On Quasiperiodic Perturbations of Ordinary Differential Equations

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CERTIFICO que la present memòria ha estat realitzada per en Àngel Jorba i Monte, i dirigida per mi, al Departament de Matemàtica Aplicada i Anàlisi de la Universitat de Barcelona.

Barcelona, Agost de 1991,

Dr. Carles Simó i Torres.

a la Monti

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CAP DE SECRETARIA

Conxita MARIN I GUERRERO

No voldria començar sense fer constar el meu agraïment a totes aquelles persones que, de una manera o altra, m'han ajudat a dur a terme aquest treball.

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Contents

Introduction

In this work we study several topics concerning quasiperiodic time-dependent perturbations of ordinary differential equations. This kind of equations appear as models in many applied problems of Celestial Mechanics, and we have used, as an illustration, the study of the behaviour near the equilateral libration points of the real Earth-Moon system. Let us introduce this problem as a motivation. As a first approximation, suppose that the Earth and Moon are revolving in circular orbits around their centre of masses, neglect the effect of the rest of the solar system and neglect the aspherical terms coming from the Earth and Moon (of course, all the effects minor than the above mentioned, as the relativistic corrections, must be neglected). With this, we can write the equations of motion of an infinitessimal particle (by infinitessimal we mean that the particle is influenced by the Earth and Moon, but it does not affect them) by means of Newton's law. The study of the motion of that particle is the so-called Restricted Three Body Problem (RTBP). Usually, in order to simplify the equations, the units of lenght, time and mass are chosen so that the angular velocity of rotation, the sum of masses of the bodies and the gravitational constant are all equal to one. With these normalized units, the distance between the bodies is also equal to one. If these equations of motion are written in a rotating frame leaving fixed the Earth and Moon (these main bodies are usually called primaries), it is known that the system has five equilibrium points (see [22] for details). Two of them can be found as the third vertex of equilateral triangles having the Earth and Moon as vertices, and they are usually called equilateral libration points.

It is also known that, when the mass parameter μ (the mass of the small primary in the normalized units) is less than the Routh critical value $\mu_R = \frac{1}{2}(1 - \sqrt{23/27}) = 0.03852\ldots$ (this is true in the Earth-Moon case) these points are linearly stable. Applying the KAM theorem to this case we can obtain that there exist invariant tori around these points. Now, if we restrict the motion of the particle to the plane of motion of the primaries we have that, inside each energy level, these tori split the phase space and this allows to prove (see [20]) that the equilateral points are stable (except for two values, $\mu = \mu_2$ and $\mu = \mu_3$ with low order resonances). In the spatial case, the invariant tori do not split the phase space and, due to the possible Arnold diffusion, these points can be unstable. But Arnold diffusion is a very slow

phenomenon and we can have small neighbourhoods of "practical stability" (see [10], [19] and [5]), that is, the particle will stay near the equilibrium point for very long time spans.

Unfortunately, the real Earth-Moon system is rather complex. In this case, due to the fact that that the motions of the Earth and the Moon are non circular (even non elliptical!) and the strong influence of the Sun, the libration points do not exist as equilibrium points, and we need to define "instantaneous" libration points as the ones forming an equilateral triangle with the Earth and the Moon at each instant. If we perform some numerical integrations starting at (or near) these points we can see that the solutions go away after a short period of time (see [17] and [13]), showing that these regions are unstable.

Two conclusions can be obtained from this fact. First: if we are interested in keeping a spacecraft there, we will need to use some kind of control. Second: the RTBP is not a good model for this problem, because the behaviour displayed by it is different from the one of the real system.

For these reasons, an improved model has been developped in order to study this problem (see [13] and [12]). This model includes the main perturbations (due to the solar effect and to the noncircular motion of the Moon), assuming that they are quasiperiodic. This is a very good approximation for time spans of some thousands of years. It is not clear if this is true for longer time spans, but this matter will not be considered in this work. This model is in good agreement with the vector field of the solar system directly computed by means of the JPL ephemeris, for the time interval for which the JPL model is available.

The study of this kind of models is the main purpose of this work.

First of all, we have focused our attention on linear differential equations with constant coefficients, affected by a small quasiperiodic perturbation. These equations appear as variational equations along a quasiperiodic solution of a general equation and they also serve as an introduction to nonlinear problems.

The purpose is to reduce those systems to constant coefficients ones by means of a quasiperiodic change of variables, as the classical Floquet theorem does for periodic systems. It is also interesting to have a way to compute this constant matrix, as well as the change of variables. The most interesting case occurs when the unperturbed system is of elliptic type. Other cases, as the hyperbolic one, have already been studied (see [3]). We have added a parameter ε in the system, multiplying the perturbation, such that if ε is equal to zero we recover the unperturbed system. In this case we have found that, under suitable hypothesis of nonresonance, analyticity and nondegeneracy with respect to ε, it is possible to reduce the system to constant coefficients, for a cantorian set of values of ε. Moreover, the proof is constructive in an iterative way. This means that it is possible to find approximations to the reduced matrix as well as to the change of variables that performs such reduction. These results are given in Chapter 1.

The nonlinear case is now going to be studied. We have then considered an elliptic equilibrium point of an autonomous ordinary differential equation, and we have added a small quasiperiodic perturbation, in such a way that the equilibrium point does not longer exist. As in the linear case, we have put a parameter (ε) multiplying the perturbation. There is some "practical" evidence (see [7] and [12]) that there exists a quasiperiodic orbit, having the same basic frequencies that the perturbation, such that, when the pertubation goes to zero, this orbit goes to the equilibrium point. Our results show that, under suitable hypothesis, this orbit exists for a cantorian set of values of ε. We have also found some results related to the stability of this orbit. These results are given in Chapter 2.

A remarkable case occurs when the system is Hamiltonian. Here it is interesting to know what happens to the invariant tori near these points when the perturbation is added. Note that the KAM theorem can not be applied directly due to the fact that the Hamiltonian is degenerated, in the sense that it has some frequencies (the ones of the perturbation) that have fixed values and they do not depend on actions in a diffeomorphic way. In this case, we have found that some tori still exist in the perturbed system. These tori come from the ones of the unperturbed system whose frequencies are nonresonant with those of the perturbation. The perturbed tori add these perturbing frequencies to the ones they already had. This can be described saying that the unperturbed tori are "quasiperiodically dancing" under the "rhythm" of the perturbation. These results can also be found in Chapter 2 and Appendix C.

The final point of this work has been to perform a study of the behaviour near the instantaneous equilateral libration points of the real Earth-Moon system. The purpose of those computations has been to find a way of keeping a spacecraft near these points in an unexpensive way. As it has been mentioned above, in the real system these points are not equilibrium points, and their neighbourhood displays unstability. This leads us to use some control to keep the spacecraft there. It would be useful to have an orbit that were always near these points, because the spacecraft could be placed on it. Thus, only a station keeping would be necessary. The simplest orbit of this kind that we can compute is the one that replaces the equilibrium point. In Chapter 3, this computation has been carried out first for a planar simplified model and then for a spatial model. Then, the solution found for this last model has been improved, by means of numerical methods, in order to have a real orbit of the real system (here, by real system we mean the model of solar system provided by the JPL tapes). This improvement has been performed for a given (fixed) time-span. That is sufficient for practical purposes. Finally, an approximation to the linear stability of this refined orbit has been computed, and a very mild unstability has been found, allowing for an unexpensive station keeping. These results are given in Chapter 3.

As the numerical methods used in Chapter 3 have some difficulties to refine an

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orbit for very long time intervals, some modifications have been introduced. These modifications can overcome those troubles and give good nominal orbits for very large time intervals. The description of these modified methods can be found inside Appendix A, as well as the results obtained.

Finally, in Appendix B the reader can find the technical details concerning the way of obtaining the models used to study the neighbourhood of the equilateral points. This has been jointly developped with Gerard Gómez, Jaume Llibre, Regina Martínez, Josep Masdemont and Carles Simó.

Chapter 1

Quasiperiodic Perturbations of Linear Equations

1.1 Introduction

In this Chapter we study linear differential equations under quasiperiodic time-dependent perturbations. First of all, let us define rigorously the concept "quasiperiodic".

Definition 1.1 *A function $f = f(t)$ is said to be a quasiperiodic function with basic frequencies $\omega_1, \ldots, \omega_r$ if $f(t) = F(\theta_1, \ldots, \theta_r)$, where F is 2π periodic in all its arguments and $\theta_j = \omega_j t$ for $j = 1, \ldots, r$.*

We assume that the quasiperiodic functions appearing in our equations are analytical. For definiteness we give the following

Definition 1.2 *A function $f = f(t)$ is said to be analytic quasiperiodic on a strip of width ρ if it is quasiperiodic and F (see Definition 1.1) is analytical for $|\mathrm{Im}\ \theta_j| \leq \rho$ for $j = 1, \ldots, r$. In this case let us denote by $\|f\|_\rho$ the norm*

$$\sup\{|F((\theta_1, \ldots, \theta_r)| \ / \ |\mathrm{Im}\ \theta_j| \leq \rho,\ 1 \leq j \leq r\}.$$

Let us consider first the following equation:

$$\dot{x} = A(t)x, \tag{1.1}$$

where $A(t)$ is an $n \times n$ matrix that depends on time in a quasiperiodic way with basic frequencies $\omega = (\omega_1, \ldots, \omega_r)^T$. A change of variables $x = P(t)y$ is said to be a Lyapunov-Perron (LP) transformation if $P(t)$ is nonsingular and $P(t)$, $P^{-1}(t)$ and $\dot{P}(t)$ are bounded for all $t \in \mathbf{R}$. Moreover, if P, P^{-1} and $\dot{P}$ are quasiperiodic, the change $x = P(t)y$ is called a quasiperiodic LP transformation.

If $x = P(t)y$ is a LP transformation, then y satisfies the equation

$$\dot{y} = B(t)y, \qquad (1.2)$$

where $B = P^{-1}(AP - \dot{P})$. Equation (1.1) is said to be reducible if there is a quasiperiodic LP transformation that transforms (1.1) to (1.2), where B is a constant matrix. Obviously if Q is periodic the reducibility in all cases is given by the classical Floquet theory (see, for example, [4] or [2]).

Classical results concerning to almost periodic systems can be found in [9] and [6]. In [3] this problem is studied for different conditions on A and Q and the ideas used here are very close to the ones found in [3]. Another source of inspiration has been the proof of KAM theorem given in [1].

A different approach to this problem can be found in [14]. To introduce it, let us define the spectrum $\Sigma = \Sigma(A)$ of (1.1) as the collection of $\lambda \in \mathbf{R}$ for which the shifted equation $\dot{x} = (A(t) - \lambda I)x$ (here, I denotes the identity matrix in $\mathbf{R}^n$) does not have an exponential dichotomy (see [6]). We will say that (1.1) satisfies the "full spectrum" hypothesis if $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i \neq \alpha_j$ for $i \neq j$. Then, in [14] is shown that if A is sufficiently smooth, its frequencies satisfy a suitable nonresonance condition and it has "full spectrum" then the system (1.1) is reducible.

In this work we will drop the "full spectrum" hypothesis and we will consider $A(t)$ analytical and close to a nonresonant constant matrix. Our system will be

$$\dot{x} = (A + \varepsilon Q(t))x, \qquad (1.3)$$

being x a d-dimensional vector. Let λ_j, $j = 1, \ldots, d$ be the eigenvalues of A and $\lambda^T = (\lambda_1, \ldots, \lambda_d)$. The greatest difficulties are found when the real parts of all λ_j are equal (perhaps zero). We present a theorem which holds in this case asking for some nonresonance conditions for the vector $\nu^T = (\lambda^T, \omega^T)$. These conditions are satisfied by a set of big relative measure in the space of the parameter ν. Under some nondegeneracy conditions we will prove that if ε_0 is small enough, there exists a cantorian subset $\mathcal{E}$ of $[0, \varepsilon_0]$ of positive measure such that, if $\varepsilon \in \mathcal{E}$ then (1.3) is reducible. Moreover, our proof is constructive using an iterative scheme with quadratic convergence with respect to ε. That is, after n steps the transformed equation looks like (1.3) with $A_n(\varepsilon)$, ε^{2^n} and $Q_n(t, \varepsilon)$ (bounded by some M_n) instead of A, ε and $Q(t)$ for ε in some cantorian set $\mathcal{E}_n$.

Similar ideas have been used in [8], but only for Hamiltonian systems.

1.2   Main Results

The average of $Q(t)$ is defined as

$$\overline{Q} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q(t)\, dt.$$

For the existence of the limit see [9]. We consider first the equation (1.3) after averaging with respect to t and some rearrangement

$$\dot{x} = (\overline{A} + \varepsilon \tilde{Q}(t))x,$$

where $\tilde{Q}(t) = Q(t) - \overline{Q}$, $\overline{A} = A + \varepsilon \overline{Q}$. Next we do the change of variables $x = (I + \varepsilon P)y$ to get

$$\dot{y} = [(I + \varepsilon P)^{-1}(\overline{A} + \varepsilon(\overline{A}P - \dot{P} + \tilde{Q})) + \varepsilon^2 (I + \varepsilon P)^{-1}\tilde{Q}P]y, \qquad (1.4)$$

We would like to have

$$(I + \varepsilon P)^{-1}(\overline{A} + \varepsilon(\overline{A}P - \dot{P} + \tilde{Q})) = \overline{A}$$

and this implies

$$\dot{P} = \overline{A}P - P\overline{A} + \tilde{Q}. \qquad (1.5)$$

Suppose now that we have a quasiperiodic solution of (1.5) with the same frequencies which appear in Q. Then, (1.4) becomes

$$\dot{y} = [\overline{A} + \varepsilon^2 (I + \varepsilon P)^{-1}\tilde{Q}P]y.$$

Now we average again and restart the process. Obviously, if we can do this until the nth step, we shall get an equation like

$$\dot{x}_n = (A_n + \varepsilon^{2^n}\tilde{Q}_n)x_n,$$

where $\|\tilde{Q}_n\|$ can be very large. We are going to see that, under suitable conditions, this method converges.

Theorem 1.1 *Consider the equation $\dot{x} = (A + \varepsilon Q(t))x$, $\varepsilon \in (0, \varepsilon_0)$ and $x \in \mathbf{R}^d$, where A is a constant matrix with different eigenvalues $\lambda_1, \ldots, \lambda_d$ and $Q(t)$ is a quasiperiodic matrix with basic frequencies $\omega_1, \ldots, \omega_r$. Suppose that*

1. Q is analytic on a strip of width ρ_0 with $\rho_0 > 0$.

2. The vector ν, where $\nu^T = (\lambda_1, \ldots, \lambda_d, \sqrt{-1}\omega_1, \ldots, \sqrt{-1}\omega_r)$ satisfies the non-resonance conditions

$$|(m, \nu)| \geq \frac{c_\nu}{|m|^{\gamma_\nu}},$$

for all $m \in \{m_1 \in \mathbf{Z}^d, |m_1| = 0 \text{ or } |m_1| = 2\} \times \{m_2 \in \mathbf{Z}^r, |m_2| \neq 0\}$, where c_ν is a positive number, $\gamma_\nu = r + d + \beta$, $\beta > -1$ and $|m| = \sum_{j=1}^{r+d} |m^j|$, where m^j denote the components of m in $\mathbf{Z}^{r+d}$.

3. Let $\overline{Q}$ be the average of Q with respect to t and let $\lambda_j^0(\varepsilon)$ be an eigenvalue of
$\overline{A} = A + \varepsilon\overline{Q}$ for $j = 1, \ldots, d$. We require

$$\left| \frac{d}{d\varepsilon}(\lambda_i^0(\varepsilon) - \lambda_j^0(\varepsilon))|_{\varepsilon=0} \right| > 2\delta > 0, \ \forall \, 1 \leq i < j \leq d.$$

Then there exists a cantorian set $\mathcal{E} \subset (0, \varepsilon_0)$ with positive Lebesgue measure such that the system $\dot{x} = (A+\varepsilon Q)x$ is reducible. If ε_0 is small enough the relative measure of $\mathcal{E}$ in $(0, \varepsilon_0)$ is close to 1. Furthermore the quasiperiodic change of variables that transforms the system to $\dot{y} = By$ (B being a constant matrix) has the same basic frequencies than Q.

Remark 1. The nonresonance condition for ν is satisfied for most of the values of ν. More concretely, if ν belongs to a ball of radius R then we have that the condition is satisfied for all ν except by a set of relative Lebesgue measure less than $4c_\nu(d+r)^{\frac{5}{2}}\frac{\zeta(2+\beta)}{R}$ where ζ denotes the Riemann zeta function. The third condition is a nondegeneracy condition, not allowing to be locked at resonance. This condition can be replaced by a higher order nondegeneracy condition but it is not so simple to state in the hypothesis.

Remark 2. We can suppose that $A = \text{diag}(\lambda_1, \ldots, \lambda_d)$. Let $\|Q\|_\rho$ be the matricial norm associated to the vector norm defined by $\|(f_1, \ldots, f_d)^T\|_\rho = \max_{1 \leq k \leq d} \|f_k\|_\rho$, where $\|f_k\|_\rho$ is the norm defined in the Introduction. Introducing a new time $\tau = st$ where

$$s = \max\left\{ \frac{\frac{\pi^2}{3}+1}{\rho_0}, \ \|Q\|_{\rho_0} \right\}$$

we can suppose $\rho_0 \geq \frac{\pi^2}{3} + 1$, $\|Q\|_{\rho_0} \leq 1$. These bounds will be used in the proof of the theorem. The scaling can change the constant c_ν and, therefore the admisible set of ε is scaled by the same factor.

1.3 Lemmas

We need some lemmas.

Lemma 1.1 *Let $N_r^m = \#\{k \in \mathbf{Z}^r \ / \ |k| = \sum_{i=1}^r |k_i| = m\}$. Then*

$$N_r^m \leq \frac{2^r}{(r-1)!}\left(m + \frac{r}{2}\right)^{r-1}, \ \forall \, r, m \geq 1.$$

Proof: As k_r ranges from $-r$ to r we have the recurrence relation

$$N_r^m = 2 \sum_{k=1}^{m-1} N_{r-1}^k + N_{r-1}^m + 2$$

and $N_1^m = 2$ for all m. This satisfies the relation given on the statement. Suppose that this relation holds for all m and some r. Then

$$
\begin{aligned}
N_{r+1}^m &= 2 \sum_{k=1}^{m-1} N_r^k + N_r^m + 2 < \\
&< 2 \int_1^m \frac{2^r}{(r-1)!} \left(x + \frac{r}{2} \right)^{r-1} dx + \frac{2^r}{(r-1)!} \left(m + \frac{r}{2} \right)^{r-1} + 2 = \\
&= \frac{2^{r+1}}{r!} \left[\left(m + \frac{r}{2} \right)^r - \left(1 + \frac{r}{2} \right)^r + \frac{r}{2} \left(m + \frac{r}{2} \right)^{r-1} + \frac{r!}{2^r} \right].
\end{aligned}
$$

But

$$
\begin{aligned}
\left(m + \frac{r}{2} \right)^r + \frac{r}{2} \left(m + \frac{r}{2} \right)^{r-1} &< \left(m + \frac{r}{2} + \frac{1}{2} \right)^r = \\
&= \left(m + \frac{r}{2} \right)^r + \frac{r}{2} \left(m + \frac{r}{2} \right)^{r-1} + \sum_{j=2}^r \frac{1}{2^j} \binom{r}{j} \left(m + \frac{r}{2} \right)^{r-j},
\end{aligned}
$$

and, using that $\frac{r!}{2^r} < (1 + \frac{r}{2})^r$ the result follows. ∎

Remark. A simpler (and worse if $r \geq 3$ and m not too small) bound as $N_r^m \leq 2rm^{r-1}$ can also be obtained by induction. There is numerical evidence that the factor $\frac{1}{2}$, which multiplies r on the statement of the Lemma can be replaced by 0.1872183, slightly larger than $(2e)^{-1}$. The bound of the Lemma is also true for $m = 0$.

Lemma 1.2 *Let*

$$p = \sum_{k \in \mathbf{Z}^r} p^k e^{(k,\omega)\sqrt{-1}t}$$

be an analytic Fourier series satisfying $|p^k| \leq A_1 |k|^\gamma e^{-\rho_1 |k|}$ *for* $k \neq 0$ *with* $\gamma > 0$. *If* $\rho_2 \in (0, \rho_1)$ *then, for* $k \neq 0$, *we have* $|p^k| \leq A_2 e^{-\rho_2 |k|}$ *where* $A_2 = A_1 \left(\frac{\gamma}{(\rho_1 - \rho_2)e} \right)^\gamma$.

Proof: We know $|p^k| \leq A_1 |k|^\gamma e^{-(\rho_1 - \rho_2)|k|} e^{-\rho_2 |k|}$. Using that the maximum of $g(x) = x^\gamma e^{-(\rho_1 - \rho_2)x}$ is reached when $x = \frac{\gamma}{\rho_1 - \rho_2}$ the proof is completed. ∎

Lemma 1.3 *We consider* $\dot{P} = AP - PA + Q$, *where* $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ *and* Q *is a quasiperiodic matrix with basic frequencies* $\omega = (\omega_1, \ldots, \omega_r)^T$ *and with zero average. Let* q_{ij} *be the elements of* Q:

$$q_{ij} = \sum_{k \in \mathbf{Z}^r \setminus \{0\}} q_{ij}^k e^{(k,\omega)\sqrt{-1}t}.$$

We suppose also $|q_{ij}^k| \leq M e^{-\rho_1 |k|}$ *and* $|\lambda_i - \lambda_j - (k,\omega)\sqrt{-1}| > \frac{c}{|k|^\gamma}$ *for all* $i, j \in \{1, \ldots, d\}$ *and all* $k \in \mathbf{Z}^r \setminus \{0\}$. *Then there exist a unique solution* P *of* $\dot{P} = AP - PA + Q$ *with the same frequencies that* Q *and satisfies* $|p_{ij}^k| \leq N e^{-\rho_2 |k|}$ *with* $\rho_2 \in (0, \rho_1)$ *and* $N = \frac{M}{c} \left(\frac{\gamma}{(\rho_1 - \rho_2)e} \right)^\gamma$.

Proof: We look for

$$p_{ij} = \sum_{k \in \mathbf{Z}^r \setminus \{0\}} p_{ij}^k e^{(k,\omega)\sqrt{-1}t},$$

and this means that we have to solve the linear system

$$\dot{p}_{ij} = \alpha_{ij} p_{ij} + q_{ij}, \quad \alpha_{ij} = \lambda_i - \lambda_j, \; 1 \leq i, j \leq d.$$

It is easy to obtain the coefficients p_{ij}^k:

$$p_{ij}^k = \frac{q_{ij}^k}{(k,\omega)\sqrt{-1} - \alpha_{ij}}.$$

From the hypothesis one has the bound

$$|p_{ij}^k| \leq |q_{ij}^k| \frac{|k|^\gamma}{c} \leq \frac{M}{c} |k|^\gamma e^{-\rho_1 |k|},$$

and using Lemma 1.2 we get

$$|p_{ij}^k| \leq \frac{M}{c} \left(\frac{\gamma}{(\rho_1 - \rho_2)e} \right)^\gamma e^{-\rho_2 |k|} = N e^{-\rho_2 |k|}. \quad \blacksquare$$

Remark. The worst situation is found when $\lambda_i - \lambda_j$ are on the imaginary axis. If they are out of it the given bounds of $|p_{ij}^k|$ are very high compared with the actual values. Therefore it is enough to restrict to the case when $\operatorname{Re}(\lambda_i - \lambda_j) = 0$ for all $i, j \in \{1, \ldots, d\}$, both for the initial matrix A and for all the matrices A_n found in the iterative process.

Lemma 1.4 *Let*

$$q(t) = \sum_{k \in \mathbf{Z}^r} q^k e^{(k,\omega)\sqrt{-1}t}$$

be such that $|q^k| \leq Me^{-\rho_1|k|}$. *Then, for* $r \geq 2$, *one has*

$$\|q\|_{\rho_2} < M \left(\frac{2}{\rho_1 - \rho_2}\right)^r \exp\left(\frac{(\rho_1 - \rho_2)r}{2}\right) \left(1 + \frac{\rho_1 - \rho_2}{\sqrt{2\pi(r-1)}}\right).$$

Proof: Let t be a complex number verifying $|\mathrm{Im}\ \theta_j| \leq \rho_2$, where $\theta_j = \omega_j t$, $1 \leq j \leq r$. Then

$$|q(t)| \leq \sum_{k \in \mathbf{Z}^r} |q^k| |e^{(k,\omega t)\sqrt{-1}}| \leq M \sum_{k \in \mathbf{Z}^r} e^{-\rho_1|k|} e^{\rho_2|k|} \leq M \sum_{k \in \mathbf{Z}^r} e^{-(\rho_1 - \rho_2)|k|}.$$

Let us define $\delta = \rho_1 - \rho_2$. This implies that

$$\|q\|_{\rho_2} \leq M \sum_{k \in \mathbf{Z}^r} e^{-\delta|k|},$$

and using Lemma 1.1

$$\|q\|_{\rho_2} \leq M \frac{2^r}{(r-1)!} \sum_{m=0}^{\infty} \left(m + \frac{r}{2}\right)^{r-1} e^{-\delta m}.$$

As the function $x \mapsto (x + \frac{r}{2})^{r-1} e^{-\delta x}$ has at most one maximum on $[0, \infty)$ the sum is bounded by the maximum plus the integral. Hence

$$
\begin{aligned}
\|q\|_{\rho_2} \quad &< \quad M \frac{2^r}{(r-1)!} \left[\left(\frac{r-1}{\delta e}\right)^{r-1} e^{\frac{\delta r}{2}} + \right.\\
&\quad + \left. \left(\frac{1}{\delta}\right)^r e^{\frac{\delta r}{2}} \int_0^\infty \left(\delta\left(x + \frac{r}{2}\right)\right)^{r-1} e^{-\delta(x + \frac{r}{2})} d\left(\delta\left(x + \frac{r}{2}\right)\right)\right] = \\
&= \quad M \frac{2^r}{(r-1)!} \frac{1}{\delta^r} e^{\frac{\delta r}{2}} \left[\frac{(r-1)^{r-1}}{e^{r-1}} \delta + (r-1)!\right] = \\
&= \quad M \left(\frac{2}{\delta}\right)^r e^{\frac{\delta r}{2}} \left[\frac{(r-1)^{r-1}}{e^{r-1}} \frac{\delta}{(r-1)!} + 1\right] < \\
&< \quad M \left(\frac{2}{\delta}\right)^r e^{\frac{\delta r}{2}} \left[1 + \frac{\delta}{\sqrt{2\pi(r-1)}}\right]. \quad \blacksquare
\end{aligned}
$$

Remark. In the statement one should replace the last factor of the bound by $\left(1 + \frac{\delta}{e}\right)$ if $r = 1$.

Lemma 1.5 *Let* $\{K_n\}_{n \in \mathbf{N}}$ *be a sequence of positive real numbers such that* $K_n \leq an^b K_{n-1}^2$. *Then*

$$K_n \leq \frac{1}{a} \left[\left(\frac{5}{3}\right)^b aK_0\right]^{2^n}.$$

Proof: It is easy to see that

$$K_n \leq a^{1+2+2^2+\cdots+2^{n-1}} \left[\prod_{i=0}^{n-1} (n-i)^{2^i} \right]^b K_0^{2^n}.$$

To bound the expression in brackets we take logarithms:

$$\ln \left(\prod_{i=0}^{n-1} (n-i)^{2^i} \right) = \sum_{i=0}^{n-2} 2^i \ln(n-i) \leq 2^n \sum_{i=0}^{\infty} \frac{\ln(i+2)}{2^{i+2}} = c2^n.$$

Then

$$
\begin{aligned}
c &= \sum_{i=0}^{\infty} \frac{\ln(i+2)}{2^{i+2}} < \sum_{i=0}^{j-1} \frac{\ln(i+2)}{2^{i+2}} + \sum_{i \geq j} \frac{\ln \left((j+2) \left(\frac{i+3}{j+2} \right)^{i-j} \right)}{2^{i+2}} = \\
&= \sum_{i=0}^{j-1} \frac{\ln(i+2)}{2^{i+2}} + \frac{\ln(j+2)}{2^{j+1}} + \ln \left(\frac{j+3}{j+2} \right) \frac{1}{2^{j+3}} \sum_{k \geq 1} \frac{k}{2^{k-1}} = \\
&= \sum_{i=0}^{j-1} \frac{\ln(i+2)}{2^{i+2}} + \frac{1}{2^{j+1}} \ln(j+3).
\end{aligned}
$$

Hence

$$\exp c < \prod_{i=0}^{j-1} (i+2)^{2^{-(i+2)}} (j+3)^{2^{-(j+1)}},$$

and taking $j = 3$ one obtains $\exp c < \frac{5}{3}$ because $2^4 \, 3^2 \, 24 < \left(\frac{5}{3} \right)^{16}$. Finally $K_n \leq a^{2^n-1} \left((\exp c)^b \right)^{2^n} K_0^{2^n}$ and the result follows. ∎

Remark: One can improve the bound on $\exp c$ but not more than three per thousand.

Lemma 1.6 *Consider the expression* $a_n = \left(\frac{(n+1)^c}{2} \right)^{2^{-n}}$ *for* $n \in \mathbb{N} \cup \{0\}$. *If* $c \geq 3$ *the maximum is obtained for* $n = 1$ *and therefore* $a_n < 2^{\frac{c-1}{2}}$.

Proof: Let $g(x) = \ln \left[\left(\frac{(x+1)^c}{2} \right)^{2^{-x}} \right] = c2^{-x}(\ln(x+1) - \ln \alpha)$, where $\alpha = 2^{\frac{1}{c}}$ and $x \geq 0$. Computing the derivative and equating to zero one should have $h(x) = \ln 2(\ln(x+1) - \ln \alpha) - \frac{1}{x+1} = 0$ to get a maximum. The function h is monotonically increasing, as $\ln \alpha \leq \frac{1}{3} \ln 2$ one has $h(1) < 0$, $h(2) > 0$ for all $c \geq 3$. To see that the maximum over the integers is attained at $n = 1$ we compare the valors for $n = 1$ and $n = 2$. One obtains $\left(\frac{2^c}{2} \right)^{\frac{1}{2}}$ and $\left(\frac{3^c}{2} \right)^{\frac{1}{4}}$ and the first one is larger than the second if $c > \frac{\ln 2}{\ln \frac{4}{3}} \approx 2.41$ ∎

Lemma 1.7 *Let M be a diagonal matrix with different eigenvalues μ_j, $j = 1, \ldots, d$, and $\alpha = \min_{i,j; i \neq j} |\mu_i - \mu_j|$. Let N be a matrix such that $(d+1)\|N\| < \alpha$ (here $\| . \|$ is the sup norm). Let ν_j, $j = 1, \ldots, d$ be the eigenvalues of $M + N$, B a suitable matrix such that $B^{-1}(M + N)B = D = \text{diag } (\nu_j)$ with condition number $C(B)$. Then*

1. $\beta = \min_{i,j; i \neq j} |\nu_i - \nu_j| \geq \alpha - 2\|N\|$.

2. $C(B) \leq \frac{\alpha + (d-3)\|N\|}{\alpha - (d+1)\|N\|}$. In particular, if $\|N\| < \frac{\alpha}{3d-1}$ then $C(B) < 2$.

Proof: From Gerschgorin Lemma (see, for example, [21]) it follows $|\mu_j - \nu_j| < \|N\|$ and hence *1* holds. Let $N = (n_{ij})$, $B = (b_{ij})$. The matrix B is made of eigenvectors of $M + N$. We choose a matrix B such that $b_{jj} = 1$, $j = 1, \ldots, d$. To determine b_{kj}, $k = 1, \ldots, d$, $k \neq j$ we have to solve a $(d - 1)$-dimensional linear system where the diagonal entries of the matrix are $\mu_k - \nu_j + n_{kk}$, $k \neq j$ and the out of diagonal entries are n_{km}, $k \neq j$, $m \neq j$. The independent term has entries $-n_{kj}$, $k \neq j$. Let b_{sj} such that $|b_{sj}| = \max_{k \neq j} |b_{kj}|$. From

$$n_{s1}b_{1j} + \cdots + n_{s\,j-1}b_{j-1\,j} + n_{s\,j+1}b_{j+1\,j} + \cdots + n_{sd}b_{dj} + (\mu_s - \nu_j)b_{sj} = -n_{sj}$$

one has

$$|b_{sj}| \leq \frac{|n_{sj}|}{|\mu_s - \nu_j| - \|N\|} \leq \frac{\|N\|}{\alpha - 2\|N\|}.$$

Therefore $B = I + B'$ with

$$\|B'\| < \frac{(d-1)\|N\|}{\alpha - 2\|N\|} < 1.$$

Then $C(B) = \|B\|\|B^{-1}\| \leq \frac{1 + \|B'\|}{1 - \|B'\|}$ and *2* follows. ∎

Lemma 1.8 *Let $\omega \in \mathbf{R}^r$ and λ_s, $s = 1, \ldots, d$ such that*

$$|\lambda_s - \lambda_j - \sqrt{-1}(k, \omega)| \geq \frac{c}{|k|^{\gamma_1}}.$$

for all s, $j \in \{1, \ldots, d\}$ and all $k \in \mathbf{Z}^r \setminus \{0\}$, where $c > 0$, $\gamma_1 > 0$. Define a resonant subset $\mathcal{R}_\mu$ as

$$\mathcal{R}_\mu = \{\varphi \in \sqrt{-1}\mathbf{R}, \; |\varphi| < \mu \; / \; \exists s, j \in \{1, \ldots, d\} \; \wedge \; \exists k' \in \mathbf{Z}^r \setminus \{0\}$$

$$\text{such that } |\varphi + \lambda_s - \lambda_j - \sqrt{-1}(k', \omega)| < \frac{\frac{c}{2}}{|k'|^{\gamma_2}}\}.$$

Let $\psi(\mu) = \frac{m(\mathcal{R}_\mu)}{2\mu}$, where m denotes the Lebesgue measure. If $\gamma_2 = \gamma_1 + r + 1$ then $\liminf_{\mu \to 0} \psi(\mu) = 0$.

