Appendix A

Homogeneous intermittency model

A.1 The β -model

The β-model was introduced by Frisch *et al.* (1978). The idea behind this model comes from the Richardson cascade, in which at each level of the cascade the energy of the large eddies L (mother) is uniformly distributed over the other eddies of size ℓ (daughters) as

$$\ell_n = L\ell^n, \tag{A.1}$$

where $n = 0, 1, 2, ..$ and $0 < \ell < 1$. The fraction P_n of this decrease of the eddies, which has a factor β $(0 < \beta < 1)$, can be defined as

$$P_n = \beta^n = \left(\frac{\ell_n}{L}\right)^{(3-D)}, \tag{A.2}$$

where

$$3 - D = \frac{\ln \beta}{\ln \ell}. \tag{A.3}$$

The parameter D is called a fractal dimension, and it expresses how the number of active eddies scales with L and is related to the number of offspring.

Frisch (1995) in his book interpreted this probability P_n, which has within three objects, a point, a curve and surface having respectively the dimension D of zero, one and two (figure A.1), and he states that the probability P_n that a sphere of radius ℓ_n (small), which is chosen in the center of the cube with a random uniform distribution, will intersect such an object is given in all cases by

$$P_n \propto \ell_n^{3-D}. \tag{A.4}$$

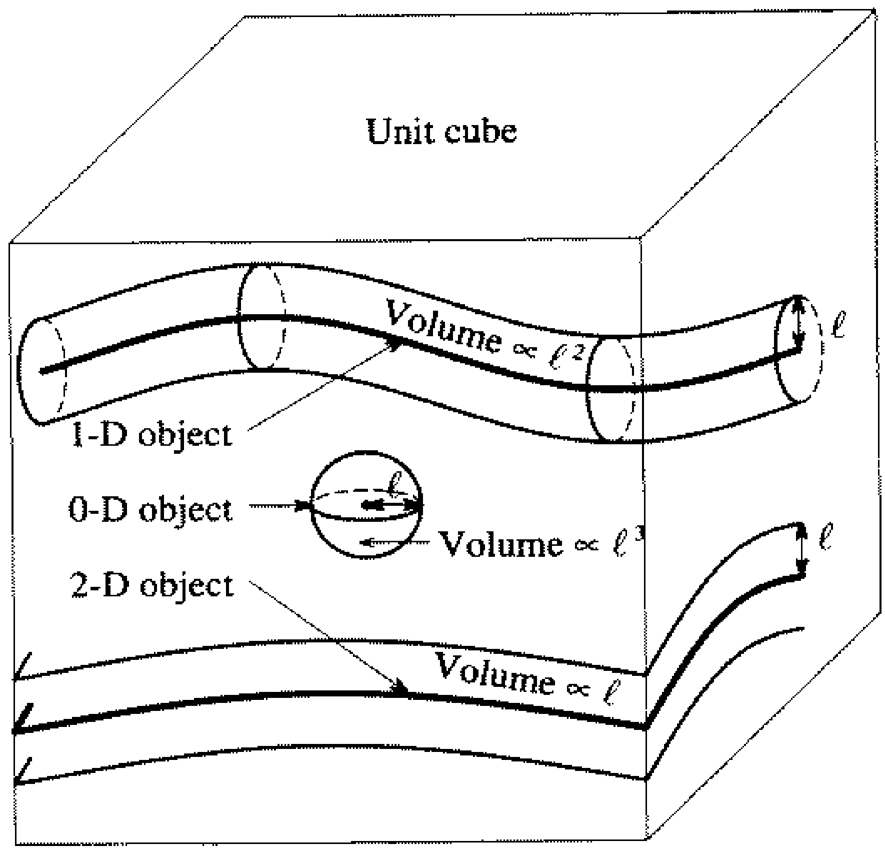

Figure A.1: The probability that a sphere of radius of ℓ encounters an object of dimension D behaves as ℓ^{3-D}.

