Chapter 2

Homogeneous turbulence

In this chapter we describe the generic Kolmogorov model for fully developed, locally homogeneous and isotropic turbulence using the concept of stationary and continuous energy cascade processes. We describe some intermittency models prominent in such a simple turbulent regime based, on a characterization of the random but homogeneous nature of the energy dissipation field. Finally, we discuss the limitation of these models in real life, environmental and industrially relevant flows where the turbulence is usually non-homogeneous, non-isotropic and non-stationary.

2.1 Kolmogorov's theory for homogeneous turbulence K41

In 1941 Kolmogorov introduced his celebrated theory (K41) for locally homogeneous, isotropic and stationary turbulence, using velocity structure functions. The velocity structure functions of order p are defined in terms of the moments of velocity differences as:

$$S_p(\ell) = \langle (u(\vec{x} + \vec{\ell}) - u(\vec{x}))^p \rangle = \langle (\delta u_\ell)^p \rangle, \tag{2.1}$$

where $\langle ... \rangle$ stands for ensemble average and u is the velocity component parallel to $\vec{\ell}$. In fact, the velocity increments refer to the amplitude of the typical turbulent fluctuations of the velocity field within a distance ℓ.

Kolmogorov's theory is based on the following similarity hypothesis:

i) For all distances ℓ=$|\vec{\ell}|$ small compared with integral scale L, $\ell << L$, the statistical properties of the velocity differences δu_ℓ for a distance ℓ are uniquely determined by the kinematic viscosity ν and the average rate of energy dissipation $\langle \epsilon \rangle$ per unit mass.

ii) In the inertial range $\eta << \ell << L$, when the distances ℓ are large compared with the scale η, where the energy is dissipated, then the kinematic viscosity ν should play no dynamical role and the velocity differences δu_ℓ are uniquely determined by the quantity $\langle \epsilon \rangle$.

This model was based on the Richardson idea of energy cascade (figure2.1), in which the energy is transferred to small scales in steps. At eddies of size L (scales are of the order of the flow width, contain most of energy and dominate the transport of momentum, mass and heat) energy is injected, then energy is transmitted to smaller and smaller eddies, until it is dissipated into heat at smallest eddies of size η (small scales responsible for most of the energy dissipation).

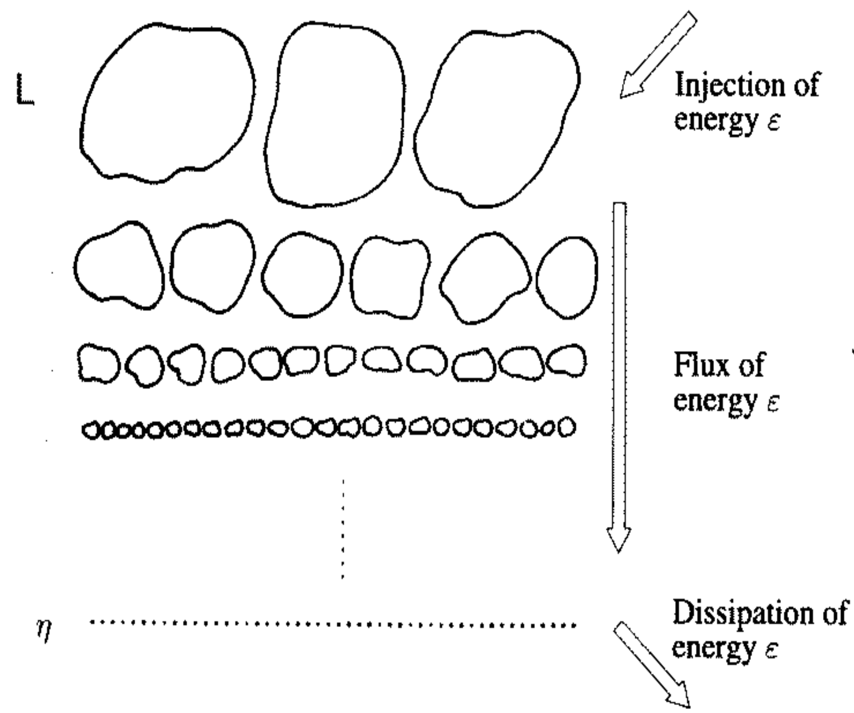

Figure 2.1: The energy cascade process following Richardson's idea.

