Essays on Networks in Economics:

Pairwise Influences and Communication Processes

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April, 2007

Submitted to
The Department of Economics
at
Universitat Autònoma de Barcelona

in Partial Fulfillment of the Requirements for the Degree of
Doctor in Economics

A la Marta, perquè tenint-la al costat tot és una mica més fàcil.
A la meva àvia, Soledad, que ha sigut per mi com una segona mare.
A la meva mare, Francesca. Perquè a ella li dec tot.

Contents

Preface

This thesis addresses two different questions. First, the economic consequences of pairwise influences, understood as externalities which intensity and sign depend on the pair of agents considered. Second, the effects of different communication processes in small groups, and its consequences for the optimal inner network structure of informal organizations.

In the first chapter "Pairwise Influences and Bargaining Among the Many" we analyze how pairwise influences affect the nature and solution of distributional conflict. In a model with pairwise influences the utility of each agent is heterogeneously influenced by the utility of other agents in the economy. In particular, there are two dimensions of heterogeneity: who influences whom, and the strength of any of such pairwise dependent influence. The pattern of bilateral influences takes the form of a weighted and directed network. This kind of models can encompass, for example, spatial externalities or social preferences. In the analysis of distributional conflict, in which agents have to divide some available resource among them, pairwise influences affect the nature of this conflict, since if the externalities they represent are negative the conflict is strengthened while if these externalities are positive the conflict is diluted. In a limit situation, there can exist a complete alignement of preferences for a uniquely determined allocation. In this last case, distributional conflict simply vanishes. Our aim in this chapter is to analyze the family of economies with pairwise influences in which there exists full distributional conflict and how the solution to this conflict maps the heterogeneous pattern of pairwise influences to shares and utilities obtained. In particular, we use the Nash bargaining solution as the solution to distributional conflict. Our results rely on network centrality indexes that measure each agent's prominence due to his position in the influences structure.

The chapter "On Pairwise Influence Models: Networks and Efficiency" extends the analysis of the indirect effects of direct pairwise influences and how these indirect effects determine the set of efficient allocations in such kind of models. The results in this chapter are developed in a social preferences framework but they immediately apply to more general setups with externalities. First, we provide a complete analysis of the mapping from pairwise influences to network externalities, that account for all levels of indirect effects generated by the pattern of influences. Then, we provide a complete characterization of the set of Pareto efficient allocations for almost every economy with pairwise influences in terms of prestige measures derived from the literature on social network analysis. We illustrate these results with the use of two different network designs, the circle and the star, that express two polar cases on the set of possible structural pattern of influences.

The chapter "Spatial Spillovers and Local Public Goods" is a first attempt to analyze some issues related to the effects of spatial spillovers in urban structure and the provision of local public goods under the light of pairwise influence models. Many public services that are distributed all along the city do not arrive in the same way to all its citizens. Hence, public services are not always pure public goods. Suppose that this services are distributed among the different neighbourhoods in which the city is divided. Then it is probably neither true that the effect of local public services is bounded to the neighbourhood level. Instead, some of these effects can extend and spill over to other neighbourhoods. The intensity of these spatial externalities probably depends on the geography and on socioeconomic characteristics of the city. The conflict that arises between neighbourhoods to obtain larger shares of the available public services when spatial externalities exist is analyzed theoretically in the first chapter of this thesis. Different divisions of the city into neighbourhoods determine different patterns of spatial externalities. These externalities are internalized in the solution to the conflict. Hence, a natural question emerges: which is the division of the city that involves larger returns in terms of social welfare of all citizens once we take into account the pattern of spatial externalities that this division entails? We do not provide a complete answer to this question. We simply illustrate, with the use of some data of Barcelona, how the tools we have previously developed provide a satisfactory benchmark in which to inscribe this analysis. Beyond that first topic, the second part of the chapter provides a public good provision game with spatial spillovers, that we analyze with some detail. Neighbourhoods choose how much they want to contribute to the provision of public services that later on are assigned to them with the use of the Nash bargaining solution. The results in this chapter are nonconclusive, but they sketch some directions for future research in which the models and tools developed in previous chapters can show useful.

In the chapter “Communication Processes: Knowledge and Decisions”, in joint work with my advisor Antoni Calvó-Armengol, we introduce a model of communication in informal organizations. We envision an organization as a team in which each member faces an individual decision problem and a common coordination problem. The decision problem expresses the desire of all individuals to choose actions that are close to the optimal, and unknown, task that should be ideally implemented. The coordination problem expresses the needs of the organization to align the actions of all its members. Each member has some private information on the real state of the world. We analyze how different communication processes of this private information impact the optimal actions of each member. Each one of these decentralized information-sharing schemes determines the way in which each member constructs his beliefs on the task to be performed, as well as what others belief is the optimal task to be performed, and the beliefs on beliefs on beliefs, and so one and so forth. The analysis provides a new concept, the knowledge index, that sums up in an idiosyncratic value these higher order beliefs for each possible communication process. Given any information structure derived from a communication process, the game has a unique Bayesian equilibrium. The equilibrium action of each agent is linear in the communication report each agent obtains, and this report is, precisely, weighted by the knowledge index. The uniqueness and linearity properties of the Bayesian equilibrium allow for clear welfare implications in our analysis. We perform some comparative statics on the different parameters that determine the relative strength of the individual decision versus the coordination problem, the informativeness of private signals, and the particular characteristics of the communication process, namely accuracies and correlations of final communication reports.

The last chapter, "On Optimal Communication Networks," also in joint work with professor Antoni Calvó-Armengol, builds on the model and results developed in the previous chapter to study a family of networked communication processes. We obtain a partial order on the set of possible networks: adding communication possibilities always increases accuracy of the task forecast, but a negative counterpart can arise if new links do not close triangles in the original network because the coordination problem can be exacerbated. Our analysis also shows that when there is one unique round of communication, and there is a fixed supply of possible links, the optimal geometric arrangement of these links maximizes a network span index, a measure of network irregularity. Instead, when the number of possible communication rounds increases, the process becomes a model with persuasion bias in which the optimal network is regular. We obtain therefore, for a wide set of parameters, a polarization result in terms of the number of available rounds to communicate.

In some sense both parts of this thesis are complementary. Both are interested in externalities, but the source of these externalities are of a different nature in each one: in the first part, we deal with heterogeneous payoff externalities; in the second part we deal with heterogeneous informational externalities. Hence, we can phrase the essence of the interests of this thesis in understanding how pairwise dependent externalities affect economic outcomes and behavior.

Acknowledgements

Vull començar donant les gràcies a en Xavier Martinez de Albèniz per donar-me a coneixer el món de l'economia teòrica i animar-me a provar d'aplicar al International Doctorate in Economic Analysis.

I have found in the IDEA doctoral program a fantastic environment to learn and to discuss about economics. All the courses have been stimulating and the faculty has permanently shown an endlessly motivation to help all the students to find our own way and to improve as researchers. I have been very fortunate to have had the opportunity to be part of this project.

I have benefited from the comments of so many different people about my research that the list would be never ending and unfairly incomplete. I want to thank all of them.

Salvador Barberà i Clara Ponsatí mereixen menció apart. Els seus comentaris en versions prèvies d'alguns dels capítols d'aquesta tesis m'han sigut de molt de profit, i vull aprofitar aquesta ocasió per agraïr una vegada més tot el seu suport durant el procés de job market.

Vull agraïr a l'Àngels, a la Sira i, especialment, a la Mercè la seva continuada disponibilitat i paciència per resoldre tots els dubtes logístics que se m'han pogut arribar a plantejar durant aquests anys. Tot el que he tingut de pesat m'ho han tornat amb simpatia.

During all this period I have had the opportunity to meet a lot of interesting and friendly people of which I have learned a lot and with which I have had some very nice moments. This includes all the students of the doctoral program, as well as some visiting students and the nice people I met during my stay in Toulouse. This made these years a rewarding experience from a more personal perspective. I thank them all for that.

Mil gràcies a l'Albert Botella, l'Albert Clop, la Carme, l'Eduard, l'Elisenda, l'Estrella, el Fran, la Grushenka, el Julio, la Laura, el Marc, la Mireia, la Miriam, el Pere, el Ricard i el Sergio per ser una font inesgotable de bons moments, indispensables per mantenir el cap fresc i els ànims renovats.

La Marta ha sigut el meu suport més important durant aquests cinc anys, i ha mostrat una increïble paciència amb els meus altibaixos. La meva mare sempre ha encertat amb el seu consell, i li agraeixo infinitament la seva obstinació en procurar-me uns bons estudis. La meva àvia m'ha mostrat sempre el més gran dels afectes, i si la bibliografia d'aquesta tesi és tant llarga potser és perque ja de ben petit ella em va enganxar l'hàbit de la lectura compulsiva. Vull agraïr a la meva família, el meu cosí Joan, el meu tiet Pep, la meva tieta Roser, la Montse, el Fructuós, la Rosario, la Modesta, la Maria i el Ramon el seu interès continuat pel desenvolupament d'aquesta tesi doctoral.

El meu darrer agraïment, no per això menys intens, ans el contrari, és pel meu mentor, en Toni. Tant des d'una perspectiva professional com personal, mereix tot el meu respecte. Els seus comentaris i consells respecte la primera part d'aquesta tesi han sigut determinants per tal que prengués la forma actual i la seva contribució és evident en la segona part. És un plaer fer recerca amb algú que mostra tanta curiositat i resolució com ell. La seva disponibilitat ha superat tot límit. La seva comprensió i encoratjament, mostrats amb la paraula adequada en cada moment, han permès que aquest viatge arribés a bon port. Tot el saber que m'ha transmès romandrà de manera perdurable en el meu record. Gràcies Toni.

PART I

Pairwise Influences

Chapter 1
Pairwise Influences and Bargaining Among the Many

1.1 Introduction

Influence in outcomes and behavior across individuals is pervasive in many social and economic settings. This influence can take many forms. It can be centralized, and result from a common influence of the environment. Else, it can be decentralized and supported by more local interactions. At its most disaggregated level, influence is mediated by pairwise interactions. The collection of such pairwise interactions can display a rich pattern that might overall influence the resulting individual outcomes.

Cross influences can be an important issue if we deal with distributional conflict. Consider a fixed resource to be divided among a set of individuals. Any possible division of this resource has a direct effect on individual outcomes. In addition to this direct effect, there can also be an indirect effect mediated by the possible interactions that might arise across some pairs of individuals. Through the chains of interactions present inside the group, these indirect effects can spread all over the population. When individuals agree collectively on the division of this common resource, all these influence effects have to be internalized adequately. The pattern of direct pairwise influences can thus play a crucial role in the solution of the distributional conflict.

When considering disaggregate pairwise influences, there are two different possible sources of heterogeneity in its underlying structure: the geometry of interactions, that expresses who influences whom, and the magnitude of each one of these influences. The aim of this paper is to analyze how the whole structure of such pairwise influences affects the bargaining outcomes.

Consider for example the case of urban crime. Crime patterns on city's neighbourhoods are interlaced and there are several reasons why this is so.

First, the division of cities into different neighbourhoods is merely an administrative issue. Criminals do not doubt in crossing the street and move from a neighbourhood to another to commit their activities. Hence, if no natural barrier imposes difficulties on this movement, this is a natural source of interdependence in crime rates across neighbourhoods. This is very much related with the so called Modifiable Area Unit Problem (M.A.U.P.) highlighted in the literature of spatial statistics in geographic information science. Different groupings into neighbourhoods lead to variability in statistical results. When aggregating for crime rates it is unlikely that there exists a univocal relation between neighbourhood characteristics and crime outcomes.

Furthermore, the embeddedness of individuals into social networks that might spread over the city, impacts also criminal behavior. Some individuals might imitate the behavior of their contacts in their network of acquaintances (see Glaeser et al., 1996). Some others might infer (maybe erroneous) information on the benefit on committing crime (see Sah, 1993). Hence, criminal behavior suffers a contagion effect inherent to the social environment. This social osmosis process also induces spatial influence in crime rates.

When the major has to decide on how to distribute resources to fight urban crime, and when a representative of each neighborhood asks for part of these resources, we come up with a distributive conflict with influences on crime rates across pairs of neighborhoods. Our analysis sheds light on how the particular neighborhood geography and the pattern of influences across neighbourhoods are internalized in the final assignment.

The environment. There are two dimensions of heterogeneity in the model. On the one hand, the particular geometry of bilateral influences. Not everybody necessarily exerts an influence on every other agent. The pattern of direct bilateral influences determines who exerts an externality on whom and how influences spread indirectly through the economy. On the other hand, the magnitude of each bilateral influence is pairwise dependent. The magnitude of an influence relation depends on exactly which agent exerts this influence and which agent receives it.

Direct influences generate indirect network effects. We call network externalities the sum of all these indirect effects. These are direct externalities derived from the assignment of resources, and measure how the level of consumption, not the utility, of an agent affects another agent's utility. Each pattern of pairwise influences determines a unique pattern of network externalities.

To better understand the spread of influences through direct bilateral influences we reinterpret at some points in the paper the model in terms of networks. The network links an agent to another whenever the second agent exerts a direct externality on the first one. Externalities spread then through the links of the network. We can keep track of all indirect influences generated from bilateral influences through paths and cycles in the associated network. The network metaphor is adequate in this setting due to asymmetric pairwise bilateral influences. Connections with well-known notions from social network analysis, such as centrality measures, arise as natural tools for the network reinterpretation of the model.

Direct bilateral influences within a finite set of agents are modeled with the use of a linear model that encompasses the possibility of positive and negative influences as well as asymmetric influences within pairs of agents. The model is characterized by a matrix that collects the possibly different levels of bilateral influences for each possible pair of agents, the primitives of the model.

A unit of a divisible resource has to be distributed among the agents of the economy. The utility that an agent obtains comes from the share of the resource that receives and also from the utility of agents that exert an influence on him, which enters his utility function proportionally to the intensity of the direct influence each of these other agents exert on him.

In the example on crime, whenever direct influences exist among crime rates in different neighbourhooods, the assignment of a part of the resources against urban crime does not only have a direct effect on this neighbourhood. Its effects also spill over the rest of neighbourhoods in an heterogeneous manner. Hence, when the different neighbourhoods are engaged in dispute for these resources, and they understand that these spillover effects exist, they show heterogeneous preferences on possible assignments. Even if probably each neighbourhood would prefer to receive all the resource, if they realize they have to achieve an agreement with the rest of neighbourhoods, they would prefer that those neighbourhoods that overall exert a larger spillover effect on it receive more resources than those that exert less.

A linear structure of influence is assumed for tractability. It ensures for, almost, every economy with influences the existence of a unique solution to the system relating utilities. This solution provides the utility in terms of the allocation, instead of in terms of others' utilities. Hence, solving the model means to characterize the effect the share of the resource an agent receives changes others' welfare, and it internalizes the indirect effects bilateral influences generate.

Still another assumption has to be argued. We suppose constant returns to the share of the resource received. This is an strong assumption. Probably the returns to direct investment on fighting against crime in a neighborhood are not constant.

However, assuming constant returns, influences become the unique underlying force that yield to differences in the results. In general, the introduction of concavities due to other reasons not related to influences would distort the analysis of the effect of influences on the bargaining outcome.

When distributional conflict exists, the bargaining outcome is given by the Nash bargaining solution. With its use we ensure a unique well-defined outcome after the resolution of the conflict. Of course, this choice comes at a cost. By using a cooperative solution we abstract from institutional or environmental restrictions that can play a role during the bargaining process. These institutional restrictions are generally introduced defining a particular non-cooperative bargaining game that takes them into account. Then each particular application should be followed by a different non-cooperative game that specifies its particularities. This way we would lose some of the general conclusions that a more stylized model can give as general features of somewhat different situations.

Results. We restrict our analysis to the more interesting case of regular economies. Regular economies are such that any allocation that exhausts resources is Pareto efficient. In regular economies, the Pareto frontier is non-degenerate.

For such economies, we characterize completely the Nash bargaining solution, providing closed-form expressions for the utilities and shares obtained.

The reason for which we restrict to regular economies is the following. In non regular economies the Pareto frontier is degenerate. This is so because the influence exerted by some given agent on others is much bigger than the influence he receives in return. Efficiency might require then that this agent receives all the resources available. If an individual in a group is so much loved by anybody else compared with other possible affective relations, which means that the influence this agent has on others' utility is very large, it could be efficient to give him all the resource.

Similarly, there might exists intermediate situations in which only some agents are allowed to receive a share of the resource for efficiency reasons. While our methods and analysis could be extended to such nonregular situations, we focus on the analysis of regular economies in which the pattern of influences excludes *de facto* some agents from the course for some part of the resources.

We first characterize regular economies. This characterization is twofold. First, regular economies are characterized by an upper bound on the aggregate level of bilateral influences every agent exerts on others. In terms of direct bilateral influences, we obtain a bound on the maximal level of aggregate direct influences an agent can exert. The economy is regular if and only if no agent exceeds this bound. Second, regular economies are characterized by conditions on the pattern of network externalities. More precisely, an economy is regular if and only if all agents are equally central in the network structure of influences. The relevant measure of network centrality is the Katz-Bonacich centrality index, pervasively used in the sociology literature, and that also arises naturally in other economic settings.

We next provide a constructive procedure to characterize the Nash bargaining solution. It is important to note that even if the economy is regular and, hence, the distributional conflict involves all agents in the economy, this does not exclude the possibility that some agents obtain finally no share of the resource. Externalities do not directly solve the bargaining problem but this does not mean that they can not be sufficiently asymmetric such that, after internalizing all influences, the Nash bargaining solution assigns nothing to some of the agents.

A geometric procedure is presented to check if the solution for a particular economy is interior, meaning all agents obtain a positive fraction of the resource, and to characterize the solution in this case.

We also devote part of our work to the analysis of α-economies. In these economies all existent pairwise influences have intensity equal to α, and whenever an agent exerts an influence on another, this other agent also exerts an influence on the first one. One of the two dimensions of heterogeneity in the model, the possibly different levels of externality intensities across individuals, is kept to a minimum. The main source of heterogeneity is the geometry of the pattern of pairwise influences.

An analysis in depth of this family of economies gives a better picture of how the particular arrangement of pairwise relations, irrespective of the intensities of these, impacts on bargaining outcomes. In particular, utility is directly related with the number of connections an agent has. Those agents that receive and exert more influences are also rewarded with larger utility levels. However, this monotonicity does not necessarily translates into receiving larger fractions of the resource.

Finally, we also study how changes on the pattern of influences distort the Nash bargaining outcome. We analyze how changes in the levels of bilateral influences change the bargaining result. Furthermore, we also discuss how our framework can be used to describe and analyze situations in which some agents that are not involved in the bargaining game can affect the bargaining outcome.

Applications. Besides the example on urban crime, our model is flexible enough to encompass other possible applications.

Consider for example government spending. When deciding how to divide the public budget within government departments a possible concern for the ministers is to take into account that outcomes related to responsibilities of one department can affect outcomes related to other departments. The result of the bargaining process should then depend on these influences generated across departments.

The department of social affairs might exert a positive externality on the department of education. If more resources are spent on social affairs, such as ameliorating life conditions in specially poor regions, this can translate into larger school attendance rates in this regions and hence an improvement in aggregate level of education of the country, a subject under the domain of the education department. Moreover, this effect on citizens' education can translate into an increase on the understanding of good and healthy habits that might imply an increase on life expectancy in the country, a fundamental issue for the health department.

Observe that the outcome improvement on health comes indirectly from an increase of resources for social affairs. This increase on resources improves the outcome for social affairs which through a direct influence improves the outcome on education, which improves the outcome on health. This is another example of indirect influences that spread due to network effects. In this setup our analysis sheds some light on how the pattern of influences across different departments maps into government bargaining agreements.

Another possible application is in the field of social preferences.[1] Following the line of seminal work from Becker (1974), altruism and envy can be interpreted as influences of a very particular kind, with its source coming from psychological reasons. An agent is altruist for another if he is better off when the other is better off. Hence, this second agent is exerting a positive influence on the first one. Inversely, an agent is envious for another agent if he is worst off when the other is better off, and in this case the influence the second agent exerts on the first is negative. Our work applies then to the analysis of a bargaining game in the presence of pure altruism and envy effects, where the pattern of altruism and envy is variable in intensity across pairs of agents.

Related Literature. Our model bears a formal resemblance with previous work on interdependent utilities by Bergstrom (1999) and Bramoullé (2001). Bramoullé also interprets this type of systems in terms of weighted and directed networks, but focuses on some qualitative features of the mapping from bilat-

[1] See Fehr and Schmidt (2002) and Sobel (2005) for very comprehensive surveys of the theoretical literature and empirical evidence on social preferences. Levine uses a linear model similar to the one we develope in our work to analyze experimental results for some classes of games. Roth (1995) is a survey of experimental evidence of social preferences in bargaining games.

eral influences to network externalities.[2] Here instead, we analyze the mapping from bilateral influences to Nash bargaining utilities and agreed shares, providing closed-form expressions for both.

A strategic model on status in networks that generates similar interdependency systems is provided in Rogers (2005). Rogers analyzes a network formation game in which agents with heterogeneous skills can choose with whom they want to contact and with which intensities they want that this contact is made. Hence, the pattern of influences is endogenously chosen. We analyze instead situations in which the structure of pairwise influences can not easily be affected by individual strategic decisions. For example, in the urban crime example no neighbourhood can do much to delimit influences among them, since these are largely determined by private decisions and actions of the population, which is an issue out of their control. A similar comment applies on the example on interdepartmental influences in a government. Also, altruism and envy are not only the result of strategic decisions but the effect of the embeddedness of individuals on a social environment they can not determine and control.

The model also resembles input-output models of linear economies (Leontief, 1951, Gale, 1960). However, input-output models only allow for positive bilateral influences, while here we do not impose sign restrictions of any sort. Of course, we also deal with different issues.

Some papers have analyzed multilateral bargaining with externalities from a non-cooperative viewpoint. Jehiel and Moldovanu (1995a, 1995b) consider a setup where one seller bargains with n potential buyers to decide which of them obtains the unit of an indivisible good. The acquisition of the good by one of the agents can exert a positive or negative externality on others. They analyze how the bargaining outcome is affected by this allocative externality.

In a political economy context, Calvert and Dietz (2004) explore how the introduction of externalities in a 3-agent economy alters the conclusions of the Baron and Ferejohn (1989) non-cooperative game of legislative bargaining. See Duggan (2004) for conditions about existence of equilibria in the n agents version of the Baron and Ferejohn game with externalities.

While our cooperative approach is less sensitive to possible particularities in the bargaining process, such as the particular mechanisms by which buyers and sellers bargain or the existence of a voting rule (the majority rule in Baron-Ferejohn models) in legislative bargaining, it allows for a general and tractable analysis of multilateral bargaining with one unique outcome prediction and an heterogeneous pattern of externalities.

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[2] For a complementary approach to this mapping and a general characterization of Pareto efficiency in such a setup, see de Martí (2006).

Our work also borrows from the very active literature on networks in economics. However, we do not deal with the formation of social and economic networks, maybe the more extensively studied issue in the field, but on games played in a fixed network.[3] Other authors have also explored the interrelation of network structure and bargaining outcomes (see Calvó-Armengol, 2001, and Corominas-Bosch, 2004). The approach in these papers is different in many respects. Just to mention a few, bargaining is not among the many and the network represents communication restrictions and delimits the possible pairs of agents that can trade.

The Katz-Bonacich centrality measure was first defined by Katz (1953) and later on developed by Bonacich (1987). It is one of the more relevant centrality measures studied in the active field of social network analysis.[4] Another game played in a network, in this case not a bargaining game, where this centrality measure naturally arises is Ballester et al. (2006). Agents play a game with pairwise dependent strategic complementarities. In the unique equilibrium of the game each agent action is proportional to his Katz-Bonacich centrality index measured on this network of complementarities.

1.2 Bilateral Influences and Network Externalities

1.2.1 Modelling Bilateral Influences

In this section we propose a simple framework that allows to model positive and negative allocative influences across individuals.

Suppose that there is an amount of a certain resource to be distributed within a group of n individuals, $\mathcal{N} = \{1, \ldots, n\}$. Let c_i be the consumption of agent $i \in \mathcal{N}$. Let $b_{ij} \in \mathbb{R}$ be the magnitude of the influence agent j exerts on agent i. Then, an increase of one unit of welfare for agent j induces an increase of b_{ij} units of welfare for agent i. Given a profile $\mathbf{c} = (c_1, \ldots, c_n)$, the utility an agent obtains , $u_i(\mathbf{c})$, is equal to

$$u_i(\mathbf{c}) = c_i + \sum_{j \neq i} b_{ij} u_j(\mathbf{c}) \qquad i = 1, \ldots, n \tag{1.1}$$

This set of equations forms what we call the bilateral influences system. Note that the relation in this system is from outcomes to outcome.

In terms of the urban crime example, b_{ij} represents how the crime rate , not the share of public budget received, in neighborhood j affects the crime rate in

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[3] See Jackson (2005) for a very extensive survey of the field of networks in economics, and for an exhaustive list of references about games played in networks, including bargaining games.

[4] For an exhaustive survey of this literature see Wasserman and Faust(1994).

neighborhood i.

Due to linearity, we can fix the sum of consumption levels to $\sum_{i=1}^{n} c_i$ to be equal to 1. Hence, we can interpret c_i as the share of the resource received by agent i.

Defining $b_{ii} = 0$ for all $i \in \mathcal{N}$, we gather all the b_{ij} in a matrix $\mathbf{B}$ of bilateral influences. An *economy* is completely characterized by its matrix of bilateral influences.

For a given economy $\mathbf{B}$, we can obtain from the structural system of bilateral influences to a reduced-form system where the utility of each agent can be directly expressed in terms of the shares profile, eliminating the dependency on other's utility.

The bilateral influence system in matrix form is equal to

$$\mathbf{u}(\mathbf{c}) = \mathbf{c} + \mathbf{B} \cdot \mathbf{u}(\mathbf{c}) \tag{1.2}$$

Hence, if $\mathbf{I}$ is the identity matrix, whenever $(\mathbf{I} - \mathbf{B})^{-1}$ exists we obtain the reduced-form system

$$\mathbf{u}(\mathbf{c}) = (\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{c} \tag{1.3}$$

The first result we provide is a genericity result. An economy is characterized by $n(n-1)$ real values. Therefore, there is a one-to-one mapping from economies to elements of $\mathbb{R}^{n(n-1)}$. From this point of view, the set of economies in $\mathbb{R}^{n(n-1)}$ for which $(\mathbf{I} - \mathbf{B})^{-1}$ does not exist has (Lebesgue) measure zero. This implies that for almost every economy $\mathbf{B}$, the associated matrix $(\mathbf{I} - \mathbf{B})^{-1}$ exists (and, of course, is unique) and the next result then follows.

Proposition 1 *For almost every economy* $\mathbf{B}$ *the associated reduced-form system is uniquely characterized.*

In words, given a structural system of bilateral influences there is no indeterminacy in the obtaining of the associated reduced-form expression, except for a negligible set of economies.[5]

Let $\mathbf{E}(\mathbf{B}) = (\mathbf{I} - \mathbf{B})^{-1}$. Each entry $e_{ij}(\mathbf{B})$ expresses the magnitude of how the utility increases, if the entry is positive, or decrease, if the entry is negative, when the level of consumption of agent j increases. We call $\mathbf{E}$ the matrix of *network externalities*. An explanation for the choice of this name follows.

[5] See Bramoullé(2001) for structural models of a similar nature for which indeterminacy in the determination of the associated reduced-form system arises.

1.2.2 From Bilateral Influences to Network Externalities

Any economy $\mathbf{B}$ can be naturally represented by a network.

A network is formed by a set of nodes and a set of links that express a relation between the pair of nodes linked. While this is an abstract object, it is a useful metaphor to represent many varied situations in applied settings. In particular, in our case this metaphor can be applied to make nodes represent the agents involved in the structural system of bilateral influences, and make links represent the pattern of bilateral influences exerted across pairs of agents. A link in such an influence network is weighted, each link has an associated value that represents the strength of the influence this link represents, as well as a particular direction, since the influence agent i exerts on j does not necessarily coincides in strength with the influence agent j exerts on agent i, and hence we have to distinguish the link from i to j and the link from j to i.

Different conventions could be adopted to express the mapping from economies to networks. We adopt the following one. We say that there is a link from agent i to agent j whenever j exerts a, positive or negative, influence on i, and the weight for this link is then equal to the coefficient $b_{ij} \in \mathbb{R}$ of the structural system of bilateral influences. Since in our model there is no self-influence we do not allow for self-loops, links from an agent to itself. The set links that begin in i point to the agents that influence agent i.

Observe the weighted and directed nature of the network defined in this way: since we have not imposed any restriction on the possible values of the coefficients in the structural influence system, the weight of a link can take any real value; also, since we have not imposed symmetry on the levels of bilateral influence, it is possible that there exist both a link from i to j and another one from j to i and that their respective weights differ. Even more, it is possible that there exist a link from i to j while there is no link from j to i.

A weighted and directed network is defined by an adjacency matrix, where the entry (i, j) in this matrix is equal to the weight of the link from i to j. This weight equals the level of bilateral influence j exerts on i. Hence, given an economy $\mathbf{B}$ the adjacency matrix of its associated network, in the way we have defined this network, is also $\mathbf{B}$.

The following equality applies

$$\mathbf{E}(\mathbf{B}) = (\mathbf{I} - \mathbf{B})^{-1} \tag{1.4}$$

Whenever $\mathbf{B}$ is a contraction[6] we have that

$$(\mathbf{I} - \mathbf{B})^{-1} = \sum_{k=0}^{+\infty} \mathbf{B}^k \tag{1.5}$$

If j exerts an influence on i with weight b_{ij} and k exerts an influence on j with weight b_{jk}, k exerts an indirect influence on i with weight equal to $b_{ij}b_{jk}$. The matrix $\mathbf{B}^2$ keeps track of these second order network influences. The entry $b_{ik}^{[2]}$ of $\mathbf{B}^2$ computes the sum of weights of all paths of length two from i to k.[7]

More generally, for any $l \geq 1$ the matrix $\mathbf{B}^l$ keeps track of the l-order network influences: each entry $b_{ik}^{[l]}$ equals the sum of weights of all paths of length l from i to k.

Therefore, whenever the expression in equation (1.1) is valid, the entry $e_{ij}(\mathbf{B})$ of $\mathbf{E}(\mathbf{B})$ is the sum of weights of *all* paths from j to i in the network represented by the economy/adjacency matrix $\mathbf{B}$. The matrix $\mathbf{E}(\mathbf{B})$ computes the sum of indirect (network) effects that the pattern of bilateral influences generates. This sum of indirect effects of any order is what we denote *network externalities*, and this is why we call matrix $\mathbf{E}(\mathbf{B})$ the matrix of *network externalities*.

Each entry $\mathbf{e}_{ij}(\mathbf{B})$ represents by how much the consumption of agent j affects the utility of agent i not only through the direct bilateral influence agent j exerts on i, represented by b_{ij}, but also through the indirect influences resulting of all possible indirect network connections from j to i.

The following example, borrowed from Bramoullé (2001), is useful to understand how important are indirect network effects for the analysis of the mapping from allocations to utilities defined in the reduced-form system.

Example 1. There are three agents, $\mathcal{N} = \{1, 2, 3\}$, and the structural system

6 The matrix $\mathbf{B}$ is a contraction if and only if all its eigenvalues have norm smaller than 1. This will be the case for example for the set of *regular* economies, that we define later, if bilateral influences are positive.

7 A *path* between i and j in network $\mathbf{G}$ is a sequence of agents $i_1, \ldots, i_K$ of N, where an agent can appear several times in this sequence, such that $i_k i_{k+1}$ is a link of $\mathbf{G}$ for every $k \in 1, \ldots, K-1$, with $i_1 = i$ and $i_K = j$. The length of such a path is equal to $K-1$, the number of links that form the path. In words, a path in g is an indirect connection from agent i to agent j through linked agents in $\mathbf{B}$. We define the *weight* of a path $i_1, \ldots, i_K$ of $\mathbf{G}$ as the product $g_{i_1 i_2} \cdots g_{i_{K-1} i_K}$. This weight is different than zero because of the definition of path. A path such that $i = j$ is called a *cycle*.

of bilateral influences relating them is

$$\begin{aligned} u_1(\mathbf{c}) &= c_1 + b_{12}u_2(\mathbf{c}) + b_{13}u_3(\mathbf{c}) \\ u_2(\mathbf{c}) &= c_2 + b_{23}u_3(\mathbf{c}) \\ u_3(\mathbf{c}) &= c_3 \end{aligned}$$

where b_{12} and b_{13} are positive but b_{23} is negative. This means that both agent 2 and 3 exert a positive direct influence on agent 1, probably with different intensities, while agent 3 exerts a negative influence on agent 2. Besides, the values for these direct bilateral influences are b_{21}, b_{13} and b_{23}.

The network that represents this situation is

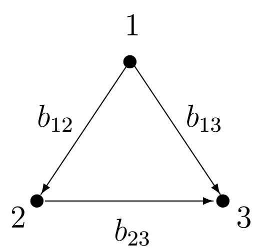


Figure 1

and the 3×3 bilateral influences matrix is

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

Hence, the 3×3 matrix of network externalities, $\mathbf{E}(\mathbf{B}) = (\mathbf{I} - \mathbf{B})^{-1}$, is equal to

$$\mathbf{E}(\mathbf{B}) = \begin{pmatrix} 1 & b_{12} & b_{13} + b_{12}b_{23} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

Observe that in this case we can easily compute the matrices of indirect network effects, $\mathbf{G}^k$ for $k \geq 2$. The matrices of higher order network effects are

$$\mathbf{B}^2 = \begin{pmatrix} 0 & 0 & b_{12}b_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\mathbf{B}^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad for\ every\ k \geq 3$$

Therefore, $\mathbf{E}(\mathbf{B}) = \mathbf{I} + \mathbf{B} + \mathbf{B}^2$. The expressions of utilities in terms of consumption are therefore

$$\begin{aligned} U_1(\mathbf{c}) &= c_1 + b_{12}c_2 + (b_{13} + b_{12}b_{23})\, c_3 \\ U_2(\mathbf{c}) &= c_2 + b_{23}c_3 \\ U_3(\mathbf{c}) &= c_3 \end{aligned}$$

The weight of the network externality agent 3 exerts on agent 1, $e_{13} = b_{13} + b_{12}b_{23}$, depends on the intensities of bilateral influences. In particular, if $-b_{12}b_{23} > b_{13}$ the externality is negative, even if the direct bilateral influence b_{13} is positive. A tension arises because even if agent 3 exerts a direct positive influence on agent 1, the negative influence agent 3 exerts on agent 2 also has an indirect (network) effect on agent 1 due to the indirect path from 3 to 1 through agent 2. This negative influence is internalized in the reduced-form system related to network externalities and makes it possible that e_{13} is negative if the bilateral negative influence 3 exerts on 2 is large enough.

We omit the dependence of $\mathbf{E}$ on $\mathbf{B}$ when no confusion is possible.

1.3 The Set of Pareto Allocations

1.3.1 Characterization

From now on we will consider that there is a certain amount of a resource that, without loss of generality, we normalize to one. Before turning to the study of distributional conflict and how agents in an economy agree to divide this unit of resource among them, we have to make a clarification about the set of Pareto efficient allocations in an economy with influences. Externalities can have severe consequences on which allocations can be Pareto efficient. Our aim in this section is to characterize the set of economies for which distributional conflict is particularly strong.

Before providing an example of the peculiar situations that can arise in economies with influences we describe the utility possibility set for any economy $\mathbf{B}$, that we denote UPS $(\mathbf{B})$. Given an economy $\mathbf{B}$, and for any feasible allocation, we have that $\mathbf{u}(\mathbf{c}) = \mathbf{E} \cdot \mathbf{c} = \sum_{i=1}^{n} c_i \mathbf{e}^{(i)}$, where $\mathbf{e}^{(i)}$ is the i-th column vector of the matrix of network externalities. Since an allocation $\mathbf{c}$ is feasible if and only if $c_i \geq 0$ for every $i \in \mathcal{N}$ and $\sum_{i=1}^{n} c_i \leq 1$, we can conclude that the utility possibility set for the economy defined by $\mathbf{B}$ is the convex hull of the columns of the matrix of network externalities $\mathbf{E}$ plus the zero vector, that is

$$\text{UPS}(\mathbf{B}) = co\left\{\mathbf{e}^{(1)}, \ldots, \mathbf{e}^{(n)}, \mathbf{0}\right\}$$

This implies that the utility possibility set for any economy $\mathbf{B}$ is a simplex, and therefore it is a convex and compact set.

Example 2. Consider the economy represented by the following network

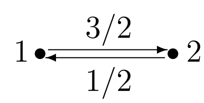


Figure 2

The 2×2 matrix of bilateral influences is

$$\mathbf{B} = \begin{pmatrix} 0 & 3/2 \\ 1/2 & 0 \end{pmatrix}$$

It follows that the matrix of network externalities for this economy is equal to

$$\mathbf{E}(\mathbf{B}) = \begin{pmatrix} 4 & 6 \\ 2 & 4 \end{pmatrix}$$

Here $\mathbf{e}^{(1)} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\mathbf{e}^{(2)} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ and the utility possibility set for this economy is the convex hull of these two vectors and the zero vector. We can depict $\mathrm{UPS}(\mathbf{B})$

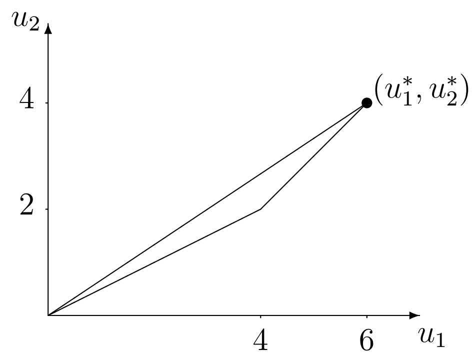


Figure 3

It turns out that the unique efficient allocation in this economy is (u_1^*, u_2^*), that corresponds to $(c_1^*, c_2^*) = (0, 1)$.[8] At the unique Pareto efficient allocation agent 2 receives all the resource. This is so because the magnitude of the externality agent 2 exerts on agent 1 is much larger than the one of the externality agent 1 exerts on agent 2. The effect of an increase in the level of consumption of agent 2 is then larger in agent's 1 utility than the effect of an increase in his own level of consumption.

When considering such kind of situations the implications on the solution to the Nash bargaining problem is immediate. Since the Nash bargaining solution has to be Pareto efficient and there is a unique Pareto efficient allocation, the Nash bargaining problem is trivially solved.

We will disregard the kind of economies that we have just described. Indeed, we will concentrate from now on in the completely opposite kind of situations. We will only consider economies where any allocation that exhausts available resources is Pareto efficient. When this happens we say that the economy is *regular*.[9] The assumption of a regular economy ensures there is a nontrivial bargaining problem and there exists competition among all agents to obtain some share of the unit of resources.

The next result provides a complete characterization of regular economies in

[8] Recall that we have normalized the total amount of resources to one. This is, of course, without lose of generality because of linearity.

[9] Observe that this is not the unique other possible situation. It could be that only a subset of agents should consume for an allocation to be Pareto efficient. In the next chapter we provide a generic characterization of all possible situations in terms of endogenous centrality measures derived from the position of each agent in the network of bilateral influences.

terms of the matrix of network externalities. Before stating it we define a useful notion for the analysis in the rest of the paper.

Definition *We say that an n-dimensional vector $\boldsymbol{\mu}$ is a strict system of weights if and only if $\mu_i > 0$ for every $i \in \mathcal{N}$ and $\sum_{i=1}^{n} \mu_i = 1$.*

Now we provide the first characterization result of regular economies.

Proposition 2 *An economy $\mathbf{B}$ is regular if and only if there exists a unique strict system of weights $\boldsymbol{\mu}$ and a positive constant $\kappa > 0$ such that $\boldsymbol{\mu} \cdot \mathbf{e}^{(i)}(\mathbf{B}) = \kappa$ for every $i \in \mathcal{N}$.*

The previous result characterizes regularity through the matrix of network externalities. Each element of column $\mathbf{e}^{(i)}$ expresses how large is the network externality agent i exerts on each agent. When we compute the weighted average of these elements we obtain a single value that expresses an overall measure of network externalities exerted by agent i. Hence, Proposition 2 says that regularity amounts to find normalized weights, which are unique and depend on the economy we are analyzing, such that this overall measure of network externalities each agent exerts is positive and equal for all agents.

