Universitat Autònoma de Barcelona

Fine Boundary Properties in Complex Analysis and Discrete Potential Theory

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Certify:

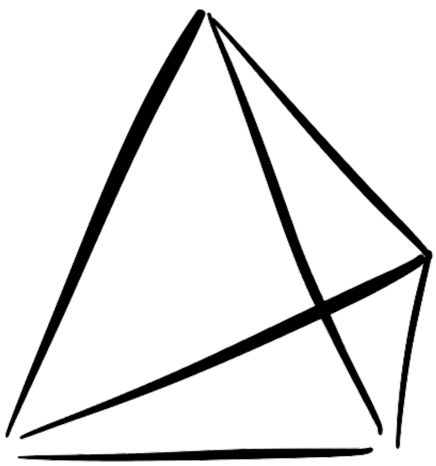
That this dissertation is presented to the Faculty of Sciences, department of Mathematics, of University of Bologna and of Autonomous University of Barcelona by Matteo Levi in fulfillment of the requirements for the degree of Doctor of Mathematics.

Bellaterra, September 25, 2019

Nicola Arcozzi　　　　Artur Nicolau

Matteo Levi

To my family.



Contents

Preface

This thesis is divided into two, quite independent, chapters. The first is a compendium of two research papers: [16], joint with Nikolaos Chalmoukis, and [5], joint with Nicola Arcozzi. The two work have a strong common pattern and range in the framework of Non Linear Potential Theory on trees. The second chapter is based insted on a joint paper with Artur Nicolau and Odí Soler i Gibert, [27], treating problems of classical Complex Analysis in the unit disc. There is, however, a general flavor which is common to the whole thesis, from which the title. In fact, all the treated problems regard fine properties of sets at the boundary of metric spaces. How to extend functions defined over a domain to its boundary is a classical and well studied problem in Analysis. For instance, the classical Fatou's theorem states that bounded analytic functions defined on the unit disc admit boundary values at almost every point of the unit circle. Equivalent results are known also for harmonic functions on trees, see for example [14]. We deal with this type of problems both in the discrete setting, in Chapter 1 and in the continuous setting, in Chapter 2. Radial limits, harmonic measures, exceptional sets are some keywords common to the two chapters. In general, we are interested in estimating the size of sets in the boundary at a fine level, which means not stopping the analysis at a measure zero scale, but using more sensible set functions such as capacities (Chapter 1) and Hausdorff contents (Chapter 2).

Trees and the unit disc are less unrelated than it seems: the firsts can be fruitfully used to provide discrete models for the latter (and for other planar domains) and to define discrete function spaces which are easier to deal with than the corresponding continuous ones. If one has at hands a good transferring technique, to solve the discrete problem can turn out to provide a solution for the correspondent continuous problem. To cite a successful example in this direction, we refer for example to [6], where the authors provide a characterization of the Carleson measures for analytic Besov spaces by solving an analogous model problem on the tree and then transferring back the result to the disc. However, analysis on trees, besides being a useful tool for getting insight in continuous problems, is an interesting subject per se, and as far as it regards this thesis we solve problems *on* trees and not *with* trees.

One problem we treat in Chapter 1 is what we call the *Equilibrium Problem*. In Electrostatics, an amount of, say positive, electric charge free to move across a conductor A in the Euclidean space will reach an equilibrium configuration μ, which at the same time: (i) minimizes the energy $\mathcal{E}(\mu)$ carried by the generated electrostatic potential; (ii) minimizes the maximum value of the potential ; (iii) makes the potential constant on all of A, but possibly for a small exceptional set. For a given system of units, there is an amount $\|\mu\|$ of charge for which the potential on (most of) A is unitary. The total charge $\|\mu\|$ is the *capacity* of the conductor and μ is the corresponding *equilibrium measure* of A. The mathematical theory of electrostatics, developed by Gauss, then put on firm mathematical foundations by Frostman, was later extended in many directions. See [12] for a survey of axiomatic linear theories which goes far beyond the scope of this thesis, and [1] for a rather general axiomatic non-linear theory. The problem we consider here, in a special instance, is that of characterizing equilibrium measures. Namely, given a positive measure μ, our "measurable", is there a way to tell whether or not it is the equilibrium measure for some conductor A? The equilibrium measures are known to satisfy a number of properties, but to the best of our knowledge a complete answer is available only for finite planar graphs, and is somehow implicit in a theorem by Schramm [34]. It is known, however, a combinatorial interpretation of the equilibrium measure μ of a *closed* subset of the boundary of a planar graph (see [10]). In Chapter 1, we characterize the equilibrium measures for Non Linear Potential Theory on trees. Our main result consists in showing that equilibrium measures are characterized by a sort of discrete integro-differential equation, which we call the *equilibrium equation*. The equation can be interpreted in several ways. On the one hand, it says that equilibrium measures on trees can be associated, in the linear case $p = 2$, to particular tilings of rectangles by squares. Alternatively, it can be reformulated in the form of a "continuous dyadic fraction". The first interpretations turns out to make available an independent proof of a tiling theorem for graphs by Benjiamini and Schram (see [10]), in the special case of a tree and, more importantly, to provide a converse result.

Another related problem treated in the first chapter is the *Dirichlet problem* on the infinite tree. There is an extensive literature on the discrete Dirichlet problem and its variations on graphs, see for example [38], [28] and [25]. In the particular case of trees we derive more precise results about the exceptional set. Our approach is based on the interplay between potential theory and probability, which is well known and today established, so that at some extent the two theories can be considered equivalent (see [26], Proposition 7.4]). We follow this path and we give a probabilistic definition of capacity in the linear case. In a sense, the fact that such an alternative probabilistic definition is available, means that the notion of capacity that we consider is the "right" one for the context. An appropriate use of the rescaling properties of capacity obtained in [5], already crucial in the Equilibrium Problem, allows us to develop a Wiener's type test for irregular points. We also discuss about the uniqueness of the solution to the Dirichlet Problem, with or without energy conditions on the solutions, and we provide a capacitary Fatou's type theorem for Sobolev functions of the tree.

Capacity is arguably the leading concept of the chapter: it allows to measure sets at a very fine scale, and it turns out to be the right tool to determine the size of the set of irregular points for the Dirichlet Problem, and to highlight its correspondence with the set of irregular points for the boundary of the tree.

In Chapter 2 we switch from the discrete to the continuous and from real valued to complex valued functions. The object of study are the analytic self maps of the unit disc, and in particular inner functions, which are those maps admitting unimodular radial limit at almost every point of the unit circle. Inner functions have a key role in holomorphic function theory. Just to cite two important applications:

1) Every function in the Hardy space H^p $(1 < p < \infty)$, and more generally every function in the Nevanlinna class $\mathcal{N}$, admits an *inner-outer factorization*, i.e. it can be factorized into the product of an inner function and a so called *outer* function. In particular, the inner factor carries all the information about the zero set of the function itself.

2) The classical problem of characterizing the invariant subspaces of the shift operator on ℓ^2, which can be rephrased in an analytic function theoretic contest using the Hardy space H^2, has a solution in terms of inner functions (the celebrated Beurling's theorem): the closed invariant subspaces of the Hardy space H^2 are exactly all the spaces of the form IH^2, I being an inner function.

Aim of the chapter is to study the metric distortion properties of sets in unit circle under the action of (boundary value maps induced by) inner functions. This type of study has already be done for inner functions fixing the origin. It is a classical result by Löwner that the Lebesgue measure of a subset E of the unit circle coincides with the one of its preimage under the action of an inner function fixing the origin. We refer to such a property as the *invariance of the measure.* More recently, in [19], Fernández and Pestana proved that, under the same assumptions, for any $0 < \alpha < 1$, the α-dimensional Hausdorff content of the preimage of E is bounded below by the one of the set E itself, times a constant only depending on the dimension α. This is what we call the *distortion of the content.* Our contribution in [27] is to extend such an analysis to the whole class of inner functions, without a priori assumptions on their fixed points. While both the aforementioned results adapt smoothly, with a standard argument of conformal invariance, to the case when the inner function f fixes a point $z \in \mathbb{D} \setminus \{0\}$ - in such a case the harmonic measure centered at z naturally takes the role of the Lebesgue measure - they do not give any information for functions with no fixed points in the unit disc. It is well known from the classical theory that any such a function must have some fixed points on the boundary of the unit disc. For this reason, we introduce a class of measures μ_p that allow to measure sets from the point of view of a chosen point p on the unit circle. In a vague sense they can be regarded as a boundary version of the harmonic measure. Using these measures we are able to prove both a μ_p–invariance theorem and a (μ_p, α)–distortion theorem. The μ_p measures are not ad hoc objects, but very natural measures, coinciding with the Lebesgue measure on the real line composed with the conformal map from the unit disc to the upper half plane mapping the point p to ∞. Beside carrying a notion of size, the μ_p measures have the special feature of giving information about the *distribution* of a set with respect to the point p: the μ_p measure of a set raises together with the concentration of the set around p, in a way that can be quantified. As a first application of our results, we present a distortion theorem for inner functions of the upper half plane fixing the point at infinity, which is hard to prove directly following the approach of Fernández and Pestana. Another application we provide is a theorem for estimating the size of the omitted values of an inner functions in terms of the size of points in the unit circle not admitting a finite angular derivative.

Chapter 1

The equilibrium problem and the Dirichlet problem on trees

In this chapter we present some results from [16] and [5]. The two papers have a strong common background in terms of approach, techniques, notation and general flavour, for which reason we have decided to merge them together in a unique chapter. The object we work on is the rooted tree, namely a connected graph with no cycles with a chosen distinguished vertex that we call the root. We work on very general trees: we do not put any restriction on the combinatorics besides local finiteness, namely from every vertex can depart only a finite number of edges. We do not ask for any uniform bound on the branching number of the tree. Also, we consider infinite trees. This generality is what makes the problem more complicated but even more interesting. The problems treated are two: the *Dirichlet problem on trees* and the *equilibrium problem on trees*

The Dirichlet problem consists, classically, in finding a harmonic function on the tree which satisfies some given boundary conditions. It is easy to construct harmonic functions extending a boundary data to the vertices of the tree, via an appropriate Poisson integral. However, when dealing with general trees, it is non trivial to study the irregular set for the problem, namely the set of boundary points where the Poisson integral of a (continuous) data fails to converge to the data itself. We introduce a concept of capacity that is the right one to measure the set of irregular points for the Dirichlet problem, which we solve in its generality. The techniques used are both probabilistic, exploiting the properties of random walks on graphs and their convergence to the boundary in the transient case, and potential theoretic, based on the behaviour of potentials, capacities and equilibrium measures with respect to the structure of the tree. Although the Dirichlet problem under exam is related to the classical Laplacian, most of the tools we use are proved in a more general non linear setting, $1 < p < +\infty$. The only step at which we are forced to settle for $p = 2$ is the probabilistic interpretation of capacity. We do not exclude that considering more general stochastic processes than the nearest neighbourhood random walk one could solve a Dirichlet problem for the $p-$Laplacian.

To each set in the boundary of the tree corresponds a unique measure supported on it and having total mass equal to the $p-$capacity of the set. It is called the *equilibrium measure* of the set. Equilibrium measures are quite mysterious objects, whose theoretical existence is assured by deep and well established results of modern Potential Theory (we refer to the treatise of Adams-Hedberg [1] for a good overview of the field), but whose nature is not well understood. The Equilibrium Problem consists in finding a description of these measures. We are able to characterize them as the set of solutions of an integro-differential equation, which accordingly we call the *equilibrium equation*.

The chapter is organized as follows: is section 1.1 we provide the fundamental terminology and give instructions to the reader for dealing with graphs, the underlying space on which we perform analysis and set up our theory. In particular, we introduce random walks having the vertices of a graph as a state space and we shortly discuss their main properties. Then, we define a notion of harmonicity for real valued functions defined on graphs, which is a key concept in the whole chapter. Finally, we present two different classical notions of boundary of an infinite graph, the Martin boundary, which is the natural boundary from a probabilistic and harmonic point of view, and the Carathéodory boundary, which is the natural notion from a combinatoric and geometric perspective. In a tree, which is the case of our interest, the two notions coincide, and give raise to what we simply call the *boundary of the tree*, a highly structured metric space which is the scenery of all our work.

Section 1.2 is devoted to set up a Non Linear Potential theory on the rooted tree, which we do following the axiomatics developed in §2.3-2.5 of the treatise of Adams-Hedberg [1]. We introduce potentials and energies and, especially, the key notion of capacity for sets on the boundary of the tree. We present some rescaling properties for capacity and associated equilibrium measures, which have been proved in [5] and are a fundamental tool in all the results to follow in the chapter. Then we discuss the interplay among flows, co-potentials of measures and $p-$harmonic functions, providing a characterization - in terms of a growth condition - for $p-$harmonic functions arising from measures.

In Section 1.3 the main results of paper [16] are presented. First, we provide a probabilistic definition of $2-$capacity, which constitute the bridge between fine boundary properties and harmonicity on the tree. Then, we present a useful result which provides a test to identify irregular points of a given set in the boundary of the tree. Finally, relying heavily on these two results, we prove that the Dirichlet problem with continuous boundary data has solution at a capacity-zero scale. We end the section by discussing the uniqueness of the solution for some class of very regular trees, and for general trees after having imposed some energy condition on the solution. We provide a counterexample which shows that in the general case this energy conditions are necessary to assure uniqueness.

Section 1.4 deals with some results of paper [5], concerning the equilibrium problem. We prove a sufficient condition for measures to be equilibrium measures, which, together with the rescaling properties of section 1.2 provides a full characterization. We give a reformulation of the result in terms of vertex functions, which can read more natural for some reader. We then discuss the regularity of boundaries, giving some examples and showing how we can build trees with regular boundaries and prescribed capacity. We also show how this can be done starting with a given general tree and modifying it as little as possible. After that, we present an application of our characterization, which, depending on the interests, could also be seen as a motivation for the whole project. Namely, we prove how it can be used to give a converse result of a well known tiling theorem from Benjamini and Schramm proved in [10]. We end the section, and the chapter, by giving a different formulation of the characterization theorem in terms of capacities and branched continued fractions.

1.1 Instructions to navigate on graphs

Let $G = (V, E)$ be a graph, where V and E are the set of vertices and edges, respectively. We write $x \sim y$ if two vertices x and y are linked by an edge α. In this case we say that x and y are the *endpoints* of α. We assume that the graph has no loops, i.e. each edge has exactly two distinct endpoints. A *path* on the graph is any sequence (finite or infinite) of vertices $\gamma = \{x_j\}$ such that $x_j \sim x_{j+1}$, for $j = 0, 1, \dots$. We will always consider G to be *connected*: for every couple of vertices x, y there always exists a path γ such that $x, y \in \gamma$. In a finite path $\{x_j\}_{j=0}^n$ the vertices x_0 and x_n are called the endpoints of γ. Such a path is called a *cycle* if it is not constant, $x_0 = x_n$ and there are no other repeated vertices. Graphs with no cycles are called *trees*. Write $|\gamma|$ for the number of vertices in the path γ. A finite path γ is a *geodesic* if $|\gamma| \leq |\gamma'|$ for any other path γ' having the same endpoints. An infinite path is called a geodesic if all its finite sub-paths are geodesics. In a tree, for any couple of vertices x, y, there exists a unique geodesic path $\gamma = [x, y]$ connecting them, while we cannot expect uniqueness on more general graphs. If x and y are endpoints of a geodesic γ, we write $d_G(x, y) = |\gamma| - 1$ for the *graph distance* between x and y (which we also call the *edge counting metric*).

We distinguish a vertex o which we call the *root* of G. We write $|x|$ for the *level* of x, which is the graph distance of x from the root. The choice of the root induces a natural partial order relation on the graph: we write $x \geq y$ if x, y lie on a same geodesic starting at the root and $|x| \geq |y|$. This can also be seen as a natural orientation for edges: if an edge α lies on some geodesic starting at the root, we write $b(\alpha)$ for the endpoint of α closer to the root (in graph distance) and $e(\alpha)$ for the other. If the edge does not lie on any such a geodesic, the two endpoints are equidistant from the root and we say that the edge is unoriented. An edge α connecting two vertices x and y is also denoted by $[x, y]$ if and only if it is oriented and $x < y$. Observe that if $[x, y]$ is an edge of the graph than $[y, x]$ it is not, and vice versa. Observe that in a rooted tree all edges are oriented. The partial order relation extends to edges in the obvious way, and also to the whole $V \cup E$: given an edge α and a vertex x we write $\alpha \leq x$ if $e(\alpha) \leq x$ and $\alpha \geq x$ if $b(\alpha) \geq x$.

We define the *sons* and the *parents* of a vertex v, respectively, as $s(v) = \{w \in V | \; w \sim v, w > v\}$ and $p(v) = \{w \in V | \; w \sim v, w < v\}$. Similarly, for an edge α we set $s(\alpha) = \{\beta \in E | \; b(\beta) = e(\alpha)\}$ and $p(\alpha) = \{\beta \in E | \; e(\beta) = b(\alpha)\}$. Observe that in a rooted tree $p(v)$ and $p(\alpha)$ are always a singletons.

We always assume that graphs are locally finite, meaning that the *branching number* $\operatorname{br}(x)$, representing the number of edges departing from the vertex x, is finite for every $x \in V$. A part from that, we do not put any restriction on the combinatorics of the graph. In particular, $\operatorname{br} : V \to \mathbb{N}$ is not asked to be bounded. Also, it is important to remark that we work with *infinite* graphs. In general, we assume that a graph G is infinite in every direction, i.e. that it has no *leaves*, which are vertices having branching number 1.

1.1.1 Random walks

A random walk (Z_n) on a graph $G = (V, E)$ is uniquely identified by (a starting point and) a function $\pi : V \times V \to \mathbb{R}_+$ such that $\sum_{y \in V} \pi(x, y) = 1$, for all $x \in V$. We call such a function a *transition probability* and $\pi(x, y)$ has to be interpreted as the probability of moving from x to y, in one step. We write $\mathbb{P}_x$ for the probability distribution of the random walk starting at x. By the Markov property we have

$$\mathbb{P}_{x_0}(Z_1 = x_1, \dots, Z_n = x_n) = \prod_{j=0}^{n-1} \pi(x_j, x_{j+1}).$$

We say that the transition probability π on a graph G is *reversible* if there exists a positive function m defined on V such that

$$m(x)\pi(x, y) = m(y)\pi(y, x) \quad \text{for all } x, y \in V. \tag{1.1}$$

A transition probability π is *simple* if it is both of nearest-neighborood type ($\pi(x, y) = 0$ if $x \not\sim y$), and locally uniform ($\pi(x, y) = 1/\operatorname{br}(x)$ if $x \sim y$). A random walk associated to a simple transition probability is called simple as well.

Observe that a simple transition probability π is reversible: in particular (1.1) holds with $m(x) = \mathrm{br}(x)$.

Let $\pi^{(n)}(x,y) := \mathbb{P}_x(Z_n = y)$, the probability that the random walk starting at x will visit y at the n-th step. We always assume random walks to be *irreducible*, meaning that there are no unreachable points, or equivalently, for every $x, y \in V$ there exists $n \in \mathbb{N}$, such that $\pi^{(n)}(x,y) > 0$.

A random walk on G is *transient* if for some (equivelently, for all) $x \in V$, $\mathbb{P}_x(Z_n \neq x$, for all $n) > 0$, or equivalently if Z_n eventually escapes to infinity with probability one. Else, the random walk is called *recurrent.* A graph G itself is called transient (recurrent) if the simple random walk on G is transient (recurrent). It is a famous result by Polya [31] that the d-dimensional lattice $\mathbb{Z}^d$ is recurrent for $d \leq 2$ and transient for $d \geq 3$.

1.1.2 Harmonic functions

A function $w : V \times V \to \mathbb{R}$ is a positive and locally finite edge weight (from now on simply *edge weight*) if $w(x,y) = w(y,x) \geq 0$, for every $x, y \in V$, it is non zero if and only if $x \sim y$ and $W(x) := \sum_{y \sim x} w(x,y) < \infty$, for every $x \in V$. If α is the edge connecting x and y, then we also write $w(\alpha)$ for $w(x,y)$. There is a natural correspondence between graphs endowed of a reversible nearest neighbourhood type transition probability and edge weighted graphs. Specifically, if π is of nearest neighbourhood type and satisfies (1.1), then $w(x,y) = m(x)p(x,y)$ is an edge weight with $W(x) = m(x)$. Conversely, given an edge weighted graph with weight w, then $\pi(x,y) = w(x,y)/W(x)$ is a nearest neighbourhood type transition probability satisfying (1.1) with $m(x) = W(x)$.

Let G be an edge weighted graph with weight w. Given a function $g : V \to \mathbb{R}$, write $dg(x,y) = g(y) - g(x)$. We define the *gradient* of g at $(x,y) \in V \times V$ to be the weighted anti-symmetric difference operator

$$\nabla g(x,y) = w(x,y) dg(x,y). \tag{1.2}$$

Since the weight w vanishes at all couples (x,y) such that $x \not\sim y$, morally the gradient is a function on (oriented) edges. If $\alpha = [x,y]$ is an edge, with the natural orientation induced by the root, then we also write $\nabla g(\alpha)$ for $\nabla g(x,y)$. The *divergence* of a function $f : V \times V \to \mathbb{R}$ at $x \in V$ is given by $\operatorname{div} f(x) = \sum_{y \in V} w(x,y) f(x,y)$.

For $p \in (1, +\infty)$, we define the *p-Laplacian* of a function $g : V \to \mathbb{R}$ as $\Delta_p g = \operatorname{div}\big(dg|dg|^{p-2}\big)$. Explicitely, for each $x \in V$ we have

$$\Delta_p g(x) = \sum_{y \sim x} \omega(x,y)\big(g(y) - g(x)\big)|g(y) - g(x)|^{p-2}.$$

We say that g is p*−harmonic* if $\Delta_p g \equiv 0$ on V. As usual, we simply call Laplacian the linear operator $\Delta := \Delta_2$ and we say that g is harmonic if $\Delta g \equiv 0$. Observe that g is harmonic if and only if

$$g(x) = \sum_{y\sim x} \pi(x,y)g(y), \qquad \text{for all } x \in V, \tag{1.3}$$

where π is the transition probability induced by the weight w. Indeed, (1.3) could be taken more generally as definition of harmonicity even on graphs with a nearest neighbourhood transition probability π which is not necessarily reversible. Notice that in the un-weighted case, namely when $w(x,y) = \delta_{xy}$, the transition probability π is simple and harmonicity coincides with the mean value property, $g(x) = \sum_{y\sim x} g(y)/\operatorname{br}(x)$, for all $x \in V$.

It is easy to see that harmonic functions satisfy the following maximum principle.

Proposition 1.1.1. *(see also, for example, [38, page 5]) Let g be harmonic on G and assume there exists a vertex x_0 such that $g(x_0) \geq g(y)$ for all $y \in V$. Then g is constant.*

Proof. By irreducibilty of the transition probability, we have that $V = \bigcup_n V_n$, where $V_n = \{y \in V : \pi^{(n)}(x_0,y) > 0\}$. Since $g(x_0) = \sum_{y\sim x} \pi(x_0,y)g(y)$, it is easily verified by induction that $g(x_0) = \sum_{y\in V_n} \pi^{(n)}(x_0,y)g(y)$ for every integer $n \geq 1$. Then, it is not possible for any $y \in V_n$ that $g(x_0) > g(y)$. This is true for every n, implying $g(y) = g(x_0)$, for every $y \in V$. □

1.1.3 The Martin boundary

There are different possible paths to follow when introducing a notion of boundary of a graph, depending on the desired applications and/or on the starting structure at hands . One possible approach, which we present in this section, is to introduce a boundary that allows to have a Herglotz-Riesz type representation theorem for non-negative harmonic functions in terms of boundary measures. This was first introduced by Martin in [29], from whom it takes the name. Of course, being harmonicity involved, such a notion makes sense for graphs with an attached transition probability. Moreover, as we will see, this boundary is also the right one to consider for assuring convergence of transient random walks to some boundary valued random variable. Out of these reasons, we can refer to it also as the *probabilistic boundary*. In the next section we will present a different type of boundary which only depends on the combinatorics of the graph itself, reason for which it is sometimes referred to as the *natural boundary*, although we will use a different name.