Proof: Take $\mu_n = \frac{c}{n^{\gamma_1}}$. For any k' with $|k'| \geq n$ and any couple s, j the measure of the resonant interval of φ is bounded by $\frac{c}{|k'|^{\gamma_2}}$. Adding for all the values of k' with $|k'| = n'$ and all s, j and using the remark following Lemma 1.1, we have

$$m(\mathcal{R}_{\mu_n}) \leq c2rd^2 \sum_{n' \geq n} \frac{1}{(n')^{\gamma_2 - r + 1}} < 2crd^2(n-1)^{-(\gamma_2 - r)}.$$

Furthermore the resonant intervals associated to $n' < n$ are disjoint with $\mathcal{R}_{\mu_n}$ if n is large enough. Hence, for n large enough, $\psi(\mu_n) < rd^2 n^{\gamma_1}(n-1)^{-(\gamma_1 + 1)}$ which goes to zero if n goes to infinity. $\blacksquare$

1.4   Proof of Theorem 1.1

First we are going to do the proof without worrying about resonances, and then we shall take out the values of ε for which the proof fails.

We suppose that we have applied the method exposed in Section 1.2 until step n, and we are going to see that we can apply it again to get the $n + 1$ step. In this way we will obtain bounds for the quasiperiodic part at the nth step and for the transformation at this step, and this allows us to prove the convergence.

Now suppose that we are at n^{th} step. This means that we have

$$\dot{x}_n = (A_n + \varepsilon^{2^n} \tilde{Q}_n)x_n,$$

where A_n is a diagonal matrix with eigenvalues $\lambda_1^n, \ldots, \lambda_d^n$ satisfying

$$|\lambda_i^n - \lambda_j^n - (k, \omega)\sqrt{-1}| > \frac{c_n}{|k|^\gamma}, \ \forall i, j,$$

with $\gamma = \gamma_\nu + r + 1$ and c_n is taken as $c_n = \frac{c_0}{(n+1)^2}$. We have $Q_n = (q_{nij})$ with

$$q_{nij} = \sum_{k \neq 0} q_{nij}^k e^{(k,\omega)\sqrt{-1}t}$$

and $|q_{nij}| \leq M_n e^{-\rho_n |k|}$ where $M_n = \|Q_n\|_{\rho_n}$. Moreover, $\{\rho_n\}_n$ is a sequence defined by $\rho_n = \rho_{n-1} - \frac{2}{n^2}$ with $\rho_0 = \frac{\pi^2}{3} + 1$, and $\overline{\rho}_n = \rho_n + \frac{1}{n^2}$.

We note that the limit value $\lim_{n \to \infty} \rho_n$ is equal to 1. Finally we suppose that Q_n has already been averaged: $\overline{Q}_n = 0$. Now we need to solve $\dot{P}_n = A_n P_n - P_n A_n + Q_n$ and we use Lemma 1.3 to get a unique $P_n = (p_{nij})_{ij}$ whose elements verify

$$p_{nij} = \sum_{k \in \mathbb{Z}^r \setminus \{0\}} p_{nij}^k e^{(k,\omega)\sqrt{-1}t}$$

and

$$|p_{nij}^k| \leq \frac{M_n}{c_n} \left(\frac{\gamma}{\frac{1}{(n+1)^2} e} \right)^\gamma e^{-\overline{\rho}_{n+1}|k|}.$$

Introducing $E = \frac{1}{c_0} \left(\frac{\gamma}{e} \right)^\gamma$ we have that $|p_{nij}^k| \leq EM_n(n+1)^{2(\gamma+1)} e^{-\bar{\rho}_{n+1}|k|}$ where E does not depend on n.

Now we can apply Lemma 1.4 to bound $\|P_n\|_{\rho_{n+1}}$:

$$\|P_n\|_{\rho_{n+1}} \leq d \max_{i,j} \|p_{nij}\|_{\rho_{n+1}}.$$

Therefore

$$\|P_n\|_{\rho_{n+1}} \leq dEM_n(n+1)^{2(\gamma+1)} \left[\left(2(n+1)^2 \right)^r e^{\frac{r}{2(n+1)^2}} \left(1 + \frac{\frac{1}{(n+1)^2}}{\sqrt{2\pi(r-1)}} \right) \right].$$

We can bound the previous expression by

$$\|P_n\|_{\rho_{n+1}} \leq LM_n(n+1)^{2(\gamma+r+1)}, \tag{1.6}$$

where

$$L = dE2^r e^{\frac{r}{2}} \left(1 + \frac{1}{\sqrt{2\pi(r-1)}} \right).$$

Of course, if $r = 1$ we replace $\sqrt{2\pi(r-1)}$ by e.

Now, remembering that $M_n = \|Q\|_{\rho_n}$ we get the bound that we were looking for:

$$\|P_n\|_{\rho_{n+1}} \leq L(n+1)^{2(\gamma+r+1)} \|Q_n\|_{\rho_n}. \tag{1.7}$$

If we change variables through $y_{n+1} = (I + \varepsilon^{2^n} P_n)x_n$ we get

$$\dot{y}_{n+1} = (A_n + \varepsilon^{2^{n+1}}(I + \varepsilon^{2^n} P_n)^{-1} Q_n P_n)y_{n+1}.$$

We suppose now that $\|\varepsilon^{2^n} P_n\| \leq \frac{1}{2}$ (we will see after that it can be achieved by selecting ε small enough). Let $Q_{n+1}^* = (I + \varepsilon^{2^n} P_n)^{-1} Q_n P_n$. We can bound now the new quasiperiodic part:

$$\|Q_{n+1}^*\|_{\rho_{n+1}} \leq \frac{1}{1 - \|\varepsilon^{2^n} P_n\|_{\rho_{n+1}}} \|Q_n\|_{\rho_{n+1}} \|P_n\|_{\rho_{n+1}},$$

and using (1.7) we obtain:

$$\|Q_{n+1}^*\|_{\rho_{n+1}} \leq 2L(n+1)^{2(\gamma+r+1)} \|Q_n\|_{\rho_n}^2.$$

At this point we introduce the following matrices: $\tilde{Q}_{n+1}^*$ (see Section 1.2) and $\overline{A}_{n+1} = A_n + \varepsilon^{2^{n+1}} \overline{Q}_{n+1}^*$ (we note that, in general, $\overline{A}_{n+1}$ has no diagonal form). We still have

$$\|\tilde{Q}_{n+1}^*\|_{\rho_{n+1}} \leq 2L(n+1)^{2(\gamma+r+1)} \|Q_n\|_{\rho_n}^2.$$

Now we have the following equation

$$\dot{y}_{n+1} = (\overline{A}_{n+1} + \varepsilon^{2^{n+1}} \tilde{Q}^*_{n+1}) y_{n+1}.$$

Let B_{n+1} be a matrix such that $B^{-1}_{n+1} \overline{A}_{n+1} B_{n+1} = A_{n+1}$ is diagonal. We choose the diagonal of B_{n+1} equal to the identity as in Lemma 1.7. Making $x_{n+1} = B_{n+1} y_{n+1}$ one obtains

$$\dot{x}_{n+1} = (A_{n+1} + \varepsilon^{2^{n+1}} Q_{n+1}) x_{n+1},$$

where $Q_{n+1} = B^{-1}_{n+1} \tilde{Q}^*_{n+1} B_{n+1}$. As $\overline{Q}_{n+1} = 0$ we only need to control the size of $\|Q_{n+1}\|_{\rho_{n+1}}$. We define the condition number $C(B) = \|B^{-1}\| \, \|B\|$ for all nonsingular constant matrices B, and we will see later that $C(B_n) \leq 2 \ \forall n$.

Now we can bound $\|Q_{n+1}\|_{\rho_{n+1}}$:

$$\|Q_{n+1}\|_{\rho_{n+1}} = \|B^{-1}_{n+1} \tilde{Q}^*_{n+1} B_{n+1}\|_{\rho_{n+1}} \leq 4L(n+1)^{2(\gamma+r+1)} \|Q_n\|^2_{\rho_n}.$$

If we suppose that the same inequality holds for $\|Q_n\|_{\rho_n}, \ldots, \|Q_1\|_{\rho_1}$ and we use Lemma 1.5 together with $\|Q_0\|_{\rho_0} = 1$ one obtains

$$\|Q_{n+1}\|_{\rho_{n+1}} \leq \frac{1}{4L} \left[\left(\frac{5}{3} \right)^b 4L \right]^{2^{n+1}},$$

where $b = 2(\gamma + r + 1)$.

At this point we are in situation to prove the convergence. The quasiperiodic part at the nth step is $\varepsilon^{2^n} Q_n$ whose norm on the strip $|\text{Im} \, z| \leq \rho_n$ is bounded by

$$\frac{1}{4L} \left[\varepsilon \left(\frac{5}{3} \right)^b 4L \right]^{2^n}.$$

This converges to 0 if the bracket is less than 1, that is, if $\varepsilon < K^{-1}$, where $K = \left(\frac{5}{3} \right)^b 4L$.

We had left without proof the fact $\|\varepsilon^{2^n} P_n\|_{\rho_{n+1}} \leq \frac{1}{2}$. Recall (1.6) and then

$$\|\varepsilon^{2^n} P_n\|_{\rho_{n+1}} \leq \frac{(n+1)^{2(\gamma+r+1)}}{4} (\varepsilon K)^{2^n}.$$

To have $\|\varepsilon^{2^n} P_n\|_{\rho_{n+1}} < \frac{1}{2}$ it is enough to take

$$\varepsilon < \left(K \max_{n \in \mathbb{N} \cup \{0\}} \left\{ \left(\frac{(n+1)^c}{2} \right)^{2^{-n}} \right\} \right)^{-1},$$

where $c = 2(\gamma + r + 1) > 2(2r + d)$. Using Lemma 1.6 it is enough to take $\varepsilon < (K 2^\gamma \sqrt{2})^{-1} = \varepsilon_1$.

To end this part we need to prove that the condition $C(B_n) \leq 2 \; \forall n$ holds if ε is sufficiently small. Let $\alpha = \min_{i \neq j} |\lambda_i^0 - \lambda_j^0|$. The succesive steps change the minimum distance between eigenvalues (see Lemma 1.7) at most by

$$2 \sum_{n \geq 0} \varepsilon^{2^{n+1}} \|\overline{Q}_{n+1}^*\|_{\rho_{n+1}} \leq \frac{1}{4L} \sum_{n \geq 0} (\varepsilon K)^{2^{n+1}} \leq \frac{1}{4L} \frac{(\varepsilon K)^2}{1 - (\varepsilon K)^2}.$$

We ask this value to be less than $\frac{\alpha}{2}$. Then $|\lambda_i^n - \lambda_j^n| > \frac{\alpha}{2}$ and the condition 2 of Lemma 1.7 to have $C(B_n) \leq 2$ is written as

$$\frac{1}{8L} (\varepsilon K)^{2^{n+1}} \leq \frac{\frac{\alpha}{2}}{3d - 1}$$

that holds for all n if it holds for $n = 0$. Hence it is enough to impose the condition

$$\varepsilon < \left[K \max \left\{ \left(\frac{4\alpha L}{3d - 1} \right)^{-\frac{1}{2}}, \left(\frac{2L\alpha}{1 + 2L\alpha} \right)^{-\frac{1}{2}} \right\} \right]^{-1} = \varepsilon_2$$

to guarantee $C(B_n) \leq 2$ for all the transformations. Hence $\|\varepsilon^{2^n} Q_n\|_{\rho=1}$ goes to zero if $\varepsilon < \min(\varepsilon_1, \varepsilon_2) = \varepsilon_3$. To see that the composition of all the transformations $B_{n+1}(I + \varepsilon^{2^n} P_n)$ is convergent we first bound the transformation at step n:

$$\|B_{n+1}(I + \varepsilon^{2^n} P_n)\|_{\rho_{n+1}} \leq$$
$$\leq \left[1 + \frac{(d-1)\varepsilon^{2^{n+1}} \|\overline{Q}_{n+1}^*\|_{\rho_{n+1}}}{\frac{\alpha}{2} - 2\varepsilon^{2^{n+1}} \|\overline{Q}_{n+1}^*\|_{\rho_{n+1}}} \right] \left[1 + \varepsilon^{2^n} \|P_n\|_{\rho_{n+1}} \right] \leq$$
$$\leq \left[1 + \frac{(d-1)\frac{(\varepsilon K)^{2^{n+1}}}{8L}}{\frac{\alpha}{2} - \frac{(\varepsilon K)^{2^{n+1}}}{4L}} \right] \left[1 + \frac{(n+1)^{2(\gamma+1)}(\varepsilon K)^{2^n}}{4} \right] =$$
$$= (1 + a_n)(1 + b_n).$$

It is clear that a_n and b_n go to zero when n goes to infinity and that the series

$$\sum_{n=0}^{\infty} a_n, \quad \sum_{n=0}^{\infty} b_n$$

are convergent if $\varepsilon < \varepsilon_3$. Then the full procedure works for $\varepsilon < \min(\varepsilon_0, \varepsilon_3) = \varepsilon_4$ provided the nonresonant condition

$$|\lambda_i^n - \lambda_j^n - (k, \omega)\sqrt{-1}| > \frac{c_n}{|k|^\gamma}$$

holds for all $i, j \in \{1, \dots, d\}$, for all $k \in \mathbb{Z}^r \setminus \{0\}$ and for all $n \in \mathbb{N} \cup \{0\}$.

To end the proof we are going to take into account the resonances. Let $\varphi_{ij}^n(\varepsilon)$ be the function that gives the values of $\lambda_i^n - \lambda_j^n$ at step n:

$$\varphi_{ij}^n(\varepsilon) = \lambda_i^0(\varepsilon) - \lambda_j^0(\varepsilon) + \varepsilon^2 d_{ij,2}^m + \varepsilon^3 d_{ij,3}^m + \cdots$$

At every step the eigenvalues and the diagonalizing matrix, B_{n+1}, depend algebraically, and therefore analytically, on ε. Hence, as

$$\left| \frac{d}{d\varepsilon} (\lambda_i^0(\varepsilon) - \lambda_j^0(\varepsilon))|_{\varepsilon=0} \right| > 2\delta$$

one has $\frac{d}{d\varepsilon}|\varphi_{ij}^n(\varepsilon)| > \delta$ if ε is small enough, $\varepsilon < \varepsilon_5$. On the other side $|\frac{d\varphi_{ij}^n}{d\varepsilon}|$ is bounded by some $\bar{\delta}$ for all i, j, n in some interval $\varepsilon \in (0,\bar{\varepsilon}) \subset (0,\varepsilon_4) \cap (0,\varepsilon_5)$. Here we use, for simplicity, the remark following Lemma 1.3 and consider all the φ_{ij}^n as purely imaginary. If we take some μ_m (see Lemma 1.8), with $\gamma_1 = \gamma_\nu$, $\gamma_2 = \gamma$, $c = c_0 = \frac{c_\nu}{2}$, such that $\frac{\mu_m}{\delta} < \bar{\varepsilon}$, when ε ranges on $(0,\bar{\varepsilon})$ then φ_{ij}^n ranges on $(-\mu_m, \mu_m)$.

To obtain the cantorian set $\mathcal{E}_0$ where the nonresonant conditions holds for $n = 0$ one should delete an infinity of intervals in the range of ε with a measure at most $\psi(\mu_m)2\mu_m\frac{1}{\delta}d^2$. The relative measure of $\mathcal{E}_0$ in $(0, \frac{\mu_m}{\delta})$ is at least $1 - \psi(\mu_m)2\bar{\delta}\frac{d^2}{\delta}$. In a similar way we obtain the set $\mathcal{E}_n \subset \mathcal{E}_{n-1}$ where the nonresonant condition holds up to n. Its relative measure in $(0, \frac{\mu_m}{\bar{\varepsilon}})$ is at least

$$1 - \psi(\mu_m)(2\bar{\delta}\frac{d^2}{\delta}) \sum_{j=0}^{n} \frac{1}{(j+1)^2} > 1 - \psi(\mu_m)\frac{\pi^2}{3}\bar{\delta}\frac{d^2}{\delta}$$

which goes to 1 if n goes to infinity. The limit set

$$\mathcal{E}_\infty = \bigcap_{n \geq 0} \mathcal{E}_n$$

is the cantorian that we were looking for. ∎

1.5 Back to the Floquet Theorem

We have seen in the previous sections that it is possible (under suitable hypothesis) to reduce a quasiperiodic linear system to constant coefficients, for a cantorian set of values of ε.

Here we want to note that the fact of eliminating "resonant" values of ε is due to technical reasons and, sometimes, such reduction can be performed even in the resonant case. For example, let us define

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(t) = \begin{pmatrix} 0 & 0 \\ \cos 2t & 0 \end{pmatrix}, \quad x \in \mathbf{R}^2,$$

ans let us consider the equation

$$\dot{x} = (A + \varepsilon Q(t))x. \tag{1.8}$$

As Q is periodic (of period π), the Floquet Theorem ensures that this equation can be reduced to constant coefficients by means of a (perhaps complex) π-periodic change of variables (see [4] or [2]). On the other hand, Theorem 1.1 can not be applied to this equation: the frequency of the perturbation is 2, the eigenvalues of the constant part A are $\pm\sqrt{-1}$, and this means that we are in resonance. Moreover, we can not avoid this resonance using the parameter ε.

Let us see this with more detail. Floquet Theorem ensures the existence of a change of variables $x = P(t,\varepsilon)y$ such that equation (1.8) becomes $\dot{y} = B(\varepsilon)y$, where $B(\varepsilon)$ does not depend on t. Now, let us try to find $B(\varepsilon)$ "analytically". If $x = P(t,\varepsilon)y$ transforms equation (1.8) into $\dot{y} = B(\varepsilon)y$, this means that $P(t,\varepsilon)$ must satisfy the equation

$$\dot{P}(t,\varepsilon) = AP(t,\varepsilon) - P(t,\varepsilon)B(\varepsilon) + \varepsilon Q(t)P(t,\varepsilon). \tag{1.9}$$

Assuming that P and B depend analytically on ε we can write $P(t,\varepsilon) = P_0(t) + \varepsilon P_1(t) + \cdots$, $B(\varepsilon) = B_0 + \varepsilon B_1 + \cdots$. Putting these expressions into (1.9) and equating the coefficients of the powers of ε we get

$$\begin{aligned}
\dot{P}_0(t) &= AP_0(t) - P_0(t)B_0, \\
\dot{P}_1(t) &= AP_1(t) - P_1(t)B_0 + Q(t)P_0(t) - B_1 P_0(t), \\
&\vdots
\end{aligned}$$

Now, let us try to compute B_0, P_0, B_1 and P_1. It is natural to choose $B_0 = A$ and $P_0 = I$ (this means that, when ε is equal to zero, the change of variable is the identity and the reduced equation is the original one). Now, to find P_1 we need to solve

$$\dot{P}_1(t) = AP_1(t) - P_1(t)A + Q(t) - B_1.$$

Note that there is no way to select the constant matrix B_1 in order to get periodic solutions of this equation. This is due to the fact that the linear operator $C \mapsto AC - CA$ has, in this case, the eigenvalues $\pm 2\sqrt{-1}$ and 0, and the matrix $Q(t)$ has the frequency 2 (otherwise, we could choose B_1 as $\overline{Q}$ and to find a π-periodic solution).

This method has failed. But, what is wrong in it?. To see it, we will look at it more carefully. The first point is the calculus of B_0 and P_0. We can compute their values using the proof of the Floquet Theorem (see [4] or [2]): let us denote by $\phi(t,\varepsilon)$ the fundamental matrix ($\phi(0,\varepsilon) = I$) of (1.8). It is easy to see that $\phi(\pi,0) = -I$, and this implies that B_0 must satisfy the equation

$$e^{\pi B_0} = -I.$$

Let us see some properties of this equation.

- One solution is $B_0 = A$.

- Define $f(x_1, x_2, x_3, x_4) = e^{\pi X}$, where

$$X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

Then, rang $f(A) = 2$.

- If C is a matrix such that $e^{\pi C} = -I$, then for all nonsingular matrix S we have

$$e^{\pi S^{-1} C S} = S^{-1} e^{\pi C} S = -I.$$

This implies that there exists a 2-parametric family of matrices $C_{\lambda,\mu}$ such that $e^{\pi C_{\lambda,\mu}} = -I$.

- Let D be a matrix near $-I$. Then, generically, there exists a countable set of matrices $\{C_n\}_{n \in \mathbf{Z}}$ such that $e^{\pi C_n} = D$.

Now let us consider again the fundamental matrix $\phi(t, \varepsilon)$, where ε is small. This implies that $\phi(\pi, \varepsilon)$ is near $-I$, and then, there exist a countable set of matrices $\{C_n(\varepsilon)\}_{n \in \mathbf{Z}}$ such that

$$e^{\pi C_n(\varepsilon)} = \phi(\pi, \varepsilon).$$

Moreover, the matrices $C_n(0)$ are elements of the set of the matrices $C_{\lambda,\mu}$ defined above.

Then, if we want the matrix $B(\varepsilon)$ of the reduced system to be continuous with respect to ε, we have to choose $B_0 = C_n(0)$ for some n. We select, as an example, $B_0 = \sqrt{-1} I$[1]. Now it is possible to compute $P_0(t)$ from the proof of the Floquet Theorem to obtain $P_0(t) = e^{At} e^{-B_0 t}$, that satisfies

$$\dot{P}_0(t) = A P_0(t) - P_0(t) B_0.$$

Making the change of variables $x = P_0(t) y$ to equation (1.8) we get

$$\dot{y} = (B_0 + \varepsilon R(t)) y,$$

where $R(t) = P_0(t)^{-1} Q(t) P_0(t)$. For simplicity we rename the vector and matrices of this equation to

$$\dot{x} = (A + \varepsilon Q(t)) x.$$

Now, starting again the process and assuming that $P(t, \varepsilon) = P_0(t) + \varepsilon P_1(t) + \cdots$, $B(\varepsilon) = B_0 + \varepsilon B_1 + \cdots$ we can get

$$\begin{aligned}
\dot{P}_0(t) &= A P_0(t) - P_0(t) B_0, \\
\dot{P}_1(t) &= A P_1(t) - P_1(t) B_0 + Q(t) P_0(t) - B_1 P_0(t),
\end{aligned}$$

$$\vdots$$

---

[1] This has been suggested by a numerical computation.

and here we can take $B_0 = A$ and $P_0(t) = I$. Using that $A = \sqrt{-1}I$ the equation to find P_1 is

$$\dot{P}_1(t) = Q(t) - B_1,$$

that can be solved easily by selecting $B_1 = \overline{Q}$. With this, the first step of the inductive process has been done.

Here, it has been shown how to overcome the resonance problems in the periodic case, showing that it is possible to find the reduced equation by means of a sequence of steps as the one described just above.

Unfortunately, the quasiperiodic case is not so simple. In this case we have to worry about not only of exact resonances but also of quasiresonances[2], and this makes the problem (of reducing a quasiperiodic system to constant coefficients for a full set of values of ε) really difficult.

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[2]In this case, as we have a lot of different perturbing frequencies, we can have resonances involving many eigenvalues.

Chapter 2

Quasiperiodic Perturbations of Elliptic Equilibrium Points

2.1 Introduction

In this Chapter we shall consider autonomous differential equations under quasiperiodic time-dependent perturbations, near an elliptic equilibrium point. The kind of equations we shall deal with is

$$\dot{x} = (A + \varepsilon Q(t,\varepsilon))x + \varepsilon g(t,\varepsilon) + h(x,t,\varepsilon),$$

where A is assumed to be elliptic, h is of second order in x and the system is autonomous when $\varepsilon = 0$.

This kind of equations appears in many problems. As an example, we can consider the equations of the motion near the Equilateral Libration points of the Earth-Moon system, including (quasiperiodic) perturbations coming from the noncircular motion of the Moon and the effect of the Sun (see [7], [13], [11] and [12]). In those works, some seminumerical methods have been applied to compute a quasiperiodic orbit replacing the equilateral relative equilibrium point (this means that, when the perturbation tends to zero, that quasiperiodic orbit tends to the libration point), but there is a lack of theoretical support to ensure that the methods used are really convergent, and the computed quasiperiodic orbit really exists. Here, the existence of that dynamical equivalent is shown for a cantorian set (of positive measure) of values of ε.

Another problem related to this is the study of the stability of that quasiperiodic solution. In order to do this a kind of Floquet theory is available (see [15] and Chapter 1 of this work), that now can be obtained as a result of the more general study presented here.

Finally, it is interesting to consider the Hamiltonian case. Here we show that

some (nonresonant) tori still persist when the quasiperiodic time-dependent perturbation is added.

A study of this kind can be found in [16], but for a slightly different problem.

2.2   A Dynamical Equivalent to Elliptic Equilibrium Points

As we have already said, we are interested in the equation

$$\dot{x} = (A + \varepsilon Q(t,\varepsilon))x + \varepsilon g(t,\varepsilon) + h(x,t,\varepsilon), \tag{2.1}$$

where the time-dependence is quasiperiodic with vector of basic frequencies $\omega = (\omega_1,\ldots,\omega_r)$ and analytic on a strip of width $\rho_0 > 0$ (see Definitions 1.1 and 1.2).

First of all, we are going to try to eliminate the independent term $(g(t))$ by means of quasiperiodic changes of variables. To do this, we shall need an scheme with quadratic convergence (otherwise the small divisors effect would make the method divergent). This kind of schemes are based in the Newton's method, that is, to linearize the problem in a known approximation of the solution, solve this linear problem and take this solution as a new (better) approximation to the solution we are looking for. These algorithms can overcome the effect of the small divisors and ensure convergence on certain regions. To apply this method to our problem we have to consider the linearized problem (we take as initial guess the zero solution, and we linearize around this point):

$$\dot{x} = (A + \varepsilon Q(t,\varepsilon))x + \varepsilon g(t,\varepsilon).$$

We are looking for a quasiperiodic solution $\underline{x}(t,\varepsilon)$ with basic frequencies the ones of g and Q such that $\lim_{\varepsilon \to 0} \underline{x}(t,\varepsilon) = 0$. At this point we note that we do not need to know $\underline{x}(t,\varepsilon)$ exactly, because an approximation of order ε is enough. This is another property of the Newton's method: we do not need to know the Jacobian matrix exactly but some approximation of it, and this approximation can be of the order of the independent term we want to make zero. In our case, this can be done by considering the linear system

$$\dot{x} = Ax + \varepsilon g(t,\varepsilon). \tag{2.2}$$

Here we need the nonresonance condition

$$|(k,\omega)\sqrt{-1} - \lambda_i| > \frac{c}{|k|^\gamma},$$

where λ_i are the eigenvalues of A. Let us call $\underline{x}(t,\varepsilon)$ the solution of (2.2) which is quasiperiodic with respect to t (with basic frequencies the ones of g) and is of order

ε. The existence of that solution will be shown by Lemma 2.5. Now we can perform the change of variables $x = \underline{x}(t, \varepsilon) + y$ to equation (2.1) to obtain

$$\dot{y} = (A + \varepsilon Q_1(t, \varepsilon))y + \varepsilon^2 g_1(t, \varepsilon) + h_1(y, t, \varepsilon), \qquad (2.3)$$

where, if $\varepsilon \neq 0$, $Q_1(t, \varepsilon) = Q(t, \varepsilon) + \frac{1}{\varepsilon} D_x h(\underline{x}(t, \varepsilon), t, \varepsilon)$, $g_1(t, \varepsilon) = \frac{1}{\varepsilon^2} h(\underline{x}(t, \varepsilon), t, \varepsilon) + \frac{1}{\varepsilon} Q(t, \varepsilon) \underline{x}(t, \varepsilon)$ and $h_1(y, t, \varepsilon) = h(\underline{x}(t, \varepsilon) + y, t, \varepsilon) - h(\underline{x}(t, \varepsilon), t, \varepsilon) - D_x h(\underline{x}(t, \varepsilon), t, \varepsilon)y$. Note that we have found troubles: now we need a solution of

$$\dot{y} = (A + \varepsilon Q_1(t, \varepsilon))y + \varepsilon^2 g_1(t, \varepsilon). \qquad (2.4)$$

with an accuracy of order ε^2, and this does not allow to take the kind of approximation of (2.2). To proceed, we need to perform a new change of variables changing the term εQ_1 by something like $\varepsilon^2 Q_2$. This can be done as follows: let us define the average of Q_1 as

$$\overline{Q_1}(\varepsilon) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q_1(t, \varepsilon) \, dt.$$

For the existence of the limit see [9]. Consider equation (2.4) after averaging with respect to t and some rearrangement

$$\dot{y} = (\overline{A}(\varepsilon) + \varepsilon \widetilde{Q}_1(t, \varepsilon))y + \varepsilon^2 g_1(t, \varepsilon),$$

where $\widetilde{Q}_1(t, \varepsilon) = Q_1(t, \varepsilon) - \overline{Q}_1(\varepsilon)$, $\overline{A}(\varepsilon) = A + \varepsilon \overline{Q}_1(\varepsilon)$. Assume now that we are able to find a quasiperiodic solution of

$$\dot{P} = \overline{A}P - P\overline{A} + \widetilde{Q}_1,$$

with the same basic frequencies than $\widetilde{Q}_1$. Then, making the change of variables $y = (I + \varepsilon P)z$ (I denotes the identity matrix) to equation (2.3) (these changes of variables have already been considered in [3] and [15], and they have been used in Chapter 1 of this work), we get the equation

$$\dot{z} = (\overline{A}(\varepsilon) + \varepsilon^2 Q_2(t, \varepsilon))z + \varepsilon^2 g_2(t, \varepsilon) + h_2(z, t, \varepsilon), \qquad (2.5)$$

where $Q_2(t, \varepsilon) = (I + \varepsilon P(t, \varepsilon))^{-1} \widetilde{Q}_1 P(t, \varepsilon)$, $g_2(t, \varepsilon) = (I + \varepsilon P(t, \varepsilon))^{-1} g_1(t, \varepsilon)$ and $h_2(z, t, \varepsilon) = (I + \varepsilon P(t, \varepsilon))^{-1} h_1((I + \varepsilon P(t, \varepsilon))z, t, \varepsilon)$. Now, using $\dot{z} = \overline{A}z + \varepsilon^2 g_2(t)$ we are able to find an approximate solution of (2.5) with an accuracy of order ε^2 that allows to proceed with the Newton's method. Note that, due to the fact that the constant matrix A changes at each step of the process, we need to have some control about its eigenvalues, to avoid resonances. This is done by means of the parameter ε and, for this reason, we ask for a nondegeneracy condition with respect to that parameter. This allows to ensure that, under suitable hypothesis, this process converges on a cantorian set of values of ε, to an equation like

$$\dot{y} = A_\infty y + h_\infty(y, t, \varepsilon),$$

where A_∞ is a constant matrix and $h_\infty(y, t, \varepsilon)$ is of second order in y. As usual, the relative measure of this cantorian is close to 1 provided that ε be small enough. That equation has the trivial solution $y = 0$ and this shows that, in the original system of equations, the origin is replaced by a quasiperiodic orbit whose basic frequencies are the ones of the perturbations, for a cantorian set of values of ε.

At this point, it is convenient to remark the following: as the equations we are dealing with are not necessarily Hamiltonian, it is possible that, in some step of the inductive process, the eigenvalues of the matrix A leave the imaginary axis. In this case, we do not need to worry about resonances anymore. As we can not know in advance if this is going to happen, we have considered during all the proof the worst case, that is, the eigenvalues are always on the imaginary axis.

In some cases it is possible that at the first step of the inductive process the eigenvalues leave the imaginary axis (this is the general case, really). Theorem 2.2 ensures that this case can be detected averaging the original system and looking for the new equilibrium point of this autonomous system. The linearized equations around that point and the "Floquet" matrix (A_∞) of the quasiperiodic orbit differ in $\mathcal{O}(\varepsilon^2)$.

2.2.1   Theorems

From now on, if $x \in \mathbf{R}^n$ we denote by $\|x\|$ the sup norm of x. If A is a matrix, $\|A\|$ denotes the corresponding sup norm.

Theorem 2.1 *Consider the differential equation*

$$\dot{x} = (A + \varepsilon Q(t, \varepsilon))x + \varepsilon g(t, \varepsilon) + h(x, t, \varepsilon), \qquad (2.6)$$

where $Q(t, \varepsilon)$, $g(t, \varepsilon)$ and $h(x, t, \varepsilon)$ depend on time in a quasiperiodic way, with basic frequencies $(\omega_1, \ldots, \omega_r)^t$ and $|\varepsilon| < \varepsilon_0$. We assume that A is a constant $d \times d$ matrix with d different eigenvalues λ_i and $\det A \neq 0$. Let us suppose that $h(x, t, \varepsilon)$ is analytic with respect to x on the ball $B_r(0)$, $h(0, t, \varepsilon) = 0$ and $D_x h(0, t, \varepsilon) = 0$. Moreover, we assume that

1. *Q, g and h are analytic with respect to t on a strip of width $\rho_0 > 0$, and they depend on ε in a bounded way.*

2. *$\|D_{xx} h(x, t, \varepsilon)\| \leq K$, where $\|x\| \leq r$, $|\varepsilon| \leq \varepsilon_0$ and t belongs to the strip defined in 1.*

3. *The vector ν, where $\nu^T = (\lambda_1, \ldots, \lambda_d, \sqrt{-1}\omega_1, \ldots, \sqrt{-1}\omega_r)$ satisfies the non-resonance conditions*

$$|(m, \nu)| \geq \frac{c_\nu}{|m|^{\gamma_\nu}},$$

for all $m \in \{m_1 \in \mathbf{Z}^d, \ 0 \leq |m_1| \leq 2\} \times \{m_2 \in \mathbf{Z}^r, \ |m_2| \neq 0\}$, *where* c_ν *is a positive number,* $\gamma_\nu = r + d + \beta$, $\beta > -1$ *and* $|m| = \sum_{j=1}^{r+d} |m^j|$, *where* m^j *denote the components of* m *in* $\mathbf{Z}^{r+d}$.

4. *Let us denote by* $\underline{x}(t, \varepsilon)$ *the unique analytical quasiperiodic solution of* $\dot{x} = Ax + \varepsilon g(t, \varepsilon)$ *such that* $\lim_{\varepsilon \to 0} \underline{x}(t, \varepsilon) = 0$ *(the existence of that solution is shown by Lemma 2.5), and define*

$$\underline{A}(\varepsilon) = A + \varepsilon \overline{Q}(\varepsilon) + \overline{D_x h(\underline{x}(t, \varepsilon), t, \varepsilon)}.$$

Let $\lambda_j^0(\varepsilon)$, $j = 1, \ldots, d$ *be the eigenvalues of* $\underline{A}$. *We require the existence of* $\overline{\delta}$, $\delta > 0$ *such that*

$$\frac{\overline{\delta}}{2}|\varepsilon_1 - \varepsilon_2| > |\lambda_i^0(\varepsilon_1) - \lambda_j^0(\varepsilon_1) - (\lambda_i^0(\varepsilon_2) - \lambda_j^0(\varepsilon_2))| > 2\delta|\varepsilon_1 - \varepsilon_2| > 0,$$

$$\frac{\overline{\delta}}{2}|\varepsilon_1 - \varepsilon_2| > |\lambda_k^0(\varepsilon_1) - \lambda_k^0(\varepsilon_2)| > 2\delta|\varepsilon_1 - \varepsilon_2| > 0,$$

for all i, j, k *satisfying* $1 \leq i < j \leq d$, $1 \leq k \leq d$ *and provided that* $|\varepsilon_1|$ *and* $|\varepsilon_2|$ *are less than some small value* ε_0.