The essential hypothesis of this model is that for high Reynolds numbers Re and ℓ smaller than the integral scale L and larger than the Kolmogorov scale η, the longitudinal velocity structure functions satisfy the relation:

$$S_p(\ell) = C_p(\langle\epsilon\rangle\ell)^{p/3}, \tag{2.2}$$

where C_p are universal constants. There is an exact dynamical relation for the third order longitudinal velocity structure function, which can be derived from the Navier-Stokes equations for homogeneous and isotropic turbulence, and which gives a compact description of much of the essential physics. This relation is called Kolmogorov's four-fifths law, and is expressed for the inertial range as:

$$\langle\delta u_\ell^3\rangle = -\frac{4}{5}\ell\langle\epsilon\rangle. \tag{2.3}$$

The universality of the relation (2.2) for $p = 3$ constitutes the most robust hydrodynamic indicator of the locally homogeneous character of transfer dynamics within the energy cascade. This relation plays an important role in experiments, for example in fixing the extent of the inertial range and in estimating energy dissipation rate per unity mass $\langle\epsilon\rangle$, in turbulent flows with less ambiguity than by means of the relation:

$$\langle\epsilon\rangle_{isotropic} = 15\nu\left\langle\left(\frac{\partial u}{\partial x}\right)^2\right\rangle, \tag{2.4}$$

valid for locally isotropic flows (Hinze (1959)).

It follows from relation (2.2) that the structure functions (2.1) have scaling behavior:

$$S_p(\ell) \propto \ell^{\xi_p}, \tag{2.5}$$

where ξ_p is called the scaling exponent of the structure function of order p. For the Kolmogorov theory (K41):

$$\xi_p = \frac{p}{3}, \tag{2.6}$$

indicating that the scaling exponents of structure functions of order p are scale-independent and universal quantities. It was assumed that there is a dynamically determined scale η which can be constructed from the average rate of energy dissipation $\langle\epsilon\rangle$ and the kinematic viscosity ν as:

$$\eta = \left(\frac{\nu^3}{\langle\epsilon\rangle}\right)^{1/4}. \tag{2.7}$$

This length-scale which would represent the smallest eddy size not damped by dissipation, is called the Kolmogorov length-scale. Similarly, Kolmogorov's time and velocity scales are defined as

$$\begin{aligned} \tau_k &= (\nu\langle\epsilon\rangle)^{1/2}, \\ v_k &= (\nu\langle\epsilon\rangle)^{1/4}. \end{aligned} \tag{2.8}$$

Therefore, the Reynolds number with reference to the two scales η and v_k is unity:

$$\frac{v_k \eta}{\nu} = 1. \tag{2.9}$$

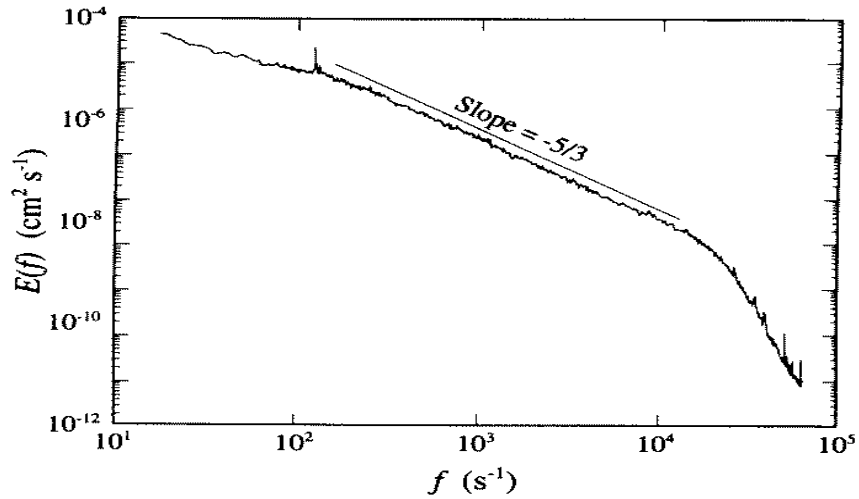

Figure 2.2: The energy spectrum in time domain in low temperature helium gas flow between counter-rotating cylinders with $R_\lambda = 1200$ (Maurer, Tabeling and Zocchi (1994)).