Write $G(x,y)$ for the expected number of times a random walk starting at x will visit y. It holds

$$G(x,y) = \sum_{n=0}^{\infty} \pi^{(n)}(x,y).$$

The function G so defined is the *Green function* associated to the given transition probability. Observe that since we consider irreducible random walks, the Green function $G(x,y) > 0$ for every $x, y \in V$. The *Martin kernel* of G is given by

$$K(x,y) = \frac{G(x,y)}{G(o,y)}, \quad x, y \in V. \tag{1.4}$$

It can be easily verified that the function $K_y(x) := K(x,y)$ takes value 1 in the root o, is bounded on V and harmonic off the diagonal (for all $x \neq y$).

On a transient graph $G(V,E)$ we can define a *probabilistic distance* as follows:

$$\rho(x,y) = \sum_{z \in V} w_z G(o,z) |K(z,x) - K(z,y)|, \quad x, y \in V, \tag{1.5}$$

where the positive weights $w_z > 0$ satisfy, $\sum_{z \in V} w_z G(z,z) < \infty$.

Denote by G^* the completion of G with respect to the probabilistic metric ρ. The *Martin Boundary* of G is defined as $G^* \setminus G$ and denoted by $\partial_M G$.

It can be shown (see for example [26], Proposition 10.13) that a sequence of vertices $\{y_n\}$ is Cauchy in (G,ρ) if and only if $\{K(x,y_n)\}_n$ is Cauchy for every $x \in V$. Hence, the Martin kernel extends continuously to a function on $G \times G^*$ by

$$K(x,\xi) = \lim_n K(x,y_n), \quad \text{for every Cauchy sequence } \xi = \{y_n\} \subseteq (V,\rho).$$

Consequently, the probabilistic distance ρ extends to the boundary: for $\xi = \{y_n\}, \nu = \{z_n\}$ Cauchy sequences of vertices (i.e. elements in the Martin boundary of T), we set $\rho(\xi,\nu) = \lim_n \rho(y_n,z_n)$. The resulting metric space (G^*,ρ) is compact.

We report here the two main results in Martin boundary theory. They show how the Martin boundary is the right concept of boundary to consider both in terms of representation of harmonic functions than in terms of limit distribution of random walks. For the proofs of these theorems and for more discussion on the topic, excellent references are section 24 of [38], sections 4 and 5 in chapter 10 of [26] and [33].

Theorem 1.1.2 (Poisson-Martin representation theorem)**.** *Let G be a transient graph and g a non-negative harmonic function on it. Then, there exists a non-negative Borel measure $\mu = \mu_g$ on the Martin boundary $\partial_M G$, such that*

$$g(x) = \int_{\partial_M G} K(x,\xi) d\mu(\xi). \tag{1.6}$$

Conversely, for every finite Borel measure μ on $\partial_M G$, the above expression defines a non-negative harmonic function on G.

The representing measure μ_g can be chosen so that it is supported on the *minimal Martin boundary*,

$$\partial_m G := \{\xi \in \partial_M G : \ K(\cdot,\xi) \text{ is minimal harmonic on } V\},$$

and in such a case it is unique. Here, a non-negative harmonic function g on G is called *minimal* if $g(o) = 1$ and for every other non-negative harmonic function g_1 on V, the relation $g_1(x) \le g(x)$ for all $x \in V$, implies that g_1 is a constant multiple of g.

Theorem 1.1.3 (The convergence theorem). *Let (Z_n) be a random walk on a transient graph G. Then, there exists a $\partial_m G$-valued random variable Z_∞ such that $Z_n \to Z_\infty$, $\mathbb{P}_x$-almost surely, for every $x \in V$.*

We define the *harmonic measure* at the vertex x as

$$\lambda_x(A) = \mathbb{P}_x(Z_\infty \in A), \quad \text{for every Borel set } A \subseteq \partial_m G.$$

Namely, λ_x is the $\mathbb{P}_x$-distribution of Z_∞. Also, given a function φ defined on ∂T and λ_x integrable on the boundary for some (all) $x \in V$, we define its *harmonic extension* (or its *Poisson integral*) to be the function $\mathcal{P}(\varphi) : V \to \mathbb{R}$ given by

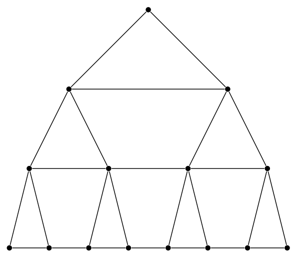
$$\mathcal{P}(\varphi)(x) := \int_{\partial T} \varphi \, d\lambda_x. \tag{1.7}$$

The harmonicity of $\mathcal{P}(\varphi)$ follows by the Markov property, since $\lambda_x = \sum_{y \sim x} p(x,y)\lambda_y$.

1.1.4 The Carathéodory boundary

While the Martin boundary depends a priori on the transition probability, we would like to realize it (or a part of it) in terms of a *natural compactification*, depending on the graph combinatorics only. A *ray* is a path without repeated vertices. We say that two infinite rays are equivalent if, roughly speaking, whenever we try to disconnect the graph in a finite number of moves, we cannot disconnect the two rays. More precisely, write $G(F)$ for the graph obtained eliminating an arbitrary finite set F of edges and vertices from G. Let $C(F)$ be the (finite) set of connected components of $G(F)$. We say that an infinite ray γ ends in $C \in C(F)$ if infinitely many of its vertices belong to the component C. Observe that such a component exists and is unique for every ray. Two infinite rays $\gamma = \{x_n\}$ and $\gamma' = \{y_n\}$ are equivalent if, for every finite F, they end in the same connected component of $G(F)$. A *prime end* of G is an equivalence class of infinite rays, and the collection of all the prime ends is the *Carathéodory boundary* of G, which we denote by $\partial_C G$. We write $\partial_C C$ for the subset of prime ends of G whose rays end in C, which is exactly the Carathéodory boundary of the subgraph C. For every graph we set $\overline{G} := G \cup \partial_C G$. Note that we always assume that the graph has no leaves, but in case it does, each leaf has to be considered a point in the Carathéodory boundary as well.

Observe that equivalent rays can still depart, in the sense that it is possible that they are definitely arbitrarily far away one from the other in graph distance. For example, let T be the binary tree ($br(x) = 2$ for every $x \in V$) and G be the planar graph obtained adding edges connecting vertices on the same level as in the picture.



Let $\gamma = \{x_n\}$ and $\gamma' = \{y_n\}$ be the only two rays starting from the root with $\mathrm{br}(x_n) = \mathrm{br}(y_n) = 4$, for all n (the leftmost and rightmost ones). They clearly belong to the same prime end - in fact the the Carathéodory boundary of G is a singleton - but $d_G(x_n, \gamma') = d_G(\gamma, y_n) = \min\{n, n^2 - 1\} \longrightarrow \infty$, as $n \to \infty$.

On $\overline{G}$ we put the topology generated by $\{\overline{C} : \ C \in C(F),\ F \text{ finite}\}$. It is clear that such a choice defines a discrete topology on G, since $\{x\} \in C(\{\alpha \in E : x \in \alpha\})$, for every $x \in V$. As seen in the example, also $\partial_C G$ can possibly contain isolated points. The space $\overline{G}$ can be clearly obtained as a finite union of open sets, and this is true also for the closure $\overline{C}$ of any component $C \in C(F)$, F being any finite subset of G, so that $\overline{G}$ is totally disconnected. It is also easy to check that with the chosen topology $\overline{G}$ is compact.

With the Carathéodory boundary at hands there is a very natural way to conceive the notion of boundary value of functions defined on the graph. Let g be a real valued function defined on the vertices of a graph G. Given a ray $\gamma = \{x_j\}_{j=1}^{\infty}$, we define the limit of g along γ as

$$\lim_{\gamma} g = \lim_{j\to\infty} g(x_j).$$

Given a prime end $\xi \in \partial_C G$, if the above limit exists and coincide for every ray $\gamma \in \xi$, we say that g admits *radial limit* at ξ and we write

$$\lim_{x\to\xi} g(x) = \lim_{\gamma} g \quad \text{for some (all) } \gamma \in \xi.$$

The *Fatou's set* of g is

$$\mathcal{F}(g) = \{\xi \in \partial_C G : \text{ there exists } \lim_{x\to\xi} g(x) \in \mathbb{R} \cup \{\pm\infty\}\}.$$

The *boundary value* of g is the map $g^* : \partial_C G \to \mathbb{R} \cup \{\pm\infty\}$ which on Fatou's points $\xi \in \mathcal{F}(g)$ is defined by

$$g^*(\xi) := \lim_{x\to\xi} g(x).$$

From now on we simply write g in place of g^* for the extension of g to the boundary since no confusion can arise.

1.1.5 The boundary of a tree

When we restrict our attention to trees, the notions of boundary we introduced become much easier and intuitive. Moreover, in a tree, the Carathéodory boundary coincides with the minimal Martin boundary (see [15, Proposition 3]). We call it simply the boundary of the tree and denote it by ∂T.

In a tree every ray is a geodesic, and for every couple of infinite rays, either one is fully contained in the other, or they eventually depart one from the other. It follows that the boundary ∂T of a tree T can be be thought of as a set of labels, each one corresponding to a half-infinite geodesic starting from the root:

$$\partial T = \left\{ \xi = \{x_j\}_{j=0}^{\infty} : \ x_0 = o, \ x_j \sim x_{j+1}, \ x_{j+1} \geq x_j \right\}.$$

Also observe that, in contrast with the Carathéodory boundary of a graph (see the example of the previous section) the boundary of a tree is a perfect set. Given an edge α, we write T_α for the subtree rooted at α having as edge set $\{\beta \in E : \beta \geq \alpha\}$, and we call it the α-*tent*. The boundary ∂T_α of the α-tent is exactly the set of rays passing through α. Observe that the topology on ∂T is the one generated by the boundary of tents, $\{\partial T_\alpha\}_{\alpha \in E}$.

It is clear that the edge counting metric does not extend properly to the boundary of an infinite tree. Hence we seek for another metric. Given two points $\xi, \eta \in V \cup \partial T$, we define their *confluent* to be the vertex $\xi \wedge \eta := \max\{[o, \xi] \cap [o, \eta]\}$. Then, we define

$$d(\xi, \eta) := e^{-|\xi \wedge \eta|}. \tag{1.8}$$

For the reader familiar with Gromov's theory of hyperbolic spaces, observe that for $\xi, \eta \in V$, expression (1.8) coincides with the Gromov product on (T, d_T), given by

$$(x, y)_o = \frac{1}{2} \left(d_T(x, o) + d_T(y, o) - d_T(x, y) \right), \quad x, y \in V.$$

In fact, one can extend such a product to points in ∂T by setting $(\xi, \eta)_o = \lim (x, y)_o$, where the limit is taken for $x \to \xi, y \to \eta$ along the rays labeled by ξ and η. With this notation we have $d(\xi, \eta) = (\xi, \eta)_o$. It is not hard to see that (1.8) defines a distance on $\overline{T}$, which in fact is an ultrametric, and that $\left(\overline{T}, d\right)$ is a compact metric space. Moreover, the topology induced by this metric is the tent topology introduced before.

There is a one-to-one correspondence between compact sets in ∂T and boundaries of trees, in the following sense.

Proposition 1.1.4. *A set $K \subseteq \partial T$ is compact if and only if there exists a subtree $T_K \subseteq T$ such that $K = \partial T_K$.*

Proof. The fact that the boundary of a subtree is compact follows directly from the definition of subtree. Conversely, if K is compact, consider the subtree $S \subseteq T$ having as edge set $E(S) = P(K) := \{\alpha \in E : \ \alpha \in \xi, \text{ for some } \xi \in \partial K\}$. Clearly $K \subseteq \partial S$. On the other hand, if $\xi \in \partial S$, by definition of boundary of a tree, $P(\{\xi\}) \subseteq E(S)$. Now, suppose by contraddiction that $\xi \notin K$. Then, by compactness, there exists and an edge $\alpha \in E$ such that $\partial T_\alpha \bigcap K = \varnothing$ and $\xi \in \partial T_\alpha$. Hence, $\beta \notin P(K)$ for $\beta \in P(\{\xi\}) \subseteq E(S)$ with $|\beta| \geq |\alpha|$, leading to a contraddiction. □

1.2 Potential Theory on the rooted tree

In this section we set up a Non Linear Potential Theory on the tree. In order to do so in a convenient way, we do attach to a chosen vertex an extra edge, not linking to any other existing vertex. In other words, we introduce a leaf. This will be the only leaf in the tree and we choose it as a root, denoting it by o. In this way, we do also have a *root edge*, the attached one, denoted by ω, which is the only one departing from o. From now on, every tree will be assumed to have this structure, to which we will refer to simply as *the rooted tree* (see the picture below). We point out that this is not a restriction at any scale, it is only a choice to make the notation less heavy, to keep the duality between objects defined on edges and vertices explicit, to guarantee a good metric normalization of the boundary, to have a more neat parallelism with the classical continuous Potential Theory and to have an easier probabilistic description in terms of random walks (see section 1.3.1).

A small remark: even technically being a leaf, in general we do not consider o to be part of the boundary of the tree. However in some situations it will be convenient to consider the so called *extended boundary*, which is $\overline{\partial T} = \partial T \cup \{o\}$.

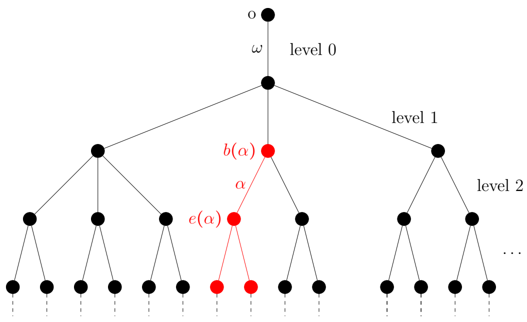


Figure 11: A rooted tree with root edge root ω and one of its tents T_α.

1.2.1 The p−capacity

Let $T = (V, E)$ be the edge weighted rooted tree, with weight w. Given a function $f : E \to \mathbb{R}$, and $p \in (1, +\infty)$, we define the *p−potential* of f, $I_p f : V \cup \partial T \to \mathbb{R} \cup \{\pm\infty\}$, by

$$I_p f(\xi) = \sum_{E \ni \alpha < \xi} \frac{f(\alpha)|f(\alpha)|^{p'-2}}{w(\alpha)}.$$

It is clear that if f is a non-negative function then $I_p f$ is defined on the whole boundary, possibly taking value $+\infty$, while in general its Fatou's set is non trivial. For $p = 2$, we drop the subscript writing I for I_2 and calling it simply the potential of f,

$$If(\xi) = \sum_{\alpha < \xi} \frac{f(\alpha)}{w(\alpha)}.$$

It is immediate to see that the following fundamental theorem of calculus on trees holds.

Proposition 1.2.1. *Let $T = (V, E)$ be a tree with root o and edge weight w. Take two functions $f : E \to \mathbb{R}$, $g : V \to \mathbb{R}$. Then, $g = If + g(o)$ on V if and only if $f = \nabla g$ on E.*

Proof. Let $g = If + g(o)$ on V. Then, for every $\alpha \in E$ we have

$$\nabla g(\alpha) = w(\alpha)\Big(g(e(\alpha)) - g(b(\alpha))\Big) = w(\alpha)\Big(\sum_{E \ni \beta \le \alpha} \frac{f(\beta)}{w(\beta)} - \sum_{E \ni \beta \lneq \alpha} \frac{f(\beta)}{w(\beta)}\Big) = f(\alpha).$$

Conversely, let $f = \nabla g$ on E. Then $If(o) = 0 = g(o) - g(o)$ and for every $x \in V \setminus \{o\}$, we have

$$If(x) = \sum_{E \ni \alpha < x} \frac{\nabla g(\alpha)}{w(\alpha)} = \sum_{E \ni \alpha < x} \big(g(e(\alpha)) - g(b(\alpha))\big) = g(x) - g(o).$$

□

Let $p \in (1, +\infty)$ be a fixed exponent and p' its Hölder conjugate, $1/p + 1/p' = 1$. The (weighted) p–norm of an edge function $f : E \to \mathbb{R}$ is given by

$$\|f\|_p^p = \sum_{\alpha \in E} \frac{|f(\alpha)|^p}{w(\alpha)}.$$

Definition 1.2.1. Suppose that $A \subseteq \partial T$ is a Borel set. We define the *p–capacity* of A,

$$c_p(A) := \inf\{\|f\|_p^p : f : E \to [0, \infty),\ If \geq 1 \text{ on } A\}.$$

Some remarks are in order. First, those functions on which we take the supremum, namely the edge functions such that $If \geq 1$ on A are called the *admissible functions* for A. Second, it is customary to say that a property holds p–capacity almost everywhere, or c_p–almost everywhere, if the set on which it does not hold has p–capacity zero. The infimum in the definition of p–capacity in general is not attained. However, one can prove [1, Theorem 2.3.10] that given a Borel set $A \subseteq \partial T$, there exists a unique function $f^A : E \to [0, +\infty)$, such that $If^A = 1, c_p$–a.e. on A and $\|f^A\|_p^p = c_p(A)$. This function is called the *p–equilibrium function* for the set A.

Definition 1.2.2. A point $\xi \in A$ such that $If^A(\xi) \neq 1$ is called *irregular* for the set A.

As a set function, p–capacity is monotone, countably subadditive and regular from inside and outside:

(i) $c_p\big(\bigcap_n (K_n)\big) = \lim_n c_p(K_n)$, for any decreasing sequence (K_n) of compact subsets of ∂T.

(ii) $c_p\big(\bigcup_n A_n\big) = \lim_n c_p(A_n)$, for any increasing sequence (A_n) of arbitrary subsets of ∂T.

Moreover, if $A = \bigcup_n A_n$, (A_n) increasing, and $c_p(A) < \infty$, then f^{A_n} converges strongly to f^A in p–norm.

Observe that without losing generality, in the definition of p–capacity the infimum can be taken over functions supported on the *predecessor set* of $\bar{A}$, $P(\bar{A}) := \{\alpha \in E :\ \alpha \in \xi, \text{ for some } \xi \in \partial \bar{A}\}$. Namely, the capacity of a compact set K only depends on the combinatorics of $P(K)$ and not on the rest of the tree, and if $T_K \subseteq T$ is the subtree having K as a boundary (see Proposition 1.1.4), we have $c_p(K) = c_p(\partial T_K)$, where the right handside is intended as the capacity when T_K is taken as the ambient space.

There exists a useful equivalent definition of capacities in terms of measures. We call *charge* a signed finite Borel measure on ∂T. The *co-potential* of a charge μ is defined by

$$I^*\mu(\alpha) = \mu(\partial T_\alpha), \ \ \alpha \in E.$$

Observe that the operator I^* is the adjoint of the potential operator I in the follong sense: for any $f : E \to \mathbb{R}$, μ charge on ∂T, we have

$$\begin{aligned}\langle If, \mu\rangle_{L^2(\partial T)} &= \int_{\partial T} If(\xi)d\mu(\xi) = \int_{\partial T} \sum_{\alpha\in E} \frac{f(\alpha)}{w(\alpha)} \chi_{\partial T_\alpha}(\xi)d\mu(\xi) \\ &= \sum_{\alpha\in E} \frac{f(\alpha)}{w(\alpha)} \int_{\partial T_\alpha} d\mu(\xi) = \sum_{\alpha\in E} \frac{f(\alpha)I^*\mu(\alpha)}{w(\alpha)} = \langle f, I^*\mu\rangle_{\ell^2(E)}.\end{aligned} \tag{1.9}$$

The *p−energy* of a charge is just

$$\mathcal{E}_p(\mu) = \|I^*\mu\|_{p'}^{p'}.$$

Observe that, taking $f(\alpha) = I^*\mu(\alpha)|I^*\mu(\alpha)|^{p'-2}$ in (1.9), we get, for every $p \in (1, +\infty)$ the relation

$$\mathcal{E}_p(\mu) = \int_{\partial T} I_p(I^*\mu)d\mu. \tag{1.10}$$

As mentioned, we have the following dual definition of capacity in terms of measures.

Theorem 1.2.2. *[see [1], Theorem 2.5.3] Suppose that $A \subseteq \partial T$ Borel. Then*

$$c_p(A) = \sup\{\mu(A)^p : \mu \geq 0, \ \operatorname{supp}(\mu) \subseteq A, \mathcal{E}_p(\mu) \leq 1\}.$$

Moreover, there exists a unique positive charge μ^A supported in $\overline{A}$, called the p−equilibrium measure of A, such that

$$\mu^A(A) = c_p(A) = \mathcal{E}_p(\mu^A),$$

and $(I^*\mu^A)^{p'-1} = f^A$.

With this definition of capacity, it is clear that if a property (P) holds c_p−a.e. on $A \subseteq \partial T$, then it holds μ−a.e. on A for every measure μ with $\mathcal{E}_p(\mu) < \infty$. To see this, let $B := \{\xi \in A : \ \neg(P)\}$, so that $c_p(B) = 0$. The measure $\nu := \dfrac{\mu|_B}{\mathcal{E}_p(\mu)^{1/p'}}$ satisfies $\mathcal{E}_p(\nu) \leq 1$. Then, $\nu(B) \leq= c_p(B) = 0$, from which it follows $\mu(B) = 0$.

Remark 1. All the potential theoretic objects we introduced, depend on the edge weight w one considers. However, to keep the notation light, we do not explicit this dependence in the associated symbols. We know that they always carry a weight and that it is a general weight, unless differently specified.

We now give a couple of toy examples in which we explicitly calculate the capacity some easy sets.

Example 1.1 (Capacity of a point)**.** As we already specified, we always assume that the trees we work with have no leaves, namely points in the boundary at a finite graph distance from the root. However, as everything else we treat, our definition of capacity applies also to trees having leaves, case that we briefly discuss for completeness. For simplicity, consider the un-weighted case, $w(\alpha) = 1$ for all $\alpha \in E$. Suppose ξ is a leaf of T. It is easy to check that the equilibrium function for the set $A = \{\xi\}$ is the function f^A taking constant value $1/|\xi|$ on all edges $\alpha < \xi$ and zero elsewhere. In fact for any other admissible function f we have

$$\|f\|_p^p \geq \sum_{\alpha<\xi} f(\alpha)^p \geq \frac{1}{|\xi|^{p-1}} \left(\sum_{\alpha<\xi} f(\beta) \right)^p = \frac{1}{|\xi|^{p-1}} If(\xi)^p \geq \frac{1}{|\xi|^{p-1}} = |\xi| \frac{1}{|\xi|^p} = \|f^A\|_p^p.$$

We deduce that the capacity of a leaf only depends on its level and is given by

$$c_p(\{\xi\}) = \|f\|_p^p = \frac{1}{|\xi|^{p-1}}.$$

Leaves are the only points in the boundary of a tree having positive capacity. To see this, let $\xi \in \partial T$ not a leaf, and suppose $c_p(\{\xi\}) = c > 0$. Then the associated equilibrium measure μ has an atom in ξ, from which $I(I^*\mu)^{p'-1} = \infty$, contradicting the properties of equilibrium functions. By subadditivity of the capacity it follows that any countable subset of ∂T has null capacity.