It is also possible to provide a characterization of regular economies in terms of the primitives of the economy, i.e. the matrix $\mathbf{B}$ of bilateral influences.

Proposition 3 *A necessary and sufficient condition for an economy $\mathbf{B}$ to be regular is that*

$$\sum_{j=1, j\neq i}^{n} b_{ji} < 1 \quad for\ every\ i \in \mathcal{N}$$

This result provides a simple and direct way to check if an economy is regular, and it helps us to understand better which are the network forces that induce regularity. It states that regularity amounts to requiring that the aggregate level of bilateral influences each agent exerts on others, $\sum_{j=1, j\neq i}^{n} b_{ji}$, is not too large, in fact not larger than 1.

For example, consider an economy where all agents are connected to each other and the level of bilateral influence across any pair of individuals is equal to certain value α. In this case the necessary and sufficient condition for such kind of economy to be regular is that $\alpha < \frac{1}{n-1}$. this condition is satisfied if $\alpha < 0$, and only if bilateral influences are positive the condition bounds α. This is quite natural. When influences exerted are negative, distributional conflict naturally arises because each agent wants the rest of the economy to receive the smallest possible share. Any allocation that exhausts resources is in this case Pareto efficient. Instead, when influences are positive and sufficiently large each agent would prefer that other agents receive all the resource since this would increase more his utility than receiving himself part of it, and the Pareto frontier might degenerate to a single point.

Consider a regular economy $\mathbf{B}$. Then we can compute easily its associated strict system of weights $\boldsymbol{\mu}$ and constant κ from Proposition 2.[10]

Let $\mathbf{B}$ be a regular economy. Define $\delta_i = 1 - \sum_{j=1, j\neq i}^{n} b_{ji}$, $i \in \mathcal{N}$, and $\delta = \sum_{i=1}^{n} \delta_i$. Then, $\kappa = \frac{1}{\delta}$ and $\mu_i = \kappa \delta_i$ for all $i \in \mathcal{N}$.

The value of δ_i is negatively related to the level of aggregate bilateral influences agent i exerts. Hence, μ_i is smaller the larger this aggregate level of bilateral influences is. The next section studies more in detail how these constants relate to the particular network structure of influences, and we provide interpretations for them.

1.3.2 Efficiency and Network Centrality

We can provide an alternative interpretation to the efficiency characterization in Proposition 2. To this end, we have to introduce some terminology derived from the literature on social networks.

Given a network we can try to measure the prominence of each agent due to her position in the network. There are several variables that can determine the prominence of an actor in a network. Furthermore, the definition of prominence may depend on the setting we are studying. It is not the same if we deal with directed or undirected networks, or with weighted or unweighted networks. Hence, there is not in the social network analysis literature a unique standard definition of prominence.

The more usual concept to analyze prominence in networks is centrality. It is fairly natural to associate prominence with connectivity and this is what centrality measures do. In the case of weighted and directed networks, as the ones we are considering in our analysis, a rough measure of centrality of agent i would be the sum of weights of the links that point to agent i, $S_i = \sum_{j\neq i} b_{ji}$.[11] This measure is called the degree centrality of agent i. In terms of our structural influence model, S_i measures the aggregate level of influence that emanates from agent i.

[10] The proof of Lemma 1 is contained in the proof of Proposition 3.

[11] This is an inner-centrality measure. Alternatively, we could define an outer-centrality measure by the sum of weights of the links that start in agent i. Since, as we will show in a moment, in our analysis the relevant centrality measure is an inner measure, we avoid this possible distinction in the text.

Note that $\delta_i = 1 - S_i$, which is a positive quantity whenever the economy $\mathbf{B}$ is regular, is then a complementary degree centrality index for agent i. Its value is smaller the largest is the degree centrality measure S_i. Therefore, μ_i is also a complementary centrality index, since it is a renormalization of δ_i to make the sum of the indices for all agents to add up to one.

While this degree centrality is informative of some kind of prominence derived by the way influences vary across pairs, it does not capture the value of how these influences spread indirectly along chains of bilateral influences.

Remember that, as we have explained before, given an economy $\mathbf{B}$ we have that for any $l \geq 1$ the matrix $\mathbf{B}^l$ keeps track of the l-order network externalities: each entry $b_{ij}^{[l]}$ equals the sum of weights of all paths of length l from i to j. Hence, to construct a more elaborate centrality measure we might include these indirect network effects subsumed in the sequence of matrices $\left\{\mathbf{B}^l\right\}_{l\geq 1}$. A natural way is to consider a decay factor $\lambda \in (0, 1]$ and weight the l-order network effects by λ^l. This is the Katz-Bonacich centrality measure. The (unweighted) Katz-Bonacich inner-centrality[12] measure vector, $\boldsymbol{\kappa}\left(\mathbf{B}; \boldsymbol{\lambda}\right)$ is defined as:

$$\boldsymbol{\kappa}\left(\mathbf{B}; \boldsymbol{\lambda}\right) = \left(\sum_{l=0}^{\infty} \lambda^l \mathbf{B}^l\right)^t \cdot \mathbf{1}$$

Whenever this vector is well-defined we can rewrite it as:

$$\boldsymbol{\kappa}\left(\mathbf{B}; \boldsymbol{\lambda}\right) = \left[\left(\mathbf{I} - \lambda\mathbf{B}\right)^{-1}\right]^t \cdot \mathbf{1}$$

A variation of this measure, called the weighted Katz-Bonacich centrality measure, is the following.

Let $\boldsymbol{\mu}$ be an strict system of weights. Then the $\boldsymbol{\mu}$-weighted Katz-Bonacich centrality measure, $\boldsymbol{\kappa}_{\boldsymbol{\mu}}\left(\mathbf{B}; \boldsymbol{\lambda}\right)$, is given by the following formula:

$$\boldsymbol{\kappa}_{\boldsymbol{\mu}}\left(\mathbf{B}; \boldsymbol{\lambda}\right) = \left[\left(\mathbf{I} - \lambda\mathbf{B}\right)^{-1}\right]^t \cdot \boldsymbol{\mu}$$

In the unweighted Katz-Bonacich centrality measure all agents count the same when considering the sum of network effects generated by each one of them. In the $\boldsymbol{\mu}$-weighted Katz-Bonacich centrality measure the network effects generated by agent i are counted with weight μ_i. Some agents count more than others when aggregating the whole matrix of network effects $\mathbf{E}\left(\mathbf{B}\right)$.

[12] It is an inner measure of centrality because it measures weights of paths and cycles that end on each agent. An outer-centrality measure could be defined without transposing in the following equation.

After this digression into the realm of social and economic networks, we can reinterpret the condition of proposition 3 making use of weighted Katz-Bonacich centrality measures. The condition is equivalent to say that there exists a unique strict system of weights $\boldsymbol{\mu}$ such that

$$\left[(\mathbf{I}-\mathbf{B})^{-1}\right]^t \cdot \boldsymbol{\mu} = \kappa \mathbf{1}$$

with κ being a positive constant. The reader can immediately recognize the $\boldsymbol{\mu}$-weighted Katz-Bonacich centrality measure, with $\lambda = 1$, in the left handside of the last equation. Hence the regularity condition says that there exists a vector of weights for which the weighted Katz-Bonacich centrality measure is equal, and positive, for all agents. This individual index measures the aggregate level of network influence effects that i generates. These are represented by the paths on the network that finish on i, and this is exactly what the Katz-Bonacich centrality index takes into account.

Two comments are in order. First, observe that our model generates endogenously the unique system of weights $\boldsymbol{\mu}$ for which this centrality condition is satisfied. This is Proposition 3. Second, the decay factor is equal to 1, and hence direct and indirect influences count the same to compute this measure of prominence. Hence, we can rewrite proposition 2 as follows

Proposition 2' *The economy $\mathbf{B}$ is regular if and only if there exists a unique strict system of weights $\boldsymbol{\mu}$ and a constant $\kappa > 0$ such that*

$$\boldsymbol{\kappa}^i_{\boldsymbol{\mu}}(\mathbf{B};1) = \kappa \qquad for\ all\ i \in \mathcal{N}$$

A general characterization, not only for regular economies, of Pareto efficiency in economies with pairwise influences by means of centrality measures can be in the next chapter.

1.4 Bargaining and Influences

1.4.1 The Bargaining Problem and its Solution

From now on, we consider only regular economies with influences. We turn to the study of distributional conflict for these economies.

We consider the classical and widely used Nash bargaining solution (Nash, 1950). Following this seminal work we define an n-person bargaining problem as a duple $\langle X, \mathbf{d}\rangle$, where $X \subset \mathbb{R}^n$ is a convex and compact set that expresses the utility possibility set in the economy, and $\mathbf{d} \in X$ is the disagreement point, that expresses the utilities each agent would obtain in case they are not able to reach an agreement. The disagreement point has to satisfy the following dominance condition: there exists $\mathbf{v} \in X$ such that $\mathbf{v}$ strictly Pareto dominates $\mathbf{d}$, i.e. $v_i > d_i$ for every $i \in \mathcal{N}$. The (symmetric)[13] Nash bargaining solution $\mathbf{x}^S = \left(x_1^S, \ldots, x_n^S\right)$ to $\langle X, \mathbf{d}\rangle$ is the solution to the following maximization problem

$$\max_{\mathbf{x} \in X} \prod_{i=1}^{n} (x_i - d_i)$$

Due to convexity of the utility possibility set X and strict convexity of the objective function this problem has a unique solution.

We want to analyze this Nash bargaining solution in the case the utility possibility set X is induced from a regular economy with influences. Observe this is possible since as we mentioned before UPS $(\mathbf{B})$ is convex and compact for any economy $\mathbf{B}$.

Given an economy $\mathbf{B}$, let $\mathbf{u}^{min} = (u_1^{min}, \ldots, u_n^{min})$ be the utility vector where each entry u_i^{min} is equal to the minimal utility agent i can obtain within the set of efficient allocations of economy $\mathbf{B}$.[14] Since we assume that the economy is regular, from Proposition 2 we know that there exist only one strict system of weights $\boldsymbol{\mu}$ and one positive constant κ such that $\boldsymbol{\mu} \cdot \mathbf{e}^{(i)} = \kappa$ for every $i \in \mathcal{N}$. Given a disagreement point $\mathbf{d}$ we relabel agents from 1 to n such that $\mu_1 \left(u^{min} - d_1\right) \geq \cdots \geq \mu_n \left(u_n^{min} - d_n\right)$. Finally, let

$$\psi^{(0)} = \frac{1}{n} \left(\kappa - \boldsymbol{\mu} \cdot \mathbf{d}\right)$$

and let

$$\psi^{(j)} = \frac{1}{n-j} \left(\kappa - \boldsymbol{\mu} \cdot \mathbf{d} - \sum_{k=1}^{j} \mu_l \left(u_k^{min} - d_k\right)\right)$$

for any $j \in \{1, \ldots, n-1\}$.

Now we have all the necessary ingredients to characterize the Nash bargaining solution for any regular economy with influences. This is done in the following result.

Proposition 4 *Consider a regular economy* $\mathbf{B}$. *Then, there exists* $j \in \{0, \ldots, n-1\}$

[13] To simplify the analysis we only consider the symmetric Nash bargaining solution. The analysis for asymmetric Nash bargaining solutions with heterogeneous bargaining power is completely analogous.

[14] In fact, $\mathbf{u}_i^{min} = \min\{e_{i1}(\mathbf{B}), \ldots, e_{in}(\mathbf{B})\}$, so this vector of minimal utilities can easily be derived from the group influence matrix.

such that the utility vector associated with the Nash bargaining solution, $\mathbf{u}^S$, is

$$u_i^S = u_i^{min} \qquad\qquad if\ i \leq j$$

and

$$u_i^S = d_i + \psi^{(j)} \frac{1}{\mu_i} \qquad if\ i > j$$

The Nash bargaining solution allocation is equal to $\mathbf{c}^S = (\mathbf{I} - \mathbf{B})\,\mathbf{u}^S$.

This result characterizes the utilities and levels of consumption of the Nash bargaining solution for any regular economy. In particular, it characterizes corner, partially corner and interior solutions.

For an individual i that obtains positive gains, $u_i^S - d_i > 0$, these gains are inversely proportional to μ_i. If both i and j obtain positive gains we have that

$$\frac{u_i^S - d_i}{u_j^S - d_j} = \frac{1 - \sum_{k \neq j} b_{kj}}{1 - \sum_{k \neq i} b_{ki}} \tag{1.6}$$

The relative gains of agent i with respect to those of agent j uniquely depend on the level of aggregate influence exerted by agent i and agent j. In particular, the largest is the magnitude of aggregate influence that emanates from agent i compared with those that emanate from agent j, the largest the relative gains of i with respect to j.

Aggregate influence levels determine relative gains for those agents that obtain positive gains. This does not mean that these levels form the unique relevant information from the structural influence model to characterize the Nash bargaining solution. The minimal utilities profile, $\mathbf{u}^{\min}$, and therefore the multiplier $\psi^{(j)}$, can not be expressed in terms of the aggregate influence levels. Indeed, minimal utilities internalize all levels of network influence effects, since u_i^{min} equals the minimal entry in i's row of matrix $\mathbf{E}(\mathbf{B})$. In non-interior solutions where some agents obtain no gains from bargaining the information from the matrix of network externalities is fundamental for the characterization of the Nash bargaining outcome.

The multiplier $\psi^{(j)}$ represents the remaining surplus, the remaining value of the available unit of resources in terms of utilities, once we subtract the minimal utilities some of the agents obtain (agents $k \leq j$). The rest of this remaining value is shared proportionally to the inverse of entries of $\boldsymbol{\mu}$.

Example 3. The analysis of the following two economies illustrate the characterization we have just provided in a 2-agents setting. Economy (a) is such that $b_{12} = 4/5$ and $b_{21} = 1/4$ while economy (b) is such that $b_{12} = b_{21} = 1/2$. The matrices of network externalities for each economy are

$$\mathbf{E}_{(a)} = \begin{pmatrix} 5/4 & 1 \\ 5/16 & 5/4 \end{pmatrix} \qquad \mathbf{E}_{(b)} = \begin{pmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{pmatrix}$$

The utility possibility set with the respective Nash bargaining solution depicted, when the disagreement point is $\mathbf{d} = \mathbf{0}$, in both cases are

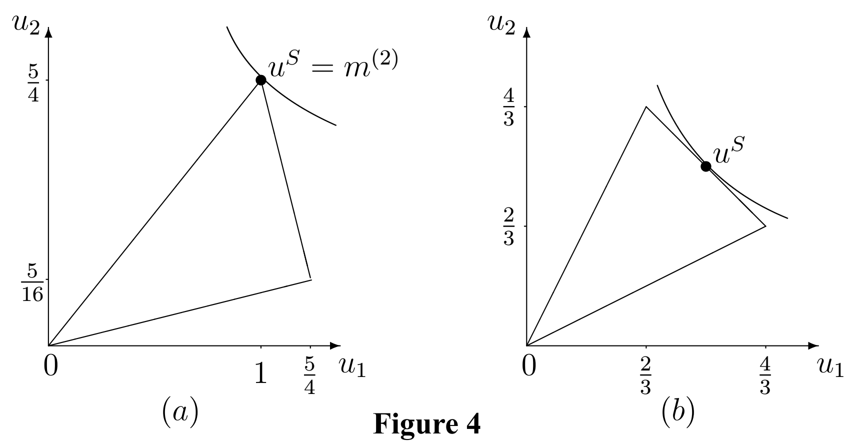


Figure 4

In example (a) we obtain a corner solution. One agent receives all the resource as the solution to the distributional conflict. We have that $\delta_1 = 3/4$, $\delta_2 = 1/5$, and therefore $\delta = 19/20$. The associated constant and strict system of weights from Proposition 2 are $\kappa = 20/19$ and $\mu_1 = 15/19$, $\mu_2 = 4/19$. The minimal utility each agent can obtain in a Pareto efficient allocation is $u_1^{min} = 1$ and $u_2^{min} = 5/16$. In this case the j from proposition 4 equals 1, and the multiplier is $\psi^{(1)} = 5/19$. In the solution, agent 1 receives nothing and agent 2 receives all the resource. Agent 1 obtains his minimal utility, $u_1^s = 1$, while agent 2, instead, obtains $u_2^S = \psi^{(1)} \frac{1}{\mu_2} = \frac{5}{19} \frac{19}{4} = \frac{5}{4}$, that equals his maximal possible utility within the set of efficient allocation.

In example (b) we obtain an interior solution. Both agents obtain a utility within their minimal and maximal utility. In this case, we have that $\delta_1 = \delta_2 = 1/2$,and therefore $\delta = 1$. The associated constant and strict system of weights from proposition 2 are therefore $\kappa = 1$ and $\mu_1 = 1/2$, $\mu_2 = 1/2$. The minimal utility each agent can obtain in an efficient situation is $u_1^{min} = u_2^{min} = 2/3$. We get that the j from proposition 4 equals 0 and the multiplier is $\psi^{(0)} = 1$. Hence, each agent obtains a utility equal to $u_1^s = u_2^s = \psi^{(0)} \frac{1}{\mu_i} = 2$.

1.4.2 Discussion

1.4.2.1 A Geometric Characterization

We can provide a graphical approach of how the Nash bargaining solution is obtained in the case of an interior solution. A similar kind of interpretation can be given for corner and semi-corner solutions but then the analysis is more involved.

In proposition 4 we have obtained a complete characterization of the Nash bargaining solution, both in terms of utilities and shares received. In particular a fundamental ingredient for this characterization is the unique strict system of weights $\boldsymbol{\mu}$ from proposition 2. Call $\boldsymbol{\mu}^{-1}$ the vector with entries the inverses of the entries of $\boldsymbol{\mu}$, i.e. $\mu_i^{-1} = 1/\mu_i$. If the Nash bargaining solution is interior, the vector of utilities agents obtain is equal to the disagreement point plus a positive multiple of vector $\boldsymbol{\mu}^{-1}$. From this construction we can derive a geometric procedure to deduce when the Nash bargaining solution is interior given a particular economy $\mathbf{B}$. We present it with the use of the two previous examples.

We depict again the utility possibility sets and the vectors $\boldsymbol{\mu}$ and $\boldsymbol{\mu}^{-1}$ for each example.[15]

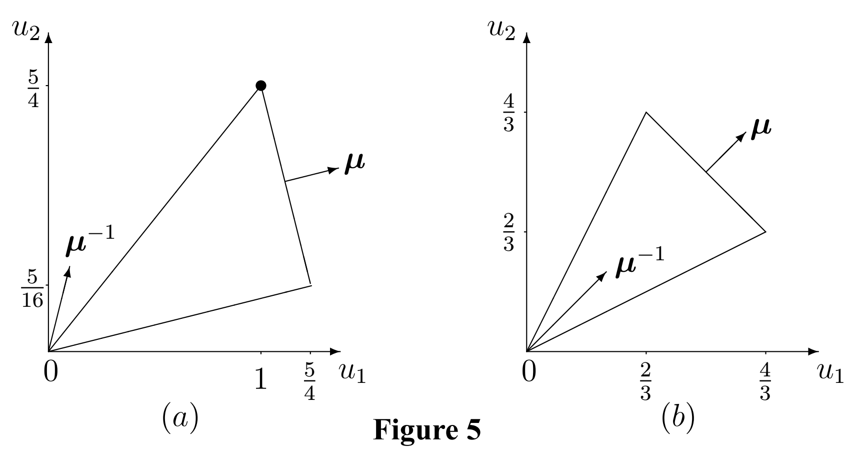


Figure 5

In a regular economy the vector $\boldsymbol{\mu}$ is orthogonal to the Pareto frontier in the utility possibility set.[16] We have depicted the vector $\boldsymbol{\mu}^{-1}$ in the point $(0, 0)$ because this

[15] We have rescaled both vectors for convenience. This does not affect at all the reasoning.

[16] The conditions on Proposition 2 determine a hyperplane with orthogonal vector $\boldsymbol{\mu}$. The utility possibility set is a subset of this hyperplane.

is the disagreement point.

In the first example, since the vector $\boldsymbol{\mu}^{-1}$ lies outside the utility possibility set, no multiple can intercept the Pareto frontier. Therefore, the solution can not be interior.

In the second example, the vector $\boldsymbol{\mu}^{-1}$ lies inside the interior of the utility possibility set and then the Nash bargaining solution can be obtained by multiplying the vector by a positive scalar until it touches a point of the Pareto frontier. This point is the Nash solution utility vector.

Just to clarify that this geometric procedure is valid for any possible disagreement point, and not only for the case in which $\mathbf{d} = \mathbf{0}$, we show here how it applies also for the first economy in the case that $\mathbf{d} = \mathbf{u}^{min}$.

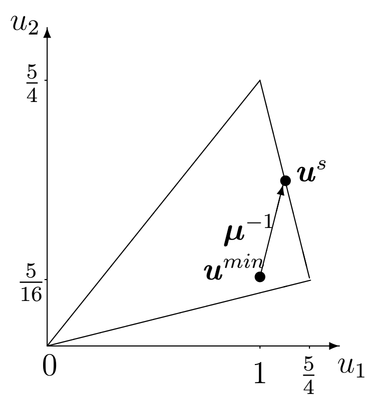


Figure 6

We locate the vector $\boldsymbol{\mu}^{-1}$, that does not change with the change of disagreement point since it only depends on the matrix $\mathbf{B}$, in the point $\mathbf{u}^{min} = (1, 5/16)$. The point where a positive rescaling of $\boldsymbol{\mu}^{-1}$ touches the Pareto frontier coincides with the Nash bargaining solution utilities profile, $\mathbf{u}^s$.

1.4.2.2 The Disagreement Point

As stressed in Binmore et al. (1996), the choice of a particular disagreement point entails also some of the features of the bargaining process when considering the Nash bargaining solution. In our case two different possible disagreement points emerge as the more natural choices.

The first one is the choice of $\mathbf{d} = \mathbf{0}$. This would be the natural choice when in the bargaining situation there are time concerns. In this case, agents know that an agreement could be infinitely delayed and therefore that they could obtain no utility at all. This time concerns are modelled by means of the choice of the zero vector as the disagreement point.

Another possibility is to choose $\mathbf{d} = \mathbf{u}^{min}$, that is that d_i coincides with the minimal utility agent i can obtain in an efficient situation, u_i^{min}. In a regular economy, this vector of minimal utilities satisfies the property of domination that the disagreement point has to satisfy. Observe that this disagreement point derives endogenously from the pattern of influences expressed by matrix $\mathbf{B}$. It might be a natural selection when considering situations in which no time concerns exist. When Pareto efficiency is a requirement of the solution to the distributional conflict, as it is in the case of the Nash bargaining solution, agent i might make recognize the rest of members in the economy that he should not obtain less than u_i^{min}.

Hence, while our cooperative approach can not capture all of the features of particular applications, some of these features can be incorporated into the model, not by changing utilities but directly through the choice of the disagreement point.

Next result provides conditions under which we can ensure that the Nash bargaining solution is interior, meaning that all agents receive some share of the resource, for the two disagreement points we have just highlighted.

Corollary 1 *The Nash bargaining solution is always interior when* $\mathbf{d} = \mathbf{u}^{\min}$. *The Nash bargaining solution is interior when* $\mathbf{d} = \mathbf{0}$ *if and only if for all* $i \in \mathcal{N}$

$$\sum_{j \neq i} b_{ij} \frac{\delta_i}{\delta_j} < 1 \tag{1.7}$$

This condition resembles the condition for regularity stated in Proposition 3. However, it is different in two aspects. First, the set of pairwise influences that appear are in this case the ones that i receives, instead of those that i exerts. Second, these bilateral influences are weighted by the quotient

$$\frac{\delta_i}{\delta_j} = \frac{\left(1 - \sum_{k \neq i} b_{ki}\right)}{\left(1 - \sum_{k \neq i} b_{kj}\right)}$$

For example, in the case that all bilateral influences are positive this quotient is larger than 1 if $\sum_{k \neq i} b_{kj} > \sum_{k \neq i} b_{ki}$. The influence j exerts on i is weighted by larger values in condition (1.7) if j exerts a larger aggregate level of direct influences on others than i. Observe that this quotient was also present when computing relative profits obtained from bargaining across pairs of individuals.

When we fix a disagreement point $\mathbf{d}$ such that for some agent $u_i^{min} > d_i$ we also impose a value to the minimal gains that this agent is going to obtain from the bargaining situation. This value is equal to $u_i^{min} - d_i$. It might be possible that these minimal gains from bargaining can not be reconciled with the conditions imposed on relative gains across individuals in (1.6) when agents obtain a positive share of the resource. The conditions in Corollary 1 exactly account for this fact, and provide the expressions that ensure that this tension does not arises.

To better understand that interiority condition when $\mathbf{d} = \mathbf{0}$, we analyze in more depth the case of two agent economies. Given an economy

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$$

the values of δ_1 and δ_2 are $\delta_1 = 1 - b_{21}$ and $\delta_2 = 1 - b_{21}$. Given the regularity condition, this two values are positive. The conditions for interiority expressed in the previous corollary are in this case,

$$\begin{aligned} \frac{1 - b_{12}}{1 - b_{21}} &> b_{12} \\ \frac{1 - b_{21}}{1 - b_{12}} &> b_{21} \end{aligned}$$

The following reasoning helps understand when we lose interiority. Fix a value $b_{21} < 1$. If $b_{12} = b_{21}$ the conditions reduce to $1 > b_{12}$ and $1 > b_{21}$ which are trivially satisfied because of regularity. When we increase b_{12} the second condition is still satisfied since the left-hand side increases. But the left-hand side of the first condition increases while the right hand side of this same condition increases. If we increase b_{12} enough it is possible that this first condition is not satisfied for the parameters. It becomes too difficult to control for both conditions. Agent 2 exerts a larger influence on agent 1 than the influence agent 1 exerts on agent 2. If this difference is sufficiently asymmetric, and this asymmetry is measured by the two conditions above, then one of the agents receives all the resource, even if the economy is regular.

1.5 Nonparticipants and the Bargaining Outcome

Until now, we have considered that all agents in the economy are involved in the bargaining problem. However, our model also provides a framework to study what would happen if some agents in the economy do not participate in the bargaining problem but care for some of the agents that indeed participate.

The following example shows in a simple economy how we can use the tools we have developed so far to clarify the effect of agents that do not participate in the bargaining problem. It is very close to the example developed in Kalai (1977).

Example 4. Consider the economy with three agents represented by the following network

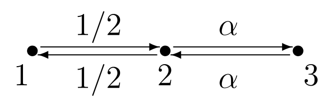


Figure 7

where α is a positive constant. Suppose only agents 1 and 2 are engaged in a bargaining problem. How does the introduction of agent 3 into the model[17] perturbs the bargaining outcome respect to the situation when we consider agent 1 and 2 in isolation? The solution to the bargaining problem without the presence of agent 3 is $\mathbf{c}^S = (1/2, 1/2)$, as we have previously described. If we take care of the existence of agent 3 we could proceed as follows.

The matrix of bilateral influences is

$$\mathbf{B} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix}$$

Hence, the matrix of network externalities when considering the 3 agents is

$$\mathbf{E}\left(\mathbf{B}\right) = \frac{4}{3 - 4\alpha^2} \begin{pmatrix} 1-\alpha^2 & 1/2 & \alpha/2 \\ 1/2 & 1 & \alpha \\ \alpha/2 & \alpha & 3/4 \end{pmatrix}$$

Each entry e_{ij} of this matrix represents the network externality magnitude agent j exerts on agent i if we consider the pattern of bilateral influences within all agents in the economy, included agent 3. Now, we could solve for the bargaining problem as we did previously in this section considering the submatrix $\mathbf{E}_{1,2}$ obtained eliminating from $\mathbf{E}\left(\mathbf{B}\right)$ the third row and column

$$\mathbf{E}_{1,2}\left(\mathbf{B}\right) = \frac{4}{3 - 4\alpha^2} \begin{pmatrix} 1-\alpha^2 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

17 In the social preferences interpretation we can interpret agent 3 as a relative of agent 2.

to characterize the Nash bargaining solution for the bargaining problem that involves only agent 1 and 2, but taking care of the initial levels of interdependence of the three agents in the economy. In particular the allocation that solves this problem is

$$\mathbf{c}^S = \left(\frac{1}{2} \cdot \frac{1 - 4\alpha^2}{1 - 2\alpha^2}, \frac{1}{2} \cdot \frac{1 - \alpha^2}{1 - 2\alpha^2} \right)$$

Observe the change in the pattern of consumption. If α is positive and sufficiently small then $c_1^S < 1/2$ and $c_2^S > 1/2$. Bilateral influences agents 2 and 3 exert on each other alters the bargaining power of agent 2 with respect to agent 1, and agent 2 obtains a larger share of the budget.

This methodology can be generalized to any number of agents. If the economy is formed by $n_1 + n_2$ agents and only the first n_1 of them are engaged into a bargaining problem, first we have to compute the matrix of network externalities $\mathbf{E}(\mathbf{B})$ for the economy as a whole, i.e. considering all the $n_1 + n_2$ agents. Then, we solve the bargaining problem using the submatrix formed by the first n_1 rows and columns, $\mathbf{E}_1(\mathbf{B})$, as if this last matrix was the matrix of network externalities of this (sub-)economy. In this way we internalize all the network effects generated by the structural influence pattern into the bargaining problem played by the first n_1 individuals, when considering all agents in the economy.

The next proposition provides a characterization of the matrix $\mathbf{E}_1(\mathbf{B})$ for such a situation. Let $\mathbf{B}_{11}$ be the matrix of bilateral influences across participants, $\mathbf{B}_{22}$ be the matrix of bilateral influences across nonparticipants, $\mathbf{B}_{12}$ be the matrix of bilateral influences from nonparticipants to participants, and $\mathbf{B}_{21}$ be the matrix of bilateral influences from participants to nonparticipants.

Proposition 5 *If $n = n_1 + n_2$, with $1 < n_1 < n$ being the number of members of the economy that participate in the bargaining game, then*

$$\mathbf{E}_1 = \left(\mathbf{I}_{n_1} - \mathbf{B}_{11} - \mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1} \mathbf{B}_{21}\right)^{-1}$$

This matrix has a natural interpretation. Observe that it is equivalent to the matrix of network externalities that we would obtain if the economy had only the n_1 agents that are involved in the distributive conflict and the matrix of bilateral influences were $\mathbf{B}_{11} + \mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1} \mathbf{B}_{21}$. Matrix $(\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1}$ equals the matrix of network externalities if only nonparticipants were in the economy, $\mathbf{E}(\mathbf{B}_{22})$. Hence, this new associated matrix of bilateral influences across participants accounts on the feedback effect, derived from influence exerted by participants on nonparticipants and viceversa, of network externalities exerted within the subeconomy formed by nonparticipants. From right to left the matrix

$$\mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1} \mathbf{B}_{21}$$

is obtained by first taking into account the direct bilateral influences participants exert on non-participants, then network effects among non-participants are computed, and finally we compute how these come back again to participants through direct influence exerted by non-participants on participants. the product of these three effects is added to the initial matrix of bilateral influences across participants to compute the solution to the Nash bargaining problem.

In our framework, if there were no influences, i.e. $b_{ij} = 0$ for all pairs ij, then the bargaining power of each agent would exactly coincide with the Nash bargaining solution share he receives. Once influences are introduced, any change in the levels of consumption of the Nash bargaining solution shares profile can be interpreted as a change in bargaining power due to the position in the network of bilateral influences. Here, network externalities, the aggregation of direct and indirect network effects, generates the pattern of bargaining power, if we want to interpret the allocation solution as the solution of an asymmetric bargaining problem without influences.

Kalai (1977) studies how non-participants in a Nash bargaining problem induce a change of bargaining power across participants, interpreting each non-participant as a replica of the participant for which this agent cares. In our case, each non-participant can care at the same time for different participants with varying intensities and we can provide the particular mapping from this interdependency structure to bargaining asymmetric outcomes.

1.6 α-economies.

In this section we study a family of networks with some particular properties.

Let $\alpha \in \mathbb{R}_+$. We say that an economy is an α-economy if whenever $b_{ij} \neq 0$, then $b_{ij} = b_{ji} = \alpha$. Hence, in an α-economy whenever there is a bilateral influence this influence is bidirectional and of same weight.[18]

In this family of economies the heterogeneity comes only from one source, the network geometry, and not from heterogeneous influence levels across pairs of agents. Therefore the analysis of this family of networks sheds some light on the isolated effect of the network geometry on the Nash bargaining outcome. In fact, as we show in the following lines, the characterization of the Nash bargaining solution becomes very transparent under some mild assumptions.

[18] Observe that in the family of α-economies $\mathbf{G} = \mathbf{B}$, since matrix $\mathbf{B}$ is symmetric.

Fixed α and an α-economy $\mathbf{B}$, we define the degree of agent i, that we denote by $deg_i(\mathbf{B})$, as

$$deg_i(\mathbf{B}) = \frac{1}{\alpha} \sum_{j \neq i} b_{ij} \tag{1.8}$$

The degree of an agent is a measure of connectivity. It equals the number of connections an agent has in the network of bilateral influences. Due to the symmetric nature of α-economies, the degree of an agent computes at the same time to how many people this agents exerts a direct influence, and from how many people this agent receives a direct influence.

Suppose that if agents do not agree in a division of the resource the disagreement outcome is that no division is implemented and hence agents receive a utility equal to 0, i.e. $\mathbf{d} = \mathbf{0}$. Then, under the regularity condition $1-(n-1)\alpha > 0$ that ensures that for a fixed α any α-economy is regular, we obtain the following characterization:

Proposition 6 *Let $\mathbf{B}$ be a regular α-economy. Then the Nash bargaining solution is interior and the utility each agent obtains is*

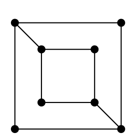
$$u_i^S = \frac{1}{n(1 - \alpha deg_i(\mathbf{B}))}$$

Hence,

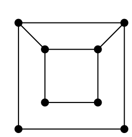
$$c_i^S = \frac{1}{n(1 - \alpha deg_i(\mathbf{B}))} - \sum_{j \neq i} b_{ij} \frac{1}{n(1 - \alpha deg_j(\mathbf{B}))}$$

Hence, for α-economies the degree of an agent is the unique relevant element from the network to determine the utility an agent obtains, whereas the share an agent obtains does not only depend on its own degree but also on the degrees of agents to which he is connected.

Example 5. Consider the following two networks.



$\mathbf{B}_a$



$\mathbf{B}_b$

In both networks there are four agents with three neighbours and four with two neighbours. However, if for example $\alpha = 0.1$, agents with three neighbours agents with three neighbours receive a larger share of the pie than in the second network, while the opposite applies for agents with two links. The following table provides the shares in both cases for the members of each class.

Nash Shares with $\alpha = 0.1$	$\mathbf{B}_a$	$\mathbf{B}_b$
Agents with 3 links	0.129	0.127
Agents with 2 links	0.121	0.123

This proves that the degree distribution is not a sufficient invariant to determine how the resource is distributed, it is also important the particular geometry of how agents are connected. This is expressed in the following corollary.

Corollary 2 *The share an agent obtains in the Nash bargaining solution increases with the number of neighbours he has and diminishes with the number of neighbours that his neighbours have.*

That is why we obtain different shares in the previous pair of networks. In the first one each agent that has three neighbours has one neighbour with three links and two with two links, while in the right one each agent with three links has two neighbours with three links and one with two. In accordance with this last corollary, this agents should obtain a smaller share in the network at the right, that is what we have observed before.

This last corollary provides in fact the main intuition on how the Nash bargaining solution internalizes influences. The share an agent receives depends on the aggregate level of bilateral influences provided at the local level. The more connected an agent is and the less connected his neighbours are, the more valuable is the share this agent receives for the spread of influence through the network. If, instead, his neighbours are also very connected, it is not necessary to give more to this agent. In this case other agents receive larger shares than before because they help more to spread the effect of influences all over the economy. The Nash bargaining solution with influences internalizes this indirect effects.

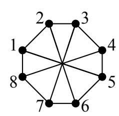
A particular case are degree-regular α-economies. In such economies all agents have the same degree. Here pattern does not matter. In fact, we have the following corollary.

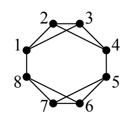
Corollary 3 *Let* $k \in \{0, \ldots, n-1\}$ *and let* $\mathbf{B}$ *be a regular* α*-economy such that* $deg_i(\mathbf{B}) = k$ *for all* $i \in \mathcal{N}$*. Then all agents obtain the same utility and consume the same quantity in the Nash bargaining solution. In particular* $c_i^S = 1/n$ *for all* $i \in \mathcal{N}$ *and*

$$u_i^S = \frac{1}{n(1-k\alpha)} \qquad \forall i \in \mathcal{N}$$

Therefore in the case of regular α-economies it does not matter how agents are connected but how many connections each agent has.

Example 6. The following two networks, with each link being bidirectional and with same weight α, differ in their particular geometry and lead to the same solution to the bargaining problem.





The network depicted in the right side has larger clustering[19] levels than the one in the left. This is not an issue in the determination of the Nash bargaining solution. This example shows that clustering plays no role in the resulting division that solves the distributional conflict under degree-regularity.

1.7 The Effect of Network Changes on Welfare and Consumption

In this section we explore how changes in the network of bilateral influences[20] translate into changes in welfare and consumption patterns. In particular, we center our attention in how a differential change on the weight of a link changes the utility and the level of consumption in equilibrium of each agent involved in the bargaining game. Hence, our results help to understand how changes in the magnitude of influences change the characteristics of the bargaining outcome.

For the sake of simplicity, we focus our attention in situations where the bargaining solution is interior, meaning that $c_i^S > 0$ and, hence, $u_i^S > u_i^{min}$ for every $i \in \mathcal{N}$.[21]

The following proposition provides conclusions on comparative statics related to the utility pattern of the Nash bargaining solution.[22]

Proposition 7 *Let $\boldsymbol{B}$ define an economy with influences such that the Nash bargaining solution is interior. Then:*

[19] Clustering measures if the agents to which an agent is connected are also connected within them.
[20] We make no distinction in defining the model in terms of networks or in terms of utilities. We refer to network changes because at some points this simplifies the necessary terminology.
[21] A more extensive analysis could be done to deal with corner solutions.
[22] If an economy is such that the Nash bargaining solution is interior, sufficiently small changes in the parameters of bilateral influences mantain interiority because of continuity. Hence, a comparative statics analysis in our setup is legitimated.

(i) $\frac{\partial u_i^S}{\partial b_{kl}} \geq 0$ *if* $l \neq i$*, with equality if and only if* $d_l = 0$.

(ii) $\frac{\partial u_i^S}{\partial b_{ki}} = sign\left(1 - \sum_{j\neq i} \delta_j d_j\right)$ *for* $k \neq i$.

The first part of the proposition states that the Nash bargaining solution utility of agent i generally increases when there is an increase on the magnitude of a bilateral influence across any two other agents, whoever these are. Increases on bilateral influences agents different than i exert on each other are beneficial for agent i. The second part of the proposition is a little bit more complex. It states that the increase on bilateral influences exerted by agent i are good for agent i if the term $\sum_{j\neq i} \delta_j d_j$ is sufficiently small (in fact, if it is smaller than one). Observe that this can happen either because the disagreement levels of agents different than i are small or because the levels of the δs are small for agents different than i. Agent j has a small level of δ_j whenever he exerts on the aggregate large positive bilateral influences on other agents. Hence, for agent i it is good to exert larger positive influences, if other agents exert large aggregate levels of bilateral influences. The intuition is that larger bilateral influences exerted by agent i can increase indirect network influences from i to himself (through cycles in the network of bilateral influences) if other agents exert sufficiently high bilateral influences as well. If not, and for example other agents exert some bilateral negative influences, the indirect network effects can be negative for agent i and imply a decrease on utility.

We move now to comparative statics results related to consumption patterns. These are provided in the following result.

