates). The second statement in Lemma 5.7.16 is a direct consequence of Lemma 5.7.7. □

Now we are ready to prove Theorem 5.7.4.

Proof of Theorem 5.7.4. Lemmas 5.7.14, 5.7.15 and 5.7.16 imply that

$$\|F(Z) - F(Z')\|_{4/3,2,\text{vec}} \lesssim G_0^{-3} \ln G_0 \llbracket \widetilde{Z} - \widetilde{Z}' \rrbracket_{1/3,1/2,\text{vec}}.$$

for $\widetilde{Z}, \widetilde{Z}' \in B(b_0 G_0^{-3}) \subset \mathcal{X}_{1/3,1/2,\text{vec}}$. Therefore, applying Lemma 5.7.11, one has

$$\llbracket \widetilde{\mathcal{F}}(\widetilde{Z}) - \widetilde{\mathcal{F}}(\widetilde{Z}') \rrbracket_{1/3,1,\text{vec}} \lesssim G_0^{-3} \ln^2 G_0 \llbracket \widetilde{Z} - \widetilde{Z}' \rrbracket_{1/3,1/2,\text{vec}}, \tag{5.107}$$

which implies

$$\llbracket \widetilde{\mathcal{F}}(\widetilde{Z}) - \widetilde{\mathcal{F}}(\widetilde{Z}') \rrbracket_{1/3,1/2,\text{vec}} \lesssim G_0^{-3/2} \ln^2 G_0 \llbracket \widetilde{Z} - \widetilde{Z}' \rrbracket_{1/3,1/2,\text{vec}}.$$

Thus, for G_0 large enough, $\widetilde{\mathcal{F}}$ is contractive from $B(b_0 G_0^{-3} \ln G_0) \subset \mathcal{X}_{1/3,1/2,\text{vec}}$ to itself with Lipschitz constant of size $\text{Lip} \lesssim G_0^{-3/2} \ln^2 G_0$ and it has a unique fixed point $\widetilde{Z}^s$. Denote now by $Z^s = \delta z + \widetilde{Z}^s$. By definition of the operator $\mathcal{F}$ we have that

$$Z^s - \mathcal{F}(0) = \mathcal{G}_A(F(Z^s) - F(0)) = \mathcal{G}_A F_1(Z^s) + \mathcal{G}_A F_2(Z^s) + \mathcal{G}_A(F_3(Z^s) - F_3(0))$$

so it follows from Lemmas 5.7.14, 5.7.15 and 5.7.16 that $Z^s - \mathcal{F}(0) \in \mathcal{X}_{1/3,1/2,\text{vec}} \subset \mathcal{Y}_{1/3,1/2,\text{vec}}$ and

$$\begin{aligned}
\|Z^s - \mathcal{F}(0)\|_{1/3,1/2,\text{vec}} &\lesssim G_0^{3/2} \ln G_0 \|F(Z^s) - F(0)\|_{1/3,1,\text{vec}} \\
&\lesssim G_0^{3/2} \llbracket \mathcal{G}_A F_1(Z^s) + \mathcal{G}_A F_2(Z^s) + \mathcal{G}_A(F_3(Z^s) - F_3(0)) \rrbracket_{1/3,1,\text{vec}} \\
&\lesssim G_0^{3/2} (G_0^{-6} \ln^2 G_0 + G_0^{-7} \ln^2 G_0 + G_0^{-6} \ln^3 G_0) \lesssim G_0^{-9/2} \ln^3 G_0.
\end{aligned}$$

Now it only remains to obtain the improved estimates for $\widetilde{\Lambda}^s$. Since it is a fixed point of $\widetilde{\mathcal{F}}$,

$$\widetilde{\Lambda}^s = \widetilde{\mathcal{F}}_2(0) + \left(\widetilde{\mathcal{F}}_2(\widetilde{Z}^s) - \widetilde{\mathcal{F}}_2(0) \right).$$

For the second term, we use (5.107) to obtain

$$\llbracket \widetilde{\mathcal{F}}_2(\widetilde{Z}^s) - \widetilde{\mathcal{F}}_2(0) \rrbracket_{1/3,1} \lesssim G_0^{-3} \ln^2 G_0 \llbracket \widetilde{Z}^s \rrbracket_{1/3,1/2} \lesssim G_0^{-6} \ln^3 G_0.$$

Using (5.93), we write the first term as $\widetilde{\mathcal{F}}_2(0) = \mathcal{G}(F_2(\delta z))$ where $F_2(\delta z) = -\partial_\gamma \mathcal{P}_1(u, \gamma, 0, \delta\eta, \delta\xi)$. Since $\langle F_2(\delta z) \rangle_\gamma = 0$ and satisfies $\|F_2(\delta z)\|_{2,3/2} \lesssim G_0^{-3}$ (see Lemma 5.B.1), one can apply item 4 of Lemma 5.7.10 one obtains

$$\llbracket \widetilde{\mathcal{F}}_2(0) \rrbracket_{1,1} \lesssim G_0^{3/2} \llbracket \widetilde{\mathcal{F}}_2(0) \rrbracket_{2,3/2} \lesssim G_0^{-3/2} \|F_2(\delta z)\|_{2,3/2} \lesssim G_0^{-9/2}.$$

□

Proof of Proposition 5.7.5

The proof of Theorem 5.7.4 can be carried out in the same way in the smaller domain

$$\widetilde{D}^u = \left\{ u \in \mathbb{C}; |\text{Im}\, u| < -\tan\beta_1 \text{Re}\, v + 1/4, |\text{Im}\, u| > \tan\beta_2 \text{Re}\, u + 1/6 - \delta \right\}$$

where all the points are at a uniform distance from the singularities $u = \pm i/3$. In this case the perturbing potential $\mathcal{P}_1$ can be easily estimated of order G_0^{-3} with the norm (5.96) with $m = 0$ for any (η_0, ξ_0) satisfying $|\eta_0|, |\xi_0| \leq 1/2$. That is, without imposing condition (5.99). Note that now the weight at the singularities $u = \pm i/3$, measured by m, is harmless since the points $\widetilde{D}^u$ are $\mathcal{O}(1)$-far from them.

This gives the estimates for the invariant manifolds. One can obtain the improved estimate for Λ and the estimate of the Lipschitz constant as has been done in the proof of Theorem 5.7.4. To obtain the estimates for the derivatives it is enough to apply Cauchy estimates. Note that in all the variables one can apply these estimates in disks of radius independent of G_0.

5.7.5 Extension of the parametrization of the unstable manifold by the flow

Theorem 5.7.4 gives a graph parameterization Z^s of the form (5.80) of $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi})$ as a formal Fourier series with analytic Fourier coefficients defined in the domain $D^s_{\kappa,\delta} \times \mathbb{T}$.

To compute the difference betweent the stable and unstable manifold, it is necessary to have the parameterizations of both manifolds defined in a common (real) domain. However, since $\widehat{y}_{\mathrm{h}}(0) = 0$, it is not possible to extend these parameterizations to a common domain containing a real interval (see (5.71)). Therefore, to compare them, we extend the stable manifold using a different parametrization. We proceed analogously as in [GMS16].

Given the paramerization $Z^s(u,\gamma) = (u, Y(u,\gamma), \gamma, \Lambda(u,\gamma), \alpha(u,\gamma), \beta(u,\gamma))$, the first step is to look for a change of variables of the form

$$\mathrm{Id} + g : (v,\xi) \mapsto (u,\gamma) = (v + g_1(v,\xi), \xi + g_2(v,\xi)), \tag{5.108}$$

such that $\mathtt{Z}^s = Z^s \circ (\mathrm{Id} + g)$ satisfies the invariance equation

$$\Phi_t\left(\mathtt{Z}^s(v,\xi)\right) = \mathtt{Z}^s\left(v + t, \xi + \frac{\nu G_0^3}{L_0^3} t\right), \tag{5.109}$$

where Φ_t is the flow associated to the Hamiltonian system (5.74). Note that the composition is understood as formal composition of formal Fourier series

$$h \circ (\mathrm{Id} + g)(v,\xi) = h(v + g_1(v,\xi), \xi + g_2(v,\xi)) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^{m} \binom{m}{n} \partial_v^{m-n} \partial_\xi^n h(v,\xi) g_1^{m-n}(v,\xi) g_2^n(v,\xi).$$

Denoting by X the associated vector field to Hamiltonian (5.74) equation (5.109) is equivalent to

$$\mathcal{L}(\mathtt{Z}^s) = X \circ \mathtt{Z}^s, \tag{5.110}$$

where the operator $\mathcal{L}$ is defined in (5.85).

We want $\mathtt{Z}^s$ to be defined in the domain $D^{\mathrm{flow}}_{\kappa,\delta} \times \mathbb{T}$, where

$$D^{\mathrm{flow}}_{\kappa,\delta} = \left\{ v \in \mathbb{C}; |\mathrm{Im}\, v| < \tan\beta_1 \mathrm{Re}\, v + 1/3 - \kappa G_0^{-3},\ |\mathrm{Im}\, v| < -\tan\beta_2 \mathrm{Re}\, v + 1/6 + \delta \right\}, \tag{5.111}$$

which can be seen in Figure 5.12.

We will relate the two types of the parameterization in the overlapping domain

$$D^{\mathrm{ovr}}_{\kappa,\delta} = D^{\mathrm{flow}}_{\kappa,\delta} \cap D^s_{\kappa,\delta}. \tag{5.112}$$

Proceeding as in [GMS16], one can obtain in this domain the change of coordinates (5.108). Abusing notation, we use the Banach space $\mathcal{Y}_{n,m}$ introduced in Section 5.7.4. Recall that the index n refers to the decay at infinity and therefore it does not give any information in the compact domain $D^{\mathrm{ovr}}_{\kappa,\delta}$ and therefore we can just take $n = 0$.

Lemma 5.7.17. *Let δ, κ and σ be the constants fixed in the statement of Theorem 5.7.4. Let $\sigma_1 < \sigma$, $\delta_1 < \delta$ and $\kappa_1 > \kappa$ such that $(\log\kappa_1 - \log\kappa)/2 < \sigma_1 - \sigma$ be fixed and consider the domain $D^{\mathrm{ovr}}_{\kappa_1,\delta_1} \times \mathbb{T}_{\sigma_1}$. Then, for G_0 big enough, there exists a (not necessarily convergent) Fourier series $g = (g_1, g_2) \in \mathcal{Y}_{0,0} \times \mathcal{Y}_{0,0}$ satisfying*

$$\|g_1\|_{0,0} \leq b_2 G_0^{-4}, \qquad \|g_2\|_{0,0} \leq b_2 G_0^{-3/2},$$

where $b_2 > 0$ is a constant independent of G_0, such that $\mathtt{Z}^s = Z^s \circ (\mathrm{Id} + g)$, satisfies (5.110).

Once we have obtained a parameterization $\mathtt{Z}^s$ which satisfies (5.110) in the overlapping domain $D^{\mathrm{ovr}}_{\kappa,\delta}$ (see (5.112)), next step is to extend this parameterization to the domain $D^{\mathrm{flow}}_{\kappa,\delta}$ in (5.111). This extension is done through a fixed point argument and follows the same lines as the flow extension of [GMS16] (Section 5.5.2). We write the parameterization $\mathtt{Z}^s$ as $\mathtt{Z}^s(v,\xi)=\mathtt{Z}_0^s(v,\xi)+\mathtt{Z}_1^s(v,\xi)$ with

$$\mathtt{Z}_0^s=\begin{pmatrix} v\\ 0\\ \xi\\ 0\\ 0\\ 0\end{pmatrix},\qquad \mathtt{Z}_1^s=\begin{pmatrix} G_0^{-1}\mathtt{U}(v,\xi)\\ \mathtt{Y}(v,\xi)\\ G_0^3\Gamma(v,\xi)\\ \Lambda(v,\xi)\\ \mathtt{A}(v,\xi)\\ \mathtt{B}(v,\xi)\end{pmatrix}. \tag{5.113}$$

The G_0–factors in the u and γ component is just to normalize the sizes. In the statement of the following lemma, abusing notation, we also use the Banach space $\mathcal{Y}_{n,m}$ introduced in Section 5.7.4 referred to the domain $D^{\mathrm{flow}}_{\kappa,\delta}$. Since the domain $D^{\mathrm{flow}}_{\kappa,\delta}$ is compact and all points are at a distance independent of G_0 from the singularities $v=\pm i/3$, we can just take $n=m=0$ (all norms $\|\cdot\|_{n,m}$ are equivalent).

Lemma 5.7.18. *Let κ_1 , δ_1, σ_1 be the constants considered in Lemma 5.7.17. Then, there exists a solution of equation* (5.110) *of the form* (5.113) *(as a formal Fourier series) for $(v,\xi)\in D^{\mathrm{flow}}_{\kappa_1,\delta_1}\times\mathbb{T}_{\sigma_1}$, whose Fourier coefficients are analytic continuation of those obtained in Lemma 5.7.17. Moreover, they satisfy $\mathtt{Z}_1^s\in\mathcal{Y}^6_{0,0}$ and*

$$\|\mathtt{U}\|_{0,0},\|\mathtt{Y}\|_{0,0},\|\Gamma\|_{0,0},\|\Lambda\|_{0,0},\|\mathtt{A}\|_{0,0},\|\mathtt{B}\|_{0,0}\lesssim G_0^{-3}.$$

The proof of this lemma is analogous to the one of Proposition 5.20 in [GMS16]. This is a standard fixed point argument in the sense that the domain $D^{\mathrm{flow}}_{\kappa,\delta}$ is "far" from the singularities $v=\pm i/3$ (the distance to these points is independent of G_0). The only issue that one has to keep in mind that we are dealing with formal Fourier series.

Once we have obtained this flow parameterizations in $D^{\mathrm{flow}}_{\kappa,\delta}$, the last step is to switch back to the graph parameterization (5.80). We want the graph parameterization to be defined in the following domain where we can compare the graph parameterizations of the stable and unstable invariant manifolds.

$$\begin{aligned} D_{\kappa,\delta}=&\{v\in\mathbb{C};\ |\mathrm{Im}\,v|<\tan\beta_1\mathrm{Re}\,v+1/3-\kappa G_0^{-3},|\mathrm{Im}\,v|<-\tan\beta_1\mathrm{Re}\,v+1/3-\kappa G_0^{-3},\\ &|\mathrm{Im}\,v|>\tan\beta_2\mathrm{Re}\,v+1/6-\delta\}\,,\end{aligned} \tag{5.114}$$

where $\kappa\in(0,1/3)$, $\delta\in(0,1/12)$ and $\beta_1,\beta_2\in(0,\pi/2)$ are fixed independently of G_0 (see Figure 5.10). Therefore, this domain is not empty provided $G_0>1$.

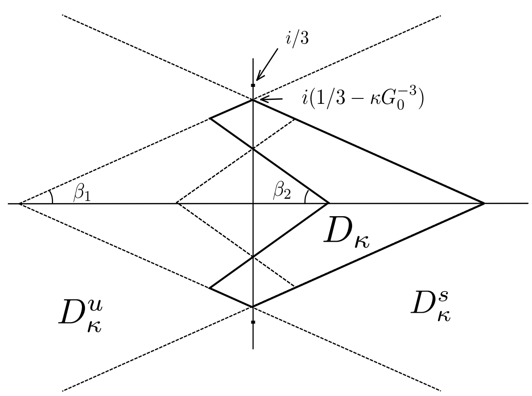


Figure 5.10: The domains $D_{\kappa,\delta}$ defined in (5.114).

Note that Theorem 5.7.4 gives already the graph parameterization Z^s in the domain $D_{\kappa,\delta}\cap D^s_{\kappa,\delta}$. Now it only remains to show that they are also defined in the domain

$$\begin{aligned}\widetilde{D}_{\kappa,\delta}=\Big\{&v\in\mathbb{C};|\mathrm{Im}\,v|<\tan\beta_1\mathrm{Re}\,v+1/3-\kappa G_0^{-3},|\mathrm{Im}\,v|>\tan\beta_2\mathrm{Re}\,v+1/6-\delta\\ &|\mathrm{Im}\,v|<-\tan\beta_2\mathrm{Re}\,v+1/6+\delta\Big\}\end{aligned} \tag{5.115}$$

(see Figure 5.11).

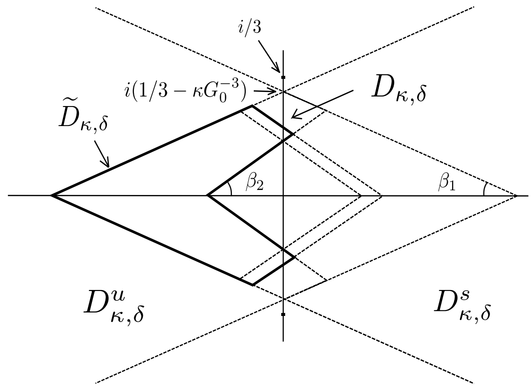


Figure 5.11: The domain $\widetilde{D}_{\kappa,\delta}$ defined in (5.115).

Indeed, it is easy to see that

$$D_{\kappa,\delta} \subset D^s_{\kappa,\delta} \cup \widetilde{D}_{\kappa,\delta}.$$

We look for a change of coordinates which transforms the flow–parameterization obtained in Lemma 5.7.18 to the graph parameterization (5.80). Note that this change is just the inverse of

$$\begin{aligned} u &= v + G_0^{-1}\mathtt{U}(v,\xi) \\ \gamma &= \xi + G_0^3\mathtt{\Gamma}(v,\xi) \end{aligned}$$

where $\mathtt{U}$ and $\mathtt{\Gamma}$ are defined in (5.113).

Lemma 5.7.19. *Consider the constants κ_1, δ_1 and σ_1 considered in Lemma 5.7.18 and any $\kappa_2 > \kappa_1$, $\delta_2 > \delta_1$ and $\sigma_2 < \sigma_1$. Then,*

- *There exists a function $h = (h_1, h_2) \in \mathcal{Y}_{0,0} \times \mathcal{Y}_{0,0}$ with*

  $$\|h_1\|_{0,0} \le b_4 \mu G_0^{-4}, \qquad \|h_2\|_{0,0} \le b_4 \mu G_0^{-1}.$$

  such that the change of coordinates $\mathrm{Id} + h$ is the inverse of the restriction of the change given by Lemma 5.7.17 to the domain $D^s_{\kappa_1,\delta_1} \cap \widetilde{D}_{\kappa_2,\delta_2}$.

- *Moreover,*

  $$Z^s = \mathtt{Z}^s \circ (\mathrm{Id} + h)$$

  defines a formal Fourier series which gives a parameterization of the stable invariant manifold as a graph, that is of the form (5.80). Then, in the domain $D_{\kappa_2,\delta_2} \times \mathbb{T}_{\sigma_2}$ this parameterization satisfies

  $$\begin{aligned} \|Y\|_{0,3/2} &\lesssim G_0^{-3} \ln G_0, & \|\Lambda\|_{0,3/2} &\lesssim G_0^{-9/2}, \\ \|\alpha e^{i\phi(u)}\|_{0,1/2} &\lesssim G_0^{-3} \ln G_0, & \|\beta e^{-i\phi(u)}\|_{0,1/2} &\lesssim G_0^{-3} \ln G_0. \end{aligned}$$

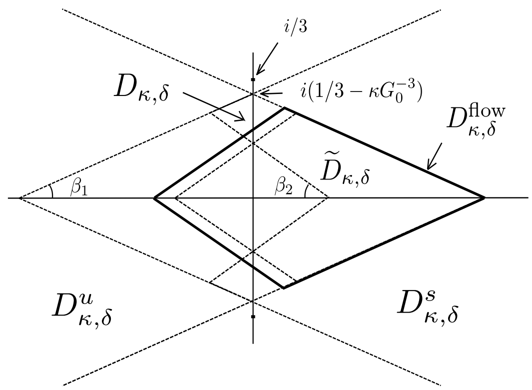


Figure 5.12: The domain $D^{\mathrm{flow}}_{\kappa,\delta}$ defined in (5.115).

Taking $\delta z = 0$ and proceeding analgously one obtains analogous results to Theorems 5.7.4 and Proposition 5.7.5 for $W^u(P_{\eta_0,\xi_0})$. For the sake of clarity, we sum up the properties of the graph parameterizations of $W^u(P_{\eta_0,\xi_0})$ and $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi})$ in the following theorem.

Theorem 5.7.20. *Let $\delta z = (0, 0, \delta\eta, \delta_\xi)$. Then, if $G_0 \gg 1$, $|\delta\eta|, |\delta\xi| \lesssim G_0^{-3}$ and*

$$|\eta_0| G_0^{3/2} \ll 1,$$

the invariant manifold $W^u(P_{\eta_0,\xi_0})$ and $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi})$ admit graph parameterizations $Z^{u,s} : D_{\kappa_2,\delta_2} \to \mathbb{C}^4$ of the form (5.80) which satisfy

$$\|Z^s - \mathcal{F}^s(0)\|_{0,1/2}, \|Z^u - \mathcal{F}^u(0)\|_{0,1/2} \lesssim G_0^{-9/2} \ln^3 G_0$$

where $\mathcal{F}^s$ is the operator defined in (5.93) and $\mathcal{F}^u$ is defined analogously but taking $\delta z = 0$. In particular, the estimates

$$\|Y^*\|_{0,3/2} \lesssim G_0^{-3} \ln G_0, \qquad \|\Lambda^*\|_{0,1} \lesssim G_0^{-9/2},$$
$$\|\alpha^* e^{i\phi(u)}\|_{0,1/2} \lesssim G_0^{-3} \ln G_0, \quad \|\beta^* e^{-i\phi(u)}\|_{0,1/2} \lesssim G_0^{-3} \ln G_0.$$

hold for $* = u, s$.

Moreover these parameterizations satisfy, that for $u \in D_{\kappa_2,\delta_2} \cap \mathbb{R}$ and $\gamma \in \mathbb{T}$,

$$|Y^s| \le G_0^{-3}, \quad |\Lambda^s| \le C G_0^{-6}, \quad |\alpha^s| \le G_0^{-3}, \quad |\beta^s| \le C G_0^{-3}$$

and for $N \ge 0$,

$$|D^N(Z^s - \mathcal{F}^s(0))|, |D^N(Z^u - \mathcal{F}^u(0))| \lesssim C(N) G_0^{-6},$$

where D^N denotes the differential of order N with respect to the variables $(u, \gamma, \eta_0, \xi_0)$ and $C(N)$ is a constant which may depend on N but independent of G_0.

5.8 Proof of Theorem 5.4.3: The difference between the invariant manifolds of infinity

This section is devoted to prove Theorem 5.4.3. Once we have obtained the parametrization of the invariant manifolds (as formal Fourier series) up to points $\mathcal{O}(G_0^{-3})$ close to the singularities $u = \pm i/3$ in Theorem 5.7.20, the next step is to study their difference. We fix (L_0, η_0, ξ_0) and consider the parameterization Z^u of the unstable manifold of the periodic orbit P_{η_0,ξ_0} and the parameterization Z^s of the stable manifold of the periodic orbit $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$.

We then define the difference vector $\Delta = \Delta(u, \gamma)$, which is 2π-periodic in γ, as

$$\Delta = (Y^u - Y^s, \Lambda^u - \Lambda^s, \alpha^u - \alpha^s, \beta^u - \beta^s)^\top . \tag{5.116}$$

The Fourier coefficients of Δ are defined in the domain $D_{\kappa,\delta}$ introduced in (5.114).

Using the equations for $(Y^*, \Lambda^*, \alpha^*, \beta^*)$ for $* = u, s$ in (5.84), we have that Δ satisfies an equation of the form

$$\widetilde{\mathcal{L}}\Delta = A\Delta + B\Delta + R\Delta, \tag{5.117}$$

where $\widetilde{\mathcal{L}}$ is the linear operator

$$\widetilde{\mathcal{L}} = \mathcal{L} + \mathcal{G}_1(Z^u)\partial_u + \mathcal{G}_2(Z^u)\partial_\gamma, \tag{5.118}$$

$\mathcal{G}_1, \mathcal{G}_2$ are the operators defined in (5.86), A is the matrix introduced in (5.87) and B and R are matrices which depend on Z^u and Z^s and its derivatives and are expected to be small compared to A. The matrix B has only one non-zero term,

$$\begin{aligned} B_{21} &= -\frac{\partial_u \Lambda^s}{G_0 \widehat{y}_{\mathrm{h}}^2} + f_1(u)\partial_\gamma \Lambda^s \\ B_{ij} &= 0 \qquad \text{otherwise.} \end{aligned} \tag{5.119}$$

where f_1 is the function introduced in (5.77). The matrix R is defined as follows

$$\begin{aligned} R(u,\gamma) = & \int_0^1 D_Z \mathcal{Q}\left(u, \gamma, sZ^u(u,\gamma) + (1-s)Z^s(u,\gamma)\right) ds \\ & - \partial_u Z^s(u,\gamma) \int_0^1 D_Z \mathcal{G}_1\left(u, \gamma, sZ^u(u,\gamma) + (1-s)Z^s(u,\gamma)\right) ds \\ & - \partial_\gamma Z^s(u,\gamma) \int_0^1 D_Z \mathcal{G}_2\left(u, \gamma, sZ^u(u,\gamma) + (1-s)Z^s(u,\gamma)\right) ds - B, \end{aligned} \tag{5.120}$$

where $\mathcal{Q}$ is the function introduced in (5.89). Note that R satisfies

$$R_{21} = 0.$$

The reason for defining the matrix B and not putting all terms together in R will be clear later. Roughly speaking, the first order of equation (5.117) is $\mathcal{L}\Delta = A\Delta$. To give an heuristic idea of the proof let us assume that Δ is a solution of this equation instead of (5.117). Then, one can easily check that Δ must be of the form

$$\Delta = \Phi_A C$$

where Φ_A is the fundamental matrix introduced in (5.91) (actually a suitable modification of it) and $C(u,\gamma)$ is a vector whose $\gamma-$ Fourier coefficients are defined (and bounded) in $D_{\kappa,\delta}$ and satisfying $\mathcal{L}C = 0$. Then, Lemma 5.8.14) will show that, for real values of the parameters, the function C minus its average with respect to γ is exponentially small.

Now, Δ is a solution of (5.117) instead of $\mathcal{L}\Delta = A\Delta$. Thus, to apply Lemma 5.8.14 we adapt these ideas. We do this in several steps. First, in Section 5.8.1 we describe the functional setting. In Section 5.8.2 we perform a symplectic change of coordinates to straighten the operator in the left hand side of (5.117). Then, in Section 5.8.3, we look for a fundamental solution of the transformed linear partial differential equation. Finally, in Section 5.8.4, we deduce the asymptotic formula of the distance between the invariant manifolds and in Section 5.8.5 we obtain more refined estimates for the average of the difference of the Λ component.

5.8.1 Weighted Fourier norms and Banach spaces

We define the Banach spaces for Fourier series with coefficients defined in $D_{\kappa,\delta}$. First, we define the Banach spaces for the Fourier coefficients as

$$\mathcal{P}_{m,q} = \{h : D_{\kappa,\delta} \to \mathbb{C} : \text{analytic}, \|h\|_{m,q} < \infty\},$$

where
$$\|h\|_{m,q} = \sup_{u\in D_{\kappa,\delta}} \left| |u-i/3|^m |u+i/3|^m e^{iq\phi_{\rm h}(u)} h(u)\right|.$$

for any $m, q \in \mathbb{R}$. Note that these definitions are the same as in Section 5.7.4 but for functions defined in $D_{\kappa,\delta}$ instead of $D^s_{\kappa,\delta}$. Now, for $\sigma > 0$, we define the Banach space for Fourier series

$$\mathcal{Q}_{m,\sigma} = \left\{ h(u,\gamma) = \sum_{\ell\in\mathbb{Z}} h^{[\ell]}(u) e^{i\ell\gamma} : h^{[\ell]} \in \mathcal{P}_{m,\ell}, \|h\|_{m,\sigma} < \infty \right\},$$

where
$$\|h\|_{m,\sigma} = \sum_{\ell\in\mathbb{Z}} \left\| h^{[\ell]} \right\|_{m,\ell} e^{|\ell|\sigma}.$$

The Banach space $\mathcal{Q}_{m,\sigma}$ satisfies the algebra properties stated in Lemma 5.7.6 (with $n = 0$). From now on in this section, we will refer to this lemma understanding the properties stated in it as properties referred to elements of $\mathcal{Q}_{m,\sigma}$ instead of elements of $\mathcal{Y}_{m,0}$.

Now, we need to define vector and matrix norms associated to the just introduced norms. Those norms inherit the structure of the norms considered in Section 5.7. We consider

$$\mathcal{Q}_{m,\sigma,\mathrm{vec}} = \mathcal{Q}_{m+1,\sigma} \times \mathcal{Q}^3_{m,\sigma}$$

with the norm
$$\|Z\|_{m,\sigma,\mathrm{vec}} = \|Y\|_{m+1,\sigma} + \|\Lambda\|_{m,\sigma} + \|e^{i\phi_h}\alpha\|_{m,\sigma} + \||e^{-i\phi_h}\beta\|_{m,\sigma}. \tag{5.121}$$

Analogously, we consider the Banach space $\mathcal{Q}_{\nu,\kappa,\delta,\sigma,\mathrm{mat}}$ of 4×4 matrices with the associated norm

$$\begin{aligned} \|\Psi\|_{m,\sigma,\mathrm{mat}} = \\ \max\Big\{ & \|\Psi_{11}\|_{m,\sigma} + \|\Psi_{21}\|_{m-1,\sigma} + \|e^{i\phi_{\rm h}(u)}\Psi_{31}\|_{m-1,\sigma} + \|e^{-i\phi_{\rm h}(u)}\Psi_{41}\|_{m-1,\sigma}, \\ & \|\Psi_{12}\|_{m+1,\sigma} + \|\Psi_{22}\|_{m,\sigma} + \|e^{i\phi_{\rm h}(u)}\Psi_{32}\|_{m,\sigma} + \|e^{-i\phi_{\rm h}(u)}\Psi_{42}\|_{m,\sigma}, \\ & \|e^{-i\phi_{\rm h}(u)}\Psi_{13}\|_{m+1,\sigma} + \|e^{-i\phi_{\rm h}(u)}\Psi_{23}\|_{m,\sigma} + \|\Psi_{33}\|_{m,\sigma} + \|e^{-2i\phi_{\rm h}(u)}\Psi_{43}\|_{m,\sigma}, \\ & \|e^{i\phi_{\rm h}(u)}\Psi_{14}\|_{m+1,\sigma} + \|e^{i\phi_{\rm h}(u)}\Psi_{24}\|_{m,\sigma} + \|e^{2i\phi_{\rm h}(u)}\Psi_{34}\|_{m,\sigma} + \|\Psi_{44}\|_{m,\sigma} \Big\}. \end{aligned} \tag{5.122}$$

Lemma 5.8.1. *The norms $\|\cdot\|_{m,\sigma,\mathrm{vec}}$ and $\|\cdot\|_{m,\sigma,\mathrm{mat}}$ introduced in* (5.121) *and* (5.122) *respectively have the following properties*

- *Consider $Z \in \mathcal{Q}_{\nu,\sigma,\mathrm{vec}}$ and a matrix $\Psi \in \mathcal{Q}_{\eta,\sigma,\mathrm{mat}}$. Then, $\Psi Z \in \mathcal{Q}_{\nu+\eta,\sigma,\mathrm{vec}}$ and*
$$\|\Psi Z\|_{\nu+\eta,\sigma,\mathrm{vec}} \lesssim \|\Psi\|_{\eta,\sigma,\mathrm{mat}} \|Z\|_{\nu,\sigma,\mathrm{vec}}.$$
- *Consider matrices $\Psi \in \mathcal{Q}_{\eta,\sigma,\mathrm{mat}}$ and $\Psi' \in \mathcal{Q}_{\nu,\sigma,\mathrm{mat}}$. Then, $\Psi\Psi' \in \mathcal{Q}_{\nu+\eta,\sigma,\mathrm{mat}}$ and*
$$\|\Psi\Psi'\|_{\nu+\eta,\sigma,\mathrm{mat}} \lesssim \|\Psi\|_{\eta,\sigma,\mathrm{mat}} \|\Psi'\|_{\nu,\sigma,\mathrm{mat}}.$$

In the present section we will need to take derivatives of and compose Fourier series.

Lemma 5.8.2. *Fix constants $\sigma' < \sigma$, $\kappa' > \kappa$ and $\delta' > \delta$ and take $h \in \mathcal{Q}_{m,\sigma}$ on the domain $D_{\kappa,\delta}$. Its derivatives, as defined in* (5.95)*, satisfy the following in $D_{\kappa',\delta'}$.*

- *$\partial_v^n h \in \mathcal{Q}_{m,\sigma'}$ and*
$$\|\partial_v^n h\|_{m,\sigma'} \le \left(\frac{\kappa'}{\kappa}\right)^m \frac{G_0^{3n} n!}{(\kappa'-\kappa)^n} \|h\|_{m,\sigma}.$$

- $\partial_\xi h \in \mathcal{Q}_{m,\sigma'}$ *and*

$$\|\partial_\xi h\|_{m,\sigma'} \leq \frac{1}{\sigma - \sigma'} \|h\|_{m,\sigma}.$$

Lemma 5.8.3. *We define the formal composition of formal Fourier series*

$$h(v + g(v,\xi), \xi) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_v^n h(v,\xi) g^n(v,\xi).$$

Fix constants $\sigma' < \sigma$, $\kappa' > \kappa$ and $\delta' > \delta$. Let $\kappa' - \kappa > \eta > 0$. Then,

- *If $h \in \mathcal{Q}_{m,\sigma}$ in $D_{\kappa,\delta}$, $g \in \mathcal{Q}_{0,\sigma'}$ in $D_{\kappa',\delta'}$ and $\|g\|_{0,\sigma'} \leq \eta G_0^{-3}$ we have that $X(v,\xi) = h(v+g(v,\xi),\xi)$ satisfies $X \in \mathcal{Y}_{m,\sigma'}$ in $D_{\kappa',\delta'}$ and*

  $$\|X\|_{m,\sigma'} \leq \left(\frac{\kappa'}{\kappa}\right)^m \left(1 - \frac{\eta}{\kappa' - \kappa}\right)^{-1} \|h\|_{m,\sigma}.$$

  Moreover, if $\|g_1\|_{0,\sigma'}, \|g_2\|_{0,\sigma'} \leq \eta G_0^{-3}$ in $D_{\kappa',\delta'}$, then $Y(v,\xi) = h(v+g_2(v,\xi),\xi) - h(v+g_1(v,\xi),\xi)$ satisfies

  $$\|Y\|_{m,\sigma'} \leq \frac{G_0^3}{\kappa' - \kappa} \left(\frac{\kappa'}{\kappa}\right)^m \left(1 - \frac{\eta}{\kappa' - \kappa}\right)^{-2} \|h\|_{m,\sigma} \|g_2 - g_1\|_{0,\sigma}.$$

- *If $\partial_v h \in \mathcal{Q}_{m,\sigma}$ in $D_{\kappa,\delta}$, $g_1, g_2 \in \mathcal{Q}_{0,\sigma'}$ in $D_{\kappa',\delta'}$ and $\|g_1\|_{0,\sigma'}, \|g_2\|_{0,\sigma'} \leq \eta G_0^{-3}$ we have that $Y \in \mathcal{Y}_{m,\sigma'}$ in $D_{\kappa',\delta'}$ and*

  $$\|Y\|_{m,\sigma'} \leq \left(\frac{\kappa'}{\kappa}\right)^m \frac{1}{1 - \frac{\eta}{\kappa'-\kappa}} \|\partial_v h\|_{m,\sigma} \|g_2 - g_1\|_{0,\sigma'}.$$

Finally we give estimates for the matrices appearing in the right hand side of (5.117).

Lemma 5.8.4. *The matrices B and R in* (5.119) *and* (5.120) *satisfy the following*

- *B_{21} satisfies $\|B_{21}\|_{1,\sigma} \lesssim G_0^{-11/2}$. Therefore $\|B\|_{2,\sigma,\mathrm{mat}} \lesssim G_0^{-11/2}$.*
- *$R \in \mathcal{Q}_{3/2,\sigma,\mathrm{mat}}$ and $\|R\|_{3/2,\sigma,\mathrm{mat}} \lesssim G_0^{-3}$.*

Proof. For B_{21} one needs the improved bounds for Λ^s given in (5.100) and the estimates in Lemma 5.7.8. The estimates for R are obtained through an easy but tedious computation using the definitions of $\mathcal{Q}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ given in (5.89) and (5.86), Lemma 5.7.8, Lemma 5.B.1 and the estimates for the Z^u, Z^s given in Theorem 5.7.20. Note that since we are dealing with formal Fourier series the compositions are understood as in Lemma 5.7.7. □

5.8.2 Straightening the differential operator

First step is to perform a symplectic change of coordinates *in phase space* so that one transforms the operator $\widetilde{\mathcal{L}}$ in (5.118) into $\mathcal{L}$. Namely, to remove the term $\mathcal{G}_1(Z^u)\partial_u \Delta + \mathcal{G}_2(Z^u)\partial_\gamma \Delta$ from the left hand side of equation (5.117).

Theorem 5.8.5. *Let σ_2, κ_2 and δ_2 be the constants considered in Lemma 5.7.19. Let $\sigma_3 < \sigma_2$, $\kappa_3 > \kappa_2$ and $\delta_3 > \delta_2$ be fixed. Then, for G_0 big enough and $|\eta_0| G_0^{3/2}$ small enough, there exists a symplectic transformation given by a (not necessarily convergent) Fourier series*

$$(u, Y, \gamma, \Lambda, \alpha, \beta) = \Phi(v, \widetilde{Y}, \gamma, \widetilde{\Lambda}, \alpha, \beta)$$

of the form

$$\Phi(v, \widetilde{Y}, \gamma, \widetilde{\Lambda}, \alpha, \beta) = \left(v + \mathcal{C}(v,\gamma), \frac{1}{1 + \partial_v \mathcal{C}(v,\gamma)} \widetilde{Y}, \gamma, \widetilde{\Lambda} - \frac{\partial_\gamma \mathcal{C}(v,\gamma)}{1 + \partial_v \mathcal{C}(v,\gamma)} \widetilde{Y}, \alpha, \beta\right) \tag{5.123}$$

where $\mathcal{C} \in \mathcal{Q}_{0,\sigma_3}$ *in* D_{κ_3,δ_3} *satisfying*

$$\|\mathcal{C}\|_{0,\sigma_3} \leq b_6 G_0^{-4} \ln G_0,$$

with $b_6 > 0$ *a constant independent of* G_0*, such that*

$$\widetilde{\Delta}(v,\gamma) = \begin{pmatrix} \frac{1}{1+\partial_v \mathcal{C}(v,\gamma)} & 0 & 0 & 0 \\ -\frac{\partial_\gamma \mathcal{C}(v,\gamma)}{1+\partial_v \mathcal{C}(v,\gamma)} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Delta\left(v + \mathcal{C}(v,\gamma), \gamma\right), \tag{5.124}$$

where Δ *is the function defined in* (5.116)*, is well defined and satisfies the equation*

$$\mathcal{L}\widetilde{\Delta} = A\widetilde{\Delta} + \mathcal{B}\widetilde{\Delta} + \mathcal{R}\widetilde{\Delta} \tag{5.125}$$

where A *is the matrix introduced in* (5.87) *and the matrices* $\mathcal{B}$ *and* $\mathcal{R}$ *satisfy*

- $\|\mathcal{B}_{21}\|_{1,\sigma_3} \lesssim G_0^{-11/2}$ *and* $\mathcal{B}_{ij} = 0$ *otherwise. Therefore* $\|\mathcal{B}\|_{2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-11/2}$
- $\mathcal{R} \in \mathcal{Q}_{3/2,\sigma_3,\mathrm{mat}}$ *in* D_{κ_3,δ_3}, $\mathcal{R}_{21} = 0$, $\|\mathcal{R}\|_{3/2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-3}$.

We devote the rest of this section to prove this theorem.

Proof of Theorem 5.8.5

We perform a change of coordinates

$$\Phi_0 : (v,\gamma) \mapsto (u,\gamma) = (v + \mathcal{C}(v,\gamma), \gamma), \tag{5.126}$$

to straighten the operator $\widetilde{\mathcal{L}}$. Clearly, the full change (5.123) is symplectic. To straighten the operator we proceed as in [GMS16]. Consider an operator of the form

$$\widetilde{\mathcal{L}} = (1 + Q_1(u,\gamma))\partial_u + \frac{\nu G_0^3}{L_0^3}(1 + Q_2(u,\gamma))\partial_\gamma.$$

and consider a change of coordinates of the form (5.126) which satisfies

$$\mathcal{L}\mathcal{C} = \frac{Q_1 \circ \Phi_0 - Q_2 \circ \Phi_0}{1 + Q_2 \circ \Phi_0}. \tag{5.127}$$

Then, if h solves the equation $\widetilde{\mathcal{L}}h = D$ for some D, the transformed $\widetilde{h} = h \circ \Phi_0$ satisfies the equation

$$\mathcal{L}\widetilde{h} = \widetilde{D} \qquad \text{where} \qquad \widetilde{D} = \frac{D \circ \Phi_0}{1 + Q_2 \circ \Phi_0}.$$

Note that all these equations and transformations have to make sense for formal Fourier series. In particular, the compositions are understood as in Lemma 5.8.3 and the fraction as

$$\frac{1}{1 + Q_2(u,\gamma)} = \sum_{q \geq 0} (-Q_2(u,\gamma))^q.$$

Proposition 5.8.6. *Let* σ_3, κ_3 *and* δ_3 *be the constants considered in Theorem 5.8.5. Then, for* G_0 *big enough and* $|\eta_0| G_0^{3/2} \ll 1$*, there exists a (not necessarily convergent) Fourier series* $\mathcal{C} \in \mathcal{Q}_{0,\sigma_3}$ *in* D_{κ_3,δ_3} *satisfying*

$$\|\mathcal{C}\|_{0,\sigma_3} \leq b_6 G_0^{-4} \ln G_0, \qquad \|\partial_v \mathcal{C}\|_{1/2,\sigma_3} \leq b_6 G_0^{-3}, \qquad \|\partial_\gamma \mathcal{C}\|_{1/2,\sigma_3} \leq b_6 G_0^{-6}$$

with $b_6 > 0$ *a constant independent of* G_0*, such that*

$$\Delta^*(v,\gamma) = \Delta\left(v + \mathcal{C}(v,\gamma), \gamma\right), \tag{5.128}$$

where Δ *is the function defined in* (5.116), *is well defined and satisfies the equation*

$$\mathcal{L}\Delta^* = A\Delta^* + \widetilde{B}\Delta^* + \widetilde{R}\Delta^* \tag{5.129}$$

where

$$\widetilde{B} = \frac{B \circ \Phi_0}{1 + \mathcal{G}_2(Z^u) \circ \Phi_0}, \qquad \widetilde{R} = \frac{A \circ \Phi_0 - A + R \circ \Phi_0}{1 + \mathcal{G}_2(Z^u) \circ \Phi_0}$$

with $\Phi_0(v, \gamma) = (v + \mathcal{C}(v, \gamma), \gamma)$. *Moreover the matrix* $\widetilde{B}$ *satisfies* $\|\widetilde{B}_{21}\|_{1,\sigma_3} \lesssim G_0^{-11/2}$, $B_{ij} = 0$ *otherwise, which imply* $\|\widetilde{B}\|_{2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-11/2}$. *The matrix* $\widetilde{R} \in \mathcal{Q}_{3/2,\sigma_3,\mathrm{mat}}$, $\widetilde{R}_{21} = 0$ *and*

$$\|\widetilde{R}\|_{3/2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-3}.$$

Using, the definition of $\widetilde{\mathcal{L}}$ in (5.118), to prove this proposition, we look for a function $\mathcal{C}$ satisfying equation (5.127) with

$$Q_1(u, \gamma) = \mathcal{G}_1(Z^u)(u, \gamma), \qquad Q_2(u, \gamma) = \frac{L_0^3}{\nu G_0^3} \mathcal{G}_2(Z^u)(u, \gamma). \tag{5.130}$$

The next lemma gives estimates for these functions.

Lemma 5.8.7. *The functions* Q_1 *and* Q_2, *in* D_{κ_2,δ_2}, *satisfy*

$$\begin{aligned} \|Q_1\|_{1/2,\sigma_2} &\lesssim G_0^{-4} \ln G_0, & \|Q_2\|_{1,\sigma_2} &\lesssim G_0^{-9/2} \\ \|\partial_u Q_1\|_{3/2,\sigma_2} &\lesssim G_0^{-4} \ln G_0, & \|\partial_u Q_2\|_{2,\sigma_2} &\lesssim G_0^{-9/2} \\ \|\partial_\gamma Q_1\|_{3/2,\sigma_2} &\lesssim G_0^{-7} \ln G_0, & \|\partial_\gamma Q_2\|_{1,\sigma_2} &\lesssim G_0^{-9/2}. \end{aligned} \tag{5.131}$$

Proof. Lemma 5.7.13 gives the estimate for Q_1. Analogous estimates can be obtained for its derivatives, differentiating (5.86) and using the estimates for Z^u and its derivatives in Theorem 5.7.20 and Lemma 5.7.8. To estimate Q_2 and its derivatives one can proceed analogously taking into account the improved estimates for Λ^u in Theorem 5.7.20. □

We obtain a solution of equation (5.127) by considering a left inverse $\widetilde{\mathcal{G}}$ of the operator $\mathcal{L}$ in the space $\mathcal{Q}_{1/2,\sigma}$ and setting up a fixed point argument.

We define the following operator acting on the Fourier coefficients as

$$\widetilde{\mathcal{G}}(h)(u, \gamma) = \sum_{q \in \mathbb{Z}} \widetilde{\mathcal{G}}(h)^{[q]}(u) e^{iq\gamma}, \tag{5.132}$$

where its Fourier coefficients are given by

$$\begin{aligned} \widetilde{\mathcal{G}}(h)^{[q]}(u) &= \int_{\overline{u}_2}^{u} e^{iq\nu G_0^3 L_0^{-3}(t-u)} h^{[q]}(t)\, dt && \text{for } q < 0 \\ \widetilde{\mathcal{G}}(h)^{[0]}(u) &= \int_{u^*}^{u} h^{[0]}(t)\, dt \\ \widetilde{\mathcal{G}}(h)^{[q]}(u) &= \int_{u_2}^{u} e^{iq\nu G_0^3 L_0^{-3}(t-u)} h^{[q]}(t)\, dt && \text{for } q > 0. \end{aligned}$$

Here $u_2 = i(1/3 - \kappa G_0^{-3})$ is the top vertex of the domain $D_{\kappa,\delta}$, $\overline{u}_2$ is its conjugate, which corresponds to the bottom vertex of the domain $D_{\kappa,\delta}$ and u^* is the left endpoint of $D_{\kappa,\delta} \cap \mathbb{R}$.

Lemma 5.8.8. *The operator* $\widetilde{\mathcal{G}}$ *in* (5.132), *in the domain* $D_{\kappa,\delta}$, *satisfies that*

- *If* $h \in \mathcal{Q}_{\nu,\sigma}$ *for some* $\nu \in (0, 1)$, *then* $\widetilde{\mathcal{G}}(h) \in \mathcal{Q}_{0,\sigma}$ *and* $\left\|\widetilde{\mathcal{G}}(h)\right\|_{0,\sigma} \leq K\|h\|_{\nu,\sigma}$.

- *If $h \in \mathcal{Q}_{1,\sigma}$, then $\widetilde{\mathcal{G}}(h) \in \mathcal{Q}_{0,\sigma}$ and $\left\|\widetilde{\mathcal{G}}(h)\right\|_{0,\sigma} \leq K \ln G_0 \|h\|_{1,\sigma}$.*
- *If $h \in \mathcal{Q}_{\nu,\sigma}$ for some $\nu > 1$, then $\widetilde{\mathcal{G}}(h) \in \mathcal{Q}_{\nu-1,\sigma}$ and $\left\|\widetilde{\mathcal{G}}(h)\right\|_{\nu-1,\sigma} \leq K\|h\|_{\nu,\sigma}$.*
- *If $h \in \mathcal{Q}_{\nu,\sigma}$ for some $\nu > 0$, then $\partial_v\widetilde{\mathcal{G}}(h) \in \mathcal{Q}_{\nu,\sigma}$ and $\left\|\partial_v\widetilde{\mathcal{G}}(h)\right\|_{\nu,\sigma} \leq K\|h\|_{\nu,\sigma}$.*
- *If $h \in \mathcal{Q}_{\nu,\sigma}$ for some $\nu > 0$ and $\langle h\rangle = 0$, then $\widetilde{\mathcal{G}}(h) \in \mathcal{Q}_{\nu,\sigma}$ and*

$$\left\|\widetilde{\mathcal{G}}(h)\right\|_{\nu,\sigma} \leq K G_0^{-3}\|h\|_{\nu,\sigma}.$$

Moreover, if h is a real-analytic Fourier series, that is $h^{[q]}(\overline{u}) = \overline{h^{[-q]}(u)}$, then so is $\widetilde{\mathcal{G}}(h)$.

This lemma can be proven as Lemma 8.3 of [**?**].

Proof of Proposition 5.8.6. We prove Proposition 5.8.6 by looking for a fixed point of the operator

$$\widetilde{\mathcal{K}} = \widetilde{\mathcal{G}} \circ \mathcal{K}, \qquad \mathcal{K}(\mathcal{C})(v,\gamma) = \left.\frac{Q_1(u,\gamma) - Q_2(u,\gamma)}{1 + Q_2(u,\gamma)}\right|_{u=v+\mathcal{C}(v,\gamma)} \tag{5.133}$$

where $\widetilde{\mathcal{G}}$ is the operator introduced in (5.132) and Q_1, Q_2 are the formal Fourier series in (5.130).

We write $\mathcal{K}(0)$ as

$$\mathcal{K}(0) = \frac{Q_1 - Q_2}{1 + Q_2} = Q_1 - \frac{Q_2(1+Q_1)}{1+Q_2}.$$

Note that, by (5.131), the second term satisfies, in D_{κ_2,δ_2},

$$\left\|\frac{Q_2(1+Q_1)}{1+Q_2}\right\|_{1,\sigma_2} \lesssim G_0^{-9/2}.$$

Now, by Lemmas 5.8.8 and 5.7.6, there exists a constant $b_6 > 0$ independent of G_0, such that, in D_{κ_3,δ_3},

$$\begin{aligned}
\left\|\widetilde{\mathcal{K}}(0)\right\|_{0,\sigma_3} = \left\|\widetilde{\mathcal{G}} \circ \mathcal{K}(0)\right\|_{0,\sigma_3} &\leq \left\|\widetilde{\mathcal{G}}(Q_1)\right\|_{0,\sigma_3} + \left\|\widetilde{\mathcal{G}}\left(\frac{Q_2(1+Q_1)}{1+Q_2}\right)\right\|_{0,\sigma_3} \\
&\leq \|Q_1\|_{1/2,\sigma_3} + \ln G_0 \left\|\frac{Q_2(1+Q_1)}{1+Q_2}\right\|_{1,\sigma_3} \\
&\leq \frac{b_6}{2} G_0^{-4} \ln G_0.
\end{aligned}$$

Now we prove that $\widetilde{\mathcal{K}}$ is a Lipschitz operator in the ball $B(b_6 G_0^{-4} \ln G_0) \subset \mathcal{Q}_{0,\sigma_3}$ in D_{κ_3,δ_3}. Take $g_1, g_2 \in B(b_6 G_0^{-4} \ln G_0) \subset \mathcal{Q}_{0,\sigma_3}$. By Lemma 5.8.3 and estimates (5.131),

$$\|\mathcal{K}(g_2) - \mathcal{K}(g_1)\|_{3/2,\sigma_3} \lesssim \left\|\partial_u\left[\frac{Q_1(u,\gamma) - Q_2(u,\gamma)}{1+Q_2(u,\gamma)}\right]\right\|_{3/2,\sigma_2} \|g_2 - g_1\|_{0,\sigma_3} \lesssim G_0^{-3}\|g_2 - g_1\|_{0,\sigma_3}.$$

Then, by Lemma 5.7.6 and 5.8.8,

$$\left\|\widetilde{\mathcal{K}}(g_2) - \widetilde{\mathcal{K}}(g_1)\right\|_{0,\sigma_3} \lesssim G_0^{3/2}\left\|\widetilde{\mathcal{K}}(g_2) - \widetilde{\mathcal{K}}(g_1)\right\|_{1/2,\sigma_3} \lesssim G_0^{3/2}\|\mathcal{K}(g_2) - \mathcal{K}(g_1)\|_{3/2,\sigma_3} \lesssim G_0^{-3/2}\|g_2 - g_1\|_{0,\sigma_3}.$$

Thus, taking G_0 large enough, the operator $\widetilde{\mathcal{K}}$ is a contractive operator $B(b_6 G_0^{-4} \ln G_0) \subset \mathcal{Q}_{0,\sigma_3}$. The fix point of the operator gives the change of coordinates provided in Proposition 5.8.6.

To obtain the estimates for $\partial_v \mathcal{C}$ it is enough to use that we have seen that for $\mathcal{C} \in B(b_6 G_0^{-4} \ln G_0) \subset \mathcal{Q}_{0,\sigma_3}$, $\mathcal{K}(\mathcal{C})$ satisfies $\mathcal{K}(\mathcal{C}) \in \mathcal{Q}_{1/2,\sigma_3}$ and $\|\mathcal{K}(\mathcal{C})\|_{1/2,\sigma_3} \lesssim G_0^{-3}$. Then, by Lemma 5.8.8, $\partial_v \mathcal{C} = \partial_v \widetilde{\mathcal{G}} \circ \mathcal{K}(\mathcal{C}) \in \mathcal{Q}_{1/2,\sigma_3}$ and satisfies $\|\partial_v \widetilde{\mathcal{G}} \circ \mathcal{K}(\mathcal{C})\|_{1/2,\sigma_3} \lesssim G_0^{-3}$. The estimates for $\partial_\gamma \mathcal{C}$ are obtained through the identity

$$\partial_\gamma \mathcal{C} = \frac{L_0^3}{\nu G_0^3} \left(\mathcal{K}(\mathcal{C}) - \partial_v \mathcal{C}\right).$$

Finally, the estimates for $\widetilde{B}$ and $\widetilde{R}$ are a direct consequence of the estimates for $\mathcal{C}$ just obtained, the estimate of R in Lemma 5.8.4, the identity (5.130), estimates (5.131), the definition of A in (5.87), the estimates of the functions f_1 and f_2 given in Lemma 5.7.9 and the condition $|\eta_0| G_0^{3/2} \ll 1$. □

Now, we are ready to prove Theorem 5.8.5.

Proof of Theorem 5.8.5. It is straightforward to check that the transformation (5.123) is symplectic. It only remains to obtain the estimates for $\mathcal{B}$ and $\mathcal{R}$. To this end, it is enough to apply the transformation

$$Y = \frac{1}{1 + \partial_v \mathcal{C}(v,\gamma)} \widetilde{Y}, \qquad \Lambda = \widetilde{\Lambda} - \frac{\partial_\gamma \mathcal{C}(v,\gamma)}{1 + \partial_v \mathcal{C}(v,\gamma)} \widetilde{Y}$$

to equation (5.129) to obtain the formulas for the coefficients $(\mathcal{B} + \mathcal{R})_{ij}$. To this end, to a 4×4 matrix M whose entries M_{ij} are functions of (v,γ) we define the following 4×4 matrix $\mathcal{J}(M)$ whose coefficients $\mathcal{J}(M)_{ij}$ are defined a

$$\begin{aligned}
\mathcal{J}(M)_{11} &= M_{11} + \frac{\partial_v \mathcal{K}(\mathcal{C})}{1 + \partial_v \mathcal{C}} - M_{12} \partial_\gamma \mathcal{C}, \quad \mathcal{J}(M)_{1j} = \left(1 + \partial_v \mathcal{C}\right) M_{1j}, \quad j = 2,3,4,\\
\mathcal{J}(M)_{21} &= \frac{M_{21} + \partial_\gamma \mathcal{C} M_{11} + \partial_\gamma \mathcal{K}(\mathcal{C}) - M_{22} \partial_\gamma \mathcal{C} - \left(\partial_\gamma \mathcal{C}\right)^2 M_{12}}{1 + \partial_v \mathcal{C}}\\
\mathcal{J}(M)_{2j} &= M_{2j} + \partial_\gamma \mathcal{C} M_{1j}, \quad j = 2,3,4\\
\mathcal{J}(M)_{i1} &= \frac{M_{i1} + M_{i2} \partial_\gamma \mathcal{C}}{1 + \partial_v \mathcal{C}}, \quad \mathcal{J}(M)_{ij} = M_{ij}, \quad i = 3,4, j = 2,3,4.
\end{aligned}$$

We split $\mathcal{B}$ and $\mathcal{R}$ as before. That is, $\mathcal{B}_{ij} = 0$ for $ij \neq 21$ and $\mathcal{R}_{21} = 0$. Then, the coefficients of the matrix $\mathcal{B}$ and $\mathcal{R}$ in Theorem 5.8.5 are defined as

$$\mathcal{B}_{21} = \frac{\widetilde{B}_{21} + \partial_\gamma \mathcal{C} \widetilde{R}_{11} + \partial_\gamma \mathcal{K}(\mathcal{C}) - \widetilde{R}_{22} \partial_\gamma \mathcal{C} - \left(\partial_\gamma \mathcal{C}\right)^2 \widetilde{R}_{12}}{1 + \partial_v \mathcal{C}}$$

and

$$\mathcal{R} = \mathcal{J}(A) - A + \mathcal{J}(\widetilde{R}) - (\mathcal{B} - \widetilde{B})$$

where A and $\widetilde{R}$ are the matrices defined in (5.87) and Proposition 5.8.6 respectively. This implies that $\mathcal{R}_{ij} = \mathcal{J}(\widetilde{R})_{ij}$ for all coefficients except $\mathcal{R}_{21} = 0$ and

$$\mathcal{R}_{i1} = \frac{-A_{i1} \partial_v \mathcal{C} + \widetilde{R}_{i1} + \partial_\gamma \mathcal{C} \left(A_{i2} + \widetilde{R}_{i2}\right)}{1 + \partial_v \mathcal{C}}$$

Then, one can obtain the estimates for the coefficients of $\mathcal{R}$ using these definitions, the estimates for $\widetilde{R}$ and $\mathcal{C}$ in Proposition 5.8.6, the estimates for the matrix A given in Lemma 5.7.9 (see the definition of A in (5.87)) and the condition $|\eta_0| G_0^{3/2} \ll 1$. For the bounds of $\partial_v \mathcal{K}(\mathcal{C})$ and $\partial_\gamma \mathcal{K}(\mathcal{C})$ one has to use the definition of $\mathcal{K}(\mathcal{C})$ in (5.133) to obtain

$$\begin{aligned}
\partial_v \mathcal{K}(\mathcal{C})(v,\gamma) &= \partial_u \left[\frac{Q_1(u,\gamma) - Q_2(u,\gamma)}{1 + Q_2(u,\gamma)} \right] \Bigg|_{u = v + \mathcal{C}(v,\gamma)} \left(1 + \partial_v \mathcal{C}(v,\gamma)\right)\\
\partial_\gamma \mathcal{K}(\mathcal{C})(v,\gamma) &= \partial_u \left[\frac{Q_1(u,\gamma) - Q_2(u,\gamma)}{1 + Q_2(u,\gamma)} \right] \Bigg|_{u = v + \mathcal{C}(v,\gamma)} \partial_\gamma \mathcal{C}(v,\gamma)\\
&\quad + \partial_\gamma \left[\frac{Q_1(u,\gamma) - Q_2(u,\gamma)}{1 + Q_2(u,\gamma)} \right] \Bigg|_{u = v + \mathcal{C}(v,\gamma)}
\end{aligned}$$

Then, using the estimates in (5.131) and Lemma 5.8.3, one has

$$\|\partial_v \mathcal{K}(\mathcal{C})\|_{3/2,\sigma_3} \lesssim G_0^{-3}, \qquad \|\partial_\gamma \mathcal{K}(\mathcal{C})\|_{1,\sigma_3} \lesssim G_0^{-11/2}.$$

□

5.8.3 The general solution for the straightened linear system

Now, we solve the linear equation (5.129) by looking for a fundamental matrix Ψ satisfying

$$\mathcal{L}\Psi = (A + \mathcal{B} + \mathcal{R})\Psi. \tag{5.134}$$

Note that in (5.91) we have obtained a fundamental matrix Φ_A of the linear equation $\mathcal{L}\Psi = A\Psi$. However, it can be easily seen that this matrix does not have good estimates with respect to the norm introduced in (5.122). Thus, we modify it slightly. Let us introduce the notation $\Phi_A = (V_1, \dots, V_4)$ where V_i are the columns of the matrix. Then, we define the new fundamental matrix

$$\widetilde{\Phi}_A = (\widetilde{V}_1, \dots \widetilde{V}_4) \qquad \text{defined as} \qquad \begin{aligned} \widetilde{V}_1 &= V_1 - \eta_0 g_1 \left(\frac{i}{3}\right) V_3 + \xi_0 g_1 \left(-\frac{i}{3}\right) V_4 \\ \widetilde{V}_2 &= V_2 - \eta_0 g_2 \left(\frac{i}{3}\right) V_3 + \xi_0 g_2 \left(-\frac{i}{3}\right) V_4 \\ \widetilde{V}_j &= V_j, j = 3, 4. \end{aligned} \tag{5.135}$$

Lemma 5.8.9. *Assume $|\eta_0| G_0^{3/2} \ll 1$. The fundamental matrix $\widetilde{\Phi}_A$ and its inverse $\widetilde{\Phi}_A^{-1}$ satisfy $\widetilde{\Phi}_A, \widetilde{\Phi}_A^{-1} \in \mathcal{Q}_{0,\sigma_3}$ in D_{κ_3,δ_3} and*

$$\left\|\widetilde{\Phi}_A\right\|_{0,\sigma_3,\mathrm{mat}} \lesssim 1, \qquad \left\|\widetilde{\Phi}_A^{-1}\right\|_{0,\sigma_3,\mathrm{mat}} \lesssim 1.$$

Moreover, the matrices Φ_A in (5.91) and $\widetilde{\Phi}_A$ in (5.135) are related as $\Phi_A = \widetilde{\Phi}_A \mathcal{J}$ where $\mathcal{J}$ is a constant matrix which satisfies

$$\mathcal{J} = \mathrm{Id} + \mathcal{O}(|\eta_0|).$$

Moreover, the $\mathcal{O}(|\eta_0|)$ terms are only present in the third and fourth row of the matrix.

The proof of this lemma is a direct consequence of the definition of $\widetilde{\Phi}_A$ in (5.135) and Lemma 5.7.9. In the next theorem we obtain a fundamental matrix of (5.134).

Theorem 5.8.10. *Let σ_3, κ_3 and δ_3 be the constants considered in Theorem 5.8.5. Then, for G_0 big enough and $|\eta_0| G_0^{3/2}$ small enough, there exists a fundamental matrix of (5.134) of the form $\Psi = \widetilde{\Phi}_A(\mathrm{Id} + \widetilde{\Psi})$ with $\widetilde{\Psi} \in \mathcal{Q}_{1/2,\sigma_3,\mathrm{mat}}$ in D_{κ_3,δ_3}, which satisfies*

$$\left\|\widetilde{\Psi}\right\|_{1/2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-3} \ln G_0.$$

Moreover,

$$\left\|\widetilde{\Psi}_{21}\right\|_{0,\sigma_3,\mathrm{mat}} \lesssim G_0^{-9/2} \ln G_0.$$

We devote the rest of this section to prove this theorem. Note that Ψ is a solution of (5.134) if an only if $\widetilde{\Psi}$ satisfies

$$\mathcal{L}\widetilde{\Psi} = \widetilde{\Phi}_A^{-1}(\mathcal{B} + \mathcal{R})\widetilde{\Phi}_A(\mathrm{Id} + \widetilde{\Psi}). \tag{5.136}$$

We solve this equation through a fixed point argument by setting up an integral equation.

The first step is to invert the operator $\mathcal{L}$. To this end, we need to use *different* integral operators depending on the components. The reason is the significantly different behavior of the components close to the singularities of the unperturbed separatrix. That is, besides the operator $\widetilde{\mathcal{G}}$ in (5.132), we define the operators

$$\mathcal{G}_\pm(h)(u,\gamma) = \mathcal{G}_\pm(h)^{[0]}(u) + \sum_{q\in\mathbb{Z}\setminus\{0\}} \widetilde{\mathcal{G}}(h)^{[q]}(u)e^{iq\gamma}, \qquad \mathcal{G}_\pm(h)^{[0]}(u) = \int_{u_\pm}^{u} h^{[0]}(t)\,dt \tag{5.137}$$

where $u_+ = u_2$ and $u_- = \bar{u}_2$, where u_2 has been introduced in (5.132). Note that equation (5.136) has many solutions which arise from the fact that the operator $\mathcal{L}$ has many left inverse operators. We choose just one solution which is convenient for us.

Lemma 5.8.11. *The operators $\mathcal{G}_\pm$ introduced in (5.137) satisfy the following. Assume $h \in \mathcal{Q}_{\nu,\sigma,\mathrm{mat}}$ in $D_{\kappa,\delta}$ with $\nu \geq 1/2$. Then $e^{\pm i\phi_\mathrm{h}(u)}\mathcal{G}_\pm(e^{\mp i\phi_\mathrm{h}(u)}h) \in \mathcal{Q}_{\nu-1,\sigma,\mathrm{mat}}$ and*

$$\begin{aligned} \left\| e^{\pm i\phi_\mathrm{h}(u)}\mathcal{G}_\pm\left(e^{\mp i\phi_\mathrm{h}(u)}h\right)\right\|_{\nu-1,\sigma} &\lesssim \|h\|_{\nu,\sigma} \qquad \textit{for } \nu > 1/2 \\ \left\| e^{\pm i\phi_\mathrm{h}(u)}\mathcal{G}_\pm\left(e^{\mp i\phi_\mathrm{h}(u)}h\right)\right\|_{-1/2,\sigma} &\lesssim \ln G_0 \|h\|_{1/2,\sigma}. \end{aligned}$$

Finally, we define an integral operator $\mathcal{G}_\mathrm{mat}$ acting on matrices in $\mathcal{Q}_{\nu,\sigma,\mathrm{mat}}$ linearly on the coefficents as follows. For $M \in \mathcal{Q}_{\nu,\sigma,\mathrm{mat}}$, we define $\mathcal{G}_\mathrm{mat}(M)$ as

$$\begin{aligned} \mathcal{G}_\mathrm{mat}(M)_{ij} &= \widetilde{\mathcal{G}}(M_{ij}) && \text{for } i=1,2, j=1,2,3,4 \\ \mathcal{G}_\mathrm{mat}(M)_{3j} &= \mathcal{G}_+(M_{ij}) && \text{for } j=1,2,3,4 \\ \mathcal{G}_\mathrm{mat}(M)_{4j} &= \mathcal{G}_-(M_{ij}) && \text{for } j=1,2,3,4. \end{aligned} \tag{5.138}$$

Lemma 5.8.12. *The operator $\mathcal{G}_\mathrm{mat}$ in (5.138) has the following properties.*

- *Assume $M \in \mathcal{Q}_{\nu,\sigma,\mathrm{mat}}$ with $\nu \geq 2$. Then $\mathcal{G}_\mathrm{mat}(M) \in \mathcal{Q}_{\nu-1,\sigma,\mathrm{mat}}$ and*

$$\begin{aligned} \|\mathcal{G}_\mathrm{mat}(M)\|_{\nu-1,\sigma} &\lesssim \|M\|_{\nu,\sigma} \qquad \textit{for } \nu > 2 \\ \|\mathcal{G}_\mathrm{mat}(M)\|_{1,\sigma} &\lesssim \ln G_0 \|M\|_{2,\sigma}. \end{aligned}$$

- *Assume $M \in \mathcal{Q}_{\nu,\sigma,\mathrm{mat}}$ with $\nu \geq 3/2$ and $M_{21} = 0$. Then $\mathcal{G}_\mathrm{mat}(M) \in \mathcal{Q}_{\nu-1,\sigma,\mathrm{mat}}$ and*

$$\begin{aligned} \|\mathcal{G}_\mathrm{mat}(M)\|_{\nu-1,\sigma} &\lesssim \|M\|_{\nu,\sigma} \qquad \textit{for } \nu > 3/2 \\ \|\mathcal{G}_\mathrm{mat}(M)\|_{1/2,\sigma} &\lesssim \ln G_0 \|M\|_{3/2,\sigma}. \end{aligned}$$

We use the operator $\mathcal{G}_\mathrm{mat}$ to look for solutions of (5.136) through an integral equation. We define the operator

$$\widetilde{\mathcal{S}}(\Psi) = \mathcal{G}_\mathrm{mat} \circ \mathcal{S}(\Psi) \qquad \text{with} \qquad \mathcal{S}(\Psi) = \widetilde{\Phi}_A^{-1}(\mathcal{B}+\mathcal{R})\widetilde{\Phi}_A(\mathrm{Id}+\Psi).$$

Lemma 5.8.13. *Consider the domain D_{κ_3,δ_3}. The affine operator $\widetilde{\mathcal{S}} : \mathcal{Q}_{1/2,\sigma_3,\mathrm{mat}} \to \mathcal{Q}_{1/2,\sigma_3,\mathrm{mat}}$ is Lipschitz and satisfies that, for any $\Psi, \Psi' \in \mathcal{Q}_{1/2,\sigma_3,\mathrm{mat}}$,*

$$\left\|\widetilde{\mathcal{S}}(\Psi) - \widetilde{\mathcal{S}}(\Psi')\right\|_{1/2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-3/2} \ln G_0 \|\Psi - \Psi'\|_{1/2,\sigma_3,\mathrm{mat}}$$

Proof. To compute the Lipschitz constant, we write

$$\mathcal{S}(\Psi) - \mathcal{S}(\Psi') = \widetilde{\Phi}_A^{-1}(\mathcal{B}+\mathcal{R})\widetilde{\Phi}_A(\Psi - \Psi').$$

The properties of $\mathcal{B}$ and $\mathcal{R}$ in Theorem 5.8.5 imply that $\|\mathcal{B}+\mathcal{R}\|_{1,\sigma_3,\mathrm{mat}} \lesssim G_0^{-3/2}$. Then, using also Lemmas 5.8.9 and Lemma 5.8.1,

$$\begin{aligned} \|\mathcal{S}(\Psi) - \mathcal{S}(\Psi')\|_{3/2,\sigma_3,\mathrm{mat}} &\lesssim \left\|\widetilde{\Phi}_A^{-1}\right\|_{0,\sigma_3,\mathrm{mat}} \|\mathcal{B}+\mathcal{R}\|_{1,\sigma_3,\mathrm{mat}} \left\|\widetilde{\Phi}_A\right\|_{0,\sigma_3,\mathrm{mat}} \|\Psi - \Psi'\|_{1/2,\sigma_3,\mathrm{mat}} \\ &\lesssim G_0^{-3/2} \|\Psi - \Psi'\|_{1/2,\sigma_3,\mathrm{mat}}. \end{aligned}$$

Thus, applying Lemma 5.8.12,

$$\left\|\widetilde{\mathcal{S}}(\Psi)-\widetilde{\mathcal{S}}(\Psi')\right\|_{1/2,\sigma_3,\mathrm{mat}} \lesssim \ln G_0 \left\|\mathcal{S}(\Psi)-\mathcal{S}(\Psi')\right\|_{3/2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-3/2}\ln G_0 \left\|\Psi-\Psi'\right\|_{1/2,\sigma_3,\mathrm{mat}}.$$

□

Then, to finish the proof of Theorem 5.8.10, it is enough to use Lemmas 5.8.8 and 5.8.12 and to use the estimates for $\mathcal{B}$ and $\mathcal{R}$ in Theorem 5.8.5 to see that

$$\widetilde{\mathcal{S}}(0)=\mathcal{G}_{\mathrm{mat}}\left[\widetilde{\Phi}_A^{-1}(\mathcal{B}+\mathcal{R})\widetilde{\Phi}_A\right]$$

satisfies

$$\begin{aligned}
\left\|\widetilde{\mathcal{S}}(0)\right\|_{1/2,\sigma_3,\mathrm{mat}} &\lesssim G_0^{3/2}\left\|\mathcal{G}_{\mathrm{mat}}\left[\widetilde{\Phi}_A^{-1}\mathcal{B}\widetilde{\Phi}_A\right]\right\|_{1,\sigma_3,\mathrm{mat}} + \left\|\mathcal{G}_{\mathrm{mat}}\left[\widetilde{\Phi}_A^{-1}\mathcal{R}\widetilde{\Phi}_A\right]\right\|_{1/2,\sigma_3,\mathrm{mat}}\\
&\lesssim G_0^{3/2}\ln G_0\left\|\widetilde{\Phi}_A^{-1}\mathcal{B}\widetilde{\Phi}_A\right\|_{2,\sigma_3,\mathrm{mat}} + \ln G_0\left\|\widetilde{\Phi}_A^{-1}\mathcal{R}\widetilde{\Phi}_A\right\|_{3/2,\sigma_3,\mathrm{mat}}\\
&\lesssim G_0^{3/2}\ln G_0\left\|\mathcal{B}\right\|_{2,\sigma_3,\mathrm{mat}} + \ln G_0\left\|\mathcal{R}\right\|_{3/2,\sigma_3,\mathrm{mat}}\\
&\lesssim G_0^{-3}\ln G_0.
\end{aligned}$$

Therefore, together with Lemma 5.8.13, one has that the operator $\widetilde{\mathcal{S}}$ has a unique fixed point $\widetilde{\Psi}$ which satisfies $\left\|\widetilde{\Psi}\right\|_{1/2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-3}\ln G_0$.

