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New Results in Averaging Theory and its Applications

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New Results in Averaging Theory and its Applications

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I dedicate this work to my nephews Felippe and Gael.

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Chapter 1

Introduction and statements of the results

The averaging theory basically consists in replacing a vector field

$$x' = F(t, x, \varepsilon), \text{ with } (t, x, \varepsilon) \in \mathbb{R} \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0),$$

by its average over the time or over an angular variable with the goal to obtain asymptotic approximations to the solutions of the original system and to obtain periodic solutions. Although this theory was originated in the $18th$ century, until 1928 it was not proved rigorously by Fatou(see [30]).

The averaging theory for finding periodic solutions consists in providing sufficient conditions for the existence of periodic solutions in a vector field by studying the equilibrium points of its associated averaged system.

This theory becomes a classical tool for studying periodic solutions of nonlinear differential systems, see for instance [28, 56, 64, 67, 86]. Moreover, remarkable contributions to it were made by Krylov and Bogoliubov [45] in the 1930s and Bogoliubov [5] in 1945. For a brief historical review, the interested reader is referred to [68, Appendix A].

In this work we will improve the averaging theory for finding periodic solutions. Then we will propose a method for studying the stability of periodic solutions that are non linearly hyperbolic. Finally, using these new results we present several applications of the theory. In particular we shall apply the new theoretical result here presented to differential systems that could not be studied with the classical results.

The system $x' = F(t, x, 0)$ is called the *unperturbed system*. Concerning the averaging theory for finding limit cycles, two main hypotheses are usually assumed: (i) F is T-periodic in the first variable; and (ii) there exists a sub-manifold $\mathcal{W} \subset \mathbb{R}^n$ such that each solution of the unperturbed system with initial condition in $\mathcal{W}$ is T-periodic. Under these hypotheses the averaging theory provides sufficient conditions for the existence of limit cycles of $x' = F(t, x, \varepsilon)$.

The classical averaging theorem for the existence of limit cycles can be stated as follows. Consider the initial value problem

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 \widetilde{F}(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{1.1}$$

and

$$\dot{\mathbf{y}} = \varepsilon g_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{1.2}$$

with $\mathbf{x}$, $\mathbf{y}$, and $\boldsymbol{x_0}$ in some open Ω of $\mathbb{R}^n$, $t \in [0, \infty)$, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. We assume that F_1, $\widetilde{F}$ are T–periodic in the variable t, and we set

$$g_1(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t, \mathbf{y})dt. \tag{1.3}$$

Theorem 1. *Assume that F_1, $\widetilde{F}$, $D_x F_1$, $D_{xx} F_1$ and $D_x \widetilde{F}$ are continuous and bounded by a constant M independent of ε in $[0, \infty) \times \Omega \times [-\varepsilon_0, \varepsilon_0]$, and that $y(t) \in \Omega$ for $t \in [0, 1/|\varepsilon|]$. Then the following statements hold:*
(a) For $t \in [0, 1/|\varepsilon|]$ we have $\mathbf{x}(t) - \mathbf{y}(t) = O(\varepsilon)$ as $\varepsilon \to 0$.
(b) If $\boldsymbol{s}$ is a singular point of system (1.2) and $\det D_{\mathbf{y}} g_1(\boldsymbol{s}) \neq 0$, then there exists a T–periodic solution $\varphi(t, \varepsilon)$ for system (1.1) which is close to $\boldsymbol{s}$ and such that $\varphi(0, \varepsilon) - \boldsymbol{s} = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$.
(c) The stability of the periodic solution $\varphi(t, \varepsilon)$ is given by the stability of the singular point.

For a proof of Theorem 1 see [81, Theorem 11.5], where it is stated on the $\varepsilon \in [0, \varepsilon_0)$ but in fact following the proof the same result works for $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ as it is stated here.

In the last decade this theory has increased immensely. Several works have been dedicated to extend the averaging theory to a wider class of differential systems. For instance, in [11], taking advantage of the Brouwer degree theory, it was developed a topological version of the first-order averaging method to study the existence of limit cycles in continuous vector fields. Their stability properties were investigated in [7], and in [54] topological version of the averaging method was extended at any order. The averaging theory has also been considered in a discontinuous context. For instance, in [54, 50], the averaging method was developed up to order 2 for discontinuous differential system, and in [40, 52] the averaging method was extend at any order for a class of discontinuous differential system.

The first result here presented (see Theorem 2) provides sufficient conditions to assure the persistence of some zeros of smooth functions $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ having the form

$$g(z, \varepsilon) = g_0(z) + \sum_{i=1}^{k} \varepsilon^i g_i(z) + \mathcal{O}(\varepsilon^{k+1}). \tag{1.4}$$

The second one (see Theorem 5) provides sufficient conditions to assure the existence of periodic solutions of the following differential system

$$x' = F(t, z, \varepsilon) = F_0(t, x) + \sum_{i=1}^{k} \varepsilon^i F_i(t, x) + \mathcal{O}(\varepsilon^{k+1}), \quad (t, z) \in \mathbb{S}^1 \times \mathcal{D}. \tag{1.5}$$

Here $\mathbb{S}^1 = \mathbb{R}/T$, for some $T > 0$, and the assumption $t \in \mathbb{S}^1$ means that the system is T-periodic in the variable t. As usual $\delta_1(\varepsilon) = \mathcal{O}(\delta_2(\varepsilon))$ means that there exists a constant $c_0 > 0$, which does not depends on ε, such that $|\delta_1(\varepsilon)| \leq c_0 |\delta_2(\varepsilon)|$ for ε sufficiently small (see [68]).

The problem of existence of periodic solutions in system (1.5) can often be reduced to the problem of persistence of zeros of equation (1.4). Usually it is assumed that either

$g(z,0)$ vanishes in a submanifold of $\mathcal{Z} \subset \mathcal{D}$, or that the unperturbed differential system $x' = F_0(t,x)$ has a submanifold $\mathcal{Z} \subset \mathcal{D}$ of T-periodic solutions. In both cases $\dim(\mathcal{Z}) \leq n$.

We assume that for some $z^* \in \mathcal{Z}$, $g(z^*,0) = 0$. We shall study the persistence of this zero for the function (1.4), $g(x,\varepsilon)$, assuming that $|\varepsilon| \neq 0$ is sufficiently small. By *persistence* we mean the existence of continuous branches $\chi(\varepsilon)$ of simple zeros of $g(x,\varepsilon)$ (that is $g(\chi(\varepsilon),\varepsilon) = 0$) such that $\chi(0) = z^*$. It is well known that if the $n \times n$ matrix $\partial_x g(z^*,0)$ (the Jacobian matrix of the function g with respect to the variable x evaluated at $x = z^*$) is nonsingular then, from a direct consequence of the Implicit Function Theorem, there exists a unique smooth branch $\chi(\varepsilon)$ of zeros of $g(x,\varepsilon)$ such that $\chi(0) = z^*$. However if the matrix $\partial_x g(z^*,0)$ is singular (has non trivial kernel) we have to use the Lyapunov–Schmidt reduction method to find branches of zeros of g (see, for instance, [23]). Here we generalize some results from [8, 9, 51], providing a collection of functions f_i, $i = 1,\ldots,k$, each one called *bifurcation function of order* i, which control the persistence of zeros contained in $\mathcal{Z}$.

The problem of existence of periodic solutions of the differential system (1.5) goes back to the works of Malkin [56] and Roseau [67]. They have studied the case $k = 1$. Let $x(t,z,\varepsilon)$ denote the solution of system (1.5) such that $x(0,z,\varepsilon) = z$. In order to find initial conditions $z \in \mathcal{D}$ such that the solution $x(t,z,\varepsilon)$ is T-periodic we may consider the function $g(z,\varepsilon) = z - x(T,z,\varepsilon)$, and then try to use the results previously obtained about the persistence of zeros. Indeed, if $\mathcal{Z} \subset \mathcal{D}$ is a submanifold of T-periodic solutions of the unperturbed system $x' = F_0(t,x)$, then $g(z,0)$ vanishes on $\mathcal{Z}$. When $\dim(\mathcal{Z}) = n$ this problem is studied at an arbitrary order of ε, see [33, 53], even for nonsmooth systems. When $\dim(\mathcal{Z}) < n$, this approach has already been used in [8], up to order 1, and in [9, 10], up to order 2. In [51] this approach was used up to order 3 relaxing some hypotheses assumed in those previous 3 works. In [34] assuming the same hypotheses of [8, 9, 10] the authors studied this problem at an arbitrary order of ε. Here, following the ideas from [53, 51], we improve the results of [34] relaxing some hypotheses and developing the method in a more general way.

In summary, we use the Lyapunov–Schmidt reduction method for studying the zeros of functions like (1.4) when the Implicit Function Theorem cannot be directly applied. Another useful tool that we shall use to deal with this problem is the Brouwer degree theory (see Appendix B), which will allow to provide estimates for these zeros. Then we apply these previous results for studying the periodic solutions of differential systems like (1.5) through their bifurcation functions, provided by the higher order averaging theory.

The results are organized as follows. In Chapter 1 we present our main results on averaging theory. In Chapter 2 we provide the proofs of the main results. Then we start apply our results to study the periodic solutions of some relevant physical systems. In Chapter 3 we study the Maxwell-Bloch system and a 3D polynomial differential system. In Chapter 4 we study 17 differential systems, including the Fitzhugh-Nagumo system, the Noose-Hover system, the Wang-Chen system and the Wei system. In Chapter 5 we study the existence and stability of periodic solutions in the Lorenz differential system and the Thomas differential system. In Chapter 6 we study the periodic solutions and invariant tori in the generalized Van der Pol - Duffing differential system using Lyapunov coefficients and averaging theory. Finally, in Chapter 7 we study the periodic solutions in a hyperchaotic Lorenz differential system.

The results presented in Chapter 1, 2 and 3 were based on the published papers [17, 14] and the preprint [18] . The results presented in Chapter 4 are published in [13] and [15]. Chapter 5 contains results from [13] and [14]. The results in Chapter 6 are submited for publication. The results in Chapter 7 are published in [16].

1.1 Statements of the main results

Before we state our main results we need some preliminary concepts and definitions. Given p, q and L positive integers, $\gamma_j = (\gamma_{j1}, \dots, \gamma_{jp}) \in \mathbb{R}^p$ for $j = 1, \dots, L$ and $\overline{z} \in \mathbb{R}^p$. Let $G : \mathbb{R}^p \to \mathbb{R}^q$ be a sufficiently smooth function, then the L-th Frechet derivative of G at $\overline{z}$ is denoted by $\partial^L G(\overline{z})$, it is a symmetric L–multilinear map, which applied to a "product" of L p-dimensional vectors denoted as $\bigodot_{j=1}^{L} \gamma_j \in \mathbb{R}^{pL}$ gives

$$\partial^L G(\overline{z}) \bigodot_{j=1}^{L} \gamma_j = \left(\sum_{i_1,\dots,i_L=1}^{p} \frac{\partial^L G^1(\overline{z})}{\partial z_{i_1} \cdots \partial z_{i_L}} \gamma_{1i_1} \cdots \gamma_{Li_L}, \cdots, \sum_{i_1,\dots,i_L=1}^{p} \frac{\partial^L G^q(\overline{z})}{\partial z_{i_1} \cdots \partial z_{i_L}} \gamma_{1i_1} \cdots \gamma_{Li_L} \right). \tag{1.6}$$

The above expression is indeed the Gâteaux derivative

$$\begin{aligned} \partial^L G(\overline{z}) \bigodot_{j=1}^{L} \gamma_j &= \frac{\partial}{\partial \tau_1 \partial \tau_2 \dots \partial \tau_L} G\left(\overline{z} + \tau_1 \gamma_1 + \tau_2 \gamma_2 + \cdots + \tau_L \gamma_L\right) \Big|_{\tau_1 = \cdots = \tau_L = 0} \\ &= \partial\Big(\dots \partial\big(\partial G(\overline{z}) \gamma_1\big) \gamma_2 \dots \Big) \gamma_L. \end{aligned}$$

We take ∂^0 as the identity operator.

1.1.1 The Lyapunov–Schmidt reduction method

We consider the function

$$g(z, \varepsilon) = \sum_{i=0}^{k} \varepsilon^i g_i(z) + \mathcal{O}(\varepsilon^{k+1}), \tag{1.7}$$

where $g_i : \mathcal{D} \to \mathbb{R}^n$ is a $\mathcal{C}^{k+1}$ function, $k \geq 1$, for $i = 0, 1, \dots, k$, being $\mathcal{D}$ an open bounded subset of $\mathbb{R}^n$. For $m < n$, let V be an open bounded subset of $\mathbb{R}^m$ and $\beta : \mathrm{Cl}(V) \to \mathbb{R}^{n-m}$ a $\mathcal{C}^{k+1}$ function, such that

$$\mathcal{Z} = \{ z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \mathrm{Cl}(V) \} \subset \mathcal{D}. \tag{1.8}$$

As usual $\mathrm{Cl}(V)$ denotes the closure of the set V.

As the main hypothesis we assume that

(H_a) the function g_0 vanishes on the m–dimensional submanifold $\mathcal{Z}$ of $\mathcal{D}$.

Using the Lyapunov–Schmidt reduction method we shall develop the bifurcation functions of order i, for $i = 1, 2, \dots, k$, which control, for $|\varepsilon| \neq 0$ small enough, the existence of branches of zeros $z(\varepsilon)$ of (1.7) bifurcating from $\mathcal{Z}$, that is from $z(0) \in \mathcal{Z}$. With this purpose we introduce some notation. The functions $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$

and $\pi^\perp : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ denote the projections onto the first m coordinates and onto the last $n-m$ coordinates, respectively. For a point $z \in \mathcal{D}$ we also consider $z = (a,b) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. We define $\partial_b^L \pi g_{i-l}(z_\alpha)$ by following the notation (1.6), taking $p = n-m$, $q = m$, $\overline{z} = \beta(\alpha)$ and $G : b \to \pi g_{i-l}(\alpha, b)$. Let S_l be the set of all l-tuples of non–negative integers $(c_1, c_2, \cdots, c_l)$ satisfying $c_1 + 2c_2 + \cdots + lc_l = l$, $L = c_1 + c_2 + \cdots + c_l$, and S_i' is the set of all $(i-1)$-tuples of non–negative integers satisfying $c_1 + 2c_2 + \cdots + (i-1)c_{i-1} = i$, $I' = c_1 + c_2 + \cdots + c_{i-1}$. From (1.6) we define

$$\partial_b^L \pi g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} = \left(\sum_{i_1,\ldots,i_L=1}^{n-m} \frac{\partial^L \pi g_{i-l}^1(a,b)}{\partial b_{i_1} \cdots \partial b_{i_L}} (\gamma_{1i_1}(\alpha))^{c_1} \cdots (\gamma_{li_l}(\alpha))^{c_l}, \cdots, \right.$$
$$\left. \sum_{i_1,\ldots,i_L=1}^{n-m} \frac{\partial^L \pi g_{i-l}^m(a,b)}{\partial b_{i_1} \cdots \partial b_{i_L}} (\gamma_{1i_1}(\alpha))^{c_1} \cdots (\gamma_{li_l}(\alpha))^{c_l} \right) \Big|_{(a,b)=z_\alpha}$$

and

$$\partial_b^L \pi^\perp g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} = \left(\sum_{i_1,\ldots,i_L=1}^{n-m} \frac{\partial^L \pi^\perp g_{i-l}^{m+1}(a,b)}{\partial b_{i_1} \cdots \partial b_{i_L}} (\gamma_{1i_1}(\alpha))^{c_1} \cdots (\gamma_{li_l}(\alpha))^{c_l}, \cdots, \right.$$
$$\left. \sum_{i_1,\ldots,i_L=1}^{n-m} \frac{\partial^L \pi^\perp g_{i-l}^n(a,b)}{\partial b_{i_1} \cdots \partial b_{i_L}} (\gamma_{1i_1}(\alpha))^{c_1} \cdots (\gamma_{li_l}(\alpha))^{c_l} \right) \Big|_{(a,b)=z_\alpha}.$$

For $i = 1, 2, \ldots, k$ we define the *bifurcation functions* $f_i : \mathrm{Cl}(V) \to \mathbb{R}^m$ of order i as

$$f_i(\alpha) = \pi g_i(z_\alpha) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} \partial_b^L \pi g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j}, \quad \text{and} \tag{1.9}$$

$$\mathcal{F}^k(\alpha, \varepsilon) = \sum_{i=1}^{k} \varepsilon^i f_i(\alpha),$$

where $\gamma_i : V \to \mathbb{R}^{n-m}$, for $i = 1, 2, \ldots, k$, are defined recurrently as

$$\begin{aligned} \gamma_1(\alpha) &= -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha) \quad \text{and} \\ \gamma_i(\alpha) &= -i! \Delta_\alpha^{-1} \left(\sum_{S_i'} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_{i-1}! (i-1)!^{c_{i-1}}} \partial_b^{I'} \pi^\perp g_0(z_\alpha) \bigodot_{j=1}^{i-1} \gamma_j(\alpha)^{c_j} \right. \\ &\quad \left. + \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} \partial_b^L \pi^\perp g_{i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} \right). \end{aligned} \tag{1.10}$$

with $\Delta_\alpha = \dfrac{\partial \pi^\perp g_0}{\partial b}(z_\alpha)$.

We clarify that $S_0 = S_0' = \emptyset$, and when $c_j = 0$, for some j, then the term γ_j does not appear in the "product" $\bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j}$.

The next theorem is the first main result of this chapter. For sake of simplicity, we take $f_0 = 0$.

Theorem 2. *Let Δ_α denote the lower right corner of the $(n-m)\times(n-m)$ matrix of the Jacobian matrix $D\,g_0(z_\alpha)$. In additional to hypothesis (H$_a$) we assume that*

(*i*) *for each $\alpha \in \mathrm{Cl}(V)$, $\det(\Delta_\alpha) \neq 0$;*

(*ii*) *for some $r \in \{1,\dots,k\}$, $f_1 = f_2 = \cdots = f_{r-1} = 0$ and f_r is not identically zero;*

(*iii*) *there exists a small parameter $\varepsilon_0 > 0$ such that for each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ there exists $a_\varepsilon \in V$ satisfying $\mathcal{F}^k(a_\varepsilon,\varepsilon) = 0$; and*

(*iv*) *there exist a constant $P_0 > 0$ and a positive integer $l \leq (k+r+1)/2$ such that*

$$\left|\partial_\alpha \mathcal{F}^k(a_\varepsilon,\varepsilon)\cdot\alpha\right| \geq P_0|\varepsilon|^l|\alpha|, \quad \text{for} \quad \alpha \in V.$$

Then, for $|\varepsilon| \neq 0$ sufficiently small, there exists $z(\varepsilon)$ such that $g(z(\varepsilon),\varepsilon) = 0$ with $|\pi^\perp z(\varepsilon) - \pi^\perp z_{a_\varepsilon}| = \mathcal{O}(\varepsilon)$ and $|\pi\, z(\varepsilon) - \pi\, z_{a_\varepsilon}| = \mathcal{O}(\varepsilon^{k+1-l})$.

In the next corollary we present a classical result in the literature, which is a direct consequence of Theorem 2.

Corollary 3. *In addiction to hypothesis (H$_a$), assume that $f_1 = f_2 = \cdots = f_{k-1} = 0$ and that for each $\alpha \in \mathrm{Cl}(V)$, $\det(\Delta_\alpha) \neq 0$. If there exists $\alpha^* \in V$ such that $f_k(\alpha^*) = 0$ and $\det(Df_k(\alpha^*)) \neq 0$, then there exists a branch of zeros $z(\varepsilon)$ with $g(z(\varepsilon),\varepsilon) = 0$ and $|z(\varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.*

Theorem 2 and Corollary 3 are proved in Section 2.1.

1.1.2 Continuation of periodic solutions

We consider the following $\mathcal{C}^{k+1}$ differential system

$$x' = F_0(t,x) + \sum_{i=1}^{k} \varepsilon^i F_i(t,x) + \mathcal{O}(\varepsilon^{k+1}), \quad (t,z,\varepsilon) \in \mathbb{S}^1 \times \mathcal{D} \times (-\varepsilon_0, \varepsilon_0). \tag{1.11}$$

Here $\mathcal{D} \subset \mathbb{R}^n$ is an open and bounded set, $\varepsilon_0 > 0$, and the prime denotes derivative with respect to the time t. We denote the right–hand side of equation (1.11) by $F(t,x,\varepsilon)$. We say that the differential system (1.11) is in the *normal form for applying the averaging theory.* Given $z \in \mathcal{D}$ we denote by $x(t,z,\varepsilon)$ the solution of the differential system (1.11) such that $x(0,z,\varepsilon) = z$. As our basic hypothesis we assume that:

(H) There exists a manifold $\mathcal{W} \subset \mathcal{D}$ such that, for each $z \in \mathcal{W}$, the solution $x(t,z,0)$ of the unperturbed system is T-periodic.

Thus we have the following result.

Lemma 4 (Fundamental Lemma)**.** *Let $x(t,z,\varepsilon)$ be the solution of the $\mathcal{C}^{k+1}$ T-periodic differential system* (1.11) *such that $x(0,z,\varepsilon) = z$. Then the equality*

$$x(t,z,\varepsilon) = x(t,z,0) + \sum_{i=1}^{k} \varepsilon^i \frac{y_i(t,z)}{i!} + \mathcal{O}(\varepsilon^{k+1}) \tag{1.12}$$

holds for $(t,z)\in\mathbb{S}^1\times\mathcal{D}$. *Here the functions* y_i *for* $1\le i\le k$, *are given recursively as*

$$\begin{aligned}
y_1(t,z) =& Y(t,z)\int_0^t Y(s,z)^{-1}F_1(s,x(s,z,0))ds,\\
y_i(t,z) =& i!\,Y(t,z)\int_0^t Y(s,z)^{-1}\Bigg(F_i(s,x(s,z,0))\\
&+\sum_{S_i'}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_{i-1}!(i-1)!^{b_{i-1}}}\partial^{I'}F_0(s,x(s,z,0))\bigodot_{j=1}^{i-1}y_j(s,z)^{b_j}\\
&+\sum_{l=1}^{i-1}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^{L}F_{i-l}(s,x(s,z,0))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\Bigg)ds,
\end{aligned}$$

where $Y(t,z)$ *is a fundamental matrix solution of the linear differential system* $y'=\partial_xF_0(t,x(t,z,0))y$, *being* $\partial_xF_0(t,x)$ *the Jacobian matrix of the function* $F_0(t,x)$.

From hypothesis (H) we see that there exists an open set $U_1\subset\mathcal{D}$ and $\varepsilon_1>0$ such that, for each $z\in\overline{U}_1$ and $\varepsilon\in[-\varepsilon_1,\varepsilon_1]$, the solution $x(t,z,\varepsilon)$ is defined on the interval $[0,t_{(z,\varepsilon)})$, with $t_{(z,\varepsilon)}>T$.

A *displacement function* $d:U_1\times(-\varepsilon_1,\varepsilon_1)\to\mathbb{R}^n$ can be defined as $d(z,\varepsilon)=x(T,z,\varepsilon)-z$. Notice that a solution (z^*,ε^*) of the equation $d(z,\varepsilon)=0$ corresponds to a T-periodic solution of the differential system (1.11) with $\varepsilon=\varepsilon^*$ and initial condition z^*. From (1.12), the displacement function reads

$$d(z,\varepsilon)=x(T,z,0)-z+\sum_{i=1}^{k}\varepsilon^i\frac{y_i(T,z)}{i!}+\mathcal{O}(\varepsilon^{k+1}).\tag{1.13}$$

The equation $d(z,\varepsilon)=0$ is equivalent to

$$g(z,\varepsilon)\stackrel{\text{def}}{=}Y(T,z)^{-1}d(z,\varepsilon)=0,\tag{1.14}$$

and from (1.13) equation (1.14) writes

$$g(z,\varepsilon)=g_0(z)+\sum_{i=1}^{k}\varepsilon^i g_i(z)+\mathcal{O}(\varepsilon^{k+1}),$$

where $g_0(z)=Y(T,z)^{-1}\left(x(T,z,0)-z\right)$ and

$$g_i(z)=Y(T,z)^{-1}\frac{y_i(T,z)}{i!},\ \ i=1,2\ldots,k,\tag{1.15}$$

are usually called the *averaged function* of order i. By abuse of notation, the function g_0 is called the averaged function of order 0. Notice that $g_0(z)=0$ if, and only if, the solution $x(t,z,0)$ of the unperturbed system is T–periodic. Therefore, from hypothesis (H), $g_0(z)=0$ for every $z\in\mathcal{Z}$.

The averaging theory for finding periodic solutions consists in providing sufficient conditions for the existence of periodic solutions of system (1.11) by studying the solutions of equation (1.14).

In [17] it was assumed that $g_0\not\equiv0$. Here we assume that $g_s\not\equiv0$ is the first nonvanishing averaged function, where $0\le s<k$. As our main hypotheses we assume that

($\mathcal{H}$) Let $g_s \not\equiv 0$, for $0 \leq s < k$, be the first nonvanishing averaged function. Assume that there exist $m < n$, V an open bounded subset of $\mathbb{R}^m$, and a $\mathcal{C}^{k+1}$ function $\beta : \overline{V} \to \mathbb{R}^{n-m}$ such that $\mathcal{Z} = \{z_\alpha = (\alpha, \beta(\alpha)) : \alpha \in \overline{V}\} \subset \mathcal{D}$, and $g_s(z_\alpha) = 0$ for every $\alpha \in \overline{V}$.

Notice that ($\mathcal{H}$) implies (H). Indeed, if $s = 0$, then (H) holds by taking $\mathcal{Z} = \mathcal{W}$. Otherwise ($H$) holds by taking $\mathcal{Z} = \mathcal{D}$.

From hypothesis ($\mathcal{H}$) and Lemma 4 equation (1.13) is equivalente to

$$h(z,\varepsilon) \stackrel{\text{def}}{=} \frac{g(z,\varepsilon)}{\varepsilon^s} = g_s(z) + \sum_{i=1}^{k-s} \varepsilon^i g_{s+i}(z) + \mathcal{O}(\varepsilon^{k-s+1}) = 0. \tag{1.16}$$

From Theorem 2 the bifurcation functions corresponding to equation (1.16) are

$$f_i(\alpha) = \pi g_{s+i}(z_\alpha) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} \partial_b^L \pi g_{s+i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j}, \tag{1.17}$$

$$\mathcal{F}^{k-s}(\alpha,\varepsilon) = \sum_{i=1}^{k-s} \varepsilon^i f_i(\alpha), \tag{1.18}$$

where $\gamma_i : V \to \mathbb{R}^{n-m}$, for $i = 1, 2, \ldots, k-s$, are defined recurrently as

$$\begin{aligned}
\gamma_1(\alpha) =& -\Delta_\alpha^{-1} \pi^\perp g_{s+1}(z_\alpha) \quad \text{and} \\
\gamma_i(\alpha) =& -i!\Delta_\alpha^{-1} \Bigg(\sum_{S_i'} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_{i-1}! (i-1)!^{c_{i-1}}} \partial_b^{I'} \pi^\perp g_s(z_\alpha) \bigodot_{j=1}^{i-1} \gamma_j(\alpha)^{c_j} \\
&+ \sum_{l=1}^{i-1} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} \partial_b^L \pi^\perp g_{s+i-l}(z_\alpha) \bigodot_{j=1}^{l} \gamma_j(\alpha)^{c_j} \Bigg),
\end{aligned}$$

with $\Delta_\alpha = \dfrac{\partial \pi^\perp g_s}{\partial b}(z_\alpha)$.

In what follows we shall state a slightly improvement of Theorem B from [17], which is suitable to a wider range of applications.

Theorem 5. *Assume hypothesis ($\mathcal{H}$) holds. Consider the Jacobian matrix*

$$\partial g_s(\mathbf{z}_\alpha) = \begin{pmatrix} \Lambda_\alpha & \Gamma_\alpha \\ B_\alpha & \Delta_\alpha \end{pmatrix},$$

where $\Lambda_\alpha = \partial_a \pi g_s(z_\alpha)$, $\Gamma_\alpha = \partial_b \pi g_s(z_\alpha)$, $B_\alpha = \partial_a \pi^\perp g_s(z_\alpha)$ and $\Delta_\alpha = \partial_b \pi^\perp g_s(z_\alpha)$. In additional to hypothesis ($\mathcal{H}$) we suppose that

(*i*) *for each $\alpha \in \overline{V}$, $\det(\Delta_\alpha) \neq 0$;*

(*ii*) *for some $r \in \{0, \ldots, k-s\}$, $f_1 = f_2 = \cdots = f_{r-1} = 0$ and f_r is not identically zero;*

(*iii*) *there exists $\overline{\varepsilon} > 0$ such that for each $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$ there exists $a_\varepsilon \in V$ satisfying $\mathcal{F}^{k-s}(a_\varepsilon, \varepsilon) = 0$; and* (*iv*) *there exist a constant* $P_0 > 0$ *and a positive integer* $l \leq (k - s + r + 1)/2$ *such that*

$$\left|\partial_\alpha \mathcal{F}^{k-s}(a_\varepsilon, \varepsilon) \cdot \alpha\right| \geq P_0|\varepsilon^l||\alpha|, \quad \text{for} \quad \alpha \in V.$$

Then for $|\varepsilon| \neq 0$ *sufficiently small there exists a* T*-periodic solution* $\varphi(t, \varepsilon)$ *of system* (1.11) *such that* $|\pi\,\varphi(0, \varepsilon) - \pi\, z_{a_\varepsilon}| = \mathcal{O}(\varepsilon^{k-s+1-l})$*, and* $|\pi^\perp \varphi(0, \varepsilon) - \pi^\perp z_{a_\varepsilon}| = \mathcal{O}(\varepsilon)$.

In the next corollary we present a classical result in the literature, which is a direct consequence of Corollary 3.

Corollary 6. *In addiction to hypothesis* $(\mathcal{H})$ *we assume that* $f_1 = f_2 = \cdots = f_{r-1} = 0$, $r = k - s$ *and that for each* $\alpha \in \mathrm{Cl}(V)$, $\det(\Delta_\alpha) \neq 0$. *If there exists* $\alpha^* \in V$ *such that* $f_r(\alpha^*) = 0$ *and* $\det(Df_r(\alpha^*)) \neq 0$*, then there exists a* T*-periodic solution* $\varphi(t, \varepsilon)$ *of* (1.11) *such that* $|\varphi(0, \varepsilon) - z_{\alpha^*}| = \mathcal{O}(\varepsilon)$.

Lemma 4, Theorem 5 and Corollary 6 are proved in Section 2.2.

It is worth to emphasize that Theorem 5 is still true when $m = n$. In fact, assuming that V is an open subset of $\mathbb{R}^n$ then $\mathcal{Z} = \mathrm{Cl}(V) \subset \mathcal{D}$ and the projections π and $\pi^\perp$ become the identity and the null operator respectively. Moreover, in this case the bifurcation functions $f_i : V \to \mathbb{R}^n$, for $i = 1, 2, \ldots, k$, are the averaged functions $f_i(\alpha) = g_i(\alpha)$ defined in (1.15). Consider $m = n$, $z_\alpha = \alpha \in \mathcal{Z}$ and the hypothesis $(\mathcal{H})$. Thus the result of Theorem 5 holds without any assumption about Δ_α. Thus we have the following corollary, which recover the main result from [53].

Corollary 7. *Assume that* $g_s \equiv 0$. *If there exists* $z^* \in \Omega$ *such that* $g_{s+1}(z^*) = 0$ *and* $Dg_{s+1}(z^*) \neq 0$*, then there exists a* T*-periodic solution* $x(t, z(\varepsilon), \varepsilon)$ *for system* (1.11) *such that* $z(0) = z^*$.

Now we use functions $\alpha(\varepsilon)$, γ_i and f_i to study the stability of the periodic solution $\varphi(t, \varepsilon)$.

1.2 Stability of the periodic solutions

A fundamental notion in qualitative theory of differential equations is the hyperbolicity. Here a constant matrix will be called *hyperbolic* if its eigenvalues lie out of the unitary circle of the complex plane, in which case its *index* is the number of eigenvalues outside the unitary circle.

Consider a matrix function $A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots + \varepsilon^k A_k$ depending on a parameter ε. If A_0 is hyperbolic of index i, then one can see that for $\varepsilon > 0$ sufficiently small $A(\varepsilon)$ will be hyperbolic with the same index i.

If A_0 is not hyperbolic the placement of the eigenvalues of $A(\varepsilon)$ may be hard to determine. To deal with this problem we use a method introduced by Murdock and Robinson in [62, 61]. The matrix $A(\varepsilon)$ is called *k-hyperbolic of index i* if for every smooth matrix function $B(\varepsilon)$ there exists an $\varepsilon_0 > 0$ such that $A(\varepsilon) + \varepsilon^k B(\varepsilon)$ is hyperbolic of index i for all ε in the interval $0 < \varepsilon < \varepsilon_0$.

The stability properties of the periodic solution $\varphi(t, \varepsilon)$ will be provided using the k-determined hyperbolicity method, as it was presented in [60, Chapter 3].

For $|\varepsilon| \neq 0$ sufficiently small let $\varphi(t,\varepsilon) = x(t, z(\varepsilon), \varepsilon)$ be a $T-$periodic solution of the differential system (1.11) given by Theorem 5 such that $z(0) = z_{\alpha^*} \in \mathcal{Z}$. The *Poincaré Map* related to $\varphi(t,\varepsilon)$ is given by

$$\Pi(z,\varepsilon) \stackrel{\text{def}}{=} x(T, z, \varepsilon) = z + d(z,\varepsilon). \tag{1.19}$$

Clearly $z(\varepsilon)$ is a fixed point of $\Pi(\cdot,\varepsilon)$. It is well known that the stability of the fixed point $z(\varepsilon)$ of the Poincaré map $\Pi(\cdot,\varepsilon)$ yields the stability of the T-periodic solution $\varphi(t,\varepsilon)$. More specifically, if the norm of each eigenvalue of $\partial_z\Pi(z(\varepsilon),\varepsilon)$ is less than 1, then the periodic solution $\varphi(t,\varepsilon)$ is stable. On the other hand, if there exists an eigenvalue of $\partial_z\Pi(z(\varepsilon),\varepsilon)$ with norm greater than 1, then the periodic solution $\varphi(t,\varepsilon)$ is unstable. From (1.19), our goal in is to show how the power series of $z(\varepsilon)$ around $\varepsilon = 0$ can be used to provide the stability of the $T-$periodic solutions $x(t, z(\varepsilon), \varepsilon)$ provided in Theorem 5. As these solutions are essentially non-hyperbolic, due to existence of a continuum of zeros of the first coefficient function of (1.16), the question about its stability can be reduced to the study of the k-determined hyperbolicity of the Jacobian matrix $\partial_z d(z(\varepsilon),\varepsilon)$.

For the sake of further applications the first result of this section is to write the formal power series of the initial condition $z(\varepsilon) = \varphi(0,\varepsilon)$ around $\varepsilon = 0$, where $\varphi(t,\varepsilon)$ is the $T-$periodic solution provided in Theorem 5.

The next result reveals how the higher order averaged functions can be used for determining the stability of the periodic solution $\mathbf{x}(t, z(\varepsilon), \varepsilon)$.

Lemma 8. *Let a_ε be the one given in hypothesis (iii) of Theorem 5 and let $x(t, z(\varepsilon), \varepsilon) = \varphi(t,\varepsilon)$ be the periodic solution of the differential system* (1.11) *provided in Theorem 5. If*

$$a_\varepsilon = \alpha_0 + \varepsilon\alpha_1 + \cdots + \varepsilon^{k-s-l}\alpha_{k-s-l} + \mathcal{O}(\varepsilon^{k-s-l+1}), \tag{1.20}$$

with $\alpha_i \in \mathbb{R}^m$ for all $0 \le i \le k-s-l$. Then we can write initial condition of the periodic orbit as

$$z(\varepsilon) = \sum_{i=0}^{k-s-l} \varepsilon^i\big(\alpha_i, \beta_i\big) + \mathcal{O}\big(\varepsilon^{k-s-l+1}\big), \tag{1.21}$$

where $\beta_0 = \beta(\alpha_0)$ and for all $1 \le i \le k-s-l$,

$$\beta_i = \gamma_i(\alpha_0) + \sum_{j=1}^{i}\sum_{S_j} \frac{1}{c_1!\, c_2!2!^{c_2}\cdots c_j!j!^{c_j}}\gamma_{i-j}^{(J)}(\alpha_0)\bigodot_{s=1}^{j}(s!\alpha_s)^{c_s}. \tag{1.22}$$

The next result provides the Taylor expansion at $\varepsilon = 0$ of the Jacobian matrix of the displacement function (1.13) evaluated at $z(\varepsilon) = \varphi(0,\varepsilon)$, where $\varphi(t,\varepsilon)$ is the T-periodic function provided in Theorem 5.

Lemma 9. *We assume that system* (1.11) *satisfies the hypotheses of Theorem 5 having the T-periodic solution $\varphi(t,\varepsilon)$. Moreover, let $z(\varepsilon) = \varphi(0,\varepsilon)$ and a_ε from statement (iii) of Theorem 5 written in the form* (1.20). *Thus the Jacobian matrix of displacement map* (1.13) *at $z = z(\varepsilon)$ can be written as*

$$\partial_z d(z(\varepsilon),\varepsilon) = \varepsilon^s A(\varepsilon) + \mathcal{O}(\varepsilon^{k-l+1}),$$

where $A(\varepsilon) = A_0 + \varepsilon A_1 + \cdots + \varepsilon^{k-s-l} A_{k-s-l}$ *where* A_j *is an* $n \times n$ *constant matrix for all* $0 \le j \le k - s - l$*. More precisely, we have* $A_0 = \partial y_s(T, z_0)$ *and*

$$A_j = \sum_{i=0}^{j} \frac{1}{(j-i)!} \sum_{S_i} \frac{1}{b_1! \cdots b_i! (i-1)!^{b_i}} \partial_z^{I+1} y_{s+j-i}(T, z_0) \bigodot_{u=1}^{i} (u! z_u)^{b_u} \,,$$

for $1 \le j \le k - s - l$*, with* $z_i = (\alpha_i, \beta_i)$ *given in* (1.21) *and* l *as in Theorem 5.*

Consequently the Jacobian matrix of the Poincaré map becomes

$$D\Pi(z, \varepsilon) \stackrel{\text{def}}{=} M(\varepsilon) + \mathcal{O}(\varepsilon^{k-l+1}), \tag{1.23}$$

with $M(\varepsilon) = Id + \varepsilon^s A(\varepsilon)$. Now we can present our result on the stability of the non-hyperbolic T-periodic solution $x(t, z(\varepsilon), \varepsilon)$ provided in Theorem 5.

Theorem 10. *We assume that system* (1.11) *has a* T*-periodic solution* $x(t, z(\varepsilon), \varepsilon)$ *as stated in Lemma* 9*, and that the Jacobian matrix of the Poincaré map at* $z(\varepsilon)$ *has the form* (1.23) *with* $M(\varepsilon)$ *hyperbolic for* $|\varepsilon|$ *sufficiently small. If there exists a matrix* $T(\varepsilon)$ *such that* $T(\varepsilon)^{-1} M(\varepsilon) T(\varepsilon) = \Lambda(\varepsilon)$*, where*

$$\Lambda(\varepsilon) = \begin{bmatrix} \lambda_1(\varepsilon) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n(\varepsilon) \end{bmatrix} = \varepsilon^{r_1} \Lambda_1 + \cdots + \varepsilon^{r_j} \Lambda_j;$$

with $r_1 < r_2 < \cdots < r_j < R = k - l + 1$ *rational numbers, and* $\Lambda_1, \ldots, \Lambda_j$ *diagonal matrices. Then there exists an* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *the eigenvalues of the Jacobian matrix* $D\Pi(z, \varepsilon)$ *are approximately equal to* $\lambda_i(\varepsilon)$ *with error* $\mathcal{O}(\varepsilon^R)$*. Consequently the matrices* $M(\varepsilon)$ *and* $D\Pi(z, \varepsilon)$ *have the same hyperbolicity type.*

The result of Theorem 10 is strongly related with the Theorem 3.7.7 of [60]. Obtaining the matrix $T(\varepsilon)$ may be the main difficulty of applying Theorem 10. In some cases it may be necessary a sequence of linear transformations and normalization in order to obtain $T(\varepsilon)$, see [60, Section 3.7]. This task always comes down to the solution of a homological equation such as

$$\mathcal{L} U_j = K_j - B_j,$$

where

$$\mathcal{L} = \mathbb{L}_Y : gl(n) \to gl(n),$$

K_j is known at the jth stage of the calculation, and B_j and U_j are to be determined and $\mathbb{L}_Y$ is the Lie operator, i.e. $\mathbb{L}_Y X = [X, Y] = XY - YX$. In this work we shall use Theorem 5 to study the Hopf or the zero–Hopf bifurcation in some three dimensional systems. Moreover Corollary 19 in Appendix 2.7 provides sufficient conditions for the existence of the matrix $T(\varepsilon)$. This will allow to use Theorem 10 for studying the stability of the bifurcated periodic orbits detected by Theorem 5.

Finally we shall show that the hypotheses of Lemma 8 are not very restrictive. We shall provide the expressions of the $\alpha_i' s$ in Lemma 8 in terms of the bifurcation functions (1.17).

Proposition 11. *Assume that $0 \le r < k$ is the first subindex such that $f_r(\alpha) \not\equiv 0$ as given by hypothesis (ii) of Theorem 5. If there exist $\alpha^* \in V \subset \mathbb{R}^m$ such that $f_r(\alpha^*) = 0$ and $\det(\partial f_r(\alpha^*)) \neq 0$. Then there exists a unique $a_\varepsilon \in V$ such that:*

(a) $a_\varepsilon = \alpha_0 + \varepsilon\alpha_1 + \cdots + \varepsilon^k\alpha_k + \mathcal{O}(\varepsilon^{k+1})$ with $\alpha_i \in \mathbb{R}^n$ for all $1 \le i \le k$ satisfying $\mathcal{F}^k(a_\varepsilon, \varepsilon) = 0$, and

(b) where the coefficients are $\alpha_0 = \alpha^*$, $\alpha_1 = -Df_r(\alpha^*)^{-1} f_{r+1}(\alpha^*)$ and for $2 \le i \le k-1$

$$\begin{aligned}\alpha_i =& \frac{-Df_r}{i!}(\alpha^*)^{-1}\Bigg(\sum_{S_i'} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_{i-1}!(i-1)!^{c_{i-1}}} f_r^{(I')}(\alpha^*) \bigodot_{j=1}^{i-1} \alpha^{(j)}(0)^{c_j} \\ &+ \sum_{l=0}^{i-1} \sum_{S_l} \frac{1}{c_1!\, c_2! 2!^{c_2} \cdots c_l! l!^{c_l}} f_{i-l+r}^{(L)}(\alpha^*) \bigodot_{j=1}^{l} \alpha^{(j)}(0)^{c_j}\Bigg),\end{aligned}$$

Proposition 11 is particularly useful to study the stability of the periodic orbits detected by Corollary 6. This result will be applied several times in this work. Thus we present now a reformulation of Corollary 6 and Theorem 10 that will be used in the applications presented in the next chapters.

Theorem 12. *Let $s \in \mathbb{R}$ such that s is the first subindex such that $g_s \not\equiv 0$. In addition to hypothesis $(\mathcal{H})$ assume that*

(i) the averaged function g_s vanishes on the manifold (1.8). That is $g_s(z_\alpha) = 0$ for all $\alpha \in \overline{V}$, and

(ii) the Jacobian matrix

$$Dg_s(\mathbf{z}_\alpha) = \begin{pmatrix} \Lambda_\alpha & \Gamma_\alpha \\ B_\alpha & \Delta_\alpha \end{pmatrix},$$

where $\Lambda_\alpha = D_a\pi g_s(z_\alpha)$, $\Gamma_\alpha = D_b\pi g_s(z_\alpha)$, $B_\alpha = D_a\pi^\perp g_s(z_\alpha)$ and $\Delta_\alpha = D_b\pi^\perp g_s(z_\alpha)$, satisfies that $\det(\Delta_\alpha) \neq 0$ for all $\alpha \in \overline{V}$.

We define the functions

$$\begin{aligned} f_1(\alpha) =& -\Gamma_\alpha \Delta_\alpha^{-1} \pi^\perp g_{s+1}(z_\alpha) + \pi g_{s+1}(z_\alpha), \\ f_2(\alpha) =& \frac{1}{2}\Gamma_\alpha \gamma_2(\alpha) + \frac{1}{2}\frac{\partial^2 \pi g_s}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + \frac{\partial \pi g_{s+1}}{\partial b}(z_\alpha)\gamma_1(\alpha) + \pi g_{s+2}(z_\alpha), \\ \gamma_1(\alpha) =& -\Delta_\alpha^{-1}\pi^\perp g_{s+1}(z_\alpha), \\ \gamma_2(\alpha) =& -\Delta_\alpha^{-1}\left(\frac{\partial^2 \pi^\perp g_s}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + 2\frac{\partial \pi^\perp g_{s+1}}{\partial b}(z_\alpha)\gamma_1(\alpha) + 2\pi^\perp g_{s+2}(\alpha)\right). \end{aligned} \tag{1.24}$$

Then the following statements hold.

(a) If there exists $\alpha^* \in V$ such that $f_1(\alpha^*) = 0$ and $\det(Df_1(\alpha^*)) \neq 0$, for $|\varepsilon| \neq 0$ sufficiently small there is an initial condition $z(\varepsilon) \in U$ such that $z(0) = z_{\alpha^*}$ and the solution $x(t, z(\varepsilon), \varepsilon)$ of system (1.11) is T-periodic.

(b) *Assume that* $f_1 \equiv 0$. *If there exists* $\alpha^* \in V$ *such that* $f_2(\alpha^*) = 0$ *and* $\det(Df_2(\alpha^*)) \neq 0$, *for* $|\varepsilon| \neq 0$ *sufficiently small there is an initial condition* $z(\varepsilon) \in U$ *such that* $z(0) = z_{\alpha^*}$ *and the solution* $x(t, z(\varepsilon), \varepsilon)$ *of system* (1.11) *is* T*-periodic.*

The next result provides the stability type of the periodic solutions detected by Theorem 12(a). Here *diagonalizable* means that the matrix has n distinct eigenvalues.

Theorem 13. *Consider* s, Γ_α, Δ_α, f_1 *and* f_2 *as defined in Theorem 12 and the Jacobian matrices* $Dy_s(T,z) = (p_{ij}(z))$ *and* $Dy_{s+1}(T,z) = (q_{ij}(z))$. *Assume that there exists* $\alpha^* \in V$ *such that* $f_1(\alpha^*) = 0$ *and* $\det(Df_1(\alpha^*)) \neq 0$. *We define the matrix function*

$$A(\varepsilon) = A_0 + \varepsilon A_1, \tag{1.25}$$

where

$$\begin{aligned} A_0 =& Dy_s(T, z_{\alpha^*}), && (1.26)\\ A_1 =& \left(Dp_{ij}(z_{\alpha^*}).z_1 + q_{ij}(z_{\alpha^*})\right), && (1.27)\\ z_1 =& \left(-Df_1(\alpha^*)^{-1} f_2(\alpha^*),\, D\beta(\alpha^*)\left(-Df_1(\alpha^*)^{-1} f_2(\alpha^*)\right) + \gamma_1(\alpha^*)\right). && (1.28) \end{aligned}$$

We assume that $A(\varepsilon)$ *satisfies the following statements:*

(s_1) A_0 *is diagonalizable and* $s > 0$, *or* $Id + A_0$ *is diagonalizable and* $s = 0$; *and*

(s_2) $Id + \varepsilon^s A_0 + \varepsilon^{s+1} A_1$ *is hyperbolic for all* ε *sufficiently small.*

Thus the Poincaré map of the periodic solution $x(t, z(\varepsilon), \varepsilon)$ *is* $s+2-$*hyperbolic.*

In other words this last result says that the hyperbolicity of the $x(t, z(\varepsilon), \varepsilon)$ can be investigated using the $\lambda_i(\varepsilon) + \mathcal{O}(\varepsilon^{s+2})$, where $\lambda_{i's}(\varepsilon)$ are the eigenvalues of $Id + \varepsilon^s A_0 + \varepsilon^{s+1} A_1$. In the next chapter we prove the results here presented.

Chapter 2

Proofs of the main results

2.1 Proof of Theorem 2 and Corollary 3

A useful tool to study the zeros of a function is the Brouwer degree (see the Appendix B for some of their properties). Let $g \in C^1(\mathcal{D})$, $\mathrm{Cl}(V) \subset \mathcal{D}$ and $\mathbf{Z}_g = \{z \in V : g(z) = 0\}$. We also assume that $J_g(z) \neq 0$ for all $z \in \mathbf{Z}_g$, where $J_g(z)$ is the Jacobian determinant of g at z. Then if V is bounded the set $\mathbf{Z}_g$ is formed by a finite number of isolated points. The Brouwer degree of g at 0 is

$$d_B(g, V, 0) = \sum_{z \in \mathbf{Z}_g} \mathrm{sign}\left(J_g(z)\right). \tag{2.1}$$

One of the main properties of the Brouwer degree is: "if $d(f, V, 0) \neq 0$ then there exists $x_0 \in V$ such that $f(x_0) = 0$"(see item (i) of Theorem 20 from Appendix B).

The next result is a key lemma for proving Theorem 2.

Lemma 14. *Let V be an open bounded subset of $\mathbb{R}^m$. Consider the continuous functions $f_i : \mathrm{Cl}(V) \to \mathbb{R}^n$, $i = 0, 1, \cdots, \kappa$, and $f, g, r : \mathrm{Cl}(V) \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ given by*

$$g(x, \varepsilon) = f_0(x) + \varepsilon f_1(x) + \cdots + \varepsilon^\kappa f_\kappa(x) \text{ and } f(x, \varepsilon) = g(x, \varepsilon) + \varepsilon^{\kappa+1} r(x, \varepsilon).$$

Let $V_\varepsilon \subset V$, $R = \max\{|r(x, \varepsilon)| : (x, \varepsilon) \in \mathrm{Cl}(V) \times [-\varepsilon_0, \varepsilon_0]\}$ and assume that $|g(x, \varepsilon)| > R|\varepsilon|^{\kappa+1}$ for all $x \in \partial V_\varepsilon$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$. Then for each $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$ we have $d_B\left(f(\cdot, \varepsilon), V_\varepsilon, 0\right) = d_B\left(g(\cdot, \varepsilon), V_\varepsilon, 0\right)$.

Proof. For a fixed $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \setminus \{0\}$, consider a continuous homotopy between $g(\cdot, \varepsilon)$ and $f(\cdot, \varepsilon)$ given by $g_t(x, \varepsilon) = g(x, \varepsilon) + t\left(f(x, \varepsilon) - g(x, \varepsilon)\right) = g(x, \varepsilon) + t\,\varepsilon^{\kappa+1} r(x, \varepsilon)$. We claim that $0 \notin g_t(\partial V_\varepsilon, \varepsilon)$ for every $t \in [0, 1]$. As usual ∂V_ε denotes the boundary of the set V_ε. Indeed, assuming that $0 \in g_{t_\varepsilon}(\partial V_\varepsilon, \varepsilon)$, for some $t_\varepsilon \in [0, 1]$, we may find $x_\varepsilon \in \partial V_\varepsilon$ such that $g_{t_\varepsilon}(x_\varepsilon, \varepsilon) = 0$ and, consequently, $g(x_\varepsilon, \varepsilon) = -t_\varepsilon \varepsilon^{\kappa+1} r(x_\varepsilon, \varepsilon)$. Thus $|g(x_\varepsilon, \varepsilon)| \leq R|\varepsilon|^{\kappa+1}$, which contradicts the hypothesis $|g(x_\varepsilon, \varepsilon)| > R|\varepsilon|^{\kappa+1}$. From Theorem 20 (iii) we conclude that $d_B(g_t(\cdot, \varepsilon), V_\varepsilon, 0)$ is constant for $t \in [0, 1]$ and then $d_B\left(f(\cdot, \varepsilon), V_\varepsilon, 0\right) = d_B\left(g(\cdot, \varepsilon), V_\varepsilon, 0\right)$. □

Lemma 14 provides a stratagem to track zeros of the perturbed function $f(x, \varepsilon)$ using a shrinking neighborhood around the zeros of $g(x, \varepsilon)$ that preserves its Brouwer degree. The way how it works can be blurry at the first moment, so to make it clear we present the following example:

Example 1. *Consider the real function $f(x,\varepsilon) = g(x,\varepsilon) + \varepsilon^2 r(x,\varepsilon)$ with $(x,\varepsilon) \in [-1,\, 1] \times [-\varepsilon_0, \varepsilon_0]$, $g(x,\varepsilon) = x^2 - \varepsilon x$, and $|r(x,\varepsilon)| \le 1/5$. The function $g(x,\varepsilon)$ has two zeros $a = 0$ and $a_\varepsilon = \varepsilon$. Taking $V_\varepsilon = (\varepsilon/2,\, 3\varepsilon/2)$ we have that, for $|\varepsilon| \neq 0$ sufficiently small, $a_\varepsilon \in V_\varepsilon$ and $d_B\left(g(\cdot,\varepsilon), V_\varepsilon, 0\right) = 1$ (see Definition (2.1)). Furthermore $\partial V_\varepsilon = \{\varepsilon/2,\, 3\varepsilon/2\}$, $|g(\varepsilon/2,\varepsilon)| = \varepsilon^2/4$, and $|g(3\varepsilon/2,\varepsilon)| = 3\varepsilon^2/4$. Thus $|g(x,\varepsilon)| > \varepsilon^2/5 \ge \varepsilon^2 \max\{|r(x,\varepsilon)| : (x,\varepsilon) \in [0,1] \times [-\varepsilon_0, \varepsilon_0]\}$. Therefore, from Lemma 14 we know that $d_B\left(f(\cdot,\varepsilon), V_\varepsilon, 0\right) = 1$. From the properties of the Brouwer degree we conclude that there exists $\alpha_\varepsilon \in V_\varepsilon$ such that $f(\alpha_\varepsilon, \varepsilon) = 0$.*

Now we recall the Faá di Bruno's Formula (see [43]) about the l^{th} derivative of a composite function.
Faá di Bruno's Formula *If u and v are functions with a sufficient number of derivatives, then*

$$\frac{d^l}{dt^l} u(v(t)) = \sum_{S_l} \frac{l!}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} u^{(L)}(v(t)) \bigodot_{j=1}^{l} v^{(j)}(t)^{b_j},$$

where S_l is the set of all l–tuples of non–negative integers $(b_1, b_2, \cdots, b_l)$ which are solutions of the equation $b_1 + 2b_2 + \cdots + l b_l = l$ and $L = b_1 + b_2 + \cdots + b_l$.

The remainder of this section consists in the proof of Theorem 2, which is divided in several claims, and the proof Corollary 3.

Proof of Theorem 2. We consider $g = (\pi g, \pi^\perp g)$, $g_i = (\pi g_i, \pi^\perp g_i)$ for $i = 0, 1, 2, \ldots, k$, and $z = (a, b) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ for $z \in \mathcal{D}$. So

$$\frac{\partial g}{\partial z}(z_\alpha, 0) = D\, g_0(z_\alpha) = \begin{pmatrix} \dfrac{\partial \pi g_0}{\partial a}(z_\alpha) & \dfrac{\partial \pi g_0}{\partial b}(z_\alpha) \\ \dfrac{\partial \pi^\perp g_0}{\partial a}(z_\alpha) & \dfrac{\partial \pi^\perp g_0}{\partial b}(z_\alpha) \end{pmatrix}.$$

We write $\Delta_\alpha = \dfrac{\partial \pi^\perp g_0}{\partial b}(z_\alpha)$. From hypotheses, $\pi^\perp g(\alpha, \beta(\alpha), 0) = \pi^\perp g_0(z_\alpha) = 0$, and

$$\det\left(\frac{\partial \pi^\perp g}{\partial b}(\alpha, \beta(\alpha), 0)\right) = \det\left(\frac{\partial \pi^\perp g_0}{\partial b}(z_\alpha)\right) = \det\left(\Delta_\alpha\right) \neq 0.$$

Thus applying the *Implicit Function Theorem* it follows that there exists an open neighborhood $U \times (-\varepsilon_1, \varepsilon_1)$ of $\mathrm{Cl}(V) \times \{0\}$ with $\varepsilon_1 \le \varepsilon_0$, and a $\mathcal{C}^{k+1}$ function $\overline{\beta} : U \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^{n-m}$ such that $\pi^\perp g(a, \overline{\beta}(a,\varepsilon), \varepsilon) = 0$ for each $(a, \varepsilon) \in U \times (-\varepsilon_1, \varepsilon_1)$, and $\overline{\beta}(\alpha, 0) = \beta(\alpha)$ for every $\alpha \in \mathrm{Cl}(V)$.

The rest of the proof is divided in the following claims.

Claim 1. *The equality $(\partial^i \overline{\beta}/\partial \varepsilon^i)(\alpha, 0) = \gamma_i(\alpha)$ holds for $i = 1, 2, \ldots, k$.*

Firstly it is easy to check that $(\partial \overline{\beta}/\partial \varepsilon)(\alpha, 0) = \gamma_1(\alpha)$. Now for some fixed $i \in \{1, 2, \ldots, k\}$ we assume by induction hypothesis that $(\partial^s \overline{\beta}/\partial \varepsilon^s)(\alpha, 0) = \gamma_s(\alpha)$ for $s = 1, \ldots, i-1$. In what follows we prove the claim for $s = i$. Consider

$$\pi^\perp g(\alpha, \overline{\beta}(\alpha,\varepsilon), \varepsilon) = \sum_{i=0}^{k} \varepsilon^i \pi^\perp g_i(\alpha, \overline{\beta}(\alpha, \varepsilon)) + \mathcal{O}(\varepsilon^{k+1}) = 0.$$

Expanding each function $\varepsilon \mapsto \pi^\perp g_i\big(\alpha, \overline{\beta}(\alpha,\varepsilon)\big)$ in Taylor series we obtain

$$\pi^\perp g(\alpha, \overline{\beta}(\alpha,\varepsilon),\varepsilon) = \sum_{i=0}^{k}\left(\varepsilon^i \sum_{l=0}^{i} \frac{1}{l!}\frac{\partial^l}{\partial\varepsilon^l}\pi^\perp g_{i-l}\left(\alpha, \overline{\beta}(\alpha,\varepsilon)\right)\Big|_{\varepsilon=0}\right) + \mathcal{O}(\varepsilon^{k+1}) = 0. \tag{2.2}$$

Applying the Faà di Bruno's formula we obtain

$$\frac{\partial^l}{\partial\varepsilon^l}\pi^\perp g_{i-l}\left(\alpha, \overline{\beta}(\alpha,\varepsilon)\right)\Big|_{\varepsilon=0} = \sum_{S_l}\left(\frac{l!}{b_1!\, b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_b^L\pi^\perp g_{i-l}\left(\alpha, \overline{\beta}(\alpha,0)\right) \bigodot_{j=1}^{l}\frac{\partial^j}{\partial\varepsilon^j}\overline{\beta}(\alpha,0)^{b_j}\right). \tag{2.3}$$

Substituting (2.3) in (2.2) we get

$$\pi^\perp g(\alpha, \overline{\beta}(\alpha,\varepsilon),\varepsilon) = \sum_{i=0}^{k}\varepsilon^i\left(\sum_{l=0}^{i}\sum_{S_l}\frac{1}{b_1!\, b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_b^L\pi^\perp g_{i-l}\left(\alpha, \overline{\beta}(\alpha,0)\right) \bigodot_{j=1}^{l}\frac{\partial^j}{\partial\varepsilon^j}\overline{\beta}(\alpha,0)^{b_j}\right) + \mathcal{O}(\varepsilon^{k+1}) = 0.$$

Since the previous equation is equal to zero for $|\varepsilon|$ sufficiently small, the coefficients of each power of ε vanish. Then for $0 \le i \le k$ and $(\alpha,\varepsilon) \in U \times (-\varepsilon_1, \varepsilon_1)$ we have

$$\sum_{l=0}^{i}\sum_{S_l}\frac{1}{b_1!\, b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_b^L\pi^\perp g_{i-l}\left(\alpha, \overline{\beta}(\alpha,0)\right) \bigodot_{j=1}^{l}\frac{\partial^j}{\partial\varepsilon^j}\overline{\beta}(\alpha,0)^{b_j} = 0.$$

This equation can be rewritten as

$$\begin{aligned}0 = &\sum_{l=0}^{i-1}\sum_{S_l}\frac{1}{b_1!\, b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_b^L\pi^\perp g_{i-l}\left(\alpha, \overline{\beta}(\alpha,0)\right) \bigodot_{j=1}^{l}\frac{\partial^j}{\partial\varepsilon^j}\overline{\beta}(\alpha,0)^{b_j}\\ &+ \sum_{S_i'}\frac{1}{b_1!\, b_2!2!^{b_2}\cdots b_{i-1}!(i-1)!^{b_{i-1}}}\partial_b^{I'}\pi^\perp g_0\left(\alpha, \overline{\beta}(\alpha,0)\right) \bigodot_{j=1}^{i-1}\frac{\partial^j}{\partial\varepsilon^j}\overline{\beta}(\alpha,0)^{b_j}\\ &+ \frac{1}{i!}\partial_b\pi^\perp g_0\left(\alpha, \overline{\beta}(\alpha,0)\right)\frac{\partial^i}{\partial\varepsilon^i}\overline{\beta}(\alpha,0).\end{aligned} \tag{2.4}$$

Here S_i' is the set of all $(i-1)$-tuples of non–negative integers satisfying $b_1 + 2b_2 + \cdots + (i-1)b_{i-1} = i$, $I' = b_1 + b_2 + \cdots + b_{i-1}$. Finally using the induction hypothesis equation (2.4) becomes

$$\frac{\partial^i\beta}{\partial\varepsilon^i}(\alpha,0) = -i!\Delta_\alpha^{-1}\left(\sum_{S_i'}\frac{1}{b_1!\, b_2!2!^{b_2}\cdots b_{(i-1)}!(i-1)!^{b_{i-1}}}\partial_b^{I'}\pi^\perp g_0(z_\alpha)\bigodot_{j=1}^{i-1}\gamma_j(\alpha)^{b_s}\right.$$

$$+\sum_{l=0}^{i-1}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial_b^L\pi^{\perp}g_{i-l}(z_\alpha)\bigodot_{j=1}^{l}\gamma_j(\alpha)^{b_s}\Bigg)=\gamma_i(\alpha).$$

This concludes the proof of Claim 1.