Many experimental studies have been done to verify the relation (2.6) predicted by Kolmogorov (1941), especially for the density of turbulent energy per unit of mass at scale ℓ for $p = 2$

$$E(\ell) = C_1'(\langle\epsilon\rangle\ell)^{2/3}, \tag{2.10}$$

and for its spectral equivalent

$$E(k) = C_2'\langle\epsilon\rangle^{2/3}k^{-5/3}. \tag{2.11}$$

Here, C_1' and C_2' in principle, are universal constants, ℓ is in the inertial sub-range and $k = 2\pi/\ell$ is the corresponding wave number. Sreenivasan (1995) shows that the constant C_2' is approximately 0.5 ± 0.05 over a wide range of Reynolds numbers Re. Many works (Grant *et al.* (1962) and Gagne (1987) have found the spectral slope to be close to 5/3, as shown in figure 2.2 as an example.

2.2 Energy dissipation random field

In fact, it is not possible to state that Kolmogorov's basic theory (K41) is consistent. Even if the turbulence is steadily locally homogeneous and relation (2.3) is verified there appear to be unequivocal departures from basic prediction (2.6) for p different from 3. A first important modification to (K41) theory was introduced in 1962 by Kolmogorov and Obukhov, after Landau's objection that energy dissipation linked with velocity gradients is in fact a random field and, therefore, the p-order moments of ϵ should also be a scale-dependent quantity like the velocity increments.

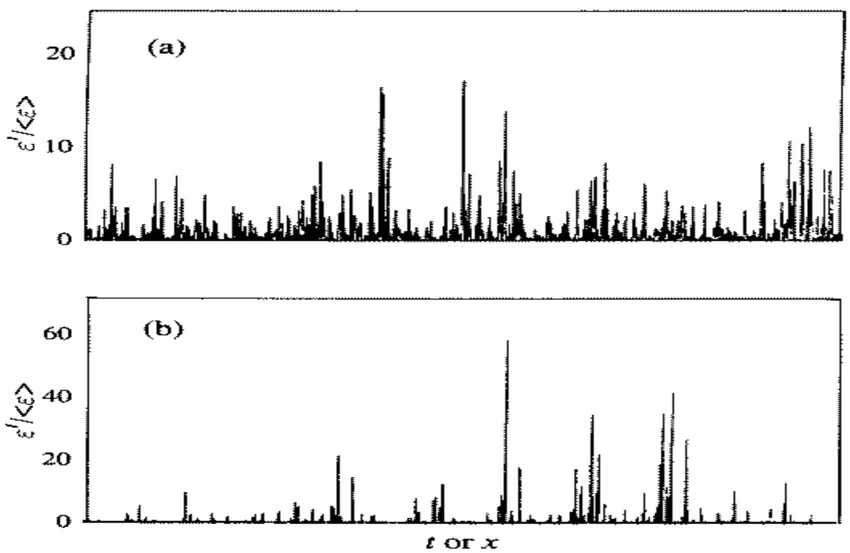

Figure 2.3: The local dissipation $\epsilon(x)$ normalized by its mean. (a) is in the laboratory boundary layer at moderate Reynolds number, and (b) is the atmospheric surface layer at a high Reynolds number (Meneveau and Sreenivasan (1991)).

In this case, C_p in equation (2.2) for $p \neq 3$ cannot be universal and it must depend on the detailed geometry of the dissipation field. Assuming *scale uniform* *variance of the random function* ϵ except for the intermittent *largest but rarest events*, the correction to the generic Kolmogorov equation (2.6) is then only induced by the deviation from scale-independent quasi-Gaussian distribution of the largest but rarest events: this is the so-called intermittency phenomenon in the framework of Kolmogorov's $K62$ theory.

However, the experimental results show strong intermittent bursts instead of a steady behavior of the energy dissipation fluctuation, indicating that ϵ fluctuates strongly. An example of this behavior is shown in figure 2.3, from the experimental results of Meneveau and Sreenivasan (1991). These results demonstrate that the energy dissipation is extremely intermittent, and that this intermittency increases strikingly with increasing Reynolds number. This phenomenon can also be shown in the probability distribution functions of the velocity increments δu_ℓ in the inertial range (figure 2.4), taken from Herweijer (1995). As it can be seen here the intermittency appears as a set of a few localized spikes of very high activity leading to the non-Gaussian distribution of the largest but rarest events.