For the estimates for $\widetilde{\Psi}_{21}$ it is enough to write $\widetilde{\Psi}_{21}$ as $\widetilde{\Psi}_{21}=\widetilde{\mathcal{S}}(0)_{21}+\left[\widetilde{\mathcal{S}}(\Psi)-\widetilde{\mathcal{S}}(\Psi')\right]_{21}$. For the first term, by Theorem 5.8.5 and Lemma 5.8.8, one has that

$$\left\|\widetilde{\mathcal{S}}(0)_{21}\right\|_{0,\sigma_3} \lesssim G_0^{-11/2}\ln G_0.$$

For the second term it is enough to use Lemma 5.8.13 to obtain

$$\left\|\left[\widetilde{\mathcal{S}}(\Psi)-\widetilde{\mathcal{S}}(\Psi')\right]_{21}\right\|_{0,\sigma_3} \leq \left\|\left[\widetilde{\mathcal{S}}(\Psi)-\widetilde{\mathcal{S}}(\Psi')\right]_{21}\right\|_{-1/2,\sigma_3} \lesssim G_0^{-3/2}\ln G_0\left\|\widetilde{\Psi}\right\|_{1/2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-9/2}\ln G_0.$$

5.8.4 Exponentially small estimates of the difference between the invariant manifolds

Last step is to obtain exponentially small bounds of the difference between invariant manifolds Δ and its first order. We first analyze $\widetilde{\Delta}$ in (5.128). Using that $\Psi=\widetilde{\Phi}_A(\mathrm{Id}+\widetilde{\Psi})$ with $\widetilde{\Psi}$ obtained in Theorem 5.8.10 is a fundamental matrix of the equation (5.134), we know that $\widetilde{\Delta}$ (which also satisfies (5.134)) is of the form

$$\widetilde{\Delta}=\widetilde{\Phi}_A(\mathrm{Id}+\widetilde{\Psi})\widehat{\Delta} \quad \text{where } \widehat{\Delta} \quad \text{satisfies } \mathcal{L}\widehat{\Delta}=0. \tag{5.139}$$

To bound the function $\widehat{\Delta}$, we use the following lemma, proven in [GMS16].

Lemma 5.8.14. *Fix $\kappa>0$, $\delta>0$ and $\sigma>0$. Let us consider a formal Fourier series $\Upsilon\in\mathcal{Q}_{0,\sigma}$ in $D_{\kappa,\delta}$ such that $\Upsilon\in\mathrm{Ker}\mathcal{L}$. Define its average*

$$\langle\Upsilon\rangle_\gamma=\frac{1}{2\pi}\int_0^{2\pi}\Upsilon(v,\gamma)d\gamma.$$

Then, the Fourier series $\Upsilon(v,\gamma)$ satisfies the following.

- *Is of the form*

$$\Upsilon(v,\gamma)=\sum_{\ell\in\mathbb{Z}}\Upsilon^{[\ell]}(v)e^{i\ell\gamma}=\sum_{\ell\in\mathbb{Z}}\widetilde{\Upsilon}^{[\ell]}e^{i\ell\left(G_0^3v+\gamma\right)}$$

  for certain constants $\widetilde{\Upsilon}^{[\ell]}\in\mathbb{C}$. In particular, its average $\langle\Upsilon\rangle_\gamma(v)=\widetilde{\Upsilon}^{[0]}$ is independent of v.

- *It defines a function for $v \in D_{\kappa,\delta} \cap \mathbb{R}$ and $\gamma \in \mathbb{T}$, whose Fourier coefficients satisfy that*

$$\left|\Upsilon^{[\ell]}(v)\right| \leq \sup_{v\in D_{\kappa,\delta}} \left|\Upsilon^{[\ell]}(v)\right| K^{|\ell|} e^{-\frac{|\ell| G_0^3}{3}} \lesssim \|\Upsilon\|_{0,\sigma} (KG_0^{3/2})^{|\ell|} e^{-\frac{|\ell| G_0^3}{3}}$$
$$\left|\partial_v\Upsilon^{[\ell]}(v)\right| \leq \sup_{v\in D_{\kappa,\delta}} \left|\Upsilon^{[\ell]}(v)\right| K^{|\ell|} G_0^3 e^{-\frac{|\ell| G_0^3}{3}} \lesssim G_0^3 \|\Upsilon\|_{0,\sigma} (KG_0^{3/2})^{|\ell|} e^{-\frac{|\ell| G_0^3}{3}}.$$

Note, nevertheless, that we do not want to bound $\widehat{\Delta}$ but its difference with respect to its first order. The first order is defined through the operators $\mathcal{F}^{u,s}$ in (5.93) (see Theorem 5.7.20) and is given by

$$\widehat{\Delta}_0 = \widetilde{\Phi}_A^{-1}\left(\mathcal{F}^u(0) - \mathcal{F}^s(0)\right). \tag{5.140}$$

Using (5.93) and the relation $\widetilde{\Phi}_A^{-1}\Phi_A = \mathcal{J}$ given in Lemma 5.8.9,

$$\widehat{\Delta}_0 = \widetilde{\Phi}_A^{-1}\Phi_A\left[\delta z + \mathcal{G}^u\left(\Phi_A^{-1}F(0)\right) - \mathcal{G}^s\left(\Phi_A^{-1}F(0)\right)\right] = \mathcal{J}\delta z + \mathcal{G}^u\left(\widetilde{\Phi}_A^{-1}F(0)\right) - \mathcal{G}^s\left(\widetilde{\Phi}_A^{-1}F(0)\right). \tag{5.141}$$

Since $\mathcal{J}\delta z$ is constant and $\mathcal{G}^{u,s}$ are both inverses of $\mathcal{L}$, $\widehat{\Delta}_0$ satisfies $\mathcal{L}\widehat{\Delta}_0 = 0$.

We write then, $\widehat{\Delta}$ as

$$\widehat{\Delta} = \widehat{\Delta}_0 + \widehat{\mathcal{E}}.$$

The next two lemmas give estimates for these functions. Recall that Θ and G_0 are related through

$$\omega = \frac{\nu G_0^3}{L_0^3}, \quad \text{with} \quad G_0 = \Theta - L_0 + \eta_0\xi_0.$$

(see (5.33)).

Lemma 5.8.15. *The function $\widehat{\Delta}_0$ in (5.141) satisfies that, for $v \in D_{\kappa_3,\delta_3} \cap \mathbb{R}$ and $\gamma \in \mathbb{T}$,*

$$\begin{aligned}
\widehat{\Delta Y}_0(v,\gamma) &= \omega\partial_\sigma\mathcal{L}(\omega v - \gamma, \eta_0, \xi_0)\\
\widehat{\Delta\Lambda}_0(v,\gamma) &= -\partial_\sigma\mathcal{L}(\omega v - \gamma, \eta_0, \xi_0)\\
\widehat{\Delta\alpha}_0(v,\gamma) &= \delta\eta - i\partial_{\xi_0}\mathcal{L}(\omega v - \gamma, \eta_0, \xi_0) + G_0^{-3}\mathcal{O}(\eta_0, G_0^{-1}\xi_0)\\
\widehat{\Delta\beta}_0(v,\gamma) &= \delta\xi + i\partial_{\eta_0}\mathcal{L}(\omega v - \gamma, \eta_0, \xi_0) + G_0^{-3}\mathcal{O}(\eta_0, G_0^{-1}\xi_0)
\end{aligned}$$

where $\mathcal{L}$ is the Melnikov potential introduced in Proposition 5.4.2.

Lemma 5.8.16. *The function $\widehat{\mathcal{E}}$ satisfies that, for $v \in D_{\kappa_3,\delta_3} \cap \mathbb{R}$ and $\gamma \in \mathbb{T}$,*

$$\begin{aligned}
|\widehat{\mathcal{E}}_Y - \langle\widehat{\mathcal{E}}_Y\rangle| &\lesssim e^{-G_0^3/3} G_0^2 \ln^2 G_0, \quad |\widehat{\mathcal{E}}_\Lambda - \langle\widehat{\mathcal{E}}_\Lambda\rangle| \lesssim e^{-G_0^3/3} G_0^{-1} \ln^2 G_0\\
|\widehat{\mathcal{E}}_\alpha - \langle\widehat{\mathcal{E}}_\alpha\rangle| &\lesssim e^{-G_0^3/3} G_0^{1/2} \ln^2 G_0, \quad |\widehat{\mathcal{E}}_\beta - \langle\widehat{\mathcal{E}}_\beta\rangle| \lesssim e^{-G_0^3/3} G_0^{1/2} \ln^2 G_0
\end{aligned} \tag{5.142}$$

and

$$|\langle\widehat{\mathcal{E}}_Y\rangle| + |\langle\widehat{\mathcal{E}}_\Lambda\rangle| + |\langle\widehat{\mathcal{E}}_\alpha\rangle| + |\langle\widehat{\mathcal{E}}_\beta\rangle| \lesssim G_0^{-6}|\ln G_0|^3.$$

Note that this lemma gives an expression of the difference between the paramerizations of the invariant manifolds $\widetilde{\Delta}$ as

$$\widetilde{\Delta} = \widetilde{\Phi}_A\left(\mathrm{Id} + \widetilde{\Psi}\right)\left(\widehat{\Delta}_0 + \widehat{\mathcal{E}}\right) \tag{5.143}$$

and the Fourier coefficients of $\widehat{\mathcal{E}}$ (except its averages) have exponentially small bounds. In the next section we improve the estimates for a certain average associated to the $\widetilde{\Lambda}$ component of $\widehat{\mathcal{E}}$.

We finish this section proving Lemmas 5.8.15 and 5.8.16.

Proof of Lemma 5.8.15. For the Y component, using the definition of $\widetilde{\Phi}_A$ in (5.135) and the properties of the matrix $\mathcal{J}$ stated in Lemma 5.8.9, one has

$$\widehat{\Delta Y}_0 = \mathcal{G}^u(F_1(0)) - \mathcal{G}^s(F_1(0)) = \mathcal{G}^u(F_1(0)) - \mathcal{G}^s(F_1(0)).$$

Then, recalling the definition of the operators $\mathcal{G}^{u,s}$ in (5.92) and of $F(0)$, one can check that $\widehat{\Delta Y}_0 = \omega\partial_\sigma\mathcal{L}$. Proceeding analogously, one can prove that $\widehat{\Delta\Lambda}_0 = -\partial_\sigma\mathcal{L}$.

For the components α and β we use that, for real values of v, the matrix $\widetilde{\Phi}_A$ satisfies $\widetilde{\Phi}_A - \mathrm{Id} = \mathcal{O}\left(G_0^{-1}\eta_0 + G_0^{-1}\xi_0\right)$. Then, using Theorem 5.7.20, for real values of (v,γ),

$$\begin{aligned}\widehat{\Delta\alpha}_0 &= \pi_\alpha(\mathcal{J}\delta z) + \mathcal{G}^u(F_3(0)) - \mathcal{G}^s(F_3(0)) + \mathcal{O}\left(G_0^{-4}\eta_0 + G_0^{-4}\xi_0\right)\\ &= \delta\eta - i\partial_{\xi_0}\mathcal{L} + \mathcal{O}\left(G_0^{-4}\eta_0 + G_0^{-4}\xi_0\right) + \mathcal{O}\left(G_0^{-3}\eta_0\right)\end{aligned}$$

(π_α denotes the projection on the α component) and analgously for β. □

Proof of Lemma 5.8.16. Using the definition of $\widehat{\Delta}_0$ in (5.140), we split the function $\widehat{\mathcal{E}}$ in (5.143) as $\widehat{\mathcal{E}} = \widehat{\mathcal{E}}^1 + \widehat{\mathcal{E}}^2 + \widehat{\mathcal{E}}^3$ with

$$\begin{aligned}\widehat{\mathcal{E}}^1 &= (\mathrm{Id} + \widetilde{\Psi})^{-1}\widetilde{\Phi}_A^{-1}\widetilde{\Delta} - \widetilde{\Phi}_A^{-1}\widetilde{\Delta} = \left[(\mathrm{Id} + \widetilde{\Psi})^{-1} - \mathrm{Id}\right]\widetilde{\Phi}_A^{-1}\widetilde{\Delta}\\ \widehat{\mathcal{E}}^2 &= \widetilde{\Phi}_A^{-1}\left(\widetilde{\Delta} - \Delta\right)\\ \widehat{\mathcal{E}}^3 &= \widetilde{\Phi}_A^{-1}\left(\Delta - (\mathcal{F}(0)^u - \mathcal{F}^s(0))\right)\end{aligned} \tag{5.144}$$

where $\widetilde{\Psi}$ is the matrix obtained in Theorem 5.8.10. We bound each term separately.

For the first term, we write the matrix as $(\mathrm{Id} + \widetilde{\Psi})^{-1} - \mathrm{Id} = \sum_{k\geq 1}(-\widetilde{\Psi})^k$. Therefore, using the estimates for $\widetilde{\Phi}_A^{-1}$ and $\widetilde{\Psi}$ in Lemma 5.8.9 and Theorem 5.8.10 respectively and the properties of the matrix norm given in Lemma 5.8.1,

$$\left\|\left[(\mathrm{Id} + \widetilde{\Psi})^{-1} - \mathrm{Id}\right]\widetilde{\Phi}_A^{-1}\right\|_{1/2,\sigma_3,\mathrm{mat}} \lesssim G_0^{-3}\ln G_0.$$

Then, using also Theorems 5.7.20 and 5.8.5 and Lemma 5.8.3, one has $\|\widetilde{\Delta}\|_{1/2,\sigma_3,\mathrm{vec}} \lesssim G_0^{-3}\ln G_0$. Thus,

$$\left\|\widehat{\mathcal{E}}^1\right\|_{1,\sigma_3,\mathrm{vec}} \lesssim G_0^{-3}\ln G_0\|\widetilde{\Delta}\|_{0,\sigma_3,\mathrm{vec}} \lesssim G_0^{-6}\ln^2 G_0.$$

For $\widehat{\mathcal{E}}_2$, we use the definition of $\widetilde{\Delta}$ in (5.128). Theorems 5.7.20 and 5.8.5 and Lemma 5.8.3 imply that

$$\|\widetilde{\Delta} - \Delta\|_{1/3,\sigma_3,\mathrm{vec}} \lesssim G_0^{-7}\ln^2 G_0.$$

Then, using this estimate and Lemmas 5.7.6 and 5.8.9,

$$\left\|\widehat{\mathcal{E}}^2\right\|_{3/2,\sigma_3,\mathrm{vec}} \lesssim \left\|\widetilde{\Phi}_A^{-1}\right\|_{0,\sigma_3,\mathrm{mat}}\|\widetilde{\Delta} - \Delta\|_{3/2,\sigma_3,\mathrm{vec}} \lesssim G_0^{-7}\ln^2 G_0.$$

Finally, using that the paramerizations of the invariant manifolds Z^*, $* = u,s$ obtained in Theorem 5.7.20 are fixed points of the operators $\mathcal{F}^*$, $* = u,s$, respectively, we write $\widehat{\mathcal{E}}^3$ as

$$\widehat{\mathcal{E}}^3 = \widetilde{\Phi}_A^{-1}\left(\mathcal{F}^u(Z^u) - \mathcal{F}^s(Z^s)\right) - \widetilde{\Phi}_A^{-1}\left(\mathcal{F}^u(0) - \mathcal{F}^s(0)\right).$$

Now, by (5.107),

$$\|\mathcal{F}^*(Z^*) - \mathcal{F}^*(0)\|_{1,\mathrm{vec},\sigma_3} \lesssim G_0^{-6}|\ln G_0|^3 \qquad * = u,s.$$

Therefore, using this estimate and the estimate for $\widetilde{\Phi}_A$ in Lemma 5.8.9, we obtain

$$\left\|\widehat{\mathcal{E}}^3\right\|_{1,\sigma_3,\mathrm{vec}} \lesssim \left\|\widetilde{\Phi}_A^{-1}\right\|_{0,\sigma_3,\mathrm{mat}}\left(\|\mathcal{F}^u(Z^u) - \mathcal{F}^u(0)\|_{1,\sigma_3,\mathrm{vec}} + \|\mathcal{F}^s(Z^s) - \mathcal{F}^s(0)\|_{1,\sigma_3,\mathrm{vec}}\right) \lesssim G_0^{-6}|\ln G_0|^3.$$

Now, for $v \in D_{\kappa_3,\delta_3} \cap \mathbb{R}$ and $\gamma \in \mathbb{T}$,

$$|\langle \widehat{\mathcal{E}} \rangle| \lesssim |\widehat{\mathcal{E}}| \lesssim \left\| \widehat{\mathcal{E}}^1 \right\|_{1,\sigma_3,\mathrm{vec}} + \left\| \widehat{\mathcal{E}}^2 \right\|_{3/2,\sigma_3,\mathrm{vec}} + \left\| \widehat{\mathcal{E}}^3 \right\|_{1,\sigma_3,\mathrm{vec}} \lesssim G_0^{-6} |\ln G_0|^3.$$

For the other harmonics, one can use $\left\| \widehat{\mathcal{E}} \right\|_{0,\sigma_3,\mathrm{vec}} \lesssim G_0^{-5/2} \ln^2 G_0$. Then, by the definition of the norm,

$$\begin{aligned} \|\widehat{\mathcal{E}}_Y\|_{0,\sigma_3} &\lesssim G_0^{1/2} \ln^2 G_0, & \|\widehat{\mathcal{E}}_\Lambda\|_{0,\sigma_3} &\lesssim G_0^{-5/2} \ln^2 G_0 \\ \|\widehat{\mathcal{E}}_\alpha\|_{0,\sigma_3} &\lesssim G_0^{-1} \ln^2 G_0, & \|\widehat{\mathcal{E}}_\beta\|_{0,\sigma_3} &\lesssim G_0^{-1} \ln^2 G_0. \end{aligned}$$

which together with Lemma 5.8.14 gives the estimates of the lemma. □

5.8.5 Improved estimates for the averaged term

The last step in the analysis of the difference between the invariant manifolds is to obtain improved estimates for the averaged term of $\widehat{\mathcal{E}}_\Lambda$. Note that for this component the first order, given by the Melnikov potential in Theorem 5.4.3, is exponentially small (see Proposition 5.4.2). Therefore, to prove that $\widehat{\Delta}_0$ is bigger than the error one has to show that $\langle \widehat{\mathcal{E}}_\Lambda \rangle$ is smaller than the corresponding Melnikov components $-\partial_\sigma \mathcal{L}$ (and thus exponentially small). We prove this fact by using the Poincaré invariant. One can proceed analgously for $\langle \widehat{\mathcal{E}}_Y \rangle$. However, this estimate is not needed because this estimate can be deduced by the the conservation of energy.

Consider the autonomous Hamiltonian system associated to (5.74). Then, it has the Poincaré invariant acting on closed loops of the phase space defined as follows. Consider a loop Γ in phase space, then

$$I(\Gamma) = \int_\Gamma (Y du + \Lambda d\gamma + \beta d\alpha)$$

is invariant under the flow associated to the Hamiltonian system. We use this invariant to improve the estimates of the γ-averaged term in $\widehat{\Delta\Lambda}$.

In Theorem 5.7.20, we have obtained parameterizations of $W^u(P_{\eta_0,\xi_0})$ and $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi})$, Z^u and Z^s. Take any loop Γ^u contained in $W^u(P_{\eta_0,\xi_0})$, homotopic to the loop $W^u(P_{\eta_0,\xi_0}) \cap \{u = u_0\}$ (and thus homotopic to P_{η_0,ξ_0}) and any loop Γ^s contained in $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi})$, homotopic to a loop $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi}) \cap \{u = u_0\}$ (and thus homotopic to $P_{\eta_0+\delta\eta,\xi_0+\delta\xi}$). More concretely, in the case of the stable manifold, take a $\mathcal{C}^1$ function $f : \mathbb{T} \to D^u_{\kappa,\delta} \cap \mathbb{R}$ and define an associated loop parameterized as

$$\Gamma^s = \{(u, Y, \gamma, \Lambda, \alpha, \beta) = (f(\gamma), Y^s(f(\gamma), \gamma), \gamma, \Lambda^s(f(\gamma), \gamma), \alpha^s(f(\gamma), \gamma), \beta^s(f(\gamma), \gamma)\} \tag{5.145}$$

where $Z^s = (Y^s, \Lambda^s, \alpha^s, \beta^s)$ is the parameterization of the invariant manifold obtained in Theorem 5.7.20.

Lemma 5.8.17. *The loops in* (5.145) *satisfy* $I(\Gamma^*) = 0$, $* = u, s$.

Proof. Call Φ_t the flow associated to the Hamiltonian $\mathcal{P}$ in (5.74) and take a loop Γ^s. Then, since the Poincaré invariant is invariant under this flow

$$I(\Gamma^s) = \lim_{t\to\infty} I(\Phi_t(\Gamma^s)).$$

Then, using that $\lim_{t\to\infty} \pi_\alpha(\Phi_t(\Gamma^s)) = \delta\eta$ and the estimates of the parameterizations of Z^s as $u \to \infty$ in Theorem 5.7.20, it is clear that $\lim_{t\to\infty} I(\Phi_t(\Gamma^s)) = 0$. □

Taking any of the loops Γ^s and Γ^u considered in Lemma 5.8.17, we have $I(\Gamma^s) - I(\Gamma^u) = 0$

Now we consider the symplectic exact transformation Φ obtained in Theorem 5.8.5 and we work in variables (v, γ). Let us fix a section $v = v_0$ and define the loops $\widetilde{\Gamma}^u = W^u(P_{\eta_0,\xi_0}) \cap \{v = v_0\}$ and $\widetilde{\Gamma}^s = W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi}) \cap \{v = v_0\}$. It is clear that $\widetilde{\Gamma}^* = \Phi^{-1}(\Gamma^*)$ for a suitable function f. Since the Poincaré invariant is invariant under exact symplectic transformations, we have that

$$I(\widetilde{\Gamma}^*) = \int_{\widetilde{\Gamma}^*} \left(\widetilde{Y} dv + \widetilde{\Lambda} d\gamma + \widetilde{\beta} d\widetilde{\alpha} \right) = I(\Gamma^*) = 0.$$

Moreover, since v is constant in the loops

$$I\left(\widetilde{\Gamma}^*\right) = \int_0^{2\pi} \left(\widetilde{\Lambda}^*(v_0,\gamma) + \widetilde{\beta}^*(v_0,\gamma)\partial_\gamma \widetilde{\alpha}^*(v_0,\gamma)\right) d\gamma = 0, \quad * = u, s$$

where $(\widetilde{Y}^*, \widetilde{\Lambda}^*, \widetilde{\alpha}^*, \widetilde{\beta}^*)$, $* = u, s$, are the parameterizations obtained in Theorem 5.8.5.

Now consider the difference $\widetilde{\Delta}$ between the parameterization of the invariant manifolds introduced in (5.128). Subtracting the Poincaré invariant for the stable and unstable loops and integrating by parts, we obtain the relation

$$0 = I(\widetilde{\Gamma}^u) - I(\widetilde{\Gamma}^s) = \int_0^{2\pi} \left(\Delta\widetilde{\Lambda}(v_0,\gamma) + h_1(v_0,\gamma)\Delta\widetilde{\alpha}(v_0,\gamma) + h_2(v_0,\gamma)\Delta\widetilde{\beta}(v_0,\gamma)\right) d\gamma \tag{5.146}$$

where

$$h_1(v_0,\gamma) = -\frac{1}{2}\left(\partial_\gamma\widetilde{\beta}^s(v_0,\gamma) + \partial_\gamma\widetilde{\beta}^u(v_0,\gamma)\right) \quad \text{and} \quad h_2(v_0,\gamma) = \frac{1}{2}\left(\partial_\gamma\widetilde{\alpha}^s(v_0,\gamma) + \partial_\gamma\widetilde{\alpha}^u(v_0,\gamma)\right).$$

Note that the functions h_i satisfy $\langle h_i\rangle_\gamma = 0$ and, by Theorem 5.7.4, $|h_i(v,\gamma)| \lesssim G_0^{-6}\ln G_0$ for $i = 1, 2$ and real values of (v,γ).

Next step is to make the transformation

$$(\widetilde{Y}, \widetilde{\Lambda}, \widetilde{\alpha}, \widetilde{\beta})^\top = \Psi(v,\gamma)(\widehat{Y}, \widehat{\Lambda}, \widehat{\alpha}, \widehat{\beta})^\top \tag{5.147}$$

where $\Psi = \widetilde{\Phi}_A(\mathrm{Id} + \widetilde{\Psi})$ is the matrix obtained in Theorem 5.8.10. The difference between the invariant manifolds in these new coordinates is the vector $\widehat{\Delta}$ introduced in (5.139), which satisfies $\mathcal{L}\widehat{\Delta} = 0$.

We analyze the Poincaré invariant relation (5.146) after performing the change of coordinates (5.147). Note that this change is not symplectic and therefore the Liouville form is not preserved. Thus, we apply the change of coordinates to (5.146) directly and we obtain

$$\int_0^{2\pi} \left((1 + \widehat{h}_0(v_0,\gamma))\widehat{\Delta}\Lambda(v_0,\gamma) + \widehat{h}_1(v_0,\gamma)\widehat{\Delta}\alpha(v_0,\gamma) + \widehat{h}_2(v_0,\gamma)\widehat{\Delta}\beta(v_0,\gamma) + \widehat{h}_3(v_0,\gamma)\widehat{\Delta}Y\right) d\gamma = 0 \tag{5.148}$$

for some functions $\widehat{h}_i$. By the definition of the fundamental matrix $\widetilde{\Phi}_A$ in (5.135) and the estimates of the matrix $\widetilde{\Psi}$ obtained in Theorem 5.8.10, one can easily check that, for real values of (v,γ) and assuming $|\eta_0|G_0^{3/2} \ll 1$, the functions $\widehat{h}_i$ satisfy

$$|\widehat{h}_i| \lesssim G_0^{-3}\ln G_0, \quad i = 0\dots 2 \qquad \text{and} \qquad |\widehat{h}_3| \lesssim G_0^{-9/2}\ln G_0. \tag{5.149}$$

We would like to use (5.148) to obtain more accurate estimates of $\langle\widehat{\Delta}\Lambda\rangle$. Assume for a moment that $\langle\widehat{h}_i\rangle = 0$ for $i = 1, 2, 3$ and let us introduce the following notation

$$\{f\}(v,\gamma) = f(v,\gamma) - \langle f\rangle(v).$$

It certainly satisfies $\langle\{f(v,\gamma)\}\rangle = 0$.

By (5.142), the four components of $\left\{\widehat{\Delta}\right\}$ are exponentially small. Under the assumption $\langle\widehat{h}_i\rangle = 0$ for $i = 1, 2, 3$ then (5.148) becomes

$$\langle\widehat{\Delta}\Lambda\rangle(v_0) = \frac{1}{2\pi(1 + \langle\widehat{h}_0\rangle)(v_0)} \int_0^{2\pi} F(v_0,\gamma)d\gamma$$

with

$$F = \left\{\widehat{h}_0\right\}\left\{\widehat{\Delta}\Lambda\right\} + \left\{\widehat{h}_1\right\}\left\{\widehat{\Delta}\alpha\right\} + \left\{\widehat{h}_2\right\}\left\{\widehat{\Delta}\beta\right\} + \left\{\widehat{h}_3\right\}\left\{\widehat{\Delta}Y\right\}.$$

Now, using the estimates given in Proposition 5.4.2 and (5.142), one would obtain

$$|F(v_0,\gamma)| \lesssim G_0^{-5/2} e^{-G_0^3/3}\ln G_0$$

therefore $\langle\widehat{\Delta}\Lambda\rangle$ would have the same estimate.

Now, this argument does not work because $\langle\widehat{h}_i\rangle \neq 0$ for $i = 1, 2, 3$. Therefore, we have to perform the close to the identity change of coordinates which depends on v but not on γ.

$$\check{\Delta}\Lambda = \widehat{\Delta}\Lambda + \frac{\langle\widehat{h}_1\rangle}{1+\langle\widehat{h}_0\rangle}\widehat{\Delta}\alpha + \frac{\langle\widehat{h}_2\rangle}{1+\langle\widehat{h}_0\rangle}\widehat{\Delta}\beta + \frac{\langle\widehat{h}_3\rangle}{1+\langle\widehat{h}_0\rangle}\widehat{\Delta}Y$$

and the other variables remain unchanged. Note that the functions h_i are small by (5.149) and therefore this change is close to the identity. Moreover, the functions $\langle\widehat{h}_i\rangle$ are independent of γ and therefore

$$\langle\check{\Delta}\Lambda\rangle = \langle\widehat{\Delta}\Lambda\rangle + \widehat{C}_1\langle\widehat{\Delta}\alpha\rangle + \widehat{C}_2\langle\widehat{\Delta}\beta\rangle + C_3\langle\widehat{\Delta}Y\rangle.$$

where

$$\widehat{C}_i = \frac{\langle\widehat{h}_i\rangle}{1+\langle\widehat{h}_0\rangle}. \tag{5.150}$$

Thus

$$\left|\left\{\check{\Delta}\Lambda\right\}\right| \lesssim G_0^{-3/2}e^{-G_0^3/3}.$$

Now the relation (5.148) becomes

$$\int_0^{2\pi}\Big((1+\widehat{h}_0(v_0,\gamma))\check{\Delta}\Lambda(v_0,\gamma) + \check{h}_1(v_0,\gamma)\widehat{\Delta}\alpha(v_0,\gamma) + \check{h}_2(v_0,\gamma)\widehat{\Delta}\beta(v_0,\gamma) + \check{h}_3(v_0,\gamma)\widehat{\Delta}Y\Big)\,d\gamma = 0$$

where

$$\check{h}_i = \widehat{h}_i - \frac{1+\widehat{h}_0}{1+\langle\widehat{h}_0\rangle}\langle\widehat{h}_i\rangle, \qquad i = 1, 2, 3,$$

and therefore satisfy $\langle\check{h}_i\rangle = 0$ and $|\check{h}_i| \lesssim G_0^{-3}\ln G_0$ for $i = 1, 2, 3$. Therefore, the argument done previously works and one can deduce that

$$\left\langle\check{\Delta}\Lambda(v_0,\gamma)\right\rangle \lesssim G_0^{-5/2}e^{-G_0^3/3}\ln G_0.$$

The just obtained results are summarized in the following lemma.

Lemma 5.8.18. *The function $\widehat{\Delta}$ introduced in* (5.139) *satisfies that $\widehat{\Delta}(v,\gamma) = N(v)\check{\Delta}(v,\gamma)$ where N is the matrix*

$$N(v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\widehat{C}_3 & 1 & -\widehat{C}_1 & -\widehat{C}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1. \end{pmatrix}$$

which satisfies

$$N = \mathrm{Id} + \mathcal{O}\left(G_0^{-3}\ln G_0\right) \qquad \text{and} \qquad N_{21} = \mathcal{O}\left(G_0^{-9/2}\ln G_0\right).$$

Moreover, for real values of $v \in D_{\kappa_3,\delta_3}\cap\mathbb{R}$, $\left\langle\check{\Delta}\Lambda(v,\gamma)\right\rangle \lesssim G_0^{-5/2}e^{-G_0^3/3}\ln G_0$.

The next two lemmas complete the estimates of the errors in Theorem 5.4.3.

Lemma 5.8.19. *The function $\widetilde{\Delta}(v,\gamma)$ in* (5.128) *can be written as*

$$\widetilde{\Delta}(v,\gamma) = \widetilde{\mathcal{N}}(v,\gamma)\left(\widehat{\Delta}_0(v,\gamma) + \widetilde{\mathcal{E}}(v,\gamma)\right)$$

where $\widehat{\Delta}_0$ is the function introduced in (5.141)*, $\widetilde{\mathcal{N}}$ is an invertible matrix satisfying*

$$\widetilde{\mathcal{N}} = \mathrm{Id} + \mathcal{O}(G_0^{-3}\ln G_0) \qquad \text{and} \qquad \widetilde{\mathcal{N}}_{21} = \mathcal{O}\left(G_0^{-9/2}\ln G_0\right),$$

and $\widetilde{\mathcal{E}}$ satisfies

$$|\widetilde{\mathcal{E}}_Y| \lesssim G_0^{-6}\ln^2 G_0, \qquad |\widetilde{\mathcal{E}}_\Lambda| \lesssim e^{-G_0^3/3}G_0^{-5/2}\ln^2 G_0, \qquad |\widetilde{\mathcal{E}}_\alpha|, |\widetilde{\mathcal{E}}_\beta| \lesssim G_0^{-6}\ln^2 G_0.$$

Moreover, $\widetilde{\mathcal{E}}_Y, \widetilde{\mathcal{E}}_\alpha, \widetilde{\mathcal{E}}_\beta \in \mathrm{Ker}\mathcal{L}$ and

$$|\widetilde{\mathcal{E}}_Y - \langle\widetilde{\mathcal{E}}_Y\rangle| \lesssim e^{-G_0^3/3}G_0^2\ln^2 G_0$$
$$|\widetilde{\mathcal{E}}_\alpha - \langle\widetilde{\mathcal{E}}_\alpha\rangle|, |\widetilde{\mathcal{E}}_\beta - \langle\widetilde{\mathcal{E}}_\beta\rangle| \lesssim e^{-G_0^3/3}G_0^{1/2}\ln^2 G_0.$$

Proof. By (5.139) and (5.8.18), one has that $\widetilde{\Delta} = \mathcal{N}\check{\Delta}$ with $\mathcal{N} = \widetilde{\Phi}_A(\mathrm{Id} + \widetilde{\Psi})N$. Then, it can be written as

$$\widetilde{\Delta} = \mathcal{N}\left(\widehat{\Delta}_0 + \widetilde{\mathcal{E}}\right) \qquad \text{with} \qquad \widetilde{\mathcal{E}} = N^{-1}\widehat{\mathcal{E}} - (\mathrm{Id} - N^{-1})\widehat{\Delta}_0 = \check{\Delta} - \widehat{\Delta}_0.$$

We estimate $\langle\widetilde{\mathcal{E}}\rangle$ and $\{\widetilde{\mathcal{E}}\}$ separately. For the Λ average component we use that

$$\langle\widetilde{\mathcal{E}}_\Lambda\rangle = \langle\check{\Delta}\Lambda\rangle = \mathcal{O}\left(G_0^{-5/2}e^{-G_0^3/3}\ln G_0\right).$$

For the other averages one has to use that $N - \mathrm{Id} = \mathcal{O}(G_0^{-3}\log G_0)$ and the estimates for $\widehat{\mathcal{E}}_\alpha$ and $\widehat{\mathcal{E}}_\beta$ in Lemma 5.8.16 and for $\widehat{\Delta}_0$ given by Lemma 5.8.15 and Proposition 5.4.2. One can proceed analogously for the no average terms $\{\widetilde{\mathcal{E}}\}$ □

Now it only remains to express the difference between the invariant manifolds in the variables (u, γ) (see Theorem 5.8.5).

Lemma 5.8.20. *The function $\Delta(u, \gamma)$ introduced in (5.116) can be written as*

$$\Delta(u, \gamma) = \mathcal{N}(u, \gamma)\left(\widehat{\Delta}_0(u, \gamma) + \mathcal{E}(u, \gamma)\right)$$

where $\widehat{\Delta}_0$ is the function introduced in (5.141), $\mathcal{N}$ is an invertible matrix satisfying

$$\mathcal{N} = \mathrm{Id} + \mathcal{O}(G_0^{-3}\log G_0) \qquad \textit{and} \qquad \mathcal{N}_{21} = \mathcal{O}\left(G_0^{-9/2}\ln G_0\right)$$

and $\mathcal{E}$ satisfies

$$|\mathcal{E}_Y| \lesssim G_0^{-6}\ln^2 G_0, \qquad |\mathcal{E}_\Lambda| \lesssim e^{-G_0^3/3}G_0^{-5/2}\ln^2 G_0, \qquad |\mathcal{E}_\alpha|, |\mathcal{E}_\beta| \lesssim G_0^{-6}\ln^2 G_0.$$

Moreover, $\mathcal{E}_Y, \mathcal{E}_\alpha, \mathcal{E}_\beta$ and

$$|\mathcal{E}_Y - \langle\mathcal{E}_Y\rangle| \lesssim e^{-G_0^3/3}G_0^2\ln^2 G_0$$
$$|\mathcal{E}_\alpha - \langle\mathcal{E}_\alpha\rangle|, |\mathcal{E}_\beta - \langle\mathcal{E}_\beta\rangle| \lesssim e^{-G_0^3/3}G_0^{1/2}\ln^2 G_0.$$

The proof of this lemma is straighforward applying the inverse of change of coordinates obtained in Theorem 5.8.5.

5.8.6 End of the proof of Theorem 5.4.3

Lemma 5.8.20, recalling the expression of $\widehat{\Delta}_0$ given in Lemma 5.8.15, completes the proof of formulas (5.37), (5.38), (5.39) in Theorem 5.4.3 (recall the relation between G_0 and Θ given in (5.33)). Note, however that it gives slightly worse estimates compared to those in Theorem 5.4.3. Indeed, Lemma 5.8.20 implies that $\mathcal{N}$ is of the form

$$\mathcal{N} = \mathrm{Id} + \mathcal{O}(\Theta^{-3}\log\Theta)$$

and $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ satisfy

$$\begin{pmatrix}\mathcal{M}_\alpha(u, \gamma, z_0, \delta z)\\ \mathcal{M}_\beta(u, \gamma, z_0, \delta z)\end{pmatrix} = \begin{pmatrix}-i\partial_{\xi_0}\mathcal{L}(\gamma - \omega u, z_0) + \mathcal{O}\left(\Theta^{-6}\ln^2\Theta\right)\\ i\partial_{\eta_0}\mathcal{L}(\gamma - \omega u, z_0) + \mathcal{O}\left(\Theta^{-6}\ln^2\Theta\right)\end{pmatrix}.$$

Below we give the sharper estimates. However, they are enough to obtain the estimates for the derivatives given in (5.41). Indeed, it is enough to apply Cauchy estimates. Indeed, the formulas (5.37), (5.38), (5.39) are valid in a domain such that $|\eta_0|\Theta^{3/2} \ll 1$ and therefore applying Cauchy estimates one loses $\Theta^{3/2}$ at each derivative.

Then, it only remains to prove the estimates in (5.36) and (5.40) and the improved estimates for (5.37), (5.38), (5.39). The first ones are a direct consequence of Theorem 5.7.20. To obtain the estimates for the derivatives of $\mathcal{M}$ in (5.40) we proceed as follows.

First note that, by Theorem 5.7.20, the paramerizations of the invariant manifolds admit an analytic continuation to the domain

$$u \in D_{\kappa,\delta} \cap \mathbb{R}, \ \gamma \in \mathbb{T}, \ |\eta_0| \leq \frac{1}{2}, \ |\xi_0| \leq \frac{1}{2}. \tag{5.151}$$

Proceeding analgously, one can also extend analytically to this domain the change of variable Φ obtained in Theorem 5.8.5. Then, one can easily check that in such domain, the associated function $\mathcal{C}$ satisfies

$$|\mathcal{C}| \lesssim \Theta^{-4}.$$

(recall (5.33)). Similarly, one can extend the matrix $\Psi = \widetilde{\Phi}_A(\mathrm{Id} + \widetilde{\Psi})$ given by Theorem 5.8.10 to the same domain, where it satisfies

$$|\widetilde{\Psi}| \lesssim \Theta^{-3}.$$

Then, one can conclude that the matrix $\mathcal{N}$ appearing in Theorem 5.4.3 can be also analytically extended to (5.151) where it satisfies

$$\mathcal{N} = \mathrm{Id} + \mathcal{O}(\Theta^{-3}).$$

(see (5.38)). This analysis also gives the improved estimates for $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ in (5.39).

Note that the derivatives of $\mathcal{C}$, $\widetilde{\Psi}$ and $\mathcal{N}$ with respect to $(\eta_0.\xi_0)$ have the same estimates in the domain

$$|\eta_0| \leq \frac{1}{4}, \ |\xi_0| \leq \frac{1}{4}.$$

Indeed, it is enough to apply Cauchy estimates. Using these estimates and the estimates of the derivatives of the parameterizations of the invariant manifolds given by Theorem 5.7.20, one can easily deduce formulas (5.40).

5.9 The homoclinic channels and the associated scattering maps

We devote this section to prove the results on the scattering maps stated in Section 5.4.3. First, in Section 5.9.1 we prove Theorem 5.4.5. That is we prove the existence of two homoclinic channels and we obtain formulas for the associated scattering maps (in suitable domains). Second, in Section 5.9.2 we prove Theorem 5.4.7 which provides the existence of an isolating block for a suitable high iterate of a combination of the two scattering maps obtained in Theorem 5.4.5.

5.9.1 The scattering maps: Proof of Theorem 5.4.5

We devote this section to prove the existence and derive formulas for the scattering maps given by Theorem 5.4.5. Consider two periodic orbits $P_{\eta_0,\xi_0}, P_{\eta_0+\delta\eta,\xi_0+\delta\xi} \in \mathcal{E}_\infty$. We fix a section $u = u^* \in (u_1, u_2)$ (which is transverse to the flow) and we analyze the intersection $W^u(P_{\eta_0,\xi_0})$ and $W^s(P_{\eta_0+\delta\eta,\xi_0+\delta\xi})$ in this section. By the expression (5.30) and Theorem 5.4.3, these invariant manifolds intersect along an heteroclinic orbit if there exists $\gamma \in \mathbb{T}$ such that

$$\begin{aligned}
Y^u(u^*, \gamma, z_0) - Y^s(u^*, \gamma, z_0, \delta z) &= 0 \\
\Lambda^u(u^*, \gamma, z_0) - \Lambda^s(u^*, \gamma, z_0, \delta z) &= 0 \\
\alpha^u(u^*, \gamma, z_0) - \alpha^s(u^*, \gamma, z_0, \delta z) &= 0 \\
\beta^u(u^*, \gamma, z_0) - \beta^s(u^*, \gamma, z_0, \delta z) &= 0
\end{aligned} \tag{5.152}$$

where $z_0=(\eta_0,\xi_0)\in\widetilde{\mathbb{D}}$ (see (5.44)) and $\delta z=(\delta\eta,\delta\xi)$ satisfies $|\delta z|\lesssim\Theta^3$.

Using (5.37) in Theorem 5.4.3, the fact that the matrix $\mathcal{N}$ is invertible and energy conservation, obtaining a zero $(\gamma,\delta z)$ of (5.152) is equivalent to obtain a zero $(\gamma,\delta z)$ of

$$\begin{aligned}\mathcal{M}_\Lambda(u^*,\gamma,z_0,\delta z)&=0\\ \delta\eta+\mathcal{M}_\alpha(u^*,\gamma,z_0,\delta z)&=0\\ \delta\xi+\mathcal{M}_\beta(u^*,\gamma,z_0,\delta z)&=0.\end{aligned}\tag{5.153}$$

We emphasize that by zeros we mean that for a given z_0 and u^* there exists γ and δz which solve these three equations.

We first analyze the second and third equations, that is

$$\delta\eta+\mathcal{M}_\alpha(u^*,\gamma,z_0,\delta z)=0,\qquad \delta\xi+\mathcal{M}_\beta(u^*,\gamma,z_0,\delta z)=0.$$

Using the asymptotic expansions for $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ given in Theorem 5.4.3, one can obtain $(\delta\eta,\delta\xi)$ in terms of (η_0,ξ_0) and γ as follows,

$$\begin{aligned}\delta\eta=\delta\eta(u^*,\gamma,z_0)&=-i\partial_{\xi_0}\mathcal{L}(\gamma-\omega u^*,\eta_0,\xi_0)+P_1(u^*,\gamma,\eta_0,\xi_0)\\ \delta\xi=\delta\xi(u^*,\gamma,z_0)&=i\partial_{\eta_0}\mathcal{L}(\gamma-\omega u^*,\eta_0,\xi_0)+P_2(u^*,\gamma,\eta_0,\xi_0)\end{aligned}\tag{5.154}$$

where by Proposition 5.4.2, the estimates of $\mathcal{M}$ in Theorem 5.4.3, implicit derivation and Cauchy estimates, the functions P_1 and P_2 satisfy

$$\begin{aligned}|\partial^i_{\eta_0}\partial^j_{\xi_0}P_i|&\le C(i,j)\Theta^{-6} &&\text{for}\quad i,j\ge 0\\ |\partial^i_{\eta_0}\partial^j_{\xi_0}\partial^k_\gamma P_i|&\le C(i,j,k)\Theta^{1/2-3(i+j)/2}\mathrm{e}^{-\frac{\tilde{\nu}\tilde{\Theta}^3}{3L_0^3}} &&\text{for}\quad i,j\ge 0,k\ge 1\end{aligned}\tag{5.155}$$

for some constants $C(i,j)$ and $C(i,j,k)$.

Now we solve the equation for the Λ component evaluated at (5.154), that is

$$\mathcal{M}_\Lambda(u^*,\gamma,z_0,\delta z(u^*,\gamma,z_0))=0.$$

Note that in the domain $\widetilde{\mathbb{D}}$ introduced in (5.44), one has that

$$\left|\mathcal{L}^{[-1]}\right|\gtrsim\Theta^{-\frac{1}{2}}.$$

Then, dividing this equation by the factor $(-2\mathcal{L}^{[-1]})$, one obtains an equation of the form

$$\sin(\omega u^*-\gamma)+\mathcal{O}(\Theta^{-1/2})=0$$

which has two solutions

$$\gamma^j=\gamma^j(u^*,z_0)=\omega u^*+(j-1)\pi+\mathcal{O}\left(\Theta^{-1/2}\right),\qquad j=1,2.\tag{5.156}$$

Moreover, one can apply Cauchy estimates reducing ϱ used in the definition (5.44) of $\widetilde{\mathbb{D}}$, to obtain that

$$|\partial^i_{\eta_0}\partial^k_{\xi_0}\gamma^j|\le C(i,k)\Theta^{-1/2-2(i+k)}\tag{5.157}$$

for $i,k\ge 0$ and some constant $C(i,k)$ independent of Θ.

Now we obtain asymptotic formulas for the scattering maps. Observe that, recalling the parameterization of the invariant manifolds in (5.30), the values $(u^*,\gamma^j(u^*,z_0),z_0,\delta z(u^*,\gamma^j(u^*,z_0),z_0))$, $u^*\in(u_1,u_2)$ and $z_0\in\widetilde{\mathbb{D}}$ (see (5.44)) solving equations (5.153) give rise to heteroclinic points

$$z^j_{\mathrm{het}}=(\lambda^j_{\mathrm{het}}(u^*,z_0),w^j_{\mathrm{het}}(u^*,z_0))\in W^u(P_{z_0})\cap W^s(P_{z_0+\delta z^j_0})$$

with $\lambda^j_{\text{het}} = \gamma^j(u^*, z_0) + \phi_{\text{h}}(u^*)$. Consequently, denoting by $\psi(t,z)$ the flow of equation (5.20), there exist $\lambda^j_\pm$ such that

$$\psi\left(t, z^j_{\text{het}}\right) - \psi\left(t, \lambda^j_-, w_0\right) \to 0 \quad \text{as} \quad t \to -\infty \tag{5.158}$$

$$\psi\left(t, z^j_{\text{het}}\right) - \psi\left(t, \lambda^j_+, w_0 + \delta w^j_0\right) \to 0 \quad \text{as} \quad t \to +\infty, \tag{5.159}$$

where $w_0 = (L_0, \eta_0, \xi_0, \infty, 0)$ and δw^j_0 is given by (5.154) with $\gamma = \gamma^j(u^*, z_0)$, in the sense that the asymptotic condition in the $\tilde{r}$ component of $\psi\left(t, z^j_{\text{het}}\right)$ means that it becomes unbounded. An important observation is that, using that the system (5.20) is autonomous, we have that, for any $s \in \mathbb{R}$

$$\begin{aligned}
\psi\left(t+s, z^j_{\text{het}}\right) - \psi\left(t+s, \lambda^j_-, w_0\right) \to 0 \quad \text{as} \quad t \to -\infty, \\
\psi\left(t+s, z^j_{\text{het}}\right) - \psi\left(t+s, \lambda^j_+, w_0 + \delta w^j_0\right) \to 0 \quad \text{as} \quad t \to +\infty,
\end{aligned}$$

and we observe that

$$\begin{aligned}
\psi\left(t+s, z^j_{\text{het}}\right) &= \psi\left(t, \psi(s, z^j_{\text{het}})\right), \\
\psi\left(t+s, \lambda^j_-, w_0\right) &= \psi\left(t, \psi\left(s, \lambda^j_-, w_0\right)\right) = (\lambda^j_- + \frac{\nu}{L_0^3} s, w_0),
\end{aligned}$$

and analgously for the other periodic orbit.

Calling $z^j_{\text{het}}(s) = \psi\left(s, z^j_{\text{het}}\right)$, and $\lambda^j_\pm(s) = \lambda^j_\pm + \frac{\nu}{L_0^3} s$, we have that

$$\begin{aligned}
\psi\left(t, z^j_{\text{het}}(s)\right) - \psi\left(t, \lambda^j_-(s), w_0\right) \to 0 \quad \text{as} \quad t \to -\infty \\
\psi\left(t, z^j_{\text{het}}(s)\right) - \psi\left(t, \lambda^j_+(s), w_0 + \delta w^j_0\right) \to 0 \quad \text{as} \quad t \to +\infty.
\end{aligned}$$

Therefore, the orbit through $z^j_{\text{het}}(s)$ is an heteroclinic orbit between the points $(\lambda^j_-(s), w_0) \in P_{z_0}$ and $(\lambda^j_+(s), w_0 + \delta w^j_0) \in P_{z_0 + \delta z^j_0}$, for any $s \in \mathbb{R}$. Analogously, given any $\lambda \in \mathbb{T}$, we can choose s through $\lambda = \lambda^j_- + \frac{\nu}{L_0^3} s$ and, abusing notation, calling again the heteroclinic point $z^j_{\text{het}}(\lambda) := z^j_{\text{het}}(s)$, we have that

$$\begin{aligned}
&\psi(t, z^j_{\text{het}}(\lambda) - \psi(t, \lambda, w_0) \to 0 \quad \text{as} \quad t \to -\infty \\
&\psi(t, z^j_{\text{het}}(\lambda)) - \psi(t, \lambda + \Delta^j, w_0 + \delta w^j_0) \to 0 \quad \text{as} \quad t \to +\infty
\end{aligned} \tag{5.160}$$

where $\Delta^j = \lambda^j_+ - \lambda^j_-$. Consequently, the scattering maps are of the form

$$\widetilde{\mathcal{S}}^j : \begin{pmatrix} L_0 \\ \lambda \\ \eta_0 \\ \xi_0 \end{pmatrix} \to \begin{pmatrix} L_0 \\ \lambda + \Delta^j \\ \eta_0 + \delta\eta^j_0 \\ \xi_0 + \delta\xi^j_0 \end{pmatrix}$$

where $(\eta_0 + \delta\eta^j, \xi_0 + \delta\xi^j) = \mathcal{S}^j(\eta_0, \xi_0)$ are independent of λ and L_0 is preserved by the conservation of the energy (recall that we are omitting the dependence on L_0 of all the functions).

Observe that $z^j_{\text{het}}(\lambda) = z^j_{\text{het}}(\lambda, \eta_0, \xi_0)$, with $(\eta_0, \xi_0) \in \widetilde{\mathbb{D}}$ and $\lambda \in \mathbb{T}$, gives a different parameterization of the homoclinic chanel introduced in (5.42).

Finally, note that to obtain formulas for $\mathcal{S}^j$ one has just to evaluate (5.154) at the solutions γ^j in (5.156), to obtain

$$\mathcal{S}^j \begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix} = \begin{pmatrix} \eta_0 - i\partial_{\xi_0}\mathcal{L}(j\pi, \eta_0, \xi_0) + \mathcal{O}\left(\Theta^{-6}\right) \\ \xi_0 + i\partial_{\eta_0}\mathcal{L}(j\pi, \eta_0, \xi_0) + \mathcal{O}\left(\Theta^{-6}\right) \end{pmatrix}. \tag{5.161}$$

Then, it is enough to use the formulas of the Melnikov potential $\mathcal{L}$ given in Proposition 5.4.2 to obtain the formulas for $\mathcal{S}^j$ in Proposition 5.4.5. The estimates for the derivatives of $\mathcal{S}^j$ are a consequence of the estimates for the derivatives of the Melnikov potential in Proposition 5.4.2 and the estimates (5.155), (5.157).

Finally, Theorem 8 in [DdlLS08] implies that the map $\widetilde{\mathcal{S}}^j$ is symplectic. Then, using the particular form of $\widetilde{\mathcal{S}}^j$ one can easily see that $\mathcal{S}^j$ is symplectic in the sense that fixing $L = L_0$ it preserves the symplectic form $d\eta_0 \wedge d\xi_0$.

Now it only remains to analyze the fixed points (η_0^j, ξ_0^j) of the scattering maps $\mathcal{S}^j$. The particular form of the fixed points given in (5.47) is just a consequence of (5.161) and the asymptotic expansions of the Melnikov potential given in Proposition 5.4.2. Note that $(\eta_0^j, \xi_0^j) \in \widetilde{\mathbb{D}}$ where $\widetilde{\mathbb{D}}$ is the domain in (5.44).

To prove the asymptotic formula (5.48) for the difference between the two fixed points one cannot use (5.161) but has to go back to equations (5.153) and analyze them when $\delta\eta = \delta\xi = 0$. In particular,

$$\mathcal{M}_\alpha(u^*, \gamma^j, \eta_0^j, \xi_0^j) = 0, \quad \mathcal{M}_\beta(u^*, \gamma^j, \eta_0^j, \xi_0^j) = 0.$$

We subtract the equalities for $j = 1$ and $j = 2$ to obtain

$$\mathcal{M}_\alpha(u^*, \gamma^2, \eta_0^2, \xi_0^2) - \mathcal{M}_\alpha(u^*, \gamma^1, \eta_0^1, \xi_0^1) = 0 \qquad \text{and} \qquad \mathcal{M}_\beta(u^*, \gamma^2, \eta_0^2, \xi_0^2) - \mathcal{M}_\beta(u^*, \gamma^1, \eta_0^1, \xi_0^1) = 0.$$

Taylor expanding, defining $\Delta\eta_0 = \eta_0^2 - \eta_0^1$, $\Delta\xi_0 = \xi_0^2 - \xi_0^1$ and using the estimates in Proposition 5.4.2 and Theorem 5.4.3, we have

$$\begin{aligned} E_\alpha + \partial_{\eta_0}\widetilde{\mathcal{M}}_\alpha(u^*, \gamma^2, \eta_0^1, \xi_0^1)\Delta\eta_0 + \partial_{\xi_0}\widetilde{\mathcal{M}}_\alpha(u^*, \gamma^2, \eta_0^1, \xi_0^1)\Delta\xi_0 + \Theta^{-3}\mathcal{O}_2(\Delta\eta_0, \Delta\xi_0) = 0 \\ E_\beta + \partial_{\eta_0}\widetilde{\mathcal{M}}_\beta(u^*, \gamma^2, \eta_0^1, \xi_0^1)\Delta\eta_0 + \partial_{\xi_0}\widetilde{\mathcal{M}}_\beta(u^*, \gamma^2, \eta_0^1, \xi_0^1)\Delta\xi_0 + \Theta^{-3}\mathcal{O}_2(\Delta\eta_0, \Delta\xi_0) = 0 \end{aligned}$$

where

$$\begin{aligned} E_\alpha &= \mathcal{M}_\alpha(u^*, \gamma^2, \eta_0^1, \xi_0^1) - \mathcal{M}_\alpha(u^*, \gamma^1, \eta_0^1, \xi_0^1) = -\frac{3i}{2}\widetilde{\nu}N_2\sqrt{\pi}L_0^{7/2}\tilde{\Theta}^{3/2}\mathrm{e}^{-\frac{\tilde{\nu}\Theta^3}{3L_0^3}}\left(1 + \mathcal{O}\left(\Theta^{-1}\right)\right) \\ E_\beta &= \mathcal{M}_\beta(u^*, \gamma^2, \eta_0^1, \xi_0^1) - \mathcal{M}_\beta(u^*, \gamma^1, \eta_0^1, \xi_0^1) = \frac{3i}{2}\widetilde{\nu}N_2\sqrt{\pi}L_0^{7/2}\tilde{\Theta}^{3/2}\mathrm{e}^{-\frac{\tilde{\nu}\Theta^3}{3L_0^3}}\left(1 + \mathcal{O}\left(\Theta^{-1}\right)\right). \end{aligned}$$

Moreover, using again the estimates in Proposition 5.4.2 and Theorem 5.4.3,

$$\begin{aligned} \partial_{\eta_0}\mathcal{M}_\alpha(u^*, \gamma^2, \eta_0^1, \xi_0^1) &= -\frac{3i}{8}\widetilde{\nu}\pi L_0^3\tilde{\Theta}^{-3}N_2 + \mathcal{O}\left(\Theta^{-5}\right) \\ \partial_{\xi_0}\mathcal{M}_\beta(u^*, \gamma^2, \eta_0^1, \xi_0^1) &= \frac{3i}{8}\widetilde{\nu}\pi L_0^3\tilde{\Theta}^{-3}N_2 + \mathcal{O}\left(\Theta^{-5}\right) \end{aligned}$$

and

$$\partial_{\xi_0}\mathcal{M}_\alpha(u^*, \gamma^2, \eta_0^1, \xi_0^1) = \partial_{\eta_0}\mathcal{M}_\beta(u^*, \gamma^2, \eta_0^1, \xi_0^1) = \mathcal{O}\left(\Theta^{-5}\right).$$

Then, recalling that $N_2 \neq 0$ (see (5.35)), it is enough to apply the Implicit Function Theorem.

5.9.2 An isolating block for an iterate of the scattering map: Proof of Theorem 5.4.7

We devote this section to construct an isolating block for a suitable iterate of the scattering map. That is, for the map $\mathcal{S} = (\mathcal{S}^1)^M \circ \mathcal{S}^2$ for a suitable large M which depends on the size of the block. To construct the block we need a "good" system of coordinates. We rely on the properties of the scattering maps obtained in Proposition 5.4.5.

The steps to build the isolating block are

1. Prove the existence of a KAM invariant curve $\mathbb{T}_*$ for $\mathcal{S}^1$. To apply KAM Theory we first have to do a finite number of steps of Birkhoff Normal Form around the elliptic point of $\mathcal{S}^1$ and consider action-angle coordinates.

2. Prove that the preimage of $\mathbb{T}_*$ by the other scattering map, that is $(\mathcal{S}^2)^{-1}(\mathbb{T}_*)$, intersects transversally $\mathbb{T}_*$.

3. Pick a small "rectangle" whose lower boundary is a piece of the torus $\mathbb{T}_*$ and one adjacent side is a piece of $(\mathcal{S}^2)^{-1}(\mathbb{T}_*)$.

4. Show that such rectangle has the isolating block property for $\mathcal{S} = (\mathcal{S}^1)^M \circ \mathcal{S}^2$.

We start with Step 1. We follow, without mentioning explicitly, all the notation used in Proposition 5.4.5. However, we restrict the scattering maps to much narrower domains (see Lemma 5.9.1 below). In fact, we consider domains which are balls centered at the fixed points z_0^j of the scattering maps obtained in Proposition 5.4.5 and exponentially small radius. We use the notation for disks introduced in (5.43). These exponentially small domains are enough to build the isolating block.

Since in this section we need to perform several symplectic transformations to the scattering maps, we denote them by Φ_i, $i = 1, 2, 3, 4$.

Lemma 5.9.1. *For $\Theta \gg 1$ large enough, there exists a symplectic change of coordinates*

$$\Phi_1 : \mathbb{D}_{\rho/2}\left(z_0^1\right) \to \mathbb{D}_\rho\left(z_0^1\right) \qquad \text{with} \qquad \rho = \tilde{\Theta}^{11/2} \mathrm{e}^{-\frac{\tilde{\nu}\tilde{\Theta}^3}{3L_0^3}}$$

such that $\Phi_1(z_0^1) = z_0^1$ and $\widetilde{\mathcal{S}}^1 = \Phi_1^{-1} \circ \mathcal{S}^1 \circ \Phi_1$ is of the form

$$\widetilde{\mathcal{S}}^1(z) = z_0^1 + e^{i\left(\omega_1 + C_1|z-z_0^1|^2 + C_2|z-z_0^1|^4\right)}(z - z_0^1) + \mathcal{O}\left(z - z_0^1\right)^7 \tag{5.162}$$

where the ω_1 has been introduced in (5.49)*, the constant $C_1 = \mathcal{T}\tilde{\Theta}^{-3} + \mathcal{O}\left(\tilde{\Theta}^{-5}\right)$ with $\mathcal{T}$ as introduced in* (5.50) *in Proposition 5.4.5, which satisfies $C_1 \neq 0$ and C_2 such that $C_2 = \mathcal{O}\left(\tilde{\Theta}^{-3}\right)$.*

Morover, Φ_1 satisfies

$$\Phi_1(z) = z + \tilde{\Theta}^{-2}\mathcal{O}\left(z^2\right) + \mathcal{O}\left(z^3\right).$$

Proof. The proof of this lemma is through the classical method of Birkhoff Normal Form by (for instance) generating functions. Fix $N \geq 1$. Note that then the small divisors which arise in the process are of the form $|k\tilde{\Theta}^{-3} - 1|$ for $k = 1 \ldots 2N - 2$. Then, taking Θ big enough, they satisfy

$$|k\tilde{\Theta}^{-3} - 1| \gtrsim \Theta^{-3}.$$

With such estimate and the estimates of the Taylor coefficients of the scattering map $\mathcal{S}^1$ given in Proposition 5.4.5, one can easily complete the proof of the lemma. □

Next step is the application of KAM Theorem. To this end, we consider action angle coordinates for $\widetilde{\mathcal{S}}^1$ (centered at the fixed point). Note that the first order of $\widetilde{\mathcal{S}}^1$ in (5.162) is integrable and therefore it only depends on the action.

Lemma 5.9.2. *Fix a parameter $\rho \in \left(0, \frac{1}{2}\tilde{\Theta}^{11/2}\mathrm{e}^{-\frac{\tilde{\nu}\tilde{\Theta}^3}{3L_0^3}}\right)$ and any $\ell \geq 4$. Consider the change of coordinates*

$$z = \Phi_2(\theta, I) = (\eta_0^1 + \rho\sqrt{I}e^{i\theta}, \xi_0^1 + \rho\sqrt{I}e^{-i\theta}).$$

Then, the map $\widehat{\mathcal{S}}^1 = \Phi_2^{-1} \circ \widetilde{\mathcal{S}}^1 \circ \Phi_2$ is symplectic with respect to the canonical form $d\theta \wedge dI$ and it is of the form

$$\widehat{\mathcal{S}}^1(\theta, I) = \begin{pmatrix} \theta + B(I) + \mathcal{O}_{\mathcal{C}^\ell}\left(\rho^6\right) \\ I + \mathcal{O}_{\mathcal{C}^\ell}\left(\rho^6\right). \end{pmatrix} \tag{5.163}$$

Moreover, for $I \in [1, 2]$, the function B satisfies

$$\begin{aligned} B(I) &= C_0\tilde{\Theta}^{-3} + \mathcal{O}\left(\tilde{\Theta}^{-4}\right) \\ \partial_I B(I) &= \rho^2\left(C_1\tilde{\Theta}^{-3} + \mathcal{O}\left(\tilde{\Theta}^{-4}\right)\right) \\ \partial_I^2 B(I) &= \mathcal{O}(\rho^4), \end{aligned}$$

where $C_1 \neq 0$ is the constant provided in Lemma 5.9.1 and $C_0 = \tilde{\nu}\pi L_0^4 A_1$ (see (5.49)*).*

We use the following KAM Theorem from [Her83, Her86] (we use the simplified version already stated in [DdlLS00])

Theorem 5.9.3. *Let $f : \mathbb{T}^1 \times [0,1] \to \mathbb{T}^1 \times [0,1]$ be an exact symplectic $\mathcal{C}^\ell$ map, $\ell \geq 4$. Assume that $f = f_0 + \delta f_1$, where $f(I,\theta) = (\theta + A(I), I)$, A is $\mathcal{C}^\ell$ and satisfies $|\partial_I A(I)| \geq M$ and $\|f_1\|_{\mathcal{C}^\ell} \leq 1$.*

Then, if $\sigma = \delta^{1/2}M^{-1}$ is sufficiently small, for a set of Diophantine numbers with $\tau = 5/4$, we can find invariant tori which are the graph of $\mathcal{C}^{\ell-3}$ functions u_ω, the motion on them is $\mathcal{C}^{\ell-3}$ conjugate to a rotation by ω and $\|u_\omega\|_{\mathcal{C}^{\ell-3}} \lesssim \delta^{1/2}$ and the tori cover the whole annulus except a set of measure smaller than constant$M^{-1}\delta^{1/2}$.

Note that the map (5.163) in $[1,2] \times \mathbb{T}$ satisfies the properties of Theorem 5.9.3 with $M \gtrsim \Theta^{-3}\rho$ and $\delta = \rho^6$ for any regularity $\mathcal{C}^\ell$ (the scattering map is actually analytic). This theorem then gives, in particular, a torus $\mathbb{T}_*$ which is invariant by $\widehat{\mathcal{S}}^1$ and is parameterized as a graph as

$$\mathbb{T}_* = \left\{ I = \Psi(\theta) = I_* + \mathcal{O}_{\mathcal{C}^1}(\rho^2), \theta \in \mathbb{T} \right\}, \qquad j = 1,2, \tag{5.164}$$

where I_* satisfies $I_* \in [1,2]$. Note that the $\mathcal{C}^1$ in the error refers to derivatives with respect to θ.

In the next lemma, we apply several steps of Birkhoff Normal Form around the torus $\mathbb{T}_*$.

Lemma 5.9.4. *There exists a symplectic change of coordinates Φ_3 satisfying*

$$(I,\theta) = \Phi_3(J,\psi) = \left(\psi + \mathcal{O}_{\mathcal{C}^1}\left(\rho^2\right), J + I_* + \mathcal{O}_{\mathcal{C}^1}\left(\rho^2\right)\right),$$

such that $\{J = 0\} = \Phi_3^{-1}(\mathbb{T}_*)$ and the map $\widehat{\mathcal{S}}^1$ becomes

$$\mathtt{S}^1(\psi, J) = \begin{pmatrix} \psi + \widetilde{B}(J) + \mathcal{O}\left(J^2\right) \\ J + \mathcal{O}\left(J^3\right) \end{pmatrix} \tag{5.165}$$

where

$$\widetilde{B}(J) = b_0 + b_1 J \quad \text{with} \quad b_0 = C_0\tilde{\Theta}^{-3} + \mathcal{O}\left(\Theta^{-4}\right), \tag{5.166}$$

$C_0 \neq 0$ is the constant provided in Lemma 5.9.2 and $b_1 \in \mathbb{R}$ satisfies $b_1 \neq 0$ for Θ large enough.

Now we express the scattering map $\mathcal{S}^2$ also in (ψ, J) coordinates to compare them. To this end, we take

$$\rho = \tilde{\Theta}^{7/2} \mathrm{e}^{-\frac{\tilde{\nu}\tilde{\Theta}^3}{3L_0^3}} \tag{5.167}$$

The exponent 7/2 is not crucial and one could take any other exponent in the interval $(3/2, 11/2)$.

Lemma 5.9.5. *Take ρ of the form* (5.167) *and Θ large enough. Then, the scattering map $\mathcal{S}^2$ expressed in the coordinates (ψ, J) obtained in Lemma 5.9.4 is of the form*

$$\mathtt{S}^2(\psi, J) = \begin{pmatrix} \mathtt{S}^2_\psi(\psi, J) \\ \mathtt{S}^2_J(\psi, J) \end{pmatrix} = \begin{pmatrix} \psi + f_1(\psi) + \mathcal{O}\left(J\right) \\ J + g_1(\psi) + \mathcal{O}\left(J\right) \end{pmatrix}$$

where

$$f_1(\psi) = \mathcal{O}_{\mathcal{C}^1}\left(\Theta^{-2}\right) \qquad \text{and} \qquad g_1(\psi) = C_2\tilde{\Theta}^{-2}\cos\psi + \mathcal{O}_{\mathcal{C}^1}\left(\Theta^{-3}\ln^2\Theta\right),$$

with some constant $C_2 \neq 0$.

Proof. We need to apply to $\mathcal{S}^2$ the changes of coordinates given in Lemmas 5.9.1 and 5.9.2. We first apply the symplectic transformation Φ_1 in Lemma 5.9.1. Then, we obtain that $\widetilde{\mathcal{S}}^2 = \Phi_1^{-1} \circ \mathcal{S}^2 \circ \Phi_1$ is of the form

$$\widetilde{\mathcal{S}}^2(z) = \Phi_1^{-1} \circ \mathcal{S}^2 \circ \Phi_1 = \widetilde{z}_0^2 + \lambda_2(z - \widetilde{z}_0^2) + \widetilde{P}_2\left(z - \widetilde{z}_0^2\right)$$

where $\widetilde{z}_0^2 = \Phi_1^{-1}(z_0^2)$, and therefore satisfies $\widetilde{z}_0^2 = z_0^2 + \mathcal{O}(z_0^2)^2$, and $\widetilde{P}_2$ is a function which satisfies

$$\left|\widetilde{P}_2\left(z - \widetilde{z}_0^2\right)\right| \leq C_3\left|z - \widetilde{z}_0^2\right|^2$$

for some constant C_3 independent of $\tilde{\Theta}$.

Now we apply the (scaled) Action-Angle transformation considered in Lemma 5.9.2. To this end, taking into account (5.48), we define

$$\Delta = \eta_0^2 - \widetilde{\eta}_0^1 = \pi_\eta \Phi_1^{-1}(z_0^1) - \pi_\eta \Phi_1^{-1}(z_0^2) = -\frac{4}{\sqrt{\pi}} L_0^{1/2} \tilde{\Theta}^{9/2} \mathrm{e}^{-\frac{\tilde{\nu}\tilde{\Theta}^3}{3L_0^3}} \left(1 + \mathcal{O}\left(\Theta^{-1}\ln^2\Theta\right)\right), \tag{5.168}$$

where we have used that Φ_1 is close to the identity (see Lemma 5.9.1) and π_η denotes the projection in the η-component.