Claim 2. *Let* $\delta: U\times(-\varepsilon_1,\varepsilon_1)\to\mathbb{R}^m$ *be the* C^{k+1} *function defined as*

$$\delta(\alpha,\varepsilon)=\pi g(\alpha,\overline{\beta}(\alpha,\varepsilon),\varepsilon).$$

Then the equality $(\partial^i\delta/\partial\varepsilon^i)(\alpha,0)=i!f_i(\alpha)$ *holds for* $i=1,2,\ldots,k$.

From (1.7) the function δ reads

$$\delta(\alpha,\varepsilon)=\sum_{j=0}^{k}\varepsilon^j\pi g_j(\alpha,\overline{\beta}(\alpha,\varepsilon))+\mathcal{O}(\varepsilon^{k+1}).$$

So computing its ith-derivative in the variable ε for $0\le i\le k$ we get

$$\frac{\partial^i\delta}{\partial\varepsilon^i}(\alpha,\varepsilon)=\sum_{j=0}^{i}\sum_{q=0}^{i}\binom{i}{q}(\varepsilon^j)^{(i-q)}\frac{\partial^q\pi g_j}{\partial\varepsilon^q}(\alpha,\overline{\beta}(\alpha,\varepsilon))+\mathcal{O}(\varepsilon).$$

Taking $\varepsilon=0$ and $l=i-j$ we obtain

$$\frac{\partial^i\delta}{\partial\varepsilon^i}(\alpha,0)=\sum_{l=1}^{i}\frac{i!}{l!}\frac{\partial^l\pi g_{i-l}}{\partial\varepsilon^l}(\alpha,\overline{\beta}(\alpha,\varepsilon))\Bigg|_{\varepsilon=0}+i!\pi g_i(z_\alpha).$$

Finally using the Faà di Brunno's formula and Claim 1 we have

$$\begin{aligned}\frac{\partial^i\delta}{\partial\varepsilon^i}(\alpha,0)&=\sum_{l=1}^{i}\frac{i!}{l!}\sum_{S_l}\frac{l!}{c_1!c_2!2!^{c_2}\ldots c_l!l!^{c_l}}\partial_b^L\pi g_{i-l}(z_\alpha)\bigodot_{s=1}^{l}\gamma_s(\alpha)^{c_s}+i!\pi g_i(z_\alpha)\\&=i!f_i(\alpha).\end{aligned}$$

This concludes the proof of Claim 2.

Using Claim 2 the function $\delta(\alpha,\varepsilon)$ can be expanded in power series of ε as

$$\delta(\alpha,\varepsilon)=\sum_{i=0}^{k}\frac{\varepsilon^i}{i!}\frac{\partial^i\delta}{\partial\varepsilon^i}(\alpha,0)+\mathcal{O}(\varepsilon^{k+1})=\mathcal{F}^k(\alpha,\varepsilon)+\mathcal{O}(\varepsilon^{k+1}),$$

and, from hypothesis (ii), we have

$$\widetilde{\delta}(\alpha,\varepsilon):=\frac{\delta(\alpha,\varepsilon)}{\varepsilon^r}=\mathcal{G}^k(\alpha,\varepsilon)+\mathcal{O}(\varepsilon^{k-r+1}),\tag{2.5}$$

where $\mathcal{G}^k(\alpha,\varepsilon)=f_r(\alpha)+\varepsilon f_{r+1}(\alpha)+\ldots+\varepsilon^{k-r}f_k(\alpha)$. Obviously the equations $\delta(\alpha,\varepsilon)=0$ and $\widetilde{\delta}(\alpha,\varepsilon)=0$ are equivalent for $\varepsilon\neq 0$.

Denote $R(\varepsilon_0)=\max\{|\widetilde{\delta}(\alpha,\varepsilon)-\mathcal{G}^k(\alpha,\varepsilon)|:(\alpha,\varepsilon)\in \mathrm{Cl}(V)\times[-\varepsilon_0,\varepsilon_0]\}$. From the continuity of the functions $\widetilde{\delta}$ and $\mathcal{G}^k$ and from the compactness of the set $\mathrm{Cl}(V)\times[-\varepsilon_0,\varepsilon_0]$ we know that $R(\varepsilon_0)<\infty$ and $R(0)=0$. In order to study the zeros of $\widetilde{\delta}(\alpha,\varepsilon)$ we use Lemma 14 for proving the following claim.

Claim 3. *Consider $a_\varepsilon \in V$ as in hypothesis (iii) and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. Then there exist $\varepsilon_0 > 0$ sufficiently small and for each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ a neighborhood $V_\varepsilon \subset V$ of a_ε such that $|\mathcal{G}^k(\overline{\alpha}, \varepsilon)| > R(\varepsilon_0)|\varepsilon^{k-r+1}|$ for all $\overline{\alpha} \in \partial V_\varepsilon$. Moreover $V_\varepsilon = B(a_\varepsilon, Q|\varepsilon|^{k+1-l})$ for some $Q > 0$.*

The parameter $\varepsilon_0 > 0$ will be chosen later on. Given $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ since $\mathcal{G}^k(\alpha, \varepsilon)$ is a $\mathcal{C}^{k+1}$ function with, $k \geq 1$, we have that

$$\mathcal{G}^k(a_\varepsilon + h, \varepsilon) = \partial_\alpha \mathcal{G}^k(a_\varepsilon, \varepsilon)h + \rho(h), \quad \rho(h) = \mathcal{O}(|h|^2), \tag{2.6}$$

for every $h \in \mathbb{R}^m$ such that $[a_\varepsilon,\, a_\varepsilon + h] \subset V$. Moreover the hypotheses (*ii*) and (*iv*) imply that

$$\left|\partial_\alpha \mathcal{G}^k(a_\varepsilon, \varepsilon) \cdot \alpha\right| \geq P_0|\varepsilon|^{l-r}|\alpha| \quad \text{for} \quad \alpha \in V. \tag{2.7}$$

Combining expressions (2.6) and (2.7) we obtain the following inequality

$$|\mathcal{G}^k(a_\varepsilon + h, \varepsilon)| \geq \left(P_0 - |\varepsilon|^{r-l}\frac{|\rho(h)|}{|h|}\right)|\varepsilon|^{l-r}|h|. \tag{2.8}$$

Take $V_\varepsilon = B(a_\varepsilon, Q|\varepsilon|^{k+1-l}) \subset V$. A point $\overline{\alpha}_\varepsilon \in \partial V_\varepsilon$ reads $\overline{\alpha}_\varepsilon = a_\varepsilon + h_\varepsilon$, where $h_\varepsilon = uQ|\varepsilon|^{k+1-l} \in \mathbb{R}^m$ and $|u| = 1$. Moreover since $\rho(h) = \mathcal{O}(|h|^2)$ we get

$$|\varepsilon|^{r-l}\frac{|\rho(h_\varepsilon)|}{|h_\varepsilon|} = |\varepsilon|^{r-l}\mathcal{O}\big(Q|\varepsilon|^{k+1-l}\big) = \mathcal{O}\big(Q|\varepsilon|^{k+r+1-2l}\big).$$

From hypothesis (*iii*) we have that $k + r + 1 - 2l \geq 0$. So in particular $\mathcal{O}\big(Q|\varepsilon|^{k+r+1-2l}\big) = \mathcal{O}(Q)$. Thus from definition of the symbol $\mathcal{O}$ there exists $c_0 > 0$, which does not depend on ε and Q, such that $|\varepsilon|^{r-l}|\rho(h_\varepsilon)|/|h_\varepsilon| \leq c_0 Q$. So the inequality (2.8) reads

$$|\mathcal{G}^k(a_\varepsilon + h_\varepsilon, \varepsilon)| \geq (P_0 - Qc_0)\, Q|\varepsilon|^{k-r+1}.$$

Note that the polynomial $\mathcal{P}(Q) = (P_0 - Qc_0)\, Q$ is positive for $0 < Q < P_0/c_0$ and reach its maximum at $Q^* = P_0/(2c_0)$. Moreover $\mathcal{P}(Q^*) = P_0^2/(4c_0)$. Since $R(0) = 0$, there exists $\varepsilon_0 > 0$ small enough in order that $R(\varepsilon_0) < P_0^2/(4c_0) = \mathcal{P}(Q^*)$. Consequently taking $Q = Q^*$ it follows that $|\mathcal{G}^k(\overline{\alpha}, \varepsilon)| > R(\varepsilon_0)|\varepsilon^{k-r+1}|$ for all $\overline{\alpha} \in \partial V_\varepsilon$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. This concludes the proof of the claim.

Applying Lemma 14 for $g = \widetilde{\delta}$, as it is defined in (2.5), $\kappa = k - r$, and $V_\varepsilon = B(a_\varepsilon, Q|\varepsilon|^{k+1-l})$ we conclude that $d_B\big(\widetilde{\delta}(\cdot, \varepsilon), V_\varepsilon, 0\big) = d_B\big(\mathcal{G}^k(\cdot, \varepsilon), V_\varepsilon, 0\big) \neq 0$. Finally, denoting $z(\varepsilon) = \big(\alpha(\varepsilon), \overline{\beta}\big(\alpha(\varepsilon),\, \varepsilon\big)\big)$ it follows that $g(z(\varepsilon), \varepsilon) = 0$.

If $z_{a_\varepsilon} = (a_\varepsilon, \beta(a_\varepsilon))$, then $|\pi z(\varepsilon) - \pi z_{a_\varepsilon}| = |\alpha(\varepsilon) - a_\varepsilon| = \mathcal{O}\left(\varepsilon^{k+1-l}\right)$ and, since $\overline{\beta}$ is Lipschitz,

$$\left|\pi^\perp z(\varepsilon) - \pi^\perp z_{a_\varepsilon}\right| = \left|\overline{\beta}(\alpha(\varepsilon), \varepsilon) - \overline{\beta}(a_\varepsilon, 0)\right| \leq L|(\alpha(\varepsilon), \varepsilon) - (a_\varepsilon, 0)| = \mathcal{O}(\varepsilon).$$

This concludes the proof of Theorem 2. □

Proof of Corollary 3. The basic idea of the proof is to show that $\mathcal{F}^k(\alpha)$ satisfies all the hypotheses of Theorem 2. From the hypotheses $\mathcal{F}^k(\alpha, \varepsilon) = \varepsilon^k f_k(\alpha)$, and $Df_k(\alpha^*) = \varepsilon^{-k}\partial_\alpha \mathcal{F}^k(\alpha^*, \varepsilon)$ is a homeomorphism on $\mathbb{R}^n$. Thus there exist constants $b, c > 0$ such that

$$b|\alpha| < |Jf_k(\alpha^*).\alpha| = \left|\frac{1}{\varepsilon^k}\partial_\alpha \mathcal{F}^k(\alpha^*, \varepsilon).\alpha\right| < c|\alpha|,$$

for all $\alpha \in \mathbb{R}^m$. Therefore $b\left|\varepsilon^k\right||\alpha| < \left|\partial_\alpha \mathcal{F}^k(\alpha^*, \varepsilon).\alpha\right| < c\left|\varepsilon^k\right||\alpha|$, which implies that $\mathcal{F}^k(\alpha^*)$ satisfies hypothesis (iii) of Theorem 2, with $l = r = k$. Indeed $(k + r + 1)/2 = k + 1/2 > k = l$. Hence the proof follows directly from Theorem 2. □

2.2 Proof of Lema 4, Theorem 5 and Collorary 6

Proof of Lemma 4. The solution $x(t, z, \varepsilon)$ can be written as

$$\begin{aligned} x(t,z,\varepsilon) &= z + \sum_{i=0}^{k} \varepsilon^i \int_0^t F_i(s, x(s,z,\varepsilon))ds + \mathcal{O}(\varepsilon^{k+1}), \quad \text{and} \\ x(t,z,0) &= z + \int_0^t F_0(s, x(s,z,0))ds. \end{aligned} \tag{2.9}$$

Moreover the result on the differentiable dependence on parameters implies that $\varepsilon \mapsto x(t, z, \varepsilon)$ is a $\mathcal{C}^{k+1}$ map. So for $i = 0, 1, \ldots, k-1$ we compute the Taylor expansion of $F_i(t, x(t, z, \varepsilon))$ around $\varepsilon = 0$ and we have

$$F_i(t, x(t,z,\varepsilon)) = F_i\left(t, x(t,z,0)\right) + \sum_{l=1}^{k-i} \frac{\varepsilon^l}{l!} \left(\frac{\partial^l}{\partial \varepsilon^l} F_i(t, x(t,z,\varepsilon)) \right)\Bigg|_{\varepsilon=0} + \mathcal{O}(\varepsilon^{k-i+1}). \tag{2.10}$$

Using the Faá di Bruno's formula to compute the l–derivatives of $F_i(t, x(t, z, \varepsilon))$ in the variable ε we get

$$\frac{\partial^l}{\partial \varepsilon^l} F_i(t, x(t,z,\varepsilon))\Bigg|_{\varepsilon=0} = \sum_{S_l} \frac{l!}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F_i(t, x(t,z,0)) \bigodot_{j=1}^{l} y_j(t,z)^{b_j}, \tag{2.11}$$

where

$$y_j(t,z) = \left(\frac{\partial^j}{\partial \varepsilon^j} x(t,z,\varepsilon) \right)\Bigg|_{\varepsilon=0}. \tag{2.12}$$

Substituting (2.11) in (2.10) the Taylor expansion of $F_i(s, x(t, z, \varepsilon))$ becomes

$$\begin{aligned} F_i(s, x(s,z,\varepsilon)) = \; & F_i\left(s, x(s,z,0)\right) \\ & + \sum_{l=1}^{k-i} \sum_{S_l} \frac{\varepsilon^l}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F_i\left(s, x(s,z,0)\right) \\ & \bigodot_{j=1}^{l} y_j(s,z)^{b_j} + \mathcal{O}(\varepsilon^{k-i+1}), \end{aligned} \tag{2.13}$$

for $i = 0, 1, \ldots, k-1$. Furthermore for $i = k$ we have

$$F_k(s, x(s,z,\varepsilon)) = F_k\left(s, x(s,z,0)\right) + \mathcal{O}(\varepsilon). \tag{2.14}$$

From (2.9), (2.13), and (2.14) we get the following equation

$$x(t,z,\varepsilon) = z + Q(t,z,\varepsilon) + \sum_{i=0}^{k} \varepsilon^i \int_0^t F_i(s, x(s,z,0))ds + \mathcal{O}(\varepsilon^{k+1}), \tag{2.15}$$

where

$$Q(t,z,\varepsilon)=\sum_{i=1}^{k-1}\varepsilon^i\sum_{l=1}^{i}\sum_{S_l}\int_0^t\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_{i-l}(s,x(s,z,0))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}ds.$$

Finally from (2.15)

$$\begin{aligned}x(t,z,\varepsilon)= \;& z+\int_0^t F_0(t,x(s,z,0))ds+\sum_{i=1}^{k-1}\varepsilon^i\Bigg(\int_0^t F_i(s,x(s,z,0))\\&+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_{i-l}(s,x(s,z,0))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\,ds\Bigg)\\&+\varepsilon^k\int_0^t F_k\left(s,x(s,z,0)\right)+\varepsilon^{k+1}\int_0^t R(s,x(s,z,\varepsilon),\varepsilon)ds+\mathcal{O}(\varepsilon^{k+1}).\end{aligned}$$

Now using this last expression of $x(t,z,\varepsilon)$ we conclude that the functions $y_i(t,z)$, defined in (2.12) for $i=1,2,\ldots,k-1$, can be computed recurrently from the following integral equation

$$\begin{aligned}y_i(t,z)=\;&\left(\frac{\partial^i x}{\partial\varepsilon^i}(t,z,\varepsilon)\right)\Bigg|_{\varepsilon=0}\\=\;&i!\int_0^t\Bigg(F_i(s,x(s,z,0))+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\\&\cdot\partial^L F_{i-l}(s,x(s,z,0))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\Bigg)ds\\=\;&\int_0^t\left(A(s)y_i(s,z)+B_i(s)\right)ds,\end{aligned}\tag{2.16}$$

where

$$\begin{aligned}A(s)=\;&\partial F_0(s,x(s,z,0)),\\B_i(s)=\;&i!\Big(F_i(s,x(s,z,0))+\sum_{S_i'}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_{i-1}!(i-1)!^{b_{i-1}}}\partial^{I'}F_0(s,x(s,z,0))\\&\bigodot_{j=1}^{i-1}y_j(s,z)^{b_j}+\sum_{l=1}^{i-1}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^L F_{i-l}(s,x(s,z,0))\\&\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\Big).\end{aligned}$$

The integral equation (2.16) is equivalent to the Cauchy problem

$$\frac{\partial}{\partial t}y_i(t,z)=A(t)y_i(t,z)+B_i(t),\quad\text{with}\quad y_i(0,z)=0,$$

which has a unique solution given by

$$
\begin{aligned}
y_i(t,z) = {} & Y(t,z)\int_0^t Y(s,z)^{-1}B_i(s)ds \\
= {} & i!\,Y(t,z)\int_0^t Y(s,z)^{-1}\Bigg(F_i(s,x(s,z,0)) \\
& +\sum_{S_i'}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_i!i!^{b_i}}\partial^{I'}F_0(s,x(s,z,0))\bigodot_{j=1}^{i-1}y_j(s,z)^{b_j} \\
& +\sum_{l=1}^{i-1}\sum_{S_l}\frac{1}{b_1!\,b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\partial^{L}F_{i-l}(s,x(s,z,0))\bigodot_{j=1}^{l}y_j(s,z)^{b_j}\Bigg)ds.
\end{aligned}
$$

Since

$$x(t,z,0)=z+\int_0^t F_0(t,x(s,z,0))ds,$$

we obtain

$$x(t,z,\varepsilon)=x(t,z,0)+\sum_{i=1}^{k}\varepsilon^i\frac{y_i(t,z)}{i!}+\mathcal{O}(\varepsilon^{k+1}).$$

This concludes the proof of the lemma □

Proof of Theorem 5. Let $x(\cdot,z,\varepsilon):[0,t_{(z,\varepsilon)})\mapsto\mathbb{R}^n$ denote the solution of system (1.11) such that $x(0,z,\varepsilon)=z$. By Theorem 8.3 of [1] there exists a neighborhood U of z and ε_1 sufficiently small such that $t_{(z,\varepsilon)}>T$ for all $(z,\varepsilon)\in U\times(-\varepsilon_1,\varepsilon_1)$. Let $h(z,\varepsilon):U\times(-\varepsilon_1,\varepsilon_1)\mapsto\mathbb{R}^n$ be the *displacement function* defined as

$$h(z,\varepsilon)=x(T,z,\varepsilon)-z. \tag{2.17}$$

Clearly $x(\cdot,\overline{z},\overline{\varepsilon})$ for some $(\overline{z},\overline{\varepsilon})\in U\times(-\varepsilon_1,\varepsilon_1)$ is a T-periodic solution of system (1.11) if and only if $h(\overline{z},\overline{\varepsilon})=0$. Studying the zeros of (2.17) is equivalent to study the zeros of

$$g(z,\varepsilon)=Y(T,z)^{-1}h(z,\varepsilon). \tag{2.18}$$

From Lemma 4 we have

$$x(t,z,\varepsilon)=x(t,z,0)+\sum_{i=1}^{k}\varepsilon^i\frac{y_i(t,z)}{i!}+\mathcal{O}(\varepsilon^{k+1}), \tag{2.19}$$

for all $(t,z)\in\mathbb{S}^1\times\mathcal{D}$, where y_i is defined in (1.12). Hence substituting (2.19) into (2.18) it follows that

$$g(z,\varepsilon)=\sum_{i=0}^{k}\varepsilon^i g_i(z)+\mathcal{O}(\varepsilon^{k+1}),$$

where $g_0(z)=Y^{-1}(t,z)\,(x(t,z,0)-z)$ and for $i=1,2,\ldots,k$ the function g_i is defined in (1.15).

From hypothesis $(\mathcal{H})$ we know that $g_s(z)$ vanishes on the manifold $\mathcal{Z}$. Moreover we have $g(z,\varepsilon)=\varepsilon^s\sum_{i=0}^{k-s}\varepsilon^i g_{s+i}+\mathcal{O}(\varepsilon^{k-s+1})$. Then we consider the function

$$h(z,\varepsilon)=\sum_{i=0}^{k-s}\varepsilon^i g_{s+i}(z)+\mathcal{O}(\varepsilon^{k-s+1})=\frac{g(z,\varepsilon)}{\varepsilon^s}. \tag{2.20}$$

The result follows from identifying (2.20) with (1.7), noticing that function (2.20) satisfies hypothesis (H_a) and applying Theorem 2. □

Proof of Corollary 6. The result follows directly from Corollary 3 applied to the function (1.18). □

2.3 Proof of Lemmas 8 and 9

Proof of Lemma 8. The proof of Theorem 5 has been obtained basically applying Theorem 2 to the function

$$g(z,\varepsilon) = g_s(z) + \varepsilon g_{s+1}(z) + \varepsilon^2 g_{s+2}(z) + \dots$$

That is we find functions $\alpha(\varepsilon)$ and $\overline{\beta}(\alpha,\varepsilon)$ such that $z_{\alpha(\varepsilon)} = (\alpha(\varepsilon), \overline{\beta}(\alpha(\varepsilon),\varepsilon))$ satisfies $g(z_{\alpha(\varepsilon)},\varepsilon) = 0$ for $|\varepsilon| \neq$ sufficiently small. Defining $\overline{k} = k - s$ whe have from Claim 1 that

$$\overline{\beta}(\alpha,\varepsilon) = \beta(\alpha) + \varepsilon\gamma_1(\alpha) + \dots + \gamma_{\overline{k}}(\alpha) + \mathcal{O}(\varepsilon^{\overline{k}+1}). \tag{2.21}$$

Furthermore we have that $\alpha(\varepsilon) = a_\varepsilon + \mathcal{O}(\varepsilon^{\overline{k}-l+1})$, and by hypothesis $a_\varepsilon = \alpha_0 + \varepsilon\alpha_1 + \dots + \varepsilon^{\overline{k}-l}\alpha_{\overline{k}-l} + \mathcal{O}(\varepsilon^{\overline{k}-l+1})$ thus we can write

$$\alpha(\varepsilon) = \alpha_0 + \varepsilon\alpha_1 + \dots + \varepsilon^{k-l}\alpha_{\overline{k}-l} + \mathcal{O}(\varepsilon^{\overline{k}-l+1}). \tag{2.22}$$

Substituting (2.22) in (2.21) and expanding the result in power series of ε around $\varepsilon = 0$ we have

$$\overline{\beta}(\alpha(\varepsilon),\varepsilon) = \beta_0 + \varepsilon\beta_1 + \dots + \varepsilon^{\overline{k}-l}\beta_{\overline{k}-l} + \mathcal{O}\big(\varepsilon^{\overline{k}-l+1}\big),$$

where the coefficients β_i up to order $k-1$ can be calculated as $\beta_0 = \overline{\beta}(\alpha(0),0) = \beta(\alpha_0)$ and for $0 < i \leq \overline{k} - l - 1$ we obtain

$$\begin{aligned}\beta_i =& \frac{1}{i!}\sum_{j=0}^{i}\sum_{q=0}^{i}\binom{i}{q}(\varepsilon^{(j)})^{(i-q)}\frac{d^q}{d\varepsilon^q}\gamma_j(\alpha(\varepsilon)) + \mathcal{O}(\varepsilon)\Big|_{\varepsilon=0}\\ =& \frac{1}{i!}\left(i!\gamma_i\big(\alpha(\varepsilon)\big) + \sum_{j=1}^{i}\binom{i}{i-j}\frac{d^{i-j}}{d\varepsilon^{i-j}}\gamma_j(\alpha(\varepsilon)) + \mathcal{O}(\varepsilon)\right)\Big|_{\varepsilon=0},\end{aligned}$$

taking $l = i - j$ and using the Faá di Bruno's forumla in the above equation we have

$$\beta_i = \gamma_i\big(\alpha(0)\big) + \sum_{l=1}^{i}\sum_{S_l}\frac{1}{c_1!\cdots c_l!(l-1)!^{c_l}}\gamma_{i-l}^{(L)}(\alpha(0))\bigodot_{s=i}^{l}\big(\alpha^{(s)}(0)\big)^{c_s}.$$

Finally, from equation (2.22) we have that $\alpha(0) = \alpha_0$ and $\alpha^{(s)}(0) = s!\alpha_s$, then using it in the above equation we obtain (1.22). This completes the proof. □

Proof of Lemma 9. We define $y_0(T,z) = x_0(T,z) - z$ and let s be the first index such that $g_s(z) \not\equiv 0$. Then the displacement map (1.13) writes

$$d(z,\varepsilon) = \varepsilon^s\sum_{i=0}^{k-s}\varepsilon^i\frac{y_{s+i}(T,z)}{(s+i)!} + \mathcal{O}(\varepsilon^{k+1}).$$

At $z = z(\varepsilon)$ this function has the Jacobian matrix

$$\partial_z d(z(\varepsilon), \varepsilon) = \varepsilon^s \sum_{i=0}^{k-s} \frac{\varepsilon^i}{(s+i)!} \partial_z y_{s+i}(T, z(\varepsilon)) + \mathcal{O}(\varepsilon^{k+1}).$$

From Lemma 8 we have that $z(\varepsilon) = \sum_{i=0}^{k-s-l} z_i + \mathcal{O}(\varepsilon^{k-s-l})$. Thus we calculate the first $k-s-l$ coefficient of the Taylor expansion of the function

$$J(\varepsilon) = \sum_{i=0}^{k-s} \frac{\varepsilon^i}{(s+i)!} \partial_z y_{s+i}(T, z(\varepsilon)) + \mathcal{O}(\varepsilon^{k-s+1}),$$

where $J(0) = \partial_z y_s(T, z_0)$. By the Leibniz rule we have

$$\begin{aligned} \frac{d^j}{d\varepsilon^j} J(\varepsilon) &= \sum_{i=0}^{k-s} \frac{\varepsilon^i}{(s+i)!} \frac{d^j}{d\varepsilon^j} \left(\partial_z y_{s+i}(T, z(\varepsilon)) \right) + \mathcal{O}(\varepsilon^{k-s-j+1}) \\ &= \sum_{i=0}^{j} \sum_{n=0}^{j} \binom{j}{n} \left(\varepsilon^i\right)^{(n)} \frac{d^{j-n}}{d\varepsilon^{j-n}} \left(\partial_z y_{s+i}(T, z(\varepsilon)) \right) + \mathcal{O}(\varepsilon) \Big|_{\varepsilon=0}. \end{aligned}$$

When $\varepsilon = 0$ the only non-vanishing terms in the above equation will be those satisfying $i = n$, then we have

$$\frac{d^j}{d\varepsilon^j} \partial_z d(z(\varepsilon), \varepsilon) = \sum_{n=0}^{j} \frac{j!}{n!(j-n)!} \frac{d^{j-n}}{d\varepsilon^{j-n}} \left(\partial_z y_{s+n}(T, z(\varepsilon)) \right). \tag{2.23}$$

Using the Faá di Bruno's Formula we have that

$$\frac{d^i}{d\varepsilon^i} \left(\partial_z y_{s+n}(T, z(\varepsilon)) \right) \Big|_{\varepsilon=0} = \sum_{S_i} \frac{i!}{b_1! \cdots b_i! (i-1)!^{b_i}} \partial_z^{I+1} y_{s+n}(T, z(\varepsilon)) \bigodot_{u=1}^{i} \left(z^{(u)}(\varepsilon) \right)^{b_u} \Big|_{\varepsilon=0}.$$

We use the above equation in (2.23) taking $i = j - n$ obtaining

$$\begin{aligned} \overline{A}_j &= \frac{d^j}{d\varepsilon^j} \partial_z d(z(\varepsilon), \varepsilon) \Big|_{\varepsilon=0} \\ &= \sum_{i=0}^{j} \frac{j!}{i!(j-i)!} \sum_{S_i} \frac{i!}{b_1! \cdots b_i! (i-1)!^{b_i}} \partial_z^{I+1} y_{s+j-i}(T, z_0) \bigodot_{u=1}^{i} (u! z_u)^{b_u}. \end{aligned}$$

Finally we take $A_j = j! \overline{A}_j$, this completes the proof. □

2.4 Proof of Theorem 10

Before the proof we present some results about $k-$determined hyperbolicity, for more details see [60, Chapter 3]. Suppose that

$$\widehat{A} = A_0 + \varepsilon A_1 + \cdots + \varepsilon^k A_k$$

is hyperbolic for all $\varepsilon > 0$. The aim of this section is to present results that help to determine under what circumstances we can say that any smooth matrix $A(\varepsilon)$ having $\widehat{A}$ as its $k-$jet will be hyperbolic with the same hyperbolicity type of $\widehat{A}$.

We assume that $A(\varepsilon)$ is diagonalizable in the following sense. There exists a fractional power series $T(\varepsilon)$, which in general contains positive and negative fractional powers of ε, such that

$$B(\varepsilon) = T(\varepsilon)^{-1}A(\varepsilon)T(\varepsilon) \tag{2.24}$$

is diagonal and is a fractional power series in ε containing only positive fractional powers. Notice that since $A(\varepsilon)$ and $B(\varepsilon)$ in (2.24) are similar for $\varepsilon > 0$, they will be of the same hyperbolicity type. We suppose that $B(\varepsilon)$ has been computed up to some fractional order, and that this portion $\widehat{B}(\varepsilon)$ of $B(\varepsilon)$ is hyperbolic and diagonal. The next theorem implies that $A(\varepsilon)$ is hyperbolic, with the same stability type as $\widehat{B}(\varepsilon)$.

Theorem 15 ([60, Theorem 3.7.7])**.** *Suppose that $C(\varepsilon)$ and $D(\varepsilon)$ are continuous matrix-valued functions defined for $\varepsilon > 0$, and that*

$$C(\varepsilon) = \Lambda(\varepsilon) + \varepsilon^R D(\varepsilon), \tag{2.25}$$

where

$$\Lambda(\varepsilon) = \begin{bmatrix} \lambda_1(\varepsilon) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda_n(\varepsilon) \end{bmatrix} = \varepsilon^{r_1}\Lambda_1 + \cdots + \varepsilon^{r_j}\Lambda_j.$$

Here $r_1 < r_2 < \cdots < r_j < R$ rational numbers, and $\Lambda_1, \ldots, \Lambda_j$ diagonal matrices. Then there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the eigenvalues of $C(\varepsilon)$ are approximately equal to the diagonal entries $\lambda_i(\varepsilon)$ of $\Lambda(\varepsilon)$, with error $\mathcal{O}(\varepsilon^R)$. Consequently the matrices $\Lambda(\varepsilon)$ and $C(\varepsilon)$ have the same hyperbolicity type.

Proof of Theorem 10. Theorem 10 is obtaining directly from applying Theorem 15 to the Jacobian matrix of the Poincaré map (1.23), identifying $\Lambda(\varepsilon)$ with $M(\varepsilon)$. □

2.5 Proof of Proposition 11

The following result will be used for proving Proposition 11.

Lemma 16. *Let $u : \mathbb{R}^n \times [0, \varepsilon_0] \to \mathbb{R}^n$ be a function of class $\mathcal{C}^k$ such that*

$$u(x, \varepsilon) = u_0(x) + \varepsilon u_1(x) + \cdots + \varepsilon^k u_k(x) + \mathcal{O}(\varepsilon^{k+1}).$$

Assume that there exists a function $v : \mathbb{R} \to \mathbb{R}^n$ of class $\mathcal{C}^k$ satisfying $u(v(\varepsilon), \varepsilon) = 0$ for $|\varepsilon| > 0$ sufficiently small and that the Jacobian matrix $\partial u_0(v(0))$ is invertible. Then

$$v(\varepsilon) = v(0) + \frac{\varepsilon}{1!}v^{(1)}(0) + \cdots + \frac{\varepsilon^k}{k!}v^{(k)}(0) + \mathcal{O}(\varepsilon^{k+1}),$$

where the Taylor's coefficients of $v(\varepsilon)$ *are written recursively as* $v^{(1)}(0) = Du_0(v(0))^{-1}u_1(v(0))$ *and for* $2 \leq i \leq k$*,*

$$v^{(i)}(0) = -\,Du_0(v(0))^{-1}\Bigg(\sum_{S_i'} \frac{1}{c_1!\,c_2!2!^{c_2}\cdots c_{i-1}!(i-1)!^{c_{i-1}}} u_0^{(I')}(v(0)) \bigodot_{j=1}^{i-1} v^{(j)}(0)^{c_j} + \sum_{l=0}^{i-1}\sum_{S_l} \frac{1}{c_1!\,c_2!2!^{c_2}\cdots c_l!l!^{c_l}} u_{i-l}^{(L)}(v(0)) \bigodot_{j=1}^{l} v^{(j)}(0)^{c_j}\Bigg).$$

Proof. For $\varepsilon >$ sufficiently small we have by hypothesis that $u(v(\varepsilon),\varepsilon) = 0$, then for $1 \leq i \leq k$ we obtain

$$\frac{d^i}{d\varepsilon^i}u(v(\varepsilon),\varepsilon) = \sum_{j=0}^{i}\sum_{q=0}^{i}\binom{i}{q}(\varepsilon^{(j)})^{(i-q)}\frac{d^q}{d\varepsilon^q}u_j(v(\varepsilon)) + \mathcal{O}(\varepsilon) = 0.$$

Taking $\varepsilon = 0$, $l = i - j$ and using the Faá di Bruno's Formula we have

$$\begin{aligned}
0 =& \sum_{l=1}^{i}\frac{i!}{l!}\frac{d^l}{d\varepsilon^l}u_{i-l}(v(0)) + i!u_i(v(0)) \\
=& \frac{d^i}{d\varepsilon^i}u_0(v(0)) + \sum_{l=1}^{i-1}\frac{i!}{l!}\frac{d^l}{d\varepsilon^l}u_{i-l}(v(0)) + i!u_i(v(0)) \\
=& \Bigg(\sum_{S_i'} \frac{i!}{c_1!\,c_2!2!^{c_2}\cdots c_{i-1}!(i-1)!^{c_{i-1}}} u_0^{(I')}(v(0)) \bigodot_{j=1}^{i-1} v^{(j)}(0)^{c_j}\Bigg) \\
&+ i!Du_0(v(0))v^{(i)}(0) + \Bigg(\sum_{l=0}^{i-1}\sum_{S_l} \frac{i!}{c_1!\,c_2!2!^{c_2}\cdots c_l!l!^{c_l}} u_{i-l}^{(L)}(v(0)) \bigodot_{j=1}^{l} v^{(j)}(0)^{c_j}\Bigg) \\
&+ i!u_i(v(0)).
\end{aligned}$$

Then we isolate $v^{(i)}(0)$ obtaining

$$v^{(i)}(0) = -Du_0(v(0))^{-1}\Bigg(\sum_{S_i'} \frac{i!}{c_1!\,c_2!2!^{c_2}\cdots c_{i-1}!(i-1)!^{c_{i-1}}} u_0^{(I')}(v(0)) \bigodot_{j=1}^{i-1} v^{(j)}(0)^{c_j} + \sum_{l=0}^{i-1}\sum_{S_l} \frac{i!}{c_1!\,c_2!2!^{c_2}\cdots c_l!l!^{c_l}} u_{i-l}^{(L)}(v(0)) \bigodot_{j=1}^{l} v^{(j)}(0)^{c_j} + u_i(v(0))\Bigg).$$

□

Proof of Proposition 11. Statement (a) follows directly from the Implicit Function Theorem. The proof of statement (b) is simply to define the function

$$u(\alpha,\varepsilon) = \mathcal{F}^{k-s}(\alpha,\varepsilon) = \sum_{i=1}^{k-s}\varepsilon^i f_i(\alpha),$$

where $u_i(\alpha) = f_i(\alpha)$ for all $0 < i < k - s$ are given in (1.17). Then from statement (a) we have $u(a_\varepsilon,\varepsilon) = 0$ and by hypothesis we know that $\partial u_0(\alpha_0)$ is invertible. Thus we apply Lemma 16 to $u(\alpha,\varepsilon)$ taking $v(\varepsilon) = \alpha_\varepsilon$, obtaining the functions shown in statement (b). □

2.6 Proof of Theorems 12 and 13

Proof of Theorem 12. Theorem 12 is just a reformulation of Corollary 6 where the bifurcation functions f_1 and f_2 were explicitly given. □

Proof of Theorem 13. Consider system (1.11) with $k = s + 2$ satisfying the hypothesis of Theorem 13. Then from (1.16) we have that the displacement function is equivalent to

$$h(z, \varepsilon) = g_s(z) + \varepsilon g_{s+1}(z) + \mathcal{O}(\varepsilon^2).$$

From (1.18) the bifurcation function in this case becomes

$$\mathcal{F}^2(\alpha, \varepsilon) = f_1(\alpha) + \varepsilon f_2(\alpha).$$

By hypothesis $f(\alpha^*) = 0$ and $Df_1(\alpha^*) \neq 0$, thus from the Implicit Function Theorem there exists

$$a_\varepsilon = \alpha^* + \varepsilon\alpha_1 + \mathcal{O}(\varepsilon^2),$$

satisfying $\mathcal{F}^2(a_\varepsilon, \varepsilon) = 0$ with $\alpha_1 = -Df_1(\alpha^*)^{-1}f_2(\alpha^*)$. Furthermore from Lemma 8 we have that

$$z(\varepsilon) = (\alpha^*, \beta(\alpha^*)) + \varepsilon\, z_1 + \mathcal{O}(\varepsilon^2),$$

with z_1 given by (1.28). Finally from Lemma 9 the Jacobian matrix of the displacement function of system writes

$$\partial_z d(z(\varepsilon)) = \varepsilon^s A_0 + \varepsilon^{s+1} A_1 + \mathcal{O}(\varepsilon^{s+2}),$$

with A_0 and A_1 given in (1.26) and (1.27) respectively. Consequently the Jacobian of the Poincaré map (1.19) at $z(\varepsilon)$ becomes $D\Pi(\mathbf{z}(\varepsilon), \varepsilon) = M(\varepsilon) + \mathcal{O}(\varepsilon^{s+2})$, with $M(\varepsilon) = Id + \varepsilon^s A_1 + \varepsilon^{s+1} A_1$.

Now we have to consider two cases, $s = 0$ and $s > 0$. If $s = 0$ then we have $M(\varepsilon) = Id + A_0 + \varepsilon A_1$, by hypothesis (i) of Theorem 12 we have $\det(A_0) = 0$. Thus we observe that $Id + A_0$ is non-hyperbolic. Indeed let v be an eigenvector of A_0 associated with the eigenvalue 0. We have $(Id + A_0)v = v$, thus 1 is an eigenvalue of $Id + A_0$. Consequently $Id + A_0$ is non-hyperbolic. By hypothesis $Id + A_0$ is diagonalizable, and by Corollary 19 there exists a matrix $T(\varepsilon)$ such that $T(\varepsilon)^{-1}M(\varepsilon)T(\varepsilon) = \varepsilon^{r_1}\Lambda_1 + \varepsilon^{r_2}\Lambda_2 + \mathcal{O}(\varepsilon^2)$ with $T(\varepsilon)^{-1}M(\varepsilon)T(\varepsilon) = \varepsilon^{r_1}\Lambda_1 + \varepsilon^{r_2}\Lambda_2$. Then the result follows from applying Theorem 15. If $s > 0$ we study the jet $A_0 + \varepsilon A_1$ separately. By hypothesis we A_0 diagonal and using Corollary 19 there exists $T(\varepsilon)$ such that $T(\varepsilon)^{-1}(A_0 + \varepsilon A_1)T(\varepsilon) = \varepsilon^{r_1}\Lambda_1 + \varepsilon^{r_2}\Lambda_2$. Thus we have $T(\varepsilon)^{-1}M(\varepsilon)T(\varepsilon) = Id + \varepsilon^{s+r_1}\Lambda_1 + \varepsilon^{s+r_2+1}\Lambda_2$ and the result follows applying Theorem 15. □

2.7 Appendix A: k-determined hyperbolicity

To find the matrix $T(\varepsilon)$ is fundamental for applying Theorem 10. Here we show sufficient conditions for the existence of a such matrix.

Definition 1. *A linear transformation $L : V \to V$ is said to be in real semisimple canonical form if V is a real vector space and L is block-diagonalizable in the form*

$$\begin{bmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_s & & & & & & \\ & & & \alpha_1 & -\beta_1 & & & & \\ & & & \beta_1 & \alpha_1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & & \alpha_t & -\beta_t \\ & & & & & & & \beta_t & \alpha_t \end{bmatrix}.$$

The task for putting the matrix series

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots$$

into the form (2.25) is related with putting $A(\varepsilon)$ in its normal form. This always comes down to the solution of a homological equation such as

$$\mathcal{L}U_j = K_j - B_j,$$

where

$$\mathcal{L} = \mathbb{L}_{A_0} : gl(n) \to gl(n),$$

K_j is known at the jth stage of the calculation, and B_j and U_j are to be determined. Here $gl(n)$ is the set of all invertible $n \times n$ real matrices.

Definition 2. *A matrix series $A(\varepsilon)$ is in semisimple normal form (to order k, or to all orders) if and only if*

1) *the leading term A_0 is semisimple; and*

2) *each term A_j (for $j \leq k$, or for all j) commutes with the leading term: $[A_j, A_0] = 0$, or $A_j \in ker\mathcal{L}$.*

Notice that only the leading term is required to be semisimple.

Consider A_0 diagonal (real or complex), it is easy to describe the normal form if we assume that all repeated eigenvalues are placed right next to each other in the diagonal. If A_0 contains a string of equal eigenvalues in its diagonal, the smallest submatrix of A_0 containing this diagonal string will be called the *block subtended by the string.* Then if r and s are integers such that the (r, s) position in A_0 lies in a block subtended by a string, if follows that $\lambda_r = \lambda_s$, and therefore $\mathcal{L}E_{rs} = 0$. Here E_{rs} is the element of the canonical basis of $gl(n)$.

Theorem 17. *If A_0 is an $n \times n$ diagonal matrix, with equal eigenvalues adjacent, then ker ($\mathcal{L}$) is the set of block diagonal $n \times n$, with blocks subtended by the strings of equal eigenvalues of A_0. A matrix series with A_0 as leading term is in semisimple normal form (up to degree k) if and only if the remaining terms (up to order k) are in this block diagonal form.*

Theorem 18. *Suppose that*

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots$$

is a real matrix series in which A_0 *is semisimple and is in real semisimple canonical form. Let*

$$T = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 1/2 & 1/2 & & & \\ & & & -i/2 & i/2 & & & \\ & & & & & \ddots & & \\ & & & & & & 1/2 & 1/2 \\ & & & & & & -i/2 & i/2 \end{bmatrix}$$

and let

$$A'(\varepsilon) = \mathbb{S}_T(A(\varepsilon)) = A'_0 + \varepsilon A'_1 + \dots.$$

Then it is possible to choose a semisimple normal form

$$B'(\varepsilon) = A'_0 + \varepsilon B'_1 + \dots$$

for the complex series $B'(\varepsilon)$ *such that the series*

$$B(\varepsilon) = \mathbb{S}_T^{-1}(B'(\varepsilon)) = A_0 + \varepsilon B_1 + \dots$$

is a real matrix series in normal form with respect to A_0.

Corollary 19. *If* A_0 *is a semisimple matrix in real semisimple canonical form, and if all of the* 1×1 *and* 2×2 *diagonal blocks are distinct, then the semisimple normal form will be in real semisimple canonical form with the same block sizes.*

2.8 Appendix B: Bifurcation functions up to order 5.

In this appendix we develop the recurrences given by Theorems 2 and 5 for computing explicitly the expressions of the bifurcation functions and the averaged functions up to order 5. As far as we know we are the first to provide these expressions.

From the recurrences (1.9) and (1.10) we explicitly develop the expressions of the bifurcation functions $f_i : V \to \mathbb{R}^m$, for $i = 1, 2, .., 5$, as stated in Theorem 2. Recall that $\Gamma_\alpha = (\partial \pi g_0/\partial b)(z_\alpha)$. So

$$\begin{aligned}
f_1(\alpha) =& \Gamma_\alpha \gamma_1(\alpha) + \pi g_1(z_\alpha),\\
\gamma_1(\alpha) =& -\Delta_\alpha^{-1} \pi^\perp g_1(z_\alpha),\\
f_2(\alpha) =& \frac{1}{2}\Gamma_\alpha \gamma_2(\alpha) + \frac{1}{2}\frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + \frac{\partial \pi g_1}{\partial b}(z_\alpha)\gamma_1(\alpha) + \pi g_2(z_\alpha),\\
\gamma_2(\alpha) =& -\Delta_\alpha^{-1}\left(\frac{\partial^2 \pi^\perp g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + 2\frac{\partial \pi^\perp g_1}{\partial b}(z_\alpha)\gamma_1(\alpha) + 2\pi^\perp g_2(\alpha)\right),
\end{aligned}$$

$$
\begin{aligned}
f_3(\alpha) =& \frac{1}{6}\Gamma_\alpha \gamma_3(\alpha) + \frac{1}{6}\frac{\partial^3 \pi g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3 + \frac{1}{2}\frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_2(\alpha)\\
&+ \frac{1}{2}\frac{\partial^2 \pi g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + \frac{1}{2}\frac{\partial \pi g_1}{\partial b}(z_\alpha)\gamma_2(\alpha) + \frac{\partial \pi g_2}{\partial b}(z_\alpha)\gamma_1(\alpha)\\
&+ \pi g_3(z_\alpha),\\
\gamma_3(\alpha) =& -\Delta_\alpha^{-1}\Bigg(\frac{\partial^3 \pi^\perp g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3 + 3\frac{\partial^2 \pi^\perp g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_2(\alpha)\\
&+ 3\frac{\partial^2 \pi^\perp g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + 2\frac{\partial \pi^\perp g_1}{\partial b}(z_\alpha)\gamma_2(\alpha) + 6\frac{\partial \pi^\perp g_2}{\partial b}(z_\alpha)\gamma_1(\alpha)\\
&+ 6\pi^\perp g_3(\alpha)\Bigg),\\
f_4(\alpha) =& \frac{1}{24}\Gamma_\alpha \gamma_4(\alpha) + \frac{1}{24}\frac{\partial^4 \pi g_0}{\partial b^4}(z_\alpha)\gamma_1(\alpha)^4 + \frac{1}{4}\frac{\partial^3 \pi g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^2\odot\gamma_2(\alpha)\\
&+ \frac{1}{8}\frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_2(\alpha)^2 + \frac{1}{6}\frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_3(\alpha)\\
&+ \frac{1}{6}\frac{\partial^3 \pi g_1}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3 + \frac{1}{2}\frac{\partial^2 \pi g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_2(\alpha) + \frac{1}{6}\frac{\partial \pi g_1}{\partial b}(z_\alpha)\gamma_3(\alpha)\\
&+ \frac{1}{2}\frac{\partial^2 \pi g_2}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + \frac{1}{2}\frac{\partial \pi g_2}{\partial b}(z_\alpha)\gamma_2(\alpha) + \frac{\partial \pi g_3}{\partial b}(z_\alpha)\gamma_1(\alpha) + \pi g_4(z_\alpha),\\
\gamma_4(\alpha) =& -\Delta_\alpha^{-1}\Bigg(\frac{\partial^4 \pi^\perp g_0}{\partial b^4}(z_\alpha)\gamma_1(\alpha)^4 + 3\frac{\partial^2 \pi^\perp g_0}{\partial b^2}(z_\alpha)\gamma_2(\alpha)^2 + 4\frac{\partial^2 \pi^\perp g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_3(\alpha)\\
&+ 6\frac{\partial^3 \pi^\perp g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^2\odot\gamma_2(\alpha) + 4\frac{\partial \pi^\perp g_1}{\partial b}(z_\alpha)\gamma_3(\alpha) + 12\frac{\partial^2 \pi^\perp g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_2(\alpha)\\
&+ 4\frac{\partial^3 \pi^\perp g_1}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3 + 12\frac{\partial \pi^\perp g_2}{\partial b}(z_\alpha)\gamma_2(\alpha) + 12\frac{\partial^2 \pi^\perp g_2}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2\\
&+ 24\frac{\partial \pi^\perp g_3}{\partial b}(z_\alpha)\gamma_1(\alpha)\Bigg),\\
f_5(\alpha) =& \frac{1}{120}\Gamma_\alpha \gamma_5(\alpha) + \frac{1}{12}\frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_2(\alpha)\odot\gamma_1(\alpha) + \frac{1}{24}\frac{\partial^2 \pi g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_4(\alpha)\\
&+ \frac{1}{8}\frac{\partial^3 \pi g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)\odot\gamma_2(\alpha)^2 + \frac{1}{12}\frac{\partial^3 \pi g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^2\odot\gamma_3(\alpha)\\
&+ \frac{1}{12}\frac{\partial^4 \pi g_0}{\partial b^4}(z_\alpha)\gamma_1(\alpha)^3\odot\gamma_2(\alpha) + \frac{1}{120}\frac{\partial^5 \pi g_0}{\partial b^5}(z_\alpha)\gamma_1(\alpha)^5\\
&+ \frac{1}{24}\frac{\partial \pi g_1}{\partial b}(z_\alpha)\gamma_4(\alpha) + \frac{1}{8}\frac{\partial^2 \pi g_1}{\partial b^2}(z_\alpha)\gamma_2(\alpha)^2 + \frac{1}{6}\frac{\partial^2 \pi g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_3(\alpha)\\
&+ \frac{1}{4}\frac{\partial^3 \pi g_1}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^2\odot\gamma_2(\alpha) + \frac{1}{24}\frac{\partial^4 \pi g_1}{\partial b^4}(z_\alpha)\gamma_1(\alpha)^4 + \frac{1}{6}\frac{\partial \pi g_2}{\partial b}(z_\alpha)\gamma_3(\alpha)\\
&+ \frac{1}{2}\frac{\partial^2 \pi g_2}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_2(\alpha) + \pi g_4(z_\alpha) + \frac{1}{6}\frac{\partial^3 \pi g_2}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3\\
&+ \frac{1}{2}\frac{\partial \pi g_3}{\partial b}(z_\alpha)\gamma_2(\alpha) + \frac{1}{2}\frac{\partial^2 \pi g_3}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2 + \frac{\partial \pi g_4}{\partial b}(z_\alpha)\gamma_1(\alpha)\\
&+ \pi g_5(z_\alpha),
\end{aligned}
$$

$$
\begin{aligned}
\gamma_5(\alpha) =& -\Delta_\alpha^{-1}\Bigg(10\frac{\partial^2\pi^\perp g_0}{\partial b^2}(z_\alpha)\gamma_2(\alpha)\odot\gamma_3(\alpha)+5\frac{\partial^2\pi^\perp g_0}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_4(\alpha)\\
&+15\frac{\partial^3\pi^\perp g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)\odot\gamma_2(\alpha)^2+10\frac{\partial^3\pi^\perp g_0}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^2\odot\gamma_3(\alpha)\\
&+10\frac{\partial^4\pi^\perp g_0}{\partial b^4}(z_\alpha)\gamma_1(\alpha)^3\odot\gamma_2(\alpha)+\frac{\partial^5\pi^\perp g_0}{\partial b^5}(z_\alpha)\gamma_1(\alpha)^5\\
&+5\frac{\partial\pi^\perp g_1}{\partial b}(z_\alpha)\gamma_4(\alpha)+15\frac{\partial^2\pi^\perp g_1}{\partial b^2}(z_\alpha)\gamma_2(\alpha)^2+20\frac{\partial^2\pi^\perp g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_3(\alpha)\\
&+30\frac{\partial^3\pi^\perp g_1}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2\odot\gamma_2(\alpha)+5\frac{\partial^4\pi^\perp g_1}{\partial b^4}(z_\alpha)\gamma_1(\alpha)^4\\
&+20\frac{\partial\pi^\perp g_2}{\partial b}(z_\alpha)\gamma_3(\alpha)+60\frac{\partial^2\pi^\perp g_2}{\partial b^2}(z_\alpha)\gamma_1(\alpha)\odot\gamma_2(\alpha)\\
&+20\frac{\partial^3\pi^\perp g_2}{\partial b^3}(z_\alpha)\gamma_1(\alpha)^3+60\frac{\partial\pi^\perp g_3}{\partial b}(z_\alpha)\gamma_2(\alpha)\\
&+60\frac{\partial^2\pi^\perp g_3}{\partial b^2}(z_\alpha)\gamma_1(\alpha)^2+120\frac{\partial\pi^\perp g_4}{\partial b}(z_\alpha)\gamma_1(\alpha)\Bigg).
\end{aligned}
$$

The averaged functions, as stated in Theorem 5, are computed as follows:

$$
g_i(z) = Y(T,z)^{-1}\frac{y_i(T,z)}{i!}.
$$

So from the recurrence (1.12) we explicitly develop the expressions of y_i, for $i = 0, 1, \dots, 5$.

$$
\begin{aligned}
y_0(t,z) =& x(t,z,0)-z,\\
y_1(t,z) =& Y(t,z)\int_0^t Y(\tau,z)^{-1}F_1(\tau,x(\tau,z,0))\mathrm{d}\tau,\\
y_2(t,z) =& Y(t,z)\int_0^t Y(\tau,z)^{-1}\Bigg[2F_2(\tau,x(\tau,z,0))+2\frac{\partial F_1}{\partial x}(\tau,x(\tau,x,0))y_1(\tau,z)\\
&+\frac{\partial^2 F_0}{\partial x^2}(\tau,x(\tau,z,0))y_1(\tau,z)^2\Bigg]\mathrm{d}\tau,\\
y_3(t,z) =& Y(t,z)\int_0^t Y(\tau,z)^{-1}\Bigg[6F_3(\tau,x(\tau,z,0))+6\frac{\partial F_2}{\partial x}(\tau,x(\tau,x,0))y_1(\tau,z)\\
&+3\frac{\partial^2 F_1}{\partial x^2}(\tau,x(\tau,z,0))y_1(\tau,z)^2+3\frac{\partial F_1}{\partial x}(\tau,x(\tau,z,0))y_2(\tau,z)\\
&+3\frac{\partial^2 F_0}{\partial x^2}(\tau,x(\tau,z,0))y_1(\tau,z)\odot y_2(\tau,z)+\frac{\partial^3 F_0}{\partial x^3}(\tau,x(\tau,z,0))y_1(\tau,z)^3\Bigg]\mathrm{d}\tau,\\
y_4(t,z) =& Y(t,z)\int_0^t Y(\tau,z)^{-1}\Bigg[24F_4(\tau,x(\tau,z,0))+24\frac{\partial F_3}{\partial x}(\tau,x(\tau,x,0))y_1(\tau,z)\\
&+12\frac{\partial^2 F_2}{\partial x^2}(\tau,x(\tau,z,0))y_1(\tau,z)^2+12\frac{\partial F_2}{\partial x}(\tau,x(\tau,z,0))y_2(\tau,z)\\
&+12\frac{\partial^2 F_1}{\partial x^2}(\tau,x(\tau,z,0))y_1(\tau,z)\odot y_2(\tau,z)+4\frac{\partial^3 F_1}{\partial x^3}(\tau,x(\tau,z,0))y_1(\tau,z)^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\frac{\partial F_1}{\partial x}(\tau, x(\tau,z,0))y_3(\tau,z) + 3\frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau,z,0))y_2(\tau,z)^2 \\
&+ 4\frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau,z,0))y_1(\tau,z) \odot y_3(\tau,z) \\
&+ 6\frac{\partial^3 F_0}{\partial x^3}(\tau, x(\tau,z,0))y_1(\tau,z)^2 \odot y_2(\tau,z) + \frac{\partial^4 F_0}{\partial x^4}(\tau, x(\tau,z,0))y_1(\tau,z)^4 \Bigg] d\tau,
\end{aligned}
$$

$$
\begin{aligned}
y_5(t,z) =& Y(t,z)\int_0^t Y(\tau,z)^{-1}\Bigg[120F_5(\tau, x(\tau,z,0)) + 120\frac{\partial F_4}{\partial x}(\tau, x(\tau,x,0))y_1(\tau,z) \\
&+ 60\frac{\partial^2 F_3}{\partial x^2}(\tau, x(\tau,z,0))y_1(\tau,z)^2 + 60\frac{\partial F_3}{\partial x}(\tau, x(\tau,z,0))y_2(\tau,z) \\
&+ 60\frac{\partial^2 F_2}{\partial x^2}(\tau, x(\tau,z,0))y_1(\tau,z) \odot y_2(\tau,z) + 20\frac{\partial^3 F_2}{\partial x^3}(\tau, x(\tau,z,0))y_1(\tau,z)^3 \\
&+ 20\frac{\partial F_2}{\partial x}(\tau, x(\tau,z,0))y_3(\tau,z) + 20\frac{\partial^2 F_1}{\partial x^2}(\tau, x(\tau,z,0))y_1(\tau,z) \odot y_3(\tau,z) \\
&+ 15\frac{\partial^2 F_1}{\partial x^2}(\tau, x(\tau,z,0))y_2(\tau,z)^2 + 30\frac{\partial^3 F_1}{\partial x^3}(\tau, x(\tau,z,0))y_1(\tau,z)^2 \odot y_2(\tau,z) \\
&+ 5\frac{\partial^4 F_1}{\partial x^4}(\tau, x(\tau,z,0))y_1(\tau,z)^4 + 5\frac{\partial F_1}{\partial x}(\tau, x(\tau,z,0))y_4(\tau,z) \\
&+ 10\frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau,z,0))y_1(\tau,z) \odot y_3(\tau,z) \\
&+ 5\frac{\partial^2 F_0}{\partial x^2}(\tau, x(\tau,z,0))y_1(\tau,z) \odot y_4(\tau,z) \\
&+ 15\frac{\partial^3 F_0}{\partial x^3}(\tau, x(\tau,z,0))y_1(\tau,z) \odot y_2(\tau,z)^2 \\
&+ 10\frac{\partial^3 F_0}{\partial x^3}(\tau, x(\tau,z,0))y_1(\tau,z)^2 \odot y_3(\tau,z) \\
&+ 10\frac{\partial^4 F_0}{\partial x^4}(\tau, x(\tau,z,0))y_1(\tau,z)^3 \odot y_2(\tau,z) + \frac{\partial^5 F_0}{\partial x^5}(\tau, x(\tau,z,0))y_1(\tau,z)^5 \Bigg] d\tau.
\end{aligned}
$$

2.9 Appendix C: Basic results on the Brouwer degree

In this appendix we follow the Browder's paper [6], and we present the existence and uniqueness result from the degree theory in finite dimensional spaces.

Theorem 20. *Let $X = \mathbb{R}^n = Y$ for a given positive integer n. For bounded open subsets V of X, consider continuous mappings $f : \mathrm{Cl}(V) \to Y$, and points y_0 in Y such that y_0 does not lie in $f(\partial V)$ (as usual ∂V denotes the boundary of V). Then to each such triple (f, V, y_0), there corresponds an integer $d(f, V, y_0)$ having the following three properties.*

(i) If $d(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If f_0 is the identity map of X onto Y, then for every bounded open set V and $y_0 \in V$, we have

$$d\left(f_0\big|_V, V, y_0\right) = \pm 1.$$

(ii) (*Additivity*) *If* $f : \mathrm{Cl}(V) \to Y$ *is a continuous map with* V *a bounded open set in* X, *and* V_1 *and* V_2 *are a pair of disjoint open subsets of* V *such that*

$$y_0 \notin f(\mathrm{Cl}(V)\backslash(V_1 \cup V_2)),$$

then

$$d\left(f_0, V, y_0\right) = d\left(f_0, V_1, y_0\right) + d\left(f_0, V_1, y_0\right).$$

(iii) (*Invariance under homotopy*) *Let* V *be a bounded open set in* X, *and consider a continuous homotopy* $\{f_t : 0 \leq t \leq 1\}$ *of maps of* $\mathrm{Cl}(V)$ *in to* Y. *Let* $\{y_t : 0 \leq t \leq 1\}$ *be a continuous curve in* Y *such that* $y_t \notin f_t(\partial V)$ *for any* $t \in [0, 1]$. *Then* $d(f_t, V, y_t)$ *is constant in* t *on* $[0, 1]$.