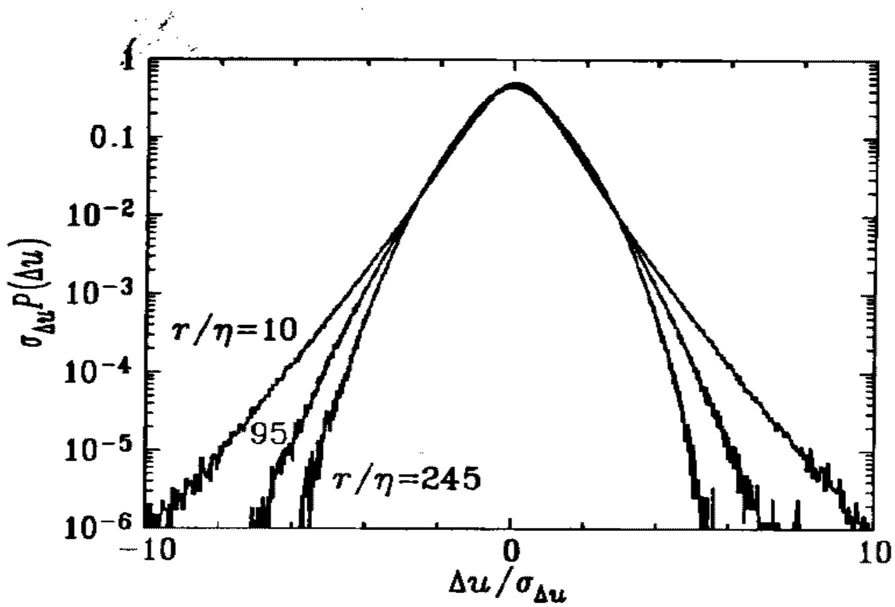

Figure 2.4: The probability distribution function in the inertial range for different scales r (Herweijer (1995)).

Obukhov (1962) suggested that the quantity $\langle\epsilon\rangle$, which plays a central role in the Kolmogorov theory (K41), could be replaced by the spatial or/ensemble averaged energy dissipation rate ϵ_ℓ locally defined as

$$\epsilon_\ell(\vec{x}) = \frac{1}{V}\int_V \epsilon(x')dx', \tag{2.12}$$

where $V = \frac{4\pi}{3}\ell^3$ is a volume sphere of radius ℓ centered on $\vec{x}$ and $\eta << \ell << L$. In this case we have,

$$\epsilon_\ell(\vec{x},t) = \epsilon'_\ell(\vec{x},t) + \epsilon_0, \tag{2.13}$$

where $\epsilon_0 = \langle\epsilon_\ell\rangle$. Clearly, the "scale-by-scale" energy dissipation rate ϵ_ℓ and the corresponding random fluctuations field ϵ'_ℓ are functions of the length-scale ℓ, position vector $\vec{x}$ and time t.

When the turbulence is locally homogeneous in the inertial energy range, which means that the amplitude of the fluctuations variance characterizing the dissipation random field is scale independent within the energy cascade range (scale-uniform dissipation random field, K62 hypothesis), the deviation from the generic K41 law is only induced by the intermittency. For non-homogeneous turbulence when ϵ_ℓ and σ_ℓ (energy transfer that will be defined in chapter 3) the variance may be scale non-uniform and this behavior is not seen in the moment of order 1 δu_ℓ^3, but it can be seen in other moments of order $p > 1$ (see figure 3.6).

Taking into account only intermittency, Kolmogorov (1962) made Obukhov's suggestion more quantitative. He introduced a refined similarity hypothesis relating the moments of the PDF of the velocity increments δu_ℓ to the moments of ϵ_ℓ. Then, the corresponding prediction for the pth-order moment of the velocity increment δu_ℓ as function of the scale separation ℓ is usually formulated as,

$$\langle\delta u_\ell^p\rangle \sim \langle\epsilon_\ell^{p/3}\rangle\ell^{p/3} \sim \ell^{\xi_p}. \tag{2.14}$$

Considering that there is a scale dependence of the dissipation as:

$$\langle\epsilon_\ell^{p/3}\rangle \sim \ell^{\tau_{p/3}}, \tag{2.15}$$

now the scaling exponents are:

$$\xi_p = p/3 + \tau_{p/3}, \tag{2.16}$$

where ϵ_ℓ is the locally defined energy dissipation per unit mass over a volume of size $\ell = \| \vec{\ell} \|$ centered at space-position $\vec{x}$, $\tau_{p/3}$ is the scaling exponent of $\langle\epsilon_\ell^{p/3}\rangle$, $\langle.\rangle$ refers to averaging over all position vectors $\vec{x}$ and $p = 1, 2...$ is the order of the statistical moment.