Example 1.2 (Capacity of the boundary of a homogeneous tree)**.** Let T^n be a homogeneous tree of degree n (i.e. a tree with constant branching number $\mathrm{br}(x) = n + 1$ for all $x \in V$) with edge root ω, and let $c(n,p) = c_p(\partial T^n)$. If f is the equilibrium function, then f^{p-1} is a flow (see (1.13)), which gives that for every $k \geq 0$ we have

$$\sum_{|\alpha|=k} |f(\alpha)|^{p-1} = f(\omega)^{p-1} = c(n,p).$$

On the other hand, for each edge function f, by Hölder inequality we have the estimate

$$\sum_{|\alpha|=k} \frac{|f(\alpha)|^{p-1}}{w(\alpha)} \leq \left(\sum_{|\alpha|=k} \frac{|f(\alpha)|^p}{w(\alpha)} \right)^{1/p'} \left(\sum_{|\alpha|=k} \frac{1}{w(\alpha)} \right)^{1/p}.$$

It follows that

$$\|f\|_p^p = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{|f(\alpha)|^p}{w(\alpha)} \geq \sum_{k=0}^{\infty} \left(\sum_{|\alpha|=k} \frac{1}{w(\alpha)} \right)^{1-p'} \left(\sum_{|\alpha|=k} \frac{|f(\alpha)|^{p-1}}{w(\alpha)} \right)^{p'}.$$

If we pick a candidate function which is constant on levels, i.e. $f(\alpha) = c(n,p)^{p'-1}/n^{k(p'-1)}$ when $|\alpha| = k$, we get equality in Hölder inequality. For any such function we clearly have that the potential If is constant on ∂T^n. If we choose properly the value at the root $f(\omega)$ we can make the constant to be 1 and f to be the equilibrium function. Imposing for $\xi \in \partial T^n$,

$$1 = If(\xi) = \sum_{\alpha<\xi} \frac{c(n,p)^{p'-1}}{w(\alpha)n^{|\alpha|(p'-1)}},$$

we get $c(n,p) = \left(\sum_{\alpha<\xi} n^{|\alpha|(1-p')}/w(\alpha)\right)^{1-p}$. In particular, in the un-weighted case $w \equiv 1$, the capacity of the boundary is

$$c(n,p) = \sum_{k=0}^{\infty} n^{k(1-p')} = \left(1 - n^{1-p'}\right)^{p-1}.$$

In section 1.4.2 we will derive the capacity of an homogeneous tree in an easier way. We conclude this chapter with a Proposition showing that in ∂T there exist compact subsets with arbitrary p–capacity.

Proposition 1.2.3. *Let T^n be a homogeneus tree of degree n. For each real number $t \in [0, c_p(\partial T^n)]$ there exists a compact subset K_t of the boundary ∂T^n such that $c_p(K_t) = t$.*

Proof. Each edge α of T^n, except the root, can be given an index $i(\alpha) \in \{0,\dots,n-1\}$ which distinguishes it from the other $n-1$ edges β such that $b(\beta) = b(\alpha)$. We can define a map $\Lambda : \partial T^n \to [0,1]$ associating to each point $\xi = \{\alpha_j\}_{j=1}^{\infty} \in \partial T^n$ the number having expansion in base n given by

$$\Lambda(\xi) = \sum_{j=1}^{\infty} i(\alpha_j) n^{-j}.$$

The map Λ is clearly onto but it fails to be injective because of the multiple representations of the rational numbers. Still, $\Lambda^{-1}(t)$ has at most two points. Moreover, Λ is continuous, since

$$|\Lambda(\xi) - \Lambda(\eta)| \le (n-1) \sum_{j=|\xi\wedge\eta|+1}^{\infty} n^{-j} \approx n^{-|\xi\wedge\eta|} \longrightarrow 0, \quad \text{as } d(\xi,\eta) \to 0.$$

Now, consider the function $\varphi : [0,1] \longrightarrow \mathbb{R}$ given by $\varphi(t) = c_p(\Lambda^{-1}[0,t])$. This is an increasing map, and we know that $\varphi(0) = 0$ and $\varphi(1) = c_p(\partial T^n)$. By the subadditivity of c_p, the continuity of Λ and the regularity of capacity, we have

$$\varphi(t+\varepsilon) - \varphi(t) \le c_p\left(\Lambda^{-1}[t, t+\varepsilon]\right) \longrightarrow c_p(\Lambda^{-1}\{t\}), \quad \text{as } \varepsilon \to 0.$$

The right handside equals zero, since the preimage of a single point under Λ is finite. By similar reasoning we estimate $\varphi(t) - \varphi(t-\epsilon)$. It follows that φ is continuous and $\varphi([0,1]) = [0, c_p(\partial T^n)]$. The result is obtained picking $K_t = \Lambda^{-1}[0, \varphi^{-1}(t)]$. ☐

1.2.2 Rescaling properties for capacities

In this section we show that equilibrium measures rescale under the change of the root in a tree, in a sense that will be more clear soon. This is a point where the behavior of trees is much simpler than the one of general graphs. We introduce a subscript notation to indicate which is the root of the tree we are referring to. For example, give the rooted tree T and some edge $\alpha \in E$, we write $I_{p,\alpha}$ for the $p-$potential operator acting on functions defined on the edges of the $\alpha-$ tent T_α, $I_{p,\alpha} f = I_p(\chi_{T_\alpha} f)$. Similarly, $c_{p,\alpha}$ will be the capacity when we consider T_α as ambient space. In the same fashion, if A is some subset of ∂T, we set $A_\alpha := A \cap \partial T_\alpha$.

A question arises: if A is a subset of ∂T and μ its $p-$equilibrium measure, which is the $p-$equilibrium measure μ_α for A_α in the tent T_α? It is natural to bet that it is a rescaling of the measure μ, i.e. $\mu_\alpha = k_\alpha \mu|_{A_\alpha}$ for some positive constant k_α. In such a case, for c_p-a.e. ξ in A_α, write M and M_α for the co-potential of μ and μ_α respectively, we would have

$$1 = I_{p,\alpha} M_\alpha(\xi) = k_\alpha^{p'-1} I_{p,\alpha} M(\xi) = k_\alpha^{p'-1}\Big(I_p M(\xi) - I_p M(b(\alpha))\Big) = k_\alpha^{p'-1}\Big(1 - I_p M(b(\alpha))\Big).$$

It follows that the only possible candidate rescaling constant is

$$k_\alpha = \Big(1 - I_p M(b(\alpha))\Big)^{1-p}. \tag{1.11}$$

We now prove that in fact our naive bet was the right one. This was already observed in [8].

Proposition 1.2.4. *Let $A \subset \partial T$, μ its $p-$equilibrium measure and $M = I^*\mu$. Then,*

$$\mu_\alpha := k_\alpha \mu|_{A_\alpha} = \frac{\mu|_{\partial T_\alpha}}{\Big(1 - I_p M(b(\alpha))\Big)^{p-1}},$$

is the $p-$equilibrium measure for $A_\alpha \subset \partial T_\alpha$.

Proof. Set $M_\alpha = I^*\mu_\alpha$. We have already seen that $(M_\alpha)^{p'-1}$ is an admissible function for $A_\alpha \subseteq \partial T_\alpha$. Suppose it is not the $p-$equilibrium function. Let φ_α be the (unique) $p-$equilibrium function of A_α in T_α. Define another edge function,

$$f = \begin{cases} \Big(1 - I_p M(b(\alpha))\Big)\varphi_\alpha & \text{on } T_\alpha \\ (M)^{p'-1} & \text{on } T \setminus T_\alpha. \end{cases}$$

Then f is admissible for $A \subseteq \partial T$, since for c_p-a.e. $\xi \in A \setminus A_\alpha$, $If(\xi) = I_p M(\xi) = 1$, while for c_p-a.e. $\xi \in A_\alpha$ it holds

$$If(\xi) = I_\alpha f(\xi) + If(b(\alpha)) = \Big(1 - I_p M(b(\alpha))\Big) I_\alpha \varphi_\alpha(\xi) + I_p M(b(\alpha)) = 1.$$

Now, using the fact that $I_pM_\alpha = k_\alpha^{p'-1}I_pM$, the relation $p(p'-1) = p'$ and the uniqueness of the p–equilibrium function, we get

$$\begin{aligned}\|f\|_p^p &= \Big(1 - I_pM(b(\alpha))\Big)^p \|\varphi_\alpha\|^p_{\ell^p(T_\alpha)} + \|(M)^{p'-1}\|^p_{\ell^p(T\smallsetminus T_\alpha)} \\ &< \|(M)^{p'-1}\|^p_{\ell^p(T_\alpha)} + = \|(M)^{p'-1}\|^p_p.\end{aligned}$$

This clearly contradicts the assumption that μ is the p–equilibrium measure for A, since we found an admissible function whose p–norm is strictly smaller than the one of $M^{p'-1}$. □

As an immediate consequence we have the following.

Corollary 1.2.5. *Let $A \subseteq \partial T$ be a set of positive capacity and $\alpha \in E$, α not the root edge of T, such that $A \subseteq \partial T_\alpha$. Then $c_{p,\alpha}(A) > c_p(A)$.*

Proof. Since $k_\alpha > 1$, we have $c_{p,\alpha}(A) = \mu_\alpha(A) = k_\alpha\mu(A) > \mu(A) = c_p(A)$. □

Observe by passing that the above corollary is supported by visual intuition, since we expect the capacity of a set to be bigger if we look at the set from a closer point of view. We can now give a necessary condition for a measure to be of equilibrium.

Theorem 1.2.6. *Let μ be the p–equilibrium measure of some set $A \subseteq \partial T$ and denote by M its co-potential. Then, for every $\alpha \in E$, M solves the following equation:*

$$M(\alpha)\Big(1 - I_pM(b(\alpha))\Big) = \sum_{\beta\geq\alpha} \frac{|M(\beta)|^{p'}}{w(\beta)}. \tag{1.12}$$

Proof. Given $\alpha \in E$, if $A_\alpha = \varnothing$ then $M(\alpha) = 0$ (since $\mathrm{supp}(\mu) \subseteq A$) and hence $M(\beta)^{p'-1} = 0$ for all $\beta \geq \alpha$, and (1.12) trivially holds. Otherwise, on one hand we have

$$c_{p,\alpha}(A_\alpha) = \mathcal{E}_{p,\alpha}(\mu_\alpha) = \frac{\mathcal{E}_{p,\alpha}(\mu)}{\Big(1 - I_pM(b(\alpha))\Big)^p},$$

and on the other, since $\mathrm{supp}(\mu) \subseteq A$,

$$c_{p,\alpha}(A_\alpha) = \mu_\alpha(A_\alpha) = k_\alpha\mu(A_\alpha) = \frac{\mu(\partial T_\alpha)}{\Big(1 - I_pM(b(\alpha))\Big)^{p-1}} = \frac{M(\alpha)}{\Big(1 - I_pM(b(\alpha))\Big)^{p-1}}.$$

Matching the two expressions we get the result. □

From now on we will refer to equation (1.12) as the *equilibrium equation.* Observe that this equation is non linear (and non local) even in Linear Potential Theory. This is not surprising, since linear combinations of equilibrium measures are only seldom equilibrium measures themselves.

1.2.3 Flows and p–harmonic functions on the tree

Let $T = (E, V)$ be the rooted tree. A function $f : E \to \mathbb{R}$ is a *flow* from o if

$$f(\alpha) = \sum_{\beta \in s(\alpha)} f(\beta), \quad \text{for all } \alpha \in E. \tag{1.13}$$

It is immediate that the co-potential of a charge defines a flow. Next proposition characterizes flows that can be obtained as co-potentials of charges.

Proposition 1.2.7. *A flow $f : E \to \mathbb{R}$ satisfies*

$$\lim_k \sum_{|\alpha|=k} |f(\alpha)| < \infty, \tag{1.14}$$

if and only if there exists a (unique) charge μ on ∂T such that $f = I^*\mu$.

Proof. Note that the limit in condition (1.14) is in fact a supremum. If f is a flow, for every integer $k \geq 0$ we have

$$\sum_{|\alpha|=k+1} |f(\alpha)| = \sum_{|\alpha|=k} \sum_{\beta \in s(\alpha)} |f(\beta)| \geq \sum_{|\alpha|=k} |f(\alpha)|,$$

from which it follows

$$\sup_k \sum_{|\alpha|=k} |f(\alpha)| = \lim_{k\to\infty} \sum_{|\alpha|=k} |f(\alpha)|.$$

For each α write $\xi(\alpha)$ for an arbitrary point in ∂T_α. For each $k \in \mathbb{N}$, define a charge,

$$\mu_k = \sum_{|\alpha|=k} f(\alpha)\delta_{\xi(\alpha)}.$$

The total variation of μ_k is given by

$$\|\mu_k\| = \sum_{|\alpha|=k} |f(\alpha)\delta_{\xi(\alpha)}(\partial T)| = \sum_{|\alpha|=k} |f(\alpha)|,$$

and from (1.14) it follows that the family of charges μ_k is uniformly bounded, so that it has a weak*-limit point μ which is a positive measure. For each edge α we have

$$I^*\mu(\alpha) = \mu(\partial T_\alpha) = \int_{\partial T} \chi_{\partial T_\alpha} d\mu = \lim_k \int_{\partial T} \chi_{\partial T_\alpha} d\mu_k = \lim_k \mu_k(\partial T_\alpha) = f(\alpha).$$

For the uniqueness part, if $f = I^*\nu$ for some other charge ν, then $\nu(\partial T_\alpha) = \mu(\partial T_\alpha)$ for each $\alpha \in E$ and hence $\mu \equiv \nu$. Conversely, let μ be a charge on ∂T and consider the flow $f = I^*\mu$. Then

$$\sum_{|\alpha|=k} |f(\alpha)| = \sum_{|\alpha|=k} |\mu^+(\partial T_\alpha) - \mu^-(\partial T_\alpha)| \leq \sum_{|\alpha|=k} |\mu|(\partial T_\alpha) = \|\mu\| < \infty.$$

□

Remark 2. Observe that if $f \geq 0$ then condition (1.14) is automatically satisfied, so that non-negative flows coincides with co-potentials of positive Borel measures.

Proposition 1.2.8. *A function $f : E \to \mathbb{R}$ is a flow if and only if $I_p f$ is a $p-$harmonic function on $V \setminus \{o\}$.*

Proof. Let $\alpha \in E$ and $x = e(\alpha)$. Then,

$$\begin{aligned}\Delta_p I_p f(x) &= w(\alpha)\Big(I_p f(b(\alpha)) - I_p f(e(\alpha))\Big)\Big|I_p f(b(\alpha)) - I_p f(e(\alpha))\Big|^{p-1} \\ &+ \sum_{\beta\in s(\alpha)} \Big(I_p f(e(\beta)) - I f_p(b(\beta))\Big)\Big|I_p f(e(\beta)) - I f_p(b(\beta))\Big|^{p-1} \\ &= -f(\alpha) + \sum_{\beta\in s(\alpha)} f(\beta).\end{aligned}$$

It follows that $\Delta_p I_p f \equiv 0$ on $V \setminus \{o\}$ if and only if (1.13) holds. □

By the above Proposition, we deduce that an easy way to produce a $p-$harmonic function on the tree is to start with a charge μ on the boundary, and take to $p-$potential of its co-potential. Next result, combining the last two propositions, completely characterizes the $p-$ harmonic functions that can be realized in terms of measures.

Corollary 1.2.9. *A function g which is $p-$harmonic on $V \setminus \{o\}$ satisfies*

$$\sup_k \sum_{|\alpha|=k} |\nabla g(\alpha)|^{p-1} < \infty, \tag{1.15}$$

if and only if there exists a charge μ such that $g = I_p(I^*\mu)$.

Proof. A function g satisfies (1.15) if and only if $f := \nabla g\,|\nabla g|^{p-2}$ satisfies (1.14) and, by Proposition 1.2.8, $g = I_p f + g(o)$ is $p-$harmonic on $V \setminus \{o\}$ if and only if f is a flow. By Proposition 1.2.7 we have the claim.

□

1.3 The Dirichlet problem on the rooted tree

1.3.1 A probabilistic interpretation of capacity

In this section we give a characterization of the $2-$capacity of the boundary of the rooted tree (with edge weight ω) in terms of behaviour of random walks on it. In this context we consider the root $o = b(\omega)$ as part of the extended boundary $\overline{\partial T} = \partial T \cup \{o\}$, and we consider a random walk (Z_n), with transition probability π induced by the weight ω, on the vertices of the rooted tree T, which *stops* when it hits the root vertex o. The random walk (Z_n) so defined is not a priori transient on V, however it is on $V \setminus \{o\}$, in the sense that it reaches the (extended) boundary with probability one. Hence, there exists a $\overline{\partial T}$–valued random variable Z_∞ such that Z_n converges to Z_∞, $\mathbb{P}_x$–almost surely for every $x \in T$, where $\mathbb{P}_x$ is the probability measure of the random walk starting at $x \in V$. Of course, in this context where we consider the extended boundary and the its first hitting time to be the stopping time, we define the harmonic measure at $x \in V$ as $\lambda_x(A) := \mathbb{P}_x(Z_\infty \in A)$, where A is a Borel subset of $\overline{\partial T}$.

Proposition 1.3.1. *Let T be the rooted tree, with edge weight . Then the $2-$capacity of ∂T equals the probability that a simple random walk starting at $e(\omega)$ will escape to the boundary before hitting the root vertex o. Formally,*

$$c_2(\partial T) = \mathbb{P}_{e(\omega)}\Big(Z_\infty \in \partial T\Big) = \lambda_{e(\omega)}(\partial T).$$

Proof. Suppose that we have a finite tree whose leaves (except the root o) are all of level $n > 0$. We can naturally identify ∂T these leaves. Then by the Markov property the function $h(x) = \lambda_x(\partial T)$ is harmonic in $\setminus\{o\}$, $h(\xi) = 1$ if $\xi \in \partial T$ and $h(o) = 0$. Since the same is true for $If^{\partial T}$, the potential of the equilibrium function of the boundary of T, by the maximum principle (1.1.1) $If^{\partial T} = h$. Then, $c_2(\partial T) = f^{\partial T}(\omega) = \nabla h(\omega) = h(e(\omega))$ and we have the result of the finite tree.

For a general tree T not necessarily finite , let T_n be the truncation of T up to level n. Then from the finite case we have that $c_2(\partial T_n) = \mathbb{P}_{e(\omega)}(\sup_i |Z_i| \geq n)$. By monotonicity of measures the last quantity converges to $\mathbb{P}_{e(\omega)}(Z_\infty \in \partial T)$, as $n \to +\infty$. It remains to show that $c_2(\partial T_n) \longrightarrow c_2(\partial T)$, as $n \to \infty$. By definition of capacity we get that $c_2(\partial T_n) \geq c_2(\partial T)$, since the equilibrium function $f^{\partial T_n}$ is an admissible function for ∂T (extend it to be zero on edges of level greater than n.)

To prove the other inequality we use the dual expression for capacity. Fix $\varepsilon > 0$. By the dual definition of capacity, there exists a positive measure μ_n on ∂T_n such that $\mathcal{E}_2(\mu_n) = \sum_{|\alpha| \leq n} I^*\mu_n(\alpha)^2 \leq 1$ and $\mu_n^2(\partial T_n) = c_2(\partial T_n) - \varepsilon$. Consider now the corresponding charges on ∂T,

$$\widetilde{\mu}_n := \sum_{|\alpha|=n} \mu_n(\partial T_\alpha)\delta_{\xi(\alpha)},$$

where $\xi(\alpha)$ is an arbitrary point in ∂T_α and $\delta_{\xi(\alpha)}$ the corresponding Dirac mass. Since $\widetilde{\mu}_n(\partial T) = \mu_n(\partial T_n) \leq 1$ we can find a weak*-limit point μ of the sequence

$\{\widetilde{\mu}_n\}$. For any $m \in \mathbb{N}$, we have that

$$\begin{aligned}\sum_{|\beta|\leq m} I^*\mu(\beta)^2 &= \lim_n \sum_{|\beta|\leq m} I^*\mu_n(\beta)^2 \\ &= \lim_n \sum_{|\beta|\leq m} I^*\widetilde{\mu}_n(\beta)^2 \\ &\leq \lim_n \sum_{|\beta|\leq n} I^*\mu_n(\beta)^2 \\ &\leq 1.\end{aligned}$$

Therefore, letting $m \to \infty$ we get that $\mathcal{E}_2(\mu) \leq 1$, and hence by the dual definition of capacity,

$$c_2(\partial T) \geq \mu(\partial T)^2 = \lim_n \widetilde{\mu}_n(\partial T)^2 = \lim_n \mu_n(\partial T_n)^2 = \lim_n c_2(\partial T_n) - \varepsilon.$$

Letting $\varepsilon \to 0$ we get the result. □

Observe that as an immediate consequence we have a characterization of transient random walks (in the classical sense introduced in section 1.1.1) in terms of capacity. This characterization was first proved by different methods in [9].

Corollary 1.3.2. *A random walk (Z_n) on the rooted tree T is transient if and only if ∂T has positive 2-capacity.*

In particular, the rooted tree is transient, namely the simple random walk is, if and only if the un-weighted capacity of its boundary is positive. This information is not empty, since there are non trivial trees having boundary of capacity zero. An example is provided by the so called 1-3 tree, see [28, Example 1.2].

1.3.2 Irregular points and the Dirichlet problem

in this section we provide an algorithm to detect irregular points in the boundary of the rooted tree T. We use here the same subscript notation introduced in section 1.2.2. The following Theorem can be seen as the analogous for trees of the classical Wiener test for irregular points (see [22, Theorem 7.1]).

Theorem 1.3.3. *A boundary point ξ is irregular for a set $A \subseteq \partial T$ of positive capacity if and only if*

$$\sum_{E\ni\alpha<\xi} c_{\alpha,p}(A_\alpha)^{p'-1} < \infty. \tag{1.16}$$

Proof. Let μ be the equilibrium measure for A, and M its co-potential. Set $\varepsilon := 1 - I_pM(\xi) \geq 0$ to be the deficit of regularity of the point $\xi \in A$. Let $\{\alpha_j\}$ be the edge-geodesic labeled by ξ, and set $t_n = \sum_{j\geq n} M(\alpha_j)^{p'-1}$. Clearly t_n is monotonically decreasing to zero, being the tail of the converging sum $I_pM(\xi)$. By Theorem 1.2.6,

$$c_n := c_{\alpha_n,p}(A_{\alpha_n})^{p'/p} = \frac{M(\alpha_n)^{p'-1}}{1 - I_pM(b(\alpha_n))} = \frac{M(\alpha_n)^{p'-1}}{\varepsilon + t_n} = \frac{t_n - t_{n+1}}{\varepsilon + t_n}.$$

Now, the sum $\sum_n c_n$ converges if and only if $\prod_n(1 - c_n) > 0$. The partial product can be explicitly calculated thanks to its telescopic structure,

$$\prod_{n=0}^{N}(1 - c_n) = \prod_{n=0}^{N} \frac{\varepsilon + t_{n+1}}{\varepsilon + t_n} = \frac{\varepsilon + t_{N+1}}{\varepsilon + t_0}.$$

Since $t_0 = \mu(\partial T)^{p'-1} > 0$, it follows that $\prod_{n=0}^{\infty}(1 - c_n) > 0$ if and only if $\varepsilon > 0$, which is, if and only if the point ξ is irregular. □

Observe that the Wiener condition (1.16) can be re-written purely in terms of capacities on the whole boundary, in the following sense.