Moreover the degree function $d(f, V, y_0)$ *is uniquely determined by the three above conditions.*

Chapter 3

Maxwell -Bloch and a 3D polynomial differential system

In this chapter we start a sequence of applications of the theoretical results presented in Chapter 1 and proved in Chapter 2. Here we study two differential systems. First we study the zero-Hopf bifurcation in the Maxwell-Bloch system, a three dimensional polynomial differential system. The computations in this chapter will be done step by step. And the proofs will be presented just after the theorems. The reason is that we want to clarify, for instance, how Theorem 12 and 13 are connected in order to provide the existence and stability of the periodic solution of the Maxwell-Bloch system. Most of the results presented in the following chapter will be done using Theorem 12 and 13. For these reason we present in Section 3.2 a system for which the classical averaging method and also Theorem 12 does not provide information about periodic solutions, for this reason in this case Theorem 13 will be necessary. This results are published in [17].

3.1 Maxwell-Bloch system

In nonlinear optics the Maxwell–Bloch equations are used to describe laser systems. For instance in [2] these equations were obtained by coupling the Maxwell equations with the Bloch equation (a linear Schrödinger like equation which describes the evolution of atoms resonantly coupled to the laser field). Recently in [48] it was identified weak foci and centers in the Maxwell-Bloch system which can be written as

$$\begin{aligned} \dot{u} &= -\,au + v, \\ \dot{v} &= -\,bv + uw, \\ \dot{w} &= -\,c(w-\delta) - 4uv. \end{aligned} \tag{3.1}$$

For $c = 0$ the differential system (3.1) has a singular line $\{(u, v, w)|u = 0, v = 0\}$; for $c \neq 0$ and $ac(\delta - ab) \leq 0$ the differential system (3.1) has one equilibrium $\mathbf{p}_0 = (0, 0, \delta)$; and for $c \neq 0$ and $ac(\delta - ab) > 0$ the differential system (3.1) has three equilibria $\mathbf{p}_\pm = \big(\pm u^*, \pm v^*, w^*\big)$ and $\mathbf{p}_0$ where

$$u^* = \sqrt{\frac{c(\delta - ab)}{4a}},\ v^* = a\sqrt{\frac{c(\delta - ab)}{4a}},\ w^* = ab.$$

Using the above strategy we shall prove the following result.

Proposition 21. *Let* $\omega \in (0, \infty)$*,* $(a, b, c) = \big(a_0 - b_1\varepsilon + a_2\varepsilon^2,\ -a_0 + b_1\varepsilon + b_2\varepsilon^2,\ c_1\varepsilon + c_2\varepsilon^2\big)$ *and* $\delta = -a_0^2 - \omega^2$ *with* $a_0(a_2 + b_2) > 0$*,* $c_1 \neq 0$ *and* ε *a small parameter. Then for* $|\varepsilon| \neq 0$ *sufficiently small the Maxwell-Bloch differential system* (3.1) *has an isolated periodic solution* $\varphi(t, \varepsilon) = \big(u(t, \varepsilon),\ v(t, \varepsilon),\ w(t, \varepsilon)\big)$ *such that*

$$\begin{aligned} u(t,\varepsilon) =& \varepsilon\,\omega\sqrt{\frac{2(a_2+b_2)}{a_0}}\ \sin t + \mathcal{O}(\varepsilon^2), \\ v(t,\varepsilon) =& \varepsilon\,\omega\sqrt{\frac{2(a_2+b_2)}{a_0}}\big(a_0 \sin t + \omega \cos t\big) + \mathcal{O}(\varepsilon^2), \ \textit{and} \\ w(t,\varepsilon) =& \delta - \varepsilon\,\frac{4\omega^2(a_2+b_2)}{c_1} + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{3.2}$$

Proof. Applying the change of variables $(u, v, w) = (\varepsilon V, \varepsilon(a_0V + \omega U), \delta + \varepsilon W)$, the differential system (3.1) reads

$$\begin{aligned} \dot U =& -\omega V + \frac{\varepsilon}{\omega}\big(VW - 2a_0b_1V - b_1\omega U\big) + \varepsilon^2\Big(\frac{a_0(a_2-b_2)V}{\omega} - b_2U\Big), \\ \dot V =& \omega U + \varepsilon b_1 V - \varepsilon^2 a_2 V, \\ \dot W =& \varepsilon\big(-c_1W - 4V(a_0V+\omega U)\big) - \varepsilon^2 c_2. \end{aligned} \tag{3.3}$$

In order to apply the strategy described above we must write the differential system (3.3) in the standard form (1.11). To this end we proceed as usual. First we consider the cylindrical change of variables $(U, V, W) = (r\cos\theta, r\sin\theta, w)$, where $r > 0$; after checking that $\dot\theta = \omega + \mathcal{O}(\varepsilon) \neq 0$, for $|\varepsilon| \neq 0$ sufficiently small, we take θ as the new independent variable. Therefore the differential system (3.3) becomes equivalent to the non-autonomous differential system

$$\frac{dz}{d\theta} = \left(\frac{\dot r}{\dot\theta}, \frac{\dot w}{\dot\theta}\right) = \varepsilon F_1(\theta, z) + \varepsilon^2 F_2(\theta, z) + \mathcal{O}(\varepsilon^3), \tag{3.4}$$

where $z = (r, w) \in \mathbb{R}^+ \times \mathbb{R}$ and $\theta \in \mathbb{S}^1$. Moreover

$$\begin{aligned} F_1(\theta,z) =& \Bigg(\frac{r}{2\omega^2}\big((w-2a_0b_1)\sin(2\theta) - 2b_1\omega\cos(2\theta)\big), \\ & -\frac{\big(c_1w + 4r^2\sin\theta(\omega\cos\theta + a_0\sin\theta)\big)}{\omega}\Bigg), \\ F_2(\theta,z) =& \Bigg(\frac{1}{2\omega^4}\big(2b_1\omega\cos\theta + (2a_0b_1 - w)\sin\theta\big)\big(2b_1\omega\cos(2\theta) + (2a_0b_1 \\ & -w)\sin(2\theta)\big) + r\omega^2\big((a_2-b_2)(\omega\cos(2\theta) + a_0\sin(2\theta)) - (a_2+b_2)\big), \\ & \frac{\big(2b_1\omega\cos\theta + (2a_0b_1 - w)\sin\theta\big)}{\omega^2}\big(c_1w + 4r^2\sin\theta(\omega\cos\theta + a_0\sin\theta)\big)\Bigg). \end{aligned} \tag{3.5}$$

For the differential system (3.4) we have that $F_0(\theta, z) = 0$. Then $x(\theta, z, 0) = (r, w)$ is the solution to the unperturbed system, and $Y(t, z) = Id$ is its corresponding fundamental matrix. In this case the averaged functions reads

$$\begin{aligned} g_1(z) = & \left(0, -\frac{2\pi\big(2a_0 r^2 + c_1 w\big)}{\omega}\right), \\ g_2(z) = & \left(\frac{\pi r\big(3a_0 r^2 + c_1 w - 2(a_2+b_2)\omega^2\big)}{2\omega^3}, \frac{\pi}{\omega^3}\big((2a_0 b_1 - w)(6a_0 r^2 + c_1 w) \right. \\ & \left. +2c_1\pi(2a_0 r^2 + c_1 w)\omega + 2\big((2b_1 + c_1)r^2 - c_2 w\big)\omega^2\big)\right). \end{aligned} \tag{3.6}$$

Now we are going to use Theorem 12 taking $s = 1$. To do this we define the function $h(z, \varepsilon) = g(z, \varepsilon)/\varepsilon$, where now $h(z, \varepsilon) = g_1(z) + \varepsilon g_2(z) + \mathcal{O}(\varepsilon^2)$. Note that the averaged function $g_1(z)$ vanishes on the manifold

$$\widetilde{\mathcal{Z}} = \left\{ z_\alpha = \left(\alpha, -\frac{2a_0\alpha^2}{c_1}\right) : \alpha > 0 \right\}.$$

Furthermore $\Delta_\alpha = -(2\pi c_1)/\omega$ is the lower right corner of the Jacobian matrix $Dg_1(z_\alpha)$ for all $z_\alpha \in \widetilde{\mathcal{Z}}$. Computing then the bifurcation function f_1 (see (1.24)) we get

$$f_1(\alpha) = \frac{\pi\alpha\big(a_0\alpha^2 - 2(a_2+b_2)\omega^2\big)}{2\omega^3}.$$

Solving the equation $f_1(\alpha) = 0$ we find

$$\alpha_0 = \omega\sqrt{\frac{2(a_2+b_2)}{a_0}}.$$

Moreover $f_1'(\alpha_0) = 2\pi(a_2+b_2)/\omega$. So it is clear that hypotheses of Theorem 12(a) are fulfilled with $s = 1$. Thus for $|\varepsilon| \neq 0$ sufficiently small it follows that there exists

$$z(\varepsilon) = \left(\omega\sqrt{\frac{2(a_2+b_2)}{a_0}}, -\frac{4\omega^2(a_2+b_2)}{c_1}\right) + \mathcal{O}(\varepsilon), \tag{3.7}$$

such that $h(z(\varepsilon), \varepsilon) = g(z(\varepsilon), \varepsilon)/\varepsilon = 0$ for every $|\varepsilon| \neq 0$ sufficiently small. Therefore we conclude that there exists a 2π-periodic solution periodic $(r(\theta, \varepsilon), w(\theta, \varepsilon))$ of the non-autonomous differential system (3.4) satisfying $(r(\theta, 0), w(\theta, 0)) = z(0)$. Since $\theta(t) = \omega t + \mathcal{O}(\varepsilon)$, this proof ends by going back through the cylindrical coordinate change of variables and then doing $(u, v, z) = \varepsilon(V, a_0 V + \omega U, W)$. □

3.1.1 Stability

We have seen that the averaged functions (3.6) up to order 2 were sufficient for detecting the existence of a periodic solution of the differential system (3.1). Now we show that the higher order averaged functions may play an important role for studying the stability of the periodic solution $\varphi(t,\varepsilon)$ provided by Theorem 12(a). Instead of applying Theorem (13) directly we are going to show why this result is realy necessary.

Clearly the stability of the periodic solution $\varphi(t,\varepsilon)$ can be derived from the eigenvalues of the Jacobian matrix of the displacement function $D_z h(z(\varepsilon),\varepsilon)$ evaluated at $z(\varepsilon) = \varphi(0,\varepsilon)$. From equation (3.7) we can write $z(\varepsilon) = z_0 + \mathcal{O}(\varepsilon^2)$. Moreover since in this case $Y(t,z) = Id$ then $D_z h(z(\varepsilon),\varepsilon) = \varepsilon D g_1(z_0) + \mathcal{O}(\varepsilon)$, where

$$Dg_1(z_0) = \begin{pmatrix} 0 & 0 \\ -8\pi\sqrt{2a_0(a_2+b_2)} & -\dfrac{2\pi c_1}{\omega} \end{pmatrix}.$$

So a first approximation of the eigenvalues $\lambda_\pm$ of the Jacobian matrix $D_z h(z(\varepsilon),\varepsilon)$ is given by

$$\lambda_+ = \mathcal{O}(\varepsilon^2), \quad \lambda_- = -\varepsilon\frac{2\pi c_1}{\omega} + \mathcal{O}(\varepsilon^2). \tag{3.8}$$

Clearly the stability of the periodic solution $\varphi(t,\varepsilon)$ cannot be completely described by these expressions. Now we show how the higher order bifurcation functions and averaging functions can be used for doing a better analyses of the stability of the periodic solution.

We recall that, after some changes of coordinates, the differential system (3.1) can be transformed into the standard form (3.4). Expanding it in power series of ε up to order 3, the differential system (3.4) becomes

$$\frac{dz}{d\theta} = \varepsilon F_1(\theta,z) + \varepsilon^2 F_2(\theta,z) + \varepsilon^3 F_3(\theta,z) + \mathcal{O}(\varepsilon^4),$$

where F_1 and F_2 are given in (3.5) and

$$\begin{aligned} F_3(\theta,z) = \Bigg(& \frac{\pi r}{4\omega^5}\Big(-3(a_0b_1 - w)\big(5a_0r^2 + c_1w\big) - 2c_1\pi(2a_0r^2 + c_1w)\omega + \big(4a_0b_1(a_2 \\ & + b_2) - 3(2b_1+c_1)r^2 - 2(a_2+b_2-c_2)w\big)\omega^2\Big), \\ & \frac{\pi}{12\omega^5}\Big(12\pi\omega\left(a_0^2\left(6r^4 - 16b_1c_1r^2\right) + 2a_0c_1w\left(7r^2 - 2b_1c_1\right) + 3c_1^2w^2\right) \\ & - 2\omega^2\left(w\left(6a_0(a_2c_1 - 2b_1c_2 - b_2c_1) + 6b_1^2c_1 - 9r^2(4b_1+3c_1) + 8\pi^2c_1^3\right)\right. \\ & \left. + a_0r^2\left(36a_0(a_2-b_2) + 108b_1^2 + 36b_1c_1 + 2\left(8\pi^2-3\right)c_1^2 - 45r^2\right) + 6c_2w^2\right) \\ & + 24\pi\omega^3\left(r^2(2a_0(a_2+b_2+c_2) - c_1(2b_1+c_1)) + 2c_1c_2w\right) \\ & - 9(w - 2a_0b_1)^2\left(10a_0r^2 + c_1w\right) + 24r^2\omega^4(c_2 - 2a_2)\Big)\Bigg). \end{aligned}$$

From (1.15) and (1.17) we compute the third averaged function and the second bifurcation function, respectively, as

$$\begin{aligned} g_3(z) = \Bigg(& \frac{\pi r}{4\omega^5}\left(\omega^2\left(4a_0b_1(a_2+b_2) - 2z(a_2+b_2-c_2) - 3r^2(2b_1+c_1)\right)\right. \\ & \left. -3(2a_0b_1 - z)\left(5a_0r^2 + c_1z\right) - 2\pi c_1\omega\left(2a_0r^2 + c_1z\right)\right), \end{aligned}$$

$$\frac{\pi}{12\omega^5}\Big(12\pi\omega\left(a_0^2\left(6r^4-16b_1c_1r^2\right)+2a_0c_1z\left(7r^2-2b_1c_1\right)+3c_1^2z^2\right)$$
$$-2\omega^2\left(z\left(6a_0(a_2c_1-2b_1c_2-b_2c_1)+6b_1^2c_1-9r^2(4b_1+3c_1)+8\pi^2c_1^3\right)\right.$$
$$+a_0r^2\left(36a_0(a_2-b_2)+108b_1^2+36b_1c_1+2\left(8\pi^2-3\right)c_1^2-45r^2\right)$$
$$\left.+6c_2z^2\right)+24\pi\omega^3\left(r^2(2a_0(a_2+b_2+c_2)-c_1(2b_1+c_1))+2c_1c_2z\right)$$
$$-9(z-2a_0b_1)^2\left(10a_0r^2+c_1z\right)+24r^2\omega^4(c_2-2a_2)\Big),$$

and

$$f_2(\alpha)=-\frac{\pi r\left(10a_0^2r^2\left(b_1c_1+r^2\right)+\omega^2\left(c_1r^2(2b_1+c_1)-4a_0(a_2+b_2)\left(b_1c_1+r^2\right)\right)\right)}{4c_1\omega^5}.$$

So $\mathcal{F}^2(\alpha,\varepsilon)=\varepsilon f_1(\alpha)+\varepsilon^2 f_2(\alpha)$. As shown in the previous subsection $a_\varepsilon=\alpha_0$ is a simple root of the function $\widetilde{f}_1(\alpha)$. Using the Implicit Function Theorem we find a branch of zeros of the equation $\mathcal{F}^2(\alpha,\varepsilon)=0$ having the form $\alpha=\overline{a}_\varepsilon=\alpha_0+\varepsilon\alpha_1+\mathcal{O}(\varepsilon^2)$, where

$$\alpha_1=\sqrt{\frac{a_2+b_2}{2a_0}}\left(\frac{8a_0^2b_1c_1+\omega^2(16a_0(a_2+b_2)+c_1(2b_1+c_1))}{2|a_0|c_1\omega}\right).$$

Note that $\overline{a}_\varepsilon$ satisfies the hypotheses (iii) and (iv) of Theorem 5 for $s=1$, $l=1$ and $k=2$. Using the relation $|\pi z(\varepsilon)-\pi z_{\overline{a}_\varepsilon}|=|\alpha(\varepsilon)-\overline{a}_\varepsilon|=\mathcal{O}\big(\varepsilon^2\big)$, provided by Theorem 5, we write $\alpha(\varepsilon)=\alpha_0+\varepsilon\alpha_1+\mathcal{O}(\varepsilon^2)$. From Claim 11 of the proof of Theorem 5 we get

$$\begin{aligned}\overline{\beta}(\alpha(\varepsilon),\varepsilon)=&\beta\big(\alpha(\varepsilon)\big)+\varepsilon\gamma_1\big(\alpha(\varepsilon)\big)+\mathcal{O}(\varepsilon^2)\\ =&\beta\big(\alpha_0+\varepsilon\alpha_1+\mathcal{O}(\varepsilon^2)\big)+\varepsilon\gamma_1\big(\alpha_0+\varepsilon\alpha_1+\mathcal{O}(\varepsilon^2)\big)+\mathcal{O}(\varepsilon^2).\end{aligned}$$

Expanding $\overline{\beta}(\alpha(\varepsilon),\varepsilon)$ in powers series of ε we have $\overline{\beta}(\alpha(\varepsilon),\varepsilon)=\beta_0+\varepsilon\beta_1+\mathcal{O}(\varepsilon^2)$, where

$$\begin{aligned}\beta_0=&\frac{(a_2+b_2)\omega^2}{c_1},\\ \beta_1=&\frac{4(a_2+b_2)\big(6a_0^2b_1c_1+\big(16a_0(a_2+b_2)+c_1(2b_1+c_1)\big)\omega^2\big)}{a_0c_1^2}.\end{aligned}$$

Finally we obtain $z(\varepsilon)=\big(\alpha(\varepsilon),\overline{\beta}\big(\alpha(\varepsilon),\varepsilon\big)\big)=z_0+\varepsilon z_1+\mathcal{O}(\varepsilon^2)$, with $z_0=(\alpha_0,\beta_0)$ and $z_1=\big(\alpha_1,\beta_1\big)$. Then we compute the Jacobian matrix of the displacement function (1.13) evaluated at $z(\varepsilon)$ as

$$\begin{aligned}D_zh(z(\varepsilon),\varepsilon)&=\varepsilon D_zg_1(z(\varepsilon))+\varepsilon^2D_zg_2(z(\varepsilon))+\mathcal{O}(\varepsilon^3)\\ &=\varepsilon D_zg_1\big(z_0+\varepsilon z_1+\mathcal{O}(\varepsilon^2)\big)+\varepsilon^2D_zg_2\big(z_0+\varepsilon z_1+\mathcal{O}(\varepsilon^2)\big)+\mathcal{O}(\varepsilon^3)\\ &=\varepsilon D_zg_1(z_0)+\varepsilon^2\left(D_z^2g_1(z_0)z_1+D_zg_2(z_0)\right)+\mathcal{O}(\varepsilon^3).\end{aligned}$$

Let $D_zg_1(z_0)=\big(p_{ij}\big)_{2\times2}$ and $D_zg_2(z_0)=\big(q_{ij}\big)_{2\times2}$, then expanding $D_zh(z(\varepsilon),\varepsilon)$ in Taylor series around $\varepsilon=0$ we have $D_zh(z(\varepsilon),\varepsilon)=\varepsilon A_1+\varepsilon^2A_2+\mathcal{O}(\varepsilon^3)$ with $A_1=D_zg_1(z_0)$ and $A_2=\big(D_zp_{ij}(z_0).z_1+q_{ij}(z_0)\big)_{2\times2}$. Therefore we may improve the approximation (3.8) of the eigenvalues $\lambda_\pm$ of $D_zh(z(\varepsilon),\varepsilon)$ as

$$\lambda_+=\varepsilon^2\frac{2\pi(a_2+b_2)}{\omega}+\mathcal{O}(\varepsilon^3), \tag{3.9}$$

$$\lambda_- = -\varepsilon\frac{2c_1\pi}{\omega} + \varepsilon^2\frac{2\pi\big(a_0b_1c_1 + \omega\big(c_1^2\pi - c_2\omega\big)\big)}{\omega^3} + \mathcal{O}(\varepsilon^3). \tag{3.10}$$

Note that we have approximations for the eigenvalues $\lambda\pm$. Then the only remaining question is: Can we use the approximations (3.9) and (3.10) to study the eigenvalues of the full matrix $D_zh(z(\varepsilon),\varepsilon)$? In general we cannot proceed in this way. The problem is that the $2-jet$ $A_0+\varepsilon A_1$ may be insufficient for recovering the information about the eigenvalues of the full matrix $D_zh(z(\varepsilon),\varepsilon)$. This problem appears for instance in the works of Murdock and Robinson [62] and Murdock [59]. Next we present an example where this problem emerges.

Example 2. *Consider the matrix* $A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 R$ *where*

$$A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix}.$$

Let λ_1 and λ_2 be the eigenvalues of $A_0+\varepsilon A_1$. Writing the Taylor series of these eigenvalues at $\varepsilon=0$ we obtain $\lambda_1 = -\varepsilon + \mathcal{O}(\varepsilon^3)$ and $\lambda_2 = -\varepsilon+\mathcal{O}(\varepsilon^3)$. Conversely computing the Taylor series of the eigenvalues of the full matrix $A(\varepsilon)$ we have the eigenvalues $\overline{\lambda}_1 = (-1+a)\varepsilon + \mathcal{O}(\varepsilon^3)$ and $\overline{\lambda}_3 = -(1+a)\varepsilon+\mathcal{O}(\varepsilon^3)$. As we can see the hyperbolicity of the matrix $A(\varepsilon)$ depends on a. Thus the hyperbolicity of the matrix $A(\varepsilon)$ cannot be studied using only A_0 and A_1.

The problem in Example 2 is that the matrix $A(\varepsilon)$ is not $2-$*hyperbolic*. Thus in order to study the eigenvalues of the matrix $D_zh(z(\varepsilon),\varepsilon)$ we need to verify the hypotheses of Theorem 12. Hence using Theorem 12 we can deduce the following statements about the stability of the periodic solution $\varphi(t,\varepsilon) = x(t,z(\varepsilon),\varepsilon)$. Recall that from the hypotheses of Proposition 21, $a_0(a_2+b_2)>0$. So

(a) If $\varepsilon c_1 < 0$ the solution $\varphi(t,\varepsilon)$ has at least one unstable direction.

(b) If $a_2+b_2>0$ and $a_0>0$, then the solution $\varphi(t,\varepsilon)$ has at least one unstable direction.

(c) If $a_2+b_2<0$, $\varepsilon c_1>0$ and $a_0<0$, then the solution $\varphi(t,\varepsilon)$ is asymptotically stable.

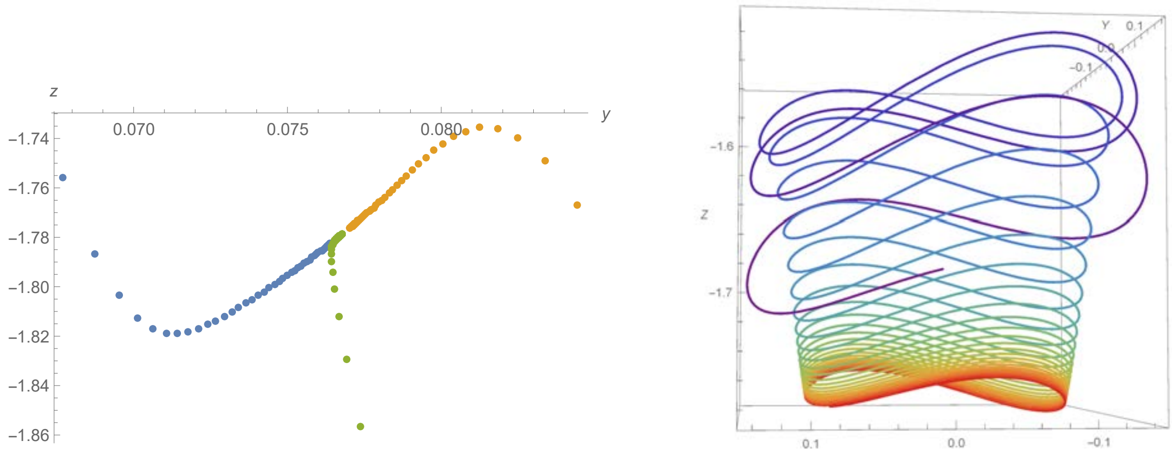
Figures 3.1 illustrate the behavior of the Maxwell–Block system (3.1) satisfying the hypotheses of Proposition 21 with $a_0=-1$, $a_2=-2$, $b_1=1$, $b_2=-2$, $c_1=2$, $c_2=1$, $\omega=1$ and $\varepsilon=1/25$.

3.2 Birth of a limit cycle in a 3D polynomial system

Consider the following 3D autonomous polynomial differential system

$$\begin{aligned}\dot{u} =& -v + \varepsilon\left(u^3 - u^2 - uv^2 - \pi v^3\right),\\ \dot{v} =& u + \varepsilon\left(\pi u^3 - 1\right),\\ \dot{w} =& w - \varepsilon u.\end{aligned} \tag{3.11}$$

In the next proposition as an application of Theorem 5 we provide sufficient conditions for the existence of an isolated periodic solution for the differential system (3.11).



(a) (b)

Figure 3.1: (A) Transversal section with $u = 0$ and $v > 0$. (B) Solution starting at $(0, \varepsilon\omega^2(2(a_2+b_2)/a_0)^{1/2}, \delta - 4\varepsilon\omega^2(a_2 + b_2)/c_1)$ being attracted by the limit cycle (3.2).

Proposition 22. *For* $|\varepsilon| > 0$ *sufficiently small system* (3.11) *has an isolated periodic solution* $\varphi(t,\varepsilon) = \big(u(t,\varepsilon),\, v(t,\varepsilon),\, w(t,\varepsilon)\big)$ *such that*

$$\begin{aligned} u(t,\varepsilon) =& \sqrt{8\,\varepsilon}\, \cos t + \mathcal{O}(\varepsilon),\\ v(t,\varepsilon) =& \sqrt{8\,\varepsilon}\, \sin t + \mathcal{O}(\varepsilon), \ \textit{and}\\ w(t,\varepsilon) =& \mathcal{O}(\varepsilon). \end{aligned}$$

Proof. Writing the differential system (3.11) in the cylindrical coordinates $(u, v, w) = (r\cos\theta, r\sin\theta, w)$ we get

$$\begin{aligned} \dot r =& \frac{\varepsilon}{4}\left(r^3 + r^2(r(\pi\sin(4\theta) + 2\cos(2\theta) + \cos(4\theta)) - 3\cos\theta - \cos(3\theta)) - 4\sin\theta\right),\\ \dot\theta =& 1 + \frac{\varepsilon}{4r}\left(r^2(\sin\theta + \sin(3\theta) - r\sin(4\theta) + \pi r\cos(4\theta) + 3\pi r) - 4\cos\theta\right),\\ \dot w =& w - \varepsilon r\cos\theta. \end{aligned}$$

Since $\dot\theta \neq 0$ for $|\varepsilon| \neq 0$ sufficiently small we can take θ as the new independent variable. So

$$\begin{aligned} \frac{dr}{d\theta} =& \ \varepsilon F_{11}(\theta, z) + \varepsilon^2 F_{21}(\theta, z) + \mathcal{O}_1(\varepsilon^3),\\ \frac{dw}{d\theta} =& \ w + \varepsilon F_{12}(\theta, z) + \varepsilon^2 F_{22}(\theta, z) + \mathcal{O}_2(\varepsilon^3), \end{aligned} \tag{3.12}$$

where $z = (r, w) \in \mathbb{R}^2$ and

$$\begin{aligned} F_{11}(\theta, z) =& \frac{1}{4}\big(r^3 + r^2(r(\pi\sin(4\theta) + 2\cos(2\theta) + \cos(4\theta)) - 3\cos\theta - \cos(3\theta))\\ & - 4\sin\theta\big),\\ F_{12}(\theta, z) =& \frac{-1}{4}\big(4\cos\theta\left(r^2 - w\right) + r^2 w(\sin\theta + \sin(3\theta) - r\sin(4\theta) + \pi r\cos(4\theta) \end{aligned}$$

$$
\begin{aligned}
&+3\pi r)\big),\\
F_{21}(\theta,z)=&\frac{-1}{16r}\big(-4\sin\theta+r^3+r^2(-3\cos\theta-\cos(3\theta)+r(\pi\sin(4\theta)\\
&+2\cos(2\theta)+\cos(4\theta)))\big)\big(r^2(\sin\theta+\sin(3\theta)-r\sin(4\theta)+\pi r\cos(4\theta)\\
&+3\pi r)-4\cos\theta\big),\\
F_{22}(\theta,z)=&\frac{1}{16r^2}\big(r^2(\sin\theta+\sin(3\theta)-r\sin(4\theta)+\pi r\cos(4\theta)+3\pi r)-4\cos\theta\big)\\
&\big(4\cos\theta\left(r^2-w\right)+r^2w(\sin\theta+\sin(3\theta)-r\sin(4\theta)+\pi r\cos(4\theta)\\
&+3\pi r\big).
\end{aligned}
$$

The differential system (3.12) is 2π-periodic in the variable θ and it is written in the standard form (1.11) with $F_0(\theta,z)=\big(0,z\big)$, $F_1(\theta,z)=\big(F_{11}(\theta,z),F_{12}(\theta,z)\big)$ and $F_2(\theta,z)=\big(F_{21}(\theta,z),F_{22}(\theta,z)\big)$. Moreover the solution of the unperturbed differential system $(3.12)_{\varepsilon=0}$ for an initial condition $z_0=(r_0,w_0)$ is given by

$$\Phi(\theta,z_0)=\big(r_0,w_0e^{\theta}\big).$$

Consider the set $\mathcal{Z}\subset\mathbb{R}^2$ such that $\mathcal{Z}=\{(\alpha,0):\alpha>0\}$. Clearly for each $z_\alpha\in\mathcal{Z}$, the averaged equation $g_0(z)=Y(T,z)^{-1}\Phi(T,z_\alpha)-z$ satisfies the hypothesis $\mathcal{H}$ with

$$Y(\theta,z)=\frac{\partial\Phi}{\partial z}(\theta,z_0)=\begin{pmatrix}1&0\\0&e^{\theta}\end{pmatrix},$$

the fundamental matrix of the unperturbed system $(3.12)_{\varepsilon=0}$. Now in order to compute the bifurcation functions (1.17) for the differential system (3.12) we first calculate

$$
\begin{aligned}
y_0(\theta,z)&=Y(\theta,z)^{-1}\big(0,\,(e^{\theta}-1)w\big),\\
y_1(\theta,z)&=Y(\theta,z)^{-1}\Bigg(\frac{r^2}{48}\left(-36\sin\theta-4\sin(3\theta)+6\pi r\sin^2(2\theta)+3r\sin(4\theta)\right)\\
&\frac{1}{48}\left(12\left(\theta r^3-4\right)+24\cos\theta\left(r^3\sin\theta+2\right)\right),\ \frac{r^2}{2}(\cos\theta-\sin\theta)\\
&-\frac{e^{\theta}r}{48}(w((36\pi\theta-3)r+16)+24)+\frac{e^{\theta}w}{48}\big(48\sin\theta+r^2(12\cos\theta\\
&+4\cos(3\theta)-3r(\pi\sin(4\theta)+\cos(4\theta)))\Bigg),\\
y_2(2\pi,z)&=Y(2\pi,z)^{-1}\Bigg(\frac{-\pi r(3r+4)}{4},\ \frac{e^{-2\pi}}{40}\big(((3-2\pi)r-6)r^2+10\big)\\
&+\frac{1}{40}\big(r^2((\pi(7+15\pi)-3)r+6)-10\big)\Bigg),
\end{aligned}
$$

and from (1.15)

$$g_i(z)=Y(2\pi,z)\frac{y_i(2\pi,z)}{i!}\quad\text{for}\quad i=0,1,2. \tag{3.13}$$

So the bifurcation functions (1.17) corresponding to the functions (3.13) become

$$f_1(\alpha)=\frac{\pi\alpha^3}{2},\quad f_2(\alpha)=\pi\alpha(3\alpha+4),\quad\text{and}\quad\mathcal{F}^2(\alpha,\varepsilon)=\varepsilon f_1(\alpha)+\varepsilon^2f_2(\alpha). \tag{3.14}$$

Now we must check that the function (3.14) satisfies the hypotheses for applying Theorem 5. So $\det(\Delta_\alpha) = \left|D_w \pi^\perp g_0(z_\alpha)\right| = 1 - e^{-2\pi} \neq 0$, and for $a_\varepsilon = \sqrt{9\varepsilon^2 + 8\varepsilon} + 3\varepsilon$ we have that

$$\mathcal{F}^2(a_\varepsilon, \varepsilon) = 0 \quad \text{and} \quad |\partial_\alpha \mathcal{F}^2(a_\varepsilon, \varepsilon)| \geq \varepsilon^2 \left(8 - \left|9\varepsilon + 3\sqrt{\varepsilon(8 + 9\varepsilon)}\right|\right).$$

Thus it is easy to find $P_0 > 0$ satisfying $|\partial_\alpha \mathcal{F}^2(a_\varepsilon, \varepsilon)| \geq \varepsilon^2 P_0$. Hence, using the notation of Theorem 5, we have $s = 1$, $k = 2$, $l = 2$, and $(k + r + 1)/2 = 2 = l$. So we can apply Theorem 5 in order to prove the existence of an isolated periodic solution $\big(r(\theta, \varepsilon), z(\theta, \varepsilon)\big)$ of the differential system (3.12) such that

$$r(0, \varepsilon) = \sqrt{9\varepsilon^2 + 8\varepsilon} + 3\varepsilon + \mathcal{O}(\varepsilon) = \sqrt{8\varepsilon} + \mathcal{O}(\varepsilon) \quad \text{and} \quad w(0, \varepsilon) = \mathcal{O}(\varepsilon).$$

Since $\theta(t) = t + \mathcal{O}(\varepsilon)$, this proof ends by going back through the cylindrical coordinate change of variables. □

Chapter 4

Fitzhugh-Nagumo and Quadratic chaotic systems

In this chapter we use Theorem 12 for studying the periodic solutions of seventeen differential systems. First we study the Fitzhugh-Nagumo system, which is related with nerve impulses in mathematical biology. The averaged equation related with this system has a continuum of zeros and we will use the method developed in Chapter 1 to detect the periodic solutions bifurcating from singular points in this set of zeros.

Furthermore we use Theorem 12 for studying the zero-Hopf and the Hopf bifurcation of other sixteen 3-dimensional differential systems that was provided by Jafari et al [41] in 2013. These systems has equilibria only for a certain choice of the parameter a, and we show that under some conditions a periodic solution emerges in these systems when the equilibria disappear. Moreover we show graphically that the periodic orbit which is born in such bifurcations is the origin of a period doubling cascade which originate the chaotic motion in those differential systems. Here we use the classical averaging theory, and the new results in this theory here developed to illustrated how the averaging theory is useful for studying the periodic orbits which bifurcate from a zero-Hopf equilibrium point, or from a Hopf bifurcation. The results here presented were published in [13] and [15].

4.1 Application to Fitzhugh-Nagumo system

The Fitzhugh-Nagumo arise in mathematical biology as a model of the transmission of electrical impulses through a nerve axon. The Fitzhugh–Nagumo equations consist in a simplified version of the Hodgkin–Huxley equations which are described using a non–linear diffusion equation coupled to an ordinary differential equation

$$u_t = u_{xx} - f(u) - v, \quad v_t = \delta(u - \gamma v), \tag{4.1}$$

where $f(u) = u(u-1)(u-a)$, $0 < a < 1/2$ is a constant, $\delta > 0$ and $\gamma > 0$ are parameters. In [38] it was stated that a single nerve impulse appears to tend as t increases to a traveling wave, i.e. a bounded solution $(u, v)(x, t) = (u, v)(\xi)$ where $\xi = x + ct$. Hence one is lead to seek for solutions of (4.1) not identically zero of the form $(u, v) = (u(\xi), v(\xi))$ for some $c \neq 0$. Substitution into (4.1) gives a set of ordinary differential equations which, after the introduction of the variables $x = u$, $y = v$ and $z = \dot{u}$ take the form

$$\begin{aligned}\dot{x} =& z,\\ \dot{y} =& b(x - dy),\\ \dot{z} =& x(x-1)(x-a) + y + cz,\end{aligned} \tag{4.2}$$

where the dot denotes the derivative with respect to ξ and $(a, b, c, d) \in \mathbb{R}^4$ are parameters. For a detailed study concerning traveling waves in (4.1) see [32]. Hereafter the differential system (4.2) will be called *Fitzhugh–Nagumo differential system.*

Proposition 23. *There are two parameter families of the Fitzhugh-Nagumo differential systems for which the origin of coordinates is a zero-Hopf equilibrium point, both families are 2–parametric. Namely*

(i) for $ad + 1 = 0$, $bd - c = 0$ and $d(1 - b^2d^3) > 0$; and

(ii) for $b = c = 0$ and $a < 0$.

Proof. A proof of this proposition can be obtained in [29]. □

Theorem 24. *Let $a = -\dfrac{1}{d} + a_1\varepsilon + a_2\varepsilon^2$, $c = b\,d + c_2\varepsilon^2$ and $\omega = \sqrt{\dfrac{1 - b^2d^3}{d}}$. Assume that $d(1 - b^2d^3) > 0$, $(d-1)a_1b \neq 0$ and $\varepsilon \neq 0$ sufficiently small. Then the Fitzhugh–Nagumo differential system (4.2) has a zero–Hopf bifurcation in the equilibrium point at the origin of coordinates, and the periodic orbit*

$$\begin{aligned}x(t,\varepsilon) =& \mathcal{O}(\varepsilon^2),\\ y(t,\varepsilon) =& \mathcal{O}(\varepsilon^2),\\ z(t,\varepsilon) =& \varepsilon\frac{a_1 d}{d-1} + \mathcal{O}(\varepsilon^2),\end{aligned} \tag{4.3}$$

born at this equilibrium when $\varepsilon = 0$.

Moreover, if

(a) $b \neq \pm\sqrt{\frac{7d\pm\sqrt{(d-191)(d+1)}+7}{12d^3(d+1)}}$ and $c_2 \neq \frac{a_1^2bd^3\left(24b^2d^3-13\right)}{24(b^2d^3-1)^2}$. Then the Fithugh–Nagumo differential system (4.2) has four periodic solutions emerging from the origin.

Theorem 24 is proved using Theorem 1 and 12. Theorem 24 is proved in section 4.3.1.

Euzébio et al. studied the zero–Hopf bifurcations of system (4.2) using the classical averaging theory (see Theorem 11 of [29]). Considering the two parameter families of zero–Hopf equilibria stated in Proposition 23(i) the authors of [29] find using the first order averaging method, a periodic solution bifurcating from the origin of the system different from the periodic solution (4.3), because in Theorem 5 of [29] the order of the periodic solution in the three variables (x, y, z) is $\mathcal{O}(\varepsilon)$ while in our case is $\mathcal{O}(\varepsilon^2)$ for x and y, and $\mathcal{O}(\varepsilon)$ for z. Moreover using second order averaging theory the authors of [29] in Theorem 6 [29] find one additional periodic solution bifurcating from the origin, while using our Theorem 24 we find four periodic solutions.

4.2 Application to a catalogue of quadratic chaotic systems

In general the equilibria of a chaotic nonlinear system play an important role in its dynamics. In fact one of the most important methods for obtaining 3-dimensional chaotic systems is the Shilnikov's method [73], which using a homoclinic orbit from the intersection of the stable and unstable manifolds of a saddle-focus equilibrium point with specified eigenvalues, provides the existence of a horseshoe in the neighborhood of this orbit and, consequently the existence of chaotic motion.

However some particularly important natural phenomena are described by nonlinear systems having no equilibria. Such as, the Noose-Hover oscillator [66], the Wei system [84] and the Wang–Chen system [82]. These nonlinear systems present chaotic behavior that cannot be detected by the Shilnikov's method.

The increasing interest in finding examples of simple chaotic flows without equilibria have been motivating many researchers in recent times, see for instance [41, 65, 80, 84, 85]. The theoretical and practical importance of these systems converted this subject in a new attractive research direction. Although there is still little knowledge about the characteristics of such systems.

In this section we shall study the existence of zero-Hopf bifurcations in 3-dimensional systems, and graphically we will show that such bifurcations sometimes are the starting bifurcation of a route to chaos. In general a *zero-Hopf bifurcation* is a codimension-two bifurcation of a 3-dimensional autonomous differential equation with a zero-Hopf equilibrium, and a *zero-Hopf equilibrium* of a 3-dimensional autonomous differential equation is an equilibrium point having two purely conjugate imaginary eigenvalues and a zero eigenvalue. Due to the lack of a general theory describing all the possible kinds of bifurcations that an unfolding of a zero-Hopf bifurcation can produce, most of the systems exhibiting these kind of bifurcations must be studied directly. Here we use averaging theory for detecting periodic solutions bifurcating from a zero-Hopf equilibrium. Furthermore using Theorem 12 we were able to detect periodic solutions in very degenerate cases, for instance when the first averaged equation has a continuum of zeros.

In 2013 Jafari et al. [41] have reported a catalogue of seventeen elementary three dimensional chaotic flows. This catalogue contains most of the elementary examples known of such systems and it includes the systems of the Noose-Hoover oscillator, the Wei system and the Wang–Chen system, listed there as system (4.4), (4.5) and (4.6), respectively. In [41] the authors used their own custom software to search for the algebraically simplest three-dimensional chaotic systems with quadratic nonlinearities and no equilibria. The search was inspired by the observation that each of the previously known examples of such systems contains a constant term (here represented by a), and that if the constant is set to zero, the resulting system is non-hyperbolic (the equilibria have eigenvalues with real part equal to zero). The method used to find these systems is that proposed in [76].

We consider the differential systems

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - zy, \\ \dot{z} &= y^2 - a. \end{aligned} \tag{4.4}$$

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + z, \\ \dot{z} &= 2y^2 + xz - a. \end{aligned} \tag{4.5}$$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= 0.1x^2 + 1.1xz - y + a. \end{aligned} \tag{4.6}$$

$$\begin{aligned}\dot{x} =& -0.1y + a,\\ \dot{y} =& x + z,\\ \dot{z} =& xz - 3y.\end{aligned} \qquad (4.7) \qquad \begin{aligned}\dot{x} =& 2y,\\ \dot{y} =& 2x - z,\\ \dot{z} =& -y^2 + z^2 + a.\end{aligned} \qquad (4.8) \qquad \begin{aligned}\dot{x} =& y,\\ \dot{y} =& z,\\ \dot{z} =& -y - xz - yz - a.\end{aligned} \qquad (4.9)$$

$$\begin{aligned}\dot{x} =& y,\\ \dot{y} =& -x + z,\\ \dot{z} =& 0.8x^2 + z^2 + a.\end{aligned} \qquad (4.10) \qquad \begin{aligned}\dot{x} =& y,\\ \dot{y} =& -x - zy,\\ \dot{z} =& xy + 0.5x^2 - a.\end{aligned} \qquad (4.11) \qquad \begin{aligned}\dot{x} =& y,\\ \dot{y} =& -x - zy,\\ \dot{z} =& -xz + 7x^2 - a.\end{aligned} \qquad (4.12)$$

$$\begin{aligned}\dot{x} =& z,\\ \dot{y} =& z - y,\\ \dot{z} =& -0.9y - xy + xz + a.\end{aligned} \qquad (4.13)$$

$$\begin{aligned}\dot{x} =& y,\\ \dot{y} =& -x + z,\\ \dot{z} =& -2xy - 1.8xz + z - a.\end{aligned} \qquad (4.14) \qquad \begin{aligned}\dot{x} =& z,\\ \dot{y} =& x - y,\\ \dot{z} =& -4x^2 + 8xy + yz + a.\end{aligned} \qquad (4.15)$$

$$\begin{aligned}\dot{x} =& -y,\\ \dot{y} =& x + z,\\ \dot{z} =& xy + xz + 0.2yz - a.\end{aligned} \qquad (4.16) \qquad \begin{aligned}\dot{x} =& y,\\ \dot{y} =& z,\\ \dot{z} =& x^2 - y^2 + xy + 0.4xz + a.\end{aligned} \qquad (4.17)$$

$$\begin{aligned}\dot{x} =& -0.8x - 0.5y^2 + xz + a,\\ \dot{y} =& -0.8y - 0.5z^2 + yx + a,\\ \dot{z} =& -0.8z - 0.5x^2 + zy + a.\end{aligned} \qquad (4.18) \qquad \begin{aligned}\dot{x} =& -y - z^2 + 2.3xy + a,\\ \dot{y} =& -z - x^2 + 2.3yz + a,\\ \dot{z} =& -x - y^2 + 2.3zx + a.\end{aligned} \qquad (4.19)$$

Each of the systems (4.4)–(4.13) have an equilibrium that undergoes a zero-Hopf bifurcation at $a = a^* = 0$, and no equilibria for $a > 0$. Each of the systems (4.14)–(4.19) have an equilibrium that undergoes a Hopf bifurcation at some $a = a^*$. The limit cycle which appears in this Hopf bifurcation later on produces a period-doubling cascade, and finally a chaotic attractor with no equilibria, i.e. the equilibrium point which exhibits the Hopf bifurcation disappears before the chaotic attractor appears.

Jafari et al. [41] have reported numerically a period doubling cascade of periodic orbits originating the route to the chaotic motion in these systems. Here we graphically observe that the first periodic orbit performing the period doubling bifurcation detected by Jafari et al. emerges in those systems at a zero-Hopf or Hopf bifurcation. This helps to understand the mechanism of chaos in these systems, and the objective of this section is to show the existence of these zero-Hopf or Hopf bifurcations using the averaging theory.

One of the contributions of this work is to show that in many cases the periodic solutions that generate (via period-doubling) the chaotic attractor started with a periodic orbit coming from a zero–Hopf or a Hopf bifurcation.

The next theorem shows that the systems considered exhibit a zero–Hopf bifurcation at $a = 0$. Although we can check that these systems have no equilibria when $a > 0$.

Theorem 25. *The following statements hold.*

(*i*) *The differential systems* (4.4)–(4.13) *exhibit a zero–Hopf bifurcation at* $a = 0$, *more precisely for* $a > 0$ *sufficiently small they have a periodic orbit which tends to a zero-Hopf equilibrium when* $a \to 0$.

(*ii*) *All the periodic solutions emerging in the zero–Hopf bifurcation are non-hyperbolic, with the exception of the differential system* (4.5) *that has a hyperbolic periodic solution.*

(*iii*) *All the periodic solutions in the zero–Hopf bifurcation emerge around the zero-Hopf equilibrium point located at the origin of coordinates, with the exception of system* (4.13), *which has the periodic solution emerging from the zero-Hopf equilibrium point* $(1, 0, 0)$.

Another interesting aspect of some the differential systems provided in [41] is that some of them have equilibria only if the parameter a belongs to convenient intervals. In these intervals a Hopf bifurcation occurs and a periodic solution emerge in the system, but as a increases the equilibria disappear and the isolated periodic solution coming from the Hopf bifurcation start its cascade of period-doubling. The differential systems having this behaviour are (4.14)–(4.19), in fact for $a < 5/36$ system (4.14) has the equilibria

$$P_{\pm} = \left(\frac{1}{18}\left(5 \pm \sqrt{25 - 180a}\right), 0, \frac{1}{18}\left(5 \pm \sqrt{25 - 180a}\right) \right),$$

such that when $a = 0$ the origin is an equilibrium point with the eigenvalues $\lambda_{1,2} = \pm i$ and $\lambda_3 = -1$. Similarly, if $a < 0$ system (4.15) has the equilibria

$$P_{\pm} = \left(\pm\frac{\sqrt{a}}{2}, \pm\frac{\sqrt{a}}{2}, 0 \right),$$

if $a = -196$ the equilibrium point P_+ has the eigenvalues $\lambda_{1,2} = \pm i\sqrt{7}$ and $\lambda_3 = -8$. The system (4.15) has the equilibria

$$P_{\pm} = \left(\pm\sqrt{a}, 0, \pm\sqrt{a}\right),$$

for $a < 0$. When $a = -25/16$ the equilibria P_+ has the eigenvalues $\lambda_{1,2} = \pm i\sqrt{2}$ and $\lambda_3 = -5/4$. System (4.15) has the equilibria

$$P_{\pm} = \left(\pm\sqrt{a}, 0, 0\right),$$

for $a < 0$ and for $a = -25$ the equilibria P_+ has the eigenvalues $\lambda_{1,2} = \pm i\sqrt{5}$ and $\lambda_3 = -2$. Finally, system (4.19) has the equilibria

$$P_{\pm} = \left(\frac{1}{13}\left(5 \pm \sqrt{25 - 130a}\right), 0, \frac{1}{13}\left(5 \pm \sqrt{25 - 130a}\right) \right),$$

for $a < 5/26$ and when $a = -560/1849$ the equilibrium point P_- has the eigenvalues $\lambda_{1,2} = \pm i\sqrt{3}$ and $\lambda_3 = -69/43$.

Theorem 26. *Consider the differential systems* (4.14)–(4.19). *The following statemens hold.*

(*i*) *Let* $a = a_2\varepsilon^2$ *with* $a_2 > 0$. *The differential system* (4.14) *has a Hopf bifurcation at* $a = 0$, *and for* $\varepsilon > 0$ *sufficiently small a periodic solution emerges from the origin of coordinates of this system.*

(*ii*) *Let* $a = -196 + a_2\varepsilon^2$ *with* $a_2 > 0$. *The differential system* (4.15) *has a Hopf bifurcation at* $a = -196$, *and for* $\varepsilon > 0$ *sufficiently small a periodic solution emerges from the equilibrium point* $(-7, -7, 0)$.

(*iii*) *Let* $a = \dfrac{-25}{16} + a_2\varepsilon^2$ *with* $a_2 > 0$. *The differential system* (4.16) *has a Hopf bifurcation at* $a = \dfrac{-25}{16}$, *and for* $\varepsilon > 0$ *sufficiently small a periodic solution emerges from the equilibrium point* $\left(-\dfrac{5}{4}, 0, \dfrac{5}{4}\right)$.

(*iv*) *Let* $a = -25 + a_2\varepsilon^2$ *with* $a_2 > 0$. *The differential system* (4.17) *has a Hopf bifurcation at* $a = -25$, *and for* $\varepsilon > 0$ *sufficiently small a periodic solution emerges from the equilibrium point* $(-5, 0, 0)$.

(*v*) *Let* $a = \dfrac{8}{25} + a_2\varepsilon^2$ *with* $a_2 > 0$. *The differential system* (4.18) *has a zero-Hopf bifurcation at* $a = \dfrac{8}{25}$, *and for* $\varepsilon > 0$ *sufficiently small a periodic solution emerges from the equilibrium point* $\left(\dfrac{4}{5}, \dfrac{4}{5}, \dfrac{4}{5}\right)$.

(*vi*) *Let* $a = -\dfrac{560}{1849} + a_2\varepsilon^2$ *with* $a_2 > 0$. *The differential system* (4.19) *has a Hopf bifurcation at* $a = -\dfrac{560}{1849}$, *and for* $\varepsilon > 0$ *sufficiently small a periodic solution emerges from the equilibrium point* $\left(-\dfrac{10}{43}, -\dfrac{10}{43}, -\dfrac{10}{43}\right)$.

To illustrate graphically the relation between the periodic solutions provided by Theorem 26 and the chaotic attractors presented in [41] we shall use system (4.19) as an example. First we observe that for $a < 5/26$ this system has the following equilibrium point

$$p_0 = \left(\frac{1}{13}\left(5 - \sqrt{25 - 130a}\right), \frac{1}{13}\left(5 - \sqrt{25 - 130a}\right), \frac{1}{13}\left(5 - \sqrt{25 - 130a}\right)\right).$$

Taking $a = -\dfrac{560}{1849} + a_2\varepsilon^2$ and $\varepsilon > 0$ sufficiently small system (4.19) has a periodic solution as stated by Theorem 26(*vi*). In this case the equilibirum point p_0 exists only if $0 < \varepsilon < \frac{23}{2}\sqrt{\frac{15}{6799}} \approx 0.54$. For instance, taking $a_2 = -\dfrac{560}{1849} + 2$ and $\varepsilon = 0.002$ it can be seen that the solution of system (4.19) starting at $(1, -1, 0)$ converges to the periodic solution, see Figure 4.1. Increasing the value of ε, for instance $\varepsilon = 0.251$ and $\varepsilon = 0.511$, the periodic solution increases its size and still remains stable, see Figures 4.2 and 4.3 respectively. For all the previously values of ε the point p_0 is an equibilibrium point of system (4.19). However for $\varepsilon = 0.691$ and $\varepsilon = 0.97$ the system has no equilibria and we can see that the periodic solution starts its cascade of period-doubling, see Figure 4.4 and 4.5. Taking $\varepsilon = 1$ the system has a strange attractor as it is reported in [41], see Figure 4.6. These solutions were plotted for $0 \leq t \leq 1000$.

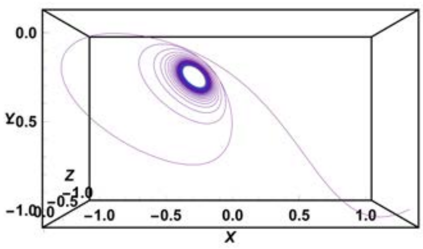


Figure 4.1: Solution of system (4.19) starting at $(1, 1, 0)$ with $\varepsilon = 0.002$.

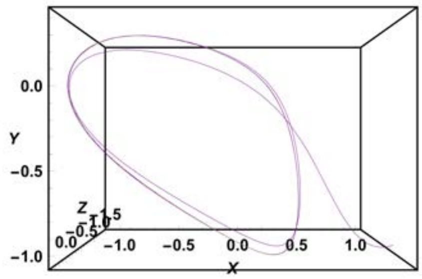


Figure 4.2: Solution of system (4.19) starting at $(1, 1, 0)$ with $\varepsilon = 0.251$.

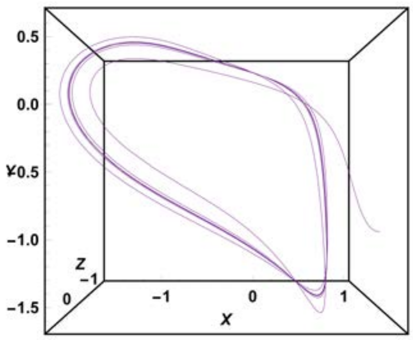


Figure 4.3: Solution of system (4.19) starting at $(1, 1, 0)$ with $\varepsilon = 0.511$.

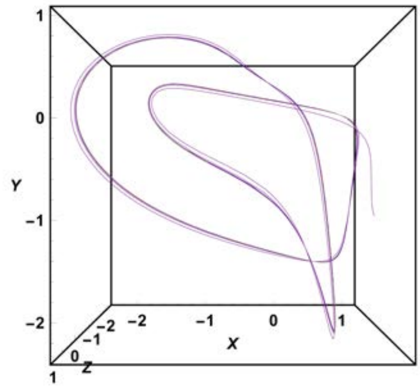


Figure 4.4: Solution of system (4.19) starting at $(1, 1, 0)$ with $\varepsilon = 0.691$.

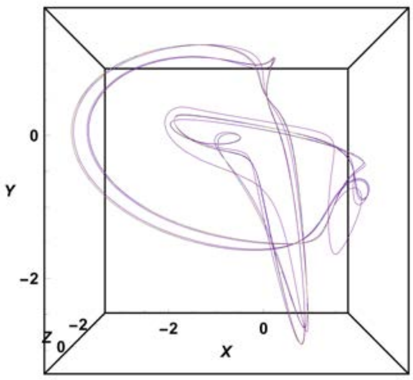


Figure 4.5: Solution of system (4.19) starting at $(1, 1, 0)$ with $\varepsilon = 0.97$.

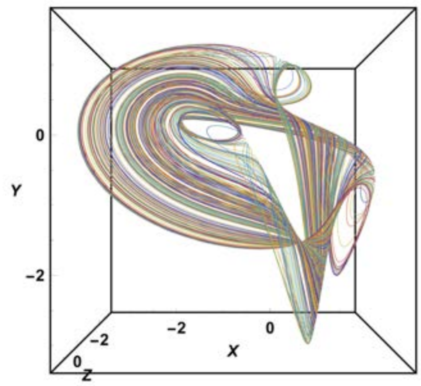


Figure 4.6: Solution of system (4.19) starting at $(1, 1, 0)$ with $\varepsilon = 1$.

4.3 Proofs

4.3.1 Proof of Theorem 24

The following lemma will be useful in the proof of Theorem 24, it was proved in [25].

Lemma 27. *Consider $p+1$ linearly independent functions $\xi_i : I \subset \mathbb{R} \to \mathbb{R}$, $i = 0, 1, \dots, p$.*

(a) Given p arbitrary values $x_i \in I$, $i = 1, 2, \dots, p$ there exist $p+1$ constants C_i, $i = 0, 1, \dots, p$ such that

$$\xi(x) := \sum_{i=0}^{p} C_i \xi_i(x) \tag{4.20}$$

is not the zero function and $\xi(x_i) = 0$ for $i = 1, 2, \dots, p$.

(b) Furthermore if all ξ_i are analytical functions on I and there exists $j \in \{1, 2, \dots, p\}$ such that $\xi_j \restriction_I$ has a constant sign, it is possible to get a function $\xi(x)$ from (4.20), such that it has at least p simple zeroes in I.