Proposition 8 *Let* ***B*** *define an economy with influences such that the Nash bargaining solution is interior. If* $\mathbf{d} = \mathbf{0}$*,*[23] *then:*

(i) $\frac{\partial c_i^S}{\partial b_{ki}} > 0$ *if* $k \neq i$

(ii) $sign\left(\frac{\partial c_i^S}{\partial b_{kl}}\right) = -sign\left(b_{il}\right)$ *if* $k \neq i \neq l \neq k$

(iii) $\frac{\partial c_i^S}{\partial b_{ij}} > 0 \Leftrightarrow b_{ij} < -\delta_j$ *for* $j \neq i$

The first part of the proposition states a very simple and natural conclusion: agent i receives a larger share whenever the level of aggregate bilateral influences

[23] We consider this case since it is the more tractable one. In the proof the interested reader can find the exact expression of each one of the derivatives, no matter which disagreement point we consider.

he exerts increases.[24] This generates a positive effect on several other agents in the economy through the spread of bilateral influences through network effects. We could also interpret this result in terms of bargaining power: since he exerts a larger aggregate level of bilateral influences his bargaining power increases, and that is why he gets a larger fraction of the resource.

The second part of the proposition states that if agent i receives a positive (resp. negative) direct bilateral influence from agent l then an increase of the direct bilateral influence agent l exerts on another agent k, different than i and l, implies that agent's i share diminishes (resp.increases). This is reminiscent of the result of the first part of the proposition, and hence the same kind of intuition applies.

Finally, the third part expresses that an increase in the weight of the direct bilateral influence agent l exerts on i implies an increase in agent i share if and only if the initial weight of this bilateral externality was sufficiently negative.

We recover example $3.(b)$ of section 4 to illustrate graphically how small changes on the levels of bilateral influences translate into changes on the utility and shares derived from the Nash bargaining solution. This example is the 2-person economy such that $b_{12} = b_{21} = 1/2$. If we increase b_{12} from the initial $b_{12} = 1/2$ to $\tilde{b}_{12} = 1/2 + \epsilon$, where $\epsilon < 1/2$ to satisfy the regularity condition, the new matrix of bilateral influences is

$$\tilde{\mathbf{B}} = \begin{pmatrix} 0 & 1/2 + \epsilon \\ 1/2 & 0 \end{pmatrix}$$

and the new matrix of network externalities is

$$\tilde{\mathbf{E}}(\mathbf{B}) = \frac{4}{3 - 2\epsilon} \begin{pmatrix} 1 & 1/2 + \epsilon \\ 1/2 & 1 \end{pmatrix}$$

The utility possibility set and the Nash bargaining solution behave as follows (dashed lines represent the initial situation and continuous lines represent the

[24] And the rest of bilateral influences do not vary.

new one; u^S is the initial Nash bargaining solution and $\tilde{u}^S$ is the new one)

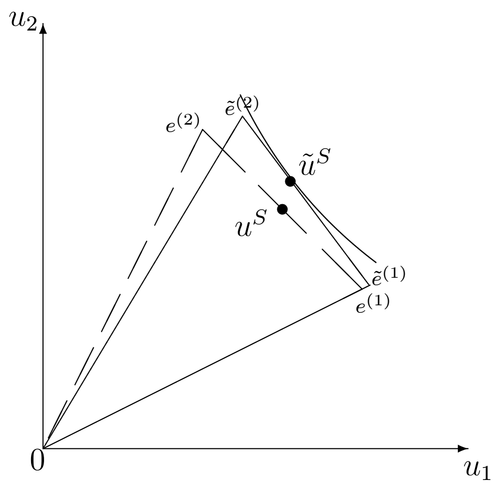


Figure 8

The increase on the level of altruism of agent 1 shifts the Pareto frontier upwards. Both agents obtain a larger utility in the new equilibrium of the bargaining game. This is consistent with the conclusions of proposition 7,[25] and now the division is no more half of the budget for each one. Indeed, the equilibrium point $\tilde{u}^S$ is closer to one of the extremes of the simplex than to the other. In particular it is closer to $\mathbf{e}^{(2)}$ which implies that c_2^S has increased while c_1^S has decreased. The increase of c_2^S is consistent with part (i) in proposition 8 while the decrease of $\mathbf{e}^{(1)}$ is consistent with part (iii), since $b_{12} = 1/2 > -1/4 = -\delta_2/2$.

1.8 Discussion

Until this point we have not discussed the suitability of the Nash bargaining solution in our setup. An initial point of debate is that with the use of this solution we abstract from the effect of coalitions when there are more than two agents. It could be possible that some agents decide to break relations with the rest of agents in the economy, and that this threat plays a role in the final division of resources. However, we do not think that in our setup this question plays a prominent role in the examples we have used to motivate our research.

Consider for example the example on urban crime. Except for the case in which the fear on crime is overwhelming, and citizens view the rest of problems

[25] Observe that the increase in u_1^S is the conclusion of part (i) while the increase of u_2^S is the conclusion of part (ii), since $\mathbf{d} = \mathbf{0}$.

associated with everyday urban life as secondary, it is difficult to imagine that this dispute on resources to fight against crime will lead to the division of the city. [26]

Similarly, in the social preferences example, we would not expect that members of a family would decide to break relations when natural daily distributional conflicts within the family arise.[27]

The omission of coalitions' role is more controversial in the case of government spending. If only the simple majority of members of the government have to agree on the division of the budget, as it is the case in the analysis provided in Baron and Ferejohn (1989), this has non-negligible strategic implications on the final agreement reached. It has been our aim in this work to analyze in detail how the effects of pairwise influences affects distributional conflict. Of course, by adding to this pairwise influences' pattern more institutional details, such as the majority voting rule in the case of government spending, we would obtain more accurate predictions of each particular example. [28]

Another consideration is the harmonization of the axioms that characterize the Nash bargaining solution and our setup with interdependent utilities. We provide here a discussion on this in terms of the alternative set of axioms proposed by Lensberg (1988). In particular, Lensberg shows that the Nash bargaining solution is the unique solution that satisfies *Pareto Efficiency*, *Anonymity* (if a utility possibility set is symmetric the solution is also symmetric) [29], *Scale Invariance* (when applying a linear transformation of utilities the solution changes accordingly to this linear transformation), and *Consistency* (if a subset of agents receive the utility they would receive in the solution, when applying the solution to the rest of the economy, the result is the same as if at the beginning we applied the solution to the economy as whole). [30]

Both *Pareto Efficiency* and *Anonymity* seem to be desirable properties of a bargaining solution. In our setup *Scale Invariance* is also desirable because when applying a linear transformation of utilities the preferences represented remain unperturbed.[31] Therefore, the axiom of *Scale Invariance* imposes that

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[26] Undoubtedly, there are cities in which this problem is real, and some neighbourhoods are introducing physical barriers to combat crime at private expenses. Is in these kind of situations in which our model would certainly not apply.

[27] But maybe it can be the case when dealing with a bequest.

[28] See Duggan(2004) for conditions about existence of equilibria in the n agents version of the Baron and Ferejohn game with externalities.

[29] More precisely, if $x \in UPS$ then $\sigma(x) \in UPS$ for any permutation σ of the entries of x.

[30] Nash's (1950) characterization substitutes Consistency by a probably more difficult to interpret axiom named Independence of Irrelevant Alternatives.

[31] This is because utilities in our model are additively separable with respect to consumption

these equivalent utility representations lead to the same solution. Hence, if we consider the three previous axioms as natural requirements of a solution, the unique axiom for the Nash bargaining solution that might deserve discussion is *Consistency*.

1.9 Conclusion

We have explored the outcome of the Nash bargaining problem with considering a simple model of interdependent behavior. Even if an economy is characterized by $n\,(n-1)$ variables, the model is tractable and we have been able to provide closed-form expressions for the bargaining outcome and comparative statics results. The network interpretation of the problem is helpful since it provides us with a set of tools that simplify the analysis and makes it more intuitive. It helps to understand the effect of heterogeneities in the model in all its dimensions, magnitude and pattern.

Part of the analysis in our work shows some similarities with previous work done by Kalai (1977).[32] Kalai interprets any agent that cares for a player in the bargaining problem but that do not participate in the bargaining problem, as a replica of this player. In our model, an agent can care for different players of the game, where this concern translates into influences as in the social preferences example in the introduction. The transmission of this concern is not done as a replication and its consequent change into the bargaining problem but through a pattern of different influences that affect players' behavior. In this sense, we allow for a more general pattern of interrelations and the transition is not done in a discrete manner, as replicas would do, but smoothly, since small changes in bilateral externality levels imply small changes in the levels of network effects.

Finally, our analysis borrows directly from the Nash bargaining solution. Different possible directions for further research are open. One possible direction could be to explore whether other cooperative solutions can be defined through some proper axioms adequate in a setting with heterogeneous influences such as the one developed in this work. Another possible direction is to go further in the study of non-cooperative bargaining models with an underlying structural pattern of bilateral influences. In particular, it might be valuable to study how the pattern of influences maps into equilibria of non-cooperative bargaining games that incorporate relevant features of particular applications, such as the voting rule in legislative bargaining, and how equilibria vary with respect to the case

levels.

[32] See also Lensberg and Thomson (1989) for some other work done with replicated agents in cooperative bargaining.

without influences.

1.10 Proofs

Proof of Proposition 1
The determinant of the matrix $\mathbf{I}-\mathbf{B}$ is a polynomial in $n\,(n-1)$ variables. The set of points of $\mathbb{R}^{n(n-1)}$ in which this polynomial vanishes forms an algebraic variety of dimension $n\,(n-1)-1$ at most, and hence it is a set with Lebesgue measure equal to zero in $\mathbb{R}^{n(n-1)}$. ■
Proof of Proposition 2
The following lemma is useful.

Given a regular economy $\mathbf{B}$, a feasible allocation $\mathbf{c}$ is Pareto efficient if and only if there exists a strict system of weights $\boldsymbol{\mu}$ such that $\boldsymbol{\mu}\cdot\mathbf{u}\,(\mathbf{c})\geq\boldsymbol{\mu}\cdot\bar{\mathbf{u}}$ for every $\bar{\mathbf{U}}\in \mathrm{UPS}\,(\mathbf{B})$.

Proof of Lemma 3 This is a slight variation of a well-known result relating Pareto efficiency to linear social welfare functions (see for example Proposition 16.E.2, pg.560, in Mas-Colell et al., 1995). The statement in terms of *strict* system of weights is valid because the shape of $\mathrm{UPS}\,(\mathbf{B})$ is a simplex, not simply a convex set. ■

Since there is no possibility of confusion we omit the dependence of $\mathbf{E}$ on $\mathbf{B}$. Observe that for any allocation $\mathbf{c}$, the vector of utilities is $\mathbf{u}\,(\mathbf{c})=\sum_{i=1}^{n}c_i\mathbf{e}^i$. If there exists a strict system of weights $\boldsymbol{\mu}$ and a strictly positive constant κ such that $\boldsymbol{\mu}\cdot\mathbf{e}^i=\kappa$ we have that for any allocation $\mathbf{c}$

$$\boldsymbol{\mu}\cdot\mathbf{u}\,(\mathbf{c})=\sum_{i=1}^{n}c_i\left(\boldsymbol{\mu}\cdot\mathbf{e}^{(i)}\right)=\kappa\sum_{i=1}^{n}c_i$$

Since $\kappa>0$, we have that $\boldsymbol{\mu}\cdot\mathbf{u}\,(\mathbf{c})$ is maximal whenever $\sum_{i=1}^{n}c_i=1$. Hence, any allocation such that $\sum_{i=1}^{n}c_i=1$ is Pareto efficient and the economy is regular.

Now, suppose any allocation such that $\sum_{i=1}^{n}c_i=1$ is Pareto efficient. Consider an interior allocation, i.e. such that $c_i>0$ for every $i\in\mathcal{N}$. The unique possible strict system of weights that can separate $\mathbf{u}\,(\mathbf{c})$ to the utility possibility set in the form of lemma 2 is the strict system of weights orthonormal to the hyperplane that contains the n columns of the matrix of network externalities. Obviously, this system of weights also separates $\mathbf{u}\,(\mathbf{c})$ to the utility possibility set when $c_i=0$ for some $i\in\mathcal{N}$. Hence, we have the unique candidate for the strict system of weights in the statement of proposition 2. From lemma 3 we know that in particular $\boldsymbol{\mu}\cdot\mathbf{u}\,(\mathbf{c})\geq\boldsymbol{\mu}\cdot\mathbf{u}\,(0)=0$. We can ensure that in fact this last inequality is strict since if it were equal to zero we would not be in a generic situation.[33]■

Proof of Proposition 3

From proposition 2 we know that there exists an strict system of weights $\boldsymbol{\mu}$ and a strictly positive constant such that $\boldsymbol{\mu} \cdot \mathbf{e}^{(i)} = \kappa$ for every $i \in \mathcal{N}$. In matrix terms this is equal to

$$\mathbf{E}^T(\mathbf{B}) \cdot \boldsymbol{\mu} = \kappa \mathbf{1}$$

where $\mathbf{E}^T(\mathbf{B})$ is the transpose matrix of $\mathbf{E}(\mathbf{B})$ and $\mathbf{1}$ is the n-dimensional vector with all entries equal to 1. Hence, we have that, since the inverse matrix of $\mathbf{E}^T(\mathbf{B})$ is equal to $(\mathbf{I}-\mathbf{B})^T$,

$$\boldsymbol{\mu} = \kappa\left((\mathbf{I}-\mathbf{B})^T \cdot \mathbf{1}\right)$$

Therefore, $\mu_i = \kappa\left(1-\sum_{j\neq i} b_{ji}\right)$. Since $\boldsymbol{\mu}$ is an strict system of weights, we have that

$$\kappa \sum_{i=1}^{n}\left(1-\sum_{j\neq i} b_{ji}\right) = 1$$

and hence

$$\kappa = \frac{1}{\sum_{i=1}^{n}\left(1-\sum_{j\neq i} b_{ji}\right)}$$

Let $\delta_i = 1-\sum_{j\neq i} b_{ji}$, and let $\delta = \sum_{i=1}^{n} \delta_i$. Then

$$\mu_i = \frac{\delta_i}{\delta}$$

Since $\boldsymbol{\mu}$ is an strict system of weights, all entries of $\boldsymbol{\mu}$ have to be strictly positive, and this can only happen if either all δ_i's are strictly positive or all δ_i's are strictly negative. However, in the latter case κ would be negative, since $\kappa = 1/(\delta)$. Hence to obtain a regular economy it is necessary that $\delta_i = 1-\sum_{j\neq i} b_{ji} > 0$ for every $i \in \mathcal{N}$.

The sufficiency result is almost immediate. Consider the weights and κ defined in Lemma 1 in the text. Then by construction these coefficients satisfy the regularity

[33] If it were equal to zero this would imply that the columns of the matrix of network externalities are linearly dependent, and hence that the determinant of $\mathbf{E}$ is equal to zero. This would mean that we were considering a non solvable system of bilateral influences.

condition in proposition 2. ■

Proof of Proposition 4
Let $\boldsymbol{\mu}$ and κ be the strict system of weights and constant from Proposition 2 associated to the economy. Let $\mathcal{J} \subseteq \mathcal{N}$ be the set of agents for which $d_j \geq u_j^{min}$. The Nash bargaining problem with network externalities is equal to

$$\max_{\mathbf{u} \in \mathrm{UPS}(\mathbf{B})} \sum_{i=1}^{n} \ln (u_i - d_i)$$

subject to

$$\sum_{i=1}^{n} \mu_i u_i = \kappa \tag{1.9}$$

$$u_i \geq d_i \qquad if\ i \in \mathcal{J} \tag{1.10}$$
$$u_i \geq u_i^{min} \qquad if\ i \notin \mathcal{J} \tag{1.11}$$

We know that the solution to this problem is unique. We denote this solution $\mathbf{u}^s$. Let $\bar{\psi}$ be the multiplier associated to restriction (1.9). The Kuhn-Tucker conditions of the problem are

$$\frac{1}{u_i^S - d_i} \leq \bar{\psi}\mu_i \qquad with\ equality\ if\ u_i^S > d_i \quad (i \in \mathcal{J}) \tag{1.12}$$

$$\frac{1}{u_i^S - d_i} \leq \bar{\psi}\mu_i \qquad with\ equality\ if\ u_i^S > u_i^{min} \quad (i \notin \mathcal{J}) \tag{1.13}$$

From (1.12) we obtain that for each $i \in \mathcal{J}$ we must have $\frac{1}{u_i^S - d_i} = \psi\mu_i$. If not, the value of the objective function in the solution would be $-\infty$. Hence, if, for simplicity, we denote $\psi = 1/\bar{\psi}$, we have

$$u_i^S = d_i + \psi \frac{1}{\mu_i} \qquad for\ every\ i \in \mathcal{J}$$

On the other hand, we obtain from (1.13) that, for every $i \notin \mathcal{J}$, u_i^S must satisfy

$$u_i^S = \max \left\{ u_i^{min}, d_i + \psi \frac{1}{\mu_i} \right\} \qquad for\ every\ i \notin \mathcal{J}$$

Observe in particular that, for every $i \notin \mathcal{J}$ it holds that $u_i^S = u_i^{min}$ if and only if $\mu_i \left(u_i^{min} - d_i \right) \geq \psi$. Using this fact, we proceed to provide and algorithm that at most in n steps provides the solution to the problem. As we stated in text, we suppose without loss of generality that $\mu_1\left(u_1^{min}-d_1\right)\geq\cdots\geq\mu_n\left(u_n^{min}-d_n\right)$

Step 0:
Suppose $u_i^S = d_i + \psi^{(0)}\frac{1}{\mu_i}$ for every $i\in\mathcal{N}$. The multiplier $\psi^{(0)}$ is equal to $\psi^{(0)} = \frac{1}{n}\left(\sum_{i=1}^n \mu_i\left(u_i^S-d_i\right)\right) = \frac{1}{n}\left(\kappa-\boldsymbol{\mu}\cdot\mathbf{d}\right)$. If $\mu_1\left(u_1^{min}-d_1\right) < \psi^{(0)}$, then $\psi = \psi^{(0)}$ and $\mathbf{u}^S$ is the utility vector associated to the Nash bargaining solution, and we are done. If not, go to step 1.

Step 1:
Suppose $u_1^S = u_1^{min}$ and $u_i^S = d_i+\psi^{(1)}\frac{1}{\mu_i}$ for every $i>1$. The multiplier $\psi^{(1)}$ is equal to $\psi^{(1)} = \frac{1}{n-1}\left(\sum_{i=2}^n \mu_i\left(u_i^S-d_i\right)\right) = \frac{1}{n-1}\left(\kappa-\boldsymbol{\mu}\cdot\mathbf{d}-\mu_1\left(u_1^S-d_1\right)\right)$. Observe that

$$(n-1)\psi^{(1)} = \kappa-\boldsymbol{\mu}\mathbf{d}-\mu_1\left(u_1^S-d_1\right)\leq n\psi^{(0)}-\psi^{(0)}$$

Hence, $\psi^{(1)}\leq\psi^{(0)}$, and therefore we know for sure that $\mu_1\left(u_1^{min}-d_1\right)\geq\psi^{(1)}$. If $\mu_2\left(u_2^{min}-d_2\right)<\psi^{(1)}$, then $\psi=\psi^{(1)}$ and $\mathbf{u}^S$ is the utility vector associated to the Nash bargaining solution, and we are done. If not, go to step 2.

Step k ($2\leq k<n$):

Suppose $u_i^S = u_i^{min}$ for $i\leq k$ and $u_i^S = d_i+\psi^{(k)}\frac{1}{\mu_i}$ for $i>k$. An analogous reasoning to the one in the previous step establishes that $\psi^{(k)}\leq\psi^{(k-1)}$. In fact

$$\begin{aligned}(n-k)\,\psi^{(k)} &= \kappa-\boldsymbol{\mu}\cdot\mathbf{d}-\sum_{l=1}^{k}\mu_l\left(u_l^S-d_l\right)\\ &\leq (n-k+1)\,\psi^{(k-1)}-\psi^{(k-1)} = (n-k)\,\psi^{(k-1)}\end{aligned}$$

The last inequality follows from the previous step of the procedure. If

$$\mu_{k+1}\left(u_{k+1}^{min}-d_{k+1}\right)<\psi^{(k)}$$

then $\mathbf{u}^S$ is the utility vector associated to the Nash bargaining solution, and we are done.

This process finishes at most in step $n-1$ since in this case we get that

$$\psi^{(n-1)} = \kappa - \boldsymbol{\mu} \cdot \mathbf{d} - \sum_{i=1}^{n-1} \mu_i \left(u_i^{min} - d_i \right) > \mu_n \left(u_n^{min} - d_n \right)$$

This last inequality follows from the fact that $\kappa = \boldsymbol{\mu} \cdot \mathbf{u}^S > \boldsymbol{\mu} \cdot \mathbf{u}^{min}$, since $\mathbf{u}^{min}$ can not be the total vector of utilities associated to an efficient allocation. Thus, if we arrive to step $n-1$, we can ensure that the utility vector associated to the Nash bargaining solution is $u_i^S = u_i^{min}$ for $i < n$ and $u_n^S = d_n + \psi^{(n-1)} \frac{1}{\mu_n} > u_n^{min}$.
■

Proof of Proposition 5
The matrix of bilateral influences is

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix}$$

The matrix of network externalities of the whole economy is $\mathbf{E}(\mathbf{B}) = (\mathbf{I} - \mathbf{B})^{-1}$ and we can decompose it as follows:

$$\mathbf{E}(\mathbf{B}) = (\mathbf{I}_n - \mathbf{B})^{-1} = \begin{pmatrix} \mathbf{E}_1 & \mathbf{E}_{12} \\ \mathbf{E}_{21} & \mathbf{E}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{n_1} - \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{I}_{n_2} - \mathbf{B}_{22} \end{pmatrix}^{-1}$$

To avoid misunderstandings, we omit the dependence on $\mathbf{B}$ for $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_{12}$ and $\mathbf{E}_{21}$. In particular, the following two conditions are satisfied:

$$\mathbf{E}_1 \cdot (\mathbf{I}_{n_1} - \mathbf{B}_{11}) + \mathbf{E}_{12} \cdot \mathbf{B}_{21} = \mathbf{I}_{n_1} \quad (1.14)$$
$$\mathbf{E}_1 \cdot \mathbf{B}_{12} + \mathbf{E}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22}) = \mathbf{0}_{n_1} \quad (1.15)$$

From the second condition, we obtain that

$$\mathbf{E}_{12} = -\mathbf{E}_1 \cdot \mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1}$$

Plugging this back into the first condition we obtain that

$$\mathbf{E}_1 \cdot \left(\mathbf{I}_{n_1} - \mathbf{B}_{11} - \mathbf{B}_{12} \cdot (\mathbf{I}_{n_2} - \mathbf{B}_{22})^{-1} \mathbf{B}_{21} \right) = \mathbf{I}_{n_1}$$

And the result follows.■

Proof of proposition 6

Fix α. The network in which the minimal utility of an agent is maximal is the complete network, where all pair of agents are connected. The matrix $\mathbf{E}$ for the complete network has entries $\frac{1-(n-2)\alpha}{(1+\alpha)(1-(n-1)\alpha)}$ in the diagonal and $\frac{\alpha}{(1+\alpha)(1-(n-1)\alpha)}$ outside the diagonal. The minimal utility an agent can obtain is the minimum of this two numbers, which coincides with the coefficient outside the diagonal, given the regularity assumption $1-(n-1)\,\alpha > 0$. Hence for any α-economy $\mathbf{B}$ we have that

$$u_i^{min}\left(\mathbf{B}\right) \leq \frac{\alpha}{\left(1+\alpha\right)\left(1-\left(n-1\right)\alpha\right)} \tag{1.16}$$

If $\mathbf{d} = \mathbf{0}$ we have that the condition to stop in the first step of the algorithm provided in the proof of Proposition 4 is

$$\left(1-deg_i\left(\mathbf{B}\right)\alpha\right)u_i^{min}\left(\mathbf{B}\right) \leq \frac{1}{n} \tag{1.17}$$

Given the regularity condition we know that $\alpha/\left(1+\alpha\right) < 1/n$ and therefore

$$\frac{\alpha}{\left(1+\alpha\right)\left(1-\left(n-1\right)\alpha\right)} < \frac{1}{n\left(1-deg_i\left(\mathbf{g}\right)\alpha\right)} \tag{1.18}$$

Hence the condition in the first step of the algorithm provided in the proof of Proposition 4 is satisfied, and we are done. ■

Proof of Proposition 7

We can rewrite the total utility an agent obtain in an interior solution as

$$u_i^s = d_i + \frac{1}{n\delta_i}\left(1-\sum_{j=1}^{n}\delta_j d_j\right) \tag{1.19}$$

If $i \neq j \neq k$, straightforward calculus yields to

$$\frac{\partial u_i^s}{\partial b_{kj}} = \frac{d_j}{n\delta_i} \tag{1.20}$$

and the result of the first part of the proposition follows, since in any regular economy $\delta_i > 0$ for all $i \in \mathcal{N}$.

If $i \neq j$ we have that

$$\frac{\partial u_i^s}{\partial b_{ji}} = \frac{1}{n\delta_i^2}\left(1-\sum_{k\neq i}\delta_k d_k\right) \tag{1.21}$$

Again, by the regularity condition, the result follows. ■

Proof of Proposition 8

When $\mathbf{d} = \mathbf{0}$ the share agent i obtains in the Nash bargaining solution when it is interior is

$$c_i^s = \frac{1}{n\delta_i} - \sum_{j \neq i} b_{ij} \frac{1}{n\delta_j} \tag{1.22}$$

Let $k \neq i$. If we differentiate the expression in (1.22) with respect to b_{ki} we obtain

$$\frac{\partial c_i^s}{\partial b_{ki}} = \frac{1}{n\delta_i^2} > 0$$

and the first part of the proposition follows. If i, j and k are pairwise different we have that

$$\frac{\partial c_i^s}{\partial b_{kj}} = -\frac{b_{ij}}{n\delta_j^2}$$

Hence, $\frac{\partial c_i^s}{\partial b_{kj}} b_{ij} \leq 0$ with equality if and only if $b_{ij} = 0$. Finally, if $i \neq j$ we have that

$$\frac{\partial c_i^s}{\partial b_{ij}} = -\frac{1}{n} \left(\frac{1}{\delta_j} + b_{ij} \frac{1}{\delta_j^2} \right)$$

Hence,

$$\frac{\partial c_i^s}{\partial b_{ij}} > 0 \Leftrightarrow \delta_j < -b_{ij}$$

■

Chapter 2
On Pairwise Influence Models: Networks and Efficiency

2.1 Introduction

There is an increasing concern within economists to introduce insights from other disciplines, such as psychology or sociology, into the realm of economics to provide a further understanding of problems that are economic in nature. The neoclassical model allows agents to be heterogeneous in tastes, each agent has her own utility function that can differ from the one of any other agent in the economy, but it neglects the social environment in which they are embedded. There are several dimensions in which our models can be enriched to provide a more general framework for economic analysis. On one side, the behavioral economics literature, and more particularly the social preferences one, escapes from the traditional homo economicus characterization of agents as completely selfish individuals and presents models where agents are concerned for others preferences, incorporating in this way features such as altruism, envy, fairness or reciprocity into the economics arena. On the other side, the literature on networks in economics enriches the underlying social structure by providing a clear topology of interrelations within agents in the economy and studying how this pattern of connections may change economic outcomes.

Our principal aim in this paper is to characterize efficient resource allocation in situations in which the members of a group can be altruists or envious with respect to other agents, and there is a well-specified network of connections that represents group structure. Hence, our work can be understood as the study of group influence on resource allocation in two different dimensions: psychological and structural. To this end, we use a model of linear $interdependent\ utilities$ where each agent has a private utility on consumption and a total utility that is decomposed as the sum of her private utility plus a weighted sum of the total utilities of the rest of agents with which she is connected. We do not make any kind of restriction neither on the weight each agent assigns to the total utility of another agent (weights can be positive, negative or zero, representing altruism, envy or indifference, respectively; and the weights for two different agents with which she may be connected can differ) nor in the shape of the network of relations (we allow for unreciprocated relations where agent i cares for agent j does not imply that agent j cares for agent i, so the network is directed).

Models with interdependent utilities generate implicit concerns through indirect connections on the network of relations: if Alice cares for Bob's utility and Bob cares for Carol's utility, Alice cares indirectly for Carol's utility, since her utility depends on Bob's utility that at the same time depends on Carol's utility. As particular examples of these indirect concerns we find recursivities: if Alice cares for Bob's utility and Bob cares for Alice's utility, Alice cares not only directly, but also indirectly, for her own utility. Of course, in a network with a substantial number of agents the number of possible indirect connections between two agents through the network can be very large, if not infinite. The first part of the paper is devoted to fully characterize the total effects of these recursivities. We provide closed-form expressions for the sum of all indirect effects, what we call *group influence*, for (almost)[34] any given network. An especially appealing property of these expressions is that they uniquely depend on simple cycles and simple paths, thus eliminating in some sense the recursivities that naturally arise in any utility interdependency setting.

The next step in our work is to characterize the Pareto efficient allocations for any given network of interdependencies. When all agents are completely selfish, the set of efficient allocations is easy to define: it is formed by all the allocations that exhausts resources, no matter how we distribute them within the group, and hence there is a plethora of efficient allocations. Whenever this situation happens for a network of interdependencies, we say that the Pareto frontier for this group is *regular*. One example of how the utility interdependency can distort the Pareto frontier from a regular situation is the following one: suppose that due to the concrete network structure and the individual levels of altruism and/or envy of each agent, and once we take into account group influence, that is, how the whole network of interdependencies may change the concern each one of the agents has for the rest of group members, there is one agent more loved than any other one, so all members of the group would like to transfer her all their resources because in this way they all would feel better off. In this situation the unique efficient allocation is that this more loved agent, the more prominent one, due to her position in the network of interdependencies, consumes everything. Another situation where the set of efficient allocations is a singleton arises when group influence generates a situation where everybody hates everybody. In this case nobody should consume nothing because the act of somebody consuming hurts all agents in the group, and we can interpret this as a situation where all agents are "nonprominent". Whenever there is a unique efficient allocation we say that the Pareto frontier is *singular*.

Our result characterizing Pareto efficiency relies on a well-known concept

[34] Except for a set of measure zero of networks.

from the social networks analysis literature, prestige, that measures the prominence of each agent due to her position in the interdependency network. We provide a particular way to measure the prestige of each agent that aggregates in one single value how much group structure influences this agent. We obtain that the more prestigious agents should be the ones that consume in an efficient situation. In particular, if all group members are equally prestigious we obtain a regular Pareto frontier, while if there is one specially prestigious agent or if everybody has negative prestige, we obtain a singular Pareto frontier, since in the first case this outstanding agent should consume everything while in the second case nobody should consume nothing. These represent two polar situations that show how extremely different the final outcome can be depending on the interdependency pattern but, indeed, there are many possible intermediate situations that we also characterize with the same methodology.

2.2 Related Literature

There is an increasing amount of experimental evidence that shows large and persistent deviations from pure selfish behavior, and that the social environment can have a large impact on economic problems. An interesting example of it is the work by Andreoni and Miller (2002), where the authors apply a revealed preference argument to experimental results and obtain that some of the agents had consistent preferences for altruism. Their conclusion is that all agents have self-interested utility functions, i.e. all agents are rational, but not all care only for their own payoff, but also for the payoff of the rest. Another example is Levine (1998), where the author studies a model with altruism and spitefulness similar to our one. Using principally results on ultimatum game experiments the author obtains a distribution of altruism in the population that works quite well in explaining other experimental results. The literature in behavioral economics has been specially imaginative in providing models that try to capture attitudes and behavioral patterns that can not easily be explained through the classical selfish rational choice models. Some examples are Rabin (1993), Levine (1998), Bolton and Ockenfels (2000), Fehr and Schmidt (1999), or Charness and Rabin (2002), just to name a few, and not including interdependent utilities models, on which we center later. A common feature of most of these models is that they are specially defined for strategic settings with two agents, and whenever the number of players is larger than two, all agents are concerned in the same way for the rest of group members. Two excellent surveys of the field of social preferences are Fehr and Schmidt (2002) and Sobel (2004).

The literature on interdependent utilities is vast. On the theory side we should highlight the work done by Bergstrom (1999) and Bramoullé (2001). Pollak

(1976) is among the first papers to provide a definition of interdependent preferences but Bergstrom (1999) is the first one rigorously defining what a system of interdependent utilities is, either for a finite or a denumerable number of agents, and devotes the rest of the paper to show that several intergenerational models that appeared previously are in fact systems of interdependent utilities. Bramoullé's work is more close in spirit to our one. The first part of his work addresses the study of some basic features of general interdependence systems not initially studied by Bergstrom. In the second part the author considers, as we do, linear interdependence systems to be able to relate social networks with interdependent utilities. He obtains some results, also relying in well-known concepts in sociology, on the relation between social structure and the induced interdependence coefficients, the coefficients that express how the private utility of an agent enters in the total utility of another one. These results are qualitative in nature and are restricted to a certain subclass of networks. On the contrary, we provide closed-form expressions of the induced interdependence coefficients and a full characterization of the quantitative effects in arbitrary social networks. We are able to obtain neat expressions for the induced interdependence coefficients for (almost) every network. Furthermore, we apply our model to obtain not only features of the interdependent utilities systems but also implications about fundamental economic concepts.

Moreover, there are many papers that use interdependent utilities in several areas of research in economics such as household economics, macroeconomics or intergenerational altruism models like Barro (1974), Becker (1974, 1981), Bergstrom (1989, 1999), Bernheim and Stark (1988), Bruce and Waldman (1991), Hori and Kanaya (1989), Kimball (1987) and Ray (1987), just to name a few[35]. However many of these works put restrictive conditions on the interdependency system to limit network externalities to the minimum, trying to eliminate in this way the recurrences and indirect links that could generate group influence. For example, the seminal work by Becker (1974) on social interactions is a system of interdependent utilities in which the network structure is a directed star. In this case there is no group influence because there is at most one path from an agent to another. This assumption simplifies the analysis but is very restrictive. Our analysis proves that a quantitative analysis can be done, at least in cases with a finite number of agents, even if the network structure is very complex.

Finally, we also partially rely in the very active literature on networks in economics. Recently, there has been an increasing attention on how non-market

[35] A more extensive covering of the literature on interdependent utilities can be found in Bergstrom(1999) and Bramoullé(2001).

interactions may take a prominent role in, for example, how people can obtain a job (Calvó-Armengol, 2004, Calvó-Armengol and Jackson, 2004), buyer-seller relations (Kranton and Minehart, 2001), strategic experimentation and public goods (Bramoullé and Kranton, 2005), how workers should be organized within a firm (Radner, 1993, Bolton and Dewatripont, 1994, Van Zandt, 1999), how people tries to coordinate to accept certain social norms and rules or to go on strike (Chwe, 2000, or Young, 2001), or in peer group effects and crime (Ballester et al., 2006). The main differential characteristic of the networks that arise in our work with respect to most of these literature is that they are both weighted and directed, while most of the times the networks that are studied in economic applications are unweighted and/or undirected.[36]

2.3 The Baseline Model and Some Examples

2.3.1 Linear Interdependent Utilities

Let $N = \{1, \ldots, n\}$ be the set of agents. Each agent has an increasing *private* utility function on his own consumption, $u_i(c_i)$. Furthermore, each agent has also a *total* utility function U_i of the form

$$U_i(c_1, \ldots, c_n) = u_i(c_i) + \sum_{j \neq i} b_{ij} U_j(c_1, \ldots, c_n) \qquad i = 1, \ldots, n$$

that is, each agent may take into account the total utility of each other agent, where b_{ij} is a real number that measures how much agent i cares for agent j. These n equations determine what we call the *interdependence system*.

The main characteristic in interdependent utilities models is that agents care for the total welfare of other agents. This model can be interpreted in several ways: agents connected by a network of social relations, members of different generations connected by a network of future and past concerns, or a network of different selves of the same individual with different concerns for selves at different periods of time. The interpretation for the first example is the following: if $b_{ij} > 0$ then agent i is altruist versus agent j while if $b_{ij} < 0$ agent i is spiteful against agent j. If $b_{ij} = 0$ agent i is indifferent to the situation of agent j.

The interdependence system can be *solved*, meaning that we can express the total utility of each agent in terms of the private utilities of each member of the group, easily. If $\mathbf{c} = (c_1, \ldots, c_n)$, $\mathbf{U}(\mathbf{c}) = (U_1(\mathbf{c}), \ldots, U_n(\mathbf{c}))$ and $\mathbf{u}(\mathbf{c}) = (u_1(c_1), \ldots, u_n(c_n))$, we can rewrite the interdependence system as

[36] Some exceptions are Rogers (2006) and Bloch and Dutta (2005).

follows

$$\mathbf{U}(\mathbf{c}) = \mathbf{u}(\mathbf{c}) + \mathbf{B} \cdot \mathbf{U}(\mathbf{c})$$

where $\mathbf{B}$ is the matrix $\mathbf{B} = (b_{ij})_{i,j}$ with zeros in the diagonal, that we call the *initial matrix* of interdependencies. Hence, the unique solution to the interdependence system is, when $(\mathbf{I} - \mathbf{B})^{-1}$ exists,

$$\mathbf{U}(\mathbf{c}) = (\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{u}(\mathbf{c})$$

Let $\mathbf{E} = (\mathbf{I} - \mathbf{B})^{-1}$. We call $\mathbf{E}$ the *group influence matrix*, and each entry e_{ij} expresses how much agent i cares for the *private* (instead of the total) utility of agent j.

Proposition 1 *The interdependence system has a unique solution generically.*

Proof: All proofs are in appendix A.

Hence, for almost every interdependence system, defined by matrix $\mathbf{B}$, we can obtain its group influence matrix $\mathbf{E}$. When this is possible we say that the interdependence system is *generic*.

2.3.2 Networks

A network is a set of agents, generally called *nodes*, $N = \{1, \dots, n\}$ and a set of links between them. These links are modeled as a set of values $g = \{g_{ij} \in \mathbf{R};\ i, j \in\ N \text{ and } i \neq j\}$. There is a link from agent i to agent j if and only if $g_{ij} \neq 0$.[37] There is a link from agent i to agent j if $g_{ij} \neq 0$. If the number of agents is n, a network is completely defined by these $n(n-1)$ coefficients in g. Hence, an equivalent tool to describe a network is its *adjacency matrix*, denoted by $\mathbf{B}(g)$, with entries $b_{ij} = g_{ij}$ if $i \neq j$ and zeros in the diagonal.

Since any system of interdependent utilities can be defined by one of such kind of matrix, we can reinterpret the interdependency matrix as the adjacency matrix of a network g that we call the *interdependency network*. This reinterpretation of a system of interdependent utilities in terms of networks is not simply a curiosity. An approach to a problem in terms of networks is useful whenever the variable(s) that define it are pairwise dependent, that is, they vary across pair of agents, and this is exactly our case, since the level of altruism/envy depends on

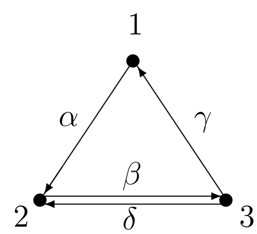
[37] This is not the more typical definition of a network found in most of the networks in economics literature. Generally, it is assumed that g_{ij} can only take value 0 or 1 (this kind of networks are generally called unweighted), and in many cases it is also assumed that $g_{ij} = g_{ji}$, that is, there is a link from agent i to agent j if and only if there is also a link from agent j to agent i (this kind of networks are called undirected).

the pair of agents we consider.[38] While the matrix $\mathbf{B}$ is our fundamental object in terms of defining a system of interdependent utilities, the network interpretation will prove very useful all around our work since it provides us with a set of tools and notions that will help in providing a more complete understanding of the kind of effects utility interdependencies generate.