It is appropriate now to explain the scaling laws of the β-model, which are derived by adapting the standard Kolmogorov theory (K41). Denote again by v_n the typical velocity difference over a distance ℓ_n within an active eddy of size ℓ. We define the energy per unit mass on scales ℓ_n as

$$E_n \propto P_n v_n^2 = v_n^2 \left(\frac{\ell_n}{L}\right)^{3-D}. \tag{A.5}$$

The rate of energy transfer from n-eddies to $(n+1)$-eddies in the inertial range is defined according to the Kolmogorov theory K41

$$\epsilon_n \propto \frac{E_n}{t_n} \propto P_n \frac{v_n^3}{\ell_n} \propto \langle\epsilon\rangle. \tag{A.6}$$

From equations (A.5) and (A.6), we obtain new form of v_n and E_n that take into account the spatial structure of the fractal dissipation field

$$v_n \propto \langle\epsilon\rangle^{\frac{1}{3}} \ell_n^{\frac{1}{3}} \left(\frac{\ell_n}{L}\right)^{-\frac{1}{3}(3-D)}, \tag{A.7}$$

and

$$E_n \propto \langle\epsilon\rangle^{\frac{2}{3}} \ell_n^{\frac{1}{3}} \left(\frac{\ell_n}{L}\right)^{\frac{1}{3}(3-D)}. \tag{A.8}$$

From equation (A.7) it is clear that the velocity field has the scaling exponent

$$h = \frac{1}{3} - \frac{3-D}{3}, \tag{A.9}$$

and the structure function of order p is written as

$$S_p(\ell_n) = P_n v_n^p \propto v_L^p \left(\frac{\ell_n}{L}\right)^{\xi_p}, \tag{A.10}$$

with

$$\xi_p = \frac{p}{3} + (3-D)(1-\frac{p}{3}). \tag{A.11}$$

Using the equation (A.11), the scaling exponent ξ_2 of the second order structure function is $\frac{2}{3} + \frac{3-D}{3}$, therefore the energy spectrum is given as

$$E(k) \propto k^{-(\frac{5}{3}+\frac{3-D}{3})}, \tag{A.12}$$

which is derived as a correction to the $k^{-\frac{5}{3}}$ law of the Kolmogorov theory (K41).

A.2 The random β-model

The random β-model was introduced by Benzi *et al.* (1984), assuming that the contraction factors β are independent random variables, and can take different values for each eddy of size ℓ_n. The $\beta_n(i)$'s are distributed according to a given probability distribution $P(\beta)$.

However, the geometrical structure of intermittency does not possess a global dilatation invariance (Benzi *et al.* (1984)) (figure A.2).