Proof of Theorem 24. Take $a = -\dfrac{1}{d} + a_1\varepsilon + a_2\varepsilon^2$, $c = bd + c_2\varepsilon^2$ and $d(1-b^2d^3) > 0$. Doing the change of variables $(x, y, z) \to \varepsilon(X, Y, Z)$ system (4.2) becomes

$$\begin{aligned}
\dot{X} =& Z,\\
\dot{Y} =& b(X - dY),\\
\dot{Z} =& -\frac{X}{d} + Y + dbZ + \varepsilon\frac{a_1 d + X(1-d)}{d}\\
&+ \varepsilon^2\left(X\left(a_2 + X(X - a_1)\right) + c_2 Z\right) - a_2 X^2 \varepsilon^3.
\end{aligned} \tag{4.21}$$

In order to write the linear part of system (4.21) in its normal real Jordan form we do the following change of variables $X = \overline{x} + \dfrac{bd}{\omega}\overline{y} + \overline{z}$, $Y = \dfrac{b}{\omega}\overline{y} + \dfrac{\overline{z}}{d}$, $Z = bd\overline{x} - \omega\overline{y}$ with $\omega = \sqrt{\dfrac{1}{d} - b^2d^2}$, obtaining

$$\begin{aligned}
\dot{\overline{x}} =& -\omega\overline{y},\\
\dot{\overline{y}} =& \omega\overline{x} + \frac{\varepsilon}{d\omega^3}\left((bd\overline{y} + (\overline{x} + \overline{z})\omega)\right)\left(b(d-1)d\overline{y} - \left(a_1 d - (d-1)(\overline{x} + \overline{z})\right)\omega\right)\\
&- \frac{\varepsilon^2}{\omega}\left(\left(a_2 + \left(\overline{x} + \overline{z} + \frac{bd\overline{y}}{\omega}\right)\left(-a_1 + \overline{x} + \overline{z} + \frac{bd\overline{y}}{\omega}\right)\right)\left(\overline{x} + \overline{z} + \frac{bd\overline{y}}{\omega}\right)\right.\\
&\left. + c_2(bd\overline{x} - \omega\overline{y})\right),\\
\dot{\overline{z}} =& -\frac{\varepsilon b}{\omega^4}(bd\overline{y} + \omega(\overline{x} + \overline{z}))(b(d-1)dy - \omega(a_1 d - (d-1)(\overline{x} + \overline{z})))\\
&+ \varepsilon^2\frac{bd}{\omega^2}\left(\left(\frac{bd\overline{y}}{\omega} + \overline{x} + \overline{z}\right)\left(\left(\frac{bd\overline{y}}{\omega} + \overline{x} + \overline{z}\right)\left(\frac{bd\overline{y}}{\omega} + \overline{x} + \overline{z} - a_1\right)\right.\right.\\
&\left.\left. + a_2\right) + c_2(bd\overline{x} - \overline{y}\omega)\right) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{4.22}$$

Using the change of variables $(\overline{x}, \overline{y}, \overline{z}) \to (u, v, w)$ where $\overline{x} = u\cos(\omega t) - v\sin(\omega t)$, $\overline{y} = v\cos(\omega t) + u\sin(\omega t)$ and $\overline{z} = w$, system (4.22) becomes

$$\begin{aligned}
\dot{u} =& \frac{\varepsilon\sin(t\omega)}{d\omega^3}(\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega)\\
&+ \omega w)((d-1)(\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega))\\
&- \omega(a_1 d - dw + w)) - \frac{\varepsilon^2\sin(t\omega)}{\omega}\left(\left(\sin(t\omega)\left(\frac{bdu}{\omega} - v\right)\right.\right.\\
&\left.\left. + \cos(t\omega)\left(\frac{bdv}{\omega} + u\right) + w\right)\left(\frac{1}{\omega^2}((\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv\right.\right.\\
&+ u\omega) + \omega w)(\omega(w - a_1) + \sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega)))\\
&\left.\left. + a_2) - c_2(\sin(t\omega)(bdv + u\omega) + \cos(t\omega)(v\omega - bdu))\right)\right) + \mathcal{O}(\varepsilon^3),\\
\dot{v} =& \frac{\varepsilon\cos(t\omega)}{d\omega^3}(\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega) + \omega w)((d\\
&- 1)(\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega)) - \omega(a_1 d - dw + w))\\
&- \frac{\varepsilon^2\cos(t\omega)}{\omega}\left(\left(\sin(t\omega)\left(\frac{bdu}{\omega} - v\right) + \cos(t\omega)\left(\frac{bdv}{\omega} + u\right) + w\right)\right.\\
&\left(\frac{1}{\omega^2}(\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega) + \omega w)(\omega(w - a_1)\right.\\
&+ \sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega)) + a_2) - c_2(\sin(t\omega)(bdv\\
&\left.\left. + u\omega) + \cos(t\omega)(v\omega - bdu))\right)\right) + \mathcal{O}(\varepsilon^3),\\
\dot{w} =& \frac{\varepsilon b}{\omega^4}(\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega) + \omega w)(\omega(a_1 d - dw + w)\\
&- (d-1)(\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega)))\\
&+ \frac{\varepsilon^2 bd}{\omega^2}\left(\left(\sin(t\omega)\left(\frac{bdu}{\omega} - v\right) + \cos(t\omega)\left(\frac{bdv}{\omega} + u\right) + w\right)\right.\\
&\left(\frac{1}{\omega^2}(\sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega) + \omega w)(\omega(w - a_1)\right.\\
&+ \sin(t\omega)(bdu - v\omega) + \cos(t\omega)(bdv + u\omega)) + a_2) - c_2(\sin(t\omega)(bdv + u\omega)\\
&\left.\left. + \cos(t\omega)(v\omega - bdu))\right)\right) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{4.23}$$

The above system is in the normal form for applying the averaging theory given in Theorems 1 and 12, where $T = \dfrac{2\pi}{\omega}$, $\mathbf{y} = (u, v, w)$ and the function (1.3) corresponding to system (4.23) is

$$\begin{aligned}
g_1(\mathbf{y}) = -\Bigg(&\frac{(a_1 d - 2(d-1)w)(bdu - v\omega)}{2d\omega^2}, \frac{(a_1 d - 2(d-1)w)(bdv + u\omega)}{2d\omega^2},\\
&\frac{b\left((d-1)\left(u^2 + v^2\right)\left(b^2 d^2 + \omega^2\right) - 2\omega^2 w(a_1 d - dw + w)\right)}{2\omega^4}\Bigg).
\end{aligned}$$

The zeros of this function are $\mathbf{s} = \left(0,\, 0,\, \dfrac{a_1 d}{d-1}\right)$, and the continuum of zeros $\mathcal{Z}_{\pm} =$

$\{(\alpha, \beta_{\pm}(\alpha)),\ \alpha \in \mathbb{R}\}$ where

$$\beta_{\pm}(\alpha) = \left(\pm\sqrt{\frac{\omega^2\left(a_1^2 d^2 - 2(d-1)^2\alpha^2\right) - 2b^2(d-1)^2 d^2\alpha^2}{2(d-1)^2\left(b^2d^2+\omega^2\right)}},\ \frac{a_1 d}{2(d-1)} \right).$$

Furthermore since $\det\left(Dg_1(\mathbf{s})\right) = -\dfrac{a_1^3 bd\left(b^2d^2+\omega^2\right)}{4\omega^6} \neq 0$, by Theorem 1 we can ensure the existence of a periodic solution of system (4.23) emerging from the origin. Consequently going back through the change of variables system (4.2) has the periodic solution (4.3).

For $\mathbf{z}_\alpha \in \mathcal{Z}_-$ we have

$$f_1(\alpha) = \frac{\pi a_1^2 d^2\sqrt{b^2d^2+\omega^2}\left(a_1^2 bd + 24c_2\omega^4\right)}{24(d-1)\omega^4\left(2(d-1)\alpha\omega\sqrt{b^2d^2+\omega^2} - bd\Lambda_\alpha\right)},$$

where $\Lambda_\alpha = \sqrt{a_1^2\left(2d-2b^2d^4\right) - \dfrac{4(d-1)^2\alpha^2}{d}}$. As $f_1(\alpha) \neq 0$ for all $\alpha \in \mathbb{R}$ we cannot apply Theorem 12 in this case. On the other hand for $\mathbf{z}_\alpha \in \mathcal{Z}_+$ we have

$$f_1(\alpha) = \frac{C_0 + C_1\alpha^2 + C_2\alpha^4 + C_3\Lambda_\alpha\alpha + C_4\Lambda_\alpha\alpha^3}{24(d-1)^2 d\omega^8\left(2(d-1)\alpha\omega\sqrt{b^2d^2+\omega^2} - bd\Lambda_\alpha\right)}, \tag{4.24}$$

where

$$\begin{aligned}
C_0 =& \pi\left(-a_1^2\right)(d-1)d^3\omega^4\left(b^2d^2+\omega^2\right)\left(a_1^2bd+24c_2\omega^4\right),\\
C_1 =& 4\pi(d-1)^3 d\omega^2\left(b^2d^2+\omega^2\right)\left(a_1^2\left(13b^3d^3-11bd\omega^2\right)+24c_2\omega^4\left(b^2d^2+\omega^2\right)\right),\\
C_2 =& -96\pi b(d-1)^5(bd-\omega)(bd+\omega)\left(b^2d^2+\omega^2\right)^2,\\
C_3 =& 2\pi d\omega^3\sqrt{b^2d^2+\omega^2}\left(a_1^2\left(b^2d^2\left(3d(d+8)\omega^2-14(d-1)^2\right)+3d(d+8)\omega^4\right.\right.\\
&\left.\left.-2(d-1)^2\omega^2\right)+24a_2(d-1)^2\omega^2\left(b^2d^2+\omega^2\right)\right),\\
C_4 =& 96\pi b^2(d-1)^4 d\omega\left(b^2d^2+\omega^2\right)^{3/2}.
\end{aligned}$$

The denominator of f_1 vanishes at $\alpha^{\pm} = \pm\sqrt{\dfrac{a_1^2b^2d^5\left(1-b^2d^3\right)}{2(d-1)^2}}$. In addition the domain of definition of Λ_α is the interval $J \subset \mathbb{R}$ such that $|\alpha| \leq \left|\dfrac{d}{(d-1)}\right|\sqrt{\dfrac{a_1^2d(1-b^2d^3)}{2}}$. Thus the domain of definition of f_1 is $I = J \setminus \{\alpha^{\pm}\}$.

In order to study the maximum number of the simple zeros of function (4.24) we are going to apply Lemma 27. Thus we define the functions $\xi_0(\alpha) = 1$, $\xi_1(\alpha) = \alpha^2$, $\xi_2(\alpha) = \alpha^4$, $\xi_3(\alpha) = \Lambda_\alpha\alpha$, $\xi_4(\alpha) = \Lambda_\alpha\alpha^3$ and $\xi(\alpha) = \sum_{i=0}^{4} C_i\xi_i(\alpha)$ with $\alpha \in I \subset \mathbb{R}$. We observe that the five functions of the set $\{C_i \mid i = 0, 1, \ldots, 4\}$ are linearly independent. Indeed due to hypothesis (a), the determinant of the Jacobian matrix $\dfrac{\partial(C_0, C_1, C_2, C_3, C_4)}{\partial(a_1, a_2, b, c_2, d)}$ is

$$\begin{aligned}
&-\tfrac{42467328\pi^5 a_1 b^2(d-1)^{14}\left(b^2d^3-1\right)^7}{d^{14}}\left(d\left(b^2(d+1)d^2\left(6b^2d^3-7\right)+2\right)\right.\\
&\left.+10\right)\left(a_1^2bd^3\left(24b^2d^3-13\right)-24c_2\left(b^2d^3-1\right)^2\right) \neq 0.
\end{aligned}$$

Thus statement (a) of Lemma 27 ensures that the function $\xi(\alpha)$ has 4 zeros. In addition, $\xi_0(\alpha)$ does not changes sign, so by statement (b) of the same lemma, $\xi(\alpha)$ has 4 simple zeros. Finally we observe that the simple zeros of $\xi(\alpha)$ are also simple zeros of $f_1(\alpha)$ so there exist values $\alpha_i \in I$ for $i = 1, 2, 3, 4$ such that $f_1(\alpha_i) = 0$, and $\det(Df_1(\alpha_i)) \neq 0$. So the conclusion of Theorem 24 follows from Theorem 12. $\square$

4.3.2 Proof of Theorem 25

The proof of Theorem 25 for systems (4.4)-(4.7) and (4.9)-(4.13) can be obtained using Corollary 7 with $s = 0$ which is equivalent with the classical averaging theory as we shall see. We start proving Theorem 25 for system (4.13).

Proof of Theorem 25 for system (4.13). We take $a = a_2\varepsilon^2$ with $a_2 > 0$ and $\varepsilon > 0$ sufficiently small. First we translate the point $p = (1, 0, 0)$ to the origin of coordinates, then we use the change of variables

$$(x, y, z) = \varepsilon\left(\frac{19X}{9} + Z, X - \frac{\sqrt{10}Y}{3}, \frac{19\sqrt{10}Y}{30}\right),$$

and the differential system (4.13) writes

$$\begin{aligned}
\dot{X} &= -\frac{3Y}{\sqrt{10}} + \frac{1}{171}\varepsilon\left(\left(10X + 3\sqrt{10}Y\right)(19X + 9Z) - 90a_2\right),\\
\dot{Y} &= \frac{3X}{\sqrt{10}} + \frac{\varepsilon\left(\left(10X + 3\sqrt{10}Y\right)(19X + 9Z) - 90a_2\right)}{57\sqrt{10}},\\
\dot{Z} &= \frac{1}{81}\varepsilon\left(90a_2 - \left(10X + 3\sqrt{10}Y\right)(19X + 9Z)\right).
\end{aligned} \tag{4.25}$$

Using the cylindrical change of variables $(X, Y, Z) = \left(\rho\cos\theta, \rho\sin\theta, z\right)$ where $\rho > 0$, system (4.25) becomes

$$\begin{aligned}
\dot{\rho} =& \frac{1}{342}\varepsilon\Big(-18a_2\left(3\sqrt{10}\sin\theta + 10\cos\theta\right) + 9\rho z\left(6\sqrt{10}\sin(2\theta) + \cos(2\theta) + 19\right)\\
&+19\rho^2\cos\theta\left(6\sqrt{10}\sin(2\theta) + \cos(2\theta) + 19\right)\Big),\\
\dot{\theta} =& \frac{3}{\sqrt{10}} + \frac{\varepsilon}{1710\rho}\left(\rho\left(3\sqrt{10}\sin\theta + 10\cos\theta\right)(19\rho\cos\theta + 9z) - 90a_2\right)\\
&\left(3\sqrt{10}\cos\theta - 10\sin\theta\right),\\
\dot{z} =& \frac{1}{81}\varepsilon\left(90a_2 - \rho\left(3\sqrt{10}\sin\theta + 10\cos\theta\right)(19\rho\cos\theta + 9z)\right).
\end{aligned}$$

This differential system can be reduced to the normal form for applying the averaging theory. Taking θ as the new independent variable we obtain the differential system

$$\begin{aligned}
\rho' =& \frac{\varepsilon}{513}\sqrt{\frac{5}{2}}\Big(\rho\left(6\sqrt{10}\sin(2\theta) + \cos(2\theta) + 19\right)(19\rho\cos\theta + 9z)\\
&-18a_2\left(3\sqrt{10}\sin\theta + 10\cos\theta\right)\Big) + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

$$z' = \frac{\varepsilon}{243}\sqrt{10}\left(90a_2 - r\left(3\sqrt{10}\sin\theta + 10\cos\theta\right)(19r\cos\theta + 9z)\right) + \mathcal{O}(\varepsilon^2).$$

Here the derivatives are taken with respect to θ. Using (1.15) we write the functions $g_0 \equiv 0$ and $g_1(z) = \sqrt{10}\pi\left(\frac{1}{3}\rho z, \frac{10}{243}(18a_2 - 19\rho^2)\right)$. The averaged function g_1 has the solutions $z_\pm = \pm\left(3\sqrt{\frac{2a_2}{19}}, 0\right)$. The result follows by taking $s = 0$ and $z^* = z_+$ and applying Corollary 7. The periodic solution is non-hyperbolic. The eigenvalues of the Jacobian matrix $Dg_1(z_+)$ are $\pm\frac{20}{9}i\sqrt{a_2}$. □

Proof of Theorem 25 for systems (4.4)-(4.7) *and* (4.9)-(4.12). The proof of Theorem 25 for systems (4.4)-(4.7) and (4.9)-(4.12) is similar to the proof of Theorem 25 for system (4.13). It can be done using Corollary 7 with $s = 0$ and doing analogous computations. The reader can check in Theorem 1.1 of [20] the proofs for these systems using classical first order averaging. The authors also provide approximations for the periodic solutions found. □

Now we prove Theorem 25 for system (4.8). This proof is not provided in [20] because the classical averaging theory does not provide information for this case. We shall prove this result using statement (b) of Theorem 12 .

Proof of Theorem 25 for system (4.8). Using the change of variables $(x, y, z) = \varepsilon\big(x + y, -y, -2z\big)$ the differential system (4.8) writes

$$\begin{aligned} \dot{X} &= -2Y + \frac{1}{2}\varepsilon\left(a_2 - Y^2 + 4Z^2\right), \\ \dot{Y} &= 2X, \\ \dot{Z} &= \frac{1}{2}\varepsilon\left(-a_2 + Y^2 - 4Z^2\right). \end{aligned} \tag{4.26}$$

Using the cylindrical change of variables $(X, Y, Z) = \big(\rho\cos\theta, \rho\sin\theta, z\big)$ where $\rho > 0$, system (4.26) becomes

$$\begin{aligned} \dot{\rho} &= \varepsilon\frac{1}{2}\cos\theta\left(a_2 - \rho^2\sin^2\theta + 4z^2\right), \\ \dot{\theta} &= 2 - \varepsilon\frac{\sin\theta\left(a_2 - \rho^2\sin^2\theta + 4z^2\right)}{2\rho}, \\ \dot{z} &= \varepsilon\frac{1}{2}\left(-a_2 + \rho^2\sin^2\theta - 4z^2\right). \end{aligned}$$

This differential system can be reduced to the normal form for applying averaging theory. Taking θ as the new independent variable we obtain the differential system

$$\begin{aligned} \rho' =& \varepsilon\frac{1}{4}\cos\theta\left(a_2 - \rho^2\sin^2\theta + 4z^2\right) + \varepsilon^2\frac{\sin\theta\cos\theta\left(a_2 - \rho^2\sin^2\theta + 4z^2\right)^2}{16\rho} \\ &+ \varepsilon^3\frac{\sin^2\theta\cos\theta\left(a_2 - \rho^2\sin^2\theta + 4z^2\right)^3}{64\rho^2} + \mathcal{O}(\varepsilon^4), \end{aligned}$$

$$
\begin{aligned}
z' =&\varepsilon\frac{1}{4}\left(-a_2+\rho^2\sin^2\theta-4z^2\right)+\varepsilon^2-\frac{\sin\theta\left(a_2-\rho^2\sin^2\theta+4z^2\right)^2}{16\rho} \\
&+\varepsilon^3\frac{\sin^2\theta\left(-a_2+\rho^2\sin^2\theta-4z^2\right)^3}{64\rho^2}+\mathcal{O}(\varepsilon^4).
\end{aligned} \tag{4.27}
$$

Here the derivatives are taken with respect to θ. Using (1.15) we write the functions

$$
\begin{aligned}
g_0(z) =&(0,0),\\
g_1(z) =&\pi\left(0,-\frac{a_2}{2}+\frac{\rho^2}{4}-2z^2\right),\\
g_2(z) =&\left(0,\frac{1}{2}\pi^2 z\left(2a_2-\rho^2+8z^2\right)\right),\\
g_3(z) =&\Big(\frac{\pi}{96\rho}\left(24a_2^2(\rho+z)+4a_2\left(-7\rho^3+48z^3+96\rho z^2-12\rho^2 z\right)+8\rho^5+384z^5\right.\\
&\left.+1152\rho z^4-192\rho^2z^3-208\rho^3z^2+15\rho^4 z\right.,\frac{\pi}{4608\rho}\left(3\rho\left(8\left(15-32\pi^2\right)a_2^2\right.\right.\\
&\left.+4\left(64\pi^2-23\right)a_2\rho^2+\left(5-64\pi^2\right)\rho^4\right)-128z\left(9a_2^2-18a_2\rho^2+10\rho^4\right)\\
&+9216z^3\left(\rho^2-a_2\right)+48\rho z^2\left(\left(60-256\pi^2\right)a_2+\left(128\pi^2-47\right)\rho^2\right)\\
&\left.-18432z^5+1152\left(5-32\pi^2\right)\rho z^4\right)\Big).
\end{aligned}
$$

Consider the graph $\mathcal{Z}=\left\{z_\alpha=(\alpha,\beta(\alpha)):\beta(\alpha)=\sqrt{\frac{\alpha^2-2a_2}{8}}\text{ and }\alpha\geq\sqrt{2a_2}\right\}$. For all $\alpha\geq\sqrt{2a_2}$ the averaged function $g_1(z_\alpha)=(0,0)$. Then taking $s=1$ in Theorem 12 we compute the bifurcation functions $f_1(\alpha)=0$ and $f_2(\alpha)=\frac{\pi\alpha^2}{64}\sqrt{\frac{\alpha^2-2a_2}{2}}$. For $\alpha^*=\sqrt{2a_2}$ we have $f_2(\alpha^*)=0$ and the derivative of f_2 goes to infinity at α^*, so it is a simple zero of f_2. Thus applying statement (b) of Theorem 12 we have that system (4.27) has a periodic solution bifurcating from point z^*_α. Consequently going back through the change of variables we have the existence of a periodic solution of system (4.8). □

4.3.3 Proof of Theorem 26

Proof of Theorem 26 statement (i). Using the change of variables $(x,y,z)=\varepsilon\big(X+Z,-Y+Z,2Z\big)$ the differential system (4.14) writes

$$
\begin{aligned}
\dot X =&-Y+\frac{\varepsilon}{10}(5a_2-2(X+Z)(5Y-14Z)),\\
\dot Y =&X+\frac{\varepsilon}{10}(2(X+Z)(5Y-14Z)-5a_2),\\
\dot Z =&Z+\frac{\varepsilon}{10}(2(X+Z)(5Y-14Z)-5a_2).
\end{aligned} \tag{4.28}
$$

Using the cylindrical change of variables $(X,Y,Z)=\big(\rho\cos\theta,\rho\sin\theta,z\big)$ where $\rho>0$, system (4.28) becomes

$$
\dot\rho =\frac{\varepsilon}{10}\varepsilon(\cos\theta-\sin\theta)\left(5a_2+28\rho z\cos\theta-10\rho\sin\theta(\rho\cos\theta+z)+28z^2\right),
$$

$$
\begin{aligned}
\dot{\theta} =& 1+\frac{\varepsilon}{10\rho}(\sin\theta+\cos\theta)\left(-5a_2-28\rho z\cos\theta+10\rho\sin\theta(\rho\cos\theta+z)-28z^2\right),\\
\dot{z} =& z+\frac{\varepsilon}{10}(2(5\rho\sin\theta-14z)(\rho\cos\theta+z)-5a_2).
\end{aligned}
$$

This differential system can be reduced to the normal form for applying the averaging theory. Taking θ as the new independent variable we obtain the differential system

$$
\begin{aligned}
\rho' =& \frac{\varepsilon}{10}(\cos\theta-\sin\theta)\left(5a_2+28z^2+28\rho z\cos\theta-10\rho\sin\theta(\rho\cos\theta+z)\right)\\
&+\varepsilon^2\frac{(\cos\theta-\sin\theta)(\sin\theta+\cos\theta)}{100\rho}\left(5a_2-5\rho^2\sin(2\theta)+28z^2-10\rho z\sin\theta\right.\\
&\left.+28\rho z\cos\theta\right)^2+\mathcal{O}(\varepsilon^3), \qquad (4.29)\\
z' =& z+\varepsilon\left(-5a_2-28z^2-28\rho z\cos\theta+10\rho\sin\theta(\rho\cos\theta+z)\right)\\
&\frac{(\rho-z(\sin\theta+\cos\theta))}{10\rho}+\varepsilon^2\frac{(\sin\theta+\cos\theta)(-\rho+z\sin\theta+z\cos\theta)}{100\rho^2}\\
&\left(5a_2-5\rho^2\sin(2\theta)+28z^2-10\rho z\sin\theta+28\rho z\cos\theta\right)^2+\mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Here the derivatives are taken with respect to θ. Using (1.15) we write the functions

$$
\begin{aligned}
g_0(z) =& \left(0,\left(1-e^{-2\pi}\right)z\right),\\
g_1(z) =& \left(\frac{1}{25}\left(e^{2\pi}-1\right)z\left(71\rho+42\left(1+e^{2\pi}\right)z\right),\right.\\
&\left.\frac{\left(e^{2\pi}-1\right)}{50\rho}\left(-25a_2\rho+10\rho^3+28\left(e^{2\pi}+e^{4\pi}\right)z^3-94e^{2\pi}\rho z^2\right)\right),\\
g_2(z) =& \left(-\frac{a_2}{7800\rho}\left(156\left(-71+71e^{2\pi}-95\pi\right)\rho^2+28\left(127-195e^{4\pi}+68e^{6\pi}\right)z^2\right.\right.\\
&\left.+3\left(1591-6474e^{2\pi}+4883e^{4\pi}\right)\rho z\right)+\frac{1}{26002860000\rho}(52005720\\
&\left(-284+284e^{2\pi}+5\pi\right)\rho^4+\left(e^{8\pi}\left(4823-2226e^{2\pi}+806e^{6\pi}\right)-3403\right)z^4\\
&8453760+38584\left(e^{6\pi}\left(1308320+1767897e^{2\pi}-1169940e^{4\pi}+90712e^{6\pi}\right)\right.\\
&\left.-1996989\right)\rho z^3-\left(759163+666540e^{4\pi}-1872448e^{6\pi}+446745e^{8\pi}\right)\\
&100011\rho^2z^2+8000880\left(e^{\pi}-1\right)\left(1+e^{\pi}\right)\left(4799+10883e^{2\pi}\right)\rho^3z\Big),\\
&-\frac{e^{3\pi}a_2^2z\sinh(\pi)}{10\rho^2}+\frac{1}{38646750675000\rho^2}\left(6956415121500\left(e^{2\pi}-1\right)\rho^5\right.\\
&+61590200e^{2\pi}\left(e^{8\pi}\left(1258803-454104e^{2\pi}-409955e^{4\pi}+230724e^{6\pi}\right)\right.\\
&\left.-625468\right)z^5+1764476e^{2\pi}\left(e^{6\pi}\left(61818120+35320311e^{2\pi}-60289650e^{4\pi}\right.\right.\\
&\left.\left.+256360e^{6\pi}\right)-37105141\right)\rho z^4-500055e^{2\pi}\left(-41423181-3714436e^{4\pi}\right.\\
&\left.+16341616e^{6\pi}+19305793e^{8\pi}+9490208e^{10\pi}\right)\rho^2z^3-3216040e^{5\pi}\rho^3z^2\\
&\sinh(\pi)(153644891-120359184\sinh(2\pi)+31576438\cosh(2\pi))\\
&\left.-204805250e^{2\pi}\left(-784488+784488e^{2\pi}+61013\pi\right)\rho^4z\right)+(-2403375\\
&\left(e^{2\pi}-1\right)\rho^3+\left(-13030+21489e^{4\pi}-4988e^{6\pi}-4147e^{8\pi}+676e^{10\pi}\right)z^3\\
&357e^{2\pi}-986e^{5\pi}\rho z^2\sinh(\pi)(19421-3834\sinh(2\pi)+12388\cosh(2\pi))
\end{aligned}
$$

$$+11310e^{2\pi}\left(-1627+1627e^{2\pi}-6375\pi\right)\rho^2 z\right)\frac{a2}{9613500\rho^2}\Big).$$

Consider the graph $\mathcal{Z}=\{z_\alpha=(\alpha,\beta(\alpha)):\beta(\alpha)=0 \text{ and } \alpha>0\}$. For all $\alpha>0$ the function $g_0(z_\alpha)=(0,0)$. Then taking $s=0$ in Theorem 12 we compute the bifurcation functions $f_1(\alpha)=0$, and

$$f_2(\alpha)=\frac{\alpha}{500}\Big(10\big(71e^{2\pi}\left(e^{2\pi}-2\right)+95\pi+71\big)a_2-\left(284e^{2\pi}\left(e^{2\pi}-2\right)-5\pi+284\right)\alpha^2\Big).$$

For $\alpha^*=\sqrt{\dfrac{5a_2}{2}\left(1+\dfrac{385\pi}{284e^{2\pi}\left(e^{2\pi}-2\right)-5\pi+284}\right)}$ we have $f_2(\alpha^*)=0$ and $Df_2(\alpha^*)\neq 0$. Thus applying statement (b) of Theorem 12 we have that system (4.29) has a periodic solution bifurcating from point z_{α^*}. Consequently going back through the change of variables we have the existence of a periodic solution for system (4.14). $\square$

Proof of Theorem 26 statement (ii). First we translate the point $(-7,-7,0)$ to the origin of coordinates. Then we use the change of variables $(x,y,z)=\varepsilon\big(8X+Z,X+\sqrt{7}Y-Z/7,-8\left(\sqrt{7}Y+Z\right)\big)$ and the differential system (4.15) writes

$$\begin{aligned}
\dot{X}=&-\sqrt{7}Y+\frac{\varepsilon}{568}\Big(7a_2-4\Big(336X^2-98\sqrt{7}XY+128XZ+98Y^2-2\sqrt{7}YZ\\
&+7Z^2\Big)\Big),\\
\dot{Y}=&\sqrt{7}X+\frac{\varepsilon}{3976\sqrt{7}}\Big(60\Big(336X^2-98\sqrt{7}XY+128XZ+98Y^2-2\sqrt{7}YZ\\
&+7Z^2\Big)-105a_2\Big),\\
\dot{Z}=&-8Z+\frac{\varepsilon}{71}\Big(4\Big(336X^2-98\sqrt{7}XY+128XZ+98Y^2-2\sqrt{7}YZ+7Z^2\Big)\\
&-7a_2\Big).
\end{aligned} \tag{4.30}$$

Using the cylindrical change of variables $(X,Y,Z)=\big(\rho\cos\theta,\rho\sin\theta,z\big)$ where $\rho>0$, system (4.30) becomes

$$\begin{aligned}
\dot{\rho}=&\frac{\varepsilon}{27832}\Big(4\rho z\left(1009\sqrt{7}\sin(2\theta)-3031\cos(2\theta)-3241\right)\\
&+28\rho^2\left(\sqrt{7}(509\sin\theta+299\sin(3\theta))-2303\cos\theta-49\cos(3\theta)\right)\\
&+7\left(a_2-4z^2\right)\left(49\cos\theta-15\sqrt{7}\sin\theta\right)\Big),\\
\dot{\theta}=&\sqrt{7}-\frac{\varepsilon}{27832\rho}\left(49\sin\theta+15\sqrt{7}\cos\theta\right)\left(7\left(a_2-4\left(31\rho^2+z^2\right)\right)-476\rho^2\cos(2\theta)\right.\\
&\left.-512\rho z\cos\theta+8\sqrt{7}\rho\sin\theta(49\rho\cos\theta+z)\right),\\
\dot{z}=&-8z+\frac{\varepsilon}{71}\left(-7a_2+476\rho^2\cos(2\theta)+868\rho^2+28z^2+512\rho z\cos\theta\right.\\
&\left.-8\sqrt{7}\rho\sin\theta(49\rho\cos\theta+z)\right).
\end{aligned}$$

This differential system can be reduced to the normal form for applying the averaging theory. Taking θ as the new independent variable we obtain the differential system

$$\begin{aligned} \rho' =& \frac{\varepsilon}{27832\sqrt{7}} \Big(28\rho^2 \Big(\sqrt{7}(509 \sin\theta + 299 \sin(3\theta)) - 2303 \cos\theta - 49 \cos(3\theta) \Big) \\ & + 4\rho z \Big(1009\sqrt{7} \sin(2\theta) - 3031 \cos(2\theta) - 3241 \Big) + 7 \left(a_2 - 4z^2 \right) \\ & \Big(49 \cos\theta - 15\sqrt{7} \sin\theta \Big) \Big) + \mathcal{O}(\varepsilon^2), \\ z' =& -\frac{8z}{\sqrt{7}} - \frac{\varepsilon}{24353\rho} \Big(\Big(49 \left(\sqrt{7}\rho + z \sin\theta \right) + 15\sqrt{7} z \cos\theta \Big) \left(a_2 - 4 \left(31\rho^2 + z^2 \right) \right. \\ & 7 - 476\rho^2 \cos(2\theta) - 512\rho z \cos\theta + 8\sqrt{7}\rho \sin\theta (49\rho \cos\theta + z) \Big) + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{4.31}$$

Here the derivatives are taken with respect to θ. Using (1.15) we compute the functions

$$\begin{aligned} g_0(z) =& \left(0, \left(1 - e^{16\pi/\sqrt{7}} \right) z \right), \\ g_1(z) =& \Bigg(\frac{97 \left(e^{-\frac{32\pi}{\sqrt{7}}} - 1 \right) z^2}{37346} - \frac{13541 \left(1 - e^{-\frac{16\pi}{\sqrt{7}}} \right) \rho z}{182896}, \frac{e^{-\frac{48\pi}{\sqrt{7}}}}{168355768\rho} \\ & \left(e^{\frac{16\pi}{\sqrt{7}}} - 1 \right) \left(90209 e^{\frac{32\pi}{\sqrt{7}}} \rho \left(4724\rho^2 - 23a_2 \right) + 1488928 z^3 + 8 e^{\frac{16\pi}{\sqrt{7}}} z^2 \right. \\ & \left. (7509965\rho + 186116 z) \right) \Bigg), \\ g_2(z) =& \left(H_1(z), H_2(z) \right), \end{aligned}$$

where the functions H_1 and H_2 are provided in the Appendix D.

Consider the graph $\mathcal{Z} = \{ z_\alpha = (\alpha, \beta(\alpha)) : \beta(\alpha) = 0 \text{ and } \alpha > 0 \}$. For all $\alpha > 0$ the function $g_0(z_\alpha) = (0, 0)$. Then taking $s = 0$ in Theorem 12 we compute the bifurcation functions $f_1(\alpha) = 0$, and

$$\begin{aligned} f_2(\alpha) =& \frac{e^{-\frac{32\pi}{\sqrt{7}}} \alpha}{2389353344} \Big(189574 e^{\frac{16\pi}{\sqrt{7}}} \left(23a_2 - 4724\alpha^2 \right) - 94787 \left(23a_2 - 4724\alpha^2 \right) \\ & + e^{\frac{32\pi}{\sqrt{7}}} \Big(26128\sqrt{7}\pi \left(23a_2 - 17684\alpha^2 \right) - 94787 \left(23a_2 - 4724\alpha^2 \right) \Big) \Big). \end{aligned}$$

The bifurcation function f_2 has the positive solution

$$\begin{aligned} \alpha^* =& \frac{1}{2} \sqrt{ \frac{23a_2 \left(e^{\frac{32\pi}{\sqrt{7}}} \left(26128\sqrt{7}\pi - 94787 \right) + 189574 e^{\frac{16\pi}{\sqrt{7}}} - 94787 \right)}{e^{\frac{32\pi}{\sqrt{7}}} \left(115511888\sqrt{7}\pi - 111943447 \right) + 223886894 e^{\frac{16\pi}{\sqrt{7}}} - 111943447} } \\ \approx & 0.0288042\sqrt{a_2}, \end{aligned}$$

such that $Df_2(\alpha^*) \approx -0.002a_2 \neq 0$. Thus applying statement (b) of Theorem 12 we have that system (4.31) has a periodic solution bifurcating from point z^*_α. Consequently, going back through the change of variables we have the existence of a periodic solution to system (4.15). □

Proof of Theorem 26 statement (iii). After translating the point $(-5/4, 0, 5/4)$ to the origin of coordinates we use the change of variables $(x, y, z) = \varepsilon\big(Y+Z, \frac{5Z}{4} - \sqrt{2}X, Y - \frac{41Z}{16}\big)$, then the differential system (4.16) writes

$$
\begin{aligned}
\dot{X} &= -\sqrt{2}Y + \frac{\varepsilon\left(320a_2 + 12\sqrt{2}X(32Y+13Z) - 320Y^2 + 20YZ + 625Z^2\right)}{912\sqrt{2}},\\
\dot{Y} &= \sqrt{2}X + \frac{\varepsilon\left(-320a_2 - 12\sqrt{2}X(32Y+13Z) + 5\left(64Y^2 - 4YZ - 125Z^2\right)\right)}{1140},\\
\dot{Z} &= -\frac{5Z}{4} + \frac{\varepsilon\left(320a_2 + 12\sqrt{2}X(32Y+13Z) - 320Y^2 + 20YZ + 625Z^2\right)}{1140}.
\end{aligned}
\tag{4.32}
$$

Using the cylindrical change of variables $(X, Y, Z) = \big(\rho\cos\theta, \rho\sin\theta, z\big)$ where $\rho > 0$, system (4.32) becomes

$$
\begin{aligned}
\dot{\rho} =& \frac{\varepsilon}{9120}\Big(5\left(64a_2 + 125z^2\right)\Big(5\sqrt{2}\cos\theta - 8\sin\theta\Big) + 32\rho^2\sin\theta\Big(-73\sqrt{2}\sin(2\theta)\\
&+20\cos(2\theta) + 100) + 2\rho z\Big(-287\sqrt{2}\sin(2\theta) + 430\cos(2\theta) + 350\Big)\Big),\\
\dot{\theta} =& \sqrt{2} + \frac{\varepsilon}{9120\rho}\Big(5\sqrt{2}\sin\theta + 8\cos\theta\Big)\left(320a_2 + 625z^2 + 4\rho\left(5\sin\theta(z - 16\rho\sin\theta)\right.\right.\\
&\left.\left.+3\sqrt{2}\cos\theta(32\rho\sin\theta + 13z)\right)\right),\\
\dot{z} =& -\frac{5z}{4} + \frac{\varepsilon}{1140}\Big(4\rho\Big(5\sin\theta(z - 16\rho\sin\theta) + 3\sqrt{2}\cos\theta(32\rho\sin\theta + 13z)\Big)\\
&+ 320a_2 + 625z^2\Big).
\end{aligned}
$$

This differential system can be reduced to the normal form for applying the averaging theory. Taking θ as the new independent variable we obtain the differential system

$$
\begin{aligned}
\rho' =& \frac{\varepsilon}{9120\sqrt{2}}\Big(- 2560a_2\sin\theta + \sqrt{2}\cos\theta\left(1600a_2 + 3125\rho^4z^2 - 1148\rho^3z\sin\theta\right.\\
&\left.-1168\rho^2\right) + 700\rho^3z + 4\rho^2\Big(292\sqrt{2}\cos(3\theta) + 50\sin\theta\left(16 - 25\rho^2z^2\right)\\
&+5\cos(2\theta)(32\sin\theta + 43\rho z))\Big) + \frac{\varepsilon^2}{83174400\sqrt{2}\rho}\Big(-2560a_2\sin\theta + \sqrt{2}\cos\theta\\
&\left(1600a_2 + 3125\rho^4z^2 - 1148\rho^3z\sin\theta - 1168\rho^2\right) + 700\rho^3z + 4\rho^2\\
&\Big(292\sqrt{2}\cos(3\theta) + 50\sin\theta\left(16 - 25\rho^2z^2\right) + 5\cos(2\theta)(32\sin\theta + 43\rho z)\Big)\Big)\\
&+\Big(\sin\theta\left(1600a_2 + 3125\rho^4z^2 + 2336\rho^2\cos(2\theta) + 736\rho^2\right) + 20\sqrt{2}\cos\theta\\
&\left(64a_2 + 125\rho^4z^2 + \rho^2\sin\theta(32\sin\theta + 43\rho z)\right) + 2\rho^3z(287\cos(2\theta) + 337)\big)\\
&+\mathcal{O}(\varepsilon^3),
\end{aligned}
\tag{4.33}
$$

$$
\begin{aligned}
z' =& -\frac{5z}{4\sqrt{2}} + \frac{\varepsilon}{36480\sqrt{2}\rho^2}\Big(1024\Big(10a_2 + \rho^2\Big(6\sqrt{2}\sin(2\theta) - 5\Big) + 5\rho^2\cos(2\theta)\Big)\\
&- 16\rho z\Big(-780a_2\sin\theta + 6\sqrt{2}\left(200a_2 - 141\rho^2\right)\cos\theta + \rho^2\Big(1265\sin\theta
\end{aligned}
$$

$$
\begin{aligned}
&+525\sin(3\theta)+534\sqrt{2}\cos(3\theta)\Big)\Big)+1875\rho^5z^3\left(13\sin\theta-20\sqrt{2}\cos\theta\right)+2\rho^4z^2\\
&\left(1221\sqrt{2}\sin(2\theta)-4875\cos(2\theta)+5515\right)\Big)+\frac{\varepsilon^2}{665395200\rho^3}\Big(\big(\sin\theta\,(1600a_2\\
&+3125\rho^4z^2+2336\rho^2\cos(2\theta)+736\rho^2\big)+20\sqrt{2}\cos\theta\,\big(64a_2+125\rho^4z^2\\
&+\rho^2\sin\theta(32\sin\theta+43\rho z)\big)+2\rho^3z(287\cos(2\theta)+337)\big)\Big(\rho\Big(5\sqrt{2}z\sin\theta\\
&\left(2496a_2+4875\rho^4z^2-4048\rho^2\right)-24z\cos\theta\left(1600a_2+3125\rho^4z^2-1128\rho^2\right)\\
&+12\rho\sin(2\theta)\left(407\rho^2z^2+1024\right)+10\sqrt{2}\rho\cos(2\theta)\left(512-975\rho^2z^2\right)\\
&-8400\sqrt{2}\rho^2z\sin(3\theta)-17088\rho^2z\cos(3\theta)\Big)+10\sqrt{2}\big(1024a_2+1103\rho^4z^2\\
&-512\rho^2\big)\Big)\Big)+\mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Here the derivatives are taken with respect to θ. Using (1.15) we write the functions

$$
\begin{aligned}
g_0(z)=&\left(0,\left(1-e^{\frac{5\pi}{2\sqrt{2}}}\right)z\right),\\
g_1(z)=&\Bigg(\frac{\rho^3z\left(-95625e^{-\frac{5\pi}{\sqrt{2}}}\sqrt{2}\rho z+95625\sqrt{2}\rho z-121264e^{-\frac{5\pi}{2\sqrt{2}}}+121264\right)}{2558160},\\
&\frac{e^{-\frac{15\pi}{2\sqrt{2}}}}{15348960\rho^2}\left(e^{\frac{5\pi}{2\sqrt{2}}}-1\right)\Big(22528e^{\frac{5\pi}{\sqrt{2}}}\left(153a_2-112\rho^2\right)-3538125\sqrt{2}\rho^5z^3\\
&-15e^{\frac{5\pi}{2\sqrt{2}}}\rho^4z^2\left(235875\sqrt{2}\rho z-240416\right)\Big)\Bigg),\\
g_2(z)=&\left(I_1(z),I_2(z)\right),
\end{aligned}
$$

where the functions I_1 and I_2 are provided in the Appendix D. Consider the graph $\mathcal{Z}=\{z_\alpha=(\alpha,\beta(\alpha)):\beta(\alpha)=0 \text{ and } \alpha>0\}$. For all $\alpha>0$ the function $g_0(z_\alpha)=(0,0)$. Then taking $s=0$ in Theorem 12 we compute the bifurcation functions $f_1(\alpha)=0$, and

$$
\begin{aligned}
f_2(\alpha)=&\frac{e^{-\frac{5\pi}{2\sqrt{2}}}}{633798675}\Big(44096\left(153\alpha a_2-112\alpha^3\right)+8e^{\frac{5\pi}{2\sqrt{2}}}\alpha\Big(5512\left(112\alpha^2-153a_2\right)\\
&+14535\pi\sqrt{2}\left(306a_2-131\alpha^2\right)\Big)\Big).
\end{aligned}
$$

The bifurcation function f_2 has the positive solution

$$
\alpha^*=3\sqrt{\frac{34a_2\left(e^{\frac{5\pi}{2\sqrt{2}}}\left(14535\sqrt{2}\pi-2756\right)+2756\right)}{e^{\frac{5\pi}{2\sqrt{2}}}\left(1904085\sqrt{2}\pi-617344\right)+617344}},
$$

such that $Df_2(\alpha^*)\approx-0.47a_2\neq0$. Thus applying statement (b) of Theorem 12 we have that system (4.33) has a periodic solution bifurcating from point z_α^*. Consequently going back through the change of variables we have the existence of a periodic solution to system (4.16). □

Proof of Theorem 26 statement (iv). First we translate the point $(-5,0,0)$ to the origin of coordinates, then using the change of variables $(x,y,z)=\varepsilon\Big(-\frac{X}{\sqrt{5}}+\frac{Y}{\sqrt{5}}-\frac{Z}{2},X+Y+Z,\sqrt{5}X-\sqrt{5}Y-2Z\Big)$ the differential system (4.17) writes

$$
\begin{aligned}
\dot{X}=&-\sqrt{5}Y-\varepsilon\frac{(2\sqrt{5}+5)}{1800\sqrt{5}}\Big(20\left(\sqrt{5}+6\right)X^2-100a_2-20\left(\sqrt{5}-6\right)Y^2\\
&+2X\left(80Y+125Z+2\sqrt{5}Z\right)+\left(250-4\sqrt{5}\right)YZ+85Z^2\Big),\\
\dot{Y}=&\sqrt{5}X-\varepsilon\frac{(2\sqrt{5}-5)}{1800\sqrt{5}}\Big(20\left(\sqrt{5}+6\right)X^2-100a_2-20\left(\sqrt{5}-6\right)Y^2\\
&+2X\left(80Y+125Z+2\sqrt{5}Z\right)+\left(250-4\sqrt{5}\right)YZ+85Z^2\Big),\\
\dot{Z}=&-2Z+\frac{1}{450}\varepsilon\Big(20\left(\sqrt{5}+6\right)X^2-100a_2+2X\left(80Y+125Z+2\sqrt{5}Z\right)\\
&-20\left(\sqrt{5}-6\right)Y^2+\left(250-4\sqrt{5}\right)YZ+85Z^2\Big).
\end{aligned}
\tag{4.34}
$$

Using the cylindrical change of variables $(X,Y,Z)=\big(\rho\cos\theta,\rho\sin\theta,z\big)$ where $\rho>0$, system (4.34) becomes

$$
\begin{aligned}
\dot{\rho}=&\frac{\varepsilon}{1800}\Big(5\left(20a_2-17z^2\right)\Big(\left(\sqrt{5}+2\right)\cos\theta-\left(\sqrt{5}-2\right)\sin\theta\Big)\\
&-2\sqrt{5}\rho^2\Big(3\left(\sqrt{5}+10\right)\sin(3\theta)+\left(37\sqrt{5}-50\right)\sin\theta-3\left(\sqrt{5}-10\right)\cos(3\theta)\\
&+\left(37\sqrt{5}+50\right)\cos\theta\Big)-2\rho z\left(240\sin(2\theta)+129\sqrt{5}\cos(2\theta)+260\right)\Big),\\
\dot{\theta}=&\sqrt{5}-\frac{\varepsilon}{1800\sqrt{5}\rho}\Big(\left(2\sqrt{5}-5\right)\cos\theta-\left(2\sqrt{5}+5\right)\sin\theta\Big)\Big(120\rho^2-100a_2\\
&+2\rho\Big(10\sqrt{5}\rho\cos(2\theta)+\cos\theta\left(80\rho\sin\theta+\left(2\sqrt{5}+125\right)z\right)\\
&+\left(125-2\sqrt{5}\right)z\sin\theta\Big)+85z^2\Big),\\
\dot{z}=&-2z+\frac{\varepsilon}{450}\Big(-100a_2+120\rho^2+2\rho\Big(10\sqrt{5}\rho\cos(2\theta)+\cos\theta\,(80\rho\sin\theta\\
&+\left(2\sqrt{5}+125\right)z\Big)+\left(125-2\sqrt{5}\right)z\sin\theta\Big)+85z^2\Big).
\end{aligned}
$$

This differential system can be reduced to the normal form for applying the averaging theory. Taking θ as the new independent variable we obtain the differential system

$$
\begin{aligned}
\rho'=&\frac{\varepsilon}{1800\sqrt{5}}\Big(5\left(20a_2-17z^2\right)\Big(\left(\sqrt{5}+2\right)\cos\theta-\left(\sqrt{5}-2\right)\sin\theta\Big)-2\sqrt{5}\rho^2\\
&\Big(3\left(\sqrt{5}+10\right)\sin(3\theta)+\left(37\sqrt{5}-50\right)\sin\theta-3\left(\sqrt{5}-10\right)\cos(3\theta)+\cos\theta\\
&\left(37\sqrt{5}+50\right)\Big)-2\rho z\left(240\sin(2\theta)+129\sqrt{5}\cos(2\theta)+260\right)\Big)+\mathcal{O}(\varepsilon^3),\\
z'=&-\frac{2\sqrt{5}z}{25}+\frac{\varepsilon}{4500\sqrt{5}\rho}\Big(10\rho+\left(2\sqrt{5}+5\right)z\sin\theta+\left(5-2\sqrt{5}\right)z\cos\theta\Big)\Big(120\rho^2
\end{aligned}
$$

$$
\begin{aligned}
&-100a_2+85z^2+2\rho\left(10\sqrt{5}\rho\cos(2\theta)+\cos\theta\left(80\rho\sin\theta+\left(2\sqrt{5}+125\right)z\right)\right.\\
&\left.\left.+\left(125-2\sqrt{5}\right)z\sin\theta\right)\right)-\varepsilon^2\frac{\left((2\sqrt{5}-5)\cos\theta-(2\sqrt{5}+5)\sin\theta\right)}{40500000\sqrt{5}\rho^2}\Big(-10\rho\\
&-\left(2\sqrt{5}+5\right)z\sin\theta+\left(2\sqrt{5}-5\right)z\cos\theta\Big)\left(120\rho^2-100a_2+85z^2+z\sin\theta\right.\\
&\left.\left(125-2\sqrt{5}\right)+2\rho\left(10\sqrt{5}\rho\cos(2\theta)+\cos\theta\left(80\rho\sin\theta+\left(2\sqrt{5}+125\right)z\right)\right)\right)^2\\
&+\mathcal{O}(\varepsilon^3).
\end{aligned}
$$

Here the derivatives are taken with respect to θ. Using (1.15) we write the functions

$$
\begin{aligned}
g_0(z)=&\left(0,(1-e^{\frac{4\pi}{\sqrt{5}}})z\right),\\
g_1(z)=&\Bigg(\frac{e^{-\frac{12\pi}{\sqrt{5}}}}{12600\rho}\left(1-e^{\frac{4\pi}{\sqrt{5}}}\right)\left(140e^{\frac{8\pi}{\sqrt{5}}}\rho\left(10a_2+\left(\sqrt{5}-12\right)\rho^2\right)+34\left(1+e^{\frac{4\pi}{\sqrt{5}}}\right)\right.\\
&\left.\left(\sqrt{5}-10\right)z^3-7\left(27\sqrt{5}+412\right)e^{\frac{4\pi}{\sqrt{5}}}\rho z^2\right)-\frac{e^{-\frac{8\pi}{\sqrt{5}}}}{25200}\left(e^{\frac{4\pi}{\sqrt{5}}}-1\right)z\\
&\left(7e^{\frac{4\pi}{\sqrt{5}}}\left(123\sqrt{5}+520\right)\rho+170\left(1+e^{\frac{4\pi}{\sqrt{5}}}\right)\left(2\sqrt{5}+1\right)z\right)\Bigg),\\
g_2(z)=&\left(J_1(z),J_2(z)\right),
\end{aligned}
$$

where the functions J_1 and J_2 are provided in the Appendix D.

Consider the graph $\mathcal{Z}=\{z_\alpha=(\alpha,\beta(\alpha)):\beta(\alpha)=0 \text{ and } \alpha>0\}$. For all $\alpha>0$ the function $g_0(z_\alpha)=(0,0)$. Then taking $s=0$ in Theorem 12 we compute the bifurcation functions $f_1(\alpha)=0$, and

$$
\begin{aligned}
f_2(\alpha)=&\frac{e^{-\frac{8\pi}{\sqrt{5}}}\alpha}{324000}\left(\left(956\sqrt{5}+5625\right)\alpha^2+2e^{\frac{4\pi}{\sqrt{5}}}\left(10\left(123\sqrt{5}+520\right)a_2-\right.\right.\\
&\left.\left(956\sqrt{5}+5625\right)\alpha^2\right)+e^{\frac{8\pi}{\sqrt{5}}}\left(\left(956\sqrt{5}+5625\right)\alpha^2+180\pi\sqrt{5}\left(8a_2-19\alpha^2\right)\right.\\
&\left.\left.-10\left(123\sqrt{5}+520\right)a_2\right)-10\left(123\sqrt{5}+520\right)a_2\right).
\end{aligned}
$$

The bifurcation function f_2 has the positive solution $\alpha^*=$

$$
\begin{aligned}
&\sqrt{\frac{10a_2\left(e^{\frac{8\pi}{\sqrt{5}}}\left(144\pi\sqrt{5}-123\sqrt{5}-520\right)-123\sqrt{5}+2e^{\frac{4\pi}{\sqrt{5}}}\left(123\sqrt{5}+520\right)-520\right)}{e^{\frac{8\pi}{\sqrt{5}}}\left(3420\pi\sqrt{5}-956\sqrt{5}-5625\right)-956\sqrt{5}+2e^{\frac{4\pi}{\sqrt{5}}}\left(956\sqrt{5}+5625\right)-5625}}\\
&\approx 0.369082\sqrt{a_2},
\end{aligned}
$$

such that $Df_2(\alpha^*)\approx -0.01a_2\neq 0$. Thus applying statement (b) of Theorem 12 we have that system (4.31) has a periodic solution bifurcating from point z^*_α. Consequently going back through the change of variables we have the existence of a periodic solution to system (4.15). □

Proof of Theorem 26 statement (v). First we translate the equilibrium point $(4/5, 4/5, 4/5)$ to the origin. Then using the change of variables $(x, y, z) = \varepsilon\Big(Z - 2Y, \sqrt{3}X + Y + ZY, \left(-\sqrt{3}\right) X + Y + Z\Big)$ the differential system (4.18) writes

$$\begin{aligned}
\dot{X} =& -\frac{1}{5}\left(4\sqrt{3}Y\right) + \frac{\varepsilon\left(3X^2 - 2\sqrt{3}X(Y - 2Z) - 3Y(Y + 4Z)\right)}{4\sqrt{3}}, \\
\dot{Y} =& \frac{4\sqrt{3}X}{5} + \frac{1}{4}\varepsilon\left(-X^2 - 2\sqrt{3}X(Y - 2Z) + Y(Y + 4Z)\right), \\
\dot{Z} =& \varepsilon\left(a_2 - 2\left(X^2 + Y^2\right) + \frac{Z^2}{2}\right).
\end{aligned} \tag{4.35}$$

Using the cylindrical change of variables $(X, Y, Z) = \left(\rho\cos\theta, \rho\sin\theta, z\right)$ where $\rho > 0$, system (4.35) becomes

$$\begin{aligned}
\dot{\rho} =& \frac{\varepsilon}{4}\rho\left(-\rho\sin(3\theta) + \sqrt{3}\rho\cos(3\theta) + 4z\right), \\
\dot{\theta} =& \frac{4\sqrt{3}}{5} + \frac{\varepsilon}{4}\left(\sqrt{3}(4z - \rho\sin(3\theta)) - \rho\cos(3\theta)\right), \\
\dot{z} =& \varepsilon\left(a_2 - 2\rho^2 + \frac{z^2}{2}\right).
\end{aligned}$$

This differential system can be reduced to the normal form for applying the averaging theory. Taking θ as the new independent variable we obtain the differential system

$$\begin{aligned}
\rho' =& \frac{5\rho\varepsilon\left(-\rho\sin(3\theta) + \sqrt{3}\rho\cos(3\theta) + 4z\right)}{16\sqrt{3}} + \mathcal{O}(\varepsilon^2), \\
z' =& \frac{5\varepsilon\left(2a_2 - 4\rho^2 + z^2\right)}{8\sqrt{3}} + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Here the derivatives are taken with respect to θ. Using (1.15) we write the functions $g_0 \equiv 0$ and $g_1(z) = \pi\left(\dfrac{5rz}{2\sqrt{3}}, \dfrac{5\left(2a_2 - 4r^2 + z^2\right)}{4\sqrt{3}}\right)$. The averaged function g_1 has the solutions $z_{\pm} = \pm\left(\omega\sqrt{\dfrac{a_2}{2}}, 0\right)$. The result follows by taking $s = 0$ and $z^* = z_+$ and applying Corollary 7. The eigenvalues of $Dg_1(z_+)$ are $\pm i5\sqrt{\dfrac{a_2}{6}}$. □

Proof of Theorem 26 statement (vi). First we translate to the origin of coordinates the point $(-10/43, -10/43, -10/43)$. Then using the change of variables $(x, y, z) = \varepsilon\big(X + Z, -\dfrac{X}{2} + \dfrac{\sqrt{3}Y}{2} + Z, -\dfrac{X}{2} - \dfrac{\sqrt{3}Y}{2} + Z\big)$ the differential system (4.19) writes

$$\begin{aligned}
\dot{X} =& -\sqrt{3}Y + \frac{\varepsilon}{40}\left(-13X^2 + 26\sqrt{3}XY + 86XZ + 13Y^2 + 86\sqrt{3}YZ\right), \\
\dot{Y} =& \sqrt{3}X + \frac{\varepsilon}{40}\left(13\sqrt{3}X^2 + 26XY - 86\sqrt{3}XZ - 13\sqrt{3}Y^2 + 86YZ\right),
\end{aligned} \tag{4.36}$$

$$\dot{Z} = -\frac{69}{43}Z + \frac{\varepsilon}{40}\left(40a_2 - 43\left(X^2+Y^2\right) + 52Z^2\right).$$

Using the cylindrical change of variables $(X,Y,Z) = \big(\rho\cos\theta, \rho\sin\theta, z\big)$ where $\rho > 0$, system (4.36) becomes

$$\begin{aligned}
\dot{\rho} =& \varepsilon\frac{\rho}{40}\left(13\sqrt{3}\rho\sin(3\theta) - 13\rho\cos(3\theta) + 86z\right),\\
\dot{\theta} =& \sqrt{3} + \frac{\varepsilon}{40}\left(13\rho\left(\sin(3\theta)+\sqrt{3}\cos(3\theta)\right) - 86\sqrt{3}z\right),\\
\dot{z} =& -\frac{69}{43}z + \varepsilon\left(a_2 + \frac{1}{40}\left(52z^2 - 43\rho^2\right)\right).
\end{aligned}$$

This differential system can be reduced to the normal form for applying the averaging theory. Taking θ as the new independent variable we obtain the differential system

$$\begin{aligned}
\rho' =& \frac{\varepsilon\rho}{40\sqrt{3}}\left(13\sqrt{3}\rho\sin(3\theta) - 13\rho\cos(3\theta) + 86z\right) + \frac{\varepsilon^2\rho}{4800}\Big(13\rho\Big(\cos(3\theta)\\
&-\sqrt{3}\sin(3\theta)\Big) - 86z\Big)\left(13\rho\left(\sin(3\theta)+\sqrt{3}\cos(3\theta)\right) - 86\sqrt{3}z\right) + \mathcal{O}(\varepsilon^3),\\
z' =& -\frac{23\sqrt{3}}{43}z + \frac{\varepsilon}{5160}\Big(43\sqrt{3}\left(40a_2 - 43\left(\rho^2+2z^2\right)\right) + 897\rho z\Big(\sin(3\theta)\\
&+\sqrt{3}\cos(3\theta)\Big)\Big) + \frac{\varepsilon^2}{206400}\Big(13\rho\Big(344\left(28z^2-5a_2\right)\left(\sin(3\theta)+\sqrt{3}\cos(3\theta)\right)\\
&+1849\rho^2\big(\sin(3\theta)+\sqrt{3}\cos(3\theta)\big) - 299\rho z\left(3\sin(6\theta)+\sqrt{3}\cos(6\theta)\right)\Big)\\
&-4\sqrt{3}z\big(41697\rho^2 + 79507z^2 - 36980a_2\big)\Big) + \mathcal{O}(\varepsilon^3).
\end{aligned} \tag{4.37}$$

Here the derivatives are taken with respect to θ. Using (1.15) we write the functions

$$\begin{aligned}
g_0(z) =& \left(0, \left(1 - e^{\frac{46\sqrt{3}\pi}{43}}\right)z\right),\\
g_1(z) =& \Bigg(\frac{1849}{1380}\left(1 - e^{-\frac{1}{43}\left(46\sqrt{3}\pi\right)}\right)\rho z, \Big(e^{\frac{46\sqrt{3}\pi}{43}}\left(40a_2 - 43\rho^2\right) - 86z^2\Big)\\
&\left(e^{\frac{46\sqrt{3}\pi}{43}} - 1\right)\frac{43e^{-\frac{1}{43}\left(92\sqrt{3}\pi\right)}}{2760}\Bigg),\\
g_2(z) =& \Bigg(\frac{1849e^{-\frac{1}{43}\left(184\sqrt{3}\pi\right)}\rho}{266008808721600}\Big(e^{\frac{184\sqrt{3}\pi}{43}}\Big(11640097\Big(240\left(46\sqrt{3}\pi - 43\right)a_2\\
&+19651z^2\Big) - 3003145026\left(46\sqrt{3}\pi - 43\right)\rho^2 + 45396995982\rho z\Big)\\
&+22989e^{\frac{138\sqrt{3}\pi}{43}}\left(130634\left(40a_2 - 43\rho^2\right) - 2456883\rho z\right) + 172180314824\\
&e^{\frac{46\sqrt{3}\pi}{43}}z^2 - 64567618059z^2 + 4557e^{\frac{92\sqrt{3}\pi}{43}}z(2432365\rho - 73810016z)\Big),\\
&\frac{e^{-\frac{1}{43}\left(230\sqrt{3}\pi\right)}}{4464148277863416508128800}\Big(e^{\frac{184\sqrt{3}\pi}{43}}\Big(-\Big(60\left(391\sqrt{3}\pi - 1849\right)a_2\\
&+75809z^2\Big)\,6719819923969996144z + 6970321671917181411381 6\rho^3
\end{aligned}$$

$$
\begin{aligned}
&+ 6786037724378612575161\,0\rho z^2 - 10446454479\rho^2 z \\
&\left(70137521867896\sqrt{3}\pi - 252393774165\right)\Big) - 4245723465 e^{\frac{138\sqrt{3}\pi}{43}} z \\
&(175587701015104 a_2 + 61893\rho(10033543\rho. + 666198742z)) \\
&-69703216719171814113816 e^{\frac{230\sqrt{3}\pi}{43}} \rho^3 + 40070454202131186256575 6z^3 \\
&+1951360992537417972 e^{\frac{92\sqrt{3}\pi}{43}} z^2(79833\rho + 329509z) \\
&-336382138219685488 e^{\frac{46\sqrt{3}\pi}{43}} z^2(144417\rho + 1588291z)\Big)\Bigg).
\end{aligned}
$$

Consider the graph $\mathcal{Z} = \{z_\alpha = (\alpha, \beta(\alpha)) : \beta(\alpha) = 0 \text{ and } \alpha > 0\}$. For all $\alpha > 0$ the function $g_0(z_\alpha) = (0, 0)$. Then taking $s = 0$ in Theorem 12 we compute the bifurcation functions $f_1(\alpha) = 0$, and

$$
f_2(\alpha) = \frac{1849 e^{-\frac{92\sqrt{3}\pi}{43}}}{3808800} \left(e^{\frac{92\sqrt{3}\pi}{43}} \left(46\sqrt{3}\pi - 43\right) + 86 e^{\frac{46\sqrt{3}\pi}{43}} - 43\right) \alpha \left(40 a_2 - 43\alpha^2\right).
$$

For $\alpha^* = 2\sqrt{\dfrac{10a_2}{43}}$ we have $f_2(\alpha^*) = 0$ and $Df_2(\alpha^*) \neq 0$. Thus applying statement (b) of Theorem 12 we have that system (4.37) has a periodic solution bifurcating from point z_α^*. Consequently, going back through the change of variables we have the existence of a periodic solution to system (4.19). $\square$

4.4 Appendix D: Functions H_i, I_i and J_i for $i = 1, 2$

Here we present the coordinate functions of g_2 that appears in the proof of statements (ii), (iii) and (iv) of Theorem 26.