Corollary 1.3.4. *A boundary point ξ is irregular for a set $A \subseteq \partial T$ of positive capacity if and only if*

$$\sum_{\alpha<\xi} \frac{c_p(A_\alpha)^{p'-1}}{1 - |\alpha| c_p(A_\alpha)^{p'-1}} < \infty.$$

Proof. Let T be any rooted tree, $A \subseteq \partial T$ and consider a tent T_α, with $|\alpha| = n$. If μ is the equilibrium measure for A_α, then the associated co-potential $M = I^*\mu$ is supported on the edges of T_α and the on the edges $\beta \leq \alpha$. Being a flow, M must be constant on the predecessors of α, namely $M(\beta) = M(\omega) = c_p(A_\alpha)$, for all $\beta \leq \alpha$. By the rescaling properties of section 1.2.2, we know that

$$M(\alpha) = c_{p,\alpha}(E_\alpha)\Big(1 - I_pM(b(\alpha))\Big)^{p-1},$$

and since $I_pM(b(\alpha)) = nc_p(A_\alpha)^{p'-1}$, we obtain

$$\begin{aligned} c_p(A_\alpha) &= \mathcal{E}_p(\mu) \\ &= nc_p(A_\alpha)^{p'} + \mathcal{E}_{p,\alpha}(\mu) \\ &= nc_p(A_\alpha)^{p'} + \Big(1 - nc_p(A_\alpha)^{p'-1}\Big)^p c_{p,\alpha}(A_\alpha). \end{aligned}$$

It follows,

$$c_{p,\alpha}(A_\alpha) = \frac{c_p(A_\alpha)}{\Big(1 - nc_p(A_\alpha)^{p'-1}\Big)^{p-1}}.$$

Substituting this expression in the Wiener condition (1.16) we get the result. □

Theorem 1.3.5. *Let $T = (V, E)$ be the rooted tree, which comes with an edge weight and an associated reversible transition probability. For any given $\varphi \in C(\partial T)$, the Poisson integral of φ satisfies*

$$\begin{cases} \Delta \mathcal{P}(\varphi) = 0 \ \ in V \setminus \{o\} \\ \lim_{x \to \xi} \mathcal{P}(\varphi)(x) = \varphi(x), \ \ if \ \xi \ \ is\ a\ regular\ point\ of\ \overline{\partial T}. \end{cases}$$

Proof. Pick $\alpha \in E$ and let $\xi = \{x_j\}_{j=0}^{\infty} \in \partial T_\alpha$. Write $\{\alpha_j\}_{j=1}^{\infty}$ for the edge geodesic labeled by ξ, so that $x_j = \alpha_j(1)$. For $n \geq |\alpha|$ we have

$$\begin{aligned} 0 \leq 1 - \lambda_{x_n}(\partial T_\alpha) &\leq \mathbb{P}_{x_n}(Z_n \text{ hits } b(\alpha) \text{ before hitting } \partial T_\alpha) \\ &= \prod_{j=|\alpha|}^{n} \mathbb{P}_{x_j}(Z_n \text{ hits } x_{j-1} \text{ before hitting } \partial T_{\alpha_j}) \\ &= \prod_{j=|\alpha|}^{n} \big(1 - c_{2,\alpha_j}(\partial T_{\alpha_j})\big). \end{aligned}$$

By the Wiener condition (1.16) we have that the right handside vanishes as $n \to +\infty$ if and only if ξ is a regular point for ∂T. Hence, for any regular point $\xi \in \partial T_\alpha$ we have $\lim_{n\to\infty} \lambda_{x_n}(\partial T_\alpha) = 1$, from which it follows that for any regular point ξ in the boundary

$$\lim_{n\to\infty} \lambda_{x_n}(\partial T_\alpha) \geq \delta_\xi(\partial T_\alpha).$$

Any open set $A \subseteq \partial T$ can be written as a disjoint union of tents $\{\partial T_{\alpha_k}\}$. Let ξ be a regular point of ∂T. Then,

$$\liminf_n \lambda_{x_n}(A) = \liminf_n \sum_k \lambda_{x_n}(\partial T_{\alpha_k}) \geq \sum_k \liminf_n \lambda_{x_n}(\partial T_{\alpha_k}) \geq \sum_k \delta_\xi(\partial T_{\alpha_k}) = \delta_\xi(A).$$

It follows that $\lambda_{x_n} \xrightarrow{w^*} \delta_\xi$, as $n \to \infty$. Therefore, for any $\varphi \in C(\partial T)$,

$$\mathcal{P}(\varphi)(x) = \int_{\partial T} \varphi \, d\lambda_x \longrightarrow \varphi(\xi), \quad \text{as } x \to \xi.$$

□

Corollary 1.3.6 (Kellog's Theorem for Trees)**.** *The set of irregular points for the Dirichlet problem has capacity zero. Which is,*

$$c_2\big(\{\xi \in \partial T: \ \textit{there exists}\ \varphi \in C(\partial T), \lim_{x\to\xi} \mathcal{P}(\varphi)(x) \neq \varphi(\xi)\}\big) = 0.$$

1.3.3 Energy conditions and uniqueness results

We give a first uniqueness result for the class of *spherically symmetric trees*, which are trees where the branching number is radial (i.e. constant on levels). Clearly homogeneous trees belong to this class.

We define the Lebesgue measure on ∂T to be the measure λ which is equidistributed among sons of any edge and is normalized with $\lambda(\partial T) = 1$. Namely, for each $\alpha \in E$, we have

$$\lambda(\partial T_\alpha) = \lambda(\partial T_{p(\alpha)}) / \deg\left(e(p(\alpha))\right) = 1 / \prod_{\beta < alpha} \deg(e(\beta)).$$

It is clear that on spherically symmetric trees $I^*\lambda$ is constant on levels. In what follows, we write $\lambda(k)$ in place of $\lambda(\partial T_\beta)$ when $|\beta| = k$. One can check that the equilibrium measure of a spherically symmetric tree is a scalar multiple of the Lebesgue measure (see section 1.3.3). In particular, by symmetry, spherically symmetric trees have no irregular boundary points.

Proposition 1.3.7. *Suppose T is a spherically symmetric tree with radial edge weights, $w(\alpha) = w(|\alpha|)$. Assume $c_2(\partial T) > 0$ and let μ be a charge on ∂T. Denote by M its potential. If $IM = 0$ Lebesgue almost everywhere on the boundary, then $\mu \equiv 0$.*

Proof. Let w be the edge weight of the tree. For a fixed $\alpha \in E$, let $s(\alpha) = \{\alpha_j\}_{j=1}^{n(\alpha)}$ and define the measures λ^{α_j} on ∂T in the following way

$$I^*\lambda^{\alpha_j}(\gamma) = \begin{cases} n(\alpha) I^*\lambda(\gamma) & \text{if } \gamma \geq \alpha_j \\ 0 & \text{if } \gamma \geq \alpha_i,\ i \neq j \\ I^*\lambda(\gamma), & \text{otherwise.} \end{cases}$$

It is clear that λ^{α_j} is absolutely continuous with respect to λ. Integrating on the α−tent, using the fact that $\lambda(\partial T_\beta)$ depends only on the level of β, for each j we get

$$
\begin{aligned}
0 &= \int_{\partial T_\alpha} IM d\lambda^{\alpha_j} \\
&= \int_{\partial T_\alpha} \sum_{\beta\in E} \frac{M(\beta)}{w(\beta)} \chi_{\partial T_\beta}(\xi) d\lambda^{a_j}(\xi) \\
&= \sum_{\beta\in E} \frac{M(\beta)}{w(\beta)} \lambda^{\alpha_j}(\partial T_\alpha \cap \partial T_\beta) \\
&= \lambda(\partial T_\alpha) \sum_{\beta<\alpha} \frac{M(\beta)}{w(\beta)} + n(\alpha) \sum_{\beta\geq\alpha_j} \lambda(\partial T_\beta) \frac{M(\beta)}{w(\beta)} \\
&= \lambda(\partial T_\alpha) \sum_{\beta<\alpha} \frac{M(\beta)}{w(\beta)} + n(\alpha) \sum_{k=|\alpha_j|}^{\infty} \frac{\lambda(k)}{w(k)} \sum_{\substack{\beta\geq\alpha_j,\\ |\beta|=k}} M(\beta) \\
&= \lambda(\partial T_\alpha) \sum_{\beta<\alpha} \frac{M(\beta)}{w(\beta)} + n(\alpha) M(\alpha_j) \sum_{k=|\alpha|+1}^{\infty} \frac{\lambda(k)}{w(k)}.
\end{aligned}
$$

Note that the last quantity is finite because the capacity of the boundary is positive. Being the same true for each j, M must be constant on $s(\alpha)$. It follows that $\mu = M(\omega)\lambda$, i.e. the measure μ is a scalar multiple of the Lebesgue measure. Hence, by (1.10), we have

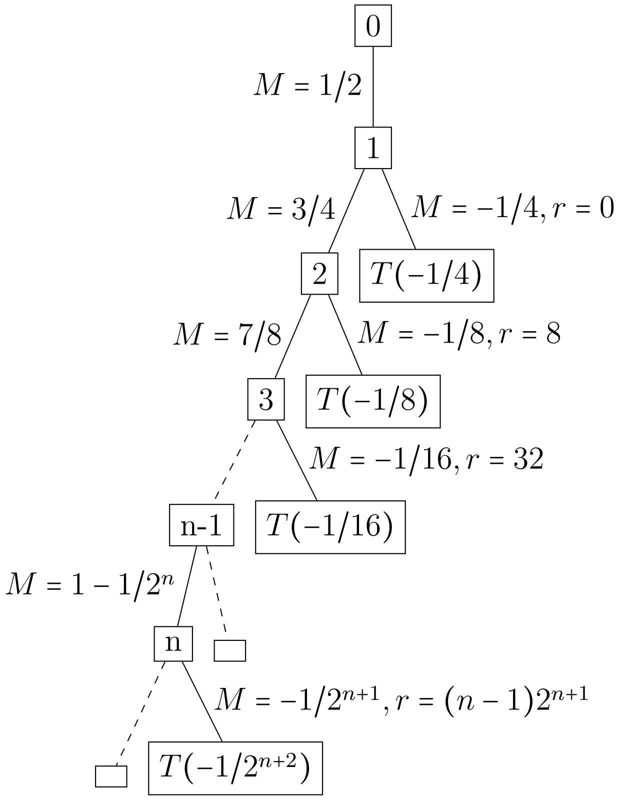
$$
0 = \int_{\partial T} IM d\mu = \sum_{\alpha\in E} \frac{M(\alpha)^2}{w(\alpha)},
$$

which gives $M \equiv 0$ on E which is the thesis.

□

The same is not true for a general tree. In fact, there exists a sub-dyadic tree T, with no irregular boundary points, and a charge μ on ∂T such that $I(I^*\mu) = 0$ everywhere except at a point, but $\mu \neq 0$, as shown in the next example.

Example 1.3. The following diagram represents an infinite sub-dyadic tree, and the values that the co-potential M of a boundary charge μ take on each edge. The number r over an edge indicates how many times the edge is repeated. Also, the label $T(a)$ means that the vertex is the root vertex of a dyadic tree which carries a total measure of a on the boundary, and the measure M is divided equally at each edge.



If ξ_0 is the leftmost point of the boundary it is clear that $IM(\xi_0) = +\infty$. If $\xi \in \partial T \smallsetminus \{\xi_0\}$ let $n = |\xi \wedge \xi_0|$. Consider the un-weighted case. Then,

$$\begin{aligned} IM(\xi) &= \sum_{i=1}^{n}\left(1-2^{-i}\right) - \frac{(n-1)2^{n+1}}{2^{n+1}} - \frac{1}{2^{n+1}}\sum_{i=0}^{\infty}2^{-i} \\ &= n - \sum_{i=1}^{n}2^{-i} - (n-1) - 2^{-n} = 0. \end{aligned}$$

By applying Wiener's test one can see that ξ_0 is a regular point of the boundary, while all other points are clearly regular by symmetry.

The situation is different if we work with charges with finite energy.

Proposition 1.3.8. *Let μ, ν be charges on ∂T, with $\mathcal{E}_p(\mu)$, $\mathcal{E}_p(\nu) < \infty$. Denote by M and V their potentials, respectively. If $I_pM = I_pV$ both μ-a.e. ad ν-a.e., then $\mu \equiv \nu$.*

Proof. Integrating both the p−potentials I_pM and I_pV with respect to both the measures and recalling the relation (1.10), we can get any of the following equalities

$$\mathcal{E}_p(\mu) = \int_{\partial T} I_pM\,d\nu = \int_{\partial T} I_pV\,d\mu = \mathcal{E}_p(\nu).$$

Setting $f(\alpha) = M(\alpha)|M(\alpha)|^{p'-2}$ and $g(\alpha) = V(\alpha)|V(\alpha)|^{p'-2}$, by (1.9), we can write

$$\begin{aligned} 0 &= \mathcal{E}_p(\mu) + \mathcal{E}_p(\nu) - \langle If, \nu\rangle_{L^2(\partial T)} - \langle Ig, \mu\rangle_{L^2(\partial T)} \\ &= \mathcal{E}_p(\mu) + \mathcal{E}_p(\nu) - \langle f, V\rangle_{\ell^2(E)} - \langle g, M\rangle_{\ell^2(E)} \\ &= \sum_{\alpha\in E} \frac{1}{w(\alpha)}\Big(f(\alpha)M(\alpha) + g(\alpha)V(\alpha) - f(\alpha)V(\alpha) - g(\alpha)M(\alpha)\Big) \\ &= \sum_{\alpha\in E} \frac{1}{w(\alpha)}(M - V)(f - g)(\alpha). \end{aligned}$$

It is now clear that the general term of the above series is positive, from which $M \equiv V$ on E. □

It is clear from the dual definition of capacity that if a property holds c_p–a.e. then it also holds μ-a.e. with respect to any charge μ of finite energy. Hence, the above result can be restated in the following slightly less general but more natural form.

Corollary 1.3.9. *Given charges μ, ν on ∂T, with $\mathcal{E}_p(\mu)$, $\mathcal{E}_p(\nu) < \infty$, if $I_pM = I_pV$, c_p–a.e. on ∂T, then $\mu \equiv \nu$.*

As a consequence, we have a partial converse of the properties of equilibrium measures given in Theorem 1.2.2.

Corollary 1.3.10. *Let μ be a Borel measure on ∂T such that $\mathcal{E}_p(\mu) < \infty$ and $I_pM = 1$ c_p–a.e. on $A = \mathrm{supp}(\mu)$. Then μ is the p–equilibrium measure for A.*

These uniqueness results can be reinterpreted in terms of functions in place of measures. The following Sobolev space naturally arises from the space of charges of finite energy,

$$W^{1,p}(T) := \{g : V \to \mathbb{R} : \ \nabla g \in \ell^p\}.$$

In fact, given a charge μ on ∂T, denoting by M its co-potential, we have

$$\mathcal{E}_p(\mu) = \|M|M|^{p'-2}\|_p^p = \|\nabla I_pM\|_p^p, \tag{1.17}$$

so that the following proposition is self evident.

Proposition 1.3.11. *Let μ be a charge on ∂T. Then $\mathcal{E}_p(\mu) < \infty$ if and only if $I_pM \in W^{1,p}(T)$.*

In terms of functions, Corollary 1.3.9 reads as follows.

Theorem 1.3.12. *Let g, h be p–harmonic functions in $W^{1,p}(T)$ satisfying* (1.15). *If $g = h$, c_p–almost everywhere on ∂T, then $g \equiv h$ on V.*

Proof. Glue together Corollary 1.2.9, Proposition 1.3.11 and Corollary 1.3.9. □

If the tree T is spherically symmetric, by Proposition 1.3.7 we know that in the linear case $p = 2$ we don't need the energy condition $g, h \in W^{1,2}(T)$ in the statement.

1.3.4 Boundary values

The uniqueness results we presented above are ment to apply to functions admitting boundary values c_p–almost everywhere. A priori it is not obvious which functions enjoy this property. Here we prove that in fact all the functions in the Sobolev spaces for which we have uniqueness results indeed admit boundary values c_p–almost everywhere. Other results of this kind can be found in [14]. To prove the existence of boundary values we follow an approach exploiting Carleson measures, which was already presented in [7] for the linear case $p = 2$. We include here the argument adapted for general Sobolev spaces. We restrict calculations in this section to the un-weighted case for semplicity.

Let $T = (V, E)$ be a rooted tree. We say that μ is a measure on $\overline{T} := V \cup \partial T$ if $\mu\big|_V$ is a function on vertices and $\mu\big|_{\partial T}$ is a measure on the boundary. Observe that if $\mu\big|_V(b(\alpha)) = I^*(\mu|_{\partial T})(\alpha)$, then it defines a measure which is not finite.

Definition 1.3.1. We say that a Borel measure μ on $\overline{T}$ is a *Carleson measure* for $W^{1,p}$ if there exists a constant $C(\mu) > 0$ such that for all $g \in W^{1,p}$

$$\int_{\overline{T}} |g|^p d\mu \le C(\mu) \|g\|_{1,p}^p. \tag{1.18}$$

These measures have been widely studied and characterized (even in the weighted case), see for example [6], [4] and [8]. In [6] it is shown that condition (1.18) can be reformulated purely in terms of the measure μ. In fact, it is shown that it is equivalent to

$$\sum_{\beta \ge \alpha} \left(\int_{\overline{T}_\beta} d\mu \right)^p \le C(\mu) \int_{\overline{T}_\alpha} d\mu, \quad (\alpha \in E). \tag{1.19}$$

Denote by $\|\mu\|_{CM}$ the best possible constant in (1.19), which for μ fully supported on ∂T reduces to

$$\|\mu\|_{CM} = \sup_{\alpha \in E} \frac{\mathcal{E}_{p,\alpha}(\mu)}{I^*\mu(\alpha)}.$$

Observe that if μ is the equilibrium measure for some set $A \subseteq \partial T$, by (1.12) it follows that, for every edge α, $\mathcal{E}_{p,\alpha}(\mu)/M(\alpha) \le 1$, with equality for $\alpha = \omega$. Hence,

$\|\mu\|_{CM} = 1$ and $c_p(A) = \mu(A)/\|\mu\|_{CM}^{p-1}$. On the other hand, for any μ supported in $A \subseteq \partial T$ we have the bound $\mathcal{E}_p(\mu) \leq \|\mu\|_{CM}\mu(A)$, from which

$$\frac{\mu(A)}{\|\mu\|_{CM}^{p-1}} \leq \frac{\mu(A)^p}{\mathcal{E}_p(\mu)^{p-1}} \leq c_p(A),$$

where the last inequality follows from the fact that the measure $\mu/\mathcal{E}_p(\mu)^{p-1}$ is admissible. We have derived the following expression of p–capacity in terms of Carleson measures of $W^{1,p}$ spaces:

$$c_p(A) = \sup\left\{\frac{\mu(A)}{\|\mu\|_{CM}^{p-1}} : \text{ supp}(\mu) \subseteq A\right\}. \tag{1.20}$$

The following proposition shows that the Fatou's set of a $W^{1,p}$ function differs from the boundary of the tree at most for a set of null capacity. The argument is taken by [7], where the result is proved for $p = 2$.

Proposition 1.3.13. *Functions in $W^{1,p}$ have boundary values c_p–a.e. on ∂T*

Proof. For $g \in W^{1,p}$ (that without loss of generality we normalize to $g(o) = 0$), define the sequence of functions $g_n^* := I\left(|\nabla g|\chi_{|\alpha|\leq n}\right)$. It is clear that g_n^* is pointwise non-decreasing and it extends to the boundary by continuity, being eventually constant. By monotonicity we have that the function $g^*(\xi) = \lim_n g_n^*(\xi)$ is well defined for every $\xi \in V \cup \partial T$. Moreover, we have the uniform bound $\|g^*\|_{1,p} \leq \|g\|_{1,p}$. Now, let μ be a Carleson measure for $W^{1,p}$ and denote by M its co-potential. By Fatou's Lemma we have

$$\begin{aligned}
\int_{\partial T} g^*(\xi)^p d\mu(\xi) &\leq \liminf_n \int_{\partial T} g_n^*(\xi)^p d\mu(\xi) \\
&= \liminf_n \sum_{|\alpha|=n} \int_{\partial T_\alpha} g_n^*(\xi)^p d\mu(\xi) \\
&= \sum_{|\alpha|=n} g(e(\alpha))^p \mu(\partial\alpha) \\
&\leq C(\mu)\|g\|_{1,p}.
\end{aligned}$$

This implies that $g^* \in L^\infty(d\mu)$, and since g^* is a bound for the radial variation of g along geodesics, by Dominated Convergence Theorem we deduce that g admits radial limit μ-a.e. on ∂T for every Carleson measure μ. In particular, the equilibrium measure μ^A of the set $A = \partial T \setminus \mathcal{F}(g)$ is a Carleson measure since $\|\mu^A\|_{CM} = 1$, from which follows that the radial limit exists c_p–a.e. □

1.4 The equilibrium problem

1.4.1 A characterization for equilibrium measures

In this section, we prove one of our main results, which, together with Theorem 1.2.6, provides a characterization of p–equilibrium measures on the rooted tree. .

Theorem 1.4.1. *Let μ be a non-negative measure on ∂T. If $M = I^*\mu$ solves the equilibrium equation* (1.12)*, namely*

$$M(\alpha)\Big(1 - I_p M(b(\alpha))\Big) = \sum_{\beta \geq \alpha} \frac{|M(\beta)|^{p'}}{w(\beta)}, \quad \text{for every } \alpha \in E,$$

then, there exists an $\mathcal{F}_\sigma$–set $A \subseteq \partial T$ such that μ is the equilibrium measure of A.

Proof. First of all, observe that a measure μ solving (1.12) has finite p–energy, since $\mathcal{E}_p(\mu) = \mu(\partial T)$. In order to guarantee signs accordance in (1.12), it must be $I_p M(b(\alpha)) \leq 1$ for each $\alpha \in E$. It follows that $I_p M(\xi) \leq 1$ for each $\xi \in \partial T$, being it the limit of the bounded sequence of its partial sums. We show that indeed $I_p M = 1$ μ-a.e. on ∂T.

Let $E_N = \{\alpha \in E : |\alpha| = N\}$. To each $N \in \mathbb{N}$ we associate a piecewise-constant function Φ_N on the boundary, $\Phi_N(\xi) = (1 - I_p M(b(\alpha)))$ for $\xi \in \partial T_\alpha$, $\alpha \in E_N$. Then we have

$$0 \leq \int_{\partial T} \Phi_N d\mu = \sum_{\alpha \in E_N} \int_{\partial T_\alpha} \Phi_N d\mu = \sum_{\alpha \in E_N} (1 - I_p M(b(\alpha)))M(\alpha) = \sum_{\alpha \in E_N} \mathcal{E}_{p,\alpha}(\mu).$$

Since

$$+\infty > \mathcal{E}_p(\mu) = \sum_{\beta \in E} \frac{|M(\beta)|^{p'}}{w(\beta)} = \sum_{|\beta| < N} \frac{|M(\beta)|^{p'}}{w(\beta)} + \sum_{|\beta| \geq N} \frac{|M(\beta)|^{p'}}{w(\beta)},$$

and

$$\sum_{|\beta| \geq N} \frac{|M(\beta)|^{p'}}{w(\beta)} = \sum_{\alpha \in E_N} \sum_{\beta \geq \alpha} \frac{|M(\beta)|^{p'}}{w(\beta)} = \sum_{\alpha \in E_N} \mathcal{E}_{p,\alpha}(\mu),$$

it follows that

$$\int_{\partial T} \Phi_N d\mu \longrightarrow 0, \text{ as } N \to +\infty.$$

Also, for each $\xi \in \partial T$, $\Phi_N(\xi) \searrow \Phi(\xi) := 1 - I_p M(\xi) \geq 0$ as $N \to +\infty$. Hence, by monotone convergence theorem, we obtain

$$\int_{\partial T} \Phi(\xi) d\mu(\xi) = \int_{\partial T} (1 - I_p M(\xi)) d\mu(\xi) = 0,$$

from which follows $I_p M(\xi) = 1$, μ–a.e on ∂T.