Denoting $(\theta_1, I_1) = \widehat{\mathcal{S}}^2(\theta, I)$, we obtain (in complex notation)

$$\begin{aligned}\rho\sqrt{I_1}e^{i\theta_1} &= -\Delta + \lambda_2\left(\Delta + \rho\sqrt{I}e^{i\theta}\right) + \mathcal{O}\left(|\Delta + \rho\sqrt{I}e^{i\theta}|^2\right) \\ &= (\lambda_2 - 1)\Delta + \lambda_2\rho\sqrt{I}e^{i\theta} + \mathcal{O}(\rho+\Delta)^2.\end{aligned}$$

Therefore,

$$\sqrt{I_1}e^{i\theta_1} = (\lambda_2 - 1)\frac{\Delta}{\rho} + \lambda_2\sqrt{I}e^{i\theta} + \frac{1}{\rho}\mathcal{O}(\rho+\Delta)^2.$$

Now, condition (5.167) and the fact that $\lambda_2 = e^{i\omega_2}$, with $\omega_2 = \omega_0\tilde{\Theta}^{-3} + \mathcal{O}(\Theta^{-4})$, implies that

$$(\lambda_2 - 1)\frac{\Delta}{\rho} \lesssim \Theta^{-2} \ll 1.$$

Therefore, using also (5.168), (5.49) and the fact $|\lambda_2| = 1$ (see Proposition 5.4.5), we obtain

$$I_1 = I\left|1 + \frac{\lambda_2 - 1}{\lambda_2\sqrt{I}}\frac{\Delta}{\rho}e^{i\theta} + \frac{1}{\rho}\mathcal{O}_{\mathcal{C}^1}(\rho+\Delta)^2\right|^2 = I\left(1 + \frac{C}{\sqrt{I}}\tilde{\Theta}^{-2}\cos\theta + \mathcal{O}_{\mathcal{C}^1}\left(\Theta^{-3}\ln^2\Theta\right)\right)$$

for some constant $C \neq 0$ independent of Θ. The notation $\mathcal{O}_{\mathcal{C}^1}$ refers to derivatives with respect to (θ, I).

The formulas for θ_1 can be obtained analogously to obtain the following expansion

$$\widehat{\mathcal{S}}^2(\theta, I) = \begin{pmatrix} \theta + \mathcal{O}_{\mathcal{C}^1}\left(\Theta^{-2}\right) \\ I + C_2\tilde{\Theta}^{-2}\sqrt{I}\cos\theta + \mathcal{O}_{\mathcal{C}^1}\left(\Theta^{-3}\ln^2\Theta\right)\end{pmatrix}$$

for some constant $C_2 \neq 0$.

Now it only remains to apply the change of coordinates Φ_3 obtained in Lemma 5.9.4, renaming the constant C_2 by $C_2\sqrt{I_*}$. □

We use the expressions of $\mathtt{S}^1$ and $\mathtt{S}^2$ given in Lemmas 5.9.4 and 5.9.5 to build the isolating block. Consider the torus $\mathbb{T}_* = \{J = 0\}$ which is invariant by $\mathtt{S}^1$. Then, we define

$$\mathbb{T}_- = \left(\mathtt{S}^2\right)^{-1}(\mathbb{T}_*)$$

and we denote by $Z_* = (\psi_*, 0)$ the intersection between $\mathbb{T}_*$ and $\mathbb{T}_-$ which satisfies

$$\psi_* = \frac{\pi}{2} + \mathcal{O}\left(\Theta^{-1}\ln^2\Theta\right). \tag{5.169}$$

This point will be one of the vertices of the block and "segments" within $\mathbb{T}_*$ and $\mathbb{T}_-$ will be two of the edges of the block.

Lemma 5.9.5 implies that

$$\partial_\psi \mathtt{S}^2_J(\psi_*, 0) = C_2\tilde{\Theta}^{-2} + \mathcal{O}\left(\Theta^{-3}\ln^2\Theta\right) \neq 0.$$

Therefore, in a neighborhood of $(\psi_*, 0)$, the torus $\mathbb{T}_-$ can be parameterized as

$$\psi = h(J) \qquad \text{for} \qquad |J| \ll 1 \qquad \text{and} \qquad h(0) = \psi_*. \tag{5.170}$$

In other words, there exists a function h satisfying $\mathtt{S}^2_J(h(J), J) = 0$.

To analyze such block we perform a last change of coordinates so that the segment of $\mathbb{T}_-$ becomes vertical.

Lemma 5.9.6. *The symplectic change of coordinates*

$$\Phi_4 : (\psi, J) = (\varphi + h(J), J) \qquad \textit{for} \quad \varphi \in \mathbb{T},\ |J| \ll 1,$$

transforms the scattering maps given in Lemmas 5.9.4 and 5.9.5 into

$$\widehat{\mathtt{S}}^1(\varphi, J) = \begin{pmatrix} \varphi + \widetilde{B}(J) + \mathcal{O}\left(J^2\right) \\ J + \mathcal{O}\left(J^3\right) \end{pmatrix} \qquad \textit{and} \qquad \widehat{\mathtt{S}}^2(\varphi, J) = \begin{pmatrix} \widehat{\mathtt{S}}^2_\varphi(\varphi, J) \\ \widehat{\mathtt{S}}^2_J(\varphi, J) \end{pmatrix} \tag{5.171}$$

which satisfies $\widehat{\mathtt{S}}^2_J(0, J) = 0$ *for* $|J| \ll 1$.

Note that, since $\widehat{\mathtt{S}}^2$ its a diffeomorphism, it satisfies

$$\mathtt{b} = \partial_\varphi \widehat{\mathtt{S}}^2_J(0,0) \neq 0. \tag{5.172}$$

We use Lemma 5.9.6 and this fact to build the isolating block.

Note that now the point $Z_* = (\psi_*, 0)$ has become $\widehat{Z}_* = (0,0)$ and the chosen two sides of the block are $J = 0$ and $\varphi = 0$. We consider the block $\mathcal{R}$ defined as

$$\mathcal{R} = \left\{(\varphi, J) :\ 0 \le \varphi \le 2\mathtt{b}^{-1}\tilde{\kappa},\ 0 \le J \le \tilde{\kappa}\right\} \quad \text{for some} \quad \tilde{\kappa} \ll 1.$$

The choice of $\varphi = 2\mathtt{b}^{-1}\tilde{\kappa}$ is for the following reason. It implies that, for $\tilde{\kappa}$ small enough,

$$\widehat{\mathtt{S}}^2_J(2\mathtt{b}^{-1}\tilde{\kappa}, J) \ge \tilde{\kappa} \qquad \text{for} \qquad J \in (0, \tilde{\kappa}).$$

Then,

$$\mathcal{R}' = \widehat{\mathtt{S}}^2(\mathcal{R}) \cap \{0 \le J \le \tilde{\kappa}\}$$

is a "rectangle" bounded by the segments $J = 0$, $J = \tilde{\kappa}$ and two other segments of the form $\varphi = h_i(J)$, $i = 1, 2$ which satisfy

$$|h_2(J) - h_1(J)| \lesssim \tilde{\kappa} \qquad \text{for} \qquad 0 \le J \le \tilde{\kappa}.$$

Now, we show that for a suitable $M \gg 1$, $\mathcal{R}$ is an isolating block for $(\widehat{\mathtt{S}}^1)^M \circ \widehat{\mathtt{S}}^2$. To this end, we must analyze $(\widehat{\mathtt{S}}^1)^M(\mathcal{R}')$. Note that M will depend on $\tilde{\kappa}$.

Consider the vertices of the rectangle $\mathcal{R}'$, Z_{ij}, $i, j = 1, 2$, with

$$Z_{i1} = (\varphi_{i1}, 0), \quad Z_{i2} = (\varphi_{i2}, \tilde{\kappa}) \qquad \text{with} \quad \varphi_{1j} < \varphi_{2j}, \qquad j = 1, 2.$$

Note that they satisfy

$$|\varphi_{ij} - \varphi_{i'j'}| \lesssim \tilde{\kappa}, \qquad i, j, i', j' = 1, 2.$$

We define

$$Z^M_{ij} = (\varphi^M_{ij}, J^M_{ij}) = \left(\widehat{\mathtt{S}}^1\right)^M (Z_{ij}),$$

which by Lemma 5.9.6 satisfy

$$J^M_{i1} = 0 \qquad \text{and} \qquad J^M_{i2} = \tilde{\kappa} + M\mathcal{O}(\tilde{\kappa}^3).$$

Choosing a suitable M satisfying

$$\frac{1}{4b_1\tilde{\kappa}} \le M \le \frac{1}{2b_1\tilde{\kappa}}, \tag{5.173}$$

where b_1 is the constant introduced in Lemma 5.9.4, we show that

$$\varphi^M_{12} - \varphi^M_{21} \gtrsim \frac{1}{8} \qquad \text{and} \qquad -\frac{1}{16} \le \varphi^M_{21} \le 0. \tag{5.174}$$

Indeed, for the first one, note that

$$\varphi^M_{12} - \varphi^M_{21} = \varphi_{12} - \varphi_{21} + Mb_1\tilde{\kappa} + M\mathcal{O}(\tilde{\kappa}^2) = \frac{1}{4} + \mathcal{O}(\tilde{\kappa}) \gtrsim \frac{1}{8}.$$

For the second estimate in (5.174) it is enough to choose a suitable M by using the particular form of the first component of $\widehat{\mathtt{S}}_1$ in Lemma 5.9.6 and the definition of b_0 in (5.166).

The estimates in (5.174) implies that $\mathcal{R}$ is an isolating block.

Now, we compute $D\widehat{\mathtt{S}} = D[(\widehat{\mathtt{S}}^1)^M \circ \widehat{\mathtt{S}}^2]$.

Lemma 5.9.7. *For $z = (\varphi, J) \in \mathcal{R}$, the matrix $D\widehat{\mathtt{S}}(z)$ is hyperbolic with eigenvalues $\lambda(z), \lambda(z)^{-1} \in \mathbb{R}$ with*

$$\lambda_{\widehat{\mathtt{S}}}(z) \gtrsim \tilde{\kappa}$$

Moreover, there exist two vectors fields $V_j : \mathcal{R} \to T\mathcal{R}$ of the form

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} V_{21}(z) \\ 1 \end{pmatrix} \quad \text{with} \quad |V_{21}(z)| \lesssim \tilde{\kappa},$$

which satisfy, for $z \in \mathcal{R}$,

$$\begin{aligned} D\widehat{\mathtt{S}}(z)V_1 &= \lambda_{\widehat{\mathtt{S}}}(z)\left(V_1 + \widehat{V}_1(z)\right) && \text{with} \quad |\widehat{V}_1(z)| \lesssim \tilde{\kappa} \\ D\widehat{\mathtt{S}}(z)V_2(z) &= \lambda_{\widehat{\mathtt{S}}}(z)^{-1}\left(V_2(\widehat{\mathtt{S}}(z)) + \widehat{V}_2(z)\right) && \text{with} \quad |\widehat{V}_2(z)| \lesssim \tilde{\kappa}. \end{aligned}$$

Proof. Note that

$$D\widehat{\mathtt{S}}^1 = \begin{pmatrix} 1 & (b_1 + \mathcal{O}(\tilde{\kappa})) \\ 0 & 1 \end{pmatrix} + \mathcal{O}\left(\tilde{\kappa}^2\right)$$

and therefore, since $M \sim \tilde{\kappa}^{-1}$,

$$D(\widehat{\mathtt{S}}^1)^M = \begin{pmatrix} 1 & Mb_1 + \mathcal{O}(1) \\ 0 & 1 \end{pmatrix} + M\mathcal{O}\left(\tilde{\kappa}^2\right) = \begin{pmatrix} 1 & Mb_1 + \mathcal{O}(1) \\ 0 & 1 \end{pmatrix} + \mathcal{O}\left(\tilde{\kappa}\right).$$

On the other hand, by Lemma 5.9.6, (5.172) and taking into account that $\widehat{\mathtt{S}}^2$ is symplectic,

$$D\widehat{\mathtt{S}}^2 = \begin{pmatrix} \eta & -\mathtt{b}^{-1} \\ \mathtt{b} & 0 \end{pmatrix} + \mathcal{O}(\tilde{\kappa})$$

for some $\eta \in \mathbb{R}$. Then,

$$D\widehat{\mathtt{S}} = \begin{pmatrix} M\mathtt{b}b_1 & -\mathtt{b}^{-1} \\ \mathtt{b} & 0 \end{pmatrix} + \mathcal{O}\left(1\right) \tag{5.175}$$

Since this matrix is symplectic (the scattering maps are, see [DdlLS08]), to prove hyperbolicity it is enough to check that the trace is bigger than 2. Indeed, for $(\varphi, J) \in \mathcal{R}$ and $\tilde{\kappa} > 0$ small enough,

$$\mathrm{tr} D\widehat{\mathtt{S}} = M\mathtt{b}b_1 + \mathcal{O}(1) \gtrsim \tilde{\kappa}^{-1}.$$

The statements for V_1 are straightforward considering the form of $D\widehat{\mathtt{S}}$ in (5.175). To obtain those for V_2 it is enough to invert the matrix $D\widehat{\mathtt{S}}$ and compute the eigenvector of large eigenvalue. □

5.10 A parabolic normal form: Proof of Theorem 5.5.2

The Theorem 5.5.2 will be an immediate consequence of the Lemmas 5.10.1, 5.10.2 and 5.10.3 below.

5.10.1 First step of normal form

The first step of the normal form transforms the "center" variables z so that its dynamics becomes much closer to the identity in a neighborhood of infinity.

Lemma 5.10.1. *For any $N \geq 0$, there exist an analytic change of variables of the form*

$$\tilde{z} = z + Z(x, y, z, t),$$

where Z is a polynomial in (x, y) of order at least 3, even in x, such that equation (5.54) becomes in the new variables

$$\begin{aligned} \dot{x} &= -x^3 y(1 + Bx^2 - By^2 + \mathcal{O}_4(x, y)), \\ \dot{y} &= -x^4(1 + (B - 4A)x^2 - By^2 + \mathcal{O}_4(x, y)), \\ \dot{\tilde{z}} &= x^6 \mathcal{O}_N(x, y). \end{aligned} \tag{5.176}$$

where the $\mathcal{O}_N(x,y)$ terms in the equations of $\dot{\hat{z}}$ as well as the $\mathcal{O}_4(x,y)$ are even functions of x. The constants A and B were introduced in (5.53).

Proof. Equation (5.54) is in the claimed form for $N=0$. We proceed by induction. Assume the claim is true for N, that is, that the equation is

$$\begin{aligned} \dot{x} &= -x^3y(1+Bx^2-By^2+\mathcal{O}_4(x,y)),\\ \dot{y} &= -x^4(1+(B-4A)x^2-By^2+\mathcal{O}_4(x,y)),\\ \dot{z} &= x^6p_N(x,y,a,b,t)+x^6\mathcal{O}_{N+1}(x,y), \end{aligned} \tag{5.177}$$

where p_N is a homogeneous polynomial in (x,y), even in x, of degree N with coefficients depending on (z,t).

First, with an averaging step, we can assume that p_N does not depend on t. Indeed, given U_N such that $\partial_t U_N = p_N - \tilde{p}_N$, where $\tilde{p}_N = \langle p_N\rangle_t$, the change

$$\tilde{z} = z + x^6 U_N(x,y,z,t),$$

transforms equation (5.177) into

$$\begin{aligned} \dot{x} &= -x^3y(1+Bx^2-By^2+\mathcal{O}_4(x,y)),\\ \dot{y} &= -x^4(1+(B-4A)x^2-By^2+\mathcal{O}_4(x,y)),\\ \dot{\tilde{z}} &= x^6\tilde{p}_N(x,y,\tilde{z})+x^6\mathcal{O}_{N+1}(x,y). \end{aligned} \tag{5.178}$$

Clearly, since p_N is even in x, so is U_n and then the parity on x of the equation remains the same.

Second, we consider the change

$$\hat{z} = \tilde{z} + Z_{N+3}(x,y,\tilde{z}),$$

where Z_{N+3} is a homogeneous polynomial in (x,y) of degree $N+3$, even in x, with coefficients depending on $\tilde{z}$. It transforms equation (5.178) into

$$\begin{aligned} \dot{x} &= -x^3y(1+Bx^2-By^2+\mathcal{O}_4(x,y)),\\ \dot{y} &= -x^4(1+(B-4A)x^2-By^2+\mathcal{O}_4(x,y)),\\ \dot{\hat{z}} &= x^3\left(x^3\tilde{p}_N(x,y,\hat{z}) - y\partial_x Z_{N+3}(x,y,\hat{z}) - x\partial_y Z_{N+3}(x,y,\hat{z})\right) + x^6\mathcal{O}_{N+1}(x,y). \end{aligned}$$

Clearly, since Z_{N+3} is even in x, the parity in x of the equation is preserved.

Since $x^3\tilde{p}_N(x,y,\hat{z})$ is an odd polynomial in x, it is in the range of the operator $L : C_{N+3} \mapsto y\partial_x C_{N+3} + x\partial_y C_{N+3}$, acting on homogeneous polynomials of degree $N+3$, even in x, the claim follows. Indeed, for any $j,\ell \geq 0$ such that $2j+\ell+1 = N+3$,

$$L\left(\sum_{i=0}^{j} a_i x^{2(j-i)} y^{\ell+2i+1}\right) = x^{2j+1}y^{\ell},$$

where

$$a_0 = \frac{1}{\ell+1} \qquad \text{and} \qquad a_i = (-1)^i \frac{(2j)\cdots(2j-2i+2)}{(\ell+1)\cdots(\ell+1+2i)}, \qquad i \geq 1.$$

□

5.10.2 Second step of normal form: straightening the invariant manifolds of infinity

Here we use the invariant manifolds of the periodic orbits P_{z_0} to find coordinates in which these manifolds are the coordinate planes.

Let $K \subset \mathbb{R}^2$ be a fixed compact set. Given $\rho > 0$, we denote by $K^{\rho}_{\mathbb{C}}$, a neighborhood of K in $\mathbb{C}^2$ such that $\operatorname{Re} z \in K$ and $|\operatorname{Im} z| < \rho$, for all $z \in K^{\rho}_{\mathbb{C}}$. Given $\delta,\sigma,\rho > 0$, we consider the domain

$$\begin{aligned} U_{\delta,\rho} = \{(q,p,z,t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^2\times\mathbb{C} \mid |\operatorname{Im} q| < \delta \operatorname{Re} q,\ |\operatorname{Im} p| < \delta \operatorname{Re} p,\\ \|(q,p)\| < \rho,\ z \in K^{\rho}_{\mathbb{C}},\ |\operatorname{Im} t| < \sigma\}. \end{aligned} \tag{5.179}$$

Lemma 5.10.2. *There exists a $\mathcal{C}^\infty$ change of variables of the form $(q,p,z) \mapsto (q,p,z)+\mathcal{O}_2(q,p)$, analytic in a domain of the form* (5.179)*, that transforms equation* (5.176) *into*

$$
\begin{aligned}
\dot q &= q((q+p)^3+\mathcal{O}_4(q,p)), & \dot z &= qp\mathcal{O}_{N+4}(q,p),\\
\dot p &= -p((q+p)^3+\mathcal{O}_4(q,p)), & \dot t &= 1.
\end{aligned}
\tag{5.180}
$$

Equation (5.180) *is analytic in $U_{\delta,\rho}$, for some $\delta,\rho>0$, and $\mathcal{C}^\infty$ at $(q,p)=(0,0)$.*

Proof. We start by straightening the tangent directions of the topological saddle in (5.176). We introduce

$$(q,p,z) = ((x-y)/2,(x+y)/2,z).$$

In these variables, equation (5.176) becomes

$$
\begin{aligned}
\dot q &= (q+p)^3\left(q-2Aq^3-6\left(A-\frac{2B}{3}\right)q^2p-6Aqp^2-2Ap^3+\mathcal{O}((q+p)^5)\right),\\
\dot p &= -(q+p)^3\left(p+2Aq^3+6\left(A-\frac{2B}{3}\right)q^2p+6Aqp^2+2Ap^3+\mathcal{O}((q+p)^5)\right)\\
\dot z &= (q+p)^6\mathcal{O}_N(q,p).
\end{aligned}
\tag{5.181}
$$

This equation is analytic and 2π-periodic in t in a neighborhood U of $\{q=p=0,\ z\in\mathbb{R}^2,\ t\in\mathbb{R}\}$ in $\mathbb{C}\times\mathbb{C}\times\mathbb{C}^2\times\mathbb{C}$.

Let $\sigma>0$, $\rho>0$ be such that $\{\|(q,p)\|<\rho\}\times K_{\mathbb{C}}^{\rho}\times\{|\mathrm{Im}\, t|<\sigma\}\subset U$. Let $z_0\in K$. We claim that that the periodic orbit P_{z_0} of (5.181) has invariant stable and unstable manifolds which admit parametrizations $\gamma^{s,u}(\cdot,t;z_0)$ of the form

$$
\begin{aligned}
(p,z) &= \gamma^u(q,t;z_0) = \left(\frac{A}{2}q^3+\mathcal{O}(q^4), z_0+\mathcal{O}(q^{N+3})\right),\\
(q,z) &= \gamma^s(p,t;z_0) = \left(\frac{A}{2}p^3+\mathcal{O}(p^4), z_0+\mathcal{O}(p^{N+3})\right),
\end{aligned}
\tag{5.182}
$$

where $\gamma^{s,u}$ are analytic in

$$
\begin{aligned}
V_{\delta,\rho}^u &= \{(q,t,z_0)\in\mathbb{C}^3 \mid |\mathrm{Im}\, q|<\delta\mathrm{Re}\, q,\ |q|<\rho,\ |\mathrm{Im}\, t|<\sigma,\ z_0\in K_{\mathbb{C}}^{\rho}\},\\
V_{\delta,\rho}^s &= \{(p,t,z_0)\in\mathbb{C}^3 \mid |\mathrm{Im}\, p|<\delta\mathrm{Re}\, p,\ |p|<\rho,\ |\mathrm{Im}\, t|<\sigma,\ z_0\in K_{\mathbb{C}}^{\rho}\},
\end{aligned}
$$

for some $\delta,\rho,\sigma>0$ and are of class $\mathcal{C}^\infty$ at $q=0$ and $p=0$, respectively.

We prove the claim for γ^u, being the one for γ^s analogous. First we remark that, if γ^u exists and is $\mathcal{C}^\infty$ at $q=0$, substituting in the vector field an imposing the invariance condition, one obtains that it must be of the form given by (5.182).

We change the sign of time in (5.181). With the introduction of the new variables

$$\tilde z = \frac{1}{q+p}(z-z_0),$$

equation (5.181) becomes

$$
\begin{aligned}
\dot q &= -q(q+p)^3+\mathcal{O}((q+p)^6),\\
\dot p &= p(q+p)^3+\mathcal{O}((q+p)^6),\\
\dot{\tilde z} &= \tilde z(q-p)(q+p)^2+\tilde z\mathcal{O}_5(q,p)+\mathcal{O}_{N+5}(q,p)
\end{aligned}
\tag{5.183}
$$

where the vector field is analytic in U.

The results in [BFdlLM07] and [GSMS17] imply that the parabolic periodic orbit $(q,p,\tilde z)=(0,0,0)$ of (5.183), which is parabolic, has an invariant stable manifold, parametrized by

$$(q,p,\tilde z) = \tilde\gamma(u,t,z_0) = (u+\mathcal{O}(u^2),\mathcal{O}(u^2),\mathcal{O}(u^2)) \tag{5.184}$$

with $\tilde{\gamma}$ analytic in $\widetilde{V}_{\delta,\rho} = \{|\mathrm{Im}\, u| < \delta|\mathrm{Re}\, u|,\ |u| < \rho,\ |\mathrm{Im}\, t| < \sigma,\ z_0 \in K_{\mathbb{C}}^{\rho}\}$, for some $\delta, \rho, \sigma > 0$ and of class $\mathcal{C}^\infty$ at $u = 0$. Since (5.183) has the time reversed, $\tilde{\gamma}$ corresponds to the unstable manifold of (5.182). We can invert the first component of (5.184), $q = \pi_q \tilde{\gamma}(u, t, z_0)$, to obtain $u = q + \mathcal{U}(q, t, z_0)$, defined and analytic in

$$V_{\delta',\rho'}^u = \{(q, t, z_0) \in \mathbb{C}^3 \mid |\mathrm{Im}\, q| < \delta' \mathrm{Re}\, q,\ |q| < \rho',\ |\mathrm{Im}\, t| < \sigma,\ z_0 \in K_{\mathbb{C}}^{\rho'}\},$$

for any $0 < \delta' < \delta$ and some $0 < \rho' < \rho$, and is $\mathcal{C}^\infty$ in $V_{\delta',\rho'}^u \cup \{0\}$[8]. Hence, the stable manifold of $(q, p, \tilde{z}) = (0, 0, 0)$ of (5.183), and therefore the unstable one of (5.182), can be written as a graph as

$$(p, \tilde{z}) = \hat{\gamma}(q, t, z_0) = \pi_{p,\tilde{z}} \tilde{\gamma}(q + \mathcal{U}(q, t, z_0), t, z_0) = \mathcal{O}(q^2).$$

We claim that $\pi_{\tilde{z}} \hat{\gamma}(q, t, z_0) = \mathcal{O}(q^{N+2})$.

Indeed, assume that $\pi_{\tilde{z}} \hat{\gamma}(q, t, z_0) = a_L q^L + \mathcal{O}(q^{L+1})$. It is clear that $L \geq 2$. But, denoting by X the vector field in (5.183), since the graph of $\hat{\gamma}$ is invariant, it satisfies

$$\begin{aligned} -La_L q^{L+3} + \mathcal{O}(q^{L+4}) &= \frac{\partial}{\partial q} \pi_{\tilde{z}} \hat{\gamma}(q, t, z_0) X_q(q, \tilde{\gamma}(q), t) \\ &= X_{\tilde{z}}(q, \tilde{\gamma}(q), t) = a_L q^{L+3} + a_L \mathcal{O}(q^{L+5}) + \mathcal{O}(q^{N+5}), \end{aligned}$$

from which the claim follows.

Going back to $z = z_0 + (q + p)\tilde{z}$ we obtain that $(p, z) = \gamma^u(q, t, z_0) = (\mathcal{O}(q^2), z_0 + \mathcal{O}(q^{N+3}))$ is a parametrization of the unstable manifold of P_{z_0}. Substituting this expression into (5.181), one obtains that $\mathcal{O}(q^2) = \frac{A}{2} q^3 + \mathcal{O}(q^4)$, which proves the claim for γ^u in (5.182).

Now we straighten the invariant manifolds using the functions γ^u and γ^s in (5.182). We claim that there exist variables (q, p, z) in which equation (5.181) becomes

$$\begin{aligned} \dot{q} &= q((q + p)^3 + \mathcal{O}_4), \\ \dot{p} &= -p((q + p)^3 + \mathcal{O}_4), \\ \dot{z} &= qp\mathcal{O}_{N+4}(q, p), \end{aligned} \tag{5.185}$$

being defined and analytic in a domain of the form (5.179) and is $\mathcal{C}^\infty$ at $(q, p) = (0, 0)$. We will apply two consecutives changes of variables, each of them straightening one invariant manifold.

Let $z_0^u(q, z, t) = z + \mathcal{O}_{N+3}(q)$ be such that

$$z = \pi_z \gamma^u(q, t; z_0^u(q, z, t)),$$

which is also analytic on $V_{\delta,\rho}^u$, $\mathcal{C}^\infty$ at $q = 0$. We define the new variables $(\tilde{q}, \tilde{p}, \tilde{z}) = \Psi_1^{-1}(q, p, z, t)$ by

$$\begin{aligned} \tilde{q} &= q, \\ \tilde{p} &= p - \pi_p \gamma^u(q, t; z_0^u(q, z, t)) = p - \frac{A}{2} q^3 + \mathcal{O}_4(q), \\ \tilde{z} &= z_0^u(q, z, t) = z + \mathcal{O}_{N+3}(q). \end{aligned} \tag{5.186}$$

Again, it is easy to see that the map

$$\Psi_1(\tilde{q}, \tilde{p}, \tilde{z}, t) = \begin{pmatrix} \tilde{q} \\ \tilde{p} + \psi_1^{\tilde{p}}(\tilde{q}, \tilde{z}, t) \\ \tilde{z} + \psi_1^{\tilde{z}}(\tilde{q}, \tilde{z}, t) \end{pmatrix} = \begin{pmatrix} \tilde{q} \\ \tilde{p} + \frac{A}{2}\tilde{q}^3 + \mathcal{O}_4(\tilde{q}) \\ \tilde{z} + \mathcal{O}_{N+3}(\tilde{q}) \end{pmatrix}$$

is analytic in the domain

$$W_{\delta,\rho}^u = \{(\tilde{q}, \tilde{p}, \tilde{z}, t) \in \mathbb{C}^4 \mid |\tilde{q}|, |\tilde{p}| \leq \rho,\ \mathrm{Re}\, \tilde{p} \geq 0,\ |\mathrm{Im}\, \tilde{q}| < \delta \mathrm{Re}\, \tilde{q},\ \tilde{z} \in K_{\mathbb{C}}^{\rho},\ t \in \mathbb{T}_\sigma\}. \tag{5.187}$$

[8] This claim can be proven as follows. $\mathcal{U}$ is trivially $\mathcal{C}^\infty$ at $u = 0$. Observe that the function $\mathcal{U}$ is the solution of the fixed point equation $\mathcal{U} = F(\mathcal{U})$, with $F(\mathcal{U})(q, t, z_0) = -\varphi(q + \mathcal{U}(q, t, z_0), t, z_0)$ and $\varphi = \pi_q \tilde{\gamma}$. Since $\varphi(u, t, z_0) = \mathcal{O}(u^2)$ and is analytic in $\widetilde{V}_{\delta_0,\rho_0}$, we have that, for any $0 < \delta' < \delta$ and $0 < \rho' < \rho$, $\partial_u \varphi(u, t, z_0) = \mathcal{O}(u)$ in $\widetilde{V}_{\delta',\rho'}$. Using this fact, it is immediate to see that F is a contraction with the norm $\|\varphi\|_2 = \sup_{(q,t,z_0) \in V_{\delta',\rho''}^u} |q^{-2}\mathcal{U}(q, t, z)|$, if ρ'' is small enough.

for some $\gamma, \rho, \sigma > 0$. Moreover, since $\Psi_{1|\tilde{q}=0}$ is the identity (and trivially it is analytic in $\{|\tilde{p}| < \rho,\ \tilde{z} \in K^{\rho}_{\mathbb{C}},\ t \in \mathbb{T}_\sigma\}$). Observe that we are considering $\tilde{q}$ with $\operatorname{Re}\tilde{q} \geq 0$.

Observe that, in these variables, the unstable manifold is given by $\tilde{p} = 0$. We claim that, in these variables, equation (5.181) becomes

$$\begin{aligned} \dot{\tilde{q}} &= \tilde{q}(\tilde{q}+\tilde{p})^3 + \mathcal{O}_4, \\ \dot{\tilde{p}} &= -\tilde{p}((\tilde{q}+\tilde{p})^3 + \mathcal{O}_4), \\ \dot{\tilde{z}} &= \tilde{p}\mathcal{O}_{N+5}(\tilde{q},\tilde{p}), \end{aligned} \tag{5.188}$$

Indeed, the claim for $\dot{\tilde{q}}$ is an immediate substitution. To see the claim for $\tilde{z}$, we observe that, on the unstable manifold, $\tilde{z}$ is constant and equal to z_0. Hence

$$\dot{\tilde{z}} = \frac{d}{dt} z_0^u(q,z,t) = \partial_q z_0^u(q,z,t)\dot{q} + \partial_z z_0^u(q,z,t)\dot{z} + \partial_t z_0^u(q,z,t) = \mathcal{O}_{N+6}(q,p) + \partial_t z_0^u(q,z,t).$$

Since $\dot{\tilde{z}}_{|\tilde{p}=0} = 0$, $\partial_t z_0^u(q,z,t) = \mathcal{O}_{N+6}(q)$ and the claim follows. Then the claim for $\dot{\tilde{p}}$ is an immediate substitution.

Observe that the composition $\gamma^s \circ \Psi_1$, where $\gamma^s(p,t;z_0)$ is the function in (5.182), is well defined and analytic in $W^u_{\delta',\rho'}$, for $0 < \delta' < \delta/3$ and ρ' small enough. Indeed, if $(\tilde{q},\tilde{p},\tilde{z},t) \in W^u_{\delta',\rho'}$, using the the function A in (5.53) is real analytic and positive for real values of $z \in K^{\rho}_{\mathbb{C}}$,

$$\pi_{\tilde{p}}\Psi_1(\tilde{q},0,\tilde{z},t) = \frac{A}{2}\tilde{q}^3 + \mathcal{O}(\tilde{q}^4) \in V^s_{\delta,\rho},$$

and $\operatorname{Re}\pi_{\tilde{p}}\Psi_1(\tilde{q},0,\tilde{z},t) > 0$, which implies that, for any $\tilde{p} \in V^s_{\delta,\rho}$ with $\operatorname{Re}\tilde{p} \geq 0$, $\tilde{p} + \frac{A}{2}\tilde{q}^3 + \mathcal{O}(\tilde{q}^4) \in V^s_{\delta,\rho}$.

It can be seen with the same type of fixed point argument that the stable manifold $(q,z) = \gamma^s(p,t,z_0)$ in (5.182) can be written in the variables $(\tilde{q},\tilde{p},\tilde{z})$ as

$$(\tilde{q},\tilde{z}) = \tilde{\gamma}^s(\tilde{p},t;z_0) = \left(\frac{A}{2}\tilde{p}^3 + \mathcal{O}(\tilde{p}^4), z_0 + \mathcal{O}(\tilde{p}^{N+3})\right), \tag{5.189}$$

analytic in

$$V^s_{\delta'',\rho''} = \{(\tilde{p},t,z_0) \in \mathbb{C}^3 \mid |\operatorname{Im}\tilde{p}| < \delta''\operatorname{Re}\tilde{p},\ |\tilde{p}| < \rho'',\ |\operatorname{Im} t| < \sigma,\ z_0 \in K^{\rho''}_{\mathbb{C}}\},$$

for some $0 < \delta'' < \delta'$, $0 < \rho'' < \rho'$.

Now, repeating the same arguments, we obtain a change of variables $(\hat{q},\hat{p},\hat{z}) = \Psi_2^{-1}(\tilde{q},\tilde{p},\tilde{z},t)$ such that

$$\Psi_2(\hat{q},\hat{p},\hat{z},t) = \begin{pmatrix} \hat{q} + \psi_2^{\hat{q}}(\hat{p},\hat{z},t) \\ \hat{p} \\ \hat{z} + \psi_2^{\hat{z}}(\hat{p},\hat{z},t) \end{pmatrix} = \begin{pmatrix} \hat{q} + \frac{A}{2}\hat{p}^3 + \mathcal{O}_4(\hat{p}) \\ \hat{p} \\ \hat{z} + \mathcal{O}_{N+3}(\hat{p}) \end{pmatrix}$$

is analytic in the domain

$$W^s_{\delta,\rho} = \{(\hat{q},\hat{p},\hat{z},t) \in \mathbb{C}^4 \mid |\hat{q}|, |\hat{p}| \leq \rho,\ \operatorname{Re}\hat{p} \geq 0,\ |\operatorname{Im}\hat{p}| < \delta\operatorname{Re}\hat{p},\ \hat{z} \in K^{\rho}_{\mathbb{C}},\ t \in \mathbb{T}_\sigma\}. \tag{5.190}$$

for some $\gamma, \rho, \sigma > 0$ (smaller than δ'' and ρ''). Moreover, $\Psi_{2|\tilde{q}=0}$ is analytic in $\{|\tilde{p}| < \rho,\ \tilde{z} \in K^{\rho}_{\mathbb{C}},\ t \in \mathbb{T}_\sigma\}$. This change is the identity on the unstable manifold. The previous arguments show that equation (5.176) in the $(\hat{q},\hat{p},\hat{z})$ variables has the form (5.180).

The change of variables $\Psi_1 \circ \Psi_2$ is then analytic in $U_{\delta,\rho}$, defined in (5.179), for some $\delta, \rho > 0$.

□

5.10.3 Third step of normal form

Next lemma provides a better control of the dynamics of $z = (a,b)$ close to $(q,p) = (0,0)$.

Lemma 5.10.3. *Let $N \geq 2$ be fixed. For any $1 \leq k < (N+1)/2$, there exists a change of variables*

$$\Phi(q,p,z,t) = (q,p,z+\mathcal{O}_{N+3}(q,p),t),$$

analytic in a domain of the form (5.179) *and of class C^{N+2} at $(q,p)=(0,0)$ such that equation* (5.180) *becomes, in the new variables,*

$$\begin{aligned} \dot{q} &= q((q+p)^3+\mathcal{O}_4),\\ \dot{p} &= -p((q+p)^3+\mathcal{O}_4),\\ \dot{z} &= q^k p^k \mathcal{O}_{N+6-2k},\\ \dot{t} &= 1, \end{aligned} \tag{5.191}$$

where $z=(a,b)$.

Proof. We prove the claim by induction. The case $k=1$ is given by Lemma 5.10.2. We observe that, since equation (5.180) is analytic in $U_{\delta,\rho}$ and $\mathcal{C}^\infty$ at $(0,0)$, the z component of the vector field, $qp\mathcal{O}_{N+4}(q,p)$, can be written as

$$qp\mathcal{O}_{N+4}(q,p) = \sum_{0\leq j<\lfloor\frac{N+2}{2}\rfloor}(qp)^{j+1}(Q_{N+4-2j}(q,z,t)+P_{N+4-2j}(p,z,t)) + (qp)^{\lfloor\frac{N+4}{2}\rfloor}\mathcal{O}_2(q,p),$$

where the functions $Q_{N+4-2j}(u,z,t), P_{N+4-2j}(u,z,t) = \mathcal{O}(u^{N+4-2j})$, $0\leq j<\lfloor\frac{N+4}{2}\rfloor$, are analytic in a domain of the form

$$V_{\delta,\rho} = \{(u,z,t)\in\mathbb{C}^3 \mid |\mathrm{Im}\,u| < \delta\mathrm{Re}\,u,\ |u|<\rho,\ |\mathrm{Im}\,t|<\sigma,\ z\in K_{\mathbb{C}}^{\rho}\},$$

for some $\delta,\rho>0$.

Assume that the equation is in the form

$$\begin{aligned} \dot{q} &= q((q+p)^3+\mathcal{O}_4),\\ \dot{p} &= -p((q+p)^3+\mathcal{O}_4),\\ \dot{z} &= q^k p^k \mathcal{O}_M(q,p), \end{aligned} \tag{5.192}$$

where $M = N+6-2k > 5$ and is analytic in $U_{\delta,\rho}$. We write the terms $\mathcal{O}_M(q,p) = R_M(q,p,z,t)$ as

$$R_M(q,p,z,t) = Q_M(q,t,z) + P_M(p,t,z) + \widetilde{R}_M(q,p,t,z),$$

where

$$Q_M(q,t,z) = R_M(q,0,z,t) = \mathcal{O}_M(q), \quad P_M(p,t,z) = R_M(0,p,z,t) = \mathcal{O}_M(p).$$

It is clear that $\widetilde{R}_M(q,p,t,z) = qp\mathcal{O}_{M-2}(q,p)$.

We perform a change a variables to get rid of the term Q_M and P_M of the form

$$\tilde{z} = z + q^k p^k (A(q,z,t) + B(p,z,t)). \tag{5.193}$$

The equation for $\tilde{z}$ becomes

$$\dot{\tilde{z}} = q^{k+1}p^{k+1}\mathcal{O}_{M-2}(q,p)$$

if

$$Q_M + h_4 A + f\partial_q A + \partial_t A = 0, \tag{5.194}$$

$$P_M + \tilde{h}_4 B + \tilde{f}\partial_q B + \partial_t B = 0, \tag{5.195}$$

where, from (5.192),

$$k(\dot{q}p + q\dot{p}) = qp(h_4(q,z,t) + \tilde{h}_4(p,z,t) + qp\hat{h}_2(q,p,z,t)) \tag{5.196}$$

with $h_4(q,z,t) = \mathcal{O}_4(q)$, $\tilde{h}_4(p,z,t) = \mathcal{O}_4(p)$ and $\hat{h}_2(q,p,z,t) = \mathcal{O}_2(q,p)$, and

$$\begin{aligned} f(q,z,t) &= \dot{q}_{|p=0} = q^4 + \mathcal{O}_5(q), \\ \tilde{f}(p,z,t) &= \dot{p}_{|q=0} = -p^4 + \mathcal{O}_5(p). \end{aligned} \tag{5.197}$$

The functions f, h_4, Q_M and $\tilde{f}, \tilde{h}_4, P_M$ are defined respectively in the sectors

$$\begin{aligned} V &= \{|\mathrm{Im}\, q| < \delta \mathrm{Re}\, q,\ |q| < \rho,\ |\mathrm{Im}\, t| < \sigma,\ z \in K_{\mathbb{C}}^{\rho}\}, \\ \widetilde{V} &= \{|\mathrm{Im}\, p| < \delta \mathrm{Re}\, p,\ |p| < \rho,\ |\mathrm{Im}\, t| < \sigma,\ z \in K_{\mathbb{C}}^{\rho}\}. \end{aligned}$$

Lemma 5.10.4. *If ρ is small enough, equations* (5.194) *and* (5.195) *admit analytic solutions A, B, defined in V and $\widetilde{V}$, such that*

$$\sup_{(q,z,t)\in V} |q^{-(M-3)} A(q,z,t)|, \sup_{(p,z,t)\in \widetilde{V}} |p^{-(M-3)} B(p,z,t)| < \infty.$$

respectively,

Proof of Lemma 5.10.4. We prove the claim for (5.194), being the proof for (5.195) analogous.

We consider the change of variables $q = q(u)$, where $q(u)$ satisfies

$$\frac{dq}{du} = f(q,z,t).$$

Since $f(q,z,t) = q^4 + \mathcal{O}_5(q)$, we have that

$$q(u) = -\frac{1}{(3u)^{1/3}}(1 + \mathcal{O}(u^{-1/3})).$$

It transforms (5.194) into

$$\mathcal{L}A = -\widehat{Q}_M - \hat{h}_4 A, \tag{5.198}$$

where

$$\mathcal{L}A = \partial_u A + \partial_t A \tag{5.199}$$

and $\widehat{Q}_M(u,z,t) = Q_M(q(u),z,t)$ and $\hat{h}_4(u,z,t) = h_4(q(u),z,t)$ are defined in

$$\widehat{V} = \{\mathrm{Re}\, u < -1/(3\rho^3),\ |\mathrm{Im}\, u| < 3 \arctan \gamma |\mathrm{Re}\, u|,\ |\mathrm{Im}\, t| < \sigma,\ z \in K_{\mathbb{C}}\}.$$

We introduce the Banach spaces

$$\mathcal{X}_\kappa = \{\alpha : \widehat{V} \to \mathbb{C} \mid \|\alpha\|_\kappa < \infty\}$$

where, writing $\alpha(u,z,t) = \sum_{\ell\in\mathbb{Z}} \alpha^{[\ell]}(u,z) e^{i\ell t}$,

$$\|\alpha\|_\kappa = \sum_{\ell\in\mathbb{Z}} \sup_{(u,z)\in U} |u^\kappa \alpha^{[\ell]}(u,z)| e^{\sigma|\ell|},$$

and

$$U = \{\mathrm{Re}\, u < -1/(3\rho^3),\ |\mathrm{Im}\, u| < 3 \arctan \gamma |\mathrm{Re}\, u|,\ z \in K_{\mathbb{C}}\}.$$

The following properties of the spaces $\mathcal{X}_\kappa$ are immediate:

1. if $\alpha \in \mathcal{X}_\kappa$ and $\tilde{\alpha} \in \mathcal{X}_{\tilde{\kappa}}$, $\alpha\tilde{\alpha} \in \mathcal{X}_{\kappa+\tilde{\kappa}}$ and

$$\|\alpha\tilde{\alpha}\|_{\kappa+\tilde{\kappa}} \le \|\alpha\|_\kappa \|\tilde{\alpha}\|_{\tilde{\kappa}},$$

2. if $\alpha \in \mathcal{X}_\kappa$, for all $\tilde{\kappa} \le \kappa$, $\alpha \in \mathcal{X}_{\tilde{\kappa}}$ and

$$\|\alpha\|_{\tilde{\kappa}} \le 3^{\kappa-\tilde{\kappa}} \rho^{3(\kappa-\tilde{\kappa})} \|\alpha\|_\kappa.$$

It is clear that $\widehat{Q}_M \in \mathcal{X}_{M/3}$ and $\hat{h}_4 \in \mathcal{X}_{4/3}$. The following lemma, whose proof is omitted, is a simplified version of Lemma 5.7.10.

Lemma 5.10.5. *Let $\beta \in \mathcal{X}_\kappa$, with $\kappa > 1$. The equation $\mathcal{L}\alpha = \beta$ has a solution $\mathcal{G}(\beta) \in \mathcal{X}_{k-1}$ with $\|\mathcal{G}(\beta)\|_{k-1} \leq \|\beta\|_k$.*

Using Lemma 5.10.5, we can rewrite equation (5.198) as the fixed point equation

$$A = \mathcal{G}(-\widehat{Q}_M - \hat{h}_4 A).$$

But, since $M > 5$, the right hand side above is a contraction in $\mathcal{X}_{M/3-1}$ if ρ is small enough, since, by 1. and 2.,

$$\begin{aligned}\|\mathcal{G}(-\widehat{Q}_M - \hat{h}_4 A) - \mathcal{G}(-\widehat{Q}_M - \hat{h}_4 \tilde{A})\|_{M/3-1} &\leq \|\hat{h}_4(A - \tilde{A})\|_{M/3} \\ &\leq \|\hat{h}_4\|_1 \|(A - \tilde{A})\|_{M/3-1} \leq 3^{1/3}\rho \|\hat{h}_4\|_{4/3} \|(A - \tilde{A})\|_{M/3-1}.\end{aligned}$$

Finally, since $u(q) = 1/(3q^3) + \mathcal{O}(q^{-2})$, the lemma follows. □

Now we can finish the proof of Lemma 5.10.3. With the choice of A and B given by Lemma 5.10.4, the change of variables (5.193) transforms (5.192) into a equation of the same form with k replaced by $k+1$ and M by $M-2$. Notice that this change of variables, since it is analytic in a sectorial domain of the form (5.179) and is of order $\mathcal{O}_{M+2k-3}(q,p) = \mathcal{O}_{N+3}(q,p)$, it is of class C^{N-2} at $(q,p) = (0,0)$. The composition of the vector field with the change is well defined in a sectorial domain of the form (5.179), with γ replaced with any $0 < \gamma' < \gamma$, if ρ is small enough. Hence the lemma is proven. □

5.11 The parabolic Lambda Lemma: Proof of Theorem 5.5.4

The proof of Theorem 5.5.4 will be a consequence of the following technical lemmas and is deferred to the end of this section. To simplify the notation in this section we denote $\Psi = \Psi_{\mathrm{loc}}$.

Let $K > 0$ be such that the terms $\mathcal{O}_k$ in (5.55) satisfy

$$\|\mathcal{O}_k\| \leq K \|(q,p)\|^k, \qquad (q,p,z,t) \in V_\rho \times W \times \mathbb{T}.$$

This bound is also true for $(q,p,z,t) \in B_\rho \times W \times \mathbb{T}$, being $B_\rho = \{(q,p) \mid |q|, |p| < \rho\}$.

We express system (5.55) in a new time in which the topological saddle is a true saddle. Indeed, since the solutions of (5.55) with initial condition $(q_0,p_0,z_0,t_0) \in B_\rho \times W \times \mathbb{T}$ with $q_0, p_0 \geq 0$ satisfy $q(t) + p(t) > 0$ while they belong to $B_\rho \times W \times \mathbb{T}$, we can write equations (5.55) in the new time s such that $dt/ds = (q+p)^{-3}$. System (5.55) becomes

$$\begin{aligned} q' &= q(1 + \mathcal{O}_1(q,p)), \\ p' &= -p(1 + \mathcal{O}_1(q,p)), \\ z' &= q^N p^N \mathcal{O}_1(q,p), \\ t' &= \frac{1}{(q+p)^3}, \end{aligned} \tag{5.200}$$

where $'$ denotes d/ds. The $\mathcal{O}_1(q,p)$ terms are uniformly bounded in terms of (q,p) in $V_\rho \times W \times \mathbb{T}$.

Given $K \subset W$, for $w_0 = (q_0,p_0,z_0,t_0) \in V_\rho \times K \times \mathbb{T}$, we define

$$s_{w_0} = \sup\{s > 0 \mid w(\tilde{s}) \in V_\rho \times W \times \mathbb{T},\ \forall \tilde{s} \in [0,s)\}, \tag{5.201}$$

where w is the solution of (5.55) with initial condition w_0.

Next lemma implies Item 1 of Theorem 5.5.4.

Lemma 5.11.1. *Let $K \subset W$ be a compact set. There exists ρ and $C > 1$ such that the solution $w = (q, p, z, t)$ of (5.200) with initial condition $w_0 = (q_0, p_0, z_0, t_0) \in V_\rho \times K \times \mathbb{T}$ satisfies*

$$\log\left(\left(\frac{\rho}{q_0}\right)^{\frac{1}{1+C\rho}}\right) \leq s_{w_0} \leq \log\left(\left(\frac{\rho}{q_0}\right)^{\frac{1}{1-C\rho}}\right).$$

Moreover, for any $0 < a \leq \rho$ and $0 < \delta < a/2$, the Poincaré map

$$\Psi : \Lambda^-_{a,\delta}(K) \to \Lambda^+_{a,\delta^{1-Ca}}(W),$$

where the sets $\Lambda^\pm_{a,\delta}(K)$ are defined in (5.57), is well defined and, if $w = (q_0, a, z_0, t_0) \in \Lambda^-_{a,\delta}(K)$ and $\Psi(w) = (a, p_1, z_1, t_1)$, then

$$\begin{aligned} q_0^{1+Ca} &\leq p_1 \leq q_0^{1-Ca}, \\ |z_1 - z_0| &\leq \frac{1}{2N} a^{N(1+3Ca)} q_0^{N(1-3Ca)}, \\ \widetilde{C}_1 q_0^{-3(1-Ca)/2} &\leq t_1 - t_0 \leq \widetilde{C}_2 q_0^{-3(1+Ca)/2} \end{aligned} \tag{5.202}$$

for some constants $\widetilde{C}_1, \widetilde{C}_2 > 0$ depending only on a.

Proof. Let $w_0 = (q_0, p_0, z_0, t_0) \in V_\rho \times K \times \mathbb{T}$ and $w = (q, p, z, t)$ the solution of (5.200) with initial condition w_0 at $s = 0$. Since $q_0, p_0 > 0$, while $w \in V_\rho \times W \times \mathbb{T}$, $q, p > 0$. Hence, there exists $C > 0$, depending only on ρ and W, such that

$$\begin{aligned} (1 - C\rho)q &\leq q' \leq (1 + C\rho)q, \\ -(1 + C\rho)p &\leq p' \leq -(1 - C\rho)p, \\ -C\rho q^N p^N &\leq z_i' \leq C\rho q^N p^N, \qquad i = 1, 2. \end{aligned} \tag{5.203}$$

Since p is decreasing and $\{p = 0\}$ is invariant, w can only leave $V_\rho \times W \times \mathbb{T}$ if $q = \rho$ or z leaves W. From (5.203), we have that for all s such $w(\tilde{s}) \in V_\rho \times W \times \mathbb{T}$ for all $\tilde{s} \in [0, s)$,

$$\begin{aligned} q_0 e^{(1-C\rho)s} &\leq q(s) \leq q_0 e^{(1+C\rho)s}, \\ p_0 e^{-(1+C\rho)s} &\leq p(s) \leq p_0 e^{-(1-C\rho)s} \end{aligned} \tag{5.204}$$

and

$$|z_i(s) - z_i(0)| \leq \frac{1}{2N} q_0^N p_0^N \left(e^{2NC\rho s} - 1\right), \qquad i = 1, 2. \tag{5.205}$$

In particular, the time $s_{q_0,q}$ to reach q from q_0 is bounded by

$$\log\left(\frac{q}{q_0}\right)^{\frac{1}{1+C\rho}} \leq s_{q_0,q} \leq \log\left(\frac{q}{q_0}\right)^{\frac{1}{1-C\rho}}, \tag{5.206}$$

but, up to this time, for $i = 1, 2$, since $0 < q_0, p_0, q < \rho < 1$,

$$|z_i(s_{q_0,q}) - z_i(0)| \leq \frac{1}{2N} q_0^N p_0^N \left(\frac{q}{q_0}\right)^{\frac{2NC\rho}{1-C\rho}} = \frac{1}{2N} q_0^{N\frac{1-3C\rho}{1-C\rho}} p_0^N q^{\frac{2NC\rho}{1-C\rho}} \leq \frac{1}{2N}\rho^{2N}. \tag{5.207}$$

Hence, taking ρ small enough depending on K, the solution through w_0 remains in $V_\rho \times W \times \mathbb{T}$ for all s such that

$$0 < s < s_{w_0} < \log\left(\frac{\rho}{q_0}\right)^{\frac{1}{1-C\rho}}.$$

Moreover, (5.207) ensures that the solution leaves $V_\rho \times W \times \mathbb{T}$ through $q = \rho$.

The time $s_{q_0,q}$ to reach q from q_0 is bounded by below by

$$s_{q_0,q} \geq \log \left(\frac{q}{q_0}\right)^{\frac{1}{1+C\rho}}, \tag{5.208}$$

putting $q = \rho$ in (5.208) we obtain the lower bound for s_{w_0}.

Consequently, for any $0 < \delta < a < \rho$, if $w_0 \in \Lambda^-_{a,\delta}(K)$, the solution through w_0 satisfies $q = a$ at a time s_* bounded by

$$\log \left(\frac{a}{q_0}\right)^{\frac{1}{1+Ca}} \leq s_* \leq \log \left(\frac{a}{q_0}\right)^{\frac{1}{1-Ca}}. \tag{5.209}$$

From (5.204) with an analogous argument replacing ρ by a, this solution satisfies

$$q_0^{1+2Ca} \leq p_0 \left(\frac{a}{q_0}\right)^{\frac{-1-Ca}{1-Ca}} \leq p_0 e^{-(1+Ca)s_*} \leq p(s_*) \leq p_0 e^{-(1-Ca)s_*} \leq p_0 \left(\frac{a}{q_0}\right)^{\frac{-1+Ca}{1+Ca}} \leq q_0^{1-2Ca}$$

and, for $i = 1, 2$,

$$|z_i(s_*) - z_i(0)| \leq \frac{1}{2N} a^{N\frac{1+Ca}{1-Ca}} q_0^{N(1-3Ca)}.$$

It only remains to estimate $t(s_*) - t_0$. Since

$$t(s_*) - t_0 = \int_0^{s_*} \frac{1}{(q+p)^3} ds,$$

$0 < q_0 < \delta$, $p_0 = a$ and $2\delta < a$, we have that

$$\begin{aligned}
t(s_*) - t_0 &\geq \int_0^{s_*} \frac{1}{(q_0 e^{(1+Ca)s} + p_0 e^{-(1-Ca)s})^3} ds \\
&= \int_0^{s_*} \frac{e^{-3Cas}}{(q_0 e^{s} + p_0 e^{-s})^3} ds \\
&\geq \left(\frac{q_0}{a}\right)^{\frac{3Ca}{1+Ca}} \int_0^{s_*} \frac{1}{(q_0 e^{s} + p_0 e^{-s})^3} ds \\
&= \left(\frac{q_0}{a}\right)^{\frac{3Ca}{1+Ca}} \frac{1}{(q_0 p_0)^{3/2}} \int_{-\log(p_0/q_0)^{1/2}}^{s_* - \log(p_0/q_0)^{1/2}} \frac{1}{(e^{\sigma} + e^{-\sigma})^3} d\sigma \\
&\geq \frac{1}{a^{3(1+Ca)/2}} \frac{1}{q_0^{3(1-Ca)/2}} \int_{-(\log 2)/2}^{(1-Ca)(\log 2)/(2(1+C)a)} \frac{1}{(e^{\sigma} + e^{-\sigma})^3} d\sigma.
\end{aligned}$$

With an analogous argument one obtains the upper bound of $t(s_*) - t_0$. □

Let $w = (q, p, z, t)$ be a solution of (5.55) with initial condition $w_0 \in V_\rho \times K \times \mathbb{T}$. We define

$$\tau = p/q \tag{5.210}$$

and, from now on, abusing notation we denote $\mathcal{O}_i = \mathcal{O}_i(q, p)$.

Clearly, $0 < \tau(s) < \infty$, for all s such that $w \in V_\rho \times K \times \mathbb{T}$. It is immediate from (5.200) that

$$\frac{d\tau}{ds} = -(2 + \mathcal{O}_1)\tau. \tag{5.211}$$

The variational equations around a solution of system (5.55) are

$$\begin{pmatrix} \dot{Q} \\ \dot{P} \\ \dot{Z} \\ \dot{T} \end{pmatrix} = \begin{pmatrix} (q+p)^2(4q+p+\mathcal{O}_2) & (q+p)^2 q(3+\mathcal{O}_1) & q\mathcal{O}_4 & q\mathcal{O}_4 \\ -(q+p)^2 p(3+\mathcal{O}_1) & -(q+p)^2(q+4p+\mathcal{O}_2) & p\mathcal{O}_4 & p\mathcal{O}_4 \\ q^{N-1}p^N\mathcal{O}_4 & q^N p^{N-1}\mathcal{O}_4 & q^N p^N \mathcal{O}_4 & q^N p^N \mathcal{O}_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q \\ P \\ Z \\ T \end{pmatrix}, \tag{5.212}$$

where $Z = (Z_1, Z_2)$.

In order to prove item 2) of Theorem 5.5.4, we will have to study the behavior of the solutions of (5.212) with initial condition $Q = Q_0 \neq 0$ along solutions of (5.55) with initial condition $w_0 = (q_0, p_0, z_0, t_0) \in U \cap \{q > 0,\ p > 0\}$ which will be taken with p_0 small but fixed and q_0 arbitrarily close to 0.

Equations (5.212) become, in the time s in which (5.200) are written and using τ in (5.210) ,

$$\begin{pmatrix} Q' \\ P' \\ Z' \\ T' \end{pmatrix} = \begin{pmatrix} \frac{4+\tau+\mathcal{O}_1}{1+\tau} & \frac{3+\mathcal{O}_1}{1+\tau} & q\mathcal{O}_1 & q\mathcal{O}_1 \\ -\frac{(3+\mathcal{O}_1)\tau}{1+\tau} & -\frac{1+4\tau+\mathcal{O}_1}{1+\tau} & p\mathcal{O}_1 & p\mathcal{O}_1 \\ q^{N-1}p^N\mathcal{O}_1 & q^Np^{N-1}\mathcal{O}_1 & q^Np^N\mathcal{O}_1 & q^Np^N\mathcal{O}_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q \\ P \\ Z \\ T \end{pmatrix}. \tag{5.213}$$

It will be convenient to perform a linear change of variables to (5.213).

Proposition 5.11.2. *There exists α^*, with $0 < \alpha^* < 6/5$ such that for any $\rho > 0$ small enough, any $w = (q, p, z, t)$, solution of (5.200) with $w_{|s=0} = w_0 = (q_0, p_0, z_0, t_0) \in V_\rho \times K \times \mathbb{T}$ and any $\alpha_0^* \in [0, \alpha^*]$, there exists a C^N function $\alpha : [0, s_{w_0}] \to \mathbb{R}$, where s_{w_0} was introduced in (5.201), with $\alpha(0) = \alpha_0^*$, such that, in the new variables*

$$\widetilde{P} = P + \alpha Q,$$

system (5.212) becomes

$$\begin{pmatrix} Q' \\ \widetilde{P}' \\ Z' \\ T' \end{pmatrix} = \begin{pmatrix} \frac{4+\tau+\mathcal{O}_1}{1+\tau} - \alpha\frac{3+\mathcal{O}_1}{1+\tau} & \frac{3+\mathcal{O}_1}{1+\tau} & q\mathcal{O}_1 & q\mathcal{O}_1 \\ 0 & -\frac{1+4\tau+\mathcal{O}_1}{1+\tau} + \alpha\frac{3+\mathcal{O}_1}{1+\tau} & p\mathcal{O}_1 & p\mathcal{O}_1 \\ q^{N-1}p^N\mathcal{O}_1 & q^Np^{N-1}\mathcal{O}_1 & q^Np^N\mathcal{O}_1 & q^Np^N\mathcal{O}_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Q \\ \widetilde{P} \\ Z \\ T \end{pmatrix}. \tag{5.214}$$

Furthermore, for $s \in (0, s_{w_0}]$,

$$0 < \alpha(s) < \frac{2\tau(s)}{1+\tau(s)}. \tag{5.215}$$

Proof. Given α and $\widetilde{P} = P + \alpha Q$, since $\tau > 0$, the equation for $\widetilde{P}$ is

$$\begin{aligned} \widetilde{P}' = &\left(-\frac{(3+\mathcal{O}_1)\tau}{1+\tau} + \alpha' + (5+\mathcal{O}_1)\alpha - \alpha^2\frac{3+\mathcal{O}_1}{1+\tau}\right) Q \\ &+ \left(-\frac{1+4\tau+\mathcal{O}_1}{1+\tau} + \alpha\frac{3+\mathcal{O}_1}{1+\tau}\right)\widetilde{P} + (p+\alpha q)\mathcal{O}_1 Z + (p+\alpha q)\mathcal{O}_1 T, \end{aligned} \tag{5.216}$$

The claim will follow finding an appropriate solution of

$$\alpha' = \nu_0 + \nu_1\alpha + \nu_2\alpha^2, \tag{5.217}$$

where

$$\nu_0 = \frac{(3+\mathcal{O}_1)\tau}{1+\tau}, \qquad \nu_1(s) = -5 + \mathcal{O}_1, \qquad \nu_2 = \frac{3+\mathcal{O}_1}{1+\tau}. \tag{5.218}$$

Let $f(w, \alpha) = \nu_0 + \nu_1\alpha + \nu_2\alpha^2$ be the right hand side of (5.217), where we have omitted the dependence of ν_i, $i = 1, 2, 3$ on w. We introduce α_0 and α_1, the nullclines of (5.217), by

$$f(w, \alpha) = \nu_2(\alpha - \alpha_0(\tau))(\alpha - \alpha_1(\tau)),$$

and R, where

$$\begin{aligned} \alpha_0(\tau) &= -\frac{\nu_1}{2\nu_2}\left(1 - \left(1 - 4\frac{\nu_0\nu_2}{\nu_1^2}\right)^{1/2}\right) \\ &= \left(\frac{5}{6} + \mathcal{O}_1\right)\left(1 + \tau - \left((1+\tau)^2 - \left(\frac{36}{25} + \mathcal{O}_1\right)\tau\right)^{1/2}\right) \\ &= \left(\frac{5}{6} + \mathcal{O}_1\right)\left(1 + \tau - \sqrt{R(\tau)}\right). \end{aligned} \tag{5.219}$$

To complete the proof of Proposition 5.11.2, we need the following two auxiliary lemmas.

Lemma 5.11.3. *The function α_0 has the following properties. For $(q,p) \in V_\rho$ (that is, $0 < \tau < \infty$),*

1. $4/5 + \mathcal{O}_1 \leq \sqrt{R(\tau)}/(1+\tau) < 1$,
2. $\lim_{\tau\to\infty} \alpha_0(\tau) = 3/5 + \mathcal{O}_1$,
3. $\lim_{\tau\to 0} \alpha_0(\tau)/\tau = 3/5 + \mathcal{O}_1$,
4. $$\frac{d}{ds}\alpha_0 = -(1+\mathcal{O}_1)\frac{\sqrt{R} - \tau + 1 + \mathcal{O}_1}{\sqrt{R}}\alpha_0,$$
5. $$-\frac{(2+\mathcal{O}_1)\alpha_0}{\sqrt{R}} \leq \frac{d}{ds}\alpha_0 \leq -\frac{(32/25+\mathcal{O}_1)\alpha_0}{\sqrt{R}}$$
6. *and* $\lim_{\tau\to 0}(d\alpha_0/ds)/\alpha_0 = -2 + \mathcal{O}_1$.

Furthermore,

$$0 < \alpha_0(\tau) < \frac{\tau}{1+\tau}. \tag{5.220}$$

Proof. Items 1 to 6 are proven in [GK12]. The rather crude bound (5.220) is a straightforward computation. □

Next lemma provides solutions of (5.217) close to the nullcline α_0.

Lemma 5.11.4. *For any $0 < \rho < 1$ small enough, the following is true. For any solution $w = (q,p,z,t)$ of (5.55) with initial condition $w_0 \in V_\rho \times K \times \mathbb{T}$, if α is a solution of (5.217) with $0 \leq \alpha(s_0) \leq 2\alpha_0(\tau(s_0))$ for some $0 < s_0 < s_{w_0}$, then $0 < \alpha(s) < 2\alpha_0(\tau(s))$ for all $s \in [s_0, s_{w_0}]$.*

Proof of Lemma 5.11.4. We only need to proof that α satisfies

$$\text{(i)}\ \ \frac{d\alpha}{ds}_{|\alpha=0} > 0, \qquad \text{(ii)}\ \ \frac{d\alpha}{ds}_{|\alpha=2\alpha_0} < 2\frac{d\alpha_0}{ds}.$$

Item (i) follows from

$$\frac{d\alpha}{ds}_{|\alpha=0} = \nu_0 > 0.$$

Now we prove (ii). Using 5 of Lemma 5.11.3, (ii) is implied by

$$\begin{aligned}\frac{d\alpha}{ds}_{|\alpha=2\alpha_0} &= \frac{3+\mathcal{O}_1}{1+\tau}\alpha_0(2\alpha_0 - \alpha_1)\\ &= \frac{3+\mathcal{O}_1}{1+\tau}\alpha_0\left(\alpha_0 - \left(\frac{5}{3}+\mathcal{O}_1\right)\sqrt{R}\right)\\ &< -2\frac{2+\mathcal{O}_1}{\sqrt{R}}\alpha_0,\end{aligned}$$

which, since $\alpha_0 > 0$, is equivalent to

$$\frac{3+\mathcal{O}_1}{1+\tau}\left(\alpha_0 - \left(\frac{5}{3}+\mathcal{O}_1\right)\sqrt{R}\right) < -\frac{4+\mathcal{O}_1}{\sqrt{R}}, \tag{5.221}$$

for $0 < \tau$. Taking into account the definition of α_0, (5.221) is equivalent to prove

$$\left(\frac{15}{2}+\mathcal{O}_1\right)R - (4+\mathcal{O}_1)(1+\tau) > \left(\frac{5}{2}+\mathcal{O}_1\right)(1+\tau)\sqrt{R},$$

for $0 < \tau$, which is equivalent to

$$\left(\left(\frac{15}{2}+\mathcal{O}_1\right)R-(4+\mathcal{O}_1)(1+\tau)\right)^2-\left(\frac{5}{2}+\mathcal{O}_1\right)^2(1+\tau)^2R>0.$$

If we disregard the $\mathcal{O}_1$ terms, which are small if ρ is small, the above inequality simply reads

$$50\tau^4-13\tau^3+\frac{826}{25}\tau^2-\frac{75}{3}\tau+6=\tau^2\left(50\tau^2-13\tau+\frac{76}{25}\right)+30\tau^2-\frac{75}{3}\tau+6>0.$$

But the above inequality holds, since $50\tau^2-13\tau+\frac{76}{25}>0$ and $30\tau^2-\frac{75}{3}\tau+6>0$. □

Let α be any solution of (5.217) with $\alpha(0)\in[0,2\alpha_0(\tau(0))]$, which, by the definition of τ and Item 2 in Lemma 5.11.3, is a nonempty interval (recall that $\tau(0)\gg 1$). By Lemma 5.11.4, α is well defined for $s\in[0,s_{w_0}]$ and $0<\alpha(s)<2\alpha_0(\tau(s))$. Then, bound (5.220) implies (5.215). System (5.214) is obtained by a straightforward computation. Observe that, by (5.220), the terms $(p+\alpha q)\mathcal{O}_1$ in (5.216) are indeed $p\mathcal{O}_1$. □

Lemma 5.11.5. *Choose $N>10$ in Theorem 5.5.2. Let W and K be the sets considered in Theorem 5.5.4. Let $w=(q,p,z,t)$ be a solution of (5.55) with initial condition, at $s=0$, $w_0\in\Lambda^-_{a,\delta}(K)$. Let $\tilde{s}_{w_0}$ be such that $w(\tilde{s}_{w_0})\in\Lambda^+_{a,\delta^{1-Ca}}(W)$. Let $W=(Q,\widetilde{P},Z,T)$ be a solution of (5.214) with initial condition, at $s=0$, $W_0=(Q_0,\widetilde{P}_0,Z_0,T_0)$.*

1. *For all $s\in[0,s_{w_0}]$,*
$$\|(Z,T)-(Z_0,T_0)\|\le Kq_0^{N-10}\|W_0\|.$$
2. *For $s=\tilde{s}_{w_0}$,*
$$|\widetilde{P}(\tilde{s}_{w_0})|\le Cq_0^{\frac{3}{5}+C\rho}(|\widetilde{P}_0|+\mathcal{O}_1\|W_0\|).$$
3. *Assume that W_0 satisfies $Q_0\neq 0$. Then, there exists δ such that for any $w_0\in\Lambda^-_{a,\delta}(K)$,*
$$|Q(\tilde{s}_{w_0})|\ge C\left(|Q_0|-Cq_0^{\frac{1}{5}+\mathcal{O}_1(\rho)}\|W_0\|\right)q_0^{-(\frac{3}{5}+\mathcal{O}_1(\rho))}.$$
4. *For any $\widetilde{P}_0$, there exists a linear map $\widetilde{Q}(Z_0,T_0)$ satisfying $|\widetilde{Q}(Z_0,T_0)|\le Cq_0^{\frac{1}{5}+\mathcal{O}_1(\rho)}\|(Z_0,T_0)\|$, such that the solution W of (5.214) with initial condition $W_0=(\widetilde{Q}(Z_0,T_0),\widetilde{P}_0,Z_0,T_0)$ satisfies $Q(\tilde{s}_{w_0})=0$.*

Proof. Here $\|\cdot\|$ will denote the sup-norm. Let α be any of the functions given by Lemma 5.11.4. Using (5.220), a direct computation shows that the spectral radius of the matrix defining system (5.214) is bounded by $7+\mathcal{O}_1(\rho)$. Hence, if ρ is small enough,

$$\|W(s)\|\le e^{8s}\|W_0\|,\qquad 0\le s\le\tilde{s}_{w_0}, \tag{5.222}$$

where, by (5.206), the time $\tilde{s}_{w_0}$ is bounded by above by

$$\tilde{s}_{w_0}\le\log\left(\frac{a}{q_0}\right)^{\frac{1}{1-C\rho}}. \tag{5.223}$$

Let $W_{Q,P}=(Q,P)$ and $W_{Z,T}=(Z,T)$. The vector $W_{Z,T}$ satisfies

$$W'_{Z,T}=q^Np^N\mathcal{O}_1W_{Z,T}+q^{N-1}p^{N-1}\mathcal{O}_1W_{Q,P}.$$

Since, by (5.204),

$$\|q^{N-1}p^{N-1}\mathcal{O}_1W_{Q,P}\|\le(q_0p_0)^{N-1}e^{2NC\rho s}\|W_{Q,P}\|\le(q_0p_0)^{N-1}e^{(8+2NC\rho)s}\|W_0\|$$

and $N>10$, we have that, for $0\le s\le \tilde s_{w_0}$, if ρ is small enough,

$$\|W_{Z,T}(s)-W_{Z,T}(0)\|\le Kq_0^{N-10}\|W_0\|,$$

for some constant K. This proves Item 1.