$$
\begin{aligned}
H_1(z) = &-\frac{13541 a_2\rho}{14840704} + \frac{\pi a_2 \rho}{568\sqrt{7}} + e^{-\frac{32\pi}{\sqrt{7}}} \left(\frac{899 a_2 z^2}{40086032\rho} - \frac{597507923 a_2 z}{47083369573856}\right. \\
&\left.-\frac{703742848467097925\rho z^2}{31898417885852553728} + \frac{3392361662905917\rho^2 z}{116684360646408632}\right) - \frac{189618713 a_2 z^2}{10586680965168\rho} \\
&+ e^{-\frac{48\pi}{\sqrt{7}}} \left(-\frac{42083 a_2 z^2}{9319261413\rho} - \frac{46840313877851 z^3}{38980159313748576} + \frac{1820956889005487\rho z^2}{115614906197860608}\right) \\
&+ e^{-\frac{16\pi}{\sqrt{7}}} \left(\frac{13541 a_2\rho}{14840704} + \frac{4046213933 a_2 z}{6557112770432} - \frac{15991921\rho^3}{85334048} - \frac{23113407207573\rho^2 z}{195893744016656}\right) \\
&- \frac{47497703966367 a_2 z}{78588870014419072} + \frac{15991921\rho^3}{85334048} - \frac{4421\pi\rho^3}{3266\sqrt{7}} - \frac{16264 e^{-16\sqrt{7}\pi} z^4}{1048694353\rho} \\
&+ \frac{247945373 z^4}{13482014602168\rho} - \frac{19453239836 e^{-\dfrac{96\pi}{\sqrt{7}}} z^3}{118452839157093} + \frac{458771914355403749512 87\rho z^2}{29937738132656054895989 76} \\
&+ \frac{19660203760820886413604205145 z^3}{21759296009409979593131149751776} + \frac{8997643056406292948 7\rho^2 z}{10119201139144117734 56}
\end{aligned}
$$

$$+e^{-\frac{64\pi}{\sqrt{7}}}\left(-\frac{1739z^4}{74243848\rho}+\frac{8088305746441z^3}{6482324361591568}-\frac{112649333870939\rho z^2}{12499380049315264}\right)$$
$$+e^{-\frac{80\pi}{\sqrt{7}}}\left(\frac{306340z^4}{14913732967\rho}-\frac{6272329328873z^3}{7986095211566962}\right),$$

$$H_2(z)=e^{-\frac{32\pi}{\sqrt{7}}}\left(-\frac{20409a_2^2z}{806731394\rho^2}-\frac{13593232675a_2z^2}{5737473674128\rho}+\frac{8622181081a_2z}{2254007514836}\right.$$
$$\left.+\frac{121148255304709\rho z^2}{171407026014574}-\frac{9933070595407\rho^2z}{207368691364912}\right)+e^{-\frac{16\pi}{\sqrt{7}}}\left(\frac{20409a_2^2z}{806731394\rho^2}\right.$$
$$+\frac{17031a_2\rho}{927544}+\frac{12143784230186761a_2z^3}{3160966085684061608 80\rho^2}+\frac{354453085049431a_2z^2}{207977542894816233\rho}$$
$$-\frac{8622181081a_2z}{2254007514836}+\frac{97\pi a_2z}{3479\sqrt{7}}-\frac{54536545\rho^3}{14724761}-\frac{24457919708879017z^5}{595620109778216080404\rho^2}$$
$$-\frac{22419812466522030773228440146 8z^4}{1027937603490897359628321099718 11\rho}$$
$$-\frac{116238165545186323672338583z^3}{269053947687891207035914104 0}-\frac{2532617336107006414115738 27\rho z^2}{7400982594047367221717209 62}$$
$$\left.+\frac{9933070595407\rho^2z}{207368691364912}-\frac{157357637\pi\rho^2z}{17043621\sqrt{7}}\right)-\frac{17031a_2\rho}{927544}$$
$$+e^{-\frac{48\pi}{\sqrt{7}}}\left(-\frac{653124039a_2z^3}{9704487615928\rho^2}+\frac{4436646543253a_2z^2}{3790211250695408\rho}+\frac{269509153422410091z^3}{348888945626512306 4}\right.$$
$$\left.-\frac{56593678003503976\rho z^2}{102098815565607553}\right)+e^{-\frac{96\pi}{\sqrt{7}}}\left(-\frac{1004634a_2z^3}{132200451845\rho^2}-\frac{9002928z^5}{104396130769\rho^2}\right.$$
$$\left.+\frac{16223479764z^4}{4620818852783\rho}+\frac{42924541078z^3}{14145448347415}\right)+e^{-\frac{80\pi}{\sqrt{7}}}\left(-\frac{93459a_2z^3}{39052264048\rho^2}\right.$$
$$\left.+\frac{15516419z^5}{267908925508\rho^2}-\frac{70253877690368467z^4}{12331001516837560228\rho}+\frac{1348260373986075z^3}{218739150863017 12}\right)$$
$$+e^{-\frac{64\pi}{\sqrt{7}}}\left(\frac{222327526a_2z^3}{5718920087111\rho^2}-\frac{227003953a_2z^2}{448946020761\rho}+\frac{654448536 65z^4}{19753624913484\rho}\right.$$
$$\left.-\frac{109952417460403592z^3}{1113807488504463291}+\frac{703608436890712\rho z^2}{3708743077506621}\right)+\frac{54536545\rho^3}{14724761}$$
$$+\frac{3232536e^{-\frac{128\pi}{\sqrt{7}}}z^5}{51386023297\rho^2}+e^{-16\sqrt{7}\pi}\left(\frac{210277z^5}{32463122538\rho^2}+\frac{874257394784z^4}{829169874099651\rho}\right),$$

$$I_1(z)=a_2\left(\frac{16\left(14535\sqrt{2}\pi-2756\right)\rho}{4142475}+\frac{2310379625\rho^3z^2}{61603278561}-\frac{19807570229\sqrt{2}\rho^2z}{584130307695}\right)$$
$$+\left(383757256080000a_2-\rho^2\left(1353367360104125\sqrt{2}\rho z+257897855245984\right)\right)$$
$$\frac{e^{-\frac{15\pi}{2\sqrt{2}}}\rho^3z^2}{50519616682304880}-\frac{e^{-\frac{5\pi}{\sqrt{2}}}\rho^2z}{18475687048500}\left(\left(251929337505\rho z+178892943548\sqrt{2}\right)\right.$$
$$\left.\rho^2-392700a_2\left(846537\sqrt{2}-2121875\rho z\right)\right)+\frac{8e^{-\frac{5\pi}{2\sqrt{2}}}\rho}{207306039612375}$$

$$\left(8415a_2\left(49013145\sqrt{2}\rho z+32779864\right)-848\rho^2\left(810462369\sqrt{2}\rho z+238117880\right)\right)$$
$$-\frac{8}{633798675}\left(1904085\sqrt{2}\pi-617344\right)\rho^3+\frac{20581329390625\rho^7z^4}{886732746729984}$$
$$-\frac{1734375e^{-\frac{35\pi}{2\sqrt{2}}}\rho^7z^4}{72939328}+\frac{775e^{-\frac{25\pi}{2\sqrt{2}}}\rho^6z^3\left(17274375\rho z+10098404\sqrt{2}\right)}{374185614528}$$
$$+\frac{25e^{-\frac{15\pi}{\sqrt{2}}}\rho^6z^3\left(376440625\rho z-293279662\sqrt{2}\right)}{2259931934304}+\frac{1929033295603982130425\rho^6z^3}{5331222913322575831953 6\sqrt{2}}$$
$$+\frac{12415915880034901\rho^5z^2}{926192972508922800}+\frac{114200436283\sqrt{2}\rho^4z}{3154303661553}+\frac{e^{-5\sqrt{2}\pi}\rho^5z^2}{297952783603200}$$
$$\left(1589668853472-625\rho z\left(18770709375\rho z+4278131824\sqrt{2}\right)\right),$$

$$I_2(z)=e^{-\frac{5\pi}{2\sqrt{2}}}\left(\frac{40832a_2^2z}{1049427\rho^2}+a_2\left(-\frac{49408\sqrt{2}}{828495\rho}-\frac{252934685856925\rho^2z^3}{2721386433710736}\right.\right.$$
$$\left.+\frac{20064998592791\sqrt{2}\rho z^2}{450364467232845}+\left(\frac{9646\sqrt{2}\pi}{165699}-\frac{925690614976}{7101967672845}\right)z\right)+\frac{2908160\sqrt{2}\rho}{51863787}$$
$$-\frac{5527550890625\rho^6z^5}{98525860747776}-\frac{347543633091000 9942475\rho^5z^4}{473886481184228962840 32\sqrt{2}}$$
$$-\frac{115148537639610717947\rho^4z^3}{881078112818013170412 00}-\frac{196462205901224732\sqrt{2}\rho^3z^2}{4154612210222995125}+\left(-\frac{2734\sqrt{2}\pi}{43605}\right.$$
$$\left.\left.+\frac{18720891867514208}{158821303067832735}\right)\rho^2z\right)-\frac{4e^{-\frac{5\pi}{\sqrt{2}}}z}{3970532576695818375\rho^2}\left(38622216259836000\right.$$
$$a_2^2-5350a_2\rho^2\left(13391448585291\sqrt{2}\rho z+24183667316248\right)$$
$$\left.+\rho^4\left(68542401952527177\sqrt{2}\rho z+117005574171963800\right)\right)$$
$$+\frac{e^{-\frac{25\pi}{2\sqrt{2}}}}{629011432051200}\rho^2z^3\left(162501300000a_2+\rho^2\left(35625\rho z\left(2487684375\rho z\right.\right.\right.$$
$$\left.\left.\left.+1986367162\sqrt{2}\right)+10845743592224\right)\right)+\frac{256\sqrt{2}\left(60409a_2-56800\rho^2\right)}{259318935\rho}+e^{-\frac{15\pi}{\sqrt{2}}}$$
$$\left(-\frac{\rho^4z^3\left(2378225\rho z\left(2055650625\rho z+1197920846\sqrt{2}\right)+63652108192902\right)}{25213764854807496}\right.$$
$$\left.+\frac{72275a_2\rho^2z^3}{2990163}\right)+\frac{e^{-5\sqrt{2}\pi}\rho z^2}{258913035496812510 00}\left(\rho^2\left(205\rho z\left(-1072261547055142\right.\right.\right.$$
$$\left.\left.+5052665015015625\sqrt{2}\rho z\right)+21417809054549248\sqrt{2}\right)-1648200a_2$$
$$\left.\left(2177119524375\rho z+379694102864\sqrt{2}\right)\right)+\frac{e^{-\frac{15\pi}{2\sqrt{2}}}\rho z^2}{3695137409700}\left(765285440625\right.$$
$$\left.a_2\rho z-342014627020\sqrt{2}a_2+426831261472\sqrt{2}\rho^2-18184717070\rho^3z\right)$$

$$+\frac{12671875e^{-10\sqrt{2}\pi}\rho^6 z^5}{109408992}-\frac{25e^{-\frac{35\pi}{2\sqrt{2}}}\rho^5 z^4\left(4858486875\rho z+2058207116\sqrt{2}\right)}{18079455474432},$$

$$J_1(z)=\frac{e^{-\frac{12\pi}{\sqrt{5}}}z^2}{3397781520000\rho}\Big(\rho\Big(21\Big(2005165908\sqrt{5}+3889750015\Big)\rho$$
$$-2465\Big(3703027\sqrt{5}+5698606\Big)z\Big)-6211800\Big(193\sqrt{5}-210\Big)a_2\Big)$$
$$\frac{17722324487957040\rho^2}{5742033134098080960000\rho}\Big(180\pi\sqrt{5}\left(8a_2-19\rho^2\right)-10\Big(123\sqrt{5}+520\Big)a_2$$
$$+\Big(956\sqrt{5}+5625\Big)\rho^2\Big)-1212044225z^2\Big(962336\Big(4577\sqrt{5}+1890\Big)a_2$$
$$-3\Big(10806179752\sqrt{5}+24675585383\Big)\rho^2\Big)+325976240\rho z\Big(3\Big(83295503384\sqrt{5}$$
$$+95313187411)\rho^2-10105\Big(13907113\sqrt{5}+9542762\Big)a_2\Big)+\Big(7\sqrt{5}+2\Big)z^4$$
$$156788319704681160+8383108\Big(1115434951621\sqrt{5}+1966086632830\Big)\rho z^3$$
$$+\frac{e^{-\frac{8\pi}{\sqrt{5}}}z}{2340848160000\rho}\Big(37600a_2\Big(12\Big(14879\sqrt{5}+149\Big)\rho+79849\sqrt{5}z\Big)+3\rho^2$$
$$\Big(-47\Big(238651676\sqrt{5}+506116445\Big)z+240\Big(30405896\sqrt{5}+66489385\Big)\rho\Big)\Big)$$
$$+\frac{e^{-\frac{4\pi}{\sqrt{5}}}}{6144012000}\Big(215a_2\Big(882\Big(123\sqrt{5}+520\Big)\rho+\Big(146029\sqrt{5}+155618\Big)z\Big)$$
$$-3\rho^2\Big(6321\Big(956\sqrt{5}+5625\Big)\rho+\Big(48206680\sqrt{5}+75128363\Big)z\Big)\Big)$$
$$-\frac{289\left(7\sqrt{5}+10\right)e^{-\frac{28\pi}{\sqrt{5}}}z^4}{10584000\rho}+\frac{17e^{-4\sqrt{5}\pi}z^3}{2302020000\rho}\Big(4930\Big(7\sqrt{5}+6\Big)z$$
$$-3\Big(107371\sqrt{5}+148485\Big)\rho\Big)-\frac{17\left(3621\sqrt{5}+6202\right)e^{-\frac{24\pi}{\sqrt{5}}}z^3}{168966000}+\frac{e^{-\frac{16\pi}{\sqrt{5}}}z^2}{12519360000\rho}$$
$$\Big(-23\Big(2645856\sqrt{5}+4997225\Big)\rho^2-3190560\sqrt{5}z^2+13600\Big(3499\sqrt{5}+4766\Big)\rho z\Big),$$

$$J_2(z)=\frac{e^{-\frac{4\pi}{\sqrt{5}}}}{3129408058083454123200000\rho^2}\Big(533259729244256z\Big(40250\Big(5\sqrt{5}+36\Big)$$
$$a_2^2+690\Big(2268\pi\sqrt{5}-2525\sqrt{5}+27760\Big)a_2\rho^2+3\Big(4223690\sqrt{5}-17388\Big(379\sqrt{5}$$
$$+410\Big)\pi+7544115\Big)\rho^4\Big)+315385039867317120\rho^3\Big(735\Big(35-11\sqrt{5}\Big)a_2+\rho^2\Big)$$
$$\Big(14247\sqrt{5}-32740\Big)-1509860806z^3\Big(\Big(295448712792\sqrt{5}+3357245264653\Big)\rho^2$$
$$+300730\Big(723208\sqrt{5}-1530333\Big)a_2\Big)-15071648\rho z^2\Big(a_2 464830\Big(14106353\sqrt{5}$$
$$-298387900\Big)+21\Big(16273193039599\sqrt{5}+23844169989280\Big)\rho^2\Big)+\Big(1583932\sqrt{5}$$
$$-3899715\Big)50341483586103z^5+\Big(1249367048959\sqrt{5}-5521411126760\Big)\rho z^4\Big)$$

$$234727024-\frac{e^{-\frac{8\pi}{\sqrt{5}}}z}{353280690000\rho^2}\left(24230500\left(5\sqrt{5}+36\right)a_2^2+14835a_2\rho\left(140\left(5552\right.\right.\right.$$
$$\left.-505\sqrt{5}\right)\rho+\left(31424\sqrt{5}+72755\right)z\Big)+21\rho^3\left(430\left(844738\sqrt{5}+1508823\right)\rho\right.$$
$$\left.\left.-23\left(30613883\sqrt{5}+55472705\right)z\right)\right)+\frac{\rho}{992250}\left(735\left(11\sqrt{5}-35\right)a_2+\left(32740\right.\right.$$
$$\left.\left.-14247\sqrt{5}\right)\rho^2\right)+\frac{e^{-\frac{12\pi}{\sqrt{5}}}z^2}{2048242140000\rho^2}\left(235a_2\left(42\left(272816\sqrt{5}+172415\right)\rho\right.\right.$$
$$\left.+5185\left(2630\sqrt{5}-6993\right)z\right)+6\rho^2\Big(47\left(26214378\sqrt{5}+238047761\right)z-35\rho$$
$$\left.\left.\left(300370783\sqrt{5}+736870135\right)\right)\right)-\frac{e^{-\frac{16\pi}{\sqrt{5}}}z^2}{5582069640000\rho^2}\Big(13340a_2\Big(14\Big(133175$$
$$\left.-9586\sqrt{5}\right)\rho+765\left(217\sqrt{5}-1104\right)z\Big)+\rho\left(-10440\rho^2\left(2887157\sqrt{5}+12737871\right)\right.$$
$$\left.\left.+79373\left(94964\sqrt{5}-408025\right)z^2+46\left(322207806\sqrt{5}+5283539111\right)\rho z\right)\right)$$
$$+\frac{17e^{-\frac{24\pi}{\sqrt{5}}}z^3}{6906060000\rho^2}\left(406\left(5\left(17\sqrt{5}-72\right)a_2+3\left(51\sqrt{5}+602\right)\rho^2\right)+8874\Big(14\sqrt{5}\right.$$
$$\left.-45\Big)z^2+1575\left(1677-274\sqrt{5}\right)\rho z\right)+\frac{e^{-4\sqrt{5}\pi}z^3}{4776135840000\rho^2}\Big(5014\Big(\Big(303480\sqrt{5}$$
$$+23965123\Big)\rho^2-2550\left(76\sqrt{5}-105\right)a_2\Big)-1102535\left(2092\sqrt{5}-4827\right)z^2+476$$
$$\left(17544628\sqrt{5}-103898005\right)\rho z\Big)+\frac{289e^{-\frac{32\pi}{\sqrt{5}}}\left(24-7\sqrt{5}\right)z^5}{10584000\rho^2}+\frac{17e^{-\frac{28\pi}{\sqrt{5}}}z^4}{277104240000\rho^2}$$
$$\left(12665\left(146\sqrt{5}-205\right)z-10496\left(424\sqrt{5}-3405\right)\rho\right).$$

Chapter 5

Lorenz and Thomas' differential systems

In this chapter we use Theorem 12 for studying the periodic solutions that emerges from a zero-Hopf bifurcation in the Lorenz differential system. Then we apply Theorem 13 for describing the stability of a such periodic solution. The same strategy will be used here for studying the Hopf and zero-Hopf bifurcation that occurs in two circulant systems, one of then known as Thomas' differential system. The results here presented where published in [14].

5.1 Application to Lorenz differential system

The Lorenz system of differential equations in $\mathbb{R}^3$ arose from the work of the meteorologist mathematician Edward N. Lorenz [55], who studied forced dissipative hydro-dynamical systems. It has become one of the most widely studied systems of ODEs because of its wide range of behaviors (see for instance [75]). Although the origins of this system lies in atmospheric modeling, the Lorenz equations also appear in other areas as in the modeling of lasers see [37], and dynamos see [44]. The *Lorenz equations* are

$$\begin{aligned} \dot{x} =& a(x-y), \\ \dot{y} =& x(b-z)-y, \\ \dot{z} =& x\,y-cz, \end{aligned} \tag{5.1}$$

with a, b, c being real coefficients.

Theorem 28. *Let $a=-1+a_2\varepsilon^2$ and $c=c_1\varepsilon$. Assume that $b>1$, $a_2<0$, $c_1\neq 0$ and $|\varepsilon|\neq 0$ sufficiently small. Then the Lorenz differential system* (5.1) *has a periodic orbit bifurcating from the origin. Furthermore for $c_1>0$ this periodic orbit is an attractor, otherwise for $c_1<0$ the periodic orbit has a stable manifold formed by two topological cylinders and an unstable manifold formed by two topological cylinders.*

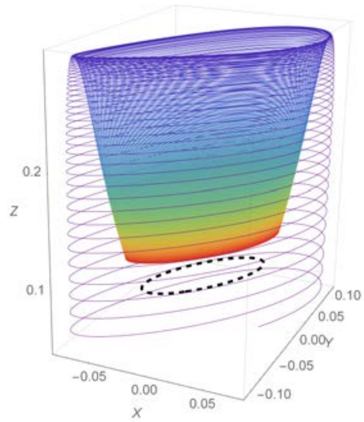


Figure 5.1: Solution of system (5.1) starting at $(0.05, -0.01, 0.05)$ being attracted by the stable periodic orbit (dashed curve) founded by Theorem 12 . The parameters of the system are $a_2 = -2$, $b = 2$ $c_1 = 1$ and $\varepsilon = 1/100$.

5.2 Application to Thomas' differential system

A circulant system is a differential system defined by a function $f(x, y, z)$ having the variables cyclically symmetric according to

$$\begin{aligned} \dot{x} &= f(x, y, z), \\ \dot{y} &= f(y, z, x), \\ \dot{x} &= f(z, x, y), \end{aligned}$$

where the function $f(u, v, w)$ is fixed and the variables are rotated. In 1999 René Thomas proposed the following two circulant systems having cyclic symmetry

$$\begin{aligned} \dot{x} &= \sin y - \beta x, \\ \dot{y} &= \sin z - \beta y, \\ \dot{z} &= \sin x - \beta z, \end{aligned} \qquad (5.2) \qquad \begin{aligned} \dot{x} &= -bx + ay - y^3, \\ \dot{y} &= -by + az - z^3, \\ \dot{z} &= -bz + ax - x^3. \end{aligned} \qquad (5.3)$$

System (5.2) is defined by the function $f(u, v, w) = -au + \sin v$ and system (5.3) is defined by $f(u, v, w) = -au + bv - v^3$. The chaotic behavior generated by these systems was presented by [79]. These systems were also studied in [77]. System (5.2) is sometimes called Thomas' system, see for instance [75, Chapter 3]. The next results give sufficient conditions for the existence of periodic solutions in these differential systems.

One can check that the origin is an equilibrium point of system (5.2), and that it has the eigenvalues $1 - \beta$, $(-1 - 2\beta - i\sqrt{3})/2$ and $(-1 - 2\beta + i\sqrt{3})/2$. When $\beta = -1/2$ the origin has a pair of complex eigenvalues on the imaginary axis and the bifurcation of a periodic orbit occurs.

Theorem 29. *Let $\beta = -1/2 + \beta_1\varepsilon + \beta_2\varepsilon^2$ where $\beta_i \in \mathbb{R}$ for $i = 1, 2$. For $\varepsilon > 0$ sufficiently small and $\beta_1 > 0$ the differential system* (5.2) *has an isolated periodic solution bifurcating from the origin. Moreover this periodic solution is unstable.*

System (5.3) has 27 steady states but we will be interested into the pair symmetric equilibrium points $\mathbf{P}_\pm = \pm\left(\sqrt{a-b}, \sqrt{a-b}, \sqrt{a-b}\right)$. Taking $a = 5\sqrt{3}\omega/6$ and $b = \sqrt{3}\omega/3$ with $\omega > 0$, these equilibrium points have the eigenvalues $-\sqrt{3}\omega$ and $\pm\omega i$. The next theorems show that periodic orbits born at $\mathbf{P}_-$ and $\mathbf{P}_+$.

Theorem 30. *Let $a = 5\sqrt{3}\omega/6 + \varepsilon a_1$, $b = \sqrt{3}\omega/3 + \varepsilon b_1$ with $\omega > 0$ and $(5b_1 - 2a_1) < 0$. Then for $\varepsilon > 0$ sufficiently small the differential system* (5.3) *has the two periodic solutions*

$$\begin{aligned}\phi_\pm(t,\varepsilon) =& \mathbf{P}_\pm + \sqrt{\varepsilon}\Big(2e^{2\sqrt{3}\pi}\xi\cos(t\omega), \frac{e^{2\sqrt{3}\pi}}{3}\xi\left(3\sin(t\omega) - \sqrt{3}\cos(t\omega)\right), \\ &-\frac{1}{3}e^{2\sqrt{3}\pi}\xi\left(3\sin(t\omega) + \sqrt{3}\cos(t\omega)\right)\Big) + \mathcal{O}(\varepsilon), \end{aligned} \tag{5.4}$$

such that $\phi_+(t,\varepsilon)$ bifurcates from $\mathbf{P}_+$, and $\phi_-(t,\varepsilon)$ bifurcates from $\mathbf{P}_-$. Here

$$\xi = \sqrt{\frac{\pi(5b_1 - 2a_1)}{-3e^{4\sqrt{3}\pi}\left(\sqrt{3} - 5\pi\right) + 6\sqrt{3}e^{2\sqrt{3}\pi} - 3\sqrt{3}}}.$$

The periodic orbit analytically found in Theorem 30 was detected numerically in [79]. In this paper it is shown that for specific values of a and b these periodic solutions produce to a strange attractors after a cascade of doubling. The following figures illustrate this phenomena. In these figures $a_1 = 6$, $b_1 = 1$ and $\omega = 1$, and the time interval varies from 0 to 1000. Figure 5.2a shows the solution starting at $(-0.8, -0.8, -0.45)$ being attracted by the periodic orbit $\phi_-(t,\varepsilon)$, see equation (5.4). As we increase ε the periodic orbit grows in size and complexity, see Figures 5.2b, 5.2c. The approximation to the periodic orbit provided by (5.4) can be seen as a dashed curve. Figures 5.2d, 5.2e and 5.2f shows the appearance of the strange attractor as ε increase.

5.3 Proofs

5.3.1 Proof of Theorem 28

Proof. The existence of a such periodic orbit is proved in Theorem 4 of [13]. Following the ideas of this proof we see that, after some changes of variables, system (5.1) can be put into the normal form for applying Theorem 12, i.e.

$$\dot{z} = \varepsilon F_1\big(z,\theta\big) + \varepsilon^2 F_2\big(z,\theta\big) + \varepsilon^3 F_3\big(z,\theta\big) + \mathcal{O}(\varepsilon^4),$$

given by equation (22) of [13], with $z = (\rho, z)$ and the derivative is with respect to θ. Thus calculating the higher order averaging functions of this system for $i = 0,1,2,3$ we have $g_i(z) = (g_{i1}(z), g_{i2}(z))$ where $g_0(z) \equiv 0$ and

$$\begin{aligned} g_{11}(z) =& 0, \\ g_{12}(z) =& \frac{\pi\left(\rho^2 - 2c_1 z\right)}{\omega}, \\ g_{21}(z) =& -\frac{\pi\rho\left(8a_2\omega^2 - 4c_1 z + 3\rho^2\right)}{8\omega^3}, \end{aligned}$$

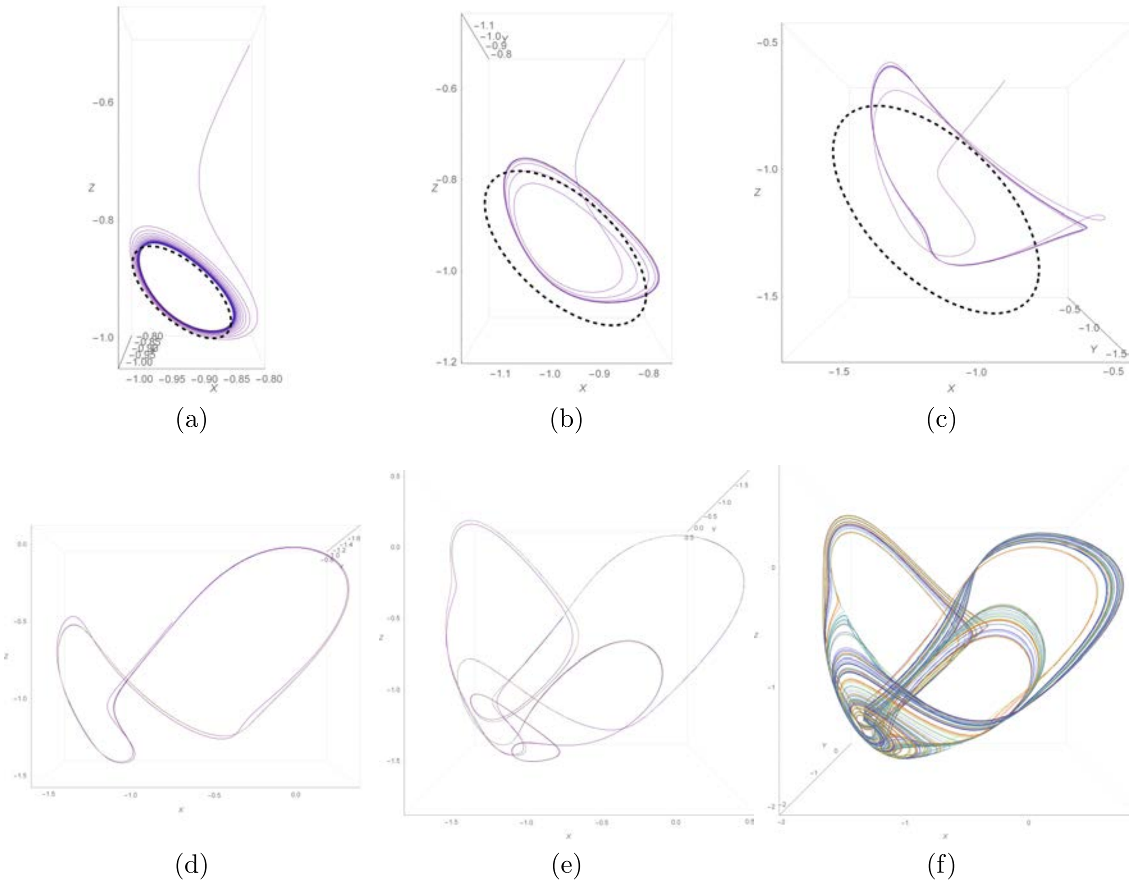


Figure 5.2: Solution $\phi_{-}(t,\varepsilon)$ for different values of ε: (a) $\varepsilon = 1/250$, (b) $\varepsilon = 1/50$, (c) $\varepsilon = 1/8$, (d) $\varepsilon = 1/6$, (e) $\varepsilon = 1/5$, (f) $\varepsilon = 1/4$

$$
\begin{aligned}
g_{22}(z) =& \frac{\pi\left(\rho^2(c_1\omega(\omega-2\pi)+3z)+2c_1z(2\pi c_1\omega-z)\right)}{2\omega^3},\\
g_{31}(z) =& -\frac{\pi\rho\left(4z\left(2a_2\omega^2+2\pi c_1^2\omega-3c_1z\right)+\rho^2(c_1\omega(3\omega-4\pi)+15z)\right)}{16\omega^5},\\
g_{32}(z) =& \frac{\pi}{96\omega^5}\Big(9\rho^4\omega(4\pi-5\omega)-8c_1z\left(12a_2\omega^4+16\pi^2c_1^2\omega^2-36\pi c_1\omega z+9z^2\right)+4\rho^2\\
& \left(3c_1\omega(9\omega-28\pi)z+45z^2\right)-2\omega^2\left(6a_2\omega(\omega+2\pi)+c_1^2\left(6\pi\omega-8\pi^2+3\right)\right)\Big).
\end{aligned}
$$

Thus we can calculate the functions $f_i(\alpha)$ for $i=1,2$ with respect to the above averaging functions and the graph

$$
\mathcal{Z}=\left\{z_\alpha=\left(\alpha,\beta(\alpha)=\frac{\alpha^2}{2c_1}\right):\alpha>0\right\},
$$

obtaining

$$
f_1(\alpha)=-\frac{\pi\alpha\left(8a_2\omega^2+\alpha^2\right)}{8\omega^3}\quad\text{and}\qquad f_2(\alpha)=-\frac{\pi\alpha^3\left(2\omega^2\left(4a_2+c_1^2\right)+5\alpha^2\right)}{32c_1\omega^5}.
$$

Under the hypothesis of Theorem 28 one can check that $\alpha^*=2\omega\sqrt{-2a_2}$ is a simple zero of the function $f_1(\alpha)$. Then we can apply Theorem 12 with $s=1$. By Proposition 11 and Lemma 8 we can write the initial point of the periodic solution as $z(\varepsilon)=z_{\alpha^*}+\varepsilon z_1$ with

$$
z_1=\left(\frac{(16a_2-c_1^2)\omega\sqrt{-2a_2}}{2c_1},4a_2\omega^2\left(\frac{12a_2}{c_1^2}-1\right)\right),
$$

and the matrix (1.25) becomes

$$
A(\varepsilon)=\begin{pmatrix}0 & 0\\ 4\pi\sqrt{-2a_2} & -\dfrac{2c_1\pi}{\omega}\end{pmatrix}+\varepsilon\begin{pmatrix} & \dfrac{6a_2\pi}{\omega} & \dfrac{\sqrt{-2a_2}c_1\pi}{\omega^2}\\ \dfrac{\pi\sqrt{-2a_2}}{\omega c_1} & (c_1^2(\omega-4\pi)-8a_2\omega) & \dfrac{2\pi(c_1^2\pi-2a_2\omega)}{\omega^2}\end{pmatrix}.
$$

The matrix $A(\varepsilon)$ has the two distinct eigenvalues

$$
\lambda_1=-\frac{2c_1\pi}{\omega}+\varepsilon\left(\frac{2c_1\pi}{\omega}\right)^2+\mathcal{O}(\varepsilon^2)\ \text{ and }\lambda_2=\varepsilon\frac{2a_2\pi}{\omega}+\mathcal{O}(\varepsilon^2).
$$

Thus we can apply Theorem 13 taking $s=1$. Since a_2 is negative by hypothesis, we have that for $\varepsilon>0$ sufficiently small if $c_1>0$, $\mathbf{Re}(\lambda_1)<\mathbf{Re}(\lambda_2)<0$, then the periodic orbit is an attractor. Otherwise, if $c_1<0$, $\mathbf{Re}(\lambda_2)<0<\mathbf{Re}(\lambda_1)$, then the periodic orbit has a stable manifold formed by two topological cylinders, and an unstable manifold formed by two topological cylinders. □

5.3.2 Proof of Theorem 29

Proof. Using the change of variables $(X,Y,Z)=\sqrt{\varepsilon}\big(x+z,(-x-\sqrt{3}y+2z)/2,(-x+\sqrt{3}y+2z)/2\big)$ the differential system (5.2) becomes

$$
\dot{X}=X\left(\frac{1}{2}-\beta_2\varepsilon^2\right)-\frac{\sin\left(\sqrt{\varepsilon}\left(-X+\sqrt{3}Y+2Z\right)/2\right)}{3\sqrt{\varepsilon}}+\frac{\sin(\sqrt{\varepsilon}(X+Z))}{3\sqrt{\varepsilon}}
$$

$$
\begin{aligned}
&+\frac{2\sin\left(\sqrt{\varepsilon}\left(X+\sqrt{3}Y-2Z\right)/2\right)}{3\sqrt{\varepsilon}},\\
\dot{Y} =& Y\left(\frac{1}{2}-\beta_2\varepsilon^2\right)+\frac{\sin(\sqrt{\varepsilon}(X+Z))}{\varepsilon\sqrt{3}}-\frac{\sin\left(\sqrt{\varepsilon}\left(-X+\sqrt{3}Y+2Z\right)/2\right)}{\sqrt{\varepsilon}\sqrt{3}},\\
\dot{Z} =& Z\left(\frac{1}{2}-\beta_2\varepsilon^2\right)-\frac{\sin\left(\sqrt{\varepsilon}\left(X+\sqrt{3}Y-2Z\right)/2\right)}{3\sqrt{\varepsilon}}+\frac{\sin(\sqrt{\varepsilon}(X+Z))}{3\sqrt{\varepsilon}}\\
&+\frac{\sin\left(\sqrt{\varepsilon}\left(-X+\sqrt{3}Y+2Z\right)/2\right)}{3\sqrt{\varepsilon}}.
\end{aligned}
\tag{5.5}
$$

We remark that for all $\delta \in \mathbb{R}$ the function $\sin(\delta\, w)/\delta$ is well defined and

$$
\lim_{\delta\to 0}\frac{\sin(\delta\, w)}{\delta}=w.
$$

Thus the above equation can also be written as

$$
\begin{aligned}
\dot{X} =& -\frac{\sqrt{3}}{2}Y+\frac{\varepsilon}{16}\Big(X^3+X^2\left(\sqrt{3}Y+2Z\right)+X\left(Y^2-4\sqrt{3}YZ+4\left(Z^2-4\beta\right)\right)\\
&+Y\left(\sqrt{3}Y^2-2YZ+4\sqrt{3}Z^2\right)\Big)+\mathcal{O}(\varepsilon^2),\\
\dot{Y} =& \frac{\sqrt{3}}{2}X+\frac{\varepsilon}{16}\Big(-\sqrt{3}X^3+X^2\left(Y-2\sqrt{3}Z\right)-X\left(\sqrt{3}Y^2+4YZ+4\sqrt{3}Z^2\right)\\
&+Y\left(Y^2+2\sqrt{3}YZ+4\left(Z^2-4\beta\right)\right)\Big)+\mathcal{O}(\varepsilon^2),\\
\dot{Z} =& \frac{3}{2}Z+\frac{\varepsilon}{24}\Big(-X^3-6X^2Z+3XY^2-2Z\left(3Y^2+2\left(6\beta+Z^2\right)\right)\Big)+\mathcal{O}(\varepsilon^2).
\end{aligned}
$$

In order to put the differential system (5.5) into the normal form for applying the averaging theory we consider the cylindrical change of variables $(X,Y,Z)=\left(\rho\cos\theta,\rho\sin\theta,w\right)$ with $\rho>0$. Then we check that $\dot{\theta}=\sqrt{3}/2+\mathcal{O}(\varepsilon^2)$ for $|\varepsilon|\neq 0$ sufficiently small. Thus taking θ as the new independent variable we obtain the differential system

$$
\dot{z}=F_0(z,\theta)+\varepsilon F_1(z,\theta)+\varepsilon^2F_2(z,\theta)+\mathcal{O}(\varepsilon^3),
\tag{5.6}
$$

with $z=(\rho,w)$, $F_0(z,\theta)=\left(0,\sqrt{3}w\right)$, and $F_i(z,\theta)=\left(F_{i1}(z,\theta),F_{i2}(z,\theta)\right)$ for $i=1,2$, where

$$
\begin{aligned}
F_{11}(z,\theta) =& \frac{\rho}{8\sqrt{3}}\left(\rho^2+2\rho w\left(\cos(3\theta)-\sqrt{3}\sin(3\theta)\right)+4\left(w^2-4\beta\right)\right),\\
F_{12}(z,\theta) =& \frac{1}{72}\Big(w\left(\sqrt{3}\left(-48\beta-3\rho^2+28w^2\right)+18rw\sin(3\theta)\right)\\
&-2\sqrt{3}\rho\cos(3\theta)\left(\rho^2-9w^2\right)\Big),\\
F_{21}(z,\theta) =& \frac{\rho}{5760}\Big(\rho\Big(-30w\sin(3\theta)\left(32\beta+\rho^2+8w^2\right)+10\sqrt{3}w\cos(3\theta)\\
&\left(-96\beta+7\rho^2+40w^2\right)+3\rho\sin(6\theta)\left(\rho^2-40w^2\right)\\
&-\sqrt{3}\rho\cos(6\theta)\left(\rho^2-120w^2\right)\Big)+20\sqrt{3}\left(-192\beta_2+\rho^4+6\rho^2\left(w^2-4\beta\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ 20w^4 - 96\beta w^2)\Big),\\
F_{22}(z,\theta) =&\frac{1}{5760\sqrt{3}}\Big(- 12w\left(960\beta_2 + 5\rho^4 + 120\beta\left(\rho^2+4w^2\right) - 50\rho^2w^2 - 228w^4\right)\\
&- 30\rho\cos(3\theta)\left(\rho^4 + 3\rho^2w^2 - 104w^4 + 96\beta w^2\right)\\
&+ \rho w\Big(\cos(6\theta)\left(360\rho w^2 - 69\rho^3\right) + 2\sqrt{3}\sin(3\theta)\left(\cos(3\theta)\left(360\rho w^2 - 23\rho^3\right)\right.\\
&+ 5w\left(-96\beta + 3\rho^2 + 104w^2\right)\Big)\Big)\Big).
\end{aligned}
$$

System (5.6) is 2π-periodic and it is into the normal form for applying Theorem 12. Furthermore for the initial condition $z_0 = (\rho_0, w_0)$ the solution of the unperturbed differential system corresponding to (5.6) is given by $\Phi(\theta, z) = \left(\rho_0, w_0 e^{\sqrt{3}\theta}\right)$. Then we consider the set $\mathcal{Z} \subset \mathbb{R}^2$ such that $\mathcal{Z} = \{(\alpha, 0) : \alpha > 0\}$. Clearly for $z_\alpha \in \mathcal{Z}$ the solution $\Phi(\theta, z_\alpha)$ can be assumed 2π-periodic, and therefore the differential system (5.6) satisfies hypothesis (H). Moreover the fundamental matrix of the variational differential system along $\Phi(\theta, z_\alpha)$ is

$$
M(\theta, z_\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\sqrt{3}\theta}\theta \end{pmatrix}.
$$

Computing the averaging functions we obtain $g_0(z) = \left(0, (e^{2\pi\sqrt{3}} - 1)w\right)$ and $g_i(z) = (g_{i1}(z), g_{i2}(z))$ for $i = 1, 2$ where

$$
\begin{aligned}
g_{11}(z) =&\frac{\rho}{12}\Big(\sqrt{3}\pi\left(\rho^2 - 16\beta\right) + \left(e^{2\sqrt{3}\pi} - 1\right)w\left(\rho + e^{2\sqrt{3}\pi}w + w\right)\Big),\\
g_{12}(z) =&\frac{1}{144}\Big(\rho^3 - e^{2\sqrt{3}\pi}\left(\rho^3 + 12\sqrt{3}\pi\rho^2 w + 28w^3 + 192\sqrt{3}\pi\beta w\right) + 28e^{6\sqrt{3}\pi}w^3\Big),\\
g_{21}(z) =&\frac{\left(1 + 16\sqrt{3}\pi + 54\pi^2\right)\rho^5}{1728} + \frac{e^{14\sqrt{3}\pi}\rho w^4}{108} + \frac{e^{10\sqrt{3}\pi}\rho w^3(15\rho - 196w)}{15120}\\
&+ \frac{e^{12\sqrt{3}\pi}\rho w^3(21\rho + 13w)}{5616} + \frac{e^{8\sqrt{3}\pi}\rho w^2\left(171\rho^2 - 700\rho w + 3192\left(w^2 - \beta\right)\right)}{229824}\\
&+ \frac{e^{2\sqrt{3}\pi}\left(\rho^2 w\left(288\beta + 48\sqrt{3}\pi\left(\rho^2 - 16\beta\right) - 19\rho^2\right) - 2\rho^5\right)}{3456} + \frac{467\rho^4 w}{169344}\\
&- \frac{5\rho^2\left(3815w^3 + 28652\beta w\right)}{1742832} - \frac{\rho^3\left(3024\pi\left(\sqrt{3} + 4\pi\right)\beta + 115w^2\right)}{18144}\\
&+ \frac{e^{6\sqrt{3}\pi}\rho w^2\left(112\beta - 42\sqrt{3}\pi\left(16\beta + \rho^2\right) + 31\rho^2 + 42\rho w\right)}{4536}\\
&- \frac{e^{4\sqrt{3}\pi}\rho w}{84672}\Big(-232\rho^3 + 105\rho^2 w + 84\sqrt{3}\pi\left(5\rho^3 - 7\rho^2 w + 80\beta\rho + 112\beta w\right)\\
&+ 96\beta\rho + 9408\beta w\Big) + \rho\left(\frac{4}{3}\pi\left(2\pi\beta^2 - \sqrt{3}\beta_2\right) - \frac{w^4}{80} + \frac{65\beta w^2}{648}\right),\\
g_{22}(z) =&e^{10\sqrt{3}\pi}\left(-\frac{\rho^2 w^3}{252} - \frac{7\rho w^4}{456} + \frac{19w^5}{480}\right) + \frac{\rho^3\left(32\beta + 12\sqrt{3}\pi\left(\rho^2 - 16\beta\right) - \rho^2\right)}{6912}\\
&+ \Big(\rho^2\Big(1071w - \left(619 + 504\sqrt{3}\pi\right)\rho\Big) - 288\beta\Big(\left(6 + 28\sqrt{3}\pi\right)\rho + 49w\Big)\Big)
\end{aligned}
$$

$$\frac{e^{6\sqrt{3}\pi}w^2}{84672}+\frac{e^{8\sqrt{3}\pi}w^2}{24192}\left(\rho\left(-21\rho^2+4\left(73-196\sqrt{3}\pi\right)\rho w+644w^2\right)\right.$$
$$\left.+112\beta\left(3\rho+28\left(1-4\sqrt{3}\pi\right)w\right)\right)+\frac{e^{4\sqrt{3}\pi}w}{22464}\left(-9984\beta^2-123\rho^4\right.$$
$$\left.+377\rho^3w-1248\beta\rho^2+78\sqrt{3}\pi\left(16\beta+\rho^2\right)^2-936\beta\rho w\right)+\frac{7e^{16\sqrt{3}\pi}w^5}{216}$$
$$-\frac{e^{12\sqrt{3}\pi}w^3\left(49\rho^2+225\rho w+196\left(4\beta+7w^2\right)\right)}{30240}+\frac{35e^{14\sqrt{3}\pi}\rho w^4}{2808}$$
$$+e^{2\sqrt{3}\pi}\left(\frac{\left(41-52\pi\left(\sqrt{3}+3\pi\right)\right)\rho^4w}{7488}-\frac{\rho^3\left(784\left(1-2\sqrt{3}\pi\right)\beta+1457w^2\right)}{169344}\right.$$
$$-\frac{\rho^2w\left(3360\left(3\left(\sqrt{3}-2\pi\right)\pi-1\right)\beta+1157w^2\right)}{60480}+\rho\left(\frac{85\beta w^2}{1764}-\frac{6085w^4}{373464}\right)$$
$$\left.-\frac{23w^5}{864}+\frac{\left(1+4\sqrt{3}\pi\right)\rho^5}{6912}+\frac{17\beta w^3}{270}+\frac{4}{9}w\left(\beta^2-3\sqrt{3}\pi\beta_2\right)\right).$$

We point out that the function $g_0(z)$ satisfies the hypothesis (i) for the graph $\mathcal{Z}=\{(\alpha,0):\alpha>0\}$. We apply Theorem 12 to system (5.6) taking $s=0$. Then we have $\Delta_\alpha=1-e^{-2\sqrt{3}\pi}\neq 0$ and the function

$$f_1(\alpha)=\frac{\pi\alpha\left(\alpha^2-16\beta_1\right)}{4\sqrt{3}},$$

has the positive simple zero $\alpha^*=4\sqrt{\beta_1}$, where $Df_1(\alpha^*)=8\pi\beta_1/\sqrt{3}$. Then system (5.6) has a 2π-periodic orbit by Theorem 12 . The periodic orbit of system (5.2) is obtained going back through the change of variables. Now we want to study the stability of this periodic orbit using Theorem 13. First using (1.24) we compute the function

$$f_2(\alpha)=\frac{\alpha}{1728}\left(2304\pi\left(2\pi\beta^2-\sqrt{3}\beta_2\right)+\left(1-2e^{2\sqrt{3}\pi}+e^{4\sqrt{3}\pi}+16\sqrt{3}\pi+54\pi^2\right)\alpha^4\right.$$
$$\left.-288\pi\left(\sqrt{3}+4\pi\right)\beta\alpha^2\right).$$

Then if $\varphi(t,\varepsilon)$ is the above periodic solution founded we can use Proposition 11 and Lemma 8 to write $\varphi(0,\varepsilon)=z_0+\varepsilon z_1+\mathcal{O}(\varepsilon^2)$, where

$$z_0=\left(4\sqrt{\beta_1},0\right),$$
$$z_1=\left(-\frac{2\left(\left(1-2e^{2\sqrt{3}\pi}+e^{4\sqrt{3}\pi}-2\sqrt{3}\pi\right)\beta^2-9\sqrt{3}\pi\beta_2\right)}{9\sqrt{3}\pi\sqrt{\beta}},\right.$$
$$\left.-\frac{4e^{2\sqrt{3}\pi}\beta^{5/2}}{27}\left(4\sqrt{3}\pi\left(1+\coth\left(\sqrt{3}\pi\right)\right)-1\right)\right).$$

Then by (1.26) and (1.27) we can write the matrix (1.25) as

$$A(\varepsilon)=\begin{pmatrix}0 & 0\\ 0 & 1-e^{-2\sqrt{3}\pi}\end{pmatrix}+\varepsilon\begin{pmatrix}\dfrac{8\pi\beta_1}{\sqrt{3}} & \dfrac{4\beta_1}{3}\left(e^{2\sqrt{3}\pi}-1\right)\\ -\dfrac{\beta_1}{3}\left(e^{2\sqrt{3}\pi}-1\right) & -\dfrac{8e^{2\sqrt{3}\pi}\pi\beta_1}{\sqrt{3}}\end{pmatrix}.$$

Now we use Theorem 13 taking $s = 0$. The eigenvalues of $Id + A_0$ are $\lambda_1 = 2 - e^{2\pi\sqrt{3}}$ and $\lambda_2 = 1$. Consequently $A(\varepsilon)$ satisfies the hypothesis ($s1$) and ($s2$). Thus the Jacobian matrix of the Poincaré map (1.19) is 2−hyperbolic. Then its eigenvalues can be written as

$$\overline{\lambda}_1 = \lambda_1 + \varepsilon\frac{8\pi\beta_1 e^{2\pi\sqrt{3}}}{\sqrt{3}} + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \overline{\lambda}_2 = 1 + \varepsilon\frac{8\pi\beta_1}{\sqrt{3}} + \mathcal{O}(\varepsilon^2).$$

Since $\beta_1 > 0$ we have $|\overline{\lambda}_i| > 1$ for all $|\varepsilon| > 0$ and the result follows. □

5.3.3 Proof of Theorem 30

Proof. We will prove the result only for the equilibrium point $\mathbf{P}_+$. The proof for the equilibrium point $\mathbf{P}_-$ follows exactly the same steps. First we translate the equilibrium point $\mathbf{P}_+$ to the origin and rescale the system using the change of variables $(X, Y, Z) = \sqrt{\varepsilon}\big(x + z, (-x - \sqrt{3}y + 2z)/2, (-x - \sqrt{3}y + 2z)/2\big)$, the differential system (5.3) becomes

$$\begin{aligned}
\dot{X} =& -\omega Y + \sqrt{\varepsilon}\Big(X^2 + 2X\left(\sqrt{3}Y + 2Z\right) - Y\left(Y + 4\sqrt{3}Z\right)\Big)\frac{3\omega\sqrt[4]{3}}{4\sqrt{2\omega}} \\
&+ \frac{\varepsilon}{8}\Big(X\left(8a_1 - 20b_1 + 3\left(Y^2 + 4\sqrt{3}YZ + 4Z^2\right)\right) + Y\Big(-3\sqrt{3}Y^2 - 6YZ \\
&- 4\sqrt{3}\left(2a_1 - 3b_1 + 3Z^2\right)\Big) + 3X^3 - 3X^2\left(\sqrt{3}Y - 2Z\right)\Big) + \mathcal{O}(\varepsilon^{3/2}), \\
\dot{Y} =& \omega X + \sqrt{\varepsilon}(\sqrt{3}X^2 - 2XY + 4\sqrt{3}XZ - \sqrt{3}Y^2 + 4YZ)\frac{3\omega\sqrt[4]{3}}{4\sqrt{2\omega}} \\
&+ \frac{\varepsilon}{8}\Big(8a_1\left(\sqrt{3}X + Y\right) - 4b_1\left(3\sqrt{3}X + 5Y\right) + 3\Big(\sqrt{3}X^3 + X^2\left(Y + 2\sqrt{3}Z\right) \\
&+ X\left(\sqrt{3}Y^2 - 4YZ + 4\sqrt{3}Z^2\right) + Y\left(Y^2 - 2\sqrt{3}YZ + 4Z^2\right)\Big)\Big) + \mathcal{O}(\varepsilon^{3/2}), \\
\dot{Z} =& -\sqrt{3}\omega Z + \sqrt{\varepsilon}(X^2 + Y^2 + 2Z^2)\frac{3\sqrt{\omega}\sqrt[4]{3}}{2\sqrt{2}} + \frac{\varepsilon}{4}\Big(8Z(b_1 - a_1) - X^3 \\
&- 6Z\left(X^2 + Y^2\right) + 3XY^2 - 4Z^3\Big) + \mathcal{O}(\varepsilon^{3/2}).
\end{aligned}$$

This system can be written into the normal form for applying the averaging theory. We use the cylindrical change of variables $(X, Y, Z) = \big(\rho\cos\theta, \rho\sin\theta, w\big)$ with $\rho > 0$. Then we check that $\dot{\theta} = \sqrt{3}/2 + \mathcal{O}(\varepsilon^{1/2})$ for $\varepsilon > 0$ sufficiently small. Then we take θ as the new independent variable obtaining the differential system

$$\dot{z} = F_0\big(z, \theta\big) + \sqrt{\varepsilon}F_1\big(z, \theta\big) + \varepsilon F_2\big(z, \theta\big) + \mathcal{O}(\varepsilon^{3/2}), \tag{5.7}$$

with $z = (\rho, w)$, $F_0\big(z, \theta\big) = \big(0, -\sqrt{3}w\big)$, and $F_i\big(z, \theta\big) = \big(F_{i1}\big(z, \theta\big), F_{i2}\big(z, \theta\big)\big)$ for $i = 1, 2$, where

$$\begin{aligned}
F_{11}\big(z, \theta\big) =& \frac{3\sqrt[4]{3}\rho\left(\sqrt{3}\rho\sin(3\theta) + \rho\cos(3\theta) + 4w\right)}{4\sqrt{2}\sqrt{\omega}}, \\
F_{12}\big(z, \theta\big) =& -\frac{3\sqrt[4]{3}\left(2\rho^2 - 8w^2 + \sqrt{3}\rho w\sin(3\theta) - 3\rho w\cos(3\theta)\right)}{4\sqrt{2}\sqrt{\omega}},
\end{aligned}$$

$$
\begin{aligned}
F_{21}(z,\theta) =& -\frac{\rho}{32\omega}\Big(3\rho\left(9\rho\cos(6\theta)+2\sqrt{3}\sin(3\theta)(3\rho\cos(3\theta)+8w)+64w\cos(3\theta)\right)\\
&+4\left(-8a_1+20b_1-3\rho^2+96w^2\right)\Big),\\
F_{22}(z,\theta) =& \frac{1}{32\omega}\Big(\rho\left(6\sqrt{3}\sin(3\theta)\left(-3\rho^2+26w^2+9\rho w\cos(3\theta)\right)+\left(46\rho^2-468w^2\right)\right.\\
&\left.\cos(3\theta)-27\rho w\cos(6\theta)\right)+2w\left(16a_1-40b_1+75\rho^2-376w^2\right)\Big).
\end{aligned}
$$

We consider the period $T=2\pi$, thus system (5.7) is in the normal form for applying Theorem 12 . Taking the initial condition $z_0=(\rho_0,w_0)$ the solution of the unperturbed differential system corresponding to (5.7) is given by $\Phi(\theta,z)=\left(\rho_0, w_0e^{-\sqrt{3}\theta}\right)$. Again we consider the set $\mathcal{Z}\subset\mathbb{R}^2$ such that $\mathcal{Z}=\{(\alpha,0):\alpha>0\}$. Thus for $z_\alpha\in\mathcal{Z}$ the solution $\Phi(\theta,z_\alpha)$ is 2π-periodic, and therefore the differential system (5.7) satisfies the hypothesis (H). Moreover the fundamental matrix of the variational differential system along $\Phi(\theta,z_\alpha)$ is

$$
M(\theta,z_\alpha)=\begin{pmatrix}1 & 0\\ 0 & e^{-\sqrt{3}\theta}\theta\end{pmatrix}.
$$

The averaging functions for this system are $g_0(z)=\left(0,(1-e^{2\pi\sqrt{3}})w\right)$ and $g_i(z)=(g_{i1}(z),g_{i2}(z))$ for $i=1,2$ where

$$
\begin{aligned}
g_{11}(z) =& \frac{3^{3/4}\left(1-e^{-2\sqrt{3}\pi}\right)\rho w}{\sqrt{2}\sqrt{\omega}},\\
g_{12}(z) =& -\frac{3^{3/4}e^{-4\sqrt{3}\pi}\left(e^{2\sqrt{3}\pi}-1\right)\left(e^{2\sqrt{3}\pi}\rho^2-4w^2\right)}{2\sqrt{2}\sqrt{\omega}},\\
g_{21}(z) =& \frac{\rho e^{-8\sqrt{3}\pi}}{112\omega}\Big(e^{8\sqrt{3}\pi}\left(28\pi\left(8a_1-5\left(4b_1+3\rho^2\right)\right)+\sqrt{3}\left(84\rho^2-168w^2-23\rho w\right)\right)\\
&-56\sqrt{3}e^{2\sqrt{3}\pi}w^2+84\sqrt{3}w^2+\sqrt{3}e^{4\sqrt{3}\pi}w(51\rho+140w)\\
&-28\sqrt{3}e^{6\sqrt{3}\pi}\rho(3\rho+w)\Big),\\
g_{22}(z) =& \frac{e^{-10\sqrt{3}\pi}}{8736\omega}\Big(-1820\sqrt{3}e^{10\sqrt{3}\pi}\rho^3+26208\sqrt{3}w^3+1092\sqrt{3}e^{2\sqrt{3}\pi}w^2(3\rho-32w)\\
&-52\sqrt{3}e^{4\sqrt{3}\pi}w^2(81\rho-658w)-39879\sqrt{3}e^{6\sqrt{3}\pi}\rho^2w+e^{8\sqrt{3}\pi}\Big(2184\pi w.\\
&\left(8a_1-20b_1-75\rho^2\right)+\sqrt{3}\left(1820\rho^3-25480w^3+936\rho w^2+39879\rho^2w\right)\Big)\Big).
\end{aligned}
$$

Function $g_0(z)$ vanishes on the the graph $\mathcal{Z}=\{(\alpha,0):\alpha>0\}$. We apply Theorem 12 to system (5.7). Here $s=0$ and $\Delta_\alpha=1-e^{-2\sqrt{3}\pi}\neq 0$. The bifurcation functions are

$$
\begin{aligned}
f_1(\alpha) =& 0,\\
f_2(\alpha) =& \frac{3\left(\sqrt{3}e^{-4\sqrt{3}\pi}\left(1-2e^{2\sqrt{3}\pi}\right)+\sqrt{3}-5\pi\right)\alpha^3+8\pi\alpha a_1-20\pi\alpha b_1}{4\omega}.
\end{aligned}
$$

Function f_2 has the positive simple zero

$$\alpha^* = 2e^{2\sqrt{3}\pi}\sqrt{\frac{\pi(5b_1 - 2a_1)}{3\sqrt{3} - 6\sqrt{3}e^{2\sqrt{3}\pi} + 3\sqrt{3}e^{4\sqrt{3}\pi} - 15e^{4\sqrt{3}\pi}\pi}},$$

where $Df_2(\alpha^*) = (10\pi b_1 - 4\pi a_1)/\omega$. By statement (b) of Theorem 12 system (5.7) has a 2π-periodic solution. The periodic solution of system (5.4) is obtained going back through the change of variables. □

Chapter 6

Generalized Van der Pol - Duffing differential system

In this work we the use averaging theory of integrability and the computation of Lyapunov coefficients to detect for the first time Hopf and zero–Hopf bifurcations in the origin of coordinates for the general Van der Pol–Duffing equations here considered. We also show, for the first time, that for certain parameter values two periodic orbits will be surrounded by a torus.

The autonomous chaotic van der Pol–Duffing oscillator is the differential system given by

$$\begin{aligned}\dot{x} =& -\nu\left(x^3-\mu x-y\right),\\ \dot{y} =& x-\alpha y-z\\ \dot{z} =& \beta y.\end{aligned} \tag{6.1}$$

Matouk and Agiza [58] used the Hopf's theorem and numerical methods to investigate the Hopf bifurcations and the existence of chaotic behavior in system (6.1). We recall that a Hopf bifurcation is the mathematical way to study the birth (or death) of a limit cycle from an equilibrium point in a family of ordinary differential equations. Their results show that there are periodic solutions and chaotic attractors bifurcating from a fixed point of this system. Later on Zhao et. al. [89] provided the general van der Pol–Duffing oscillator given by

$$\begin{aligned}\dot{x} =& -\nu\left(x^3-\mu x-y\right),\\ \dot{y} =& -hz+kx-\alpha y\\ \dot{z} =& \beta y,\end{aligned} \tag{6.2}$$

where α, h, β, k, ν and μ are real positive parameters.

Motivated by the works of Leonov and Kuznetsov, they found the occurrence of hidden chaotic attractors besides periodic orbits and chaotic attractors of system (6.2).

The Hopf bifurcation analysis done in [89] discuss only the periodic orbits bifurcating from the pair of symmetric equilibria

$$P_{\pm}=\left(\pm\sqrt{\mu},0,\pm\sqrt{\mu}\frac{k}{h}\right).$$

However we are going to show that a rich bifurcating phenomena may emerge at the origin of coordinates $O(0,0,0)$. Although the origin of coordinates is an unstable equilibrium point of system (6.2), we will analytically prove that multiple periodic orbits may bifurcate from this point. Some of these periodic orbits are stable guaranteeing a very controlled behavior of the system. We will study analytically all possible classical and degenerate Hopf bifurcations as well as the zero-Hopf bifurcations at the origin of coordinates. Some of these techniques were also used for other chaotic systems (see for instance [12, 63]). The results of this chapter are presented in [19] and submitted for publication.