We denote by $\mathcal{G}_n$ the set of all networks with n nodes. We can provide a graphical representation of any network $g \in \mathcal{G}_n$. The nodes are represented by circles, and there is a row from node i to node j if and only if $g_{ij} \neq 0$. For example, the graphical representation of the network $g \in \mathcal{G}_3$ that has adjacency matrix

$$\mathbf{B}(g) = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & \delta & 0 \end{pmatrix}$$

where α, β, γ and δ are different than zero, is

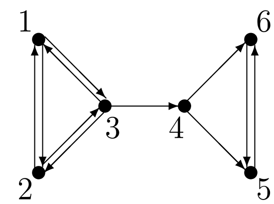


A $path$ from agent i to agent j is an ordered set of agents $p = (i, i_1, \ldots, i_k, j)$ of $\mathcal{N}$, where an agent can appear several times, such that $i \neq j$. We say that the path p belongs to the network g if $g_{ii_1} g_{i_1 i_2} \cdots g_{i_k j} \neq 0$. We say that a path is *simple* if all nodes of the path are different. In words, a path in g is an indirect connection from agent i to agent j through linked agents in g . We denote by $\mathcal{P}_{ij}(g)$ the set of *simple paths* from i to j in g. Given a path $p \in \mathcal{P}_{ij}(g)$ we define its weight, that we denote $w(p)$, as the product of weights of the links involved in the path: $w(p) = g_{ii_1} g_{i_1 i_2} \cdots g_{i_k j}$. A path that do not belong to g has weight equal zero.

Similarly, a *cycle* is defined like a path but with the simple difference that in this case $i = j$. Hence, a cycle is an indirect connection from agent i to himself through a chain of linked agents in g. We say that a cycle is a *simple*

[38] And on the order we consider these pair of agents; that is why networks we consider are generally directed.

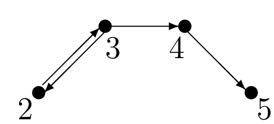
cycle if the only agent that appears twice in the cycle is the initial-final agent. Observe that a simple path is a path that does not contain any cycle. $C(g)$ denotes the set of simple cycles in g. Like with paths, we define the weight of a cycle $c = (i, i_1, \ldots, i_k, i) \in C(g)$, denoted by $w(c)$, as the product of weights of the links involved in the cycle, that is, $w(c) = g_{ii_1} g_{i_1 i_2} \cdots g_{i_k i}$. We say that a cycle c belongs or is in network g if its weight is different than 0.
For example, consider the following network g



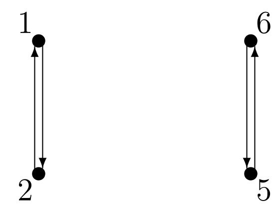
$(1, 2, 3, 4)$ is a simple path in g from 1 to 4, $(1, 2, 1, 2, 3, 4)$ is a path in g that is not simple, and $(5, 4)$ is a path that do not belong to g. Similarly, $(1, 2, 3, 1)$ is simple cycle in g, $(1, 2, 1, 2, 3, 1)$ is a cycle in g that is not simple, and $(3, 4, 3)$ is a cycle that do not belong to g.

Given a network g and a subset $S \subset N$, we denote by $g \backslash S$ the network that is obtained eliminating from g all the nodes in S and all the links that involve a node in S, and this new network $g \backslash S$ has as adjacency matrix the principal submatrix of $\mathbf{B}(g)$ obtained by eliminating the rows and columns indexed by S. In particular, given a simple path $p \in P_{ij}(g)$ (resp. cycle $c \in C(g)$), $\{p\}$ (resp. $\{c\}$) denotes the set of nodes involved in this path (resp. cycle), and $g \backslash \{p\}$ (resp. $g \backslash \{c\}$) is the network obtained eliminating this set of nodes.

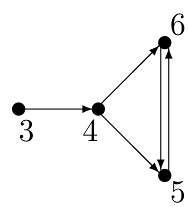
For example, if we consider the previous network g in figure 2, the network obtained by eliminating the nodes 1 and 6, $g \backslash \{1, 6\}$, is



the network obtained by eliminating the path $p = (3, 4)$ is



and the network obtained by eliminating the cycle $(1, 2, 1)$ is



To provide the characterization of group influence in a later section we only need one more ingredient. Any permutation of n elements, $\sigma \in S_n$, decomposes the set of n agents into different cycles in a unique way, and we denote this set of cycles $c(\sigma)$. The cardinality of the set $c(\sigma)$ is denoted by $\sharp\sigma$. Given any $\sigma \in S_n$ we define its weight, $w(\sigma)$, as the product of weights of the cycles in which σ decomposes the network, i.e. $w(\sigma) = \prod_{c \in c(\sigma)} w(c)$. All this is explained with greater detail in the last section of this chapter.

2.4 The General Case: Group Influence

In some cases[39] we can express $\mathbf{E} = (\mathbf{I} - \mathbf{B})^{-1}$ as

$$(\mathbf{I} - \mathbf{B})^{-1} = \sum_{k \geq 0} \mathbf{B}^k = \mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + \cdots \tag{2.1}$$

Each matrix $\mathbf{B}^k$ computes the indirect effects of order k, that is, the entry $\mathbf{B}^k_{ij}$ is equal to the sum of weights of all paths of length k that connect agent i to agent j.[40] Hence, the entry $\mathbf{E}_{ij}$ is equal to the *sum of weights of all the paths of any length* that connect i to j. This sum of indirect effects of any order is

[39] Whenever the infinite sum of powers of $\mathbf{B}$ expressed below is a well-defined matrix, this matrix is the inverse of $\mathbf{B}$. In fact, the necessary and sufficient condition for (4) to hold is that $\mathbf{B}$ has to be a contraction, or, what is equivalent, that all the eigenvalues of $\mathbf{B}$ have absolute value smaller than 1.

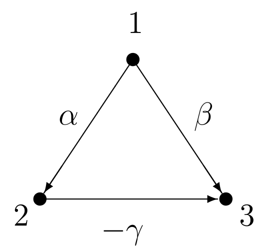
[40] For a proof of this assertion, see Bramoullé(2001).

what we denote *group influence*. Our aim in this section is to completely characterize, for any interdependency network, group influence. We begin presenting two examples that provide an sketch of the forces that interdependence systems generate.

2.4.1 Understanding Group Influence

2.4.1.1 Paths can generate Sentimental Contradictions

There are three agents, $N = \{1, 2, 3\}$, and the network g relating them is the following



where α, β and γ are strictly positive parameters. This means that agent 1 loves both agent 2 and agent 3, while agent 2 hates agent 3. Hence, the interdependence system is

$$\begin{aligned} U_1(\mathbf{c}) &= u(c_1) + \alpha U_2(\mathbf{c}) + \beta U_3(\mathbf{c}) \\ U_2(\mathbf{c}) &= u(c_2) - \gamma U_3(\mathbf{c}) \\ U_3(\mathbf{c}) &= u(c_3) \end{aligned}$$

and the initial interdependence matrix $\mathbf{B} = \mathbf{B}(g)$ is

$$\mathbf{B} = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & -\gamma \\ 0 & 0 & 0 \end{pmatrix}$$

In this case we can easily compute the matrices of indirect effects, $\mathbf{B}^k$ for $k \geq 2$. The matrix of second order effects is

$$\mathbf{B}^2 = \begin{pmatrix} 0 & 0 & -\alpha\gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

while

$$\mathbf{B}^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad for\ every\ k \geq 3$$

We obtain that the induced interdependence matrix $\mathbf{E} = \mathbf{E}(g)$ is

$$\mathbf{E} = \mathbf{I} + \mathbf{B} + \mathbf{B}^2 = \begin{pmatrix} 1 & \alpha & \beta - \alpha\gamma \\ 0 & 1 & -\gamma \\ 0 & 0 & 1 \end{pmatrix}$$

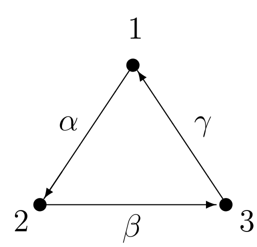
Hence, the expressions of the total utilities in terms of the private ones are

$$\begin{aligned} U_1(\mathbf{c}) &= u_1(c_1) + \alpha u_2(c_2) + (\beta - \alpha\gamma) u_3(c_3) \\ U_2(\mathbf{c}) &= u_2(c_2) - \gamma u_3(c_3) \\ U_3(\mathbf{c}) &= u_3(c_3) \end{aligned}$$

Observe that the induced coefficient of agent 1 related to the utility of agent 3, $e_{13} = \beta - \alpha\gamma$, depends on the intensities of the interdependence system. If $\alpha\gamma > \beta$ the coefficient is negative, even if in the interdependence system $b_{13} = \beta$ was positive. A tension arises between the affection agent 1 has for the two other agents and the hate that agent 2 shows for agent 3: the hate of agent 2 is internalized in the interdependence system and, if agent 1 cares much more for agent 2 than for agent 3, it makes it possible that finally e_{13} be negative.

2.4.1.2 Cycles can generate Sentimental Reinforcement

CASE 1: There are three agents $N = \{1, 2, 3\}$, and the initial situation is the following one



where $\alpha, \beta, \gamma \in (0, 1)$. This means that agent 1 loves agent 2, agent 2 loves agent 3, and agent 3 loves agent 1. How does this virtuous cycle affects the induced interdependence coefficients? Take for example agent 1. e_{11} expresses all the direct and indirect effects that start and finish in agent 1. The unique possibility of indirectly arriving to agent 1 if we start in agent 1 is through the cycle $(1, 2, 3, 1)$, and the weight of this cycle is $g_{12}g_{23}g_{31} = \alpha\beta\gamma$. Hence the sum of indirect effects from agent 1 to himself is

$$\alpha\beta\gamma + (\alpha\beta\gamma)^2 + (\alpha\beta\gamma)^3 + \cdots$$

that is, the sum of the indirect effects of passing one time through the cycle, two times through the cycle, three times, etc. Therefore, e_{11}, which is equal to the sum of the direct effect[41] equal to 1 plus the indirect effects, is

$$e_{11} = 1 + \sum_{k=1}^{\infty} (\alpha\beta\gamma)^k = \frac{1}{1 - \alpha\beta\gamma} > 1$$

The cycle provides a reinforcement of the concern agent 1 has for himself, since, initially loved himself with coefficient 1, but, finally, his private utility enters into his total utility with coefficient $e_{11} > 1$. Obviously, given the symmetry of the network, the same happens with the rest of agents.

But this cycle not only reinforces the concern an agent has for himself. Initially, agent 1 loves agent 2 with coefficient equal to $g_{12} = \alpha$, but the same cycle $c = (1, 2, 3, 1)$ generates indirect effects from agent 1 to agent 2: agent 1 can arrive directly to agent 2 through the link 1→2, but also indirectly through the paths (1,2,3,1,2), (1,2,3,1,2,3,1,2), etc. The weight of these paths is $\alpha\beta\gamma\alpha = \alpha(\alpha\beta\gamma)$, $\alpha\beta\gamma\alpha\beta\gamma\alpha = \alpha(\alpha\beta\gamma)^2$, etc. Hence the sum of direct and indirect effects from agent 1 to agent 2, e_{12}, is equal to

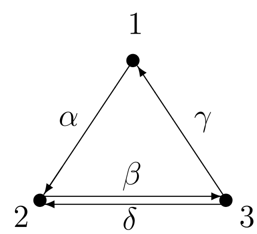
$$e_{12} = \alpha \sum_{k=0}^{\infty} (\alpha\beta\gamma)^k = \frac{\alpha}{1 - \alpha\beta\gamma} > \alpha$$

The cycle not only provides self-reinforcement, it also reinforces the links between agents in the network. Later we will see that this is a general fact: part of the effect cycles generate is global and homogeneous, it affects in the same way all the relations in the network, reinforcing or weakening all of them in the same way.

CASE 2: The situation is the same than in the previous case but now there is

[41] In the system of interdependent utilities everybody loves himself with coefficient 1.

a new link from agent 3 to agent 2, with positive weight equal to δ. Graphically,



To the previous existing relations, we add a new one: now the love agent 2 showed for agent 3 is reciprocal, and agent 3 loves agent 2 with an intensity equal to δ. Does this friendship relation between agent 2 and agent 3 can benefit in any way agent 1? As we will see in just a moment, the answer is yes. This proves that even cycles where an agent is not involved can be beneficial for him. The network structure generates externalities effects: how the rest of people is connected affects the final concerns of an agent. This effects are not always positive. For example, if δ had been negative, the hate agent 3 would profess for agent 2 would indirectly hurt agent 1.

Let's see how this new friendship can benefit agent 1's concern for himself. The reason is clear: this new friendship generates new indirect effects from agent 1 to himself. To the indirect effects derived from the cycle $(1, 2, 3, 1)$ we have to add the ones that involve the new cycle $(2, 3, 2)$. For example there is a new indirect effect generated by $(1, 2, 3, 2, 3, 1)$, that has weight equal $\alpha\beta\delta\beta\gamma > 0$. As a matter of fact, it is obvious that all the new indirect effects have positive weight, since all the initial coefficients, α, β, γ and δ, are positive.

In the following subsection we obtain expressions for the induced coefficients for any generic network. The expression we obtain for e_{11} is in this case equal to

$$e_{11} = \frac{1 - \beta\delta}{1 - \alpha\beta\gamma - \beta\delta}$$

Observe that $e_{11} = \frac{1-\beta\delta}{1-\alpha\beta\gamma-\beta\delta} > \frac{1}{1-\alpha\beta\gamma}$ whenever $1 - \alpha\beta\gamma - \beta\delta > 0$, but that $e_{11} = \frac{1-\beta\delta}{1-\alpha\beta\gamma-\beta\delta} < 0$ if $1 - \alpha\beta\gamma - \beta\delta < 0$. Hence, we are confronted with two opposite situations: if the sum of weights of the cycles $(1, 2, 3, 1)$ and $(2, 3, 2)$ is smaller than one the situation described in this case is still better than the one described in case 1, and hence the new friendship formed is beneficial for agent 1;

however, if the sum of weights is larger than one, the situation becomes dramatic. The excessive love intensities generate pernicious effects: agent 1's concern for himself becomes negative!

2.4.2 A Generic Characterization of Group Influence

We provide in this section a characterization of the group influence matrix $\mathbf{E}(g)$ for any generic interdependency network. To this purpose first we have to define a set of aggregation values for any network g. Let

$$\Lambda(g) = \left[\sum_{\sigma \in S_n} (-1)^{\sharp\sigma} W(\sigma) \right]^{-1}$$

and let

$$\Gamma_{ij}(g) = \sum_{p \in P_{ij}} w(p) \frac{1}{\Lambda(g \backslash \{p\})} \quad for\ every\ i, j \in N$$

In particular, $\Gamma_{ii} = \frac{1}{\Lambda(g \backslash \{i\})}$. Observe that this $n^2 + 1$ values only depend on simple cycles and simple paths of the network g. This set of values is the tool we need to completely characterize group influence.

Theorem 1 *Let g be a generic interdependency network. The entries e_{ij} of $\mathbf{E}(g)$ are equal to*

$$e_{ij}(g) = \Lambda(g) \Gamma_{ij}(g) \qquad for\ every\ i, j \in N$$

This result is specially appealing because it provides a structural characterization of peer effects for any generic interdependence system. It provides closed-form expressions of how finally each individual cares for the private utility of any other agent expressed in terms of the weights of simple paths and simple cycles of the associated network structure. We can almost *see* which are the effects generated by the interdependency network.

As stated, each entry $e_{ij}(g)$ is decomposed as the product of two terms, $\Lambda(g)$ and $\Gamma_{ij}(g)$. the first term is common in all cases while the second depends on the pair of agents we consider. We can interpret this decomposition as an expression of two different forces that play a role in the determination of group influence, and that are very related to the two previous examples. In what follows we explain how.

2.4.2.1 The network multiplier

We call $\Lambda(g)$ the network multiplier. It is a global and homogeneous effect generated by the network structure. It only depends on the simple cycles in the network. In fact, *the inverse of* $\Lambda(g)$ can be rewritten as

$$1-\sum_{c_1\in C(g)} w(c_1)+\sum_{\substack{c_1,c_2\in C(g)\\ c_1,c_2\ disj.}} w(c_1)\,w(c_2)-\sum_{\substack{c_1,c_2,c_3\in C(g)\\ c_1,c_2,c_3\ disj.}} w(c_1)\,w(c_2)\,w(c_3)+\cdots$$

This network multiplier is a pure reinforcement/weakening effect that affects in the same way all the agents in the network. It subsumes the kind of reinforcement or weakening effect that cycles generate and that are exemplified in subsection .

The explanation why in the expression of $e(g)=1/\Lambda(g)$ they appear not only the weights of the cycles but also the product of combinations of cycles with different signs, is that cycles have not a pure effect. The effects of different cycles are entangled and thus a correction term has to be introduced to the pure effects expression $1-\sum_{c_1\in C(g)} w(c_1)$.[42]

2.4.2.2 Local externalities

We call $\Gamma_{ij}(g)$ the *local externality* generated by the interdependency network in the concern agent i has for agent j because they take into account not only the direct concern agent i can have with agent j but also all the simple indirect concerns of agent i with agent j. That is why $\Gamma_{ij}(g)$ depends on the simple paths that go from agent i to agent j. Local externalities, in contrast with the network multiplier, are mainly local and heterogeneous effects generated by the interdependency network.

Observe that $\Gamma_{ii}(g)$ is equal to the inverse of the network multiplier of the subnetwork of g obtained eliminating agent i. Therefore the group influence coefficient of agent i with respect to himself, e_{ii}, is equal to the ratio $\Lambda(g)/\Lambda(g\backslash\{i\})$. There are several possibilities. For example, if both multipliers are positive and $\Lambda(g)>\Lambda(g\backslash\{i\})$ then the social structure reinforces the concern agent i has for himself, $e_{ii}>1$. This comparison between network multipliers can be understood as checking how valuable is agent i for the rest of the group. A large network multiplier of the whole network compared to the network multiplier associated to the network $g\backslash\{i\}$ means that agent i, specially the way she is connected to other agents, exerts a positive externality on the rest of the group, because if we eliminate her from the network the network multiplier decreases.

Similarly, the induced coefficient of interdependence of agent i with respect

[42] To have an analogy, it is much like what happens with the probability of intersection of events in probability theory.

to agent j if $i \neq j$ is

$$e_{ij}(g) = \sum_{p \in P_{ij}(g)} w(p) \frac{\Lambda(g)}{\Lambda(g \backslash \{p\})}$$

It depends on the direct paths from i to j, but the effect of each these paths is not only its weight. The weight of a path p can be reinforced if the agents that belong to it are valuable, where valuable has a similar meaning than in the previous discussion. For example, if $\Lambda(g)$ is larger than $\Lambda(g \backslash \{p\})$, the social network multiplier of the network obtained by removing the agents on path p, and both multipliers are positive, the effect of path p in the coefficient of group influence e_{ij} its intensified since it is larger than its weight. This happens because the connections of the agents that belong to the path are important for the reinforcement effects generated by the whole network structure, and this is measured through the network multipliers.

2.5 The General Case: Efficiency, A Matter of Prestige

From now on, the setting is the following one: there is a quantity π of a certain good that has to be distributed within a group of n agents, and we assume that the private utility over consumption of each agent is $u_i(c_i) = c_i$. We assume that private returns to direct consumption are linear because we don't want that possible concavities of these functions interfere with the effects that are generated by the network structure, that are the effects in which we want to concentrate in this work. Latter on I will provide some hints of the consequences of the introduction of concavities in the private return functions $u_i(c_i)$.

The question we want to address in this section is how does the network structure determines efficiency in this setting. Hence, we want to determine which allocations are Pareto efficient when we have to distribute the budget π of the good within n agents that are linked by a interdependency network. Our characterization relies on a well-known concept in the literature of social network analysis, *prestige*. The prestige of an agent embedded in a network of relations provides a measure of her relevance for the rest of agents given his position in the network structure of the group.

2.5.1 Prestige in Social and Economic Networks

Not all the agents in a social network are necessarily equally important, whatever important may mean. There are several variables that can determine the importance or prominence of an actor in the network. Furthermore, the definition of prominence may depend on the setting we are studying. It is not the same if we deal with directed or undirected networks, or with weighted or unweighted networks. Hence, there is not in the social networks analysis literature a unique standard definition of prominence.

Sociologists have defined importance or prominence in networks mainly through two different concepts: $centrality$ and $prestige$. Both concepts are related to $connectivity$: centrality of an agent is related to the paths that start in this agent, while prestige is related to the paths that finish in this agent. It depends on the particular setting, but, roughly speaking, centrality is related to what you give and prestige is related to what you receive. For example, an agent can be important in the structure of a firm because he can transmit easily information to many other workers; in this case the relevant notion to study would be centrality. However, in our case an agent is important depending on how much he is loved, that is, on how much love or affection he receives. Hence prestige is the relevant concept we have to deal with. There is a huge literature in sociology about prominence in networks and several centrality and prestige measures have been defined.[43] Our work is not the first economic model relying in concepts related to prominence in social networks. For an example of an application of network theory to economics in which centrality plays a key role see Ballester et al.(2004).

As we have previously stated, in our model an agent is more important than another one if he is more loved than the other: the private utilities of agents j_1 and j_2 enter into the total utility of agent i with coefficients e_{ij_1} and e_{ij_2} respectively; given there does not exist any concavities , agent j_1 is more important than agent j_2 for agent i if and only if $e_{ij_1} > e_{ij_2}$. Each column contains in its entries the basic information about prestige of each agent. To obtain an aggregate value of prestige we do it in the simplest possible way as a weighted sum of the entries of each column. First, we define what is an strict system of weights.

Definition 1 *An strict system of weights is a vector* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ *such that* $\mu_i > 0$ *for all* $i \in \mathcal{N}$ *and* $\sum\limits_{i \in \mathcal{N}} \mu_i = 1$.[44]

Given a network g and an strict system of weights $\boldsymbol{\mu}$, we define the centrality of agent i, that we denote $\kappa_i(\boldsymbol{\mu}; \mathbf{g})$ as $\kappa_i(\boldsymbol{\mu}; \mathbf{g}) = \boldsymbol{\mu} \cdot \mathbf{e}^{(i)}$.

2.5.2 Pareto Efficiency

The following theorem provides a complete characterization of the Pareto frontier in an interdependent utilities setting given any interdependency network.

Theorem 2 *A vector of consumption* $\mathbf{c} = (c_1, \ldots, c_n)$ *is Pareto efficient*

43 See chapter 5 in Wasserman and Faust(1994) for a survey of this literature and references.

44 For amore detailed explanation about this centrality measure of prestige, see section 3 of chapter one.

(i) when $\sum_{i=1}^{n} c_i = \pi$*, if and only if there exists a strict system of weights* $\boldsymbol{\mu}$ *and a constant* $\kappa > 0$ *such that* $\kappa_i(\boldsymbol{\mu}; g) = \kappa$ *if* $c_i > 0$ *and* $\kappa_i(\boldsymbol{\mu}; g) \leq \kappa$ *if* $c_i = 0$

(ii) when $\sum_{i=1}^{n} c_i < \pi$*, if and only if there exists a strict system of weights* $\boldsymbol{\mu}$ *such that* $\kappa_i(\boldsymbol{\mu}; g) = 0$ *if* $c_i > 0$ *and* $\kappa_i(\boldsymbol{\mu}; g) \leq 0$ *if* $c_i = 0$.

Hence, the measure of prominence of the agents in the network obtained through prestige measures allows us to provide a general characterization of the Pareto efficient allocations for any system of interdependent utilities. In particular two polar situations are included in the characterization: the Pareto frontier is regular if and only if there exists a strict system of weights $\boldsymbol{\mu}$ and a positive constant κ for which $\kappa_i(\boldsymbol{\mu}; g) = \kappa$ for every $i \in N$; the Pareto frontier is singular with agent j consuming everything if for every strict system of weights $\boldsymbol{\mu}$ there exists a positive constant κ for which $\kappa_j(\boldsymbol{\mu}; g) = \kappa$ and $\kappa_i(\boldsymbol{\mu}; g) < \kappa$ for any other $i \neq j$, and the Pareto frontier is singular and nobody should consume if for every strict system of weights $\boldsymbol{\mu}$, $\kappa_i(\boldsymbol{\mu}; g) < 0$. We are in a regular situation if there is homogeneity in prestige, while the Pareto frontier is singular if either there is a privileged agent or everybody is negatively prestigious, no matter which strict system of weights we use to compute prestige.

Corollary 1 (Singular Pareto frontier) *The Pareto frontier is singular with nobody consuming if and only if* $\mathbf{E}(g)$ *is a negative matrix.*

Proposition 2 (Regular Pareto frontier) *The Pareto frontier is regular if and only if* $\sum_{j=1, j\neq i}^{n} b_{ji} < 1$ *for every* $i \in N$.

Each agent puts an initial weight of 1 to her private utility in the expression of her total utility. $\sum_{j=1, j\neq i}^{n} b_{ji}$ is the sum of concerns of the rest of agents in the group with agent i. To obtain a regular Pareto frontier nobody can be excessively loved, meaning that this sum of concerns with each agent i can not exceed the own concern of agent i with herself.

It is interesting to note that the result of the first corollary, on singular Pareto frontiers, is valid for a more general family of private utility functions $u_i(c_i)$ than $u_i(c_i) = c_i$. As far as $u_i(c_i)$ is an increasing function of c_i, the result still holds.

2.6 A Simple Case: α-homogeneous Networks

Let α be a *positive* real number. We define the set of α-homogeneous networks

$Hom(\alpha)$ as the set of networks such that whenever $b_{ij} \neq 0$ then $b_{ij} = b_{ji} = \alpha$. Hence, an α-homogeneous network is undirected, because if there exists a link from agent i to agent j there also exists a link from agent j to agent i, and the weight of all the network links is positive and equal to α.

In this family of networks both levels of heterogeneity are present but controlled: the network of interdependencies is undirected and the level of altruism is homogenized to a single value. This larger control on the structure of interdependencies provides a simple picture of some of the more general results we have obtained. Furthermore, it includes classical networks in the literature, such as the circle and star, that are stylized examples of different kinds of possible social structures.

Let $deg_i(g) = |\{j \in N \setminus \{i\} \ s.t. \ b_{ij} \neq 0\}|$ for each $i \in N$. We call $deg_i(g)$ the *degree* of agent i, and it measures the number of agents with whom agent i is linked. We denote by $deg_{min}(g) = \min_{k \in N} deg_k(g)$ and $deg_{max}(g) = \max_{k \in N} deg_k(g)$ the *minimal* and *maximal* network degrees, respectively.

Again, suppose there is a quantity π of available resources that the group can share. The following result characterizes, given a certain α-homogeneous network of interdependencies, the values of α for which we obtain a *regular* Pareto frontier. Just to remember, this means that the set of efficient allocations is formed by all the allocations that exhaust resources and that these resources are distributed anyway within the members of the group.

Proposition 3 *Let $g \in Hom(\alpha)$. The Pareto frontier is regular if and only if $\alpha < \frac{1}{deg_{max}(g)}$.*

Hence, connectivity, evaluated here as the number of connections each agent has, provides us the criteria to evaluate if an α-homogeneous network has a regular Pareto frontier. The next result states an analogous result for a singular Pareto frontier.

In this case this result is not a complete characterization of singularity. It provides a region of the α-space for which we can ensure singularity but it does not characterizes all the α's for which the Pareto frontier is singular with nobody consuming.

Proposition 4 *Let $g \in Hom(\alpha)$ be connected and let $\lambda_1(g)$ be the maximal eigenvalue of the interdependency matrix $\mathbf{B}(g)$. There exists $\varepsilon \in (0, +\infty)$ such that for every $\alpha \in \left(\frac{1}{\lambda_1(g)}, \frac{1}{\lambda_1(g)} + \varepsilon\right)$ the unique efficient allocation is that nobody consumes.*

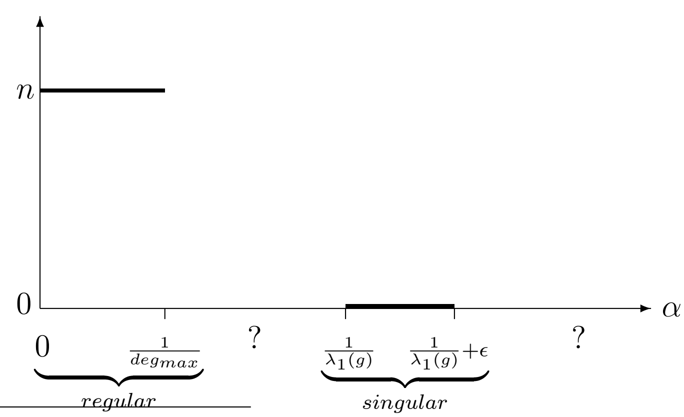
The geometry of the network considered characterizes regularity and provides sufficient conditions for singularity. In particular, connectivity and spectral properties of the network are crucial. This is specially interesting since there is a deep knowledge in the field of graph theory on these network structural properties.[45] A lot is known about the properties of the spectra of graphs, and we can apply it in this case, and in the general one.

This proposition seems to extend the result of Bergstrom (1999) on the apparent paradox that when two lovers that express more love for the other than for themselves, i.e. $b_{ij} > 1$ if $i \neq j$, want to share some spaghettis they have prepared for dinner, the unique efficient allocation is that they do not eat nothing at all. The analysis of these kind of two agents situations with positive externalities and its implications on resource allocation is extended in Vadasz (2005). We have proved therefore that this kind of extreme result on resource allocations is general, no matter which kind of private utilities $u_i(c_i)$ each agent shows and the number of agents in the economy, at least for some small small of interval of αs just after the critical point $\frac{1}{\lambda_1(g)}$. However, as we will see in a few lines, when there are more than two agents, other situations than a singular Pareto frontier with nobody consuming can emerge when positive externalities are severe.

From proposition 3 we derive a simple consequence.

Corollary 2 *For any connected society homogeneously altruistic there exist some levels of altruism for which the unique efficient situation is worst than if the society were totally selfish. An upper bound for the smallest of these values is* $\frac{1}{deg_{min}}$.

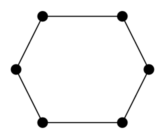
The following figure subsumes what we know from propositions 2 and 3.



[45] See for example Cvetkovik et al. (1980).

With the use of two classical examples, the circle and the star, we can obtain a further understanding of what can happens in the rest of the α space.

A circle is a network where each agent is connected to two neighbors, one at right and the other at left. It is the prototypical example of a situation very regular in terms of social structure since all agents are indistinguishable in terms of their position in the network.

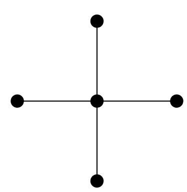


Circle (n=6)

Proposition 5 (The Circle) *Let $g \in Hom(\alpha)$ be a circle with $n \geq 2$ agents. The Pareto frontier is regular if and only if $\alpha < \frac{1}{2}$, and singular with nobody consuming for every $\alpha > \frac{1}{2}$.*

Hence, there are situations in which the two critical points collapse into one. In fact, this happens whenever $deg_{max}(g) = \lambda_1(g)$, and this happens if and only if the network is regular, i.e. $deg_1(g) = \cdots = deg_n(g)$. Moreover, the characterization in the case of the circle is bipolar and ϵ in proposition 3 is set to $+\infty$.

The star is quite a different situation with respect to the circle. In a star one agent (the central agent) is connected to all the rest of agents of the group (the periphery), and all the agents in the periphery are only connected to the central agent.



Star (n=5)

The central agent plays a very differentiated role with respect to the ones in the periphery since he provides cohesion to the group. This kind of social structure has a more complex efficiency pattern than the circle.

Proposition 6 (The Star) *Let $g \in Hom(\alpha)$ be a star with $n \geq 3$ agents. The Pareto frontier is*

(i) regular if $\alpha < \frac{1}{n-1}$

(ii) neither regular nor singular for $\alpha \in \left(\frac{1}{n-1}, \beta(n)\right)$, where $\beta(n) = \frac{-1+\sqrt{4n-7}}{2n-4}$ is an increasing function in n that takes values in $\left(\frac{1}{n-1}, \frac{1}{\sqrt{n-1}}\right)$, and the efficient allocations are such that the central agent and only one agent in the periphery share the total available resources

(iii) singular, with the central agent consuming everything, if $\alpha \in \left(\beta(n), \frac{1}{\sqrt{n-1}}\right)$

(iv) singular, with nobody consuming, if $\alpha \in \left(\frac{1}{\sqrt{n-1}}, \frac{1}{\sqrt{n-2}}\right)$

(v) neither regular nor singular for $\alpha > \frac{1}{\sqrt{n-2}}$, and the efficient allocations are such that only some of the agents in the periphery can share the available resources (not necessarily exhausting them), and with the number of allowed agents to consume increasing with α.

Here, within the two critical points different possibilities coexist. Moreover, the ϵ in proposition 3 tends to 0 as the number of group members increases, and the situation is nonsingular for any α greater than $\lambda_1(g) + \epsilon$. Observe that, while in the case of the circle it is true that whenever externalities are sufficiently positive the unique possible efficient allocation is always that nobody consumes, this is not necessarily the case in a more centralized structure like the star. In this case, even if externalities are positive and large, the efficient allocations include the possibility that nobody consumes but it does not include this possibility. Hence, the paradox on true lovers that hate spaghettis partially vanishes when we increase the number of agents in the economy. Observe in particular that in the case of two-agents economies, whenever both agents show love for each other the associated network structure can be understood as a circle or as a star. When we have at least three agents this possible confusion of structures disappears.

2.7 Conclusion

We have considered a model with two elements: (i) individuals are concerned for others' total utility (interdependent utilities), (ii) each individual is concerned for a different subgroup of agents and with different intensity for each agent (social networks). The intensity of each link can be positive, negative or zero, allowing for altruism, envy, of selfishness. The interdependence system or, alternatively, the social network, is determined by a set of $n\,(n-1)$ real values, the intensities with which each agent cares for each other. But, even if the model is very rich, we have been able to obtain closed-form expressions of the utility possibility set under interdependent utilities for any generic situation. Moreover, we have obtained several results about the Pareto frontier under interdependent utilities thanks to n values, the prestige measures of each of the n agents, that aggregate all the information the $n\,(n-1)$ initial values contain. Hence, to study efficiency, we reduce the dimension of complexity of the problem from the initial dimension of the interdependence system, which is of order n^2, to the number of agents in the economy, n.

We have concentrated in this work in the study of situations where there are no concavities on the private returns on direct investment. This way we can concentrate only on the study of network effects, without other effects interfering in the results. While the assumption of linearity in private returns is restrictive, we believe our conclusions are of some value also in the case private function returns are concave, as can be seen for example in the result of Proposition 4, which holds whenever each function $u_i(c_i)$ is increasing in c_i. With linearity, we can measure how valuable is an agent for the rest of the group simply by means of prestige, but if private returns are concave an agent is not only valuable by his position in the network: an individual will be more valuable the more prestigious he is, but also the larger it is his marginal return to an extra marginal unit of the resources. It seems that then, concavities, limit the possibilities that an extreme situation in which one agent receives all the resources is Pareto efficient. Probably, due to marginal private returns it would be optimal for this agent to transfer part of the resources to other agents that are less prestigious,

From our study of distributional conflicts we derive that prestige also plays a role in the determination of the Nash bargaining solution. In this work we have considered the interdependency network as given, taking a social planner's perspective. In some environments it could be of interest to study the following game of strategic network formation:[46] if the total available resources are distributed using the Nash bargaining solution and before the distribution agents can strategically choose with who they want to be linked and with which intensity, which are the equilibria of this game? As a conjecture, it seems that the Nash equilibria of this game should internalize prestige, centrality or other notions related to the geometry of the network, in the strategy each agent would play.

[46] See Jackson(2003) for a survey of this literature.

Our methodology seems sufficiently general to be applied in other situations where the effects generated by interdependent utilities can be preponderant, such as public goods games. Utility interdependencies can have significant effects on individual contributions. Altruistic agents can be disposed to increment their contributions while envious agents maybe prefer to decrease their ones. A conjecture might be that, since in this case it seems that the behavior of each agent may depend not only on the affection he receives but also on the one he provides, both centrality and prestige can affect the equilibria of the game.

Linear interdependent utilities are specially meaningful if we want to model pure altruism or envy. However, it would be of interest to extend the work we have done here to more general systems of interdependent utilities that include other patterns of behavior such as fairness.

Finally, we have only studied situations with a finite number of agents. While this is a very plausible assumption if we deal with social groups, several models of interdependent utilities from macroeconomics are defined for an infinite number of agents. Due to the interlinkage we have found between interdependent utilities and spectra of graphs, and that there is an increasing knowledge of properties on infinite graphs spectra, it is plausible that, at least for some families of networks, there can be obtained for infinite systems of interdependent utilities some partially analogous results to the ones we have obtained in the present work.

2.8 Proofs

Proof of Theorem 1:
Let $\bar{\mathbf{E}}$ be the matrix that has as entries $\bar{e}_{ij} = \Gamma_{ij}$ and let $e(g) = 1/\Lambda(g)$. We have to prove that

$$(\mathbf{I} - \mathbf{B})\,\bar{\mathbf{E}} = e\mathbf{I}$$

First, let's consider the entries (i,i) of the matrix $(\mathbf{I} - \mathbf{B})\,\bar{\mathbf{E}}$, that are equal to

$$\sum_{k=1}^{n} (\mathbf{I} - \mathbf{B})_{ik}\,\bar{e}_{ki} = e\left(g\backslash\{i\}\right) - \sum_{j=1, j\neq i}^{n} b_{ij} \sum_{p\in P_{ji}} w(p)\, e\left(g\backslash\{p\}\right) \qquad (2.2)$$

Observe that the weight of any simple cycle $c = (i, i_1, \ldots, i_k, i) \in C(g)$ is equal to $b_{ii_1} b_{i_1 i_2} \cdots b_{i_k i} = b_{ii_1} w(p)$ where p is the simple path equal to $(i_1, \ldots, i_k, i)$. Since the elements that form the cycle c are the same that form the path p, we can rewrite the second term of the right-hand side of (2.2) as

$$\sum_{j=1,j\neq i}^{n} b_{ij} \sum_{p\in P_{ji}} w(p)\, e(g\backslash\{p\}) = \sum_{\substack{c\in C(g)\\ i\in c}} w(c)\, e(g\backslash\{c\}) \tag{2.3}$$

Thus, using the alternative formulation of $e(g)$ in text for the network $g\backslash\{p\}$, we have that the right hand side of (2.2) is equal to

$$\left(1-\sum_{c_1\in C(g\backslash\{i\})} w(c_1)+\sum_{\substack{c_1,c_2\in C(g\backslash\{i\})\\ c_1,c_2 \text{ disjoint}}} w(c_1)\,w(c_2)-\cdots\right)- \tag{2.4}$$

$$-\sum_{c\in C(g),i\in c} w(c)\left(1-\sum_{\substack{c_1\in C(g)\\ c,c_1 \text{ disjoints}}} w(c_1)+\sum_{\substack{c_1,c_2\in C(g)\\ c,c_1,c_2 \text{ disjoints}}} w(c_1)\,w(c_2)-\cdots\right)$$

but we can reorder the terms in this expression as follows

$$1-\left(\sum_{c_1\in C(g\backslash\{i\})} w(c_1)+\sum_{c\in C(g),i\in c} w(c)\right)+ \tag{2.5}$$

$$+\left(\sum_{\substack{c_1,c_2\in C(g\backslash\{i\})\\ c_1,c_2 \text{ disjoint}}} w(c_1)\,w(c_2)+\sum_{\substack{c_1,c\in C(g)\\ c_1,c \text{ disjoint}}} w(c)\,w(c_1)\right)-$$

$$-\left(\sum_{\substack{c_1,c_2,c_3\in C(g\backslash\{i\})\\ c_1,c_2,c_3\ disj.}} w(c_1)\,w(c_2)\,w(c_3)+\sum_{\substack{c,c_1,c_2\in C(g),i\in c\\ c,c_1,c_2\ disj.}} w(c)\,w(c_1)\,w(c_2)\right)+\cdots$$

and this expression is equal to

$$1- \quad \left(\sum_{c_1\in C(g)} w(c_1)\right) + \left(\sum_{\substack{c_1,c_2\in C(g)\\ c_1,c_2\ disj.}} w(c_1)\,w(c_2)\right) -$$
$$-\left(\sum_{\substack{c_1,c_2,c_3\in C(g)\\ c_1,c_2,c_3\ disj.}} w(c_1)\,w(c_2)\,w(c_3)\right) + \cdots \tag{2.6}$$

which is equal to the alternative expression of $e(g)$ we have provided in text.