Now we prove Item 2. The equation for $\tilde P$ is

$$\widetilde{P}'=A\widetilde{P}+p\mathcal{O}_1W_{Z,T},$$

where

$$A=-\frac{1+4\tau+\mathcal{O}_1}{1+\tau}+\alpha\frac{3+\mathcal{O}_1}{1+\tau}.$$

Using the bounds on α given by Lemma 5.11.4 and (5.215), a straightforward computation shows that, if ρ is small enough, for all $\tau>0$, $A<-3/5$. Hence, using again (5.204), $|\widetilde{P}(s)|\le(|\widetilde{P}_0|+\mathcal{O}_1\|W_0\|)e^{-\frac{3}{5}s}$, which, taking into account the bound of $\tilde s_{w_0}$, implies Item 2.

To prove Item 3, observe that the equation for Q is

$$Q'=\widetilde{A}Q+\widetilde{B}\widetilde{P}+q\mathcal{O}_1W_{Z,T},$$

where

$$\widetilde{A}=\frac{4+\tau+\mathcal{O}_1}{1+\tau}-\alpha\frac{3+\mathcal{O}_1}{1+\tau},\qquad \widetilde{B}=\frac{3+\mathcal{O}_1}{1+\tau}.$$

Again, a straightforward computation shows that, if ρ is small enough, $\widetilde{A}>3/5$. Defining $u(s)=\exp\int_0^s\widetilde{A}(\sigma)\,d\sigma$, we have that

$$Q(s)=u(s)\left[Q_0+\int_0^s u(-\sigma)(\widetilde{B}(\sigma)\widetilde{P}(\sigma)+q(\sigma)\mathcal{O}_1W_{Z,T}(\sigma))\,d\sigma\right] \tag{5.224}$$

We bound the terms in the integral in the following way. First we observe that, using (5.211),

$$0\le\widetilde{B}(\sigma)=\frac{3+\mathcal{O}_1}{1+\tau}\le\frac{q_0}{p_0}\frac{3+\mathcal{O}_1}{\frac{q_0}{p_0}+e^{-(2+\mathcal{O}_1(\rho))s}}<\frac{q_0}{p_0}(3+\mathcal{O}_1)e^{(2+\mathcal{O}_1(\rho))s}.$$

Hence, by the previous bound on $\widetilde{P}$, for some constant $K>0$,

$$\begin{aligned}\left|\int_0^s u(-\sigma)\widetilde{B}(\sigma)\widetilde{P}(\sigma)\,d\sigma\right|&\le(|\widetilde{P}_0|+\mathcal{O}_1\|W_0\|)(3+\mathcal{O}_1)\frac{q_0}{p_0}\int_0^s e^{(\frac{4}{5}+\mathcal{O}_1(\rho))\sigma}\,d\sigma\\&\le K(|\widetilde{P}_0|+\mathcal{O}_1\|W_0\|)q_0e^{(\frac{4}{5}+\mathcal{O}_1(\rho))s}.\end{aligned}$$

Using (5.204) to bound $q(s)$, the other term in the integral can be bounded as

$$\begin{aligned}\left|\int_0^s u(-\sigma)q(\sigma)\mathcal{O}_1W_{Z,T}(\sigma)\,d\sigma\right|&\le\mathcal{O}_1(\rho)\|W_0\|q_0\int_0^s e^{(\frac{2}{5}+\mathcal{O}_1(\rho))\sigma}\,d\sigma\\&\le\mathcal{O}_1(\rho)\|W_0\|q_0e^{(\frac{2}{5}+\mathcal{O}_1(\rho))s}.\end{aligned}$$

That is, since $0<s<\tilde s_{w_0}$ and using (5.223),

$$\left|Q_0+\int_0^s u(-\sigma)(\widetilde{B}(\sigma)\widetilde{P}(\sigma)+q(\sigma)\mathcal{O}_1W_{Z,T}(\sigma))\,d\sigma\right|\ge|Q_0|-Kq_0^{\frac{1}{5}+\mathcal{O}_1(\rho)}\|W_0\|.$$

Since $0<q_0<\delta$, substituting this bound into (5.224) and evaluating at $s=\tilde s_{w_0}$, we obtain 3.

Item 4 follows immediately from the bounds of the terms inside the brackets in (5.224). □

Proof of Theorem 5.5.4. Lemma 5.11.1 proves that the Poincaré map $\Psi : \Lambda^-_{a,\delta}(K) \to \Lambda^+_{a,\delta^{1-Ca}}(W)$ is well defined for any compact $K \subset W$ if $0 < 2\delta < a$ are small enough and also implies the estimates of Item 1 of Theorem 5.5.4.

Given $I \subset \mathbb{R}$, an interval, let $\gamma(u) = (q_0(u), a, z_0(u), t_0(u))$, $u \in I$ be a C^1 curve in $\Lambda^-_{a,\delta}(K)$ with $0 < q(u) < \delta$, and $\tilde{\gamma} = \Psi \circ \gamma = (a, p_1, z_1, t_1)$, which is well defined if δ is small enough. Along this proof we choose different curves $\gamma(u)$.

Let us compute $\tilde{\gamma}'(u)$. Let $X = (X_q, X_p, X_z, X_t)$ denote the vector field in (5.55) and $w = (q, p, z, t)$. Since $\tilde{\gamma}(u) = \varphi_{t_1(u)-t_0(u)}(\gamma(u))$, we have that

$$\tilde{\gamma}'(u) = \begin{pmatrix} 0 \\ p_1'(u) \\ z_1'(u) \\ t_1'(u) \end{pmatrix} = D_w \varphi_{t_1(u)-t_0(u)}(\gamma(u))\gamma'(u) + X(\tilde{\gamma}(u))(t_1'(u) - t_0'(u))$$

$$= \begin{pmatrix} Q_{|t_1(u)-t_0(u)} + X_q(\tilde{\gamma}(u))(t_1'(u) - t_0'(u)) \\ P_{|t_1(u)-t_0(u)} + X_p(\tilde{\gamma}(u))(t_1'(u) - t_0'(u)) \\ Z_{|t_1(u)-t_0(u)} + X_z(\tilde{\gamma}(u))(t_1'(u) - t_0'(u)) \\ T_{|t_1(u)-t_0(u)} + X_t(\tilde{\gamma}(u))(t_1'(u) - t_0'(u)) \end{pmatrix}, \quad (5.225)$$

where (Q, P, Z, T) is the solution of (5.212) along $\varphi_{t-t_0(u)}(\gamma(u))$ with initial condition $\gamma'(u)$.

From the first component of (5.225),

$$t_1'(u) - t_0'(u) = -\frac{Q_{|t_1(u)-t_0(u)}}{X_q(\tilde{\gamma}(u))}. \quad (5.226)$$

We observe that $X_q(\tilde{\gamma}(u)) = a(a + \mathcal{O}(q_0(u)^{1-\mathcal{O}_1(a)})^3 + \mathcal{O}(a^4))$.

We choose α in Lemma 5.11.4 such that $\alpha(0) = 0$. We apply the change of variables of Proposition 5.11.2 and consider $(Q, \tilde{P}, Z, T)$, the corresponding solution of (5.214). By the choice of α, $(Q, P, Z, T)_{|s=0} = (Q, \tilde{P}, Z, T)_{|s=0}$.

Now we prove Item 2 of Theorem 5.5.4. Assume $q(u) = u$, $0 < u < \delta$. Let $W_u^0 = (Q_u^0, P_u^0, Z_u^0, T_u^0) = \gamma'(u)$. Let (Q_u, P_u, Z_u, T_u) be the solution of (5.213) with initial condition W_u^0 and $(Q_u, \widetilde{P}_u, Z_u, T_u)$, the solution of (5.214) with the same initial condition. If δ is small enough, $\sup_{0<u<\delta} \|\gamma'(u)\| = \sup_{0<u<\delta} \|W_u^0\| < 2\|W_0^0\|$. Hence, if δ is small enough, by item 3 of Lemma 5.11.5,

$$|Q_u(\tilde{s}_{w_0})| \geq C\left(|Q_u^0| - Cu^{\frac{1}{5}+\mathcal{O}_1(\rho)}\|W_0^0\|\right)u^{-(\frac{3}{5}+\mathcal{O}_1(\rho))} \geq \widetilde{C}|Q_u^0|u^{-(\frac{3}{5}+\mathcal{O}_1(\rho))}.$$

In the case we are considering, $Q_u^0 = 1$. This inequality, combined with (5.226), implies

$$|t_1'(u) - t_0'(u)| \geq Cu^{-(\frac{3}{5}+\mathcal{O}_1(\rho))}. \quad (5.227)$$

Hence, by item 2 of Lemma 5.11.5, the bound of α given by Lemma 5.11.4, bound (5.220), (5.226) and the facts that $T' = 0$, $T = t_0'(u)$, $|X_p(\tilde{\gamma}(u))| \leq Cq(u)^{1-Ca}$ and $|X_q(\tilde{\gamma}(u))| \leq C$,

$$\begin{aligned}
\left|\frac{p_1'(u)}{t_1'(u)}\right| &= \frac{|P_u(\tilde{s}_{\gamma(u)}) + X_p(\tilde{\gamma}(u))(t_1'(u) - t_0'(u)|}{|t_1'(u)|} \\
&\leq \frac{|P_u(\tilde{s}_{\gamma(u)}) + X_p(\tilde{\gamma}(u))(t_1'(u) - t_0'(u)|}{|t_1'(u) - t_0'(u))|}\left(1 + \frac{|t_0'(u)|}{|t_1'(u)|}\right) \\
&\leq C\frac{|\widetilde{P}_u(\tilde{s}_{\gamma(u)}) - \alpha(\tilde{s}_{\gamma(u)})Q_u(\tilde{s}_{\gamma(u)}) + X_p(\tilde{\gamma}(u))(t_1'(u) - t_0'(u))|}{|t_1'(u) - t_0'(u)|} \\
&= C\frac{|\widetilde{P}_u(\tilde{s}_{\gamma(u)}) + [(\alpha(\tilde{s}_{\gamma(u)})X_q(\tilde{\gamma}(u)) + X_p(\tilde{\gamma}(u))](t_1'(u) - t_0'(u))|}{|t_1'(u) - t_0'(u)|} \\
&= C\left(\frac{|\widetilde{P}_u(\tilde{s}_{\gamma(u)})|}{|t_1'(u) - t_0'(u)|} + \left|(\alpha(\tilde{s}_{\gamma(u)})X_q(\tilde{\gamma}(u)) + X_p(\tilde{\gamma}(u))\right|\right) \\
&\leq \widetilde{C}u^{1-Ca}.
\end{aligned}$$

And, analogously, using Item 1 of Lemma 5.11.5 and the fact that $N > 10$, we obtain that

$$\begin{aligned}
\left|\frac{z_1'(u)}{t_1'(u)}\right| &\leq \frac{|Z_u(\tilde{s}_{w_0}) + X_z(\tilde{\gamma}(u))(t_1'(u) - t_0'(u)|}{|t_1'(u) - t_0'(u))|}\left(1 + \frac{|t_0'(u)|}{|t_1'(u)|}\right) \\
&\leq C\left(\frac{|Z_u(\tilde{s}_{w_0}))|}{|t_1'(u) - t_0'(u)|} + |X_z(\tilde{\gamma}(u))|\right) \\
&\leq C u^{\frac{3}{5} - Ca},
\end{aligned}$$

which proves Item 2 of Theorem 5.5.4.

Now we prove Item 3 of Theorem 5.5.4. We first observe that, using (5.222), (5.223) and (5.226),

$$|t_1'(u) - t_0'(u)| \leq C|Q_u(\tilde{s}_{\gamma(u)})| \leq Cq(u)^{-8-Ca}\|W_u\|.$$

Then, using Item 1 of Lemma 5.11.5 and, since $N > 10$, $|X_z(\tilde{\gamma}(u))| \leq Cq(u)^{N-Ca}$,

$$\begin{aligned}
|z_1'(u) - z_0'(u)| &= |Z_{|t_1(u)-t_0(u)} - Z_{|0} + X_z(\tilde{\gamma}(u))(t_1'(u) - t_0'(u))| \\
&\leq Cq(u)^{N-10}\|W_u\|.
\end{aligned}$$

Hence, Item 3 is proven.

We finally prove Item 4. Let $\tilde{q}_0 \in (0, \delta)$ and $\tilde{w}_0 = (\tilde{q}_0, a, \tilde{z}_0, \tilde{t}_0) \in \Lambda_\delta^-(K)$. Taking into account (5.227), which also holds in this case, by the Implicit Function Theorem, the equation

$$t_1(q, a, z, t) - t = t_1(\tilde{q}_0, a, \tilde{z}_0, \tilde{t}_0) - \tilde{t}_0$$

defines a function $q = q_0(z, t)$, with (z, t) in neighborhood of (z_0, t_0). Given $(z_0(u), t_0(u))$, any curve in $K \times \mathbb{T}$ with $z_0(0) = \tilde{z}_0$ and $t_0(0) = \tilde{t}_0$, let $\gamma(u) = (q_0(z_0(u), t_0(u)), a, z_0(u), t_0(u))$ and $\tilde{\gamma}(a, p_1, z_1, t_1) = \Psi \circ \gamma$. Since, by the definition of the function q_0, $(t_1(u) - t_0(u))' = 0$, from (5.225) we obtain that

$$\tilde{\gamma}'(u) = \begin{pmatrix} Q_{|t_1(u)-t_0(u)} \\ P_{|t_1(u)-t_0(u)} \\ Z_{|t_1(u)-t_0(u)} \\ T_{|t_1(u)-t_0(u)} \end{pmatrix},$$

where (Q, P, Z, T) is the solution of (5.212) along $\varphi_{t-t_0(u)}(\gamma(0))$ with initial condition $\gamma'(0)$. In particular, $Q_{|t_1(u)-t_0(u)} = 0$. But, then, this implies that $Q_{|0}$ has to be the value $\widetilde{Q}_0$ given by Item 4 of Lemma 5.11.5. Moreover, this implies $P_{|t_1(u)-t_0(u)} = \tilde{P}_{|t_1(u)-t_0(u)}$. From the bounds of Lemma 5.11.5 follow 4 of Theorem 5.5.4. □

5.12 Conjugation with the Bernouilli shift

We devote this section to prove Propositions 5.6.4 and 5.6.5. This is done in several steps. First, in Section 5.12.1, we analyze the differential of the return maps $\widetilde{\Psi}_{i,j}$ defined in (5.64). In particular, we analyze its expanding, contracting and center directions. Then, in Section 5.12.2, we prove Proposition 5.6.4. In Section 5.12.3, we analyze the differential of $\widetilde{\Psi}$, the high iterate of the return map defined in (5.67). Finally, in Section 5.12.4, we use these cone fields to prove Proposition 5.6.5.

5.12.1 The differential of the intermediate return maps

In the following lemma, we write a vector $v \in T_\omega \mathcal{Q}_\delta^i$ in the basis $\frac{\partial}{\partial p}, \frac{\partial}{\partial \tau}, \frac{\partial}{\partial z}$, where $\frac{\partial}{\partial z}$ stands for $(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2})$, given by the coordinates defined by A_i.

Lemma 5.12.1. *Let N be fixed. Assume δ small enough.*

1. *The vector field*

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

satisfies that, for any $\omega = (p, \tau, z) \in \mathcal{Q}_\delta^i \cap \widetilde{\Psi}_{i,j}^{-1}(\mathcal{Q}_\delta^j)$,

$$D\widetilde{\Psi}_{i,j}(\omega)v_1 = \lambda_1^i(\omega)(v_1 + \tilde{v}_1^{i,j}(\omega)), \qquad 1 \le i, j \le 2, \tag{5.228}$$

where $\widetilde{\Psi}_{i,j}$ *is the return map defined in (5.64) written in coordinates* (p, τ, z) *and*

$$\lambda_1^i(\omega) \gtrsim \tau^{-\frac{3}{5}+\tilde{C}a},$$
$$|\tilde{v}_{1,p}^{i,j}(\omega)| \le \mathcal{O}\left(\tau^{1-\tilde{C}a}\right), \quad \tilde{v}_{1,\tau}^{i,j}(\omega) = 0, \quad \|\tilde{v}_{1,z}^{i,j}(\omega)\| \le \mathcal{O}\left(\tau^{\frac{3}{5}-\tilde{C}a}\right).$$

Moreover, for any vector

$$\hat{v}_1 = \begin{pmatrix} b \\ 1 \\ c \end{pmatrix}$$

for $|b|, |c| \lesssim 1$ *one has*

$$D\widetilde{\Psi}_{i,j}(\omega)\hat{v}_1 = \lambda_1^i(\omega)(v_1 + \hat{v}_1^{i,j}(\omega)), \qquad 1 \le i, j \le 2, \tag{5.229}$$

for some vector $\hat{v}_1^{i,j}$ *satisfying*

$$|\hat{v}_{1,p}^{i,j}(\omega)| \le \mathcal{O}\left(\tau^{1-\tilde{C}a}\right), \quad |\hat{v}_{1,\tau}^{i,j}(\omega)|, \|\hat{v}_{1,z}^{i,j}(\omega)\| \le \mathcal{O}\left(\tau^{\frac{3}{5}-\tilde{C}a}\right) \quad \text{and} \quad \|\lambda_1^i(\omega)\hat{v}_{1,z}^{i,j}(\omega)\| \le \mathcal{O}(1).$$

2. *There exist* C^0 *vector fields* $v_2^{i,j} : \mathcal{Q}_\delta^i \to T\mathcal{Q}_\delta^i$, $i, j = 1, 2$, *of the form*

$$v_2^{i,j}(\omega) = \begin{pmatrix} 1 \\ \tilde{v}_{2,\tau}^{i,j}(\omega) \\ v_{2,z}^i(z) + \tilde{v}_{2,z}^{i,j}(\omega) \end{pmatrix}, \tag{5.230}$$

where the functions $v_{2,z}^i$ *depend only on* z *and satisfy* $\|v_{2,z}^i(z)\| = \mathcal{O}(1)$, *and*

$$|\tilde{v}_{2,\tau}^{i,j}(\omega)| = \mathcal{O}(p) + \mathcal{O}\left(\tau^{\frac{3}{5}-\tilde{C}a}\right), \qquad \|\tilde{v}_{2,z}^{i,j}(\omega)\| \le \mathcal{O}\left(\tau^{\frac{3}{5}-\tilde{C}a}\right), \qquad i = 1, 2,$$

such that for any $\omega = (p, \tau, z) \in \mathcal{Q}_\delta^i \cap \widetilde{\Psi}_{i,j}^{-1}(\mathcal{Q}_\delta^j)$, *the following holds.*

$$D\widetilde{\Psi}_{i,j}(\omega)v_2^{i,j}(\omega) = \lambda_2^{i,j}(\omega)(v_2^{i,j}(\widetilde{\Psi}_{i,j}(\omega)) + \hat{v}_2^{i,j}(\omega)), \qquad 1 \le i, j \le 2, \tag{5.231}$$

where

$$\lambda_2^{i,j}(\omega)^{-1} \gtrsim \tau^{-\frac{3}{5}+\tilde{C}a},$$
$$\hat{v}_{2,p}^{i,j}(\omega) = 0, \quad |\hat{v}_{2,\tau}^{i,j}(\omega)| \le \mathcal{O}\left(\tau^{1-\tilde{C}a}\right), \quad \|\hat{v}_{2,z}^{i,j}(\omega)\| \le \mathcal{O}\left(\tau^{\frac{3}{5}-\tilde{C}a}\right).$$

3. *For any* $v_z(z)$, C^0 *vector field in* $\mathbb{R}^2$, $i, j = 1, 2$, *there exist vector fields*

$$v^{i,j}(\omega) = \begin{pmatrix} 0 \\ 0 \\ v_z(z) \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{v}_\tau^{i,j}(\omega) \\ 0 \end{pmatrix},$$

with $|\tilde{v}_\tau^{i,j}| \le \mathcal{O}(\tau^{1/5-Ca})$, *such that the following holds.*

$$D\widetilde{\Psi}_{i,j}(\omega)v^{i,j}(\omega) = \begin{pmatrix} 0 \\ 0 \\ D\widehat{\mathtt{S}}_i(z)v_z(z) \end{pmatrix} + \hat{v}^{i,j}(\omega), \qquad \|\hat{v}^{i,j}\| \le \mathcal{O}(\tau^{1/5-Ca}, a). \tag{5.232}$$

Proof. We start with v_1. For any $\omega=(p,\tau,z)\in\mathcal{Q}^i_\delta$, $z=(z_1,z_2)$, we have that, in view of (5.62),

$$\begin{aligned}
D\widetilde{\Psi}_{\mathrm{glob},i}(\omega)v_1&=\begin{pmatrix}\mathcal{O}_1(\tau) & \nu_1^i(z)+\mathcal{O}_1(p,\tau) & \mathcal{O}_1(\tau)\\ \nu_2^i(z)+\mathcal{O}_1(p,\tau) & \mathcal{O}_1(p) & \mathcal{O}_1(p)\\ \widetilde{S}_{i,p}(z)+\mathcal{O}_1(p,\tau) & \widetilde{S}_{i,\tau}(z)+\mathcal{O}_1(p,\tau) & D\widehat{\mathtt{S}}_i(z)+\mathcal{O}_1(p,\tau)\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\\
&=(\nu_1^i(z)+\mathcal{O}_1(p,\tau))\begin{pmatrix}1\\ \mathcal{O}_1(p,\tau)\\ \widetilde{S}_{i,\tau}(z)\nu_1^i(z)^{-1}+\mathcal{O}_1(p,\tau)\end{pmatrix},
\end{aligned}$$

where $\widetilde{S}_{i,p}(z)=\partial_p\pi_z\Psi_{\mathrm{glob},i}(0,0,z)$ and $\widetilde{S}_{i,\tau}(z)=\partial_\tau\pi_z\Psi_{\mathrm{glob},i}(0,0,z)$. Hence, using that $(q^*,\sigma^*,z^*)=\widetilde{\Psi}_{\mathrm{glob},i}(\omega)$ satisfies $q^*=\tau\nu_1^i(z)(1+\mathcal{O}_1(p,\tau))$, by Item 2 of Theorem 5.5.4, we have that, using (5.61) and (5.60)

$$\begin{aligned}
D\widetilde{\Psi}_{i,j}(\omega)v_1&=DA_j(\Psi_{i,j}(A_i^{-1}(\omega)))D\Psi_{\mathrm{loc},i,j}(B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))DB_i^{-1}(\widetilde{\Psi}_{\mathrm{glob},i}(\omega))D\widetilde{\Psi}_{\mathrm{glob},i}(\omega)v_1\\
&=(\nu_1^i(z)+\mathcal{O}_1(p,\tau))DA_j(\Psi_{i,j}(A_i^{-1}(\omega)))D\Psi_{\mathrm{loc},i,j}((B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))\begin{pmatrix}1\\ \mathcal{O}_0(p,\tau)\\ \mathcal{O}_0(p,\tau)\end{pmatrix}\\
&=\lambda_1(\omega)(\nu_1^i(z)+\mathcal{O}_1(p,\tau))\begin{pmatrix}P^*_{i,j}\\1\\Z^*_{i,j}\end{pmatrix},
\end{aligned}$$

where, for some $C>0$,

$$\lambda_1(\omega)\gtrsim\tau^{-\frac{3}{5}+Ca},\qquad |P^*_{i,j}|\le\mathcal{O}\left(\tau^{1-Ca}\right),\qquad \|Z^*_{i,j}\|\le\mathcal{O}\left(\tau^{\frac{3}{5}-Ca}\right).$$

This proves the claim for v_1, taking $\lambda_1^i(\omega)=\lambda_1(\omega)(\nu_1^i(z)+\mathcal{O}_1(p,\tau))$. The proof of the second statement of Item 1 follows exactly the same lines.

We now prove Item 2. Let $v^i_{2,z}(z)=\widehat{S}_{i,\sigma}(\mathtt{S}_i(z))$, where $\widehat{S}_{i,\sigma}(z)=\partial_\sigma\pi_z\widetilde{\Psi}^{-1}_{\mathrm{glob},i}(0,0,z)$. We observe that, since $\widetilde{\Psi}_{i,j}=\widetilde{\Psi}_{\mathrm{loc},i,j}\circ\widetilde{\Psi}_{\mathrm{glob},i}$,

$$\begin{aligned}
\left(D\widetilde{\Psi}_{i,j}(\omega)\right)^{-1}&=D(\widetilde{\Psi}^{-1}_{i,j})(\widetilde{\Psi}_{i,j}(\omega))=D(\widetilde{\Psi}^{-1}_{\mathrm{glob},i})(\widetilde{\Psi}^{-1}_{\mathrm{loc},i,j}\circ\widetilde{\Psi}_{i,j}(\omega))D(\widetilde{\Psi}^{-1}_{\mathrm{loc},i,j})(\widetilde{\Psi}_{i,j}(\omega))\\
&=D(\widetilde{\Psi}^{-1}_{\mathrm{glob},i})(\widetilde{\Psi}_{\mathrm{glob},i}(\omega))D(\widetilde{\Psi}^{-1}_{\mathrm{loc},i,j})(\widetilde{\Psi}_{i,j}(\omega)).
\end{aligned}$$

First of all, we notice that, denoting $\widetilde{\Psi}_{i,j}(\omega)=(\hat{p},\hat{\tau},\hat{z})$, by (5.62) and Item 1 of Theorem 5.5.4,

$$(\tau\nu_1^i(z))^{1+Ca}(1+\mathcal{O}_1(p,\tau))\le\hat{p}\le(\tau\nu_1^i(z))^{1-Ca}(1+\mathcal{O}_1(p,\tau)).\tag{5.233}$$

Then, since

$$\widetilde{\Psi}^{-1}_{\mathrm{loc},i,j}=B_i\circ\Psi^{-1}_{\mathrm{loc},i,j}\circ A_j^{-1},$$

applying Item 2 of Theorem 5.5.4 to $(\Psi_{\mathrm{loc},i,j})^{-1}$ (that is, applying Item 2 of Theorem 5.5.4, changing the sign of the vector field), evaluated at $A_j^{-1}\circ\widetilde{\Psi}_{i,j}(\omega)$, and to the vector $v_2^{i,j}$ in (5.230),

$$D(\widetilde{\Psi}^{-1}_{\mathrm{loc},i,j})(\widetilde{\Psi}_{i,j}(\omega))v_2^{i,j}(\widetilde{\Psi}_{i,j}(\omega))=\tilde{\lambda}_2^{i,j}(\omega)\begin{pmatrix}\hat{Q}^*_{i,j}\\1\\\hat{Z}^*_{i,j}\end{pmatrix},$$

where, for some $\tilde{C}>0$,

$$\tilde{\lambda}_2^{i,j}(\omega)\gtrsim\tau^{-\frac{3}{5}+\tilde{C}a},\qquad |\hat{Q}^*_{i,j}|\le\mathcal{O}\left(\tau^{1-\tilde{C}a}\right),\qquad \|\hat{Z}^*_{i,j}\|\le\mathcal{O}\left(\tau^{\frac{3}{5}-Ca}\right).$$

Hence, by (5.63),

$$
\begin{aligned}
D\widetilde{\Psi}_{i,j}(\omega)^{-1}v_2^j(\widetilde{\Psi}_{i,j}(\omega)) &= D(\widetilde{\Psi}_{\mathrm{glob},i}^{-1})(\widetilde{\Psi}_{\mathrm{glob},i}(\omega))D(\widetilde{\Psi}_{\mathrm{loc},i,j}^{-1})(\widetilde{\Psi}_{i,j}(\omega))v_2^j(\widetilde{\Psi}_{i,j}(\omega)) \\
&= \tilde{\lambda}_2^{i,j}(\omega)D(\widetilde{\Psi}_{\mathrm{glob},i}^{-1})(\widetilde{\Psi}_{\mathrm{glob},i}(\omega))\begin{pmatrix}\hat{Q}_{i,j}^*\\ 1\\ \hat{Z}_{i,j}^*\end{pmatrix} \\
&= \tilde{\lambda}_2^{i,j}(\omega)\begin{pmatrix}\mathcal{O}_1(\tau) & \mu_1^i(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau) & \mathcal{O}_1(\tau)\\ \mu_2^i(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau) & \mathcal{O}_1(p) & \mathcal{O}_1(p)\\ \widehat{S}_{i,q}(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau) & \widehat{S}_{i,\sigma}(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau) & D(\widehat{\mathtt{S}}_i^{-1})(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau)\end{pmatrix}\begin{pmatrix}\hat{Q}_{i,j}^*\\ 1\\ \hat{Z}_{i,j}^*\end{pmatrix} \\
&= \tilde{\lambda}_2^{i,j}(\omega)(\mu_1^i(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau))\begin{pmatrix}1\\ \bar{T}_{i,j}^*\\ \widehat{S}_{i,\sigma}(\widehat{\mathtt{S}}_i(z))+\bar{Z}_{i,j}^*\end{pmatrix},
\end{aligned}
$$

where $\widehat{S}_{i,q}(z)=\partial_q\pi_z\widetilde{\Psi}_{\mathrm{glob},i}^{-1}(0,0,z)$ and $\widehat{S}_{i,\sigma}(z)=\partial_\sigma\pi_z\widetilde{\Psi}_{\mathrm{glob},i}^{-1}(0,0,z)$ and

$$
|\bar{T}_{i,j}^*|\le\mathcal{O}_1(p)+\mathcal{O}\left(\tau^{\frac{3}{5}-\tilde{C}a}\right),\ \|\bar{Z}_{i,j}^*\|\le\mathcal{O}\left(\tau^{\frac{3}{5}-\tilde{C}a}\right).
$$

The claim follows taking $(\lambda_2^{i,j})^{-1}(\omega)=\tilde{\lambda}_2^{i,j}(\omega)(\mu_1^i(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau))$.

Finally, we prove Item 3. In order to find the vector fields, we look for $v^{i,j}=v_0+\tilde{v}^{i,j}(p,\tau,z)$, with $v_0=(0,0,v_z)^\top$ and $\tilde{v}^{i,j}=(0,\tilde{v}_{\tau,1}^{i,j}+\tilde{v}_{\tau,2}^{i,j},0)^\top$. Note that both corrections appear in the τ-direction. We write down them separated since they will play different roles. Roughly speaking $\tilde{v}_{\tau,1}^{i,j}$ will be obtained by applying Item 4 in Theorem 5.5.4 whereas $\tilde{v}_{\tau,2}^{i,j}$ is obtained by applying Item 2 in the same theorem.

We have that

$$
D\widetilde{\Psi}_{\mathrm{glob},i}(\omega)v^{i,j}(\omega)=\begin{pmatrix}\nu_1^i(z)\tilde{v}_{\tau,1}^{i,j}+\mathcal{O}_1(p,\tau)v^{i,j}\\ \mathcal{O}_1(p,\tau)v^{i,j}\\ D\widehat{\mathtt{S}}_i(z)v_z+\widetilde{S}_{i,\tau}(z)\tilde{v}_{\tau,1}^{i,j}+\mathcal{O}_1(p,\tau)v^{i,j}\end{pmatrix}+\begin{pmatrix}\nu_1^i(z)\tilde{v}_{\tau,2}^{i,j}\\ 0\\ \widetilde{S}_{i,\tau}(z)\tilde{v}_{\tau,2}^{i,j}\end{pmatrix}.
$$

Hence, by (5.61),

$$
DB_i^{-1}(\widetilde{\Psi}_{\mathrm{glob},i}(\omega))D\widetilde{\Psi}_{\mathrm{glob},i}(\omega)v^{i,j}(\omega)=w_1(\omega)+w_2(\omega) \tag{5.234}
$$

with

$$
w_1(\omega)=\begin{pmatrix}\nu_1^i(z)\tilde{v}_{\tau,1}^{i,j}+\mathcal{O}_1(p,\tau)v^{i,j}\\ -\beta_{i,2}(\widehat{\mathtt{S}}_i(z))DS_i(z)v_z-\left[\beta_{i,1}(\widehat{\mathtt{S}}_i(z))\nu_1^i(z)+\beta_{i,2}(\widehat{\mathtt{S}}_i(z))\widetilde{S}_{i,\tau}(z)\right]\tilde{v}_{\tau,1}^{i,j}+\mathcal{O}_1(p,\tau)v^{i,j}\\ D\widehat{\mathtt{S}}_i(z)v_z+\widetilde{S}_{i,\tau}(z)\tilde{v}_{\tau,1}^{i,j}+\mathcal{O}_1(p,\tau)v^{i,j}\end{pmatrix} \tag{5.235}
$$

$$
w_2(\omega)=\begin{pmatrix}\nu_1^i(z)\tilde{v}_{\tau,2}^{i,j}\\ -\left[\beta_{i,1}(\widehat{\mathtt{S}}_i(z))\nu_1^i(z)+\beta_{i,2}(\widehat{\mathtt{S}}_i(z))\widetilde{S}_{i,\tau}(z)\right]\tilde{v}_{\tau,2}^{i,j}\\ \widetilde{S}_{i,\tau}(z)\tilde{v}_{\tau,2}^{i,j}\end{pmatrix} \tag{5.236}
$$

where the functions $\beta_{i,1}(z)=-\frac{\partial\widetilde{w}_i^u}{\partial q}(0,0,z)$ and $\beta_{i,2}=-(t_i^u)'(z)-\frac{\partial\widetilde{w}_i^u}{\partial z}(0,0,z)$ (see (5.61)).

Since $\pi_q(\widetilde{\Psi}_{\mathrm{glob},i}(\omega))=\nu_1^i(z)\tau(1+\mathcal{O}_1(p,\tau))$, (see (5.62)), by Item 4 of Lemma 5.11.5 with $\widetilde{P}_0=0$, there exists a linear map $\widetilde{Q}(B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))$, with $\|\widetilde{Q}(B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))\|\le\mathcal{O}(\tau^{1/5})$, such that if w_1 satisfies

$$
\pi_q w_1(\omega)=\widetilde{Q}(B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))\left[\pi_{t,z}w_1(\omega)\right] \tag{5.237}
$$

then

$$
\begin{gathered}
\pi_t D\Psi_{\mathrm{loc},i,j}(B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))w_1(\omega)=\pi_t w_1(\omega),\\
|\pi_p D\Psi_{\mathrm{loc},i,j}(B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))w_1(\omega)|\le\mathcal{O}(\tau^{3/5-Ca})\|\pi_{t,z}w_1(\omega)\|,\\
\|\pi_z D\Psi_{\mathrm{loc},i,j}(B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))w_1(\omega)-\pi_z w_1(\omega)\|\le\mathcal{O}(\tau)\|w_1(\omega)\|.
\end{gathered}
$$

We use this fact to choose a suitable w_1 (by choosing a suitable $\tilde{v}^{i,j}_{\tau,1}$).

Moreover, since ν^i_1 does not vanish, from Item 2 of Theorem 5.5.4,

$$D\Psi_{\mathrm{loc},i,j}(B_i^{-1}\circ\widetilde{\Psi}_{\mathrm{glob},i}(\omega))w_2(\omega)=\lambda(\omega)\nu^i_1(z)\tilde{v}^{i,j}_{\tau,2}(\omega)\begin{pmatrix}P^*\\1\\Z^*\end{pmatrix}$$

with

$$\lambda(\omega)\gtrsim\tau^{-(3/5-Ca)},\qquad |P^*|\le\mathcal{O}(\tau^{1-Ca}),\qquad |Z^*|\le\mathcal{O}(\tau^{3/5-Ca}).$$

Then, if we assume for a moment that (5.237) is satisfied, cumbersome but straightforward computations lead to

$$D\widetilde{\Psi}_{i,j}(\omega)v^{i,j}=\begin{pmatrix}\lambda(\tau)\nu^i_1(z)\tilde{v}^{i,j}_{\tau,2}(\omega)P^*+\mathcal{O}(\tau^{3/5-Ca})v^{i,j}\\ \widetilde{T}_0(\omega)+\mathcal{O}_1(p,\tau)v^{i,j}+\lambda(\omega)\widetilde{T}_1(\omega)\tilde{v}^{i,j}_{\tau,2}(\omega)\\ D\widehat{\mathtt{S}}_i(z)v_z+\widetilde{S}_{i,p}(z)\tilde{v}^{i,j}_{\tau,1}+\mathcal{O}_1(p,\tau)v^{i,j}+\lambda(\omega)\nu^i_1(z)\tilde{v}^{i,j}_{\tau,2}(\omega)Z^*\end{pmatrix}\tag{5.238}$$

where

$$\begin{aligned}\widetilde{T}_0(\omega)=&\left[\alpha_{i,2}(\widehat{\mathtt{S}}_i(z))-\beta_{i,2}(\widehat{\mathtt{S}}_i(z))\right]D\widehat{\mathtt{S}}_i(z)v_z-((\beta_{i,1}(\widehat{\mathtt{S}}_i(z))-\alpha_{i,1}(\widehat{\mathtt{S}}_i(z)))\nu^i_1(z)\\&+(\beta_{i,2}(\widehat{\mathtt{S}}_i(z))-\alpha_{i,2}(\widehat{\mathtt{S}}_i(z)))\widetilde{S}_{i,p}(z))\tilde{v}^{i,j}_{\tau,1}\\ \widetilde{T}_1(\omega)=&\nu^i_1(z)\left[1+(\alpha_{i,1}(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau))P^*+(\alpha_{i,2}(\widehat{\mathtt{S}}_i(z))+\mathcal{O}_1(p,\tau))Z^*\right]\end{aligned}$$

where $\alpha_{i,1}=-\partial_p\tilde{w}^s_i(p,z)$ and $\alpha_{i,2}=-\partial_z t^s_i(z)-\partial_z\tilde{w}^s_i(p,z)$. Note that for small (p,τ), $\widetilde{T}_1(\omega)\gtrsim 1$.

We choose

$$\tilde{v}^{i,j}_{\tau,2}(\omega)=-\frac{1}{\lambda(\tau)\widetilde{T}_1(\omega)}\widetilde{T}_0(\omega).\tag{5.239}$$

Observe that this choice of $\tilde{v}^{i,j}_{\tau,2}$ is linear in $v^{i,j}_{\tau,1}$ and satisfies

$$\tilde{v}^{i,j}_{\tau,2}(\omega)=\mathcal{O}(\tau^{3/5-Ca})+\mathcal{O}(\tau^{3/5-Ca})\tilde{v}^{i,j}_{\tau,1}(\omega).$$

Inserting this choice of $\tilde{v}^{i,j}_{\tau,2}$ in (5.237), we obtain the fixed point equation for $\tilde{v}^{i,j}_{\tau,1}$

$$\tilde{v}^{i,j}_{\tau,1}=\frac{1}{\nu^i_1(z)}\left(\widetilde{Q}(B_i^{-1}\circ\Psi_{\mathrm{glob},i}(\omega))\left(\begin{pmatrix}\widetilde{T}(\omega)\\ D\widehat{\mathtt{S}}_i(z)v_z+\widetilde{S}_{i,p}(z)\tilde{v}^i_\tau\end{pmatrix}+\mathcal{O}_1(p,\tau)\tilde{v}^{i,j}\right)\right).$$

It clearly has a solution $\tilde{v}^{i,j}_{\tau,1}=\mathcal{O}(\tau^{1/5})\|v_z\|$. Then, taking into account (5.238) and (5.239), we have that

$$\pi_p D\widetilde{\Psi}_{i,j}(\omega)v^{i,j}=\lambda(\tau)\nu^i_1(z)\tilde{v}^{i,j}_{\tau,2}(\omega)P^*+\mathcal{O}(\tau^{3/5-Ca})v^{i,j}=\mathcal{O}(\tau^{3/5-Ca})v_z$$

and

$$\pi_\tau D\widetilde{\Psi}_{i,j}(\omega)v^{i,j}=\mathcal{O}_1(p,\tau)v^{i,j}=\mathcal{O}_1(p,\tau)v_z.$$

This completes the proof of Item 3. □

5.12.2 Horizontal and vertical strips: Proof of Proposition 5.6.4

Proposition 5.6.4 is direct consequence of the following lemma.

Lemma 5.12.2. *For any* horizontal surface*,*

$$S_h=\{(p,\tau,\varphi,J)\in\mathcal{Q}^2_\delta\mid (p,J)=(h_1(\tau,\varphi),h_2(\tau,\varphi)),\ (\tau,\varphi)\in(0,\delta)\times(0,\tilde{\kappa})\}\subset\Sigma_2,$$

with h a $\mathcal{C}^1$ function satisfying

$$\sup_{(\tau,\varphi)\in(0,\delta)\times(0,\tilde{\kappa})}\|\partial_\tau h_1(\tau,\varphi)\|<\mathcal{O}(1),\qquad\sup_{(\tau,\varphi)\in(0,\delta)\times(0,\tilde{\kappa})}\|\partial_\varphi h_1(\tau,\varphi)\|\quad<\mathcal{O}(\delta),\tag{5.240}$$

$\widetilde{\Psi}(S_h)\cap\mathcal{Q}^2_\delta$ *has an infinite number of connected components. Moreover,* $\widetilde{\Psi}(S_h)\cap\mathcal{Q}^2_\delta$ *contains a countable union of horizontal surfaces,* S_{h_n}*, with* $\lim_{n\to\infty}h_{n,1}=0$*, in the* $\mathcal{C}^0$ *topology and*

$$\begin{aligned}|\partial_\tau h_{n,1}|&\lesssim\mathcal{O}(\delta), & |\partial_\varphi h_{n,1}|&\lesssim\mathcal{O}(\tilde\kappa\delta),\\ |\partial_\tau h_{n,2}|&\lesssim\mathcal{O}(\delta)+\mathcal{O}(\tilde\kappa), & |\partial_\varphi h_{n,2}|&\lesssim\mathcal{O}(\tilde\kappa).\end{aligned}$$

Analogously, for any vertical surface,

$$S_v=\{(p,\tau,\varphi,J)\in\mathcal{Q}^2_\delta\mid(\tau,\varphi)=(v_1(p,J),v_2(p,J)),\ (p,J)\in(0,\delta)\times(0,\tilde\kappa)\}\subset\Sigma_2,$$

with v *a* $\mathcal{C}^1$ *function satisfying*

$$\sup_{(p,J)\in(0,\delta)\times(0,\tilde\kappa)}\|\partial_p v_1(\tau,\varphi)\|<\mathcal{O}(1),\quad\sup_{(p,J)\in(0,\delta)\times(0,\tilde\kappa)}\|\partial_J v_1(\tau,\varphi)\|\quad<\mathcal{O}(\delta),\tag{5.241}$$

$\widetilde{\Psi}^{-1}(S_v)\cap\mathcal{Q}^2_\delta$ *has an infinite number of connected components. Moreover,* $\widetilde{\Psi}^{-1}(S_v)\cap\mathcal{Q}^2_\delta$ *contains a countable union of vertical surfaces,* S_{v_n}*, with* $\lim_{n\to\infty}v_n=0$*, in the* $\mathcal{C}^0$ *topology and*

$$\begin{aligned}|\partial_p v_{n,1}|&\lesssim\mathcal{O}(\delta), & |\partial_J h_{n,1}|&\lesssim\mathcal{O}(\tilde\kappa\delta),\\ |\partial_p h_{n,2}|&\lesssim\mathcal{O}(1), & |\partial_J h_{n,2}|&\lesssim\mathcal{O}(\tilde\kappa).\end{aligned}$$

In particular, $\operatorname{Lip}h_n\lesssim\mathcal{O}(\delta)+\mathcal{O}(\tilde\kappa)$ *and* $\operatorname{Lip}v_n\lesssim\mathcal{O}(1)$*, uniformly in* n*.*

Proof. In this proof we denote $z=(z_1,z_2)=(\varphi,J)$. Let $h:(0,\delta)\times(0,\tilde\kappa)\to(0,\delta)\times(0,\tilde\kappa)$ be a function satisfying (5.240). Let

$$\Lambda_0(\tau,z_1)=(h_1(\tau,z_1),\tau,z_1,h_2(\tau,z_1))^\top\qquad\text{and}\qquad\widetilde{\Lambda}_1(\tau,z_1)=\widetilde{\Psi}_{2,1}\circ\Lambda_0(\tau,z_1)=(\tilde h_1,\widetilde{T},\widetilde{Z})^\top(\tau,z_1).\tag{5.242}$$

By Item 1 of Theorem 5.5.4, the definition of $\widetilde{\Psi}_{2,1}$ in (5.64) and the expression of $\widetilde{\Psi}_{\mathrm{glob},2}$ in (5.62), we have that

$$\widetilde{T}(\tau,z_1)\gtrsim\tau^{-3/2+Ca},\tag{5.243}$$

for all $z_1\in(0,\tilde\kappa)$. Hence, for any $n\in\mathbb{N}$, sufficiently large, there exist $\tau^-_{1,n}<\tau^+_{1,n}$ such that $\widetilde{T}(\tau^+_{1,n},z_1)\le n<n+\delta\le\widetilde{T}(\tau^-_{1,n},z_1)$, for all $z_1\in(0,\tilde\kappa)$ and, moreover, $\tau^\pm_{1,n}\to0$ as $n\to+\infty$.

By Item 1 of Lemma 5.12.1,

$$\partial_\tau\widetilde{\Lambda}_1(\tau,z_1)=D\widetilde{\Psi}_{2,1}\circ\Lambda_0(\tau,z_1)\begin{pmatrix}\partial_\tau h_1(\tau,z_1)\\1\\0\\\partial_\tau h_2(\tau,z_1)\end{pmatrix}=\begin{pmatrix}\lambda(\tau,z_1)\varepsilon_1(\tau,z_1)\\\lambda(\tau,z_1)\\Z_1(\tau,z_1)\\Z_2(\tau,z_1)\end{pmatrix}\tag{5.244}$$

where,

$$|\lambda(\tau,z_1)|\gtrsim C\tau^{-3/5+Ca},\qquad|\varepsilon_1(\tau,z_1)|\lesssim\tau^{1-Ca},\qquad\|(Z_1(\tau,z_1),Z_2(\tau,z_1))\|\lesssim\mathcal{O}(1).\tag{5.245}$$

In particular, $|\partial_\tau\widetilde{T}(\tau,z_1)|=|\lambda(\tau,z_1)|>C\tau^{-3/5+Ca}$. Hence, the equation

$$\widetilde{T}(\tau,z_1)=T$$

defines a function $\hat\tau_1(T,z_1)$, for $T>0$ large enough and $z_1\in(0,\tilde\kappa)$, such that $\tau^-_{1,n}\le\hat\tau_1(T,z_1)\le\tau^+_{1,n}$ for $T\in[n,n+\delta]$, with $\partial_T\hat\tau_1(T(\tau,z_1),z_1)=\lambda^{-1}(\hat\tau_1(T,z_1),z_1)$. Taking $\tau^+_{1,n}$ small enough by taking n large enough, we can assume that $\hat\tau_1(T,z_1)^{1/5-Ca}\lesssim\mathcal{O}(\delta)$ for $T\in[n,n+\delta]$. Moreover $\sup_{T\in[n,n+\delta]}\hat\tau_1(T,z_1)\to0$ as $n\to+\infty$.

We define

$$\Lambda_1(T,z_1)=\widetilde{\Lambda}_1(\hat\tau_1(T,z_1),z_1).$$

By construction, $\pi_\tau \Lambda_1(T, z_1) = T$ and, by Item 1 of Theorem 5.5.4, $\sup_{T\in[n,n+\delta]} \pi_p \Lambda_1(T, z_1) \to 0$ as $n \to +\infty$. In view of (5.244), it satisfies

$$\partial_T \Lambda_1(T, z_1) = \partial_\tau \widetilde{\Lambda}_1(\hat{\tau}_1(T, z_1), z_1)\partial_T \hat{\tau}_1(T, z_1) = \begin{pmatrix} \mathcal{O}(\delta) \\ 1 \\ \mathcal{O}(\delta) \\ \mathcal{O}(\delta) \end{pmatrix}.$$

Moreover, by (5.62) and Item 1 of Theorem 5.5.4, the third component of Λ_1, satisfies

$$\begin{aligned} &|\pi_{z_1} \Lambda_1(T, z_1) - \pi_{z_1} \widehat{\mathtt{S}}^2(z_1, h_2(\hat{\tau}_1(T, z_1), z_1))| \\ &\qquad = |\pi_{z_1} \widetilde{\Psi}_{2,1} \circ \Lambda_0(\hat{\tau}_1(T, z_1), z_1) - \pi_{z_1} \widehat{\mathtt{S}}^2(z_1, h_2(\hat{\tau}_1(T, z_1), z_1))| \lesssim \mathcal{O}(\delta). \end{aligned} \tag{5.246}$$

Now we compute the derivatives of Λ_1 with respect to z_1. We have that

$$\begin{aligned} \partial_{z_1} \Lambda_1(T, z_1) &= \partial_{z_1} \left[\widetilde{\Lambda}_1(\hat{\tau}_1(T, z_1), z_1)\right] \\ &= D\widetilde{\Psi}_{2,1} \circ \Lambda_0(\hat{\tau}_1(T, z_1), z_1) \left[\partial_\tau \Lambda_0(\hat{\tau}_1(T, z_1), z_1)\partial_{z_1} \hat{\tau}_1(T, z_1) + \partial_{z_1} \Lambda_0(\hat{\tau}_1(T, z_1), z_1)\right]. \end{aligned}$$

We write the vector $\partial_{z_1} \Lambda_0(\hat{\tau}_1(T, z_1), z_1)$ as

$$(\partial_{z_1} \Lambda_0)(\hat{\tau}_1(T, z_1), z_1) = \begin{pmatrix} \partial_{z_1} h_1 \\ 0 \\ 1 \\ \partial_{z_1} h_2 \end{pmatrix} = \partial_{z_1} h_1 \begin{pmatrix} 1 \\ b \\ c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0 \\ d \\ 1 - \partial_{z_1} h_1 c_1 \\ \partial_{z_1} h_2 - \partial_{z_1} h_1 c_2 \end{pmatrix} - (d + \partial_{z_1} h_1 b) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{5.247}$$

where

- b, $c = (c_1, c_2)$ are the functions given by Item 2 of Lemma 5.12.1 such that

  $$|b(\tau, z_1)| \lesssim \mathcal{O}(\delta, \hat{\tau}_1^{3/5-Ca}), \qquad \|(c_1, c_2)(\tau, z_1)\| \lesssim \mathcal{O}(1)$$

  and

  $$D\widetilde{\Psi}_{2,1} \circ \Lambda_0(\hat{\tau}_1(T, z_1), z_1)(1, b, c)^\top = \mu(\hat{\tau}_1, z_1)(1, b^*, c^*)^\top$$

  with

  $$|\mu(\hat{\tau}_1, z_1)| \lesssim \mathcal{O}(\hat{\tau}_1^{3/5-Ca}), \qquad |b^*(\hat{\tau}_1, z_1)| \lesssim \mathcal{O}(\delta), \qquad \|(c_1^*, c_2^*)(\tau, z_1)\| \lesssim \mathcal{O}(1)$$

- $d = \mathcal{O}(\hat{\tau}_1^{1/5-Ca})$ is given by Item 3 of Lemma 5.12.1 satisfies

  $$D\widetilde{\Psi}_{2,1} \circ \Lambda_0(\hat{\tau}_1(T, z_1), z_1) \begin{pmatrix} 0 \\ d \\ 1 - \partial_{z_1} h_1 c_1 \\ \partial_{z_1} h_2 - \partial_{z_1} h_1 c_2 \end{pmatrix} = \begin{pmatrix} d_1^* \\ d_2^* \\ D\widehat{\mathtt{S}}^2(z_1, h_2(\hat{\tau}_1, z_1)) \begin{pmatrix} 1 - \partial_{z_1} h_1 c_1 \\ \partial_{z_1} h_2 - \partial_{z_1} h_1 c_2 \end{pmatrix} + d_3^* \end{pmatrix}$$

  with $\|(d_1^*, d_2^*, d_3^*)\| = \mathcal{O}(\hat{\tau}_1^{1/5-Ca})$.

Then, applying Item 1 of Lemma 5.12.1 to $(d+\partial_{z_1} h_1 b)(0, 1, 0, 0)^\top$, using formula (5.247) and the previous bounds for μ, b^* and d_2^*, we obtain

$$\pi_T D\widetilde{\Psi}_{2,1} \circ \Lambda_0(\hat{\tau}_1(T, z_1), z_1)\partial_{z_1} \Lambda_0(\hat{\tau}_1(T, z_1), z_1) = \lambda(\hat{\tau}_1, z_1)(d + \partial_{z_1} h_1 b) + \mathcal{O}(\hat{\tau}_1^{1/5-Ca}). \tag{5.248}$$

By using the second part of Item 1 of Lemma 5.12.1 (see (5.229)),

$$\pi_T D\widetilde{\Psi}_{2,1} \circ \Lambda_0(\hat{\tau}_1(T, z_1), z_1)\partial_\tau \Lambda_0(\hat{\tau}_1(T, z_1), z_1)\partial_{z_1} \hat{\tau}_1(T, z_1) = \lambda(\hat{\tau}_1, z_1) \left(1 + \mathcal{O}(\hat{\tau}_1^{3/5-Ca})\right) \partial_{z_1} \hat{\tau}_1(T, z_1). \tag{5.249}$$

Now, since $\pi_T \Lambda_1(T, z_1) = T$, we have that $\pi_T \partial_{z_1} \Lambda_1(T, z_1) = 0$. Then, combining (5.248) and (5.249) we obtain that

$$\partial_{z_1} \hat{\tau}_1(T, z_1) = \mathcal{O}(\delta).$$

This bound is not good enough. In order to improve it, we introduce $A = \partial_\tau h_1 \partial_{z_1} \hat{\tau}_1 + \partial_{z_1} h_1 = \mathcal{O}(\delta)$ and rewrite $\partial_{z_1} [\Lambda_0(\hat{\tau}_1(T, z_1), z_1)]$ as

$$\partial_{z_1} [\Lambda_0(\hat{\tau}_1(T, z_1), z_1)] = A \begin{pmatrix} 1 \\ \tilde{b} \\ \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{d} \\ 1 - A\tilde{c}_1 \\ \partial_\tau h_2 \partial_{z_1} \hat{\tau}_1 + \partial_{z_1} h_2 - A\tilde{c}_2 \end{pmatrix} - (\tilde{d} + A\tilde{b} - \partial_{z_1} \hat{\tau}_1) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

where, again, $\tilde{b} = \mathcal{O}(\delta)$, $\tilde{c} = (\tilde{c}_1, \tilde{c}_2) = \mathcal{O}(1)$ are the functions given by Item 2 of Lemma 5.12.1 and $\tilde{d} = \mathcal{O}(\hat{\tau}_1^{1/5-Ca})$ is given by Item 3 of Lemma 5.12.1. Then,

$$\begin{aligned} &D\widetilde{\Psi}_{2,1} \circ \Lambda_0(\hat{\tau}_1(T, z_1), z_1) \partial_{z_1} [\Lambda_0(\hat{\tau}_1(T, z_1), z_1)] \\ &= \mu A \begin{pmatrix} 1 \\ \mathcal{O}(\delta) \\ \mathcal{O}(1) \\ \mathcal{O}(1) \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\delta) \\ \mathcal{O}(\delta) \\ D\widehat{\mathtt{S}}^2(z_1, h_2(\hat{\tau}_1, z_1)) \begin{pmatrix} 1 - A\tilde{c}_1 \\ \partial_\tau h_2 \partial_{z_1} \hat{\tau}_1 + \partial_{z_1} h_2 - A\tilde{c}_2 \end{pmatrix} + \mathcal{O}(\delta) \end{pmatrix} - (\tilde{a} + A\tilde{b} - \partial_{z_1} \hat{\tau}_1) \begin{pmatrix} \lambda \mathcal{O}(\delta) \\ \lambda \\ \mathcal{O}(1) \\ \mathcal{O}(1) \end{pmatrix}, \end{aligned}$$

where $|\mu(\hat{\tau}_1, z_1)| \lesssim \mathcal{O}(\hat{\tau}_1^{3/5-Ca}) = \mathcal{O}(\delta)$. Hence, using again that $\pi_T \Lambda_1(T, z_1) = T$, we obtain that $\tilde{a} + A\tilde{b} - \partial_{z_1} \hat{\tau}_1 = \mathcal{O}(\delta)/\lambda$. This implies that

$$\pi_p \partial_{z_1} \Lambda_1(T, z_1) = \mu A + \mathcal{O}(\delta) - (\tilde{a} + A\tilde{b} - \partial_{z_1} \hat{\tau}_1) \lambda \mathcal{O}(\delta) = \mathcal{O}(\delta).$$

Summarizing,

$$\partial_T \Lambda_1(T, z_1) = \begin{pmatrix} \mathcal{O}(\delta) \\ 1 \\ \mathcal{O}(\delta) \\ \mathcal{O}(\delta) \end{pmatrix}, \quad \partial_{z_1} \Lambda_1(T, z_1) = \begin{pmatrix} \mathcal{O}(\delta) \\ \mathcal{O}(\delta) \\ D\widehat{\mathtt{S}}^2(z_1, h_2(\hat{\tau}_1, z_1)) \begin{pmatrix} 1 \\ \partial_{z_1} h_2 \end{pmatrix} + \mathcal{O}(\delta) \end{pmatrix}.$$

Now we proceed by induction, defining $\widetilde{\Lambda}_j = \widetilde{\Psi}_{1,1} \circ \Lambda_{j-1}$, for $2 \le j \le M$. With the same argument, $\pi_\tau \widetilde{\Lambda}_j(\tau, z_1) = T$ defines a function $\hat{\tau}_j(T, z_1)$, with T large enough, such that $\Lambda_j(T, z_1) = \widetilde{\Lambda}_j(\hat{\tau}_j(T, z_1), z_1)$ satisfies

$$\partial_T \Lambda_j(T, z_1) = \begin{pmatrix} \mathcal{O}(\delta) \\ 1 \\ \mathcal{O}(\delta) \\ \mathcal{O}(\delta) \end{pmatrix}, \quad \partial_{z_1} \Lambda_j(T, z_1) = \begin{pmatrix} \mathcal{O}(\delta) \\ \mathcal{O}(\delta) \\ D((\widehat{\mathtt{S}}^1)^{j-1} \circ \widehat{\mathtt{S}}^2)(z_1, h_2(\hat{\tau}_j, z_1)) \begin{pmatrix} 1 \\ \partial_{z_1} h_2 \end{pmatrix} + \mathcal{O}(\delta) \end{pmatrix}$$

and

$$|\pi_{z_1} \Lambda_j(T, z_1) - \pi_{z_1} (\widehat{\mathtt{S}}^1)^{j-1} \circ \widehat{\mathtt{S}}^2(z_1, h_2(\hat{\tau}_j, z_1))| \lesssim \mathcal{O}(\delta). \tag{5.250}$$

Of course, the $\mathcal{O}(\delta)$ terms depend on j.

In the last step, we define $\widetilde{\Lambda}_{M+1}(\tau, z_1) = \widetilde{\Psi}_{1,2} \circ \Lambda_M(\tau, z_1)$, defined for $\tau \in (0, \delta)$. With the same argument to obtain (5.243), we have that

$$\widetilde{T}_{M+1}(\tau, z_1) = \pi_T \widetilde{\Lambda}_{M+1}(\tau, z_1) \ge C\tau^{-3/2+Ca}$$

and, from (5.244) and (5.245),

$$|\partial_\tau \widetilde{T}_{M+1}(\tau, z_1)| = |\lambda(\tau, z_1)| > C\tau^{-3/5+Ca}.$$

Hence, the equation

$$\widetilde{T}_{M+1}(\tau, z_1) = T$$

defines a function $\hat{\tau}_{M+1}(T, z_1)$, for T large enough and $z_1 \in (0, \tilde{\kappa})$, strictly decreasing in τ with $\lim_{T\to\infty} \hat{\tau}_{M+1}(T, z_1) = 0$, uniformly in z_1, with $\partial_T \hat{\tau}_{M+1}(T(\tau, z_1), z_1) = \lambda(\hat{\tau}_{M+1}(T, z_1), z_1)^{-1}$. With the previous arguments, $\Lambda_{M+1}(T, z_1) = \widetilde{\Lambda}_{M+1}(\hat{\tau}_{M+1}(T, z_1), z_1)$

$$|\pi_{z_1}\Lambda_{M+1}(T, z_1) - \pi_{z_1}\widehat{\mathtt{S}}(z_1, h_2(\hat{\tau}_{M+1}, z_1))| \lesssim \mathcal{O}(\delta), \tag{5.251}$$

and

$$\partial_T \Lambda_{M+1}(T, z_1) = \begin{pmatrix} \mathcal{O}(\delta) \\ 1 \\ \mathcal{O}(\delta) \\ \mathcal{O}(\delta) \end{pmatrix}, \ \partial_{z_1}\Lambda_{M+1}(T, z_1) = \begin{pmatrix} \mathcal{O}(\delta) \\ \mathcal{O}(\delta) \\ D\widehat{\mathtt{S}}(z_1, h_2(\hat{\tau}_{M+1}, z_1)) \begin{pmatrix} 1 \\ \partial_{z_1} h_2 \end{pmatrix} + \mathcal{O}(\delta) \end{pmatrix}, \tag{5.252}$$

where $\widehat{\mathtt{S}} = (\widehat{\mathtt{S}}^1)^M \circ \widehat{\mathtt{S}}^2$ was introduced in of Theorem 5.4.7. Since, by Item 3 of Theorem 5.4.7,

$$\left| \pi_{z_1} D\widehat{\mathtt{S}}(z_1, h_2(\hat{\tau}_{M+1}, z_1)) \begin{pmatrix} 1 \\ \partial_{z_1} h_2 \end{pmatrix} \right| \gtrsim \tilde{\kappa}^{-1}, \tag{5.253}$$

the equation $\pi_{z_1}\Lambda_{M+1}(T, z_1) = Z_1$ defines a function $\hat{z}_{M+1}(T, Z_1)$, with T large enough and $Z_1 \in (0, \tilde{\kappa})$. Using (5.252) and (5.253), we immediately have

$$|\partial_T \hat{z}_{M+1}(T, Z_1)| \lesssim \mathcal{O}(\delta\tilde{\kappa}), \qquad |\partial_{Z_1} \hat{z}_{M+1}(T, Z_1)| \lesssim \mathcal{O}(\tilde{\kappa}).$$

Then, $\Lambda(T, Z_1) = \Lambda_{M+1}(T, \hat{z}_{M+1}(T, Z_1))$ is defined for T large enough and $Z_1 \in (0, \tilde{\kappa})$, satisfies

$$\pi_T \Lambda(T, Z_1) = T, \qquad \pi_{Z_1}\Lambda(T, Z_1) = Z_1,$$

and, denoting $\tilde{h} = (\tilde{h}_1, \tilde{h}_2) = (\pi_p \Lambda, \pi_{z_2}\Lambda)$,

$$\partial_T \tilde{h} = \begin{pmatrix} \mathcal{O}(\delta) \\ \mathcal{O}(\delta) + \mathcal{O}(\tilde{\kappa}) \end{pmatrix}, \qquad \partial_{Z_1} \tilde{h} = \begin{pmatrix} \mathcal{O}(\delta\tilde{\kappa}) \\ \mathcal{O}(\tilde{\kappa}) \end{pmatrix}.$$

Moreover, if we denote by $\tilde{h}_n = (\tilde{h}_{n,1}, \tilde{h}_{n,2})$ the restriction of $\tilde{h}$ to $T \in [n, n+\delta]$, it satisfies that

$$\lim_{n\to\infty} h_{n,1} = 0.$$

This proves the claim for the horizontal bands. The claim for the vertical one is proven analogously. □

5.12.3 The differential of the high iterate of the return map

In this section we analyze the differential of the map $\widetilde{\Psi}$ in (5.67). Note that this map is a composition of the return maps $\widetilde{\Psi}_{i,j}$ (see (5.64)), whose differentials have been studied in Section 5.12.1. In that section we have obtained a good basis at each point of the tangent space which captures the expanding, contracting and center direction for each map $\widetilde{\Psi}_{i,j}$. Note however that the basis depends on the map (and certainly on the point!). Therefore, one has to adjust these bases so that they capture the expanding/contracting behavior for the differential of $\widetilde{\Psi}$. This is done in Proposition 5.12.4 below.

First we state a lemma, which is an immediate consequence of Lemma 5.12.1 and matrix products.

Lemma 5.12.3. *Consider the vector fields $\{v_1, v_2^{i,j}, v_3^{i,j}, v_4^{i,j}\}$, $i = 1, 2$, where v_1 and $v_2^{i,j}$ are the vectors in Items 1 and 2 of Lemma 5.12.1 and*

$$v_3^{i,j} = e_3 + \tilde{v}_3^{i,j}, \qquad v_4^{i} = e_4 + \tilde{v}_4^{i,j},$$

where $e_3 = (0, 0, 1, 0)^\top$, $e_4 = (0, 0, 0, 1)^\top$ and $\tilde{v}_3^{i,j}$ and $\tilde{v}_4^{i,j}$ are such that $v_3^{i,j}$ and $v_4^{i,j}$ satisfy Item 3 of Lemma 5.12.1. They form a basis of $T_\omega \mathcal{Q}_\delta^i$ at any $\omega = (p, \tau, z) \in \mathcal{Q}_\delta^i \cap \Psi_{i,j}^{-1}(\mathcal{Q}_\delta^j)$. Let $C_{i,j}(\omega)$ denote the *matrix of the change of coordinates from the standard basis to* $\{v_1, v_2^{i,j}, v_3^{i,j}, v_4^{i,j}\}$. *Then,*

$$
\begin{aligned}
\mathcal{M}_{2,1}(\omega) &= C_{1,1}(\widetilde{\Psi}_{2,1}(\omega))^{-1} D\widetilde{\Psi}_{2,1}(\omega) C_{2,1}(\omega) = \begin{pmatrix} \lambda^{2,1} & \mu^{2,1}\varepsilon_1^{2,1} & \varepsilon_2^{2,1} \\ \lambda^{2,1}\varepsilon_3^{2,1} & \mu^{2,1} & \varepsilon_4^{2,1} \\ \lambda^{2,1}\varepsilon_5^{2,1} & \mu^{2,1}(a_{2,1}+\varepsilon_6^{2,1}) & D\widehat{\mathtt{S}}^2(z)+\varepsilon_7^{2,1} \end{pmatrix}, \\
\mathcal{M}_{1,1}(\omega) &= C_{1,1}(\widetilde{\Psi}_{1,1}(\omega))^{-1} D\widetilde{\Psi}_{1,1}(\omega) C_{1,1}(\omega) = \begin{pmatrix} \lambda^{1,1} & \mu^{1,1}\varepsilon_1^{1,1} & \varepsilon_2^{1,1} \\ \lambda^{1,1}\varepsilon_3^{1,1} & \mu^{1,1} & \varepsilon_4^{1,1} \\ \lambda^{1,1}\varepsilon_5^{1,1} & \mu^{1,1}\varepsilon_6^{1,1} & D\widehat{\mathtt{S}}^1(z)+\varepsilon_7^{1,1} \end{pmatrix}, \\
\widetilde{\mathcal{M}}_{1,1}(\omega) &= C_{1,2}(\widetilde{\Psi}_{1,1}(\omega))^{-1} D\widetilde{\Psi}_{1,1}(\omega) C_{1,1}(\omega) = \begin{pmatrix} \tilde\lambda^{1,1} & \tilde\mu^{1,1}\delta_1^{1,1} & \delta_2^{1,1} \\ \tilde\lambda^{1,1}\delta_3^{1,1} & \tilde\mu^{1,1} & \delta_4^{1,1} \\ \tilde\lambda^{1,1}\delta_5^{1,1} & \tilde\mu^{1,1}(a_{1,2}+\delta_6^{1,1}) & D\widehat{\mathtt{S}}^1(z)+\delta_7^{1,1} \end{pmatrix}, \\
\mathcal{M}_{1,2}(\omega) &= C_{2,1}(\widetilde{\Psi}_{1,2}(\omega))^{-1} D\widetilde{\Psi}_{1,2}(\omega) C_{1,2}(\omega) = \begin{pmatrix} \lambda^{1,2} & \mu^{1,2}\varepsilon_1^{1,2} & \varepsilon_2^{1,2} \\ \lambda^{1,2}\varepsilon_3^{1,2} & \mu^{1,2} & \varepsilon_4^{1,2} \\ \lambda^{1,2}\varepsilon_5^{1,2} & \mu^{1,2}(a_{1,1}+\varepsilon_6^{1,2}) & D\widehat{\mathtt{S}}^1(z)+\varepsilon_7^{1,2} \end{pmatrix},
\end{aligned}
\tag{5.254}
$$

where

$$
\begin{aligned}
|\varepsilon_k^{i,j}(\omega)|, |\delta_k^{1,1}(\omega)|, &\le \mathcal{O}(\tau^{\frac{1}{5}-Ca}, \delta), \qquad k = 1, \dots, 7, \\
\lambda^{i,j}(\omega), \tilde\lambda^{1,1}(\omega), \mu^{i,j}(\omega)^{-1}, \tilde\mu^{1,1}(\omega)^{-1} &\gtrsim \tau^{-\frac{3}{5}+Ca}
\end{aligned}
$$

and $a_{i,j} = a_{i,j}(z)$, $i, j = 1, 2$ *satisfy* $|a_{i,j}| \le \mathcal{O}(1)$.

This lemma provides formulas for the differential of the intermediate return maps in "good bases". The next proposition provides a good basis for the high iterate of the return map $\widetilde\Psi$ in (5.67).

Proposition 5.12.4. *Consider* $\tilde\kappa$ *given by Theorem 5.4.7 and* $\delta > 0$ *small enough. Consider also the map* $\widetilde\Psi = \widetilde\Psi_{1,2} \circ \widetilde\Psi_{1,1}^{M-1} \circ \widetilde\Psi_{2,1}$ *defined in* $\mathcal{Q}_\delta^2 \subset \Sigma_2$. *There exists* $C : \mathcal{Q}_\delta^2 \cap \widetilde\Psi^{-1}(\mathcal{Q}_\delta^2) \to \mathcal{M}_{4\times 4}(\mathbb{R})$ *of the form*

$$C(\omega) = C_{2,1}(\omega)\widetilde{C}(\omega)C_{\widehat{\mathtt{S}}}(\omega)$$

where $C_{2,1}(\omega)$ *is the matrix introduced in Lemma 5.12.3,*

$$\widetilde{C}(\omega) = \begin{pmatrix} 1 & a(\omega) & b(\omega) \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \qquad a(\omega), b(\omega) = \mathcal{O}(\tau^{3/5-Ca}\delta^{1/5}),$$

and, for $\omega = (p, \tau, z)$,

$$C_{\widehat{\mathtt{S}}}(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & V_{2,1}(z) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $V_{2,1}$ *is given by Item 3 of Theorem 5.4.7, such that*

$$C(\widetilde\Psi(\omega))^{-1} D\widetilde\Psi(\omega) C(\omega) = \begin{pmatrix} \lambda & \mu\varepsilon_5 & \varepsilon_7 & \varepsilon_{10} \\ \lambda\varepsilon_2 & \mu(1+\varepsilon_6) & \varepsilon_8 & \varepsilon_{11} \\ \lambda\varepsilon_3 & \mu\tilde c_1 & \lambda_{\widehat{\mathtt{S}}} & \lambda_{\widehat{\mathtt{S}}}^{-1}\varepsilon_{12} \\ \lambda\varepsilon_4 & \mu\tilde c_2 & \lambda_{\widehat{\mathtt{S}}}\varepsilon_9 & \lambda_{\widehat{\mathtt{S}}}^{-1} \end{pmatrix}, \tag{5.255}$$

where

$$\lambda(\omega) \gtrsim \tau^{-\frac{3}{5}+\tilde{C}a}\delta^{-3M/5}, \qquad \mu(\omega)^{-1} \gtrsim \tau^{-\frac{3}{5}+\tilde{C}a}, \qquad \tilde c_1, \tilde c_2 = \mathcal{O}(1), \qquad |\varepsilon_j| \lesssim \mathcal{O}(\delta^{1/5}),$$

for $j = 2, \dots, 11$, $j \neq 9$,

$$|\varepsilon_9|, |\varepsilon_{12}| \lesssim \tilde\kappa$$

and $\lambda_{\widehat{\mathtt{S}}} \geq \tilde{\kappa}^{-1}$ was introduced in Item 3 of Theorem 5.4.7.