6.1 Application to General Van der Pol - Duffing differential system

More precisely our first main result is the following one.

Theorem 31. *Consider system* (6.2) *with*

$$\alpha = \varepsilon\alpha_1, \quad \beta = \frac{k\nu + \omega^2}{h} + \varepsilon\beta_1 \quad \text{and} \quad \mu = \varepsilon\mu_1,$$

where $(\alpha_1, \beta_1, \mu_1) \in \mathbb{R}^3_+$ *and* $\varepsilon > 0$ *is a small parameter. Let* $\rho = (2k\mu_1\nu^2 - \alpha_1\omega^2)/k$. *Then there exist* $\varepsilon_0 > 0$ *sufficiently small such that for* $0 < \varepsilon < \varepsilon_0$ *the following statements hold.*

(a) If $\rho \leq 0$, *system* (6.2) *has the periodic solution*

$$\varphi_0(t,\varepsilon) = 2\sqrt{\frac{\varepsilon\,(\alpha_1\omega^2 + k\mu_1\nu^2)}{3k}}\left(-\frac{\cos(\omega t)}{\nu}, -\frac{\sin(\omega t)}{\nu^2}, \frac{(k\nu+\omega^2)\cos(\omega t)}{h\nu^2}\right) + \mathcal{O}(\varepsilon),$$

bifurcating from the origin of coordinates. This periodic solution has a stable manifold formed by two topological cylinders and an unstable manifold also formed by two topological cylinders.

(b) If $\rho > 0$, *system* (6.2) *has 3 simultaneous periodic solutions bifurcating from the origin coordinates, mainly*

$$\begin{aligned}\varphi_\pm(t,\varepsilon) = \sqrt{\frac{\varepsilon}{5}}\Bigg(&\pm\frac{1}{\nu}\sqrt{\frac{2\alpha_1\omega^2 + k\mu_1\nu^2}{k}} - \frac{2\cos(t\omega)}{3\nu}\sqrt{\frac{6k\mu_1\nu^2 - 3\alpha_1\omega^2}{k}}, \\ &\frac{2\omega\sin(t\omega)}{\nu^2}\sqrt{\frac{2k\mu_1\nu^2 - \alpha_1\omega^2}{3k}}, \pm\frac{1}{h\nu}\sqrt{k\,(2\alpha_1\omega^2 + k\mu_1\nu^2)} \\ &-\frac{2\sqrt{3}\,(k\nu+\omega^2)}{3h\nu^2}\cos(t\omega)\sqrt{\frac{2k\mu_1\nu^2 - \alpha_1\omega^2}{k}}\Bigg) + \mathcal{O}(\varepsilon)\end{aligned}$$

and $\varphi_0(t,\varepsilon)$. *Moreover* $\varphi_\pm(t,\varepsilon)$ *are symmetric and stable, and* $\varphi_0(t,\varepsilon)$ *has a stable manifold formed by two topological cylinders and an unstable manifold also formed by two topological cylinders.*

The choice of the parameters in Theorem 31 comes from the fact that if $\alpha = 0$, $\beta = (k\nu + \omega^2)/h$ and $\mu = 0$ the origin is a *zero-Hopf equilibrium point*, i.e., the Jacobian matrix of system (6.2) at $O(0,0,0)$ has the eigenvalues $\lambda_{\pm} = \pm i\omega$ and $\lambda_0 = 0$. In Figure 6.1 we have plotted a solution converging to the periodic orbit obtained in Theorem 31.

From now on we relax the condition over the coefficients of system (6.2) by taking $(\alpha, \beta, k, h, \nu) \in \mathbb{R}^5$. In the next result we show that an invariant torus bifurcate from the periodic orbits $\varphi_{\pm}(t)$ given in Theorem 31. The proof uses the fact that we can show the existence of a Hopf bifurcation for the averaged system of system (6.2) (see the proof of Theorem 31).

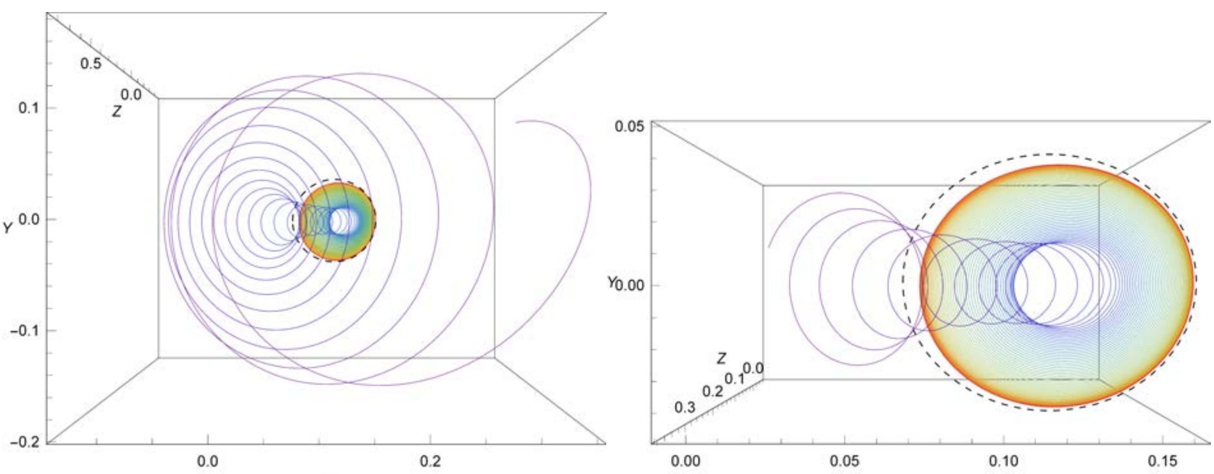
Theorem 32. *Consider system* (6.2) *and* $0 < \varepsilon < \varepsilon_0$ *as stated in Theorem 31. Take*

$$\alpha_1 = \frac{8\bar{k}\nu^2\omega^2 - 3\bar{k}\nu^4}{4\omega^4}, \quad \mu_1 = \frac{3\nu^2 + 2\omega^2}{2\omega^2}, \quad \text{where } \bar{k} = \frac{8\nu\omega^4}{3\nu^4 - 8\nu^2\omega^2} + \delta$$

and assume that

$$l = \frac{9\nu^3\left(3\nu^4 - 8\omega^4\right)\left(81\nu^8 + 216\nu^6\omega^2 - 896\nu^4\omega^4 + 800\nu^2\omega^6 - 256\omega^8\right)}{20\left(8\omega^3 - 3\nu^2\omega\right)} \neq 0.$$

For $|\delta| > 0$ *sufficiently small each one of the periodic solutions* $\varphi_{\pm}(t,\varepsilon)$ *will be surrounded by an invariant torus. If* $\bar{k}$ *is in the region where* $\varphi_{\pm}(t,\varepsilon)$ *is unstable (stable), then the torus will be stable (unstable).*



(a) (b)

Figure 6.1: (a) The solution starting at $(0.3, 0.1, 0.6)$. Here $\varepsilon = 1/20$, $h = 1$, $k = 2.04772$, $\alpha_1 = 1$, $\beta_1 = 1$, $\mu_1 = 0.332958$, $\nu = 1$ and $\omega = 1$. (b) The aproximation $\varphi_+(\varepsilon, t)$ in detail.

Figure 6.2 shows a solution converging to the torus around the periodic solution founded in Theorem 32.

In the proof of Theorem 32 we will see that the first and second Lyapunov coefficients vanish but the third coefficient is different from zero. This will lead to the statement of the theorem.

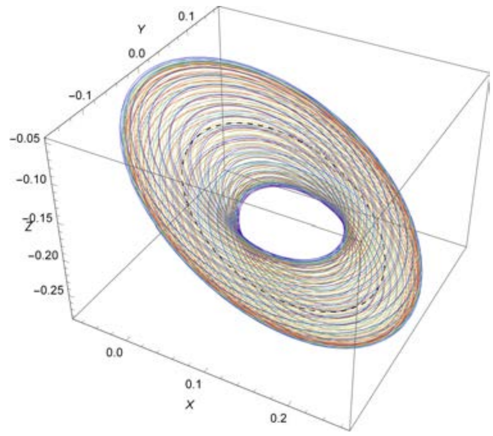
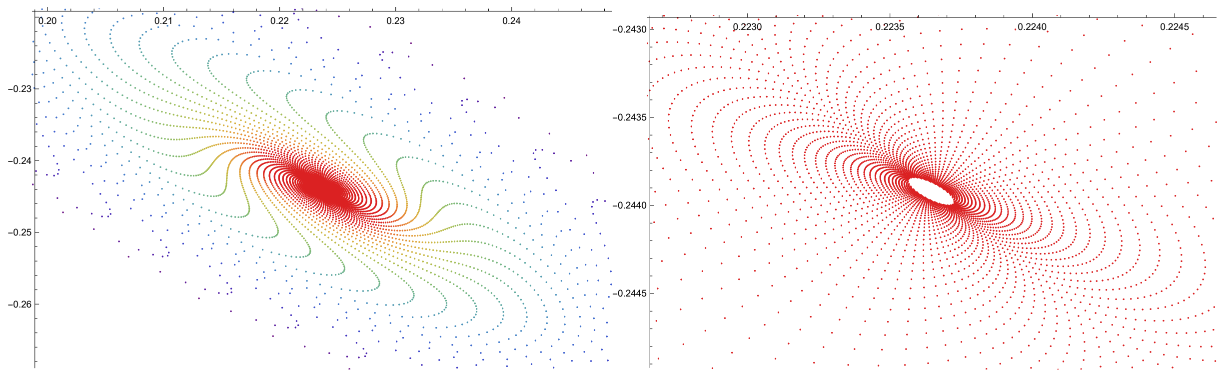


Figure 6.2: Solution of system (6.2) starting at $(0.229319, 0.0210973, -0.26471)$. Here $\varepsilon = 1/80$, $h = 1$, $\alpha_1 = -1.595$, $\mu_1 = -1.99375$, $\nu = 5/2$, $\beta_1 = 1$, $\delta = 0.005$ and $\omega = 1$. The dashed curve represent an approximation of the solution provided by $\varphi_+(\varepsilon, t)$.



(a) (b)

Figure 6.3: (a) it shows the transverse section, (b) The transverse section in detail.

Figure 6.3 show the transversal section of the periodic solution presented in Theorem 32. There we can see how the solution converge to the small torus around the periodic solution. Finally we show that system (6.2) also has a Hopf equilibrium bifurcation at the origin of coordinates.

Proposition 33. *Let* $\bar{k} = -\dfrac{\alpha\,(\alpha\mu\nu - \beta h - \mu^2\nu^2)}{\nu(\alpha - \mu\nu)}$ *and assume that* $(\beta h\mu\nu)(\mu\nu - \alpha) > 0$ *and* $\mathbf{x}_0 = O(0,0,0)$. *Then* $(\mathbf{x}_0, \bar{k})$ *is a Hopf point (see Section 6.2.1) of system* (6.2). *Moreover its corresponding eigenvalues are*

$$\pm i\sqrt{\frac{\beta h\mu\nu}{\mu\nu - \alpha}} \quad \text{and} \quad \alpha - \mu\nu.$$

Proof. The proof of Proposition (33) is done by direct computations, and we omit it here. □

Theorem 34. *Consider system* (6.2) *as in Proposition 33. Let*

$$(\alpha, \beta, k, h, \nu) \in \mathbb{R}^5 \quad \textit{with} \quad \alpha = \mu\nu \left(1 - \frac{\beta h}{\omega^2}\right) \quad \textit{and} \quad \omega > 0.$$

Then for k sufficiently close to the bifurcation value $\bar{k}$ the following statements hold:

(i) If $(\omega^2 - \beta h)(\beta h \mu^2 \nu^2 - \omega^4) > 0$, system (6.2) *has a supercritical Hopf bifurcation at the origin of coordinates,*

(ii) If $(\omega^2 - \beta h)(\beta h \mu^2 \nu^2 - \omega^4) < 0$, system (6.2) *has a subcritical Hopf bifurcation at the origin of coordinates.*

We have added an Appendix E where we show how to obtain similar conclusions to the ones given in Theorem 34 but using the averaging theory described in Theorem 12.

6.2 Proofs

Remark 1. *From the averaging theory we know that there is a coordinate transformation*

$$\mathbf{x} = \mathbf{y} + \varepsilon \mathbf{u}(\mathbf{y}, t),$$

T–periodic in t that carries the solutions of the original system (1.11) *to the solutions of the full averaged system of system* (1.11), *i.e.*

$$\dot{\mathbf{y}} = \varepsilon g_1(\mathbf{y}) + \varepsilon^2 \widetilde{g}(\mathbf{y}, t, \varepsilon), \tag{6.3}$$

where $\widetilde{g}$ is T–periodic in t. Taking $\tau = \varepsilon t$ we obtain the system

$$z' = g_1(z) + \varepsilon \widetilde{g}(z, \tau/\varepsilon, \varepsilon). \tag{6.4}$$

with $' = d/d\tau$. For a fixed $\varepsilon > 0$ sufficiently small it is well known that if the (guiding) system $z' = g_1(z)$ has a periodic solution due to a Hopf bifurcation, then system (6.4) *also has a periodic solution (cf. [57, Theorem* 7.1*, pg. 250]). Thus the full averaged system* (6.3) *will have a periodic solution of period $\mathcal{O}(1/\varepsilon)$. In this case a torus will emerge from the periodic solution in the original system* (1.11)*. This bifurcation is called Neimark-Sacker bifurcation. For more information about Neimark-Sacker bifurcation see [46]. For details about Neimark-Sacker bifurcation due to a Hopf bifurcation in the averaged system see [68, Chapter C] and [3]. Similar ideas were also used in [27].*

6.2.1 Lyapunov coefficients

In this section we present some basic notions about the Hopf bifurcations and Lyapunov coefficients. The theory of the Lyapunov coefficients can be found in [46, Chapters 3 and 10]. We also refer the reader to [74] where the Lyapunov coefficient are calculated in great detail up to order 4.

Consider the differential equation

$$x' = f(\mathbf{x}, \boldsymbol{\mu}), \tag{6.5}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\boldsymbol{\mu} \in \mathbb{R}^m$ are respectively vectors representing phase phase variables and control parameters.

Here a *Hopf point* $(\mathbf{x}_0, \boldsymbol{\mu}_0)$ is an equilibrium point of (6.5) where the Jacobian matrix $A = f_{\mathbf{x}}(\mathbf{x}_0, \boldsymbol{\mu}_0)$ has a pair of purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega$, $\omega > 0$ and admits no other eigenvalues with zero real part.

Denoting the variable $\mathbf{x} - \mathbf{x}_0$ also by $\mathbf{x}$ we write

$$F(\mathbf{x}) = f(\mathbf{x}, \boldsymbol{\mu}_0),$$

as

$$F(\mathbf{x}) = A\mathbf{x} + \frac{1}{2}B(\mathbf{x}, \mathbf{x}) + \frac{1}{6}C(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \mathcal{O}(||\mathbf{x}||^4),$$

where $A = f_{\mathbf{x}}(0, \boldsymbol{\mu}_0)$,

$$B_i(\mathbf{x}, \mathbf{y}) = \sum_{j,k=1}^{n} \left. \frac{\partial^2 F_i(\boldsymbol{\xi})}{\partial \xi_j \partial \xi_k} \right|_{\boldsymbol{\xi}=0} x_j y_k,$$

and

$$C_i(\mathbf{x}, \mathbf{y}, z) = \sum_{j,k,l=1}^{n} \left. \frac{\partial^3 F_i(\boldsymbol{\xi})}{\partial \xi_j \partial \xi_k \partial \xi_l} \right|_{\boldsymbol{\xi}=0} x_j y_k z_l.$$

Let p, $q \in \mathbb{C}^n$ be vectors such that

$$Aq = i\omega q, \quad A^{\intercal} p = -i\omega p, \quad \bar{q}.q = \bar{p}.q = 1,$$

where $A^{\intercal}$ is the transposed of the matrix A. We define the *first Lyapunov coefficient* as

$$l_1 = \frac{1}{2\omega} \mathrm{Re} \left(\bar{p}.C(q, q, \bar{q}) - 2\bar{p}.B(q, A^{-1}.B(q, \bar{q})) + \bar{p}B(\bar{q}, (2\omega i I_n - A)^{-1} B(q, q)) \right), \quad (6.6)$$

where I_n is the $n \times n$ identity matrix.

A Hopf point is called *transversal* if the parameter controlling the complex eigenvalues cross the imaginary axis with non-zero derivative. We have the following lemma.

Lemma 35. *Consider the differential system* (6.5) *having the Hopf point* $(\mathbf{x}_0, \boldsymbol{\mu}_0)$ *and assume that* $l_1 \neq 0$ *and* $\mathrm{Re}(\lambda_{\pm}(\boldsymbol{\mu}_0)) \neq 0$. *Then following statements hold.*

(i) If $l_1 > 0$, *the differential system* (6.5) *has a supercritical Hopf bifurcation at* $\mathbf{x}_0$.

(ii) If $l_1 < 0$, *the differential system* (6.5) *has a subcritical Hopf bifurcation at* $\mathbf{x}_0$.

To study the co-dimensions two and three Hopf bifurcations, we need to compute the pertinent Lyapunov stability coefficients.

Here we consider a cubic polynomial system such that the differential system (6.5) writes

$$x' = f(\mathbf{x}, \boldsymbol{\mu}_0) = A\mathbf{x} + \frac{1}{2}B(\mathbf{x}, \mathbf{x}) + \frac{1}{6}C(\mathbf{x}, \mathbf{x}, \mathbf{x}). \quad (6.7)$$

The two-dimensional center manifolds associated to the eigenvalues $\lambda_{1,2}$ can be parameterized by the variables w and $\bar{w}$ by the immersion of the form $x = H(w, \bar{w})$, where $H : \mathbb{C}^2 \mapsto \mathbb{R}^n$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \leq j+k \leq 6} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + \mathcal{O}(|\omega|^7), \quad (6.8)$$

with $h_{jk} \in \mathbb{C}^3$. Then substituting this expression in (6.7) we get the following equation

$$H_w \dot{w} + H_{\bar{w}} \dot{\bar{w}} = f(H(w, \bar{w})). \tag{6.9}$$

The vectors h_{jk} are obtained solving the linear systems defined by the coefficients in (6.8) by taking into account the coefficients of (6.7). System (6.9) on the chart w for a central manifold, is written as

$$\dot{w} = i\omega w + \frac{1}{2} G_{21} w|w|^2 + \frac{1}{12} G_{32} w|w|^4 + \frac{1}{144} G_{43} w|w|^6 + \mathcal{O}(|\omega|^8),$$

with $G_{jk} \in \mathbb{C}$. Thus the first three Lyapunov coefficients are

$$l_1 = \frac{1}{2}\mathrm{Re}(G_{21}), \quad l_2 = \frac{1}{12}\mathrm{Re}(G_{12}) \quad \text{and} \quad l_3 = \frac{1}{2}\mathrm{Re}(G_{43}).$$

More precisely, we have

$$\begin{aligned} h_{11} =& A^{-1} B(q, \bar{q}), \\ h_{20} =& (2i\omega * I_3 - A)^{-1} B(q, \bar{q}), \end{aligned}$$

where I_n is the $n \times n$ identity matrix. From the coefficients of the terms w^3 in (6.9), we have

$$h_{30} = (3i\omega I_n - A)^{-1}(3B(q, h_{20}) + c(q, q, q)).$$

From the coefficients of the terms $w^2\bar{w}$ in (6.9), we obtain the singular system for h_{21}

$$\begin{pmatrix} i\omega I_n - A & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{21} \\ s \end{pmatrix} = \begin{pmatrix} H_{21} - G_{21} q \\ 0 \end{pmatrix},$$

where $H_{21} = B(\bar{q}, h_{20}) + 2B(q, h_{11}) + C(q, q, \bar{q})$ and $G_{21} = \bar{p}.H_{21}$.

From the coefficients of the terms w^4, $w^3\bar{w}$ and $w^2\bar{w}^2$ in (6.9) one obtains respectively

$$\begin{aligned} h_{40} =& (4i\omega I_n - A)^{-1}(3B(h_{20}, h_{20}) + 4B(q, h_{30}) + 6C(q, q, h_{20}), \\ h_{31} =& (2i\omega I_n - A)^{-1}\big(B(\bar{q}, h_{30}) + 3B(h_{20}, h_{11}) + 3B(q, h_{21}) + 3C(q, \bar{q}, h_{20}) \\ &+ 3C(q, q, h_{11}) - 3G_{21} h_{20}\big), \\ h_{22} =& -A^{-1}(B(\bar{h}_{20}, h_{h20}) + 2B(q, \bar{h}_{21}) + 2B(\bar{q}, h_{21}) + 2B(h_{11}, h_{11}) \\ &+ C(q, q, \bar{h}_{20}) + 4C(q, \bar{q}, h_{11}) + C(\bar{q}, \bar{q}, h_{20})). \end{aligned}$$

Defining

$$\begin{aligned} H_{32} =& B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, h_{20}) + 2B(\bar{q}, h_{31}) + 6B(h_{11}, h_{21}) + 3B(q, h_{22}) \\ &+ 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{21}) + 6C(\bar{q}, h_{20}, h_{11}) + 6C(q, \bar{q}, h_{21}) \\ &+ C(\bar{q}, \bar{q}, h_{30}) + 6C(q, h_{11}, h_{11}) - 3\bar{G}_{21} h_{21} - 6G_{21} h_{21}, \end{aligned}$$

we have $G_{32} = \bar{p}.H_{32}$. The complex vector h_{32} can be found solving the non-singular system

$$\begin{pmatrix} i\omega I_n - A & q \\ \bar{p} & 0 \end{pmatrix} \begin{pmatrix} h_{32} \\ s \end{pmatrix} = \begin{pmatrix} H_{32} - G_{32} q \\ 0 \end{pmatrix},$$

Finally from the terms $w^4\bar{w}$, $w^4\bar{w}^2$ and $w^3\bar{w}^3$ in (6.9), one has respectively

$$
\begin{aligned}
h_{41} =&(3i\omega I_n - A)^{-1}\big(B(\bar{q}, h_{40}) + 4B(h_{11}, h_{30}) + 6B(h_{20}, h_{21}) + 4B(q, h_{31}) \\
&+ 3C(\bar{q}, h_{20}, h_{20}) + 4C(q, \bar{q}, h_{30}) + 12C(q, h_{11}, h_{20}) + 6C(q, q, h_{21}) \\
&- 6G_{21}h_{30}\big), \\
h_{42} =&(2i\omega I_n - A)^{-1}\big(B(\bar{h}_{20}, h_{40}) + 4B(\bar{h}_{21}, h_{30}) + 2B(\bar{q}, h_{41}) + 8B(h_{11}, h_{31}) \\
&+ 6B(h_{20}, h_{22}) + 6B(h_{21}, h_{21}) + 4B(q, h_{32}) + 3C(h_{20}, h_{20}, \bar{h}_{20}) \\
&+ 4C(q, \bar{h}_{20}, h_{30}) + 12C(q, h_{20}, \bar{h}_{21}) + 8C(\bar{q}, h_{11}, h_{30}) + 12C(\bar{q}, h_{20}, h_{21}) \\
&+ 8C(q, \bar{q}, h_{31}) + C(\bar{q}, \bar{q}, h_{40}) + 12C(h_{11}, h_{11}, h_{20} + 24C(q, h_{11}, h_{21})) \\
&+ 6C(q, q, h_{22}) - 4(\bar{G}_{21}h_{31} + 3G_{21}h_{31} + G_{32}h_{20})\big), \\
h_{33} =& - A^{-1}(3B(\bar{h}_{20}, h_{31}) + 9B(h_{21}, \bar{h}_{21}) + B(\bar{h}_{30}, h_{30}) + 3B(h_{20}, \bar{h}_{31}) \\
&+ 3B(q, \bar{h}_{32}) + 3B(\bar{q}, h_{32}) + 9B(h_{11}, h_{22}) + 9C(\bar{q}, \bar{h}_{20}, h_{21}) \\
&+ 3C(\bar{q}, \bar{h}_{20}, h_{30}) + 9C(h_{11}, \bar{h}_{20}, h_{20}) + 9C(q, \bar{h}_{20}, h_{21}) \\
&+ 18C(q, h_{11}, \bar{h}_{21}) + 3C(q, h_{20}, \bar{h}_{30}) + 3C(q, q, \bar{h}_{31}) \\
&+ 18C(\bar{q}, h_{11}, h_{21}) + 9C(q, \bar{q}, h_{22}) + 3C(\bar{q}, \bar{q}, h_{31}) \\
&+ 6C(h_{11}, h_{11}, h_{11}) - 9h_{22}(\bar{G}_{21} + G_{21}) - 3h_{11}(\bar{G}_{32} + G_{32})), \\
H_{43} =&3B(\bar{h}_{20}, h_{41}) + 12B(\bar{h}_{21}, h_{31}) + B(\bar{h}_{30}, h_{40}) + 4B(h_{30}, \bar{h}_{31}) \\
&+ 6B(h_{20}, \bar{h}_{32}) + 3B(\bar{q}, h_{42}) + 12B(h_{11}, h_{32}) + 18B(h_{21}, h_{22}) + 4B(q, h_{33}) \\
&+ 3C(\bar{q}, \bar{h}_{20}, h_{40}) + 12C(h_{11}, \bar{h}_{20}, h_{30}) + 18C(h_{20}, \bar{h}_{20}, h_{21}) \\
&+ 12C(q, \bar{h}_{20}, h_{31}) + 12C(\bar{q}, \bar{h}_{21}, h_{30}) + 36C(h_{11}, h_{20}, \bar{h}_{21}) \\
&+ 36C(q, h_{21}, \bar{h}_{21}) + 3C(h_{20}, h_{20}, \bar{h}_{30}) + 4C(q, h_{30}, \bar{h}_{30})) \\
&+ 12C(q, h_{20}, \bar{h}_{31} + 6C(q, q, \bar{h}_{32}) + 24C(\bar{q}, h_{11}, h_{31})) \\
&+ 18C(\bar{q}, h_{20}, h_{22}) + 18C(\bar{q}, h_{21}, h_{21}) + 12C(q, \bar{q}, h_{32})) \\
&+ 3C(\bar{q}, \bar{q}, h_{41}) + 36C(h_{11}, h_{11}, h_{21}) + 36C(q, h_{11}, h_{22}) \\
&- 6(2\bar{G}_{21}h_{32} + \bar{G}_{32}h_{21} + 3G_{21}h_{32} + 2G_{32}h_{21}),
\end{aligned}
$$

obtaining $G_{43} = \bar{p}.H_{43}$.

6.2.2 Proof of Theorem 31

Proof. First we do the reescaling $(x, y, z) = \varepsilon(\bar{x}, \bar{y}, \bar{z})$ obtaining the system

$$
\begin{aligned}
\dot{\bar{x}} =&\bar{y}\nu + \varepsilon\left(\mu_1 - \bar{x}^2\right), \\
\dot{\bar{y}} =&\bar{x}k - h\bar{z} - \varepsilon\bar{y}, \\
\dot{\bar{z}} =&\frac{\bar{y}\left(k\nu + \omega^2\right)}{h} + \varepsilon\beta_1\bar{y},
\end{aligned}
$$

In order to write the linear part of this system in its Jordan real normal form, we do the linear change of variables

$$
(\bar{x}, \bar{y}, \bar{z}) = \left(Z - \frac{X\nu}{\omega}, Y, \frac{1}{h}\left(k\left(z - \frac{x\nu}{\omega}\right) - X\omega\right)\right).
$$

Then the previous system writes

$$\begin{aligned}
\dot{X} =& -\omega Y + \frac{\varepsilon}{\omega^4}\left(k\nu(\nu X - \omega Z)\left(\nu^2X^2 - 2\nu X\omega Z + \omega^2\left(Z^2 - \mu_1\right)\right) - \beta_1 h\omega^3 Y\right), \\
\dot{Y} =& \omega X - \varepsilon\alpha_1 Y, \qquad\qquad (6.10)\\
\dot{Z} =& \frac{\varepsilon}{\omega^5}\left(k\nu(X\nu - Z\omega)\left(X^2\nu^2 - 2XZ\nu\omega + \left(Z^2 - \mu_1\right)\omega^2\right) + \omega^2\big(- hY\beta_1\omega\right.\\
&\left. +(X\nu - Z\omega)\big(X^2\nu^2 - 2XZ\nu\omega + \left(Z^2 - \mu_1\right)\omega^2\big)\big)\right).
\end{aligned}$$

Now we use cylindrical coordinates $X = r\cos\theta$, $Y = r\sin\theta$ and $Z = z$ in system (6.10). Thus taking θ as the new independent variable we finally obtain a system in the normal form for applying the averaging theorem

$$\begin{aligned}
r' =& \frac{\varepsilon}{\omega^5}\left(-\beta_1 hr\omega^3\sin(\theta)\cos(\theta) + k\nu\cos(\theta)(\nu r\cos(\theta) - \omega z)\left(\nu r\cos(\theta)(\nu r\cos(\theta)\right.\right.\\
&\left.\left. -2\omega z) + \omega^2\left(z^2 - \mu_1\right)\right) + \alpha_1(-r)\omega^4\sin^2(\theta)\right) + \mathcal{O}(\varepsilon^2) = F_{11}(\theta, r, z) + \mathcal{O}(\varepsilon^2),\\
z' =& \frac{\varepsilon}{\omega^6}\left(\nu\left(k\nu + \omega^2\right)(\nu r\cos(\theta) - \omega z)\left(\nu r\cos(\theta)(\nu r\cos(\theta) - 2\omega z) + \omega^2\left(z^2 - \mu_1\right)\right)\right.\\
&\left. -\beta_1 h\nu r\omega^3\sin(\theta)\right) + \mathcal{O}(\varepsilon^2) = F_{12}(\theta, r, z) + \mathcal{O}(\varepsilon^2), \qquad (6.11)
\end{aligned}$$

where here $' = d/d\theta$. System (6.11) is 2π-periodic in θ, and we can use the first order averaging method to write its averaged system

$$\begin{aligned}
r' =& \varepsilon\frac{r\left(-4\alpha_1\omega^4 + 3k\nu^4 r^2 + 4k\nu^2\omega^2\left(3z^2 - \mu_1\right)\right)}{8\omega^5},\\
z' =& -\varepsilon\frac{\nu z\left(k\nu + \omega^2\right)\left(3\nu^2 r^2 + 2\omega^2\left(z^2 - \mu_1\right)\right)}{2\omega^5}.
\end{aligned}$$

The equilibrium point of the averaged system satisfying $r > 0$ are

$$S_\pm = \left(\frac{2\omega}{\nu}\sqrt{\frac{\rho}{15}}, \pm\sqrt{\frac{\rho}{5}}\right) \quad \text{and} \quad S_0 = \left(\frac{2\omega}{\nu^2}\sqrt{\frac{\alpha_1\omega^2 + k\mu_1\nu^2}{3k}}, 0\right),$$

where $\rho = (2k\mu_1\nu^2 - \alpha_1\omega^2)/k$. Consequently the following statements hold.

(*i*) If $\rho \leq 0$ the only equilibrium point of system (6.11) is S_0.

(*ii*) If $\rho > 0$ system (6.11) has three equilibrium points $S_\pm$ and S_0.

For analyzing the stability of these equilibria we study their eigenvalues. First the eigenvalues of the Jacobian matrix of system (6.11) at S_0 are $\lambda_- < 0 < \lambda_+$, where

$$\lambda_- = -\frac{\left(k\nu + \omega^2\right)\left(2\alpha_1\omega^2 + k\mu_1\nu^2\right)}{k\nu\omega^3} \quad \text{and} \quad \lambda_+ = \frac{\alpha_1\omega^2 + k\mu_1\nu^2}{\omega^3}.$$

Thus S_0 always has one stable and one unstable direction.

For analyzing the stability of $S_\pm$ we assume $\rho > 0$. Then we write the characteristic polynomial of its Jacobian matrix

$$C(\lambda) = \lambda^2 + b\lambda + c,$$

where

$$b = \frac{4\alpha_1\omega^2 + 5\alpha_1 k\nu + 2k\mu_1\nu^2}{5k\nu\omega} \text{ and } c = \frac{2\rho\,(k\nu+\omega^2)\,(2\alpha_1\omega^2 + k\mu_1\nu^2)}{5\nu\omega^6}.$$

Let λ_1 and λ_2 be the solutions of $C(\lambda) = 0$ then the following statement hold.

(i) If $b^2 - 4c \leq 0$ then $Re(\lambda_1) = Re(\lambda_2) = -b/2 < 0$.

(ii) If $b^2 - 4c > 0$ then $\lambda_1 < \lambda_2 < 0$.

Statement (i) follows directly from the fact that $b > 0$. Furthermore the graph of $C(\lambda)$ is a parabola opening upwards cutting the ordinate axis at $C(0) = c > 0$ and since $C'(0) = b > 0$, it is increasing at $C(0)$, then its roots must be at the left side of the abscissa axis. This graph analysis justifies statement (ii). □

6.2.3 Proof of Theorem 32

Proof. Let

$$\alpha_1 = \frac{8k\nu^2\omega^2 - 3k\nu^4}{4\omega^4} \quad \text{and} \quad \mu_1 = \frac{3\nu^2 + 2\omega^2}{2\omega^2}.$$

To prove the Neimark-Sacker bifurcation occurring in system (6.2) we use Remark 1. Thus we use the averaged system (1.2) to write the guiding system as

$$\begin{aligned} r' &= \frac{3k\nu^4 r^3}{8\omega^5} + r\left(\frac{3k\nu^2 z^2}{2\omega^3} - \frac{3k\nu^2\,(\nu^2+4\omega^2)}{8\omega^5}\right), \\ z' &= \frac{\nu z\,(3\nu^2+2\omega^2)\,(k\nu+\omega^2)}{2\omega^5} - \frac{3\nu^3 r^2 z\,(k\nu+\omega^2)}{2\omega^5} - \frac{\nu z^3\,(k\nu+\omega^2)}{\omega^3}. \end{aligned} \tag{6.12}$$

We are going to show that the guiding system has Hopf bifurcations at $S_\pm = (1, \pm 1)$. Due to the symmetry of the system, the proof will be done only for the Hopf bifurcation at S_+.

We translate S_+ to the origin of coordinates, and do the change of coordinates

$$(r, z) = \left(\frac{6\sqrt{5}\nu^3\zeta}{8\omega^3 - 3\nu^2\omega}, \frac{6\sqrt{5}\nu^3\rho}{3\nu^2\omega - 8\omega^3}\right),$$

obtaining the system

$$\begin{pmatrix}\dot\rho \\ \dot\zeta\end{pmatrix} = \begin{pmatrix}0 & -\lambda \\ \lambda & 0\end{pmatrix}\begin{pmatrix}\rho \\ \zeta\end{pmatrix} + \begin{pmatrix}h_1(\rho,\zeta) \\ h_2(\rho,\zeta)\end{pmatrix}, \tag{6.13}$$

where $\lambda = \dfrac{6\sqrt{5}\nu^3}{\omega\,(3\nu^2 - 8\omega^2)}$, and

$$\begin{aligned} h_1(\rho,\zeta) &= -\frac{3\nu^3\left(\nu^2(\rho+1)\left(\rho^2 + 2\sqrt{5}\rho\zeta + 5\zeta^2\right) + 4\rho\omega^2\left(\rho^2+\rho-2\sqrt{5}\zeta\right)\right)}{4\omega^3\,(8\omega^2 - 3\nu^2)}, \\ h_2(\rho,\zeta) &= \frac{3\nu^3}{16\sqrt{5}\omega^5\,(8\omega^2-3\nu^2)}\Big(4\nu^2\omega^2\left((7\rho+10)\rho^2 + 5(\rho-2)\zeta^2 + 8\sqrt{5}(\rho+1)\rho\zeta\right) \end{aligned}$$

$$+\nu^4\left(\rho^3+3\sqrt{5}\rho^2\zeta+15\rho\zeta^2+5\sqrt{5}\zeta^3\right)+16\rho\omega^4\left((\rho-5)\rho-2\sqrt{5}\zeta\right)\Big).$$

In order to calculate the first Lyapunov coefficient of system (6.13) we compute the multilinear functions

$$A=\begin{pmatrix}0 & -\lambda\\ \lambda & 0\end{pmatrix},$$

$$B(\mathbf{x},\mathbf{y})=\Bigg(\frac{3\nu^3}{6\nu^2\omega^3-16\omega^5}\Big(x_1y_1\left(\nu^2+4\omega^2\right)+\sqrt{5}x_1y_2\left(\nu^2-4\omega^2\right)$$
$$+x_2\left(\sqrt{5}y_1\left(\nu^2-4\omega^2\right)+5\nu^2y_2\right)\Big),\frac{3\nu^5}{\sqrt{5}\omega^3\left(8\omega^2-3\nu^2\right)}$$
$$\left(5x_1y_1+2\sqrt{5}x_1y_2+2\sqrt{5}y_1x_2-5x_2y_2\right)-6\nu^3\omega^2\Big(5x_1y_1$$
$$+\sqrt{5}x_1y_2+\sqrt{5}y_1x_2\Big)\Bigg),\quad\text{and}$$

$$C(\mathbf{x},\mathbf{y},z)=\Bigg(\frac{36\nu^3x_1y_1z_1\omega^2}{6\nu^2\omega^3-16\omega^5}+\frac{3\nu^5}{6\nu^2\omega^3-16\omega^5}\Big(x_1\Big(3y_1z_1+2\sqrt{5}y_1z_2$$
$$+2\sqrt{5}z_1y_2+5y_2z_2\Big)+x_2\left(2\sqrt{5}y_1z_1+5y_1z_2+5z_1y_2\right)\Big),$$
$$\frac{3}{40\omega^5\left(8\omega^2-3\nu^2\right)}\Big(48\sqrt{5}\nu^3x_1y_1z_1\omega^4+3\nu^7\Big(x_1\Big(y_1\left(\sqrt{5}z_1+5z_2\right)$$
$$+5y_2\left(z_1+\sqrt{5}z_2\right)\Big)+5x_2\left(y_1z_1+\sqrt{5}y_1z_2+\sqrt{5}z_1y_2+5y_2z_2\right)\Big)$$
$$+4\nu^5\omega^2\Big(x_1\left(21\sqrt{5}y_1z_1+40y_1z_2+40z_1y_2+5\sqrt{5}y_2z_2\right)$$
$$+5x_2\left(8y_1z_1+\sqrt{5}y_1z_2+\sqrt{5}z_1y_2\right)\Big)\Big)\Bigg),$$

as defined in Section 6.2.1. We take the eigenvectors

$$p=q=\frac{1}{\sqrt{2}}(1,-i).$$

In order to calculate the first Lyapunov coefficient we compute the quantities

$$h_{11}=\left(-\frac{8\omega^4}{3\nu^4+8\omega^4},-\frac{3\nu^4}{3\nu^4+8\omega^4}\right),$$

$$h_{20}=\left(\frac{8i\omega^2\left(-6\sqrt{5}\nu^2+\left(11\sqrt{5}+10i\right)\omega^2\right)}{15\left(3\nu^4+8\omega^4\right)},\frac{\nu^2\left(3\left(25+4i\sqrt{5}\right)\nu^2+16\left(-5-2i\sqrt{5}\right)\omega^2\right)}{15\left(3\nu^4+8\omega^4\right)}\right),$$

$$h_{30}=\Bigg(\frac{4\omega^2\left(27\left(1-i\sqrt{5}\right)\nu^4+12\left(-17+7i\sqrt{5}\right)\nu^2\omega^2+4\left(53-13i\sqrt{5}\right)\omega^4\right)}{5\nu^2\left(3\nu^4+8\omega^4\right)\sqrt{\frac{24\omega^4}{\nu^4}+9}},$$
$$\frac{\sqrt{3}\nu^2\left(3\left(17+8i\sqrt{5}\right)\nu^4+8\left(-14-11i\sqrt{5}\right)\nu^2\omega^2+16\left(1+4i\sqrt{5}\right)\omega^4\right)}{5\left(3\nu^4+8\omega^4\right)^{3/2}}\Bigg),$$

$$H_{21} = \left(\frac{8i\sqrt{3}\nu^3\omega\left(3\left(4\sqrt{5}+5i\right)\nu^4+16\left(\sqrt{5}+5i\right)\nu^2\omega^2+8\left(4\sqrt{5}+35i\right)\omega^4\right)}{5\left(3\nu^2-8\omega^2\right)\left(3\nu^4+8\omega^4\right)^{3/2}}, \right.$$
$$\left. \frac{6\sqrt{3}\nu^5\left(3\left(5+13i\sqrt{5}\right)\nu^4+32i\sqrt{5}\nu^2\omega^2+8\left(-5+3i\sqrt{5}\right)\omega^4\right)}{5\left(8\omega^3-3\nu^2\omega\right)\left(3\nu^4+8\omega^4\right)^{3/2}} \right),$$
$$G_{21} = \frac{48i\nu^3\left(3\nu^4+4\nu^2\omega^2+8\omega^4\right)}{\sqrt{5}\omega\left(8\omega^2-3\nu^2\right)\left(3\nu^4+8\omega^4\right)}.$$

Thus we have $l_1 = 0$.

Now we perform the computations to obtain the second Lyapunov coefficients

$$h_{21} = \left(\sqrt{\frac{3}{5}}\frac{2i\omega^2\left(8\omega^4-3\nu^4\right)}{\left(3\nu^4+8\omega^4\right)^{3/2}}, \frac{i\sqrt{3}\left(\sqrt{5}+5i\right)\left(3\nu^6-8\nu^2\omega^4\right)}{10\left(3\nu^4+8\omega^4\right)^{3/2}} \right),$$
$$h_{40} = \left(\frac{64\omega^2}{1125\left(3\nu^4+8\omega^4\right)^2}\left(72\left(35-13i\sqrt{5}\right)\nu^6+3\left(-6595+1016i\sqrt{5}\right)\nu^4\omega^2 \right.\right.$$
$$\left. +4\left(9715-47i\sqrt{5}\right)\nu^2\omega^4+\left(-21445-1324i\sqrt{5}\right)\omega^6\right), \frac{1}{375\left(3\nu^4+8\omega^4\right)^2}$$
$$\left(9\left(1775+3128i\sqrt{5}\right)\nu^8+96\left(-325-1441i\sqrt{5}\right)\nu^6\omega^2\right.$$
$$\left.\left. +64\left(-1525+3044i\sqrt{5}\right)\nu^4\omega^4+4096\left(25-17i\sqrt{5}\right)\nu^2\omega^6\right)\right),$$
$$h_{31} = \left(\frac{4\omega^2}{75\left(3\nu^4+8\omega^4\right)^2}\left(18\left(15-19i\sqrt{5}\right)\nu^6+\left(-345+441i\sqrt{5}\right)\nu^4\omega^2\right.\right.$$
$$\left. +16\left(-70+29i\sqrt{5}\right)\nu^2\omega^4+8\left(-85+89i\sqrt{5}\omega^6\right)\right), \frac{\nu^2}{150\left(3\nu^4+8\omega^4\right)^2}$$
$$\left(9\left(205+31i\sqrt{5}\right)\nu^6+48\left(95-4i\sqrt{5}\right)\nu^4\omega^2+8\left(965+179i\sqrt{5}\right)\nu^2\omega^4\right.$$
$$\left.\left. +128\left(-125-23i\sqrt{5}\right)\omega^6\right)\right),$$
$$h_{22} = \left(-\frac{16\omega^4\left(93\nu^4-176\nu^2\omega^2+188\omega^4\right)}{15\left(3\nu^4+8\omega^4\right)^2}, -\frac{\nu^4\left(141\nu^4-352\nu^2\omega^2+496\omega^4\right)}{5\left(3\nu^4+8\omega^4\right)^2} \right),$$
$$H_{32} = \left(\frac{4\sqrt{3}\nu^3\omega}{25\left(3\nu^2-8\omega^2\right)\left(3\nu^4+8\omega^4\right)^{5/2}}\left(9\left(665+101i\sqrt{5}\right)\nu^8 \right.\right.$$
$$+48\left(160-47i\sqrt{5}\right)\nu^6\omega^2+16\left(-1000+689i\sqrt{5}\right)\nu^4\omega^4$$
$$\left. +128\left(340-47i\sqrt{5}\right)\nu^2\omega^6+64\left(-2665+101i\sqrt{5}\right)\omega^8\right),$$
$$\frac{6\sqrt{3}\nu^5}{25\left(3\nu^2\omega-8\omega^3\right)\left(3\nu^4+8\omega^4\right)^{5/2}}\left(9\left(-360-461i\sqrt{5}\right)\nu^8\right.$$
$$+24\left(35+129i\sqrt{5}\right)\nu^6\omega^2+8\left(815-563i\sqrt{5}\right)\nu^4\omega^4$$

$$+ 64\left(-25+69i\sqrt{5}\right)\nu^2\omega^6 + 64\left(195+94i\sqrt{5}\right)\omega^8\Big)\Bigg),$$

$$G_{32} = \frac{48i\sqrt{5}\delta\nu^3\left(3\nu^4-8\nu^2\omega^2+8\omega^4\right)\left(3\nu^4+4\nu^2\omega^2+8\omega^4\right)}{\left(8\omega^3-3\nu^2\omega\right)\left(3\nu^4+8\omega^4\right)^2}.$$

Obtaining again $l_2 = 0$.

Finally we have to calculate

$$h_{32} = \Bigg(\frac{\sqrt{3}\omega^2}{25\left(3\nu^4+8\omega^4\right)^{5/2}}\Big(\left(-297-999i\sqrt{5}\right)\nu^8 + 48\left(1+6i\sqrt{5}\right)\nu^6\omega^2$$
$$+ 2608\nu^4\omega^4 + 128\left(1-6i\sqrt{5}\right)\nu^2\omega^6 + 192\left(-11+37i\sqrt{5}\right)\omega^8\Big),$$
$$\frac{\sqrt{3}\nu^2}{50\left(3\nu^4+8\omega^4\right)^{5/2}}\Big(81\left(-29+8i\sqrt{5}\right)\nu^8 + 24\left(29-7i\sqrt{5}\right)\nu^6\omega^2$$
$$- 1304i\left(\sqrt{5}-i\right)\nu^4\omega^4 + 64\left(-31+5i\sqrt{5}\right)\nu^2\omega^6 + 192\left(98-13i\sqrt{5}\right)\omega^8\Big)\Bigg),$$

$$h_{41} = \Bigg(\frac{8\omega^2}{125\sqrt{3}\left(3\nu^4+8\omega^4\right)^{5/2}}\Big(27\left(211-140i\sqrt{5}\right)\nu^8$$
$$+ 36\left(-848+199i\sqrt{5}\right)\nu^6\omega^2 + 12\left(271-536i\sqrt{5}\right)\nu^4\omega^4$$
$$+ 32\left(-19+536i\sqrt{5}\right)\nu^2\omega^6 + 16\left(3857-619i\sqrt{5}\right)\omega^8\Big),$$
$$\frac{\sqrt{3}\nu^2}{125\left(3\nu^4+8\omega^4\right)^{5/2}}\Big(9\left(1328+455i\sqrt{5}\right)\nu^8 + 24\left(283-419i\sqrt{5}\right)\nu^6\omega^2$$
$$+ 72\left(-29+149i\sqrt{5}\right)\nu^4\omega^4 + 576\left(-183-83i\sqrt{5}\right)\nu^2\omega^6 +$$
$$128\left(259+397i\sqrt{5}\right)\omega^8\Big)\Bigg),$$

$$h_{42} = \Bigg(\frac{8\omega^2}{1875\left(3\nu^4+8\omega^4\right)^3}\Big(54\left(4445-3154i\sqrt{5}\right)\nu^{10} + 45\left(-2615+11036i\sqrt{5}\right)\nu^8\omega^2$$
$$+ 1200\left(-511-107i\sqrt{5}\right)\nu^6\omega^4 + 80\left(-15700+3653i\sqrt{5}\right)\nu^4\omega^6$$
$$+ 1280\left(5+44i\sqrt{5}\right)\nu^2\omega^8 + 64\left(11005+13399i\sqrt{5}\right)\omega^{10}\Big)\Bigg),$$

$$h_{33} = \Bigg(\frac{4\omega^2}{25\left(3\nu^4+8\omega^4\right)^3}\Big(-2187i\sqrt{5}\nu^{10} + 9\left(-5653-270i\sqrt{5}\right)\nu^8\omega^2$$
$$+ 24\left(4547+477i\sqrt{5}\right)\nu^6\omega^4 + 16\left(-8782+351i\sqrt{5}\right)\nu^4\omega^6$$
$$+ 64\left(2657-234i\sqrt{5}\right)\nu^2\omega^8 + 64\left(-1783+36i\sqrt{5}\right)\omega^{10}\Big),$$
$$\frac{3\nu^2}{50\left(3\nu^4+8\omega^4\right)^3}\Big(9\left(-1963+162i\sqrt{5}\right)\nu^{10} + 96\left(1103+135i\sqrt{5}\right)\nu^8\omega^2$$

$$+ 32\left(-6596 - 711i\sqrt{5}\right)\nu^6\omega^4 + 128\left(1531 - 243i\sqrt{5}\right)\nu^4\omega^6$$
$$+ 64\left(-2533 + 786i\sqrt{5}\right)\nu^2\omega^8 - 9216i\left(\sqrt{5} - 5i\right)\omega^{10}\Big)\Bigg),$$

$$H_{43} = \Bigg(\frac{24\sqrt{3}\nu^3\omega}{625\left(3\nu^2 - 8\omega^2\right)\left(3\nu^4 + 8\omega^4\right)^{7/2}}\Big(27\left(101960 + 11141i\sqrt{5}\right)\nu^{12}$$
$$+ 72\left(22270 + 13269i\sqrt{5}\right)\nu^{10}\omega^2 + 600i\left(2025\sqrt{5} + 18337i\right)\nu^8\omega^4$$
$$+ 320\left(69785 - 30641i\sqrt{5}\right)\nu^6\omega^6 + 4800i\left(4215\sqrt{5} + 10117i\right)\nu^4\omega^8$$
$$+ 1536\left(43005 - 5252i\sqrt{5}\right)\nu^2\omega^{10} + 512\left(-254120 + 10241i\sqrt{5}\right)\omega^{12}\Big),$$
$$\frac{6\sqrt{3}\nu^5}{625\left(3\nu^2\omega - 8\omega^3\right)\left(3\nu^4 + 8\omega^4\right)^{7/2}}\Big(\left(-7787205 - 6311547i\sqrt{5}\right)\nu^{12}$$
$$+ 216\left(39995 + 53957i\sqrt{5}\right)\nu^{10}\omega^2 + 3600\left(3289 - 11246i\sqrt{5}\right)\nu^8\omega^4$$
$$+ 640i\left(66413\sqrt{5} + 39685i\right)\nu^6\omega^6 + 3200\left(10024 - 911i\sqrt{5}\right)\nu^4\omega^8$$
$$+ 512\left(12115 - 62099i\sqrt{5}\right)\nu^2\omega^{10} + 112128\left(535 + 501i\sqrt{5}\right)\omega^{12}\Big)\Bigg),$$

$$G_{43} = \frac{36\nu^3}{625\left(8\omega^3 - 3\nu^2\omega\right)\left(3\nu^4 + 8\omega^4\right)^3}\Big(81\left(3375 + 4697i\sqrt{5}\right)\nu^{12}$$
$$+ 72\left(10125 - 17716i\sqrt{5}\right)\nu^{10}\omega^2 + 120\left(-31275 + 35639i\sqrt{5}\right)\nu^8\omega^4$$
$$+ 160\left(4725 - 33449i\sqrt{5}\right)\nu^6\omega^6 + 1280\left(5625 + 3566i\sqrt{5}\right)\nu^4\omega^8$$
$$+ 256\left(-28125 - 15857i\sqrt{5}\right)\nu^2\omega^{10} + 6144\left(375 + 1193i\sqrt{5}\right)\omega^{12}\Big).$$

Hence

$$l_3 = \frac{\mathrm{Re}(G_{43})}{144} = \frac{9\nu^3\left(3\nu^4 - 8\omega^4\right)\left(81\nu^8 + 216\nu^6\omega^2 - 896\nu^4\omega^4 + 800\nu^2\omega^6 - 256\omega^8\right)}{20\left(8\omega^3 - 3\nu^2\omega\right)\left(3\nu^4 + 8\omega^4\right)^3}$$

and by the arguments given in Section 6.2.1 we know that the guiding system (6.12) has a Hopf bifurcation. Figure 6.4 numerically shows the periodic solution for system (6.12). □

6.2.4 Proof of Theorem 34

Proof. The proof will be provided using Lemma 35. In order to simplify the computations we take $\alpha = \mu\nu\left(1 - \frac{\beta h}{\omega^2}\right)$ with $\omega > 0$. Then the bifurcation coefficient becomes $\bar{k} = -\frac{\left(\omega^2 - \beta h\right)\left(\mu^2\nu^2 + \omega^2\right)}{\nu\omega^2}$, and the eigenvalues are $\pm i\omega$ and $\frac{\beta h\mu\nu}{\omega^2}$.

The characteristic polynomial of the Jacobian matrix of system (6.2) at the origin is

$$\beta h\mu\nu + \lambda\left(\nu\left(k + \mu^2\nu\right) - \frac{\beta h\left(\mu^2\nu^2 + \omega^2\right)}{\omega^2}\right) + \lambda^2\frac{\beta h\mu\nu}{\omega^2} - \lambda^3 = 0.$$

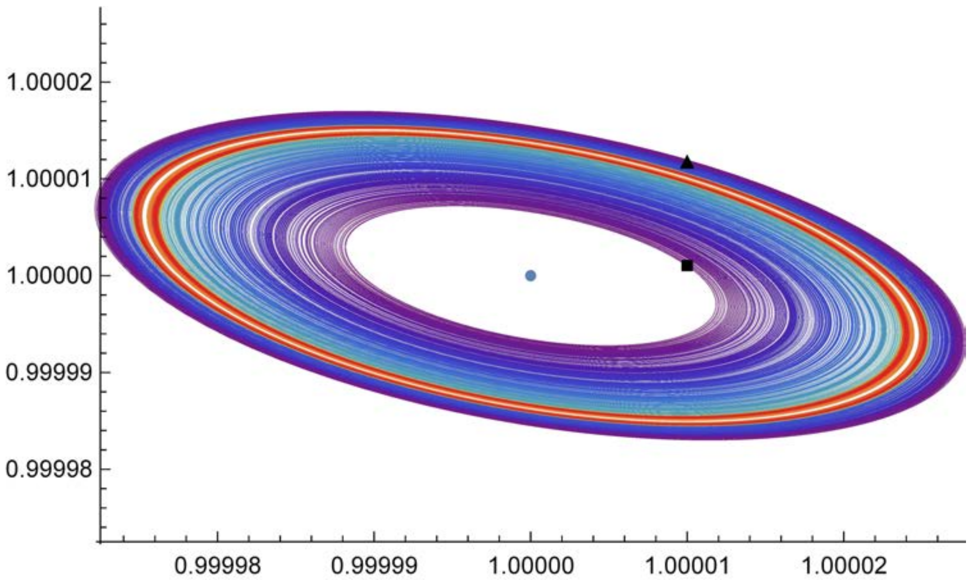


Figure 6.4: Two solutions converging to the limit cycle resulting of the Hopf bifurcation at S_+. Here $k = -1.595$, $\nu = 1$ and $\omega = 1$. The square and triangle represent the initial points $(1.00001, 1.000002)$ and $(1.00001, 1.00002)$ respectively.

Assume that $\lambda(k)$ is a solution of the characteristic polynomial. It depends continuously on k and we can write

$$\frac{d\lambda}{dk}(k) = \frac{\nu\omega^2\lambda(k)}{\beta h\left(\mu^2\nu^2+\omega^2\right) - 2\beta h\mu\nu\lambda(k) + 3\omega^2\lambda(k)^2 - \nu\omega^2\left(k+\mu^2\nu\right)}. \tag{6.14}$$

Let $\lambda(\bar{k}) = i\omega$ and taking $k = \bar{k}$ in (6.14) we have that

$$\operatorname{Re}\left(\frac{d\lambda}{dk}(\bar{k})\right) = -\frac{\beta h\mu\nu^2\omega^2}{2\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)} \neq 0.$$

Thus in order to apply Lemma 35 accordingly with the arguments of Section 6.2.1, we calculate

$$A = \begin{pmatrix} \mu\nu & \nu & 0 \\ \dfrac{\left(\beta h-\omega^2\right)\left(\mu^2\nu^2+\omega^2\right)}{\nu\omega^2} & \mu\nu\left(\dfrac{\beta h}{\omega^2}-1\right) & -h \\ 0 & \beta & 0 \end{pmatrix},$$

and the multilinear functions $B(\mathbf{x},\mathbf{y}) = (0,0,0)$ and $C(\mathbf{x},\mathbf{y},z) = (-6x_1y_1z_1, 0, 0)$. We also obtain the eigenvectors

$$p = \sigma_1\left(-\frac{i\beta\left(\beta h-\omega^2\right)(\mu\nu-i\omega)^2}{2\nu\omega\left(\beta h\mu\nu-i\omega^3\right)}, \frac{\beta\omega(\omega+i\mu\nu)}{2\beta h\mu\nu-2i\omega^3}, \frac{\beta h(\mu\nu-i\omega)}{2\beta h\mu\nu-2i\omega^3}\right),$$

$$q = \sigma_2\left(-\frac{i\nu\omega}{\operatorname{sgn}(\beta)(\mu\nu-i\omega)}, \frac{i\omega}{\operatorname{sgn}(\beta)}, |\beta|\right),$$

where

$$\sigma_1 = \sqrt{\frac{\omega^2\left(\dfrac{\nu^2}{\mu^2\nu^2+\omega^2}+1\right)}{\beta^2}+1}, \quad \sigma_2 = \sqrt{\frac{\mu^2\nu^2+\omega^2}{\beta^2\left(\mu^2\nu^2+\omega^2\right)+\left(\mu^2+1\right)\nu^2\omega^2+\omega^4}}.$$

In this case the first Lyapunov coefficient (see (6.6))

$$l_1 = \frac{1}{2\omega}\mathrm{Re}\,(\bar{p}.C(q,q,\bar{q}))$$

becomes

$$l_1 = \frac{3\nu^3\omega\left(\omega^2-\beta h\right)\left(\beta h\mu^2\nu^2-\omega^4\right)}{2\left(\beta^2\left(\mu^2\nu^2+\omega^2\right)+\left(\mu^2+1\right)\nu^2\omega^2+\omega^4\right)\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)}.$$

So the result follows directly from Lemma 35. □

6.3 Appendix E: Hopf bifurcation via Theorem 12

In this Appendix, we show that the Hopf bifurcation detected by the computation of the Lyapunov coefficients in Theorem 34 can also be detected with the averaging method (see Theorem 12). More precisely we have the following result.