In a self-similar situation corresponding to a homogeneous and isotropic turbulence characterized by *scale uniform random dissipation field*, $\tau_1 = 0$ and the correction $\tau_{p/3}$ for $p \neq 3$ in relation (2.16) is only induced by the the intermittency correction. Then, relation (2.16) guarantees the basic result $\xi_3 = 1$ for locally homogeneous and isotropic turbulence.

2.3 Scale uniform dissipation random field: intermittency models

After the publication of the Kolmogorov theory K41, many efforts have been devoted to the modeling of the scaling laws of the velocity structure functions in turbulent flows. Most of these models have used the scenario of energy cascade. The success of these models can be evaluated basically on how well they agree with experiments. Some of these well known models will be discussed in Appendix A as the β model, and the random β model. In the following, we will describe the Log-normal model K62 and the She Leveque model. The choice of these two models was made because of their interest on our study of non-homogeneous and non-isotropic flows in the next chapters.

2.3.1 Log-normal model K62

The Kolmogorov theory (K41) is approximate because it ignores the fluctuations of the energy dissipation in the energy cascade. Assuming homogeneous and isotropic turbulence, the energy dissipation random field ϵ_ℓ is scale uniform in the inertial range except for largest but rarest fluctuations that are assumed to have a log-normal distribution. Therefore, the refined similarity hypothesis leads to writing equation (2.2) as,

$$S_p(\ell) = C_p \langle \epsilon_\ell^{\frac{p}{3}} \rangle \ell^{\frac{p}{3}}. \tag{2.17}$$

The scaling exponents for the energy dissipation are conventionally defined as,

$$\langle \epsilon_\ell^p \rangle / \langle \epsilon \rangle^p \propto (\ell/L)^{\beta(p)}. \tag{2.18}$$

The proportionality constants, omitted here, are not expected to be universal. The reasons for writing this relation are explained pragmatically by (Novikov (1971), Chhabra and Sreenivasan (1992)) in terms of the so-called breakdown coefficients or multipliers. Assuming homogeneity $\langle\epsilon_\ell\rangle = \langle\epsilon\rangle$, equation (2.17) can be rewritten as:

$$S_p(\ell) \propto \langle\epsilon\rangle^{\frac{p}{3}} \ell^{\frac{p}{3}} (\ell/L)^{\gamma(p)}, \tag{2.19}$$

where ℓ is a distance within the inertial range. The refined similarity hypothesis (2.17) implies that $\gamma(p) = \beta(p/3)$. Kolmogorov (1962) suggested that ϵ_ℓ should have a log-normal distribution. If the scaling of (2.18) is accepted with $\beta(1) = 0$, this distribution gives:

$$\beta(p) = \mu p(1-p)/2 \tag{2.20}$$

and

$$\gamma(p) = \mu p(3-p)/18. \tag{2.21}$$

Finally, using equations (2.19) and (2.21), it follows that for the log-normal model the absolute scaling exponents ξ_p of the structure functions can be written as:

$$\xi_p = \frac{p}{3} + \frac{1}{18}\mu p(3-p), \tag{2.22}$$

where μ is called the *intermittency exponent*, or *intermittency parameter*, which describes the intermittency of the fluctuation of ϵ_ℓ. μ is the exponent in the inertial range power law behavior of autocorrelation function of the dissipation rate,

$$\langle\epsilon(x)\epsilon(x+\ell)\rangle \sim (\frac{L}{\ell})^{\mu}, \tag{2.23}$$

where L is the integral scale. Usually, the sixth-order scaling exponent of the velocity structure function gives a direct way to calculate the intermittency exponent μ, using the relation,

$$\mu = 2 - \xi_6. \tag{2.24}$$

There have been many attempts to measure the intermittency exponent μ in both high and low Reynolds number flows, but they have given widely varying results. A variety of authors (Antonia *et al.* (1981, 1982), Chambers and Antonia (1984) and Sreenivasan and Kailasnath (1993)) have studied this exponent and confirmed that at high Reynolds numbers *Re*, most of experiments are consistent with a value of $\mu = 0.25 \pm 0.05$.