Analogously, there exists $\widehat{C} : \mathcal{Q}_\delta^2 \cap \Psi(\mathcal{Q}_\delta^2) \to \mathcal{M}_{4\times 4}(\mathbb{R})$ of the form

$$\widehat{C}(\widetilde{\Psi}(\omega)) = C_{2,1}(\widetilde{\Psi}(\omega))\widetilde{C}^*(\widetilde{\Psi}(\omega))C^*_{\widehat{\mathtt{S}}}(\omega)$$

with

$$\widetilde{C}^*(\widetilde{\Psi}(\omega)) = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{a}(\omega) & 1 & \tilde{b}(\omega) \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \qquad \tilde{a}(\omega), \tilde{b}(\omega) = \mathcal{O}(\delta^{4/5-Ca}),$$

and, for $\omega(p,\tau,z)$,

$$C^*_{\widehat{\mathtt{S}}}(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & V_{2,1}((\widehat{\mathtt{S}})^{-1}(z)) \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that

$$\widehat{C}(\omega)^{-1}D(\widetilde{\Psi})^{-1}(\widetilde{\Psi}(\omega))\widehat{C}(\widetilde{\Psi}(\omega)) = \begin{pmatrix} \tilde{\mu}\tilde{\varepsilon}_1 & \tilde{\lambda}\tilde{\varepsilon}_5 & \tilde{\varepsilon}_8 & \tilde{\varepsilon}_{11} \\ \tilde{\varepsilon}_2 & \tilde{\lambda} & \tilde{\varepsilon}_9 & \tilde{\varepsilon}_{12} \\ \tilde{\varepsilon}_3 & \tilde{\lambda}\tilde{\varepsilon}_6 & \lambda_{\widehat{\mathtt{S}}}^{-1} & \lambda_{\widehat{\mathtt{S}}}^{-1}\tilde{\varepsilon}_{13} \\ \tilde{\varepsilon}_4 & \tilde{\lambda}\tilde{\varepsilon}_7 & \lambda_{\widehat{\mathtt{S}}}\tilde{\varepsilon}_{10} & \lambda_{\widehat{\mathtt{S}}} \end{pmatrix}, \tag{5.256}$$

where

$$\tilde{\lambda}(\omega) \gtrsim \delta^{-3(M+1)/5}, \qquad \tilde{\mu}(\omega)^{-1} \gtrsim \delta^{-\frac{3}{5}}, \qquad |\tilde{\varepsilon}_j| \lesssim \mathcal{O}(\delta^{1/5}),$$

for $j = 1, \ldots, 12$, $j \neq 10, 13$,

$$|\tilde{\varepsilon}_{10}|, |\tilde{\varepsilon}_{13}| \lesssim \tilde{\kappa},$$

Proof. In view of Lemma 5.12.3, we write

$$C_{2,1}(\tilde{\Psi}(\omega))^{-1}D\tilde{\Psi}(\omega)C_{2,1}(\omega) = \mathcal{M}_M(\omega)\cdots\mathcal{M}_0(\omega), \tag{5.257}$$

where the matrices

$$\begin{aligned} \mathcal{M}_0(\omega) &= \mathcal{M}_{2,1}(\omega) \\ \mathcal{M}_j(\omega) &= \mathcal{M}_{1,1}(\widetilde{\Psi}_{1,1}^{j-1} \circ \widetilde{\Psi}_{2,1}(\omega)), \qquad j = 1, \ldots, M-2, \\ \mathcal{M}_{M-1}(\omega) &= \widetilde{\mathcal{M}}_{1,1}(\widetilde{\Psi}_{1,1}^{M-2} \circ \widetilde{\Psi}_{2,1}(\omega)), \\ \mathcal{M}_M(\omega) &= \mathcal{M}_{1,2}(\widetilde{\Psi}_{1,1}^{M-1} \circ \widetilde{\Psi}_{2,1}(\omega)), \end{aligned}$$

are given by Lemma 5.12.3. The product of the matrices in (5.257), in the current form, is difficult to control. We find an adapted basis in which this product of matrices has a more convenient expression. We proceed in the following way. We claim that, for any $0 \leq j \leq M$, there exists a matrix

$$C_j = \begin{pmatrix} 1 & \alpha_j & \beta_j \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}, \qquad |\alpha_j|, \|\beta_j\| \leq K_M \tau^{3/5-Ca}\delta^{(1+3j)/5}, \tag{5.258}$$

where K_M is a constant depending only on M, such that, if we define $\widetilde{C}_j = \widetilde{C}_{j-1}C_j$ with $j \geq 0$, $\widetilde{C}_{-1} = \mathrm{Id}$, satisfies

$$\mathcal{M}_j \ldots \mathcal{M}_0 \widetilde{C}_j = \begin{pmatrix} \tilde{\lambda}_j & 0 & 0 \\ \tilde{\lambda}_j\varepsilon_{j,1} & \mu(1+\varepsilon_{j,2}) & \varepsilon_{j,3} \\ \tilde{\lambda}_j\varepsilon_{j,4} & \mu c_j & \widetilde{\mathcal{S}}_j + \varepsilon_{j,5} \end{pmatrix} \tag{5.259}$$

with

$$\widetilde{\mathcal{S}}_j(z) = D((\widehat{\mathtt{S}}^1)^{j-1} \circ \widehat{\mathtt{S}}^2)(z) \tag{5.260}$$

and

$$\tilde{\lambda}_j \gtrsim \tau^{-3/5+Ca}\delta^{-3(j-1)/5}, \qquad \mu^{-1} \gtrsim \tau^{-3/5+Ca}, \qquad \varepsilon_{j,k} = \mathcal{O}(\delta^{1/5}), \qquad c_j = \mathcal{O}(1). \tag{5.261}$$

The constants involved in the above equalities depend only on M.

We prove this claim by induction. The case $j = 0$ follows from the expression of $\mathcal{M}_0 = \mathcal{M}_{2,1}$ given Lemma 5.12.3, taking $\widetilde{C}_0 = C_0$ as in (5.258) with

$$\tilde{\lambda}_0 = \lambda^{2,1}, \qquad \mu = \mu^{2,1}, \qquad \alpha_0 = -\frac{\mu^{2,1}\varepsilon_1^{2,1}}{\lambda^{2,1}}, \qquad \beta_0 = -\frac{\varepsilon_2^{2,1}}{\lambda^{2,1}}.$$

Now assume that (5.259) holds for $j-1$, with $1 \le j \le M-2$, that is, there exists $\tilde{C}_{j-1}$ such that

$$\begin{aligned}
&\mathcal{M}_j\left(\mathcal{M}_{j-1}\ldots\mathcal{M}_0\tilde{C}_{j-1}\right)\\
&\qquad = \begin{pmatrix} \lambda_j & \mu_j\tilde{\varepsilon}_{j,1} & \tilde{\varepsilon}_{j,2} \\ \lambda_j\tilde{\varepsilon}_{j,3} & \mu_j & \tilde{\varepsilon}_{j,4} \\ \lambda_j\tilde{\varepsilon}_{j,5} & \mu_j\tilde{\varepsilon}_{j,6} & \mathcal{S}_j+\tilde{\varepsilon}_{j,7}\end{pmatrix}\begin{pmatrix} \tilde{\lambda}_{j-1} & 0 & 0 \\ \tilde{\lambda}_{j-1}\varepsilon_{j-1,1} & \mu(1+\varepsilon_{j-1,2}) & \varepsilon_{j-1,3} \\ \tilde{\lambda}_{j-1}\varepsilon_{j-1,4} & \mu c_{j-1} & \widetilde{\mathcal{S}}_{j-1}+\varepsilon_{j-1,5}\end{pmatrix}
\end{aligned}$$

with

$$\begin{aligned}
(\lambda_j,\mu_j) &= (\lambda^{1,1},\mu^{1,1})\circ\widetilde{\Psi}_{1,1}^{j-1}\circ\widetilde{\Psi}_{2,1}\\
\mathcal{S}_j &= D\widehat{\mathbb{S}}^1\circ\pi_z\widetilde{\Psi}_{1,1}^{j-1}\circ\widetilde{\Psi}_{2,1}\\
\tilde{\varepsilon}_{j,k} &= \varepsilon_k^{1,1}\circ\widetilde{\Psi}_{1,1}^{j-1}\circ\widetilde{\Psi}_{2,1}, \qquad k=1,\ldots,7.
\end{aligned} \tag{5.262}$$

where π_z is the projection onto the z component. Observe that, by hypothesis, $\widetilde{\Psi}_{1,1}^{j-1}\circ\widetilde{\Psi}_{2,1}(\omega) \in \mathcal{Q}_\delta^1$ and therefore $\pi_\tau\widetilde{\Psi}_{1,1}^{j-1}\circ\widetilde{\Psi}_{2,1}(\omega), \pi_p\widetilde{\Psi}_{1,1}^{j-1}\circ\widetilde{\Psi}_{2,1}(\omega) \in (0,\delta)$. In particular, by Lemma 5.12.3,

$$|\lambda_j|, |\mu_j|^{-1} \gtrsim \delta^{-3/5} \qquad \text{and} \qquad |\tilde{\varepsilon}_{j,k}| \lesssim \delta^{1/5}.$$

Then, the elements of the top row of $\mathcal{M}_j\mathcal{M}_{j-1}\ldots\mathcal{M}_0\tilde{C}_{j-1}$ are

$$\begin{aligned}
\tilde{\lambda}_j &= \lambda_j\tilde{\lambda}_{j-1}\left(1+\frac{1}{\lambda_j}\left(\mu_j\tilde{\varepsilon}_{j,1}\varepsilon_{j-1,1}+\tilde{\varepsilon}_{j,2}\varepsilon_{j-1,4}\right)\right)\\
&\gtrsim \delta^{-3/5}\tau^{-3/5+Ca}\delta^{-3(j-2)/5} \gtrsim \tau^{-3/5+Ca}\delta^{-3(j-1)/5},
\end{aligned} \tag{5.263}$$

and

$$D_1 = \mu\left(\mu_j\tilde{\varepsilon}_{j,1}(1+\varepsilon_{j-1,2})+\tilde{\varepsilon}_{j,2}c_{j-1}\right), \qquad D_2 = \mu_j\tilde{\varepsilon}_{j,1}\varepsilon_{j-1,3}+\tilde{\varepsilon}_{j,2}(\widetilde{\mathcal{S}}_{j-1}+\varepsilon_{j-1,5}).$$

Hence, taking

$$\alpha_j = -\frac{D_1}{\tilde{\lambda}_j} = \mathcal{O}\left(\tau^{2(3/5-Ca)}\delta^{(3j+1)/5}\right), \qquad \beta_j = -\frac{D_2}{\tilde{\lambda}_j} = \mathcal{O}\left(\tau^{3/5-Ca}\delta^{(3j+1)/5}\right),$$

we have that

$$\mathcal{M}_j\mathcal{M}_{j-1}\ldots\mathcal{M}_0\tilde{C}_{j-1}C_j = \begin{pmatrix} \tilde{\lambda}_j & 0 & 0 \\ \tilde{\lambda}_j\varepsilon_{j,1} & \mu(1+\varepsilon_{j,2}) & \varepsilon_{j,3} \\ \tilde{\lambda}_j\varepsilon_{j,4} & \mu c_j & \widetilde{\mathcal{S}}_j+\varepsilon_{j,5}\end{pmatrix},$$

for some $\varepsilon_{j,k}$, $k = 1\ldots 5$ and c_j.

Note that $\varepsilon_{j,1}$ and $\varepsilon_{j,4}$ do not depend on the choice of α_j and β_j. Indeed, using the equality in the first row in (5.263), they satisfy

$$\begin{aligned}
\varepsilon_{j,1} &= \frac{\lambda_j\tilde{\lambda}_{j-1}}{\tilde{\lambda}_j}\left(\tilde{\varepsilon}_{j,3}+\frac{1}{\lambda_j}(\mu_j\varepsilon_{j-1,1}+\tilde{\varepsilon}_{j,4}\varepsilon_{j-1,4})\right) = \mathcal{O}(1)\mathcal{O}(\delta^{1/5}),\\
\varepsilon_{j,4} &= \frac{\lambda_j\tilde{\lambda}_{j-1}}{\tilde{\lambda}_j}\left(\tilde{\varepsilon}_{j,5}+\frac{1}{\lambda_j}(\mu_j\tilde{\varepsilon}_{j,6}\varepsilon_{j-1,1}+(\mathcal{S}_j+\tilde{\varepsilon}_{j,7})\varepsilon_{j-1,4})\right) = \mathcal{O}(1)\mathcal{O}(\delta^{1/5}).
\end{aligned}$$

Clearly, α_j and β_j satisfy the inequalities in (5.258). Moreover,

$$\tilde{\lambda}_j \varepsilon_{j,1} \alpha_j = -D_1 \varepsilon_{j,1} = \mu \mathcal{O}(\delta^{1/5}), \qquad \tilde{\lambda}_j \varepsilon_{j,1} \beta_j = -D_2 \varepsilon_{j,1} = \mathcal{O}(\delta^{1/5}). \tag{5.264}$$

The bounds of the elements of $\mathcal{M}_j \mathcal{M}_{j-1} \dots \mathcal{M}_0 \tilde{C}_{j-1} C_j$ can be computed immediately from Lemma 5.12.3 and the induction hypotheses. Indeed,

$$\begin{aligned}
\varepsilon_{j,2} &= \frac{\tilde{\lambda}_j \varepsilon_{j,1} \alpha_j}{\mu} + \mu_j (1 + \varepsilon_{j-1,2}) + \tilde{\varepsilon}_{j,4} c_{j-1} = \mathcal{O}(\delta^{1/5}), \\
\varepsilon_{j,3} &= \tilde{\lambda}_j \varepsilon_{j,1} \beta_j + \mu_j \varepsilon_{j-1,3} + \tilde{\varepsilon}_{j,4} (\widetilde{\mathcal{S}}_{j-1} + \varepsilon_{j-1,5}) = \mathcal{O}(\delta^{1/5}), \\
c_j &= \frac{\tilde{\lambda}_j \varepsilon_{j,4} \alpha_j}{\mu} + \mu_j \tilde{\varepsilon}_{j,6} (1 + \varepsilon_{j-1,2}) + (\mathcal{S}_j + \tilde{\varepsilon}_{j,7}) c_{j-1} = \mathcal{O}(1), \\
\varepsilon_{j,5} &= -\widetilde{\mathcal{S}}_j + \tilde{\lambda}_j \varepsilon_{j,4} \beta_j + (\mathcal{S}_j + \tilde{\varepsilon}_{j,7})(\widetilde{\mathcal{S}}_{j-1} + \varepsilon_{j-1,5}) + \mu_j \tilde{\varepsilon}_{j,6} \varepsilon_{j-1,3} = \mathcal{S}_j \widetilde{\mathcal{S}}_{j-1} - \widetilde{\mathcal{S}}_j + \mathcal{O}(\delta^{1/5})
\end{aligned}$$

where we have used (5.264) in the bounds of $\varepsilon_{j,2}$, $\varepsilon_{j,3}$, c_j.

To get small estimates for $\varepsilon_{j,5}$ note that, by Statement 1 of Proposition 5.5.4, one has that for $j = 1 \dots M-2$ and $\omega = (p, \tau, z) \in \mathcal{Q}^2_\delta$,

$$\left| \pi_z \widetilde{\Psi}_{1,1}^{j-1} \circ \widetilde{\Psi}_{2,1}(\omega) - \widehat{\mathtt{S}}_1^{j-1} \circ \widehat{\mathtt{S}}_2(z) \right| \leq \mathcal{O}(\delta).$$

Then, by the definition of $\mathcal{S}_j$ and $\widetilde{\mathcal{S}}_j$ in (5.262) and (5.260) respectively,

$$\left| \mathcal{S}_j \widetilde{\mathcal{S}}_{j-1} - \widetilde{\mathcal{S}}_j \right| \leq \mathcal{O}(\delta).$$

This implies that $|\varepsilon_{j,5}| \leq \mathcal{O}(\delta^{1/5})$.

The obtained estimates prove the claim for $0 \leq j \leq M-2$. The cases $j = M-1, M$ can be treated exactly in the same way. They only differ in $\varepsilon_{j,4}$, c_j and $\varepsilon_{j,5}$, where $\tilde{\varepsilon}_{j-1,6}$ should be substituted by some $\tilde{c}_j = \mathcal{O}(1)$ coming from Lemma 5.12.3. Their final bounds remain the same.

Observe that, from (5.258),

$$\widetilde{C}_M = C_0 \cdots C_M = \begin{pmatrix} 1 & \sum_{0 \leq j \leq M} \alpha_j & \sum_{0 \leq j \leq M} \beta_j \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$$

where, from the bounds in (5.258),

$$\left| \sum_{0 \leq j \leq M} \alpha_j \right|, \left\| \sum_{0 \leq j \leq M} \beta_j \right\| \leq \mathcal{O}(\tau^{3/5 - C a} \delta^{1/5}).$$

Hence, from (5.259) with $j = M$, we have that, for any $\omega \in \mathcal{Q}^2_\delta \cap \widetilde{\Psi}^{-1}(\mathcal{Q}^2_\delta)$,

$$\begin{aligned}
&\widetilde{C}_M^{-1}(\Psi(\omega)) C_{2,1}(\tilde{\Psi}(\omega))^{-1} D\tilde{\Psi}(\omega) C_{2,1}(\omega) \widetilde{C}_M(\omega) \\
&\qquad = \begin{pmatrix} 1 & -\sum_{0 \leq j \leq M} \alpha_j & -\sum_{0 \leq j \leq M} \beta_j \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_M & 0 & 0 \\ \tilde{\lambda}_M \varepsilon_{M,1} & \mu(1 + \varepsilon_{M,2}) & \varepsilon_{M,3} \\ \tilde{\lambda}_M \varepsilon_{M,4} & \mu c_M & \widetilde{\mathcal{S}}_M + \varepsilon_{M,5} \end{pmatrix} \\
&\qquad = \begin{pmatrix} \tilde{\lambda}_M (1 + \mathcal{O}(\tau^{3/5 - C a} \delta^{2/5})) & \mu \mathcal{O}(\tau^{3/5 - C a} \delta^{1/5}) & \mathcal{O}(\tau^{3/5 - C a} \delta^{1/5}) \\ \tilde{\lambda}_M \varepsilon_{M,1} & \mu(1 + \varepsilon_{M,2}) & \varepsilon_{M,3} \\ \tilde{\lambda}_M \varepsilon_{M,4} & \mu c_M & \widetilde{\mathcal{S}}_M + \varepsilon_{M,5} \end{pmatrix}.
\end{aligned}$$

Finally, the claim follows from the properties of $V_{2,1}$ in Theorem 5.4.7.

The proof of (5.256) is completely analogous, considering matrices D_j of the form

$$D_j = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_j & 1 & \beta_j \\ 0 & 0 & \mathrm{Id} \end{pmatrix},$$

which are also a group. □

5.12.4 Stable and unstable cone fields: Proof of Proposition 5.6.5

The analysis of the differential of the map $\widetilde{\Psi}$ (see (5.67)) performed in Section 5.12.3 allows us to set up stable and unstable cone fields and prove Proposition 5.6.5.

The only difficulty in the study of the behavior of the cone fields is that, of the two expanding factors $\lambda, \lambda_{\widehat{\mathbb{S}}}$ (see Proposition 5.12.4), which satisfy $\lambda > \lambda_{\widehat{\mathbb{S}}} > 1$, λ is unbounded in $\mathcal{Q}_\delta$ (as $\tau \to 0$). Hence the error terms $\lambda\varepsilon_i$ in (5.255) are not necessarily small and, in fact, can be large.

We start by considering the unstable cone field S^u_{ω,κ_u}, introduced in (5.70). Let $\mathcal{M}$ be the matrix in (5.255) that is,

$$\mathcal{M}(\omega) = C(\widetilde{\Psi}(\omega))^{-1} D\widetilde{\Psi}(\omega) C(\omega).$$

For $x \in S^u_{\omega,\kappa_u}$, let $y = C(\omega)x$, where C is given by Lemma 5.12.4. We will denote $y = (y_u, y_s)$, where $y_u = (y_1, y_3)$ and $y_s = (y_2, y_4)$, and use the norms $\|y_u\| = \max\{|y_1|, |y_3|\}$ and $\|y_s\| = \max\{|y_2|, |y_4|\}$. Given $\kappa > 0$, we denote

$$\widetilde{S}^u_{\omega,\kappa} = \{y;\ \|y_s\| \leq \kappa \|y_u\|\}.$$

We also denote $(\mathcal{M}y)_u = \mathcal{M}_{u,u} y_u + \mathcal{M}_{u,s} y_s$ and $(\mathcal{M}y)_s = \mathcal{M}_{s,u} y_u + \mathcal{M}_{s,s} y_s$, where

$$\begin{aligned} \mathcal{M}_{u,u} &= \begin{pmatrix} \lambda & \varepsilon_7 \\ \lambda\varepsilon_3 & \lambda_{\widehat{\mathbb{S}}} \end{pmatrix}, \qquad & \mathcal{M}_{u,s} &= \begin{pmatrix} \mu\varepsilon_5 & \varepsilon_{10} \\ \mu\tilde{c}_1 & \lambda_{\widehat{\mathbb{S}}}^{-1}\varepsilon_{12} \end{pmatrix}, \\ \mathcal{M}_{s,u} &= \begin{pmatrix} \lambda\varepsilon_2 & \varepsilon_8 \\ \lambda\varepsilon_4 & \lambda_{\widehat{\mathbb{S}}}\varepsilon_9 \end{pmatrix}, \qquad & \mathcal{M}_{s,s} &= \begin{pmatrix} \mu(1+\varepsilon_6) & \varepsilon_{11} \\ \mu\tilde{c}_2 & \lambda_{\widehat{\mathbb{S}}}^{-1} \end{pmatrix}. \end{aligned} \tag{5.265}$$

with $|\varepsilon_i| \lesssim \mathcal{O}(\delta^{1/5})$, $i \neq 9, 12$ and $|\varepsilon_9|, |\varepsilon_{12}| \lesssim \mathcal{O}(\tilde{\kappa})$.

Recall that $C(\omega) = C_{2,1}(\omega)\widetilde{C}(\omega)C_{\widehat{\mathbb{S}}}(\omega)$ (see Proposition 5.12.4). The form of $\widetilde{C}$ and $C_{\widehat{\mathbb{S}}}$ has been given in Proposition 5.12.4. Lemmas 5.12.1 and 5.12.3 imply that $C_{2,1}$ is of the form

$$C_{2,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \mathcal{O}(\delta) & \mathcal{O}(\delta) & \mathcal{O}(\delta) \\ 0 & \tilde{a} & 1 & 0 \\ 0 & \tilde{b} & 0 & 1 \end{pmatrix}.$$

Let $\hat{a} = \sup_{\omega \in \mathcal{Q}_\delta} |\tilde{a}(\omega)| \lesssim \mathcal{O}(1)$ and $\hat{b} = \sup_{\omega \in \mathcal{Q}_\delta} |\tilde{b}(\omega)| \lesssim \mathcal{O}(1)$. Then, it is immediate to check that

$$C(\omega) S^u_{\omega,\kappa_u} \subset \widetilde{S}^u_{\omega,\widetilde{\kappa}_u}, \qquad \widetilde{\kappa}_u \geq \frac{(1+\mathcal{O}(\delta^{1/5}))\kappa_u + \hat{b}}{1+\hat{a}} = \mathcal{O}(1), \tag{5.266}$$

and, for $\beta > 0$, small enough,

$$C(\omega)^{-1} \widetilde{S}^u_{\omega,\beta} \subset S^u_{\omega,\widetilde{\beta}}, \qquad \widetilde{\beta} \leq \frac{(1+\hat{b})\beta}{1 - \mathcal{O}(\delta^{1/5}) - (1+\hat{a}+\mathcal{O}(\delta))\beta} = \mathcal{O}(\beta). \tag{5.267}$$

We claim that

1. if $y \in \widetilde{S}^u_{\omega,\widetilde{\kappa}_u}$, $\|(\mathcal{M}y)_u\| \geq \lambda_{\widehat{\mathbb{S}}}(1+\mathcal{O}(\delta^{1/5}))\|y_u\|$ and
2. $\mathcal{M}(\omega)\widetilde{S}^u_{\omega,\widetilde{\kappa}_u} \subset \widetilde{S}^u_{\omega,\hat{\kappa}_u}$, with $\hat{\kappa}_u = \mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa})$.

Indeed, let $y \in \widetilde{S}^u_{\omega,\widetilde{\kappa}_u}$. We first observe that $\|\mathcal{M}_{u,u}^{-1}\| \leq \lambda_{\widehat{\mathbb{S}}}^{-1}$. This implies that

$$\|\mathcal{M}_{u,u} y_u\| \geq \|\mathcal{M}_{u,u}^{-1}\|^{-1} \|y_u\| \geq \lambda_{\widehat{\mathbb{S}}} \|y_u\|.$$

Hence, using that $\|y_s\| \leq \widetilde{\kappa}_u \|y_u\|$,

$$\|(\mathcal{M}y)_u\| \geq \|\mathcal{M}_{u,u} y_u\| - \|\mathcal{M}_{u,s} y_s\| \geq (\lambda_{\widehat{\mathbb{S}}} - \mathcal{O}(\delta^{1/5})\widetilde{\kappa}_u)\|y_u\|. \tag{5.268}$$

Taking into account the bound on $\widetilde{\kappa}_u$ given by (5.266), this last inequality implies Item 1. However, this is the minimum expansion in the unstable directions. If $\|y_u\| = |y_1|$, the expansion is much larger, as follows from

$$\begin{aligned}\|(\mathcal{M}y)_u\| &\geq |(\mathcal{M}y)_1| = |\lambda y_1 + \varepsilon_7 y_3 + \mu\varepsilon_5 y_2 + \varepsilon_{10} y_4| \\ &\geq (\lambda - \mathcal{O}(\delta^{1/5}) - \mathcal{O}(\delta^{1/5})\widetilde{\kappa}_u)|y_1| = \lambda(1 - \mathcal{O}(\delta^{1/5})/\lambda - \mathcal{O}(\delta^{1/5})\widetilde{\kappa}_u/\lambda)\|y_u\|. \quad (5.269)\end{aligned}$$

Also, if $\|y_u\| = |y_1|$ and $|y_1| \geq (\lambda_{\widetilde{\mathrm{S}}}/\lambda)|y_3|$, we have that

$$|(\mathcal{M}y)_1| = |\lambda y_1 + \varepsilon_7 y_3 + \mu\varepsilon_5 y_2 + \varepsilon_{10} y_4| \geq \lambda(1 - \mathcal{O}(\delta^{1/5})/\lambda_{\widetilde{\mathrm{S}}} - \mathcal{O}(\delta^{1/5})\widetilde{\kappa}_u)|y_1|. \quad (5.270)$$

Now we prove Item 2. We first claim that, if $y \in \widetilde{S}^u_{\omega,\widetilde{\kappa}_u}$,

$$\|\mathcal{M}_{s,u} y_u\| \leq \begin{cases} \lambda(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|y_u\|, & \text{if } \|y_u\| = |y_1|, \\ \lambda(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))|y_1|, & \text{if } \|y_u\| = |y_3| \text{ and } |y_1| \geq (\lambda_{\widetilde{\mathrm{S}}}/\lambda)|y_3|, \\ \lambda_{\widetilde{\mathrm{S}}}(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|y_u\|, & \text{if } \|y_u\| = |y_3| \text{ and } |y_1| \leq (\lambda_{\widetilde{\mathrm{S}}}/\lambda)|y_3|. \end{cases} \quad (5.271)$$

Indeed, if $\|y_u\| = |y_1|$, by the definition of $\mathcal{M}_{s,u}$ in (5.265) and the fact that $\lambda_{\widetilde{\mathrm{S}}}/\lambda < 1$,

$$\|\mathcal{M}_{s,u} y_u\| \leq \lambda\mathcal{O}(\delta^{1/5})|y_1| + \lambda_{\widetilde{\mathrm{S}}}\mathcal{O}(\tilde{\kappa})|y_3| \leq \lambda(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|y_u\|.$$

In the case $\|y_u\| = |y_3|$ and $|y_1| \geq (\lambda_{\widetilde{\mathrm{S}}}/\lambda)|y_3|$, we have that

$$\|\mathcal{M}_{s,u} y_u\| \leq \lambda\mathcal{O}(\delta^{1/5})|y_1| + \lambda_{\widetilde{\mathrm{S}}}\mathcal{O}(\tilde{\kappa})|y_3| \leq \lambda(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))|y_1|.$$

Finally, if $\|y_u\| = |y_3|$ and $|y_1| \leq (\lambda_{\widetilde{\mathrm{S}}}/\lambda)|y_3|$,

$$\|\mathcal{M}_{s,u} y_u\| \leq \lambda\mathcal{O}(\delta^{1/5})|y_1| + \lambda_{\widetilde{\mathrm{S}}}\mathcal{O}(\tilde{\kappa})|y_3| \leq \lambda_{\widetilde{\mathrm{S}}}(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|y_u\|,$$

which proves (5.271).

Hence, if $y \in \widetilde{S}^u_{\omega,\widetilde{\kappa}_u}$ and $\|y_u\| = |y_1|$, by the first inequality in (5.271), using that $\|\mathcal{M}_{s,s}\| \leq \mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa})$ and (5.269), we have that

$$\|(\mathcal{M}y)_s\| \leq \|\mathcal{M}_{s,u} y_u\| + \|\mathcal{M}_{s,s} y_s\| \leq \lambda(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|y_u\| \leq (\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|(\mathcal{M}y)_u\|.$$

In the case $\|y_u\| = |y_3|$ and $|y_1| \geq (\lambda_{\widetilde{\mathrm{S}}}/\lambda)|y_3|$, by (5.268), the second inequality in (5.271) and (5.270),

$$\begin{aligned}\|(\mathcal{M}y)_s\| &\leq \|\mathcal{M}_{s,u} y_u\| + \|\mathcal{M}_{s,s} y_s\| \leq \lambda(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))|y_1| + (\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|y_s\| \\ &\leq (\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))|(\mathcal{M}y)_1| + (\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|(\mathcal{M}y)_u\| \leq (\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|(\mathcal{M}y)_u\|.\end{aligned}$$

Finally, in the case $\|y_u\| = |y_3|$ and $|y_1| \leq (\lambda_{\widetilde{\mathrm{S}}}/\lambda)|y_3|$, by the third inequality in (5.271) and (5.268),

$$\begin{aligned}\|(\mathcal{M}y)_s\| &\leq \|\mathcal{M}_{s,u} y_u\| + \|\mathcal{M}_{s,s} y_s\| \leq \lambda_{\widetilde{\mathrm{S}}}(\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|y_u\| + (\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|y_s\| \\ &\leq (\mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa}))\|(\mathcal{M}y)_u\|.\end{aligned}$$

This proves Item 2. Then, taking $\beta = \mathcal{O}(\delta^{1/5}) + \mathcal{O}(\tilde{\kappa})$ in (5.267), the claim for the unstable cones follows.

The proof of the claim for the stable cones is completely analogous. It is only necessary to use (5.256) instead of (5.255). We simply emphasize that (5.267) is replaced by (5.266).

5.A Proof of Proposition 5.7.1

We devote this section to proof Proposition 5.7.1. Separating the linear and non-linear terms, the invariance equation (5.83) can be rewritten as

$$\mathcal{L}Z = D_Z X_Z^0(x,0)Z + F(x,Z)$$

where

$$F(x,Z) = X_Z^0(x,Z) - X_Z^0(x,0) - DX_Z^0(x,0)Z + X_Z^1(x,Z) - DZ\left(X_x^0(x,Z) - \Omega + X_x^1(x,Z)\right)$$

with $\Omega = (1, \nu G_0^3/L_0^3)^\top$.

Observe that, using the definition of Q_0 in (5.76) and defining $q = \Lambda - \alpha\xi_0 - \eta_0\beta - \alpha\beta$, one has

$$\begin{aligned}
&DZ\left(X_x^0(x,Z) - \Omega + X_x^1(x,Z)\right)\\
&= \partial_u Z\left(\partial_Y Q_0(u,Y,q) - 1\right) + \partial_\gamma Z\left(\frac{G_0^3\nu}{(L_0+\Lambda)^3} - \frac{G_0^3\nu}{L_0^3} + \partial_q Q_0(u,Y,q) + \partial_\Lambda \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta)\right)\\
&= \partial_u Z\left(\frac{Y}{G_0 y_{\rm h}^2(u)} + f_1(u)q_1\right) + \partial_\gamma Z\left(\frac{G_0^3\nu}{(L_0+\Lambda)^3} - \frac{G_0^3\nu}{L_0^3} + f_1(u)Y + f_2(u)q + \partial_\Lambda \mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta)\right).
\end{aligned}$$

Therefore,

$$DZ\left(X_x^0(x,Z) - \Omega + X_x^1(x,Z)\right) = \partial_u Z\mathcal{G}_1(u,\gamma,Z) + \partial_\gamma Z\mathcal{G}_2(u,\gamma,Z)$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are the functions introduced in (5.86). Moreover, $X_Z^0(x,0) = 0$ and

$$A(u) = DX_Z^0(u,\gamma,0) = \begin{pmatrix} -\frac{\partial^2\mathcal{P}_0}{\partial Y\partial u} & -\frac{\partial^2\mathcal{P}_0}{\partial\Lambda\partial u} & -\frac{\partial^2\mathcal{P}_0}{\partial\alpha\partial u} & -\frac{\partial^2\mathcal{P}_0}{\partial\beta\partial u}\\ 0 & 0 & 0 & 0\\ -i\frac{\partial^2\mathcal{P}_0}{\partial Y\partial\beta} & -i\frac{\partial^2\mathcal{P}_0}{\partial\Lambda\partial\beta} & -i\frac{\partial^2\mathcal{P}_0}{\partial\alpha\partial\beta} & -i\frac{\partial^2\mathcal{P}_0}{\partial\beta\partial\beta}\\ i\frac{\partial^2\mathcal{P}_0}{\partial Y\partial\alpha} & i\frac{\partial^2\mathcal{P}_0}{\partial\Lambda\partial\alpha} & i\frac{\partial^2\mathcal{P}_0}{\partial\alpha\partial\alpha} & i\frac{\partial^2\mathcal{P}_0}{\partial\beta\partial\alpha}\end{pmatrix}(u,\gamma,0) = \begin{pmatrix} 0 & 0\\ \mathcal{A}(u) & \mathcal{B}(u)\end{pmatrix}.$$

We obtain the expression of $\mathcal{A}(u)$ and $\mathcal{B}(u)$ using the formula for $\mathcal{P}_0$ in (5.75) and (5.76), (5.77). Defining

$$\mathcal{Q}(Z) = X_Z^0(x,Z) - X_Z^0(x0) - AZ + X_Z^1(x,Z)$$

we obtain the formulas (5.89).

5.B Estimates for the perturbing potential

The goal of this appendix is to give estimates for the Fourier coefficents of the potential $\mathcal{P}_1$ introduced in (5.78). This estimates are thoroughly used in the proof of Theorem 5.7.4 given in Section 5.7.4 and in the analysis of the Melnikov potential given in Appendix 5.C.

Using (5.18), (5.19) and (5.71), the potential $\mathcal{P}_1$ satisfies

$$\begin{aligned}
\mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) &= G_0^3\widetilde{W}\left(\gamma+\phi_{\rm h}(u), L_0+\Lambda, e^{i\phi_{\rm h}(u)}(\eta_0+\alpha), e^{-i\phi_{\rm h}(u)}(\xi_0+\beta), G_0^2\widehat{r}_{\rm h}(u)\right)\\
&= \frac{\widetilde{\nu}G_0}{\widehat{r}_{\rm h}(u)}\left(\frac{m_0}{\left|1+\frac{\widetilde{\sigma}_0}{G_0^2}\frac{\widetilde{\rho}e^{iv}}{\widehat{r}_{\rm h}(u)}\sqrt{\frac{\eta_0+\alpha}{\xi_0+\beta}}e^{i\phi_{\rm h}(u)}\right|} + \frac{m_1}{\left|1-\frac{\widetilde{\sigma}_1}{G_0^2}\frac{\widetilde{\rho}e^{iv}}{\widehat{r}_{\rm h}(u)}\sqrt{\frac{\eta_0+\alpha}{\xi_0+\beta}}e^{i\phi_{\rm h}(u)}\right|} - (m_0+m_1)\right),
\end{aligned}$$

where the function $\widetilde{\rho}(\ell,L,\Gamma)e^{iv(\ell,L,\Gamma)}$ is evaluated at

$$\ell = \gamma - \frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta}, \quad L = L_0+\Lambda, \quad \Gamma = L_0+\Lambda-(\eta_0+\alpha)(\xi_0+\beta).$$

By (6.8) and (6.9), $\widetilde{\rho}e^{iv}$ can be written also as function of (ℓ, L, e_c), where e_c is the function introduced in (6.7). In coordinates (5.71), e_c can be written as

$$e_c = \mathcal{E}(\Lambda,\alpha,\beta)\sqrt{(\eta_0+\alpha)(\xi_0+\beta)} \quad \text{where} \quad \mathcal{E}(\Lambda,\alpha,\beta) = \frac{\sqrt{2(L_0+\Lambda)-(\eta_0+\alpha)(\xi_0+\beta)}}{L_0+\Lambda}. \tag{5.272}$$

Next lemma gives some crucial information about the Fourier expansion of the perturbed Hamiltonian $\mathcal{P}_1$ in the domains $D^u_{\kappa,\delta}$, $D^s_{\kappa,\delta}$ in (5.81).

Lemma 5.B.1. *Assume $|\alpha_0| < \zeta_0$, $|\beta_0| < \zeta_0$, where $\zeta_0 G_0^{3/2} \ll 1$, $1/2 \leq L_0 \leq 2$. Then, there exists $\sigma > 0$ such that for $u \in D^u_{\kappa,\delta} \cup D^s_{\kappa,\delta}$, $\gamma \in \mathbb{T}_\sigma$, $|\Lambda| \leq 1/4$, $|\alpha| \leq \zeta_0/4$, $|\beta| \leq \zeta_0/4$, the function $\mathcal{P}_1$ can be written as*

$$\mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) = \sum_{q\in\mathbb{Z}} \mathcal{P}_1^{[q]}(u,\Lambda,\alpha,\beta)e^{iq\gamma} \qquad \text{where} \qquad \mathcal{P}_1^{[q]}(u,\Lambda,\alpha,\beta) = \widehat{\mathcal{P}}_1^{[q]}(u,\Lambda,\alpha,\beta)e^{iq\phi_{\mathrm{h}}(u)}$$

for some coefficients $\widehat{\mathcal{P}}_1^{[q]}$ satisfying

$$\begin{aligned}
\left|\widehat{\mathcal{P}}_1^{[q]}\right| &\leq \frac{K}{G_0^3|\widehat{r}_{\mathrm{h}}(u)|^3}e^{-|q|\sigma} & \left|\partial_u\widehat{\mathcal{P}}_1^{[q]}\right| &\leq \frac{K}{G_0^3|\widehat{r}_{\mathrm{h}}(u)|^4}|\widehat{y}_{\mathrm{h}}(u)|e^{-|q|\sigma} \\
\left|\partial_\gamma\widehat{\mathcal{P}}_1^{[q]}\right| &\leq \frac{K}{G_0^3|\widehat{r}_{\mathrm{h}}(u)|^3}e^{-|q|\sigma} & \left|\partial_\Lambda\widehat{\mathcal{P}}_1^{[q]}\right| &\leq \frac{K}{G_0^3|\widehat{r}_{\mathrm{h}}(u)|^3}e^{-|q|\sigma} \\
\left|e^{-i\phi_{\mathrm{h}}(u)}\partial_\alpha\widehat{\mathcal{P}}_1^{[q]}\right| &\leq \frac{K}{G_0^3|\widehat{r}_{\mathrm{h}}(u)|^3}e^{-|q|\sigma} & \left|e^{i\phi_{\mathrm{h}}(u)}\partial_\beta\widehat{\mathcal{P}}_1^{[q]}\right| &\leq \frac{K}{G_0^3|\widehat{r}_{\mathrm{h}}(u)|^3}e^{-|q|\sigma}.
\end{aligned} \tag{5.273}$$

We devote the rest of this appendix to prove this lemma.

Proof of Lemma 5.B.1. To estimate the Fourier coefficients of $\mathcal{P}_1$ it is convenient to analyze first the Newtonian potential in Delaunay coordinates. Indeed, we consider the potential $\widetilde{W}$ in Delaunay coordinates,

$$V(\ell,L,\phi,\Gamma,\widetilde{r}) = \widetilde{W}\left(\ell+\phi, L, \sqrt{L-\Gamma}e^{i\phi}, \sqrt{L-\Gamma}e^{-i\phi}, \widetilde{r}\right)$$

which reads

$$V(\ell,L,\phi,\Gamma,\widetilde{r}) = \frac{\widetilde{\nu}}{\widetilde{r}}\left(\frac{m_0}{|1+\widetilde{\sigma}_0 n e^{i\phi}|} + \frac{m_1}{|1-\widetilde{\sigma}_1 n e^{i\phi}|} - (m_0+m_1)\right), \tag{5.274}$$

where $n = n(\ell,L,\phi,\Gamma,\widetilde{r}) = \frac{\widetilde{\rho}}{\widetilde{r}}e^{iv}$.

This potential can be rewritten as $V(\ell,L,\phi,\Gamma,\widetilde{r}) = \sum_{q\in\mathbb{Z}} V^{[q]}e^{iq\ell}$ with

$$\begin{aligned}
V^{[q]}(L,\phi,\Gamma,\widetilde{r}) &= \frac{1}{2\pi}\int_0^{2\pi} V(\ell,L,\phi,\Gamma,\widetilde{r})e^{-iq\ell}d\ell \\
&= \frac{\widetilde{\nu}}{2\pi\widetilde{r}}\int_0^{2\pi}\left(\frac{m_0}{|1+\widetilde{\sigma}_0 n e^{i\phi}|} + \frac{m_1}{|1-\widetilde{\sigma}_1 n e^{i\phi}|} - (m_0+m_1)\right)e^{-iq\ell}d\ell.
\end{aligned}$$

To estimate these integrals, we perform the change to the excentric anomaly

$$\ell = E - e_c\sin E, \qquad d\ell = (1-e_c\cos E)dE \tag{5.275}$$

and use that

$$\widetilde{\rho}e^{iv} = L^2\left(a^2e^{iE} - e_c + \frac{e_c^2}{4a^2}e^{-iE}\right), \qquad a = \frac{\sqrt{1+e_c}+\sqrt{1-e_c}}{2} \tag{5.276}$$

where $e_c = \frac{1}{L}\sqrt{L^2-\Gamma^2}$. Then we do a second change of variables $E+\phi = s$ to obtain

$$\begin{aligned}
V^{[q]} &= \frac{\widetilde{\nu}}{\widetilde{r}}\int_0^{2\pi}\left(\frac{m_0}{|1+\widetilde{\sigma}_0\widetilde{n}e^{i\phi}|} + \frac{m_1}{|1-\widetilde{\sigma}_1\widetilde{n}e^{i\phi}|} - (m_0+m_1)\right)e^{-iq(s-\phi-e_c\sin(s-\phi))}(1-e_c\cos(s-\phi))ds \\
&= e^{iq\phi}\frac{\widetilde{\nu}}{\widetilde{r}}\int_0^{2\pi}\left(\frac{m_0}{|1+\widetilde{\sigma}_0\widetilde{n}e^{i\phi}|} + \frac{m_1}{|1-\widetilde{\sigma}_1\widetilde{n}e^{i\phi}|} - (m_0+m_1)\right)e^{-iq(s-e_c\sin(s-\phi))}(1-e_c\cos(s-\phi))ds \\
&= e^{iq\phi}\widehat{V}^{[q]}(L,\phi,\Gamma,\widetilde{r})
\end{aligned}$$

where

$$\widetilde{n}(s,L,\phi,\Gamma,\widetilde{r})e^{i\phi} = n(s-\phi-e_c\sin(s-\phi),L,\phi,\Gamma,\widetilde{r})e^{i\phi} = \frac{1}{\widetilde{r}}L^2\left(a^2e^{is} - e_ce^{i\phi} + \frac{e_c^2}{4a^2}e^{-i(s-2\phi)}\right).$$

Now we relate these Fourier coefficients to those of $\mathcal{P}_1$. To this end, we relate Delaunay coordinates to the coordinates introduced in (5.71). One can see that the $(\ell, L, \phi, \Gamma, \widetilde{r}, \widetilde{y}) \to (u, Y, \gamma, \Lambda, \alpha, \beta)$ is given by

$$\begin{aligned}
\ell &= \gamma - \frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta} & L &= L_0+\Lambda \\
\phi &= \phi_{\rm h}(u) + \frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta} & \Gamma &= L-(\eta_0+\alpha)(\xi_0+\beta) \\
\widetilde{r} &= G_0^2\widehat{r}_{\rm h}(u) & \widehat{y} &= \frac{\widehat{y}_{\rm h}(u)}{G_0} + \frac{Y}{G_0^2\widehat{y}_{\rm h}(u)} + \frac{\Lambda-(\eta_0+\alpha)(\xi_0+\beta)+\eta_0\xi_0}{G_0^2\widehat{y}_{\rm h}(u)(\widehat{r}_{\rm h}(u))^2}.
\end{aligned} \tag{5.277}$$

Then,

$$\begin{aligned}
&\mathcal{P}_1(u,\gamma,\Lambda,\alpha,\beta) \\
&= G_0^3 V\left(\gamma - \frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta}, L_0+\Lambda, \phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta}, L_0+\Lambda-(\eta_0+\alpha)(\xi_0+\beta), G_0^2\widehat{r}_{\rm h}(u)\right) \\
&= \sum_{q\in\mathbb{Z}} e^{iq(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})} e^{iq(\gamma-\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})}\widehat{\mathcal{P}}_1^{[q]}(u,\Lambda,\alpha,\beta) \\
&= \sum_{q\in\mathbb{Z}} e^{iq(\phi_{\rm h}(u)+\gamma)}\widehat{\mathcal{P}}_1^{[q]}(u,\Lambda,\alpha,\beta)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\mathcal{P}}_1^{[q]}(u,\Lambda,\alpha,\beta) &= G_0^3\widehat{V}^{[q]}\left(L_0+\Lambda, \phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta}, L_0+\Lambda-(\eta_0+\alpha)(\xi_0+\beta), G_0^2\widehat{r}_{\rm h}(u)\right) \\
&= \frac{\widetilde{\nu}G_0}{2\pi\widehat{r}_{\rm h}(u)}\int_0^{2\pi}\left(\frac{m_0}{\left|1+\widetilde{\sigma}_0\widehat{n}e^{i(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})}\right|} + \frac{m_1}{\left|1-\widetilde{\sigma}_1\widehat{n}e^{i(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})}\right|} - (m_0+m_1)\right) \\
&\quad e^{-iq(s-e_c\sin(s-(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})))}\left(1-e_c\cos\left(s-\left(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta}\right)\right)\right)ds
\end{aligned}$$

where now

$$\widehat{n}e^{i(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})} = \frac{1}{G_0^2\widehat{r}_{\rm h}(u)}(L_0+\Lambda)^2\left(a^2e^{is} - e_ce^{i(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})} + \frac{e_c^2}{4a^2}e^{-i(s-2(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta}))}\right).$$

The first important observation is that, using the expresion for the eccentricity in (5.272) one has

$$e_ce^{\frac{1}{2}\log\frac{\eta_0+\alpha}{\xi_0+\beta}} = (\eta_0+\alpha)\mathcal{E}, \qquad e_ce^{-\frac{1}{2}\log\frac{\eta_0+\alpha}{\xi_0+\beta}} = (\xi_0+\beta)\mathcal{E}$$

which implies

$$\begin{aligned}
e_c\sin(s-(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta}))) &= \frac{\mathcal{E}}{2i}\left((\xi_0+\beta)e^{i(s-\phi_{\rm h}(u))} - (\eta_0+\alpha)e^{-i(s-\phi_{\rm h}(u))}\right) \\
e_c\cos(s-(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta}))) &= \frac{\mathcal{E}}{2}\left((\xi_0+\beta)e^{i(s-\phi_{\rm h}(u))} + (\eta_0+\alpha)e^{-i(s-\phi_{\rm h}(u))}\right)
\end{aligned}$$

and

$$\widehat{n}e^{i(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})} = \frac{(L_0+\Lambda)^2}{G_0^2\widehat{r}_{\rm h}(u)}\left(a^2e^{is} - (\eta_0+\alpha)\mathcal{E}e^{i\phi_{\rm h}(u)} + \mathcal{E}^2\frac{(\eta_0+\alpha)^2}{4a^2}e^{-i(s-2\phi_{\rm h}(u))}\right).$$

Now we observe that, taking into account that $1/2 \leq a \leq 2$, the asymptotics provided by Lemma 5.4.1 for $\widehat{r}_{\rm h}$ and $\phi_{\rm h}$, and the fact that $|\eta_0+\alpha|+|\xi_0+\beta| \leq \zeta_0$ imply

$$\left|(|\eta_0+\alpha|+|\xi_0+\beta|)\mathcal{E}e^{\pm i\phi_{\rm h}(u)}\right| \leq K\zeta_0 G_0^{3/2} \ll 1$$

we have that

$$\left|\widehat{n}e^{i(\phi_{\rm h}(u)+\frac{1}{2i}\log\frac{\eta_0+\alpha}{\xi_0+\beta})}\right|\leq\frac{K}{G_0^2|\widehat{r}_{\rm h}(u)|}\leq G_0^{-1/2}\leq\frac{1}{2}$$

under the hypotheses of the lemma. Therefore we have that, using the cancellations (observe that $\widetilde{\sigma}_0 m_0-\widetilde{\sigma}_1 m_1=0$),

$$\left|\widehat{\mathcal{P}}_1^{[q]}(u,\Lambda,\alpha,\beta)\right|\leq K\frac{1}{G_0^3|\widehat{r}_{\rm h}(u)|^3}.$$

The bounds for the derivatives can be made analogously differentiating the expressions for $\mathcal{P}_1$. □

5.C The Melnikov potential: Proof of Proposition 5.4.2

We devote this section to prove Proposition 5.4.2 which gives estimates for the Melnikov potential $\mathcal{L}$ introduced in (5.32). Note that $\mathcal{L}$ can be rewritten in terms of the perturbing potential $\mathcal{P}_1$ introduced in (5.78) as

$$\mathcal{L}(\sigma,\eta_0,\xi_0)=\int_{-\infty}^{+\infty}\mathcal{P}_1(s,\sigma+\omega s,0,0,0)\,ds\qquad\text{where}\qquad\omega=\frac{\nu G_0^3}{L_0^3}\quad\text{(see (5.33)).}$$

First, we obtain estimates for the harmonics different from $0,\pm1$ of the potential $\mathcal{L}$.

Lemma 5.C.1. *The q-Fourier coefficient of the Melnikov potential* (5.32) *with* $|q|\geq 2$ *can be bounded as*

$$\left|\mathcal{L}^{[q]}\right|\leq K^q G_0^{\frac{3|q|+1}{2}}\mathrm{e}^{-\frac{|q|\tilde{\nu}G_0^3}{3L_0^3}},\tag{5.278}$$

for some $K>0$ independent of G_0 and q.

Proof. The proof of this lemma is straighforward if one writes the Fourier coefficients of $\mathcal{L}$ in terms of the Fourier coefficients of the perturbing Hamiltonian $\mathcal{P}_1$ introduced in (5.78) (which can be also expressed in terms of the function $\widehat{\mathcal{P}}_1^{[q]}$ introduced in Lemma 5.B.1) as

$$\mathcal{L}^{[q]}=\int_{-\infty}^{+\infty}\mathcal{P}_1^{[q]}(u,0,0,0)\mathrm{e}^{i\frac{q\tilde{\nu}G_0^3}{L_0^3}u}du=\int_{-\infty}^{+\infty}\widehat{\mathcal{P}}_1^{[q]}(u,0,0,0)\mathrm{e}^{iq\phi_{\rm h}(u)}\mathrm{e}^{i\frac{q\tilde{\nu}G_0^3}{L_0^3}u}du.$$

Then, it is enough to change the path of the integral to $\operatorname{Im}u=\frac{1}{3}-G_0^3$ and use the bounds of Lemma 5.B.1 and that, by Lemma 5.4.1,

$$|\mathrm{e}^{qi\phi_{\rm h}(u)}|\leq K^q G_0^{\frac{3|q|}{2}}.$$

□

Now we give asymptotic formulas for the harmonics $q=0,\pm1$. It is easy to check that

$$\mathcal{L}^{[q]}(\eta_0,\xi_0)=\overline{\mathcal{L}^{[-q]}}(\xi_0,\eta_0)$$

and therefore it is enough to compute $q=0,1$.

Lemma 5.C.2. *The Fourier coefficients $\mathcal{L}^{[0]}$ and $\mathcal{L}^{[1]}$ satisfy*

$$\mathcal{L}^{[0]}(\eta_0,\xi_0)=\frac{\tilde{\nu}\pi}{8}L_0^4G_0^{-3}\left[N_2\left(1+\frac{3}{L_0}\eta_0\xi_0-\frac{3}{2L_0^2}(\eta_0\xi_0)^2\right)-\frac{15N_3L_0^2}{8\sqrt{2L_0}}G_0^{-2}(\eta_0+\xi_0)+\eta_0\xi_0\mathcal{O}_1(\eta_0,\xi_0)\right]$$

$$\mathcal{L}^{[1]}(\eta_0,\xi_0)=\frac{\tilde{\nu}\sqrt{\pi}}{4}\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}L_0^4G_0^{3/2}\left[\left(\frac{N_3L_0^2}{8\sqrt{2}}G_0^{-2}-\frac{3N_2}{\sqrt{L_0}}\eta_0\right)+\mathcal{O}\left(G_0^{-5/2},G_0^{-3/2}\eta_0,\eta_0\xi_0,\eta_0^2G_0,\xi_0^2G_0\right)\right],$$

where N_2 and N_3 are given in (5.35)

Proof. Proceeding as in the proof of Lemma 5.B.1, we use Delaunay coordinates. To this end, we first compute expansions of the potential V introduced in (5.274) in powers of $\widetilde{r}$ as $V = V_1 + V_2 + V_\geq$ with

$$V_1(\ell, L, \phi, \Gamma, \widetilde{r}) = N_2 \widetilde{\nu} \frac{\widetilde{\rho}^2}{4\widetilde{r}^3} \left(3\cos 2(v+\phi) + 1\right),$$
$$V_2(\ell, L, \phi, \Gamma, \widetilde{r}) = -N_3 \widetilde{\nu} \frac{\widetilde{\rho}^3}{8\widetilde{r}^4} \left(3\cos(v+\phi) + 5\cos 3(v+\phi)\right),$$

(see (5.35)) and $V_\geq$ is of the form $V_\geq = \frac{1}{\widetilde{r}} E$ and E is a function of $z = \frac{1}{\widetilde{r}} \widetilde{\rho} e^{i(v+\phi)}$ of order 4. Accordingly, we write the potential $\mathcal{P}_1(u, \gamma, \Lambda, \alpha, \beta)$ in (5.78) as

$$\mathcal{P}_1 = \mathcal{P}_{11} + \mathcal{P}_{12} + \mathcal{P}_{1\geq}.$$

Now we compute formulas for the Fourier coefficents of $\mathcal{P}_1^{[q]}$ with $q = 0, 1$. For the coefficients $\mathcal{P}_{1\geq}^{[q]}$, proceeding as in Lemma 5.B.1, one can prove that

$$\mathcal{P}_{1\geq}^{[q]}(u, 0, 0, 0) = \widehat{\mathcal{P}}_{1\geq}^{[q]}(u, 0, 0, 0) e^{iq\phi_{\mathrm{h}}(u)} \qquad \text{with} \qquad \left|\widehat{\mathcal{P}}_{1\geq}^{[q]}(u, 0, 0, 0)\right| \lesssim \frac{1}{G_0^7 |\widehat{r}_{\mathrm{h}}^5(u)|}. \tag{5.279}$$

For $\mathcal{P}_{11}$ and $\mathcal{P}_{12}$ we have explicit formulas. To compute their Fourier coefficients, we introduce the coefficients $C_q^{n,m}$, defined, following [DKdlRS19], by

$$\widetilde{\rho}^n(\ell, L, \Gamma) \mathrm{e}^{imv(\ell, L, \Gamma)} = \sum_{q\in\mathbb{Z}} C_q^{n,m}(L, e_c) \mathrm{e}^{iq\ell} = \sum_{q\in\mathbb{Z}} \left(\frac{\xi_0}{\eta_0}\right)^{q/2} C_q^{n,m}(L_0, e_c) \mathrm{e}^{iq\gamma} \tag{5.280}$$

where $\ell = \gamma - \frac{1}{2i}\log(\eta_0/\xi_0)$, $L = L_0$, $\Gamma = L_0 - \eta_0\xi_0$ (see (5.277)).

The coefficients $C_q^{n,m}$ depend on L and e_c is the eccentricity (see (5.272)). Then, recalling also $\phi = \phi_{\mathrm{h}}(u) + \frac{1}{2i}\ln\left(\frac{\eta_0}{\xi_0}\right)$, we obtain

$$\begin{aligned}
\mathcal{P}_{11}^{[q]}(u, 0, 0, 0) &= \frac{N_2\widetilde{\nu}}{4G_0^3 \widehat{r}_{\mathrm{h}}^3(u)} \left(C_q^{2,0} + \frac{3}{2} C_q^{2,2} e^{2i\phi} + \frac{3}{2} C_q^{2,-2} e^{-2i\phi}\right) \left(\frac{\xi_0}{\eta_0}\right)^{q/2} \\
&= \frac{N_2\widetilde{\nu}}{4G_0^3 \widehat{r}_{\mathrm{h}}^3(u)} \left(C_q^{2,0} + \frac{3}{2} C_q^{2,2} e^{2i\phi_{\mathrm{h}}(u)} \left(\frac{\xi_0}{\eta_0}\right)^{-1} + \frac{3}{2} C_q^{2,-2} e^{-2i\phi_{\mathrm{h}}(u)} \left(\frac{\xi_0}{\eta_0}\right)\right) \left(\frac{\xi_0}{\eta_0}\right)^{q/2} \\
\mathcal{P}_{12}^{[q]}(u, 0, 0, 0) &= -\frac{N_3\widetilde{\nu}}{8G_0^5 \widehat{r}_{\mathrm{h}}^4(u)} \left(\frac{3}{2} C_q^{3,1} e^{i\phi} + \frac{3}{2} C_q^{3,-1} e^{-i\phi} + \frac{5}{2} C_q^{3,3} e^{3i\phi} + \frac{5}{2} C_q^{3,-3} e^{-3i\phi}\right) \left(\frac{\xi_0}{\eta_0}\right)^{q/2} \\
&= -\frac{N_3\widetilde{\nu}}{8G_0^5 \widehat{r}_{\mathrm{h}}^4(u)} \left(\frac{3}{2} C_q^{3,1} e^{i\phi_{\mathrm{h}}(u)} \left(\frac{\xi_0}{\eta_0}\right)^{-1/2} + \frac{3}{2} C_q^{3,-1} e^{-i\phi_{\mathrm{h}}(u)} \left(\frac{\xi_0}{\eta_0}\right)^{1/2}\right. \\
&\qquad \left. + \frac{5}{2} C_q^{3,3} e^{3i\phi_{\mathrm{h}}(u)} \left(\frac{\xi_0}{\eta_0}\right)^{-3/2} + \frac{5}{2} C_q^{3,-3} e^{-3i\phi_{\mathrm{h}}(u)} \left(\frac{\xi_0}{\eta_0}\right)^{3/2}\right) \left(\frac{\xi_0}{\eta_0}\right)^{q/2},
\end{aligned}$$

where N_2 and N_3 are given by in (5.35).

The functions $\mathcal{L}^{[q]}$, $q = 0, 1$, can be written as $\mathcal{L}^{[q]} = \mathcal{L}_1^{[q]} + \mathcal{L}_2^{[q]} + \mathcal{L}_\geq^{[q]}$ with

$$\mathcal{L}_i^{[q]} = \int_{-\infty}^{+\infty} \mathcal{P}_{1i}^{[q]}(s, 0, 0, 0) e^{iq\omega s} ds, \qquad i = 1, 2, \geq.$$

We first give estimates for the last terms. For $\mathcal{L}_\geq^{[1]}$ it is enough to change the path of integration to $\operatorname{Im} s = 1/3 - G_0^{-3}$ and use (5.279) and the properties in Lemma 5.4.1. For $\mathcal{L}_\geq^{[0]}$ one can estimate the integral directly. Then, one can obtain

$$\left|\mathcal{L}_\geq^{[0]}\right| \lesssim G_0^{-7} \qquad \text{and} \qquad \left|\mathcal{L}_\geq^{[1]}\right| \lesssim G_0^{-1} \mathrm{e}^{-\frac{\widetilde{\nu} G_0^3}{3L_0^3}}. \tag{5.281}$$

Thus, it only remains to obtain formulas for $\mathcal{L}_i^{[q]}$, $q=0,1$, $i=1,2$. Using the formulas in (5.24), they are given in terms of the integrals

$$\int_{-\infty}^{+\infty}\frac{1}{\widehat{r}_{\rm h}^{2l-k+1}(t)}\mathrm{e}^{-ik\phi_{\rm h}(t)}\mathrm{e}^{iq\omega t}dt=\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{iq\omega(\tau+\frac{\tau^3}{3})}}{(\tau-i)^{2l-2k}(\tau+i)^{2l}}d\tau\qquad\text{for } l\geq k\geq 0$$

$$\int_{-\infty}^{+\infty}\frac{1}{\widehat{r}_{\rm h}^{k-2l+1}(t)}\mathrm{e}^{-ik\phi_{\rm h}(t)}\mathrm{e}^{iq\omega t}dt=\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{iq\omega(\tau+\frac{\tau^3}{3})}}{(\tau-i)^{-2l}(\tau+i)^{2k-2l}}d\tau\qquad\text{for } l\leq k\leq -1.$$

Therefore, following the notation of [DKdlRS19], if we introduce

$$N(q,m,n)=\frac{2^{m+n}}{G_0^{2m+2n-1}}\binom{-1/2}{m}\binom{-1/2}{n}\int_{-\infty}^{+\infty}\frac{e^{iq\omega(\tau+\frac{\tau^3}{3})}}{(\tau-i)^{2m}(\tau+i)^{2n}}d\tau,\tag{5.282}$$

one can write these functions as

$$\begin{aligned}
\mathcal{L}_1^{[0]}=&\frac{N_2\widetilde{\nu}}{4}\left(C_0^{2,0}N(0,1,1)+\left(\frac{\xi_0}{\eta_0}\right)^{-1}C_0^{2,2}N(0,2,0)+\left(\frac{\xi_0}{\eta_0}\right)C_0^{2,-2}N(0,0,2)\right)\\
\mathcal{L}_1^{[1]}=&\frac{N_2\widetilde{\nu}}{4}\left(\left(\frac{\xi_0}{\eta_0}\right)^{1/2}C_1^{2,0}N(1,1,1)+\left(\frac{\xi_0}{\eta_0}\right)^{-1/2}C_1^{2,2}N(1,2,0)+\left(\frac{\xi_0}{\eta_0}\right)^{3/2}C_1^{2,-2}N(1,0,2)\right)\\
\mathcal{L}_2^{[0]}=&\frac{N_3\widetilde{\nu}}{8}\left(\left(\frac{\xi_0}{\eta_0}\right)^{-1/2}C_0^{3,1}N(0,2,1)+\left(\frac{\xi_0}{\eta_0}\right)^{1/2}C_0^{3,-1}N(0,1,2)\right.\\
&\left.+\left(\frac{\xi_0}{\eta_0}\right)^{-3/2}C_0^{3,3}N(0,3,0)+\left(\frac{\xi_0}{\eta_0}\right)^{3/2}C_0^{3,-3}N(0,3,0)\right)\\
\mathcal{L}_2^{[1]}=&\frac{N_3\widetilde{\nu}}{8}\left(C_1^{3,1}N(1,2,1)+\left(\frac{\xi_0}{\eta_0}\right)C_1^{3,-1}N(1,1,2)\right.\\
&\left.+\left(\frac{\xi_0}{\eta_0}\right)^{-1}C_1^{3,3}N(1,3,0)+\left(\frac{\xi_0}{\eta_0}\right)^{2}C_1^{3,-3}N(1,0,3)\right)
\end{aligned}\tag{5.283}$$

It can be easly check that the functions N satisfy

$$N(0,0,k)=N(0,k,0)=0,\quad\text{for}\quad k\geq 2\qquad\text{and}\qquad N(-q,m,n)=N(q,n,m).$$

The leading terms of the integrals in (5.282) are given in [DKdlRS19] (see Lemma 30 and the proof of Lemma 36),

$$\begin{aligned}
N(0,1,1)&=\frac{\pi}{2}G_0^{-3} & N(1,2,1)&=\frac{1}{4}\sqrt{\frac{\pi}{2}}G_0^{-\frac{1}{2}}\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}\left(1+\mathcal{O}(G_0^{-3/2})\right)\\
N(0,2,1)&=\frac{3\pi}{8}G_0^{-5} & N(1,2,0)&=\sqrt{\frac{\pi}{2}}G_0^{\frac{3}{2}}\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}\left(1+\mathcal{O}(G_0^{-3/2})\right)\\
N(0,1,2)&=\frac{3\pi}{8}G_0^{-5} & N(1,3,0)&=\frac{1}{3}\sqrt{\frac{\pi}{2}}G_0^{\frac{5}{2}}\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}\left(1+\mathcal{O}(G_0^{-3/2})\right)
\end{aligned}$$

and

$$\begin{aligned}
N(1,1,1)&=\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}\mathcal{O}\left(G_0^{-3/2}\right), & N(1,0,2)&=\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}\mathcal{O}\left(G_0^{-9/2}\right)\\
N(1,1,2)&=\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}\mathcal{O}\left(G_0^{-7/2}\right), & N(1,0,3)&=\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}\mathcal{O}\left(G_0^{-13/2}\right).
\end{aligned}$$

Now it remains to estimate some of the coefficients $C_q^{n,m}$ in (5.280), defined as

$$C_q^{n,m}(L_0,e_c)=\frac{L_0^{2n}}{2\pi}\int_0^{2\pi}\widetilde{\rho}^n(\ell,L_0,e_c)e^{imv(\ell,L_0,e_c)}e^{-iq\ell}d\ell.$$

Proceeding as in (5.275), we change variables to the eccentric anomaly. Then, using (6.8) and (6.9) to express $\widetilde{\rho}$ and v in terms of the eccentric anomaly, one obtains

$$C_q^{n,m}(L_0,e_c) = \frac{L_0^{2n}}{2\pi}\int_0^{2\pi}\left(a^2 e^{iE} + \frac{e_c^2}{4a^2}e^{-iE} - e_c\right)^m (1-e_c\cos E)^{n+1-m}\, e^{-iq(E-e_c\sin E)}dE, \qquad (5.284)$$

where a has been defined in (5.276). These formulas easily imply the symmetries

$$C_q^{n,m}(e_c) = C_{-q}^{n,-m}(e_c) = \overline{C}_q^{n,m}(e_c) \qquad \text{and} \qquad C_q^{n,m}(e_c) = (-1)^{q+m}C_q^{n,m}(-e_c).$$

One can also compute them, as was already done in [DKdlRS19], to obtain

$$\begin{aligned}
C_0^{2,0} &= L_0^4\left(1+\frac{3}{L_0}\eta_0\xi_0 - \frac{3}{2L_0^2}(\eta_0\xi_0)^2\right)\\
C_0^{3,-1}\left(\frac{\xi_0}{\eta_0}\right)^{1/2} &= -\frac{5}{\sqrt{2L_0}}L_0^6\left(\xi_0 + \mathcal{O}(\xi_0^2\eta_0)\right)\\
C_0^{3,1}\left(\frac{\xi_0}{\eta_0}\right)^{-1/2} &= -\frac{5}{\sqrt{2L_0}}L_0^6\left(\eta_0 + \mathcal{O}(\eta_0^2\xi_0)\right)\\
C_1^{3,1} &= L_0^6\left(1+\mathcal{O}(\eta_0\xi_0)\right)\\
\left(\frac{\xi_0}{\eta_0}\right)^{-1/2} C_1^{2,2} &= -3L_0^4\sqrt{\frac{2}{L_0}}\eta_0 + \mathcal{O}\left(\eta_0^2\xi_0\right).
\end{aligned}$$

They also can be easily bounded, switching the integration path in (5.284) to either $\operatorname{Im} E = \log e_c$ (if $q-m>0$) or $\operatorname{Im} E = -\log e_c$ (if $q-m<0$), as

$$|C_q^{n,m}(e_c)| \lesssim e_c^{|m-q|}.$$

Using this estimate, one obtains

$$\begin{aligned}
&\left(\frac{\xi_0}{\eta_0}\right)^{1/2} C_1^{2,0} = \mathcal{O}(\xi_0), \qquad \left(\frac{\xi_0}{\eta_0}\right)^{3/2} C_1^{2,-2} = \mathcal{O}\left(\xi_0^3\right), \qquad \left(-\frac{\xi_0}{\eta_0}\right) C_1^{3,-1} = \mathcal{O}\left(\xi_0^2\right)\\
&\left(\frac{\xi_0}{\eta_0}\right)^{-1} C_1^{3,3} = \mathcal{O}\left(\eta_0^2\right), \qquad \left(\frac{\xi_0}{\eta_0}\right)^{2} C_1^{3,-3} = \mathcal{O}\left(\xi_0^4\right).
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{L}_1^{[0]} &= \frac{N_2\widetilde{\nu}\pi}{8}L_0^4 G_0^{-3}\left(1+\frac{3}{L_0}\eta_0\xi_0 - \frac{3}{2L_0^2}(\eta_0\xi_0)^2\right)\\
\mathcal{L}_1^{[1]} &= -\frac{N_2\widetilde{\nu}}{4}\sqrt{\frac{\pi}{L_0}}G_0^{\frac{3}{2}}\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}3L_0^4\eta_0\left(1+\mathcal{O}(\eta_0\xi_0, G_0^{-3/2})\right)\\
\mathcal{L}_2^{[0]} &= -\frac{15N_3\widetilde{\nu}\pi L_0^6}{64\sqrt{2L_0}}G_0^{-5}\left(\eta_0+\xi_0+\eta_0\xi_0\mathcal{O}_1(\eta_0,\xi_0)\right)\\
\mathcal{L}_2^{[1]} &= \frac{N_3\widetilde{\nu}}{32}\sqrt{\frac{\pi}{2}}L_0^6 G_0^{-\frac{1}{2}}\mathrm{e}^{-\frac{\tilde{\nu}G_0^3}{3L_0^3}}\left(1+\mathcal{O}\left(\eta_0\xi_0, \xi_0^2G_0^{-3}, G_0^{-3/2}, \eta_0^2G_0^3\right)\right).
\end{aligned}$$

These formulas and (5.281) give the asymptotic expansions stated in Lemma 5.C.2. □

To complete the proof of Proposition 5.4.2 it is enough to use the relation between Θ, $\tilde{\Theta}$ and G_0 in (5.34), (5.33).

Chapter 6

Global instability in the 3 Body Problem: The Melnikov approximation

Abstract: The 3 Body Problem is a Hamiltonian system which models the motion of three bodies interacting via Newtonian gravitation. Understanding its global dynamics is one of the more challenging question in dynamics.

This is the first of a series of papers devoted to prove the existence of *global instability* in the 3 Body Problem on negative energy levels. We focus on the *hierarchical regime* where one body is far away from the other two, whose instantaneous, relative motion, happens on an ellipse, and, in particular, are interested in the existence of orbits for which the angular momentum of the far body is transferred to the binary, elliptic system, resulting in a substantial change of its eccentricity (from nearly circular to almost collision).