Then there exists a cantorian set $\mathcal{E} \subset (0, \varepsilon_0)$ *with positive Lebesgue measure such that the equation (2.6) can be transformed into*

$$\dot{y} = A_\infty y + h_\infty(y, t, \varepsilon),$$

where A_∞ *is a constant matrix and* $h_\infty(y, t, \varepsilon)$ *is of second order in* y. *If* ε_0 *is small enough the relative measure of* $\mathcal{E}$ *in* $(0, \varepsilon_0)$ *is close to 1. Furthermore the quasiperiodic change of variables that performs this transformation is analytic with respect to* t *and it has the same basic frequencies than* Q, g *and* h.

Remark: During the proof of this Theorem, it will be supposed that $\rho_0 \geq 1 + \pi^2/6$. This condition can be achieved introducing a new time $\tau = st$, where

$$s = \max\left\{\frac{1 + \frac{\pi^2}{6}}{\rho_0}, \ 1\right\}.$$

This scaling may change the constant c_ν and, therefore the set $\mathcal{E}$ is scaled by the same factor.

Corollary 2.1 *Under the hypothesis of Theorem 2.1, there exists a cantorian set* $\mathcal{E} \subset (0, \varepsilon_0)$ *with positive Lebesgue measure such that the equation (2.6) has a quasiperiodic solution* $x_\varepsilon(t)$ *with basic frequencies* $(\omega_1, \ldots, \omega_r)$, *such that*

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathcal{E}}} \|x_\varepsilon\| = 0.$$

Theorem 2.2 *Let us consider the equation (2.6) and let us assume that all the hypothesis of Theorem 2.1 hold. Moreover, let us assume that the nonlinear part $h(x,t,\varepsilon)$ is of class C^2 with respect to ε and $h(x,t,0) \equiv h(x)$. Then, if ε is sufficiently small, the averaged system*

$$\dot{y} = (A + \varepsilon\overline{Q})y + \varepsilon\overline{g} + \overline{h}(y,\varepsilon) \tag{2.7}$$

has an equilibrium point $x_0(\varepsilon)$ such that

1. $\displaystyle\lim_{\varepsilon \to 0} \|x_0(\varepsilon)\| = 0.$

2. The matrix A_{x_0} of the linearized system around $x_0(\varepsilon)$ and the matrix A_∞ obtained in Theorem 2.1 satisfy $\|A_{x_0} - A_\infty\| = \mathcal{O}(\varepsilon^2)$.

Corollary 2.2 *Let us define $\lambda_i^{x_0}$, $1 \le i \le d$, as the eigenvalues of the matrix A_{x_0} defined in Theorem 2.2. Then, under the hypothesis of Theorem 2.2, an equivalent version of the hypothesis 4 in Theorem 2.1 is obtained if λ_i^0 are replaced by $\lambda_i^{x_0}$.*

2.2.2 Previous Lemmas

In this section the reader will find the lemmas used during the proofs.

Lemma 2.1 *Let $\delta \in]0, e^{r+\alpha-1}]$ and $\alpha > 0$. Then*

$$\sum_{k \in \mathbf{Z}^r} |k|^\alpha e^{-\delta|k|} \le \frac{2^{r+1}(r + \alpha - 1)^{r+\alpha-1}}{\delta^{r+\alpha}}$$

Proof: Using that $\#\{k \in \mathbf{Z}^r \ / \ |k| = \displaystyle\sum_{i=1}^{r} |k_i| = m\} \le 2^r m^{r-1}$ we obtain

$$\sum_{k \in \mathbf{Z}^r} |k|^\alpha e^{-\delta|k|} \le 2^r \sum_{m=0}^{\infty} m^{r+\alpha-1} e^{-\delta m} = (\Delta)$$

As the unique maximum of $g(x) = x^{r+\alpha-1} e^{-\delta x}$ is reached when $x = \frac{r+\alpha-1}{\delta}$, we can bound the sum above by this maximum plus the integral:

$$
\begin{aligned}
(\Delta) \ \le \ & 2^r \left[\left(\frac{r+\alpha-1}{\delta e} \right)^{r+\alpha-1} + \int_0^\infty m^{r+\alpha-1} e^{-\delta m} \, dm \right] = \\
= \ & 2^r \left[\left(\frac{r+\alpha-1}{\delta e} \right)^{r+\alpha-1} + \frac{1}{\delta^{r+\alpha}} \Gamma(r+\alpha) \right] = \\
= \ & 2^r \frac{(r+\alpha-1)^{r+\alpha-1}}{\delta^{r+\alpha}} \left[\frac{\delta}{e^{r+\alpha-1}} + \frac{\Gamma(r+\alpha)}{(r+\alpha-1)^{r+\alpha-1}} \right].
\end{aligned}
$$

Now, using that the value between brackets is bounded by 2, the result follows. ∎

Lemma 2.2 *Let* $h : U \subset \mathbf{R}^d \to \mathbf{R}^d$ *be a function of class* $\mathcal{C}^2$ *on a ball* $B_r(0)$, *satisfying that* $h(0) = 0$, $D_x h(0) = 0$ *and* $\|D_{xx} h(x)\| \leq K$, *where* $x \in B_r(0)$. *Then* $\|h(x)\| \leq \frac{K}{2} \|x\|^2$ *and* $\|D_x h(x)\| \leq K \|x\|$.

Proof: This follows from Taylor's formula:

$$\|h(x)\| = \|h(0) + D_x h(0)x + R_2(x)\| = \|R_2(x)\| \leq \frac{K}{2} \|x\|^2,$$

$$\|D_x h(x)\| = \|D_x h(x) - D_x h(0)\| \leq \|D_{xx} h(\xi)\| \, \|x\| \leq K \|x\|,$$

that works if $x \in B_r(0)$. ∎

Lemma 2.3 *Let* M *be a diagonal matrix with* d *diferent nonzero eigenvalues* μ_j, $j = 1, \ldots, d$, *and* $\alpha = \min\{\min_{i,j; i \neq j} |\mu_i - \mu_j|, \min_i |\mu_i|\}$. *Let* N *be a matrix such that* $(d+1)\|N\| < \alpha$. *Let* ν_j, $j = 1, \ldots, d$ *be the eigenvalues of* $M + N$, B *a suitable matrix such that* $B^{-1}(M + N)B = D = \mathrm{diag}(\nu_j)$ *with condition number* $C(B)$. *Then*

1. $\beta = \min\{\min_{i,j; i \neq j} |\nu_i - \nu_j|, \min_i |\nu_i|\} \geq \alpha - 2\|N\|$.

2. $C(B) \leq \frac{\alpha + (d-3)\|N\|}{\alpha - (d+1)\|N\|}$. *In particular, if* $\|N\| < \frac{\alpha}{3d-1}$ *then* $C(B) < 2$.

Proof: It is essentially the same that the one of Lemma 1.7. ∎

Lemma 2.4 *Let* A_0 *be a* $d \times d$ *matrix such that* $\mathrm{Spec}(A_0) = \{\lambda_1^0, \ldots, \lambda_d^0\}$, $|\lambda_i^0| > 2\mu$, $|\lambda_i^0 - \lambda_j^0| > 2\mu$, $i \neq j$, *where* $\mu > 0$. *Then, there exists* $\alpha > 0$ *and* $\beta > 1$ *such that, if* A *verifies* $\|A - A_0\| < \alpha$, *the following conditions hold:*

1. $\mathrm{Spec}(A) = \{\lambda_1, \ldots, \lambda_d\}$, *and* $|\lambda_i| > \mu$, $|\lambda_i - \lambda_j| > \mu$, $i \neq j$.

2. *There exists a nonsingular matrix* B *such that* $B^{-1}AB = \mathrm{diag}(\lambda_1, \ldots, \lambda_d)$ *satisfying* $\|B\| < \beta$ *and* $\|B^{-1}\| < \beta$.

Proof: Let us define B_0 to be a nonsingular $d \times d$ matrix satisfying that $B_0^{-1} A_0 B_0 = D_0 = \mathrm{diag}(\lambda_1^0, \ldots, \lambda_d^0)$. Let A be a matrix, and we write $A = A_0 + (A - A_0)$. Then $B_0^{-1} A B_0 = D_0 + N$, where $N = B_0^{-1}(A - A_0)B_0$. Here we can apply Lemma 2.3 to obtain *1*, if $\|A - A_0\|$ is small enough. Note that Lemma 2.3 states that the condition number of the matrix C that diagonalizes $D_0 + N$ is less than 2, provided that $\|A - A_0\|$ is sufficiently small. In this case, the matrix that diagonalizes A can be obtained multiplying B_0 by C. Hence, its norm can be bounded by $2\|B_0\|$. ∎

Lemma 2.5 *Let us consider the equation* $\dot{x} = Ax + \varepsilon g(t)$, *where* A *is a* $d \times d$ *matrix belonging to the ball* $B_\alpha(A_0) \subset \mathcal{L}(\mathbf{R}^d, \mathbf{R}^d)$ *with* α *as given by Lemma 2.4,*

$g(t) = (g_i(t))_{1 \leq i \leq d}$ and $g_i(t)$ *is an analytic quasiperiodic function on a strip of width* ρ_1:

$$g_i(t) = \sum_{k \in \mathbf{Z}^r} g_i^k e^{(k,\omega)\sqrt{-1}t}.$$

Let us assume that $|(k,\omega)\sqrt{-1} - \lambda_i| > \frac{c}{|k|^\gamma}$ $\forall \lambda_i \in \mathrm{Spec}(A)$, *where* $c \leq c_0$ *(see remark below). Let* ρ_2 *such that* $0 < \rho_2 < \rho_1$ *and* $\rho_1 - \rho_2 \leq 1$. *Then, there exists a unique quasiperiodic solution of* $\dot{x} = Ax + \varepsilon g(t)$ *having the same basic frequencies than* g *and satisfying*

$$\|x\|_{\rho_2} \leq \varepsilon \|g\|_{\rho_1} \frac{L_1}{c(\rho_1 - \rho_2)^{r+\gamma}},$$

where $L_1 = \frac{c_0 \beta^2}{\mu} + \beta^2 2^{r+1}(r + \gamma - 1)^{r+\gamma-1}$ *and* μ *is defined in Lemma 2.4.*

Remark 1. The condition $c \leq c_0$ comes from the fact that, during the proof of the Theorem 2.1, the value of c is decreased at each step of the inductive process. Thus, this condition allows to give simpler bounds.

Remark 2. In this Lemma as well as in the forthcoming ones, we consider Q, g, h depending also on ε (see Theorem 2.1) but, for simplicity, we do not write this explicitly.

Proof: Let B the matrix found in Lemma 2.4. Making the change of variables $x = By$ and defining $h(t) = B^{-1}g$ the equation becomes

$$\dot{y} = Dy + \varepsilon h(t).$$

As D is a diagonal matrix, we can handle this equation as d unidimensional equations, that can be solved easily: if $y = (y_i)_{1 \leq i \leq d}$ and

$$y_i(t) = \sum_{k \in \mathbf{Z}^r} y_i^k e^{(k,\omega)\sqrt{-1}t},$$

the coefficients must be $y_i^k = \varepsilon \frac{h_i^k}{(k,\omega)\sqrt{-1}-\lambda_i}$, and they can be bounded by

$$|y_i^k| \leq \begin{cases} \varepsilon \frac{\|h\|_{\rho_1}}{\mu} & \text{if } k = 0 \\ \varepsilon \|h\|_{\rho_1} \frac{|k|^\gamma}{c} e^{-\rho_1|k|} & \text{if } k \neq 0. \end{cases}$$

Now, we need to bound the norm $\|y\|_{\rho_2}$. Let t be a complex value such that $|\mathrm{Im}\,\omega_i t| \leq \rho_2$ (for all i). Then

$$|y_i(t)| \leq \sum_{k \in \mathbf{Z}^r} |y_i^k| \, |e^{(k,\omega t)\sqrt{-1}}| \leq \varepsilon \frac{\|h\|_{\rho_1}}{\mu} + \sum_{k \neq 0} \varepsilon \|h\|_{\rho_1} \frac{|k|^\gamma}{c} e^{-\rho_1|k|} e^{\rho_2|k|}.$$

We define $\delta = \rho_1 - \rho_2$. Now we can use Lemma 2.1 to bound the sum above:

$$|y_i(t)| \leq \varepsilon \|h\|_{\rho_1} \left[\frac{1}{\mu} + \frac{2^{r+1}(r+\gamma-1)^{r+\gamma-1}}{c\delta^{r+\gamma}} \right] =$$

$$= \varepsilon \frac{\|h\|_{\rho_1}}{c\delta^{r+\gamma}} \left[\frac{c\delta^{r+\gamma}}{\mu} + 2^{r+1}(r+\gamma-1)^{r+\gamma-1} \right].$$

Here we can use that $\delta \leq 1$, $c \leq c_0$ and define $L_1' = \frac{c_0}{\mu} + 2^{r+1}(r+\gamma-1)^{r+\gamma-1}$ to get

$$|y_i(t)| \leq \varepsilon \frac{\|h\|_{\rho_1}}{c\delta^{r+\gamma}} L_1'.$$

Finally,

$$\|x\|_{\rho_2} \leq \|B\| \, \|y\|_{\rho_2} \leq \varepsilon \frac{\|B\| \, \|B^{-1}\| \, \|g\|_{\rho_1}}{c\delta^{r+\gamma}} L_1',$$

and defining L_1 as $\beta^2 L_1'$ (see Lemma 2.4) the result follows. ∎

Lemma 2.6 *Let us consider the equation* $\dot{P} = AP - PA + Q$, *where* $A \in B_\alpha(A_0)$, $Q = (q_{ij})$ *where* $q_{ij}(t)$ *are analytic quasiperiodic functions on a strip of width* ρ_1:

$$q_{ij}(t) = \sum_{k \in \mathbf{Z}^r} q_{ij}^k e^{(k,\omega)\sqrt{-1}t}.$$

We also assume that Q has average equal to 0 and $|(k,\omega)\sqrt{-1} - \lambda_i + \lambda_j| > \frac{c}{|k|^\gamma}$ $\forall \lambda_i \in \mathrm{Spec}(A)$, where $c \leq c_0$. Let ρ_2 such that $0 < \rho_2 < \rho_1$ and $\rho_1 - \rho_2 \leq 1$. Then, there exists a unique quasiperiodic solution of $\dot{P} = AP - PA + Q$ having the same basic frequencies than Q and satisfying

$$\|P\|_{\rho_2} \leq \frac{\|Q\|_{\rho_1}}{c\delta^{r+\gamma}} L_2,$$

where $L_2 = \beta^4 2^{r+1}(r+\gamma-1)^{r+\gamma-1}$.

Proof: Let B be the matrix found in Lemma 2.4. Making the change of variables $P = BSB^{-1}$ and defining $R = B^{-1}QB$, the equation becomes

$$\dot{S} = DS - SD + R.$$

As D is a diagonal matrix, we can handle this equation as d^2 unidimensional equations, that can be solved easily: if $S = (s_{ij})$ and

$$s_{ij}(t) = \sum_{k \in \mathbf{Z}^r} s_{ij}^k e^{(k,\omega)\sqrt{-1}t},$$

the coefficients must be $s^k_{ij} = \frac{r^k_{ij}}{(k,\omega)\sqrt{-1}-\lambda_i+\lambda_j}$, and they can be bounded by

$$|s^k_{ij}| \leq \|r_{ij}\|\frac{|k|^\gamma}{c}e^{-\rho_1|k|}$$

Now, we need to bound the norm $\|S\|_{\rho_2}$. Let t be a complex value such that $|\text{Im } \omega_i t| \leq \rho_2$ (for all i):

$$|s_{ij}(t)| \leq \sum_{k\in\mathbf{Z}^r} |s^k_{ij}| \, |e^{(k,\omega t)\sqrt{-1}}| \leq \sum_{k\neq 0} \|s_{ij}\|_{\rho_1}\frac{|k|^\gamma}{c}e^{-\rho_1|k|}e^{\rho_2|k|}.$$

We define $\delta = \rho_1 - \rho_2$. Now we can use Lemma 2.1 to bound the sum above:

$$|s_{ij}(t)| \leq \frac{\|r_{ij}\|_{\rho_1}}{c}\left[\frac{2^{r+1}(r+\gamma-1)^{r+\gamma-1}}{\delta^{r+\gamma}}\right] \leq \frac{\|r_{ij}\|_{\rho_1}}{c\delta^{r+\gamma}}L'_2,$$

where $L'_2 = 2^{r+1}(r+\gamma-1)^{r+\gamma-1}$. Finally,

$$\|P\|_{\rho_2} \leq \|B\| \, \|B^{-1}\|\frac{\|R\|_{\rho_1}}{c\delta^{r+\gamma}}L'_2 \leq \beta^4 L'_2\frac{\|Q\|_{\rho_1}}{c\delta^{r+\gamma}}.$$

If L_2 is defined as $\beta^4 L'_2$ we get the bound we were looking for. ∎

Lemma 2.7 *Let us consider* $\dot{x} = (A + \varepsilon Q(t))x + \varepsilon g(t) + h(x,t)$, *where the time dependence is assumed to be analytic quasiperiodic on a strip of width* ρ_1. *We also assume that* $h(x,t)$ *is analytic with respect to* x *on the ball* $B_r(0)$ *and that satisfies* $\|D_{xx}h(x,t)\|_{\rho_1} \leq K, \forall x \in B_r(0)$. *Moreover,* $A \in B_\alpha(A_0)$ *and* $|(k,\omega)\sqrt{-1}-\lambda_i| > \frac{c}{|k|^\gamma}$ $\forall \lambda_i \in \text{Spec}(A)$, *where* $c \leq c_0$. *Let* ρ_2 *such that* $0 < \rho_2 < \rho_1$ *and* $\rho_1 - \rho_2 \leq 1$. *Then, there exists a change of variables* $x = y + \underline{x}(t)$ *that transforms the initial equation in*

$$\dot{y} = (\overline{A} + \varepsilon\widetilde{Q}_1)y + \varepsilon^2 g_1(t) + h_1(x,t),$$

where $\widetilde{Q}$ *has zero average and the following bounds hold:*

1. $\|\widetilde{Q}_1\|_{\rho_2} \leq \|Q\|_{\rho_1} + \|g\|_{\rho_1}\dfrac{KL_1}{c(\rho_1-\rho_2)^{r+\gamma}}$, *where* L_1 *was defined in Lemma 2.5.*

2. $\|g_1\|_{\rho_2} \leq \dfrac{1}{c^2(\rho_1-\rho_2)^{2r+2\gamma}}\left[\dfrac{KL_1^2\|g\|_{\rho_1}^2}{2} + c_0L_1\|Q\|_{\rho_1}\|g\|_{\rho_1}\right].$

3. $\|\overline{A}\| \leq \|A\| + \varepsilon\|Q_1\|_{\rho_1}.$

4. $\|D_{yy}h_1(y,t)\|_{\rho_2} \leq K.$

5. $\|\underline{x}(t)\|_{\rho_2} \leq \varepsilon\|g\|_{\rho_1}\dfrac{L_1}{c(\rho_1-\rho_2)^{r+\gamma}},$

where $y \in B_{\tau_1}(0)$, $\tau_1 = \tau - \|\underline{x}(t)\|_{\rho_2}$ and ε is small enough.

Proof: Let $\underline{x}$ be such that $\dot{\underline{x}} = A\underline{x} + \varepsilon g$. In Lemma 2.5 we obtained

$$\|\underline{x}\|_{\rho_2} \leq \varepsilon \|g\|_{\rho_1} \frac{L_1}{c(\rho_1 - \rho_2)^{r+\gamma}},$$

where L_1 is a constant. Making the change of variables $x = y + \underline{x}(t)$ we get

$$\dot{y} = (A + \varepsilon Q + D_x h(\underline{x}(t), t))y + h(\underline{x}(t), t) + \varepsilon Q \underline{x}(t) + h_1(y, t),$$

where $h_1(y, t) = h(\underline{x}(t) + y, t) - h(\underline{x}(t), t) - D_x h(\underline{x}(t), t)y$. Defining $Q_1 = Q + \frac{1}{\varepsilon} D_x h(\underline{x}(t), t)$ and $g_1 = \frac{1}{\varepsilon^2} h(\underline{x}(t), t) + \frac{1}{\varepsilon} Q \underline{x}(t)$ ($\varepsilon \neq 0$), the equation is then as follows:

$$\dot{y} = (A + \varepsilon Q_1(t))y + \varepsilon^2 g_1(t) + h_1(y, t).$$

To end up, the terms of this equation must be bounded. Let us start with Q_1. Using Lemma 2.2 we get

$$\|Q_1\|_{\rho_2} \leq \|Q\|_{\rho_2} + \frac{1}{\varepsilon} K \|\underline{x}(t)\|_{\rho_2} \leq \|Q\|_{\rho_1} + \|g\|_{\rho_1} \frac{KL_1}{c(\rho_1 - \rho_2)^{r+\gamma}}.$$

Now, let us bound $\|g_1\|_{\rho_2}$, again by means of Lemma 2.2:

$$
\begin{aligned}
\|g_1\|_{\rho_2} &\leq \frac{1}{\varepsilon^2} \frac{K}{2} \|\underline{x}(t)\|_{\rho_2}^2 + \frac{1}{\varepsilon} \|Q\|_{\rho_1} \|\underline{x}(t)\|_{\rho_2} \leq \\
&\leq \frac{1}{c^2(\rho_1 - \rho_2)^{2r+2\gamma}} \left[\frac{KL_1^2 \|g\|_{\rho_1}^2}{2} + \|Q\|_{\rho_1} \|g\|_{\rho_1} L_1 c(\rho_1 - \rho_2)^{r+\gamma} \right].
\end{aligned}
$$

Using $c \leq c_0$ and $\rho_1 - \rho_2 \leq 1$ the bound is obtained. Now is the turn of $D_{yy}h_1(y, t)$:

$$\|D_{yy}h_1(y, t)\|_{\rho_2} = \|D_{xx}h(\underline{x}(t) + y, t)\| \leq K.$$

To do this we have to require $y \in B_{\tau_1}(0)$, where $\tau_1 = \tau - \|\underline{x}(t)\|_{\rho_2}$ (ε is supposed small enough). Now, using that $Q_1(t) = \overline{Q}_1 + \tilde{Q}_1(t)$ and defining $\overline{A} = A + \varepsilon \overline{Q}_1$ we obtain

$$\dot{y} = (\overline{A} + \varepsilon \tilde{Q}_1(t))y + \varepsilon^2 g_1(t) + h_1(y, t).$$

Finally,

$$\|\overline{A}\| \leq \|A\| + \varepsilon \|\overline{Q}_1\|_{\rho_2},$$

and taking into account that $\|\overline{Q}_1\|_{\rho_2} \leq \|Q_1\|_{\rho_2}$ and that $\|\tilde{Q}_1(t)\|_{\rho_2} \leq \|Q_1\|_{\rho_2}$ the proof is finished. ∎

Lemma 2.8 *Let us consider* $\dot{x} = (A + \varepsilon Q(t))x + \varepsilon^2 g(t) + h(x,t)$, *where the time dependence is assumed to be analytic quasiperiodic on a strip of width* ρ_1 *and* Q *has zero average. We also assume that* $h(x,t)$ *is analytic with respect to* x *on the ball* $B_r(0)$ *and that satisfies* $\|D_{xx}h(x,t)\|_{\rho_1} \leq K$, $\forall x \in B_r(0)$. *Moreover,* $A \in B_\alpha(A_0)$ *and* $|(k,\omega)\sqrt{-1} - \lambda_i + \lambda_j| > \frac{c}{|k|^\gamma}$ $\forall \lambda_i, \lambda_j \in \mathrm{Spec}(A)$, *where* $c \leq c_0$. *Let* ρ_2 *such that* $0 < \rho_2 < \rho_1$ *and* $\rho_1 - \rho_2 \leq 1$. *Then, there exists a change of variables* $x = (I + \varepsilon P(t))y$, *where* I *is the identity* $d \times d$ *matrix and* $P(t)$ *is analytic quasiperiodic on a strip of width* ρ_2, *that transforms the initial equation in*

$$\dot{y} = (\overline{A} + \varepsilon^2 \widetilde{Q}_1)y + \varepsilon^2 g_1(t) + h_1(y,t),$$

where $\widetilde{Q}_1$ *has zero average and the following bounds hold:*

1. $\|\widetilde{Q}_1\|_{\rho_2} \leq \dfrac{\|P\|_{\rho_2}}{1 - \varepsilon\|P\|_{\rho_2}}\|Q\|_{\rho_1}$, *where* $\|P\|_{\rho_2} \leq \dfrac{\|Q\|_{\rho_1}}{c\delta^{r+\gamma}}L_2$ *and* L_2 *was defined in Lemma 2.6.*

2. $\|g_1\|_{\rho_2} \leq \dfrac{1}{1 - \varepsilon\|P\|_{\rho_2}}\|g\|_{\rho_1}$.

3. $\|D_{yy}h_1(y,t)\|_{\rho_2} \leq K\dfrac{(1 + \varepsilon\|P\|_{\rho_2})^2}{1 - \varepsilon\|P\|_{\rho_2}}$.

4. $\|\overline{A}\| \leq \|A\| + \varepsilon\dfrac{\|P\|_{\rho_2}}{1 - \varepsilon\|P\|_{\rho_2}}\|Q\|_{\rho_1}$,

where $y \in B_{\tau_1}(0)$, $\tau_1 = \dfrac{\tau}{1 + \varepsilon\|P\|_{\rho_2}}$ *and* ε *is small enough.*

Proof: Using Lemma 2.6 we can solve $\dot{P} = AP - PA + Q$. The solution that we have found verifies

$$\|P\|_{\rho_2} \leq \frac{\|Q\|_{\rho_1}}{c\delta^{r+\gamma}}L_2.$$

Now, by means of the change of variables $x = (I + \varepsilon P)y$ and introducing the notation $Q_1 = (I + \varepsilon P)^{-1}QP$, $g_1 = (I + \varepsilon P)^{-1}g$ and $h_1(y,t) = (I + \varepsilon P)^{-1}h((I + \varepsilon P)y, t)$ we obtain the following equation

$$\dot{y} = (A + \varepsilon^2 Q_1(t))y + \varepsilon^2 g_1(t) + h_1(y,t).$$

Now we are going to bound the terms of this equation. For this purpose we need the bound of $\|P\|_{\rho_2}$ provided by Lemma 2.6 and displayed above.

$$\|Q_1\|_{\rho_2} \leq \left(\sum_{i=0}^{\infty}\varepsilon^i\|P\|_{\rho_2}^i\right)\|Q\|_{\rho_2}\|P\|_{\rho_2} \leq \frac{\|P\|_{\rho_2}}{1 - \varepsilon\|P\|_{\rho_2}}\|Q\|_{\rho_1},$$

$$\|g_1\|_{\rho_2} \le \frac{1}{1 - \varepsilon\|P\|_{\rho_2}} \|g\|_{\rho_1},$$

$$\|D_{yy}h_1(y,t)\|_{\rho_2} \le \frac{1}{1 - \varepsilon\|P\|_{\rho_2}} K \|I + \varepsilon P\|_{\rho_2}^2 \le K \frac{(1 + \varepsilon\|P\|_{\rho_2})^2}{1 - \varepsilon\|P\|_{\rho_2}}.$$

Of course, we require $y \in B_{\tau_1}(0)$, where $\tau_1 = \tau/(1 + \varepsilon\|P\|_{\rho_2})$ and ε is small enough. To end up this, we rewrite the equation using $Q_1(t) = \overline{Q}_1 + \widetilde{Q}_1(t)$ and $\overline{A} = A + \varepsilon^2 \overline{Q}_1$ and we obtain

$$\dot{y} = (\overline{A} + \varepsilon^2 \widetilde{Q}_1)y + \varepsilon^2 g_1(t) + h_1(y,t),$$

and we only need to bound $\overline{A}$:

$$\|\overline{A}\| \le \|A\| + \varepsilon^2 \|Q_1\|_{\rho_2}. \quad \blacksquare$$

Lemma 2.9 *Let $\{K_n\}_n$ be a sequence of positive real numbers satisfying $K_n \le ab^n n^\gamma K_{n-1}^2$, where $b > 1$. Then*

$$K_n \le \frac{1}{a}\left[aK_0 b^2 \left(\frac{5}{3}\right)^\gamma\right]^{2^n}.$$

Proof: It is easy to see that

$$K_n \le a^{1+2+2^2+\cdots+2^{n-1}} b^{n+2(n-1)+2^2(n-2)+\cdots+2^{n-1}} \left[\prod_{i=0}^{n-1}(n-i)^{2^i}\right]^\gamma K_0^{2^n}.$$

To obtain the exponent of b we use the identity

$$\frac{x - x^{n+1}}{1 - x} = \sum_{i=0}^{n-1} x^{i+1}.$$

We compute the derivatives of both sides and put $x = 1/2$. Then

$$\sum_{i=0}^{n-1} 2^i(n-i) = 2^{n+1} - n - 2.$$

To end up, we need to bound the expression between brackets in the bound of K_n displayed above. For this purpose we take logarithms:

$$\ln\left(\prod_{i=0}^{n-1}(n-i)^{2^i}\right) = \sum_{i=0}^{n-2} 2^i \ln(n-i) \le 2^n \sum_{i=0}^{\infty} \frac{\ln(i+2)}{2^{i+2}} = c2^n.$$

In Lemma 1.5 it has been shown that $\exp c < 5/3$. Finally

$$K_n \le \frac{1}{a} a^{2^n} (b^2)^{2^n} b^{-(n+2)} \left[\left(\frac{5}{3}\right)^\gamma\right]^{2^n} K_0^{2^n},$$

and taking into account that $b > 1$ the result follows. $\blacksquare$

Lemma 2.10 *Let* $\{a_n\}_n$ *be a sequence of positive real numbers satisfying* $a_n \in$ $]0,1]$, $\prod_{n=0}^{\infty} a_n = a \in]0,1]$. *Let* $\{b_n\}_n$ *be another sequence of positive real numbers satisfying* $\sum_{n=0}^{\infty} b_n = b < +\infty$. *Consider the new sequence* $\{\tau_n\}_n$ *defined by* $\tau_{n+1} = a_n \tau_n - b_n$. *Then the sequence* $\{\tau_n\}_n$ *converges to a limit value* τ_{∞} *satisfying* $\tau_{\infty} \geq a\tau_0 - b$.

Proof: It is easy to see that

$$\tau_{n+1} = \left(\prod_{i=0}^{n} a_i\right) \tau_0 - \sum_{i=0}^{n-1} \left[\left(\prod_{j=i+1}^{n} a_j\right) b_i\right] - b_n.$$

As all the terms appearing in this expression converge, τ_n does. Moreover, using that

$$\prod_{j=i+1}^{n} a_j \leq 1$$

for all n, the result follows. ∎

Lemma 2.11 *Let* $\omega \in \mathbf{R}^r$ *and* λ_i, $i = 1, \ldots, d$ *such that*

$$|\lambda_i - \sqrt{-1}(k, \omega)| \geq \frac{c}{|k|^{\gamma_1}}.$$

for all $i \in \{1, \ldots, d\}$ *and all* $k \in \mathbf{Z}^r \setminus \{0\}$, *where* $c > 0$, $\gamma_1 > 0$. *We define a resonant subset* $\mathcal{R}_\mu$ *as*

$$\mathcal{R}_\mu = \{\varphi \in \sqrt{-1}\mathbf{R}, \; |\varphi| < \mu \; / \; \exists i \in \{1, \ldots, d\} \; \wedge \; \exists k' \in \mathbf{Z}^r \setminus \{0\}$$

$$such \; that \; |\varphi + \lambda_i - \sqrt{-1}(k', \omega)| < \frac{\frac{c}{2}}{|k'|^{\gamma_2}}\}.$$

Let $\psi(\mu) = \frac{m(\mathcal{R}_\mu)}{2\mu}$, *where* m *denotes the Lebesgue measure. If* $\gamma_2 = \gamma_1 + r + 1$ *then* $\liminf_{\mu \to 0} \psi(\mu) = 0$.

Proof: Take $\mu_n = \frac{c}{n^{\gamma_1}}$. For any k' with $|k'| \geq n$ and any i, the measure of the resonant interval of φ is bounded by $\frac{c}{|k'|^{\gamma_2}}$. Adding for all the values of k' with $|k'| = n'$ and all i, we have

$$m(\mathcal{R}_{\mu_n}) \leq c2rd \sum_{n' \geq n} \frac{1}{(n')^{\gamma_2 - r + 1}} < 2crd(n-1)^{-(\gamma_2 - r)}.$$

Furthermore the resonant intervals associated to $n' < n$ are disjoint with $\mathcal{R}_{\mu_n}$ if n is large enough. Hence, for n large enough, $\psi(\mu_n) < rdn^{\gamma_1}(n-1)^{-(\gamma_1+1)}$ which goes to zero if n goes to infinity. ∎

2.2.3 Proof of Theorem 2.1

First we are going to do the proof without worrying about resonances, and then we shall take out the values of ε for which the proof fails.

First of all, let us denote by A_0 the initial matrix A (see Theorem 2.1) corresponding to the averaged linear part of the differential system. Let μ be a real value such that, if $\text{Spec}(A_0) = \{\lambda_1^0, \ldots, \lambda_d^0\}$, then $|\lambda_i^0| > 2\mu$, $|\lambda_i^0 - \lambda_j^0| > 2\mu$ for all $i \neq j$. Then Lemma 2.4 can be applied to obtain values α and β such that all the matrices contained inside the ball $B_\alpha(A_0) = \{A \,/\, \|A - A_0\| < \alpha\}$ can be diagonalized. Moreover, the matrix B of the diagonalizing change of variables satisfies $\|B\| < \beta$, $\|B^{-1}\| < \beta$. During the proof we shall see that, if ε is small enough, all the matrices A_n that appear during the inductive process are inside that ball.

As we assume that the dependence of Q, g and h with respect to ε is bounded, every time we compute some norm we mean, without explicit mention, that we look not only for the maximum with respect to t in the suitable strip, but also with respect to ε in the allowed range.