Thus, we have established from equations (2.2) to (2.6) that for any network g and for any agent $i \in N$

$$\begin{aligned} e(g) - e(g\backslash\{i\}) &= -\sum_{j=1,j\neq i}^{n} b_{ij} \sum_{p\in P_{ji}} w(p)\,e(g\backslash\{p\}) \\ &= -\sum_{\substack{c\in C(g)\\ i\in c}} w(c)\,e(g\backslash\{c\}) \end{aligned} \tag{2.7}$$

Now we have to prove that the entries (i,j) with $i \neq j$ of the matrix $(\mathbf{I}-\mathbf{B})\,\bar{\mathbf{E}}$ are equal to 0. The entry (i,j) of this product is equal to

$$\sum_{k=1}^{n} (\mathbf{I}-\mathbf{B})_{ik}\,\bar{e}_{kj} = \underbrace{\bar{e}_{ij}}_{[A]} - \underbrace{b_{ij}e(g\backslash\{j\})}_{[B]} - \underbrace{\sum_{k=1;k\neq i,j}^{n} b_{ik}\bar{e}_{kj}}_{[C]} \tag{2.8}$$

Let's develop each term in the right hand side of (2.8) individually.

$[A]$:

$$\bar{e}_{ij} = \sum_{p\in P_{ij}(g)} w(p)\,e(g\backslash\{p\}) \tag{2.9}$$

$[B]$:

$$b_{ij}e(g\backslash\{j\}) = \underbrace{b_{ij}e(g\backslash\{i,j\})}_{[B1]} + \underbrace{b_{ij}\left[e(g\backslash\{j\}) - e(g\backslash\{i,j\})\right]}_{[B2]} \tag{2.10}$$

$[C]$:

$$\sum_{\substack{k=1\\k\neq i,j}}^{n} b_{ik}\bar{e}_{kj} = \underbrace{\sum_{\substack{k=1\\k\neq i,j}}^{n} b_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} w(p)\, e(g\backslash\{p\})}_{[C1]} + \underbrace{+\sum_{\substack{k=1\\k\neq i,j}}^{n} b_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\in p}} w(p)\, e(g\backslash\{p\})}_{[C2]} \tag{2.11}$$

Furthermore, $[C1]$ is equal to

$$\underbrace{\sum_{\substack{k=1\\k\neq i,j}}^{n} b_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} w(p)\, e(g\backslash\{p,i\})+}_{[C1.a]} \underbrace{+\sum_{\substack{k=1\\k\neq i,j}}^{n} b_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} w(p)\left[e(g\backslash\{p\}) - e(g\backslash\{p,i\})\right]}_{[C1.b]} \tag{2.12}$$

where $g\backslash\{p,i\}$ means the network obtained removing not only all agents in the path p but also agent i. Observe that $[C1.a]$ can be rewritten as follows

$$\sum_{\substack{k=1\\k\neq i,j}}^{n} b_{ik} \sum_{\substack{p\in P_{kj}(g)\\i\notin p}} w(p)\, e(g\backslash\{p,i\}) = \sum_{\substack{p\in P_{ij}(g)\\p\neq(i,j)}} w(p)\, e(g\backslash\{p\}) \tag{2.13}$$

It follows from this last expression of $[C1.a]$ that $[A]-[C1.a]-[B1]=0$. Therefore, to be able to conclude that $[A]-[B]-[C]=0$, we still have to prove that $[B2]+[C1.b]+[C2]=0$. First, observe that, if $k\neq i,j$ and $p=(k,i_1,\ldots,i_l,i,i_{l+1},\ldots,i_{l+s},j)\in P_{kj}(g)$ such that $i\in p$, we have that

$$\begin{aligned} b_{ik}w(p) &= b_{ik}\left(b_{ki_1}\cdots b_{i_li}b_{ii_{l+1}}\cdots b_{i_{l+s}j}\right) \\ &= \left(b_{ik}b_{ki_1}\cdots b_{i_li}\right)b_{ii_{l+1}}\cdots b_{i_{l+s}j} = w(c)\,w(\bar{p}) \end{aligned} \tag{2.14}$$

where

$$\bar{p} = (i, i_{l+1}, \ldots, i_{l+s}, j) \in P_{ij}(g)$$

and

$$c = (i, k, i_1, \ldots, i_l, i) \in C(g \backslash (\{\bar{p}\} \setminus \{i\}))$$

is such that $i \in c$. Hence, taking into account that the elements in p are the same that the union of elements of $\bar{p}$ and c, we can rewrite $[C2]$ as follows:

$$\sum_{\substack{k=1 \\ k \neq i,j}}^{n} b_{ik} \sum_{\substack{p \in P_{kj}(g) \\ i \in p}} w(p)\, e(g \backslash \{p\}) = \\ = \sum_{\bar{p} \in P_{ij}(g)} w(\bar{p}) \sum_{\substack{c \in C(g \backslash (\{\bar{p}\} \setminus \{i\})) \\ i \in c}} w(c)\, e(g \backslash \{\bar{p}, c\}) \qquad (2.15)$$

Now, we rewrite $[C1.b]$ and $[B2]$ applying equation (2.7) to the network $g \backslash \{p\}$. $[C1.b]$ is equal to

$$\sum_{\substack{k=1 \\ k \neq i,j}} b_{ik} \sum_{\substack{p \in P_{kj}(g) \\ i \notin p}} w(p) \left[e(g \backslash \{p\}) - e(g \backslash \{p, i\}) \right] = \\ = \sum_{\substack{\bar{p} \in P_{ij}(g) \\ \bar{p} \neq (i,j)}} \left[- \sum_{\substack{c \in C(g \backslash (\{\bar{p}\} \setminus \{i\})) \\ i \in c}} w(c)\, e(g \backslash \{\bar{p}, c\}) \right] \qquad (2.16)$$

while, similarly, $[B2]$ is equal to

$$b_{ij} \left[e(g \backslash \{j\}) - e(g \backslash \{i, j\}) \right] = -b_{ij} \sum_{\substack{c \in C(g \backslash \{j\}) \\ i \in c}} w(c)\, e(g \backslash \{i, j\}) \qquad (2.17)$$

Hence, we obtain that

$$[C1.b] + [B2] = - \sum_{\bar{p} \in P_{ij}(g)} w(\bar{p}) \left[\sum_{\substack{c \in C(g \backslash (\{\bar{p}\} \setminus \{i\})) \\ i \in c}} w(c)\, e(g \backslash \{\bar{p}, c\}) \right] \\ = -[C2] \qquad (2.18)$$

that is what we wanted to prove. ■

Proof of Theorem 2:

It is partially a consequence of the following result:

An allocation $\mathbf{c}$ is Pareto efficient if and only if there exists a strict system of weights $\boldsymbol{\mu}$ such that $\boldsymbol{\mu} \cdot \mathbf{U}(\mathbf{c}) \geq \boldsymbol{\mu} \cdot \bar{\mathbf{U}}$ for every $\bar{\mathbf{U}} \in UPS(g;\pi)$.[47]

This lemma derives from an application of the separating hyperplane theorem. Given an allocation $\mathbf{c} = (c_1, \ldots, c_n)$ then $\mathbf{U}(\mathbf{c}) = \sum_{k=1}^{n} c_k \mathbf{e}^{(k)}$. We distinguish two different situations:

(i) $\mathbf{c}$ *is Pareto efficient and* $\sum_{k=1}^{n} c_k = \pi$: in this situation the associated utility vector is a convex combination of extreme points of the simplex all different from the zero vector, since

$$\mathbf{U}(\mathbf{c}) = \sum_{k=1}^{n} \frac{c_k}{\pi} \pi \mathbf{e}^{(k)}$$

Since $\mathbf{c}$ is Pareto efficient, we know that the above lemma applies for a certain strict system of weights $\boldsymbol{\mu}$. This implies that $\boldsymbol{\mu} \cdot \mathbf{e}^{(i)} = \boldsymbol{\mu} \cdot \mathbf{e}^{(j)}$ for every i, j such that $c_i, c_j > 0$. Suppose not: then, without lost of generality we can assume that $\boldsymbol{\mu} \cdot \mathbf{e}^{(i)} < \boldsymbol{\mu} \cdot \mathbf{e}^{(j)}$, but transferring the quantity c_i from agent i to agent j would generate a new feasible allocation, name it $\bar{\mathbf{c}}$, such that $\boldsymbol{\mu} \cdot \mathbf{U}(\mathbf{c}) < \boldsymbol{\mu} \cdot \mathbf{U}(\bar{\mathbf{c}})$, contradicting the lemma. Observe that since $\pi \boldsymbol{\mu} \cdot \mathbf{e}^{(i)} = \pi \boldsymbol{\mu} \cdot \mathbf{e}^{(j)}$ for every i, j such that $c_i, c_j > 0$ and since

$$\boldsymbol{\mu} \cdot \mathbf{U}(\mathbf{c}) = \sum_{k=1}^{n} \frac{c_k}{\pi} \left(\pi \boldsymbol{\mu} \cdot \mathbf{e}^{(k)} \right)$$

it follows that $\pi \boldsymbol{\mu} \cdot \mathbf{e}^{(i)} = \boldsymbol{\mu} \cdot \mathbf{U}(\mathbf{c})$ for every i such that $c_i > 0$.
The zero vector is a feasible allocation. Its associated utility vector is also the zero vector. This implies that $\boldsymbol{\mu} \cdot \mathbf{U}(\mathbf{c}) \geq 0$. If we let $K = \frac{1}{\pi}(\boldsymbol{\mu} \cdot \mathbf{U}(\mathbf{c})) \geq 0$ we can subsume all we have obtained in terms of prestige measures as follows: $\boldsymbol{\kappa}_k(\boldsymbol{\mu}; g) = \kappa$ whenever $c_k > 0$ and $\boldsymbol{\kappa}_k(\boldsymbol{\mu}; g) \leq \kappa$ whenever $c_k = 0$.

(ii) $\mathbf{c}$ *is Pareto efficient and* $\sum_{k=1}^{n} c_k < \pi$: the prove is analogous to the one for case one if we observe that, if $\tilde{c} = \pi - \sum_{k=1}^{n} c_k$, now the expression of

[47] This is a slight variation of a well-known result relating Pareto efficiency to linear social welfare functions (see for example Proposition 16.E.2, pg.560, in Mas-Colell et al.). The statement in terms of *strict* system of weights is valid because the shape of $UPS(g;\pi)$ is a simplex, not simply a convex set.

$\mathbf{U}(\mathbf{c})$ as a convex combination is

$$\mathbf{U}(\mathbf{c}) = \sum_{k=1}^{n} \frac{c_k}{\pi} \pi \mathbf{e}^{(k)} + \frac{\tilde{c}}{\pi} \mathbf{0}$$

Hence, $\pi \boldsymbol{\mu} \cdot \mathbf{e}^{(i)} = \boldsymbol{\mu} \cdot \mathbf{0} = 0$ whenever $c_i > 0$, and $\boldsymbol{\mu} \cdot \mathbf{U}(\mathbf{c}) = 0$. Therefore, the lemma implies that $\pi \boldsymbol{\mu} \cdot \mathbf{e}^{(i)} \leq 0$ if $c_i = 0$. We can rewrite all this in terms of prestige measures as: $\kappa_k(\boldsymbol{\mu}; g) = 0$ if $c_k > 0$ and $\kappa_k(\boldsymbol{\mu}; g) \leq 0$ if $c_k = 0$. ■

Proof of Proposition 2:

See proof of proposition 3 in chapter 1. ■

Proof of Proposition 3:

It is an immediate consequence of proposition 2. ■

Proof of Proposition 4:

We can order the eigenvalues of the $n \times n$ matrix $\mathbf{B}$ as follows: $\lambda_1(\mathbf{B}) > \lambda_2(\mathbf{B}) \geq \cdots \geq \lambda_n(\mathbf{B})$.[48] Hence, and taking into account the following facts,

- *i)* the determinant of a matrix is equal to the product of its eigenvalues
- *ii)* the matrix $\mathbf{E}(\mathbf{B})$ can be found computing the determinants of all $(n-1) \times (n-1)$ submatrices of $\mathbf{B}$ and the determinant of $\mathbf{B}$
- *iii)* the largest eigenvalue of any submatrix of $\mathbf{B}$ is strictly smaller than $\lambda_1(\mathbf{B})$.[49]
- *iv)* the eigenvalues of a matrix are continuous functions in terms of the entries of the matrix

we can reason as follows. The eigenvalues of the matrix $\mathbf{E}(\mathbf{B})$ are equal to $\frac{1}{1-\lambda_1(\mathbf{B})}, \ldots, \frac{1}{1-\lambda_n(\mathbf{B})}$. Hence, using that $det(\mathbf{B}) = det(\mathbf{E}(\mathbf{B}))^{-1}$ we have, using *(i)*, that

$$det(\mathbf{B}) = \left(\prod_{i=1}^{n} \frac{1}{1-\lambda_i(\mathbf{B})} \right)^{-1}$$

With a simple continuity argument making use of *(iv)* we obtain that there exist ϵ_1 such that for every $\alpha \in \left(\frac{1}{\lambda_1(g)}, \frac{1}{\lambda_1(g)} + \epsilon_1 \right)$ we have that $det(\mathbf{B})$ is strictly

48 These eigenvalues are real because the matrix is real and symmetric. Moreover, it is a classical result that the largest eigenvalue of a matrix is simple and hence strictly larger than all the rest.

49 This is true if the matrix is irreducible which in networks terms means that there are not two different groups of agents such that the members of one group are not connected to members of the other one. This assumption is plausible in our study. If this happens we could study in isolation each group.

negative.[50] Similarly, making use of a similar argument we can conclude (making use of *(ii)*, *(iii)* and *(iv)*) that there exists ϵ_2 such that for every $\alpha \in \left(\frac{1}{\lambda_1(g)}, \frac{1}{\lambda_1(g)} + \epsilon_2\right)$ the determinant of any $(n-1) \times (n-1)$ submatrix of $\mathbf{B}$ is strictly positive. All together, and if $\epsilon = min\{\epsilon_1, \epsilon_2\}$, we obtain that for every $\alpha \in \left(\frac{1}{\lambda_1(g)}, \frac{1}{\lambda_1(g)} + \epsilon\right)$ all the entries of $\mathbf{E}(\mathbf{B})$ are strictly negative and hence the Pareto frontier is singular with nobody consuming. ■

Proof of Proposition 5:

Since the circle is a regular network where all agents are equal, we can not distinguish one agent from another, there are only two possibilities: either everybody can consume (regular Pareto frontier) or nobody should consume (singular Pareto frontier). We know from proposition 3 that the Pareto frontier is regular if and only if $\alpha \leq \frac{1}{deg_{max}}$ which in this case is equivalent to $\alpha \leq \frac{1}{2}$ since $deg_{max} = deg_{min} = 2$. Therefore the Pareto frontier is singular, with nobody consuming, if and only if $\alpha > \frac{1}{2}$. ■

Proof of Proposition 6:

Applying theorem 1 we obtain that the group influence matrix of a star is, if agent 1 is the central agent and agents from 2 to n are the peripheral ones, equal to the following $n \times n$ matrix

$$\frac{1}{1-(n-1)\alpha^2}\begin{pmatrix} 1 & \alpha & \alpha & \cdots & \alpha \\ \alpha & 1-(n-2)\alpha^2 & \alpha^2 & \cdots & \alpha^2 \\ \alpha & \alpha^2 & 1-(n-2)\alpha^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \alpha^2 \\ \alpha & \alpha^2 & \cdots & \alpha^2 & 1-(n-2)\alpha^2 \end{pmatrix}$$

The result follows then by inspection applying theorem 2. Each possibility in the statement of the proposition represent a situation where, associated with a particular strict system of weights, the prestige of the agents that can consume is equal and larger than the ones that do not consume. A careful inspection of the different possibilities associated with the above matrix leads the result. ■

2.9 Grouping people into cycles

The principal mathematical tool we use in this section are permutations. This subsection presents the basic results on permutations and provides a graph-theoretical representation via subgraphs of this combinatorial tool.

[50] This is because $\lambda_1(\mathbf{B}) = \alpha\lambda_1(g)$ and because $\lambda_1(\mathbf{B})$ is strictly larger than the rest of eigenvalues of $\mathbf{B}$.

A permutation σ over the set $N = \{1, \ldots, n\}$ is simply a one-to-one function from N to N. We denote the set of permutations over N by S_n. To give a concrete permutation σ, we have to say which are the images of each element in N. An easy way of doing this is writing each element of N in a row and then write below the respective images

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix} \tag{2.19}$$

A *cyclic permutation of order* r is a permutation of the form

$$\begin{pmatrix} n_1 & n_2 & \cdots & n_r \\ n_2 & n_3 & \cdots & n_1 \end{pmatrix} \tag{2.20}$$

where $\{n_1, \ldots, n_r\}$ is a subset of different elements of N.[51] For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \tag{2.21}$$

represents the cyclic permutation where 3 is the image of 1, 2 is the image of 3, and 1 is the image of 2.

It follows from the definition of permutation that the composition of two different permutations σ and τ of S_n is also a permutation of S_n. Generally, the composition is called *product* of permutations, and we denote it by $\sigma\tau$. We have the following result.

Result. Each permutation σ can be decomposed as a product of disjoint cyclic permutations.[52] Furthermore, this decomposition is unique, except for the order in the product.

Given a permutation σ, we denote by $c(\sigma)$ the set of cyclic permutations (of order larger than 1)[53] that belong to the decomposition into cyclic permutations of σ, and by $\sharp\sigma$ the cardinality of this set, that is the number of cyclic permutations

[51] To simplify notation, when defining a permutation σ, it is usual to not express the image of the elements $x \in N$ such that $\sigma(x) = x$. Hence, for example, when defining a cyclic permutation, we suppose that $\sigma(x) = x$ for every $x \in \mathcal{N} \setminus \{n_1, \ldots, n_r\}$.

[52] By disjoint we mean that there are no common elements in two different cyclic permutations of the decomposition.

[53] A cyclic permutation of order one lets unaltered the element that belongs to it. Therefore, from now on, when we refere to a cyclic permutation we mean a cyclic permutation of order at least equal to two.

in the decomposition of σ. For example, the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 5 & 1 & 6 \end{pmatrix} \tag{2.22}$$

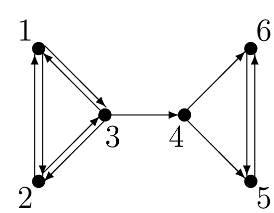
is decomposed into cyclic permutations as follows

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 6 & 5 \end{pmatrix} \tag{2.23}$$

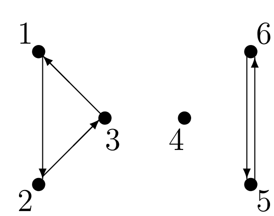
and hence $c(\sigma) = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 6 \\ 6 & 5 \end{pmatrix} \right\}$ and $\sharp\sigma = 2$.

Coming back again to networks, there is an obvious one-to-one correspondence between cyclic permutations and network cycles. Hence, each permutation is, in fact, dividing the set N of agents into disjoint cycles.

For example, let's revisit the network in figure 1



The permutation σ in S_6 defined in (19) divides this network into cycles as follows



Consequently, we can define in a natural way the weight of a permutation σ,

$w(\sigma)$, as the product of weights of the simple cycles in $c(\sigma)$, i.e.

$$w(\sigma) = \prod_{c \in c(\sigma)} w(c) \tag{2.24}$$

Chapter 3
Spatial Spillovers and Local Public Goods

3.1 Introduction

The government of any region provides public facilities to satisfy the needs of its population. Most of the times it is assumed that this public facilities reaches everyone in the same way. However, this is not always necessarily the case. As Olson (1969) notes:

> *One or more of the following of the three following logically possible relationships between the "boundaries" of a collective good and the boundaries of the government that provides it will apply: (1) the collective good reaches beyond the boundaries of the government that provides it; (2) the collective good reaches only a part of the constituency that provides it; or (3) the boundaries of the collective good are the same as those of the jurisdiction that provides it.*

In this chapter we deal with a variation of the second possibility. The task for the government is easy if geography is not an important factor for citizens to enjoy these facilities. This would be the case with a *pure* public good, expressed in (1) and (3). However if the effects of these facilities are of a local nature, then its allocation all around the geography of the city becomes a more difficult task. Any allocation will imply that some agents receive a more direct benefit than others, but a second important feature of this kind of local public goods is that they can show spillover effects: even if some agents do not receive an immediate benefit from a particular allocation of resources, they can receive some positive (or negative) externalities from it. Hence, if the government realizes that these spillover effects exist, it has to take them into account and the task of deciding between different possible allocations becomes a more complex task.

A possible example is the case of a city and its division into neighbourhoods. An activity in which spillover effects are clearly documented is the case of urban crime. There is a large amount of works in the criminology literature that highlights that there exist significant correlations among crime rates of different neighbourhoods of a city and its relation with social and geographic characteristics of the city (see, for example, Anselin and Messner, 1994, Anselin *et* *al.*, 2000, Bowers and Johnson, 2003, Morenoff *et al.*, 2001, Mears and Bhati, 2006). This suggests that there will also exist significant positive spillovers when local resources to combat urban crime are implemented. Therefore, how these resources are allocated across the geography of the city becomes an important factor in the combat of urban crime and its positive implications for the citizenship. Those that live closer to where some resources are allocated will benefit more directly from them. Anyhow, because of the existence of such spillover effects, more distant individuals can also benefit from them.

Generally, when the jurisdiction is divided into smaller districts and these districts can have local representatives that ask for part of these resources, a natural set of possible solutions is to divide the available resources among them according to some bargaining procedure. The analysis of this class of bargaining problems has been performed in chapter one with the use of the Nash bargaining solution.

In the first part of this chapter we build on this work to analyze, in a very simplified way, how two different divisions of a city entail different levels of social welfare when the government decides to distribute resources making use of this decentralized bargaining game. Any two different divisions generate different patterns of indirect spillover effects which are internalized in a different manner both in the final division of resources and welfare of each neighbourhood. Hence, its social welfare implications are not trivial. This implies that a problem about the optimal division of the city exists. We do not solve this problem here, but we use some data on two different divisions of Barcelona to build corresponding patterns of spatial spillovers and analyze its possible welfare implications.

A second important question related to local public goods and spatial spillovers is where do the resources implemented come from. We analyze in the second part of this chapter a public good provision game in which neighbourhoods provide part of their wealth to generate the resources that later on are divided following the Nash bargaining solution. We suppose there are two possible sources of heterogeneity: the pattern of spatial spillovers that the public good generates, and the distribution of wealth across the population. Hence, neighbourhoods internalize in their equilibrium decisions both their level of wealth compared to that of the rest of the city and the positive benefits they receive both from the direct effects of the share of public good that is directly allocated in their neighbourhood as well as the spillover effects generated by the allocation of resources across the rest of the city. The analysis is again not exhaustive but pretends to show some features relating the social and geographic structure of the city, reflected in the pattern of spatial spillovers, with the incentives of neighbourhoods to contribute to the public good.

3.2 On the optimal division of a city

Suppose that in a city the government decides how to divide public resources among neighbourhoods following a particular fixed rule, such as the Nash bargaining solution. Furthermore suppose that this government is aware that there exists a clear pattern of spillovers across neighbourhoods, and that the division rule internalizes the effects of this pattern of spillovers. Therefore, a crucial question is how to optimally divide the city into different neighbourhoods to maximize the effects of the allocation of public resources. This problem is complex since different city divisions lead to different patterns of externalities which lead to different levels of social welfare. Therefore the optimal solution has to take into account all these effects. We illustrate the possibility of using the tools developed in previous chapters comparing, in a somewhat simplistic manner, two different divisions of the city of Barcelona.

The population in Barcelona in year 2006 was $1.605.602$. There are different divisions in which this population is disaggregated. The next figure shows the actual division of the city of Barcelona into neighbourhoods, called $districtes$.

These $districtes$ are the main representative sub-units of the city. To illustrate the way we can map this division to some more algebraic representation of the city, we present its adjacency matrix, G^d . The entry ij of this matrix equals 1 if the limits of neighbours i and j partially overlap, and it is equal to 0 otherwise:

$$G^d = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

In this division the mean connectivity of neighbourhoods is 3.6. The less connected is Nou Barris, that only is connected to two other neighbourhoods, while the more connected neighbourhood is the Eixample, that is connected to seven other neighbourhoods. The Eixample is an especially central neighbourhood of Barcelona.

In the same way we can construct more elaborate algebraic structures that represent some features of the city, such as the matrix of bilateral influences $\mathbf{B}^d$

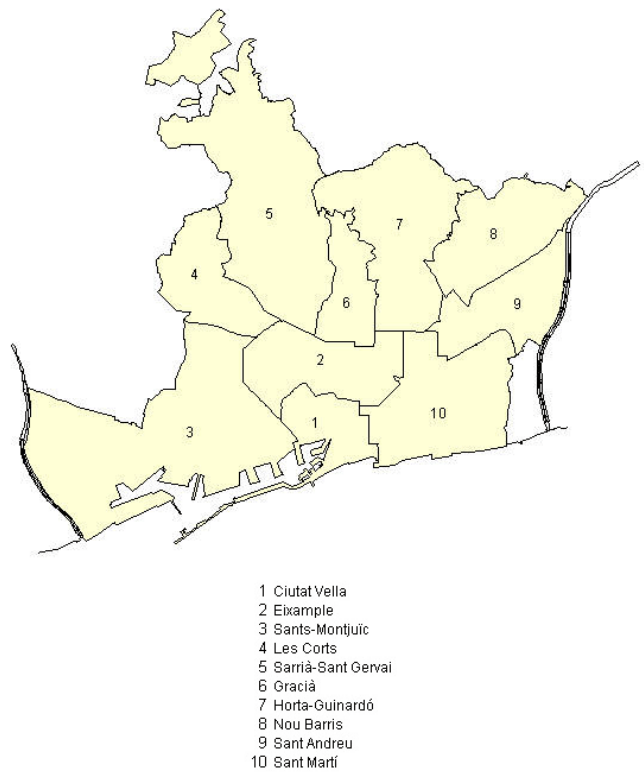


Figure 3.1 Barcelona: Districtes

associated to this division. To this end we should know which are the relevant geographic and socioeconomic variables that determine the different magnitudes of pairwise dependent spatial externalities.

A more disaggregate division is obtained with the use of $zones\ estadstiques$, that are sub-units of the districtes. The next figure provides a graphical representation of this division:

The mean connectivity for each neighbourhood, that we will call simply $zones$, is in this division equal to $4,8$. The more connected zones are Sant Gervasi and Guinardó, while the less connected zone is Barri Besos. Note that none of the more connected zones belong to the Eixample, the more connected district.

The statistical data for these divisions can be obtained in the webpage http://www.bcn.cat/estadistica/angl

Figure 3.2 Barcelona: Zones Estadístiques

In what follows we provide a particular function that shapes the levels of direct externality from one neighbourhood to another with the use of these statistics. These direct externalities will determine the pattern of spatial spillovers when the local public good is allocated within the different neighbourhoods.

For the sake of simplicity, we assume that the level of direct spatial externality from one neighbourhood to another only depends on the distribution of the population among the different neighbourhoods, abstracting from other socioeconomic variables that could also determine these levels of direct externalities. In particular we assume the following closed-form expression:

Assumption The level of influence of neighbourhood j on neighbourhood i depends on the population distribution in the city in the following way

$$b_{ij} = \frac{p_i}{\sum_{k \in N_j} p_k} \cdot \frac{p_j}{p}$$

This function of influence shows two characteristics. First, each neighbourhood exerts an influence that is proportional to its population. Second, the level of influence of a neighbourhood on another directly depends on the population that receive this direct influence compared with the overall population of the neighbourhoods that receive direct influence from the neighbourhood.

In both cases we obtain that the economy defined by this function of influences, and the population data collected during 2006 at Barcelona for both divisions, is regular and with the Nash bargaining solution being interior. We can then immediately apply the results obtained in chapter 1 to compute the social welfare derived from the division associated to the Nash bargaining solution, under the additional assumption that the disagreement outcome is the absence of allocation of resources, i.e. $\mathbf{d} = \mathbf{0}$. We obtain that the social welfare when there is a unit of resource to distribute is 1.34362 in the case of $districtes$, and 1.29601 in the case of $zones$.

Remark 1 *Under the assumptions specified above, the division of Barcelona in* $districtes$ *is better than the one in* $zones\ estadstiques$*.*

If there where no externalities, the social welfare would be in both cases equal to 1. The increase on social welfare with respect to the no-externality example is larger on average in the case of districts compared with that of statistical zones. Hence, representation of the population at the district level seems to be more beneficial for the city as a whole than representation at the level of statistical zones.

This is, by no means, a definitive conclusion with the data provided from city statistics. It is important to have correct direct influence functions that take into account all the relevant characteristics of neighbourhoods for the spillover effects of certain local public goods. However, the exercise we have provided shows that the model proposed and results obtained in chapter one can prove useful to attempt to better understand how the particular geography of the city models such spillover effects and that we can derive from this analysis a welfare assessment derived from different geographical partitions of the city into neighbourhoods.

3.3 On the governance of local public goods

In this section we model a public good contribution game among the neighbourhoods of a city. Each neighbourhood has some idiosyncratic characteristics, in this case represented by different levels of mean income. There exist a well-defined pattern of spatial spillovers among them, and neighbourhoods are aware of it. They choose which part of their income they dedicate to the provision of a public good that later on would be assigned across the city following the Nash bargaining solution. Hence, the way in which neighbourhoods internalize the pattern of pairwise influences, in this case spatial spillovers, is reminiscent of the use of this bargaining solution and its properties, that we have extensively analyzed in chapter 1.

The closest model to the one we present here is Bloch and Zenginobuz, 2006, in which the authors also analyze a public good provision game with local spillovers but of a different nature. In their model each district provides an independent public good. Each district benefits from its public good, and of the others' with some level of positive externality. Hence, the questions they have in mind are somewhat different from those we have exposed here.

3.3.1 The model

Agents

The city is divided into a finite number of neighbourhoods, n. We denote by $\mathcal{N}$ the set of neighbourhoods.

Preferences

Let w_i denote the income level of each neigbourhood, and let $\bar{w}$ denote the mean income level at the city level, i.e. $\bar{w} = (w_1 + \cdots + w_n)/n$. Each neighbourhood can decide to invest part of its income to the provision of a public good, c. We differ from the standard approach to public goods in which everybody enjoys in the same way the public good. In our case, the public good is divided and allocated through the different neighbourhoods of the city. Each neighbourhood enjoys a direct effect of the share of public good it receives and also can receive positive spillovers from the allocation of the public good in the rest of neighbourhoods. These spillovers can be of different magnitude depending on the pair of neighbourhoods we consider. Each neighbourhood can potentially exert a different level of direct spillover to each other neighbourhood, if it exerts any kind of spillover at all.

Hence, each neighbourhood obtains utility from two sources: the income that maintains once it has invested in the creation of the public good, if it has invested at all, and the utility derived from the allocation of the public good through the city and the positive spillovers derived from it. Let τ_i denote the contribution of neighbourhood i to the public good, and let c denote the level of public good produced. The production function of the public good given a contributions' profile $\boldsymbol{\tau}$ is

$$c(\boldsymbol{\tau}) = \ln\left(1 + \sum_i \tau_i\right)$$

The level if public good is an strictly concave function on the sum of contributions of all neighbourhoods. Individual contributions are therefore perfect substitutes of each other.

The utility derived from wealth is equal to $\ln(1 + w_i - \tau_i)$. It is a decreasing concave function with respect to τ_i.

If the public good is divided among the different neighbourhoods in such a form that each one receives a part of it equal to $c_i(\boldsymbol{\tau})$, we denote by

$$u_i(c_1(\boldsymbol{\tau}), \ldots, c_n(\boldsymbol{\tau}))$$

the utility neighbourhood i obtains from the local public good.

The total utility each neighbourhood obtains, that we denote v_i, is equal to the sum of the utility derived from wealth and the utility derived from the public good:

$$\tilde{v}_i(\boldsymbol{\tau}) = \ln(1 + w_i - \tau_i) + u_i(c(\boldsymbol{\tau}))$$

Spatial Spillovers

Furthermore, we suppose that the component of the utility that measures the impact of the public good on each neighbourhood follows a pairwise influence model. Each neighbourhood j exerts a, positive or negative, direct externality on each other neighbourhood i that equals $b_{ij} \in \mathbb{R}$. The utility on the public good of a neighbourhood i is related to that of the rest of neighbourhoods as follows

$$u_i(c(\boldsymbol{\tau})) = c_i(\boldsymbol{\tau}) + \sum_{j \neq i} b_{ij} u_j(c(\boldsymbol{\tau}))$$

We can gather all the information of this pairwise influence system into a matrix $\mathbf{B} = (b_{ij})_{i,j}$ with zeros in the diagonal.

A city $\mathcal{C}$ is therefore characterized by its set of neighbourhoods $\mathcal{N}$, the wealth distribution among neighbourhoods $\mathbf{w} = (w_1, \ldots, w_n)$, and the pattern of spatial spillovers among neighbourhoods given by the matrix $\mathbf{B}$.

Let $b_i = \sum_{j \neq i} b_{ji}$ equal the sum of direct spillovers i exerts on the rest of neighbourhoods, and denote by $\delta_i = 1 - b_i$. This constant δ_i will be important in the analysis. It is a complementary measure of direct spillovers exerted by neighbourhood i. We assume that $\delta_i > 0$ for all $i \in \mathcal{N}$. These condition imposes that aggregate direct spillovers exerted by each neighbourhood are bounded from above by 1, and it ensures that we are in a regular economy that does not show degenerated outcomes derived from the pattern of spillovers of the kind we have studied in the previous chapter. Moreover, we assume that

$$\sum_{j \neq i} b_{ji}\delta_j < \delta_i \quad \forall i \in \mathcal{N}$$

This second condition is important for reasons that will become more apparent in few lines that are related to the allocation rule of the public good among the different neighbourhoods.

3.3.2 The game.

The public good provision game has two stages. In the first one each neighbourhood chooses a level of contribution to the public good, $\tau_i \in [0, w_i]$.

In the second stage of the game neighbourhoods bargain on how to allocate the level of public good obtained from individual contributions, $\ln\left(1 + \sum_{i=1}^{n} \tau_i\right)$, among them. If they do not reach an agreement the public good is not divided and the unique utility agents obtain derives from the remaining wealth once discounted its contribution. We assume that the solution of this bargaining procedure is given by the Nash bargaining solution. Because of the no-division disagreement outcome, the disagreement point is equal to

$$\mathbf{d}(\boldsymbol{\tau}) = (\ln(1 + w_1 - \tau_1), \ldots, \ln(1 + w_1 - \tau_1))$$

The previous conditions on the matrix $\mathbf{B}$ ensure that the Nash bargaining solution will be interior for any city $\mathcal{C}$ given this disagreement point. Since the Nash bargaining solution satisfies the property of *Scale Invariance,* we can immediately apply the results obtained in the first chapter to characterize the utilities neighbourhoods obtain in this second stage of the game. We do that in the form of a lemma.

Lemma The final utilities of the public good provision game are

$$v_i(\boldsymbol{\tau}) = \ln(1 + w_i - \tau_i) + \frac{c(\boldsymbol{\tau})}{n\delta_i}$$

We differ from the analysis of Bloch and Zenginobuz in two different aspects: first, that in their model each jurisdiction/neighbourhood creates its own public good, from which other jurisdictions can benefit; second, we allow for heterogeneity in wealth. Hence, there are two dimensions of possible heterogeneity, spillovers and wealth distribution.

3.3.3 The Equilibria

Let $\tau_{-i} = \sum_{j \neq i} \tau_j$. The function v_i is strictly concave in τ_i.Since

$$\frac{\partial v_i}{\partial \tau_i} = -\frac{1}{1 + w_i - \tau_i} + \frac{1}{n\delta_i \left(1 + \tau\right)}$$

we obtain that the best-reply function of agent i is

$$BR_i\left(\tau_{-i}\right) = \begin{cases} 0 & \text{if } 1 + \tau_{-i} \geq \frac{1+w_i}{n\delta_i} \\ \frac{1+w_i-n\delta_i}{1+n\delta_i} - \frac{n\delta_i}{1+n\delta_i}\tau_{-i} & \text{if } 1 + \tau_{-i} \in \left(\frac{1+w_i}{n\delta_i}, \frac{1}{n\delta_i}\right) \\ w_i & \text{if } 1 + w_i + \tau_{-i} \leq \frac{1}{n\delta_i} \end{cases}$$

We can clearly observe the dependences of individual decisions on the two dimensions of heterogeneity of the problem (the wage distribution and the pattern of spatial spillovers). For example, a neighbourhood decides to not contribute at all if it is not wealthy enough or aggregate externalities it directly exerts on the rest of neighbourhoods are small (measured by the parameter δ_i). This last consequence is reminiscent of the use of the Nash bargaining solution on the solution of distributional conflict.

Observe that an equilibrium does not need to be interior, meaning that the contribution of each agent is strictly positive and strictly smaller then his total welfare. The following proposition characterizes the unique possible interior equilibrium in terms of the city's characteristics.

Proposition 1 *If the public good provision game admits an interior equilibrium, it is unique within the set of interior allocations, and of the form*

$$\tau_i^* = \frac{1 + w_i - n\delta_i}{1 + n^2\bar{\delta}} + \frac{n^2}{1 + n^2\bar{\delta}} \left[\left(1 + \bar{w}\right)\left(\bar{\delta} - \delta_i\right) - \bar{\delta}\left(\bar{w} - w_i\right)\right]$$

We can decompose the contribution of each neighbourhood into two different components. The first one, that equals $\frac{1+w_i-n\delta_i}{1+n^2\bar{\delta}}$, is essentially, with the exception of the common renormalization $\frac{1}{1+n^2\bar{\delta}}$ to all neighbourhoods, an idiosyncratic component that only takes into account individual income, w_i, and aggregate direct externalities exerted by the neighbourhood, measured by δ_i.

The second term is more involved and comprises more information about the whole city. Four elements are important: again individual income, w_i, and the aggregate level of direct externalities exerted by the neighbourhood, δ_i, and city's mean income and mean externalities exerted.

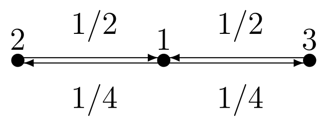
The contribution of a neighbourhood increases the more wealthy it is with respect to the city wealth's mean. This effect is larger the larger is the mean of δ's, i.e. this effect is larger the smaller are the direct externalities neighbourhoods exert in mean. This is not surprising. When spillovers are weak, neighbourhoods have to invest more on the provision of the public good to receive its effects. Also, the contribution of a neighbourhood increases the smaller is its associated δ compared with the mean of δ's, i.e. the larger are the aggregate direct spillovers this neighbourhood exerts compared with the mean of aggregate direct spillovers a neighbourhood in the city exerts in mean. Furthermore, this effect is larger the larger is the mean income level of the city. This effect is reminiscent of the use of the Nash bargaining solution with the disagreement outcome of no public good assignment, but it also seems to be quite natural. When a neighbourhood gives more than what it receives, it knows that this will probably diminish the incentives of the rest of neigbourhoods to contribute to the public good provision, and has to compensate this annoying consequence by contributing itself more.

Example Consider the following two cities, $\mathcal{C}_1$ and $\mathcal{C}_2$. Both cities have three neighbourhoods, $\mathcal{N} = \{1, 2, 3\}$. The matrix of spatial spillovers is the same for both

$$\mathbf{B} = \begin{pmatrix} 0 & 1/4 & 1/4 \\ 1/2 & 0 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}$$

The unique difference between these two cities is in the wealth distribution. The wealth distribution of $\mathcal{C}_1$ is $\mathbf{w}^1 = (2, 3/2, 3/2)$, while the one for $\mathcal{C}_2$ is $\mathbf{w}^2 = (2, 1, 1)$.

To get a more graphical representation of these two cities we depict in form of a network the flows and intensities of direct spillovers from one neighbourhood to another:



For the city $\mathcal{C}_1$ it is easy to check that the unique equilibrium of the public good provision game is interior. In particular. $\boldsymbol{\tau}^* = (6/11, 1/22, 1/22)$. The difference in income translates into large differences between the contributions of the poor neighbourhoods compared to that of neighbourhood 1, but it is not sufficiently large to preclude the participation of any of them.