Now we want to deal with the *irregular points* for μ, i.e. with the μ–measure zero set

$$\mathcal{I}(\mu) = \{\xi \in \operatorname{supp}(\mu) : \ I_pM(\xi) < 1\}. \tag{1.21}$$

Let $B_n = \{\xi \in \operatorname{supp}(\mu) : \ I_pM(\xi) \le 1 - 1/2^n\}$. Clearly $B_n \subseteq B_{n+1}$ and $\mathcal{I}(\mu) = \bigcup_n B_n$. Fix $\varepsilon > 0$, and choose a collection of edges $(\alpha_j^n)_{n\in\mathbb{N}, j\in\mathcal{J}_n}$, such that $\{\partial T_{\alpha_j^n}\}_{j\in\mathcal{J}_n}$ is an open cover of B_n. Without loss of generally, we can assume that $(\alpha_j^n)_{n\in\mathbb{N}, j\in\mathcal{J}_n}$ satisfies the following:

(i) $B_n \subseteq \bigcup_{j\in\mathcal{J}_n} \partial T_{\alpha_j^n}$

(ii) $T_{\alpha_j^n} \bigcap T_{\alpha_i^l} = \varnothing$, for $(j,n) \neq (i,l)$

(iii) $|\mathcal{J}_n| = m_n \in \mathbb{N}$

(iv) $\sum_{j\in\mathcal{J}_n} M(\alpha_j^n) < \varepsilon/2^n$.

In fact, if the intersection in (ii) was not empty, one of the two would be contained in the other and could be replaced by it in the covering family. Moreover, all the sublevel sets B_n are compact, since the potential I_pM is clearly continuous. Hence, for each n, we can extract some finite subcover so that $B_n \subseteq \bigcup_{j\in\mathcal{J}_n} \partial T_{\alpha_j^n}$ with $|\mathcal{J}_n| = m_n \in \mathbb{N}$, which is the finiteness condition on the index set in (iii). Finally, condition (iv) is because the measure μ is outer regular, i.e. $0 = \mu(B_n) = \inf\{\mu(\partial T_\alpha) : \alpha \in E, \partial T_\alpha \supseteq B_n\}$, so there exist sequences $(\alpha_j^n)_j$ such that $\mu(\partial T_{\alpha_j^n}) \to 0$ and we can assume we properly extract each subcover from one of those.

Write $\partial T = F_\varepsilon \bigcup G_\varepsilon$, where $G_\varepsilon := \bigcup_{j,n} \partial T_{\alpha_j^n}$ and $F_\varepsilon = \partial T \setminus G_\varepsilon$. Observe that,

$$\mu(\partial T) \ge \mu(F_\varepsilon) = \mu(\partial T) - \mu(G_\varepsilon) = \mu(\partial T) - \sum_{j,n} M(\alpha_j^n) \ge \mu(\partial T) - \sum_n \varepsilon/2^n = \mu(\partial T) - \varepsilon.$$

Hence we have,

$$\mu(\partial T) = \lim_{\varepsilon\to 0} \mu(F_\varepsilon).$$

Now, $I_pM \equiv 1$ on $A_\varepsilon := \operatorname{supp}(\mu) \cap F_\varepsilon$, so that $M^{p'-1}$ is a p–admissible function for A_ε. Then by definition of capacity we have

$$c_p(A_\varepsilon) \le \mathcal{E}_p(\mu|_{F_\varepsilon}) \le \mathcal{E}_p(\mu) = \mu(\partial T). \tag{1.22}$$

On the other hand, the measure $\nu^\varepsilon := \dfrac{\mu|_{F_\varepsilon}}{\mathcal{E}_p(\mu|_{F_\varepsilon})^{1/p'}}$ is admissible for A_ε, since $\mathcal{E}_p(\nu^\varepsilon) = 1$. By the dual definition of capacity it follows that

$$c_p(A_\varepsilon) \ge \nu^\varepsilon(A_\varepsilon)^p \ge \Big(\frac{\mu(F_\varepsilon)}{\mu(\partial T)^{1/p'}}\Big)^p. \tag{1.23}$$

We are now ready to build up the candidate $\mathcal{F}_\sigma$ set. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\varepsilon_k \searrow 0$ as $k \to +\infty$. Define $A := \bigcup_k A_{\varepsilon_k}$, which is clearly an $\mathcal{F}_\sigma$ set. Observe that we can assume that the covers related to each choice of ε are taken so that $G_{\varepsilon_{k+1}} \subset G_{\varepsilon_k}$. Therefore we have $A_{\varepsilon_k} \nearrow A$ as $k \to +\infty$. It follows that

$$\mu(A) = \lim_{k\to\infty} \mu(A_{\varepsilon_k}) = \mu(\partial T) - \lim_{k\to\infty} \mu(G_{\varepsilon_k}) \geq \mu(\partial T) - \lim_{k\to\infty} \sum_n \frac{\varepsilon_k}{2^n} = \mu(\partial T),$$

while the reverse inequality is trivially true. Using this together with (1.22) and (1.23) and the regularity of the $p-$capacity, we obtain

$$c_p(A) = \lim_{k\to+\infty} c_p(A_{\varepsilon_k}) = \mu(\partial T) = \mu(A).$$

□

It is clear from the proof that the situation is much easier for measures with no irregular points.

Corollary 1.4.2. *Suppose that the co-copotential M of a non-negative measure μ on ∂T solves (1.12) and $I_p(I^*\mu) \equiv 1$ on* $\mathrm{supp}(\mu)$*. Then, μ is the $p-$equilibrium measure of* $\mathrm{supp}(\mu)$*.*

In principle, equation (1.12) can also be interpreted as an equation in the function M, with no a priori assumption on weather such a function is the co-potential of a measure.

Proposition 1.4.3. *Let $M : E \to \mathbb{R}$, $I_pM < 1$ on V, be a solution of* (1.12)*. Then M is a non-negative flow on T.*

Proof. First of all observe that M solving (1.12) and such that $I_pM < 1$ automatically implies that $M \geq 0$ on E. We want to show that M satisfies (1.13) for every $\alpha \in E$. If $M(\alpha) = 0$ for some edge α, then the right handside of equation (1.12) says that $M(\beta) = 0$ on all edges $\beta \geq \alpha$ and then clearly (1.13) holds in α. Now, consider edges α such that $M(\alpha) \neq 0$. We have

$$\begin{aligned}\sum_{\beta\geq\alpha} \frac{M(\beta)^{p'}}{w(\beta)} - \frac{M(\alpha)^{p'}}{w(\alpha)} &= \sum_{\beta\in s(\alpha)} \sum_{\gamma\geq\beta} \frac{M(\beta)^{p'}}{w(\beta)} = \sum_{\beta\in s(\alpha)} M(\beta)\Big(1 - I_pM(b(\beta))\Big) \\ &= \Big(1 - I_pM(e(\alpha))\Big) \sum_{\beta\in s(\alpha)} M(\beta) \\ &= \Big(1 - I_pM(e(\alpha))\Big)\Big(- M(\alpha) + \sum_{\beta\in s(\alpha)} M(\beta)\Big) \\ &+ \sum_{\beta\geq\alpha} \frac{M(\beta)^{p'}}{w(\beta)} - \frac{M(\alpha)^{p'}}{w(\alpha)}.\end{aligned}$$

Since $1 - I_pM(e(\alpha)) \neq 0$, for every α, it follows that M is forward additive. □

By Remark 2 and the above Proposition we have the following slightly reinforced statement for Theorem 1.4.1.

Corollary 1.4.4. *Suppose that a function $M : E \to \mathbb{R}$ satisfies* (1.12) *and $I_pM < 1$ on V. Then, there exists an $\mathcal{F}_\sigma$ set A such that $M = I^*\mu$, where μ is the $p-$equilibrium measure of A.*

A similar calculation as in the proof of Proposition 1.4.3 can be used to show that, if M is a (not necessarily positive) flow and (1.12) holds for $|\alpha|$ large, then it holds everywhere. We give here the explicit details.

Proposition 1.4.5. *Let $\alpha \in E$ and suppose that M is a flow solving* (1.12) *for $\beta \in s(\alpha)$. Then M solves* (1.12) *also in α.*

Proof. The proof is just an easy direct calculation.

$$
\begin{aligned}
\sum_{\beta\geq\alpha}\frac{|M(\beta)|^{p'}}{w(\beta)} = \frac{|M(\alpha)|^{p'}}{w(\alpha)} + \sum_{\beta\in s(\alpha)}\sum_{\gamma\geq\beta}\frac{|M(\beta)|^{p'}}{w(\beta)} &= \frac{|M(\alpha)|^{p'}}{w(\alpha)} + \sum_{\beta\in s(\alpha)} M(\beta)\Big(1 - I_pM(e(\alpha))\Big) \\
&= \frac{M(\alpha)^2|M(\alpha)|^{p'-2}}{w(\alpha)} + M(\alpha)\Big(1 - I_pM(e(\alpha))\Big) \\
&= M(\alpha)\Big(\frac{M(\alpha)|M(\alpha)|^{p'-2}}{w(\alpha)} + 1 - I_pM(e(\alpha))\Big) \\
&= M(\alpha)\Big(1 - I_pM(b(\alpha))\Big).
\end{aligned}
$$

□

If we consider functions defined on vertices instead, we have an alternative formulation of Corollary 1.4.4, which reads as follows.

Corollary 1.4.6. *Let $g : V \to \mathbb{R}$, $g < 1$ on V, be a solution of the equation*

$$\nabla g|\nabla g|^{p-2}\Big(1 - g(b(\alpha))\Big) = \|\nabla g\|^{p'}_{W^{1,p'}(T_\alpha)}, \quad \textit{for all } \alpha \in E.$$

Then, g is $p-$harmonic and there exists an $\mathcal{F}_\sigma$ set A such that $g = I_p(I^*\mu)$, where μ is the $p-$equilibrium measure of A.

Proof. If g solves the above equation, then the function $M = \nabla g|\nabla g|^{p-2}$ solves (1.12). By Proposition 1.4.3, M is a non-negative flow, so that $g = I_pM$ is $p-$harmonic. The conclusion is given by Corollary 1.4.4. □

We end the section writing for once the full Characterization Theorem for equilibrium measures in a compact form.

Theorem 1.4.7. *Let $T = (E, V)$ be a rooted tree with an edge weight w.*

(i) If μ is the equilibrium measure of some set $A \subseteq \partial T$, then its co-potential $M = I^*\mu$ solves the equilibrium equation (1.12).

(ii) If $M : E \to \mathbb{R}^+$ is a solution of the equilibrium equation (1.12)*, then there exists an $\mathcal{F}_\sigma$–set $A \subseteq \partial T$ such that M is the co-potential of a the equilibrium measure of A.*

1.4.2 Regularity of boundaries

Potential Theory on trees presented in section 1.2.1 provides us with a notion of regularity for boundaries of trees (or their subsets). Let T be a rooted tree, $A \subseteq \partial T$ and $\mu = \mu^A$ the equilibrium measure for A. Denote as usual with M the co-potential of μ. We define the set of *p–irregular points* of E as the set

$$\mathcal{I}(E) = \{\xi \in E : \ I_p M(\xi) < 1\}.$$

Using the same terminology as in the proof of Theorem 1.4.1, the irregular points of A are the irregular points for its equilibrium measure μ^A, i.e. $\mathcal{I}(A) = \mathcal{I}(\mu^A)$. By definition of equilibrium measure, the set of irregular points has always null capacity. Conversely, every point of a set of null capacity is irregular. We say that the boundary ∂T of a tree is regular if $\mathcal{I}(\partial T) = \varnothing$. Intuitive examples of trees with regular boundaries are finite and spherically symmetric trees, defined in section 1.3.3. For finite trees, regularity follows from the fact that each point is a leaf, which we know to have positive capacity (see Example 1.1). In spherically symmetric trees, all the quantities of our interest are constant on levels (or radial), which simplifies all the calculations. We write $\deg(k)$ and $c_p(k)$ to indicate respectively $\mathrm{br}(e(\alpha)) - 1$ and $c_{p,\alpha}(\partial T_\alpha)$, for $|\alpha| = k$. It is clear that also the co-potential M of the equilibrium measure μ of the boundary of a spherically symmetric is radial, i.e. $M(\alpha) = M(\omega)/\operatorname{card}\{|\beta| = k\}$ for all $|\alpha| = k$. A way to see this, is to observe that the rescaling property of Proposition 1.2.4 provides a recursive formula for the co-potential M: for any edge α of level k we have

$$M(\alpha) = c_p(k)\left(1 - \sum_{\beta < \alpha} \frac{M(\beta)^{p'-1}}{w(\beta)}\right)^{p-1}. \tag{1.24}$$

It follows that $M(\alpha)$ is constant on levels and that boundaries of spherically symmetric trees are regular: the intuition arising from the symmetry (if a point is irregular then by symmetry all the points would be), is supported by the fact that the equilibrium function for the boundary is constant on levels, which gives $If^{\partial T} \equiv 1$ on the boundary.

The radial structure of the equilibrium measure provides an easy way to express the capacity of spherically symmetric trees with radial weights. In fact, writing μ for the equilibrium measure of the boundary we have

$$c_p(\partial T) = \mathcal{E}_p(\mu) = \sum_{k=0}^{\infty} \sum_{|\beta|=k} \frac{1}{w(k)} \left(\frac{M(\omega)}{\text{card}\{|\beta| = k\}} \right)^{p'} = c_p(\partial T)^{p'} \sum_{k=0}^{\infty} \frac{1}{w(k)\,\text{card}\{|\beta| = k\}^{p'-1}}.$$

Solving the equation we obtain

$$c_p(\partial T) = \left(\sum_{k=0}^{\infty} \frac{\text{card}\{|\beta| = k\}^{1-p'}}{w(k)} \right)^{1-p}. \tag{1.25}$$

In particular, if we write $c(n,p)$ for the un-weighted p–capacity of the boundary of a homogeneous tree of order n, we recover the quantity calculated in Example 1.2,

$$c(n,p) = \left(1 - n^{1-p'}\right)^{p-1}.$$

Observe that as $n \to \infty$ we have $c(n,p) \to 1$, which is by definition an upper bound for every tree capacity. This, togheter with Proposition 1.2.3, tells us that for any given real number $c \in (0,1)$ we can find a tree T such that $c_p(\partial T) = c$. However, the construction in the proof of Proposition 1.2.3 does not provide a regular tree (the top right point of the boundary is irregular). It turns out that one can construct a regular tree of prescribed (arbitrary) capacity.

Lemma 1.4.8. *If $B < 1$, for every real number $\lambda > 0$ there exists a sequence of integers $n_0, n_1, \dots$ such that*

$$\lambda = \sum_{j=0}^{\infty} n_j B^j.$$

Proof. Set $n_0 = \lfloor \lambda \rfloor$, so that there exists a reminder $r_0 < 1$ such that $\lambda = n_0 + r_0$. Setting $n_1 = \lfloor r_0/B \rfloor$, we have $\lambda = n_0 + n_1 B + r_1$, with $r_1 < B$, and proceeding like this one gets

$$\lambda = \sum_{j=0}^{k} n_j B^j + r_k, \quad \text{with } n_k = \lfloor r_{k-1}/B^k \rfloor,\ r_k < B^k.$$

Since $B^k \to 0$, as $k \to \infty$, taking the limit we obtain the result. □

Theorem 1.4.9. *For any real number $c \in (0, c(2,p))$ there exists a subdyadic tree T with regular boundary such that $c_p(\partial T) = c$.*

Proof. Given $0 \le c \le c(2,p)$, let $\lambda = c^{1-p'}$ and $n_0, n_1, \dots$ non-negative integer coefficients such that

$$\lambda = \sum_{j=0}^{\infty} \frac{n_j}{2^{(p'-1)j}}.$$

Let T be a spherically symmetric and *purely subdyadic* tree, i.e. $\deg(\alpha) \leq 2$ for every edge α. Let $\xi \in \partial T$, $\{\alpha_j\}_{j=0}^{\infty}$ be the associated edge geodesic and $\{\beta_j\}_{j=0}^{\infty}$ the subgeodesic obtained extracting all the edges of degree 2. We can choose the tree so that $|\beta_0| = n_0$ and $|\beta_j| - |\beta_{j-1}| = n_j$. Then by (1.25) we have

$$c(2,p) \geq c_p(\partial T) = \left(\sum_{k=0}^{\infty} \frac{n_k}{2^{(p'-1)k}}\right)^{1-p} = \lambda^{1-p} = c.$$

□

Although irregular points are a concrete obstacle when working with capacities, we have some regularization methods. Given any pair of rooted trees S, T, write $\langle S, T\rangle$ for their *biggest common subtree*, namely the tree having the property that any tree Z which is a subtree of both S and T, is also a subtree of $\langle S, T\rangle$. It is clear that such a subtree maximizes the capacity: $c_p(\partial Z) \leq c_p(\partial\langle S, T\rangle)$, for any Z subtree of S and T.

Theorem 1.4.10 (Approximation of capacity via regular boundaries)**.** *For any rooted tree T there exists a tree R having regular boundary such that*

$$c_p(\partial R) = c_p(\partial T) := c.$$

Moreover, the tree R can be choosed so that it agrees with T at a arbitrary scale: for every $\varepsilon > 0$ there exists R so that,

$$c_p(\partial\langle T, R\rangle) \geq c - \varepsilon.$$

Proof. Let μ be the equilibrium measure for ∂T, so that it solves (1.12). Write M for $I^*\mu$. Choosen $\varepsilon > 0$ we can associate it the same family of tents $T_{\alpha_j^n}$ constructed in the proof of Theorem 1.4.1. By Theorem 1.4.9, as small as ε is, we can choose a regular tree T_j^n such that $c_p(\partial T_j^n) = c_{\alpha_j^n,p}(\partial T_{\alpha_j^n})$. Construct a new tree $R = R(\varepsilon)$ from T substituting each tent $T_{\alpha_j^n}$ with the tree T_j^n rooted in α_j^n. Write ν_j^n for the equilibrium measure of ∂T_j^n, and N_j^n for the associated co-potential. Write S_ε for the subtree $T \setminus \bigcup_{j,n} T_j^n$ and, as in the proof of Theorem 1.4.1, $F_\varepsilon = \partial S_\varepsilon$ for $\partial T \setminus \bigcup_{j,n} \partial T_{\alpha_j^n}$, which equals $\partial R \setminus \bigcup_{j,n} \partial T_j^n$.

Set $k(\alpha_j^n) := 1 - I_p M(b(\alpha_j^n))$. We want to define a measure μ^ε on ∂R such that

(a) $\mu^\varepsilon|_{\partial T_j^n}$ is the p−equilibrium measure of ∂T_j^n as a subset of ∂R, i.e., by Proposition 1.2.4, $\mu^\varepsilon|_{\partial T_j^n} = k(\alpha_j^n)^{p-1}\nu_j^n$.

(b) $\mu^\varepsilon|_{F_\varepsilon} = \mu$.

These requests identify a unique measure μ^ε, since for each edge β of R, we have that ∂R_β can be expressed as a disjoint union of sets which are fully contained in F_ε or in some ∂T_j^n, whose measure is defined by (a) and (b) respectively. Observe that with this measure, the size of the boundary of each trasplanted tree T_j^n is the same as the size of the boundary of the removed tent $\partial T_{\alpha_j^n}$ with respect to μ:

$$M^\varepsilon(\alpha_j^n) = \mu^\varepsilon(\partial T_j^n) = \mu(\partial T_{\alpha_j^n}) = M(\alpha_j^n),$$

where $M^\varepsilon = I^*\mu^\varepsilon$. As a consequence, the co-potentials of μ and μ^ε coincide on the edges which are common to T and R. In fact, for any edge β of S_ε, we have

$$\begin{aligned} M^\varepsilon(\beta) &= \sum_{j,n:\ \alpha_j^n \geq \beta} M^\varepsilon(\alpha_j^n) + \mu^\varepsilon\Big(\partial T_\beta^\varepsilon \setminus \bigcup_{j,n} \partial T_j^n\Big) \\ &= \sum_{j,n:\ \alpha_j^n \geq \beta} M(\alpha_j^n) + \mu\Big(\partial T_\beta^\varepsilon \setminus \bigcup_{j,n} \partial T_{\alpha_j^n}\Big) = M(\beta). \end{aligned}$$

In particular,

$$\mu^\varepsilon(\partial R) = \mu(\partial T). \tag{1.26}$$

We claim that μ^ε is the equilibrium measure for ∂R. To see this, first of all we verify that it solves equation (1.12). If α belongs to one of the trasplanted trees, namely $\alpha \geq \alpha_j^n$, we have

$$\begin{aligned} \mathcal{E}_{p,\alpha}(\mu^\varepsilon) &= k(\alpha_j^n)^p \mathcal{E}_{p,\alpha}(\nu_j^n) = k(\alpha_j^n)^p\Big(1 - I_{p,\alpha_j^n} N_j^n(b(\alpha_j^n))\Big) N_j^n(\alpha) \\ &= k(\alpha_j^n)\Big(1 - I_{p,\alpha_j^n} N_j^n(b(\alpha_j^n))\Big) M^\varepsilon(\alpha) \\ &= \Big(k(\alpha_j^n) - I_{p,\alpha_j^n} M^\varepsilon(b(\alpha_j^n))\Big) M^\varepsilon(\alpha) \\ &= \Big(1 - I_p M^\varepsilon(b(\alpha))\Big) M^\varepsilon(\alpha). \end{aligned} \tag{1.27}$$

Hence, μ^ε solves (1.12) for every edge α of T_j^n. Observe that in particular, since $M^\varepsilon(\alpha_j^n) = M(\alpha_j^n)$, equation (1.27) together with equation (1.12) for μ give

$$\mathcal{E}_{\alpha_j^n,p}(\mu^\varepsilon) = k(\alpha_j^n) M(\alpha) = \mathcal{E}_{\alpha_j^n,p}(\mu).$$

In fact, the measure μ^ε was constructed so that its energy on the trees T_j^n equals the energy of μ on the corresponding tents $T_{\alpha_j^n}$. This is true more generally for any tent rooted in one of the edges that T and R have in common. Namely, for any edge α of S_ε we have

$$\begin{aligned} \mathcal{E}_{p,\alpha}(\mu^\varepsilon) &= \sum_{\beta \in S_\varepsilon \cap R_\alpha} \frac{M^\varepsilon(\beta)^{p'}}{w(\beta)} + \sum_{\alpha_j^n \geq \alpha} \mathcal{E}_{\alpha_j^n,p}(\mu^\varepsilon) \\ &= \sum_{\beta \in S_\varepsilon \cap T_\alpha} \frac{M^\varepsilon(\beta)^{p'}}{w(\beta)} + \sum_{\alpha_j^n \geq \alpha} \mathcal{E}_{\alpha_j^n,p}(\mu) = \mathcal{E}_{p,\alpha}(\mu). \end{aligned}$$

Again by equation (1.12) for the measure μ on the original tree T, for any edge $\alpha \in S_\varepsilon$ we get

$$\mathcal{E}_{p,\alpha}(\mu^\varepsilon) = \mathcal{E}_{p,\alpha}(\mu) = \Big(1 - I_p M(b(\alpha))\Big) M(\alpha) = \Big(1 - I_p M^\varepsilon(b(\alpha))\Big) M^\varepsilon(\alpha). \tag{1.28}$$

By (1.27) and (1.28) together we have that equation (1.12) holds for the measure μ^ε and every edge $\alpha \in R$.Computing the p−potential of M^ε on ∂R, for $\xi \in \partial T_j^n$ we get

$$I_p M^\varepsilon(\xi) = I_{p,\alpha_j^n} M^\varepsilon(\xi) + I_p M(b(\alpha_j^n)) = \Big(1 - I_p M(b(\alpha_j^n))\Big) I_{p,\alpha_j^n} N_j^n(\xi) + I_p M(b(\alpha_j^n)),$$

while for $\xi \in F_\varepsilon$ it holds $I_p M^\varepsilon(\xi) = I_p M(\xi) \equiv 1$.