To begin the proof, we suppose that we have applied the method exposed before up to step n, and we are going to see that we can apply it again to get the $n + 1$ step. In this way we shall obtain bounds for the quasiperiodic part at the nth step and for the transformation at this step, and this allows us to prove the convergence.

Now suppose that we are at n^{th} step. This means that the equation we have is

$$\dot{x}_n = (A_n(\varepsilon) + \varepsilon^{2^n}Q_n(t,\varepsilon))x_n + \varepsilon^{2^n}g_n(t,\varepsilon) + h_n(x_n,t,\varepsilon), \qquad (2.8)$$

where A_n belongs to $B_\alpha(A_0)$, its eigenvalues λ_i verify the nonresonance condition

$$|(k,\omega)\sqrt{-1} - \lambda_i| > \frac{c_n}{|k|^\gamma},$$

where $\gamma = \gamma_\nu + r + 1$ and $c_n = c_0/(n+1)^2$. Due to the fact that we need to reduce the width of the analyticity strip of the quasiperiodic functions, we define $\rho_n = \rho_{n-1} - 1/n^2$ and $\sigma_n = \rho_n - 1/(2(n+1)^2)$, with $\rho_0 = 1 + \pi^2/6$. During the proof we shall see that the analyticity ball (with respect to x) of $h_n(x,t)$ has to be reduced at each step of the inductive process, and we shall found that, by selecting ε small enough, the limit radius of this ball is positive. Let us define τ_n as this radius at step n. Now we can apply Lemma 2.7 to transform equation (2.8) into

$$\dot{y}_n = (\widehat{A}_n(\varepsilon) + \varepsilon^{2^n}\widehat{Q}_n(t,\varepsilon))y_n + \varepsilon^{2^{n+1}}\widehat{g}_n(t,\varepsilon) + \widehat{h}_n(y_n,t,\varepsilon), \qquad (2.9)$$

where the width of the analyticity strip has been reduced to σ_n. Now, assuming that the nonresonance condition

$$|(k,\omega)\sqrt{-1} - \lambda_i + \lambda_j| > \frac{c_n}{|k|^\gamma},$$

holds for all λ_i, $\lambda_j \in \text{Spec}(\widehat{A}_n(\varepsilon))$ we can apply Lemma 2.8 to equation (2.9) and to get

$$\dot{x}_{n+1} = (A_{n+1}(\varepsilon) + \varepsilon^{2^{n+1}} Q_{n+1}(t,\varepsilon))x_{n+1} + \varepsilon^{2^{n+1}} g_{n+1}(t,\varepsilon) + h_{n+1}(x_{n+1},t,\varepsilon). \quad (2.10)$$

Now, the width of the analyticity strip has been reduced to ρ_{n+1}. The next step of the proof is to obtain bounds of the terms appearing in equation (2.10) depending on the bounds of the terms of equation (2.8). Using Lemma 2.8, and a condition on $\|P_n\|_{\rho_{n+1}}$ (see below) we get

$$\|Q_{n+1}\|_{\rho_{n+1}} \leq M_1(n+1)^{2r+2\gamma+2}\|\widehat{Q}_n\|_{\sigma_n}^2,$$

where $M_1 = 2^{r+\gamma+1}L_2/c_0$. Here we need Lemma 2.7 to bound the expression above, but the bound provided by this Lemma has an "uncontrolled" term, that is, the bound of the second derivative of h_n. Let us call this value K_n. Note that it is "modified" at each step by Lemma 2.8. In order to bound it, we shall assume that ε is small enough to ensure that $\varepsilon^{2^n}\|P_n\|_{\rho_{n+1}}$ is less than $1/2$. This implies that the value of ε will be reduced at each step, if necessary, to guarantee that condition. We will see that this condition is achieved from a certain step onwards, without modifying ε anymore. Therefore, we assume that $K_n \leq (9/2)^n K_0$ (when the convergence will be proved, we shall give a more realistic bound of K_n, that converges to a real number) and Lemma 2.7 states that

$$\|\widehat{Q}_n\|_{\sigma_n} \leq \|Q_n\|_{\rho_n} + M_2(n+1)^{2r+2\gamma+2}\left(\frac{9}{2}\right)^n \|g_n\|_{\rho_n},$$

where $M_2 = 2^{r+\gamma}L_1K_0/c_0$. Now we bound the norm of g_{n+1}:

$$\|g_{n+1}\|_{\rho_{n+1}} \leq 2\|\widehat{g}_n\|_{\sigma_n},$$

and from Lemma 2.7

$$\|g_{n+1}\|_{\rho_{n+1}} \leq M_3(n+1)^{4r+4\gamma+4}\left[M_4\left(\frac{9}{2}\right)^n \|g_n\|_{\rho_n}^2 + M_5\|Q_n\|_{\rho_n}\|g_n\|_{\rho_n}\right],$$

where $M_3 = 2^{2r+2\gamma+1}/c_0^2$, $M_4 = K_0L_1^2/2$ and $M_5 = c_0L_1$. For simplicity reasons, let us denote $\alpha_n = \|Q_n\|_{\rho_n}$ and $\beta_n = \|g_n\|_{\rho_n}$. This means that we have obtained the following bounds:

$$\alpha_{n+1} \leq M_1(n+1)^{2r+2\gamma+2}\left(\alpha_n + M_2\left(\frac{9}{2}\right)^n (n+1)^{2r+2\gamma+2}\beta_n\right)^2,$$

$$\beta_{n+1} \leq M_3(n+1)^{4r+4\gamma+4}\left(M_4\left(\frac{9}{2}\right)^n \beta_n^2 + M_5\alpha_n\beta_n\right).$$

To bound α_n and β_n we define $\eta_n = \max\{\alpha_n, \beta_n\}$. After some rearranging we get

$$\alpha_{n+1} \leq M_6(n+1)^{6r+6\gamma+6} \left(\frac{9}{2}\right)^n \eta_n^2,$$

$$\beta_{n+1} \leq M_7(n+1)^{4r+4\gamma+4} \left(\frac{9}{2}\right)^n \eta_n^2,$$

where M_6 and M_7 are constants such that

$$M_1 \left(\frac{(\frac{2}{9})^n}{(n+1)^{2r+2\gamma+2}} + M_2\right)^2 \leq M_6,$$

$$M_3 \left(M_4 + \left(\frac{2}{9}\right)^n M_5\right) \leq M_7.$$

Defining $M_8 = \max\{M_6, M_7\}$ we obtain a bound for η_{n+1}:

$$\eta_{n+1} \leq M_8 \left(\frac{9}{2}\right)^n (n+1)^{6r+6\gamma+6} \eta_n^2,$$

and, using Lemma 2.9 we obtain

$$\eta_{n+1} \leq \frac{1}{M_8} \left[M_8\eta_0 \left(\frac{9}{2}\right)^2 \left(\frac{5}{3}\right)^{6r+6\gamma+6}\right]^{2^{n+1}}.$$

Let us denote $M_9 = M_8\eta_0(9/2)^2(5/3)^{6r+6\gamma+6}$. We have proved that

$$\|Q_n\|_{\rho_n} \leq \frac{1}{M_8}M_9^{2^n}, \quad \|g_n\|_{\rho_n} \leq \frac{1}{M_8}M_9^{2^n}.$$

Note that this bound allows to ensure that, if $\varepsilon < \varepsilon_1 = M_9^{-1}$

$$\lim_{n\to\infty} \varepsilon^{2^n}\|Q_n\|_{\rho_n} = \lim_{n\to\infty} \varepsilon^{2^n}\|g_n\|_{\rho_n} = 0.$$

The next step is to bound $\|P_n\|_{\rho_{n+1}}$. For this purpose we use Lemma 2.8:

$$\|P_n\|_{\rho_{n+1}} \leq \frac{2^{r+\gamma}L_2}{c_0}(n+1)^{2r+2\gamma+2}\|\widehat{Q}_n\|_{\sigma_n} \leq$$

$$\leq \frac{2^{r+\gamma}L_2}{c_0}(n+1)^{2r+2\gamma+2}\left[\|Q_n\|_{\rho_n} + M_2\left(\frac{9}{2}\right)^n (n+1)^{2r+2\gamma+2}\|g_n\|_{\rho_n}\right] \leq$$

$$\leq \frac{2^{r+\gamma}L_2}{c_0}\left(\frac{9}{2}\right)^n (n+1)^{4r+4\gamma+4}\left[\frac{(\frac{2}{9})^n}{(n+1)^{2r+2\gamma+2}} + M_2\right]\eta_n.$$

Let M_{10} be a real number such that

$$\frac{2^{r+\gamma}L_2}{c_0}\left[\frac{(\frac{2}{9})^n}{(n+1)^{2r+2\gamma+2}} + M_2\right]\frac{1}{M_8} \leq M_{10}.$$

This implies that

$$\|P_n\|_{\rho_{n+1}} \leq M_{10} \left(\frac{9}{2}\right)^n (n+1)^{4r+4\gamma+4} M_9^{2^n},$$

and due to the fact that $\varepsilon < \varepsilon_1 = M_9^{-1}$ we have proved that

$$\lim_{n\to\infty} \varepsilon^{2^n} \|P_n\|_{\rho_{n+1}} = 0.$$

This allows to have the condition $\varepsilon^{2^n} \|P_n\|_{\rho_{n+1}} < 1/2$, without reducing the value of ε at each step. Now we are going to bound $\|\underline{x}_n(t)\|_{\sigma_n}$:

$$\|\underline{x}_n(t)\|_{\sigma_n} \leq \frac{2^{r+\gamma}L_1}{c_0}(n+1)^{2r+2\gamma+2}\varepsilon^{2^n}\|g_n\|_{\rho_n} \leq M_{11}(n+1)^{2r+2\gamma+2}(\varepsilon M_9)^{2^n},$$

where $M_{11} = 2^{r+\gamma}L_1/(c_0 M_8)$. When the changes of coordinates have been bounded, we can estimate the decrease of the radius τ_n of the ball where h_n is analytic with respect to x. It has been shown that

$$\tau_{n+1} = \frac{\widehat{\tau}_n}{1 + \varepsilon^{2^n}\|P_n\|_{\rho_{n+1}}} = \frac{1}{1 + \varepsilon^{2^n}\|P_n\|_{\rho_{n+1}}}\tau_n - \frac{\|\underline{x}_n(t)\|_{\rho_n}}{1 + \varepsilon^{2^n}\|P_n\|_{\rho_{n+1}}}.$$

Now, we define

$$a_n = \frac{1}{1 + \varepsilon^{2^n}\|P_n\|_{\rho_{n+1}}}, \qquad b_n = \frac{\|\underline{x}_n(t)\|_{\rho_n}}{1 + \varepsilon^{2^n}\|P_n\|_{\rho_{n+1}}}.$$

It is easy to prove that $\prod_{n=0}^{\infty} a_n$ converges:

$$\left|\ln \prod_{n=0}^{N} a_n\right| \leq \sum_{n=0}^{N} |\ln a_n| < \sum_{n=0}^{N} \varepsilon^{2^n}\|P_n\|_{\rho_{n+1}} < \infty \quad \forall N \in \mathbf{N}.$$

As $\sum b_n$ is also convergent, we can apply Lemma 2.10 to get $\tau_\infty \geq a\tau_0 - b$, that is positive if ε is taken small enough.

Now, let us bound $\|A_n\|$:

$$\begin{aligned}
\|A_{n+1}\| &\leq \|\widehat{A}_n\| + \varepsilon^{2^n}\frac{\|P_n\|_{\rho_{n+1}}}{1 - \varepsilon^{2^n}\|P_n\|_{\rho_{n+1}}} \\
&\leq \|\widehat{A}_n\| + \varepsilon^{2^n}\|Q_n\|_{\rho_n} + \varepsilon^{2^n}\frac{\|P_n\|_{\rho_{n+1}}}{1 - \varepsilon^{2^n}\|P_n\|_{\rho_{n+1}}}.
\end{aligned}$$

Using the bounds found above we can write that

$$\|A_{n+1}\| \leq \|A_n\| + \kappa_n,$$

where

$$\kappa_n = 2M_{10} \left(\frac{9}{2}\right)^n (n+1)^{4r+4\gamma+4} (\varepsilon M_9)^{2^n} + \frac{1}{M_8}(\varepsilon M_9)^{2^n}.$$

As $\sum \kappa_n$ is convergent we can ensure that, if ε is selected small enough, the matrices A_n are always inside the ball $B_\alpha(A_0)$ defined before.

Consider now the value K_n. We have used above the pesimistic bound $K_n \leq (9/2)^n K_0$. Note that this bound does not allow to guarantee the convergence of the functions $h_n(x_n, t)$ to an analytic function $h_\infty(x_\infty, t)$ with respect to x. Now we can use a more accurate bound of that value to get this: From Lemma 2.8 we know that

$$K_{n+1} \leq K_n \frac{(1 + \varepsilon^{2^n} \|P_n\|_{\rho_{n+1}})^2}{1 - \varepsilon^{2^n} \|P_n\|_{\rho_{n+1}}},$$

and, by means of the inequality $1/(1-x) \leq 1 + 2x$ if $0 \leq x \leq 1/2$, we get

$$K_{n+1} \leq \left(1 + 2\varepsilon^{2^n} \|P_n\|_{\rho_{n+1}}\right)^3 K_n.$$

And, using the bounds of $\|P_n\|_{\rho_{n+1}}$ that we already know, it is easy to see that the (bound of the) value K_n converges.

To end up the proof we are going to take into account the resonances. Let $\varphi_i^n(\varepsilon)$, $1 \leq i \leq d^2 + d$ be the functions that give the values of λ_j^n and $\lambda_{j_1}^n - \lambda_{j_2}^n$ at step n. At every step the eigenvalues depend on ε in an unknown but controlled way. As $|\varphi_i^0(\varepsilon_1) - \varphi_i^0(\varepsilon_2)| > 2\delta|\varepsilon_1 - \varepsilon_2|$ and $|\varphi_i^n(\varepsilon) - \varphi_i^0(\varepsilon)| = \mathcal{O}(\varepsilon^2)$ one has $|\varphi_i^n(\varepsilon_1) - \varphi_i^n(\varepsilon_2)| > \delta|\varepsilon_1 - \varepsilon_2|$ if $|\varepsilon_1|$, $|\varepsilon_2|$ are small enough. On the other side $|\varphi_i^n(\varepsilon_1) - \varphi_i^n(\varepsilon_2)|$ is bounded by $\bar\delta$ for all i, j, n if ε_1, ε_2 belong to some interval $(0, \bar\varepsilon)$. Here we suppose, for simplicity, that all the φ_i^n are purely imaginary. If we take some μ_m (see Lemma 2.11), with $\gamma_1 = \gamma_\nu$, $\gamma_2 = \gamma$, $c = c_0 = \frac{c_*}{2}$, such that $\frac{\mu_m}{\delta} < \bar\varepsilon$, when ε ranges on $(0, \bar\varepsilon)$ then φ_i^n ranges on $(-\mu_m, \mu_m)$.

To obtain the cantorian set $\mathcal{E}_0$ where the nonresonance condition holds for $n = 0$ one should delete an infinity of intervals in the range of ε with a measure at most $\psi(\mu_m) 2\mu_m \frac{1}{\delta}(d^2 + d)$. The relative measure of $\mathcal{E}_0$ in $(0, \frac{\mu_m}{\delta})$ is at least $1 - \psi(\mu_m) 2\bar\delta \frac{d^2+d}{\delta}$. In a similar way we obtain the set $\mathcal{E}_n \subset \mathcal{E}_{n-1}$ where the nonresonance condition holds up to n. Its relative measure in $(0, \frac{\mu_m}{\bar\varepsilon})$ is at least

$$1 - \psi(\mu_m)(2\bar\delta \frac{d^2 + d}{\delta}) \sum_{j=0}^{n} \frac{1}{(j+1)^2} > 1 - \psi(\mu_m) \frac{\pi^2}{3} \bar\delta \frac{d^2 + d}{\delta}$$

which goes to 1 if n goes to infinity. The limit set

$$\mathcal{E}_\infty = \bigcap_{n \geq 0} \mathcal{E}_n$$

is the cantorian that we were looking for. ∎

2.2.4   Proof of Theorem 2.2

As $\det A \neq 0$, the Contraction Lemma (see [21]) ensures that, if ε is small enough, there exists a function $x_0(\varepsilon)$ such that

$$(A + \varepsilon\overline{Q})x_0(\varepsilon) + \varepsilon\overline{g} + \overline{h}(x_0(\varepsilon), \varepsilon) = 0,$$

and verifying $x_0(\varepsilon) = \mathcal{O}(\varepsilon)$. Let us define

$$A_{x_0} = A + \varepsilon\overline{Q} + D_x\overline{h}(x_0(\varepsilon), \varepsilon),$$

and let $\underline{x}(t, \varepsilon)$ be such that

$$\underline{\dot{x}} = A\underline{x} + \varepsilon g(t, \varepsilon). \tag{2.11}$$

(The existence of that solution was shown in Lemma 2.5 and we recall that is $\mathcal{O}(\varepsilon)$). The terms of order ε of the matrix A_∞ are provided by Lemma 2.7 at the first step of the inductive process. This modified matrix $\hat{A}$ is

$$\hat{A} = A + \varepsilon\overline{Q} + \overline{D_x h(\underline{x}, t, \varepsilon)}.$$

Then,

$$\|A_{x_0} - \hat{A}\| = \|D_x\overline{h}(x_0(\varepsilon), \varepsilon) - \overline{D_x h(\underline{x}(t, \varepsilon), t, \varepsilon)}\|.$$

But

$$\overline{D_x h(\underline{x}(t, \varepsilon), t, \varepsilon)} = \overline{(C + \varepsilon R(t))\underline{x}(t)} + \mathcal{O}(\varepsilon^2) = C\overline{\underline{x}} + \mathcal{O}(\varepsilon^2),$$

where $C = \frac{1}{2}D_{xx}h(0, t, 0)$ is a constant matrix by hypothesis. Moreover, it is also easy to obtain that

$$D_x\overline{h}(x_0(\varepsilon), \varepsilon) = (C + \varepsilon\overline{R})x_0(\varepsilon) + \mathcal{O}(\varepsilon^2) = Cx_0(\varepsilon) + \mathcal{O}(\varepsilon^2)$$

We have obtained that

$$\|A_{x_0} - \hat{A}\| = C(x_0(\varepsilon) - \overline{\underline{x}}) + \mathcal{O}(\varepsilon^2).$$

Now, averaging (2.11) we get that $A\overline{\underline{x}} + \varepsilon\overline{g} = 0$, and using $Ax_0(\varepsilon) + \varepsilon\overline{g} = -(\varepsilon\overline{Q}x_0(\varepsilon) + \overline{h}(x_0(\varepsilon), \varepsilon)) = \mathcal{O}(\varepsilon^2)$, we obtain $\|x_0(\varepsilon) - \overline{\underline{x}}\| = \mathcal{O}(\varepsilon^2)$, that ends the proof. ∎

2.3   The Neighbourhood of an Elliptic Equilibrium Point of a Hamiltonian System

It is also interesting to consider the Hamiltonian case. Let us suppose that the origin is a fixed point of the vector field for all the values of ε less that a certain value.

Then, we can take action-angle variables and write the equations as:

$$\dot{p}_1 = -\frac{\partial H^\varepsilon}{\partial q_1},$$

$$\dot{p}_2 = -\frac{\partial H^\varepsilon}{\partial q_2},$$

$$\dot{q}_1 = \varphi,$$

$$\dot{q}_2 = \frac{\partial H^\varepsilon}{\partial p_2},$$

where q_1 are the angles of the perturbation, p_1 are the corresponding actions, q_2, p_2 are the angles and actions of the unperturbed system and $\varphi = (\varphi_1, \ldots, \varphi_r)$ is a constant vector of frequencies. Note that the first equation is not relevant. It has been added to get a Hamiltonian form. The corresponding Hamiltonian is:

$$H^\varepsilon(p_1, p_2, q_1, q_2) = (\varphi, p_1) + H_0(p_2) + \varepsilon H_1(p_2, q_1, q_2).$$

As this Hamiltonian is degenerate, we can not use the KAM theorem directly to conclude the existence of invariant tori near the origin for ε small enough. We have considered this case in Theorem 2.3, and we have found that the proof of the classical KAM theorem (see [1]) still works, because the perturbing frequencies are not modified in any step of the inductive process, and we only have to worry about the proper frequencies of the Hamiltonian, that can be controlled provided that the nondegeneracy condition

$$\det\left(\frac{\partial^2 H_0}{\partial(p_2)^2}\right) \neq 0,$$

holds. The result obtained is that there exist invariant tori near the origin for ε small enough. The frequencies of these tori are the ones of the unperturbed tori plus the ones of the perturbation. The tori whose frequencies are in resonance with the ones of the perturbation are destroyed.

Finally, in case that the origin is not a fixed point of the perturbed Hamiltonian, we can reduce to this case performing a change of variables transforming the quasiperiodic orbit that replaces the equilibrium point (we recall that this orbit exists for a cantorian set of values of ε) in a fixed point.

Theorem 2.3 *Let us consider the Hamiltonian*

$$H(p_1, p_2, q_1, q_2) = (\varphi, p_1) + H_0(p_2) + H_1(p_2, q_1, q_2),$$

where q_1 are the angles of the perturbation, p_1 are the corresponding actions, q_2, p_2 are the angles and actions of the unperturbed system and $\varphi = (\varphi_1, \ldots, \varphi_{n_1})$ is a constant vector of frequencies satisfying the nonresonance condition

$$|<k, \varphi>| > \frac{c}{|k|^\gamma}, \quad \forall k \in \mathbf{Z}^{n_1} \setminus \{0\}, \quad \gamma > n_1 - 1.$$

Let G^1 be a compact domain of $\mathbf{R}^{n_1}$, let G^2 be a compact domain of $\mathbf{R}^{n_2}$ and let G be $G^1 \times G^2$. Suppose now that this Hamiltonian function $H(p_1, p_2, q_1, q_2) = H(p, q)$ is analytic on the domain $F = \{(p, q) \,/\, p = (p_1, p_2) \in G, |\mathrm{Im}\ q| \leq \rho\}$ and has period 2π with respect to the variables q. Let us assume that, in the domain F,

$$\det \left| \frac{\partial^2 H_0}{\partial (p_2)^2} \right| \neq 0. \tag{2.12}$$

Then for any $\kappa > 0$ there exists $M = M(\kappa, \rho, G, H_0) > 0$ such that if in F we have

$$|H_1| \leq M, \tag{2.13}$$

then the motion defined by the Hamiltonian equations

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{q}_1 = \varphi,$$
$$\dot{p}_2 = -\frac{\partial H}{\partial q_2}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2}, \tag{2.14}$$

has the following properties:

1. *There exists a decomposition* $\mathrm{Re}\ F = F_1 + F_2$*, where F_1 is invariant and F_2 is small: mes $F_2 \leq \kappa$ mes F.*

2. *F_1 is composed of invariant n-dimensional analytic tori I_ϕ, defined parametrically by the equations*

$$p = p_\phi + f_\phi(Q), \quad q = Q + g_\phi(Q), \tag{2.15}$$

where f_ϕ, g_ϕ are analytic functions of period 2π in the variables Q, and ϕ is a parameter determining the torus I_ϕ. In fact ϕ consists of all the frequencies, i.e., the ones of the external excitation and the proper frequencies: $\phi = (\varphi_1, \ldots, \varphi_{n_1}, \omega_1, \ldots, \omega_{n_2})$.

3. *The invariant tori I_ϕ differ little from the tori $p = p_\phi$:*

$$|f_\phi(Q)| < \varkappa, \quad |g_\phi(Q)| < \varkappa. \tag{2.16}$$

4. *The motion (2.14) on the invariant torus I_ϕ is quasiperiodic with n frequencies $\varphi_1, \ldots, \varphi_{n_1}, \omega_1, \ldots, \omega_{n_2}$ ($n = n_1 + n_2$):*

$$Q = \phi, \quad \omega = \left. \frac{\partial H_0}{\partial p_2} \right|_{p_\phi}. \tag{2.17}$$

2.3.1 Sketch of the Proof of Theorem 2.3

The proof of this Theorem is essentially the same of the KAM Theorem contained in [1], and its technical details can be found inside Appendix C. Here we show the idea of that proof.

Let us define p and q as the vectors p_1, p_2 and q_1, q_2 respectively. Now, the Hamiltonian we have is

$$H^\varepsilon = (\varphi, p_1) + H_0(p_2) + \varepsilon \overline{H}_1(p_2) + \varepsilon \widetilde{H}_1(p_2, q), \qquad (2.18)$$

and let us consider the generating function $S(P, q) = \sum_{k \neq 0} S^k(P_2) e^{(k,q)\sqrt{-1}}$. If we perform the canonical change of variables

$$p_1 = P_1 + \varepsilon \frac{\partial S}{\partial q_1},$$

$$p_2 = P_2 + \varepsilon \frac{\partial S}{\partial q_2},$$

$$Q_1 = q_1,$$

$$Q_2 = q_2 + \varepsilon \frac{\partial S}{\partial P_2},$$

to (2.18) we obtain

$$H^\varepsilon = (\varphi, P_1) + H_0(P_2) + \varepsilon \overline{H}_1(P_2) + \varepsilon F + \varepsilon^2 R(P_2, q),$$

where $F = (\varphi, S_{q_1}) + (\omega(P_2), S_{q_2}) + \widetilde{H}_1(P_2, q)$ and $\omega(p_2) = \frac{\partial H_0}{\partial p_2}(p_2)$. Let $\phi(P_2)$ be the vector $\varphi, \omega(P_2)$. We ask for the condition $F = 0$:

$$(\phi(P_2), S_q) + \widetilde{H}_1(P_2, q) = 0.$$

Now, using that $\widetilde{H}_1(P_2, q) = \sum_{k \neq 0} h_1^k(P_2) e^{(k,q)\sqrt{-1}}$, the coefficients of the Fourier expansion for the generating function S can be obtained easily:

$$S^k(P_2) = \frac{h_1^k}{(\phi(P_2), k)} \sqrt{-1}.$$

To ensure the convergence of that series, we need the usual nonresonance condition

$$|(\phi(P_2), k)| \geq \frac{c}{|k|^\gamma}, \qquad (2.19)$$

that allows to proof the convergence in a smaller strip than the one on which $\widetilde{H}_1$ is analytic. With this the Hamiltonian takes the form

$$H^\varepsilon = (\varphi, P_1) + H_1^\varepsilon(P_2) + \varepsilon^2 \overline{H}_3(P_2) + \varepsilon^2 \widetilde{H}_3(P_2, Q).$$

This new Hamiltonian is very similar to (2.18), but with ε^2 instead of ε. Note that the difference between this proof and the one contained in [1] is the condition (2.19). Due to the fact that the first components of $\phi(P_2)$ are the ones of φ, *that are constant in all the inductive process*, we only have to worry about the last ones, $\omega(P_2)$. These components are different at each step of the process, but they can be controlled by the nondegeneracy condition

$$\det\left(\frac{\partial^2 H_0}{\partial (p_2)^2}\right) \neq 0.$$

The way to do that is exactly the same one showed in [1]. Note that, to get an rigorous proof of this theorem, we only need to copy the proof contained in [1], but adding the "parameter" φ. The unique difference is that now, the nonresonance condition is stronger, in the sense that we must eliminate a bigger set of (resonant) tori.

In order to bound the measure of the resonant set of frequencies, we give the following lemma.

Lemma 2.12 *Let us define the set* $\Omega = \Omega(\varphi, \rho, c_2, n_1, n_2, \gamma_2)$ *of the following way:*

$$\Omega = \{\omega \in B_\rho(0) \subset \mathbf{R}^{n_2} \quad / \quad \exists (k,l) \in \mathbf{Z}^{n_1} \times (\mathbf{Z}^{n_2} \setminus \{0\}) \ such$$

$$that \mid < k, \varphi > + < l, \omega > \mid \leq \frac{c_2}{(|k| + |l|)^{\gamma_2}} \ \},$$

where $\gamma_2 > n_1 + n_2 - 1$, $n_1 > 0$, $n_2 > 0$ *and* $B_\rho(0)$ *denotes the elements whose Euclidean norm is less than* ρ. *Then, the Lebesge measure of* Ω *can be bounded by*

$$\mu(\Omega) \leq c_2 \alpha(n_1, n_2, \gamma_2) \rho^{n_2 - 1}.$$

Proof: Let (k, l) be an element of $\mathbf{Z}^{n_1} \times (\mathbf{Z}^{n_2} \setminus \{0\})$. The corresponding resonant hyperplane is defined as the set of ω's satisfying

$$< k, \varphi > + < l, \omega >= 0.$$

The elements of Ω are contained in strips of width

$$\frac{c_2}{\|l\|(|k| + |l|)^{\gamma_2}}$$

around these hyperplanes (here, $\|.\|$ states for the usual Euclidean norm). By intersecting those hyperplanes with the ball $B_\rho(0)$ we obtain the bound

$$\mu(\Omega) \leq \sum_k \left(\sum_{l \neq 0} \frac{2c_2 S_{n_2-1} \rho^{n_2-1}}{\|l\|(|k| + |l|)^{\gamma_2}}\right) \leq 2c_2 n_2 S_{n_2-1} \rho^{n_2-1} \sum_k \sigma(k),$$

where we have used $\|l\| \geq |l|/n_2$, S_m is the measure of the m-dimensional unit ball and $\sigma(k)$ is defined as

$$\sigma(k) = \sum_{l \neq 0} \frac{1}{|l|(|k| + |l|)^{\gamma_2}}.$$

Now, let us bound $\sigma(k)$. For this purpose we consider two cases:

1. $k = 0$.

$$\sigma(0) = \sum_{l \neq 0} \frac{1}{|l|^{\gamma_2 + 1}}.$$

Now, using $\#\{l \in \mathbf{Z}^{n_2} \,/\, |l| = i\} \leq 2^{n_2} i^{n_2 - 1}$ it is easy to get that

$$\sigma(0) \leq 2^{n_2} \zeta(\gamma_2 - n_2 + 2),$$

where ζ states for the Riemann zeta function that is well defined because $\gamma_2 - n_2 + 2 > 2$.

2. $k \neq 0$.

$$\sigma(k) = \sum_{l \neq 0} \frac{1}{|l|(|k| + |l|)^{\gamma_2}} \leq 2^{n_2} \sum_{i \geq 1} \frac{i^{n_2 - 2}}{(|k| + i)^{\gamma_2}} = 2^{n_2} \sum_{i \geq |k| + 1} \frac{(i - |k|)^{n_2 - 2}}{i^{\gamma_2}} \leq$$

$$\leq 2^{n_2} \sum_{i \geq |k| + 1} \frac{1}{i^{\gamma_2 - n_2 + 2}} \leq 2^{n_2} \int_{|k|}^{\infty} \frac{dx}{x^{\gamma_2 - n_2 + 2}} = \frac{2^{n_2}}{(\gamma_2 - n_2 + 1)|k|^{\gamma_2 - n_2 + 1}}.$$

Let us bound $\sum \sigma(k)$:

$$\sum_k \sigma(k) \leq 2^{n_2} \zeta(\gamma_2 - n_2 + 2) + \frac{2^{n_2}}{\gamma_2 - n_2 + 1} \sum_{k \neq 0} \frac{1}{|k|^{\gamma_2 - n_2 + 1}} \leq$$

$$\leq 2^{n_2} \zeta(\gamma_2 - n_2 + 2) + \frac{2^{n_1 + n_2}}{\gamma_2 - n_2 + 1} \zeta(\gamma_2 - n_1 - n_2 + 2).$$

Now, defining

$$\alpha(n_1, n_2, \gamma_2) = 2 n_2 S_{n_2 - 1} \left[2^{n_2} \zeta(\gamma_2 - n_2 + 2) + \frac{2^{n_1 + n_2}}{\gamma_2 - n_2 + 1} \zeta(\gamma_2 - n_1 - n_2 + 2) \right],$$

we obtain the bound we were looking for. ∎

Chapter 3

A Semianalytical Study of the Equilateral Libration Points of the Earth-Moon System

3.1 Introduction

The main purpose of this Chapter is to find quasiperiodic solutions near the equilateral libration points of the Earth-Moon system.

The equations of motion near these points can be written as (see Appendix B)

$$\ddot{x} = P(7)\left[-\frac{x - x_E}{r_{PE}^3}(1 - \mu) - \frac{x + x_E}{r_{PM}^3}\mu - x_E(1 - 2\mu)\right] +$$
$$+P(1) + P(2)x + P(3)y + P(4)z + P(5)\dot{x} + P(6)\dot{y}, \tag{3.1}$$

$$\ddot{y} = P(7)\left[-\frac{y - y_E}{r_{PE}^3}(1 - \mu) - \frac{y - y_E}{r_{PM}^3}\mu - y_E\right] + P(8) + P(9)x +$$
$$+P(10)y + P(11)z + P(12)\dot{x} + P(13)\dot{y} + P(14)\dot{z}, \tag{3.2}$$

$$\ddot{z} = P(7)\left[-\frac{z}{r_{PE}^3}(1 - \mu) - \frac{z}{r_{PM}^3}\mu\right] + P(15) + P(16)x +$$
$$+P(17)y + P(18)z + P(19)\dot{y} + P(20)\dot{z}, \tag{3.3}$$

where r_{PE}, r_{PM} denote the distances from the particle to the Earth and Moon, respectively, given by $r_{PE}^2 = (x - x_E)^2 + (y - y_E)^2 + z^2$, $r_{PM}^2 = (x + x_E)^2 + (y - y_E)^2 + z^2$. We recall that $x_E = -1/2$, $y_E = -\sqrt{3}/2$ for L_4 and $x_E = -1/2$, $y_E = \sqrt{3}/2$ for L_5.

The functions $P(i)$ that appear in the equations are of the form:

$$P(i) = A_{i,0} + \sum_{j=1}^{m} A_{i,j}\cos\theta_j + \sum_{j=1}^{m} B_{i,j}\sin\theta_j,$$

with $\theta_j = \nu_j t + \varphi_j$, where t denotes the normalized time. Values for all of the coefficients, frequencies and phases can be found in Appendix B.

An expansion of the nonlinear terms of the equations of motion, in power series of the coordinates (x, y, z), can be done easily using Legendre polynomials as it will be described in the next section.

Another possibility is to produce those terms by means of a recurrence based on the one of the Legendre polynomials.