Our approach relies on the existence of a (topological) Normally Hyperbolic Invariant Cylinder $\mathcal{E}_\infty$, located "at infinity" and can be seen as a rather non trivial extension of the previous results [DKdlRS19, GPS23b] for the restricted case. In this first paper we describe the mechanism and introduce the so-called *Melnikov approximation*, a crucial tool for proving the existence of transverse intersections between $\mathcal{E}_\infty$. The validity of the Melnikov approximation and the construction of a transition chain of periodic orbits in $\mathcal{E}_\infty$ leading to topological instability, will be the subject of a future work.

6.1 Introduction

The 3 Body Problem models the motion of three bodies interacting via Newtonian gravitation. We consider the planar case, in which the three bodies move on the same plane. Its dynamics is given by the Hamiltonian system

$$H_{3\mathrm{BP}}(q,p) = \sum_{i=0}^{2} \frac{|p_i|^2}{2m_i} - \sum_{0\le i<j\le 2} \frac{m_i m_j}{|q_i - q_j|} \qquad (q,p) \in \mathbb{R}^{12} \setminus \Delta \tag{6.1}$$

where $m_i > 0$, $i = 0, 1, 2$, and $\Delta = \{q_i = q_j \text{ for some } 0 \le i < j \le 2\}$ corresponds to the collision set. In the so-called *hierarchical regime*, that is, the region of the phase space where one body is far from the other two, the Hamiltonian (6.1) can be studied as a small perturbation of two uncoupled Kepler problems: one describing the dyamics of the binary system and one describing the motion of the far away body q_3 with respect to the center of mass of the other two q_1, q_2. In this *nearly integrable setting*, one can try to understand the mechanisms giving raise to stable and unstable motions making use of the techniques of Hamiltonian perturbation theory.

The first major achievement towards understanding this picture, was obtained by Arnold in [Arn63], who gave a master application of the KAM techniques to prove the existence of a positive measure set of quasiperiodic motions in the coplanar 3 Body Problem. The proof was later extended to case of $N \geq 3$ bodies in the work of Féjoz and Herman [Fej04] (see also [Rob95, CP11]). On the other hand, in accordance with the general belief that the N Body Problem, although strongly degenerate, displays the main features of a "typical" Hamiltonian system, in his ICM address, Herman conjectured [Her98] that the set of nonwandering points for the flow of the N Body Problem is nowhere dense on every energy level for $N \geq 3$. This would imply topological instability for the N Body Problem in a very strong sense.

At the moment, this conjecture seems largely out of reach, and the very few rigourous examples of topological instability in Celestial Mechanics were given quite recently in [CG18, DKdlRS19, GPS23b] for the Restricted 3 Body Problem and in [CFG22, CFG23] for the spatial 4 Body Problem. In all these works, the underlying mechanism, is the so called *Arnold diffusion mechanism*. This mechanism, proposed by Arnold in his seminal study of topological instability in nearly integrable Hamiltonian systems (see [Arn64]), is based on the existence of a *transition chain of invariant tori*, that is, a sequence of partially hyperbolic invariant tori connected by transverse heteroclinic orbits between them. In modern language, the Arnold diffusion mechanism relies on the existence of a Normally Hyperbolic Invariant Manifold (NHIM) whose stable and unstable manifolds intersect transversally along a homoclinic manifold. Then, if the inner dynamics on the NHIM contains "sufficient" quasiperiodic invariant tori (or other invariant objects such as Aubry-Mather sets), one can combine the outer excursions along the homoclinic manifold with quasiperiodic inner dynamics (or orbits shadowing the Aubry-Mather sets) to obtain a transition chain leading to topological instability.

When studying the existence of Arnold diffusion in concrete models, for example in Celestial Mechanics, usually, one of the main difficulties is to prove that the invariant manifolds of the NHIM intersect transversally. In regular perturbation frameworks, i.e. when there exist no different time scales, this problem can be tackled by means of the so-called Poincaré-Melnikov theory, which gives an asymptotic formula for the distance (measured along a suitable section) between these invariant manifolds (see, for example, [DdlLS06, DdlLS08], and see [DKdlRS19] for an application to Celestial Mechanics). However, when there exist different time scales, one needs much more extra work to prove that the distance between the invariant manifolds can be approximated by a (modified) Melnikov function (see [LMS03, GPS23b]).

This paper is the first of a series of two papers devoted to study the existence of Arnold diffusion in the 3 Body Problem. Here, we identify a (topological) NHIM for the 3BP and introduce the Melnikov approximation for studying the existence of transverse intersections between its stable and unstable manifolds. The second paper will be devoted to justify rigourously the Melnikov approximation as well as the construction of unstable motions.

6.1.1 A degenerate Arnold diffusion mechanism for the Restricted 3BP

The 3 Body Problem is called *restricted* if two bodies, the *primaries*, have strictly positive masses, and the third one has zero mass. Our approach to study the existence of Arnold Diffusion in the 3 Body Problem is strongly based on the previous works [DKdlRS19, GPS23b], where the authors considered the Restricted Planar Elliptic 3 Body Problem. In this model, the primaries move, according to Kepler laws, on ellipses of eccentricity $\epsilon_0 \in (0,1)$. The authors in [DKdlRS19, GPS23b] show the existence of a *transition chain of periodic orbits* located in the region of the phase space where the third body is far from the other two. In this hierarchical regime, the R3BP can be studied as a periodic perturbation of the 2 Body Problem describing the motion of the massless body with respect to the center of mass of the primaries. The constructed transition chain of periodic orbits leads to topological instability in the following sense: there exist orbits along which the angular momentum $G = |q \wedge p|$ of the massless body, a conserved quantity in the 2 Body Problem approximation, experiences arbitrarily long variations provided the eccentricity of the primaries orbit is positive, but sufficiently small.

More concretely, for the R3BP, there exists a 3-dimensional "invariant manifold at infinity" $\mathcal{P}_\infty$, corresponding to the ω-limit set (resp. α-limit set) of the points which lead to forward (resp. backwards) parabolic motions (when the massless body tends to infinity with asymptotic zero velocity). The manifold $\mathcal{P}_\infty$ is strongly degenerate in the sense that it is completely foliated by periodic orbits. Although the linearized vector field vanishes on the normal directions to $\mathcal{P}_\infty$, this manifold possesses 4-dimensional stable and unstable manifolds $W^{s,u}(\mathcal{P}_\infty)$, which are indeed the set of forward and backwards parabolic motions. For $G_* \gg 1$, the submanifolds $W^{u,s}(\mathcal{P}_\infty \cap \{G \geq G_*\})$ are contained in the *hierarchical region* (third body far from the primaries) and, if the eccentricity of the primaries is small enough, these manifolds intersect transversally along two different homoclinic manifolds, which are moreover diffeomorphic to $\mathcal{P}_\infty^* = \mathcal{P}_\infty \cap \{G_* < G < \epsilon^{-1/3}\}$ (where ϵ is the eccentricity of the primaries orbit). Therefore, one can define two different *global scattering maps* on $\mathcal{P}_\infty^*$ encoding the dynamics along the two different homoclinic manifolds. Since the inner dynamics on $\mathcal{P}_\infty$ is trivial, one cannot rely on the classical approach of combining inner quasiperiodic dynamics and the outer dynamics of one scattering map to prove the existence of drifting orbits. However, since both scattering maps are defined globally, by analyzing the dynamics of the iterated function system given by them, one can show the existence of orbits along which the angular momentum grows from $G \leq G_*$ to $G \geq \epsilon^{-1/3}$.

6.1.2 The parabolic-elliptic regime and Arnold Diffusion in the 3BP

We will see in Section 6.2 that, for the 3BP, on each constant, negative energy hypersurface, there exists a 3-dimensional invariant submanifold at infinity $\mathcal{E}_\infty$. It corresponds to the ω-limit set (resp. α-limit set) of the points which lead to forward (resp. backwards) orbits along which the motion of one body is parabolic and the motion the other two bodies is elliptic. Since at $\mathcal{E}_\infty$ the distance between one body (the one performing the parabolic motion) and the other two (the binary, elliptic, system) is infinite, the coupling in the hierarchical approximation vanishes identically on $\mathcal{E}_\infty$. Thus, the dynamics on $\mathcal{E}_\infty$ is completely integrable. Moreover, due to the so-called super integrability of the 2 Body Problem, $\mathcal{E}_\infty$ is foliated by periodic orbits.

It is known (see [BFM20c]), that $\mathcal{E}_\infty$ possesses 4-dimensional stable and unstable invariant manifolds. Therefore, one can try to extend the techniques developed in [DKdlRS19, GPS23b] to prove the existence of a transition chain of periodic orbits in the 3BP. Indeed, due to the robustness of the mechanism, one could directly prove that the transition chain of heteroclinic orbits constructed in [DKdlRS19, GPS23b] for the R3BP can be continued to the 3BP if the mass m_2 is sufficiently small. Therefore, one can deduce that, in the 3BP, if m_2 is sufficiently small, there exist orbits along which the angular momentum G of the third body experiences significant variations, while the eccentricity of the inner bodies remains small.

In the present work, we are interested in proving the existence of Arnold Diffusion in the 3BP for any choice of the masses $m_0, m_1, m_2 > 0$. The substantial difference is that, due to the conservation of the total angular momentum, as the angular momentum of the third body grows, so does the eccentricity of the orbit of the binary system [1]. However, in order to construct orbits along which this transfer of angular momentum is significant, one cannot make use of the arguments developed in [DKdlRS19, GPS23b], since they strongly rely on the hypothesis that the eccentricity of the primaries orbit is small enough. Thus, new techniques have to be developed to, in particular, analyze the existence of transverse intersections between the invariant manifolds of $\mathcal{E}_\infty$ (see also Section 6.4 and, in particular, the discussion at the end of Section 6.4.2 where we outline the main technical difficulties).

Our goal in this series of papers can now be stated clearly: We want to construct a transition chain of periodic orbits contained in $\mathcal{E}_\infty$, along which the angular momentum of the third body is transferred to the binary system, resulting in a substantial change of its eccentricity. In particular, we want to construct orbits which transition from close to circular orbits to highly eccentric ellipses (i.e., with close to collision points) This first paper is devoted to analyze the so called Melnikov approximation of the distance between the invariant manifolds of $\mathcal{E}_\infty$.

Remark 6.1.1. *Notice that, in [DKdlRS19, GPS23b], the variations of the angular momentum G of the massless body are bounded above by the value of $1/\epsilon$. This limitation, as already commented, is due to the fact that we were able to prove the existence of scattering maps only in the submanifold $\mathcal{P}_\infty^* = \mathcal{P}_\infty \cap \{G_* < G < \epsilon^{-1/3}\}$ (see Section 6.1.1). With the techniques that we develop in these series of papers, we can remove that limitation and prove, that, for any $\epsilon \in (0,1)$, there exist orbits of the RPE3BP which present an* unbounded growth *of the angular momentum of the massless body.*

[1]The change of eccentricity due to the transfer of angular momentum goes to zero with $m_2 \to 0$.

The parabolic-elliptic regime and hyperbolic sets in the 3BP

The parabolic-elliptic regime has recently been considered in [GMPS22] to establish the existence of non trivial hyperbolic sets in the 3BP. To prove this result, the authors focus on an invariant submanifold $\mathcal{E}_{\infty,\mathrm{circ}} \subset \mathcal{E}_\infty$ corresponding to nearly circular motion of the bodies q_0, q_1, and prove that its stable and unstable invariant manifolds intersect transversally. Then, they prove the existence of a non trivial hyperbolic set for the return map to a suitable section close to the transverse intersection. The construction is rather involved due to the existence of center directions.

Notice, however, that, since the hyperbolic set is contained in the region of the phase space where the binary, elliptic system, performs close to circular motions, it does not lead to topological instability, in the sense that it does not contain orbits along which the angular momentum of the third body is transferred to the binary system, whose eccentricity is always close to zero for all the orbits in the hyperbolic set.

A remarkably interesting, but even more challenging question, is to study how the eccentricities associated to initial conditions lying on a sufficiently small neighbourhood of $\mathcal{E}_\infty$ distribute after sufficiently long time. This would require to combine, and extend considerably, the ideas from Sections 6.1.2 and 6.1.2. We hope to come back to this question in the future.

6.1.3 The Melnikov approximation

We focus, for $G_* \gg 1$, on the invariant submanifold $\mathcal{E}_\infty \cap \{G \geq G_*\}$ where G stands for the angular momentum of the third body. The reason is that its stable and unstable manifolds are contained in the hierarchichal region and therefore, can be analyzed perturbatively. In Section 6.4, we give an heuristic argument to obtain an asymptotic formula, up to polynomially small relative errors in $1/G_*$, for the distance between $W^u(\mathcal{E}_\infty \cap \{G \geq G_*\})$ and $W^s(\mathcal{E}_\infty \cap \{G \geq G_*\})$. This first order is the so-called Melnikov approximation.

Our perturbative setting corresponds, however, to a singular perturbation framework, since there exist *different time scales*: the fast dynamics of the binary system and the slow dynamics of the parabolic motion of the third body, whose interaction with the binary system is weak due the decay of Newtonian gravitation with distance. The effect of the coupling averages then to an exponentially small reminder in $1/G_*$, [2] which, as a matter of fact, bounds the distance between $W^u(\mathcal{E}_\infty \cap \{G \geq G_*\})$ and $W^s(\mathcal{E}_\infty \cap \{G \geq G_*\})$. Therefore, in order for the Melnikov approximation to yield a valid asymptotic formula, one needs to show that the errors in the Melnikov approximation are also exponentially small in $1/G_*$.

In this work, we only compute the Melnikov approximation of the distance between $W^u(\mathcal{E}_\infty \cap \{G \geq G_*\})$ and $W^s(\mathcal{E}_\infty \cap \{G \geq G_*\})$. The rigourous justification of this approximation, and the construction of a transition chain of periodic orbits contained in $W^u(\mathcal{E}_\infty \cap \{G \geq G_*\}) \cap W^s(\mathcal{E}_\infty \cap \{G \geq G_*\})$ will be the subject of a separate paper.

6.1.4 Organization of the article

In Section 6.2 we introduce a suitable coordinate system in which the dynamics between the binary (elliptic) system and the motion of the third (parabolic) body are uncoupled up to higher order interactions. In Section 6.3 we study the dynamics associated to the two uncoupled 2 Body Problems which gives the first order dynamics of the parabollic-elliptic regime. In particular, we show that there exists an unperturbed homoclinic manifold to $\mathcal{E}_\infty$. In Section 6.4 we introduce an adapted coordinate system in a neighbourhood of the unperturbed homoclinic manifold, which is suitable for analyzing the existence of transverse intersections between the perturbed manifolds and give an heuristic justification of the Melnikov approximation. In Section 6.5 we state our main result: namely, we give an asymptotic formula of the so-called *Melnikov potential* and outline the proof of this result. Sections 6.6 and 6.7 are devoted to complete the proof of the main result.

[2] The quantity $1/G_*$ can be taken as a measure of the ratio between the two time scales of the parabolic-elliptic regime.

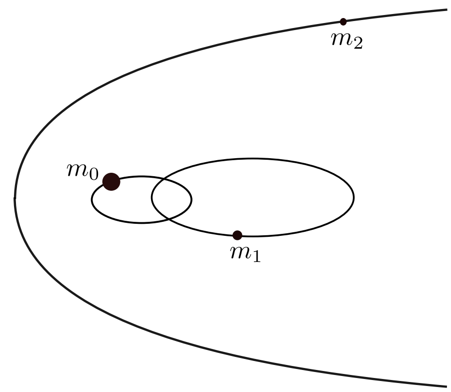


Figure 6.1: Sketch of an instantaneous configuration in the parabolic-elliptic regime.

6.2 A good coordinate system for the parabolic-elliptic regime

In this section we introduce a coordinate system suitable for describing the region of the phase space where, *up to higher order interactions*, the bodies $q_0, q_1 \in \mathbb{R}^2$ form a binary system with negative energy, i.e. they revolve around each other in Keplerian *ellipses*, and the third body $q_2 \in \mathbb{R}^2$ is located far from q_0, q_1 and performs a zero energy, i.e. *parabolic*, motion with respect to the center of mass of q_0, q_1 (see Figure 6.2).

6.2.1 Symplectic reduction of the planar 3 Body Problem

The Hamiltonian $H_{3\text{BP}}$ in (6.1) defines a six degrees of freedom Hamiltonian system. We start by reducing it by translations with the classical Jacobi coordinates to obtain a four degrees of freedom Hamiltonian system. That is, we define the symplectic transformation

$$\begin{aligned} Q_0 &= q_0 & P_0 &= p_0 + p_1 + p_2 \\ Q_1 &= q_1 - q_0 & P_1 &= p_1 + \frac{m_1}{m_0 + m_1} p_2 \\ Q_2 &= q_2 - \frac{m_0 q_0 + m_1 q_1}{m_0 + m_1} & P_2 &= p_2. \end{aligned}$$

These coordinates allow to reduce by the total linear momentum since now P_0 is a first integral. Assuming $P_0 = 0$, the Hamiltonian of the 3 Body Problem becomes

$$H^*(Q_1, P_1, Q_2, P_2) = \sum_{j=1}^{2} \frac{|P_j|^2}{2\mu_j} - V^*(Q_1, Q_2)$$

where

$$\frac{1}{\mu_1} = \frac{1}{m_0} + \frac{1}{m_1}, \qquad \frac{1}{\mu_2} = \frac{1}{m_0 + m_1} + \frac{1}{m_2}$$

and

$$V^*(Q_1, Q_2) = \frac{m_0 m_1}{|Q_1|} + \frac{m_0 m_2}{|Q_2 + \sigma_0 Q_1|} + \frac{m_1 m_2}{|Q_2 + \sigma_1 Q_1|}$$

with

$$\sigma_0 = \frac{m_1}{m_0 + m_1}, \qquad \sigma_1 = \frac{-m_0}{m_0 + m_1} = \frac{-1}{1 + \sigma_0}. \tag{6.2}$$

Next step is to express the Hamiltonian H^* in polar coordinates. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we consider the symplectic transformation

$$Q_1 = \varrho e^{i\theta}, \qquad Q_2 = r e^{i\alpha}, \qquad P_1 = z e^{i\theta} + i \frac{\Gamma}{\varrho} e^{i\theta}, \qquad P_2 = y e^{i\alpha} + i \frac{G}{r} e^{i\alpha}$$

which leads to the Hamiltonian

$$H_{\rm pol}(r,\varrho,\alpha,\theta,y,z,G,\Gamma)=\frac{1}{\mu_1}\left(\frac{z^2}{2}+\frac{\Gamma^2}{2\varrho^2}\right)+\frac{1}{\mu_2}\left(\frac{y^2}{2}+\frac{G^2}{2r^2}\right)-V_{\rm pol}\left(r,\varrho,\theta-\alpha\right),$$

where

$$\begin{aligned}V_{\rm pol}\left(r,\varrho,\theta-\alpha\right)=V^*\left(\varrho e^{i\theta},re^{i\alpha}\right)&=\frac{m_0m_1}{\varrho}+\frac{m_0m_2}{|re^{i\alpha}+\sigma_0\varrho e^{i\theta}|}+\frac{m_1m_2}{|re^{i\alpha}+\sigma_1\varrho e^{i\theta}|}\\&=\frac{m_0m_1}{\varrho}+\frac{1}{r}\left(\frac{m_0m_2}{|1+\sigma_0\frac{\varrho}{r}e^{i(\theta-\alpha)}|}+\frac{m_1m_2}{|1+\sigma_1\frac{\varrho}{r}e^{i(\theta-\alpha)}|}\right).\end{aligned}$$

Focus now on the region of the phase space where $r\gg\varrho$, in which the third body is far from the other two. Then, we have

$$H_{\rm pol}(r,\varrho,\alpha,\theta,z,y,G,\Gamma)=\frac{1}{\mu_1}\left(\frac{z^2}{2}+\frac{\Gamma^2}{2\varrho^2}\right)+\frac{1}{\mu_2}\left(\frac{y^2}{2}+\frac{G^2}{2r^2}\right)-\frac{m_0m_1}{\varrho}-\frac{m_2(m_0+m_1)}{r}+\mathcal{O}\left(\frac{\varrho^2}{r^3}\right).$$

Thus, *at first order*, we have two uncoupled Hamiltonians, one for $(\varrho,\theta,z,\Gamma)$ and the other for (r,α,y,G), [3]

$$\begin{aligned}H_{\rm el}(\varrho,\theta,z,\Gamma)&=\frac{1}{\mu_1}\left(\frac{z^2}{2}+\frac{\Gamma^2}{2\varrho^2}\right)-m_0m_1\frac{1}{\varrho}\\H_{\rm par}(r,\alpha,y,G)&=\frac{1}{\mu_2}\left(\frac{y^2}{2}+\frac{G^2}{2r^2}\right)-m_2(m_0+m_1)\frac{1}{r}.\end{aligned}\tag{6.3}$$

To have the first order Hamiltonians $H_{\rm el}$ and $H_{\rm par}$ independent of the masses, we make the following scaling to the variables

$$\varrho=\frac{1}{\mu_1m_0m_1}\widetilde{\varrho},\quad z=\mu_1m_0m_1\widetilde{z},\quad r=\frac{1}{\mu_2m_2(m_0+m_1)}\widetilde{r}\quad\text{and}\quad y=\mu_2m_2(m_0+m_1)\widetilde{y},$$

which is conformally symplectic. Then, after time rescaling, we obtain the Hamiltonian

$$\widetilde{H}_{\rm pol}(\widetilde{r},\widetilde{\varrho},\alpha,\theta,\widetilde{y},\widetilde{z},\Gamma,G)=\nu\left(\frac{\widetilde{z}^2}{2}+\frac{\Gamma^2}{2\widetilde{\varrho}^2}-\frac{1}{\widetilde{\varrho}}\right)+\left(\frac{\widetilde{y}^2}{2}+\frac{G^2}{2\widetilde{r}^2}-\frac{1}{\widetilde{r}}\right)-\widetilde{V}(\widetilde{\varrho},\widetilde{r},\theta-\alpha).$$

with

$$\widetilde{V}(\widetilde{\varrho},\widetilde{r},\theta-\alpha)=\frac{\widetilde{\nu}}{\widetilde{r}}\left(\frac{m_0}{|1+\widetilde{\sigma}_0\frac{\widetilde{\varrho}}{\widetilde{r}}e^{i(\theta-\alpha)}|}+\frac{m_1}{|1+\widetilde{\sigma}_1\frac{\widetilde{\varrho}}{\widetilde{r}}e^{i(\theta-\alpha)}|}-(m_0+m_1)\right),\tag{6.4}$$

and

$$\nu=\frac{\mu_1m_0^2m_1^2}{\mu_2m_2^2(m_0+m_1)^2},\qquad\widetilde{\nu}=(m_0+m_1)m_2^2\qquad\text{and}\qquad\widetilde{\sigma}_i=\frac{\mu_2m_2(m_0+m_1)}{\mu_1m_0m_1}\sigma_i.\tag{6.5}$$

Note that the potential $\widetilde{V}$ only depends on the angles through $\theta-\alpha$ due to the rotational symmetry of the system. Now, we change the polar variables $(\widetilde{\varrho},\theta,\widetilde{z},\Gamma)$ to the classical Delaunay coordinates (see, for instance, [Win41])

$$(\widetilde{\varrho},\theta,\widetilde{z},\Gamma)\mapsto(\ell,L,g,\Gamma).\tag{6.6}$$

This change is symplectic. As usual, from the Delaunay actions, which are the square of the semimajor axis L and the angular momentum Γ, one can compute the eccentricity

$$\epsilon(L,\Gamma)=\sqrt{1-\frac{\Gamma^2}{L^2}}.\tag{6.7}$$

[3]The notation $H_{\rm el}$ and $H_{\rm par}$ is used to emphasize that, up to (small) variations caused by the coupling terms $\mathcal{O}\left(\frac{\varrho^2}{r^3}\right)$, we will work in the region of the phase space where $H_{\rm el}<0$ and $H_{\rm par}=0$ (see also Section 6.3).

The position variables $(\widetilde{\varrho}, \theta)$ can be expressed in terms of Delaunay variables as

$$\widetilde{\varrho} = \widetilde{\varrho}(\ell, L, \Gamma) = L^2(1 - \epsilon \cos E) \qquad \text{and} \qquad \theta = \theta(\ell, L, g, \Gamma) = f(\ell, L, \Gamma) + g, \tag{6.8}$$

where the angles true anomaly f and eccentric anomaly E are defined in terms of the mean anomaly l and eccentricity ϵ as

$$l = E - \epsilon \sin E \qquad \text{and} \qquad \tan \frac{f}{2} = \sqrt{\frac{1+\epsilon}{1-\epsilon}} \tan \frac{E}{2}. \tag{6.9}$$

One could also write an expression for $\tilde{z}$, but it is not necessary to obtain the new Hamiltonian

$$\widehat{H}(\widetilde{r}, \alpha, g, \ell, \widetilde{y}, G, \Gamma, L) = -\frac{\nu}{2L^2} + \left(\frac{\widetilde{y}^2}{2} + \frac{G^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}} \right) - \widetilde{V}(\widetilde{\varrho}(\ell, L, \Gamma), \widetilde{r}, f(\ell, L, \Gamma) + g - \alpha),$$

where $\widetilde{V}$ is the potential introduced in (6.4). Now, by (6.8), the distance condition corresponds to $\widetilde{r} \gg L^2$ and the *first order* uncoupled Hamiltonians are

$$\widehat{H}_{\mathrm{el}}(L) = -\frac{\nu}{2L^2} \qquad \text{and} \qquad \widehat{H}_{\mathrm{par}}(\widetilde{r}, y, G) = \frac{\widetilde{y}^2}{2} + \frac{G^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}} \tag{6.10}$$

whereas $\widetilde{V} = \mathcal{O}(\frac{\widetilde{\varrho}^2}{\widetilde{r}^3}) = \mathcal{O}(\frac{L^4}{\widetilde{r}^3})$.

Now, we make the last reduction which uses the rotational symmetry. We define the new angle $\phi = g - \alpha$ and the total angular momentum $\Theta = G + \Gamma$. To have a symplectic change of coordinates, we consider the transformation

$$(\widetilde{r}, \alpha, g, \ell, y, G, \Gamma, L) = (\widetilde{r}, \alpha, g - \alpha, \ell, \widetilde{y}, G + \Gamma, \Gamma, L) \tag{6.11}$$

Then, we obtain the following Hamiltonian, which is independent of α,

$$\begin{aligned} \mathcal{H}(\widetilde{r}, \phi, \ell, \widetilde{y}, \Gamma, L; \Theta) =& \widehat{H}(\widetilde{r}, \alpha, \phi + \alpha, \ell, \widetilde{y}, \Theta - \Gamma, \Gamma, L) \\ =& -\frac{\nu}{2L^2} + \left(\frac{\widetilde{y}^2}{2} + \frac{(\Theta - \Gamma)^2}{2\widetilde{r}^2} - \frac{1}{\widetilde{r}} \right) - \widetilde{V}(\widetilde{\varrho}(\ell, L, \Gamma), \widetilde{r}, f(\ell, L, \Gamma) + \phi). \end{aligned} \tag{6.12}$$

Since this Hamiltonian is independent of α, the total angular momentum Θ is a conserved quantity which can be taken as a parameter of the system. The Hamiltonian $\mathcal{H}$ induces a well defined Hamiltonian flow on the symplectic manifold $(\mathcal{M}, \mathrm{d}\sigma)$ where

$$\mathcal{M} = \{(\widetilde{r}, \ell, \phi, \widetilde{y}, \Gamma, L) \in \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^3 \colon 0 < \Gamma < L\} \tag{6.13}$$

and $\sigma = \widetilde{y}\mathrm{d}\widetilde{r} + \Gamma \mathrm{d}\phi + L\mathrm{d}\ell$.

6.2.2 The parabolic manifold at infinity

Fix any $L_0 \in \mathbb{R}_+$ and any $\Theta > L_0$. From the expression of $\mathcal{H}$ in (6.12), we observe that

$$\mathcal{E}_\infty = \{(\widetilde{r}, \phi, \ell, \widetilde{y}, \Gamma, L) = (\infty, \phi, \ell, 0, L_0, \Gamma) \colon (\phi, \ell) \in \mathbb{T}^2, \ L = L_0, \ 0 < \Gamma < L_0\} \tag{6.14}$$

is an "invariant manifold at infinity" contained in the energy level $\mathcal{H}^{-1}(-\nu/2L_0^2)$ [4]. We call $\mathcal{E}_\infty$ the *parabolic-elliptic infinity*, since it corresponds to the ω-limit set (resp. α-limit set) of the forward (resp. backwards) parabolic-elliptic motions. The dynamics on $\mathcal{E}_\infty$ is simply given by the (integrable) resonant linear flow on $\mathbb{T}^2 \times \{0 < \Gamma < L_0\}$

$$\dot{\phi} = 0, \qquad \dot{\ell} = \frac{\nu}{L_0^3}, \qquad \dot{\Gamma} = 0.$$

[4] To analyze this manifold properly, one should introduce the McGehee transformation $\tilde{r} = 2/x^2$, (see, for instance [McG73]).

In other words, $\mathcal{E}_\infty$ is foliated by periodic orbits $\gamma_{\phi,\Gamma}$ with frequency $(\omega_\phi,\omega_\ell) = (0,\nu/L_0^3)$. It will be convenient for us, to define, for

$$G_0 \in \mathcal{G}_{\Theta,L_0} = \{G \in \mathbb{R}_+ \colon \Theta - L_0 < G_0 < \Theta\}, \tag{6.15}$$

the invariant torus [5] $\mathcal{T}_{G_0} \subset \mathcal{E}_\infty$ given by

$$\mathcal{T}_{G_0} = \bigcup_{\phi\in\mathbb{T}} \gamma_{\phi,\Theta-G_0} = \mathcal{E}_\infty \cap \{\Gamma = \Theta - G_0\}. \tag{6.16}$$

Observe that $\mathcal{E}_\infty = \bigcup_{G_0\in\mathcal{G}_{\Theta,L_0}} \mathcal{T}_{G_0}$.

Despite being degenerate (the linearized vector field on the normal directions vanishes), the tori $\mathcal{T}_{G_0}$ possess 3-dimensional stable and unstable invariant manifolds (see [BFM20c]) [6]

$$\begin{aligned} W^s(\mathcal{T}_{G_0}) =& \{x \in \mathcal{H}^{-1}(-\nu/(2L_0^2)) \colon \exists x_+ \in \mathcal{T}_{G_0} \text{ s.t. } \lim_{t\to+\infty} |\phi^t(x) - \phi^t(x_+)| \to 0\} \\ W^u(\mathcal{T}_{G_0}) =& \{x \in \mathcal{H}^{-1}(-\nu/(2L_0^2)) \colon \exists x_- \in \mathcal{T}_{G_0} \text{ s.t. } \lim_{t\to-\infty} |\phi^t(x) - \phi^t(x_-)| \to 0\}, \end{aligned} \tag{6.17}$$

where ϕ^t denotes the time t flow associated to the Hamiltonian $\mathcal{H}$ in (6.12). The union of these manifolds gives rise to the (4-dimensional) stable and unstable manifolds of the cylinder $\mathcal{E}_\infty$

$$\begin{aligned} W^s(\mathcal{E}_\infty) =& \{x \in \mathcal{H}^{-1}(-\nu/(2L_0^2)) \colon \exists x_+ \in \mathcal{E}_\infty \text{ s.t. } \lim_{t\to+\infty} |\phi^t(x) - \phi^t(x_+)| \to 0\} = \bigcup_{G_0\in\mathcal{G}_{\Theta,L_0}} W^s(\mathcal{T}_{G_0}) \\ W^u(\mathcal{E}_\infty) =& \{x \in \mathcal{H}^{-1}(-\nu/(2L_0^2)) \colon \exists x_- \in \mathcal{E}_\infty \text{ s.t. } \lim_{t\to-\infty} |\phi^t(x) - \phi^t(x_-)| \to 0\} = \bigcup_{G_0\in\mathcal{G}_{\Theta,L_0}} W^u(\mathcal{T}_{G_0}). \end{aligned} \tag{6.18}$$

As already discussed in the introduction, our approach to prove the existence of drfiting orbits in the 3 Body Problem, is to construct a transition chain of heteroclinic orbits contained in $W^s(\mathcal{E}_\infty) \pitchfork W^u(\mathcal{E}_\infty)$.

6.3 The 2 Body Problem

We have seen in Section 6.2.1 that the Hamiltonian $\mathcal{H}$ in (6.12) corresponds, up to $\mathcal{O}(L^4/\tilde{r}^3)$, to the sum of the two uncoupled Hamiltonians H_{par} and H_{el} in (6.10). These are 2 Body Problem Hamiltonians expressed in different coordinate systems, each of them suited to describe the dynamics on different energy levels:

$$\{(\phi,\ell,\Gamma,L) \in \mathbb{T}^2 \times \mathbb{R}_+^2 \colon H_{\text{el}}(g,\ell,\Gamma,L) < 0\} \quad \text{and} \quad \{(\tilde{r},\alpha,\tilde{y},G) \in \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R}^2 \colon H_{\text{par}}(\tilde{r},\alpha,\tilde{y},G) = 0\}. \tag{6.19}$$

Remark 6.3.1. *It is well known that the 2 Body Problem can be symplectically reduced to a Hamiltonian system with one degree of freedom. However, since later we will study the dynamics of the 3 Body Problem as a perturbation of two uncoupled 2 Body Problems, we prefer to describe the dynamics of H_{el} and H_{par} in their full phase space. The integrability of course reflects in the existence of many integrals of motion (see* (6.19) *and Remark 6.3.3 below).*

The Hamiltonian H_{el} introduces a linear flow on $(\phi,\ell,\Gamma,L) \in \mathbb{T}^2 \times \mathbb{R}_+^2 \cap \{0 < \Gamma < L\}$ given by

$$\dot{\phi} = 0, \qquad \dot{\ell} = \frac{\nu}{L_0^3}, \qquad \dot{\Gamma} = 0, \qquad \dot{L} = 0.$$

We inmediately observe the existence of a foliation by two dimensional resonant tori

$$\mathtt{T}_{\Gamma_0,L_0} = \{(\phi,\ell,\Gamma,L) = (\phi,\ell,\Gamma_0,L_0)\}. \tag{6.20}$$

The energy level $H_{\text{par}} = 0$ corresponds to the parabolic motions of the 2 Body Problem. These are described in the next section.

[5] The reason why we parametrize the tori in terms of the value of G instead of using the value of Γ will be explained later (see Remark 6.4.1).

[6] Note that $\pi_r \circ \phi^t(x) = \infty$ as $t \to \infty$ (resp. $t \to -\infty$) for $x \in W^s(\mathcal{T}_{G_0})$ (resp. $x \in W^u(\mathcal{T}_{G_0})$).

6.3.1 The parabolic homoclinic manifold of the 2BP

One inmediately checks that $\mathcal{P}_\infty = \{(\tilde{r},\alpha,\tilde{y},G) = (\infty,\alpha,0,G)\colon (\alpha,G)\in\mathbb{T}\times\mathbb{R}\}$ is a 2 dimensional invariant cylinder for the Hamiltonian $H_{\rm par}$, contained in the zero energy level. Moreover, $\mathcal{P}_\infty$ possesses stable and unstable manifolds which indeed coincide along the 3-dimensional homoclinic submanifold $W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty)$ [7]

$$W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty) = \{x\in\mathbb{R}_+\times\mathbb{T}\times\mathbb{R}^2\colon \exists z\in\mathcal{P}_\infty \quad \text{for which} \quad \lim_{t\pm\infty}|\phi^t_{\rm par}(x)-\phi^t_{\rm par}(z)|=0\}, \tag{6.21}$$

where $\phi^t_{\rm 2BP}$ is the flow associated to the Hamiltonian $H_{\rm par}$ in (6.10). The following lemma gives a parametrization of the homoclinic manifold $W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty)$. A proof can be found in [MP94].

Lemma 6.3.2. *There exist real-analytic functions $r_{\rm h}(u), \alpha_{\rm h}(u)$ and $y_{\rm h}(u)$ such that*

$$W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty) = \left\{\Gamma_{\rm 2BP}(u,\beta) = (G^2 r_{\rm h}(u), \beta+\alpha_{\rm h}(u), G^{-1}y_{\rm h}(u), G)\in\mathbb{R}_+\times\mathbb{T}\times\mathbb{R}^2\colon u\in\mathbb{R},\ \beta\in\mathbb{T},\ G\in\mathbb{R}\backslash\{0\}\right\}$$

and, if we denote by $X_{\rm 2BP}$ the vector field associated to the Hamiltonian $H_{\rm par}$,

$$X_{\rm 2BP}\circ\Gamma_{\rm 2BP} = D\Gamma_{\rm 2BP}\ \Upsilon \qquad \textit{with} \qquad \Upsilon = (G^{-3},0).$$

The functions $r_{\rm h}, y_{\rm h}$ and $\alpha_{\rm h}$ admit a unique analytic extension to $\mathbb{C}\setminus\{u=is\colon s\in(-\infty,-1/3]\cup[1/3,\infty)\}$ and satisfy the asymptotic behavior

$$r_{\rm h}(u)\sim u^{2/3} \qquad \exp(i\alpha_{\rm h}(u))\sim 1 \qquad y_{\rm h}(u)\sim u^{-1/3} \qquad \textit{as}\quad \operatorname{Re} u\to\pm\infty$$

and

$$r_{\rm h}(u)\sim(u\pm i/3)^{1/2} \qquad \exp(i\alpha_{\rm h}(u))\sim\left(\frac{u\pm i/3}{u\mp i/3}\right)^{1/2} \qquad y_{\rm h}(u)\sim(u\pm i/3)^{-1/2} \qquad \textit{as}\quad u\to\pm i/3.$$

Moreover, $y_{\rm h}(u)=0$ if and only if $u=0$ and $r_{\rm h}(u)\geq 1/2$ for all $u\in\mathbb{R}$.

Remark 6.3.3. *The integrability of the Hamiltonian $H_{\rm par}$ (see Remark 6.3.1) reflects in the conservation of the angular momentum G. Indeed, one can check that, for any $G_*\in\mathbb{R}\setminus\{0\}$, $W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty\cap\{G=G_*\})$ are invariant submanifolds homoclinic to the invariant torus at infinity $\mathcal{P}_\infty\cap\{G=G_*\}$.*

6.4 Adapted coordinates in a neighbourhood of the unperturbed homoclinic manifold

The last statement in Lemma 6.3.2 implies that

$$\pi_{\tilde{r}}(W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty\cap\{G=G_*\}))\geq G_*^2/2.$$

Thus, by choosing $G_*\gg 1$, the homoclinic manifold $W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty\cap\{G=G_*\})$ is contained in the region of the phase space where $\tilde{r}\gg L_0^2$. Consequently, on a neighbourhood of $W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty\cap\{G=G_*\})$, the Hamiltonian $\mathcal{H}$ in (6.12) can be studied as a perturbation of the sum of the two uncoupled Hamiltonians $H_{\rm el}$ and $H_{\rm par}$. Then, one expects that, for $G_0\gg 1$, the invariant manifolds $W^{u,s}(\mathcal{T}_{G_0})$ in (6.17) are close to the product $W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty\cap\{G=G_0\})\times\mathtt{T}_{\Theta-G_0,L_0}\subset\mathcal{M}$ and can be studied perturbatively.

Remark 6.4.1. *The reason why we parametrize the tori* (6.16) *in terms of the value of G_0 is that we already have a parametrization, in terms of G_0 of the invariant manifold $W^{\rm h}_{\rm 2BP}(\mathcal{P}_\infty\cap\{G=G_0\})$.*

[7] Note that for the $\tilde{r}$ component $\pi_{\tilde{r}}\circ\phi^t_{\rm par}(x)\to\infty$ as $t\to\pm\infty$.

We now fix any

$$L_0 \in \mathbb{R}_+ \qquad \text{and} \qquad \Theta \gg L_0. \tag{6.22}$$

Then, by the definition of the set $\mathcal{G}_{\Theta,L_0}$ in (6.15), for any $G_0 \in \mathcal{G}_{\Theta,L_0}$ we have that $G_0 \gg 1$. In order to analyze the dynamics in a neighbourhood of $W^{\mathrm{h}}_{2\mathrm{BP}}(\mathcal{P}_\infty \cap \{G = G_0\}) \times \mathtt{T}_{\Theta-G_0,L_0} \subset \mathcal{M}$, we introduce the change of coordinates

$$\eta_{G_0} : (u, \beta, \lambda, Y, J, \Lambda) \mapsto (r, \phi, \ell, y, \Gamma, L) \tag{6.23}$$

given by

$$r = G_0^2 r_{\mathrm{h}}(u), \qquad \phi = \beta + \alpha_{\mathrm{h}}(u), \qquad \ell = \lambda, \qquad y = G_0^{-1} y_{\mathrm{h}}(u) + \frac{Y - r_{\mathrm{h}}^2(u)J}{G_0^2 y_{\mathrm{h}}(u)}, \qquad \Gamma = \Gamma_0 + J, \qquad L = L_0 + \Lambda.$$

where $r_{\mathrm{h}}, \alpha_{\mathrm{h}}, y_{\mathrm{h}}$ are the functions introduced in Lemma 6.3.2 and $\Gamma_0 = \Theta - G_0$. The proof of the following result is a straightforward computation.

Lemma 6.4.2. *Let $(\mathcal{M}, \mathrm{d}\sigma)$ be the exact symplectic manifold in (6.13). Let $(M, \mathrm{d}\tau)$ be the exact symplectic manifold*

$$M = \left\{(u, \beta, \lambda, Y, J, \Lambda) \in \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^3\right\} \qquad \textit{and} \qquad \tau = Y\mathrm{d}u + J\mathrm{d}\beta + \Lambda\mathrm{d}\lambda$$

The change of variables $\eta_{G_0} : M \setminus \{u = 0\} \to \mathbb{R}_+ \times \mathbb{T}^2 \times \mathbb{R}^3$ defined in (6.23) satisfies

$$\eta_{G_0}^* \sigma - \tau = \frac{2}{G_0^2 r_{\mathrm{h}}(u)} \mathrm{d}u + G_0 \mathrm{d}\beta.$$

In particular, η_{G_0} is a symplectic change of variables between $(\mathcal{M}, \mathrm{d}\sigma)$ and $(M \setminus \{u = 0\}, \mathrm{d}\tau)$.

Remark 6.4.3. *The map η_{G_0} is not defined at $u = 0$ since $y_{\mathrm{h}}(u) = 0$ (see Lemma 6.3.2).*

The reason for introducing the coordinate transformation (6.23) is that, in the new coordinates, the flow is almost linear on the region $\{|Y|, |J|, |\Lambda| \ll 1\}$. Indeed, up to a constant rescaling, the Hamiltonian in the new coordinates reads

$$H(u, \beta, \lambda, J, \Lambda; L_0, \Gamma_0, G_0) = -\frac{-\nu G_0^3}{2(L_0 + \Lambda)^2} + Y + \frac{(Y - r_{\mathrm{h}}^2(u)J)^2}{2y_{\mathrm{h}}^2(u)G_0} + \frac{J^2}{2r_{\mathrm{h}}^2(u)G_0} - V(u, \beta, \lambda, J, \Lambda; L_0, \Gamma_0, G_0), \tag{6.24}$$

where we have introduced the *perturbative potential*

$$V(u, \beta, \lambda, Y, J, \Lambda; L_0, \Gamma_0, G_0) = G_0^3 \widetilde{V}\big(G_0^2 r_h(u), \varrho(\lambda, L_0 + \Lambda, \Gamma_0 + J), f(\lambda, , L_0 + \Lambda, \Gamma_0 + J) + \beta + \alpha_h(u)\big) - \frac{G_0}{r_h(u)}. \tag{6.25}$$

Since, as we will see in Section 6.5.1,

$$V(u, \beta, \lambda, 0, 0; L_0, \Gamma_0, G_0) = \mathcal{O}(G_0^{-3} r_{\mathrm{h}}^{-3}(u)),$$

we obtain

$$\begin{aligned}
&\dot{u} = 1 + \mathcal{O}(|Y|, |J|, |\Lambda|), && \dot{\beta} = \mathcal{O}(G_0^{-3}, |Y|, |J|, |\Lambda|), && \dot{\lambda} = -\nu \frac{G_0^3}{L_0^3} + \mathcal{O}(G_0^{-3}, |Y|, |J|, |\Lambda|) \\
&\dot{Y} = \mathcal{O}(G_0^{-3}, |Y|, |J|, |\Lambda|), && \dot{J} = \mathcal{O}(G_0^{-3}, |Y|, |J|, |\Lambda|), && \dot{\Lambda} = \mathcal{O}(G_0^{-3}, |Y|, |J|, |\Lambda|).
\end{aligned}$$

6.4.1 Parametrization of the invariant manifolds

We now want to obtain a parametrization of the invariant manifolds associated to the invariant torus $\mathcal{T}_{G_0} \subset \mathcal{E}_\infty$. Since these manifolds are Lagrangian, there exist functions (u_0 is a positive real number)

$$S^u : (-\infty, -u_0) \times \mathbb{T}^2 \to \mathbb{R} \qquad S^s : (u_0, +\infty) \times \mathbb{T}^2 \to \mathbb{R}, \tag{6.26}$$

solutions to the Hamilton-Jacobi equation

$$H(q,\nabla S^{u,s}(q)) = -\frac{\nu G_0^3}{L_0^2}, \qquad q=(u,\beta,t), \tag{6.27}$$

where H is the Hamiltonian (6.24), such that

$$W^u_{\mathrm{loc}}(\mathcal{T}_{G_0}) = \{(q,\nabla S^u(q))\colon q=(u,\beta,t)\in(-\infty,-u_0)\times\mathbb{T}^2\}$$
$$W^s_{\mathrm{loc}}(\mathcal{T}_{G_0}) = \{(q,\nabla S^s(q))\colon q=(u,\beta,t)\in(u_0,+\infty)\times\mathbb{T}^2\}.$$

Therefore, if one can extend the functions $S^{u,s}$ in (6.26) to a suitable common domain, one can measure the distance between the manifolds $W^u_{\mathrm{loc}}(\mathcal{T}_{G_0})$ and $W^s_{\mathrm{loc}}(\mathcal{T}_{G_0})$ simply by studying the gradient of the so-called *splitting potential* (see [Eli94, DG00, Sau01])

$$\Delta S(u,\beta,t;L_0,\Gamma_0,G_0) = S^u(u,\beta,t;L_0,\Gamma_0,G_0) - S^s(u,\beta,t;L_0,\Gamma_0,G_0). \tag{6.28}$$

6.4.2 Approximation of the splitting potential

From the expression for H in (6.24), the Hamilton-Jacobi equation (6.27) reads

$$\partial_u S^{u,s} + \nu\frac{G_0^3}{L_0^3}\partial_\lambda S^{u,s} + \frac{(\partial_u S^{u,s} - r_{\mathrm{h}}^2(u)\partial_\beta S^{u,s})^2}{2y_{\mathrm{h}}^2(u)G_0} + \frac{(\partial_\beta S^{u,s})^2}{2r_{\mathrm{h}}^2(u)G_0}$$
$$+ \left(\frac{\nu G_0^3}{L_0^2} - \frac{\nu G_0^3}{(L_0+\partial_\lambda S^{u,s})^2} - \frac{\nu G_0^3}{L_0^3}\partial_\lambda S^{u,s}\right) - V(u,\beta,\lambda,\partial_\beta S^{u,s},\partial_\lambda S^{u,s};L_0,\Gamma_0,G_0) = 0.$$

Thus, one expects that, up to first order, the generating functions $S^{u,s}$, are approximated by the *half Melnikov potentials* $L^{u,s}$ which are defined as the unique solutions to

$$\partial_u L^{u,s} + \nu\frac{G_0^3}{L_0^3}\partial_\lambda L^{u,s} - V(u,\beta,\lambda,0,0;L_0,\Gamma_0,G_0) = 0 \qquad \lim_{\mathrm{Re}\ u\to-\infty} L^u = 0, \quad \lim_{\mathrm{Re}\ u\to\infty} L^s = 0$$

and, that the splitting potential ΔS in (6.28) is given, up to first order, by the so-called *Melnikov potential*

$$\widetilde{L}(u,\beta,\lambda;L_0,\Gamma_0,G_0) = L^u(u,\beta,\lambda;L_0,\Gamma_0,G_0) - L^s(u,\beta,\lambda;L_0,\Gamma_0,G_0). \tag{6.29}$$

In view of this heuristic argument, the first step towards proving the existence of transverse intersections between $W^u(\mathcal{T}_{G_0})$ and $W^s(\mathcal{T}_{G_0})$, is to study the Melnikov potential (6.29). To that end, we invert the linear operator $\mathcal{L} = \partial_u + \nu(G_0/L_0)^3\partial_\lambda$ and express the half Melnikov potentials as [8]

$$L^u(u,\beta,\lambda;L_0,\Gamma_0,G_0) = \int_{-\infty}^0 V(u+s,\beta,\lambda+\nu(G_0/L_0)^3 s,0,0;L_0,\Gamma_0,G_0)\mathrm{d}s$$
$$L^s(u,\beta,\lambda;L_0,\Gamma_0,G_0) = -\int_0^{+\infty} V(u+s,\beta,\lambda+\nu(G_0/L_0)^3 s,0,0;L_0,\Gamma_0,G_0)\mathrm{d}s,$$

so

$$\tilde{L}(u,\beta,\lambda;L_0,\Gamma_0,G_0) = \int_{-\infty}^{+\infty} V(u+s,\beta,\lambda+\nu(G_0/L_0)^3 s,0,0;L_0,\Gamma_0,G_0)\mathrm{d}s.$$

It turns out that, up to rescaling, $V(u,\beta,\lambda,0,0;L_0,\Gamma_0,G_0)$ only depends on the parameters L_0,Γ_0,G_0 through the quantities

$$I = \frac{G_0}{L_0} \qquad \text{and} \qquad \epsilon_0 = \sqrt{1-\frac{\Gamma_0^2}{L_0^2}}. \tag{6.30}$$

[8]The linear operator $\mathcal{L}$ admits a right inverse in a suitable space of real analytic functions, defined in a complex neighborhood of $(-\infty,-u_0)\times\mathbb{T}^2$, which decay sufficiently fast for $u\to-\infty$. An analogous statment holds for real analytic functions, defined on a complex neighborhood of $(-\infty,-u_0)\times\mathbb{T}^2$, which decay sufficiently fast for $u\to\infty$ (See, for example, [GPS23b]).

Based on this observation, it will be convenient to introduce the function $\widetilde{U}(u, \beta, \lambda; I, \epsilon_0)$ defined by

$$\tilde{U}\big(u, \beta, \lambda; I(G_0, L_0), \epsilon_0(\Gamma_0, L_0)\big) = V(u, \beta, \lambda, 0, 0; L_0, \Gamma_0, G_0), \tag{6.31}$$

and use the following, more compact, notation for the Melnikov potential defined in (6.29)

$$L\big(\beta, I, \lambda - \nu I^3 u; \epsilon_0\big) = \int_{\mathbb{R}} \tilde{U}\big(s, \beta, \lambda - \nu I^3 u + \nu I^3 s; I, \epsilon_0\big) \mathrm{d}s. \tag{6.32}$$

This heuristic approximation can be made rigourous, without too much extra work, for approximating the splitting potential (6.28) by the Melnikov potential (6.32) in the C^0 topology. The approximation in the C^1 topology (recall that transverse intersections between $W^u(\mathcal{T}_{G_0})$ and $W^s(\mathcal{T}_{G_0})$ correspond to non degenerate critical points of the splitting potential) requires however considerably more extra work. The reason is the following. Due to the existence of different time scales and the real-analyticity of H in (6.24), the splitting between $W^u(\mathcal{T}_{G_0})$ and $W^s(\mathcal{T}_{G_0})$ is highly anisotropic. A more or less standard, averaging procedure, can be used to prove that the splitting (measured along a suitable section) between $W^u(\mathcal{T}_{G_0})$ and $W^s(\mathcal{T}_{G_0})$ is polynomially small (in $1/G_*$) in the direction conjugated to the resonant angle β, and exponentially small (in $1/G_*$) in a direction close to the direction conjugated to the fast angle λ (see [Nei84, LMS03]). This argument, however, only yields a (non sharp) upper bound for the gradient of the splitting potential. In order to obtain an asymptotic formula (or a sharp estimate), one would need to carry on this averaging procedure with an "optimal loss of analyticity". Alternatively, following the ideas of Lazutkin [Laz87], one can try to extend the splitting potential (or some vector parametrization of the invariant manifolds) to a complex domain which gets "sufficiently close" to the complex singularities of the perturbing potential V. Then, a standard argument can be used to obtain sharp estimates on the decay of the Fourier coefficients of the splitting potential ΔS, what, in turn, yields a sharp estimate of each component of its gradient. This was the approach in [GMS16, GPS23b]. However, in these works, where the eccentricity ϵ was assumed to be either zero or small enough:

- The singularities of the perturbing term V are "sufficiently close" to that of the parametrization of the unperturbed homoclinic manifold in Lemma 6.3.2.
- The domain of analyticity of the perturbing term V can be written as a direct product.

This is not the case when the normal form (6.23) is centered around an orbit with arbitrary eccentricity $\epsilon \in (0, 1)$. As we will see in Section 6.5.4, the singularities of V move away from those of the unperturbed homoclinic manifold in Lemma 6.3.2. Moreover, the domain of analyticity of the perturbing term V cannot be written as a direct product. These facts complicate, heavily, the asymptotic analysis of the Melnikov potential L in (6.32) and the justification of the Melnikov approximation. In the rest of the paper we introduce new tools to obtain an asymptotic analysis of the Melnikov function L and which, we hope, can be of interest for the analysis of Melnikov functions in other Hamiltonian systems.

Finally, notice that we have only concerned ourselves with the analysis of the stable and unstable manifolds of the same torus. However, in order to construct a transition chain leading to Arnold Diffusion, we need to understand how the manifolds associated to pairs of different tori intersect. This requires substantially more work and has been studied for the first time, in the exponentially small setting, in [GPS23b].

Remark 6.4.4. *We fix once and for all the values of $m_0, m_1, m_2 \neq 0$ such that $m_0 \neq m_1$. We also fix $\epsilon_0 \in (0, 1)$. In the following, when we introduce a real constant C and say that $C \neq 0$, we mean that for any choice of these parameters as above, the constant does not vanish.*

6.5 The Melnikov potential

In this section we present our main result. Namely, for any fixed value of the masses $m_0, m_1, m_2 > 0$, any eccentricity $\epsilon_0 \in (0, 1)$ and sufficiently large I, we obtain an asymptotic formula for the Melnikov potential L in (6.32). To this end, the introduction of some notation is in order. Given a value $0 < I_* < \infty$, we define the unbounded cylinder

$$\Lambda(I_*) = \{(\beta, I) \in \mathbb{T} \times \mathbb{R} \colon I_* \leq I\}.$$

Theorem 6.5.1. *There exists $I^* \gg 1$ such that the Melnikov potential L in (6.32) satisfies the following properties for all $(\beta, I, \sigma) \in \Lambda(I_*) \times \mathbb{T}$:*

- *It can be expressed as an absolutely convergent Fourier series*

$$L(\beta, I, \sigma; \epsilon_0) = \sum_{l \in \mathbb{Z}} L^{[l]}(\beta, I; \epsilon_0) e^{il\sigma}, \qquad L^{[l]}(\beta, I; \epsilon_0) = \frac{1}{2\pi} \int_{\mathbb{T}} L(\beta, I, \sigma; \epsilon_0) e^{-il\sigma} \mathrm{d}\sigma.$$

- *For $l = 0$,*

$$L^{[0]}(\beta, I; \epsilon_0) = \frac{\tilde{\nu}\pi}{2} \Bigg(\left(1 + \frac{3\epsilon_0^2}{2}\right)(m_0 \tilde{\sigma}_0^2 + m_1 \tilde{\sigma}_1^2) I^{-3} + \frac{15}{4}(1 + ...)(m_0 \tilde{\sigma}_0^3 + m_1 \tilde{\sigma}_1^3)\ \epsilon_0 I^{-5} \cos\beta + E(\beta, I) \Bigg),$$

  with

$$|E(\beta, I)| \lesssim I^{-7}.$$

- *For $l = 1, 2$ there exist explicit real constants $\mathcal{A}_l \neq 0$ such that* [9]

$$L^{[l]}(\beta, I; \epsilon_0) = \mathcal{A}_l I^{-1} \exp(-l\nu I^3/3) \Bigg(\frac{m_0}{\sqrt{\tilde{\sigma}_0}}(1 + \mathcal{O}(I^{-1}))\ \exp(l q_0(\beta)) + \frac{m_1}{\sqrt{|\tilde{\sigma}_1|}}(1 + \mathcal{O}(I^{-1}))\ \exp(l q_1(\beta)) \Bigg) + T_l(\beta; I, \epsilon_0) \tag{6.33}$$

  where

$$q_0(\beta) = -\frac{2}{3}\nu \tilde{\sigma}_0 \epsilon_0 I(\cos\beta - i \sin\beta) + \mathcal{O}(1), \qquad q_1(\beta) = -\frac{2}{3}\nu \tilde{\sigma}_1 \epsilon_0 I(\cos\beta - i \sin\beta) + \mathcal{O}(1)$$

  and

$$|T_l(\beta, I; \epsilon_0)| \lesssim I^{-9/8} \exp(-l\nu I^3/3) \Big(1 + \mathcal{O}\left(\exp(l \mathrm{Re}\ q_0(\beta))\right) + \mathcal{O}\left(\exp(l \mathrm{Re}\ q_1(\beta))\right) \Big).$$

- *The sum of the higher coefficients*

$$L_{\geq 3}(\beta, I, \sigma; \epsilon_0) = \sum_{|l| \geq 3} L^{[l]}(\beta, I; \epsilon_0) e^{il\sigma}$$

  satisfies

$$|L_{\geq 3}| \lesssim \exp(-3\nu I^3/4).$$

6.5.1 The Melnikov potential as an infinite sum of fast oscillatory integrals

The Melnikov potential L in (6.32) is a real integral of a real-analytic function $\tilde{U}$ (see (6.31)) whose phase in the 2π-periodic variable σ oscillates rapidly. Therefore, one expects that the size of the Fourier coefficients of the function $\sigma \mapsto L$ decays exponentially fast with $|l|$. In order to see this, we first obtain an explicit formula for the function $\tilde{U}(u, \beta, \lambda; I, \epsilon_0)$. The following lemma was proved in [MP94].

Remark 6.5.2. *The reader will forgive us for keep using the notation r_h, α_h and ρ, f after the change of variables introduced in Lemmas 6.5.3 and 6.5.4.*

[9] Recall that $\tilde{\sigma}_1 < 0$ so $\min_{\beta \in \mathbb{T}}(\mathrm{Re}\ q_0(\beta), \mathrm{Re}\ q_1(\beta)) < 0$. Therefore, (6.33) provides asymptotic expression of $L^{[l]}$, with $l = 1, 2$, for all $\beta \in \mathbb{T}$.

Lemma 6.5.3. *Let $r_h(u)$ and $\alpha_h(u)$ be the functions defined in Lemma 6.3.2. Then, under the real-analytic change of variables $u = (\tau + \tau^3/3)/2$ we have that*

$$r_h(\tau) = \frac{\tau^2+1}{2} \qquad \text{and} \qquad e^{i\alpha_h(\tau)} = \frac{\tau - i}{\tau + i}.$$

Introduce also the function $\rho(\lambda)$ given by

$$\varrho(\lambda, L_0, \Gamma_0) = L_0^2 \rho(\lambda; \epsilon_0(\Gamma_0, L_0)). \tag{6.34}$$

where ϱ was defined in (6.8) and ϵ_0 in(6.30). We will also need the following lemma (see, for instance, ...)

Lemma 6.5.4. *Let $\rho(\lambda)$ and $f(\lambda)$ be the functions defined in* (6.34) *and* (6.8)*. Then, under the real-analytic change of variables given by the Kepler equation $\lambda = \xi - \epsilon_0 \sin \xi$, we have*

$$\rho(\xi) = 1 - \epsilon_0 \cos \xi \qquad \text{and} \qquad \rho(\xi) e^{if(\xi)} = \frac{\epsilon}{2\kappa_\epsilon}\left(e^{i\xi} - 2\kappa_\epsilon + \kappa_\epsilon^2 e^{-i\xi}\right)$$

where

$$\kappa_\epsilon = \frac{\epsilon_0}{1 + \sqrt{1 - \epsilon_0^2}}. \tag{6.35}$$

Consider the function $\tilde{U}(u, \beta, \lambda; I, \epsilon_0)$ defined in (6.31) and denote by

$$U(\tau, \beta, \xi; I, \epsilon_0) = \tilde{U}(u(\tau), \beta, \lambda(\xi); I, \epsilon_0).$$

Then, we obtain that

$$U(\tau, \beta, \xi; I, \epsilon_0) = \frac{m_0 \tilde{\nu} I}{\left| r_h(\tau) + \frac{\tilde{\sigma}_0}{I^2} \rho(\xi) e^{i(f(\xi) + \beta + \alpha_h(\tau))} \right|} + \frac{m_1 \tilde{\nu} I}{\left| r_h(\tau) + \frac{\tilde{\sigma}_1}{I^2} \rho(\xi) e^{i(f(\xi) + \beta + \alpha_h(\tau))} \right|} - \frac{(m_0 + m_1)\tilde{\nu} I}{r_h(\tau)}, \tag{6.36}$$

where $\tilde{\nu}$ and $\tilde{\sigma}_\star$, $\star = 0, 1$, are defined in (6.5). It is straightforward to check that, for

$$I > \sqrt{2(1 + \epsilon_0) \max(\tilde{\sigma}_0, |\tilde{\sigma}_1|)},$$

(6.36) is a real-analytic function of all its arguments. Therefore, for all $l \in \mathbb{Z}$, the Fourier coefficients $U^{[l]}(\tau, \beta; I, \epsilon_0)$ of the function $\lambda \mapsto \tilde{U}(u(\tau), \beta, \lambda; I, \epsilon_0)$ are well defined and given by the expression

$$U^{[l]}(\tau, \beta; I, \epsilon_0) = \frac{1}{2\pi} \int_{\mathbb{T}} U(\tau, \beta, \xi; I, \epsilon_0) \rho(\xi) e^{-il\lambda(\xi)} \mathrm{d}\xi, \tag{6.37}$$

Moreover, the series $U(\tau, \beta, \xi; I, \epsilon_0) = \sum_{l \in \mathbb{Z}} U^{[l]}(\tau, \beta; I, \epsilon_0) e^{il\lambda(\xi)}$ is absolutely convergent for any $(\tau, \beta, \xi) \in \mathbb{R} \times \mathbb{T}^2$. On the other hand, a trivial expansion of the denominators in (6.36) in powers of I^{-2}, together with the fact that

$$m_0 \tilde{\sigma}_0 + m_1 \tilde{\sigma}_1 = 0 \tag{6.38}$$

and the expressions for $r_h(\tau), \alpha_h(\tau)$ in Lemma 6.5.3 show that $\tau^2 U(\tau, \beta, \xi; I, \epsilon_0)$ is absolutely integrable on $\mathbb{R}$ as a function of τ (see also the proof of Lemma 6.5.5 below). Thus, we can express the Melnikov potential in (6.32) as the absolutely convergent infinite sum of (convergent) improper integrals

$$\begin{aligned} L(\beta, I, \lambda - \nu I^3 u; \epsilon_0) =& \frac{1}{2} \int_{\mathbb{R}} \tilde{U}\big(u(s), \beta, \lambda - \nu I^3 u + \nu I^3 u(s); I, \epsilon_0\big)(s^2+1) \mathrm{d}s \\ =& \frac{1}{4\pi} \sum_{l \in \mathbb{Z}} e^{il(\lambda - \nu I^3 u)} \int_{\mathbb{R}} U^{[l]}\big(s, \beta; I, \epsilon_0\big) e^{il\nu I^3 u(s)} (s^2+1) \mathrm{d}s \\ =& \frac{1}{2} \sum_{l \in \mathbb{Z}} e^{il(\lambda - \nu I^3 u)} L^{[l]}(\beta, I; \epsilon_0), \end{aligned}$$

where by $u(s)$ we mean $u(s) = (s + s^3/3)/2$, the functions $U^{[l]}$ are defined in (6.37) and

$$L^{[l]}(\beta, I; \epsilon_0) = \frac{1}{2\pi} \int_{\mathbb{R}} U^{[l]}\big(s, \beta; I, \epsilon_0\big) e^{il\nu I^3 u(s)} (s^2+1) \mathrm{d}s. \tag{6.39}$$

For $|l| \geq 1$ and $I \gg 1$, (6.39) are fast oscillatory integrals with phase $l\nu I^3 u(s)$.

6.5.2 Asymptotic analysis of $L^{[0]}$

In this section we provide an asymptotic expression, for $I \gg 1$, of the function $L^{[0]}$ defined in (6.39).

Lemma 6.5.5. *There exists $I_* \gg 1$ such that for all $(\beta, I) \in \Lambda(I_*)$*

$$L^{[0]}(\beta, I; \epsilon_0) = \frac{\tilde{\nu}\pi}{2}\left(\left(1+\frac{3\epsilon_0^2}{2}\right)(m_0\tilde{\sigma}_0^2+m_1\tilde{\sigma}_1^2)I^{-3}+\frac{15}{4}(1+...)(m_0\tilde{\sigma}_0^3+m_1\tilde{\sigma}_1^3)\ \epsilon_0 I^{-5}\cos\beta+E(\beta,I)\right), \tag{6.40}$$

with

$$|E(\beta, I)| \lesssim I^{-7}.$$

Proof. Let $\star = 0, 1$ and denote by (make use of Lemma 6.5.3)

$$\Delta_{\star,\pm}(\tau,\beta,\xi;I,\epsilon_0) = \frac{\tilde{\sigma}_\star\rho(\xi)}{I^2 r_h(\tau)}\exp\big(\pm i(\beta+\alpha_h(\tau)-f(\xi))\big) = \frac{2\tilde{\sigma}_\star\rho(\xi)}{I^2(\tau\pm i)^2}\exp\big(\pm i(\beta-f(\xi))\big). \tag{6.41}$$

Then, in view of (6.36), the potential $U(\tau,\beta,\xi;I,\epsilon_0)$ is given by the explicit expression

$$U = \frac{\tilde{\nu}I}{r_h(\tau)}\left(\frac{m_0}{\sqrt{(1+\Delta_{0,+})(1+\Delta_{0,-})}}+\frac{m_1}{\sqrt{(1+\Delta_{1,+})(1+\Delta_{1,-})}}-(m_0+m_1)\right). \tag{6.42}$$

Since for $(\tau,\beta,\xi) \in \mathbb{R}\times\mathbb{T}^2$ we have

$$|\Delta_{\star,\pm}| \lesssim I^{-2}, \qquad \star = 0, 1,$$

we can expand the terms $(1+\Delta_{\star,\pm})^{-1/2}$ up to order 3 in $\Delta_{\star,\pm}$, use Lemma 6.5.3 and (6.38) to obtain

$$\begin{aligned} U(\tau,\beta,\xi;I,\epsilon_0) =& \frac{\tilde{\nu}I}{r_h(\tau)}\Bigg(\big(m_0\tilde{\sigma}_0^2+m_1\tilde{\sigma}_1^2\big)\frac{\rho^2(\xi)}{I^4(\tau^2+1)^2}+ \\ &+\big(m_0\tilde{\sigma}_0^3+m_1\tilde{\sigma}_1^3\big)\frac{3\rho^3(\xi)}{2I^6}\left(\frac{e^{i(\beta-f(\xi))}}{(\tau-i)^2(\tau+i)^4}+\frac{e^{-i(\beta-f(\xi))}}{(\tau+i)^2(\tau-i)^4}\right)\Bigg) \\ &+E_0(\tau,\beta,\xi;I,\epsilon_0)+\mathcal{O}(|I^{-7}r_h^{-5}(\tau)|), \end{aligned} \tag{6.43}$$

where the term

$$E_0 = \frac{3\tilde{\nu}I}{8r_{\mathrm{h}}(\tau)}\big(\Delta_{0,+}^2+\Delta_{0,-}^2+\Delta_{1,+}^2+\Delta_{1,-}^2-\frac{5}{6}\big(\Delta_{0,+}^3+\Delta_{0,-}^3+\Delta_{1,+}^3+\Delta_{1,-}^3\big)\big),$$

satisfies that that

$$\int_{\mathbb{R}}(\tau^2+1)E_0\mathrm{d}\tau = 0.$$

Recall that by (6.39), for $l = 0$, $L^{[0]} = \int_{\mathbb{R}}\int_{\mathbb{T}} U(\tau,\beta,\xi;I,\epsilon_0)\rho(\xi)\mathrm{d}\xi\mathrm{d}\tau$. Then, the proof of the lemma is completed d making use of the formulas

$$\frac{1}{2\pi}\int_{\xi\in\mathbb{T}}\rho^3(\xi)\mathrm{d}\xi = 1+\frac{3\epsilon_0^2}{2} \qquad \frac{1}{2\pi}\int_{\xi\in\mathbb{T}}\rho^4(\xi)e^{\pm if(\xi)}\mathrm{d}\xi = -\frac{5\epsilon_0}{2}+...$$

and

$$\int_{\tau\in\mathbb{R}}(\tau^2+1)^{-2}\mathrm{d}\tau = \frac{\pi}{2} \qquad \int_{\tau\in\mathbb{R}}(\tau\pm i)^{-2}(\tau\mp i)^{-4}\mathrm{d}\tau = -\frac{\pi}{4}.$$

□

6.5.3 Estimates for $L^{[l]}$ with $|l| \geq 3$

To analyze $L^{[l]}$ we begin by analyzing the behavior of $U^{[l]}$ for complex values of τ. Let $C > 0$ be a sufficiently large constant and introduce the complex disks

$$D = \{\tau \in \mathbb{C}\colon |\tau - i| \leq CI^{-1}\} \qquad \bar{D} = \{\tau \in \mathbb{C}\colon \bar{\tau} \in D\}. \tag{6.44}$$

Lemma 6.5.6. *There exists $I_* \gg 1$ such that, for all $(\beta, I) \in \Lambda(I_*)$ and all $l \in \mathbb{Z}$, the function*

$$\tau \mapsto U^{[l]}(\tau, \beta; I, \epsilon_0) : \mathbb{C} \setminus (D \cup \overline{D}) \to \mathbb{C} \tag{6.45}$$

defined in (6.37)*, is analytic. Moreover, for all $l \in \mathbb{Z}$ and $\tau \in \mathbb{C} \setminus (D \cup \bar{D})$, we have that*

$$|U^{[l]}(\tau, \beta; I, \epsilon_0)| \lesssim I^2 \min(1, |\tau|^{-6}). \tag{6.46}$$

Proof. Let $\Delta_{\star,\pm}$ be the functions defined in (6.41). Choosing C, in the definition of D in (6.44), large enough, for all $(\beta, \xi) \in \mathbb{T}^2$ and all $\tau \in \mathbb{C} \setminus (D \cup \bar{D})$ we have that

$$|\Delta_{\star,\pm}| \leq 1/2 \qquad \star = 0, 1.$$

It then follows that, for all $l \in \mathbb{Z}$, for all $\beta \in \mathbb{T}$ and all $\tau \in \mathbb{C} \setminus (D \cup \bar{D})$ the function $U^{[l]}(\tau, \beta)$ defined by (6.37) is analytic. We now show how to obtain the estimate (6.46). For $\tau \in \partial D \cup \partial \bar{D}$ we have that $|\Delta_{\star,\pm}| \leq 1/2$. Therefore, the expansion (6.43) we used in the proof of Lemma 6.5.5 shows that

$$\max_{\tau \in \partial D \cup \partial \bar{D}} |U(\tau, \beta, \xi)| \lesssim I^2.$$

On the other hand, for all τ such that $|\tau| = 4$, $|\Delta_{\star,\pm}| \lesssim I^{-2}$ and we obtain that

$$\max_{|\tau|=4} |U(\tau, \beta, \xi)| \lesssim I^{-3}.$$

It then follows from direct application of the maximum principle that

$$\max_{\tau \in \{|\tau| \leq 4\} \setminus (D \cup \bar{D})} |U(\tau, \beta, \xi)| \lesssim I^2.$$

Finally, for τ such that $|\tau| \geq 4$ we have that $|\Delta_{\star,\pm}| \lesssim \tau^{-2} I^{-2}$ and using again (6.38) we obtain that

$$\max_{\tau \in \{|\tau| \geq 4\}} |U(\tau, \beta, \xi)| \lesssim \tau^{-6} I^{-3}.$$

To complete the proof it is enough to use the definition of $U^{[l]}$ in (6.37). □

As already pointed out at the beginning of the present section, since the function $\sigma \to \mathcal{L}(\beta, \sigma; I, \epsilon_0)$ is periodic and real analytic, the size of its Fourier coefficients decays exponentially fast as $|l|$ increases. This is shown in the next lemma where we obtain (non- sharp) estimates on the decay for the harmonics $L^{[l]}(\beta; I, \epsilon_0)$ with $|l| \geq 3$.