Since N_j^n is the co-potential of the p−equilibrium measure of a regular boundary, then $I_{p,\alpha_j^n} N_j^n(\xi) \equiv 1$ on ∂T_j^n. It follows that $I_p M^\varepsilon \equiv 1$ on $\operatorname{supp}(\mu^\varepsilon) = \partial R$. Then ∂R is regular and by Corollary 1.4.2, μ^ε must be its equilibrium measure. It follows from equation (1.26) that

$$c_p(\partial R) = c_p(\partial T).$$

To end the proof, we have to show that the p−capacity of the tree $\langle R, T \rangle$ is arbitrary large. By monotonicity and subadditivity we have,

$$c_p(\partial \langle R, T \rangle) \geq c_p(\partial F_\varepsilon) \geq c - c_p\left(\bigcup_{j,n} \partial T_j^n\right) \geq c - \sum_{j,n} c_p(\partial T_j^n).$$

By the Rescaling Property of Proposition 1.2.4, and the relation

$$N_j^n(\alpha_j^n) = M(\alpha_j^n)/k(\alpha_j^n)^{p-1},$$

we have

$$\sum_{j,n} c_p(\partial T_j^n) = \sum_{j,n} N_j^n(\alpha_j^n) k(\alpha_j^n)^{p-1} = \sum_{j,n} M(\alpha_j^n) < \varepsilon,$$

which gives, $c_p(\langle R, T \rangle) > c - \varepsilon$. □

1.4.3 Infinite tilings and an inverse tiling theorem

In [13] Brooks, Smith, Stone and Tutte considered the problem of tiling a rectangle with a finite number of squares and proved that to any finite connected planar graph G can be associated such a packing. The same graph can produce different packings. Chosing any two vertices in G, they show how the associated packing can be built in such a way to reflect this choice. See [13] for more details. In [10] Benjamini and Schramm extended this result to the infinite case, showing that infinite graphs can produce infinite packings. Theorem 1.4.7 can be reformulated, for $p = 2$, in terms of square packings of a rectangle. With this reformulation, part (i) of the theorem is essentially equivalent to the infinite packing theorem by Benjamini and Schramm in the special case when G is a rooted tree T, hence providing a new and different proof of it. More importantly, part (ii) provides a converse result, in a sense that will be more clear once introduced the proper terminology.

Given a rectangle R (a closed planar region whose boundary is a rectangle), we say that a family of squares $Q = \{Q_j\}$ is a *square tiling* of R if $\mathrm{int}(Q_i) \bigcap \mathrm{int}(Q_j) = \varnothing$, for $i \neq j$, and $R = \bigcup_j Q_j$. We denote by $|Q_j|$ the area of the rectangle Q_j. By rotation invariance of the problem we always think rectangles and squares to have sides parallel to the coordinates axes of $\mathbb{R}^2$, and we talk about upper (lower) and left (right) sides (as well as horizontal and vertical sides) in the obvious way. We write $B(j)$ and $E(j)$ for the upper and lower side of Q_j, respectively.

We say that *the combinatorics of a family Q of squares in the plane are prescribed by a tree T* if the followings are true.

(i) The squares in the family are indexed by the edges of the tree, $Q = \{Q_\alpha : \alpha \in E\}$.

(ii) $B(\alpha) \subseteq E(\beta)$ whenever $b(\alpha) = e(\beta)$.

The un-weighted and linear ($p = 2$) version of Theorem 1.4.7, can be reformulated as follows.

Theorem 1.4.11. *(i) Let $T = (V, E)$ be a rooted tree and $\mu = \mu^A$ be the equilibrium measure (for $p = 2$) for some set $A \subseteq \partial T$. Then, there exists a square tiling $\{Q_\alpha\}_{\alpha \in E}$ of the rectangle $R = [0, c_2(A)] \times [0, 1]$, where the combinatorics of the tiling are prescribed by T and the square Q_α has side of length $|Q_\alpha|^{1/2} = \mu^A(\partial T_\alpha)$.*

(ii) Conversely, suppose a rectangle $R = [0, c] \times [0, 1]$ is square-tiled by $\{Q_\alpha\}_\alpha$ with combinatorics given by a rooted tree T. Then there exists an F_σ subset A of ∂T such that the measure $\mu(\partial T_\alpha) = |Q_\alpha|^{1/2}$, is the equilibrium measure of A, and then $c_2(A) = c$.

Proof. (i) Given a tree T with root edge ω and a set $A \subseteq \partial T$, let $\{Q_\alpha\}_{\alpha \in E}$ be a family of squares such that Q_α has side of length $\ell(\alpha) = \mu(\partial T_\alpha)$, being $\mu = \mu^A$ the equilibrium measure of A. By the additivity of μ we can place the squares on the plane in such a way that,

$$E(\beta) = \bigcup_{\beta \in s(\alpha)} B(\alpha).$$

With this choice, the combinatorics is prescribed by T. Moreover, it is clear that the interiors of the squares in the family are pairwise disjoint and that $\bigcup_\alpha Q_\alpha$ is both vertically and horizontally convex (its intersection with any vertical (horizontal) line is either empty, or a point, or a line segment). Now, let R be the rectangle having vertical sides of length 1 and upper side coinciding with the upper side of $Q(\omega)$, so being of length $\mu(\partial T) = \mu(A) = c_2(A)$. Then,

$$|R| = c_2(A) = I^*\mu(\omega) = \mathcal{E}_2(\mu) = \sum_{\alpha \in E} \mu(\partial T_\alpha)^2 = \sum_{\alpha \in E} |Q_\alpha| = \Big|\bigcup_\alpha Q_\alpha\Big|.$$

It follows that it is enough to show that the family of squares is contained in R to prove that it is a tiling. It is clear that all the family $\{Q_\alpha\}$ lies in between the two vertical sides of R, and that the horizontal room is fully filled, by additivity of the measure. Moreover, being μ an equilibrium measure, for every $\xi \in \partial T$ it must hold

$$1 \geq I(I^*\mu)(\xi) = \sum_{\beta < \xi} \mu(\partial T_\beta) = \sum_{\beta < \xi} \ell(\beta).$$

It follows that $\bigcup_\alpha Q_\alpha \subseteq R$ and $\{Q_\alpha\}_\alpha$ is a tiling.
(ii) Let the rectangle R be tiled according to the combinatorics of the given tree T, as described above. Then for each $\alpha \in E$, we have:

$$\sum_{\beta \geq \alpha} |Q(\beta)| = \ell(\alpha)\Big(1 - \sum_{\beta < \alpha} \ell(\beta)\Big).$$

Hence, it is immediate that if we define a measure on ∂T by $\mu(\partial T_\alpha) = \ell(\alpha)$, it solves equation (1.12). By Theorem 1.4.7, it must be the equilibrium measure of some F_σ subset A of ∂T. □

Recall that, given any real number $c \in (0,1)$, it is possible to build a tree (even with regular boundary) with $c_2(\partial T) = c$ (see 1.4.9). It follows that we can perform square tilings of rectangles with any ratio of sides.

Observe also that in (i), if we replace T by its subtree obtained keeping only the edges α with $\mu^E(\partial T_\alpha) > 0$, then tiling remains, in fact, the same (it is the same without degenerate squares). Hence from now on we can well assume that A is dense in ∂T.

It might be interesting to informally discuss some features of the tiling, and its relation to the set A. The example below can provide a useful illustration of what we are here saying in general terms. For each ξ in ∂T, let $\{\alpha_n\}$ be the associated edge geodesic with the edge $|\alpha_n| = n$, and choose a point x_{n+1} in Q_{α_n}. Then, it is immediate that $\lim_{n\to\infty} x_n =: \lambda(\xi)$ exists in R, and that it does not depend on the choice of the x_n's. Let $\pi(\xi)$ be the orthogonal projection of $\lambda(\xi)$ onto the lower side of R, identified with $[0, c_2(A)]$. The following facts are easy to check:

(i) let $A' = \{\xi \in \partial T : I(I^*\mu^A)(\xi) = 1\}$: then $\mu^{A'} = \mu^A$, hence they induce the same tiling;

(ii) π is injective but possibly at countably many points and surjective from ∂T onto $[0, c_2(A)]$;

(iii) $\mu^A(\pi^{-1}(B)) = \ell(B)$ for all measurable subsets $B \subseteq [0, c_2(A)]$ (where ℓ denotes length measure on $[0, c_2(A)]$);

(iv) let $\mathrm{Ex}(A) := \{\xi \in \partial T : \pi(\xi) \neq \lambda(\xi)\}$: then, $\mathrm{Ex}(A) = \partial T \setminus A$ (if $A = A'$);

(v) by passing to a subtree of T, we can always assume that $c_2(A \cap \partial T_\alpha) > 0$ for all edges α.

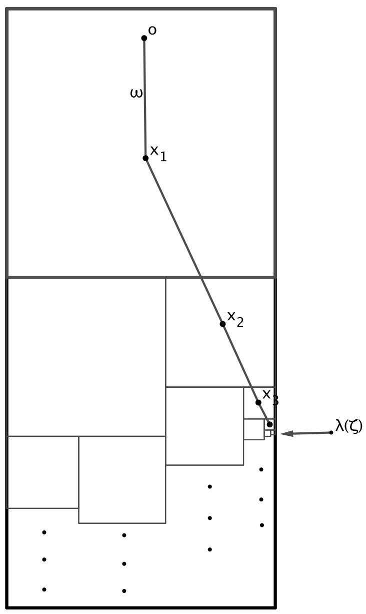


Figure 12: For the boundary point $\xi = \{x_j\} \in \partial T$, $\lambda(\xi)$ does not lie on the bottom side of the rectangle. i.e. $\xi \in \mathrm{Ex}(A)$

The combinatorics of the tree are not, by themselves, enough to determine a unique rectangle R and a square tiling of it. They are, if we assume that the set A in the Theorem 1.4.11 is *closed*, but they are not in general. This is in striking contrast with the case of finite trees, or even finite graphs. However, under the assumption $A = A'$ of (iv), if $c_2(A) < c_2(\partial T)$, then a price has to be paid. In fact, in that case

$$c_2(\mathrm{Ex}(A)) \geq c_2(\partial T) - c_2(A) > 0 :$$

the exceptional set $\mathrm{Ex}(A)$ is rather large, although, clearly, $0 = \mu^A(\mathrm{Ex}(A)) = \ell(\pi(\mathrm{Ex}(A)))$. In the next example we construct a tree with these features.

Example 1.4 (A regular set of dyadic combinatorics and arbitrarily small capacity with positive capacity in every subtree). Let $\varepsilon > 0$ be any small number, and $T = T_2$ a dyadic tree with edge root ω. Let $n = n(\omega)$ be the number of steps one has to move to the left, starting from the root, before finding an edge α_n^ω such that $c_p(\partial T_{\alpha_n^\omega}) \le \varepsilon/2$. Let $\{\alpha_1^\omega, \dots, \alpha_n^\omega\}$ be the geodesic from the root to α_n^ω, and β_j be the right brother of α_j^ω, i.e. the only edge with $b(\beta_j) = b(\alpha_j^\omega)$. In each subtree T_{β_j}, $j = 1, \dots, n$, starting from the root β_j, move to the left, say $n(\beta_j)$ steps, until you find an edge $\alpha_{n(\beta_j)}^{\beta_j}$ such that $c_p(\partial T_{\alpha_{n(\beta_j)}^{\beta_j}}) \le \varepsilon/(2^2 n)$. Then we iterate the process: $\{\alpha_1^{\beta_j}, \dots, \alpha_{n(\beta_j)}^{\beta_j}\}$ is the geodesic from β_j to $\alpha_{n(\beta_j)}^{\beta_j}$ and $\gamma_i = \gamma_i(j)$ the right brother of $\alpha_i^{\beta_j}$. In each subtree T_{γ_i} we individuate as before, always moving to the left, subtrees with $c_p(\partial T_{\alpha_{n(\gamma_i)}^{\gamma_i}}) \le \varepsilon/(2^3 n(\beta_j))$, and so on. Let

$$A_1 = \partial T_{\alpha_n^\omega}, \quad A_2 = \bigcup_{j=1}^{n} \partial T_{\alpha_{n(\beta_j)}^{\beta_j}}, \quad A_3 = \bigcup_{j=1}^{n} \bigcup_{i=1}^{n(\beta_j)} \partial T_{\alpha_{n(\gamma_i)}^{\gamma_i}}, \quad \dots, \quad \text{and set} \quad A = \bigcup_k A_k.$$

By construction, for every $\alpha \in E$ the tree T_α contains a tent with boundary in A. Since tents have positive capacity (for example by the rescaling property of Proposition 1.2.4), it follows that $c_p(A \cap \partial T_\alpha) > 0$ for every $\alpha \in E$. On the other hand,

$$c_p(A) \le \sum_k c_p(A_k) \le \sum_k \frac{\varepsilon}{2^k} = \varepsilon.$$

For the regularity, observe that by construction for every point $\xi \in A$, there exists some edge α such that $\xi \in \partial T_\alpha \subseteq A$. Therefore, if μ, μ^α are the equilibrium measures for A and ∂T_α respectively, and M, M^α their co-potentials, we have

$$I_p M(\xi) = I_{p,\alpha} M(\xi) + I_p M(b(\alpha)) = \big(1 - I_p M(b(\alpha))\big) I_{p,\alpha} M^\alpha(\xi) + I_p M(b(\alpha)) = 1,$$

by the regularity of homogeneous trees.

1.4.4 Branched continued fractions

Theorem 1.4.7 can be reformulated in terms of *branched continued fractions*. Besides adding further interesting structure to the class of equilibrium measures, this provides a recursive formula for concretely calculating capacity of sets. An accessible survey on branched continued fractions is in [11]. In the whole section we disregard weights (we set $w(\alpha) = 1$ for all edges α) to make it more readable.

Proposition 1.4.12. *Let $M : E \to \mathbb{R}^+$ be any non-negative function such that $I_p M < 1$ on V, and consider the associated rescaled function defined by*

$$c(\alpha) = \frac{M(\alpha)}{\left(1 - I_p M(b(\alpha))\right)^{p-1}}. \tag{1.29}$$

Then, M is the potential of a measure μ on ∂T if and only if c is defined, for each edge α which is not a leaf, by the following recursive formula

$$c(\alpha) = \frac{\sum_{\beta \in s(\alpha)} c(\beta)}{\left(1 + \left(\sum_{\beta \in s(\alpha)} c(\beta)\right)^{p'-1}\right)^{p-1}}. \tag{1.30}$$

Proof. By (1.29), $M(\omega) = c(\omega)$. Denote by α^- the father of α, i.e. the only edge α^- such that $\alpha \in s(\alpha^-)$. For every $\alpha \neq \omega$, we have

$$\begin{aligned} M(\alpha)^{p'-1} &= c(\alpha)^{p'-1}\left(1 - I_p M(b(\alpha))\right) = c(\alpha)^{p'-1}\left(1 - I_p M(b(\alpha^-)) - M(\alpha^-)^{p'-1}\right) \\ &= c(\alpha)^{p'-1}\left(\frac{M(\alpha^-)^{p'-1}}{c(\alpha^-)^{p'-1}} - M(\alpha^-)^{p'-1}\right) = c(\alpha)^{p'-1} M(\alpha^-)^{p'-1} \frac{1 - c(\alpha^-)^{p'-1}}{c(\alpha^-)^{p'-1}}. \end{aligned}$$

Iterating we obtain,

$$M(\alpha) = c(\alpha) \prod_{\gamma < \alpha} \left(1 - c(\gamma)^{p'-1}\right)^{p-1}. \tag{1.31}$$

Hence, for any chosen edge α which is not a leaf, it holds

$$\sum_{\beta \in s(\alpha)} M(\beta) = \prod_{\gamma \leq \alpha} \left(1 - c(\gamma)^{p'-1}\right)^{p-1} \sum_{\beta \in s(\alpha)} c(\beta).$$

Now, M is the co-potential of a measure if and only if the flow condition (1.13) holds. Namely, using (1.31), (1.13) holds if and only if

$$c(\alpha) = \left(1 - c(\alpha)^{p'-1}\right)^{p-1} \sum_{\beta \in s(\alpha)} c(\beta),$$

which is equivalent (1.30), as can be seen solving with respect to $c(\alpha)$. □

By the rescaling properties of equilibrium measure (Proposition 1.2.4), we have that if μ is the p–equilibrium measure for a set $A \subseteq \partial T$, then $c(\alpha) = c_{p,\alpha}(A_\alpha)$. This gives us an algorithm to calculate the capacity of a set in ∂T in terms of successive tents capacities. Moreover, by relation (1.30) we deduce that capacities can be expressed by means of branched continued fractions. For example, by (1.30) we obtain the expression

$$c_2(\partial T) = \cfrac{1}{1 + \cfrac{1}{\sum\limits_{\beta \in s(\omega)} \cfrac{1}{1 + \sum\limits_{\gamma \in s(\beta)} \cfrac{1}{1 + \dots}}}} .$$

In [36, p. 57] the same structure was observed for the *total resistence* R of an infinite tree without edges of degree 1. In particular, for such a class of trees, we obtain the relation

$$c_2(\partial T) = \frac{1}{1+R}.$$

To end the section, we give a reformulation of Theorem 1.4.7 which provides a characterization of equilibrium measures by means of an equation for capacities.

Theorem 1.4.13. *Let μ be a measure on ∂T such that $I_p(I^*\mu) < 1$ on V. Let $M = I^*\mu$ and write $c(\alpha)$ for the rescaled potential obtained from $M(\alpha)$ by means of (1.29). Then, μ is the equilibrium measure for some set $E \subseteq \partial T$ if and only if*

$$c(\alpha)\left(1 - c(\alpha)^{p'-1}\right) = \sum_{\beta>\alpha} c(\beta)^{p'} \prod_{\alpha \le \gamma < \beta} \left(1 - c(\gamma)^{p'-1}\right)^p . \tag{1.32}$$

Proof. By (1.29), for each edge α we have

$$M(\alpha)\big(1 - I_p M(b(\alpha))\big) = \frac{M(\alpha)^{p'}}{c(\alpha)^{p'-1}}.$$

By Theorem 1.4.7, μ is an equilibrium measure if and only if (1.12) holds, i.e. if and only if

$$M(\alpha)^{p'}\left(1 - c(\alpha)^{p'-1}\right) = c(\alpha)^{p'-1} \sum_{\beta>\alpha} M(\beta)^{p'}.$$

But by Proposition 1.4.12, we know that $c(\alpha)$ solves (1.30), or equivalently, $M(\alpha)$ is defined by (1.31). Hence, sobstituting above we get

$$c(\alpha)^{p'} \prod_{\gamma<\alpha} \left(1 - c(\gamma)^{p'-1}\right)^p \left(1 - c(\alpha)^{p'-1}\right) = c(\alpha)^{p'-1} \sum_{\beta>\alpha} c(\beta)^{p'} \prod_{\gamma<\beta} \left(1 - c(\gamma)^{p'-1}\right)^p ,$$

which is (1.32). □

Chapter 2

Boundary behavior of inner functions

In this chapter we present some new results concerning the boundary behavior of inner functions, which are analytic self-maps of the unit disc $\mathbb{D}$, mapping almost every point of the unit circle to the unit circle. Every inner function f induces a boundary map defined (a.e.) on $\partial\mathbb{D}$, which we denote by the same symbol f with no risk of confusion. This boundary map lacks the regularity of the inner function itself and it is actually discontinuous at every point $\xi \in \partial\mathbb{D}$ where f does not extend analytically. For this reason, it is not easy to study the behavior of an inner function as a mapping from the unit circle to itself.

A classical result due to Löwner (see, for instance, page 12 of [3]) says that Lebesgue measure λ on the unit circle is invariant under the action of any inner function f fixing the origin: for every set $E \subseteq \partial\mathbb{D}$,

$$\lambda(f^{-1}(E)) = \lambda(E). \tag{2.1}$$

In the 90's Fernández and Pestana pushed such an analysis to a deeper level, studying the distortion of the α-dimensional Hausdorff content

$$M_\alpha(E) = \inf \sum_j \lambda(I_j)^\alpha$$

of a set $E \subseteq \partial\mathbb{D}$ under an inner function f fixing the origin. In [19], they prove that for any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$ such that

$$M_\alpha(f^{-1}(E)) \geq C_\alpha M_\alpha(E). \tag{2.2}$$

Both the results adapt smoothly, with a standard argument of conformal invariance, to the case when f fixes a point $z \in \mathbb{D} \setminus \{0\}$. In such a case the harmonic measure λ_z centered at z, naturally takes the role of the Lebesgue measure, as we will see later. However, inner functions are defined as well (a.e.) on the unit circle, and they can present boundary fixed points, i.e. points $\xi \in \partial\mathbb{D}$ such that $f(\xi) = \xi$, and no interior fixed points. In such a case, the conformal invariant extension of results (2.1) and (2.2) do not give any information on distortion of measure and content for inner functions with no interior fixed points. This is the original observation from which our work departs. We look for a more general class of measures which allow us to get results in the spirit of (2.1) and (2.2) but applying also to inner functions with no interior fixed points.

The chapter is divided into two sections. The first one contains all the required preliminaries while the second contains our results. More in detail, in section 2.1 we begin by introducing some basics concepts about analytic self-maps of the unit disc, such as angular derivatives and their connection with boundary fixed points. We then define inner functions giving some examples and presenting their canonical factorization. We introduce the powerful tool of Clark measures which allows us to give an alternative definition of inner function and an extra characterization of points admitting finite angular derivatives. We then talk about the analytic continuation of inner functions across the boundary and we show, roughly speaking, that boundary points are either extremely regular or extremely irregular (Theorem 2.1.7). This is the only original result in the first section, which is meant to be introductory to the problem.

Section 2.2 is devoted to present our results. We state Löwner Lemma and Fernández-Pestana theorem in a conformal invariant version, so that they apply to all inner functions with interior fixed points. We introduce a new measure μ_p fitting with our purpouse, which is to extend the aforementioned results to inner having only boundary fixed points. The approach we adopt is to "push" the known results to the boundary, obtaining the desired results as some sort of limit cases of (2.1) and (2.2). Once our results are proved, we provide two applications. The first one concerns a smoothness property of inner functions which omit large sets of the unit disc and it is inspired by a nice result in [19]. In the second application we obtain analogous results on distortion of sets in the real line under inner mappings of the upper half plane.

All the original results in this chapter have been obtained together with Artur Nicolau and Odí Soler i Gibert and are essentially contained in [27].

2.1 Self maps of the unit disc

2.1.1 Fixed points and angular derivatives

Our object of study are holomorphic functions defined on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which, by means of the Riemann mapping theorem, can be though of as a model space for every simply connected domain of the complex plane. The class of bounded analytic functions on the unit disc is classically denoted by H^∞. An *analytic self map* of the unit disc is a function $f \in H^\infty$ such that $f(\mathbb{D}) \subseteq \mathbb{D}$. We denote by $\mathrm{Aut}(\mathbb{D})$ the group of the automorphisms of the unit disc, i.e. the class of univalent self maps of the unit disc such that $f(\mathbb{D}) = \mathbb{D}$. They are exactly the Möbius transformations of the form $\lambda\varphi_w$, where $w \in \mathbb{D}, |\lambda| = 1$ and φ_w is the conformal bijection of $\mathbb{D}$ mapping w to 0 and vice versa:

$$\varphi_w(z) := \frac{w - z}{1 - \overline{w}z}.$$

We say that a sequence of points $\{z_n\} \subseteq \mathbb{D}$ converges non-tangentially to $p \in \partial\mathbb{D}$ if for each n the distance of z_n from p is comparable to its distance from the boundary. Namely, $\{z_n\}$ tends to p and there exists some $0 < \beta < 1$ such that $\{z_n\} \subset \Gamma_\beta(p) = \{z \in \mathbb{D} : |z - p| < \beta(1 - |z|)\}$. The region $\Gamma_\beta(p)$ is called the Stolz angle with opening β and vertex at p. We say that a function $f : \mathbb{D} \to \mathbb{C}$ admits angular limit at $p \in \partial\mathbb{D}$ if there exists $\eta \in \mathbb{C}_\infty$ such that $f(z_n) \longrightarrow \eta$ for every sequence of points $\{z_n\}$ converging non-tangentially to p. We write $\angle \lim_{z\to p} f(z) = \eta$ or simply $\eta = f(p)$. It is a classical result by Fatou that bounded holomorphic functions on the unit disc admit angular limit at almost every point of the boundary.