Theorem 36. *Consider the differential system* (6.2) *with*

$$k = -\frac{\left(\omega^2-\beta h\right)\left(\mu^2\nu^2+\omega^2\right)}{\nu\omega^2} - \varepsilon\frac{\alpha_1\left(\beta h\mu^2\nu^2+\omega^4\right)}{\beta h\mu\nu^2},\ \ \alpha = \frac{\mu\nu\omega^2-\beta h\mu\nu}{\omega^2}+\varepsilon\alpha_1,$$

and assume that

$$\beta h\mu\nu^2\left(\beta h-\omega^2\right)\left(\omega^4-\beta h\mu^2\nu^2\right) > 0.$$

Then for $|\varepsilon| > 0$ *sufficiently small system* (6.2) *has a periodic solution bifurcating from the origin of coordinates.*

Proof. Following the algorithm used in the proof of Theorem 31, we first do a reescaling of the system by doing $(x,y,z) = \varepsilon(\bar{x},\bar{y},\bar{z})$. Then we do the linear change of coordinates

$$\begin{aligned}(\bar{x},\bar{y},\bar{z}) = \Bigg(&X+Z, -\frac{-\beta h\mu\nu Z+\mu\nu X\omega^2+\omega^3Y+\mu\nu\omega^2Z}{\nu\omega^2},\\ &\frac{\beta hX\omega-\beta h\mu\nu Y+\beta h\omega Z-\omega^3Z}{h\nu\omega}\Bigg),\end{aligned}$$

obtaining the system

$$\begin{aligned}
\dot{X} =& -\omega Y+\varepsilon\frac{\alpha_1\left(\beta^2h^2\mu^2\nu^2Z-\beta h\mu\nu\omega^3Y+\omega^6(X+Z)\right)}{\beta^2h^2\mu^2\nu^2+\omega^6}\\
&-\varepsilon^2\frac{\nu(X+Z)^3\left(\beta h-\omega^2\right)\left(\beta h\mu^2\nu^2-\omega^4\right)}{\beta^2h^2\mu^2\nu^2+\omega^6},\\
\dot{Y} =& \omega X+\varepsilon\frac{\left(\alpha_1\beta^2h^2\mu^2\nu^2\omega^3Z-\alpha_1\beta h\mu\nu\omega^6Y+\alpha_1X\omega^9+\alpha_1\omega^9Z\right)}{\beta^3h^3\mu^3\nu^3+\beta h\mu\nu\omega^6}\\
&+\varepsilon^2\frac{\mu\nu^2\omega(X+Z)^3\left(\omega^4-\beta^2h^2\right)}{\beta^2h^2\mu^2\nu^2+\omega^6},\\
\dot{Z} =& \frac{\beta h\mu\nu}{\omega^2}Z+\varepsilon\left(\frac{\alpha_1\beta h\mu\nu X\left(\beta h\mu\nu\cos(Y)+\omega^3\sin(Y)\right)}{\beta^2h^2\mu^2\nu^2+\omega^6}-\alpha_1(X\cos(Y)+Z)\right)
\end{aligned} \tag{6.15}$$

$$
-\varepsilon^2 \frac{\beta h \nu \omega^2 \left(\mu^2\nu^2+\omega^2\right)\left(X\cos(Y)+Z\right)^3}{\beta^2 h^2 \mu^2 \nu^2 + \omega^6}.
$$

Then we take the cylindrical coordinates $X = r\cos\theta$, $Y = r\sin\theta$ and $Z = z$ in system (6.15) and take θ as the new independent variable obtaining a differential system

$$
(r', z') = F_0(r,z) + \varepsilon F_1(r,z,\theta) + \varepsilon^2 F_2(r,z,\theta) + \mathcal{O}(\varepsilon^3), \tag{6.16}
$$

where

$$
\begin{aligned}
F_0(r,z) =& \left(0, \frac{\beta h \mu \nu z}{\omega^3}\right),\\
F_1(r,z,\theta) =& \Bigg(\Big(z\big(\beta^2h^2\mu^2\nu^2+\omega^6\big) - \beta h\mu\nu r\omega^3\sin(\theta) + r\omega^6\cos(\theta)\Big)\\
& \frac{\alpha_1\left(\beta h\mu\nu\cos(\theta)+\omega^3\sin(\theta)\right)}{\beta h\mu\nu\omega\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)}, -\Big(z\big(\beta^2h^2\mu^2\nu^2+\omega^6\big)\\
& -\beta h\mu\nu r\omega^3\sin(\theta) + r\omega^6\cos(\theta)\Big)\Big(\omega^3(r+z\cos(\theta))\\
& -\beta h\mu\nu z\sin(\theta)\Big)\frac{\alpha_1}{r\omega^4\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
F_2(r,z,\theta) =& \Bigg(\frac{1}{h^2r\beta^2\mu^2\nu^2\omega^2\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)^2}\Big(-hr^2\beta\mu\nu\omega\big(\alpha_1^2\omega^{14}+hr^2\beta\mu\nu^2\omega^{12}\\
& -h^2r^2\beta^2\mu\nu^2\left(\mu^2\nu^2+\omega^2\right)\omega^8+2h^3r^2\beta^3\mu^3\nu^4\omega^6-h^4r^2\beta^4\mu^3\nu^4\left(\mu^2\nu^2+\omega^2\right)\omega^2\\
& +h^5r^2\beta^5\mu^5\nu^6\big)\cos^4(\theta)+r\omega\big(r\omega\big(-\alpha_1^2\omega^{16}+h^2\beta^2\mu^2\nu^2\left(3\alpha_1^2+r^2\mu\nu^2\right)\omega^{10}\\
& +h^4r^2\beta^4\mu^3\nu^4(\mu\nu-\omega)(\mu\nu+\omega)\omega^4-h^6r^2\beta^6\mu^5\nu^6\big)\sin(\theta)-hz\beta\mu\nu\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\\
& \big(2\alpha_1^2\omega^8+3hr^2\beta\mu\nu^2\omega^6-3h^2r^2\beta^2\mu\nu^2\left(\mu^2\nu^2+\omega^2\right)\omega^2+3h^3r^2\beta^3\mu^3\nu^4\big)\big)\cos^3(\theta)\\
& -\omega\big(h\beta\mu\nu\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\big(\alpha_1^2\omega^8+3hr^2\beta\mu\nu^2\omega^6+h^2\beta^2\mu\nu^2\big(\alpha_1^2\mu\\
& -3r^2\left(\mu^2\nu^2+\omega^2\right)\big)\omega^2+3h^3r^2\beta^3\mu^3\nu^4\big)z^2+r\omega\sin(\theta)\big(3hr\alpha_1^2\beta\mu\nu\big(h^2\beta^2\mu^2\nu^2\\
& -\omega^6\big)\sin(\theta)\omega^7+z\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\big(2\alpha_1^2\omega^{10}-h^2\beta^2\mu^2\nu^2\left(4\alpha_1^2+3r^2\mu\nu^2\right)\omega^4\\
& +3h^4r^2\beta^4\mu^3\nu^4\big)\big)\big)\cos^2(\theta)+\big(\sin(\theta)\big(hr\alpha_1^2\beta\mu\nu\sin(\theta)\big(4z\omega^{12}+hr\beta\mu\nu\big(h^2\beta^2\mu^2\nu^2\\
& -3\omega^6\big)\sin(\theta)\omega^3-2h^4z\beta^4\mu^4\nu^4\big)\omega^3+z^2\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\big(-\alpha_1^2\omega^{12}\\
& +3h^2r^2\beta^2\mu^3\nu^4\omega^6+h^4\beta^4\mu^3\nu^4\left(\alpha_1^2\mu-3r^2\omega^2\right)\big)\big)-h^2rz^3\beta^2\mu^2\nu^3\omega\left(h\beta-\omega^2\right)\\
& \left(h\beta\mu^2\nu^2-\omega^4\right)\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\big)\cos(\theta)+h\beta\mu\nu\omega^2\sin(\theta)\big(\alpha_1^2\omega\big(h^2rz\beta^2\mu^2\nu^2\\
& \sin(2\theta)\omega^6+\sin(\theta)\left(z\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)-hr\beta\mu\nu\omega^3\sin(\theta)\right)^2\Big)-hrz^3\beta\mu^2\nu^3\\
& \left(h^2\beta^2-\omega^4\right)\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\big)\Big), \frac{1}{hr^2\beta\mu\nu\omega^5\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)^2}\Big(r^2z\omega^2\big(\alpha_1^2\omega^{16}\\
& -h^2r^2\beta^2\mu^3\nu^4\omega^{10}+h^4r^2\beta^4\mu^3\nu^4\left(\omega^2-\mu^2\nu^2\right)\omega^4+h^6r^2\beta^6\mu^5\nu^6\big)\cos^4(\theta)\\
& -r\omega\big(-\left(r^2+2z^2\right)\alpha_1^2\omega^{17}+h^2\beta^2\mu\nu^2\big(\left(\mu^2\nu^2+\omega^2\right)r^4+z^2\mu\left(3r^2\mu\nu^2-2\alpha_1^2\right)\big)\omega^{11}\\
& +h^4r^2\beta^4\mu^3\nu^4\big(\left(r^2+3z^2\right)\mu^2\nu^2+\left(r^2-3z^2\right)\omega^2\big)\omega^5-3h^6r^2z^2\beta^6\mu^5\nu^6\omega+hrz\beta\mu\nu\\
& \big(4\alpha_1^2\omega^{14}+hr^2\beta\mu\nu^2\omega^{12}-h^2r^2\beta^2\mu\nu^2\left(\mu^2\nu^2+\omega^2\right)\omega^8+2h^3r^2\beta^3\mu^3\nu^4\omega^6\\
& -h^4r^2\beta^4\mu^3\nu^4\left(\mu^2\nu^2+\omega^2\right)\omega^2+h^5r^2\beta^5\mu^5\nu^6\big)\sin(\theta)\big)\cos^3(\theta)
\end{aligned}
$$

$$+\omega\left(z\omega\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\left(\left(2r^2+z^2\right)\alpha_1^2\omega^{10}+h^2\beta^2\mu\nu^2\left(z^2\mu\left(\alpha_1^2-3r^2\mu\nu^2\right)\right.\right.\right.$$
$$\left.-3r^4\left(\mu^2\nu^2+\omega^2\right)\right)\omega^4+3h^4r^2z^2\beta^4\mu^3\nu^4\Big)+3hr\beta\mu\nu\sin(\theta)\left(-\left(r^2+2z^2\right)\alpha_1^2\omega^{14}\right.$$
$$-hr^2z^2\beta\mu\nu^2\omega^{12}+2hrz\alpha_1^2\beta\mu\nu\sin(\theta)\omega^{11}+h^2z^2\beta^2\mu\nu^2\left(r^2\left(\mu^2\nu^2+\omega^2\right)-2\alpha_1^2\mu\right)\omega^8$$
$$\left.\left.-2h^3r^2z^2\beta^3\mu^3\nu^4\omega^6+h^4r^2z^2\beta^4\mu^3\nu^4\left(\mu^2\nu^2+\omega^2\right)\omega^2-h^5r^2z^2\beta^5\mu^5\nu^6\right)\right)\cos^2(\theta)$$
$$+\omega\left(-4h^3r^2z\alpha_1^2\beta^3\mu^3\nu^3\sin^3(\theta)\omega^8+3h^2r\alpha_1^2\beta^2\mu^2\nu^2\left(r^2\omega^6+2z^2\left(\omega^6\right.\right.\right.$$
$$\left.\left.+h^2\beta^2\mu^2\nu^2\right)\right)\sin^2(\theta)\omega^5+rz^2\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\left(\alpha_1^2\omega^{10}-h^2\beta^2\mu\nu^2\left(\left(3r^2+z^2\right)\right.\right.$$
$$\left.\left.\mu^2\nu^2+3r^2\omega^2-\alpha_1^2\mu\right)\omega^4+h^4z^2\beta^4\mu^3\nu^4\right)\omega-hz\beta\mu\nu\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)$$
$$\left(2\left(2r^2+z^2\right)\alpha_1^2\omega^8+3hr^2z^2\beta\mu\nu^2\omega^6+h^2z^2\beta^2\mu\nu^2\left(2\alpha_1^2\mu-3r^2\left(\mu^2\nu^2+\omega^2\right)\right)\omega^2\right.$$
$$\left.+3h^3r^2z^2\beta^3\mu^3\nu^4\right)\sin(\theta)\Big)\cos(\theta)-h\beta\mu\nu\left(hr^2z^3\beta\nu\left(\mu^2\nu^2+\omega^2\right)\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\omega^6\right.$$
$$+\sin(\theta)\left(r\omega\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\left(\alpha_1^2\omega^8+hz^2\beta\mu\nu^2\omega^6-h^2\beta^2\mu\nu^2\left(z^2\left(\mu^2\nu^2+\omega^2\right)\right.\right.\right.$$
$$\left.\left.-\alpha_1^2\mu\right)\omega^2+h^3z^2\beta^3\mu^3\nu^4\right)z^2+h\alpha_1^2\beta\mu\nu\sin(\theta)\left(hr\beta\mu\nu\omega^3\sin(\theta)\left(r^2\omega^6\right.\right.$$
$$\left.-hrz\beta\mu\nu\sin(\theta)\omega^3+2z^2\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)\right)-z\left(\omega^6+h^2\beta^2\mu^2\nu^2\right)$$
$$\left.\left.\left.\left(\left(2r^2+z^2\right)\omega^6+h^2z^2\beta^2\mu^2\nu^2\right)\right)\right)\right)\Bigg)\Bigg).$$

Thus we use (1.15) to obtain the the following averaged functions of system (6.16):

$$g_0(r,z)=\left(0,\left(1-e^{-\frac{2\pi\beta h\mu\nu}{\omega^3}}\right)z\right),$$

$$g_1(r,z)=\left(\frac{e^{\frac{2\pi\beta h\mu\nu}{\omega^3}}\left(\alpha_1\beta^2h^2\mu^2\nu^2\omega^2z-\alpha_1\omega^8z\right)}{2\beta^3h^3\mu^3\nu^3+2\beta h\mu\nu\omega^6}-\frac{\alpha_1\beta^2h^2\mu^2\nu^2\omega^2z-\alpha_1\omega^8z}{2\beta^3h^3\mu^3\nu^3+2\beta h\mu\nu\omega^6},\right.$$
$$\left.e^{\frac{2\pi\beta h\mu\nu}{\omega^3}}\left(\frac{\alpha_1\beta h\mu\nu\omega^2z^2}{\beta^2h^2\mu^2\nu^2r+r\omega^6}-\frac{3\pi\alpha_1z}{2\omega}\right)-\frac{\alpha_1\beta h\mu\nu\omega^2z^2e^{\frac{4\pi\beta h\mu\nu}{\omega^3}}}{\beta^2h^2\mu^2\nu^2r+r\omega^6}\right),$$

$$g_2(r,z)=\left(\frac{\alpha_1^2\omega^4z^2e^{\frac{8\pi\beta h\mu\nu}{\omega^3}}\left(-\frac{8}{\beta^2h^2\mu^2\nu^2+\omega^6}+\frac{18}{4\beta^2h^2\mu^2\nu^2+\omega^6}-\frac{1}{\beta^2h^2\mu^2\nu^2}\right)}{8r}\right.$$
$$-\frac{\omega^2z^2e^{\frac{6\pi\beta h\mu\nu}{\omega^3}}}{r\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)\left(9\beta^2h^2\mu^2\nu^2+\omega^6\right)}\left(3\beta^3h^3\mu^3\nu^4rz-\beta^2h^2\mu\nu^2\omega^2\right.$$
$$\left.\left(6\alpha_1^2\mu+rz\left(3\mu^2\nu^2+4\omega^2\right)\right)+3\beta h\mu\nu^2r\omega^6z+\omega^8\left(2\alpha_1^2+\mu\nu^2rz\right)\right)$$
$$+\frac{1}{24}e^{\frac{2\pi\beta h\mu\nu}{\omega^3}}\left(-\frac{24\beta^2h^2\mu^2\nu^2z\left(2\alpha_1^2\omega^4+3\mu\nu^2r^2\left(\beta^2h^2-\omega^4\right)\right)}{\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}\right.$$
$$+\frac{81\beta^2h^2\mu\nu^2r^2z-54\beta h\mu\nu^2r^2\omega^2z-3\omega^4z\left(8\alpha_1^2+9\mu\nu^2r^2\right)}{\beta^2h^2\mu^2\nu^2+\omega^6}$$
$$+\frac{z\left(-9\beta^2h^2\mu\nu^2r^2-18\beta h\mu\nu^2r^2\omega^2+\omega^4\left(32\alpha_1^2+27\mu\nu^2r^2\right)\right)}{\beta^2h^2\mu^2\nu^2+9\omega^6}$$
$$\left.+\frac{10\alpha_1^2\omega^4z}{\beta^2h^2\mu^2\nu^2}+\frac{30\alpha_1^2\omega^4z}{\beta^2h^2\mu^2\nu^2+4\omega^6}\right)$$
$$+\frac{1}{24}ze^{\frac{4\pi\beta h\mu\nu}{\omega^3}}\left(\frac{864\alpha_1^2\beta^2h^2\mu^2\nu^2\omega^4}{\left(4\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}-\frac{275\alpha_1^2\omega^4}{4\beta^2h^2\mu^2\nu^2+9\omega^6}\right.$$
$$-\frac{3\alpha_1^2\omega\left(144\pi\beta h\mu\nu+79\omega^3\right)}{4\beta^2h^2\mu^2\nu^2+\omega^6}+\frac{6\left(3\beta^2h^2\mu\nu^2r^2z-3\beta h\mu\nu^2r^2\omega^2z+4\alpha_1^2\omega^4(r-z)\right)}{r\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)}$$
$$+\frac{2\alpha_1^2\omega^4(4r+3z)}{\beta^2h^2\mu^2\nu^2r}+\frac{12\beta^2h^2\mu^2\nu^2\left(4\alpha_1^2\omega^4+3\mu\nu^2rz\left(\omega^4-\beta^2h^2\right)\right)}{\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}+\frac{72\pi\alpha_1^2\omega}{\beta h\mu\nu}$$

$$-\frac{18r\omega^2 z}{\beta h\mu}+\frac{18rz}{\mu}\Bigg)+\frac{1}{24}\Bigg(\frac{z\left(9\beta^2h^2\mu\nu^2r^2+18\beta h\mu\nu^2r^2\omega^2+\omega^4\left(-\left(32\alpha_1^2+27\mu\nu^2r^2\right)\right)\right)}{\beta^2h^2\mu^2\nu^2+9\omega^6}$$

$$+\frac{3}{r\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)}\left(\beta^2h^2\mu\nu^2rz\left(-27r^2-6rz+4z^2\right)+2\beta h\nu r^2\omega\left(\pi\left(4\alpha_1^2\mu\right.\right.\right.$$

$$\left.\left.\left.+3r^2\left(\mu^2\nu^2+\omega^2\right)\right)+3\mu\nu\omega z(3r+z)\right)+\omega^4z\left(9\mu\nu^2r^3-4\mu\nu^2rz^2+8\alpha_1^2z\right)\right)$$

$$+\frac{36z^2\left(-3\beta^2h^2\mu\nu^2rz+2\beta h\mu\nu^2r\omega^2z+\omega^4\left(2\alpha_1^2+\mu\nu^2rz\right)\right)}{r\left(9\beta^2h^2\mu^2\nu^2+\omega^6\right)}-\frac{3\alpha_1^2\omega^4z(6r+z)}{\beta^2h^2\mu^2\nu^2r}$$

$$+\frac{3\alpha_1^2\omega^4z(79r-18z)}{r\left(4\beta^2h^2\mu^2\nu^2+\omega^6\right)}+\frac{36\beta^2h^2\mu^3\nu^4rz(2r+z)\left(\beta^2h^2-\omega^4\right)}{\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}-\frac{864\alpha_1^2\beta^2h^2\mu^2\nu^2\omega^4z}{\left(4\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}$$

$$-\frac{30\alpha_1^2\omega^4z}{\beta^2h^2\mu^2\nu^2+4\omega^6}+\frac{275\alpha_1^2\omega^4z}{4\beta^2h^2\mu^2\nu^2+9\omega^6}-\frac{24\pi\alpha_1^2r\omega}{\beta h\mu\nu}+\frac{18r\omega^2z^2}{\beta h\mu}-\frac{18\pi\nu r^3}{\omega}-\frac{18rz^2}{\mu}\Bigg),$$

$$\frac{\alpha_1^2\omega^4z^3e^{\frac{10\pi\beta h\mu\nu}{\omega^3}}\left(\frac{8}{\beta^2h^2\mu^2\nu^2+\omega^6}-\frac{9}{4\beta^2h^2\mu^2\nu^2+\omega^6}\right)}{4r^2}+\frac{1}{12r^2}e^{\frac{8\pi\beta h\mu\nu}{\omega^3}}\Bigg(\frac{12\alpha_1^2\beta^2h^2\mu^2\nu^2r\omega^4z^2}{\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}$$

$$+\frac{3z^2\left(-6\beta^2h^2\mu\nu^2rz^2+4\beta h\mu\nu^2r\omega^2z^2+\omega^4\left(-11\alpha_1^2r+2\mu\nu^2rz^2+8\alpha_1^2z\right)\right)}{9\beta^2h^2\mu^2\nu^2+\omega^6}$$

$$+\frac{r\left(6\beta^2h^2\mu\nu^2z^4+\omega^4z^2\left(17\alpha_1^2-6\mu\nu^2z^2\right)\right)}{\beta^2h^2\mu^2\nu^2+\omega^6}-\frac{2\alpha_1^2\omega^4z^3}{\beta^2h^2\mu^2\nu^2}-\frac{150\alpha_1^2\omega^4z^3}{9\beta^2h^2\mu^2\nu^2+4\omega^6}$$

$$-\frac{16\alpha_1^2\beta^4h^4\mu^4\nu^4r\omega^4z^2}{\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^3}\Bigg)+\frac{1}{8}z^2e^{\frac{6\pi\beta h\mu\nu}{\omega^3}}\Bigg(\frac{2\alpha_1^2\omega^4z}{\beta^2h^2\mu^2\nu^2r^2}$$

$$+\frac{\alpha_1^2\omega\left(144\pi\beta h\mu\nu r+\omega^3(29r+40z)\right)}{r^2\left(4\beta^2h^2\mu^2\nu^2+\omega^6\right)}-\frac{288\alpha_1^2\beta^2h^2\mu^2\nu^2\omega^4}{r\left(4\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}+\frac{75\alpha_1^2\omega^4}{4\beta^2h^2\mu^2\nu^2r+9r\omega^6}$$

$$+\frac{4\beta^2h^2\mu\nu^2r^2z+6\beta h\mu\nu^2r^2\omega^2z-2\omega^4\left(5\mu\nu^2r^2z+2\alpha_1^2(7r+2z)\right)}{r^2\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)}$$

$$+\frac{12\beta^2h^2\mu^2\nu^2\left(\beta^2h^2\mu\nu^2rz-\omega^4\left(\alpha_1^2+\mu\nu^2rz\right)\right)}{r\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}+\frac{16\alpha_1^2\beta^4h^4\mu^4\nu^4\omega^4}{r\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^3}-\frac{4z}{\mu}\Bigg)$$

$$+\frac{1}{72}ze^{\frac{4\pi\beta h\mu\nu}{\omega^3}}\Bigg(-\frac{126\alpha_1^2\omega^4}{\beta^2h^2\mu^2\nu^2}+\frac{324\alpha_1^2\beta^2h^2\mu^2\nu^2\omega^4}{\left(\beta^2h^2\mu^2\nu^2+4\omega^6\right)^2}-\frac{50\alpha_1^2\omega^4}{\beta^2h^2\mu^2\nu^2+16\omega^6}$$

$$-\frac{36\alpha_1^2\omega\left(9\pi\beta h\mu\nu+\omega^3\right)}{\beta^2h^2\mu^2\nu^2+4\omega^6}+\frac{27z\left(3\beta^2h^2\mu\nu^2r^2+6\beta h\mu\nu^2r^2\omega^2+\omega^4\left(-\left(8\alpha_1^2+9\mu\nu^2r^2\right)\right)\right)}{r\left(\beta^2h^2\mu^2\nu^2+9\omega^6\right)}$$

$$+\frac{-297\beta^2h^2\mu\nu^2r^2z+54\beta h\mu\nu r\omega\left(4\pi\alpha_1^2+\nu r\omega z\right)+\omega^4\left(27\mu\nu^2r^2z-16\alpha_1^2(r-27z)\right)}{r\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)}$$

$$+\frac{24\beta^2h^2\mu^2\nu^2\left(18\beta^2h^2\mu\nu^2r^2z-\omega^4\left(18\mu\nu^2r^2z+\alpha_1^2(7r+3z)\right)\right)}{r\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^2}+\frac{324\pi\alpha_1^2\omega}{\beta h\mu\nu}\Bigg)$$

$$+\frac{\left(r\omega^4\left(-9\alpha_1^2\omega^{12}+\beta^4h^4\mu^3\nu^4r^2\left(\mu^2\nu^2+\omega^2\right)+\beta^2h^2\mu\nu^2\omega^6\left(7r^2\left(\mu^2\nu^2+\omega^2\right)-\alpha_1^2\mu\right)\right)\right)}{\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)^2\left(\beta^2h^2\mu^2\nu^2+9\omega^6\right)}$$

$$+e^{\frac{2h\pi\beta\mu\nu}{\omega^3}}\left(-82944z\left(z^2-21r^2\right)\alpha_1^2\omega^{79}+2737152h\pi r^2z\alpha_1^2\beta\mu\nu\omega^{76}+576h^2\beta^2\mu\nu^2\right.$$

$$\left(864r^2\omega^2z^3+\mu\left(2160r^2\mu\nu^2z^3+\left(1728r^3+71340zr^2-9504z^2r-7901z^3\right)\alpha_1^2\right)\right)\omega^{73}$$

$$-1728h^3rz\beta^3\mu^2\nu^3\left(144z\left(4r^2+3zr+4z^2\right)\mu\nu\omega+\pi\left(432r^3\left(5\mu^2\nu^2+4\omega^2\right)-35863r\alpha_1^2\mu\right)\right)$$

$$\omega^{70}+16h^4\beta^4\mu^3\nu^4\left(\mu\left(\left(1329264r^3+23794914zr^2-6029928z^2r-3487801z^3\right)\alpha_1^2\right.\right.$$

$$\left.-108r\left(448r^4-3648z^2r^2-16969z^3r-2304z^4\right)\mu\nu^2\right)-216r\left(224r^4-1152z^2r^2\right.$$

$$\left.\left.-3221z^3r-288z^4\right)\omega^2\right)\omega^{67}-48h^5rz\beta^5\mu^4\nu^5\left(36z\left(13012r^2+9663zr+8276z^2\right)\mu\nu\omega\right.$$

$$\left.+\pi\left(540r^3\left(3221\mu^2\nu^2+2548\omega^2\right)-11180393r\alpha_1^2\mu\right)\right)\omega^{64}+4h^6\beta^6\mu^5\nu^6$$

$$(24r(-172568r^4+864288z^2r^2+969223z^3r+107496z^4)\omega^2+\mu((41952144r^3$$
$$+464944986zr^2-139496600z^2r-73417955z^3)\alpha_1^2+12r(-345136r^4+2812368z^2r^2$$
$$+5588507z^3r+1108800z^4)\mu\nu^2))\omega^{61}-12h^7rz\beta^7\mu^6\nu^7(45877728\pi\omega^2r^3+275\pi\mu$$
$$(214860r^2\mu\nu^2-713851\alpha_1^2)r+20z(808332r^2+590865zr+344300z^2)\mu\nu\omega)\omega^{58}$$
$$+h^8\beta^8\mu^7\nu^8(24r(-5460008r^4+26093184z^2r^2+15442711z^3r+1225400z^4)\omega^2$$
$$+\mu((637246224r^3+5389071552zr^2-1872743384z^2r-834561355z^3)\alpha_1^2$$
$$+12r(-10920016r^4+89088048z^2r^2+104835947z^3r+23108800z^4)\mu\nu^2))\omega^{55}$$
$$X-15h^9rz\beta^9\mu^8\nu^9(4z(14108748r^2+10063185zr+4174060z^2)\mu\nu\omega$$
$$+33\pi(20r^3(304945\mu^2\nu^2+229632\omega^2)-11713659r\alpha_1^2\mu))\omega^{52}+h^{10}\beta^{10}\mu^9\nu^{10}$$
$$(6r(-83286888r^4+365438784z^2r^2+115456041z^3r-4477000z^4)\omega^2$$
$$+\mu(2(625825998r^3+4913764632zr^2-2025504589z^2r-746523730z^3)\alpha_1^2$$
$$+3r(-166573776r^4+1361639088z^2r^2+1105664037z^3r+241489600z^4)\mu\nu^2))\omega^{49}$$
$$-15h^{11}rz\beta^{11}\mu^{10}\nu^{11}(z(137546100r^2+94444911zr+29330708z^2)\mu\nu\omega$$
$$+11\pi(r^3(42929505\mu^2\nu^2+30684264\omega^2)-50917669r\alpha_1^2\mu))\omega^{46}+h^{12}\beta^{12}\mu^{11}\nu^{12}$$
$$(12r(-82398971r^4+303747408z^2r^2+26173642z^3r-23709015z^4)\omega^2$$
$$+\mu((1288254084r^3+11447055852zr^2-5409517286z^2r-1732543015z^3)\alpha_1^2$$
$$+6r(-164797942r^4+1351749306z^2r^2+842060099z^3r+172562280z^4)\mu\nu^2))\omega^{43}$$
$$-15h^{13}rz\beta^{13}\mu^{12}\nu^{13}(8z(24502461r^2+15874413zr+3909656z^2)\mu\nu\omega$$
$$+85\pi(r^3(7470312\mu^2\nu^2+4865133\omega^2)-5788106r\alpha_1^2\mu))\omega^{40}+2h^{14}\beta^{14}\mu^{13}\nu^{14}$$
$$(\mu((351310338r^3+4119249822zr^2-2174003283z^2r-633570970z^3)\alpha_1^2$$
$$+6r(-85931879r^4+709119027z^2r^2+368414548z^3r+70105100z^4)\mu\nu^2)$$
$$-3r(171863758r^4-408252954z^2r^2+147006769z^3r+94369160z^4)\omega^2)\omega^{37}$$
$$-30h^{15}rz\beta^{15}\mu^{14}\nu^{15}(z(83693688r^2+49771869zr+10278412z^2)\mu\nu\omega$$
$$+51\pi(r^3(4879595\mu^2\nu^2+2658624\omega^2)-2608451r\alpha_1^2\mu))\omega^{34}+h^{16}\beta^{16}\mu^{15}\nu^{16}$$
$$(\mu((205660764r^3+3433683732zr^2-2013222818z^2r-560898505z^3)\alpha_1^2$$
$$+6r(-95841042r^4+798782166z^2r^2+364880469z^3r+65358040z^4)\mu\nu^2)$$
$$-6r(95841042r^4+4332234z^2r^2+246373401z^3r+88818140z^4)\omega^2)\omega^{31}$$
$$-15h^{17}rz\beta^{17}\mu^{16}\nu^{17}(82112550\pi\omega^2r^3+55\pi\mu(4128552r^2\mu\nu^2-1616617\alpha_1^2)r$$
$$+8z(10774395r^2+5676759zr+1033208z^2)\mu\nu\omega)\omega^{28}+h^{18}\beta^{18}\mu^{17}\nu^{18}$$
$$(\mu(4(7902804r^3+190191111zr^2-126071166z^2r-36161645z^3)\alpha_1^2$$
$$+3r(-58044436r^4+490872468z^2r^2+201693707z^3r+34597200z^4)\mu\nu^2)$$
$$-12r(14511109r^4+64576773z^2r^2+77941252z^3r+22346940z^4)\omega^2)\omega^{25}$$
$$-15h^{19}rz\beta^{19}\mu^{18}\nu^{19}(z(26424924r^2+11863599zr+1979252z^2)\mu\nu\omega$$
$$+11\pi(3r^3(1797515\mu^2\nu^2+61828\omega^2)-1644041r\alpha_1^2\mu))\omega^{22}+h^{20}\beta^{20}\mu^{19}\nu^{20}$$
$$(\mu((2320704r^3+78896208zr^2-62328280z^2r-20571781z^3)\alpha_1^2$$
$$+12r(-2340796r^4+20186748z^2r^2+7485977z^3r+1246960z^4)\mu\nu^2)$$
$$-6r(4681592r^4+60737934z^2r^2+48587531z^3r+12350470z^4)\omega^2)\omega^{19}$$
$$-3h^{21}rz\beta^{21}\mu^{20}\nu^{21}(20z(1149684r^2+424149zr+66748z^2)\mu\nu\omega$$
$$+33\pi(25r^3(51412\mu^2\nu^2-30771\omega^2)-323252r\alpha_1^2\mu))\omega^{16}-24h^{22}\beta^{22}\mu^{21}\nu^{22}$$
$$(r(91512r^4+2977674z^2r^2+1926057z^3r+456610z^4)\omega^2+\mu$$
$$((-2592r^3-148536zr^2+130412z^2r+60051z^3)\alpha_1^2+2r(45756r^4-404028z^2r^2$$

$$-134577z^3r-22000z^4\right)\mu\nu^2\right)\Big)\omega^{13}-12h^{23}rz\beta^{23}\mu^{22}\nu^{23}\left(4z\left(125604r^2+37233zr\right.\right.$$
$$\left.\left.+5644z^2\right)\mu\nu\omega+3\pi\left(r^3\left(248220\mu^2\nu^2-508339\omega^2\right)-54396r\alpha_1^2\mu\right)\right)\omega^{10}+144h^{24}\beta^{24}$$
$$\mu^{23}\nu^{24}\left(3\mu\left(z\left(144r^2-112zr-87z^2\right)\alpha_1^2+4r\left(-36r^4+324z^2r^2+99z^3r+16z^4\right)\mu\nu^2\right)\right.$$
$$\left.-2r\left(216r^4+21582z^2r^2+11979z^3r+2726z^4\right)\omega^2\right)\omega^7-432h^{25}rz\beta^{25}\mu^{24}\nu^{25}$$
$$\left(-3705\pi\omega^2r^3+108\pi\mu\left(5r^2\mu\nu^2-\alpha_1^2\right)r+4z\left(108r^2+27zr+4z^2\right)\mu\nu\omega\right)\omega^4$$
$$-10368h^{26}rz^2\left(18r^2+9zr+2z^2\right)\beta^{26}\mu^{25}\nu^{26}\omega^3+46656h^{27}\pi r^4z\beta^{27}\mu^{26}\nu^{27}\Big)$$
$$\Big(12h^2r^2\beta^2\mu^2\nu^2\left(\omega^6+4h^2\beta^2\mu^2\nu^2\right)^2\left(\omega^6+9h^2\beta^2\mu^2\nu^2\right)\left(4\omega^6+h^2\beta^2\mu^2\nu^2\right)^2$$
$$\left(4\omega^6+9h^2\beta^2\mu^2\nu^2\right)\left(9\omega^6+h^2\beta^2\mu^2\nu^2\right)\left(9\omega^6+4h^2\beta^2\mu^2\nu^2\right)\left(16\omega^6+h^2\beta^2\mu^2\nu^2\right)$$
$$\left.\left(\omega^7+h^2\beta^2\mu^2\nu^2\omega\right)^3\Big)^{-1}\right).$$

Function $g_0(z)$ vanishes on the the graph $\mathcal{Z}=\{(r,0):r>0\}$. We apply Theorem 12 to system (6.16). Here $\Delta_\alpha=1-e^{-\frac{2\pi\beta h\mu\nu}{\omega^3}}\neq 0$. Computing the bifurcation functions we have

$$f_1(r)=0,$$
$$f_2(r)=-\frac{\pi r}{4\beta h\mu\nu\omega\left(\beta^2h^2\mu^2\nu^2+\omega^6\right)}\Big(4\alpha_1^2\omega^8+3\beta^3h^3\mu^3\nu^4r^2+\beta^2h^2\mu\nu^2\omega^2$$
$$\left(\mu\nu\left(4\alpha_2\mu+4k_2-3\mu\nu r^2\right)-3r^2\omega^2\right)+\beta h\mu\nu\omega^6\left(4\alpha_2+3\nu r^2\right)\Big).$$

Let
$$r_0=\beta h\mu\nu^2\left(\beta h-\omega^2\right)\left(\omega^4-\beta h\mu^2\nu^2\right).$$

If $r_0>0$ the equation $f_2(r)=0$ has the positive solution $\bar{r}=\dfrac{2\alpha_1\omega_4}{\sqrt{3r_0}}$. Furthermore,

$$f_2'(\bar{r})=\frac{2\pi\alpha_1^2\omega^7}{\beta^3h^3\mu^3\nu^3+\beta h\mu\nu\omega^6}\neq 0$$

and consequently, by Theorem 12 (b), system (6.16) has a periodic solution bifurcating from the origin. □

Chapter 7

Zero-Hopf Bifurcations in a Hyperchaotic Lorenz System

In recent times a so-called *hyperchaotic Lorenz* system was introduced; see for instance [4, 21, 26, 31, 39, 42, 72, 71, 70, 78, 83, 87, 90] and the references therein. MathSciNet presently lists 32 papers on *hyperchaotic Lorenz* systems. We observe that not all these hyperchaotic Lorenz systems are similar, since they can vary in one or two terms. However these systems are autonomous differential systems in a phase space of dimension at least four, with a dissipative structure, and at least two unstable directions, such that at least one is due to a nonlinearity. The hyperchaotic systems has a dynamics hard to predict or control, for this reason such systems are as well of use in secure communications systems see, for instance [88].

Our aim in this work is to study, from a dynamical point of view, the 4–*dimensional zero–Hopf equilibria* in the hyperchaotic Lorenz system. Here, a 4–dimensional zero–Hopf equilibrium means an equilibrium point with two zeros and a pair of pure conjugate imaginary numbers as eigenvalues. Using the averaging theory and convenient changes of variables and parameters we can analyse the zero–Hopf bifurcations. More precisely we study the zero–Hopf bifurcations of the following hyperchaotic Lorenz system (given in [26, 42])

$$\begin{aligned} \dot{x} &= a(y-x)+w, \\ \dot{y} &= cx-y-xz, \\ \dot{z} &= -bz+xy, \\ \dot{w} &= dw-xz, \end{aligned} \tag{7.1}$$

for appropriate choices of the parameters a, b, c and d.

There are several works studying zero–Hopf bifurcation see for instance Guckenheimer [35], Guckenheimer and Holmes [36], Han [38], Kuznetsov [46], Llibre [49], Marsden, Scheurle [69]. . . . It has been shown that, under specific conditions, some elaborated invariant sets of the unfolding could be bifurcated from a zero–Hopf equilibrium and hence, in some cases, a zero–Hopf equilibrium is the local birth of "chaos" (see [22, 69]). Also, recently Li and Wang [47] published a paper on a Hopf bifurcation in a 3-dimensional Lorenz-type system. Due to the complexity related to the high dimensionality, there is very little work done on the n–dimensional zero–Hopf bifurcation with $n > 3$.

The characterization of the zero-Hopf bifurcation at the origin was recently study by Cid-Montiel, Llibre and Stoica in [24]. In this work we are going to complete this characterization analyzing all singular points for system (7.1). The results here presented were published in [16].

7.1 Application to a Hyperchaotic Lorenz System

First we are going to compute the equilibrium points of the hyperchaotic Lorenz system (7.1). One may verify that for any choice of the parameters, the origin of coordinates of $\mathbb{R}^4$ is always an equilibrium point for this system. Moreover if $ad \neq 1$ and $abd(1-c)(c-ad) > 0$, system (7.1) will have two additional equilibrium points

$$\boldsymbol{p}_\pm = \left(\pm\frac{\sqrt{abd(1-c)}}{\sqrt{c-ad}}, \pm\frac{\sqrt{abd(1-c)(c-ad)}}{1-ad}, \frac{ad(1-c)}{1-ad}, \pm\frac{a(1-c)\sqrt{abd(1-c)}}{(1-ad)\sqrt{c-ad}} \right).$$

Considering $b = 0$ then all the z–axis is filled of equilibria. And if $b = 0$ and $ad(1-c)(1-ad) \neq 0$ we have the additional equilibrium point

$$\boldsymbol{p} = \left(0, 0, \frac{ad(1-c)}{1-ad}, 0\right).$$

We observe that the two equilibria $\boldsymbol{p}_\pm$ tends to the equilibrium points $\boldsymbol{p}$ when $b \to 0$. In short, the equilibrium point of system (7.1) can be $\boldsymbol{p}_+$, $\boldsymbol{p}_-$, $\boldsymbol{p}$ and the origin.

Note that system (7.1) is invariant by the symmetry $(x, y, z, w) \to (-x, -y, z, -w)$, i.e. it is invariant under the symmetry with respect to the z-axis. Therefore we can study $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$ simultaneously using only one of these points. Due to that in what follows we consider only the equilibrium $\boldsymbol{p}_+$ in order to study when it will be a zero–Hopf equilibrium for some values of the parameter, and clearly the same will occur for the other equilibrium $\boldsymbol{p}_-$.

In the next result we characterize when the equilibria $\boldsymbol{p}$, $\boldsymbol{p}_\pm$ and the origin are zero–Hopf equilibria. To simplify the expressions we define

$$D_a = \sqrt{a^6 + 2a^5 - 3a^4 - 14a^3 - 14a^2 - 4a + 1}.$$

Proposition 37. *The following statements hold.*

(*a*) *The origin is a zero–Hopf equilibrium if and only if* $a = -1$, $b = d = 0$ *and* $c > 1$.

(*b*) *Assume that* $ad(1-c)(1-ad) \neq 0$ *and* $b = 0$. *The equilibrium point* $\boldsymbol{p}$ *is a zero–Hopf equilibrium if and only if* $d = a + 1$, *and*

(*b*.1) *either* $\dfrac{1 + d^3 - d^4 + (d^2 - 1)\,c}{d^2 - d - 1} > 0$;

(*b*.2$_+$) *or* $c_+ = \dfrac{-1 + a(1+a)^2(a^2 + 2a + 3) + (a^2 + a - 1)D_a}{2a(a^2 + 3a + 3)}$ *and* $\dfrac{-4 - 9a - 10a^2 - 5a^3 - a^4 + (2 + a)D_a}{3 + 3a + a^2} > 0$;

(*b*.2$_-$) *or* $c_- = \dfrac{-1 + a(1+a)^2(a^2 + 2a + 3) - (a^2 + a - 1)D_a}{2a(a^2 + 3a + 3)}$ *and* $\dfrac{-4 - 9a - 10a^2 - 5a^3 - a^4 - (2 + a)D_a}{3 + 3a + a^2} > 0$;

(*c*) *The equilibria* $\boldsymbol{p}_\pm$ *never are zero–Hopf equilibria.*

Note that despite z–axis being filled of equilibria when $b = 0$, its only zero–Hopf equilibria are the origin and the equilibrium point $\boldsymbol{p}$. Furthermore, corresponding to statement (a) of the Proposition 37 there is one 1-dimensional parametric family for which the origin is a zero–Hopf equilibrium point and there are three parametric families for which the equilibrium point $\boldsymbol{p}$ is a zero–Hopf equilibrium of the hyperchaotic Lorenz system, one 2–dimensional parametric family corresponding to conditions ($b.1$) and two 1–dimensional parametric families corresponding to conditions ($b.2_+$) and ($b.2_-$).

If (a) holds the eigenvalues at the origin are 0, 0 and

$$\pm\omega i = \pm\sqrt{c-1}\,i.$$

If ($b.1$) holds the eigenvalues at $\boldsymbol{p}$ are 0, 0 and

$$\pm\omega_0 i = \pm\sqrt{\frac{1+d^3-d^4+(d^2-1)\,c}{d^2-d-1}}\,i. \tag{7.2}$$

If ($b.2_+$) holds the eigenvalues at $\boldsymbol{p}$ are 0, 0 and

$$\pm\omega_+ i = \pm\sqrt{\frac{-4-9a-10a^2-5a^3-a^4+(2+a)D_a}{2(3+3a+a^2)}}\,i. \tag{7.3}$$

If ($b.2_-$) holds the eigenvalues at $\boldsymbol{p}$ are 0, 0 and

$$\pm\omega_- i = \pm\sqrt{\frac{-4-9a-10a^2-5a^3-a^4-(2+a)D_a}{2(3+3a+a^2)}}\,i. \tag{7.4}$$

The next results characterizes when periodic orbits bifurcate from these zero–Hopf equilibrium points.

Theorem 38. *(i) Consider system* (7.1) *with*

$$a = -1+\varepsilon a_1, \quad b = \varepsilon b_1, \quad c = 1+c_0^2, \quad \text{and} \quad d = \varepsilon d_1. \tag{7.5}$$

For the zero–Hopf equilibrium at the origin we have:

(a) If $a_1 b_1 \neq 0$, $a_1 \neq d_1$ *and* $c_0 > 0$, *then there exists an* $\varepsilon_1 > 0$ *such that when* $|\varepsilon| < \varepsilon_1$ *the hyperchaotic Lorenz system* (7.1) *has a periodic solution* $\big(x(t,\varepsilon),\, y(t,\varepsilon),\, z(t,\varepsilon),\, w(t,\varepsilon)\big) =$

$$\varepsilon\left(\sqrt{2a_1b_1}c_0\sin(c_0t),\ \sqrt{2a_1b_1}c_1\big(\sin(c_0t)-c_0\cos(c_0t)\big),\ a_1c_0^2,\ 0\right)+\mathcal{O}(\varepsilon^2), \tag{7.6}$$

bifurcating from the origin. The periodic solution (7.6) *is stable if* $b_1 > 0$, $a_1 < 0$ *and* $d_1 < a_1$.

(b) If $b_1\, d_1 \neq 0$, $a_1 \neq d_1$ and $c_0 > 0$, then there is a convenient choice of ε such that for either $\varepsilon \in (-\varepsilon_1,\, 0)$, or $\varepsilon \in (0,\, \varepsilon_1)$ there are two additional periodic solutions $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big)_{\pm} =$

$$\varepsilon \left(\pm c_0 \sqrt{\frac{b_1 d_1}{1+c_0^2}},\ \pm c_0 \sqrt{b_1 d_1 (1+c_0^2)},\ c_0^2 d_1,\ \pm c_0^3 \sqrt{\frac{b_1 d_1}{1+c_1^2}} \right) + \mathcal{O}(\varepsilon^2) \qquad (7.7)$$

emerging from the origin. These solutions are stable if $b_1 > 0$, $d_1 > 0$ and $d_1 < a_1$.

(ii) Considering system (7.1) with

$$a = d - 1 + \varepsilon a_1 \quad \text{and} \quad b = \varepsilon^2 b_1. \qquad (7.8)$$

If $a_1\, b_1 \neq 0$, $c \neq 1$, $d \notin \{0, \pm 1\}$ and $\omega_0 \in \mathbb{R}^*$, then there exists an $\varepsilon_1 > 0$ such that when $|\varepsilon| < \varepsilon_1$ the hyperchaotic Lorenz system (7.1) has a periodic solution $\big(x(t,\varepsilon),\, y(t,\varepsilon),\, z(t,\varepsilon),\, w(t,\varepsilon)\big) =$

$$\begin{aligned}&\Bigg(\varepsilon \sqrt{\frac{2b_1(1-c)(d-1)d(d^2-1)}{d(d-1)-1}} \sin(\omega_0 t),\\ &\varepsilon \sqrt{\frac{2b_1(1-c)(d-1)d}{(d(d-1)-1)(d^2-1)}} (\omega_0 \cos(\omega_0 t) - \sin(\omega_0 t)),\\ &\frac{(c-1)(d-1)d}{d^2-d-1} + \varepsilon \frac{a_1\,(-cd^2 - cd + c + d^4 - d^3 + d - 1)}{(d^2-d-1)^2},\\ &\varepsilon d(d-1) \sqrt{\frac{2b_1(1-c)d(d-1)}{(d(d-1)-1)(d^2-1)}} (d \sin(\omega_0 t) + \omega_0 \cos(\omega_0 t)) \Bigg) + \mathcal{O}(\varepsilon^{\mathbf{2}}), \end{aligned} \qquad (7.9)$$

bifurcating from the zero–Hopf equilibrium point $\boldsymbol{p}$. The periodic solution (7.9) is unstable if $a_1 < 0$ or $b_1(c-1)(d-1)d < 0$. Furthermore there is a convenient choice of ε such that for either $\varepsilon \in (-\varepsilon_1,\, 0)$ or $\varepsilon \in (0,\, \varepsilon_1)$ there are two additional periodic solutions $\big(x(t,\varepsilon),\, y(t,\varepsilon),\, z(t,\varepsilon),\, w(t,\varepsilon)\big)_1 =$

$$\begin{aligned}&\Bigg(-\varepsilon \sqrt{\frac{b_1(c-1)(1-d)d}{c+d-d^2}},\ \varepsilon \sqrt{\frac{b_1(c-1)(1-d)d(c+d-d^2)}{d(d-1)-1}},\\ &\frac{(c-1)(d-1)d}{d(d-1)-1} - \varepsilon \frac{a_1(c-1)d}{(1-d-d^2)^2},\ \frac{\varepsilon}{d(d-1)-1} \sqrt{\frac{b_1 d(c-1)^3(1-d)^3}{c+d-d^2}} \Bigg) + \mathcal{O}(\varepsilon^2)\end{aligned} \qquad (7.10)$$

and $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big)_2 =$

$$\Bigg(\varepsilon \sqrt{\frac{b_1(c-1)(1-d)d}{c+d-d^2}},\ -\varepsilon \sqrt{\frac{b_1(c-1)(1-d)d(c+d-d^2)}{d(d-1)-1}},$$

$$\frac{(c-1)(d-1)d}{d(d-1)-1} - \varepsilon\frac{a_1(c-1)d}{(1-d-d^2)^2},\ \frac{-\varepsilon}{d(d-1)-1}\sqrt{\frac{b_1 d(c-1)^3(1-d)^3}{c+d-d^2}}\Bigg) + \mathcal{O}(\varepsilon^2) \tag{7.11}$$

emerging from $\boldsymbol{p}$. *These orbits are unstable if* $a_1 < 0$ *or* $b_1(c-1)(d-1)d > 0$.

(iii) Considering system (7.1) *with*

$$b = \varepsilon^2 b_1, \quad c = c_+ + \varepsilon c_1 \quad \text{and} \quad d = 1 + a + \varepsilon d_1. \tag{7.12}$$

If $b_1 d_1 \neq 0$, $a \notin \left\{-2,\ -1,\ 0,\ \frac{1\pm\sqrt{5}}{2},\ \frac{-3\pm\sqrt{5}}{2}\right\}$ *and* $\omega_+ \in \mathbb{R}^*$, *then there exists* $\varepsilon_1 > 0$ *such that when* $|\varepsilon| < \varepsilon_1$ *the hyperchaotic Lorenz system* (7.1) *has a periodic solution* $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big) =$

$$\begin{aligned}
&\Bigg(\varepsilon\frac{\sqrt{(2+a)}}{(1+a)\omega_+}\left(\frac{a(1+a)^3}{(3+a(3+a))^2}(1+a(16+a(45+a(59+2a(20\right.\\
&\left.+a(7+a))))) + D_a)\right)^{\frac{1}{2}}\sin(\omega_+ t),\\
&\frac{\varepsilon}{a(1+a)\omega_+\sqrt{2+a}}\left(\frac{a(1+a)^3}{(3+a(3+a))^2}(1+a(16+a(45+a(59\right.\\
&\left.+2a(20+a(7+a))))) + D_a)\right)^{\frac{1}{2}}\big(\omega_+\cos(\omega_+ t) - \sin(\omega_+ t)\big),\\
&\frac{(a+1)\left(a^3+3a^2+4a+D_a+1\right)}{2\left(a^2+3a+3\right)} + \varepsilon\frac{(a+1)}{2\left(a^2+a-1\right)\left(a(a+3)+3\right)}\\
&(2a(a(a+3)+3)c_1 - a(a(a+5)+8)d_1 + d_1 D_a - 5d_1),\\
&\frac{\varepsilon}{\omega_+\sqrt{2+a}}\left(\frac{a(1+a)^3}{(3+a(3+a))^2}(1+a(16+a(45+a(59\right.\\
&\left.+2a(20+a(7+a))))) + D_a)\right)^{\frac{1}{2}}\big(\omega_+\cos(\omega_+ t) + (1+a)\sin(\omega_+ t)\big)\Bigg) + \mathcal{O}(\varepsilon^2),
\end{aligned} \tag{7.13}$$

bifurcating from the zero–Hopf equilibrium point $\boldsymbol{p}$. *Furthermore there is a convenient choice of* ε *such that for either* $\varepsilon \in (-\varepsilon_1, 0)$ *or* $\varepsilon \in (0, \varepsilon_1)$ *there are two additional periodic solutions* $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big)_1 =$

$$\begin{aligned}
&\Bigg(-\varepsilon\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}},\ \frac{\varepsilon(1-a(1+a)(2+a)+Da)}{3+3a+a^2)}\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}},\\
&\frac{(1+a)(1+4a+3a^2+a^3+Da)}{2(3+a(3+a))} + \frac{\varepsilon}{2(a^2+a-1)(3+a(3+a))}\\
&\Big(2a(1+a)(3+a(3+a))c_0 - d_1 - a(4+a(3+a))d_1 - d_1 Da\Big),\\
&-\frac{\varepsilon(1+4a+3a^2+a^3+Da)}{2(3+3a+a^2)}\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}}\Bigg) + \mathcal{O}(\varepsilon^2)
\end{aligned} \tag{7.14}$$

and $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big)_2 =$

$$\Bigg(\varepsilon\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}},\ -\frac{\varepsilon(1-a(1+a)(2+a)+Da)}{3+3a+a^2)}\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}},$$
$$\frac{(1+a)(1+4a+3a^2+a^3+Da)}{2(3+a(3+a))}+\frac{\varepsilon}{2(a^2+a-1)(3+a(3+a))}$$
$$\Big(2a(1+a)(3+a(3+a))c_0-d_1-a(4+a(3+a))d_1-d_1Da\Big),$$
$$\frac{\varepsilon(1+4a+3a^2+a^3+Da)}{2(3+3a+a^2)}\sqrt{\frac{b_1(Da+a^3+a^2-1)}{2}}\Bigg)+\mathcal{O}(\varepsilon^2), \tag{7.15}$$

emerging from $\boldsymbol{p}$*. These orbits are unstable if* $d_1>0$ *or if the eigenvalues* (7.28) *are non-zero real numbers.*

(iv) Consider system (7.1) *with*

$$b=\varepsilon^2 b_1,\quad c=c_-+\varepsilon c_1\quad \text{and}\quad d=1+a+\varepsilon d_1. \tag{7.16}$$

If $b_1d_1\neq 0$*,* $a\notin\Big\{-2,\,-1,\,0,\,\frac{-1\pm\sqrt{5}}{2},\,\frac{-3+\sqrt{5}}{2}\Big\}$ *and* $\omega_-\in\mathbb{R}^*$*, then there exists* $\varepsilon_1>0$ *such that when* $|\varepsilon|<\varepsilon_1$ *the hyperchaotic Lorenz system* (7.1) *has a periodic solution* $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big) =$

$$\Bigg(\varepsilon\frac{2+a}{a(1+a)(3+a(3+a))\omega_-}\big(a(1+a)^3b_1(1+a(16+a(45+a(59+2a(20$$
$$+a(7+a)))))-D_a)\big)^{\frac{1}{2}}|a|\sin(\omega\_t),\ \frac{\varepsilon}{\sqrt{2+a}(3+a(3+a))|a|(a+a^2)\omega_-}$$
$$+\big(a(1+a)^3b_1(1+a(16+a(45+a(59+2a(20a(7+a)))))-D_a)\big)^{\frac{1}{2}}\big(\omega_-|a|\cos(\omega\_t)$$
$$-a\sin(\omega\_t)\big),\ \frac{(a+1)(a^3+3a^2+4a-D_a+1)}{2(a^2+3a+3)}+\frac{\varepsilon(a+1)}{2(3+a(3+a))(a^2+a-1)}$$
$$\big(2a(3+a(3+a))c_1-(5+a(8+a(5+a))d_1-d_1D_a)\big),\ \frac{\varepsilon}{\sqrt{2+a}(3+a(3+a))|a|\omega_-}$$
$$\big(a(1+a)^3b_1(1+a(16+a(45+a(59+2a(20+a(7+a)))))-D_a)\big)^{\frac{1}{2}}$$
$$\big(\cos(\omega\_t)+a(1+a)\sin(\omega\_t)\big)\Bigg)+\mathcal{O}(\varepsilon^2) \tag{7.17}$$

bifurcating from the zero–Hopf equilibrium point $\boldsymbol{p}$*. Furthermore there is a convenient choice of* ε *such that for either* $\varepsilon\in(-\varepsilon_1,\,0)$*, or* $\varepsilon\in(0,\,\varepsilon_1)$ *there are two additional periodic solutions* $\big(x(t,\varepsilon), y(t,\varepsilon), z(t,\varepsilon), w(t,\varepsilon)\big)_\pm =$

$$\Bigg(\pm\varepsilon\sqrt{\frac{b_1(a^3+a^2-1-D_a)}{2}},\ \pm\varepsilon\frac{a(1+a)(2+a-1+D_a)}{2a(a^2+3a+3)}\sqrt{\frac{b_1(a^3+a^2-1-D_a)}{2}}$$
$$\frac{(1+a)(1+a(4+a(3+a))-D_a)}{2(a^2+3a+3)}+\frac{\varepsilon}{2(a^3+a^2-1)(a^2+3a+3)}$$
$$\Big(2a(1+a)(a^2+3a+3)c_0-d_1-a(4+a(3+a))d_1+d_1D_a\Big),$$

$$\pm\varepsilon\frac{1+a(4+a(3+a))-D_a}{2(a^2+3a+3)}\sqrt{\frac{b_1(a^3+a^2-1-D_a)}{2}}\Bigg)+\mathcal{O}(\varepsilon^2), \qquad (7.18)$$

emerging from $\boldsymbol{p}$.

Theorem 38 is proved in Section 7.2.

In statements (iii) and (iv) of Theorems 38 we do not provide the type of linear stability for the solutions (7.13), (7.17) and (7.18) because the expressions of the eigenvalues are huge.

7.2 Proofs

In this section we prove our results.

7.2.1 Proof of Proposition 37

Proof. First we assume that $b=0$. The characteristic polynomial $P(\lambda)$ of the linear part of the differential system (7.1) at the equilibrium point $(0,0,z_0,0)$ is

$$\lambda^4+(a-d+1)\lambda^3+(-ca-da+z_0a+a-d+z_0)\lambda^2+(-ad+acd-az_0d+z_0)\lambda.$$

Clearly an equilibrium point is a zero–Hopf equilibrium if and only if $P(\lambda)=\lambda^2\ (\lambda^2+\omega^2)$ with $\omega>0$. Hence solving the equation $P(\lambda)=\lambda^2\ (\lambda^2+\omega^2)$, with respect to the parameters a, b, c, d and ω, we get only two real solutions:

$$S_1:\ \ \omega=\sqrt{c-1},\qquad z_0=0,\qquad d=0,\qquad a=-1;$$

$$S_2:\ \ \omega=\sqrt{\frac{(2+a)z_0}{1+a}-(1+a)^2},\quad z_0=\frac{(a^2+a)(c-1)}{a^2+a-1},\quad d=a+1.$$

The solution S_1 says when the equilibrium point located at the origin is zero–Hopf, proving statement (a), and it is easy to check that the solution S_2 corresponds to the equilibrium $\boldsymbol{p}$.

Now we shall provide necessary and sufficient conditions under which either $\boldsymbol{p}_+$ if $b\neq 0$, or $\boldsymbol{p}$ if $b=0$, is a zero–Hopf equilibrium point. The Jacobian matrix of system (7.1) evaluated at $\boldsymbol{p}_+$ is

$$A=\begin{pmatrix} -a & a & 0 & 1\\ \dfrac{ad-c}{ad-1} & -1 & \dfrac{\sqrt{a\,b\,d\,(1-c)}}{\sqrt{c-ad}} & 0\\ \dfrac{\sqrt{a\,b\,d\,(1-c)\,(c-ad)}}{ad-1} & \dfrac{\sqrt{a\,b\,d\,(1-c)}}{-\sqrt{c-ad}} & -b & 0\\ \dfrac{ad(1-c)}{ad-1} & 0 & \dfrac{\sqrt{a\,b\,d\,(1-c)\,(c-ad)}}{\sqrt{c-ad}} & d \end{pmatrix},$$

and its characteristic polynomial is $P(\lambda)=\lambda^4+\sigma_3\lambda^3+\sigma_2\lambda^2+\sigma_1\lambda+\sigma_0$ with

$$\sigma_0=-\,2abd(c-1),$$

$$
\begin{aligned}
\sigma_1 =& \frac{-a^2d^2 - 2a^3d^2 - a^3d^3 + ac - dc + adc + a^2dc + 2ad^2c + 3a^2d^2c + 2a^3d^2c}{(ad-1)(ad-c)} \\
&+ \frac{-a^2d^3c - ac^2 - 2adc^2 - a^2dc^2}{(ad-1)(ad-c)}, \\
\sigma_2 =& \frac{ad^2 - a^2d - a^2bd + abd^2 + a^3bd^2 - a^2d^3 - a^3d^3 - a^2bd^3 + ac + bc + abc - dc}{(ad-1)(ad-c)} \\
&+ \frac{a^2dc - bdc - 2abdc - a^2bdc + ad^2c + 2a^2d^2c + abd^2c + a^2bd^2c - ac^2 - adc^2}{(ad-1)(ad-c)}, \\
\sigma_3 =& 1 + a + b - d.
\end{aligned}
$$

The expressions for the matrix A and for its characteristic polynomial also work for the equilibrium $\boldsymbol{p}$ taking $b = 0$.

Forcing that $P(\lambda) = \lambda^2\,(\lambda^2 + \omega^2)$, i.e. we must solve the following system: $\sigma_3 = 0$, $\sigma_2 = \omega^2$, $\sigma_1 = 0$ and $\sigma_0 = 0$. Obtaining the following three real solutions:

$$
\begin{aligned}
S^1 :& \omega = \omega_0,\ b = 0 \text{ and } a = d - 1; \\
S^2 :& \omega = \omega_-, b = 0,\ d = 1 + a \text{ and } c = c_-; \\
S^3 :& \omega = \omega_+,\ b = 0,\ d = 1 + a \text{ and } c = c_+;
\end{aligned}
$$

where $c_\pm$ are defined in the statement of Proposition 37, ω_0 in (7.2) and $\omega_\pm$ in (7.3) and (7.4).

The solution S^1 says that $\boldsymbol{p}$ is a zero–Hopf equilibrium if condition $(b.1)$ holds. While the solutions S^2 and S^3 correspond to the fact that $\boldsymbol{p}$ is a zero–Hopf equilibrium when conditions $(b.2+)$ and $(b.2-)$ hold. This completes the proof of statement (b).