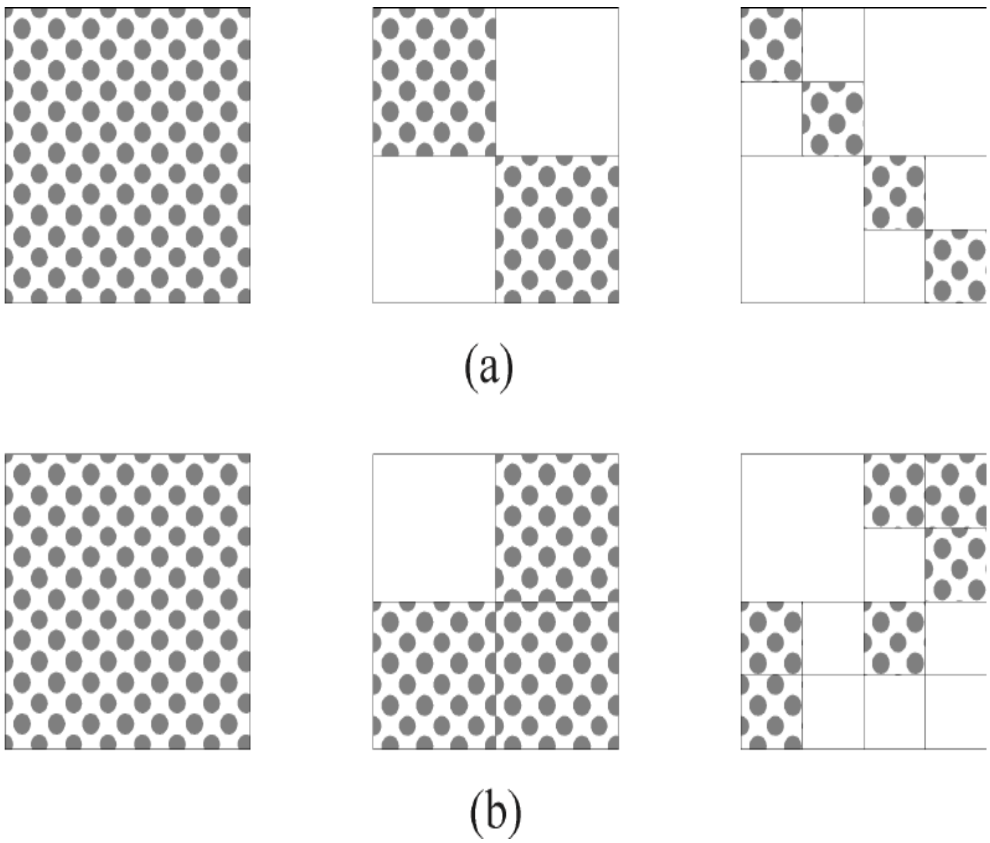

Figure A.2: (a) Schematic view of the β-model and compared with the random β-model (b).

Considering then that an eddy of size $\ell_{j+1} = \ell_j/2$ has a fraction $B_j = \sum_i \beta_j(i)/N_j$ of the volume occupied by its hypercube "mother", it follows that the number of active eddies after n steps is given by

$$N_n = 2^{3n} \prod_{j=1}^{n} \beta_j \tag{A.13}$$

if we denote by $\ell_n(k)$, $k = 1, .., N_n$, the N_n active eddies at the nth step. Each $\ell_n(k)$ generated eddies of size $\ell_{n+1}(k)$, where k indicates their origin. Taking into account that the energy dissipation rate for $\ell_n(k)$ and $\ell_{n+1}(k)$ eddies is constant then,

$$v_n^3(k)/\ell_n(k) = \beta_{n+1}(k) v_{n+1}^3(k)/\ell_{n+1}(k). \tag{A.14}$$

Iterating this equation with a particular history of random β's $(\beta_1, ..., \beta_n)$ leads to

$$v_n \sim \ell_n^{-1/3} (\prod_{i=1}^{n} \beta_i)^{-1/3}, \tag{A.15}$$

from equation (A.15) the structure function can be written as:

$$S_p(\ell_n) \sim \ell_n^{p/3} \int \prod_{i=1}^{n} \beta_i^{1-p/3} P(\beta_1, ..., \beta_n) d\beta_i, \tag{A.16}$$

and because there are no correlations between different steps of the fragmentation process, it follows that

$$P(\beta_1, ..., \beta_n) = \prod_{i=1}^{n} P(\beta_i). \tag{A.17}$$

Therefore, the structure function is rewritten as

$$S_p(\ell_n) \sim \ell_n^{p/3} \prod_{i=1}^{n} \int \beta_i^{1-p/3} P(\beta_i) d\beta_i = \ell_n^{p/3} \langle \beta^{1-p/3} \rangle^n, \tag{A.18}$$

where $\langle ... \rangle$ is the average over the distribution $P(\beta)$. Taking the last step $\ell_n = 2^{-2}$, then

$$S_p(\ell_n) \sim \ell_n^{\frac{p}{3} - \log_2 \langle \beta^{1-\frac{p}{3}} \rangle}. \tag{A.19}$$

From equation (A.19), it follows that the scaling exponents ξ_p for the random β-model can be written as,

$$\zeta_p = \frac{p}{3} - \log_2 \langle \beta^{1-\frac{p}{3}} \rangle. \tag{A.20}$$

This scaling exponents ξ_p in general will be a nonlinear function of p. The probability distribution $P(\beta)$ is based in principle on the knowledge of all the β moments and all the scaling exponents ξ_p.

It was assumed by Benzi *et al.* (1984) that an active eddy can generate either velocity sheets ($\beta = 0.5$) or space-filling eddies ($\beta = 1$). Therefore, the probability distribution $P(\beta)$ was supposed to have a form such as,

$$P(\beta) = \chi\delta(\beta - 0.5) + (1 - \chi)\delta(\beta - 1), \tag{A.21}$$

where χ is a free parameter. Benzi *et al.* (1984) found that this probability distribution function $P(\beta)$ leads, with $\chi = \frac{1}{8}$, to a good fit to the available experimental data.