Let f be an analytic self map of the unit disc having unimodular angular limit at ξ, i.e. $f(\xi) \in \partial\mathbb{D}$. We say that f has a finite *angular derivative* at ξ if one of the following equivalent conditions hold:

(i) $\displaystyle \angle \lim_{z\to\xi} \frac{f(\xi) - f(z)}{\xi - z}$ exists and is finite.

(ii) $\displaystyle \liminf_{z\to\xi} \frac{1 - |f(z)|}{1 - |z|} = \delta > 0.$

(iii) $\displaystyle \angle \lim_{z\to\xi} f'(z)$ exists and is finite.

The equivalence of the three conditions above is guaranteed by the classical Julia-Carathéodory theorem (see for example Chapters IV and V of [35]). The value of the limits in (i) and (iii) coincide and are denoted by $f'(\xi)$, namely the angular derivative of f at ξ. Moreover it holds

$$f'(\xi) = \overline{\xi} f(\xi)\delta, \tag{2.3}$$

so that in particular $\delta = |f'(\xi)|$. As a convention, we set $|f'(\xi)| = +\infty$ if the function f does not have a finite angular derivative at ξ.

Angular derivatives may help in identifying and describing fixed points. If f is a disc automorphism (not the identity and not a rotation fixing only the origin), solving the second degree fixed point equation $f(z) = z$ one can see that either f has a fixed point $z_1 \in \mathbb{D}$ and another one $z_2 \in \mathbb{C} \setminus \mathbb{D}$ (in which case f is called *elliptic*), or it has two (possibly coinciding) fixed points ξ_1, ξ_2 on $\partial\mathbb{D}$. Moreover, if f is an automorphism and $f(z_0) = z_0$, it holds $|f'(z_0)| = 1$. More generally, as an immediate consequence of the Schwarz-Pick Lemma, any analytic self map f of the unit disc can have at most one fixed point z_0 in the open unit disc, at which it holds $|f'(z_0)| \leq 1$, with equality if and only if f is a disc automorphism. Of course, f can possibly have also some boundary fixed point, in the sense that there exists some point $\xi_0 \in \partial\mathbb{D}$, at which f admits angular limit $f(\xi_0)$, and $f(\xi_0) = \xi_0$. By (2.3) it follows immediately that $|f'(\xi_0)| > 0$ at boundary fixed points. Unfortunately, we cannot say much about the number of boundary fixed points of f when the function is not a disc automorphism. However, the following celebrated theorem assures that there is at most one fixed point which is attracting for the dynamics of the iterates of f.

Theorem 2.1.1 (Denjoy-Wolff, see e.g. Chapter V of [35].). *Let f be an analytic self map of the unit disc which is not an elliptic automorphism. Then, there exists a unique point $p \in \overline{\mathbb{D}}$, called the Denjoy-Wolff fixed point of f, such that the iterates f^n tend to p uniformly on compact sets of $\mathbb{D}$. Moreover, p is the unique fixed point of f in $\overline{\mathbb{D}}$ with $0 < |f'(p)| \leq 1$.*

So, given an analytic self map of the unit disc f, either it has a fixed point z_0 in the interior of $\mathbb{D}$, which is the Denjoy-Wolff fixed point, or it has no interior fixed points, in which case the Denjoy-Wolff fixed point lies on the boundary and it is the only fixed point ξ_0 in $\partial\mathbb{D}$ for which the angular derivative satisfies $0 < |f'(\xi_0)| \leq 1$.

2.1.2 Inner Functions

Let $f : \mathbb{D} \to \mathbb{D}$ be analytic. We denote by $\mathcal{F}(f)$ the set of Fatou's points of f, i.e. points $\xi \in \partial\mathbb{D}$ at which the angular limit $f(\xi)$ exists and is unimodular. Our object of study are those functions whose set of Fatou's points have full measure in the unit circle.

Definition 2.1.1. An analyitic function $f : \mathbb{D} \to \mathbb{D}$ is called *inner* if $|f(\xi)| = 1$ for a.e. $\xi \in \partial\mathbb{D}$, i.e. if almost every boundary point is a Fatou's point for f.

It is clear that every disc automorphism is an inner function for which all boundary points are Fatou's, and the same is true for every finite product of automorphisms. Less evident is the fact that even some infinite products of automorphisms may produce inner functions. A Möbius transformation is called a *Blaschke factor* if it is positive at 0 or if it vanishes but has a positive derivative at 0. Namely, Blashke factors are exactly the disc automorphisms of the form

$b_a(z) = (\bar{a}/a)\varphi_a(z)$, plus the identity function z. An infinite product of Blaschke factors converges to an inner function if and only if the zeros of the factors satisfy some natural asymptotic condition. Accordingly, the produced inner function takes the name of (infinite) *Blaschke product*.

Proposition 2.1.2 (see, e.g. [24], pag. 63)**.** *Let $\{z_j\}$ be a sequence of points in $\mathbb{D} \smallsetminus \{0\}$ satisfying the Blaschke condition*

$$\sum_j (1 - |z_j|) < \infty. \tag{2.4}$$

Then, the Blaschke product of the sequence $\{z_j\}$,

$$B_{\{z_j\}}(z) = \prod_{j=1}^{+\infty} \frac{\overline{z_j}}{|z_j|} \frac{z_j - z}{1 - \overline{z_j} z},$$

converges uniformly on compact subsets of $\mathbb{D}$ and defines an inner function.

Condition (2.4) is satisfied by the zeros of any bounded analytic map of the unit disc (in fact, any function in the Nevanlinna class, see [21]). It follows that for every $f \in H^\infty$ we can write $f = z^k B_{\{z_j\}} g$, where $k \in \mathbb{N}$ is the multiplicity (possibly zero) of the zero of f at the origin, B is the Blaschke product of the other zeros of f and g is a zero free bounded analytic map of the unit disc. Assume, without loss of generality, that $\|f\| \le 1$, and observe that $\log |g|^{-1}$ is a positive harmonic function. Hence by the Herglotz-Riesz representation theorem, it is the Poisson integral of a positive measure μ on the unit circle. Writing the Poisson kernel as the real part of the Herglotz kernel, we have

$$\log |g(z)|^{-1} = \mathrm{Re}\,(-\log g(z)) = \mathrm{Re} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta). \tag{2.5}$$

Hence, there exists $\lambda \in \partial\mathbb{D}$ such that

$$g(z) = \lambda \exp\left(- \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right). \tag{2.6}$$

Now, f is an inner function if and only if the angular limit of $\log |g|^{-1}$ vanishes almost everywhere on the unit circle, from which follows that the measure μ in (2.6) is singular. Accordingly, an analytic self map of the unit disc having the form (2.6) with μ a positive singular measure is called a *singular inner function* and will be denoted by S_μ. We have just proved the following classical factorization theorem.

Theorem 2.1.3. *Every inner function f has the form $f = BS_\mu$, where $B = z^k B_{\{z_j\}}$ is the Blaschke product of the zeros of f and S is the singular inner function associated to a singular measure μ.*

A Blaschke product is the prototype of inner function one has to have in mind. In fact, not only they are one of the building blocks of every inner function as the just stated factorization theorem proves. It turns out that every non constant inner function is in fact a Blaschke product, up to composition with a disc automorphism.

Theorem 2.1.4 (Frostman, see e.g. [21], theorem 6.4)**.** *Let f be a non constant inner function. Then, for every $w \in \mathbb{D} \setminus K$, where K is a set of logarithmic capacity zero, the function*

$$f_w(z) = \frac{f(z) - w}{1 - \overline{w} f(z)},$$

is a Blashke product.

It is worth mentioning that Ahern and Clark have provided a full characterization of points where an inner function admits a finite angular derivative in terms of the distribution of its zeros and the singular measure of its singular inner factor.

Theorem 2.1.5 ([2], Theorem 2)**.** *For an inner function $f = z^k B_{\{z_j\}} S_\mu$ and a point $\xi_0 \in \partial\mathbb{D}$, it holds*

$$|f'(\xi_0)| = \sum_{n=0}^{\infty} \frac{1 - |a_n|^2}{|\xi_0 - a_n|^2} + 2 \int_{\partial\mathbb{D}} \frac{1}{|\xi - \xi_0|^2} d\mu(\xi).$$

The case when f is a Blaschke product was already observed by Frostman in [20].

2.1.3 Clark measures and analytic continuation

Fix an analytic self-map of the unit disc f. For any point $\alpha \in \partial\mathbb{D}$, the associated function

$$z \mapsto \operatorname{Re} \frac{\alpha + f(z)}{\alpha - f(z)} = \frac{1 - |f(z)|^2}{|\alpha - f(z)|^2}, \tag{2.7}$$

is harmonic and non-negative on $\mathbb{D}$. Hence, by Herglotz-Riesz representation theorem, there exists a finite and non-negative measure μ_α on $\partial\mathbb{D}$ such that:

$$\operatorname{Re} \frac{\alpha + f(z)}{\alpha - f(z)} = \int_{\partial\mathbb{D}} \frac{1 - |z|^2}{|\xi - z|^2} d\mu_\alpha(\xi) = \int_{\partial\mathbb{D}} \operatorname{Re} \frac{\xi + z}{\xi - z} d\mu_\alpha(\xi) =: P[\mu_\alpha](z), \quad z \in \mathbb{D}. \tag{2.8}$$

The elements of the family $\{\mu_\alpha\}_{\alpha \in \partial\mathbb{D}}$ are called *Clark measures* (for the function f). See [32] and [30] for good surveys on these measures and their applications in complex analysis and operator theory. If we denote by ν_α the Lebesgue absolutely continuous part of the Clark measure μ_α, then it is classical that $P(\mu_\alpha)(r\xi) \to \nu_\alpha(\xi)$ as $r \to 1^-$, for a.e. $\xi \in \partial\mathbb{D}$. On the other hand, it is clear by (2.7) that

$\lim_{r\to 1^-} P(\mu_\alpha)(r\xi) = 0$, at least at every Fatou's point $\xi \notin f^{-1}(\alpha)$, from which follows the following alternative definition of inner function.

Theorem 2.1.6 (see [32], theorem 2.2)**.** *An analytic self-map f of the unit disc is an inner function if and only if the associated Clark measure μ_α is strictly singular for some (equivalently, for all) $\alpha \in \partial\mathbb{D}$.*

Moreover (see again [32], theorem 2.2) the singular part of any Clark measure μ_α is supported in $f^{-1}(\alpha)$. It follows that when f is an inner function,

$$\mu_\alpha \left(\partial\mathbb{D} \setminus \{\xi \in \partial\mathbb{D} : \angle \lim_{z\to\xi} f(z) = \alpha\} \right) = 0. \tag{2.9}$$

It is worth to mention that Clark measures also provide a further characterization of the points at which an analytic self map of the unit disc f admits finite angular derivative. In particular, f admits finite angular derivative at $\xi_0 \in \partial\mathbb{D}$ if and only if the Clark measure $\mu_{f(\xi_0)}$ has a point mass at ξ_0 (see [32, Theorem 3.1]). No other Clark measure can have a point mass at ξ_0: $\mu_\alpha(\{\xi_0\}) > 0$ implies $\alpha = f(\xi_0)$ and $|f'(\xi_0)| < \infty$.

There are different ways to classify points of the unit circle with respect to an inner function f. We have seen that we can distinguish points which are Fatou's for f and points which are not, or points which have finite angular derivative and points which have not. We introduce now a further classification method. Every inner function f partitions the unit circle into two sets: the points at which f can be extended analytically and the points at which it cannot. The latter is called the set of *singular points* of f, denoted by $\mathrm{Sing}(f)$, and it can be proved (see [21], theorem 6.1 and 6.2) that, if $f = z^k B_{\{z_j\}} S_\mu$, then

$$\mathrm{Sing}(f) = \overline{\{z_j\}} \cup \mathrm{supp}(\mu).$$

Observe that singular points can well admit finite angular derivative. For example, consider a Blaschke product B with zeros $\{a_n\}$ tending tangentially to 1 in a way such that

$$\sum_{n=0}^{\infty} \frac{1-|a_n|^2}{|1-a_n|^2} < +\infty. \tag{2.10}$$

Then, by Theorem 2.1.5, at the singular point 1 we have $|B'(1)| < \infty$.

Nevertheless, the behavior of an inner function at singular points is very wild, in the following sense: let f be an inner function and $\xi_0 \in \partial\mathbb{D}$. Then, either f extends analytically in an open neighbourhood of ξ_0, or (see [21], theorem 6.6),

$$C_{\mathbb{D}}(f, \xi_0) = \overline{\mathbb{D}}. \tag{2.11}$$

Here, $C_{\mathbb{D}}(f,\xi_0)$ is the *cluster set* of f at ξ_0, which for any function f which is meromorphic and sigle-valued on the unit disc and for any $\xi_0 \in \partial\mathbb{D}$ is defined as

$$C_{\mathbb{D}}(f,\xi_0) = \left\{\alpha \in \mathbb{C} \cup \{\infty\} : \text{ there exists } \{z_n\} \subseteq \mathbb{D} \text{ s.t. } \lim_n z_n = \xi_0 \text{ and } \lim_n f(z_n) = \alpha\right\}.$$

In fact, we can show that this bahavior holds in a stronger sense for the boundary map induced by f.

Theorem 2.1.7. *Let f be an inner function and $\xi_0 \in \partial\mathbb{D}$ a singular point. Then, for every arc I containing ξ_0 it holds $f(I \setminus \{\xi_0\}) = \partial\mathbb{D}$.*

Proof. Let ξ_0 be a singular point for f and fix $\alpha \in \partial\mathbb{D}$. For $\delta > 0$, let $I_\delta = \{\xi \in \partial\mathbb{D} : |\xi - \xi_o| < \delta\}$. We want to show that for any $\delta > 0$ there exists a Fatou's point $\xi \in I_\delta$, such that $f(\xi) = \alpha$. By (2.9) it suffices to show that $\mu_\alpha(I_\delta) > 0$ for any $\delta > 0$. The Clark measure μ_α satisfies (2.8), hence the analytic function

$$g(z) = \frac{\alpha + f(z)}{\alpha - f(z)} - \int_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z} d\mu_\alpha(\xi),$$

has zero real part. Therefore, $g \equiv iC$, for some real constant C. In particular, since $\mu_\alpha(\partial\mathbb{D}) = P[\mu_\alpha](0)$, we get $C = \operatorname{Im} \frac{\alpha+f(0)}{\alpha-f(0)}$.

Now, suppose that $\mu_\alpha(I_\delta) = 0$ for some $\delta > 0$. Then,

$$\frac{\alpha + f(z)}{\alpha - f(z)} - iC = \int_{\partial\mathbb{D}} \frac{\xi + z}{\xi - z} d\mu_\alpha(\xi) = \int_{\partial\mathbb{D} \setminus I_\delta} \frac{\xi + z}{\xi - z} d\mu_\alpha(\xi),$$

implying that the left handside extends analytically through the singular point ξ_0. But by (2.11), we know that there exists a sequence of points $\{z_n\} \subseteq \mathbb{D}$ tending to ξ_0 for which $f(z_n) \to \alpha$. Hence the denominator in the right hand side can get arbitrarily small as $z \to \xi_0$, leading to a contradiction.

□

2.2 Distribution and distortion of sets under inner functions

2.2.1 Motivation and statement of results

Denote by λ the normalized Lebesgue measure on $\partial\mathbb{D}$ and by λ_z the harmonic measure from the point $z \in \mathbb{D}$, given by

$$\lambda_z(E) = \int_E \frac{1 - |z|^2}{|\xi - z|^2} d\lambda(\xi),$$

for any measurable set $E \subseteq \partial\mathbb{D}$. A classical result due to Löwner (see, for instance, page 12 of [3]) says that Lebesgue measure is invariant under the action of any inner function fixing the origin. More generally, the following conformally invariant version of Löwner's Lemma holds.

Theorem 2.2.1 (conformal invariant version of [3], theorem 1.6)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be an inner function and $z \in \mathbb{D}$. Then,*

$$\lambda_z(f^{-1}(E)) = \lambda_{f(z)}(E)$$

for any measurable set $E \subseteq \partial\mathbb{D}$.

Observe that, if $z \in \mathbb{D}$ is a fixed point of f, Theorem 2.2.1 says that λ_z is invariant under the action of f. So, morally the above theorem says that if $f(z) = z$ then if one looks at the sets E and $f^{-1}(E)$ from the base point z they appear to have the same size However, it may be the case that f has no fixed points in $\mathbb{D}$ but only on $\partial\mathbb{D}$. For example, consider the function

$$f(z) = \frac{2z^3 + 1}{2 + z^3}.$$

It is easy to check that f is an inner function and that its only fixed points are $+1$ and -1. In particular $+1$ is the Denjoy-Wolff point, since $|f'(1)| = 1$ while $|f'(-1)| = 9$.

A question arises naturally: does it exists a measure which is invariant under the action of an inner function with no interior fixed points? In other words, can we find an analogue of Theorem 2.2.1 when $z \in \partial\mathbb{D}$? The idea is to look for a measure on $\partial\mathbb{D}$ which allows to measure sets with respect to boundary points. This can be seen as an analogue of the harmonic measure in which the base point can lie on the boundary. To this goal we consider a measure introduced by Doering and Mañé in [17]. Fix a point $p \in \overline{\mathbb{D}}$ and consider the positive measure μ_p on $\partial\mathbb{D}$ defined by

$$\mu_p(E) = \int_E \frac{1}{|\xi - p|^2}\, d\lambda(\xi),$$

for any measurable set $E \subseteq \partial\mathbb{D}$. Observe that for a point $p \in \partial\mathbb{D}$ the measure μ_p is not finite, while for $p \in \mathbb{D}$, it is just a scalar multiple of the harmonic measure given by $\mu_p = (1 - |p|^2)^{-1}\lambda_p$. Also, it is clear that μ_p is absolutely continuous with respect to the Lebesgue measure: $\mu_p(E) = 0$ for some (equivalently, for all) p if and only if $\lambda((E) = 0$. A very natural interpretation of the measure μ_p when $p \in \partial\mathbb{D}$ is the following. Let $\omega_p\colon \mathbb{D} \to \mathbb{H}$ be the conformal map from the disc into the upper half-plane $\mathbb{H}$ such that $\omega_p(p) = \infty$ and $\omega_p(0) = i/2$. Then, for any measurable set $E \subseteq \partial\mathbb{D}$, we have that $\mu_p(E) = |\omega_p(E)|$, where we denote by $|A|$ the Lebesgue measure of a set $A \subseteq \mathbb{R}$. Roughly speaking, for a point $p \in \partial\mathbb{D}$, the measure μ_p gives information about the size and the distribution of a set around the point p. Sets having large μ_p measure are those that are highly concentrated around the point p. In particular, if E is an open neighbourhood of p, then $\mu_p(E) = \infty$. Our first result is the following analogue of Theorem 2.2.1.

Theorem 2.2.2. *Let $f: \mathbb{D} \to \mathbb{D}$ be an inner function and let $p \in \partial\mathbb{D}$ be a boundary Fatou point of f.*

(a) Assume $|f'(p)| < \infty$. Then

$$\mu_p(f^{-1}(E)) = |f'(p)| \mu_{f(p)}(E)$$

for any measurable set $E \subseteq \partial\mathbb{D}$.

(b) If $|f'(p)| = \infty$ and $E \subseteq \mathbb{D}$ is a measurable set, then $\mu_p(f^{-1}(E)) = \infty$ if $\lambda(E) > 0$ and $\mu_p(f^{-1}(E)) = 0$ if $\lambda(E) = 0$.

As we can see, we still have a general relation between the measure of a set and its preimage under f, independent of the set. Nonetheless, in this case, a distortion factor appears and it is given by the size of the angular derivative at the point p. If $p \in \partial\mathbb{D}$ is the Denjoy-Wolff fixed point of f, this result was previously proved in [17].

In [19], Fernández and Pestana studied the distortion of Hausdorff contents under inner functions. For any fixed $z \in \mathbb{D}$ and $0 < \alpha < 1$, consider the Hausdorff content defined as

$$M_\alpha(\lambda_z)(E) = \inf \sum_j \lambda_z(I_j)^\alpha,$$

where the infimum is taken over all collections of arcs $\{I_j\}$ of the unit circle such that $E \subseteq \bigcup I_j$. Thus $M_\alpha(\lambda_0)(E)$ is the standard Hausdorff content of E, which is denoted by $M_\alpha(E)$. Observe that if $z \in \mathbb{D}$ and τ is the automorphism of $\mathbb{D}$ which interchanges z and 0, then $M_\alpha(\lambda_z)(E) = M_\alpha(\tau^{-1}(E))$ for any $E \subseteq \partial\mathbb{D}$. Fernández and Pestana proved the following result, analogous to Theorem 2.2.1 for Hausdorff contents, stated here in a conformally invariant way.

Theorem 1. For any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$ such that, if $f: \mathbb{D} \to \mathbb{D}$ is an inner function and $z \in \mathbb{D}$, we have

$$M_\alpha(\lambda_z)(f^{-1}(E)) \geq C_\alpha M_\alpha(\lambda_{f(z)})(E)$$

for any Borel set $E \subseteq \partial\mathbb{D}$.

It is also shown in [19] that there exists an inner function f such that the preimage of a single point has Hausdorff dimension 1. Hence, the converse estimate in Theorem 1 is false. It is worth mentioning that a related result for sets $E \subseteq \mathbb{D}$ was established in [23].

For $0 < \alpha < 1$ and $p \in \partial\mathbb{D}$, we define the (p,α)-Hausdorff content of a Borel set $E \subseteq \partial\mathbb{D}$ as

$$M_\alpha(\mu_p)(E) \coloneqq \inf \sum_j \mu_p(I_j)^\alpha,$$

where the infimum is taken over all collections of arcs $\{I_j\}$ of the unit circle such that $E \setminus \{p\} \subseteq \bigcup I_j$. Our second result is the following analogue of Theorem 1 when $z \in \partial\mathbb{D}$.

Theorem 2.2.3. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be an inner function and let $p \in \partial\mathbb{D}$ be a boundary Fatou point of f.*

(a) Assume $|f'(p)| < \infty$. Then for any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$, independent of f, such that

$$M_\alpha(\mu_p)(f^{-1}(E)) \geq C_\alpha |f'(p)|^\alpha M_\alpha(\mu_{f(p)})(E)$$

for any Borel set $E \subseteq \partial\mathbb{D}$.

(b) Assume $|f'(p)| = \infty$. Then we have that $M_\alpha(\mu_p)(f^{-1}(E)) = \infty$ for any Borel set $E \subseteq \partial\mathbb{D}$ such that $M_\alpha(E) > 0$.

The proofs of Theorem 2.2.2 and Theorem 2.2.3 are given in Section 2.2.2.

2.2.2 Boundary distortion theorems

In this section we prove our main results. We start with some elementary properties of the measure μ_p and the content $M_\alpha(\mu_p)$. Recall that a sequence of points $\{p_n\} \subseteq \mathbb{D}$ converges non-tangentially to a point $p \in \partial\mathbb{D}$ if $\lim p_n = p$ and there exists $\beta > 0$ such that $\{p_n\} \subseteq \Gamma_\beta(p)$.

Lemma 2.2.4. *Let $p \in \partial\mathbb{D}$. For every sequence of points $\{p_n\} \subseteq \mathbb{D}$ converging non-tangentially to p, we have*

$$\mu_{p_n}(E) \longrightarrow \mu_p(E), \qquad \text{as } n \to \infty,$$

for any measurable set $E \subseteq \partial\mathbb{D}$.