Lemma 6.5.7. *There exists $I_* \gg 1$ such that, for all $(\beta, I) \in \Lambda(I_*)$,*

$$\sum_{|l| \geq 3} \left| L^{[l]}(\beta, I; \epsilon_0) \right| \lesssim I^2 \exp\left(-3\nu I^3/4\right).$$

Proof. We have shown in Lemma 6.5.6 that for all $l \in \mathbb{Z}$ the function $\tau \to U^{[l]}(\tau, \beta, I)$ is analytic on $\mathbb{C} \setminus (D \cup \overline{D})$ and satisfies the estimate (6.46). Therefore, due to the absolute convergence of the integral (6.39) defining $L^{[l]}(\beta, I)$, one can change the integration contour from the real line to a suitable homotopic path in the complex plane. For $l > 0$ we choose the curve $\Gamma = \Gamma_0 \cup \Gamma_{\rm st}$ defined by (see Figure 6.5.3)

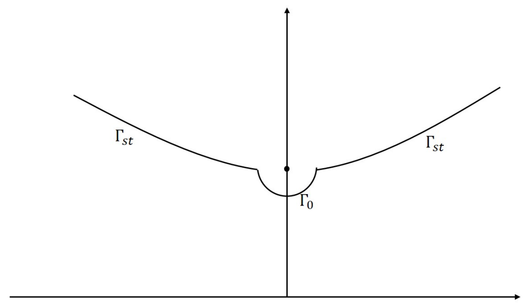


Figure 6.2: Sketch of the integration path for the estimation of the size of $|L^{[l]}(\varphi, I)|$ with $l > 3$.

$$\Gamma_0 = \{\tau \in \mathbb{C}\colon |\tau - i| = c,\ -3\pi/2 \leq \arg(u(\tau) - i/3) \leq \pi/2\}$$
$$\Gamma_{\mathrm{st}} = \{\tau \in \mathbb{C}\colon \mathrm{Im}\ u(\tau) \geq u^*,\ \arg(u(\tau) - i/3) \in \{-3\pi/2, \pi/2\}\}$$

with c being the unique positive solution of the equation

$$\frac{c^2}{2} + \frac{c^3}{6} = \frac{1}{12}$$

and $u^* = \mathrm{Im}(u(\tau^*))$ for τ^* such that $|\tau^* - i| = c$ and $\arg(u(\tau^*) - i/3) = \pi/2$. Thanks to our choice of c, on Γ_0 we have

$$|u(\tau) - i/3| = \left| \frac{i(\tau - i)^2}{2} + \frac{(\tau - i)^3}{6} \right| \leq \frac{|\tau - i|^2}{2} + \frac{|\tau - i|^3}{6} = \frac{1}{12}.$$

Therefore,

$$\min_{\tau \in \Gamma_0} \big(\mathrm{Im}(u(\tau))\big) \geq \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

and we have that

$$\max_{\tau \in \Gamma_0} \left|\exp(il\nu I^3 u(\tau))\right| \leq \exp\big(-l\nu I^3/4\big).$$

Thus, using the bound (6.46) we obtain

$$\left| \int_{\Gamma_0} U^{[l]}(\tau) e^{il\nu I^3 u(\tau)} (\tau^2 + 1) \mathrm{d}\tau \right| \lesssim I^2 \exp\big(-l\nu I^3/4\big).$$

On the other hand, we have chosen Γ_{st} to be the union of two steepest descent paths for $u(\tau)$. Hence, using that $2\mathrm{d}u = (\tau^2 + 1)\mathrm{d}\tau$ and writing $x = \mathrm{Im}\ u - 1/4$, we deduce from (6.46)

$$\begin{aligned} \left| \int_{\Gamma_{\mathrm{st}}} U^{[l]}(\tau) e^{ilI^3 u(\tau)} (\tau^2 + 1) \mathrm{d}\tau \right| =& 2 \left| \int_{u(\Gamma_{\mathrm{st}})} U^{[l]}(\tau(u)) e^{ilI^3 u} \mathrm{d}u \right| \\ \lesssim& I^2 \exp\big(-l\nu I^3/4\big) \int_0^\infty e^{-l\nu I^3 x} \mathrm{d}x \\ \lesssim& I^{-1} \exp\big(-l\nu I^3/4\big). \end{aligned}$$

Joining both computations we obtain that for $l \geq 1$

$$\left| L^{[l]}(\beta; I, \epsilon_0) \right| \lesssim I^2 \exp\big(-l\nu I^3/4\big).$$

Actually, since $\sigma \mapsto L(\beta, \sigma; I, \epsilon_0)$ is a real analytic function, we have that for $|l| \geq 1$

$$\left| L^{[l]}(\beta; I, \epsilon_0) \right| \lesssim I^2 \exp\big(-|l|\nu I^3/4\big).$$

Then, we conclude that

$$\left|\sum_{|l|\geq 3} L^{[l]}(\beta; I, \epsilon_0)\right| \lesssim I^2 \exp\big(- 3\nu I^3/4\big) \sum_{|l|\geq 3} \exp\big(- (|l|-3)\nu I^3/4\big) \lesssim I^2 \exp\big(- 3\nu I^3/4\big),$$

as was to be shown. □

6.5.4 The Fourier coefficients $L^{[1]}$ and $L^{[2]}$

In Lemmas 6.5.6 and 6.5.7 we have exploited the fact that for $|l| \geq 1$, $L^{[l]}(\beta, I)$ are oscillatory integrals (with phase given by $\nu I^3 u(\tau)$) of a real-analytic function to deduce that they decay exponentially fast with $|l|$ for $|l| \geq 3$. It is clear that, making use of the estimates in Lemma 6.5.6, the argument used in the proof of Lemma 6.5.7 also yields exponentially small bounds for the Fourier coefficients $L^{[l]}(\beta, I)$ with $|l| = 1, 2$. However, in order to obtain an asymptotic formula for $L^{[l]}(\beta, I)$ $|l| = 1, 2$, one has to integrate along a path which reaches the complex singularity $\tau = \tau'$ of the (analytic continuation of) the function $\tau \mapsto U^{[l]}(\tau, \beta; I, \epsilon_0)$ in (6.37), for which:

- $\mathrm{Im}(u(\tau'))$ is closest to zero,
- τ' is contained in the real-analytic branch associated to the function $u = (\tau + \tau^3/3)/2$.

So far we know, from Lemma 6.5.6, that $U^{[l]}$ in (6.37) defines an analytic function for $\tau \in \mathbb{C} \setminus (D \cup \bar{D})$ (see 6.44). We now locate all the complex singularities of the analytic continuation of $U^{[l]}$ in (6.37) which are relevant for our analysis. Then, we obtain local expansions close to each of these complex singularities. This will allow us to deduce an asymptotic formula for $L^{[1]}(\beta, I)$ and $L^{[2]}(\beta, I)$.

It will be convenient to write U in (6.36)) as $U = U_0 + U_1$ with

$$U_\star(\tau, \beta, \xi; I, \epsilon_0) = \frac{m_\star \tilde{\nu} I}{\left|r_h(\tau) + \frac{\tilde{\sigma}_\star}{I^2}\rho(\xi) e^{i(f(\xi)+\beta+\alpha_h(\tau))}\right|} - \frac{m_\star \tilde{\nu} I}{r_h(\tau)} \qquad \star = 0, 1 \tag{6.47}$$

and define, for $(\tau, \beta) \in \mathbb{R} \times \mathbb{T}$,

$$U_\star^{[l]}(\tau, \beta; I, \epsilon_0) = \frac{1}{2\pi} \int_0^{2\pi} \rho(\xi) U_\star(\tau, \beta, \xi; I, \epsilon_0) e^{-il\lambda(\xi)} \mathrm{d}\xi \qquad \star = 0, 1. \tag{6.48}$$

Of course, the same argument in Lemma 6.5.6 shows that (6.48) defines an analytic function for $(\tau, \beta) \in \mathbb{C} \setminus \big(D \cup \bar{D}\big) \times \mathbb{T}$. In the following, when we speak about the function element $U_\star^{[l]}$ in (6.48), we implicitely refer to the pair (6.48) and the region $\mathbb{C} \setminus (D \cup \bar{D})$.

Remark 6.5.8. *Until further notice, the variable β as well as the parameters I, ϵ_0 will be kept constant. Consequently, we omit the dependence of any function on them.*

Remark 6.5.9. *As we have already seen in Lemma 6.5.7, in order to analyze the coefficients $L^{[l]}$ with $l > 0$ we only need to study (6.48) for $\tau \in H$ where H is the upper half plane*

$$H = \{\tau \in \mathbb{C} \colon \mathrm{Im}\ \tau \geq 0\}. \tag{6.49}$$

Recall that due to real-analyticity, it is only necessary to study $L^{[l]}$ with $l > 0$.

Let $\star = 0, 1$, denote by $\tau_\pm^\star(\beta; I, \epsilon_0) \in \mathbb{C}$ the unique points satisfying

$$(\tau_\pm^\star - i)^2 + 2\tilde{\sigma}_\star \epsilon_0 I^{-2} e^{-i\beta} = 0, \tag{6.50}$$

and introduce the punctured disk

$$D_\star^* = D \setminus \{i, \tau_+^\star, \tau_-^\star\}. \tag{6.51}$$

where the disk D is defined in (6.44).

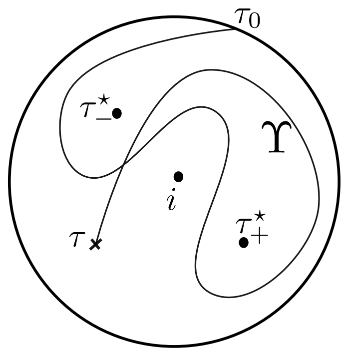


Figure 6.3: An example of a curve $\Upsilon : [0,1] \to D_\star^*$ joining a point $\Upsilon(0) = \tau_0 \in \partial D$ with a poing $\Upsilon(1) = \tau \in D_\star^*$. The position of $\tau_\pm^\star$ corresponds to an arbitrary value of β.

Remark 6.5.10. *Until further notice, the variable β as well as the parameters I, ϵ_0 will be kept constant. Consequently, we omit the dependence of any function on them.*

Also, we fix any value of C in the definition of D in (6.44) large enough, so Lemmas 6.5.6 and 6.5.7 hold and the points $\tau_\pm^\star$ are well inside the disk D.

Proposition 6.5.11. *Let $\star = 0, 1$ and fix any $\tau_0 \in \partial D$. Let $\tau \in D_\star^*$ and let $\Upsilon : [0,1] \to D_\star^*$ be any continuous curve joining $\Upsilon(0) = \tau_0$ to $\Upsilon(1) = \tau$. Then, the function element $U_\star^{[l]}$ in (6.48) admits an analytic continuation along Υ wich we denote by $U_\star^{[l]}(\tau;\Upsilon)$.*

Moreover, if $\Upsilon, \Upsilon' : [0,1] \to D_\star^*$ are two continuous curves with $\Upsilon(0) = \Upsilon'(0) = \tau_0$, $\Upsilon(1) = \Upsilon'(1) = \tau$ and such that the closed curve $\widetilde{\Upsilon} = \Upsilon'\Upsilon^{-1}$ is contractible (in $D_\star^*$) to a point, or homotopic (in $D_\star^*$) to ∂D, then $U_\star^{[l]}(\tau;\Upsilon) = U_\star^{[l]}(\tau, \Upsilon')$.

The proof of this result is deferred to Section 6.7. Proposition 6.5.11 shows that, for $\Upsilon : [0,1] \to D_\star^*$ as above, the only candidates to be singularities of the analytic continuation of $U_\star^{[l]}$ in (6.48) along Υ are $\{i, \tau_+^\star, \tau_-^\star\}$. We will see in Proposition 6.5.12 (whose proof is contained in Section 6.7) that they are indeed singularities. The second statement in Proposition 6.5.11 can thus be understood as giving partial information about the monodromy of the analytic continuation of (6.48) around these singularities.

Although Proposition 6.5.11 holds for a rather large family of curves, in order to prove Theorem 6.5.1, it is only necessary to study the behavior of the continuation of $U_\star^{[l]}$ in (6.48) along the following family of curves. Let $\star = 0, 1$, fix any $\tau_0 \in \partial D$ and let $\tau \in D_\star^*$. Then, we define the family of paths (see Figure 6.5.4)

$$\mathcal{X}_\tau = \{\Upsilon \in C([0,1], D_\star^*) : \Upsilon(0) = \tau_0,\ \Upsilon(1) = \tau,\ \mathrm{card}\{\Upsilon \cap J_\star\} \leq 1\}, \tag{6.52}$$

where $J_\star$ is the segment

$$J_\star = \{\tau \in \mathbb{C} \colon \tau = \lambda\tau_+^\star + (1-\lambda)\tau_-^\star,\ \lambda \in [0,1]\}. \tag{6.53}$$

Introduce now the punctured disks (see also Figure 6.5.4)

$$C_i = \left\{\tau \in D_\star^* \colon 0 < |\tau - i| \leq I^{-5/4}\right\} \qquad C_\pm^\star = \left\{\tau \in D_\star^* \colon 0 < |\tau - \tau_+^\star| \leq I^{-3/2}\right\}. \tag{6.54}$$

From the definition of D in (6.44), that of the points $\tau_\pm^\star$ in (6.50) and Remark 6.5.10, one easily checks that $C_i, C_\pm^\star \subset D_\star^*$ and $C_i \cap C_\pm^\star = \emptyset$, $C_+^\star \cap C_-^\star = \emptyset$.

Proposition 6.5.12. *Let $\star = 0, 1$, $\tau \in D_\star^*$, $\Upsilon \in \mathcal{X}_\tau$ and let $U_\star^{[l]}(\tau;\Upsilon)$ be the function element obtained in Proposition 6.5.11 by analytic continuation of (6.48) along Υ. Then, for $l = 1, 2$,*

- *For $\tau \in C_\pm^\star$ the asymptotic formula*

$$U_\star^{[l]}(\tau;\Upsilon) = \tilde{U}_\star^{[l]}(\tau;\Upsilon) + S_{l,\star}(\tau;\Upsilon) + R_{l,\star}(\tau)$$

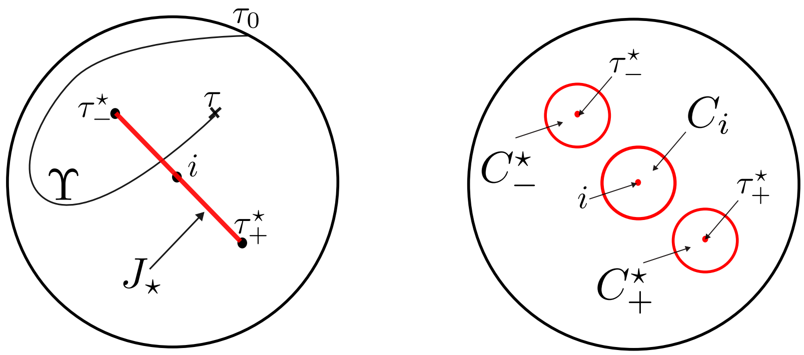


Figure 6.4: On the left, for a given $\tau \in D_\star^*$, example of a curve $\Upsilon \subset \mathcal{X}_\tau$ such that $\mathrm{card}\{\Upsilon \cap J_\star\} = 1$. On the right we sketch in red the boundary of the sets C_i, C_+ and C_-. Again the situation depicted correspond to an arbitrarily chosen value of β.

holds with [10]

$$\tilde{U}_\star^{[l]}(\tau;\Upsilon) = m_\star A_l \sqrt{\frac{e^{i\beta}}{\tilde{\sigma}_\star}}\; I^2(1+\mathcal{O}(I^{-1}))\; \ln((1-2b_\star)(\tau;\Upsilon)) \qquad\qquad b_\star(\tau) = \frac{2\tilde{\sigma}_\star \epsilon_0 e^{-i\beta}}{I^2(\tau-i)^2}, \quad (6.55)$$

where $A_l \neq 0$ *are explicit, purely imaginary, constants,* [11]

$$|S_{l,\star}(\tau;\Upsilon)| \lesssim I^{15/8}$$

and $R_{l,\star}(\tau)$ *is independent of* Υ *(and therefore analytic for all* $\tau \in C_\pm^\star \cup \{\tau_\pm\}$*).*

- *For* $\tau \in C_i$ *there exists a constant* $B_{l,\star} \in \mathbb{C}$ *such that,*

$$U_\star^{[l]}(\tau;\Upsilon) = B_{l,\star} + E_{l,\star}(\tau;\Upsilon)$$

with

$$|E_{l,\star}(\tau;\Upsilon)| \lesssim I^{15/8}.$$

- *For* $\tau \in D \setminus (C_i \cup C_+^\star \cup C_-^\star)$ *we have*

$$\left|U_\star^{[l]}(\tau;\Upsilon)\right| \lesssim I^{9/4}.$$

Remark 6.5.13. *It is admittedly akward that the estimate for* $\tau \in D \setminus (C_i \cup C_+^\star \cup C_-^\star)$ *(far from the singularities) is worse than the estimates for* $\tau \in C_\pm^\star$ *and* $\tau \in C_i$*. This is only because the analysis we have made away from the singularities is far less refined than the analysis close to the singularities. A more detailed analysis would lead to better estimates far from the singularities, however, that will not be necessary for our purposes.*

The proof of Proposition 6.5.12 is postponed until Section 6.7. The following lemma, proved also in Section 6.7, will be important to obtain an asymptotic expression for $L^{[1]}$ and $L^{[2]}$.

Lemma 6.5.14. *Let* $\tilde{U}_\star^{[l]}$ *be the function defined in* (6.55)*. Then,* $\partial_\tau \tilde{U}^{[l]}$ *is a meromorphic function on* $C_\pm^\star \cup \{\tau_\pm\}$ *with a pole of order one at* $\tau = \tau_\pm$*. Moreover, there exist explicit, purely imaginary, constants* $\hat{A}_l \neq 0$ [12] *such that*

$$\operatorname*{Res}_{\tau=\tau_\pm^\star} \left(\partial_\tau \tilde{U}_\star^{[l]}(\tau)\right) = m_\star \hat{A}_l \sqrt{\frac{e^{i\beta}}{\tilde{\sigma}_\star}}\; I^2\big(1+\mathcal{O}(I^{-1})\big).$$

[10]The term $1-2b_\star$ vanishes for $\tau = \tau_\pm^\star$. Therefore, one needs to take into account the argument of $(\tau-\tau_+^\star)(\tau-\tau_-^\star)$ or, what is the same, the dependence on the path Υ. We use the real analytic branch of the logarithm function.

[11]The constants A_l are computed explicitely in Section 6.7, see (6.100).

[12]The constants $\hat{A}_l$ are computed explicitely in Section 6.7.4, see (6.102).

We now show how to use Proposition 6.5.12 and Lemma 6.5.14 to obtain an asymptotic expression for $L^{[1]}$ and $L^{[2]}$. We write $L^{[l]}$ in (6.39) as $L^{[l]} = L_0^{[l]} + L_1^{[l]}$ with

$$L_\star^{[l]} = \frac{1}{2\pi}\int_{\mathbb{R}} U^{[l]}(s)e^{il\nu I^3 u(s)}(s^2+1)\mathrm{d}s \qquad \star = 0,1. \tag{6.56}$$

Proposition 6.5.15. *Let $l = 1,2$. Then, there exists explicit real constants $\tilde{A}_{l,\star} \neq 0$ [13] such that*

$$L_\star^{[l]}(\beta, I) = \frac{m_\star}{\sqrt{\tilde{\sigma}_\star}}\ \tilde{A}_l I^{-1}(1+\mathcal{O}(I^{-1}))\ \exp(-l\nu I^3(1/3+p_\star(\beta,I)+h_\star(\beta,I))) + T_{l,\star}(\beta,I) + T_{l,\star,\exp}(\beta,I) \tag{6.57}$$

with

$$p_\star(\beta,I) = \frac{2}{3}\tilde{\sigma}_\star\epsilon_0 I^{-2}(\cos\beta - i\sin\beta) \qquad |h_\star(\beta,I)| \lesssim I^{-3}$$

and

$$|T_{l,\star}(\beta,I)| \lesssim I^{-9/8}\exp(-l\nu(1/3+\mathrm{Re}\ p_\star)) \qquad |T_{l,\star,\exp}(\beta,I)| \lesssim I^{-9/8}\exp(-l\nu(1/3)).$$

Remark 6.5.16. *We point out that* (6.57) *only gives an asymptotic formula of $L_\star^{[l]}$ for $\beta \in \mathbb{T}$ such that* Re $p_\star < 0$ *that is $\beta \in (\pi/2, 3\pi/2)$.*

Since $L^{[l]} = L_0^{[l]} + L_1^{[l]}$, the asymptotic formulas stated in Theorem 6.5.1 are straightforward from the ones in Proposition 6.5.15. This concludes the proof of Theorem 6.5.1.

6.6 Proof of Proposition 6.5.15

In order to prove Proposition 6.5.15, we change the integration contour in (6.56) to a combination of steepest descent paths [14] which visit the singularities $\tau = \tau'$ of the function $\tau \mapsto U^{[l]}(\tau,\beta)$ for which Im $u(\tau')$ is closest to zero and which are contained in the real-analytic branch (see Remark 6.6.1) of the function $u(\tau) = (\tau + \tau^3/3)/2$.

Remark 6.6.1. *The complex plane $\tau \in \mathbb{C}$ can be divided in in three disjoint open connected which are all mapped bijectively by the polynomial $u(\tau) = (\tau+\tau^3/3)/2$ onto $\mathbb{C}\setminus\{u = is\colon s \in (-\infty,-1/3]\cup[1/3,+\infty)\}$ (see Figure 6.6). We will denote by real-analytic branch the (unique) open connected component containing the real line.*

In Propositions 6.5.11 and 6.5.12, we have seen that $\tau = i$ and $\tau = \tau_\pm^\star(\beta)$, defined in (6.50), are the unique singularities of the the analytic continuation of $U^{[l]}(\tau,\beta)$ along paths in $\Upsilon : [0,1] \to D_\star^*$. Since these singularities move as we change the value of the angle β, the integration contour that we choose to compute (6.56), will be different for different values of β. The first observation is that $\beta \mapsto \tau_\pm^\star(\beta)$ are 4π-periodic functions. Hence, throughout this section, we are forced to consider $\beta \in \mathtt{T} = \mathbb{R}/4\pi\mathbb{Z}$.

Remark 6.6.2. *For $\tau \in \mathbb{R}$, $U^{[l]}(\tau,\beta)$ is a 2π-periodic function of β. Of course, since $L^{[l]}(\beta)$ in* (6.56) *is defined as an integral over $\tau \in \mathbb{R}$, the function $L^{[l]}(\beta)$ is also 2π periodic in β. However, to compute $L^{[l]}(\beta)$, we will change the integration contour from the real line to a path which enters the region $\tau \in D$. In this region, due to the existence of branching points, it is not true anymore that the analytic continuation of the function $U^{[l]}(\tau,\beta)$ is a 2π-periodic function of β and one should instead study $U^{[l]}(\tau,\beta)$ as a function of $\beta \in \mathtt{T}$.*

[13] The constants $\tilde{A}_l$ are computed explicitely in Section 6.6, see (6.67).

[14] By steepest descent path we mean a segment in the $\tau \in \mathbb{C}$ plane where Re $u(\tau) = $ const.

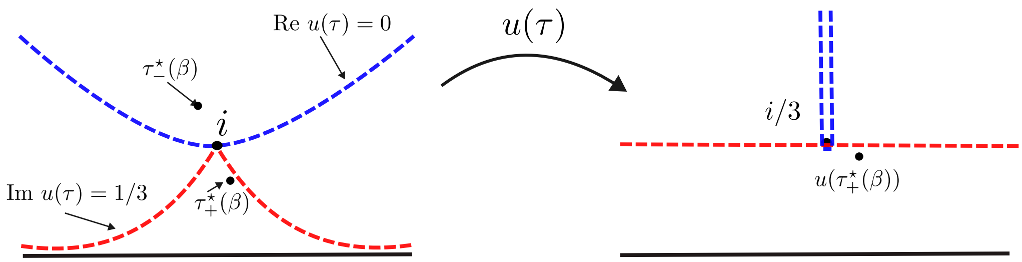


Figure 6.5: On the left the plane $\tau \in \mathbb{C}$. The position of the points $\tau_\pm(\beta)$ correspond to the case $\beta \in \mathtt{T}_{+,\star}$. On the right, the image under the real-analytic transformation $u(\tau)$. The dashed blue (red) curve on the left is sent onto the dashed blue (red) curve on the right.

Define now the subsets (understood as mod 4π)

$$\begin{aligned} \mathtt{T}_{+,0} &= [\pi/3, 5\pi/3], & \mathtt{T}_{-,0} &= [\pi/3+2\pi, 5\pi/3+2\pi], & \mathtt{T}_{i,0} &= \mathtt{T} \setminus (\mathtt{T}_{+,0} \cup \mathtt{T}_{-,0}) \\ \mathtt{T}_{+,1} &= [4\pi/3, 8\pi/3], & \mathtt{T}_{-,1} &= [4\pi/3+2\pi, 8\pi/3+2\pi], & \mathtt{T}_{i,1} &= \mathtt{T} \setminus (\mathtt{T}_{+,1} \cup \mathtt{T}_{-,1}) \end{aligned} \tag{6.58}$$

and write $u(\tau) = (\tau + \tau^3/3)/2$ as

$$u(\tau) - \frac{i}{3} = \frac{i}{3}(\tau - i)^2 + \frac{1}{6}(\tau - i)^3 \tag{6.59}$$

We distinguish 3 situations:

1. $\beta \in \mathtt{T}_{+,\star}$: For these values of β one easily checks that [15] $u(\tau_+^\star(\beta))$ is contained in the real-analytic branch of the function $u(\tau) = (\tau + \tau^3/3)/2$ and $u(\tau_-^\star(\beta))$ is not. Moreover, from (6.59) and the definition of $\tau_+^\star(\beta)$ in (6.50), we obtain that

$$\frac{1}{3}\left(1 - 2|\tilde{\sigma}_\star|\epsilon_0 I^{-2}\right) + \mathcal{O}(I^{-3}) \leq \text{Im } u(\tau_+^\star(\beta)) \leq \frac{1}{3}\left(1 + |\tilde{\sigma}_\star|\epsilon_0 I^{-2}\right) + \mathcal{O}(I^{-3}).$$

   We will see that, in this case, the main contribution to the integral $L_\star^{[l]}$ defined in (6.56) is given by the singularities $\tau = \tau_+^\star(\beta)$ and/or $\tau = i$.

2. $\beta \in \mathtt{T}_{-,\star}$: For these values of β one easily checks that $u(\tau_-^\star(\beta))$ is contained in the real-analytic branch of the function $u(\tau) = (\tau + \tau^3/3)/2$ and $u(\tau_+^\star(\beta))$ is not. From (6.59) and the definition of $\tau_-(\beta)$ in (6.50), we obtain that

$$\frac{1}{3}\left(1 - 2|\tilde{\sigma}_\star|\epsilon_0 I^{-2}\right) + \mathcal{O}(I^{-3}) \leq \text{Im } u(\tau_-^\star(\beta)) \leq \frac{1}{3}\left(1 + |\tilde{\sigma}_\star|\epsilon_0 I^{-2}\right) + \mathcal{O}(I^{-3}).$$

   We will see that, in this case, the main contribution to the integral $L_\star^{[l]}$ defined in (6.56) is given by the singularities $\tau = \tau_-(\beta)$ and/or $\tau = i$.

3. $\beta \in \mathtt{T}_{i,\star}$. In this case, from (6.59) and the definition of $\tau_\pm(\beta)$ in (6.50)

$$\text{Im } u(\tau_\pm^\star(\beta)) = 1/3\left(1 + 2|\tilde{\sigma}_\star|\epsilon_0 I^{-2}\cos\beta\right) + \mathcal{O}(I^{-3}) \geq 1/3(1 + |\tilde{\sigma}_\star|\epsilon_0 I^{-2}) + \mathcal{O}(I^{-3}) > 1/3.$$

   We will see that, in this case, the main contribution to the integral $L_\star^{[l]}$ defined in (6.56) is given by the singularity $\tau = i$.

We now describe in full detail the case $\beta \in \mathtt{T}_{+,\star}$ and sketch later the changes needed to analyze the other two situations.

[15] We arbitrarily define $\tau_+^\star(\beta)$ to be the solution of (6.50) which is contained in the real-analytic branch of $u(\tau) = (\tau + \tau^3/3)/2$ for $\beta \in \mathtt{T}_{+,\star}$.

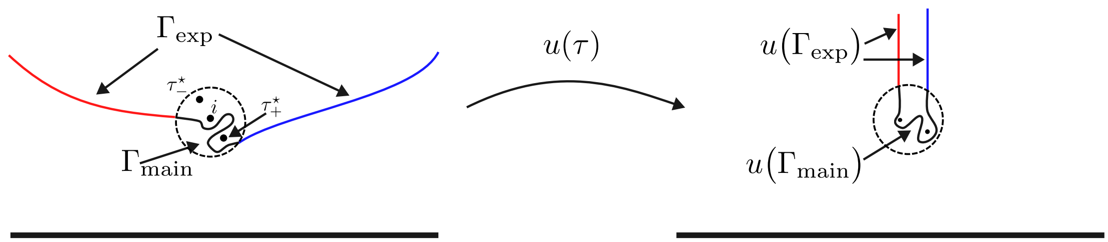


Figure 6.6: Sketch of the curve Γ. On red and blue, the two connected components of Γ_{exp}. In black an sketch of Γ_{main}, which will be defined precisely later (see also Figure 6.6.1.)

6.6.1 Case (1): $\beta \in \mathtt{T}_{+,\star}$

We will change the integration contour in (6.56) from the real line to a path (H denotes the upper half plane)

$$\Gamma \subset \mathbb{C}_\star \equiv H \setminus \{i, \tau_+^*(\beta), \tau_-^*(\beta)\}$$

which is a suitable combination of steepest descent paths visiting the singularities $\tau = i$ and $\tau_+^\star(\beta)$ of the (analytic continuation) of $U^{[l]}(\tau, \beta)$. To that end, we first observe that, for $\beta \in \mathtt{T}_+$, expression (6.59) implies that

$$-7\pi/6 \leq \arg(u(\tau_+(\beta)) - i/3) \leq \pi/6.$$

Just to avoid technicalities, we focus on the case where $\arg(u(\tau_+(\beta)) - i/3) > -\pi/2$ and indicate, at the end of the section, how the obtained result extends to the full interval $\beta \in \mathtt{T}_{+,\star}$. Let $\beta \in \mathtt{T}_{+,\star}$ be such that $\arg(u(\tau_+(\beta)) - i/3) > -\pi/2$.

Fix a value of $\beta \in \mathtt{T}_{+,\star}$ for which $\arg(u(\tau_+(\beta)) - i/3) > -\pi/2$ (we now omit the dependence on β until further notice). Then, we consider

$$\Gamma = \Gamma_{\text{exp}} \cup \Gamma_{\text{main}} \subset \mathbb{C}_\star \tag{6.60}$$

where $\Gamma_{\text{exp}}, \Gamma_{\text{main}}$ are defined as follows (see Figure 6.6.1). The former one is defined as

$$\begin{aligned} \Gamma_{\text{exp}} = &\{\tau \in \mathbb{C} \setminus D \colon \text{Im } u(\tau) > 1/3,\ \arg\left(u(\tau) - i/3\right) = -3\pi/2\} \\ &\cup \left\{\tau \in \mathbb{C} \setminus D \colon \text{Im } u(\tau) > \text{Im } u(\tau_+^\star),\ \arg\left(u(\tau) - u(\tau_+^\star)\right) = \pi/2\right\} \end{aligned} \tag{6.61}$$

The curve $\Gamma_{\text{main}} \subset D$ is such that Γ is connected and homotopic to the real line in $\mathbb{C}_\star$. More concretely, we will also choose Γ_{main} as an union of steepest descent paths (for $u(\tau)$) contained in D and visiting $\tau = i$ and $\tau = \tau_+^\star$. To that end, in the following technical lemma, we study the behavior of the imaginary part of $u(\tau)$ along different steepest descent paths for $u(\tau)$ and contained in D. We first define the constants

$$\hat{u}_\star = \min\left(1/3,\ \text{Im}\left(u\left(\tau_+^\star\right)\right)\right) + I^{-11/4}, \qquad \tilde{u}_\star = \max\left(1/3,\ \text{Im}\left(u\left(\tau_+^\star\right)\right)\right) + I^{-9/4}. \tag{6.62}$$

Lemma 6.6.3. *Let $\tilde{\Gamma} \subset D$ be a connected segment. Then,*

- *If it satisfies* $\text{Re } u(\tau) = 0$ *and* $\text{Im } u(\tau) \geq 1/3$,

$$\max_{\tau \in \tilde{\Gamma} \cap C_i} \text{Im } u(\tau) < \tilde{u}_\star \qquad \min_{\tau \in \tilde{\Gamma} \setminus C_i} \text{Im } u(\tau) \geq \hat{u}_\star.$$

- *If it satisfies* $\text{Re } u(\tau) = \text{Re } u(\tau_+^\star)$ *and* $\text{Im } u(\tau) \geq \text{Im } u(\tau_+^\star)$,

$$\max_{\tau \in \tilde{\Gamma} \cap C_+} \text{Im } u(\tau) < \tilde{u}_\star \qquad \min_{\tau \in \tilde{\Gamma} \setminus C_+} \text{Im } u(\tau) \geq \hat{u}_\star.$$

Proof. Expanding $u(\tau) = (\tau + \tau^3/3)/2$ in Taylor series around $\tau = i$

$$u(\tau) - \frac{i}{3} = \frac{i}{2}(\tau - i)^2 + \frac{1}{6}(\tau - i)^3 = \frac{i}{2}(\tau - i)^2 \left(1 + \mathcal{O}|\tau - i|\right)$$

and the first statement follows from the definition of C_i. The Taylor expansion of $u(\tau)$ around $\tau = \tau_+^\star$ yields

$$u(\tau) - u(\tau_+^\star) = 2iI^{-1}\sqrt{\epsilon\tilde{\sigma}_\star e^{-i\beta}}(\tau - \tau_+^\star)\left(1 + \mathcal{O}|\tau - \tau_+^\star|\right).$$

Then, the second statement follows from the definition of $C_+^\star$.

□

Lemma 6.6.3 provides us with suitable information on the behavior of Im u along steepest descent paths visiting either $\tau = i$ or $\tau_+^\star$. This information is crucial to define Γ_{main} as a union of paths along which Imu is sufficiently large (so their contribution will be exponentially smaller) and paths contained in either C_i or $C_+^\star$, for which we can perform an asymptotic analysis. More concretely, we choose Γ_{main} (see (6.60)) as the union

$$\Gamma_{\mathrm{main}} = \Gamma_i \cup \Gamma_{\mathrm{join}} \cup \Gamma_+$$

where

$$\Gamma_i = \tilde{\Gamma}_i \cup \Gamma_{i,\mathrm{exp}}, \qquad \Gamma_+ = \tilde{\Gamma}_+ \cup \Gamma_{i,\mathrm{exp}},$$

with

- $\Gamma_{i,\mathrm{exp}}$ a combination of two steepest descent paths visiting the disk C_i

$$\begin{aligned}\Gamma_{i,\mathrm{exp}} =& \{\tau \in D \setminus C_i \colon \mathrm{Im}\ 1/3 < u(\tau),\ \arg(u(\tau) - i/3) = -3\pi/2\} \\ & \cup \{\tau \in D \setminus C_i \colon \mathrm{Im}\ 1/3 < u(\tau) \le \tilde{u}_\star,\ \arg(u(\tau) - i/3) = \pi/2\}\end{aligned} \tag{6.63}$$

- $\Gamma_{+,\mathrm{exp}}$ a combination of two steepest descent paths visiting the disk $C_+^\star$

$$\begin{aligned}\Gamma_{+,\mathrm{exp}} =& \{\tau \in D \setminus C_+^\star \colon \mathrm{Im}\ u(\tau_+^\star) \le \mathrm{Im}\ u(\tau) \le \tilde{u}_\star,\ \arg(u(\tau) - u(\tau_+^\star)) = -3\pi/2\} \\ & \cup \{\tau \in D \setminus C_+^\star \colon \mathrm{Im}\ u(\tau_+^\star) \le \mathrm{Im}\ u(\tau),\ \arg(u(\tau) - u(\tau_+^\star)) = \pi/2\}\end{aligned} \tag{6.64}$$

- Γ_{join} a segment joining $\Gamma_{i,\mathrm{exp}}$ and $\Gamma_{+,\mathrm{exp}}$

$$\Gamma_{\mathrm{join}} = \{\tau \in D \setminus (C_i \cup C_+^\star) \colon 0 \le \mathrm{Re}\left(u\left(\tau\right)\right) \le \mathrm{Re}\left(u\left(\tau_+^\star\right)\right),\ \mathrm{Im}\left(u\left(\tau\right)\right) = \tilde{u}_\star\}.$$

- $\tilde{\Gamma}_i, \tilde{\Gamma}_+$ are contained in C_i and C_+ respectively, and are such that Γ is connected and is homotopic in $\mathbb{C}_\star$ to the real line (see Figure 6.6.1). They will be defined more precisely later.

We notice that for each $\tau \in \Gamma$, there exist a unique (up to homotopy) path $\Upsilon_\tau \in \mathcal{X}_\tau$, where $\mathcal{X}_\tau$ is the set of paths defined in (6.52). Therefore, in the following, in order to simplify the notation, for each $\tau \in \Gamma$ we will simply write

$$U_\star^{[l]}(\tau) = U_\star^{[l]}(\tau; \Upsilon_\tau)$$

to denote the analytic continuation of (6.48) along Υ_τ. The above discussion shows that $L_\star^{[l]}$ in (6.56) is equivalent to

$$L_\star^{[l]} = \frac{1}{2\pi}\int_\Gamma (\tau^2 + 1)U_\star^{[l]}(\tau)e^{il\nu I^3 u(\tau)}\mathrm{d}\tau. \tag{6.65}$$

We are now in position to obtain an asymptotic formula for (6.65) for case (I), i.e. $\beta \in \mathtt{T}_{+,\star}$. The idea behind the definition of Γ is that, in view of Lemma 6.6.3, the contribution of the segments $\Gamma_{\mathrm{exp}}, \Gamma_{i,\mathrm{exp}}, \Gamma_{+,\mathrm{exp}}$ and Γ_{join} to the integral (6.56), will be exponentially smaller than the contribution of the segments $\tilde{\Gamma}_i, \tilde{\Gamma}_+$.

We first bound the contribution to (6.65) of the segments, $\Gamma_{\mathrm{exp}}, \Gamma_{i,\mathrm{exp}}, \Gamma_{+,\mathrm{exp}}$ and Γ_{join}. For $\tau \in \Gamma_{\mathrm{exp}}$, defined in (6.61), we have from Lemma 6.5.6 that

$$\left|U_\star^{[l]}\right| \lesssim I^2.$$

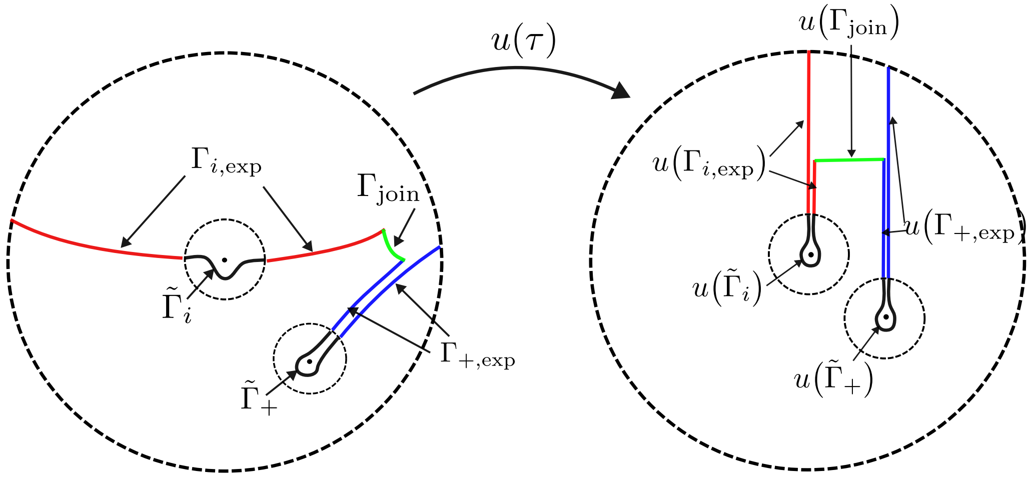


Figure 6.7: Sketch of the curve Γ_{main}. On red (blue), the two connected components of $\Gamma_{i,\text{exp}}$ ($\Gamma_{+,\text{exp}}$). In black an sketch of $\tilde\Gamma_i$ and $\tilde\Gamma_+$, which will be defined precisely later.

We now notice that Γ_{exp} is the union of two segments which are steepest descent paths for the variable u. Moreover, from Lemma 6.6.3 we know that, for all $\tau \in \Gamma_{\text{exp}}$, Im $u(\tau) \geq \hat u_\star$ where $\hat u_\star$ has been defined in (6.62). Then, using that $\mathrm{d}u = \frac{1}{2}(\tau^2+1)\mathrm{d}\tau$, and denoting by $u(\Gamma_{\text{exp}})$ the image of Γ_{exp} under the real analytic change of variables $u(\tau) = (\tau + \tau^3/3)/2$ introduced in Lemma 6.5.3 we obtain that

$$\int_{\Gamma_{\text{exp}}} (\tau^2+1) U_\star^{[l]}(\tau) e^{il\nu I^3 u(\tau)} \mathrm{d}\tau = 2 \int_{u(\Gamma_{\text{exp}})} U_\star^{[l]}(\tau(u)) e^{il\nu I^3 u} \mathrm{d}u.$$

Then, recalling that Re $u(\tau) = 0$ at Γ_{exp}, if we write $x = \text{Im } u - \hat u_\star$, we obtain

$$\begin{aligned}\left|\int_{\Gamma_{\text{exp}}} (\tau^2+1) U_\star^{[l]}(\tau) e^{il\nu I^3 u(\tau)} \mathrm{d}\tau\right| &= 2\left|\int_{u(\Gamma_{\text{exp}})} U_\star^{[l]}(\tau(u)) e^{il\nu I^3 u} \mathrm{d}u\right| \\ &\lesssim I^2 \exp(-l\nu I^3 \hat u_\star) \int_0^\infty e^{-l\nu I^3 x} \mathrm{d}x \lesssim I^{-1} \exp(-l\nu I^3 \hat u_\star).\end{aligned}$$

The segments $\Gamma_{i,\text{exp}}, \Gamma_{+,\text{exp}}$ (see (6.63) and (6.64)) are also steepest descent paths for the variable u. Therefore, Lemma 6.6.3, the estimate for $|U_\star^{[l]}|$ when $\tau \in D \setminus (C_i \cup C_+^\star \cup C_-^\star)$ in Proposition 6.5.12 and the very same argument used to bound the contribution of Γ_{exp}, show that

$$\begin{aligned}\left|\int_{\Gamma_{i,\text{exp}}} (\tau^2+1) U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau\right| &\lesssim I^{9/4} \exp(-l\nu I^3 \hat u_\star) \int_0^\infty e^{-l\nu I^3 x} \mathrm{d}x \lesssim I^{-3/4} \exp(-l\nu I^3 \hat u_\star) \\ \left|\int_{\Gamma_{+,\text{exp}}} (\tau^2+1) U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau\right| &\lesssim I^{9/4} \exp(-l\nu I^3 \hat u_\star) \int_0^\infty e^{-l\nu I^3 x} \mathrm{d}x \lesssim I^{-3/4} \exp(-l\nu I^3 \hat u_\star).\end{aligned}$$

In order to bound the contribution from the segment Γ_{join} we simply use that length$(\Gamma_{\text{join}}) \lesssim I^{-1}$, the fact that $|\tau^2+1| \lesssim I^{-1}$ for $\tau \in \Gamma_{\text{join}}$ and the estimate for $|U_\star^{[l]}(\tau)|$ when $\tau \in D \setminus (C_i \cup C_+^\star \cup C_-^\star)$ in Proposition 6.5.12 to obtain

$$\left|\int_{\Gamma_{\text{join}}} (\tau^2+1) U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau\right| \lesssim I^{5/4} \exp(-l\nu I^3 \tilde u_\star) \text{length}(\Gamma_{\text{join}}) \lesssim I^{1/4} \exp(-l\nu I^3 \tilde u_\star).$$

The following step in the proof is to bound the contribution of the segment $\tilde\Gamma_i \subset C_i$. According to Proposition 6.5.12, there exists a constant $B_{l,\star} \in \mathbb{C}$ such that, for $\tau \in C_i$,

$$U_\star^{[l]}(\tau) = B_{l,\star} + E_{l,\star}(\tau)$$

with $|E_{l,\star}(\tau)| \lesssim I^{7/4}$. Due to the analyticity of $U_\star^{[l]}$ on the region C_i we can choose $\tilde\Gamma_i$ to be given by

$$\begin{aligned}\tilde\Gamma_i = \tilde\Gamma_{i,\mathrm{st}} \cup \tilde\Gamma_{i,\varepsilon} \equiv &\{\tau \in C_i \colon \mathrm{Im}\ u(\tau) > 1/3 + \delta(\varepsilon),\ \arg(u(\tau) - i/3) \in \{-3\pi/2, \pi/2\}\} \\ &\cup \{\tau \in C_i \colon |\tau - i| = \varepsilon,\ \arg(u(\tau) - i/3) \in (-3\pi/2, \pi/2)\}\end{aligned}$$

for arbitrarily small $0 < \varepsilon \ll 1$ and $\delta(\varepsilon)$ such that $\tilde\Gamma_i$ is connected. The integral of the constant term is trivially seen to be zero, so we only have to bound the integral of the term $E_{l,\star}(\tau)$. The uniform bounds for $|E_{l,\star}|$ in Proposition 6.5.12 imply that the contribution of $\tilde\Gamma_{i,\varepsilon}$ is proportional to ε, thus, arbitrarily small. On the other hand, $\tilde\Gamma_{i,\mathrm{st}}$ is again the union of two steepest descent paths for the variable u. Arguing as above, defining $x = \mathrm{Im}\ u - 1/3$ and making use of the estimate $|E_{l,\star}| \lesssim I^{15/8}$, we obtain

$$\left| \int_{\tilde\Gamma_{i,\mathrm{st}}} (\tau^2+1) U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau \right| \lesssim I^{15/8} \exp(-l\nu I^3/3) \int_0^\infty e^{-l\nu I^3 x} \mathrm{d}x \lesssim I^{-9/8} \exp(-l\nu I^3/3).$$

Now we analyze the contribution of $\tilde\Gamma_+$. We have shown in Proposition 6.5.12 that, for $\tau \in C_+^\star$,

$$U^{[l]}(\tau) = \tilde U_\star^{[l]}(\tau) + S_{l,\star}(\tau) + R_{l,\star}(\tau)$$

where $\tilde U^{[l]}$ is defined in (6.55), $|S_{l,\star}(\tau)| \lesssim I^{15/8}$ and $R_{l,\star}(\tau)$ is analytic for $\tau \in C_+^\star \cup \{\tau_+^\star\}$. The contribution of $R_{l,\star}(\tau)$ to the integral is zero due to analyticity. To bound the contribution of $S_{l,\star}$ we deform the path $\tilde\Gamma_+$ in a similar way to the one used to bound the contribution of $\tilde\Gamma_i$, that is, we choose $\tilde\Gamma_+$ to be a combination of two steepest descent paths $\tilde\Gamma_{+,\mathrm{st}}$, starting at τ_+ and an arbitrarily small circumference $\tilde\Gamma_{+,\varepsilon}$ around τ_+ closing the path. The contribution of $\tilde\Gamma_{+,\varepsilon}$ is proportional to ε, thus, arbitrarily small. Defining $x = \mathrm{Im}\ u - 1/3$, we obtain that

$$\begin{aligned}\left| \int_{\tilde\Gamma_{+,\exp}} (\tau^2+1) S_{l,\star}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau \right| &\lesssim I^{15/8} \exp(-l\nu I^3 \mathrm{Im}\ u(\tau_+^\star)) \int_0^\infty e^{-l\nu I^3 x} \mathrm{d}x \\ &\lesssim I^{-9/8} \exp(-l\nu I^3 \mathrm{Im}\ u(\tau_+^\star)).\end{aligned}$$

Finally, we evaluate the integral of $\tilde U_\star^{[l]}(\tau)$ on the whole $\tilde\Gamma_+$ directly. To this end, we integrate by parts and obtain

$$\int_{\tilde\Gamma_+} (\tau^2+1) \tilde U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau = \frac{2}{il\nu I^3} \tilde U_\star^{[l]}(\tau) e^{il\nu I^3 u(\tau)} \Big|_{\tau_b}^{\tau_a} - \frac{2}{il\nu I^3} \int_{\tilde\Gamma_+} \partial_\tau \tilde U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau$$

where τ_a, τ_b are the endpoints of $\tilde\Gamma_+$. Since $|\tilde U_\star^{[l]}(\tau)| \lesssim I^2 \ln(I)$ for $\tau \in C_+$ and, in view of Lemma 6.6.3,

$$\left| e^{il\nu I^3 u(\tau_a)} \right|, \left| e^{il\nu I^3 u(\tau_b)} \right| \lesssim \exp(-l\nu I^3 \hat u_\star),$$

we obtain that

$$\left| \tilde U_\star^{[l]}(\tau) e^{il\nu I^3 u(\tau)} \Big|_{\tau_b}^{\tau_a} \right| \lesssim I^2 \ln(I) \exp(-l\nu I^3 \hat u_\star).$$

On the other hand, we have seen in Lemma 6.5.14 that $\tau = \tau_+^\star$ is a simple pole of the function $\partial_\tau \tilde U_\star^{[l]}(\tau)$. Therefore, a direct application of the residue theorem shows that

$$\int_{\tilde\Gamma_+} \partial_\tau U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau = -2\pi i \operatorname*{Res}_{\tau=\tau_+^\star} \left(\partial_\tau \tilde U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \right).$$

Thus,

$$\begin{aligned}\int_{\tilde\Gamma_+} (\tau^2+1) U_\star^{[l]}(\tau) e^{ilI^3 u(\tau)} \mathrm{d}\tau = &\frac{4}{l\nu I^3} \operatorname*{Res}_{\tau=\tau_+^\star} \left(\partial_\tau \tilde U_\star^{[l]}(\tau) \right) \exp(-(l\nu I^3 u(\tau_+^\star)) \\ &+ \mathcal{O}\left(I^{-1} \ln(I) \exp(-l\nu I^3 \tilde u_\star),\ I^{-9/8} \exp(-l\nu I^3 \mathrm{Im}\ u(\tau_+^\star)) \right).\end{aligned}$$

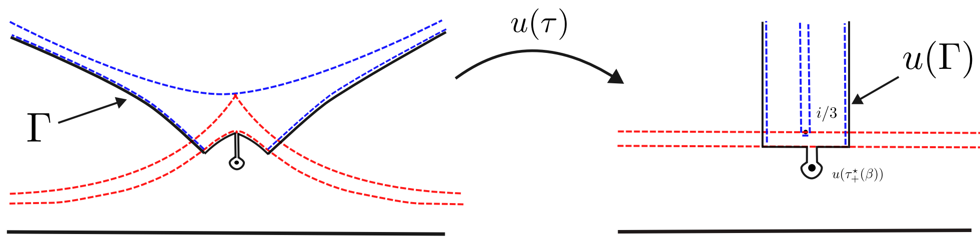


Figure 6.8: In black, sketch of the curve Γ for values of $\beta \in \mathtt{T}_{+,\star}$ such that $\arg(u(\tau_+^\star(\beta)) - i/3) > -\pi/2$. Red (blue) dashed lines correspond to lines for which the imaginary (real) part of $u(\tau)$ is constant.

From the formula for the residue given in Lemma 6.5.14 it is straightforward to check that

$$\frac{4}{l\nu I^3}\operatorname*{Res}_{\tau=\tau_+^\star}\left(\partial_\tau \tilde{U}_\star^{[l]}(\tau)\right)\exp(il\nu I^3 u(\tau_+^\star)) = \tilde{A}_l \frac{m_\star}{\sqrt{\tilde{\sigma}_\star}}\ I^{-1}(1+\mathcal{O}(I^{-1}))\ \exp\big(-l\nu I^3(1/3+p_\star(\beta)) + h_+^\star(\beta)\big), \tag{6.66}$$

where

$$\tilde{A}_l = \frac{4}{l\nu}\hat{A}_l, \tag{6.67}$$

and

$$p_\star(\beta) = \frac{2}{3}\tilde{\sigma}_\star\epsilon_0 I^{-2}e^{-i\beta} \qquad\qquad h_+^\star(\beta) = -i\left(\frac{\beta}{2} - \frac{l\nu}{6}(2\tilde{\sigma}_\star\epsilon_0)^{3/2}e^{-i3\beta/2}\right).$$

Combining this asymptotic computation with the estimates obtained for the contribution of $\Gamma_{\exp}$, $\Gamma_{i,\exp}$, $\Gamma_{+\exp}$, Γ_{join} and Γ_i, we obtain that, for $\beta \in \mathtt{T}$ corresponding to the situation (I), i.e. $\beta \in \mathtt{T}_{+,\star}$,

$$\begin{aligned} L_\star^{[l]}(\beta) =& \tilde{A}_l \frac{m_\star}{\sqrt{\tilde{\sigma}_\star}}\ I^{-1}(1+\mathcal{O}(I^{-1}))\ \exp\big(-l\nu I^3(1/3+p_\star(\beta)) + h_+^\star(\beta)\big) \\ &+ \mathcal{O}\left(I^{-9/8}\exp(-l\nu I^3/3), I^{-9/8}\exp(-l\nu\ \operatorname{Im}\ u(\tau_+^\star)(\beta))\right). \end{aligned} \tag{6.68}$$

Remark 6.6.4. *Without saying, we have chosen I large enough so the exponentially small errors are much smaller than the polynomial ones.*

We have given, a complete, analytical description of the path Γ used to obtain the asymptotic formula (6.68) for the case where $\beta \in \mathtt{T}_{+,\star}$ and $\arg(u(\tau_+^\star(\beta)) - i/3) > -\pi/2$. An important feature of the chosen path Γ is that even when the two singularities $\tau = i$ and $\tau_+^\star(\beta)$ have the same imaginary part [16], the only significant contributions to the integral $L_\star^{[l]}$ in (6.56) come from the local segments $\tilde{\Gamma}_i$ and $\tilde{\Gamma}_+$. However, when $\beta \in \mathtt{T}_{+,\star}$ and $\arg(u(\tau_+^\star(\beta)) - i/3) \to -\pi/2$, the path Γ is not well defined since Re $u(\tau_+^\star(\beta)) \to 0$.

Rather than providing lengthy formulas describing how to extend the path across β such that $\arg(u(\tau_+^\star(\beta)) - i/3) \to -\pi/2$ we briefly discuss how one can deform the integration path Γ to extend the asymptotic expression (6.68) for all $\beta \in \mathtt{T}_{+,\star}$. The main observation is that as $\arg(u(\tau_+^\star(\beta)) - i/3) \to -\pi/2$ we have

$$\operatorname{Im}\ u(\tau_+^\star(\beta)) = \frac{1}{3}(1 - 2|\tilde{\sigma}_\star|\epsilon_0 I^{-2}) + \mathcal{O}(I^{-3}) < 1/3.$$

Therefore, it is enough to consider a path Γ which visits only the singularity $\tau_+^\star(\beta)$ (see Figure 6.6.1). One can check that an argument similar to that of the present section yields the asymptotic fomula (6.68) also for $\arg(u(\tau_+^\star(\beta)) - i/3) \to -\pi/2$.

6.6.2 Cases (2): $\beta \in \mathtt{T}_{-,\star}$ and (3):$\beta \in \mathtt{T}_{i,\star}$

In order to obtain an asymptotic formula for the values of $\beta \in \mathtt{T}$ corresponding to the situation (2), i.e. $\beta \in \mathtt{T}_{-,\star}$ the path Γ is chosen in the same way as we did for situation (1) but with $\tau_-^\star(\beta)$ replacing $\tau_+^\star(\beta)$.

[16] This happens for $\beta \sim \pi$ when $\star = 0$ and $\beta \sim 2\pi$ when $\star = 1$.

We obtain that for $\beta \in \mathtt{T}_{-,\star}$ corresponding to the situation (2),

$$
\begin{aligned}
L_\star^{[l]}(\beta) =& \tilde{A}_l \frac{m_\star}{\sqrt{\tilde{\sigma}_\star}} \, I^{-1}(1+\mathcal{O}(I^{-1})) \, \exp\big(-l\nu I^3(1/3+p_\star(\beta))+h_-^\star(\beta)\big) \\
&+ \mathcal{O}\left(I^{-9/8}\exp(-l\nu I^3/3), I^{-9/8}\exp(-l\nu \text{ Im } u(\tau_+^\star)(\beta))\right)
\end{aligned}
$$

where

$$
h_-^\star(\beta) = -i\left(\frac{\beta+2\pi}{2} - \frac{l\nu}{6}(2\tilde{\sigma}_\star\epsilon_0)^{3/2} e^{-i3(\beta+2\pi)/2}\right).
$$

For values of $\beta \in \mathtt{T}$ corresponding to the situation (3), i.e. $\beta \in \mathtt{T}_{i,\star}$, we have already seen that

$$
\text{Im } u(\tau_\pm^\star(\beta)) \geq \frac{1}{3}(1+|\tilde{\sigma}_\star|\epsilon_0 I^{-2}) + \mathcal{O}(I^{-3}) > 1/3.
$$

Moreover, one can check that

$$
-\frac{\sqrt{3}}{2}|\tilde{\sigma}_\star|\epsilon_0 I^{-2} + \mathcal{O}(I^{-3}) \leq \text{Re } u(\tau_\pm^\star(\beta)) \leq \frac{\sqrt{3}}{2}|\tilde{\sigma}_\star|\epsilon_0 I^{-2} + \mathcal{O}(I^{-3}).
$$

We thus define the path $\Gamma = \Gamma_{\text{exp}} \cup \Gamma_{\text{join}} \cup \Gamma_i$ where

$$
\Gamma_{\text{exp}} = \{\tau \in \mathbb{C} \colon \text{Im } u(\tau) \geq \frac{1}{2}\big(1/3 + \text{Im } u(\tau_\pm^\star(\beta))\big), \ \text{Re } u(\tau) \in \{-|\tilde{\sigma}_\star|\epsilon_0 I^{-2}, |\tilde{\sigma}_\star|\epsilon_0 I^{-2}\}\},
$$

the path Γ_{join} is given by

$$
\Gamma_{\text{join}} = \{\tau \in \mathbb{C} \colon \text{Im } u(\tau) = \frac{1}{2}\big(1/3 + \text{Im } u(\tau_\pm^\star(\beta))\big), \ \text{Re } u(\tau) \in [-|\tilde{\sigma}_\star|\epsilon_0 I^{-2}, |\tilde{\sigma}_\star|\epsilon_0 I^{-2}]\}
$$

and $\Gamma_i = \tilde{\Gamma}_i \cup \Gamma_{i,\text{exp}}$ with

$$
\Gamma_{i,\text{exp}} = \{\tau \in \mathbb{C} \setminus D \colon \text{Im } u(\tau) \leq \frac{1}{2}\big(1/3 + \text{Im } u(\tau_\pm^\star(\beta))\big), \ \arg(u(\tau) - i/3) \in \{-3\pi/2, \pi/2\}\}
$$

and $\tilde{\Gamma}_i$ is such that Γ is connected and homotopic in $\mathbb{C}_\star$ to the real line.

Then, an argument completely analogous to the one used in Section 6.6.1 shows that for $\beta \in \mathtt{T}_{i,\star}$

$$
|L_\star^{[l]}(\beta)| \lesssim I^{-9/8}\exp(-l\nu I^3/3). \tag{6.69}
$$

To complete the proof of Proposition 6.5.15 we simply define $h_\star = h_+$ for $\beta \in \mathtt{T}_{+,\star}$, $h_\star = h_-$ for $\beta \in \mathtt{T}_{-,\star}$ and $h_\star = 0$ for $\beta \in \mathtt{T}_{i,\star}$.

Remark 6.6.5. *We point out that, for $\beta \in \mathtt{T}_{+,\star}$ we have*

$$
h_+(\beta) = h_-(\beta + 2\pi)
$$

and, for $\beta \in \mathtt{T}_{-,\star}$, we have

$$
h_-(\beta) = h_+(\beta - 2\pi).
$$

Thus, expression (6.57) *is understood as an asymptotic expression of $L_\star^{[l]}$ for $\beta \in \mathbb{T}$ such that* $\text{Re } p_\star(\beta) \leq 0$ *and as an estimate for $L_\star^{[l]}$ when $\beta \in \mathbb{T}$ is such that* $\text{Re } p_\star(\beta) \leq 0$.

6.7 The Fourier coefficients of the potential U: Proof or Propositions 6.5.11 and 6.5.12 and Lemma 6.5.14

The rest of the paper is devoted to the proof of Propositions 6.5.11 and 6.5.12 and Lemma 6.5.14. We omit the dependence of all functions on the variable $\beta \in \mathbb{T}$ and the parameters I, ϵ_0.

6.7.1 The analytic continuation of $\tau \mapsto U_\star^{[l]}(\tau)$

Let D be the disk introduced in (6.44). In Lemma 6.5.6 we have seen that, for all $l \in \mathbb{Z}$, the expression of $U_\star^{[l]}$ in (6.48) defines an analytic function for $\tau \in \mathbb{C} \setminus (D \cup \bar{D})$. We recall that, when we speak about the function element $U_\star^{[l]}$ in (6.48), we implicitly refer to the pair (6.48) and the region $\mathbb{C} \setminus (D \cup \bar{D})$. We now want to extend analytically the function element (6.48) along curves Υ which connect an arbitrary point $\tau_0 \in \mathbb{C} \setminus (D \cup \bar{D})$ and a point τ in the region D. To that end, it will be convenient to perform the change of variables $z = e^{i\xi}$, and, abusing notation, write $\lambda(z) = \lambda(\xi(z))$ and $\rho(z) = \rho(\xi(z))$, where $\lambda(\xi)$ and $\rho(\xi)$ are the functions in Lemma 6.5.4. Then, if we denote by

$$W_\star(\tau, z) = U_\star(\tau, \lambda(z)) \tag{6.70}$$

the expression (6.48) is equivalent to

$$U_\star^{[l]}(\tau) = \frac{-i}{2\pi} \int_{\gamma_1} \rho(z) W_\star(\tau, z) z^{-1} e^{il\lambda(z)} \mathrm{d}z. \tag{6.71}$$

where γ_1 is the circumference $\{|z| = 1\}$ with positive orientation. The first step towards studying the analytic continuation of the function element $U_\star^{[l]}$ in (6.71) (defined on the region $\mathbb{C} \setminus (D \cup \bar{D})$) is to identify the complex singularities of the function $z \to W_\star(\tau, z)$ in (6.70).

Remark 6.7.1. *From now on we will omit the subscript $\star = 0, 1$ and simply write $U^{[l]}, W, m$ and $\tilde{\sigma}$ instead of $U_\star^{[l]}, W_\star, m_\star$ and $\tilde{\sigma}_\star$. All the results we state in the following are valid for $\star = 0, 1$.*

Lemma 6.7.2. *Let κ_ϵ be the constant defined in (6.35) and, for $\tau \in \mathbb{R}$, define*

$$a(\tau) = \kappa_\epsilon \frac{b(\tau) - 1 - \sqrt{1 - 2b(\tau)}}{b(\tau)} \qquad a_\epsilon(\tau) = \frac{\kappa_\epsilon^2}{a(\tau)} \qquad b(\tau) = \frac{\tilde{\sigma}\epsilon e^{-i\beta}}{I^2(\tau - i)^2} \tag{6.72}$$

$$c(\tau) = \frac{1}{\kappa_\epsilon} \frac{\tilde{b}(\tau) - 1 + \sqrt{1 - 2\tilde{b}(\tau)}}{\tilde{b}(\tau)} \qquad c_\epsilon(\tau, \varphi) = \frac{1}{\kappa_\epsilon^2 c(\tau)} \qquad \tilde{b}(\tau) = \frac{\tilde{\sigma}\epsilon e^{i\beta}}{I^2(\tau + i)^2}. \tag{6.73}$$

Then, for $(\tau, z) \in \mathbb{R} \times \{|z| = 1\}$, the function $W(\tau, z)$ defined in (6.70) can be expressed as

$$W(\tau, z) = \frac{mI^3}{\epsilon\tilde{\sigma}} \frac{z}{\sqrt{(z - a(\tau))(z - a_\epsilon(\tau))(z - c(\tau))(z - c_\epsilon(\tau))}}. \tag{6.74}$$

Proof. Making use of the expressions for $\rho(\xi), f(\xi)$ in Lemma 6.5.4 we have that (by abuse of notation we write $\rho(z) = \rho(\lambda(z))$ and $f(z) = f(\lambda(z))$)

$$\rho(z) e^{if(z)} = \frac{\epsilon}{2\kappa_\epsilon} \left(z - 2\kappa_\epsilon + \frac{\kappa_\epsilon^2}{z} \right) \qquad \rho(z) e^{-if(z)} = \frac{\kappa_\epsilon \epsilon}{2} \left(z - \frac{2}{\kappa_\epsilon} + \frac{1}{z\kappa_\epsilon^2} \right), \tag{6.75}$$

The proof follows after a tedious, but straightforward, algebraic manipulation.

□

Lemma 6.7.2 shows that the points $z = a(\tau), a_\epsilon(\tau), c(\tau)$ and $c_\epsilon(\tau)$ are branching points with exponent $-1/2$ of the function $z \mapsto W(\tau, z)$. In the following lemmas we obtain some information about the functions $a(\tau), a_\epsilon(\tau), c(\tau)$ and $c_\epsilon(\tau)$ in (6.72).

Lemma 6.7.3. *Fix any $\tau_0 \in \mathbb{R}$ and let $\tau_\pm \in \mathbb{C}$ be the points defined in (6.50). Let $\tau \in \mathbb{C}$ and let $\Upsilon : [0, 1] \to \mathbb{C}$ be such that $\Upsilon(0) = \tau_0$, $\Upsilon(1) = \tau$ and $\Upsilon([0, 1)) \subset \mathbb{C} \setminus \{\tau_+, \tau_-\}$. Then, a, a_ϵ defined in (6.72) admit a unique continuation along Υ which we denote by $a(\tau; \Upsilon), a_\epsilon(\tau; \Upsilon)$. This continuation is analytic if and only if $\tau \in \mathbb{C} \setminus \{\tau_+, \tau_-\}$. Moreover, for any two curves $\Upsilon, \Upsilon' \subset \mathbb{C} \setminus \{\tau_+, \tau_-\}$ sharing the same endpoints, the analytic continuations along them coincide if and only if the sum of the indexes of the closed curve $\Upsilon'\Upsilon^{-1}$ with respect to $\tau = \tau_+$ and $\tau = \tau_-$ belongs to $2\mathbb{Z}$.*

An analogous statment holds for c, c_ϵ replacing $\tau_\pm$ by $\bar{\tau}_\pm$. We denote their analytic continuations along a curve Υ by $c(\tau; \Upsilon), c_\epsilon(\tau; \Upsilon)$

Proof. To prove the result for a, a_ϵ, as $b(\tau) \neq 0$ for all $\tau \in \mathbb{C}$, we only have to check that $1 - 2b = 0$ if and only if $\tau \in \{\tau_+, \tau_-\}$, but this is straightforward from the definition of b in (6.72). The result for c, c_ϵ follows analogously. □

We now study the behavior of the analytic continuation of a, a_ϵ, c and c_ϵ. As already pointed out, we can reduce our study to curves contained in the upper half plane

$$H = \{\tau \in \mathbb{C} \colon \operatorname{Im} \tau \geq 0\}.$$

In view of Lemma 6.7.3, the analytic continuations $c(\tau; \Upsilon), c_\epsilon(\tau; \Upsilon)$ along curves $\Upsilon \subset H$ do not depend on the choice of Υ so we will simply write c, c_ϵ. For the analytic continuations of a, a_ϵ along a path Υ we write

$$a(\tau; \Upsilon) = \kappa_\epsilon \frac{b(\tau) - 1 - \sqrt{(1-2b)(\tau; \Upsilon)}}{b(\tau)} \qquad a_\epsilon(\tau; \Upsilon) = \frac{\kappa_\epsilon^2}{a(\tau; \Upsilon)}. \tag{6.76}$$

Remark 6.7.4. *The notation* $(1-2b)(\tau; \Upsilon)$ *is used to emphasize that we keep track of the argument of* $1 - 2b$ *along the path* Υ.

Lemma 6.7.5. *Fix any* $\tau_0 \in \mathbb{R}$. *Let* $\tau \in H$ *and let* $\Upsilon : [0,1] \to H$ *be such that* $\Upsilon(0) = \tau_0$, $\Upsilon(1) = \tau$. *Then, for all* $\tau \in H$

$$|c(\tau)| \lesssim I^{-2} \qquad |c_\epsilon(\tau)| \gtrsim I^2$$

Let $a(\tau; \Upsilon), a_\epsilon(\tau; \Upsilon)$ *be the analytic continuation of* (6.72) *along* $\Upsilon : [0,1] \to H \setminus \{\tau_+, \tau_-\}$ *and let* $J \subset H$ *be the segment defined in* (6.53). *Then, there exists a constant* $\widetilde{C} > \kappa_\epsilon$ *such that, if* $\tau \in D \setminus \{\tau_+, \tau_-\}$ *(see* (6.44) (6.50)*) and* $\operatorname{card}\{\Upsilon \cap J\} = 0$

$$\kappa_\epsilon < |a(\tau; \Upsilon)| \leq \widetilde{C}. \tag{6.77}$$

Moreover, for any $\tau \in H \setminus \{\tau_+, \tau_-\}$ *and two different curves* Υ, Υ' *such that* $\operatorname{card}\{\Upsilon \cap J\} = 0$ *and* $\operatorname{card}\{\Upsilon' \cap J\} = 1$

$$a(\tau, \Upsilon') = a_\epsilon(\tau; \Upsilon) \qquad a_\epsilon(\tau, \Upsilon') = a(\tau; \Upsilon). \tag{6.78}$$

Proof. For $\tau \in H$ we have that $I^2|\tau + i| \gg 1$, and therefore $\tilde{b}(\tau)$ is small, in fact $|\tilde{b}| \leq I^{-2}$. The result for the functions $c(\tau), c_\epsilon(\tau)$ follows by expanding these functions in power series in the variable $\tilde{b}(\tau)$.