Since in the three solutions we have $b = 0$ it follows that the equilibrium $\boldsymbol{p}_+$ never can be a zero–Hopf equilibrium, proving statement (c) and consequently proving Proposition 37. □

7.2.2 Proof of statement (i) of Theorem 38

Proof. First we assume condition (7.5) and the new scale $(x, y, z, w) = \varepsilon\,(\overline{x}, \overline{y}, \overline{z}, \overline{w})$ to system (7.1), obtaining

$$
\begin{aligned}
\dot{\overline{x}} =& (-1 + \varepsilon a_1)\,(\overline{y} - \overline{x}) + \overline{w}, \\
\dot{\overline{y}} =& (1 + c_0^2)\overline{x} - \overline{y} - \overline{x}\,\overline{z}, \\
\dot{\overline{z}} =& -\varepsilon b_1 \overline{z} + \varepsilon \overline{x}\,\overline{y}, \\
\dot{\overline{w}} =& \varepsilon d_1 \overline{w} - \varepsilon \overline{x}\,\overline{z}.
\end{aligned}
\tag{7.19}
$$

We now do the linear change of variables

$$
\begin{aligned}
\overline{x} =& \frac{\widetilde{z}}{c_0^2} + \widetilde{y}, \\
\overline{y} =& \widetilde{z}\left(\frac{1 + c_0^2}{c_0^2}\right) - c_0\widetilde{x} + \widetilde{y},
\end{aligned}
$$

$$\begin{aligned}\overline{z} =& \widetilde{w},\\ \overline{w} =& \widetilde{z},\end{aligned}$$

in order to write the unperturbed part of system (7.19) in its real Jordan normal form in accordance with the matrix

$$\mathrm{J} = \begin{pmatrix} 0 & -c_0 & 0 & 0 \\ c_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus omitting the tilde we obtain the new system

$$\begin{aligned}\dot{x} =& -c_0 y + \varepsilon\left(\frac{z(a_1+d_1)}{c_0} - a_1 x\right),\\ \dot{y} =& c_0 x + \varepsilon\left(a_1(z - c_0 x) + \frac{c_0^2(wy - d_1 z) + wz}{c_0^4}\right),\\ \dot{z} =& \varepsilon\left(d_1 z - w\left(\frac{z}{c_0^2} + y\right)\right),\\ \dot{w} =& \varepsilon\left(\left(\frac{z}{c_0^2} + y\right)\left(\frac{z}{c_0^2} - c_0 x + y + z\right) - b_1 w\right).\end{aligned}$$

Finally we use the generalized cylindrical coordinates to write the previous system in the form

$$\begin{aligned}\dot{r} =& \varepsilon\Big(\frac{z}{c_0^4}\big(c_0^3(a_1+d_1)\cos\theta + (a_1c_0^4 - c_0^2 d_1 + w)\sin\theta\big) +\\ & r\Big(\frac{w\sin\theta^2}{c_0^2} - a_1\cos\theta^2 - a_1 c_0\cos\theta\sin\theta\Big)\Big),\\ \dot{\theta} =& c_0 + \varepsilon\big(\cos\theta\big((a_1c_0^4 - c_0^2 d_1 + w)z + c_0^2 r(a_1 c_0^2 + w)\sin\theta\big)\\ & - a_1 c_0^5 r\cos\theta^2 - c_0^3(a_1+d_1)z\sin\theta\big),\\ \dot{z} =& \frac{\varepsilon}{c_0^2}\big(c_0^2 d_1 z - wz - c_0^2 rw\sin\theta\big),\\ \dot{w} =& \varepsilon\Big(-b1w + \Big(\frac{z}{c_0^2} + r\sin\theta\Big)\Big(z + \frac{z}{c_0^2} - c_0 r\cos(\theta + r\sin\theta)\Big)\Big).\end{aligned} \tag{7.20}$$

We also take θ as a new independent variable, obtaining the system

$$\begin{aligned}\frac{dr}{d\theta} =& \varepsilon\left(\frac{a_1c_0^4 r\cos^2(\theta) - \sin\theta\left(z\left(a_1c_0^4 - c_0^2 d_1 + w\right) + c_0^2 rw\sin(\theta)\right)}{-c_0^5}\right.\\ & \left. + \frac{c_0^3\cos(\theta)\left(a_1 c_0^2 r\sin\theta + z(-a_1 - d_1)\right)}{-c_0^5}\right) + \varepsilon^2 F_{21}(\theta, \mathbf{y}, \varepsilon),\\ \frac{dz}{d\theta} =& \varepsilon\left(-\frac{wz}{c_0^3} + \frac{d_1 z}{c_0} - \frac{rw\sin\theta}{c_0}\right) + \varepsilon^2 F_{22}(\theta, \mathbf{y}, \varepsilon),\\ \frac{dw}{d\theta} =& \varepsilon\left(-\frac{b_1 w}{c_0} + \frac{z^2}{c_0^5} + \frac{2rz\sin\theta}{c_0^3} + \frac{z^2}{c_0^3} - \frac{rz\cos\theta}{c_0^2} + \frac{r^2\sin^2(\theta)}{c_0}\right.\end{aligned} \tag{7.21}$$

$$+\frac{rz\sin\theta}{c_0}-r^2\sin\theta\cos\theta\Big)+\varepsilon^2F_{23}(\theta,\mathbf{y},\varepsilon),$$

where $\mathbf{F}_2(\theta,\mathbf{y},\varepsilon)=\big(F_{21}(\theta,\mathbf{y},\varepsilon),\,F_{22}(\theta,\mathbf{y},\varepsilon),\,F_{23}(\theta,\mathbf{y},\varepsilon)\big)$ is a 2π–periodic function in θ, and $\mathbf{y}=(r,z,w)$.

To study the periodic orbits of system (7.21) we compute the averaged function (1.3) of Theorem 5 corresponding to system (7.1), and we get

$$g_1(\mathbf{y})=\left(\frac{r\,(w-a_1c_0^2)}{2c_0^3},\ \frac{z\,(c_0^2d_1-w)}{c_0^3},\ \frac{r^2-2b_1w}{2c_0}+\frac{(c_0^2+1)\,z^2}{c_0^5}\right).$$

solving the non-linear system $g(\mathbf{y})=0$, we have

$$\mathbf{s}_0=\left(c_0\sqrt{2a_1b_1},\,0,\,a_1c_0^2\right),$$

$$\mathbf{s}_1^0=\left(0,\,\frac{\sqrt{b_1d_1}c_0^3}{\sqrt{1+c_0^2}},\,\sqrt{1+c_0^2}\right),$$

$$\mathbf{s}_2^0=\left(0,\,-\frac{\sqrt{b_1d_1}c_0^3}{\sqrt{1+c_0^2}},\,\sqrt{1+c_0^2}\right).$$

The solution $\mathbf{s}_0$ has the Jacobian

$$\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_0)\right)=\frac{a_1b_1(a_1-d_1)}{c_0^3},$$

which is non-zero under conditions (a), then by Theorem 5 we know that there is a periodic solution $\mathbf{\Phi}(t,\varepsilon)$ close to $\mathbf{s}_0$ such that $\mathbf{\Phi}(0,\varepsilon)=\mathbf{s}_0+\mathcal{O}(\varepsilon)$. Going back through the change of coordinates, it provides the periodic solution (7.6) of system (7.1).

We also notice that the eigenvalues of $\mathbf{s}_0$ are $\dfrac{-b_1\pm\sqrt{b_1(4a_1+b_1)}}{2c_0}$ and $\dfrac{d_1-a_1}{c_0}$, we use Theorem 5 (c) to study the stability of the periodic solution (7.6). Here we divide the analysis in two cases:

When $\dfrac{-b_1\pm\sqrt{b_1(4a_1+b_1)}}{2c_0}\in\mathbb{R}$: In this case the solution is stable if $d_1<a_1$, $b_1>0$ and $\dfrac{-b_1}{4}\le a_1<0$.

When $\dfrac{-b_1\pm\sqrt{b_1(4a_1+b_1)}}{2c_0}\in\mathbb{C}$: In this case the solution is stable if $d_1<a_1$, $b_1>0$ and $a_1<\frac{-b_1}{4}$.

In summary this periodic solution is stable if $b_1>0$, $a_1<0$ and $d_1<a_1$.

The solutions $\mathbf{s}_1^0$ and $\mathbf{s}_2^0$ are such that $\det\left(\dfrac{\partial g}{\partial y}(\mathbf{s}_i^0)\right)=\dfrac{b_1d_1(d_1-a_1)}{c_0^3}$, for $i=1,2$. By hypothesis (b) they also provide two additional periodic solution for system (7.1) if $r=\varepsilon r_1+\mathcal{O}(\varepsilon^2)>0$, this is possible restricting ε to one of the half intervals $\varepsilon\in(-\varepsilon_1,0)$, or $\varepsilon\in(0,\varepsilon_1)$.

We notice that $\mathbf{s}_1^0$ and $\mathbf{s}_2^0$ have the same eigenvalues $\dfrac{b_1\pm\sqrt{b_1(b_1-8d_1)}}{-2c_0}$ and $\dfrac{d_1-a_1}{2c_0}$. Thus following the previous analysis we can say by Theorem 5 (c) that the periodic solution (7.7) is stable if $b_1>0$, $d_1>0$ and $a_1>d_1$. □

7.2.3 Proof of statement (ii) of Theorem 38

Proof. Assuming conditions (7.8), system (7.1) has two equilibrium points $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$, when $\varepsilon \to 0$ these equilibria tends to

$$\boldsymbol{p} = \left(0, 0, \frac{(d-1)d(c-1)}{(d-1)d-1}, 0\right).$$

We now are studying the bifurcation of periodic orbits from this point. First we translate $\boldsymbol{p}$ to the origin of coordinates doing $(x, y, z, w) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) + \boldsymbol{p}$, then we introduce the scaling $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = \varepsilon\,(X, Y, Z, W)$. With these changes of variables the hyperchaotic Lorenz system (7.1) becomes

$$\begin{aligned}
\dot{X} &= (1-d)X + (d-1)Y + W + \varepsilon a_1 (Y - X),\\
\dot{Y} &= \frac{-d^2+d+c}{-d^2+d+1} X - Y - \varepsilon XZ,\\
\dot{Z} &= \varepsilon\left(\frac{b_1 d(d(-c)+d+c-1)}{(d-1)d-1} + XY\right) - \varepsilon^2 b_1 Z,\\
\dot{W} &= d\left(\frac{(d(-c)+d+c-1)}{(d-1)d-1} X + W\right) - \varepsilon XZ.
\end{aligned} \tag{7.22}$$

We do the linear change of variables

$$\begin{aligned}
X =& \frac{(d^2-1)}{\omega_0}\widetilde{y} + \widetilde{z},\\
Y =& -\frac{1}{\omega_0}\widetilde{y} - \frac{(c-d^2+d)}{(d-1)d-1}\widetilde{z} + \widetilde{x},\\
Z =& \widetilde{w},\\
W =& \frac{(d^2(d-1))}{\omega_0}\widetilde{y} + \frac{(c-1)(d-1)}{(d-1)d-1}\widetilde{z} + (d-1)d\widetilde{x},
\end{aligned}$$

in order to write the unperturbed part of system (7.22) in its real Jordan normal form, in accordance with the matrix

$$\mathrm{J} = \begin{pmatrix} 0 & -\omega_0 & 0 & 0\\ \omega_0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus omitting the tilde, we obtain a system of the form

$$\dot{\mathbf{x}} = \mathrm{J}\,\mathbf{x} + \varepsilon G_1(t, \mathbf{x}) + \varepsilon^2 G_2(t, \mathbf{x}), \tag{7.23}$$

where $\mathbf{x} = (x, y, z, w)$,

$$G_1(t, \mathbf{x}) = \begin{pmatrix} G_{11}(t, \mathbf{x})\\ G_{12}(t, \mathbf{x})\\ G_{13}(t, \mathbf{x})\\ G_{14}(t, \mathbf{x}) \end{pmatrix} \quad \text{and} \quad G_2(t, \mathbf{x}) = \begin{pmatrix} 0\\ 0\\ -b_1 z\\ 0 \end{pmatrix},$$

with

$$
\begin{aligned}
G_{11}(t,\mathbf{x}) =& \frac{(a_1(c-1)(d-1)z - a_1\left((d-2)d^2+1\right)x + d\left(-d^2+d+1\right)wz)}{\left((d-1)d-1\right)\left(d^2-1\right)} \\
&+ \frac{\left(d^4-2d^3+d\right)y(a_1 d-(d+1)w)}{\left((d-1)d-1\right)\left(d^2-1\right)\omega_0}, \\
G_{12}(t,\mathbf{x}) =& \frac{\left(-d^2+d+1\right)^2 W\left(\left(d^2-1\right)Y+\omega_0 Z\right)}{\left(d^2-1\right)\left(-cd^2+c+(d-1)d^3-1\right)} \\
&- \frac{a_1}{\left(-d^2+d+1\right)^2\left(d^2-1\right)\left(-cd^2+c+(d-1)d^3-1\right)} \\
&\left((d-2)d^2+1\right)\left(-c(d+1)+(d-1)d^2+1\right)\left(\omega_0\left((c-1)Z\right.\right. \\
&\left.\left.+\left(-d^2+d+1\right)X\right)+\left((d-1)d-1\right)d^2Y\right), \\
G_{13}(t,\mathbf{x}) =& \frac{a_1 d\omega_0\left((c-1)Z+\left(-d^2+d+1\right)X\right)+a_1((d-1)d-1)d^3Y}{\omega_0\left((d-1)\left(cd+c-d^3\right)+1\right)} \\
&+ \frac{\left(-d^2+d+1\right)^2 W\left(\left(d^2-1\right)Y+\omega_0 Z\right)}{\omega_0\left((d-1)\left(cd+c-d^3\right)+1\right)}, \\
G_{14}(t,\mathbf{x}) =& \frac{-(b_1(c-1)(d-1)d+Z(cZ-(d-1)d(X+Z))+XZ)}{((d-1)d-1)} \\
&+ \frac{Y^2\left(-\left(c\left(d^2-1\right)+d\left(-d\left(\omega_0^2+2\right)+\omega_0^2+1\right)+\omega_0^2+2\right)\right)}{((d-1)d-1)\omega_0^2} \\
&+ \frac{Y\left(\left(d^2-1\right)X-\left(\omega_0^2+2\right)Z\right)}{\omega_0}.
\end{aligned}
$$

We now use generalized cylindrical coordinates obtaining system (7.32). Taking θ as the new independent variable we have

$$
\begin{aligned}
\frac{dr}{d\theta} =& \frac{\varepsilon}{(d(d-1)-1)^2\omega_0}\Big(\frac{z}{d^2-1}\big((d(d-1)-1)(a_1(c-1)(d-1)+d(d \\
&-d^2-1)w)\omega_0\cos\theta+(-a_1(c-1)(d-1)(c+cd+d^2-d^3-1) \\
&-(d(d-1)-1)^3w)\sin\theta\big)+\big((d(d-1)-1)^2r\omega_0(a_1(d(d \\
&-1)-1)\omega_0^2\cos^2\theta+\varepsilon^2F_{21}(\theta,\mathbf{y},\varepsilon), \\
\frac{dz}{d\theta} =& \frac{\varepsilon}{(d(d-1)-1)\omega_0}\Big((a_1(c-1)d+(1+d-d^2)w)z\omega_0+(d(d-1)-1)r \\
&(-a_1d\omega_0\cos\theta+(a_1d^3+w+d(1+(d-2)d(1+d))w)\sin\theta)\Big)+\varepsilon^2F_{22}(\theta,\mathbf{y},\varepsilon), \\
\frac{dw}{d\theta} =& \frac{\varepsilon}{\omega_0^2}\left(\frac{-b1(c-1)(d-1)d\omega_0}{d(d-1)-1}+\big(z\omega_0+(d^2-1)r\sin\theta\big)\left(\frac{-(c+d-d^2)z}{d(d-1)-1}\right.\right. \\
&\left.\left.+\ r\cos\theta-\frac{r\sin\theta}{\omega_0}\right)\right)+\varepsilon^2F_{23}(\theta,\mathbf{y},\varepsilon),
\end{aligned}
\tag{7.24}
$$

where $\mathbf{F}_2(\theta,\mathbf{y},\varepsilon) = \big(F_{21}(\theta,\mathbf{y},\varepsilon),\ F_{22}(\theta,\mathbf{y},\varepsilon),\ F_{23}(\theta,\mathbf{y},\varepsilon)\big)$ is a 2π–periodic function in θ, and $\mathbf{y} = (r,z,w)$.

We are going to applying Theorem 5, thus we compute the averaged function $g_1(\mathbf{y}) = (g_{11}(\mathbf{y}),\, g_{12}(\mathbf{y}),\, g_{13}(\mathbf{y}))$ corresponding to system (7.24) where

$$\begin{aligned}
g_{11}(\mathbf{y}) =& -\frac{r(a_1(1-d+d^3-d^4+c(d^2-d-1))+(1+d-d^2)w)}{2(d(d-1)-1)\omega_0^3},\\
g_{12}(\mathbf{y}) =& \frac{(a_1(c-1)d+(1+d-d^2)^2w)z}{(d(d-1)-1)\omega_0^3},\\
g_{13}(\mathbf{y}) =& \frac{-2b_1(c-1)(d-1)d-2(c+d-d^2)z^2}{2(d(d-1)-1)\omega_0}\\
&+\frac{(d(d-1)-1)^2(d^2-1)r^2}{2(d(d-1)-1)(d^3(d-1)-cd+c-1)\omega_0},
\end{aligned}$$

solving the non-linear system $g_1(\mathbf{y}) = 0$, we obtain the solutions

$$\begin{aligned}
\mathbf{s}_1 =& \left(\sqrt{\frac{2b_1d(d-cd+c-1)}{(d^2-1)(d(d-1)-1)}},\, 0,\, \frac{a_1(d-1+(d-1)d^3-c(d^2+d-1))}{(d(d-1)-1)}\right),\\
\mathbf{s}_1^1 =& \left(0,\, \frac{\sqrt{(1-c)(d-1)db_1}}{\sqrt{c+d-d^2}},\, -\frac{a_1(c-1)d}{(1+d-d^2)^2}\right),\\
\mathbf{s}_2^1 =& \left(0,\, -\frac{\sqrt{(1-c)(d-1)db_1}}{\sqrt{c+d-d^2}},\, -\frac{a_1(c-1)d}{(1+d-d^2)^2}\right).
\end{aligned}$$

The solution $\mathbf{s}_1$ has the Jacobian

$$\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_1)\right) = -\frac{a_1b_1(c-1)(d-1)d}{\omega_0^5},$$

which is non-zero, then by Theorem 5 we know that there is a periodic solution $\mathbf{\Phi}(t,\varepsilon)$ close to $\mathbf{s}_1$ such that $\mathbf{\Phi}(0,\varepsilon) = \mathbf{s}_1 + \mathcal{O}(\varepsilon)$. Going back through the change of variables it provides the periodic solution (7.9) of system (7.1).

We also notice that the eigenvalues of $\mathbf{s}_1$ are $\pm\frac{\sqrt{-b_1(c-1)(d-1)d}}{\omega_0^2}$ and $-\frac{a_1}{\omega_0}$. By Theorem 5 (c) the periodic solution (7.9) is unstable if $a_1 < 0$ or $b_1(c-1)(d-1)d < 0$.

The solutions $\mathbf{s}_1^1$ and $\mathbf{s}_2^1$ are such that $\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_i^1)\right) = \frac{a_1b_1(c-1)(d-1)d}{\omega_0^5}$, for $i = 1, 2$. They also provide two additional periodic solution for system (7.1) if $r = \varepsilon r_1 + \mathcal{O}(\varepsilon^2) > 0$, this is possible restricting ε to one of the half intervals $\varepsilon \in (-\varepsilon_1, 0)$, or $\varepsilon \in (0, \varepsilon_1)$.

We notice that $\mathbf{s}_1^1$ and $\mathbf{s}_2^1$ have the same eigenvalues $\pm\frac{\sqrt{2b_1(c-1)(d-1)d}}{\omega_0^2}$ and $-\frac{a_1}{2\omega_0}$. Thus by Theorem 5 (c) the periodic solutions (7.10) and (7.11) are unstable if $a_1 < 0$ or $b_1(c-1)(d-1)d > 0$. □

7.2.4 Proof of statement (iii) of Theorem 38

Proof. Assuming conditions (7.12) system (7.1) has two equilibrium points $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$, when $\varepsilon \to 0$ these equilibria tends to

$$\boldsymbol{p} = \left(0, 0, \frac{(a+1)\,(a^3+3a^2+4a+D_a+1)}{2\,(a^2+3a+3)}, 0\right).$$

We now are studying the bifurcation of periodic orbits from this point. First we translate $\boldsymbol{p}$ to the origin of coordinates doing $(x, y, z, w) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) + \boldsymbol{p}$, then we introduce the scaling $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = \varepsilon\,(X, Y, Z, W)$. With these changes of variables system (7.1) becomes

$$\begin{aligned}
\dot{X} =& a(Y - X) + W,\\
\dot{Y} =& \frac{(-1 + a(1 + a)(2 + a) - D_a)X}{2a(3 + a(3 + a))} - Y + X(c_1 - Z),\\
\dot{Z} =& \varepsilon\left(XY - \frac{(1 + a)b_1(1 + a(4 + a(3 + a)) + D_a)}{2(3 + a(3 + a))}\right) - \varepsilon^2 b_1 Z,\\
\dot{W} =& W + aW - \frac{(1 + a)(1 + a(4 + a(3 + a)) + D_a)X}{2(3 + a(3 + a))} + \varepsilon\big(d_1 W - XZ\big).
\end{aligned} \tag{7.25}$$

We do the linear change of variables

$$\begin{aligned}
X =& \widetilde{z} + \frac{(2 + a)\widetilde{y}}{(1 + a)\omega_+},\\
Y =& \frac{\widetilde{x}}{a + a^2} + \frac{(a(1 + a)(2 + a) - D_a - 1)\widetilde{z}}{2a(3 + a(3 + a))} - \frac{\widetilde{y}}{(a + a^2)\omega_+},\\
Z =& \widetilde{w},\\
W =& \widetilde{x} + \frac{(1 + a(4 + a(3 + a)) + D_a)\widetilde{z}}{2(3 + a(3 + a))} + \frac{(1 + a)\widetilde{y}}{(a + a^2)\omega_+},
\end{aligned}$$

in order to write the unperturbed part of system (7.25) in its real Jordan normal form, in accordance with the matrix

$$\mathrm{J} = \begin{pmatrix} 0 & -\omega_+ & 0 & 0\\ \omega_+ & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus omitting the tilde we obtain a system of the form

$$\dot{\mathbf{x}} = \mathrm{J}\,\mathbf{x} + \varepsilon H_1(t, \mathbf{x}) + \varepsilon^2 H_2(t, \mathbf{x}), \tag{7.26}$$

with $\mathbf{x} = (x, y, z, w)$. Using cylindrical coordinates we obtain system (7.34). In order to put system (7.34) in the normal form (1.11), we take θ as the new independent variable and then we have

$$\begin{aligned}
\frac{dr}{d\theta} =& -\varepsilon\Big((1 + a)(3 + a(3 + a))d_1\left(\sqrt{2}\omega_+\right)^3 r\cos\theta^2\\
& - \frac{\sin\theta}{3 + a(3 + a)}\Big(- a(1 + a)(3 + a(3 + a))(4 + a(9 + a(10 + a(5 + a))))c_1\\
& + a(16 + a(45 + a(59 + 2a(7 + a))))d_1 + (a(1 + a)(2 + a)(3 + a(3 + a))c_1\\
& + d_1)D_a + d_1 + (-1 + a + a^2)(3 + a(3 + a))(4 + a(9 + a(10 + a(5\\
& + a)) - D_a) - 2D_a)w\Big)z + 2(3 + a(3 + a))^2
\end{aligned}$$

$$\sqrt{\frac{-a(9+a(10+a(5+a))-D_a)}{3+a(3+a)}}r\Big((1+a)(a(2+a)c_1-(1+a)d_1)$$
$$-(2+a)(-1+a+a^2)w\Big)\sin\theta\Big)+\varepsilon^2F_{21}(\theta,\mathbf{y},\varepsilon),$$

$$\begin{aligned}\frac{dz}{d\theta}=&\varepsilon\Big(\sqrt{2}(3+a(3+a))\varepsilon\big((1+a)\sqrt{2}\omega_+(a(6c_1-4d_1)-d_1(1+D_a)+a^2(12c_1\\&-3d_1-10w)+a^3(8c_1-d_1-8W)+2a^4(c_1-w)+6w)z+2(3\\&+a(3+a))r(-(1+a)d_1\sqrt{2}\omega_+\cos\theta-\sqrt{2}((1+a)(d_1+a(-(2+a)c_1\\&+d_1))+(2+a(a^2+a-1)w)\sin\theta)\big)\Big)+\varepsilon^2F_{22}(\theta,\mathbf{y},\varepsilon),\end{aligned}\tag{7.27}$$

$$\begin{aligned}\frac{dw}{d\theta}=&-\varepsilon\Big(2(1+a)^3\big(ab_1(1+a(16+a(45+a(59+2a(20+a(7+a))))))+D_a)\\&+(1-a(1+a)(a^2+a-1)(3+a(3+a))+D_a+a(5+a(4+a))D_a)z^2\big)\\&+(3+a(3+a))r\big(-2(1+a)(4+a(9+a(10+a(5+a))-D_a)z\cos\theta\\&+2(2+a)(3+a(3+a))r\cos(2\theta)+\sqrt{2}(1+a)(a(2+a)(3+a)-3-D_a)\\&-2(4+D_a))\sqrt{2}\omega_+z\sin\theta+(2+a)(3+a(3+a))r\big(\sqrt{2}\sqrt{2}\omega_+\sin(2\theta)\big)\big)\Big)\Big/\\&\Bigg(a(1+a)^2(3+a(3+a))(4+a(9+a(10+a(5+a))-D_a)\\&-2D_a)\sqrt{\frac{-2(4+a(9+a(10+a(5+a))))+2(2+a)D_a}{3+a(3+a)}}\Bigg)\\&+\varepsilon^2F_{23}(\theta,\mathbf{y},\varepsilon),\end{aligned}$$

with $\mathbf{F}_2(\theta,\mathbf{y},\varepsilon)=\big(F_{21}(\theta,\mathbf{y},\varepsilon),\ F_{22}(\theta,\mathbf{y},\varepsilon),\ F_{23}(\theta,\mathbf{y},\varepsilon)\big)$ a 2π–periodic function in θ.

Now we can apply Theorem 5 and calculate the averaged function

$$g_1(\mathbf{y})=(\mathbf{g}_{11}(\mathbf{y}),\mathbf{g}_{12}(\mathbf{y}),\mathbf{g}_{13}(\mathbf{y}))$$

of (7.27) where

$$\begin{aligned}\mathbf{g}_{11}(\mathbf{y})=&\frac{1}{4\big(3+a(3+a)\big)\omega_+^3}\Big((1+a)(2a(3+a(3+a))c_1-5d_1\\&-a\big(8+a(5+a)\big)d_1+d_1D_a)r-2\big(a^2+a-1\big)\big(3+a(3+a)\big)rw\Big),\\ \mathbf{g}_{12}(\mathbf{y})=&\frac{-1}{2(3+a(3+a))\omega_+}z\Big(2(a(a+3)+3)(a(a+1)(c_1-w)+w)\\&-(a(a(a+3)+4)+1)d_1-d_1D_a\Big),\\ \mathbf{g}_{13}(\mathbf{y})=&-\frac{1}{4\big(3+a(3+a)\big)\omega_+}\Bigg(-2(1+a)b_1\big(1+a(4+a(3+a))+D_a\big)\\&+\frac{(2+a)\big(3+a(3+a)\big)\big(2(2+D_a)+a(9+a(10+a(5+a))+D_a)\big)r^2}{a(1+a)^5\big(1+a(3+a)\big)}\\&+\frac{2\big(-1+a(1+a)(2+a)-D_a\big)z^2}{a}\Bigg).\end{aligned}$$

The non-linear system $g(\mathbf{y}) = 0$ has the solutions

$$\mathbf{s}_2 = \left(\sqrt{\frac{a(1+a)^3 b_1\big(1+a(16+a(45+a(59+2a(20+a(7+a)))))+D_a\big)}{(2+a)\big(3+a(3+a)\big)^2}}, \right.$$
$$\left. 0, \frac{(1+a)\big(2a(3+a(3+a))c_1-(5+a(8+a(5+a)))d_1+d_1D_a\big)}{2(a^2+a-1)\big(3+a(3+a)\big)} \right),$$
$$\mathbf{s}_1^2 = \left(0, \frac{\sqrt{(a^3+a^2-1+D_a)\,b}}{\sqrt{2}}, \right.$$
$$\left. \frac{-2a(1+a)(3+a(3+a)c_1+d_1+a(4+a(3+a))d_1+D_ad_1)}{2(a^2+a-1)(3+a(3+a))} \right),$$
$$\mathbf{s}_1^2 = \left(0, -\frac{\sqrt{(a^3+a^2-1+D_a)\,b}}{\sqrt{2}}, \right.$$
$$\left. \frac{-2a(1+a)(3+a(3+a)c_1+d_1+a(4+a(3+a))d_1+D_ad_1)}{2(a^2+a-1)(3+a(3+a))} \right).$$

The solution $\mathbf{s}_2$ has the Jacobian

$$\det\left(\frac{\partial \mathbf{g}}{\partial y}(\mathbf{s}_2)\right) = \frac{(1+a)(a^2+a-1)b_1d_1(1+a(4+a(3+a))-D_a)}{2(3+a(3+a))\omega_+^5}$$

which is non-zero, then by Theorem 5, there is a periodic solution $\mathbf{\Phi}(t,\varepsilon)$ close to $\mathbf{s}_2$ such that $\mathbf{\Phi}(0,\varepsilon) = \mathbf{s}_2 + \mathcal{O}(\varepsilon)$. Going back through the change of variables, it provides the periodic solution (7.13) of system (7.1).

The solutions $\mathbf{s}_1^2$ and $\mathbf{s}_2^2$ are such that

$$\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_i^2)\right) = -\frac{(a^2+a-1)b_1d_1(a(1+a)(2+a)-D_a-1)(D_a+a^2+a^3-1)}{4a(3+3a+a^2)\omega_+^5},$$

for $i = 1, 2$. They also provide two additional periodic solutions for system (7.1) if $r = \varepsilon r_1 + \mathcal{O}(\varepsilon^2) > 0$, this is possible restricting ε to one of the half intervals $\varepsilon \in (-\varepsilon_1, 0)$, or $\varepsilon \in (0, \varepsilon_1)$.

We notice that $\mathbf{s}_1^2$ and $\mathbf{s}_2^2$ has the same eigenvalues $\dfrac{d_1}{2\omega_+}$ and

$$\frac{\pm 1}{a(3+a(3+a))^3\omega_+^5}\Big(-a^2(a+1)\left(a^2+a-1\right)(a(a+3)+3)^2 b_1\Big(a\left(a^{13}+15a^{12}\right.$$
$$+103a^{11}+428a^{10}+1202a^9+2427a^8+3699a^7+4487a^6+4581a^5+4038a^4$$
$$+2948a^3-(a+1)^2(a+2)^2\left(a^2+a+1\right)(a(a+3)+3)(a(a+4)+5)D_a$$
$$\left.+1614a^2+573a+100\right)+D_a+1\Big)\Big)^{\frac{1}{2}}. \tag{7.28}$$

Thus by Theorem 5 (*c*) the periodic solutions (7.14) and (7.15) are unstable if $d_1 > 0$, or if the eigenvalues (7.28) are non-zero real numbers. □

7.2.5 Proof of statement (iv) of Theorem 38

Proof. of statement (iv) of Theorem 38. Assuming conditions (7.16) system (7.1) has two equilibrium points $\boldsymbol{p}_+$ and $\boldsymbol{p}_-$, when $\varepsilon \to 0$ these equilibria tends to

$$\boldsymbol{p} = \left(0,\ \frac{(a+1)(a^3+3a^2+4a-D_a+1)}{2(a^2+3a+3)},\ 0\right).$$

We now are studying the bifurcation of periodic orbits from this point. First we translate $\boldsymbol{p}$ to the origin of coordinates doing $(x,y,z,w) = (\overline{x},\overline{y},\overline{z},\overline{w}) + \boldsymbol{p}$, then we introduce the scaling $(\overline{x},\overline{y},\overline{z},\overline{w}) = \varepsilon\,(X,Y,Z,W)$. With these changes of variables system (7.1) becomes

$$\begin{aligned}
\dot{X} =& W + a(Y-X),\\
\dot{Y} =& \frac{a(1+a)(2+a)+D_a-1}{2a\big(3+a(3+a)\big)}X - Y + \varepsilon(c_1-Z)X,\\
\dot{Z} =& \varepsilon\left(XY - \frac{(1+a)b_1\big(1+a(4+a(3+a))-D_a\big)}{2\big(3+a(3+a)\big)}\right) - \varepsilon^2 b_1 Z,\\
\dot{W} =& (1+a)W - \frac{(1+a)\,(1+a(4+a(3+a))-D_a)}{2\big(3+a(3+a)\big)}X + \varepsilon(d_1W - XZ).
\end{aligned} \tag{7.29}$$

We do the linear change of variables

$$\begin{aligned}
X =& \widetilde{w} + \frac{\sqrt{2}(2+a)}{(1+a)\sqrt{2}\omega_-}\widetilde{y} + \widetilde{z},\\
Y =& \frac{1}{2a(1+a)\big(3+a(3+a)\big)\sqrt{2}\omega_-}\Big(-2\sqrt{2}\big(3+a(3+a)\big)\widetilde{y} + \sqrt{2}\omega_-\big((1+a)(-1\\
&+a(1+a)(2+a)+D_a)\widetilde{w} + 2\big(3+a(3+a)\big)\widetilde{x} + (1+a)\big(-1+a(1\\
&+a)(2+a)+D_a\big)\widetilde{z}\Big),\\
Z =& \widetilde{w},\\
W =& \widetilde{x} + \frac{\sqrt{2}}{\sqrt{2}\omega_-}\widetilde{y} + \frac{1}{2\big(3+a(3+a)\big)\sqrt{2}\omega_-}\Big(2\sqrt{2}a\big(3+a(3+a)\big)\widetilde{y} + \big(1+a(4\\
&+a(3+a))-D_a\big)\sqrt{2}\omega_-(\widetilde{w}+\widetilde{z})\Big),
\end{aligned}$$

in order to write the unperturbed part of system (7.29) in its real Jordan normal form, in accordance with the matrix

$$\mathrm{J} = \begin{pmatrix} 0 & -\omega_- & 0 & 0\\ \omega_- & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus omitting the tilde we obtain a system of the form

$$\dot{\mathbf{x}} = \mathrm{J}\,\mathbf{x} + \varepsilon I_1(t,\mathbf{x}) + \varepsilon^2 I_2(t,\mathbf{x}), \tag{7.30}$$

with $\mathbf{x}$=(x, y, z, w). Using cylindrical coordinates we obtain system (7.33) and taking θ as the new independent variable we have

$$
\begin{aligned}
\frac{dr}{d\theta} =& -\frac{\varepsilon\sqrt{2}}{(2+a)\big(4+a(9+a(10+a(5+a))-D_a)-2D_a\big)^2} \\
& \Big((1+a)\sqrt{2}\omega_-\big(a\big(3+a(3+a)\big)c_1\big(4+a(9+a(10+a(5+a))-D_a)-2Da\big) \\
& + d_1\big(1+a(16+a(45+a(59+2a(20+a(7+a)))))+D_a\big) \\
& -(1+a)\big(3+a(3+a)\big)\big(4+a(9+a(10+a(10+a(5+a))-D_a) \\
& -2D_a\big)w\big)z\cos\theta-(1+a)\big(3+a(3+a)\big)^2 d_1\sqrt{2}\omega_-^3 r\cos\theta^2 \\
& -\sin\theta\big(\sqrt{2}(1+a)\big(-a(1+a)\big(3+a(3+a)\big)\big(4+a(9+a(10+a(5 \\
& +a)))\big)c_1+d_1+a\big(16+a(59+2a(20+a(7+a))))\big)d_1+\big(a(1+a)(2+a) \\
& \big(3+a(3+a)\big)c_1+d_1\big)D_a+(a^2+a-1)\big(3+a(3+a)\big)\big(4+a(9+a(10 \\
& +a(5+a))-D_a)w\big)z+2\big(3+a(3+a)\big)^2\sqrt{2}\omega_-\big)r\big((1+a)\big(a(2+a)c_1 \\
& -(1+a)d_1\big)-2(2+a)\big(a^2+a-1\big)w\big)\sin\theta+2^{\frac{-1}{2}}\big(3+a(3+a)\big) \\
& \big(4+a(9+a(10+a(5+a))-D_a)-2D_a\big)r\big(a\big((2+a)c_1+d_1+ad_1\big) \\
& -(1+a)(1+a(2+a)w)\big)\sin(2\theta)\Big)+\varepsilon^2 F_{21}(\theta,\mathbf{y},\varepsilon), \qquad (7.31)\\
\frac{dz}{d\theta} =& \frac{-\varepsilon\sqrt{2}\big(3+a(3+a)\big)}{(1+a)\big(4+a(9+a(10+a(5+a))-D_a)-2D_a\big)^2}\Big((1+a)\sqrt{2}\omega_-\big(a(6c_1 \\
& -4d_1)-d_1(1+D_a)+a^2(12c_1-3d_1-10w)+a^3(8c_1-d_18w)+2a^4(c_1-w) \\
& +6w\big)z2\big(3+a(3+a)\big)r\big((1+a)\sqrt{2}\omega_-\cos\theta+\sqrt{2}\big((1+a)(d_1 \\
& +a(-(2+a)c_1+d_1))+(2+a)\big(a^2+a-1\big)w\big)\sin\theta\big)\Big)+\varepsilon^2 F_{22}(\theta,\mathbf{y},\varepsilon), \\
\frac{dw}{d\theta} =& \frac{-\varepsilon\sqrt{2}\big(3+a(3+a)\big)}{(1+a)\big(4+a(9+a(10+a(5+a))-D_a)-D_a\big)^2}\Big((1+a)\sqrt{2}\omega_-\big(a(6c_1-4d_1) \\
& -(1+D_a)+a^2(12c_1-3d_1-10w)+a^3(8c_1-d_1-8w)+2a^4(c_1-w) \\
& +6w\big)z-2\big(3+a(3+a)\big)r\big((1+a)d_1\sqrt{2}\omega_-\cos\theta+\sqrt{2}\big((1+a)(d_1 \\
& +a(-(2+a)c_1+d_1))+(2+a)(a^2+a-1)w\big)\sin\theta\big)\Big)+\varepsilon^2 F_{23}(\theta,\mathbf{y},\varepsilon),
\end{aligned}
$$

with $\mathbf{F}_2(\theta,\mathbf{y},\varepsilon)=\big(F_{21}(\theta,\mathbf{y},\varepsilon),\ F_{22}(\theta,\mathbf{y},\varepsilon),\ F_{23}(\theta,\mathbf{y},\varepsilon)\big)$ a 2π–periodic function in θ. Now we can apply Theorem 5 and calculate the averaged function $g(\mathbf{y})=(\mathbf{g}_1(\mathbf{y}),\mathbf{g}_2(\mathbf{y}),\mathbf{g}_3(\mathbf{y}))$ corresponding to system (7.31), where

$$
\begin{aligned}
\mathbf{g}_1(\mathbf{y}) =& \frac{-1}{2\big(3+a(3+a)\big)\omega_-^3}\Big(-(1+a)\big(2a\big(3+a(3+a)\big)c_1 \\
& -5d_1-a\big(8+a(5+a)\big)d_1+d1D_a\big)r+2(a^2+a-1)\big(3+a(3+a)\big)rw\Big), \\
\mathbf{g}_2(\mathbf{y}) =& \frac{-1}{\big(4+a(9+a(10+a(5+a))-D_a)-2D_a\big)^2}\Big(\big(3+a(3+a)\big)2\omega_- \\
& \big((-1-a(4+a(3+a)))d_1-d_1D_a+2\big(3+a(3+a)\big)\big(a(1+a)(c_1-w)
\end{aligned}
$$

$$
\begin{aligned}
&+w))z\Big),\\
\mathbf{g}_3(\mathbf{y}) =& \frac{1}{4a\big(3+a(3+a)\big)\omega_-}\Big(-2a(1+a)b_1\big(1+a(4+a(3+a))+D_a\big)\\
&+\frac{(2+a)\big(3+a(3+a)\big)\big(2(2+D_a)+a(9+a(10+a(5+a))+D_a)\big)r^2}{(1+a)^5\big(1+a(3+a)\big)}\\
&+2\big(-1+a(1+a)(2+a)-D_a\big)z^2\Big).
\end{aligned}
$$

The non-linear system $g(\mathbf{y})=0$ has the solutions

$$
\begin{aligned}
\mathbf{s}_3 =& \Bigg(\sqrt{\frac{a(1+a)^3b_1\big(1-D_a+a(16+a(45+a(59+2a(20+a(7+a)))))\big)}{(2+a)\big(3+a(3+a)\big)^2}},\\
&0, \frac{(1+a)\big(2a\big(3+a(3+a)\big)c_1-\big(5+a(8+a(5+a))\big)d_1+d_1D_a\big)}{2(a^2+a-1)\big(3+a(3+a)\big)}\Bigg),\\
\mathbf{s}_1^3 =& \Bigg(0, \frac{2a(1+a)(3+a(3+a))c_1-(1+a(4+a(3+a)))d_1+D_ad_1}{2(a^2+a-1)(3+a(3+a))},\\
&\frac{\sqrt{(a^3+a^2-1-D_a)\,b_1}}{\sqrt{2}}\Bigg),\\
\mathbf{s}_2^3 =& \Bigg(0, \frac{2a(1+a)(3+a(3+a))c_1-(1+a(4+a(3+a)))d_1+D_ad_1}{2(a^2+a-1)(3+a(3+a))},\\
&-\frac{\sqrt{(a^3+a^2-1-D_a)\,b_1}}{\sqrt{2}}\Bigg).
\end{aligned}
$$

The solution $\mathbf{s}_3$ has the Jacobian

$$
\begin{aligned}
\det\left(\frac{\partial \mathbf{g}}{\partial y}(\mathbf{s}_3)\right) =& \frac{a}{|a|2(3+a(3+a))^2\omega_-^7}\Big((1+a)(a^2+a-1)b_1d_1(1+a(16+a(45\\
&+a(59+2a(20+a(7+a)))))-D_a)\Big)
\end{aligned}
$$

which is non-zero, then by Theorem 5 we know that there is a periodic solution $\mathbf{\Phi}(t,\varepsilon)$ close to $\mathbf{s}_3$ such that $\mathbf{\Phi}(0,\varepsilon)=\mathbf{s}_3+\mathcal{O}(\varepsilon)$. Going back through the change of variables, it provides the periodic solution (7.17) of system (7.1).

The solutions $\mathbf{s}_1^3$ and $\mathbf{s}_2^3$ are such that

$$
\begin{aligned}
\det\left(\frac{\partial g}{\partial y}(\mathbf{s}_i^3)\right) =& \frac{-a}{|a|2(3+a(3+a))^2\omega_-^7}\Big((1+a)(a^2+a-1)b_1d_1(1+a(16+a(45\\
&+a(59+2a(20+a(7+a)))))-D_a)\Big),
\end{aligned}
$$

for $i=1,2$. They also provide two additional periodic solution for system (7.1) if $r=\varepsilon r_1+\mathcal{O}(\varepsilon^2)>0$, this is possible restricting ε to one of the half intervals $\varepsilon\in(-\varepsilon_1,0)$ or $\varepsilon\in(0,\varepsilon_1)$. $\square$

7.3 Appendix F

Let $R_a = (d-1)(cd + c - d^3) + 1$ and $R_b = d^2 - d - 1$ then system (7.23) in cylindrical coordinates writes

$$
\begin{aligned}
\dot{r} =& \varepsilon\Big(- \frac{r\left(a_1(d(2R_a + R_b(R_b+2)) + R_b(R_a + R_b + 1)) + R_b^2 w(d+R_b)\right)}{2R_a(d+R_b)} \\
&+ \frac{z\cos\theta(a_1(-c(R_b+1) + R_b(d+R_b+2) + R_a + 1) - dR_b w)}{R_b(d+R_b)} \\
&+ \frac{r\cos(2\theta)\left(a_1(R_b(d(R_b+2) + R_b + 1) + R_a(R_b+2)) + R_b^2 w(d+R_b)\right)}{2R_a(d+R_b)} \\
&- \frac{z\sin\theta\left(a_1\left(c(R_a-1) + d\left(R_a + R_b^2 + R_b\right) - 2R_a + 1\right) + R_b^3 w\right)}{\sqrt{R_a}R_b^{3/2}(d+R_b)} \\
&+ \frac{r\sin(2\theta)(a_1(dR_b(R_b+2) + R_a) - (d+1)R_b(R_b+1)w)}{2\sqrt{R_a}\sqrt{R_b}(d+R_b)}\Big), \\
\dot{\theta} =& \omega_0 + \varepsilon\Big(\frac{a_1\cos^2(\theta)(dR_b + R_a)}{\sqrt{R_a}\sqrt{R_b}(d+R_b)} + \frac{\sqrt{R_b}(R_b+1)\sin^2(\theta)(-a_1 d + dw + w)}{\sqrt{R_a}(d+R_b)} \\
&- \frac{z\sin\theta(a_1(-c(R_b+1) + R_b(d+R_b+2) + R_a + 1) - dR_b w)}{rR_b(d+R_b)} \\
&\cos\theta\Big(- \frac{z\left(a_1\left(c(R_a-1) + d\left(R_a + R_b^2 + R_b\right) - 2R_a + 1\right) + R_b^3 z\right)}{r\sqrt{R_a}R_b^{3/2}(d+R_b)} \\
&\frac{\sin\theta\left(a_1(R_b(d(R_b+2) + R_b + 1) + R_a(R_b+2)) + R_b^2 w(d+R_b)\right)}{R_a(d+R_b)}\Big)\Big), \\
\dot{z} =& \varepsilon\Big(\frac{z\left(a_1(R_b(-c+d+R_b+2) + R_a) + R_b^2 w\right)}{R_a} - \frac{a_1 drR_b\cos\theta}{R_a} \\
&+ \frac{rR_b^{3/2}\sin\theta(a_1(d(R_b+2) + R_b + 1) + R_b w(d+R_b))}{R_a^{3/2}}\Big), \\
\dot{w} =& \varepsilon\Big(\frac{Z\left(a_1(R_b(-c+d+R_b+2) + R_a) + R_b^2 w\right)}{R_a} - \frac{a_1 drR_b\cos\theta}{R_a} \\
&\frac{rR_b^{3/2}\sin\theta(a_1(d(R_b+2) + R_b + 1) + R_b w(d+R_b))}{R_a^{3/2}}\Big).
\end{aligned}
\tag{7.32}
$$

System (7.26) in cylindrical coordinates.

$$
\begin{aligned}
\dot{r} =& \varepsilon\Big(- \frac{(a+1)d_1 r(a(a(a(a+5)+10) - D_a + 9) - 2D_a + 4)\cos^2(\theta)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^2} \\
&+ \sin\theta\Big(\frac{\sqrt{2}(a+1)d_1 Z(a(a(a(2a(a(a+7)+20)+59)+45)+16) + D_a + 1)}{(a+2)(a(a+3)+3)^2\sqrt{2}\omega_+^3} \\
&- \frac{(a+1)Z(a(a(a(a+5)+10) - D_a + 9) - 2D_a + 4)(a(a+1)(c_1 - W) + W)}{(a+2)(a(a+3)+3)\omega_+^3}\Big) \\
&+ \sin\theta^2\Big(\frac{2a(a+1)c_1 r - 2\left(a^2 + a - 1\right)rW}{\sqrt{2}\omega_+^2} - \frac{2(a+1)^2 d_1 r}{(a+2)\sqrt{2}\omega_+^2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\cos\theta\Big(-\frac{(a+1)d_1Z(a(a(a(2a(a(a+7)+20)+59)+45)+16)+D_a+1)}{(a+2)(a(a+3)+3)^2\sqrt{2}\omega_+^2}\\
&-\frac{(a+1)Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(ac_1-(a+1)W)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^2}\\
&-\sin\theta\Big(\frac{\sqrt{2}a(a+1)d_1r(a(a(a(a+5)+10)-D_a+9)-2D_a+4)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^3}\\
&+\frac{\sqrt{2}r(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(ac_1-(a+1)W)}{(a(a+3)+3)\sqrt{2}\omega_+^3}\Big)\Big)\Big),\\
\dot{\theta}=&\omega_++\varepsilon\Big(\frac{\sqrt{2}(a+1)d_1(a(a(a(a+5)+10)-D_a+9)-2D_a+4)\cos^2(\theta)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^3}\\
&+\sin\theta\Big(\frac{(a+1)d_1Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)}{2(a+2)(a(a+3)+3)^2\sqrt{2}\omega_+^2r}\\
&\frac{(a(a(a+3)+4)+D_a+1)}{2(a+2)(a(a+3)+3)^2\sqrt{2}\omega_+^2r}\\
&+\frac{(a+1)Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(ac_1-(a+1)W)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^2r}\Big)\\
&+\sin\theta^2\Big(\frac{\sqrt{2}(a+1)^2d_1(a(a(a(a+5)+10)-D_a+9)-2D_a+4)}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^3}\\
&+\frac{(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(ac_1-(a+1)W)}{(a(a+3)+3)\omega_+^3}\Big)\\
&+\cos\theta\Big(\frac{1}{4(a+2)(a(a+3)+3)^2\omega_+^3r}\big((a+1)d_1Z(a(a(a(a+5)\\
&+10)-D_a+9)-2D_a+4)(a(a(a+3)+4)+D_a+1)\big)\\
&-\frac{(a+1)Z(a(a(a(a+5)+10)-D_a+9)-2D_a+4)(a(a+1)(c_1-W)+W)}{(a+2)(a(a+3)+3)\omega_+^3r}\Big)\\
&+\sin\theta\Big(\frac{(a+1)d_1(a(a(a+1)(a+2)-D_a-3)-2(D_a+1))}{(a+2)(a(a+3)+3)\sqrt{2}\omega_+^2}\\
&+\frac{2(a(a+1)(c_1-W)+W)}{\sqrt{2}\omega_+^2}\Big)\Big),
\end{aligned}
\tag{7.33}
$$

$$
\begin{aligned}
\dot{z}=&-\frac{\varepsilon}{(a+1)(a(a+3)+3)\sqrt{2}\omega_+^3}\Big((1+a)\sqrt{2}\omega_+\Big(a(6c_1-4d_1)-d_1(1+D_a)\\
&+a^2(12c_1-3d_1-10w)+a^3(8c_1-d_1-8w)+2a^4(c_1-w)+6w\Big)z-2(3+a(3\\
&+a))r\Big((1+a)d_1\sqrt{2}\omega_+\cos\theta+\sqrt{2}\Big((1+a)(d_1+a(-(2+a)c_1+d_1))+(2+a)(a^2\\
&+a-1)w\Big)\sin\theta\Big)\Big)+b_1w\varepsilon^2,\\
\dot{w}=&\varepsilon\Big(\frac{1+a}{a(3+a(3+a))(4+a(9+a(10+a(5+a))-D_a)-2D_a)}\Big(-ab_1\Big(1+a\big(16
\end{aligned}
$$

$$
\begin{aligned}
&+ a(45 + a(59 + 2a(20 + a(7 + a))))\Big) + D_a\Big) - z^2 + \big(a(1 + a)\big(a^2 + a \\
&- 1\big)(3 + a(3 + a)) - \big(1 + a(5 + a(4 + a))\big)D_a\big)z^2\Big) \\
&+ \frac{\sqrt{2}\omega_+ rZ(a(-a(a + 2)(a + 3) + D_a + 3) + 2(D_a + 4))\sin\theta}{\sqrt{2}a(a + 1)(a(a(a(a + 5) + 10) - D_a + 9) - 2D_a + 4)} \\
&+ \cos\theta\left(\frac{rZ}{a^2 + a} - \frac{\sqrt{2}(a + 2)(a(a + 3) + 3)\sqrt{2}\omega_+ r^2\sin\theta}{a(a + 1)^2(a(a(a(a + 5) + 10) - D_a + 9) - 2D_a + 4)}\right) \\
&+ \frac{2(a + 2)(a(a + 3) + 3)r^2\sin^2(\theta)}{a(a + 1)^2(a(a(a(a + 5) + 10) - D_a + 9) - 2D_a + 4)}.
\end{aligned}
$$

System (7.30) in cylindrical coordinates.

$$
\begin{aligned}
\dot{r} =& \frac{\varepsilon}{2(a(a + 3) + 3)(a + 2)^2 D_a + 2(a(a + 3) + 3)(a(a(a(a + 5) + 10) + 9) + 4)(a + 2)} \\
&\Big(2(a + 1)(a(a + 3) + 3)(a(a(a(a + 5) + 10) + 9) + 4)d_1 r\cos^2(\theta) \\
&2\cos\theta\Big((1 + a)(d_1 - 12w + a((3 + a(3 + a))(4 + a(9 + a(10 + a(5 + a))))c_1 \\
&+ (16 + a(45 + a(59 + 2a(20 + a(7 + a)))))d_1 - (51 + a(100 \\
&+ a(115 + a(82 + a(36 + a(9 + a))))))w))(w + z) - \sqrt{2}(3 + a(3 \\
&+ a))^2\sqrt{2}\omega_- r(a((2 + a)c_1 + d_1 + ad_1) - (1 + a)(2 + a)w)\sin\theta\Big) \\
&- (3 + a(3 + a))\sin\theta\big(\sqrt{2}(1 + a)\sqrt{2}\omega_-(6w - d_1 + a(2(1 + a(3 + a(3 + a))c_1 \\
&- (4 + a(3 + a))d_1 - 2a(5 + a(4 + a))w))(w + z) + 4(3 + a(3 + a))r((1 + a)(a(2 \\
&+ a)c_1 - (1 + a)d_1) - (2 + a)(a^2 + a - 1)w)\sin\theta\big) + D_a\big(2(1 + a)(a(2 + a)(3 \\
&+ a(3 + a))c_1 - d_1 - 6w - a(3 + a(5 + a(3 + a))w)(w + z)\cos\theta \\
&+ (1 + a)(2 + a)(3 + a(3 + a))d_1 r\cos\theta^2 \\
&- \sqrt{2}(1 + a)(3 + a(3 + a))d_1)\sqrt{2}\omega_-(w + z)\sin\theta\big)\Big), \\
\dot{\theta} =& \omega_- + \frac{\varepsilon}{2(2 + a)(3 + 2a + a^2)^2\sqrt{2}\omega_-^3 r}\Big(2\sqrt{2}(1 + a)(3 + a(3 + a))d_1(2(2 + D_a) \\
&+ a(9 + a(10 + a(5 + a)) + D_a))r\cos\theta^2 + (1 + a)(2(2 + D_a) \\
&+ a(9 + a(10 + a(5 + a)) + D_a))\sqrt{2}\omega_-(d_1 - d_1 D_a + a^2(6c_1 + 3d_1 - 8w) \\
&+ 2a(3c_1 + 2d_1 - 6w) + a^3(2c_1 + d_1 - 2w) - 6w)(w + z)\sin\theta \\
&+ 2\sqrt{2}(3 + a(3 + a))(2(2 + D_a) + a(9 + a(10 + a(5 + a)) + D_a))r(d_1 \\
&+ a(2 + a)(c_1 + d_1) - 2w - a(3 + a)w)\sin\theta^2 + \cos\theta(-\sqrt{2}(1 + a)(2(2 \\
&+ D_a) + a(9 + a(10 + a(5 + a)) + D_a))(a(6c_1 - 4d_1) + d_1(D_a - 1) \\
&+ a^2(12c_1 - 3d_1 - 10w) + a^3(8c_1 + d_1 - 8w) + 2a^4(c_1 - w) + 6w)(w + z) \\
&+ 2(3 + a(3 + a))\sqrt{2}\omega_- r(2d_1(D_a - 1) + a^3(28c_1 + 5d_1 - 26w) \\
&+ a^2(30c_1 + d_1(D_a - 1) - 20w) + 4a^4(3c_1 + d_1 - 3w) + a^5(2c_1 + d_1 - 2w) \\
&+ 12w + a(12c_1 - 5d_1 + 3c_1 D_a + 6w))\sin\theta)\Big),
\end{aligned}
$$

$$
\begin{aligned}
\dot{z} =&\varepsilon\Big(2\left(-(a+1)(a(a+1)(a(a+3)+3)(a(a(a(a(a(a+5)+7)-4)-17)-12)\right.\\
&-3)-1)Z^2+a(a(a+3)+3)(a(a+1)(a(a+3)+3)(a(a(a(a+5)+10)+9)\\
&+4)c_1-(a(a(a(2a(a(a+7)+20)+59)+45)+16)+1)d_1)Z\\
&+a(a+1)(a(a(a(a(a(a(a(a(a+12)+65)+208)+435)+623)+621)\\
&+424)+183)+40)+1)b_1)\,(a+1)^2+(-2(a(a(a(a(a(a(a(a(2a(a+11)\\
&+107)+296)+495)+470)+141)-219)-312)-180)-46)-1)(a+1)^2\\
&-2(a(a+1)(a+2)(a(a+3)+3)(a(2a(a+3)+5)-1)+1)D_a(a+1)^2\big)\,W^2\\
&+W\left((a+1)D_a\left(2(a+1)(a(a(a+3)+3)(a(a+1)(a+2)(a(a+3)+3)c_1\right.\right.\\
&+d_1-a(a(a+3)(3a(a+3)+14)+17)Z-2Z)-2Z)+(a+2)(a(a+3)\\
&+3)r\left(\sqrt{2}(a+2)(a(a+1)(a+2)-1)\sqrt{2}\omega\_\sin\theta-2(a(a(a(a+5)+10)+9)\right.\\
&+4)\cos\theta))+(a+1)\left(2(a+1)(a(a(a+3)+3)(a(a+1)(a(a+3)+3)(a(a(a\right.\\
&+5)+10)+9)+4)c_1-(a(a(a(2a(a(a+7)+20)+59)+45)+16)+1)d_1)\\
&+a(56-a(a(a(a(a(a(a(a(3a+32)+149)+386)+576)+411)-107)\\
&-519)-501)-243))Z+2Z)+(a(a+3)+3)r\left(\sqrt{2}(a(a(a(a(a(a(3a+26)\right.\\
&+93)+168)+134)-26)-131)-84)-14)\sqrt{2}\omega\_\sin\theta-2(a(a(a(a(a(a(a\\
&+8)+27)+50)+58)+52)+45)+30)+10)\cos\theta)))+\frac{1}{2}(a(a+3)\\
&+3)r\left(-2(a+2)(a(a+3)+3)(a(a(a(a+5)+10)+9)+4)r+2(a+2)(a(a\right.\\
&+3)+3)(a(a(a(a+5)+10)+9)+4)\cos(2\theta)r-4(a+1)(a(a+1)(a(a+3)\\
&+3)(a(a(a(a+5)+10)+9)+4)d_1+(a(a(a(a(a(a(a(a+8)+27)+50)+58)\\
&+52)+45)+30)+10)Z)\cos\theta+\sqrt{2}\sqrt{2}\omega\_\left((a+2)(a(a+3)+3)(a(a(a(a\right.\\
&+5)+10)+9)+4)r\sin(2\theta)-2(a+1)\left(2a(a+1)(a(a+2)c_1\right. \qquad (7.34)\\
&-(a+1)d_1)(a(a+3)+3)^2-a(a(a(a(a(a(a(a+8)+25)+34)+2)-56)\\
&-77)-48)Z+14Z)\sin\theta))+D_a\left(-2(a(a(a(a(a(a(a+8)+27)+49)\right.\\
&+51)+31)+11)+1)Z^2(a+1)^3+2a\left(a(a+1)^2(a+2)^2(a(a+3)+3)\right.\\
&-1)\,b_1(a+1)^3+2a(a(a+3)+3)(a(a+1)(a+2)(a(a+3)+3)c_1\\
&+d_1)Z(a+1)^2-a(a(a+5)+9)(a(a(a+5)+9)+12)r^2\\
&-36r^2+\frac{1}{2}(a+2)(a(a+3)+3)r\left((a+2)\left(2(a(a+3)+3)r\cos(2\theta)\right.\right.\\
&+\sqrt{2}\sqrt{2}\omega\_(2(a(a(a(a+4)+5)+1)-1)Z\sin\theta+(a(a+3)+3)r\sin(2\theta))\Big)\\
&-4(a+1)(a(a+1)(a(a+3)+3)d_1+(a(a(a(a+5)+10)+9)+4)Z)\cos\theta))\Big)\\
&\Big/\big(2a(1+a)^2(3+a(3+a))(10+8D_a+a(30+22D_a+a(52+20D_a\\
&+a(58+7D_a+a(50+a(27+a(8+a))+Da))))\big)+b_1w\varepsilon^2,\\
\dot{w} =&\frac{\varepsilon}{(2a(1+a)^2(3+a(3+a))(2(2+Da)+a(9+a(10+a(5+a))+Da)))}\\
&\Big((a+1)^2(a(a(a(a+5)+10)+D_a+9)+2(D_a+2))\left(a^5b_1+4a^4b_1\right.
\end{aligned}
$$

$$+ a^3\left(7b_1 - (W+Z)^2\right) - a^2\left(b_1(D_a - 5) + 3(W+Z)^2\right) - a\left(b_1(D_a - 1)\right.$$
$$\left. +2(W+Z)^2\right) - (D_a - 1)(W+Z)^2\Big) + \sqrt{2}(a(a+3)+3)(a+1)\sqrt{2}\omega\_r(a(a(a$$
$$+2)(a+3) + D_a - 3) + 2(D_a - 4))\sin\theta(W+Z) - 4(a+2)(a(a+3)$$
$$+3)^2r^2\sin^2(\theta)\Big) - wb_1\varepsilon^2.$$

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