3.1.1   Expansion of the Equations

We consider the potential $V = \frac{1-\mu}{r_{PE}} + \frac{\mu}{r_{PM}}$. Let ψ be the angle between the vectors $(x_E, y_E, 0)$ and (x, y, z) and ρ^2 be $x^2 + y^2 + z^2$. Then

$$\frac{1}{r_{PE}} = \frac{1}{\sqrt{1 - 2\rho\cos\psi + \rho^2}} = \sum_{n=0}^{\infty} \rho^n P_n(\cos\psi), \tag{3.4}$$

where P_n is the Legendre polynomial of degree n:

$$P_n(\omega) = \frac{1}{2^n n!} \frac{d^n}{d\omega^n}(\omega^2 - 1)^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \alpha_{n,k}\omega^{n-2k}.$$

Here $\left[\frac{n}{2}\right]$ denotes the integer part of $\frac{n}{2}$ and

$$\alpha_{n,k} = \frac{(-1)^k}{2^n n!}\binom{n}{k}\frac{(2n - 2k)!}{(n - 2k)!}.$$

Substituting these expressions in (3.4) one obtains

$$\begin{aligned}
\frac{1}{r_{PE}} &= \sum_{n=0}^{\infty} \rho^n \sum_{n=0}^{\left[\frac{n}{2}\right]} \alpha_{n,k}\cos^{n-2k}\psi = \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \alpha_{n,k}(xx_E + yy_E)^{n-2k}(x^2 + y^2 + z^2)^k.
\end{aligned} \tag{3.5}$$

Finally, in order to obtain the expansion for L_4 (the L_5 case can be done with an identical process) we put $x_E = -\frac{1}{2}$, $y_E = -\frac{\sqrt{3}}{2}$ and (3.5) becomes

$$\frac{1}{r_{PE}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\left[\frac{n}{2}\right]} \beta_{n,k}(x + \sqrt{3}y)^{n-2k}(x^2 + y^2 + z^2)^k, \tag{3.6}$$

where

$$\beta_{n,k} = \frac{(-1)^{n-k}(2n - 2k)!}{4^{n-k}(n - k)!k!(n - 2k)!}.$$

To perform the expansion (3.6) by algebraic manipulation it is better to change the order of the sums:

$$\frac{1}{r_{PE}} = \sum_{k=0}^{[\frac{N}{2}]}(x^2 + y^2 + z^2)^k \sum_{n=2k}^{N} \beta_{n,k}(x + \sqrt{3}y)^{n-2k},$$

where N denotes the expansion order wanted. With a similar computation it can be found that:

$$\frac{1}{r_{PM}} = \sum_{k=0}^{[\frac{N}{2}]}(x^2 + y^2 + z^2)^k \sum_{n=2k}^{N} \beta_{n,k}(-x + \sqrt{3}y)^{n-2k}$$

and, finally,

$$V = \frac{1-\mu}{r_{PE}} + \frac{\mu}{r_{PM}} =$$
$$= \sum_{k=0}^{[\frac{N}{2}]}(x^2 + y^2 + z^2)^k \sum_{n=2k}^{N} \beta_{n,k}\left[(1-\mu)(x + \sqrt{3}y)^{n-2k} + \right. \tag{3.7}$$
$$\left. + \mu(-x + \sqrt{3}y)^{n-2k}\right].$$

Now, using a program which takes advantage of the particularities of the latter expression, the expansions of $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ are obtained. This program has been written to give the final solution up to order nine. This is enough for practical purposes, because the contribution of higher order terms is much less than the threshold used to select the coefficients $A_{i,j}$, $B_{i,j}$. Now let us see a short description of that program as well as of its routines:

Subroutine DERIV Given a polynomial of three variables x, y and z, this routine returns three polynomials which are its derivatives with respect to x, y and z. This is used at the end of the computation of (3.7) to obtain the expansion wanted.

Subroutine PRODE This routine performs the product of two polynomials of three variables. It is used to multiply the term $(x^2 + y^2 + z^2)^k$ by the result of the inner summatory of (3.7).

Subroutine XYZK Given a polynomial S, this routine returns $S \times (x^2 + y^2 + z^2)$. It is used to obtain the different powers of $(x^2 + y^2 + z^2)$ starting from $S = 1$.

Subroutine PART2V This routine computes the result of the inner summatory of (3.7), taking advantage of the cancellations that occur in this formula

Subroutine INIT This routine fills a real array with the powers of $\sqrt{3}$ (they will be used inside PART2V) and a couple of integer arrays containing information about the way to store polynomials (in fact this is not necessary, but it allows to increase the speed of the program).

Main Program This routine initializes the parameters used by the program (x_E, y_E and μ) and it uses the formula (3.7) to compute V. Then the final result is obtained (by derivation) and the result is written in the files EXPAN.1 (expansion of the first equation), EXPAN.2 (second one) and EXPAN.3 (third one).

3.2   Idea of the Resolution Method

The equations of motion can be written as

$$\ddot{x} = f_1(t, x, y, z, \dot{x}, \dot{y}, \dot{z}),$$
$$\ddot{y} = f_2(t, x, y, z, \dot{x}, \dot{y}, \dot{z}),$$
$$\ddot{z} = f_3(t, x, y, z, \dot{x}, \dot{y}, \dot{z}).$$

Define $G = (G_1, G_2, G_3)$, where $G_1 = f_1 - \ddot{x}$, $G_2 = f_2 - \ddot{y}$, and $G_3 = f_3 - \ddot{z}$. We are looking for quasiperiodic solutions of the form

$$x = \sum x_k \exp((k, \omega) t \sqrt{-1}),$$
$$y = \sum y_k \exp((k, \omega) t \sqrt{-1}),$$
$$z = \sum z_k \exp((k, \omega) t \sqrt{-1}),$$

where $\omega = (\omega_1, \omega_2, ..., \omega_r)^t$ is a known set of basic frequencies, the ones that appear as basic frequencies, in the developments of the functions $P(i)$, and (k, ω) denotes the inner product of $k = (k_1, k_2, ..., k_r)^t$ and ω. We consider x, y and z truncated up to some order. After substitution, G may be considered as a function of x_k, y_k and z_k, which means that looking for quasiperiodic solutions of this problem is more or less equivalent (this depends on the order of approximation taken for the trigonometric expansions of x, y, and z) to look for a zero of the function $G : \mathbf{R}^m \to \mathbf{R}^m$, where m denotes the total number of coefficients. To solve this equation we use a Newton method. In order to calculate the Jacobian matrix, the chain rule is used in the following way: $\frac{\partial G}{\partial x_k} = \frac{\partial G}{\partial x} \cdot \frac{\partial x}{\partial x_k}$, and so on.

3.3   The Planar Case

In order to do a preliminar study, the following simplifications are introduced:

i	j	A_{ij}	B_{ij}	i	j	A_{ij}	B_{ij}	i	j	A_{ij}	B_{ij}
0	0	1.0047	0.0000	3	4	0.0000	-0.0165	7	1	0.0335	0.0000
0	1	0.0315	0.0000	3	6	0.0000	-0.0018	7	2	-0.0009	0.0000
0	2	-0.0008	0.0000	4	1	0.0000	-0.0169	7	3	0.0165	0.0000
0	3	0.1644	0.0000	4	3	0.0000	-0.1079	7	4	0.0102	0.0000
0	4	0.0266	0.0000	4	4	0.0000	-0.0295	7	5	0.0135	0.0000
0	5	0.0134	0.0000	4	5	0.0000	-0.0088	7	6	0.0023	0.0000
0	6	0.0042	0.0000	4	6	0.0000	-0.0038	8	0	-2.0000	0.0000
1	0	0.0014	0.0000	5	0	2.0000	0.0000	8	1	-0.0382	0.0000
1	1	0.0000	0.0016	5	1	0.0382	0.0000	8	2	0.0011	0.0000
1	4	0.0000	-0.0142	5	2	-0.0011	0.0000	8	3	-0.2176	0.0000
1	6	0.0000	-0.0016	5	3	0.2176	0.0000	8	4	-0.0429	0.0000
2	0	1.0076	0.0000	5	4	0.0429	0.0000	8	5	-0.0148	0.0000
2	1	0.0315	0.0000	5	5	0.0148	0.0000	8	6	-0.0053	0.0000
2	2	-0.0008	0.0000	5	6	0.0053	0.0000	9	1	0.0000	-0.0164
2	3	0.1650	0.0000	6	0	0.0025	0.0000	9	2	0.0000	0.0005
2	4	0.0266	0.0000	6	1	0.0017	0.0000	9	3	0.0000	-0.1079
2	5	0.0135	0.0000	6	4	-0.0143	0.0000	9	4	0.0000	-0.0295
2	6	0.0043	0.0000	6	6	-0.0016	0.0000	9	5	0.0000	-0.0088
3	1	0.0000	0.0018	7	0	1.0076	0.0000	9	6	0.0000	-0.0038

Table 3.1: Fourier coefficients of the perturbations.

1. The problem is considered planar.

2. Only the two more relevant frequencies are retained, namely

    (a) the frequency of the mean elongation of the Sun (let us call it ψ).

    (b) the difference between the frequencies of the mean longitude of the Moon and the mean longitude of the lunar perigee (let us call it M).

After these simplifications we obtain a system that contains the same basic difficulties as the original one, but it is easier to compute.

Finally, introducing a new notation, the equations that we have studied are:

$$\ddot{x} = q_0 \left[-\frac{x - x_E}{r_{PE}^3}(1 - \mu) - \frac{x + x_E}{r_{PM}^3}\mu - x_E(1 - 2\mu) \right] + q_1 + q_2 x + q_3 y +$$
$$+ q_4 \dot{x} + q_5 \dot{y},$$

$$\ddot{y} = q_0 \left[-\frac{y - y_E}{r_{PE}^3}(1 - \mu) - \frac{y - y_E}{r_{PM}^3}\mu - y_E \right] + q_6 + q_7 y + q_8 \dot{x} + q_9 \dot{y},$$

(3.8)

where x_E, y_E, r_{PE} and r_{PM} are defined as above, $q_0, \ldots, q_9$ are functions of time:

$$q_i = \sum_{j=0}^{6} A_{ij} \cos \theta_j + B_{ij} \sin \theta_j,$$

with $\theta_0 = 0$, $\theta_1 = 2\psi - M$, $\theta_2 = \psi$, $\theta_3 = M$, $\theta_4 = 2\psi$, $\theta_5 = 2M$ and $\theta_6 = 2\psi + M$. The values of ψ and M are given by $\psi = 0.9251959855t + 5.0920835091$ and $M = 0.9915452215t + 2.2415337977$. We recall that the origin of time is in the year 2000.0, and that 2π units of it are equivalent to a sidereal periode of the Moon. The coefficients A_{ij}, B_{ij} are given in Table 3.1. As usual, we shall use an expanded form for those equations, that can be obtained in the same way used for the spatial case (see section 3.1.1). We shall take this expansion up to order 4 and, as we shall see later, this will be enough to have an accuracy similar (even better) to the accuracy of the perturbations. Hence, the final expanded equations are

$$\ddot{x} = q_0 \left[\sum_{i+j \le 4} \bar{\alpha}_{ij} x^i y^j \right] + q_1 + q_2 x + q_3 y + q_4 \dot{x} + q_5 \dot{y}$$

$$\ddot{y} = q_0 \left[\sum_{i+j \le 4} \bar{\beta}_{ij} x^i y^j \right] + q_6 + q_7 x + q_8 \dot{x} + q_9 \dot{y}$$

(3.9)

where the coefficients $\bar{\alpha}_{ij}$ and $\bar{\beta}_{ij}$ are given in Table 3.2.

3.3.1   The Method

We want to find a quasiperiodic solution of the preceeding system, i.e., a solution expressed in terms of cosine and sine of angles depending on ψ and M in the following form:

$$x = \sum_{j,k=-\infty}^{\infty} x_1(j,k) \cos(j\psi + kM) + x_2(j,k) \sin(j\psi + kM),$$

$$y = \sum_{j,k=-\infty}^{\infty} y_1(j,k) \cos(j\psi + kM) + y_2(j,k) \sin(j\psi + kM).$$

(3.10)

The problem is now to find the coefficients x_1, x_2, y_1, y_2 of these series that satisfy the equations of motion.

Our approach to the problem is semianalytical, this is, we shall not find exactly these coefficients but some numerical aproximations of them.

The method that we have used is essentially to substitute expresions (3.10) of x and y in the equations (3.9), make the computation of the operations analytically (by means of an algebraic manipulator) and then solve the resultant system numerically.

The first problem of this method is to find a good expression of the series x, y that allows us to reduce the number of coefficients to a manageable one without loosing significant information.

i	j	$\bar{\alpha}_{ij}$	$\bar{\beta}_{ij}$
0	0	0.00000000D+00	0.00000000D+00
1	0	−0.25000000D+00	0.12674707D+01
0	1	0.12674707D+01	0.12500000D+01
2	0	0.12806055D+01	−0.32475953D+00
1	1	−0.64951905D+00	−0.40247600D+01
0	2	−0.20123800D+01	−0.97427858D+00
3	0	−0.11562500D+01	−0.13202820D+01
2	1	−0.39608460D+01	0.38437500D+01
1	2	0.38437500D+01	0.71295227D+01
0	3	0.23765076D+01	0.93750000D−01
4	0	−0.43830247D+00	0.19282597D+01
3	1	0.77130388D+01	0.32777402D+01
2	2	0.49166103D+01	−0.14005255D+02
1	3	−0.93368364D+01	−0.84611433D+01
0	4	−0.21152858D+01	0.11163609D+01

Table 3.2: Coefficients of the expansions in (3.9).

The biggests coefficients of these series are always located near the diagonal $(n\psi, -nM)$ and that suggested us to use the following expressions for x and y:

$$
\begin{aligned}
x = {} & x(1) + \sum_{i=1}^{N_M}(x(2i)\cos(iM) + x(2i+1)\sin(iM)) + \\
& + \sum_{j=1}^{N_\psi}(\sum_{i=-N_M-j}^{N_M-j}(x(4j(N_M+1)+2i)\cos(j\psi+iM) + \\
& + x(4j(N_M+1)+2i+1)\sin(j\psi+iM))), \\
y = {} & y(1) + \sum_{i=1}^{N_M}(y(2i)\cos(iM) + y(2i+1)\sin(iM)) + \\
& + \sum_{j=1}^{N_\psi}(\sum_{i=-N_M-j}^{N_M-j}(y(4j(N_M+1)+2i)\cos(j\psi+iM) + \\
& + y(4j(N_M+1)+2i+1)\sin(j\psi+iM))),
\end{aligned}
\tag{3.11}
$$

where N_M and N_ψ are some values that we can fix arbitrarily, and they represent, respectively the "dispersion" allowed for the coefficients from the diagonal and the longitude of it.

Now, the problem can be reduced to the search of the solution of the following system

$$
G_1(x, y) = f_1(x, y, \dot{x}, \dot{y}) - \ddot{x} = 0,
$$

$$G_2(x,y) = f_2(x,y,\dot{x},\dot{y}) - \ddot{y} = 0,$$

where

$$f_1(x,y,\dot{x},\dot{y}) = q_0\left[\sum_{i+j\leq L}\bar{\alpha}_{ij}x^iy^j\right] + q_1 + q_2x + q_3y + q_4\dot{x} + q_5\dot{y},$$

$$f_2(x,y,\dot{x},\dot{y}) = q_0\left[\sum_{i+j\leq L}\bar{\beta}_{ij}x^iy^j\right] + q_6 + q_7x + q_8\dot{x} + q_9\dot{y}.$$

Because x and y are considered as a function of its coefficients, we can see our problem as the search of the zero of a function:

$$
\begin{aligned}
F : \mathbf{R}^{2p} &\longrightarrow \mathbf{R}^{2p}\\
(x(1),\ldots,x(p),y(1),\ldots,y(p)) &\longmapsto (f_1 - \ddot{x}, f_2 - \ddot{y})
\end{aligned}
\tag{3.12}
$$

where $p = (2N_\psi + 1)(2N_M + 1)$ and in the computation of f_1 and f_2 we keep the same kind of terms used for x, y.

The numerical method that we have used to solve this equation is a Newton continuation method.

Thus, the general scheme of the method is the following:

We consider a m-th approximation of the system (3.12), $F^{(m)}$. Then, we compute the Jacobian of the system as usual, using the following expressions

$$\frac{\partial F_k^{(m)}}{\partial x(j)} = \frac{\partial F_k^{(m)}}{\partial x}\cdot\frac{\partial x}{\partial x(j)} + \frac{\partial F_k^{(m)}}{\partial \dot{x}}\cdot\frac{\partial \dot{x}}{\partial x(j)} + \frac{\partial F_k^{(m)}}{\partial \ddot{x}}\cdot\frac{\partial \ddot{x}}{\partial x(j)},$$

$$\frac{\partial F_k^{(m)}}{\partial y(j)} = \frac{\partial F_k^{(m)}}{\partial y}\cdot\frac{\partial y}{\partial y(j)} + \frac{\partial F_k^{(m)}}{\partial \dot{y}}\cdot\frac{\partial \dot{y}}{\partial y(j)} + \frac{\partial F_k^{(m)}}{\partial \ddot{y}}\cdot\frac{\partial \ddot{y}}{\partial y(j)}.$$

The relationship between ψ, M and t and (4.2) allows finally to write the corresponding expressions for $\frac{\partial \dot{x}}{\partial x(j)}$, $\frac{\partial \ddot{x}}{\partial x(j)}$, $\frac{\partial \dot{y}}{\partial y(j)}$, $\frac{\partial \ddot{y}}{\partial y(j)}$, in each case.

Actually, we took a first degree aproximation to the system, and the solution of this system is found in one iteration. The following step consisted in adding the next order terms. By means of a continuation method, using as the initial condition of every iteration the solution obtained in the preceeding one it has been possible to get the solution when quadratic terms were included in the equation. The third degree terms and the quartic ones could be added without needing of continuation, and finally we have obtained a solution of the fourth degree aproximation to the system that is also a good aproximation to the solution of the system. We shall see this further on.

Before finishing this point we must make an important remark. Using the method explained below, the matrix of the system we must solve in every step was of $2p \times 2p$. Because the final values for N_ψ and N_M are, respectively, 70 and 30 and not all the

coefficients are significant, we need to solve a smaller system containing only the most significant coefficients.

The central point of this is how to choose these coefficients. After each iteration we have two series, $\tilde{x}$, $\tilde{y}$, which are approximations to the solutions of the system. If we substitute now these series $\tilde{x}$, $\tilde{y}$ in $F^{(m)}$ we obtain a series $F^{(m)}(\tilde{x}, \tilde{y})$ from which we can choose its biggests coefficients and add them to the ones that take part into the Newton method. In this way the list of elements to keep for x and y is modified on line as required.

It is important to note that all the operations involved in the computation of the function have been made with all the coefficients, and it is only in the resolution of the linear system where we use the selected ones. This is because we can not choose, a priori, which ones are the most significant and, on the other hand, the operations involved in the function (such as product of series) produced many new components. If we had worked only with a small number of coefficients in all the operations we would have lost much information. With this method all the operations are done to lose as few information as possible. Using the results above, we corrected the coefficients dynamically and accurately.

3.3.2 The Manipulator

In order to do all the computations in an easy way we have implemented an algebraic manipulator that can deal with Fourier series of the type explained below. This manipulator has to do four main operations: product and derivatives (of first and second orders) of the series and conversions between series with selected coefficients and series with all the coefficients, in both ways.

These four operations have been increased with two more: the square of a series and the product of a series by an unitary series (only one coefficient different from 0). These last operations were not necessary, but they must be done very often and, in order to optimize the computations, we built them.

The nucleus of the manipulator consists in two routines that allow us to separate the operations explained before from the actual disposition of the terms inside the vectors that we used to store them. These routines are, in fact, the only ones that know this disposition. One of these routines returns, given an index of the vector, the coefficient of ψ and M and if the component is either a sine or a cosine. The other routine returns the index of the vector of a $\sin(j\psi + kM)$ or $\cos(j\psi + kM)$.

The other routines of the manipulator simply apply the formulae for each operation and they call these routines to know the information about the computation of the vectors they are working with.

There is another routine of the manipulator which compares all the components of the vector with a threshold value, dropping all of them that are less than this tolerance. This routine is called after each operation of the manipulator and it allows

us to pay attention only to the meaningful components, preventing the program from an uncontrolled growth of the vectors.

3.3.3   Results

In this part the results obtained with these computations are presented. The dominant coefficients of the fourth order solution are given in Table 3.3.

First we check the quality of the linear approximation. We consider the semianalytical solution of the linearized system and we take the value of this solution at $t = 0$ as an initial condition for numerical integration of the model equations (3.8). This is shown on Figure 3.1. The same process is made for the semianalytical solutions of the equations of 2nd, 3th and 4th order and the results are shown in Figures 3.2, 3.3 and 3.4. These plots show that the solution of the linear and quadratic systems are not good approximations to the numerical solution of the model. For this reason we consider the euclidean norm of the difference between numerical and semianalytical solutions. This allows to see that, in the quartic case, we have a degree of accuracy similar to the accuracy of the perturbations. Plots of the error in this case can be seen in Figure 3.5. Note that the error shows an exponential behaviour due to the existence of an unstable direction.

To compute the local behaviour associated to this orbit we observe that, after 192 days, the solution passes near (in the four-dimensional phase space) the initial conditions and it suggests to compute a pseudo-monodromy matrix, obtain its eigenvalues (shown in Table 3.4) and use the corresponding eigenvectors to get a linear approximation to the invariant manifolds. Figures 3.6 to 3.9 represent, in linear approximation, the behaviour inside each one of these manifolds. These results seem to show the existence of three invariant manifolds associated with this orbit. One of them is slightly unstable and gives this nature to the orbit, another one is stable and the last one looks like a central manifold.

i	j	k	$x(i,j,k)$	$y(i,j,k)$
0	0	0	0.115257D−01	−0.877293D−02
0	1	0	0.226069D−01	−0.132180D−01
0	1	1	−0.529571D−02	−0.117557D−01
0	2	0	0.165964D−02	−0.121897D−02
0	2	1	−0.322429D−03	−0.114187D−02
2	−4	0	−0.571189D−03	0.595989D−03
2	−4	1	−0.335783D−03	−0.421593D−03
2	−3	0	−0.795810D−02	0.600651D−02
2	−3	1	−0.362805D−02	−0.425411D−02
2	−2	0	−0.294875D−01	0.162597D−01
2	−2	1	0.989340D−02	−0.984482D−02
2	−1	0	−0.285269D−02	0.133790D−01
2	−1	1	0.200583D−01	−0.760429D−02
2	0	0	−0.191681D−02	0.148359D−01
2	0	1	0.174897D−01	−0.314588D−02
2	1	0	0.137323D−04	0.135286D−02
2	1	1	0.141434D−02	−0.435312D−03
4	−5	0	0.526449D−03	−0.239412D−04
4	−5	1	−0.424884D−03	0.689134D−03
4	−4	0	0.132698D−02	−0.226398D−03
4	−4	1	−0.359551D−02	0.273513D−02
4	−3	0	0.728474D−03	−0.335751D−03
4	−2	1	0.563396D−03	−0.136330D−03

Table 3.3: Fourier coefficients of the quasiperiodic final solution which satisfy that $\sqrt{x(i,j,k)^2 + y(i,j,k)^2} \geq 5 \times 10^{-4}$. The values of i and j are the coefficients of ψ and M (see section 2), and k indicates if it refers to a cosine or a sine (0 = cosine, 1 = sine).

−0.562891D+00	+	0.821943D+00$\sqrt{-1}$
−0.562891D+00	−	0.821943D+00$\sqrt{-1}$
0.607630D+01		
0.170180D+00		

Table 3.4: Eigenvalues of the pseudo-monodromy matrix.
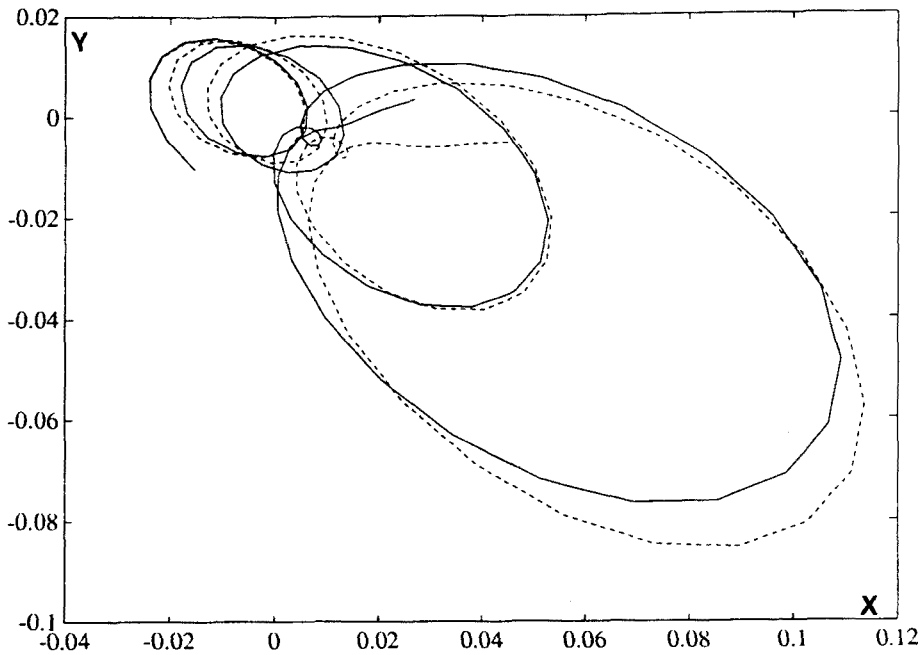
Figure 3.1 Continuous line: semianalytical solution of the linearized system. Dotted line: numerical solution of (3.8) with the same initial conditions as the semianalytical one. Initial epoch: year 2000.0. Time interval: 90 days. Projections on the (x, y) plane with normalized units (1 unit $=$ distance Earth-Moon at the epoch). A polygonal is plotted with a time interval of 1 day between consecutive vertices.
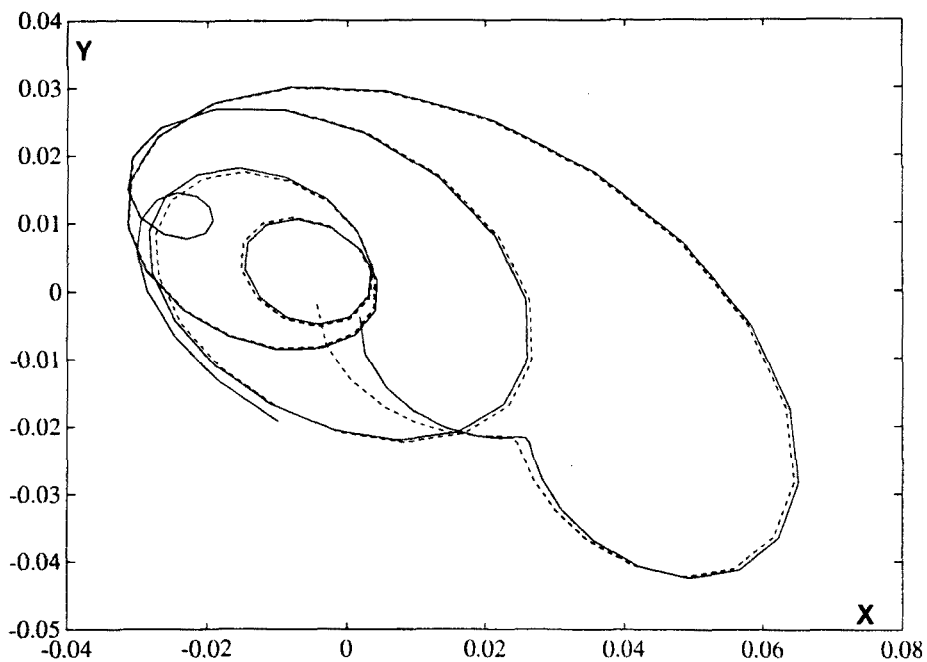
Figure 3.2 Same as Figure 3.1 but the semianalytical solution corresponds to the system up to second order.
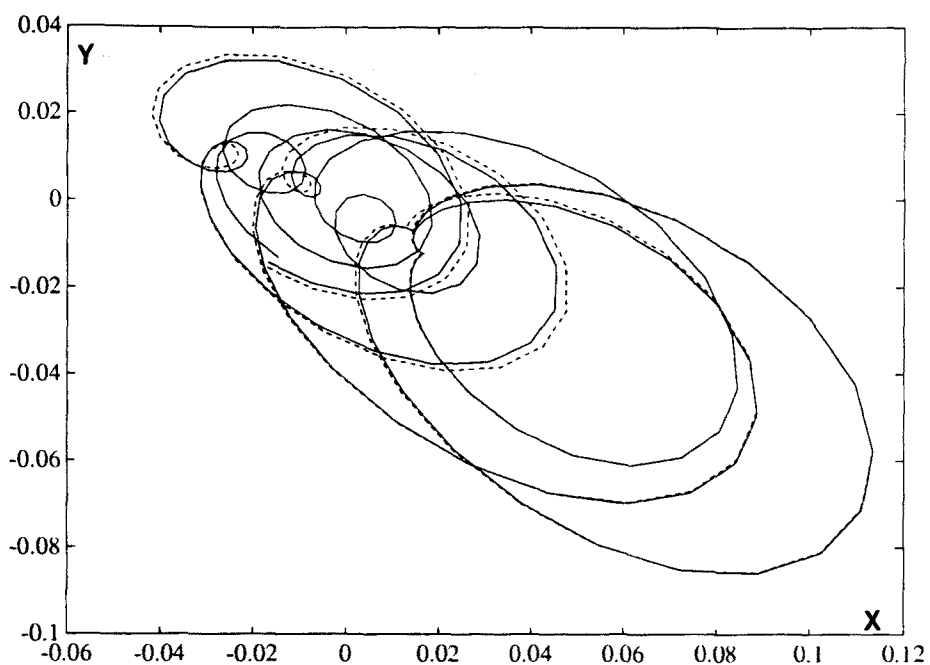
Figure 3.3 Same as Figure 3.1 but the semianalytical solution corresponds to the system up to third order. Time interval: 192 days.
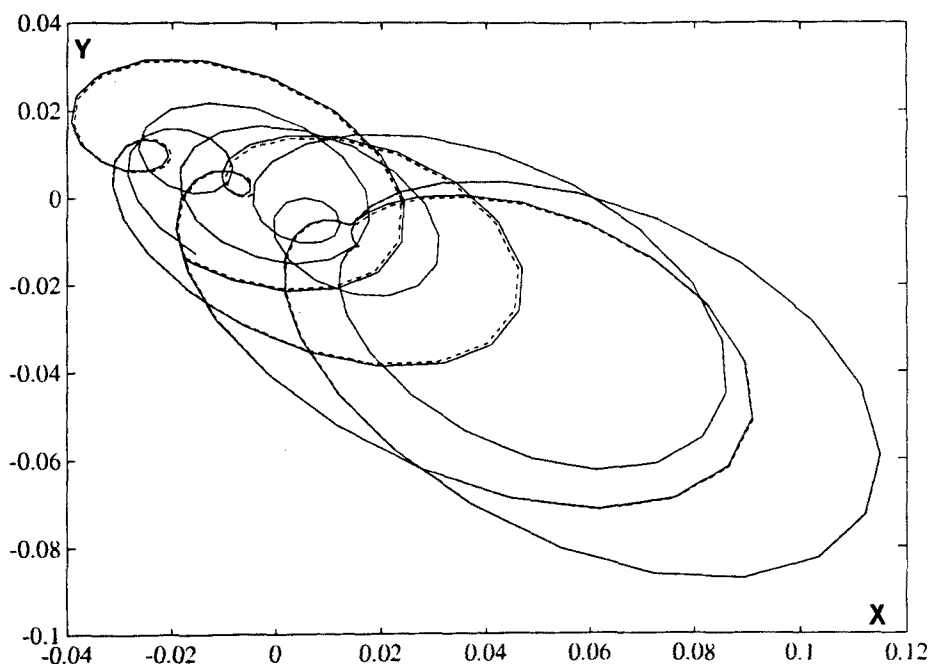
Figure 3.4 Same as Figure 3.1 but the semianalytical solution corresponds to the system up to fourth order. Time interval: 192 days.
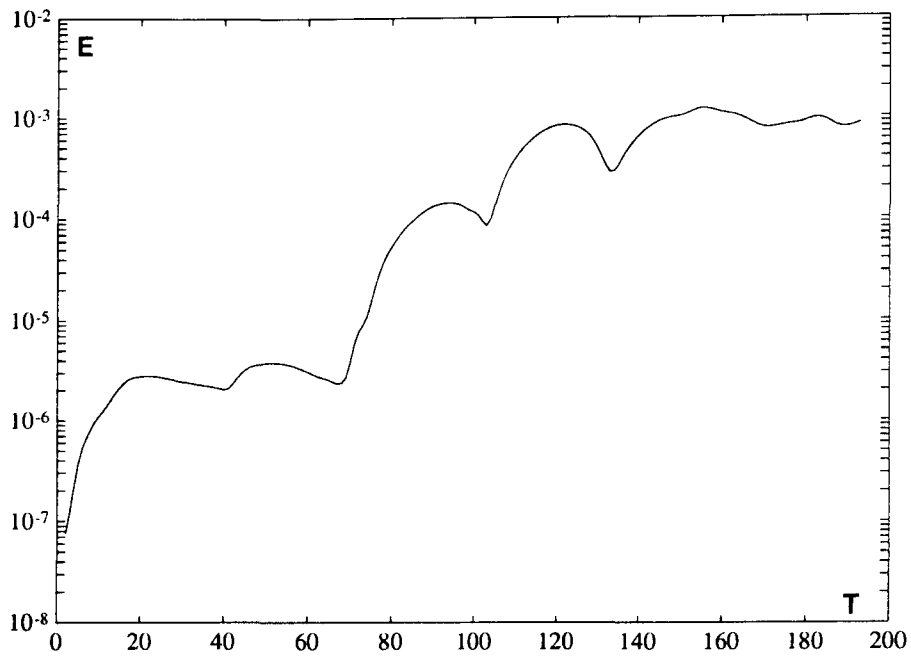
Figure 3.5 Euclidean norm of the error between the solutions plotted in Figure 3.4 in the phase space versus time in days.
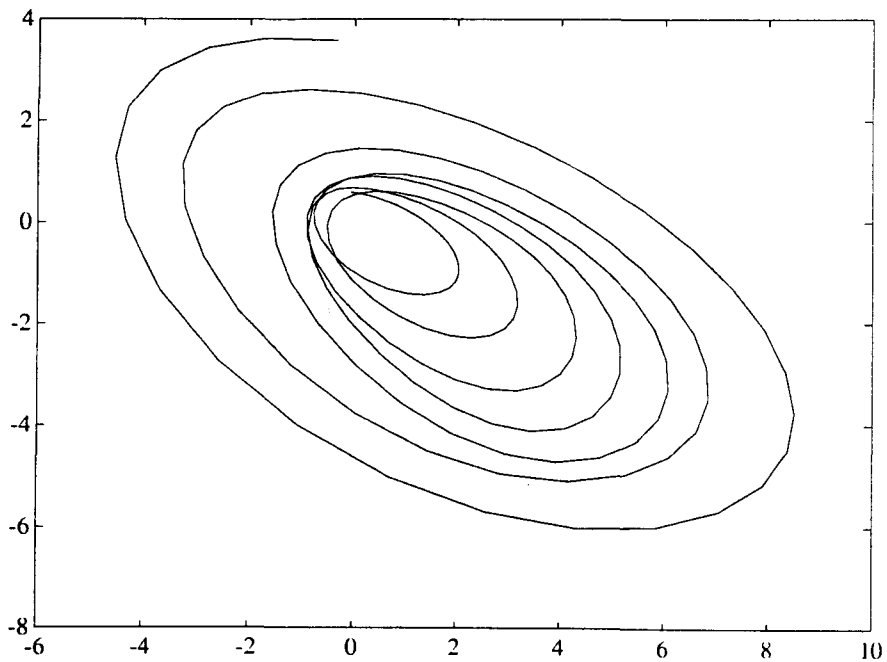
Figure 3.6 Projection on the (x, y) plane of the unstable eigenvector of the pseudo-monodromy matrix for the time span of Figure 3.4. The variational equations associated to (3.8) and the initial conditions of Figure 3.4 have been used. The initial vector is unitary (in the phase space).
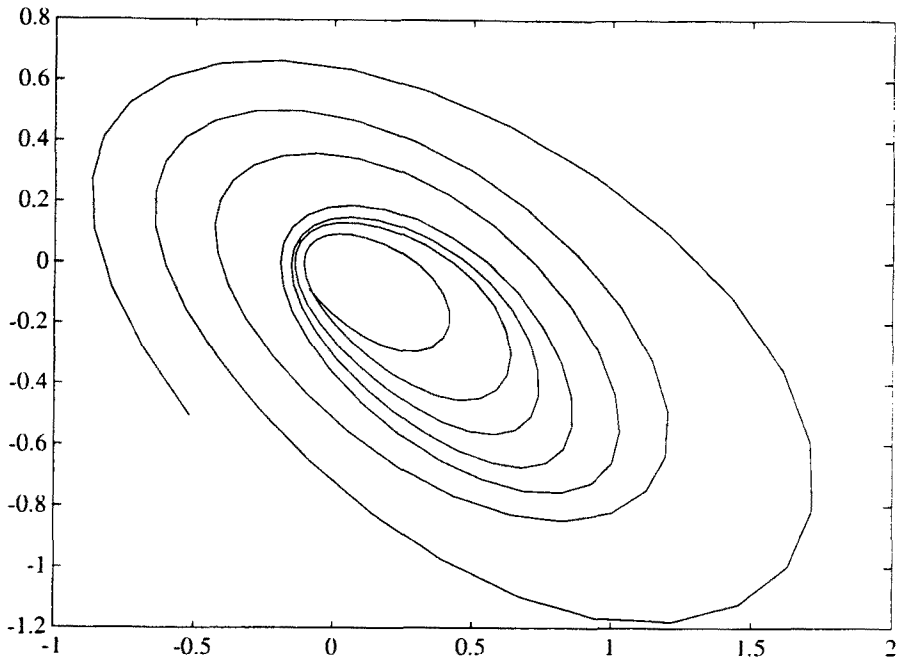
Figure 3.7 Same as Figure 3.6 but for the stable eigenvector.
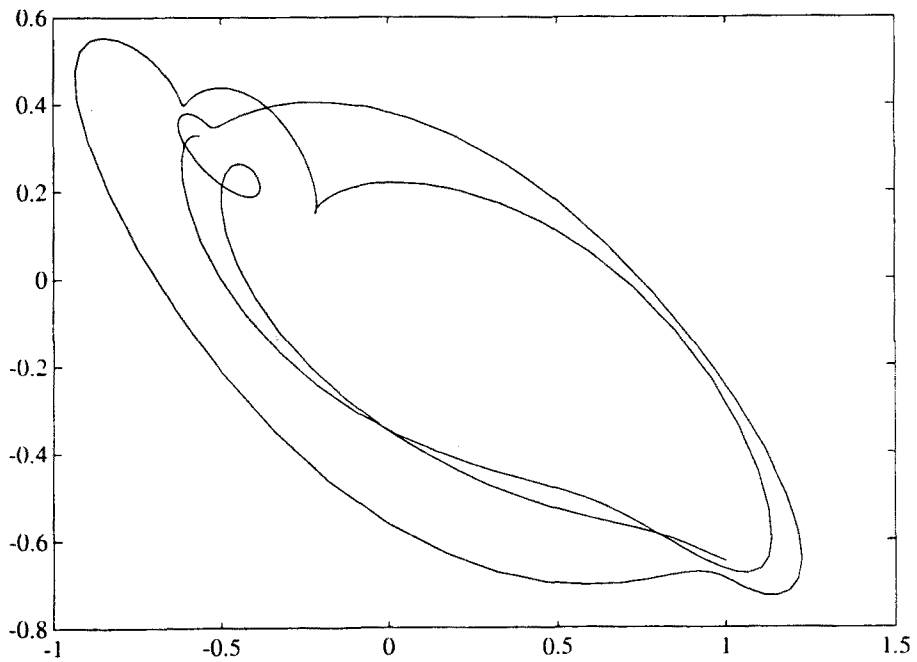
Figure 3.8 Same as Figure 3.6 but for the real part of the eigenvector with complex eigenvalue (of modulus close to 1).
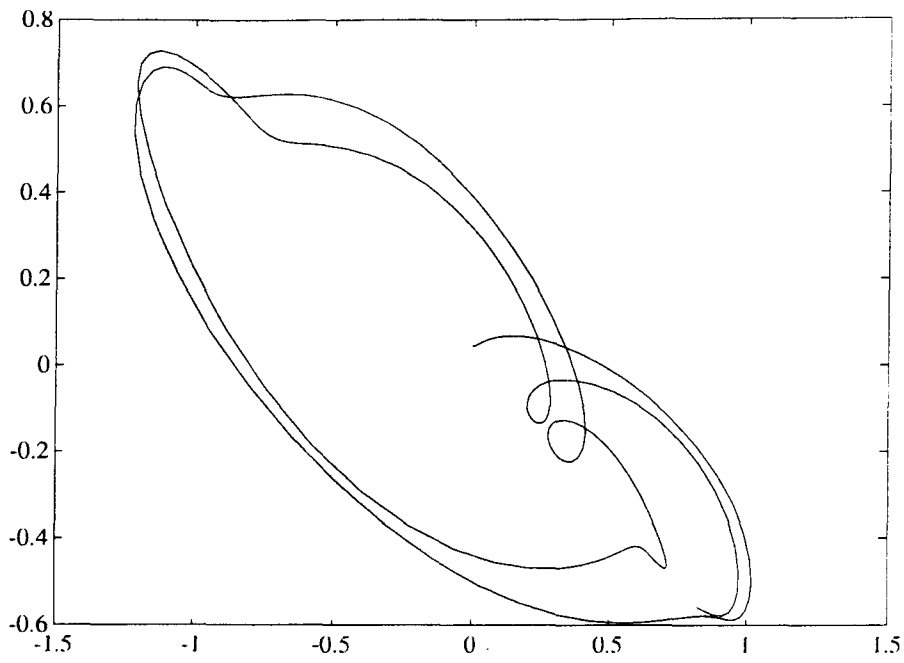
Figure 3.9 Same as Figure 3.8 but for the imaginary part.