Proof. Let $\{p_n\}_n \subseteq \mathbb{D}$ be any sequence of points approaching p, and write $\mu_n = \mu_{p_n}$ for every $n \geq 1$. By Fatou's Lemma, we have

$$\liminf_n \mu_n(E) \geq \int_E \lim_n \frac{1}{|\xi - p_n|^2}\, d\lambda(\xi) = \mu_p(E),$$

from which it follows that the result is true when $\mu_p(E) = \infty$. So assume $\mu_p(E) < \infty$. Fix $\varepsilon > 0$ and consider an arc I centred at p and such that $\mu_p(E \cap I) < \varepsilon$. Since $p_n \to p$ non-tangentially, there exists a constant $C > 0$ such that $|\xi - p_n| \geq C|\xi - p|$ for every $\xi \in \partial\mathbb{D}$ and every $n \geq 1$. Hence, we have that $\mu_n(E \cap I) \leq C^{-2}\varepsilon$ for every n. On the other hand, by dominated convergence, we have that

$$\mu_n(E \cap (\partial\mathbb{D} \setminus I)) \longrightarrow \mu_p(E \cap (\partial\mathbb{D} \setminus I)), \qquad \text{as } n \to \infty,$$

from which the result follows. □

Observe that the assumption on the non-tangential convergence of the sequence $\{p_n\}$ to p only enters into play if $p \in \overline{E}$. If $p \notin \overline{E}$, the result holds true for any approaching sequence. However, as the following example shows, Lemma 2.2.4 fails badly if p_n approaches p tangentially. Fix a point $p \in \partial\mathbb{D}$ and consider a sequence of points $\{\xi_n\} \subseteq \partial\mathbb{D}$ such that $|\xi_n - p| = 1/(2n)$ for every $n \geq 1$. Consider as well the sequence of pairwise disjoint arcs $\{I_n\}$ such that I_n is centred at ξ_n and $\lambda(I_n) = 1/(4n^4)$ for every $n \geq 1$. Now, let $E \coloneqq \bigcup_n I_n$, $p_n = (1 - \lambda(I_n))\,\xi_n$, and $\mu_n = \mu_{p_n}$, for every $n \geq 1$. Since $(1 - |p_n|)/|p - p_n| \leq 1/n^3 \longrightarrow 0$, the sequence $\{p_n\}$ converges to p tangentially. For $\xi \in I_n$, we have $|p_n - \xi| \leq 2\lambda(I_n)$ and $\mu_n(I_n) \geq (4\lambda(I_n))^{-1} = n^4$. Now, on one hand we have $\mu_n(E) \geq \mu_n(I_n) \longrightarrow \infty$, as $n \to \infty$. On the other hand since $|p - \xi| \leq 1/n$ for any $\xi \in I_n$, we have $\mu_p(I_n) \leq n^2\lambda(I_n) = 1/4n^2$ and we deduce

$$\mu_p(E) = \sum_n \mu_p(I_n) < \infty.$$

For $0 < \alpha < 1$ and $z \in \mathbb{D}$ consider the (z,α)-Hausdorff content of a Borel set $E \subseteq \partial\mathbb{D}$ defined as

$$M_\alpha(\mu_z)(E) = \inf \sum_j \mu_z(I_j)^\alpha,$$

where the infimum is taken over all collections of arcs $\{I_j\}$ such that $E \subseteq \bigcup I_j$.

Lemma 2.2.5. *Given $p \in \partial\mathbb{D}$ and $\beta > 0$, let $\Gamma_\beta(p)$ be the Stolz angle of opening β with vertex at p. Then there exists a constant $C = C(\beta) > 0$ such that*

$$\mu_z(A) \leq C\mu_p(A)$$

for any measurable set $A \subseteq \partial\mathbb{D}$ and any $z \in \Gamma_\beta(p)$. Consequently, for any $0 < \alpha < 1$ we also have $M_\alpha(\mu_z)(A) \leq C^\alpha M_\alpha(\mu_p)(A)$ for any set $A \subseteq \partial\mathbb{D}$ and any $z \in \Gamma_\beta(p)$.

Proof. Observe that there exists a constant $C = C(\beta) > 0$ such that $|\xi - z| \geq C|\xi - p|$ for any $z \in \Gamma_\beta(p)$ and any $\xi \in \partial\mathbb{D}$. Hence, $\mu_z(A) \leq C^{-2}\mu_p(A)$ for any measurable set $A \subseteq \partial\mathbb{D}$ and any $z \in \Gamma_\beta(p)$. This last estimate also gives $M_\alpha(\mu_z)(A) \leq C^{-2\alpha}M_\alpha(\mu_p)(A)$. □

The corresponding result to Lemma 2.2.4 for Hausdorff contents reads as follows.

Lemma 2.2.6. *Let* $0 < \alpha < 1$ *and* $p \in \partial\mathbb{D}$. *For any sequence of points* $\{p_n\} \subseteq \mathbb{D}$ *converging non-tangentially to* p, *we have*

$$\lim_{n\to\infty} M_\alpha(\mu_{p_n})(E) = M_\alpha(\mu_p)(E) \tag{2.12}$$

for any set $E \subseteq \partial\mathbb{D}$.

Proof. Write $\mu_n = \mu_{p_n}$ for every $n \geq 1$. Assume that $M_\alpha(\mu_p)(E) < \infty$. In this case, we split the proof of the result into two parts. First we show that

$$\limsup_{n\to\infty} M_\alpha(\mu_n)(E) \leq M_\alpha(\mu_p)(E), \tag{2.13}$$

and then we prove that

$$\liminf_{n\to\infty} M_\alpha(\mu_n)(E) \geq M_\alpha(\mu_p)(E), \tag{2.14}$$

from which (2.12) follows immediately. To prove (2.13), given $\varepsilon > 0$, take a covering of the set $E \setminus \{p\}$ by open arcs $\{I_j\}$ such that

$$\sum_j \mu_p(I_j)^\alpha \leq M_\alpha(\mu_p)(E) + \varepsilon.$$

Now, by Lemma 2.2.5, for each interval I_j and for every $n \geq 1$ we have that

$$\mu_n(I_j) \leq C\mu_p(I_j).$$

Thus, by Lemma 2.2.4 and dominated convergence, we get that

$$\sum_j \mu_n(I_j)^\alpha \longrightarrow \sum_j \mu_p(I_j)^\alpha, \qquad \text{as } n \to \infty.$$

By definition, $M_\alpha(\mu_n)(E) \leq \sum_j \mu_n(I_j)^\alpha$ and, thus (2.13) follows immediately.

We prove inequality (2.14) by considering two cases. Assume first that $p \notin \overline{E}$. Pick $\varepsilon > 0$ and a covering of E by open arcs $\{I_j\}$, such that $\operatorname{dist}(I_j, p) \geq \operatorname{dist}(\overline{E}, p)/2$ for every arc I_j. Observe that, in this situation, there exists $n_0 > 0$ such that if $n > n_0$, we have that

$$\mu_n(I_j) \geq (1-\varepsilon)^{1/\alpha}\mu_p(I_j)$$

for every arc I_j in our covering. Thus, for any such covering of $E \setminus \{p\}$, if $n > n_0$ we have that

$$\sum_j \mu_n(I_j)^\alpha \geq (1-\varepsilon) M_\alpha(\mu_p)(E).$$

Observe that the infimum of $\sum_j \mu_n(I_j)^\alpha$ when ranging over all coverings $\{I_j\}$ of $E \setminus \{p\}$ by open arcs satisfying that $\operatorname{dist}(I_j, p) \geq \operatorname{dist}(\overline{E}, p)/2$ is, precisely, $M_\alpha(\mu_n)(E)$. Hence, equation (2.14) follows in the case that $p \notin \overline{E}$, and therefore equation (2.12) as well in this situation.

In the case that $p \in \overline{E}$, since we assumed that $M_\alpha(\mu_p)(E) < \infty$, given $\varepsilon > 0$ we can choose $\delta = \delta(\varepsilon) > 0$ such that $M_\alpha(\mu_p)(E \cap I(p,\delta)) < \varepsilon$, where $I(p,\delta)$ denotes the arc centred at p of length δ. Let us denote $E_\delta = E \setminus I(p,\delta)$. Since $p \notin \overline{E_\delta}$, we already know that

$$\lim_{n\to\infty} M_\alpha(\mu_n)(E_\delta) = M_\alpha(\mu_p)(E_\delta) \geq M_\alpha(\mu_p)(E) - \varepsilon.$$

Hence, for any given $\varepsilon > 0$, we have

$$\liminf_{n\to\infty} M_\alpha(\mu_n)(E) \geq \lim_{n\to\infty} M_\alpha(\mu_n)(E_\delta) \geq M_\alpha(\mu_p)(E) - \varepsilon.$$

This concludes the proof whenever $M_\alpha(\mu_p)(E) < \infty$.

Assume now that $M_\alpha(\mu_p)(E) = \infty$. In this case, for any $N > 0$ we can find $\delta = \delta(N) > 0$ such that $M_\alpha(\mu_p)(E_\delta) > N$, where again $E_\delta = E \setminus I(p,\delta)$. Since $p \notin \overline{E_\delta}$, we have that

$$\lim_{n\to\infty} M_\alpha(\mu_n)(E_\delta) = M_\alpha(\mu_p)(E_\delta) > N.$$

Hence, there exists $n_0 > 0$ such that if $n > n_0$, then $M_\alpha(\mu_n)(E_\delta) > N$. Using that $M_\alpha(\mu_n)(E) \geq M_\alpha(\mu_n)(E_\delta)$, we get (2.12) in the case in which $M_\alpha(\mu_p)(E) = \infty$ as well. □

We will use the following immediate consequence of the Julia-Carathéodory theorem.

Lemma 2.2.7. *Let f be a holomorphic self map of the unit disc. Let $\{p_n\}$ be a sequence of points in $\mathbb{D}$ converging non-tangentially to a point $p \in \partial\mathbb{D}$. If $|f'(p)| < \infty$, then $\{f(p_n)\}$ converges to $f(p) \in \partial\mathbb{D}$ non-tangentially.*

Proof. Since $|f'(p)| < \infty$ we have that $f(p) \in \partial\mathbb{D}$. Write

$$\frac{1-|f(p_n)|}{|f(p)-f(p_n)|} = \frac{1-|f(p_n)|}{1-|p_n|}\frac{1-|p_n|}{|p-p_n|}\frac{|p-p_n|}{|f(p)-f(p_n)|}.$$

Also because $|f'(p)| < \infty$, by Julia-Carathéodory Theorem, the first and third terms converge respectively to $|f'(p)|$ and $|f'(p)|^{-1}$, and therefore

$$\liminf_n \frac{1-|f(p_n)|}{|f(p)-f(p_n)|} = \liminf_n \frac{1-|p_n|}{|p-p_n|} > 0.$$

□

Note that the assumption of finite angular derivative is necessary in the above statement, even if we ask for the function f to be inner. In fact, it can be proved that there exist inner functions mapping a given Stolz angle to a tangential region (see [18]).

We are now ready to prove our main results.

Proof of Theorem 2.2.2. We can choose a sequence of points $\{p_n\}$ in $\mathbb{D}$ approaching p non-tangentially such that

$$\lim_{n\to\infty} \frac{1-|f(p_n)|^2}{1-|p_n|^2} = |f'(p)| > 0. \tag{2.15}$$

By Theorem 2.2.1, we have that

$$\mu_{p_n}(f^{-1}(E)) = \frac{1-|f(p_n)|^2}{1-|p_n|^2}\mu_{f(p_n)}(E). \tag{2.16}$$

Lemma 2.2.4 gives that $\mu_{p_n}(f^{-1}(E)) \to \mu_p(f^{-1}(E))$ as $n \to \infty$. If $|f'(p)| < \infty$, applying Lemma 2.2.7 one deduces that $f(p_n)$ converges to $f(p)$ non-tangentially. Thus, Lemma 2.2.4 gives that $\mu_{f(p_n)}(E) \to \mu_{f(p)}(E)$ as $n \to \infty$. Therefore, equations (2.15) and (2.16) prove the statement (a). Assume now that $|f'(p)| = \infty$. If $\mu_{f(p)}(E) = 0$, we have $\lambda(E) = 0$. Hence, by Theorem 2.2.1, we have that $\lambda(f^{-1}(E)) = 0$ and it follows that $\mu_p(f^{-1}(E)) = 0$. Finally assume $\mu_{f(p)}(E) > 0$. Observe that for any $n \geq 1$ we have $\mu_{f(p_n)}(E) > \lambda(E)/4 > 0$. Thus, since $|f'(p)| = \infty$, the right-hand side of equation (2.16) tends to infinity and, by Lemma 2.2.4, we deduce that $\mu_p(f^{-1}(E)) = \infty$. □

Proof of Theorem 2.2.3. We will use Theorem 1 in the following form. For $z \in \mathbb{D}$ we have that

$$M_\alpha(\mu_z)(f^{-1}(E)) \geq C_\alpha \left(\frac{1-|f(z)|^2}{1-|z|^2}\right)^\alpha M_\alpha(\mu_{f(z)})(E) \tag{2.17}$$

for any Borel set $E \subseteq \partial\mathbb{D}$. We can choose a sequence of points $\{p_n\}$ in $\mathbb{D}$ approaching p non-tangentially such that

$$\lim_{n\to\infty} \frac{1-|f(p_n)|^2}{1-|p_n|^2} = |f'(p)| > 0. \tag{2.18}$$

Assume $|f'(p)| < \infty$. Applying Lemma 2.2.6 and equation (2.17), we get

$$\begin{aligned} M_\alpha(\mu_p)(f^{-1}(E)) &= \lim_{n\to\infty} M_\alpha(\mu_{p_n})(f^{-1}(E)) \\ &\geq \limsup_{n\to\infty} C_\alpha \left(\frac{1-|f(p_n)|^2}{1-|p_n|^2}\right)^\alpha M_\alpha(\mu_{f(p_n)})(E) \\ &= C_\alpha |f'(p)|^\alpha \limsup_{n\to\infty} M_\alpha(\mu_{f(p_n)})(E). \end{aligned}$$

By Lemma 2.2.7, $f(p_n)$ tends to $f(p)$ non-tangentially as $n \to \infty$ and hence, Lemma 2.2.6 gives that

$$\lim_{n\to\infty} M_\alpha(\mu_{f(p_n)})(E) = M_\alpha(\mu_{f(p)})(E),$$

which finishes the proof of part (a). Assume now $|f'(p)| = \infty$. We can assume $f(p) \notin E$. Since $M_\alpha(\mu_{f(p)})(E) > 0$, there exists an arc I centred at $f(p)$ such that $M_\alpha(\mu_{f(p)})(E \setminus I) > 0$. Write $E^* = E \setminus I$. Then there exists $n_0 > 0$ such that $M_\alpha(\mu_{f(p_n)})(E^*) > M_\alpha(\mu_{f(p)})(E^*)/2$ if $n > n_0$. Now,

$$\begin{aligned} M_\alpha(\mu_p)(f^{-1}(E^*)) &= \lim_{n\to\infty} M_\alpha(\mu_{p_n})(f^{-1}(E^*)) \\ &\geq C_\alpha \limsup_{n\to\infty} \left(\frac{1-|f(p_n)|^2}{1-|p_n|^2} \right)^\alpha M_\alpha(\mu_{f(p_n)})(E^*) = \infty. \end{aligned}$$

Hence $M_\alpha(\mu_p)(f^{-1}(E)) = \infty$. □

2.2.3 Omitted values

In this section we aim to present a first application of our results. A classical result by Frostman says that any inner function f can omit at most a set of logarithmic capacity zero, that is, $\mathbb{D} \setminus f(\mathbb{D})$ has logarithmic capacity zero (it follows from theorem 2.1.4). Conversely, given a relatively compact set K of the unit disc having logarithmic capacity zero, the universal covering map $f: \mathbb{D} \to \mathbb{D} \setminus K$ is an inner function (see page 323 of [37]). Given a set $E \subseteq \mathbb{D}$, its *non-tangential closure* on $\partial\mathbb{D}$, denoted by E^{NT}, is the set of points $\xi \in \partial\mathbb{D}$ for which there exists a sequence $\{z_n\} \subseteq E$ such that $z_n \to \xi$ non-tangentially. We first state an auxiliary result which may have independent interest.

Lemma 2.2.8. *Let $f: \mathbb{D} \to \mathbb{D}$ be an inner function and let $E = \mathbb{D} \setminus f(\mathbb{D})$ be the set of its omitted points. Then*

$$f^{-1}(E^{NT}) \subseteq \{\xi \in \partial\mathbb{D}: |f'(\xi)| = \infty\}.$$

Proof. Consider a point $\xi \in \partial\mathbb{D}$ such that the angular derivative of f at ξ exists and it is finite, and let $\xi = f(\xi)$. In other words assume that

$$\lim_{\Gamma_\beta(\xi) \ni z \to \xi} \frac{\xi - f(z)}{\xi - z} = A \tag{2.19}$$

is finite. We want to see that, in this situation, for any opening $\gamma > 1$, there is $0 < s = s(\gamma) < 1$ such that the truncated cone

$$\Gamma_{\gamma,s}(\xi) = \{w \in \mathbb{D}: |\xi - w| < \gamma(1-|w|), |\xi - w| < s\}$$

does not intersect E, that is, $\Gamma_{\gamma,s}(\xi) \subseteq f(\mathbb{D})$. So fix $\gamma > 1$ and consider $\Gamma_{\gamma,s}(\xi)$ with $0 < s < 1$ to be determined. Fix $w_0 \in \Gamma_{\gamma,s}(\xi)$. We want to see that there is $z_0 \in \mathbb{D}$ such that $f(z_0) = w_0$. By equation (2.19), we can express

$$f(z) = \xi + A(z - \xi) + o(|z - \xi|),$$

where $o(|z-\xi|)/|z-\xi| \to 0$ as $z \to \xi$ non-tangentially. Consider $\Gamma_{\beta,r}(\xi)$ with $\beta > 2\gamma$ and $0 < r < 1$ to be determined. Observe that there exists $0 < r_0 < 1$ such that, if $r < r_0$ and $0 < s < |A|r/2$, then for any $z \in \partial\Gamma_{\beta,r}(\xi)$ we have that

$$|(f(z) - w_0) - (\xi + A(z - \xi) - w_0)| < |\xi + A(z - \xi) - w_0|.$$

Thus, by Rouché's Theorem, the functions $f(z)-w_0$ and $g(z)-w_0 = \xi+A(z-\xi)-w_0$ have the same number of zeroes in $\Gamma_{\beta,r}(\xi)$. But $g(z)$ is a degree 1 polynomial and $g(\Gamma_{\beta,r}(\xi)) = \Gamma_{\beta,|A|r}(\xi) \supseteq \Gamma_{\gamma,s}(\xi)$, and thus $g(z) - w_0$ has a single zero on $\Gamma_{\beta,r}(\xi)$. Therefore, there is $z_0 \in \Gamma_{\beta,r}(\xi)$ such that $f(z_0) = w_0$, which completes the proof. □

As an application of Theorem 2.2.3 and Lemma 2.2.8, we have the following result.

Corollary 2.2.9. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be an inner function and let $E = \mathbb{D} \setminus f(\mathbb{D})$ be the set of its omitted points. Let p be a boundary Fatou point of f.*

(a) Assume $|f'(p)| < \infty$. Then for any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$, independent of f, such that

$$M_\alpha(\mu_p)\left(\{\xi \in \partial\mathbb{D} : |f'(\xi)| = \infty\}\right) \geq C_\alpha |f'(p)|^\alpha M_\alpha(\mu_{f(p)})(E^{NT}). \tag{2.20}$$

(b) Assume $|f'(p)| = \infty$. Then, whenever $M_\alpha(\mu_{f(p)})(E^{NT}) > 0$,

$$M_\alpha(\mu_p)\left(\{\xi \in \partial\mathbb{D} : |f'(\xi)| = \infty\}\right) = \infty.$$

2.2.4 Inner functions in the upper half plane

Let $\mathbb{H} = \{w \in \mathbb{C} : \Im(w) > 0\}$ be the upper half plane. A holomorphic mapping $g\colon \mathbb{H} \to \mathbb{H}$ is an *inner function* of the upper half plane if $\lim_{y\to 0} g(x+iy) \in \mathbb{R}$ for a.e. $x \in \mathbb{R}$. This natural definition agrees with conformal changes of coordinates: given $p \in \partial\mathbb{D}$ denote by w_p the Möbius transformation mapping $\mathbb{D}$ onto $\mathbb{H}$, the point p to ∞ and, say, the origin to $i/2$. Then, g is an inner function of the upper half plane if and only if $f = w_p^{-1} \circ g \circ w_p$ is an inner function of the unit disc $\mathbb{D}$. Observe that $g(\infty) = \lim_{t\to+\infty} g(it) = \infty$ if and only if $f(p) = p$. A holomorphic mapping g from $\mathbb{H}$ into $\mathbb{H}$ has a finite angular derivative at ∞ if

$$g'(\infty) = \lim_{t\to+\infty} \frac{it}{g(it)}$$

exists and is finite. Otherwise, we write $|g'(\infty)| = \infty$. Observe that g has a finite angular derivative at infinity if and only if $f = w_p^{-1} \circ g \circ w_p$ has a finite angular derivative at p. Let w denote $w_p(z)$. Moreover, the identity $|g'(\infty)| = |f'(p)|$ holds in the sense that both quantities coincide when they are finite, and if one of them is infinite so is the other. This fact easily follows from the identity

$$\frac{w}{g(w)} = \frac{p+z}{p+f(z)} \frac{p-f(z)}{p-z}.$$

Let $|A|$ denote the Lebesgue measure of a measurable set $A \subseteq \mathbb{R}$ and, for $0 < \alpha < 1$, let $M_\alpha(A)$ denote its α-Hausdorff content. We now state the versions of Theorems 2.2.2 and 2.2.3 in this setting.

Corollary 2.2.10. *Let $g\colon \mathbb{H} \to \mathbb{H}$ be an inner function and assume that $g(\infty) = \infty$.*

(a) Assume $|g'(\infty)| < \infty$. Then

$$|g^{-1}(A)| = |g'(\infty)||A| \tag{2.21}$$

for any measurable set $A \subseteq \mathbb{R}$. Moreover, for any $0 < \alpha < 1$ there exists a constant $C_\alpha > 0$, independent of g, such that

$$M_\alpha(g^{-1}(A)) \geq C_\alpha |g'(\infty)|^\alpha M_\alpha(A) \tag{2.22}$$

for any Borel set $A \subseteq \mathbb{R}$.

(b) If $|g'(\infty)| = \infty$ and $A \subseteq \mathbb{R}$ is a measurable set, then $|g^{-1}(A)| = \infty$ if $|A| > 0$ and $|g^{-1}(A)| = 0$ if $|A| = 0$. Moreover, $M_\alpha(g^{-1}(A)) = \infty$ for any Borel set $A \subseteq \mathbb{R}$ such that $M_\alpha(A) > 0$.

Proof. Note that for any measurable set $A \subseteq \mathbb{R}$ we have

$$|A| = \mu_p(w_p^{-1}(A)), \qquad p \in \partial\mathbb{D}. \tag{2.23}$$

Hence, $|g^{-1}(A)| = \mu_p(w_p^{-1}(g^{-1}(A))) = \mu_p(f^{-1}(w_p^{-1}(A)))$. Applying Theorem 2.2.2 and (2.23) we deduce that $|g^{-1}(A)| = |f'(p)|\mu_p(w_p^{-1}(A)) = |g'(\infty)||A|$ which is (2.21). It follows from (2.23) and w_p being a Möbius map that

$$M_\alpha(\mu_p)(E) = M_\alpha(w_p(E)), \quad E \subseteq \partial\mathbb{D}. \tag{2.24}$$

Thus, the previous argument shows that (2.22) holds. Part (b) follows from similar considerations. □

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