In city $\mathcal{C}_2$ income inequality is larger between the central neighbourhood and those at the periphery. With the use of the best-replies functions specified above we can easily derive that the unique equilibrium of the public good provision game for this city is that neighbourhood one contributes with half of its income and that the two other neighbourhoods do not contribute at all, i.e. $\boldsymbol{\tau}^* = (1, 0, 0)$. All neighbourhoods are interested in the benefit that the public good generates but neighbourhoods two and three are not wealthy enough to contribute to its provision.

3.3.4 The social optimum

We proceed to study the socially optimal contribution profile to the public good. We characterize this profile in case it is interior, and we proceed to compare it to the equilibrium of the public good provision game in case it is also interior. To describe this optimal provision rule, we first define the harmonic mean of a set of numbers. Let $\mathbf{x}$ be an n-dimensional real vector. The harmonic mean of the elements of this vector, that we denote $H(\mathbf{x})$, is equal to

$$H(\mathbf{x}) = \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$

Proposition 2 *If the social optimum is interior the socially optimal level of contribution of agent i is*

$$\tau_i^s = \frac{1 + w_i - H(\boldsymbol{\delta})}{1 + nH(\boldsymbol{\delta})} + \frac{nH(\boldsymbol{\delta})}{1 + nH(\boldsymbol{\delta})}(w_i - \bar{w})$$

The optimal level of provision of each neighbourhood is formed by an intrinsic component that simply depends on its own level of income and the (harmonic) mean of the δ's of all neighbourhoods. This intrinsic component is larger the larger is its own income level and the smaller it is the harmonic mean of δ's, i.e. whenever the direct externalities are larger. The second component depends on the wealth distribution on the whole city. The larger is the income level of one neighbourhood compared with the city's mean income level, the more the neighbourhood contributes. This effect is more severe when the harmonic mean of δ's is larger. Again, the intuition for this result is simple.

Proposition 3 *For the case of homogeneous spillovers and equally wealthy districts, there is always underinvestment with respect to the socially optimal level, and underinvestment decreases in δ for any $\delta > n^{-\frac{3}{2}}$. Furthermore, the difference between individual utility at the social optimum and individual utility at equilibrium is decreasing in δ for any $\delta > 0$.*

This results highlights that the collective action problem still exists even if there are no differences in wealth or in the pattern of spatial spillovers. As Olson (1971) notes this is a natural feature on players' actions in most games of public good provision. In particular, the presence of positive externalities do not dilute the underinvestment effect. To the contrary the difference between the socially optimal action and the equilibrium action increases when aggregate direct externalities increase. An this is also true when we compare differences in utilities.

3.4 Conclusions

Both sections of this chapter highlight an aspect that economic analysis has tended to neglect but that other social sciences are taking into account when analyzing questions of importance for citizens well-being, such as the case of criminal activity. Namely, taking as an example the case of a city, that even if public services are assigned at a very local level, its effects can exceed the fictitious institutional boundaries by which the city is divided into neighbourhoods. These institutional boundaries have a role on the possible magnitude of this spillover effects since they determine both the possible divisions of available resources the government can choose and the interneighbourhoods spillovers that any division will generate.

In particular we have focused on two different questions. First, from a social planner perspective an important choice, before public services are implemented, is in which form should districts be designed to optimally take into account the possible externalities that these public services will generate. That is, how a government should take into account geographical dispersion and its social consequences.

Second, a particular division has also consequences on the incentives districts have to provide the necessary inputs to produce such public services. We have followed in this case the perspective of a public good provision game, in which each neighbourhood, in the form of a local representative, chooses a level of provision taking into account spatial spillovers as well as other socioeconomic determinants, such as the distribution of wealth between the different neighbourhoods of the city.

We have only provided some tentative answers to these questions. And, in Olson's words:

> *Reality is almost always too complex to permit policy recommendations derived solely from a single model, much less a premodel of the sort alumbrated here.* Olson (1969)

The answers we have provided are surely not definitive solid conclusions to that questions. However, we think that our model can show up some features relevant to express and discuss these issues.

Several different approaches could also have been taken to study these issues. First, when comparing the two different divisions into neighbourhoods of the city of Barcelona we have made very strong assumptions on the form of direct spillovers from one neighbourhood to another. In particular we have assumed that these direct spillovers only depend on the population distribution among the different neighbourhoods, following a closed-form expression that depends on this distribution. Of course, it is difficult to provide a correct theoretical measure that incorporates this and other socioeconomic variables into account. Empirical work should be addressed to test if some particular closed-form expressions perform sufficiently well to try to analysis much more rigorously this question. As mentioned in the introduction there is a lot of work on spatial econometrics related with criminal activity. Maybe this literature can be an initial source of inspiration for such an attempt (see Anselin, 2003, for an introduction to this literature with some applications relevant to public economic issues for local governments).

With respect to the second section of this chapter, an alternative approach to the production of public services could be to select heterogenous taxation levels for the different neighbourhoods of the city. This would be another alternative model to analyze an intermediate approach in the theory of fiscal federalism from the two polar perspectives of purely homogeneous level of taxation at the city level and complete decentralization of the control of public services to the different neighbourhoods (Oates, 1972, is a classical reference on this issue, and Besley and Coates, 2003, and Lockwood, 2002, are modern approaches that incorporate externalities to this analysis). The introduction of spatial spillovers in a taxation model with externalities flowing from one neighbourhood to another would enrich the possible conclusions in such issues.

Of course, our model is based in several assumptions that require further consideration. We have here imposed a particular timing on the game of provision and distribution of the public good. And in the second stage of this game we have considered a particular solution to the distributional conflict for resources. Moreover, we have neglected the role of free mobility of citizens. The possibility of a voting-with–their-feet mechanism (see Tiebout, 1956) is an important issue to be considered when dealing with local public good provision.

To study the robustness of our results to other game specifications as well as how to include other aspects, such as reallocation, into the model we have sketched here are important questions that we leave for future research.

3.5 Proofs

Proof of Proposition 1:

If there exists an interior equilibrium, in which $\tau_i^* \in (0, w_i)$ for all i, it solves the following system of equations:

$$\tau_i^* + \frac{n\delta_i}{1+n\delta_i}\tau_{-i}^* = \frac{1+w_i-n\delta_i}{1+n\delta_i} \qquad i \in \mathcal{N}$$

Let $s_i = \frac{1+w_i-n\delta_i}{1+n\delta_i}$, and let $a_i = \frac{n\delta_i}{1+n\delta_i}$. Let $\mathbf{A}$ be the matrix with ones in the diagonal and $a_{ij} = a_i$ if $j \neq i$. Then the interior equilibrium is equal to

$$\boldsymbol{\tau}^* = \mathbf{A}^{-1} \cdot \mathbf{s}$$

It is a simple exercise of linear algebra to prove that

$$\mathbf{A}_{ij}^{-1} = \begin{cases} (1+n\delta_i)\frac{1+n\sum\limits_{k\neq i}\delta_k}{1+n^2\overline{\delta}} & \text{if } i = j \\ -(1+n\delta_j)\frac{n\delta_i}{1+n^2\overline{\delta}} & \text{if } i \neq j \end{cases}$$

Therefore, $\tau_i^* = \sum\limits_{j=1}^{n} \mathbf{A}_{ij}^{-1} s_j = \frac{1+w_i-n\delta_i}{1+n^2\overline{\delta}} + \frac{n}{1+n^2\overline{\delta}} \sum\limits_{j\neq i} \left[\delta_j\left(1+w_i\right) - \delta_i\left(1+w_j\right)\right].$

Some algebra shows that

$$\sum_{j\neq i}\left[\delta_j\left(1+w_i\right)-\delta_i\left(1+w_j\right)\right]=n\left[\left(1+\bar{w}\right)\left(\bar{\delta}-\delta_i\right)-\bar{\delta}\left(\bar{w}-w_i\right)\right]$$

■

Proof of Proposition 2:

The first-order conditions of the social planner problem, in the case that the social optimum is interior are:

$$\tau_i^s+\frac{H\left(\boldsymbol{\delta}\right)}{1+H\left(\boldsymbol{\delta}\right)}\tau_{-i}^s=\frac{1+w_i-H\left(\boldsymbol{\delta}\right)}{1+H\left(\boldsymbol{\delta}\right)}\qquad i\in\mathcal{N}$$

where $\tau_{-i}^s=\sum\limits_{j\neq i}\tau_j^s$. These first-order conditions form a system of linear equations with a unique solution. Let $r_i=\frac{1+w_i-H(\boldsymbol{\delta})}{1+H(\boldsymbol{\delta})}$, and let $\alpha=\frac{H(\boldsymbol{\delta})}{1+H(\boldsymbol{\delta})}$. Let $\mathbf{Z}$ be the matrix with $z_{ii}=1$ for all $i\in\mathcal{N}$

and $z_{ij}=\alpha$ whenever $i\neq j$. The solution to the above system of linear equations is:

$$\boldsymbol{\tau}^s=\mathbf{Z}^{-1}\cdot\mathbf{r}$$

Some algebra leads to:

$$\mathbf{Z}_{ij}^{-1}=\begin{cases}\frac{(1+H(\boldsymbol{\delta}))(1+(n-1)H(\boldsymbol{\delta}))}{(1+nH(\boldsymbol{\delta}))} & \text{if } i=j\\ -\frac{(1+H(\boldsymbol{\delta}))H(\boldsymbol{\delta})}{(1+nH(\boldsymbol{\delta}))} & \text{if } i\neq j\end{cases}$$

Plugging this back in $\tau_i^s=\sum\limits_{j=1}^{n}\mathbf{Z}_{ij}^{-1}r$, and rearranging terms we obtain the desired result. ■

Proof of Proposition 3:

We omit the proof. The result follows from direct comparison of the expressions obtained. ■

PART II

Communication Processes

Chapter 4
Communication Processes: Knowledge and Decisions

with

Antoni Calvó-Armengol

4.1 Introduction

Communication between individuals fosters cooperation within the group and improves decision-making in uncertain environments. Organizations, in fact, are often described as an authority structure whose role is precisely to assign communication capabilities and decision-making tasks among its different hierarchy levels. According to this view, it is the formal authority structure of the organization that determines its performance. Organizations are optimal when their hierarchy architecture is suitably designed for that purpose.

However, beyond the formal working relationships institutionalized in the organization chart, informal working relationships also emerge and stabilize spontaneously. Such informal ties result from social processes in the workplace and often arise in response to specific needs encountered by organization members. Informal organizations, that overlap with the formal chart, have always been present in the modern corporation. With the development of communication technologies and new forms of group work that bring together geographically dispersed collaborators around a common project, informal organizations are becoming ubiquitous.

The aim of this paper is precisely to analyze informal organizations, to evaluate their performance and to find out their optimal inner configuration.

The environment

Our model of informal organizations has two main ingredients.

First, we envision informal organizations as a group of individual facing a common task with cooperative work. More precisely, individuals exert their action for the group while facing both an external and an internal concern. The external concern corresponds to the common task ascribed to the group, and whose characteristics display some degree of uncertainty. The internal concern reflects the coordination benefit from aligning ones' action with the actions of one's collaborators.

Second, informal communication is pervasive inside groups, be it by E-mail, face-to-face meetings, etc. Our premise is that informal communication is, precisely, the most important working relationship operating inside the group.

Altogether, an informal organization is characterized by its information-processing needs and its information-processing capabilities. Information-processing needs correspond to the nature of the information uncertainty about the task to be performed, and to the exact balance between the external and the internal concern faced by each member of the group. Information-processing capabilities correspond to the communication process used to transmit information.

The results

The internal coordination problem together with the uncertain external concern define an incomplete information game. The communication process disseminates within the group any private information held by individuals about the task. The form and characteristics of this communication process determines the information structure available to all organization members and, ultimately, their choices. We map this decentralized information-sharing scheme to individual optimal choices, and relate changes in the communication structure to variations in individual and aggregate payoffs. We provide a closed-form expression for the unique Bayes-Nash equilibrium, which is linear in individual information, for a rich class of correlated signals. We also provide monotone comparative statics results about the aggregate equilibrium payoffs with respect to the individual preferences and to the information structure, and compare it to the social optimum. In the next chapter we apply this general results to the study of a particular family of networked communication processes.

At a Bayes-Nash equilibrium, individuals best-respond to their own assessment of other's choices. Equilibrium also requires that everybody correctly anticipates each others' choices. This presumes that players can keep track of own beliefs about the task to be performed, as well as own beliefs about others' beliefs, own beliefs about others' beliefs about both own and others' beliefs, and so on. The communication process shapes the information held by each individual as well as the information overlap across different individuals. This information overlap is crucial to build cross inferences about each others' information and, ultimately, to compute the (arbitrary) high order beliefs that enter the equilibrium determination.

In this setup, we characterize the unique Bayes-Nash equilibrium for general communication processes. The equilibrium strategies are linear and depend on the knowledge index of each agent, computed for the information structure induced by the prevailing communication process. The knowledge index, which we introduce, measures the (arbitrary) high-order beliefs each agent entertains about every other agents' information, and thus reflects the communication possibilities available to everyone. Beyond reflecting the geometry of communication possibilities, the knowledge index also depends on the exact balance between internal and external concerns for the organization members −their information-processing needs− that enters the fixed-point equilibrium calculation. This index is formally reminiscent of standard centrality measures in sociology, but is computed with an information correlation matrix rather than with socio-metric data.

Altogether, we provide a closed-form expression for equilibrium play where the role of information needs and communication possibilities is apparent. To establish uniqueness and linearity of the equilibrium play we rely on a central result in team theory due to Radner (1962).

We provide a closed-form expression for equilibrium payoffs and work out their comparative statics with respect to the exogenous payoff parameters, comprising the balance of the internal versus the external concern and the accuracy of the private signals about the task to be performed, and with respect to the accuracy and correlations of private reports that characterize the information structure derived from a particular communication process.

4.2 Related Literature

This paper uses team theory to analyze the role of communication in organizations. We analyze a bayesian game with common interests with a unique equilibrium. This unicity result relies on the seminal work of Radner (1962) on the theory of team decisions.

Team theory complements the principal-agent view of organizations and it has been used to answer a variety of questions on the theory of organizations. For example, building on Marschak and Radner (1972), there is an extensive literature that analyzes in a team-theoretical framework the optimal inner structure of an organization that begins with Crémer (1980) and is followed by Aoki (1986), Geanakoplos an Milgrom (1991), Dessein and Santos (2006) and Dessein et al. (2006).

In particular, Dessein and Santos and Dessein et al. highlight some communicational aspects of organizations, as we try to do in this work. However, the communication structure in these set of papers differs from ours and we answer complementary questions to those they analyze. A main difference is that in these works communication is strategic. In our case, communication is non-strategic due to the complete alignment of preferences of all agents.

Another question analyzed in the team-theoretical literature of organizations, and also closely related to our work, is on the optimal information structure of an organization. Crémer (1993) analyzes, in a two agents family of quadratic payoff functions, when shared knowledge, in which all agents receive the same signal about the state of the world, is superior to decentralized knowledge, in which each agent receives a different signal of the state of the world. Building on this work, Prat (2002) extends this analysis to more general setups, allowing for any finite number of agents and general team payoff structures.

Our work also relates to some of the recent literature on the social value of information. Morris and Shin (2002) analyze in a game with a similar payoff structure with the one we analyze here the impact of public information on social welfare. They show that in this game with strategic complementarities public information can reduce welfare. Morris and Shin (2006) complements this analysis with the introduction of semi-public signals that act as a restricted communication process, and provide some comparisons between public and semi-public information with respect to welfare. Angeletos and Pavan (2007) extends the analysis of the social value of information to a broader class of games that include the possibility of both strategic complementarities and substitutability. Van Zandt and Vives (2007) provide general results on the equilibria of bayesian games with complementarities making use of tools from lattice theory.

While the payoff structure of our model is similar to that in Morris and Shin (2002, 2006) we depart from this literature in two different directions: first, we suppose there is a finite number of agents, while this literature has assumed a continuum of players, and we allow for heterogeneous accuracies and correlations of information between pairs of players, while this literature has assumed, if any, several restrictions on the possible heterogeneity on the information structure. Our model therefore highlights how distributional aspects of information, accuracies and correlations, impact players' actions and social welfare. This is reflected in the knowledge index, that subsumes informational heterogeneity and characterizes idiosyncratic equilibrium actions.

From a more abstract perspective, our work also relates to the literature on global games, higher-order beliefs and common knowledge in games (Rubinstein, 1989; Geanakoplos, 1992, Morris and Shin, 2003). The knowledge index we introduce provides a tractable tool to aggregate all higher order beliefs into a scalar value that measures how informative is his communication report to strategically internalize both the decision problem and the coordination motive.

Finally, there are alternative approaches to communication in organizations that are interested in different questions than those we pursue here. Crémer et al. (2007), building in some early discussions by Arrow on communication in organizations (see Arrow, 1974), study the language with which information is transmitted in organizations. They analyze which is the optimal organizational language and how it affects firm structure. Dewatripont and Tirole (2005) analyze a model of communication in which there are strategic interactions between the communication efforts of the members of an organization. Efforts determine if communication is informative or not. We abstract from this question and suppose that when agents communicate information, this information is hard, meaning that there is no loss or noise in the communication process.

4.3 The game

4.3.1 Actions and payoffs

Consider a population of n players, each choosing an action $a_i \in \mathbb{R}$. Payoffs depend on own and others actions, and on some exogenous parameter $\theta \in \mathbb{R}$. More precisely, we focus on payoffs that reflect two different concerns. On one hand, players want to match their action a_i to the value of θ. On the other hand, pressed by conformity, they all want to align their choice with that of the others.

We consider the following quadratic payoffs:[54]

$$u_i(a_1, ..., a_n; \theta) = -(1-r)(a_i - \theta)^2 - r\frac{1}{n-1}\sum_{j \neq i}(a_i - a_j)^2, \ i = 1, ..., n. \tag{4.1}$$

The first term is a quadratic loss between own's action and the target −the external concern. The second term is the average discrepancy (or distance) between own's and others' actions for all possible pair-wise comparisons −the internal concern. The parameter $0 \leq r \leq 1$ measures the balance of the external target concern (that binds at $r = 0$) versus the internal coordination concern (that binds at $r = 1$).

If the exact value of θ is known to everybody, $a_i^* = \theta$ for all players i is the unique (first-best) Nash equilibrium. We analyze instead the case of incomplete information, where the exact value of θ is not known.

4.3.2 Information structure

In the incomplete information case, the value of θ is determined by nature, $\theta \sim N(\theta_0, \phi_0)$.

Players don't know the exact realization of θ. There is a private signal for each of the players, conditionally independent. Players share their private signals

[54] An alternative measure of mis-coordination is the quadratic loss between own action and the *average* of others' action. Formally, $v_i(a_1, ..., a_n; \theta) = -(1-r)(a_i - \theta)^2 - r(a_i - A_{-i})^2$, $i = 1, ..., n$,where $A_{-i} = \left(\sum_{j \neq i} a_j\right) / (n-1)$ which is reminiscent of the discrete population counterpart of the beauty contest game for a continuum of agents in Morris and Shin (2002). One can readily check that the Bayes-Nash equilibria of the incomplete information game with these payoffs is the same than that with payoffs (4.1). Equilibrium payoffs, however, differ.

according to some communication process, which delivers some output signal $y_i = \theta + \varepsilon_i$. We assume that the vector of output signal follows a multinormal distribution:

$$\mathbf{y}|\theta \sim N\left(\theta \mathbf{1}_n, \mathbf{\Sigma}\right), \tag{4.2}$$

where $\mathbf{1}_n$ is the $n-$dimensional vector of 1s, and $\mathbf{\Sigma} = [\sigma_{ij}]$ is a general ($n-$square and symmetric) variance-covariance matrix.

This general specification of output signals allows for any correlation pattern across individual signals. For instance, when output signals are (conditionally) independent, we have:

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_{nn} \end{bmatrix},$$

with $\sigma_{11} = \cdots = \sigma_{nn}$ if signals are identically distributed. Instead, when output signals have a common public component, we have:

$$\mathbf{\Sigma} = \sigma \begin{bmatrix} 1 & \cdots & \rho \\ \vdots & \ddots & \vdots \\ \rho & \cdots & 1 \end{bmatrix},$$

where $0 \leq \rho \leq 1$ is the correlation across signals induced by the public component.

More generally, the variance-covariance matrix $\mathbf{\Sigma}$ can take any value. In Section 6, we analyze a particular family of communication processes by which players mix their conditionally independent private signals to obtain individual communication reports (the output signals) that can exhibit arbitrary correlation patterns.

Denote by $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot|\widehat{y}_i]$ the expectation operator by player i conditional on his signal realization being $\widehat{y}_i$. Standard algebra on normal distributions leads to:[55]

$$\mathbb{E}_i[\theta] = (1 - f_i)\,\theta_0 + f_i \widehat{y}_i \qquad \text{and} \qquad \mathbb{E}_i[y_j] = (1 - \omega_{ji})\,\theta_0 + \omega_{ji}\widehat{y}_i, \tag{4.3}$$

where

$$f_i(\mathbf{\Sigma}) = \frac{\phi_\theta}{\phi_\theta + \sigma_{ii}} \qquad \text{and} \qquad \omega_{ji}(\mathbf{\Sigma}) = \frac{\phi_\theta + \sigma_{ij}}{\phi_\theta + \sigma_{ii}},$$

for all $i = 1, ..., n$ and $j \neq i$.

[55] See the appendix for more details.

We gather together all the possible values (4.3) for pair-wise inferences about each other's signals into an $n-$square matrix:

$$\mathbf{\Omega}\left(\mathbf{\Sigma}\right) = \left[\begin{array}{ccc} 0 & & \omega_{ij}\left(\mathbf{\Sigma}\right) \\ & \ddots & \\ \omega_{ij}\left(\mathbf{\Sigma}\right) & & 0 \end{array}\right].$$

We set $\omega_{ii} = 0$, for all i. Each of the $n\left(n-1\right)$ out-of-diagonal cells in $\mathbf{\Omega}\left(\mathbf{\Sigma}\right)$ gives the factor by which the row player i multiplies his own report to compute his forecast of every other column player j's report. Notice that the matrix $\mathbf{\Omega}\left(\mathbf{\Sigma}\right)$ is symmetric only when the diagonal cells of $\mathbf{\Sigma}$ are identical, i.e, $\sigma_{11} = \cdots = \sigma_{nn}$. In words, i need not make the same inferences about j's reports than j about i.

We omit the parameter $\mathbf{\Sigma}$ when no confusion is possible.

4.4 High-order beliefs and the knowledge index

4.4.1 Best-responses and high-order beliefs

A strategy maps output signals to actions. At equilibrium, each player maximizes his expected payoffs, where the expectation is taken over the target parameter θ as well as others' actions, and is conditional on own information.

Let $\rho = r/\left(n-1\right)$. Given that payoffs are concave in own action, player i's best response is obtained from $\partial\mathbb{E}_i\left[u_i\left(a;\theta\right)\right]/\partial a_i = 0$. We get:

$$BR_i\left(a_{-i}\right) = \left(1-r\right)\mathbb{E}_i\left[\theta\right] + \rho\sum_{j\neq i}\mathbb{E}_i\left[a_j\right], \text{ for all } i = 1, ..., n. \tag{4.4}$$

The Bayes-Nash equilibria are given by the signal functions or strategies $a_i\left(\widehat{y}_i\right)$ that solve (4.4). It turns out that the linearity of both best-responses and the Gaussian information structure imply that the only signal functions that solve (4.4) are linear in own information.

Before stating and proving formally this result, we provide a simple heuristic that suggests that equilibrium strategies should indeed be linear, and that paves the way towards our closed-form equilibrium characterization.

Nesting best-responses into each other in (4.4) gives:

$$BR_i\left(a_{-i}\right) = \left(1-r\right)\mathbb{E}_i\left[\theta\right] + \left(1-r\right)\rho\sum_{j\neq i}\mathbb{E}_i\mathbb{E}_j\left[\theta\right] + \rho^2\sum_{j\neq i}\sum_{k\neq j}\mathbb{E}_i\mathbb{E}_j\left[a_k\right],$$

so that optimal actions depend on first and second-order iterated expectations about θ.

Recursively iterating the process, one can easily check that $BR_i(\cdot)$ can be written as an infinite sum of arbitrarily high-order iterated expectations of the target value θ:

$$\mathbb{E}_{i_1}\mathbb{E}_{i_2}\cdots\mathbb{E}_{i_p}[\theta], \tag{4.5}$$

which we denote as $\mathbb{E}_{i_1,i_2,\ldots,i_p}[\theta]$, weighted by the geometrically decaying factor ρ^p.

For the sake of illustration, suppose that $\theta_0 = 0$. Then, using (4.3) recursively, we can write these iterated expectations as a function of the inferences players make about each others' output signals:

$$\mathbb{E}_{i_1,i_2,\ldots,i_p}[\theta] = f_{i_p}\omega_{i_p i_{p-1}}\ldots\omega_{i_2 i_1}\widehat{y}_{i_1}. \tag{4.6}$$

Two comments are in order.

First, arbitrarily high-order iterated expectations by player i can be expressed as a linear function of his own output report $\widehat{y}_i$, which suggests a linear solution to (4.4). Indeed, we characterize below a linear equilibrium, which turns out to be the unique equilibrium of the game (provided $r < 1$).

Second, high-order iterated expectations (4.6) depend both on the number of iterations and on the ordered identities of the players along the chain of iterated expectations (expectations about whom expectations, about whom expectations ...). This is not surprising. Indeed, the entries of $\mathbf{\Omega}$ allow for up to $n(n-1)$ different inferences technologies of each others's signals depending on the identity of the two involved players, the inferring one and the inferred one.

In general, we cannot invoke nor construct some average belief operator to compute high-order beliefs, neither expect symmetric behavior from the part of the players. Rather, high-order iterated expectations are an-isotropic, and change with the particular ordered chain of pair-wise inferences. This an-isotropy very likely sustains asymmetric choices across players.[56] This an-isotropy rules out symmetric equilibria, in general.

When $\theta_0 = 0$, using the expression for high-order iterated expectations (4.6), we define high-order beliefs about the value of θ by agent i as follows:

- order one beliefs are $\mathbb{E}_i^1[\theta] = \mathbb{E}_i[\theta] = f_i y_i$;
- order two beliefs are $\mathbb{E}_i^2[\theta] = \sum_{j\neq i}\mathbb{E}_{i,j}[\theta] = \sum_{j\neq i} f_j\omega_{ji}y_i$

[56] Note that $\mathbb{E}_{i,j}[\theta] = f_j\omega_{ji}\widehat{y}_i \neq f_i\omega_{ij}\widehat{y}_j = \mathbb{E}_{j,i}[\theta]$, unless $\widehat{y}_i = \widehat{y}_j$ (notice that $f_j\omega_{ji} = f_i\omega_{ij}$). More generally, $\mathbb{E}_{i_1,i_2,\ldots,i_p}[\theta] \neq \mathbb{E}_{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_p)}[\theta]$ for all non-trivial permutation σ of $\{i_1,\ldots,i_p\}$. Therefore, beyond the identity of the players in the chain of iterated expectations, the order also matters.

- order $p \geq 2$ beliefs are $\mathbb{E}_i^p[\theta] = \sum_{i_p \neq \cdots \neq i_2 \neq i} \mathbb{E}_{i,i_2,\ldots,i_p}[\theta] = \sum_{i_p \neq \cdots \neq i_2 \neq i} f_{i_p} \omega_{i_p i_{p-1}} \ldots \omega_{i_2 i} y_i$

4.4.2 The knowledge index

We now define a *knowledge index* that proves useful to provide a closed-form expression for the equilibrium.

The inference matrix $\mathbf{\Omega}$ keeps track of the pair-wise inference coefficients ω_{ij} across all pairs of players. The pth power $\mathbf{\Omega}^p = \mathbf{\Omega}^{(p\ times)} \overset{\ldots}{} \mathbf{\Omega}$ keeps track of the inference coefficients $\omega_{i_p i_{p-1}} \ldots \omega_{i_2 i_1}$ for all $p-$order chains of iterated inferences that enter the calculation of order p beliefs.

The coordinates of $\mathbf{\Omega 1}_n$ sum, for all row player, the pair-wise inference coefficients about every other player's signal, $\omega_{i1} + ... + \omega_{in}$. More generally, the coordinates of $\mathbf{\Omega}^p \mathbf{1}_n$ sum, for all row player, all the compound inference coefficients that enter the calculation of $p-$order iterated expectations (4.6).

Consider the following infinite sum:

$$\mathbf{1}_n + \rho \mathbf{\Omega 1}_n + \rho^2 \mathbf{\Omega}^2 \mathbf{1}_n + \rho^3 \mathbf{\Omega}^3 \mathbf{1}_n + \cdots = \sum_{p=0}^{+\infty} \rho^p \mathbf{\Omega}^p \mathbf{1}_n = \left[\mathbf{I}_n - \rho \mathbf{\Omega}\right]^{-1} \mathbf{1}_n, \quad (4.7)$$

where $\mathbf{I}_n$ is the $n-$identity matrix. The coefficients of the vector $\left[\mathbf{I}_n - \rho \mathbf{\Omega}\right]^{-1} \mathbf{1}_n$ sum all compound inference coefficients that enter the calculation of arbitrary high-order iterated expectations for each player, weighted by the geometrically decaying factor ρ^p.[57]

Notice that (4.7) is well-defined for low enough values of ρ. It turns out that an exact strict upper bound for convergence to obtain is the inverse of the largest eigenvalue of $\mathbf{\Omega}$ (Debreu and Herstein, 1953). This largest eigenvalue is sometimes called the spectral index of the matrix.

Suppose that $\sigma_{ij} \leq \sigma_{ii}$, for all $i \neq j$. Then, $0 \leq \omega_{ij} \leq 1$, for all $i \neq j$. Together with the fact that $\omega_{ii} = 0$, for all i, we conclude that an upper bound for the spectral index of $\mathbf{\Omega}$ is $n-1$. Therefore, the infinite series (4.7) converges when $\rho < 1/\left(n-1\right)$ or, equivalently, $r < 1$.

Definition 1 *Let $r < 1$ and $\mathbf{\Sigma}$ such that $\sigma_{ij} \leq \sigma_{ii}$, for all $i \neq j$. The vector of individual knowledge indexes is:*

$$\mathbf{k}(r, \mathbf{\Sigma}) = (1-r)\left[\mathbf{I}_n - \rho \mathbf{\Omega}\left(\mathbf{\Sigma}\right)\right]^{-1} \mathbf{1}_n. \quad (4.8)$$

[57] The first term $\mathbf{1}_n$ corresponds to the individual forecast of θ based on own information. The higher order terms correspond to individual forecast of θ that involve inferences about some other players' reports.

Suppose first that output signals are perfectly informative about the true value of θ, that is, $\mathbf{\Sigma} = \mathbf{0}_{n\times n}$. Then, $\omega_{ij} = 1$ for all $i \neq j$ and $k_i(r, \mathbf{0}_{n\times n}) = 1$, for all i. With complete information, the knowledge index is one for all players.

Under incomplete information, instead, the knowledge index takes a value smaller than one for every player, $0 \leq k_i(r, \mathbf{\Sigma}) \leq 1$, for all i. The fact that $\mathbf{k}(r, \mathbf{\Sigma})$ is non-negative derives from the fact that the matrix $[\mathbf{I}_n - \rho\mathbf{\Omega}]^{-1}$ is non-negative when the spectral index condition holds (Debreu and Herstein, 1953), which is true in this case by the discussion above. When there is no coordination concern, the knowledge index is one for all players, $k_i(0, \mathbf{\Sigma}) = 1$, for all i.

In general, the knowledge index varies across players in a way that reflects the anisotropy of high-order iterated expectations. This index is formally reminiscent of standard centrality measures in sociology, but is computed with an information correlation matrix rather than with socio-metric data.

The knowledge index also depends on the value of r. Indeed, it is the coordination concern that triggers the high-order iterated expectations that boil down to (4.8). How the knowledge changes with r is not trivial, as $[\mathbf{I}_n - \rho\mathbf{\Omega}]^{-1}$ is an increasing (infinite) polynomial in r whereas the multiplicative factor $(1 - r)$ decreases with r. The next result shows that the latter effect dominate. However, because of these two conflicting effects, the elasticity of the knowledge index with respect to r is smaller than one.

Proposition 1 $\mathbf{k}(r, \mathbf{\Sigma})$ *is non-increasing with* r.

4.5 Equilibrium

We are now ready to state the next result that establishes uniqueness of the Bayes-Nash equilibrium, fully characterized in terms of the knowledge index.

Theorem 1 *Let* $r < 1$ *and* $\mathbf{\Sigma}$ *such that* $\sigma_{ij} \leq \sigma_{ii}$, *for all* $i \neq j$. *The unique Bayes-Nash equilibrium has strategies linear in output signals given by:*

$$a_i^* (\widehat{y}_i) = (1 - k_i)\,\theta_0 + k_i \mathbb{E}_i [\theta] = (1 - k_i)\,\theta_0 + k_i f_i \widehat{y}_i, \tag{4.9}$$

for all $i = 1, ..., n$, *where* $k_i(r, \mathbf{\Sigma})$ *is the knowledge index defined in (4.8).*

When $r = 0$, players face a simple decision problem, $u_i(a; \theta) = -(a_i - \theta)^2$. The optimal action is their individual forecast $\mathbb{E}_i [\theta] = f_i \widehat{y}_i$; the knowledge index is $k_i (0, \mathbf{\Sigma}) = 1$.

When $0 < r < 1$, instead, payoffs are interdependent. Players now need to conciliate the decision problem with the coordination concern. At equilibrium, they rely on their own signal in proportion to their knowledge index. Otherwise, the mean prior acts as a focal action. The equilibrium behavior reflects the anisotropy of the chains of pair-wise inferences, itself related to the details of the variance-covariance matrix of individual signals.

Notice that the focal action (here, the mean prior θ_0 of the target value) serves very well the purpose of minimizing the coordination loss. However, it induces an individual decision loss equal to $-(1-r)\,\phi_\theta$, proportional to the variance of the prior distribution. This is exactly the same loss that players would incur if no output signal were available. Instead, if players use some of the information contained in their output signal, they can reduce this decision loss. On top of that, and even more when individual signals are correlated, players can also use individual signals to draw inferences about others' information. Output signals can thus also be useful on the coordination front. The knowledge index, which depends both on the salience of the coordination problem r, and on the variance-covariance matrix of output signals $\mathbf{\Sigma}$, reflects the optimal equilibrium use of private information to reduce both decision and coordination losses. The focal action is used in proportion with the lack of knowledge.

The uniqueness and linearity result follow from a central theorem by Radner (1962) on teams, and the fact that our quadratic game payoffs admit a potential that represents common (team) interests for all players, as pointed out by Ui (2004). The particular closed-form for the equilibrium strategy, that involves explicitly the knowledge index, exploits the intimate connection between quadratic games and centrality indexes in sociology established by Ballester $et\ al.$ (2006).

Combining Theorem and Proposition1, we can conclude that when the coordination problem becomes more acute (that is, r increases), players shift weight from their information to the focal action. Note, however, that the decrease in the information weight is smaller than the increase in the coordination concern. Indeed, the elasticity of $\mathbf{k}$ with respect to r is smaller than one. This is because, as explained above, information also serves (partly) the purpose of coordinating actions.

Theorem 1 establishes existence, uniqueness and linearity of the Bayes-Nash equilibrium when $r < 1$ and $\sigma_{ij} \leq \sigma_{ii}$, for all $i \neq j$. This last condition on $\mathbf{\Sigma}$ is always true if signal accuracies are all the same, $\sigma_{11} = \cdots = \sigma_{nn}$. Otherwise, it imposes bounds on the signal correlations. Indeed, writing covariances as $\sigma_{ij} = \gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}$, where $0 \leq \gamma_{ij} \leq 1$ is the correlation coefficient, the condition becomes $\gamma_{ij}^2 \leq \sigma_{jj}/\sigma_{ii}$, for all i, j.

However, this condition on $\mathbf{\Sigma}$ can be relaxed, as stated below.

Remark 1 *Let r and $\mathbf{\Sigma}$ such that the largest eigenvalue of $\rho\mathbf{\Omega}(\mathbf{\Sigma})$ is strictly* *smaller than one. Then, (4.8) is well-defined and the unique Bayes-Nash equilibrium has strategies linear in output signals given by (4.9).*

4.6 Welfare

From now on, we suppose that $\mathbf{\Sigma}$ such that $\sigma_{ij} \leq \sigma_{ii}$, for all $i \neq j$, unless otherwise specified. We also assume that $\theta_0 = 0$ to simplify computations without altering the qualitative conclusions.

We compute the aggregate ex ante equilibrium payoffs given by:

$$U^* (r, \mathbf{\Sigma}) = \mathbb{E}_\theta \mathbb{E}_{(y_1,...,y_n)|\theta} [\sum_{i=1}^{n} u_i (a^* (y)) |\theta]. \tag{4.10}$$

Define the following diagonal matrix of forecast coefficients (4.3), with zeros off-diagonal:

$$\mathbf{F} (\mathbf{\Sigma}) = \begin{bmatrix} f_1 (\mathbf{\Sigma}) & & 0 \\ & \ddots & \\ 0 & & f_n (\mathbf{\Sigma}) \end{bmatrix}.$$

Straight algebra leads to the following expression.

Proposition 2 *Let $r < 1$. The aggregate ex ante equilibrium payoffs (4.10) are:*

$$U^* (r, \mathbf{\Sigma}) = \phi_\theta (1 - r) \left[\mathbf{k}^t (r, \mathbf{\Sigma}) \mathbf{F} (\mathbf{\Sigma}) \mathbf{k} (r, \mathbf{\Sigma}) - n\right] \leq 0. \tag{4.11}$$

There are two sources of welfare losses, coordination losses and decision losses.

Decision losses correspond to the external target concern. They reflect the inaccuracies of the individual information to correctly forecast the true value of θ. When $r = 0$, the expression for ex ante payoffs (4.11) becomes:

$$U^* (0, \mathbf{\Sigma}) = \phi_\theta \left[\sum_{i=1}^{n} f_i (\mathbf{\Sigma}) - n\right],$$

which is an increasing function of the forecast coefficients f_i, $i = 1, ..., n$. The forecast coefficients are monotone increasing with the individual signal accuracies $1/\sigma_{ii}$.

Coordination losses correspond to the internal coordination concern. Miscoordination reflects the heterogeneity in equilibrium play, itself related to the heterogeneity in knowledge indexes. Indeed, straightforward algebra shows that coordination losses are proportional to $\sum_i f_i \left(k_i - k_i^2\right)$. Coordination losses are small if k_i is close to zero for all i, that is, all players take an action very close to the focal action. Coordination losses are also small when k_i is close to one for all i, that is, either coordination concerns vanish ($r = 0$) or there is no incomplete information ($\mathbf{\Sigma} = \mathbf{0}_{n\times n}$).

Information structure and welfare

Expression (4.11) implies that $U^*\left(r, \mathbf{\Sigma}\right)$ is an increasing function of

$$\mathbf{k}^t\left(r, \mathbf{\Sigma}\right)\mathbf{F}\left(\mathbf{\Sigma}\right)\mathbf{k}\left(r, \mathbf{\Sigma}\right)$$

from which we derive the following result.

Theorem 2 *Let* $\mathbf{\Sigma}, \mathbf{\Sigma}'$ *such that* $\mathbf{F}\left(\mathbf{\Sigma}'\right) \geq \mathbf{F}\left(\mathbf{\Sigma}\right)$ *and* $\mathbf{\Omega}(\mathbf{\Sigma}') \geq \mathbf{\Omega}\left(\mathbf{\Sigma}\right)$. *Then,* $U^*\left(r, \mathbf{\Sigma}'\right) \geq U^*\left(r, \mathbf{\Sigma}\right)$.

This general comparative statics result of equilibrium welfare with respect to the variance-covariance matrix has a number of implications that we now explore.

We first establish comparative statics result with respect to information accuracy and correlations.