3.4 The Spatial Case

Here, the spatial case is considered, and the model given in section 3.1 is studied. As before, we are interested in a quasiperiodic solution of the equations of motion having the same basic frequencies that the perturbation. The algorithm will be the one described in Section 3.2 and used for the planar case. As the algebraic manipulator used for the planar problem is not enough to deal with this case, a new one has been built. It is going to be described in the next sections, as well as the results obtained with it.

3.4.1 The Algebraic Manipulator

Now we are going to describe the algebraic manipulator specially built (in FOR-TRAN 77 language) to solve the latter problem. Here, a trigonometric series is stored as a vector of real numbers (each one corresponding to a coefficient), a vector of integers (of the same dimension) and an integer containing this dimension. This integer is only used in some operations to check an "overflow trigonometric series" condition. An auxiliary table, with the coefficients of the linear combinations of the basic frequencies, will also be used by the manipulator.

Storing Fourier Series

We need to store trigonometric series and handle them in a fast way. Moreover, we want the size of the vectors that contain them to be as short as possible. In order to get the latter condition, we have decided to store only the non-zero coefficients, positioned in a contiguous way (as a stack). This means that we must have a way to recognize those coefficients, because their position inside the vector does not allow to identify them.

For this reason we have taken an auxiliar integer table of frequencies which contains, for each column, different integer values. For example, if a column contains the integers $(k_1, \ldots, k_r)$ it means that it refers to the frequency $k_1\omega_1 + \cdots + k_r\omega_r$, where $(\omega_1, \ldots, \omega_r)$ is the vector of basic frequencies (obviously, the number of rows must be at least the number of frequencies). This table will be used in two different ways:

1. Given a position, to find the corresponding coefficients of the frequencies.

2. Given an integer vector, to search for its position inside the table.

The first use is easy and fast to do, but not the second one. At this point, it would be useful that elements of the table were given in some order, because we could use a binary search. One could think that it would be enough to fill the table at the beginning of the program using the selected order, but this does not work very well,

due to the fact that when we start the program we do not know yet which frequencies
will appear in the intermediate computations. We need to add frequencies to the
table during the process, and we need to restart the order in a quick way. For this
reason we have added to the table an integer vector which refers to the table in
an ordered way and, when a new frequency is added (at the end of the table), the
program only needs to modify this vector to get the table in order again. The order
selected has been lexicographic:

$$(k_1^1, \ldots, k_r^1) < (k_1^2, \ldots, k_r^2) \Leftrightarrow \begin{cases} k_1^1 < k_1^2 & \text{if } k_1^1 \neq k_1^2 \\ (k_2^1, \ldots, k_r^1) < (k_2^2, \ldots, k_r^2) & \text{if } k_1^1 = k_1^2. \end{cases}$$

Now, let us see how we can use this. We suppose that the real vector of coefficients
is called x and we define an integer vector called nx. The first component of x
is always used to store the independent term (the average). The first component
of nx is used to store the number of non-zero coefficients of the series (the last
meaningful component of vector x). Now, let j be an integer less than or equal to
nx(1) and we consider the coefficient x(j). If nx(j) is even, x(j) corresponds to
a cosinus, otherwise it corresponds to a sinus. In the column m=nx(j)/2 (integer
division) of the table described above we can find the frequencies corresponding to
this coefficient.

Now we are going to see how all of these things have been programmed. In what
refers to storing frequencies in the table, we restrict ourselves to the ones that have
the first integer coefficient (k_1 using the above notation) greater than or equal to
zero. If we have a frequency with $k_1 < 0$, we change its sign and, in the case that
it corresponds to a sinus, the sign of the coefficient is also changed. If $k_1 = 0$ we
impose that the latter condition holds for k_2. If $k_2 = 0$ we impose it to k_3, and so
on.

All of these arrays have been included inside a common called taules:

<div style="text-align:center">common /taules/ taufre,ntau,ordtau,busc</div>

taufre has been declared as integer*2 and ntau, ordtau and busc as integer*4.
taufre(r,n) contains the table of frequencies (n is the maximum number of them
and r the number of basic frequencies), ntau is the actual number of frequencies in
the table, ordtau(n) is the vector that holds the order of taufre and busc(2,m)
is an array used to get a quicker binary search. Let us see how it works. Given a
frequency $(k_1, \ldots, k_r)$, $k_1 \geq 0$ the numbers busc(1,k_1+1) and busc(2,k_1+1) refer
to the first and last place (of ordtau) where this k_1 appears, and then the search
field is shorter. If there do not exist any frequencies with this k_1 the corresponding
values of busc are -1.

Finally, we are going to comment on some points about another system of storing
Fourier series used in some intermediate computations. As we will see later, in many
operations the manipulator uses a real array as a workspace. This vector has twice

more components that the table of frequencies and it is used to hold series in such a way that it allows to make sums faster. This is due to the fact that the position of a coefficient in this vector plays the same role that the index vector seen before. For example, with the same notation as before, the coefficient x(j) is put in the component nx(j) (this is only true for j > 1, x(1) is put in the first component of the working vector). Note that if we have two vectors of this kind, the sum of them can be done by adding, component by component, the two vectors. The dissadvantage of having many vectors like this is the waste of memory that they require, because many of this components may be zero. For this reason we prefer to keep the Fourier series in the packed form seen before. We have only one working vector and we pass it to routines that make use of it.

Basic Subroutines

The following routines have been built with the aim of having an easy-to-use system to handle the tables of frequencies.

Subroutine FREQ The first parameter is an **integer*4** variable coming from a Fourier series (for example, using the above notation, some value nx(k)) that contains information relative to a coefficient. Unchanged on exit. The second parameter is an **integer*4** array that will contain on exit the frequencies corresponding to n, and the third parameter will contain 0 if it refers to a cosinus or 1 if it refers to a sinus.

Subroutine POSI This routine is inverse of the former one: it returns an integer which contains information about the frequency (following with the example, information to be stored in the vector nx). The first parameter is an **integer*4** vector of frequencies and the second one is an integer containing 0 or 1. These parameters are unchanged on exit. The routine returns an integer value n to be stored in an integer vector corresponding to a Fourier series. The routine performs a binary search of fv in the table of frequencies and, assuming that fv is found in the place j, makes n=2*j+i. If fv is not found (i. e., it is not in the table) the routine calls the subroutine ACTUAL.

Subroutine ACTUAL This routine adds a frequency to the table. The first parameter is the vector containing the frequency to be added and the second one (lloc) is the place (of ordtau) where this frequency must be (this value has been found during the binary search of the routine POSI). The elements which previously were stored in ortau, in places from lloc up to the end, are shifted one place to the right. Both parameters are unchanged on exit.

Subroutine EMPAQ As we have already seen, in many operations the manipulator uses a real array as a workspace. This routine packs a trigonometric series

contained in the working vector.

Subroutine NORMAL The first parameter is an integer vector (f) containing a frequency, the second one is an integer saying how many frequencies we have. With this, the routine puts f in normal form (the first component non equal to zero must be positive) and returns is=-1 if it has changed the sign of f, or is=1 otherwise. If all of the components of f are equal to 0 it returns is=0. This routine is called during some operations (for example, a product of series).

Subroutine WFREC This routine returns the numerical value of the frequency corresponding to a place (given by means of a parameter) of the working vector.

Subroutine INITAU This routine initializes the tables used by the manipulator. It must be called at the beginning of the program, before making any other call to a routine of the manipulator. It has no parameters.

Subroutine AVALUA This routine computes the numerical value of a trigonometric series at a given time value t.

Subroutine LLEGEIX Given a channel (already opened), this routine reads a Fourier series through this channel and returns it in packed form.

Subroutine ESCRIU Given a channel (already opened), this routine writes a Fourier series given in packed form.

Making Operations

Now we are going to describe the main operations performed by the manipulator, which include sums, multiplications of Fourier series and multiplications of Fourier series by real numbers, differentiation and some special operations that we shall describe later. To keep only the meaningful terms of the series, we have included a threshold that allows to drop terms less (in absolute value) than this threshold, which is in a common named control.

Subroutine SUMA This routine performs the sum of two Fourier series. The method consists in expanding the first series inside the working array and then add the second series putting the results in the same working vector. Finally, the result is packed (this routine calls SUMAF).

Subroutine SUMAF Given a Fourier series in packed form and a Fourier series in expanded form, this routine adds them and puts the result in the same vector that contains the series in expanded form (the vector in expanded form

is the working vector mentioned in section 3.4.1). This routine has been built to increase the speed of operations of type $a = a + b$.

Subroutine PROD This routine obtains the product of two series. The way to do this is to fill a working vector with zeros and to call the routine PRODF (see below). This routine returns the result inside the working vector and finally it is packed.

Subroutine PRODF Given two Fourier series in packed form (we can call them a and b) and a Fourier series in expanded form called w, this routine performs the operation $w = w + a \times b$. To do this, each term of the first series is multiplied by each term of the second one. If the absolute value of the resultant coefficient is less than the threshold, it is dropped, otherwise the frequencies of the terms that are multiplying are computed. With this, the result is added to the corresponding position of the working array. As SUMAF, this routine has been built to increase the speed of some operations.

Subroutine PR1C This routine performs the product of a series containing only one term (whose coefficient is equal to one) by a Fourier series in packed form. The result is returned also in packed form. The unitary series is given by a number displaying the frequencies and if it is a sinus or a cosinus (this number is of the same kind as the components of the index vector (nx) seen before). The routine works with the same algorithm as PROD. This operation is needed a lot of times during the calculus of the Jacobian matrix. This routine can be avoided, but it is easy to write and allows to simplify the main algorithm (we do not need to fill a trigonometric array with an unitary series and to call PROD).

Subroutine PR1CF This routine makes the same operation as PR1C, but adding the result to the series contained in the working vector. This allows us to get more speed during some operations.

Subroutine PRESC This routine multiplies a real number by a trigonometric series. This is a very common operation in the main algorithm and we preferred to write it instead of using the routine PROD.

Subroutine DF This routine computes the derivative of a Fourier series as usual: for each term the integer vector k containing the frequency is obtained, and the coefficient is multiplied by (k, ω) (the vector ω is stored in a common called freque) and if it is a cosinus, its sign is changed.

Subroutine MDF2 This routine performs the second derivative of a trigonometric series, and multiplies it by -1. Note that it is faster to compute this directly

than to call routine DF twice and then to call PRESC. The algorithm is of the same kind that in the latter routine.

Subroutine NORMA This routine returns the norm of a Fourier series. The norm that we have selected is the euclidean norm of the vector containing its coefficients.

3.4.2   The Newton Method

Now, using the manipulator described above, it is possible to write a program that looks for a zero of the function G defined before, by means of a Newton method. For this purpose we need an initial condition close enough to the solution that we are looking for. Due to the fact that the perturbation is too big to take the zero solution as initial condition, we are going to use a Newton continuation method. For this reason we have added a continuation parameter (h) multiplying the non constant terms of the perturbations. Thus, when h is equal to 0 we have an autonomous differential system with an equilibrium point near the origin (the equilibrium point is not the origin itself because the constant terms of the perturbations are not multiplied by h). This point is close enough to the origin to be found by the Newton process starting from the zero solution. Then we increase h and this point becomes a small quasiperiodic orbit (see Chapter 2) that can be found by the Newton method (otherwise the value of h must be reduced to an intermediate value). Now the value of h is increased again and the Newton process is started from the solution found for the last value of the parameter. If no problems appear (that is, bifurcations and/or turning points), this process can continue until h reaches the value 1. We shall come back to this point later.

Some Remarks about the Jacobian Matrix

Due to the big quantity of coefficients appearing in the Fourier series, it is impossible to include all of them in the Newton method. For this reason we have selected some of them. The selected elements are stored in integer vectors (we store the indices corresponding to them as the vector nx seen before) and put in a common called frenew. Because of this, we have two different values of the function G: the ordinary value and the one obtained taking into account only the terms included in the Newton method. Note that we are solving with respect to this last value.

Adding terms to the Newton Process

The criterion to select the coefficients that will take part in the Newton process is based on the size of the amplitudes of the residual acceleration. At the beginning only the terms appearing in the initial condition are selected (if this is constant

and equal to zero, only the independent terms take part in the process). When the reduced equation (that is, taking into account only the terms of the Newton process) is solved, the program checks the size of the amplitudes of the terms of the residual acceleration that are outside the Newton process. If it finds that some of them are larger than a given threshold, then they must be added to the Newton process, else this step of the continuation method is finished and we can start the next one.

We want to make a remark about adding terms to the Newton process. If we add too many terms to it, we can find out that the neighbourhood of convergence is too small and we are out of it. In order to avoid this, we have a control on the maximum number of terms going in the Newton process, and if this number is reached, the threshold is increased. We have also implemented a similar procedure if the program is going to take a few new terms for the Newton method. Of course, in this case the corresponding threshold is reduced.

High Level Routines

We start writing a routine that evaluates the function G. This is not a problem, using the algebraic manipulator and a series expansion for the terms coming from the RTBP. We also have continuation parameters in the terms of each degree and an integer variable containing the actual degree of the power expansion. These parameters are stored in a common called `conti`. This allows to select the degree at which the solution is wanted.

Then, we need a routine that computes the Jacobian matrix of G. This is easy using the chain rule as mentioned in 3.2 and the polynomial expansion of the equations. Note that in this part we also have the continuation parameters for the expansion. Moreover, in order to increase the speed, we realize that some powers and products of series that we need in this part have already been done during the evaluation of G, so, we keep these results in memory.

Once the Jacobian matrix is filled up, we need to solve a linear system with a big matrix (note that this matrix is not sparse at all). This system involves only the terms selected before. For this purpose, we have written a routine adapted to a paged environment. The method used is based on the Gaussian elimination.

Due to the fact that we do not know the stepsize for the continuation parameter, we have implemented an automatic stepsize control. This works in the following way: when the Newton method does not find the zero in a few iterations (this number is supplied by a control input file), it assumes that the continuation parameter has been increased too much. Then the program recovers the solution corresponding to the last value of the parameter (it has been stored previously by the program in a disk file), assigns to the parameter an intermediate value and starts the process again. Of course, if the difference between this intermediate value and the last successful one is less than a fixed threshold (given also by a control file) the program stops

issuing the corresponding error message.

Subroutine INIPOT This routine computes the required powers $x^i y^j z^k$, where (x, y, z) is a vector of Fourier series, and stores the result (in the common /pow/) to be used by the routines that compute the function G and its Jacobian matrix. If n is the actual degree of the expansion we are working with, then the powers are computed and stored (except those of degree one) up to order $n - 1$. This is due to the fact that the powers of order n are only needed once (during the computation of G), while the ones of order less than n are used many times (in the computations of G and its Jacobian matrix). The way to obtain the powers of x, y and z is the following: powers of orders less or equal than three are computed "explicitely" (using the algebraic manipulator, we have written directly the expressions for order two and, using these results, the ones of order three). The powers of degree greater or equal than four are obtained as a product of two powers previously computed. This routine calls POSAP, TREUP and SPLIT.

Subroutine POSAP This routine stores a Fourier series $a = x^i y^j z^k$ in the common /pow/. The parameters are the Fourier series a, ia, na (see section 3.4.1) and i, j, k. The routine puts the Fourier series in the vectors real*8 pot and integer*4 npot. This common also has an integer*4 value containing the first free component of the vectors, in order to add the Fourier series at the end of the series already contained in the vectors. Once the series is added, this value is updated. To recover these series we need to write down where they are. This is done by means of the routine LLP (see below): given the integer vector of exponents i, j and k, this routine returns an integer value l. Then, in the column l of the array integer*4 inpo(2,161) we store the first and the last component of pot and npot in which we can find that Fourier series.

Subroutine LLP Given the integer vector (i, j, k), the routine returns the column of inpo in which we can find the first and the last components of the arrays pot and npot containing the value $x^i y^j z^k$. More concretely, if $n = i + j + k$, then the returned value is $(((n+3)n+2)n-24)6+i(n+1)-((i-1)i)/2+j+1$.

Subroutine TREUP This routine extracts the Fourier series of $x^i y^j z^k$. In order to know where it is stored, a call to the routine LLP is done.

Subroutine TREUPP This routine works exactly as TREUP but returning the result multiplied by a given real*8 variable. It is used by the routine MONOMI (see below).

Subroutine SPLIT Given an integer vector (i, j, k), this routine returns two integer vectors (i_1, j_1, k_1) and (i_2, j_2, k_2) such that $i_1 + i_2 = i$, $j_1 + j_2 = j$, $k_1 + k_2 = k$

and satisfying $i_1 + j_1 + k_1 = E(n/2)$ (E=integer part), where $n = i + j + k$. This routine is used to know which powers we must multiply to get the power we want. This is used to obtain powers that are not in the common /pow/ (this happens with the powers of maximum order and during the filling of pow).

Subroutine MONOMI Given an integer vector (i, j, k), a real number p, the Fourier series x, y and z and a working vector w containing a Fourier series, this routine returns $w = w + p \times x^i y^j z^k$, where w is the working vector. Before calling this routine a previous call to INIPOT (to compute the required powers of x, y and z) is needed.

Subroutine CORXET1 This routine computes the value between brackets ([ ]) in equation (3.1), by means of calls to MONOMI. The value is returned in expanded form inside a working vector.

Subroutine CORXET2 Like CORXET1, but for equation (3.2).

Subroutine CORXET3 Like CORXET1, but for equation (3.3).

Subroutine FUNC Routine that computes the value of function G (see page 54), calling routines CORXET1, CORXET2, CORXET3 and using the manipulator to compute the other terms appearing in the equations.

Subroutine DFUNC This routine computes the Jacobian of function G, calling the following routines:

1. D1X: it computes $\dfrac{\partial G_1}{\partial x}$.

2. D1Y: it computes $\dfrac{\partial G_1}{\partial y}$.

3. D1Z: it computes $\dfrac{\partial G_1}{\partial z}$.

4. D2X: it computes $\dfrac{\partial G_2}{\partial x}$.

5. D2Y: it computes $\dfrac{\partial G_2}{\partial y}$.

6. D2Z: it computes $\dfrac{\partial G_2}{\partial z}$.

7. D3X: it computes $\dfrac{\partial G_3}{\partial x}$.

8. D3Y: it computes $\dfrac{\partial G_3}{\partial y}$.

9. D3Z: it computes $\dfrac{\partial G_3}{\partial z}$.

Subroutine SISTEMA This routine solves a linear system by means of LU decomposition (see [21]). The operations are done taking into account that FORTRAN stores the matrices as a sequence of columns, and trying to avoid big jumps inside the matrix.

3.4.3 The Program

The program uses many data files in order to read the values of the basic frequencies (the values $\omega = (\omega_1, \ldots, \omega_r)^t$), some phases related to them, control parameters (threshold to drop terms and to stop Newton method, maximum number of iterations and continuation parameters) and coefficients of the polynomial expansions of the equations. The program also reads a file containing the initial condition to start all the process (usually, it is taken equal to zero). During the execution, the program creates a file (called NEWTON.TMP) in which the actual solution is stored. This is due to the fact that if the new step of continuation does not succeed, the program can recover the last succesful solution (for the last continuation parameter, of course) to start again with a smaller parameter. This file is usually small (less than 20K). During the run, the program writes an execution log in a file called QPL4.CON. If the execution is succesful, the result is stored in a file called QPL4.RES.

Subroutine INIEXP This routine reads the coefficients of the power expansion of the vectorfield from the files EXPAN.1, EXPAN.2 and EXPAN.3 for the first, second and third equations, respectively.

Subroutine LLEQ This routine reads one perturbing function P (from the data file QUS.COE), and multiplies the non-constant terms of it by the continuation parameter h.

Subroutine INQUS This routine reads all the perturbing functions stored in the file QUS.COE and multiplies the non-constant terms of them by the continuation parameter. To do this, the routine LLEQ is used.

Subroutine INIQP This routine initializes all the program. It reads the values of the basic frequencies (and phases) of the perturbations (from the file FREQUE.VAL), the control parameters for the Newton method (from the file CONTROL.QP), the coefficients of the power expansion of the vectorfield, the values of the perturbations and the initial condition (from QPL4.DAD). This routine calls INIEXP and INQUS.

Subroutine QAF This routine returns the number of terms of the residual acceleration with amplitude larger (in absolute value) than a given threshold.

Subroutine AFE This routine adds the terms of the residual acceleration whose amplitude is larger (in absolute value) than a given threshold to the set of variables and to the equations to be solved by Newton's method.

Subroutine CHECK This routine is called before adding terms to the Newton process in order to avoid having out-of-range arrays.

Subroutine CTI From the residual acceleration (the value of G), this routine computes the independent term of the linear system that appears in the Newton method. This step needs a "conversion" from Fourier series to an array, using the vectors of coefficients that take part in the Newton process.

Subroutine NOUP This routine computes the new approximation to the solution of $G = 0$, using the solution of the linear system. As in CTI, here we need a "conversion" from an array to Fourier series.

Subroutine SCOOBY This routine looks inside the residual acceleration and searches for amplitudes larger (in absolute value) than a threshold real*8 tol, and adds them to the Newton process. The number of terms added must be between the integer*4 values mitn and matn, modifying the value of tol if necessary. This routine calls the routine AFE.

Subroutine CNFNA This routine controls all the Newton process. It checks the actual step of the process and takes the following decisions:

- If the residual acceleration is not small enough, the new independent term is computed.

- If the value of G is small enough, it looks for "big terms" inside the residual acceleration (calling SCOOBY), adding them if necessary.

- If G is small enough and there are not terms inside the residual acceleration big enough to be added to the process, the continuation parameter is moved.

- If the number of iterations with this value of the continuation parameter is too large, it sets a flag telling the routine NWTA to recover the last succesful solution, and the continuation parameter step is halved.

The routine returns a flag containing what it has done, because the routine NWTA (this is the one that calls CNFNA) needs to know what is happening (see below).

Subroutine NWTA This routine performs the Newton method. It computes G and then calls CNFNA in order to know what to do. Depending on the value of the flag returned by this routine, the action taken can be:

- To continue with the process computing the Jacobian matrix (DFUNC), the independent term (CTI), solving the linear system (SISTEMA), obtaining the new approximation (NOUP) and computing again G.

- To overwrite the last succesful solution stored in the disk file NEW-TON.TMP with the solution obtained for the actual value of the parameter and to compute G again (the continuation parameter has been increased inside CNFNA).

- To recover the last succesful solution from NEWTON.TMP and to compute G again (the continuation parameter has been moved inside CNFNA).

- To return the control to the main program because the final solution has been found.

Main Program Here we can find all the declarations of the arrays. This program calls INIQP to initialize the execution and then it calls NWTA to perform the Newton method. Finally, it opens the file QPL4.RES and writes in it the final result.

Test of the Program

In order to check all of these routines, we have solved again the planar problem (see section 3.3).

We start by setting the continuation parameter of the perturbation (h) to 1, and we select the degree of the power expansion equal to one. With this, we get a linear problem that the Newton method can solve easily. Then we add the quadratic terms (without need of any kind of continuation), and finally the third and fourth order terms. We have made another tests starting with the power expansion up to order four and moving the continuation parameter h from 0 to 1. All the results obtained with these tests are in good agreement with the ones mentioned in 3.3

3.4.4   Results of the Algebraic Manipulator

We have done several runs of this software package. Initially, we tried to use the same scheme as in [7], that is, to fix the continuation parameter h to 1, and the degree n of the power expansion to 1. With this we got a linear problem that the Newton process solved without troubles. Then we added the second degree terms and by using continuation we could solve this problem, but the number of terms taking part in the Newton process was very large and we had many troubles with the memory and swap needed by the program. Then the third degree terms could not be added directly (this is because they are meaningful in this problem) and we had to use continuation again. During this continuation the number of terms inside the Newton process became so high that we started to have troubles with the capacity

of our computer. Nevertheless, the solution obtained by this method is good enough to be refined numerically by means of a parallel shooting (see [21] for an explanation of this numerical method).

Due to the fact that this method does not seem to be the best to solve this problem (computer requirements are very high) we have implemented continuation for the perturbation (as it has already been mentioned during the description of the software). With this, we have obtained better results than the ones mentioned in the last paragraph. As an example, starting with the vectorfield expanded up to order six and giving to the Newton method a threshold $tol = 5 \times 10^{-5}$ (less than the accuracy of the model) to select the coefficients that will take part in the process, the program goes from continuation parameter $h = 0$ to $h = 1$ without any troubles, and the amount of time and memory needed are much less than before.

Taking as initial condition the one supplied by this analytical approximation, we have integrated the model equations and Newton's equations of motion (using the JPL Ephemeris). In Figure 3.10 the solution given by the algebraic manipulator is compared with the integration of the model equations. The integration starts at the epoch 2000.0 and lasts for 1 year. Plots of the solution obtained integrating Newton's equations with the same initial conditions, for a time span of 2 and 5 years, are given in Figures 3.11 and 3.12 respectively. From now on, we shall call this the planar solution because it is close to the (x, y) plane.

Note that after two years the solution is still contained in a domain of moderate size around L_4, but after five years it has grown up to a rather large domain. This is due to the fact that the initial conditions are not good enough at this step and to the inherent instability of the motion around L_4. Furthermore one can guess (specially in Figure 3.12, (x, z) projections) a sequence of Lissajous patterns. They suggest the existence of invariant unstable tori.

Now, if we try to improve the planar solution, supplying this solution as an initial contition for the manipulator, setting the continuation parameter h to 1, selecting the degree of the approximation to six (the program can handle up to degree nine, but the terms of degree larger than six can be neglected due to the actual size of the solution) and fixing the criterion to take terms inside the Newton process to $tol = 1 \times 10^{-5}$ (we stress that this improvement is relative to the model equations, not to the JPL model), we find out that, though the residual acceleration of the initial condition is very small, the method diverges. If we try to proceed as for the planar solution (starting from $h = 0$) but with $tol = 1 \times 10^{-5}$, we find out that the Newton process starts to decrease the continuation step until the program stops because the continuation step is too small (this happens for a continuation value near 0.98). Now if we take the intermediate solution in which the process has stopped as an initial condition, we set again $h = 0$ but we fix the value $tol = 1000$ (we do not allow more terms inside the Newton process than the ones of the initial condition) and we start again the program (with this we are dealing with a system

of algebraic equations of degree six and with fixed dimension) we find that near the point $h = 0.98$ the Jacobian of G has a change of sign. This is related to some bifurcation or turning point.