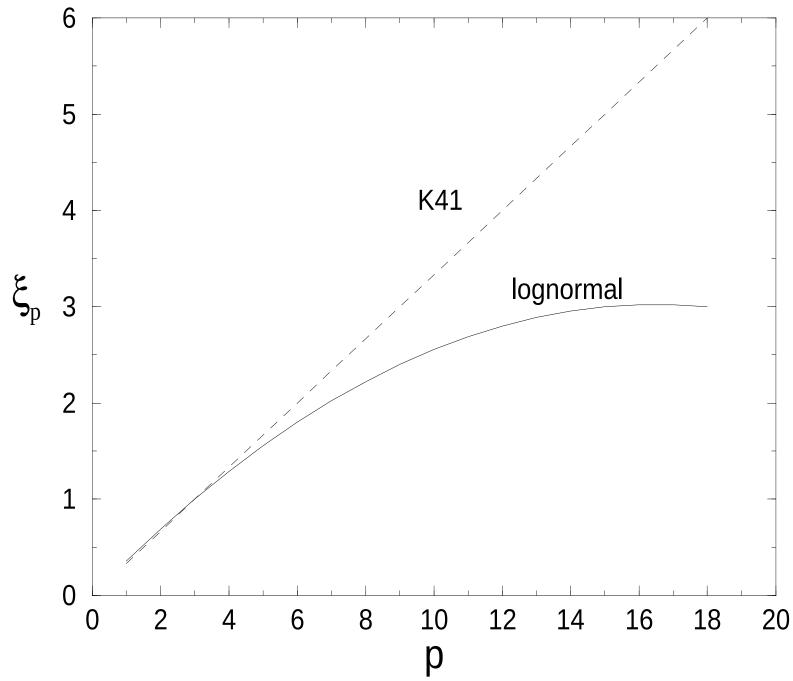

Figure 2.5: The comparison between the log-normal model and K41 theory.

2.3.2 She-Leveque model

This model was introduced by She and Leveque (hereafter SL) (1994), who predicted a simple shape for the scaling exponents of the structure functions. The phenomenological theory of the SL model is based on the idea that moments of the energy dissipation field, $\langle\epsilon\rangle$, are characterized by a hierarchical structure. The intensity of the p order dissipation structures $\epsilon_\ell^{(p)}$ is expressed as:

$$\epsilon_\ell^{(p)} = \frac{\langle\epsilon_\ell^{p+1}\rangle}{\langle\epsilon_\ell^{p}\rangle}, \tag{2.25}$$

where ℓ is in the inertial range and $\epsilon_\ell^{(p)}$ is a monotonous increasing function of p. The two extreme characteristic structures of the hierarchy $\epsilon_\ell^{(0)}$ and $\epsilon_\ell^{(\infty)}$ are defined as

$$\epsilon_\ell^{(0)} = \lim_{p\to 0} \frac{\langle \epsilon_\ell^{p+1}\rangle}{\langle \epsilon_\ell^{p}\rangle} \tag{2.26}$$

$$\epsilon_\ell^{(\infty)} = \lim_{p\to \infty} \frac{\langle \epsilon_\ell^{p+1}\rangle}{\langle \epsilon_\ell^{p}\rangle}, \tag{2.27}$$

where $\epsilon_\ell^{(0)}$ is the mean dissipation rate (by definition is uniform in scale) and $\epsilon_\ell^{(\infty)}$ is the relative contribution to the transfer of the most intermittent structures at scale ℓ. The scaling behavior of $\epsilon_\ell^{(\infty)}$ is estimated by a kinetic energy divided by time

$$\epsilon_\ell^{(\infty)} \sim \delta E^\infty / t_\ell, \tag{2.28}$$

where δE^∞ is the kinetic energy and t_ℓ is the time scale. $\epsilon_\ell^{(\infty)}$ was chosen to have no anomalous scaling in t_ℓ, which amounts to setting a uniform time scale for the dissipation of various intensities. This yields $t_\ell \sim \langle \epsilon_\ell \rangle^{-\frac{1}{3}} \ell^{\frac{2}{3}}$.