We write covariances as $\sigma_{ij} = \gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}$, where $0 \leq \gamma_{ij} \leq 1$ is the correlation coefficient. In other words, the variance-covariance matrix $\mathbf{\Sigma}$ is uniquely defined by a vector of accuracies:

$$\boldsymbol{\sigma} = \left(\sigma_{11}, ..., \sigma_{nn}\right)$$

and a symmetric correlation matrix:

$$\mathbf{\Gamma} = \begin{bmatrix} 1 & & \gamma_{ij} \\ & \ddots & \\ \gamma_{ij} & & 1 \end{bmatrix}.$$

Proposition 3 $U^*\left(r, \mathbf{\Sigma}\right)$ *is non-decreasing with* $\mathbf{\Gamma}$ *and non-increasing with* $\boldsymbol{\sigma}$ *(for the component-wise partial order)*

Aggregate welfare increases when individual information becomes more accurate and/or correlation increases.

Remark 2 *Suppose that equilibrium existence is obtained under the general spectral index condition of Remark1. Then, the monotonicity with respect to* $\mathbf{\Gamma}$ *always holds. The monotonicity with respect to* $\boldsymbol{\sigma}$ *holds if correlations are not* *too high, that is,* $\gamma_{ij} \leq 2\sqrt{\sigma_{ii}/\sigma_{jj}}$*, for all* i, j*. A sufficient condition is that individual accuracies are not too different, that is,* $\max_i\{\sigma_{ii}\} \leq 4\min_i\{\sigma_{ii}\}$.

We now establish a comparative statics result with respect to the correlation pattern.

Proposition 4 *Suppose that* $\sigma_{11} = \cdots = \sigma_{nn}$*. Let* j, l *such that* $k_l(r, \mathbf{\Sigma}) > k_j(r, \mathbf{\Sigma})$ *and* $i \neq j, l$ *such that* $\gamma_{ij} > \gamma_{il}$*. Let* $\mathbf{\Sigma}'$ *obtained from* $\mathbf{\Sigma}$ *by swapping* γ_{ij} *and* γ_{il}*. Then* $\mathbf{k}(r, \mathbf{\Sigma}') > \mathbf{k}(r, \mathbf{\Sigma})$ *and* $U^*(r, \mathbf{\Sigma}') > U^*(r, \mathbf{\Sigma})$.

In words, the knowledge index increases for all players when players with a higher knowledge index make better inferences about every other player's information than players with a lower knowledge index. As a consequence, aggregate welfare is higher.

4.6.1 Payoffs and welfare

Payoffs are a weighted sum of an external target concern and an internal coordination concern, with weights $1 - r$ and r, respectively. The external target concern induces a loss that depends on the forecast accuracy of the value of θ. This loss increases when players receive noisier information. The coordination concern induces a loss that depends on the discrepancy of actions across players. When the dispersed information available to players differs widely across them, the coordination loss is higher the more individual actions are sensitive to private information. As a matter of fact, Proposition 1 together with (4.9) imply that players shift weight from their information towards the focal action when r increases. If the information available to them is not very accurate to start with, this allows players to decrease coordination losses while alleviating the burden of the decision loss.

Proposition 5 $U^*(r, \mathbf{\Sigma})$ *is monotone increasing with* r *whenever* $\sigma_{ii} \geq \underline{\sigma}$*, for all* $i = 1, ..., n$ *for some* $\underline{\sigma}$*.*

In words, when the information available to players is noisy, decreasing the forecast problem and increasing the coordination problem reduces equilibrium welfare loss.

The social optimum

We now compare the equilibrium use of information with the efficient use of information. The next result characterizes the optimal actions a social planner would implement.

Proposition 6 *Let $r < 1$. The socially optimal action for each player is:*

$$a_i^S(\widehat{y}_i) = \left(1 - k_i\left(\frac{2r}{1+r}, \mathbf{\Sigma}\right)\right)\theta_0 + k_i\left(\frac{2r}{1+r}, \mathbf{\Sigma}\right)\mathbb{E}_i[\theta].$$

The social optimal action is equal to the equilibrium action of a game where the weight assigned to the coordination loss is increased from r to $2r/(1+r)$. Notice that Proposition 1 implies that $\mathbf{k}(2r/(1+r), \mathbf{\Sigma}) \leq \mathbf{k}(r, \mathbf{\Sigma})$. At equilibrium, agents overuse their private information compared to the social optimum. Indeed, at the socially optimal actions each player relies more on the prior information and less on the private information than at the equilibrium action.

4.7 Discussion

We have developed a model of communication in organizations that encompasses many different possible communication processes. A communication process determines how individual information on the common task to be performed flows between organization's members, and therefore the quality of the final report each agent handles (accuracy) as well as the level of common information between pairs of individuals (correlations). We have been able to characterize for the game that models the common interests of all agents its unique Bayes-Nash equilibrium. This characterization provides a neat picture of how higher-order beliefs are formed and how this higher-order beliefs are embedded in the equilibrium actions profile. This embeddedness is subsumed in the knowledge index of each agent, that determines the heterogeneity on agents' actions. The knowledge index measures how much an agent can rely on the information subsumed in his communication report to strategically internalize the double concern, the decision problem and the coordination motive, to infer correctly the state of the world and the actions of the rest of players. Furthermore, knowledge indexes prove useful not only to characterize equilibrium actions but also equilibrium payoffs. Finally, we have been able to provide comparative statics results on the different parameters that characterize communication processes, and compare the socially optimal actions to the equilibrium actions.

We believe that our methodology can be extended to more general payoff structures, with either strategic complementarities or substitutabilities. As far as we maintain quadratic payoffs, a natural assumption in the literature of games with complementarities, we can probably ensure unicity for some range of parameters, and the linearity of best-responses suggests that similar knowledge index structures could be obtained.

The literature of information aggregation in political economy (see, for example, Sobel, 2006 and references therein) provides a rich set of questions in which communication is a crucial element before deliberation. For example, different aspects of decisions made by committees have been studied (Li et al., 2001 and Swank and Visser, 2007). Adapting our model to analyze decisions by committees, we could study how different communication protocols affect the final decision of the committee.

Industrial organization and finance are some other areas in which information aggregation is an important element (Vives, 1988). Vives (2007) provides an exhaustive survey of the literature on information aggregation in markets. Our tools could be applied for example to analyze in an oligopoly setup how different forms of collaboration structures, derived from needs to communicate information about uncertain aspects of markets, impacts firms' decisions and market outcomes.

In some of these other environments there can exist an strategic component on communication that in our model is missing. To which extend some of these related questions can be addressed with a similar methodology that the one we have developed here is a question that we plan to pursue in future work.

4.8 Proofs

From (4.2) we deduce that $(\theta, y_1, ..., y_n)$ follows the following multinormal distribution:

$$(\theta, y_1, ..., y_n) \sim N(\mathbf{0}_{n+1}, \begin{bmatrix} \phi_\theta & \phi_\theta \mathbf{1}_n^t \\ \phi_\theta \mathbf{1}_n & \mathbf{\Sigma} \end{bmatrix}),$$

where $\mathbf{0}_{n+1}$ is the $(n+1)-$dimensional vector of all zeros. The previous expression leads to:

$$(y_1, ..., y_n) \sim N\left(\mathbf{0}_n, \phi_\theta \mathbf{J}_n + \mathbf{\Sigma}\right), \tag{4.12}$$

where $\mathbf{J}_n$ is the $n-$square matrix of all ones.

Proof of Proposition 1: Let $\mathbf{M} = [\mathbf{I}_n - \rho\mathbf{\Omega}]^{-1}$. Noticing that $\mathbf{M}\cdot[\mathbf{I}_n - \rho\mathbf{\Omega}] = \mathbf{I}_n$ and differentiating with respect to ρ leads to:

$$\frac{\partial}{\partial \rho}\mathbf{M} = \mathbf{\Omega} \cdot \mathbf{M}^2,$$

and thus

$$\frac{\partial}{\partial r}\mathbf{M} = \frac{1}{n-1}\mathbf{\Omega} \cdot \mathbf{M}^2. \tag{4.13}$$

From $\mathbf{k} = (1-r)\,\mathbf{M}\cdot\mathbf{1}_n$ and (4.13) we obtain:

$$\frac{\partial}{\partial r}\mathbf{k} = -\frac{1}{1-r}\mathbf{k} + \frac{1-r}{n-1}\mathbf{\Omega}\mathbf{M}\mathbf{k},$$

and thus, since $[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{\Omega} = \mathbf{\Omega}\,[\mathbf{I}_n - \rho\mathbf{\Omega}]$ and $[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{M} = \mathbf{I}_n$,

$$\begin{aligned}[\mathbf{I}_n - \rho\mathbf{\Omega}]\frac{\partial}{\partial r}\mathbf{k} &= -\frac{1}{1-r}[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{k} + \frac{1-r}{n-1}[\mathbf{I}_n - \rho\mathbf{\Omega}]\,\mathbf{\Omega}\mathbf{M}\mathbf{k} \\ &= -\mathbf{1}_n + \frac{1-r}{n-1}\mathbf{\Omega}\mathbf{k}.\end{aligned}$$

We know that $\mathbf{0}_n \leq \mathbf{k} \leq \mathbf{1}_n$, and that $\frac{1}{n-1}\mathbf{\Omega}\mathbf{1}_n \leq \mathbf{1}_n$ (from the fact that $0 \leq \omega_{ij} \leq 1$ for all i, j). Therefore, $\frac{1-r}{n-1}\mathbf{\Omega}\mathbf{k} \leq \mathbf{1}_n$, that is,

$$[\mathbf{I}_n - \rho\mathbf{\Omega}]\frac{\partial}{\partial r}\mathbf{k} \leq \mathbf{0}. \tag{4.14}$$

We show that $\partial k_i/\partial r \leq 0$, for all i. Suppose not. Let $i^* \in \arg\max\{\partial k_i/\partial r : i = 1, ..., n\}$. By assumption, $\partial k_{i^*}/\partial r > \mathbf{0}$. We have:

$$\rho\mathbf{\Omega}\frac{\partial}{\partial r}\mathbf{k} \leq \frac{\partial k_{i^*}}{\partial r}\rho\mathbf{\Omega}\mathbf{1}_n \leq r\frac{\partial k_{i^*}}{\partial r}\mathbf{1}_n.$$

The $i^* - th$ coordinate of the left-hand side of (4.14) is thus bounded from below by:

$$(1-r)\frac{\partial k_{i^*}}{\partial r} > 0,$$

which contradicts (4.14). ■

Proof of Theorem 1: We look for a linear equilibrium strategy, $a_i(\widehat{y}_i) = \alpha_i + \beta_i\widehat{y}_i$. Plugging back into (4.4), we obtain a linear system of equations with unknowns $\{\alpha_i, \beta_i\}_{i=1,\ldots,n}$:

$$\alpha_i + \beta_i\widehat{y}_i = (1-r)\,\mathbb{E}_i[\theta] + \rho\sum_{j\neq i}\left(\alpha_j + \beta_j\mathbb{E}_i[y_j]\right), \quad \text{for all } \widehat{y}_i \in \mathbb{R} \text{ and } i = 1, ..., n. \tag{4.15}$$

We first show that (4.9) is indeed a Bayes-Nash equilibrium. Plugging back (4.3) into (4.15), and subtracting θ_0 from both sides of (4.15), we get the following equilibrium conditions:

$$\alpha_i = \rho \sum_{j \neq i} \alpha_j, \text{ for all } i = 1, ..., n,$$

Therefore, $\alpha_i = 0$, for all i, and:

$$\beta_i = (1-r) f_i + \rho \sum_{j \neq i} \beta_j \omega_{ji}, \text{ for all } i = 1, ..., n. \tag{4.16}$$

Notice that $f_j \omega_{ji} = f_i \omega_{ij}$, for all i,j. Let $\gamma_i = \beta_i / f_i$. Dividing equation i by f_i in (4.16) leads to:

$$\gamma_i = (1-r) + \rho \sum_{j \neq i} \omega_{ij} \gamma_j, \text{ for all } i = 1, ..., n.$$

Notice also that $c(i,j) \leq g_j$, and thus $\omega_{ij} \leq 1$, for all i, j. Denote by $\mu(\mathbf{A})$ the largest eigenvalue of a matrix $\mathbf{A}$. Then, $\mu(\mathbf{\Omega}) \leq \mu(\mathbf{I}_n - \mathbf{J}_n) = n-1$. Therefore $\rho\mu(\mathbf{\Omega}) < 1$ is equivalent to $r < 1$. From Theorem 1 in Ballester et $al.$ (2006) we deduce that $\beta_i = f_i k_i(r, \mathbf{\Sigma})$.

We now show that the linear equilibrium identified in (4.9) is the unique equilibrium. We proceed in two steps.

First, define the following payoff function:

$$V(a_1, ..., a_n) = -(1-r) \sum_{i=1}^{n} (a_i - \theta)^2 - \frac{1}{2} \rho \sum_{i=1}^{n} \sum_{j \neq i} (a_i - a_j)^2. \tag{4.17}$$

It is readily checked that $\partial \mathbb{E}[u_i(a) | y_i] / \partial a_i = \partial \mathbb{E}[V(a) | y_i] / \partial a_i$, for all action profile and information. In words, $V(\cdot)$ is a potential for the game payoffs u_i.[58] Therefore, the Bayesian Nash equilibria coincide with the team person-by-person maximal decisions for the team objective function (4.17) and the information structure (4.2).

Second, Theorem 4 in Radner (1962) gives a sufficient condition for uniqueness of the optimal and the person-by-person maximal team decision functions (which, then, necessarily coincide) when team payoffs are quadratic as in (4.17). This condition boils down to the $n-$square matrix of cross derivatives $\mathbf{Q} =$

[58] See Ui (2004) for a formal definition and a general existence result of a (Bayesian) potential for Bayesian quadratic games with symmetric cross-derivatives of payoffs.

$[q_{ij}] = [\partial^2 V(a) / \partial a_i \partial a_j]$ being negative definite. We have:

$$q_{ij} = \begin{cases} -1, \text{ if } i = j \\ \rho, \text{ if } i \neq j \end{cases} .$$

We compute the determinant of $\mathbf{Q}$:

$$\begin{aligned}
\det(\mathbf{Q}) &= (-1)^n \begin{vmatrix} 1 & -\rho & \cdots & -\rho \\ -\rho & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\rho \\ -\rho & \cdots & -\rho & 1 \end{vmatrix} \\
&= (-1)^n (1-r) \begin{vmatrix} 1 & -\rho & \cdots & -\rho \\ 1 & \ddots & \ddots & \vdots \\ \vdots & -\rho & \ddots & -\rho \\ 1 & & -\rho & 1 \end{vmatrix} \\
&= (-1)^n (1-r) \begin{vmatrix} 1 & -\rho & \cdots & -\rho \\ 0 & 1+\rho & 0 & \\ \vdots & \cdots & \cdots & -\rho \\ 0 & \cdots & 0 & 1+\rho \end{vmatrix} \\
&= (-1)^n (1-r)(1+\rho)^{n-1}
\end{aligned}$$

The first equality is obtained by adding up all the columns to the first one, and then factorizing by the common term $1 - r$. The second equality is obtained by subtracting the first row to every other row. We are left with an upper triangular matrix; the determinant is just the product of the diagonal terms.

The determinant of $\mathbf{Q}$ has thus the same sign than $(-1)^n$. Mutatis mutandis, we deduce that the minors of order p of $\mathbf{Q}$ have the sign of $(-1)^p$. The matrix $\mathbf{Q}$ is thus definite negative, and this concludes the proof. ■

Proof of Proposition 2: Developing the square terms and summing over all is gives:

$$\begin{aligned}
\sum_{i=1}^{n} u_i(a^*(y)) &= -(1-r)\left[n\theta^2 + \sum_{i=1}^{n} a_i^{*2} - 2\theta \sum_{i=1}^{n} a_i^*\right] \\
&\quad -\rho\left[2(n-1)\sum_{i=1}^{n} a_i^{*2} - 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^* a_j^*\right] .
\end{aligned} \tag{4.18}$$

Then, noticing that $\mathbb{E}[\theta] = 0$ and $\mathbb{E}\left[\theta^2\right] = \phi_0$, we obtain the following identities:

$$\begin{aligned}
\mathbb{E}_\theta \mathbb{E}_{(y_1,\ldots,y_n)|\theta}\left[a_i^*|\theta\right] &= \phi_\theta f_i k_i(r,\mathbf{\Sigma}) \\
\mathbb{E}_\theta \mathbb{E}_{(y_1,\ldots,y_n)|\theta}\left[a_i^{*2}|\theta\right] &= \phi_\theta f_i k_i^2(r,\mathbf{\Sigma}) \\
\mathbb{E}_\theta \mathbb{E}_{(y_1,\ldots,y_n)|\theta}\left[a_i^* a_j^*|\theta\right] &= \phi_\theta f_i \omega_{ij} k_i(r,\mathbf{\Sigma}) k_j(r,\mathbf{\Sigma}),\, i \neq j
\end{aligned}$$

Using (4.18) and the previous identities, we obtain:

$$\begin{aligned}
U^*(r,\mathbf{\Sigma}) &= -(1-r)\, n\phi_\theta + 2\rho\phi_\theta \sum_{i=1}^n \sum_{j=1}^n f_i \omega_{ij} k_i(r,\mathbf{\Sigma}) k_j(r,\mathbf{\Sigma}) - \\
&\quad -(1+r)\,\phi_\theta \sum_{i=1}^n f_i k_i^2(r,\mathbf{\Sigma}) + 2\phi_\theta \sum_{i=1}^n f_i k_i(r,\mathbf{\Sigma}).
\end{aligned}$$

Next, notice that:

$$\sum_{i=1}^n \sum_{j=1}^n f_i \omega_{ij} k_i(r,\mathbf{\Sigma}) k_j(r,\mathbf{\Sigma}) = (\mathbf{F}\mathbf{k})^t\, \mathbf{\Omega}\mathbf{k} = \mathbf{k}^t \mathbf{F} \mathbf{\Omega} \mathbf{k} = \frac{1}{\rho} \mathbf{k}^t \mathbf{F}\left[\mathbf{k} - \mathbf{1}_n\right].$$

Plugging back into the previous expression for $U^*(r,\mathbf{\Sigma})$ and simplifying gives the desired result. ■

Proof of Theorem 2: It is clear that $U^*(r,\mathbf{\Sigma})$ increases with $\mathbf{F}$ and with $\mathbf{k}$ (for the component-wise ordering). It is also clear that $\mathbf{F}(\mathbf{\Sigma})$ increases with $\mathbf{\Sigma}$. We show that $\mathbf{k}(r,\mathbf{\Sigma})$ increases with $\mathbf{\Omega}$. Let $\mathbf{\Omega}' \geq \mathbf{\Omega}$. We write $\mathbf{\Omega}' = \mathbf{\Omega} + \mathbf{D}$, with $d_{ij} \geq 0$ for all i, j. Then,

$$\left[\mathbf{I}_n - \rho\mathbf{\Omega}\right]\mathbf{k} = \frac{1}{1-r} = \left[\mathbf{I}_n - \rho\mathbf{\Omega}'\right]\mathbf{k}' = \left[\mathbf{I}_n - \rho\mathbf{\Omega}\right]\mathbf{k}' - \rho\mathbf{D}\mathbf{k}',$$

from which we obtain:

$$\mathbf{k}' - \mathbf{k} = \rho\left[\mathbf{I}_n - \rho\mathbf{\Omega}\right]^{-1}\mathbf{D}\mathbf{k}',$$

and thus $\mathbf{k}' - \mathbf{k} \geq \mathbf{0}$ from the fact that $\left[\mathbf{I}_n - \rho\mathbf{\Omega}\right]^{-1}$ is a non-negative matrix when the spectral index condition holds (Debreu and Herstein, 1953). ■

Proof of Proposition 3: The proof uses the comparative statics of the knowledge index with respect to $\mathbf{\Omega}$ established in the proof of Theorem 2, and the following calculations:

$$\begin{aligned}\frac{\partial\omega_{ji}}{\partial\sigma_{ii}} &= \frac{\partial}{\partial\sigma_{ii}}\left(\frac{\phi_\theta+\gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}}{\phi_\theta+\sigma_{ii}}\right)\\ &= \frac{1}{(\phi_\theta+\sigma_{ii})^2}\left[\frac{1}{2}\gamma_{ij}\sqrt{\frac{\sigma_{jj}}{\sigma_{ii}}}(\phi_\theta+\sigma_{ii})-\left(\phi_\theta+\gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}\right)\right]\\ &= \frac{1}{(\phi_\theta+\sigma_{ii})^2}\left[\frac{1}{2}\phi_\theta\gamma_{ij}\sqrt{\frac{\sigma_{jj}}{\sigma_{ii}}}-\frac{1}{2}\gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}-\phi_\theta\right]\end{aligned}$$

Let

$$F(x)=\frac{1}{2}x\gamma_{ij}\sqrt{\frac{\sigma_{jj}}{\sigma_{ii}}}-\frac{1}{2}\gamma_{ij}\sqrt{\sigma_{ii}\sigma_{jj}}-x.$$

A sufficient condition for $\partial\omega_{ji}/\partial\sigma_{ii}\leq 0$ is that $F(x)\leq 0$ for all $x\geq 0$. Notice that $F(0)\leq 0$. So, it suffices to check that $F'(x)\leq 0$ for all $x\geq 0$, that is:

$$\gamma_{ij}\sqrt{\frac{\sigma_{jj}}{\sigma_{ii}}}\leq 2\Leftrightarrow\sigma_{ij}\leq 2\sigma_{ii},\text{ for all } i,j,$$

which is true under the conditions of Theorem . ■

Proof of Proposition 4: Let $\mathbf{\Omega}(\lambda)$ be a matrix identical to $\mathbf{\Omega}$ but for cells i,j and k,l where the new coefficients are, respectively, $(1-\lambda)\omega_{ij}+\lambda\omega_{kl}$ and $(1-\lambda)\omega_{kl}+\lambda\omega_{ij}$, with $0\leq\lambda\leq 1$. Notice that $\mathbf{\Omega}(0)=\mathbf{\Omega}$. We have:

$$\frac{\partial}{\partial\lambda}\mathbf{M}=\rho\mathbf{M}\left(\frac{\partial}{\partial\lambda}\mathbf{\Omega}\right)\mathbf{M},$$

and thus

$$\frac{\partial}{\partial\lambda}\mathbf{k}=\rho\mathbf{M}\left(\frac{\partial}{\partial\lambda}\mathbf{\Omega}\right)\mathbf{k}.$$

Notice that $\partial\mathbf{\Omega}/\partial\lambda$ is a matrix with all cells equal to zero but for cells i,j and k,l equal, respectively, to $-(\omega_{ij}-\omega_{kl})$ and $\omega_{ij}-\omega_{kl}$. When $i=k$, we have:

$$\frac{\partial}{\partial\lambda}\mathbf{k}=\rho(\omega_{ij}-\omega_{il})(k_l-k_j)\begin{bmatrix}m_{1i}\\ \vdots\\ m_{ni}\end{bmatrix},$$

and thus the sign of $\partial\mathbf{k}/\partial\lambda$ is that of $(\omega_{ij}-\omega_{il})(k_l-k_j)$. The result follows from the fact that swapping is equivalent to setting $\lambda=1$. ■

Proof of Proposition 5: Recall that equilibrium payoffs are

$$U^*(r,\mathbf{\Sigma}) = \phi_0 (1-r) \left[\mathbf{k}^t \mathbf{F}\mathbf{k} - n\right].$$

Therefore:

$$\frac{1}{\phi_\theta}\frac{\partial}{\partial r}U^*(r,\mathbf{\Sigma}) = n - \mathbf{k}^t\mathbf{F}\mathbf{k} + (1-r)\left(\frac{\partial}{\partial r}\mathbf{k}\right)^t \mathbf{F}\mathbf{k}. \tag{4.19}$$

Notice that $n - \mathbf{k}^t\mathbf{F}\mathbf{k} \geq 0$ while $(\partial\mathbf{k}/\partial r)^t\,\mathbf{F}\mathbf{k} \leq 0$, so that (4.19) is the sum of a positive and a negative term, with ambiguous overall sign. However, $\mathbf{\Omega}$, $\mathbf{M}$, and $\mathbf{k}$ all take values in a compact set when $\mathbf{\Sigma} \in \mathbb{R}^{n\times n}$, with $\sigma_{ij} \leq \sigma_{ii}$, for all i, j. By (4.14), so does $\partial\mathbf{k}/\partial r$. Also, $\lim_{(\sigma_{11},\ldots,\sigma_{nn})\uparrow+\infty}\mathbf{F}(\mathbf{\Sigma}) = \mathbf{0}_{n\times n}$. Therefore,

$$\lim_{(\sigma_{11},\ldots,\sigma_{nn})\uparrow+\infty}\frac{1}{\phi_\theta}\frac{\partial}{\partial r}U^* = n.$$

■

Proof of Proposition 6: The expression in (4.18) is valid for any profile of actions. Renormalizing it we have:

$$\widetilde{U} = -(1+r)\sum_{i=1}^{n} a_i^2 + 2(1-r)\,\theta\sum_{i=1}^{n} a_i + 2\rho\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j.$$

Let's compute the optimal linear social actions of each agent, $a^S(y_i) = \alpha_i^S + \beta_i^S f_i y_i$, that maximize ex ante social welfare $\mathbb{E}_\theta\mathbb{E}_{y|\theta}\widetilde{U}$. Straightforward algebra gives:

$$\begin{aligned}\mathbb{E}_\theta\mathbb{E}_{y|\theta}\widetilde{U} \;=\; & -(1+r)\sum_i\left[\alpha_i^2 + 2\alpha_i\beta_i f_i\theta_0 + \beta_i^2 f_i^2\left(\sigma_{ii}+\phi_\theta+\theta_0^2\right)\right] + \\ & +2(1-r)\sum_i\left[\alpha_i\theta_0 + \beta_i f_i\left(\phi_\theta+\theta_0^2\right)\right] + \\ & +2\rho\sum_i\sum_{j\neq i}\left[\alpha_i\alpha_j + \alpha_i\beta_j f_j\theta_0 + \alpha_j\beta_i f_i\theta_0 + \beta_i f_i\beta_j f_j\left(\sigma_{ij}+\phi_\theta+\theta_0\right)\right]\end{aligned} \tag{4.20}$$

The first-order conditions to maximize $\mathbb{E}_\theta\mathbb{E}_{y|\theta}\widetilde{U}$ are

$$\frac{\partial \mathbb{E}_\theta \mathbb{E}_{y|\theta} \widetilde{U}}{\partial \alpha_i} = -\left(1+r\right)\left(2\alpha_i + 2\beta_i f_i \theta_0\right)$$
$$+2\left(1-r\right)\theta_0 + 4\rho \sum_{j \neq i} \left(\alpha_j + \beta_j f_j \theta_0\right) = 0$$

$$\frac{\partial \mathbb{E}_\theta \mathbb{E}_{y|\theta} \widetilde{U}}{\partial \beta_i} = -\left(1+r\right)\left[\alpha_i f_i \theta_0 + \beta_i f_i^2 \left(\sigma_{ii} + \phi_\theta + \theta_0^2\right)\right] + \left(1-r\right) f_i \left(\phi_\theta + \theta_0^2\right) \quad (4.21)$$
$$+2\rho \sum_{j \neq i} \left[\alpha_j f_i \theta_0 + \beta_j f_i f_j \left(\sigma_{ij} + \phi_\theta + \theta_0^2\right)\right] = 0 \quad (4.22)$$

We can rewrite the first-order conditions with respect to the αs as

$$a_i^S\left(\theta_0\right) = \frac{1-r}{1+r}\theta_0 + \frac{2r}{1+r}\frac{1}{n-1}\sum_{j \neq i} a_j^S\left(\theta_0\right). \quad (4.23)$$

The social optimal action of each agent evaluated at θ_0 is a weighted sum of θ_0 and the mean of the other player's actions evaluated at θ_0.

Similarly, the first-order conditions with respect to the βs can be rewritten, whenever $\theta_0 \neq 0$, as

$$a_i^S\left(\theta_0 + \frac{1}{\theta_0}\left(\phi_\theta + \sigma_{ii}\right)\right) = \frac{1-r}{1+r}\left(\theta_0 + \frac{\phi_\theta}{\theta_0}\right) \quad (4.24)$$
$$+\frac{2r}{1+r}\frac{1}{n-1}\sum_{j \neq i} a_j^S\left(\theta_0 + \frac{1}{\theta_0}\left(\phi_\theta + \sigma_{ij}\right)\right).$$

Subtracting (4.23) from (4.24) we obtain the following linear system of equations for the βs:

$$\beta_i^S f_i = \frac{1-r}{1+r} f_i + \frac{2r}{1+r}\frac{1}{n-1}\sum_{j \neq i} \beta_j^S f_j \omega_{ji}, \text{ for all } i = 1, \ldots, n. \quad (4.25)$$

Notice that $f_j \omega_{ji} = f_i \omega_{ij}$, for all i, j. Dividing equation i by f_i in (4.25) leads to:

$$\begin{aligned}\beta_i^S &= \frac{1-r}{1+r} + \frac{2r}{1+r}\frac{1}{n-1}\sum_{j\neq i}\omega_{ij}\beta_j^S \\ &= \left(1-\frac{2r}{1+r}\right) + \frac{2r}{1+r}\frac{1}{n-1}\sum_{j\neq i}\omega_{ij}\beta_j^S\end{aligned}$$

Hence, we obtain:

$$\beta_i^S = k_i\left(\frac{2r}{1+r}, \mathbf{\Sigma}\right).$$

Plugging this back in (4.23), and with the use of some straightforward algebra, we obtain:

$$\alpha_i^S = \left(1 - k_i\left(\frac{2r}{1+r}, \mathbf{\Sigma}\right) f_i\left(\mathbf{\Sigma}\right)\right)\theta_0$$

The result then follows.

■

Chapter 5
On Optimal Communication Networks

with

Antoni Calvó-Armengol

5.1 Introduction

In this chapter we model informal interactions by means of a network. The network nodes are the individual members of the organization, and the network links keep track of who communicates with whom. The network of communications represents the pattern of ongoing communications inside the group.

We build on the model we have developed in the previous chapter to try to provide a theory of optimal communication structures in informal organizations. An informal organization is characterized by its information-processing needs and its information-processing capabilities. Information-processing needs correspond to the nature of the information uncertainty about the task to be performed, and to the exact balance between the external and the internal concern faced by each member of the group. Information-processing capabilities correspond to the network of communications.

The communication network disseminates within the group any private information held by individuals about the task. The geometry of this communication network determines the information structure available to all organization members and, ultimately, their choices. We map the network geometry to individual optimal choices, and relate changes in the communication structure to variations in individual and aggregate payoffs.

At the unique Bayes-Nash equilibrium we have characterized in the previous chapter, individuals best-respond to their own assessment of other's choices. Equilibrium also requires that everybody correctly anticipates each others' choices. This presumes that players can keep track of own beliefs about the task to be performed, as well as own beliefs about others' beliefs, own beliefs about others' beliefs about both own and others' beliefs, and so on. Beyond delimiting who communicates with whom, the communication network also determines how much information any two given individuals have in common by singling out their common interlocutors. Altogether, the communication network shapes the information held by each individual as well as the information overlap across different individuals. This information overlap is crucial to build cross inferences about each others' information and, ultimately, to compute the (arbitrary) high order beliefs that enter the equilibrium determination.

More precisely, consider three individuals that can all communicate with each other. The communication network is a close triad –a triangle. In this case, all agents have access to the same information. They entertain similar beliefs about the task characteristics, as well as similar beliefs about such beliefs and so on. High order beliefs are then easy to compute, and coincide with the average high order beliefs for the group.

Suppose now that two individuals among the three cannot communicate with each other anymore. The communication network is then an open triad with a hub and two spokes –a star. Because all players do not have access to the same information, the hub and the spokes have different (order one) beliefs about the task to perform. Also, the hub is aware of every piece of information privately held by a given spoke agent (he is his sole interlocutor), while the other spoke agent is not. Therefore, the hub and any spoke agent have different (order two) beliefs about the (order one) beliefs of the other spoke agent. For similar reasons, any spoke agent has different (order two) beliefs about the (order one) beliefs of the hub and the other spoke agent. And this is also true for higher order beliefs.

To summarize, the kind of high order inferences involved in the computation of high order beliefs rely on chains of pair-wise inferences for agents having interlocutors in common. Beyond the identity of the agents in these chains of pair-wise inferences, their order also matters. As such, high order beliefs are anisotropic and vary across individuals with the exact geometry of the ego-centred communication possibilities available to them. We give a closed-form expression for these high order beliefs that reflects the ability of each individual to infer the information held by the other group members after communication has occurred and as a function of their location in the communication network.

The comparative statics of the equilibrium payoffs with respect to the communication network is directly inherited from that of the knowledge index.

Add some communication links to a given network. We get a denser network, which includes the original one as a subset. We say that this denser network is a closure of the original one if the impact of the added links is to increase the number of close triads (triangles) in the network and to decrease the number of open triads, both simultaneously. It turns out that a network closure increases the knowledge index of every agent, and thus increases equilibrium payoffs from them all. The intuition is straightforward. The external concern requires more accurate estimations of the characteristics of the common task. A denser network aggregates better all the available information for all agents, and can only increase this accuracy. The internal coordination concern, instead, requires that individuals anticipate each others' efforts better. Within close triads, agents fully share all their private information with each other, which allows them to build more accurate cross inferences about each others knowledge. A network closure thus also improves on coordination.

One implication of monotone equilibrium payoffs with respect to network closure is that the optimal communication structure is the complete graph, where everyone can communicate with everyone else.

However, communication urbi et orbi need not be costless or even possible and, in fact, empirical evidence tracking down the structure of informal communication structures shows that the complete network is a very rare exception rather than the rule. We inquiry about the optimal communication network for a fixed supply of communication links. Because network closure defines a preorder that does not allow to rank all networks unambiguously, we cannot make use of our previous comparative statics result to single out the optimal geometric arrangement of a fixed supply of network links.

When the internal coordination problem is not too overwhelming (compared to the external concern), and the uncertainty about the characteristics of the group task to be performed is large enough, the optimal fit between information-processing needs and capabilities is achieved by a centralized and clustered network. More precisely, the optimal geometric arrangement is the one that maximizes a network span index that we define.[59] Instead, when the internal coordination concern is more demanding, the optimal network is regular (and thus distributed).

Finally, we discuss some extensions of the model that check the robustness of our results to a different specification of the coordination problem, and to the inclusion of publicly shared information about the external concern.

5.2 Related Literature

Besides the more general literature relating communication and information structures to the performance of organizations surveyed in the previous chapter, an increasing amount of research in economics and related areas is devoted to analyze more precisely the particular inner network structure of organizations.

The experimental psychology literature has long ago documented the crucial role of communication time and communication pattern for information aggregation purposes is reported in the seminal work by Bavelas (1950) followed by Leavitt (1951). These works initiated a plethora of empirical research comparing centralized versus decentralized network organizational structures but it was not

[59] For instance, the star is the network with maximum span among the minimally connected networks – the trees.

accompanied by much theoretical development. Shaw (1964) provides a review of this literature.

Organizational theorists have recently emphasized the important role of networks within organizations, and try to analyze with the use of tools from social network analysis (see Wasserman and Faust, 1994, for a formal approach to this topic) which are the relevant variables of the network structure of an organization in management. Krackhardt and Hanson (1993), Ahuja and Carley (1999) and Cummings and Cross (2003) are some relevant examples in which the particular networked communication structure of real-world organizations is sketched. A common finding of this literature is that the network structure of communication substantially differs from the formal chart of the organization. Furthermore, communication networks tend to be clearly incomplete.

The literature on the optimal formal organization has highlighted the role of hierarchies, a particular form of network structures, to reduce costs associated to communication transmission and information processing. Sah and Stiglitz (1987) is an early example on this direction, in which the authors compare two different organizational structures, polyarchies and hierarchies, and its respective benefits to reduce possible errors when processing the information the organization receives and communicates.

Besides, building on Radner (1993), the work by Bolton and Dewatripont (1994), Van Zandt (1999a), Garicano (2000), Guimerà et al. (2002) and Doots et al. (2003) highlights the importance of hierarchies, and more general network structures, to diminish the costs related to processing information that flows through the network of contacts. This literature is surveyed by Van Zandt (1999b) and Ioannides (2003).

From a more theoretical perspective, our work also relates to the literature of games of incomplete information played in a network, such as Morris (2000) and Chwe (2000). In particular, Chwe (2000) is closer in spirit to our work. Chwe analyzes a game in which agents want to coordinate their binary decisions and guess the action of others from the local information that agents communicate to their neighbours in the network. Anyhow, our model, and the procedure of our analysis, differs in several points from Chwe. We analyze a game with a continuum of possible actions and agents do not only pursue to coordinate, but also to attain the real state of the world. These differences reflect in the way agents rely on the network to form higher order expectations in both models.

5.3 A network communication process

So far, the communication process is characterized by the distribution over its output signals. We now describe a particular instance of a communication process for which we can explicitly compute this distribution over output signals.

The networked class of decentralized communication devices we consider is the following. Agents connected through a given network, and only them, communicate in pairs and for a fixed number of rounds. At each round, agents average the stream of signals previously received from their network contacts, and communicate this average signal back to them. Averaging reduces the volume of information prior to its retransmission, and thus saves on information-processing costs. However, from round two onwards, this simple heuristic that treats signals in the stream as de facto mutually independent, fails to adjust properly for redundant information from a common third-party. In fact, DeMarzo *et al.* (2003) show that the converging beliefs dynamics for this simple rule over-weight the private information of more "central" agents in the communication network. For this reason, asymptotic beliefs are not correct when the underlying network is irregular.

Here, we analyze how this decentralized information-sharing scheme shapes individual and organization decisions and outcomes when the network geometry varies and for arbitrary communication rounds.

5.3.1 A class of communication processes, $P^t(g)$

Players receive a conditionally independent private signal, $x_i|\theta \sim N(\theta, \phi_\varepsilon)$.

We model the communication possibilities by a network g. We set $g_{ij} = g_{ji} = 1$ if i and j communicate with each other, and $g_{ij} = 0$ otherwise. Of course, $g_{ii} = 1$, while $g_i = g_{i1} + \cdots + g_{in}$ is the total number of interlocutors to player i, including oneself.

This communication process disseminates idiosyncratic signals in the population. After the first communication round the information available to player i is:

$$x_i^1(g) = \frac{1}{g_i}\sum_{j=1}^{n} g_{ij}x_j. \tag{5.1}$$

The communication report $x_i^1(g)$ averages private signals across all information sources available to i, which include communication partners in the network g and oneself.

More generally, denote by $\mathbf{G} = [g_{ij}]$ the network adjacency matrix of communication links, and by $\overline{\mathbf{G}} = [g_{ij}/g_i]$ its row normalization. A compact notation for the communication report after one communication round is $\mathbf{x}^1 = \overline{\mathbf{G}}\mathbf{x}$.

When players communicate repeatedly with their network interlocutors in the network g, and average the incoming stream of signals before sending it back to them, the resulting communication reports after t completed rounds of communication are $\mathbf{x}^t = \overline{\mathbf{G}}^t \mathbf{x}$. We denote this communication process by $\mathbb{P}^t(g)$.

The covariance between the output signals x_i^t and x_j^t is then readily computed:

$$\sigma_{ij}^t(g) = \sum_{k=1}^{n} \overline{g}_{ik}^{[t]} \overline{g}_{jk}^{[t]}, \text{ for all } i, j,$$

where $\overline{\mathbf{G}}^t = \left[\overline{g}_{ij}^{[t]}\right]$. In matrix notation, we have $\boldsymbol{\Sigma}^{t+1}(g) = \overline{\mathbf{G}}^{t+1}\overline{\mathbf{G}}^{t+1\prime} = \overline{\mathbf{G}}\boldsymbol{\Sigma}^t(g)\overline{\mathbf{G}}'$. The equilibrium actions can then be readily computed from Theorem 1.

5.3.2 One communication round, $P^1(g)$

Let $\alpha = \phi_\varepsilon/\phi_\theta$. Then, it is readily checked that

$$f_i(g) = \frac{1}{1+\frac{\alpha}{g_i}} \qquad \text{and} \qquad \omega_{ij}(g) = f_j(g)\left(1+\alpha\frac{c_{ij}(g)}{g_i g_j}\right),$$

where $c_{ij}(g) = \#\{k : g_{ik}g_{kj} = 1\}$ is the number of common interlocutors to both i and j in the network of communication g.[60] In particular, $c_{ij}(g) = 0$ for all two players that do not share any common information source with each other, and $c_{ij}(g) \geq 1$, otherwise.