This last point seems related to the following fact: take the model equations and average them with respect to time. Then, this autonomous system has a periodic solution (see Figure 3.13) whose frequency is rather close to the difference between the first and the third basic frequencies of the model equations (they are, respectively, the mean longitude of the Moon (equal to one, because of the choice of the units) and the mean longitude of the ascending node of the Moon). This orbit seems to be related to the vertical oscilations of the RTBP. When the non-constant part of the perturbation is added, this periodic orbit bifurcates to a quasiperiodic orbit having the same basic frequencies than the ones of the perturbation. This leads us to the fact that the problem of finding a quasiperiodic solution near L_4 has lots of solutions, but probably only a few of them have as basic set of frequencies the one of the excitation. Furthermore only a part of the last ones can be obtained as a natural continuation of L_4.
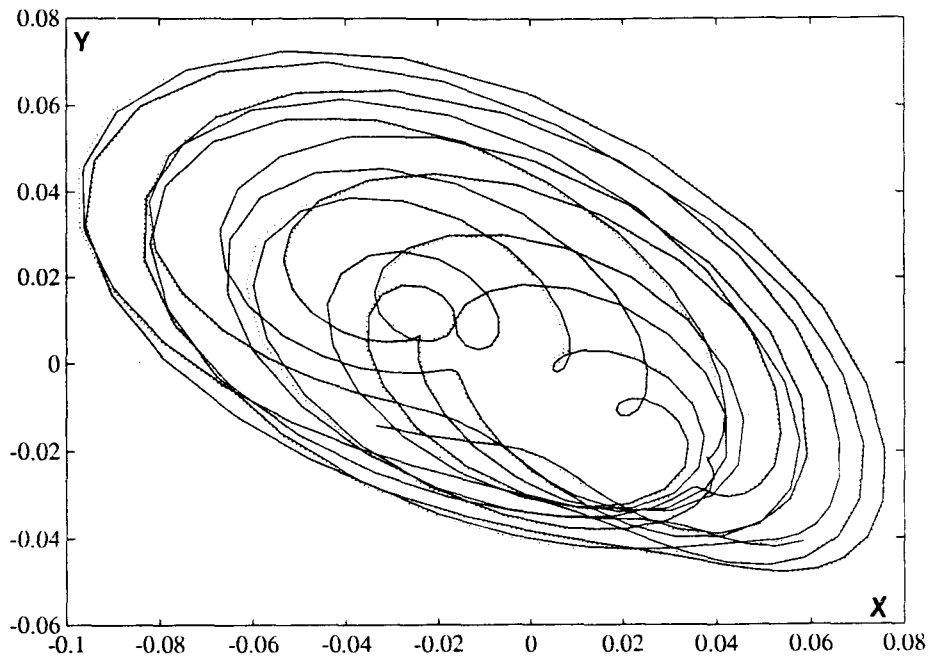
Figure 3.10 (a) (x, y) projection of the solution obtained integrating the model equations (dotted line) and the values of the solution provided by the manipulator (solid line).
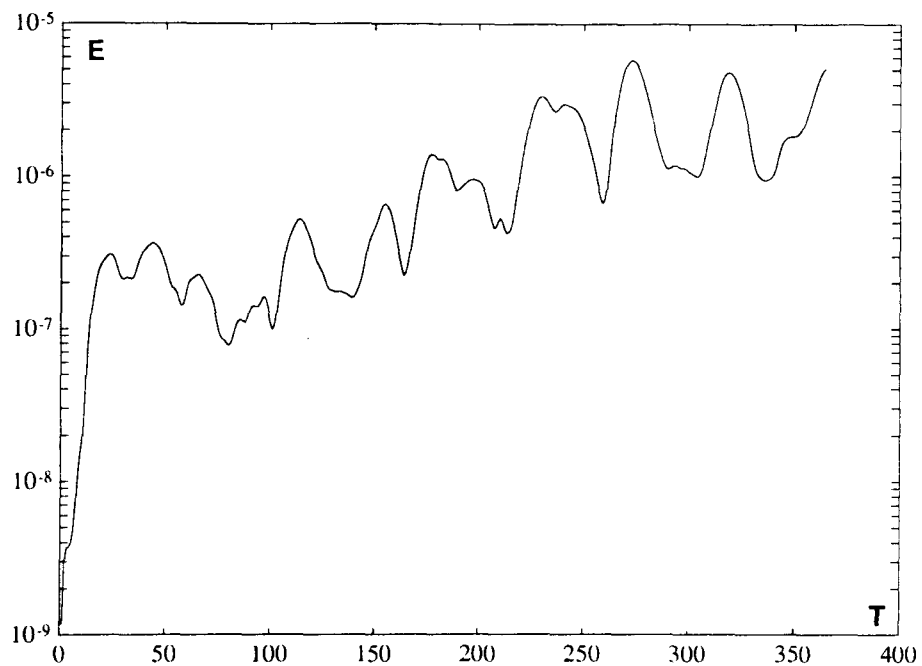
Figure 3.10 (b) Euclidean norm of the differences (in the phase space) between the solution obtained by integration and the one given by the manipulator.
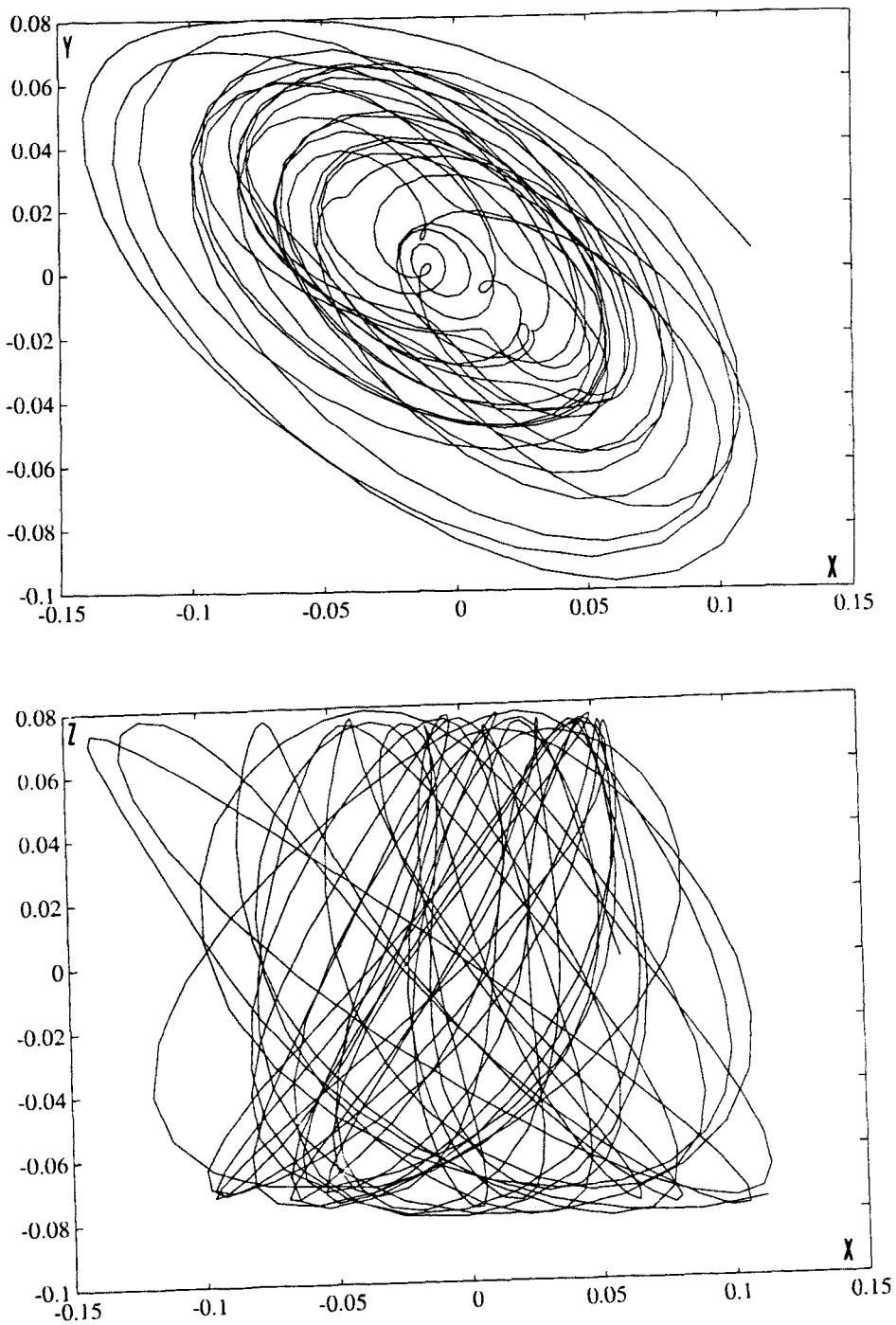
Figure 3.11 Projections of the solution obtained integrating Newton's equations of motion for a time span of 2 years. The initial condition is the one given by the manipulator and it has not been refined.
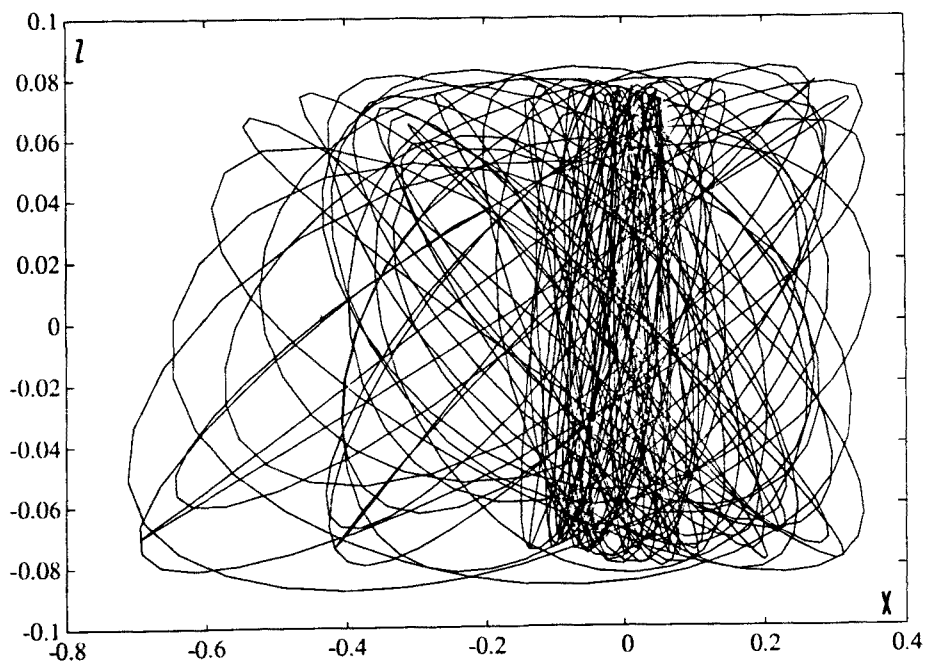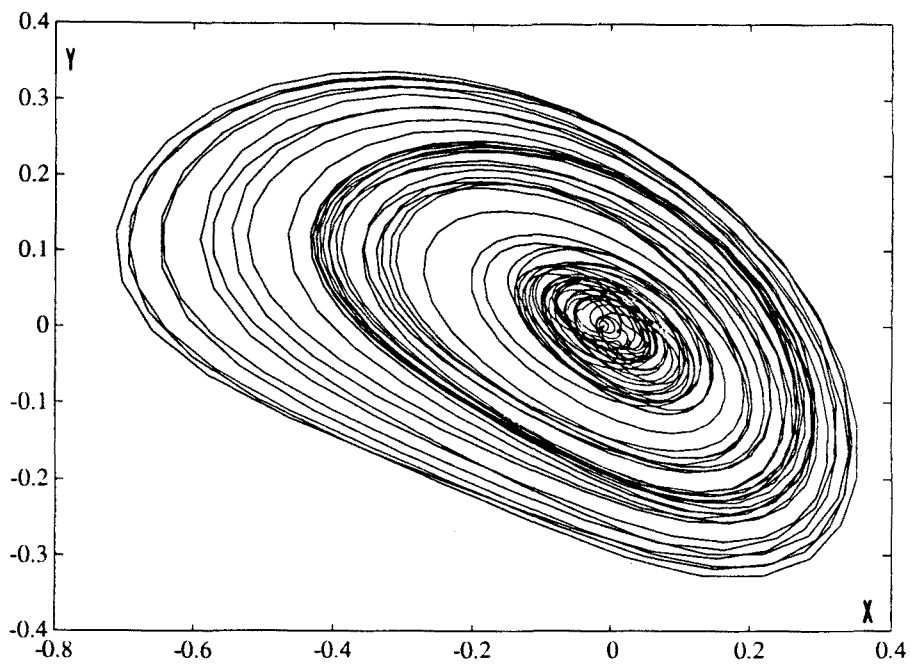
Figure 3.12 Same as Figure 3.11 but for a time span of 5 years.
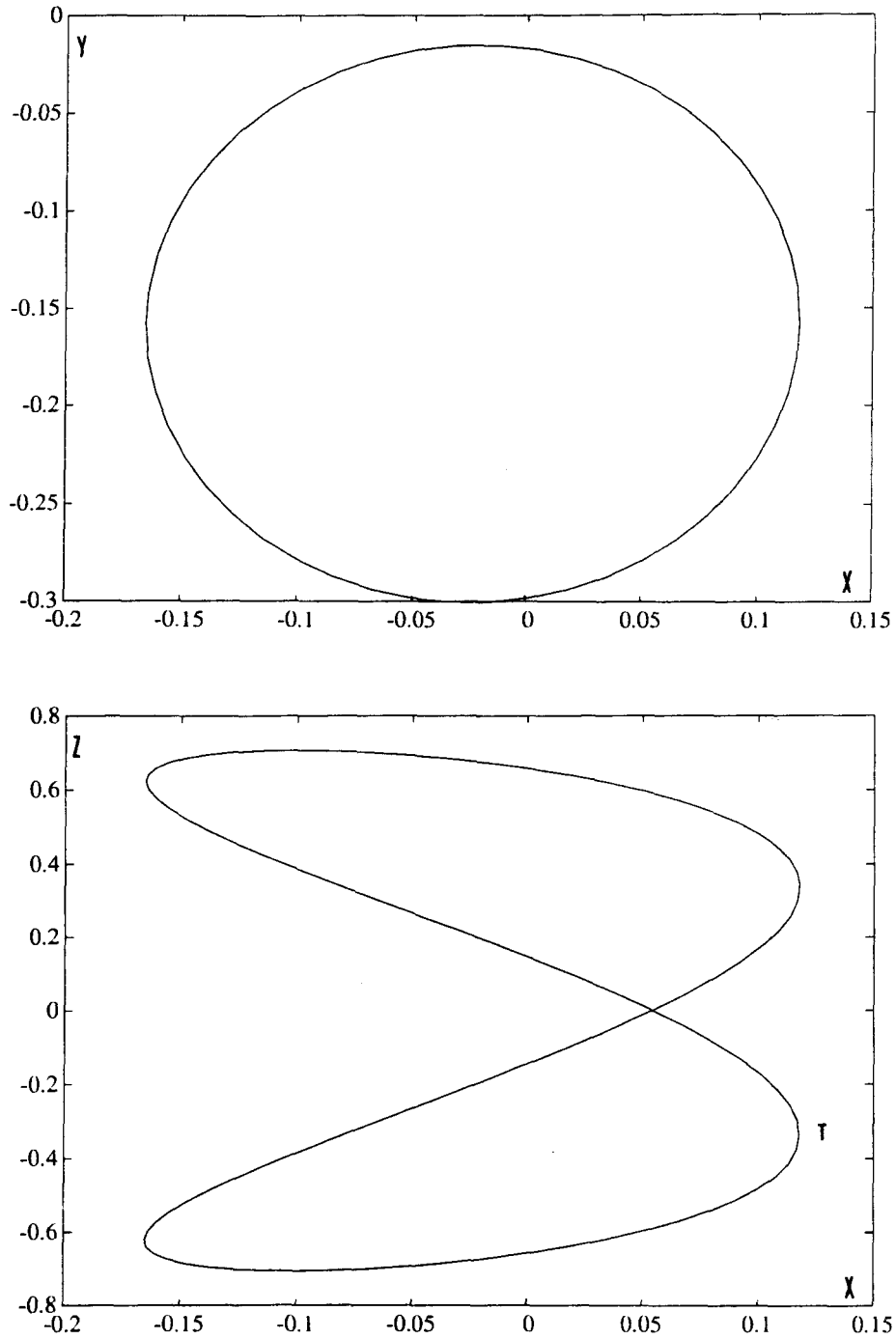
Figure 3.13 Projection of the periodic orbit of averaged model equations.

3.5 Numerical Refinement

In order to obtain a good nominal orbit we have implemented a parallel shooting method (see [21]) to get a solution of the JPL model very similar to the one found with the algebraic manipulator.

Now, let us see some details about this parallel shooting. First of all, we split the time span $[a, b]$ in which we want to find the nominal orbit in several pieces $[t_i, t_{i+1}]$, $i \in \{0, \ldots, n-1\}$, verifying that $t_0 = a$, $t_n = b$ and $h = t_{i+1} - t_i = (b-a)/n$ (different time intervals can be used but we have chosen here all of them equal). Now let i be a value between 1 and $n - 1$ and let $\mathcal{F}_i$ be the function defined as follows: if x_i is a point of the phase space, then $\mathcal{F}_i(x_i) = y_{i+1}$, where y_{i+1} is the value of the solution defined by the initial conditions (t_i, x_i) at time t_{i+1}, under the flow of the JPL model. Now we consider the vector $(x_0, \ldots, x_n)$, where x_i are coordinates in the phase space corresponding to time t_i. We can define $\mathcal{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_{n-1})$ in the usual way:

$$(x_0, \ldots, x_n) \overset{\mathcal{F}}{\longmapsto} (y_1, \ldots, y_n),$$

and note that, if the values x_i are all from the same solution of the JPL model, then we should have that $y_i = x_i$, $1 \le i \le n$. Then we impose these conditions and this leads us to solve a nonlinear system of $6n$ equations and $6n + 6$ unknowns. To do this, we have used again a Newton method and we have taken as initial guess the values provided by the planar solution obtained using the algebraic manipulator. Due to the fact that we have more unknowns than equations, we have added six more conditions: we have fixed the initial position (the three first components of x_0) and the final one (the three first components of x_n). With this, the system we deal with has $6n$ equations and the same number of unknowns. Finally, we note that the linear system that appears at each iteration of the Newton method has a band matrix. For this reason we have written a routine adapted to this kind of matrix. Another way to perform the parallel shooting procedure is the following: instead of adding to the set of $6n$ equations the above mentioned six conditions (which in some sense can force the solution in a non natural way), replace them trying to minimize, with respect to some norm, the total corrections to be done at each step. The results of this procedure can be found in Appendix A.

Subroutine SISBAN This routine solves a linear system of equations with a banded matrix, using only the band of the matrix plus three diagonals. The method is based in Gaussian elimination with partial pivoting.

Subroutine OMPP This routine computes the initial guess for the Newton method, from a Fourier series corresponding to an approximation obtained by means of the algebraic manipulator. This guess is stored in a vector. This routine uses the routines AVALUA and DF of the manipulator.

Subroutine COIN This routine changes coordinates, from the normalized ones to ecliptic ones (JPL), to the vector of initial conditions obtained by OMPP.

Subroutine FUN This routine computes the value of the function we want to make zero. For this purpose it performs a numerical integration of the equations of motion in the solar system using the JPL tapes to obtain the position of the bodies. The routine also integrates the variational equations and fills the Jacobian matrix of the function. The device used in this numerical integration has been the Shampine-Gordon method (see [18]).

Subroutine COOF This routine performs a change of coordinates, from the ecliptic ones to normalized ones, to the vector of final conditions obtained by the Newton method.

Main Program This routine reads the basic frequencies of the perturbations from the file FREQUE.VAL, the initial Fourier series (to obtain the initial condition for the Newton method) from the file QPL4.RES, some control parameters from the file TPL4.CTL (initial time, time step, number of time steps for the Newton process as well as thresholds for the numerical integration and a tolerance to know when the Newton process has finished) and the model of the solar system used from the file MODEL.DAT. With this, the program performs a Newton method (using the routines mentioned above), writing in the console some information about what it is doing. When the solution is found, the program writes the initial condition of the final orbit in the file TPL4.RES.

3.5.1 Final Results

Now, using the program described above, it is possible to find an orbit for the JPL model with a behaviour similar to the one of the planar solution obtained by the manipulator. In Figure 3.14 we show the numerical refinement of the planar orbit obtained in Section 3.4.4 for a time span of five years starting at the year 2000. The time step used in the parallel shooting has been of 1 day. To refine this orbit for longer time spans is very difficult (the parallel shooting does not converge) due to the fact that there exists a resonance (already mentioned in section 3.4.4) implying that this solution is not an exact solution of the system but a minimum of the residual acceleration. Nevertheless, the solution found for five years has a very slow "diffusion" and it goes away of L_4 slowly. As an example of this, Figure 3.15 shows its behaviour for more than seven years, and it looks like quasiperiodic.
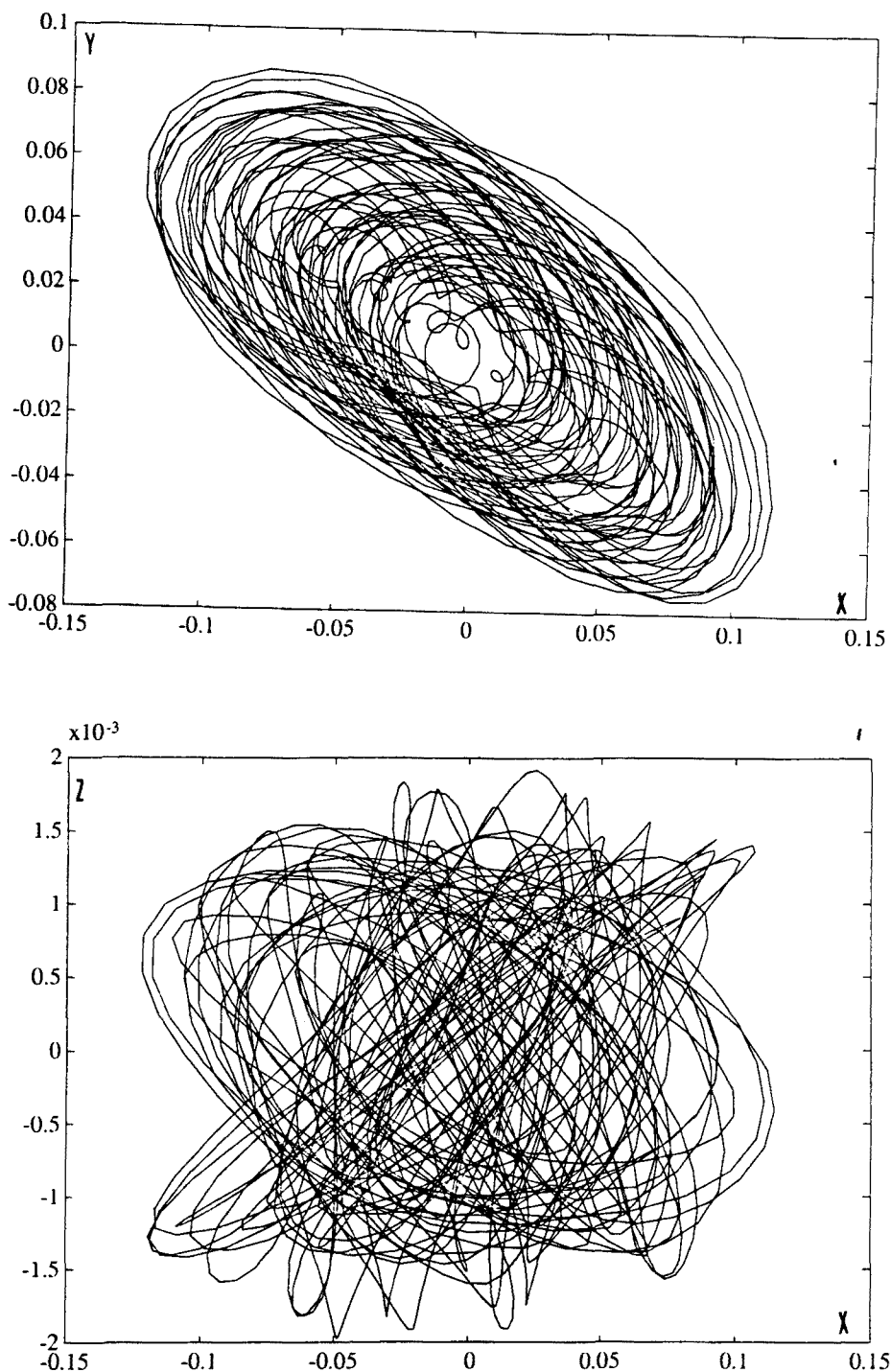
Figure 3.14 Numerical refinement of the planar orbit for a time span of 5 years.
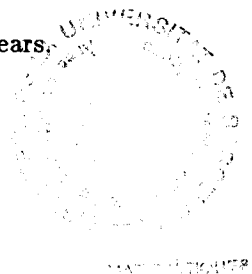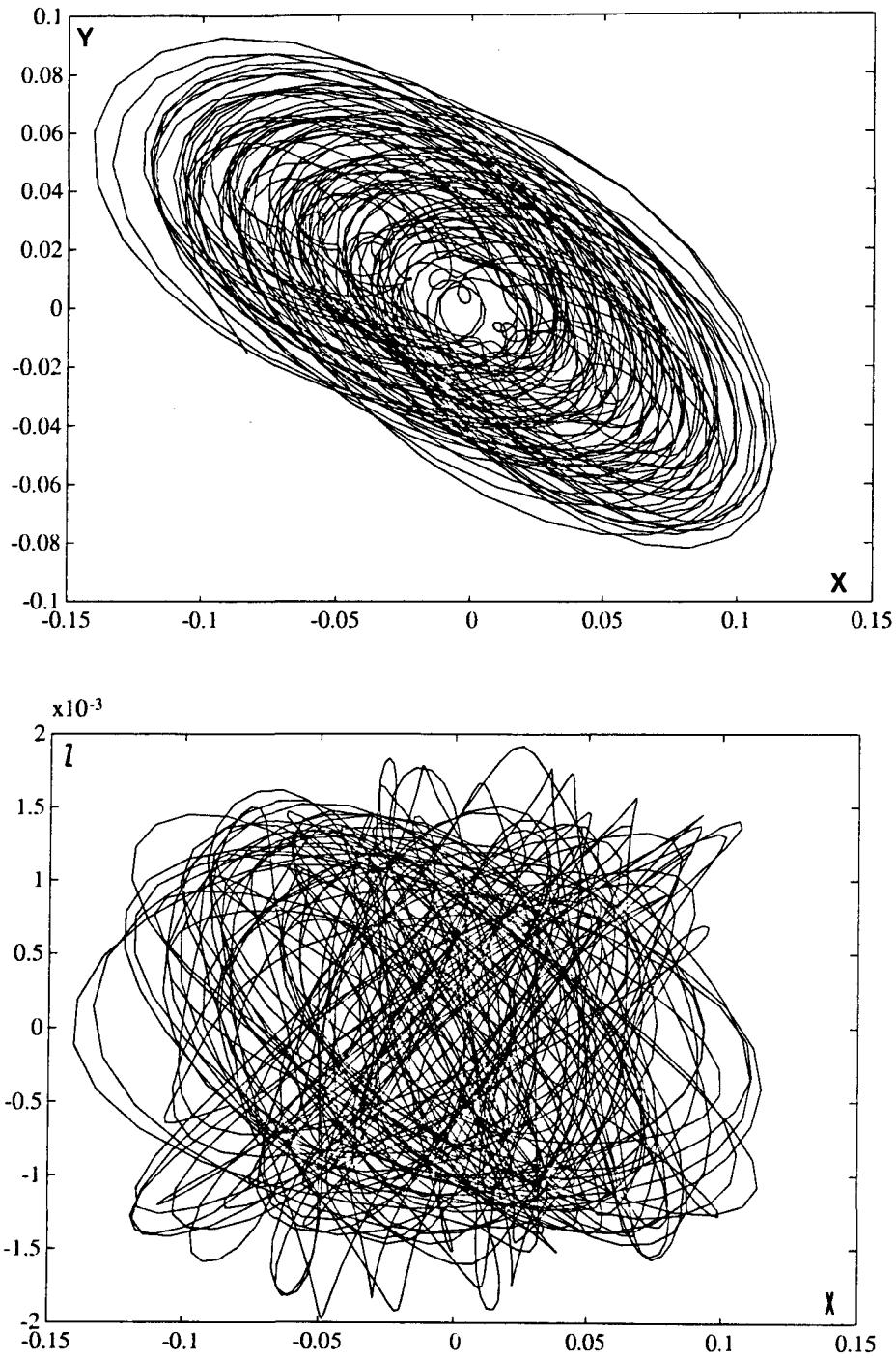
Figure 3.15 The same orbit of Figure 3.14 but integrated for a larger time span, which goes from 500 days before 2000.0 to 2100 days after 2000.0.

3.6 The Behaviour Near the Computed Nearly Quasiperiodic Solution

To study both the stability properties of the computed solution and the relative motion of particles running in orbits close to that one, we have performed a variational study. Of course, in this case we can not talk about monodromy matrix because we never return to the initial conditions or, even if this happens, the subsequent motion will be different. But we can look for points returning close to the initial conditions on the phase space. Figure 3.16 gives the distance to the initial conditions in adimensional units as a function of time. One can see a typical quasiperiodic pattern and after, roughly, 1625 days, we are not too far from the initial point. Now we can consider the variational matrix after that time interval as if it were a monodromy matrix.

On the other hand, during the integration of the variational equations (using the JPL model, of course), we can record the spectral radius of the instantaneous variational matrix. That value has been plotted as a function of time in Figure 3.17, during the full period 2000.0 to 2005.0. Letting aside some local (in time) irregularities on the first 600 days, the pattern is quite neat. These irregularities can be due to the strong constraints on the parallel shooting method which forces the initial and final positions to coincide with the one given by the semianalytical approximation. From the end of the second year on, there are large oscillations with half a year period with superimposed smaller oscillations with period roughly equal to one half of the lunar sinodic period. This can be better seen in Figure 3.18, where we have plotted the quotient $\ln \rho(t)/t$. Here t is the time elapsed since the initial epoch 2000.0 and $\rho(t)$ is the spectral radius (i.e. the maximum of the moduli of the eigenvalues) of the variational matrix at the current time. To skip the initial irregular behaviour this has been plotted starting at the year 2002.0 and the abscisae in the figure denote days since 2002.0. The limit of $\ln \rho(t)/t$ should give the maximal Lyapunov exponent. This seems to be close to 3.65×10^{-3} if the time unit is one day. This implies an average increasing of the distance to the computed orbit at a rate of $\exp(365 \times 3.65 \times 10^{-3}) \approx 3.79$ per year. This is an extremely mild instability and offers no problems concerning station keeping.

We return to the approximate monodromy matrix after 1625 days. Let λ_i, $i = 1, \ldots, 6$ the eigenvalues of this matrix. The numerical values are

$$\lambda_1 = -30.664197, \quad \lambda_3 = -0.079456 + 0.996838i, \quad \lambda_5 = -0.511657,$$
$$\lambda_2 = -1.954433, \quad \lambda_4 = -0.079456 - 0.996838i, \quad \lambda_6 = -0.032611.$$

They appear almost exactly in reciprocal pairs ($\lambda_j \lambda_{7-j} = 1$) as it happens for autonomous Hamiltonians. Note that the chosen epoch (2000.0 + 1625 days) is particularly good. For that epoch $\ln \rho(t)/t$ equals 2.11×10^{-3}. But we recall that the limit

value 3.65×10^{-3}, say, is an average. The eigenvalue λ_1 (resp. λ_6) is related to a mild instability (resp. stability). Then λ_2 (resp. λ_5) to a very mild instability (resp. stability), and λ_3, λ_4 are related to neutral behaviour (on the average during that time interval). The corresponding eigenvalues, e_1 to e_6, are given in the following Table, with modulus normalized to 1.

e_1	e_2	e_3, e_4	e_5	e_6
-0.163767	-0.286769	$0.169256 \mp 0.395523i$	0.321951	0.375073
0.964713	0.940385	$0.946713 \pm 0.000000i$	0.930086	0.904416
0.072496	0.053245	$-0.110340 \pm 0.913048i$	-0.033994	-0.074290
0.189793	0.162855	$0.208013 \pm 0.002104i$	0.160132	0.176994
-0.032347	-0.059263	$0.028111 \mp 0.093038i$	0.061770	0.064810
-0.013662	-0.023880	$0.137305 \pm 0.035283i$	-0.025978	-0.017493

We note that the neutral modes have a stronger relation with the vertical direction $(z, \dot{z})$ that the other ones. This is related to the suspected resonance between the proper vertical frequency of the restricted problem around L_4 and some of the exciting frequencies.

To see the behaviour of a nearby particle we have computed the following vectors: let $A(t)$ the solution of the variational equations starting with $A(0) = \mathrm{Id}$, where time is counted since the epoch 2000.0. Then $e_j(t) = A(t)e_j$, $j = 1,\ldots,6$ will give the relative evolution of a particle with respect to the basic orbit (or of one particle with respect to another one if they are located initially at $p \pm \alpha e_j$, where p is the initial condition corresponding to the quasiperiodic solution and α is a small quantity). The vectors $e_j(t)$, $j = 1,\ldots,6$, with $e_j(0)$ of modulus 1, have been projected on the (x, y) plane. Figures 3.19, 3.20, 3.21, 3.22, 3.23 and 3.24 present the results for the modes $e_1(t)$ to $e_6(t)$ respectively. To see the dominant exponentially increasing (resp. decreasing) character of the modulus of $e_1(t)$ (resp. $e_6(t)$) we present the figures 3.19 and 3.24. Figure 3.19 has been obtained as follows: let $(r(t), \theta(t))$ be the polar coordinates of the (x, y) projection of $e_1(t)$. Compute $\rho(t) = \mathrm{argsinh}(r(t))$ and plot the polar coordinates $(\rho(t), \theta(t))$. Figure 3.24 has been obtained with the same procedure that 3.19, but with the scaling $\rho(t) = \mathrm{argsinh}(20r(t))/\mathrm{argsinh}(20)$. Figures 3.20, 3.21, 3.22 and 3.23 have not any kind of scaling. In figures 3.19 and 3.24 one can see the superposition of two dominant periodic contributions added to the exponential behaviour.