In this model the quantity $\langle \epsilon_\ell \rangle / \epsilon_\ell^{(\infty)}$ satisfies the relation

$$\frac{\langle \epsilon_\ell \rangle}{\epsilon_\ell^{(\infty)}} = \frac{\epsilon_0}{\epsilon_\ell^{(\infty)}} \sim \ell^\alpha, \tag{2.29}$$

where α is a positive constant assumed by She and Leveque to be $2/3$. ϵ_0 is the continuous mean transfer kept constant by the mean dissipation rate at negligibly scales $\ell \to 0$ in the flow domain. This implies that $\langle \epsilon_\ell \rangle$ has scale divergent behavior due to the presence of intermittent structures, responsible for the anomalous of the scaling laws of the velocity structure functions.

The scaling of $\epsilon_\ell^{(p)}$ characterizes how singular the p order structures are. Both in physical space and time, structures of intermittency $\epsilon_\ell^{(p-1)}$ are expected to occur before the formation of structures of p order having slightly higher intensity. This procedure implies that the scaling behavior of $\epsilon_\ell^{(p)}$ is related to that of $\epsilon_\ell^{(p-1)}$ and $\epsilon_\ell^{(\infty)}$. Therefore, the $(p+1)$ order fluctuation structure $\epsilon_\ell^{(p+1)}$ is written as a superposition of $\epsilon_\ell^{(p)}$ and $\epsilon_\ell^{(\infty)}$,

$$\epsilon_\ell^{(p+1)} \sim \epsilon_\ell^{(p)\beta} \epsilon_\ell^{(\infty)(1-\beta)}. \tag{2.30}$$

It follows from relations (2.16), (2.25) and (2.30) that

$$\tau_{p+2} - (1+\beta)\tau_{p+1} + \beta\tau_p + \frac{2}{3}(1-\beta) = 0. \tag{2.31}$$

Introducing a function $f(p)$ with $f(\infty)=0$, such that,

$$\tau_p = -\frac{2}{3}p + 2 + f(p), \tag{2.32}$$

the second order homogeneous difference equation for $f(p)$ is then written as,

$$f(p+2) - (1+\beta)f(p+1) + \beta f(p) = 0, \tag{2.33}$$

supplemented with the boundary conditions $\tau_0 = 2+f(0) = 0$ and $\tau_1 = \frac{4}{3}+f(1) = 0$. The unique solution of (2.33) is $f(p) = \alpha\beta^p$. The condition $\tau_0 = 0$ is derived from the assumption that the energy dissipation does not concentrate on a singular measure in the zero viscosity limit, while $\tau_1 = 0$ is exact.

The final results are

$$\tau_p = -\frac{2}{3}p + 2\left(1 - \left(\frac{2}{3}\right)^p\right), \tag{2.34}$$

and using the relation (2.16) of the Kolmogorov refined similarity hypothesis (1962), the p order scaling exponent ζ_p of the velocity structure function is written by She and Leveque as

$$\xi_p = \frac{p}{9} + 2\left(1 - \left(\frac{2}{3}\right)^{\frac{p}{3}}\right). \tag{2.35}$$

The intermittency correction to the Kolmogorov (1941) energy spectrum $E(k) \sim k^{-\frac{5}{3}}$ is given by

$$E(k) \sim k^{-\frac{5}{3}-0.03}. \tag{2.36}$$

On the other hand, the relation (2.35) can also be written as

$$\xi_p = \frac{p}{3}[1-\alpha] + \alpha\frac{1-\beta^{p/3}}{1-\beta}, \tag{2.37}$$

where α and β are in general adjustable parameters characterizing the specificity of each flow (conservation laws, forcing, degree of homogeneity). Table 2.1 shows the values of some scaling exponents of the SL model (with $\alpha = 2/3$ and $\beta = 2/3$ in (2.37) compared with the two Kolmogorov theories K41 (1941) and K62 (1962).

p	*K41*	*K62*	*SL*
2	*0.66*	*0.70*	*0.70*
4	*1.33*	*1.28*	*1.28*
5	*1.66*	*1.53*	*1.53*
6	*2.00*	*1.78*	*1.78*

Table 2.1: Comparison of the scaling exponents of the SL model with respect to the Kolmogorov theories K41 and K62.