Consider, for instance, a regular communication networks such that $g_i = d$ for every i. Then, $\sum_{j=1}^n c_{ij} = d^2$, for all i,[61] and thus $\sum_j \omega_{ij} = \varpi(d)$ for all i, where

$$\varpi(d) = \frac{n-1+\alpha(d-1)}{d+\alpha}.$$

Therefore, each of the terms in the infinite sum (4.7) can be written as $\rho^p\boldsymbol{\Omega}^p = \rho^p\varpi^p(\boldsymbol{\Omega}/\varpi)^p = \rho^p\varpi^p(\boldsymbol{\Omega}/\varpi)$, where the last equality uses the fact that $\boldsymbol{\Omega}/\varpi$ is a double-stochastic matrix, that is, a non-negative matrix with all rows and columns adding up to one. The knowledge index in a regular network is then readily computed:

$$k_i(\rho, d) = \frac{1-r}{1-\rho\varpi(d)}, \text{ for all } i. \tag{5.2}$$

[60] Given our assumption that $g_{ii} = 1$, $c_{ii}(g) = g_i$, for all i, while $c_{ij}(g) = 2$ for all two players who are one-link away from each other in the network of communication ($g_{ij} = 1$).

[61] The adjacency matrix of a regular network of degree d is $\mathbf{G} = d\mathbf{M}$, where $\mathbf{M}$ is a double stochastic matrix. Hence, $\mathbf{G}^2 = d^2\mathbf{M}^2$, where $\mathbf{M}^2$ is also a double stochastic matrix.

5.4 On the Optimal Communication Structure

5.4.1 Ordering Networks

In general, the impact of adding new links on both the knowledge index and the equilibrium payoffs depends very critically on the geometry of both the original and the resulting network. To fix ideas, let $n = 3$, and consider the three following networks in Figure 1.

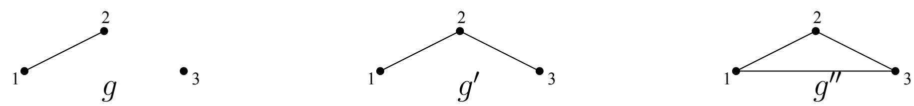


Figure 1.

Adding link 23 leads from g to g', while adding link 13 leads from g' to g''.

Consider first the change from g' to g''. With the new link 13, both the number of information sources for players 2 and 3, and the sources overlap for all three players increase. As a result, all the inference coefficients increase, that is, $\mathbf{\Omega}'' \geq \mathbf{\Omega}'$, and so do the knowledge indexes of each player, that is, $\mathbf{k}(\rho, \mathbf{\Omega}'') \geq \mathbf{k}(\rho, \mathbf{\Omega}')$.

Consider now the change from g to g'. With the new link 23, players 2 and 3 widen their information sources, thus making better inferences about others' reports. Formally, $\omega'_{2i} \geq \omega_{2i}$ and $\omega'_{3i} \geq \omega_{3i}$, for all i. Instead, player 1 retrieves information from the same sources in g and in g', but loses grip on 2's true value of the report in the new network g' compared to the old one g. Formally,

$$\omega'_{12} = \frac{1 + \frac{\alpha}{3}}{1 + \frac{\alpha}{2}} < 1 = \omega_{12}.$$

As the previous example illustrates, the inference matrix need not be monotonic to link addition. New links that close triples (e.g., from g' to g'') create a common grounding in information sources, and thus allow for better cross inferences. Instead, new links that lead to open triples (e.g., from g to g') spread away information sources and dampen the accuracy of cross reports by adding noise on each others' awareness. The impact of link addition on the knowledge index and equilibrium payoffs is not clear (see Proposition) and depends on the relative balance of both forces.[62]

To keep track of the impact of link addition on the number of close and open

[62] For the case of $n = 3$ players, one can readily check that $\mathbf{k}(\rho, \mathbf{\Omega}') \geq \mathbf{k}(\rho, \mathbf{\Omega})$ although $\omega'_{12} < \omega_{12}$. Below we give an example where link addition actually *decreases* the knowledge index.

triples, we define:

$$\tau(g) = \#\{(i,j,l) : g_{ij}g_{jl}g_{li} = 1, i \neq j \neq l \neq i\},$$

and

$$\iota(g) = \#\{(i,j,l) : g_{ij}g_{jl} = 1, g_{li} = 0, i \neq j \neq l \neq i\}.$$

By definition, $\tau(g)$ (resp. $\iota(g)$) gives the number of close (resp. open) triples in g.

Definition 1 *Let g, g' two networks. We say that g' is a closure of g, denoted $g' \trianglerighteq g$, when both $g' \supseteq g$ and $(\tau(g'), -\iota(g')) \geq (\tau(g), -\iota(g))$.*

In words, a network closure amounts to adding links such that the number of close triples (weakly) decreases whereas that of open triples (weakly) decreases.

Suppose, for instance, that g is a star with one hub and $n-1$ spokes. Then, $\tau(g) = 0$ and $\iota(g) = (n-1)(n-2)$. Let g' deduced from g by adding ℓ spoke-to-spoke links. Then, $\tau(g') = 2\ell$ and $\iota(g') = \iota(g) - 2\ell$. Therefore, $g' \trianglerighteq g$, that is, g' is a closure of g.

We have the following result.

Proposition 1 *If $g' \trianglerighteq g$, then $\mathbf{k}(r, \mathbf{\Sigma}^1(g')) \geq \mathbf{k}(r, \mathbf{\Sigma}^1(g))$ and*

$$U^*\left(r, \mathbf{\Sigma}^1(g')\right) \geq U^*\left(r, \mathbf{\Sigma}^1(g)\right)$$

In particular, denoting by g^N the complete network where all communication links are active, $U^*\left(r, \mathbf{\Sigma}^1\left(g^N\right)\right) \geq U^*(r, \mathbf{\Sigma}^1(g))$, for all network g.

The next example shows that adding links without satisficing the closure condition can indeed be detrimental for the knowledge index of some players.

Consider the two networks in Figure 2, where g' is deduced from g by adding the link 14.

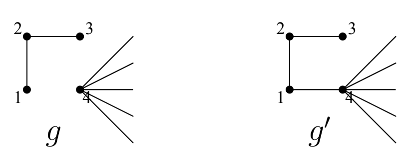


Figure 2.

Using (4.7) and (4.8), we can write, at a first order in r:

$$\mathbf{k}\left(r, \mathbf{\Sigma}^1(g')\right) - \mathbf{k}\left(r, \mathbf{\Sigma}^1(g)\right) = \rho(1-r)\left[\mathbf{\Omega}^1(g') - \mathbf{\Omega}^1(g)\right]\mathbf{1}_n + o(r).$$

In particular, $k_1(r, \mathbf{\Sigma}^1(g')) - k_1(r, \mathbf{\Sigma}^1(g))$ is proportional to $(\omega'_{12} - \omega_{12}) + (\omega'_{14} - \omega_{14}) + o(r)$, where:

$$\begin{aligned}
\omega'_{12} - \omega_{12} &= -\frac{\frac{\alpha}{9}}{1+\frac{\alpha}{3}} < 0 \\
\omega'_{14} - \omega_{14} &= \frac{1+\frac{2\alpha}{3(g_4+1)}}{1+\frac{\alpha}{g_4+1}} - \frac{1}{1+\frac{\alpha}{g_4}} \underset{g_4\to+\infty}{\to} 0
\end{aligned}$$

Therefore, for a small enough r and a high enough g_4, we have $k_1(r, \mathbf{\Sigma}^1(g')) < k_1(r, \mathbf{\Sigma}^1(g))$ despite $g \subset g'$. Note that $\tau(g) = \tau(g')$, while $\iota(g') > \iota(g)$, implying that g' is not a closure of g.

5.4.2 Optimal communication networks for $\mathbb{P}^1(g)$

Proposition 1 suggests that the complete network is the optimal communication network. But, as Marshak and Radner (1972) warn: "Ideally, one would like to compare information structures on the basis of net value of information, namely gross value of information minus the cost of both the information and the associated best decision function. Therefore, any comparison between the gross values of two information structures is meaningful only in the context of some assumption about the relative costs of the two structures" (p. 224).

In our model, information structures are fully determined by the underlying communication network. Abstracting away from cognitive decision costs, we assume in what follows that communication links are costly (to set up and maintain). More precisely, we fix to some value $\gamma \in \{n-1, ..., n(n-1)/2\}$ the total number of communication links available, and we solve for the following optimal network design problem:

$$\max_g \{U^*\left(r, \mathbf{\Sigma}^1(g)\right) : \sum_{i,j} g_{ij} \leq 2\gamma\}. \tag{5.3}$$

This is a finite optimization problem, that admits at least one solution. For instance, when $\gamma = n(n-1)/2$, the complete network g^N solves (5.3). More generally, the optimal network corresponds to the geometric arrangement of the available communication resources that best accounts for the weighted external and internal concern of all players.

Although a full characterization of the solutions to (5.3) is not available, the next discussion clarifies the forces at play.

Suppose first that α takes a very small value. Then, private signals allow for a very accurate estimation of the true value of θ, and the more so do communication reports. The external target concern and its associated welfare loss are then of secondary importance. Instead, the internal coordination concern becomes more (relatively) demanding. Therefore, the optimal network architecture is the one that primarily solves the coordination problem, namely, a regular or distributed network.[63]

Suppose now that α takes a large value. Then, private signals are very noisy. More inaccurate predictions of the underlying exogenous parameter increase the welfare loss from the external target concern. Suppose further that r is very small. Then, the external target concern is the driver for the optimal network solving (5.3). In this case, two opposite forces are at work. On one hand, a distributed or regular network homogenizes the forecasts about θ across players, but sets a common upper bound $2\gamma/n$ on the information sources available to every player. On the other hand, a centralized or irregular network leads to heterogeneous forecast about θ. But, by allocating most of the available information sources to a handful of players, it allows those ones to make a more accurate predictions that in the dispersed network.

The optimal network is the one that solves this trade-off optimally. Formally, given a network g, let $c_i(g) = \sum_{j=1}^{n} c_{ij}(g)$, the number of two-link away contacts of player i in the network. Then, $c_i(g)/g_i$ is the ratio of two-link away contacts per direct link. It measures the range of indirect contacts. A high (resp. low) value of the ratio $c_i(g)/g_i$ corresponds to a long-range (resp. a short-span) of indirect contacts.

Consider for instance a star encompassing n players, where player 1 is the hub and players $2, ..., n$ are the spokes. Through a spoke-to-hub link, a spoke gains indirect access to every other spoke in the star. Instead, the spoke-to-hub link does not add any indirect contact to the hub but for the spoke itself. A spoke-to-hub link thus warrants a long-range of indirect contacts to the spoke but a short-span to the hub. Formally, $c_2(g)/g_2 = n/2 > 2(n-1)/n = c_1(g)/g_1$.

Define now:

$$S(g) = \sum_{i=1}^{n} \frac{c_i(g)}{g_i}.$$

This is the aggregate span or range of indirect contacts per network link. This span index increases with the number of super-connectors in the network, that give access to a wide range of indirect contacts to the nodes appending to them. It also increases with the number of triangles in the network. In other words, a high (resp. low) span index is tantamount to an irregular (resp. distributed) and clustered (resp. open-knitted) network geometry.

[63] Or, to be more precise, the most regular network available given the link resource constraint γ.

We have the following result.

Proposition 2 *Fix γ. There exists $\underline{\alpha} > 0$ and $0 \leq \overline{r} < 1$ such that, for all $\alpha \geq \underline{\alpha}$ and $r \leq \overline{r}$, the optimal network that solves (5.3) is $g^* \in \arg\max_g \{S(g) : \sum_{i,j} g_{ij} \leq 2\gamma\}$;*

Fix the total number of network links available. Different geometric arrangements of these links lead to different network structures.

When the internal coordination problem is not too overwhelming and the signals about the exogenous target value are sufficiently noisy, the optimal geometric arrangement maximizes the aggregate network span. Namely, optimal networks are clustered and display an irregular distribution of connectivities.

We illustrate this point below with two examples.

Suppose first that $\gamma = n - 1$. This is the minimal number of links required to get a connected network, where each player is indirectly linked to every other one. Minimally connected networks are known as trees. The line and the star are two examples of trees.

Remark 1 *For sufficiently high α and low r, the optimal tree is the star.*

In words, the tree with maximal span is the most irregular one (for the distribution of connectivities).

In a star, spoke players play the same strategy, but the hub a different one. This mismatch induces a coordination loss. When r is small, this coordination loss is not much of a concern, and the optimal geometry is the one that reduces the welfare loss induced by external target concern.

The hub of a star makes as accurate as possible a prediction about the exogenous target value than any other node in any alternative tree configuration. This is because the hub has access trough the communication channels available to him to every other private signal in the population. On the contrary, the prediction made by spokes is as inaccurate as possible −they only have access to one additional source of information. When private signals are sufficiently noisy, the gain in accuracy for the hub compensates for the loss in accuracy for the spokes, and the star is the optimal tree.

Consider, for example, the two network architectures in Figure with $n = 4$ players and $\gamma = 4$ links, that we name the kite and the wheel.

kite *wheel*

$Figure\,3.$

One can readily checked that $S\,(g)$ is worth 38/3 for the kite, and 12 for the wheel. According to Proposition 2, the kite is the optimal network (yielding higher aggregate payoffs) for high enough values of α and low enough values of r.

5.4.3 Infinite communication rounds, $P^{\infty}\,(g)$

Suppose now that players communicate an infinite number of rounds, $t \to +\infty$. Writing $\overline{\mathbf{G}}^{t+1} = \overline{\mathbf{G}}\overline{\mathbf{G}}^{t}$, one can view $\overline{\mathbf{G}}$ as the Markov transition matrix for the row probability vectors $\left(\overline{g}_{i1}^{[t]}, ..., \overline{g}_{in}^{[t]}\right)$ of the row-normalized matrix $\overline{\mathbf{G}}$. We thus have $\lim_{t\to+\infty} \overline{g}_{j}^{[t]} = \overline{g}_{j}^{\infty}$, where $\overline{\mathbf{g}}_{j}^{\infty}$ is the unique invariant distribution of the irreducible and aperiodic Markov process with transitions $\overline{\mathbf{G}}$. In turn, the fact that all row vectors of are identical implies that long-run beliefs for $\mathbb{P}^{\infty}\,(g)$ are common to all players, that is:

$$x_i^{\infty} = x^{\infty} = \overline{g}_1^{\infty} x_1 + \cdots + \overline{g}_n^{\infty} x_n, \text{ for all } i,$$

a weighted sum of private input signals.

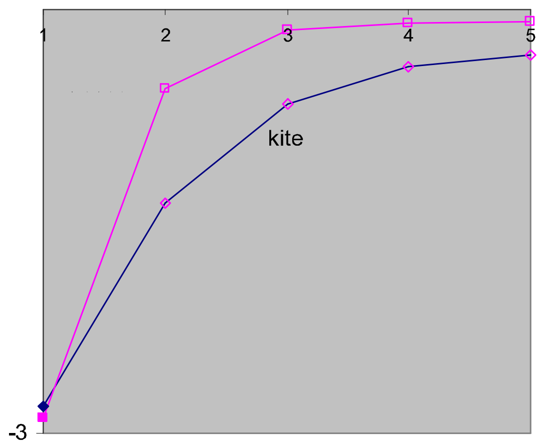
We compute the weights. With an undirected network we have $g_i \overline{g}_{ij}^{[t]} = \overline{g}_{ji}^{[t]} g_j$ from simple algebra, from which we obtain $g_i \overline{g}_j^{\infty} = \overline{g}_i^{\infty} g_j$ at the limit, and thus $\overline{g}_i^{\infty} = g_i / \left(g_1 + \cdots + g_n\right)$. Because averages of incoming signal streams at each communication round do not discount properly for redundant information from common sources, better connected players in the communication network end up credited with a higher weight in the emergent long-run consensual beliefs.

When all players share the same beliefs, the knowledge index hits its upper bound of one. Aggregate long-run payoffs then take the following simple form:

$$U^{*}\left(r, \mathbf{\Sigma}^{\infty}\,(g)\right) = n\,(1-r)\,\phi_{\theta} \left[\frac{1}{1+\alpha \frac{g_1^2+\cdots+g_n^2}{(g_1+\cdots+g_n)^2}} - 1\right].$$

Fix the total supply of links, $\left(g_1 + \cdots + g_n\right)/2$. Then, these payoffs are maximal when $g_1^2 + \cdots + g_n^2$ is minimal, namely, on a regular network. Aggregate payoffs are thus higher for the wheel than for the kite in Figure 3.

We can compare the corresponding equilibrium payoffs for these two different network geometries, the kite and the wheel, and for various communication rounds to get a sense of the cost for the installed communication capacity. The next graphic shows the comparison of welfare from the kite (the blue line) and the while (the red line) for the case in which $r = 1/2$ and $\alpha = 5$ and with the number of periods ranging from 1 to 5:



In this graphic we can clearly observe the kind of polarization result with respect to the number of communication rounds. When there is only one round, the kite, the more irregular network between the two, performs better than the wheel. Instead, as the number of rounds increases the wheel, a regular network with the same number of links than the kite, becomes the preferred choice.

5.5 Extensions

5.5.1 On the payoffs

The payoffs considered so far are a weighted average of an external concern and an internal coordination concern. The coordination payoffs in (4.1) add up the quadratic losses between own action and every other players' action, and then average them all.

An alternative measure of mis-coordination is the quadratic loss between own action and the $average$ of others' action.

Formally,

$$v_i\left(a_1, ..., a_n; \theta\right) = -\left(1-r\right)\left(a_i - \theta\right)^2 - r\left(a_i - A_{-i}\right)^2, \ i = 1, ..., n, \quad (5.4)$$

where $A_{-i} = \left(\sum_{j \neq i} a_j\right) / \left(n-1\right)$. The expression (5.4) are the discrete population counterpart of the beauty contest game for a continuum of agents in Morris and Shin (2002).

Straight algebra leads to:

$$v_i(a;\theta) = u_i(a;\theta) + \frac{r}{n-1}\sum_{j\neq i}(a_j - A_{-i})^2.$$

In particular, this implies that the Bayes-Nash equilibria of the incomplete information game with communication and payoffs (5.4) are characterized in Proposition . This equilibrium is thus unique, linear and given by:

$$a_i^*(y_i) = f_i(g)\, k_i(\rho, \mathbf{\Omega}(g)) y_i, \text{ for all } i = 1, ..., n.$$

The new payoffs, however, lead to a different expression for equilibrium payoffs.

Remark 2 *Let $r < 1$. The aggregate ex ante equilibrium payoffs are now:*

$$V^*(g) = \phi_0 (1-r)\left[\frac{1}{n-1}\mathbf{k}^t(\rho,\mathbf{\Omega})\mathbf{F}\mathbf{k}(\rho,\mathbf{\Omega}) + \frac{n-2}{n-1}\mathbf{k}^t(\rho,\mathbf{\Omega})\mathbf{F}\mathbf{1} - n\right] \leq 0.$$

Observe that if n is large enough, this expression is close to

$$\phi_0(1-r)\left[\sum_i f_i k_i - n\right]$$

that coincides with the expression of welfare we have obtained with the payoffs analyzed in the previous sections except that the term on k_i is linear and not quadratic.

5.5.2 Public and private signals

Beyond receiving a private signal x_i, organization members also receive a public signal z about the true value of the task characteristics. We have $z|\theta \sim N(\theta, \phi_z)$, independent of the private signals x_i. This public signal is shared by all organization members and has no connection with the details of the communication network.

The best-response is still given by (4.4):

$$a_i = (1-r)\,\mathbb{E}_i[\theta] + \rho\sum_{j\neq i}\mathbb{E}_i[a_j], \text{ for all } i = 1, ..., n,$$

except that now the expected values are computed based on the information $\{y_i(g), z\}$ available to each agent, that is, $\mathbb{E}_i[\cdot] = \mathbb{E}_i[\cdot|y_i(g), z]$.

We first compute $\mathbb{E}\left[\theta|y_i\left(g\right),z\right]$. This is a weighted average, with weights proportional to the precisions of three distributions: the prior distribution of θ, the posterior distribution of $y_i\left(g\right)$, and the posterior distribution of z. We have:

$$\mathbb{E}\left[\theta|y_i\left(g\right),z\right]=\frac{1}{\frac{1}{\phi_0}+\frac{g_i}{\phi}+\frac{1}{\phi_z}}\left(\frac{g_i}{\phi}y_i\left(g\right)+\frac{1}{\phi_z}z\right)$$

Let $\alpha=\phi/\phi_0$ and $\alpha_z=\phi/\phi_z$. Define:

$$f_i^z\left(g\right)=\frac{g_i}{g_i+\alpha_z+\alpha}\quad\text{and}\quad h_i\left(g\right)=\frac{\alpha_z}{g_i+\alpha_z+\alpha}$$

Straightforward algebra gives:

$$\mathbb{E}\left[\theta|y_i\left(g\right),z\right]=f_i^z\left(g\right)y_i\left(g\right)+h_i\left(g\right)z.$$

The more agent i has contacts with, the more valuable is the communication report to him, and the less valuable the common signal (in relative terms). We omit the parameter g when there is no risk of confusion.

For all player $i=1,...,n$, define:

$$\pi_i=h_i\left[1+\frac{\rho}{1-r}\sum_{j\neq i}f_j^z k_j(\rho,\mathbf{\Omega})\left(1-\frac{c_{ij}}{g_i}\right)\right].$$

Let $\mathbf{\Pi}$ be the matrix with a zero diagonal and out-of-diagonal entries given by $\pi_{ij}=\pi_j/\pi_i$.

Proposition 3 *When $r<1$, the incomplete information game with communication network g has a unique Bayes-Nash equilibrium strategy linear in private reports and in the public signal, and given by:*

$$a_i^*\left(y_i,z\right)=f_i^z k_i(\rho,\mathbf{\Omega})\,y_i+\pi_i k_i(\rho,\mathbf{\Pi})\,z\ \text{, for all } i=1,...,n. \tag{5.5}$$

The public signal translates into an increase from α to $\alpha+\alpha_z$ as captured by f_i^z with respect to the case without public signal. Therefore the entries of $\mathbf{\Omega}$ decrease, and the coordinates of the knowledge index also decrease. In words, the public signal helps in building cross inferences about each others information, and thus dampens the value of the communication network to build these cross inferences. The more precise this public signal, the less important the information conveyed by the communication reports $y_i\left(g\right)$.

The public signal has an added effect captured by $\pi_i k_i(\rho,\mathbf{\Pi})$. Indeed, the public signal acts as a focal point for the coordination problem and thus has a multiplier effect on players' actions.

5.6 Conclusion

In this chapter we have extended the analysis of the previous one by analyzing a family of network communication processes. Any element of this family does not only add some geometric structure to the personal relations exhibited in an organization but also introduces two other important dimensions of communication processes. First, a temporal dimension expressed by the number of rounds of communication. Second, information processing technologies, by limiting the ability of agents to process the mean of signals he receives at each period and not the whole string of signals.

These three dimensions of a communication process seem sufficient to represent a variety of situations. In particular, an interesting avenue of future research would be to explore which are the reasons and incentives to communicate instead of obtaining better information about the real state of the world through individual experimentation. This trade-off is not trivial: while communication helps to share information and hence to partially avoid coordination motive, it can be optimal for some individuals to individually experiment for some periods and to communicate this information in the future. This information obtained is therefore more precise and when communicated to the rest of individuals in the future it can be understood by everybody as a superior source of information by each one that receives it and also help to avoid the coordination motive, as well as, because of the precision of this signal, help to obtain a more close approximation to the state of the world.

Other situations can involve a more strategical concern for the communication of information. Due to the common interest feature of the game we analyze, every agent finds optimal to communicate truthfully his information. However this is not necessarily the case when instead of a game with strategic complements we were dealing with a game with strategic substitutes, or in which competition for superior information is an issue. This could be the case for example when different sections of a firm can share through some partially decentralized mechanisms their information. This is the case for example in Dessein et al. in which they analyze in a two sections firm the possible trade-off's between centralization and decentralization of information in a similar setup than ours. Our model and tools developed can be helpful to extend this analysis to firms with more than two sections and intermediate specifications of intersectional communication are possible.

Our model of networked communication might also be applied to the analysis of networks of collaboration. In an oligopoly setup it can be optimal for some firms to share some of the informations they have about the industry. How these firms form networks of communication and how this network structure impacts firms' decisions an market outcomes is a question for future research.

Finally, there are some political economy questions in which communication is relevant. How voters communicate among them information about the candidates in an election, added to the public information voters can receive from different sources as newspapers and polls, and how this spread of information affects the final voting decisions of the population is a relevant question that has been initially analyzed in Myatt (2006). Also, how leadership form in political parties is an issue that involves communication and information transmission. This question is addresses in Dewan and Myatt (2007). How our model could be adopted to extend the analysis of such issues is an open question that deserves future consideration.

5.7 Proofs

Proof of Proposition 1: Let $ij \notin g$. We have:

$$\omega_{ij}(g) = \frac{g_j}{\alpha + g_j}\left[1 + \alpha\frac{c_{ij}(g)}{g_i g_j}\right],$$

and

$$\omega_{ij}(g+ij) = \frac{g_j + 1}{\alpha + g_j + 1}\left[1 + \alpha\frac{c_{ij}(g) + 2}{(g_i + 1)(g_j + 1)}\right].$$

Tedious algebra then gives:

$$\begin{aligned}\omega_{ij}(g+ij) - \omega_{ij}(g) &= \frac{\alpha}{g_i(g_i+1)(\alpha+g_j)(\alpha+g_j+1)} \cdot \\ &\quad \cdot \left[g_i(g_i + 1 + 2g_j + \alpha) - c_{ij}(g)(g_i + 1 + g_j + \alpha)\right].\end{aligned}$$

Noticing that $c_{ij}(g) \leq g_i$, we can conclude that the term in brackets is nonnegative, and thus:

$$\omega_{ij}(g+ij) \geq \omega_{ij}(g).$$

Let now x, y such that $i, j \notin \{x, y\}$. Then, clearly, $\omega_{xy}(g+ij) = \omega_{xy}(g)$.

Finally, let $k \neq j$. We have:

$$\omega_{ik}(g) = \frac{g_k}{\alpha + g_k}\left[1 + \alpha\frac{c_{ik}(g)}{g_i g_k}\right],$$

and

$$\omega_{ik}(g+ij) = \frac{g_k}{\alpha + g_k}\left[1 + \alpha\frac{c_{ik}(g+ij)}{(g_i + 1) g_k}\right].$$

We distinguish two cases.

First, $g_{jk} = 1$. Then, $c_{ik}(g + ij) = c_{ij}(g) + 1$, and we can easily check that $\omega_{ik}(g + ij) \geq \omega_{ik}(g)$.

Second, $g_{jk} = 0$. Then, $c_{ik}(g + ij) = c_{ik}(g)$, and thus $\omega_{ik}(g + ij) < \omega_{ik}(g)$, for all $\alpha > 0$.

Therefore, let g' obtained from g by adding some links, that is, $g \subset g'$. Then, if any new link between two new partners i and j is such that $g_{ik} = 1$ implies that $g_{jk} = 1$, for all k, and for all pair of newly linked partners i, j, then $\mathbf{\Omega}(g') \geq \mathbf{\Omega}(g)$. Otherwise, the inequality need not hold. ■

Proof of Proposition 2: At a first order in r, we have:

$$k_i^2(r, \mathbf{\Sigma}^1(g)) = 1 - 2r + 2\rho \sum_{j=1}^{n} \omega_{ij}(g) + o(r). \tag{5.6}$$

In particular, if g is a regular network with degree d we have $\sum_{j=1}^{n} \omega_{ij}(g) = \varpi(d)$.

For large values of α, (5.6) can be rewritten as:

$$k_i^2(r, \mathbf{\Sigma}^1(g)) = 1 - 2r + 2\rho \frac{c_i(g)}{g_i} + O(1/\alpha) + o(r).$$

Denoting by $f_i(d)$ the forecast coefficient in a regular network of degree d, we have:

$$\frac{f_i(g)}{f_i(d)} = 1 + O(1/\alpha).$$

Denoting by $U^*(d)$ the ex ante aggregate equilibrium payoffs in a regular network, we have:

$$\frac{U^*(r, \mathbf{\Sigma}^1(g))}{U^*(d)} = \frac{1 - 2r + 2\frac{\rho}{n}\sum_{i=1}^{n}\frac{c_i(g)}{g_i}}{1 - 2r + 2\rho\varpi(d)} + O(1/\alpha) + o(r),$$

and the result follows. ■

Proof of Remark 1: We establish the result by induction on the size n of the population.

When $n = 2, 3$ the only trees are the star, so the result holds trivially in these cases.

When $n = 4$, the only two possible trees are the star and the line. Straight algebra gives $S(g_{n=4}^{star}) = 23/2 > 31/3 = S(g_{n=4}^{wheel})$. So the result also holds for $n = 4$. Notice that, strictly speaking, the induction argument does not require to check the validity of the result for $n = 4$ given that it already holds for $n = 2, 3$. However, the induction argument that follows is established for $n \geq 5$, and so the case $n = 4$ needs to be worked out separately.

Suppose thus that $S\left(g_n^{star}\right) \in \arg\max\{S\left(g\right) : g \text{ a tree on } n \text{ players}\}$. Straight algebra leads to:

$$S\left(g_n^{star}\right) = \frac{n\left(n-1\right)}{2} + \frac{2\left(n-1\right)}{n}.$$

Consider an arbitrary tree g_{n+1} on $n+1$ players. Let $i \in \{1, ..., n+1\}$ such that $g_{ij,n+1} = 1$ for some unique $j \neq i$. In words, the link ij is a loose-end of the tree g_{n+1}. Notice that g_{n+1} being a tree implies that at least two such loose-ends exist.

Given that ij is a loose-end, g_{n+1}^{-ij} (the network deduced from g_{n+1} by eliminating the link ij) is a tree on n players. Noticing that $c_k\left(g_{n+1}\right) - c_k\left(g_{n+1}^{-ij}\right) = g_{kj}$, for all $i \neq j \neq k$, we have:

$$S\left(g_{n+1}\right) = S\left(g_{n+1}^{-ij}\right) + \frac{c_i\left(g_{n+1}\right)}{g_{i,n+1}} + \frac{c_j\left(g_{n+1}\right)}{g_{j,n+1}} - \frac{c_j\left(g_{n+1}^{-ij}\right)}{g_{j,n+1}-1} + \sum_{j \neq k \neq i} \frac{g_{kj,n+1}}{g_{k,n+1}}.$$

Notice that $c_j\left(g_{n+1}^{-ij}\right) = c_j\left(g_{n+1}\right) - 2$ and that $c_i\left(g_{n+1}\right)/g_{i,n+1} = g_{j,n+1}/2$. Also, by the induction hypothesis, $S\left(g_{n+1}^{-ij}\right) \leq S\left(g_n^{star}\right)$. Therefore, $S\left(g_{n+1}\right) \leq S\left(g_{n+1}^{star}\right)$ is equivalent to:

$$\frac{g_{j,n+1}}{2} + \frac{2}{g_{j,n+1}-1} - \frac{c_j\left(g_{n+1}\right)}{g_{j,n+1}\left(g_{j,n+1}-1\right)} + \sum_{j \neq k \neq i} \frac{g_{kj,n+1}}{g_{k,n+1}} \leq$$

$$\leq S\left(g_{n+1}^{star}\right) - S\left(g_n^{star}\right) = n + \frac{2}{n\left(n+1\right)}.$$

Notice that $g_{k,n+1} \geq 2$, for all k, and thus $\sum_{j \neq k \neq i} g_{kj,n+1}/g_{k,n+1} \leq \left(n-1\right)/2$. It thus suffices to show that:

$$\frac{g_{j,n+1}}{2} + \frac{2}{g_{j,n+1}-1} + \frac{n-1}{2} \leq n + \frac{2}{n\left(n+1\right)}.$$

It is easily checked that

$$\frac{g_{j,n+1}}{2} + \frac{2}{g_{j,n+1}-1} \leq \frac{n}{2} + \frac{2}{n-1}, \text{ for all } n \geq 3.$$

We are thus left to show that

$$\frac{2}{n-1} \leq \frac{1}{2} + \frac{2}{n(n+1)},$$

which is true for all $n \geq 5$. ■

Proof of Remark 2: Let $\sigma_i = \sum_{j \neq i} (a_j - A_{-i})^2$. The following equality is straightforward

$$\sum_{j \neq i} (a_i - a_j)^2 = (n-1)(a_i - A_{-i}) + \sigma_i$$

Hence,

$$\rho \sum_{j \neq i} (a_i - a_j)^2 = r(a_i - A_{-i}) + \rho \sigma_i$$

This implies that $v_i = u_i + \rho \sigma_i$ and therefore, if $\sigma = \sum_i \sigma_i$,

$$V^* = U^* + \frac{r}{n-1} \mathbb{E}[\sigma]$$

We have that $\sigma_i = \sum_{j \neq i} a_j^2 - (n-1) A_{-i}^2$. Since

$$A_{-i}^2 = \frac{1}{(n-1)^2} \left(\sum_{j \neq i} a_j^2 + \sum_{k \neq l; k,l \neq i} a_k a_l \right)$$

we obtain that:

$$\sigma_i = \left(1 - \frac{1}{n-1}\right) \sum_{j \neq i} a_j^2 - \frac{1}{n-1} \sum_{k \neq l; k,l \neq i} a_k a_l$$

Therefore,

$$\begin{aligned} \sigma &= \left(\frac{n-2}{n-1}\right)(n-1) \sum_i a_i^2 - \frac{1}{n-1}(n-2) \sum_{k \neq l} a_k a_l \\ &= (n-2)\left(\sum_i a_i^2 - \frac{1}{n-1} \sum_{k \neq l} a_k a_l \right) \end{aligned}$$

We obtain that $\mathbb{E}[\sigma] = (n-2)\phi_0$. Rearrenging, we obtain

$$\mathbb{E}[\sigma] = (n-2)\phi_0 \frac{1-r}{r} \left(\sum_i f_i \left(k_i - k_i^2\right) \right)$$

Hence,

$$V^* = U^* + \frac{n-2}{n-1}\phi_0 (1-r) \left(\sum_i f_i \left(k_i - k_i^2\right) \right)$$

Finally, we get that

$$V^* = \phi_0 (1-r) \left[\left(\frac{1}{n-1} \right) \sum_i f_i k_i^2 + \frac{n-2}{n-1} \sum_i f_i k_i - n \right]$$

or, in vector notation,

$$V^*(g) = \phi_0 (1-r) \left[\frac{1}{n-1} \mathbf{k}^t(\rho, \mathbf{\Omega}) \mathbf{F} \mathbf{k}(\rho, \mathbf{\Omega}) + \frac{n-2}{n-1} \mathbf{k}^t(\rho, \mathbf{\Omega}) \mathbf{F} \mathbf{1} \right] \leq 0.$$

Proof Proposition 3: Suppose that the optimal equilibrium action of agent i is of the following linear form:

$$a_i^* = \alpha_i + \beta_i y_i + \gamma_i z$$

The equilibrium actions are then obtained from the system described by the n equations in (4.4). Observe that, if $i \neq j$,

$$\mathbb{E}\left[a_j^* | y_i, z\right] = \mathbb{E}\left[\alpha_j + \beta_j y_j + \gamma_j z | y_i, z\right] = \alpha_j + \gamma_j z + \beta_j \mathbb{E}\left[y_j | y_i, z\right]$$

We now compute $\mathbb{E}[y_j|y_i, z]$. We know that $(z, y_1, ..., y_n)\,|\theta \sim N\left(\theta \mathbf{1}_{n+1}, \phi \tilde{\mathbf{\Sigma}}\right)$ where $\tilde{\Sigma}$ is the following $(n+1) \times (n+1)$ matrix

$$\tilde{\mathbf{\Sigma}} = \begin{bmatrix} 1/\alpha_z & \mathbf{0}_n^t \\ \mathbf{0}_n & \mathbf{\Sigma} \end{bmatrix}$$

where $\mathbf{0}_n$ is the n-dimensional vector with all entries equal to 0, and $\mathbf{\Sigma}$ is the matrix defined in (4.2). From this, we deduce that$(\theta, z, y_1, ..., y_n) \sim N(\mathbf{0}_{n+2}, \mathbf{\Xi})$, where $\mathbf{\Xi}$ is the following $(n+2) \times (n+2)$ matrix:

$$\mathbf{\Xi} = \begin{bmatrix} \phi_0 & \phi_0 \cdot \mathbf{1}_{n+1}^t \\ \phi_0 \cdot \mathbf{1}_{n+1} & \phi_0 \mathbf{J}_{n+1} + \phi \tilde{\mathbf{\Sigma}} \end{bmatrix},$$

where $\mathbf{J}_{n+1}$ is the $(n+1) \times (n+1)$ matrix with all entries equal to 1.

From this, we deduce that the unconditional distribution of the information vector $(z, y_1, ..., y_n)$ is $(z, y_1, ..., y_n) \sim N(\mathbf{0}_{n+1}, \mathbf{\Psi})$ where

$$\mathbf{\Psi} = \begin{bmatrix} \phi_0 + \phi_z & \phi_0 \cdot \mathbf{1}_n^t \\ \phi_0 \cdot \mathbf{1}_n & \phi_0 \mathbf{J}_n + \phi \mathbf{\Sigma} \end{bmatrix}$$

Now, making use of the identities about the multinormal we obtain that:

$$\mathbb{E}\left[y_j|y_i,z\right]=\frac{\alpha_z\left(\sigma_{ii}-\sigma_{ij}\right)}{1+\left(\alpha+\alpha_z\right)\sigma_{ii}}z+\frac{1+\left(\alpha+\alpha_z\right)\sigma_{ij}}{1+\left(\alpha+\alpha_z\right)\sigma_{ii}}y_i=\zeta_{ji}z+\omega^z_{ji}y_i$$

Observe that this is consistent with the results obtained without public signal. If the variance of z tends to $+\infty$ (and therefore public information becomes irrelevant) then α_z tends to 0 and we recover the expression for $\mathbb{E}\left[y_j|y_i\right]$ in (**??**).

We can now solve (4.4), that we rewrite as:

$$\begin{aligned}\alpha_i+\beta_i y_i+\gamma_i z &= \left(1-r\right)\left(f_i y_i+h_i z\right)+\\ &+\rho\sum_{j\neq i}\left(\alpha_j+\beta_j\left(\omega^z_{ij}y_i+\zeta_{ij}z\right)+\gamma_j z\right), \qquad i\in N\end{aligned} \tag{5.7}$$

This system can be divided into three different ones: one that relates independent terms, another one that involves the coefficients of y's, and the other one for the coefficients of z. We solve them sequentially.

The system for independent terms is quite simple:

$$\alpha_i=\rho\sum_{j\neq i}\alpha_j, \quad i\in N \tag{5.8}$$

The unique solution to this system is $\alpha_i=0$ for all $i\in N$.

The system for terms that multiply y_i's is

$$\beta_i=\left(1-r\right)f_i+\rho\sum_{j\neq i}\beta_j\omega^z_{ij} \tag{5.9}$$

Therefore, $\beta_i=f_i k_i(\rho,\mathbf{\Omega})$.

Finally, the system that involves the terms that multiply z is

$$\gamma_i=\left(1-r\right)h_i+\rho\sum_{j\neq i}\left(\beta_j\zeta_{ij}+\gamma_j\right), \quad i\in N \tag{5.10}$$

Since we know the values of h_i, β_j,[64] and ζ_{ji} for all i,j, we can define the constant $\pi_i = h_i+\rho\left(\sum_{j\neq i}\beta_j\zeta_{ij}\right)/\left(1-r\right)$. The system in (5.10) is then equivalent to

$$\gamma_i=\left(1-r\right)\pi_i+\rho\sum_{j\neq i}\gamma_j \tag{5.11}$$

[64] Observe that it is important the order in which we solve the three systems.

Let $\mathbf{\Pi}$ be the matrix with a zero diagonal and out-of-diagonal entries given by $\pi_{ij} = \pi_j/\pi_i$. Then, the solution to (5.11) is $\gamma_i = \pi_i k_i(\rho, \mathbf{\Pi})$. ■

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