We now prove the results for a, a_ϵ. The idea is to study the shape of the level sets of the function $|a(\tau; \Upsilon)|$ for $\tau \in H \setminus J$ and $\Upsilon : [0,1] \to H \setminus J$. First we notice that, in view of Lemma 6.7.3, the continuation along two different paths (sharing the same endpoint) which do not cross J is the same. Therefore we drop the dependence on Υ. For τ such that $I|\tau - i| \gg 1$ we have

$$|a(\tau)| \sim \kappa_\epsilon \frac{I^2 |\tau - i|^2}{|\tilde{\sigma}| \epsilon_0} \gg 1. \tag{6.79}$$

We now want to study what happens as τ approaches J. To that end, it will be convenient to introduce the variable $\zeta(\tau) = \kappa_\epsilon / a(\tau)$. After some manipulations we arrive to the expression

$$\zeta(\tau) = \frac{\sqrt{1 - 2b(\tau)} - 1}{\sqrt{1 - 2b(\tau)} + 1}.$$

Writing $x(\tau) = \sqrt{1 - 2b(\tau)}$, the map $x \mapsto \zeta$ is a Möbius transformation whose inverse is given by

$$x = \frac{1 + \zeta}{1 - \zeta}. \tag{6.80}$$

We study the image of the curve $\zeta(\theta) = re^{i\theta}$ under the map (6.80) for $\theta \in \mathbb{T}$ and fixed $r < 1$ (in view of (6.79), we have $|\zeta| \ll 1$ far from J). We have that

$$x(\theta; r) = \frac{1 + re^{i\theta}}{1 - re^{i\theta}}$$

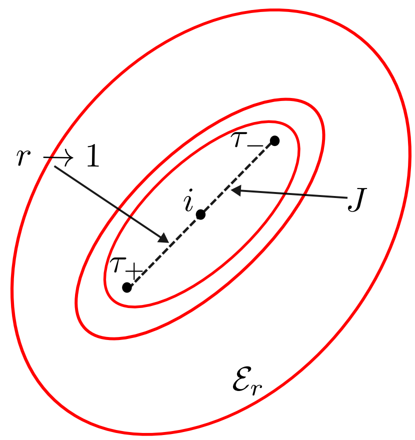


Figure 6.9: The family of ellipses $\{\mathcal{E}_r\}_{r\geq 1}$ defined in (6.81). The dashed segment corresponds to J.

Substituting now $x = \sqrt{1-2b(\tau)}$ we obtain the curve in the $\tau \in \mathbb{C}$ plane

$$(\tau(\theta; r) - i)^2 + \frac{\tilde{\sigma}\epsilon_0 e^{-i\beta}}{2} \frac{(1-re^{i\theta})^2}{re^{i\theta}} = 0.$$

This curve corresponds indeed to the ellipse

$$\mathcal{E}_r = \left\{ \tau \in \mathbb{C}\colon \frac{|\tau(\theta;r)-i|^2 \cos^2\beta}{p_r^2} + \frac{|\tau(\theta;r)-i|^2 \sin^2\beta}{q_r^2} = 1, \theta \in \mathbb{T} \right\} \tag{6.81}$$

where

$$p_r = \sqrt{\frac{\tilde{\sigma}\epsilon_0}{2I^2}} \frac{1+r}{\sqrt{r}} \qquad\qquad q_r = \sqrt{\frac{\tilde{\sigma}\epsilon_0}{2I^2}} \frac{1-r}{\sqrt{r}}.$$

Consider now the family of ellipses $\mathcal{E}_r$ in (6.81) for $r<1$ (see Figure 6.7.1). For all $r<1$ the semimajor axis corresponds to a segment centered at $\tau = i$, in the direction of J and of length $p_r > \sqrt{2\tilde{\sigma}\epsilon_0} I^{-1} =$ (length J)/2. Also for $r<1$, the semiminor axis has strictly positive length. When $r \to 1$ these ellipses collapse to the segment J. We conclude that for all $\tau \in H \setminus J$, we have $|\zeta(\tau)| < 1$ what implies that for all $\tau \in H \setminus J$, $|a(\tau)| > \kappa_\epsilon$. Finally, let r_* large enough so the disk D in (6.44) is contained in the bounded component among the two connected components in which $\mathcal{E}_{r_*}$ divides the complex plane and set $\widetilde{C} = \kappa_\epsilon / r_*$.

The last item in the lemma is trivial from the definition in (6.76).

□

The following result will be useful for later computations.

Lemma 6.7.6. *Fix any $\tau_0 \in \mathbb{R}$. Let $\tau \in H$ and let $\Upsilon : [0,1] \to H$ be such that $\Upsilon(0) = \tau_0$, $\Upsilon(1) = \tau$. Let $a(\tau;\Upsilon), a_\epsilon(\tau;\Upsilon)$ be given in (6.76). Then*

$$a(\tau;\Upsilon) - a_\epsilon(\tau;\Upsilon) = \frac{-2\kappa_\epsilon\sqrt{(1-2b)(\tau;\Upsilon)}}{b(\tau;\Upsilon)}, \tag{6.82}$$

with $b(\tau)$ the function defined in (6.72).

Remark 6.7.7. *One can easily check that $a(\tau;\Upsilon) = a_\epsilon(\tau;\Upsilon)$ if and only if $\tau = i$ (for which $|b| = \infty$) or $\tau = \tau_\pm$ (for which $1-2b = 0$). The notation $(1-2b)(\tau;\Upsilon)$ and $b(\tau;\Upsilon)$ is used to keep track of the argument of these quantities along Υ. In this way we can keep track of the argument of $a(\tau;\Upsilon) - a_\epsilon(\tau;\Upsilon)$ along Υ. This will be important in the forthcoming discussion.*

Proof. By the definition of a, a_ϵ in (6.76) we obtain that

$$a(\tau;\Upsilon) - a_\epsilon(\tau;\Upsilon) = \kappa_\epsilon \left(\frac{b(\tau) - 1 - \sqrt{(1-2b)(\tau;\Upsilon)}}{b(\tau;\Upsilon)} - \frac{b(\tau;\Upsilon)}{b(\tau) - 1 - \sqrt{(1-2b)(\tau;\Upsilon)}} \right) = \frac{-2\kappa_\epsilon \sqrt{1-2b(\tau;\Upsilon)}}{b(\tau;\Upsilon)}.$$

□

We now come back to the problem of defining the analytic continuation of $U^{[l]}$ in (6.48). To this end, we notice that, in view of Lemma 6.7.5, for $\tau \in \mathbb{R}$, the function $z \mapsto W(\tau, z)$ is analytic on the annulus $\{z \in \mathbb{C} \colon \kappa_\epsilon < |z| < 1\}$. Therefore, we can change the integration contour in (6.71) from γ_1 to γ_{κ_ϵ}, defined as the curve $\{|z| = \kappa_\epsilon\}$ with positive orientation. We obtain that, for $\tau \in \mathbb{R}$, expression (6.71) is equivalent to

$$U_\star^{[l]}(\tau) = \frac{-i}{2\pi} \int_{\gamma_{\kappa_\epsilon}} \rho(z) W_\star(\tau, z) z^{-1} e^{il\lambda(z)} \mathrm{d}z. \tag{6.83}$$

In fact, (6.83) defines an analytic function for any $\tau \in H \backslash J$. Indeed, for any $\tau \in H \backslash J$, and any $\Upsilon : [0,1] \to H \backslash J$, such that $\Upsilon(0) \in \mathbb{R}$ and $\Upsilon(1) = \tau$, we have that $\{|z| = \kappa_\epsilon\} \cap \{a(\Upsilon(t);\Upsilon), a_\epsilon(\Upsilon(t);\Upsilon), c(\Upsilon(t)), c_\epsilon(\Upsilon(t))\} = \emptyset$ for all $t \in [0,1]$.
We now embed this idea in a more general framework, which will allow us, in Proposition 6.7.10, to define the analytic continuation of (6.83) along curves $\Upsilon \subset H$ which do cross the segment J. Fix any $\tau_0 \in D \setminus J$ and for any $\tau \in D$ define the family of curves

$$\mathcal{Y}_\tau = \{\Upsilon \in C([0,1], D) \colon \Upsilon(0) = \tau_0,\ \Upsilon(1) = \tau,\ \Upsilon([0,1)) \subset D \setminus \{\tau_+, \tau_-\}\}. \tag{6.84}$$

Then, we introduce the "collision set":

$$\texttt{Col} = \{(\tau;\Upsilon) \in D \times \mathcal{Y}_\tau \colon \ \{a(\tau;\Upsilon), c_\epsilon(\tau)\} \cap \{a_\epsilon(\tau;\Upsilon), c(\tau), 0 \neq \emptyset\}\}. \tag{6.85}$$

The set Col is described in the following lemma and will be related to the singularities of the analytic continuation of (6.83) in Lemma 6.7.10.

Lemma 6.7.8. *The collision set* Col *defined in* (6.85) *satisfies*

$$\texttt{Col} = \left\{ \{i\} \times \mathcal{Y}_i, \{\tau_+\} \times \mathcal{Y}_{\tau_+}, \{\tau_-\} \times \mathcal{Y}_{\tau_-} \right\}$$

at which we have that $a(\tau;\Upsilon) = a_\epsilon(\tau;\Upsilon)$ (independently of the curve Υ).

Proof. We deduce from Lemma 6.7.6 that $a(\tau;\Upsilon) = a_\epsilon(\tau;\Upsilon)$ if and only if $\frac{1}{b} = 0$ or $1 - 2b = 0$, which corresponds to the points $\tau = i$ and $\tau = \tau_+, \tau_-$, respectively. The existence of other possible "collisions" is excluded using the bounds obtained in Proposition 6.7.5. □

We now consider the closed curve $\{|z| = \kappa_\epsilon\}$. It divides the z-complex plane $\mathbb{C}$ in two connected components: we denote by B_0 the bounded one and by B_∞ the unbounded one. From Lemma 6.7.5, we observe that, for $\tau \in H \setminus J$ and $\Upsilon \in \mathcal{Y}_\tau$ such that $\Upsilon \cap J = \emptyset$, we have that

$$\{0, a_\epsilon(\tau;\Upsilon), c(\tau)\} \subset B_0 \qquad \{a(\tau;\Upsilon), c_\epsilon(\tau)\} \subset B_\infty$$

so no singularity of the integrand in (6.83) (see also (6.74)) is located on the curve $\{|z| = \kappa_\epsilon\}$. Therefore, as already discussed, (6.83) defines an analytic function.

Let now $\tau \in D$ and take a curve $\Upsilon \in \mathcal{Y}_\tau$ such that $\mathrm{card}\{\Upsilon \cap J\} = 1$. From Lemma 6.7.5 we observe that now

$$\{0, a(\tau;\Upsilon), c(\tau)\} \subset B_0 \qquad \{a_\epsilon(\tau;\Upsilon), c_\epsilon(\tau)\} \subset B_\infty,$$

and the expression (6.83) does not make sense since a singularity of the integrand has crossed the integration contour. Of course this is only a matter of how we have defined (6.83). Indeed, thanks to analyticity, as long as the path Υ along we want to continue (6.83) satisfies $\Upsilon \cap \widetilde{\texttt{Col}} = \emptyset$, where

$$\widetilde{\texttt{Col}} = \{i, \tau_+, \tau_-\},$$

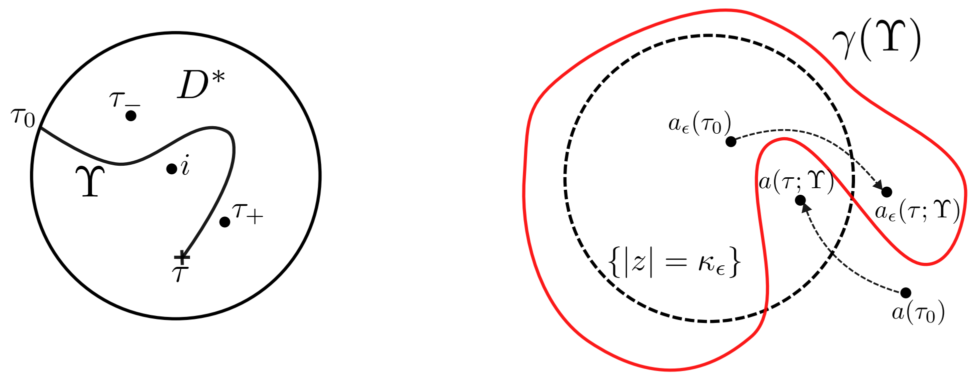


Figure 6.10: Sketch, in red, of the curve $\gamma(\Upsilon)$ obtained by continuous deformation of the curve $\{|z|=\kappa_\epsilon\}$ (pictured in dashed lines).

we have freedom to change the integration contour in (6.83) as we travel Υ to avoid having singular points on it (see Figure 6.7.1). This is the content of Proposition 6.7.10 below. We now introduce some notation which clarifies its statement. We introduce the punctured plane

$$\mathcal{C}_\tau = \{z \in \mathbb{C}\colon z \notin \{a(\tau;\Upsilon), a_\epsilon(\tau;\Upsilon), c(\tau), c_\epsilon(\tau)\}\},$$

and the sets

$$D^* = D \setminus \widetilde{\texttt{Col}} \qquad \mathcal{D} = \{(\tau, z) \in D^* \times \mathbb{C}\colon z \in C_\tau\}. \tag{6.86}$$

Definition 6.7.9. *Let $(\tau_0, z_0) \in (D\setminus J)\times\{|z| = \kappa_\epsilon\}$ be fixed and let $(\tau, z) \in \mathcal{D}$. We say that a continuous curve $\Psi : [0,1] \to \mathbb{C}^2$ is an admissible path from (τ_0, z_0) to (τ, z) if $\Psi(0) = (\tau_0, z_0)$, $\Psi(1) = (\tau, z)$ and $\Psi([0,1]) \subset \mathcal{D}$.*

We are now ready to continue analytically $U^{[l]}$ in (6.83) along paths crossing J. Given a closed loop γ we denote by $B_0^\gamma, B_\infty^\gamma$ the bounded and unbounded connected components in which γ divides the z-complex plane $\mathbb{C}$.

Proposition 6.7.10. *Let τ_0 in $D \setminus J$, take any $\tau \in D^*$ and let $\Upsilon : [0,1] \to D^*$ be a continuous curve joining them. Then, there exists a closed curve $\gamma(\Upsilon) \in \mathcal{C}_\tau$ satisfying*

$$\{0, a_\epsilon(\tau;\Upsilon), c(\tau;\Upsilon)\} \subset B_0^\gamma \qquad \{a(\tau;\Upsilon), c_\epsilon(\tau;\Upsilon)\} \subset B_\infty^\gamma$$

and such that the analytic continuation of (6.83) along Υ is given by

$$\mathcal{U}^{[l]}(\tau;\Upsilon) = \frac{-i}{2\pi}\int_{\gamma(\Upsilon)} \rho(z) W(\tau, z; \Psi_{\tau,z}) z^{-1} e^{-il\lambda(z)} \mathrm{d}z, \tag{6.87}$$

where, for all $z \in \gamma(\Upsilon)$, $\Psi_{\tau,z}$ is any admissible curve such that $\widetilde{\Psi} : [0,1] \times \gamma(\Upsilon) \to \mathcal{D}$ defined by $(t,z) \mapsto \Psi_{\tau,z}(t)$ is an homotopy between $\{\tau_0\} \times \{|z| = \kappa_\epsilon\}$ and $\{\tau\} \times \gamma(\Upsilon)$.

Moreover, for $\Upsilon, \Upsilon' \subset D^*$ with $\Upsilon(0) = \Upsilon'(0) \in D \setminus J$, $\Upsilon(1) = \Upsilon'(1) = \tau$, and such that the closed curve $\tilde{\Upsilon} = \Upsilon\Upsilon^{-1}$ is contractible to a point or homotopic to ∂D then $\mathcal{U}^{[l]}(\tau;\Upsilon) = \mathcal{U}^{[l]}(\tau;\Upsilon')$.

Proof. The first part of the lemma follows from standard arguments in complex analysis and the discussion preceding the proposition. We now prove the second part of the lemma. Take now $\Upsilon, \Upsilon' \subset D^*$ with $\Upsilon(0) = \Upsilon'(0) \in \partial D$, $\Upsilon(1) = \Upsilon'(1) = \tau$, and define the closed loop $\tilde{\Upsilon} = \Upsilon'\Upsilon^{-1}$. Introduce the function $q(t) : [0,1] \to \mathbb{R}$ given by

$$q(t;\tilde{\Upsilon}) = \arg\left(1 - \frac{a_\epsilon(\tilde{\Upsilon}(t);\tilde{\Upsilon})}{a(\tilde{\Upsilon}(t);\tilde{\Upsilon})}\right).$$

If $q(0,\Upsilon) = q(1,\Upsilon)$ then, the curves $\gamma(\Upsilon)$ and $\gamma(\Upsilon')$ obtained in the first part of the lemma must be homotopic in $\mathcal{C}_\tau$ where $\tau = \Upsilon(1) = \Upsilon'(1)$. Thus, it is clear that, if $q(0,\Upsilon) = q(1,\Upsilon)$, then $\mathcal{U}^{[l]}(\tilde{\Upsilon}(0);\Upsilon) =$

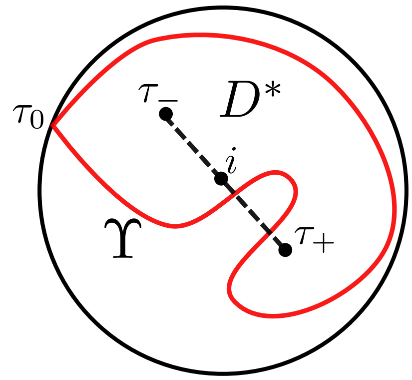


Figure 6.11: Sketch of a curve Υ, in red, homotopic in D^* to ∂D.

$\mathcal{U}^{[l]}(\tilde{\Upsilon}(1), \Upsilon)$. But this condition is met for any closed curve $\tilde{\Upsilon} \subset D^*$ which is homotopic to ∂D. Indeed, consider first the case where $\tilde{\Upsilon} \cap J = \emptyset$. In view of Lemma 6.7.5 we have that $|a(\tilde{\Upsilon}(t), \tilde{\Upsilon})/a_\epsilon(\tilde{\Upsilon}(t), \tilde{\Upsilon})| < 1$ for all $t \in [0,1]$. Therefore, we must have $q(1, \tilde{\Upsilon}) = q(0, \tilde{\Upsilon})$. Due to the uniqueness of analytic continuation on simply connected domains we can drop the assumption $\tilde{\Upsilon} \cap J = \emptyset$ (see Figure 6.7.1). □

Proposition 6.7.10 shows that the points $\tau = i, \tau_+, \tau_-$ are the unique canditates to be singular points of the analytic continuation of (6.83) along curves $\Upsilon \subset D$. We will see in the forthcoming sections, where we study the quantitative behavior of (6.87), along a suitable family of paths Υ, that $\tau = i, \tau_+, \tau_-$ are indeed singular points of the analytic continuation of (6.83).

From now on, since the singularities c and c_ϵ play no role when we restrict to $\tau \in H$, we make use of the more convenient expression for the function W (see (6.74))

$$W(\tau, z) = \frac{C_\beta}{(\tau + i)\sqrt{1 + h(\tau, z)}} \frac{I^2}{\sqrt{z - (a + a_\epsilon)(\tau) + \frac{\kappa_\epsilon^2}{z}}} \tag{6.88}$$

where

$$C_\beta = \frac{-im\tilde{\nu}}{\pi}\sqrt{\frac{\kappa_\epsilon e^{i\beta}}{\tilde{\sigma}\epsilon_0}} \qquad\qquad h(\tau, z) = \frac{2\tilde{\sigma}e^{i\beta}}{I^2(\tau + i)^2}\rho(z)e^{-if(z)}. \tag{6.89}$$

and, denote its continuation along any admissible path $\Psi_{\tau,z} : [0,1] \to \mathbb{C}^2$, as $W(\tau, z; \Psi_{\tau,z})$.

Remark 6.7.11. *In the forthcoming sections, we will always assume the following without mentioning:*

- *We fix an arbitrary* $\tau_0 \in H \setminus J$.
- *Given a point* $\tau \in D^*$ *we will denote by* $\Upsilon : [0,1] \to D^*$ *any continuous curve such that* $\Upsilon(0) = \tau_0$ *and* $\Upsilon(1) = \tau$.
- $\tilde{\gamma}(\Upsilon)$ *is any curve in the homotopy class (in* $\mathcal{C}_\tau$*) of the curve* $\gamma(\Upsilon)$ *obtained in Proposition 6.7.10.*
- *Given* $\tilde{\gamma}(\Upsilon)$ *as above, for all* $z \in \tilde{\gamma}(\Upsilon)$, $\Psi_{\tau,z}$ *is any admissible curve such that* $\widetilde{\Psi} : [0,1] \times \tilde{\gamma}(\Upsilon) \to \mathcal{D}$ *defined by* $(t, z) \mapsto \Psi_{\tau,z}(t)$ *is an homotopy between* $\{\tau_0\} \times \{|z| = \kappa_\epsilon\}$ *and* $\{\tau\} \times \tilde{\gamma}(\Upsilon)$.
- *For a given* $\Psi_{\tau,z}$ *as above, and, understanding* $z \mapsto \Psi_{\tau,z}(1)$ *as a parametrization of the curve* $\tilde{\gamma}(\Upsilon)$, *we will simply write* $W(\tau, z)$ *instead of* $W(\tau, z; \Psi_{\tau,z})$.

6.7.2 Behavior of $U^{[l]}(\tau;\Upsilon)$ for $\tau \in D \setminus (C_i \cup C_+ \cup C_-)$

We first choose a curve in the homotopy class of the curve $\gamma(\Upsilon)$ which is suitable for obtaining bounds for $\left|U^{[l]}(\tau;\Upsilon)\right|$ when $\tau \in D \setminus (C_i \cup C_+ \cup C_-)$ with C_i, C_+ and C_- the punctured disks defined in (6.54). We recall that, in the present, and forthcoming sections, we include curves $\Upsilon : [0,1] \to D^*$ such that $\mathrm{card}\{\Upsilon \cap J\} \in \{0,1\}$, where J is the segment in (6.53).

Lemma 6.7.12. *If $\tau \in D\setminus(C_i\cup C_+\cup C_-)$ the closed curve $\gamma = \gamma(\Upsilon)$ in Proposition 6.7.10 can be chosen such that*

- $\mathrm{length}(\gamma) \lesssim 1$,
- *For $z \in \gamma$ we have that*

$$|z| \sim 1 \qquad\qquad |z - a(\tau;\Upsilon)|, |z - a_\epsilon(\tau;\Upsilon)| \gtrsim I^{-1/4}$$

Proof. In order to prove the lemma we only need to verify that there exists a suitable lower bound on the distance between $a(\tau;\Upsilon)$ and $a_\epsilon(\tau;\Upsilon)$ and, in order to bound the length of the curve, give an estimate on how far these points are from the origin in the $z \in \mathbb{C}$ plane. From expression (6.82) we obtain that

$$\big(a(\tau;\Upsilon) - a_\epsilon(\tau;\Upsilon)\big)^2 = 16\kappa_\epsilon^2\, \frac{1-2b(\tau)}{b^2(\tau)}$$

does not depend on the curve Υ. That is, the function $f(\tau) \equiv (a-a_\epsilon)^2(\tau)$ is an analytic function on $D \subset \mathbb{C}$. Then, we can apply the maximum principle to $f(\tau)$ to obtain that (making use of the definition of $C_i, C_\pm$ in (6.54))

$$\min_{\tau\in D\setminus(C_i\cup C_\pm)} |a - a_\epsilon| \gtrsim I^{-1/4}.$$

We now want to obtain a lower bound on $|a(\tau,\Upsilon)|$. To that end, we use that $aa_\epsilon = \kappa_\epsilon^2$ and apply the maximum principle on the subset of the Riemann surface associated to the analytic continuation of a_ϵ which projects onto D. We know from Lemma 6.7.5 that, there exists $\tilde{C} > \kappa_\epsilon$ such that

$$\max\{|a(\tau;\Upsilon)|\colon \tau\in\partial D,\ \mathrm{card}\{\Upsilon\cap J\}=0\} \le \widetilde{C} \qquad \max\{|a_\epsilon(\tau;\Upsilon)|\colon \tau\in\partial D,\ \mathrm{card}\{\Upsilon\cap J\}=0\} < \kappa_\epsilon.$$

Combining these estimates with the last item in Lemma 6.7.5, we obtain

$$\max\{|a_\epsilon(\tau;\Upsilon)|\colon \tau\in\partial D,\ \mathrm{card}\{\Upsilon\cap J\}=1\} \le \widetilde{C} \qquad \max\{|a(\tau;\Upsilon)|\colon \tau\in\partial D,\ \mathrm{card}\{\Upsilon\cap J\}=1\} < \kappa_\epsilon.$$

Therefore, we can conclude that

$$\max\{|a_\epsilon(\tau;\Upsilon)|\colon \tau\in D,\ \mathrm{card}\{\Upsilon\cap J\}\in\{0,1\}\} \le \max\{|a_\epsilon(\tau;\Upsilon)|\colon \tau\in\partial D,\ \mathrm{card}\{\Upsilon\cap J\}\in\{0,1\}\} \le \tilde{C}$$

and, consequently, since $aa_\epsilon = \kappa_\epsilon^2$,

$$\min\{|a(\tau;\Upsilon)|\colon \tau\in D,\ \mathrm{card}\{\Upsilon\cap J\}\in\{0,1\}\} \ge \min\{|a(\tau;\Upsilon)|\colon \tau\in\partial D,\ \mathrm{card}\{\Upsilon\cap J\}\in\{0,1\}\} \ge \kappa_\epsilon^2/\widetilde{C}.$$

Then, in view of the preceeding discussion, given the curve $\gamma(\Upsilon)$ of Lemma 6.7.10, which we already know that exists and satisfies

$$\{0, a_\epsilon(\tau;\Upsilon), c(\tau;\Upsilon)\} \subset B_0^\gamma \qquad\qquad \{a(\tau;\Upsilon), c_\epsilon(\tau;\Upsilon)\} \subset B_\infty^\gamma,$$

we can always find $\tilde{\gamma}(\Upsilon)$ homotopic to $\gamma(\Upsilon)$ in $\mathcal{C}_\tau$ satisfying the requirements in the lemma. □

Lemma 6.7.13. *Let $U^{[l]}(\tau;\Upsilon)$ be the analytic continuation along Υ of* (6.83) *obtained in Proposition 6.7.10. Then, for $\tau \in D\setminus(C_i\cup C_+\cup C_-)$ we have that*

$$\left|U^{[l]}(\tau;\Upsilon)\right| \lesssim I^{9/4}.$$

Proof. Change the curve $\gamma(\Upsilon)$, obtained in Proposition 6.7.10, to a curve $\tilde{\gamma}(\Upsilon)$ satisfying the properties stated in Lemma 6.7.12. Therefore, making use of the formulas for $\rho(z)$ and $e^{il\lambda(z)}$ in (6.75), it is clear that for $z \in \tilde{\gamma}(\Upsilon)$ we have

$$\left|\rho(z)z^{-1}e^{il\lambda(z)}\right| \lesssim 1.$$

On the other hand, from the bounds in Lemma 6.7.12, and expression (6.88), and the definition of $C_i, C_\pm$ in (6.54), we obtain that, for $\tau \in D \setminus (C_i \cup C_+ \cup C_-)$ and $z \in \tilde{\gamma}(\Upsilon)$

$$|W(\tau, z)| \lesssim I^{9/4}$$

and the result follows.

□

6.7.3 Behavior of $U^{[l]}(\tau; \Upsilon)$ for $\tau \in C_i \cup C_\pm$

Define

$$f_l(\tau, z) = \frac{C_\beta I^2 \rho(z) e^{il\lambda(z)}}{(\tau + i)\sqrt{z(1 + h(\tau, z))}}, \tag{6.90}$$

where the constant C_β is defined in (6.89) and the expression for $\rho(z)e^{il\lambda(z)}$ can be deduced from (6.75). We then write $U^{[l]}(\Upsilon)$ in (6.87) as

$$\begin{aligned} U^{[l]}(\tau; \Upsilon) =& \int_{\gamma(\Upsilon)} \rho(z) W(\tau, z) z^{-1} e^{il\lambda(z)} \mathrm{d}z \\ =& f_l(\tau, a(\tau; \Upsilon)) \int_{\gamma(\Upsilon)} \frac{z^{-1}\mathrm{d}z}{\sqrt{z^2 + (a + a_\epsilon)(\tau)z + \kappa_\epsilon^2}} + \int_{\gamma(\Upsilon)} \frac{(f_l(\tau, z) - f_l(\tau, a(\tau; \Upsilon)))z^{-1}\mathrm{d}z}{\sqrt{z^2 + (a + a_\epsilon)(\tau)z + \kappa_\epsilon^2}}. \end{aligned} \tag{6.91}$$

It is now convenient to introduce the function $\zeta(\tau; \Upsilon)$ (we already used this function in the proof of Lemma 6.7.5), defined by

$$\zeta^2(\tau; \Upsilon) = \frac{a_\epsilon(\tau; \Upsilon)}{a(\tau; \Upsilon)} = \frac{\kappa_\epsilon^2}{a^2(\tau; \Upsilon)}. \tag{6.92}$$

Lemma 6.7.14. *Let*

$$K(\zeta) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - \zeta^2 \sin^2\theta}},$$

be the complete elliptic integral of the first kind. Then, we have that

$$\int_{\gamma(\Upsilon)} \frac{z^{-1}\mathrm{d}z}{\sqrt{z^2 + (a + a_\epsilon)(\tau)z + \kappa_\epsilon^2}} = 4\kappa_\epsilon^{-1}\zeta(\tau; \Upsilon) K(\zeta(\tau; \Upsilon)).$$

Proof. We consider $\tau \in \mathbb{R}$ and Υ such that $\operatorname{card}\{\Upsilon \cap J\} = 0$, so we know that

$$|a(\tau; \Upsilon)| > 1, \qquad |a_\epsilon(\tau; \Upsilon)| < 1,$$

and we can simply choose

$$\int_{\gamma(\Upsilon)} \frac{z^{-1}\mathrm{d}z}{\sqrt{z^2 + (a + a_\epsilon)(\tau)z + \kappa_\epsilon^2}} = \int_{\gamma_1} \frac{z^{-1}\mathrm{d}z}{\sqrt{z^2 + (a + a_\epsilon)(\tau)z + \kappa_\epsilon^2}}, \tag{6.93}$$

where γ_{κ_ϵ} is the circumference $\{|z| = \kappa_\epsilon\}$ with positive orientation. Moreover, taking into account the decay of the integrand for $|z| \to \infty$, we can change the integration contour in (6.93) to the curve $\gamma = \gamma_{\text{down}}\, \gamma_{\text{up}}$ where

$$\begin{aligned} \gamma_{\text{up}} =& \{z \in \mathbb{C} \colon z = a(\tau; \Upsilon)s,\ s \in (1, \infty)\} \\ \gamma_{\text{down}} =& \{z \in \mathbb{C} \colon z = a(\tau; \Upsilon)se^{-i2\pi},\ s \in (1, \infty)\}. \end{aligned}$$

A straightforward computation then shows that

$$\int_{\gamma_1} \frac{z^{-1}\mathrm{d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}} = \frac{2}{a(\tau)}\int_1^\infty \frac{s^{-1}\mathrm{d}s}{\sqrt{(s-1)(s-\zeta^2(\tau;\Upsilon))}}.$$

Changing the integration variable to $x=1/s$ and then to θ where $x=\sin^2\theta$ we obtain

$$\int_1^\infty \frac{s^{-1}\mathrm{d}s}{\sqrt{(s-1)(s-\zeta^2)}} = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x)(1-\zeta^2 x)}} = 2\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-\zeta^2\sin^2\theta}}.$$

Thus, for all $\tau\in\mathbb{R}$, we have that

$$\int_{\gamma(\Upsilon)} \frac{z^{-1}\mathrm{d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}} = 4\kappa_\epsilon^{-1}\zeta(\tau;\Upsilon)K(\zeta(\tau;\Upsilon)).$$

The result follows since $K(\zeta(\tau;\Upsilon))$ can be continued analytically along any curve for which $\zeta^2(\tau;\Upsilon)\neq 1$. However, we know from Lemma 6.7.8 and the definition of D^* that $\zeta^2(\tau;\Upsilon)\neq 1$ for $\tau\in D^*$. □

If we denote by

$$E_l(\tau;\Upsilon) = \int_{\gamma(\Upsilon)} \frac{(f_l(\tau,z)-f_l(\tau,a(\tau;\Upsilon)))z^{-1}\mathrm{d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}}. \tag{6.94}$$

(6.91) and Lemma 6.7.14 imply that

$$U^{[l]}(\tau;\Upsilon) = 4\kappa_\epsilon^{-1} f_l(\tau,a(\tau;\Upsilon))\ \zeta(\tau;\Upsilon)K(\zeta(\tau;\Upsilon)) + E_l(\tau;\Upsilon). \tag{6.95}$$

In order to prove Proposition 6.5.12 it remains to obtain local expansions of $K(\zeta(\tau;\Upsilon))$ and study the quantitative behavior of $E_l(\tau;\Upsilon)$ for $\tau\in C_\pm$ and $\tau\in C_i$. The proof of the following lemma can be found in, for example, [?].

Lemma 6.7.15. *For all $\zeta\in\mathbb{C}$ such that $0<1-\zeta^2\ll 1$ there exists a constant $c\in\mathbb{C}$ such that*

$$K(\zeta) = \frac{1}{2}\ln(1-\zeta^2)+c+\mathcal{O}(|1-\zeta^2|).$$

In the next lemma, whose proof is straightforward, we provide some technical information about the function $f_l(\tau,z)$ introduced in (6.90).

Lemma 6.7.16. *Fix any two constants $0<c<C$. Then, for all $z\in\mathbb{C}$ such that $c\le|z|\le C$, we have*

$$|f_l(\tau,z)|, |\partial_\tau f_l(\tau,z)|, |\partial_z f_l(\tau,z)| \lesssim I^2.$$

Moreover, there exists $\tilde f_l(\tau,z)$ such that $f_l(\tau,z)=(z-\kappa_\epsilon)\tilde f_l(\tau,z)$, $\tilde f_l(\tau,z)$ is analytic in $c\le|z|\le C$ and satisfies $|\tilde f_l(\tau,z)|\lesssim I^2$.

The following lemma will also be useful. It shows that for $\tau\in C_i$ and $\tau\in C_\pm$, the singular points $a(\tau;\Upsilon)$ and $a_\epsilon(\tau;\Upsilon)$ are close to $|z|=\kappa_\epsilon$. It also gives a lower bound on their distance.

Lemma 6.7.17. *Let $\zeta(\tau;\Upsilon)$ be defined in* (6.92). *Then, for $\tau\in C_\pm$,*

$$a(\tau;\Upsilon) = -\kappa_\epsilon+\mathcal{O}(I|\tau-\tau_\pm|^{1/2}), \qquad 1-\zeta^2(\tau;\Upsilon) = 4\sqrt{(1-2b)(\tau;\Upsilon)}\left(1+\mathcal{O}(I^{1/2}|\tau-\tau_\pm|^{1/2})\right) \tag{6.96}$$

and, for $\tau\in C_i$,

$$a(\tau;\Upsilon) = \kappa_\epsilon+\mathcal{O}(I|\tau-i|) \qquad |1-\zeta^2(\tau;\Upsilon)|\sim I|\tau-i|. \tag{6.97}$$

Proof. The expressions in (6.96) are obtained expanding (6.76) and (6.82) in powers of $1-2b$. The expressions in (6.97) are obtained expanding (6.76) and (6.82) in powers of $1/b$. □

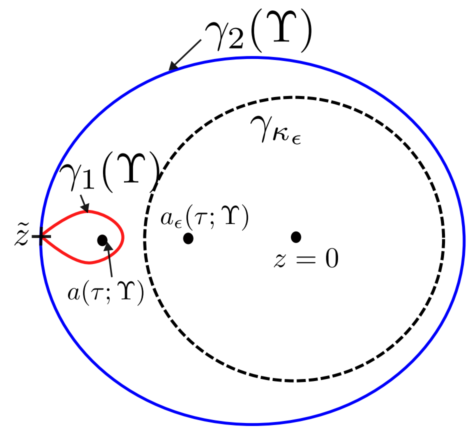


Figure 6.12: Deformation of γ_{κ_ϵ} into the union $\tilde{\gamma}(\Upsilon) = \gamma_1(\Upsilon)\gamma_2(\Upsilon)$ where $\gamma_1(\Upsilon)$ is depicted in red and $\gamma_2(\Upsilon)$ is depicted in blue.

We now study the function $E_l(\tau;\Upsilon)$ introduced in (6.94). To that end, it is important to choose a suitable path $\tilde{\gamma}(\Upsilon)$ in the homotopy class of $\gamma(\Upsilon)$.

Recall the framework introduced in Remark 6.7.11. Let $\operatorname{ind}(z,\gamma)$ stand for the index of a closed curve $\gamma \subset \mathbb{C}$ around a point $z \in \mathbb{C}$. The following observations will be important in the proof of Lemma 6.7.20. Let $\tau \in D \setminus J$ and let Υ be such that $\Upsilon \cap J = \emptyset$. Then, as already explained, $\gamma(\Upsilon)$ can be chosen to be the oriented circumference γ_{κ_ϵ}. Fix any point $\tilde{z} \in \mathcal{C}_\tau$. Then, the curve γ_{κ_ϵ} can be deformed homotopically in $\mathcal{C}_\tau$ to the union $\tilde{\gamma}(\Upsilon)$ of two closed loops $\gamma_i(\Upsilon)$, $i = 1, 2$, starting at $\tilde{z}$ and such that (see Figure 6.7.3)

- $\operatorname{ind}(a(\tau;\Upsilon), \gamma_1(\Upsilon)) = -1$, $\operatorname{ind}(a_\epsilon(\tau;\Upsilon), \gamma_1(\Upsilon)) = 0$ and $\operatorname{ind}(0, \gamma_1(\Upsilon)) = 0$
- $\operatorname{ind}(a(\tau;\Upsilon)\gamma_2(\Upsilon)) = \operatorname{ind}(a_\epsilon(\tau;\Upsilon), \gamma_2(\Upsilon)) = \operatorname{ind}(0, \gamma_2(\Upsilon)) = 1$.

The idea behind this decomposition is that only the curve $\tilde{\gamma}_1(\Upsilon)$ is "trapped" between the singularities $a(\tau;\Upsilon), a_\epsilon(\tau;\Upsilon)$. By choosing the point $\tilde{z}$ far enough from $a(\tau;\Upsilon), a_\epsilon(\tau;\Upsilon)$, the curve $\tilde{\gamma}_2(\Upsilon)$ can be chosen to be sufficiently far from $a(\tau;\Upsilon), a_\epsilon(\tau;\Upsilon)$. Therefore, the contribution of the segment $\tilde{\gamma}_2(\Upsilon)$ to the integral (6.94) will be small. On the other hand, if $\tilde{z}$ is not too far from $a(\tau;\Upsilon), a_\epsilon(\tau;\Upsilon)$ the contribution of the segment $\tilde{\gamma}_2(\Upsilon)$ to the integral (6.94) can be analyzed asymptotically.

When $\tau \in D^*$, defined in (6.86), and Υ is an arbitrary curve $\Upsilon \subset D^*$, the situation is slightly more complicated, since the points $a(\tau;\Upsilon)$ and $a_\epsilon(\tau;\Upsilon)$ may have revolved around themselves in a complicated way, entangling thus, the geometry of the curve $\gamma(\Upsilon)$ (see Figure 6.7.3). However, by construction of the curve $\gamma(\Upsilon)$ in Proposition 6.7.10, since

$$\operatorname{ind}(a(\tau_0), \gamma_{\kappa_\epsilon}) = 0, \qquad \operatorname{ind}(a_\epsilon(\tau_0), \gamma_{\kappa_\epsilon}) = 1, \qquad \operatorname{ind}(0, \gamma_{\kappa_\epsilon}) = 1,$$

the curve $\gamma(\Upsilon)$ must satisfy

$$\operatorname{ind}(a(\tau;\Upsilon), \gamma(\Upsilon)) = 0, \qquad \operatorname{ind}(a_\epsilon(\tau;\Upsilon), \gamma(\Upsilon)) = 1, \qquad \operatorname{ind}(0, \gamma(\Upsilon)) = 1.$$

Then, the discussion above generalizes as follows (see Figure 6.7.3).

Lemma 6.7.18. *Let $\gamma(\Upsilon)$ be the curve obtained in Proposition 6.7.10 and fix any point $\tilde{z} \in \mathcal{C}_\tau$. Then, there exists an even number $k(\Upsilon) \geq 2$ and a family of loops $\{\gamma_i(\Upsilon)\}_{1 \leq i \leq k(\Upsilon)}$ starting at $\tilde{z}$ satisfying*

- *For $i = 1$*

$$\operatorname{ind}(a(\tau;\Upsilon), \gamma_i(\Upsilon)) = -1 \qquad \operatorname{ind}(a_\epsilon(\tau;\Upsilon), \gamma_i(\Upsilon)) = 0, \qquad \operatorname{ind}(0, \gamma_i(\Upsilon)) = 0.$$

- *For $1 < i \leq k(\Upsilon)/2$,*

$$\operatorname{ind}(a(\tau;\Upsilon), \gamma_i(\Upsilon)) = -1, \qquad \operatorname{ind}(a_\epsilon(\tau;\Upsilon), \gamma_i(\Upsilon)) = -1, \qquad \operatorname{ind}(0, \gamma_i(\Upsilon)) = 0.$$

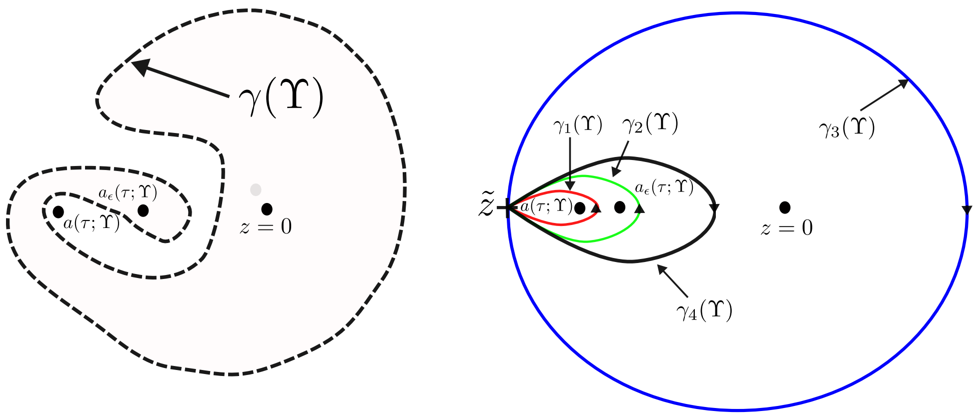


Figure 6.13: On the left, along an arbitrary curve Υ, the points $a(\tau;\Upsilon)$ and $a_\epsilon(\tau;\Upsilon)$ may have revolved around each other in a complicated manner, entangling, thus, the shape of the curve $\gamma(\Upsilon)$ (in dashed lines). On the right, the loop decomposition associated to the curve $\gamma(\Upsilon)$ depicted on the right: $\gamma_1(\Upsilon)$ in red, $\gamma_2(\Upsilon)$ in green, $\gamma_3(\Upsilon)$ in blue and $\gamma_4(\Upsilon)$ in black.

- *For* $i = k(\Upsilon)/2 + 1$

$$\operatorname{ind}(a(\tau;\Upsilon),\gamma_i(\Upsilon)) = 1, \qquad \operatorname{ind}(a_\epsilon(\tau;\Upsilon),\gamma_i(\Upsilon)) = 1, \qquad \operatorname{ind}(0,\gamma_i(\Upsilon)) = 1.$$

- *For* $k(\Upsilon)/2 < i \le k(\Upsilon)$*,*

$$\operatorname{ind}(a(\tau;\Upsilon),\gamma_i(\Upsilon)) = 1, \qquad \operatorname{ind}(a_\epsilon(\tau;\Upsilon),\gamma_i(\Upsilon)) = 1, \qquad \operatorname{ind}(0,\gamma_i(\Upsilon)) = 0.$$

such that the composition $\tilde\gamma(\Upsilon) = \gamma_1(\Upsilon)\cdots\gamma_i(\Upsilon)\cdots\gamma_{k(\Upsilon)}(\Upsilon)$ *is homotopic in* $\mathcal{C}_\tau$ *to* $\gamma(\Upsilon)$*.*

Remark 6.7.19. *Notice that, in particular, the closed curve* $\tilde\gamma(\Upsilon)$ *in Lemma 6.7.18 satisfies* $\operatorname{ind}(a(\tau;\Upsilon),\tilde\gamma(\Upsilon)) = 0$*,* $\operatorname{ind}(a_\epsilon(\tau;\Upsilon),\tilde\gamma(\Upsilon)) = 1$ *and* $\operatorname{ind}(0,\tilde\gamma(\Upsilon)) = 1$*.*

The number $k(\Upsilon)$ *increases with the total number of times that* $a(\tau;\Upsilon)$ *and* $a_\epsilon(\tau;\Upsilon)$ *have turned around themselves. That is,* $k(\Upsilon)$ *reflects the monodromy along different curves* Υ,Υ' *sharing the same endopoints. In the particular case in which* $\Upsilon\cap J = \emptyset$ *so* $\gamma(\Upsilon)$ *is homotopic to* γ_{κ_ϵ}*, Lemma 6.7.18 holds with* $k(\Upsilon) = 2$ *(meaning that only items 1 and 3 are present in that case).*

As before, the idea now is that only the curve $\gamma_1(\Upsilon)$ is "trapped" between the singularities $a(\tau,\Upsilon), a_\epsilon(\tau;\Upsilon)$ (recall that, although the origin is also a singularity of the integrand in (6.94), Lemma 6.95 ensures that for $\tau \in C_\pm$ and for $\tau \in C_i$ both $a(\tau;\Upsilon), a_\epsilon(\tau;\Upsilon)$ are far from the origin). Therefore, by choosing $\tilde z$ properly, the contribution of $\gamma_1(\Upsilon)$ can be analyzed asymptotically and the contribution of $\gamma_i(\Upsilon)$ for $i \ge 2$ can be shown to be smaller. Because of this we will write

$$\tilde\gamma(\Upsilon) = \gamma_{\text{sing}}(\Upsilon)\gamma_{\text{reg}}(\Upsilon), \qquad \text{where} \qquad \gamma_{\text{sing}}(\Upsilon) = \gamma_1(\Upsilon) \qquad \gamma_{\text{reg}}(\Upsilon) = \gamma_2(\Upsilon)\cdots\gamma_i(\Upsilon)\cdots\gamma_{k(\Upsilon)}(\Upsilon).$$

Let us point out one more important observation for the analysis of the integral $E_l(\tau;\Upsilon)$ in (6.94). Split the integral (6.94) as

$$E_l(\tau;\Upsilon) = \int_{\gamma_{\text{sing}}(\Upsilon)} \frac{(f_l(\tau,z) - f_l(\tau,a(\tau;\Upsilon)))z^{-1}\mathrm{d}z}{\sqrt{z^2 + (a+a_\epsilon)(\tau)z + \kappa_\epsilon^2}} + \int_{\gamma_{\text{reg}}(\Upsilon)} \frac{(f_l(\tau,z) - f_l(\tau,a(\tau;\Upsilon)))z^{-1}\mathrm{d}z}{\sqrt{z^2 + (a+a_\epsilon)(\tau)z + \kappa_\epsilon^2}} \tag{6.98}$$

and focus on the second term. Taking into account that $1 - 2b \to 0$ as $\tau \to \tau_\pm$,

$$z^2 - (a + a_\epsilon)z + \kappa_\epsilon^2 = (z + \kappa_\epsilon)^2 - 2\kappa_\epsilon z(2 - 1/b) \sim (z + \kappa_\epsilon)^2.$$

Therefore, since, in view of Lemma 6.7.17, for $\tau \to \tau_\pm$ we have $a(\tau;\Upsilon), a_\epsilon(\tau;\Upsilon) \to -\kappa_\epsilon$, if all z in $\gamma_{\rm reg}$ are sufficiently far from $a(\tau,\Upsilon), a_\epsilon(\tau;\Upsilon)$, one expects that

$$\frac{(f_l(\tau,z) - f_l(\tau,a(\tau;\Upsilon)))z^{-1}}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}} \sim \frac{(f_l(\tau,z) - f_l(\tau,a(\tau;\Upsilon)))z^{-1}}{z+\kappa_\epsilon}$$

and therefore that

$$\int_{\gamma_{\rm reg}(\Upsilon)} \frac{(f_l(\tau,z) - f_l(\tau,a(\tau;\Upsilon)))z^{-1}{\rm d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}} \sim \int_{\gamma_{\rm reg}(\Upsilon)} \frac{f_l(\tau,z)z^{-1}{\rm d}z}{(z+\kappa_\epsilon)} - f_l(\tau;a(\tau;\Upsilon))\int_{\gamma_{\rm reg}(\Upsilon)} \frac{z^{-1}{\rm d}z}{(z+\kappa_\epsilon)}. \quad (6.99)$$

The crucial remark now is that the integrals in the right hand side of (6.99) do not depend on Υ. Indeed, all the integrals along the loops forming $\gamma_{\rm reg}(\Upsilon)$, except the one with ${\rm ind}(a(\tau;\Upsilon),\gamma_i(\Upsilon)) = {\rm ind}(a_\epsilon(\tau;\Upsilon),\gamma_i(\Upsilon)) = {\rm ind}(0,\gamma_i(\Upsilon)) = 1$, cancel out. On the other hand, as we show in Lemma 6.7.20, the error commited in the approximation (6.99) is sufficiently small compared to the leading term in (6.95).

Lemma 6.7.20. *Let $\tau \in C_\pm$. Then, there exist $E_{{\rm reg},l}(\tau), \widetilde{E}_{\rm reg}(\tau)$ and $E_{{\rm sing},l}(\tau;\Upsilon)$ such that*

$$E_l(\tau;\Upsilon) = E_{{\rm reg},l}(\tau) + f_l(\tau;a(\tau,\Upsilon))\widetilde{E}_{\rm reg}(\tau) + E_{{\rm sing},l}(\tau;\Upsilon)$$

where $E_{{\rm reg},l}(\tau), \widetilde{E}_{\rm reg}(\tau)$ do not depend on the choice of Υ (i.e. they are analytic functions for $\tau \in C_\pm \cup \{\tau_\pm\}$), $E_{{\rm sing},l}$ is analytic on $C_\pm$ and satisfies

$$|E_{{\rm sing},l}(\tau;\Upsilon)| \lesssim I^{15/8}.$$

Proof. Fix $\delta = I^{-1/8}$. Given $\gamma(\Upsilon)$ in the definition of $E_l(\tau;\Upsilon)$ in (6.94), we let $\tilde{z} = a(\tau;\Upsilon) + \delta(a(\tau;\Upsilon) - a_\epsilon(\tau;\Upsilon))/|(a(\tau;\Upsilon) - a_\epsilon(\tau;\Upsilon)|$, deform $\gamma(\Upsilon)$ to the composition of closed loops $\tilde{\gamma}(\Upsilon)$ in Lemma 6.7.18 and denote by $\gamma_{\rm sing}(\Upsilon)$ and $\gamma_{\rm reg}(\Upsilon)$ its singular and regular part. According to the discussion preceeding the lemma, we write

$$E_{{\rm reg},l}(\tau;\Upsilon) = \int_{\gamma_{\rm reg}(\Upsilon)} \frac{f_l(\tau,z)z^{-1}{\rm d}z}{(z+\kappa_\epsilon)}$$
$$\widetilde{E}_{\rm reg}(\tau;\Upsilon) = \int_{\gamma_{\rm reg}(\Upsilon)} \frac{z^{-1}{\rm d}z}{(z+\kappa_\epsilon)},$$

and

$$E_{{\rm sing},l,a}(\tau;\Upsilon) = \int_{\gamma_{\rm sing}(\Upsilon)} \frac{(f_l(\tau,z) - f_l(\tau,a(\tau;\Upsilon)))z^{-1}{\rm d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}}$$
$$E_{{\rm sing},b}(\tau;\Upsilon) = -\int_{\gamma_{\rm reg}(\Upsilon)} \frac{z^{-1}}{(z+\kappa_\epsilon)}\left(1 - \left(1 + \frac{2\kappa_\epsilon z(1-2b)(\tau;\Upsilon)}{(z+\kappa_\epsilon)^2 b(\tau)}\right)^{-1/2}\right){\rm d}z$$
$$E_{{\rm sing},l,c}(\tau;\Upsilon) = -\int_{\gamma_{\rm reg}(\Upsilon)} \frac{f_l(\tau,z)z^{-1}}{(z+\kappa_\epsilon)}\left(1 - \left(1 + \frac{2\kappa_\epsilon z(1-2b)(\tau;\Upsilon)}{(z+\kappa_\epsilon)^2 b(\tau)}\right)^{-1/2}\right){\rm d}z,$$

so

$$E_l(\tau;\Upsilon) = E_{{\rm reg},l}(\tau) + f_l(\tau;a(\tau,\Upsilon))\widetilde{E}_{\rm reg}(\tau) + E_{{\rm sing},l,a}(\tau;\Upsilon) + f_l(\tau,a(\tau;\Upsilon))E_{{\rm sing},b}(\tau;\Upsilon) + E_{{\rm sing},l,c}(\tau;\Upsilon).$$

The integrals $E_{{\rm reg},l}$ and $\widetilde{E}_{\rm reg}$ do not depend on Υ. We now bound $E_{{\rm sing},l,a}$. Let $s^* = \delta/|a(\tau;\Upsilon) - a_\epsilon(\tau;\Upsilon)|$, then the loop $\gamma_{\rm sing}(\Upsilon)$ can be deformed to

$$\gamma'_{\rm sing}(\Upsilon) = \left\{z \in \mathbb{C} \colon z = a(\tau;\Upsilon) + (a(\tau;\Upsilon) - a_\epsilon(\tau;\Upsilon))s,\ s \in \left(0,s^*\right) \cup \left(0, e^{-i2\pi}s^*\right)\right\}.$$

Then, we write

$$f_l(\tau,z)-f_l(\tau,a(\tau;\Upsilon))=(z-a(\tau;\Upsilon))\int_0^1\partial_z f_l\big(\tau,a(\tau;\Upsilon)+x(z-a(\tau;\Upsilon))\big)\mathrm{d}x$$

and use that $|z|\sim 1$ for all $z\in\gamma'_{\mathrm{sing}}$ and Lemma 6.7.16 to show that for $z\in\gamma'_{\mathrm{sing}}$

$$|f_l(\tau,z)-f_l(\tau,a(\tau;\Upsilon))|\lesssim I^2|z-a(\tau;\Upsilon)|.$$

Therefore, writing $z(s)=a(\tau;\Upsilon)+(a(\tau;\Upsilon)-a_\epsilon(\tau;\Upsilon))s$,

$$\begin{aligned}\big|E_{\mathrm{sing},l,a}(\tau;\Upsilon,\delta)\big|\lesssim& I^2|a(\tau;\Upsilon)-a_\epsilon(\tau;\Upsilon)|\int_0^{s^*}\frac{|z(s)-a(\tau,\Upsilon)|\mathrm{d}s}{\sqrt{z^2(s)+(a+a_\epsilon)(\tau)z(s)+\kappa_\epsilon^2}}\\ \lesssim& I^2|a(\tau;\Upsilon)-a_\epsilon(\tau;\Upsilon)|\int_0^{s^*}\sqrt{\frac{s}{s+1}}\mathrm{d}s\lesssim\delta I^2=I^{15/8}.\end{aligned}$$

It only remains to bound $E_{\mathrm{sing},b}$ and $E_{\mathrm{sing},l,c}$. To that end we notice that, for all $\tau\in C_\pm$ and all $z\in\gamma_{\mathrm{reg}}$, with γ_{reg} as above, by the assumption on δ,

$$\left|\frac{(1-2b)}{(z+\kappa_\epsilon)^2b}\right|\lesssim\delta^{-2}I|\tau-\tau_\pm|\le\delta^{-2}I^{-1/2}\ll 1.$$

It therefore follows that

$$|E_{\mathrm{sing},b}|\lesssim\delta^{-2}I|\tau-\tau_\pm|\lesssim\delta^{-2}I^{-1/2}=I^{-1/4}.$$

and

$$|E_{\mathrm{sing},l,c}|\lesssim\delta^{-2}I^3|\tau-\tau_\pm|\lesssim\delta^{-2}I^{3/2}=I^{7/4}.$$

The conclusion of the lemma follows by writing

$$E_{\mathrm{sing},l}=E_{\mathrm{sing},l,a}(\tau;\Upsilon)+f_l(\tau,a(\tau;\Upsilon))E_{\mathrm{sing},b}(\tau;\Upsilon)+E_{\mathrm{sing},l,c}(\tau;\Upsilon).$$

□

We now study the behavior of the function $E(\tau;\Upsilon)$ in (6.98), for $\tau\in C_i$.

Lemma 6.7.21. *Let $\tau\in C_i$. Then, there exist a constant $B_l\in\mathbb{C}$ and $E_{\mathrm{sing},l}(\tau;\Upsilon)$ such that*

$$E(\tau;\Upsilon)=C+E_{\mathrm{sing},l}(\tau;\Upsilon)$$

with

$$|E_{\mathrm{sing},l}(\tau;\Upsilon)|\lesssim I^{15/8}.$$

Proof. Fix $\delta=I^{-1/8}$. Given $\gamma(\Upsilon)$ in the definition of $E(\tau;\Upsilon)$ in (6.94), we let $\tilde z=a(\tau;\Upsilon)+\delta(a(\tau;\Upsilon)-a_\epsilon(\tau;\Upsilon))/|(a(\tau;\Upsilon)-a_\epsilon(\tau;\Upsilon)|$, we deform $\gamma(\Upsilon)$ to the composition of closed loops $\tilde\gamma(\Upsilon)$ in Lemma 6.7.18 and denote by $\gamma_{\mathrm{sing}}(\Upsilon)$ and $\gamma_{\mathrm{reg}}(\Upsilon)$ its singular and regular part. We then write $E=E_{l,a}+E_{l,b}$ where

$$E_{l,a}=\int_{\gamma_{\mathrm{sing}}(\Upsilon)}\frac{(f_l(\tau,z)-f_l(\tau,a(\tau;\Upsilon)))z^{-1}\mathrm{d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}}$$

and further write $E_{l,b}=E_{l,c}+E_{l,d}-f_l(i,a(\tau;\Upsilon))E_{l,e}$ where

$$\begin{aligned}E_{l,c}=&\int_{\gamma_{\mathrm{reg}}(\Upsilon)}\frac{\big((f_l(\tau,z)-f_l(\tau,a(\tau;\Upsilon)))-(f_l(i,z)-f_l(i,a(\tau;\Upsilon)))\big)z^{-1}\mathrm{d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}}\\ E_{l,d}=&\int_{\gamma_{\mathrm{reg}}(\Upsilon)}\frac{f_l(i,z)z^{-1}\mathrm{d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}}\\ E_{l,e}=&\int_{\gamma_{\mathrm{reg}}(\Upsilon)}\frac{z^{-1}\mathrm{d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}}.\end{aligned}$$

The integral $E_{l,a}$ is bounded exactly as we did in Lemma 6.7.20. On the other hand, since, in view of Lemma 6.7.16,

$$\big|(f_l(\tau,z)-f_l(\tau,a(\tau;\Upsilon)))-(f_l(i,z)-f_l(i,a(\tau;\Upsilon)))\big|\lesssim I^2|\tau-i|,$$

one easily obtains that

$$|E_{l,c}|\lesssim \delta^{-1}I^2|\tau-i|\lesssim \delta^{-1}I^{3/4}=I^{7/8}.$$

For $E_{l,d}$ we write

$$\begin{aligned}E_{l,d}&=\int_{\gamma_{\mathrm{reg}}(\Upsilon)}\frac{f_l(i,z)z^{-1}\mathrm{d}z}{\sqrt{z^2+(a+a_\epsilon)(\tau)z+\kappa_\epsilon^2}}\\&=\int_{\gamma_{\mathrm{reg}}(\Upsilon)}\frac{f_l(i,z)z^{-1}\mathrm{d}z}{z-\kappa_\epsilon}+\int_{\gamma_{\mathrm{reg}}(\Upsilon)}\frac{f_l(i,z)z^{-1}}{z-\kappa_\epsilon}\left(1-\left(1+\frac{b(\tau)}{(z-\kappa_\epsilon)^2}\right)^{-1/2}\right)\mathrm{d}z.\end{aligned}$$

The first term is analytic for $\tau\in C_i\cup\{i\}$ and, moreover, does not depend on τ. Let B_l denote its value. On the other hand, since, for $\tau\in C_i$ and $z\in\gamma_{\mathrm{reg}}$, by the definition of δ,

$$|b(\tau)(z-\kappa_\epsilon)^{-2}|\lesssim \delta^{-2}I^2|\tau-i|^2\lesssim\delta^{-2}I^{-1/2}\ll 1,$$

and, by Lemma6.7.16 $|f_l(i,z)|\lesssim I^2$, we can bound

$$\left|\int_{\gamma_{\mathrm{reg}}(\Upsilon)}\frac{f_l(i,z)z^{-1}}{z-\kappa_\epsilon}\left(1-\left(1+\frac{b(\tau)}{(z-\kappa_\epsilon)^2}\right)^{-1/2}\right)\mathrm{d}z\right|\lesssim I^2\delta^{-1}|b(\tau)|\lesssim I^4\delta^{-1}|\tau-i|^2=I^{13/8}.$$

Finally, making use of Lemma 6.7.17, one deduces that

$$|f_l(i,a(\tau;\Upsilon)|\lesssim I^2|a(\tau;\Upsilon)-\kappa_\epsilon|\lesssim I^3|\tau-i|,$$

so

$$|f_l(i,a(\tau;\Upsilon)E_{l,e}(\tau;\Upsilon)|\lesssim\delta^{-1}I^3|\tau-i|\lesssim\delta^{-1}I^{7/4}=I^{15/8}.$$

The lemma now follows combining all the estimates. □

We finally sum up Lemmas 6.7.14, 6.7.20 and 6.7.21 in the following proposition.

Proposition 6.7.22. *Let $\tau\in C_\pm$. Then,*

$$U^{[l]}(\tau;\Upsilon)=4\kappa_\epsilon{-}1f_l(\tau_\pm,a(\tau_\pm))\ln\big((1-2b)(\tau;\Upsilon)\big)+S_{l,\pm}(\tau;\Upsilon)+R_{l,\pm}(\tau)$$

where $S_l(\tau;\Upsilon)$ is analytic in $C_\pm$, satisfies

$$|S_{l,\pm}(\tau;\Upsilon)|\lesssim I^{15/8},$$

and $R_{l,\pm}(\tau)$ does not depend on the choice of Υ (i.e. it is an analytic function for $\tau\in C_\pm\cup\{\tau_\pm\}$). On the other hand, for $\tau\in C_i$, there exists a constant $B_l\in\mathbb{C}$ and a function $\widehat{E}_l$, analytic in C_i, such that

$$U^{[l]}(\tau;\Upsilon)=B_l+\widehat{E}_l(\tau;\Upsilon)$$

with

$$|\widehat{E}_l(\tau;\Upsilon)|\lesssim I^{15/8}.$$

Proof. We first prove the asympotic formula for $\tau\in C_\pm$. According to Lemma 6.7.14,

$$K(\zeta(\tau;\Upsilon))=\frac{1}{2}\ln\big(1-\zeta^2(\tau;\Upsilon)\big)+c+\mathcal{O}\big(|1-\zeta^2(\tau;\Upsilon)|\big),$$

for some $c \in \mathbb{C}$. Also, in view of Lemma 6.7.15,

$$\ln(1-\zeta^2(\tau;\Upsilon)) = \frac{1}{2}\ln((1-2b)(\tau;\Upsilon)) + \ln(4) - 2\ln\left(1+\sqrt{(1-2b)(\tau;\Upsilon)}\right).$$

Thus, for $\tilde{c} = c + \ln(4)$, we have

$$K(\zeta(\tau;\Upsilon)) = \frac{1}{4}\ln(1-2b(\tau;\Upsilon)) + \tilde{c} + \mathcal{O}\big(|1-2b(\tau;\Upsilon)|^{1/2}\big).$$

We therefore write (recall that, from Lemma 6.7.17, $\zeta(\tau;\Upsilon) = -1 + \mathcal{O}(I^{1/2}|\tau-\tau_\pm|^{1/2})$ for $\tau \in C_\pm$)

$$\begin{aligned}4\kappa_\epsilon^{-1}\zeta(\tau;\Upsilon)f_l(\tau,a(\tau;\Upsilon))K(\zeta(\tau;\Upsilon)) =& -\kappa_\epsilon^{-1}f_l(\tau_\pm,a(\tau_\pm))\ln(1-2b(\tau;\Upsilon)) - 4\tilde{c}\kappa_\epsilon^{-1}f_l(\tau_\pm,a(\tau_\pm))\\ &-4\kappa_\epsilon^{-1}f_l(\tau_\pm,a(\tau_\pm))\big(K(\zeta(\tau;\Upsilon)) - \frac{1}{4}\ln(1-2b(\tau;\Upsilon)) - \tilde{c}\big)\\ &-4\kappa_\epsilon^{-1}(\zeta(\tau;\Upsilon)f_l(\tau,a(\tau;\Upsilon)) - f_l(\tau_\pm,a(\tau_\pm)))\ K(\zeta(\tau;\Upsilon)).\end{aligned}$$

Then, we define

$$\begin{aligned}R_{l,\pm}(\tau) =& E_{\mathrm{reg},l}(\tau) + f_l(\tau_\pm,a(\tau_\pm))\widetilde{E}_{\mathrm{reg}}(\tau) - 4\tilde{c}\kappa_\epsilon^{-1}f_l(\tau_\pm,a(\tau_\pm))\\ S_{l,\pm}(\tau;\Upsilon) =& E_{\mathrm{sing},l}(\tau;\Upsilon) + \big(f_l(\tau,a(\tau;\Upsilon)) - f_l(\tau_\pm,a(\tau_\pm))\big)\widetilde{E}_{\mathrm{reg}}(\tau)\\ &-4\kappa_\epsilon^{-1}f_l(\tau_\pm,a(\tau_\pm))\big(K(\zeta(\tau;\Upsilon)) - \frac{1}{4}\ln(1-2b(\tau;\Upsilon)) - \tilde{c}\big)\\ &-4\kappa_\epsilon^{-1}(\zeta(\tau;\Upsilon)f_l(\tau,a(\tau;\Upsilon)) - f_l(\tau_\pm,a(\tau_\pm)))K(\zeta(\tau;\Upsilon)),\end{aligned}$$

where $E_{\mathrm{reg},l}, \widetilde{E}_{\mathrm{reg}}$ and E_{sing} are the functions obtained in Lemma 6.7.20. The desired bounds for $S(\tau;\Upsilon)$ follow from the fact that, for $\tau \in C_\pm$,

$$|(1-2b)(\tau;\Upsilon)| \lesssim I|\tau-\tau_\pm|,$$

and, from Lemma 6.7.16,

$$\begin{aligned}|f_l(\tau_\pm,a(\tau_\pm)) - f_l(\tau,a(\tau;\Upsilon))| \lesssim& \sup_{\tilde{\tau}\in C_i}|\partial_\tau f_l(\tilde{\tau},a(\tau_\pm))||\tau-\tau_\pm| + \sup_{\tilde{\tau}\in C_\pm}|\partial_z f_l(\tau,a(\tilde{\tau};\Upsilon))||a(\tau;\Upsilon)-a(\tau_\pm)|\\ &+ I^2\big(|\tau-\tau_\pm| + |a(\tau;\Upsilon)-a(\tau_\pm)|\big) \lesssim I^2\left(|\tau-\tau_\pm| + I^{1/2}|\tau-\tau_\pm|^{1/2}\right) \lesssim I^{7/4}.\end{aligned}$$

Indeed, combining all the previous estimates and the bound for $E_{\mathrm{sing},l}$ in Lemma 6.7.20,

$$|S_{l,\pm}(\tau;\Upsilon)| \lesssim I^{15/8}.$$

We now analyze the case $\tau \in C_i$. We have shown tha,t for $\tau \in C_i$ there exists $B_l \in \mathbb{C}$ such that

$$U^{[l]}(\tau;\Upsilon) = 4\kappa_\epsilon^{-1}f_l(\tau,a(\tau;\Upsilon))K(\zeta(\tau;\Upsilon)) + B_l + E_{\mathrm{sing}}(\tau;\Upsilon)$$

where

$$|E_{\mathrm{sing}}(\tau;\Upsilon)| \lesssim I^{15/8}.$$

On the other hand, from Lemmas 6.7.16 and 6.7.17,

$$|f_l(\tau,a(\tau;\Upsilon))| \lesssim I^2|a(\tau;\Upsilon)-\kappa_\epsilon| \lesssim I^3|\tau-i| \lesssim I^{7/4}.$$

Then, we obtain that

$$|U^{[l]}(\tau;\Upsilon) - B_l| \lesssim I^{15/8} + I^{7/4} \lesssim I^{15/8},$$

as was to be shown. □

Observe now that

$$f_{l,\star}(\tau_\pm^\star, a(\tau_\pm^\star)) = \frac{I^2 C_{\beta,\star}\rho(-\kappa_\epsilon)e^{il\lambda(-\kappa_\epsilon)}}{(\tau_\pm + i)\sqrt{-\kappa_\epsilon(1+h_\star(\tau_\pm,-\kappa_\epsilon))}} = \frac{I^2\tilde{\nu}m_\star\rho(-\kappa_\epsilon)e^{il\lambda(-\kappa_\epsilon)}}{2i}\sqrt{\frac{e^{i\beta}}{\tilde{\sigma}_\star\kappa_\epsilon\epsilon_0}} + \mathcal{O}(I).$$

Then, Proposition 6.5.12 now follows by setting, for $\tau \in C_\pm$,

$$\tilde{U}_\star^{[l]}(\tau;\Upsilon) = I^2 A_l m_\star\sqrt{\frac{e^{i\beta}}{\tilde{\sigma}_\star}}\left(1+\mathcal{O}(I^{-1})\right)\ \ln((1-2b)(\tau;\Upsilon)),$$

with

$$A_l = \frac{-2i\tilde{\nu}\rho(-\kappa_\epsilon)e^{il\lambda(-\kappa_\epsilon)}}{\kappa_\epsilon\sqrt{\kappa_\epsilon\epsilon_0}}, \tag{6.100}$$

and joining the results in Lemma 6.7.13 and Proposition 6.7.22.

6.7.4 Evaluation of the residues. Proof of Lemma 6.5.14

A trivial computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(1-2b_\star) = \frac{1}{1-2b_\star}\frac{\mathrm{d}b_\star}{\mathrm{d}\tau} = \frac{-4I^{-2}\tilde{\sigma}_\star\epsilon_0 e^{-i\beta}}{(\tau-\tau_+^\star)(\tau-\tau_-^\star)(\tau-i)} \tag{6.101}$$

from where Lemma 6.5.14 follows by defining (notice that, the residue at $\tau = \tau_\pm$ does not depend on β and I and is the same for both $\star = 0, 1$)

$$\begin{aligned}\hat{A}_l =& A_l \operatorname*{Res}_{\tau=\tau_+^\star}\left(\frac{-4I^{-2}\tilde{\sigma}_\star\epsilon_0 e^{-i\beta}}{(\tau-\tau_+^\star)(\tau-\tau_-^\star)(\tau-i)}\right) = A_l \operatorname*{Res}_{\tau=\tau_-^\star}\left(\frac{-4I^{-2}\tilde{\sigma}_\star\epsilon_0 e^{-i\beta}}{(\tau-\tau_+^\star)(\tau-\tau_-^\star)(\tau-i)}\right) = -A_l\frac{4I^{-2}\tilde{\sigma}_\star\epsilon_0 e^{-i\beta}}{2(\tau_\pm^\star - i)^2} \\ =& -A_l.\end{aligned} \tag{6.102}$$

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