In this model the intermittency parameter μ in the inertial range which is expressed in terms of the sixth-order scaling exponent ξ_6 by relation (2.24), is equal to 2/9; this is in good agreement with the experimental measurements of Anselmet *et al.* (1984) and Vincent *et al.* (1991), who found that μ lies between 0.2 and 0.25.

2.4 The search for more general models

In this chapter we have provided a basic description of some intermittency models for fully developed turbulence, with special emphasis on the role of the energy distribution amongst eddies of different size, in scaling laws of the velocity increments for the inertial range. However it is shown that the scaling exponents of the velocity structure functions for these models deviate from the linear scaling $p/3$ predicted by Kolmogorov (1941), who assumed that the energy dissipation ϵ changes smoothly with space and time, or that it is constant.

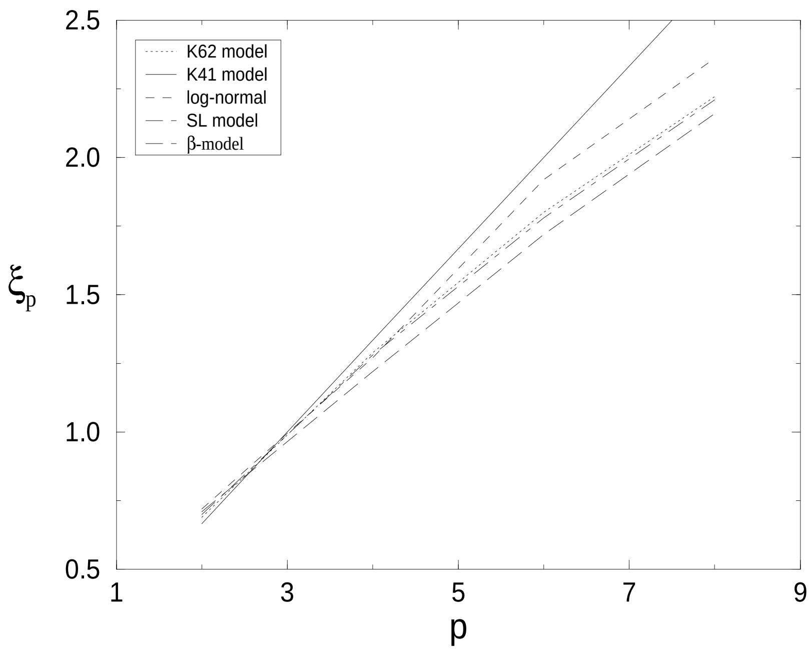

Figure 2.6: Scaling exponents ξ_p of different models for homogeneous turbulence.

Figure 2.6 shows the scaling exponents in the inertial range for different models: $\xi_p = p/3$ for K41 theory, $\xi_p = \frac{p}{3} + \frac{1}{18}\mu p(3-p)$ with $\mu = 0.2$ for the log-normal model, the She-Leveque model $\xi_p = \frac{p}{3}[1-\alpha] + \alpha\frac{1-\beta^{p/3}}{1-\beta}$ with ($\alpha = \beta = 2/3$), and finally the β-model and the random β-model described in Appendix A. It shows that the scaling exponents of these models deviate from the Kolmogorov law $p/3$. However, it is not possible to choose one model over the other, because each model agrees with some experiments but does not work quite so well with other experiments.

Our aim is to investigate and to call attention to the validity of these models in non-homogeneous and non-isotropic turbulence, which is ubiquitous in real life, both in industrial and environmental turbulent flows.

A second reason for the non-negligible deviation from Kolmogorov's universality and self-similarity is when the local stationarity or homogeneity in the energy cascade range is perturbed by the large scale motion and/or turbulent energy sources i.e. the dynamics is nonlocal, and then the exponents characterizing the velocity structure function behavior are scale-dependent quantities. For example, this effect is well known in the atmospheric boundary layers (Kailasnath *et al.* (1992) and Praskovsky and Oncley (1994)).

In Chapter 3 we will present further models, Dubrulle and BDF model, that allow us to investigate more complex flows. Because the complexity of the flow increases as it becomes non-homogeneous, this study is of interest in practical applications.