Finally if we consider two nearby particles located at symmetrical positions with respect to the quasiperiodic solutions, we suggest the posibility of putting them along the unstable direction. This is because the line joining the particles is sweeping all the directions close to the ecliptic. Assume the relative distance is initially 1 Km. Then, after one month, their distance will increase, on the average, by $\exp(30\ln(\lambda_1)/1625) - 1 \approx 0.065$ Km.
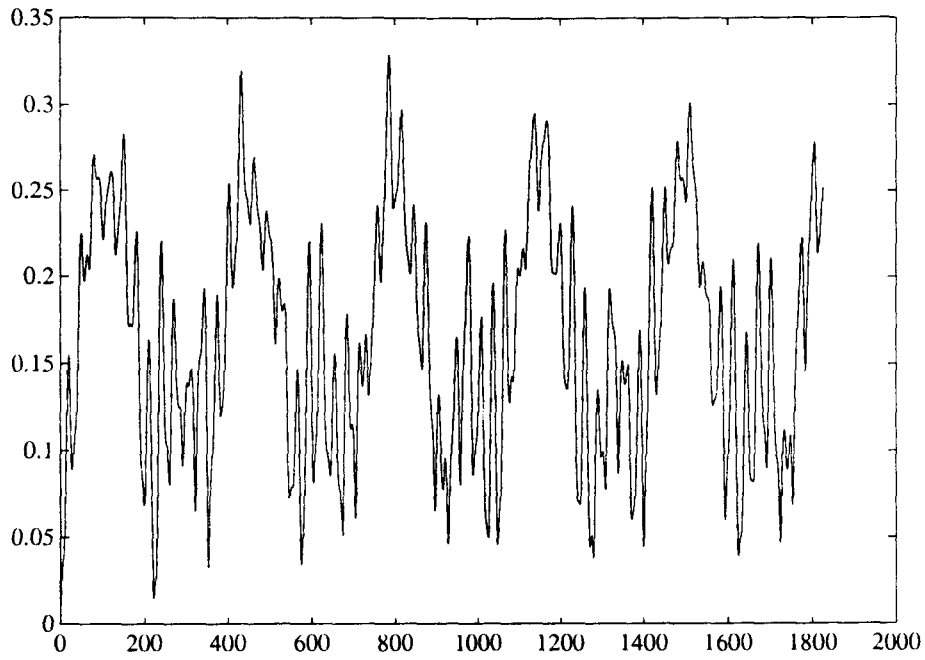
Figure 3.16 Distance to the initial conditions in adimensional units as a function of time (in days) from 2000.0 to 2005.0.
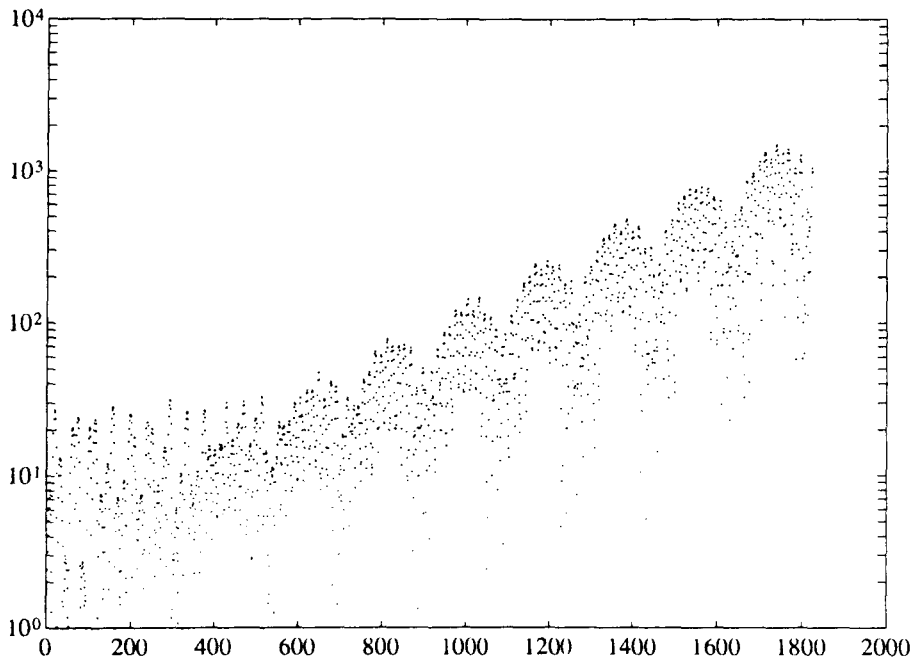
Figure 3.17 Spectral radius of the variational matrix as a function of time (in days) from 2000.0 to 2005.0.
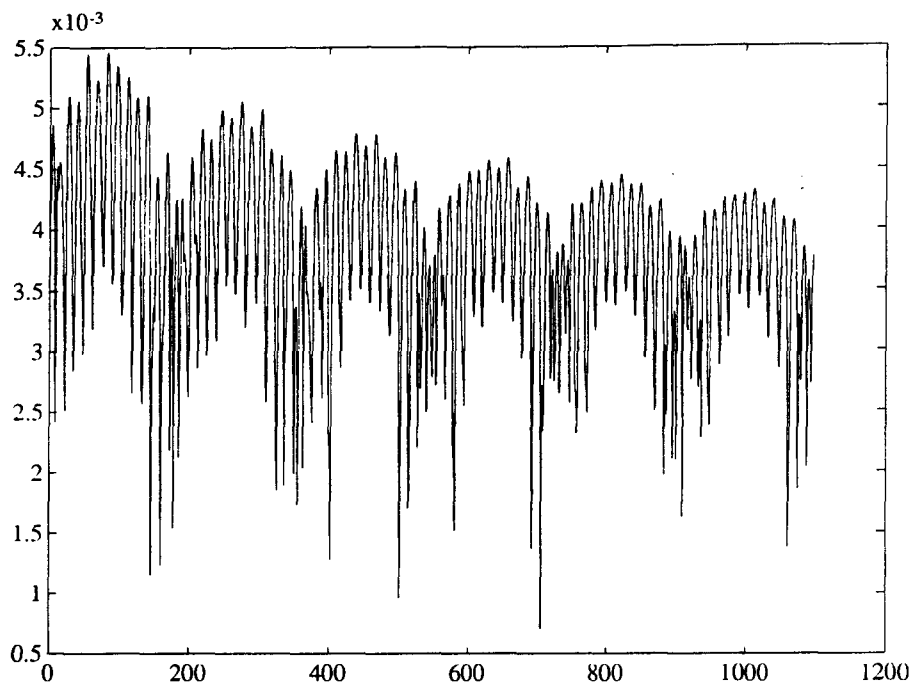
Figure 3.18 Values of $\ln \rho(t)/t$, where the time t (in days) ranges from 2002.0 to 2005, and $\rho(t)$ is the spectral radius of the variational matrix.
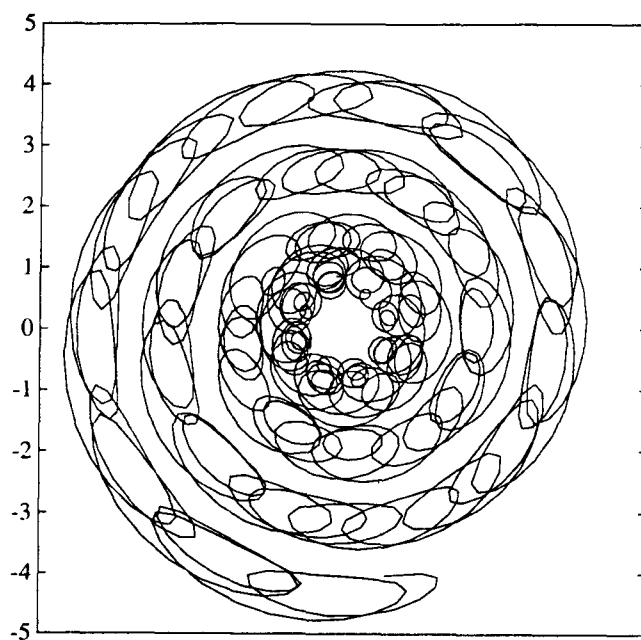
Figure 3.19 (x, y) projection of the mode $e_1(t)$. Polar coordinates have been used, with the scaling $r \sim \text{argsinh}(r)$. Time span: 1625 days.
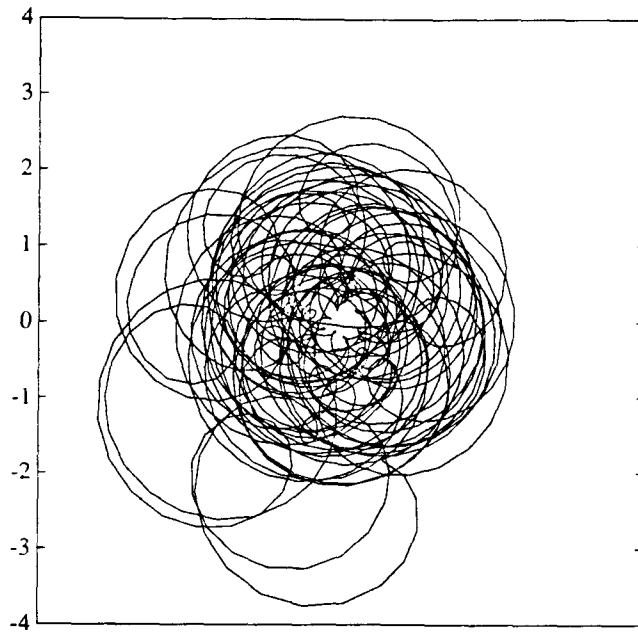
Figure 3.20 (x, y) projection of the mode $e_2(t)$. Time span: 1625 days.
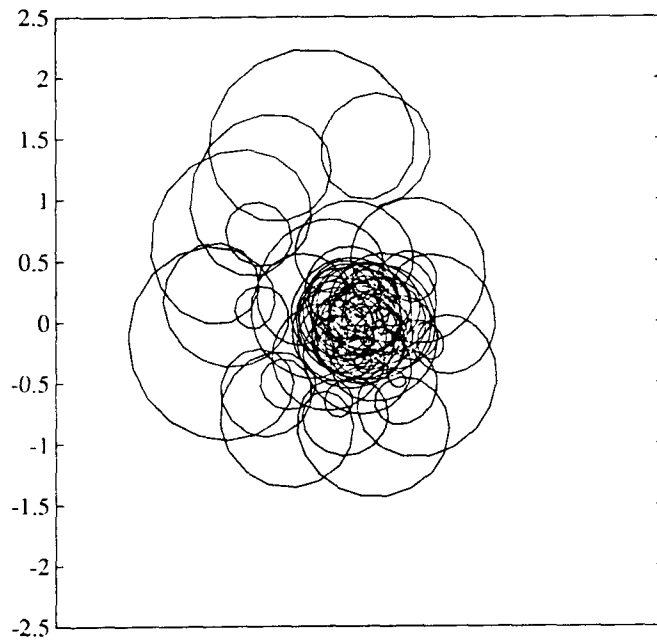
Figure 3.21 (x, y) projection of the real part of the mode $e_3(t)$ (Note that $\text{Re}(e_3) = \text{Re}(e_4)$). Time span: 1625 days.
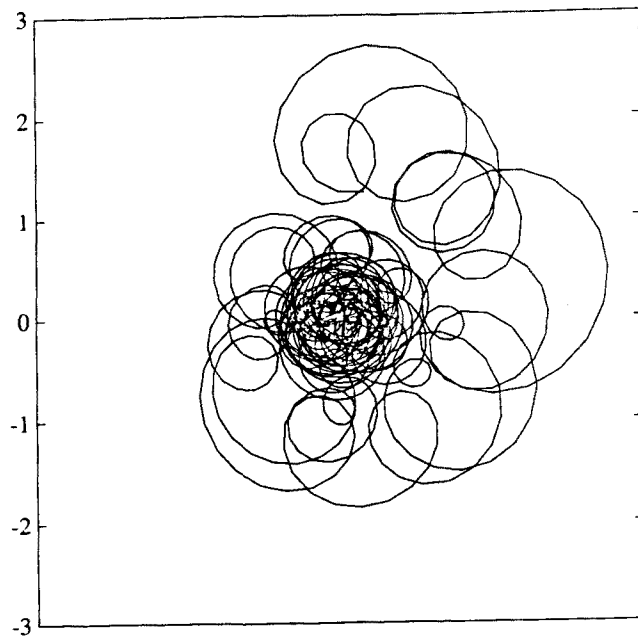
Figure 3.22 (x, y) projection of the imaginary part of the mode $e_3(t)$ (Note that $\text{Im}(e_3) = -\text{Im}(e_4)$). Time span: 1625 days.
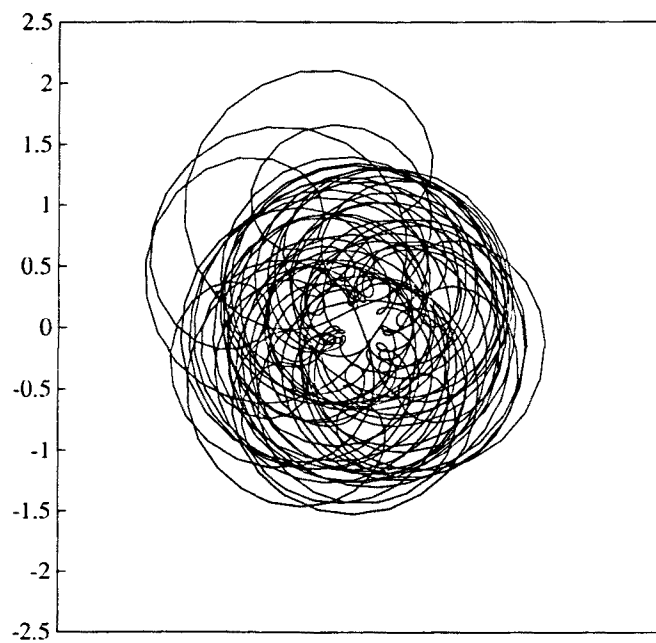
Figure 3.23 (x, y) projection of the mode $e_5(t)$. Time span: 1625 days.
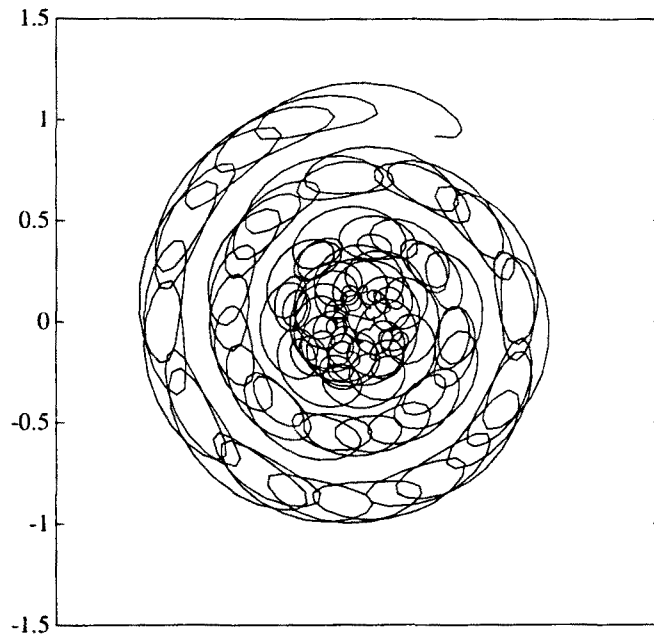
Figure 3.24 (x, y) projection of the mode $e_6(t)$. Polar coordinates have been used, with the scaling $r \sim \mathrm{argsinh}(20r)/\mathrm{argsinh}(20)$. Time span: 1625 days.

3.7   Problems and Extensions

Looking for the behaviour of the orbits near the geometrical triangular points of the Earth-Moon system we are faced with several problems. If we consider first the RTBP the phase space is filled up by periodic orbits, two-dimensional and three-dimensional tori as well as stochastic zones. The last ones are extremely small near the equilibrium point but the size increases when the distance to the equilibrium point does. To this simplified model we add relatively very important perturbations, mainly due to the effect of the Sun and to the non circular motion of the Moon and the Earth. Those perturbations can be assumed to be quasiperiodic for moderate time intervals. This produces strong resonances in some domains of the phase space and therefore large deviations from the (relatively) clear situation found for the RTBP. For small perturbations we have just to add a kind of quasiperiodic swinging to the previous structure. But the actual size of the perturbations is big enough to produce bifurcations loosing uniqueness. Therefore this is a source of problems for the analytical approximate solutions and for the subsequent numerical refinement as stated in 3.4.4 and 3.5.

The global problem, that is, the study with great detail of the real behaviour in a full neighbourhood of the former equilibrium point is worthy. It is useful for the design of future missions. From the abstract point of view there is a lack of supporting theory. From the analytical and numerical points of view this is a formidable task. We think that the methodology to be used to deal with this problem would be useful in many other contexts.

Appendix A

The Modified Parallel Shooting Method

A.1 Introduction

As it has been mentioned before (see Section 3.5), the parallel shooting method needs some extra conditions, because the matching ones are not enough to provide a system with the same number of equations and variables. To overcome this difficulty is usual to add some boundary conditions to the initial and final points of the orbit, but sometimes these conditions can force the solution in a non natural way. In order to avoid this we shall not add extra conditions, and we shall apply Newton's method directly. This leads to a linear system with more unknowns than equations, having (in the general case) an hyperplane of solutions. Then, we select one of these solutions to proceed. Now, the criterion to select this solution has to be fixed. It is natural to look for the solution nearest (with respect to some norm) to the initial condition of the Newton's method, and this leads us to choose the point of the hyperplane nearest to the origin. We have used the euclidean distance to select that point, but it is possible to use other distances.

A.1.1 The Program

This is essentially the same program that the one described in Section 3.5, but with some modifications inside its routines. The biggest one is inside routine SISBAN, because it is the one that looks for the solution of minimum norm. There are several ways of computing that solution, but due to the fact that the linear system we deal with has a band matrix and the dimension is very large, we have look at that point as the orthogonal projection of the origin on the hyperplane of solutions (as the distance is the euclidean one, the scalar product is the euclidean one). With

this, it easy to see that the point we are looking for is

$$x = a + \sum_{i=0}^{s} c_i \vec{v}_i,$$

where a is a point of the hyperplane, s is the dimension of that hyperplane, $\vec{v}_i$ are its director vectors and c_i are the components of the array c defined as $Mc = b$, where $M = (m_{ij})$, $b = (b_i)$, $m_{ij} = <\vec{v}_i, \vec{v}_j>$, $b_i = - <a, \vec{v}_i>$ and $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, s\}$. In the actual case we have that s is equal to 6 (see Section 3.5). Finally, in order to compute the point a and the vectors $\vec{v}_i$, the program uses Gaussian elimination with partial pivoting.

A.2   Results

Here the results obtained with this parallel shooting are shown. In order to compare both methods we have done the same computations that are in Chapter 3. The results obtained are in Section A.2.1. Then we have refined the orbit for a longer time interval (3000 days), (see Section A.2.2), in order to show that this modified method can overcome the convergence problems that the old one has.

A.2.1   Results for a Time Interval of 5 Years

Here we show the results of the numerical refinement of the planar orbit (provided by the algebraic manipulator, as it has already seen in Chapter 3) for a time-span of five years. Figure A.1 shows this refinement for a time interval starting at the year 2000.0 and lasting for five years. The time step used in the parallel shooting has been of 1 day, but it is possible to use longer time steps. Figure A.2 shows the orbit of Figure A.1, but for more than seven years, and its behaviour is still quasiperiodic.

Stability

We have done the same study that in Chapter 3 and the corresponding plots of that Chapter have been included in the next pages (Figures from A.3 to A.11). The technical details are not repeated here, and only the results are given. Concerning to the eigenvalues of the monodromy matrix after 1625 days, we include the following table.

$$\lambda_1 = -30.138752, \quad \lambda_3 = -0.072544 + 0.997365i, \quad \lambda_5 = -0.901362,$$
$$\lambda_2 = -1.109433, \quad \lambda_4 = -0.072544 - 0.997365i, \quad \lambda_6 = -0.033180.$$

The corresponding eigenvectors are

e_1	e_2	e_3, e_4	e_5	e_6
−0.160431	−0.114138	0.111424 ∓ 0.334360i	0.007236	0.290550
0.964742	0.978770	0.959823 ± 0.000000i	0.985659	0.934313
0.070793	0.018856	−0.084227 ± 0.938259i	0.003793	−0.071302
0.193770	0.165579	0.209092 ± 0.003455i	0.166308	0.185854
−0.027942	−0.022039	0.021572 ∓ 0.082562i	0.002511	0.052081
−0.013992	−0.026892	0.122659 ± 0.032316i	−0.027310	−0.017370

We recall the fact that the central directions have a stronger relation with the vertical direction than the stable and unstable ones. In the next figures one can see the (linear) behaviour inside each one of those manifolds. As in Chapter 3, they have been plotted by projecting the vectors $e_i(t)$ on the (x, y) plane, where $e_i(t) = A(t)e_i$, $i = 1, \ldots, 6$, $A(t)$ is the variational matrix at instant t and the vectors e_i are the eigenvectors of the table given above.
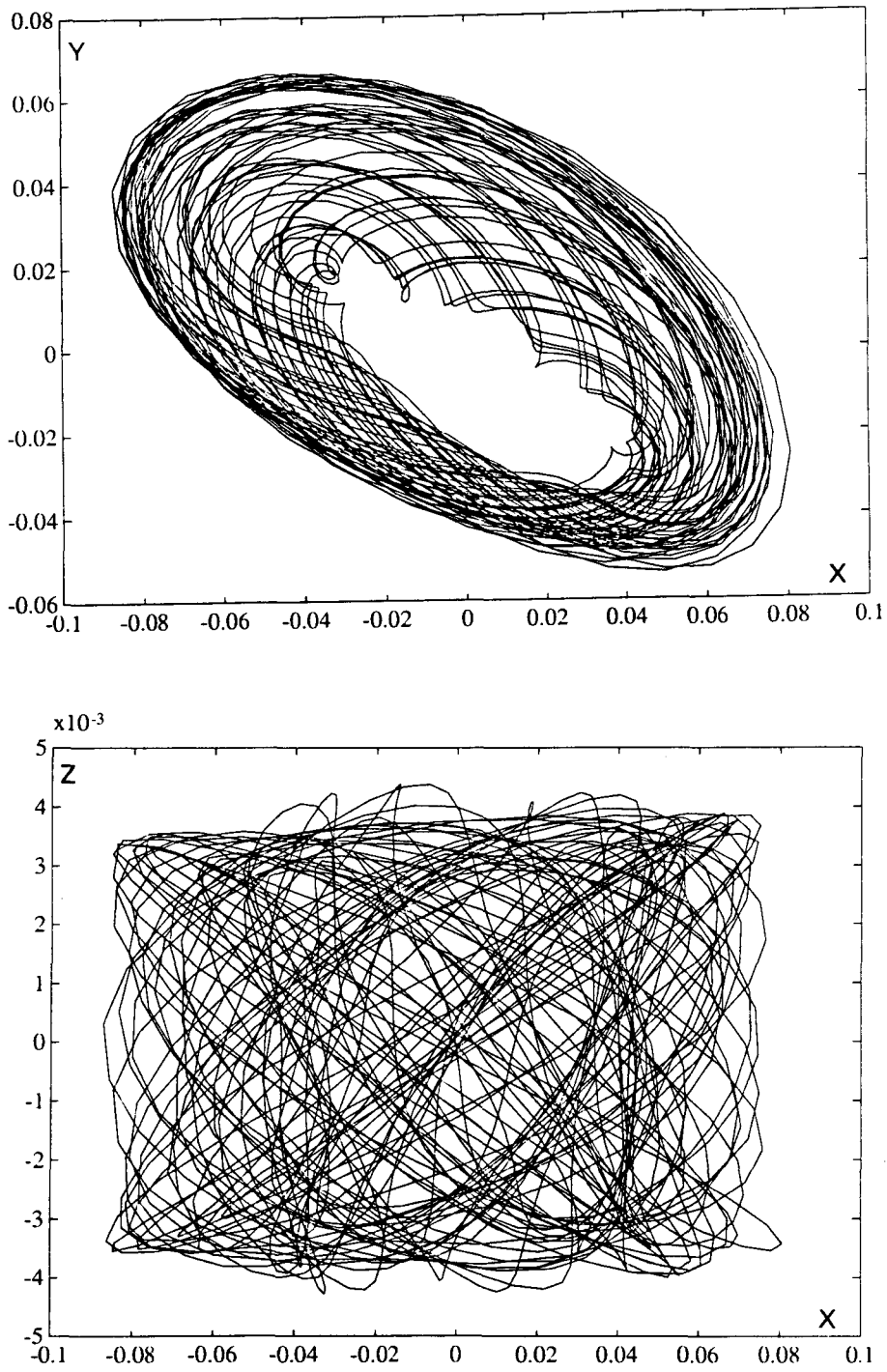
Figure A.1 Numerical refinement of the planar orbit for a time span of 5 years.
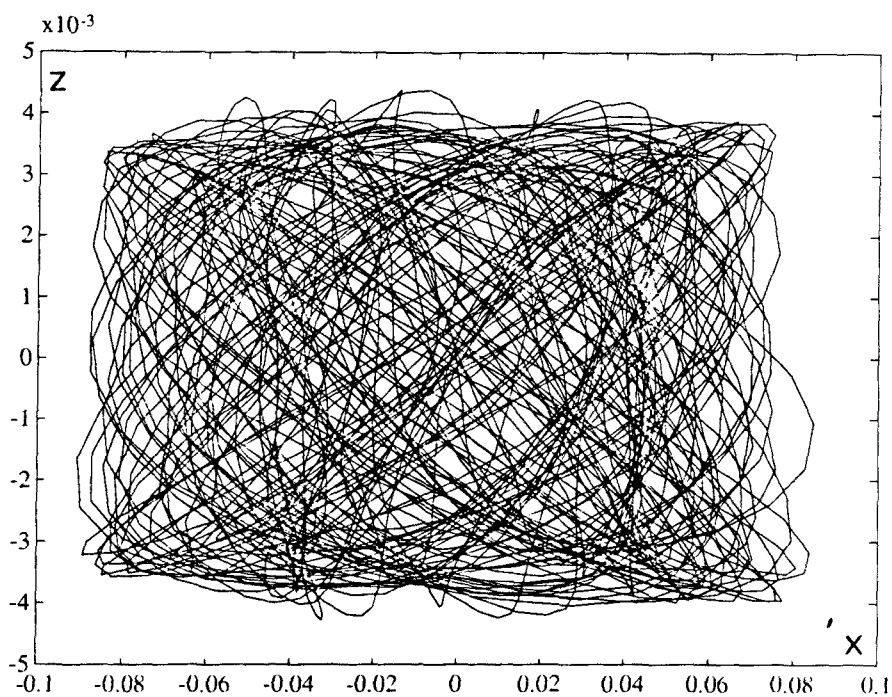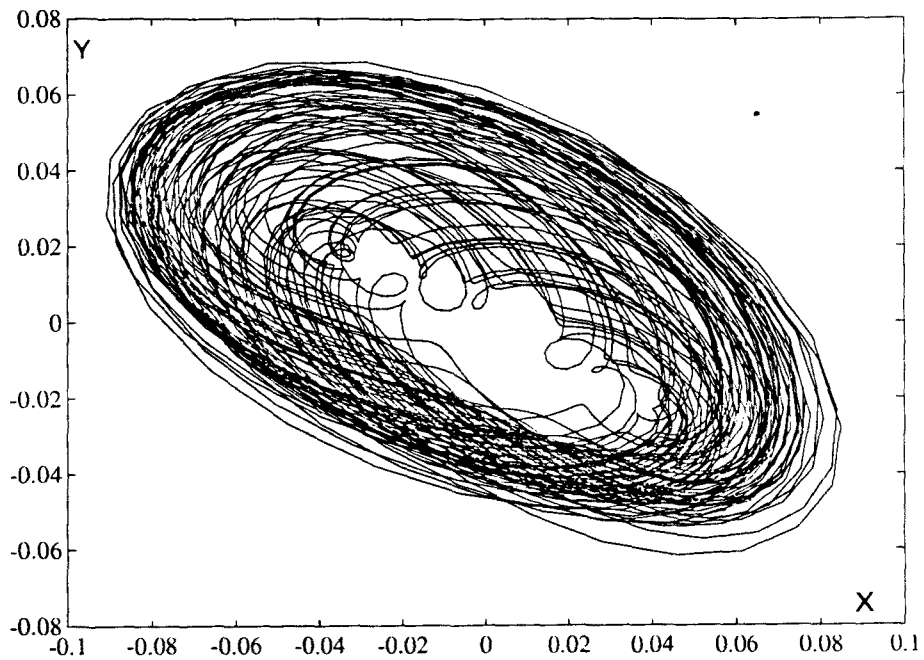
Figure A.2 The same orbit of Figure 1.14 but integrated for a larger time span, which goes from 500 days before 2000.0 to 2100 days after 2000.0.
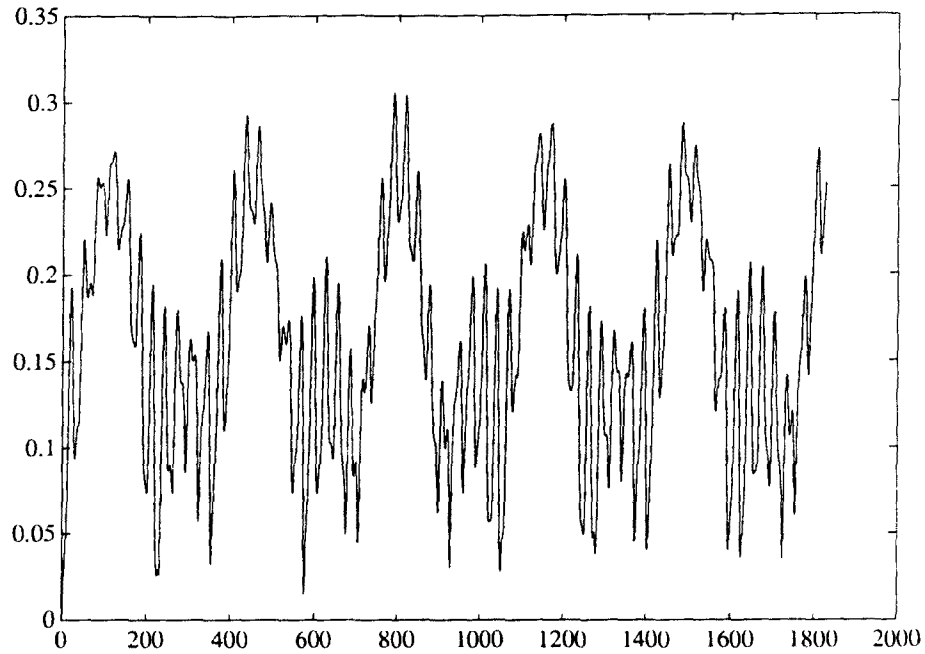
Figure A.3 Distance to the initial conditions in adimensional units as a function of time (in days) from 2000.0 to 2005.0.
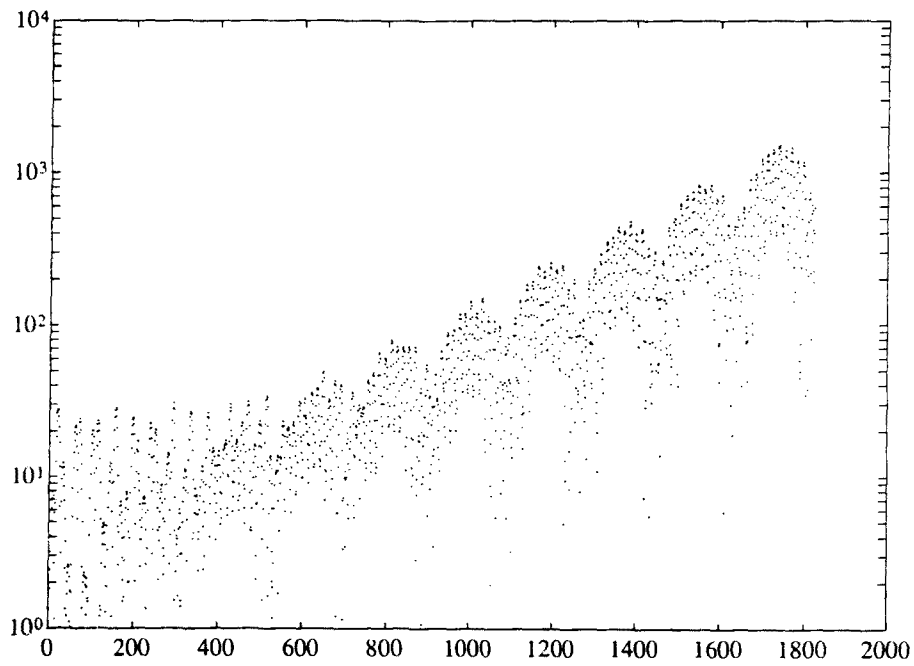
Figure A.4 Spectral radius of the variational matrix as a function of time (in days) from 2000.0 to 2005.0.
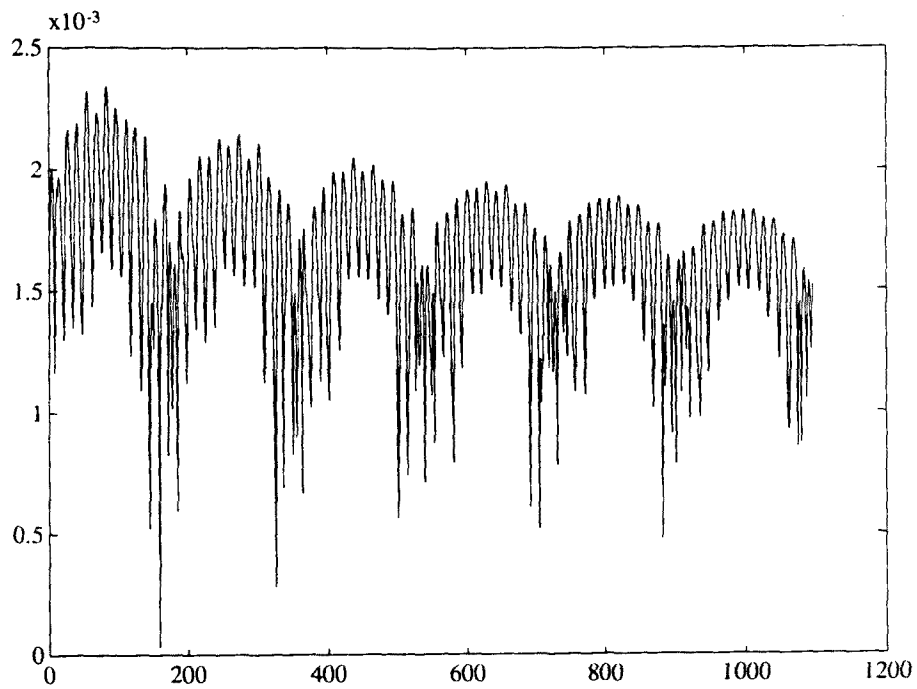
Figure A.5 Values of $\ln \rho(t)/t$, where the time t (in days) ranges from 2002.0 to 2005, and $\rho(t)$ is the spectral radius of the variational matrix.
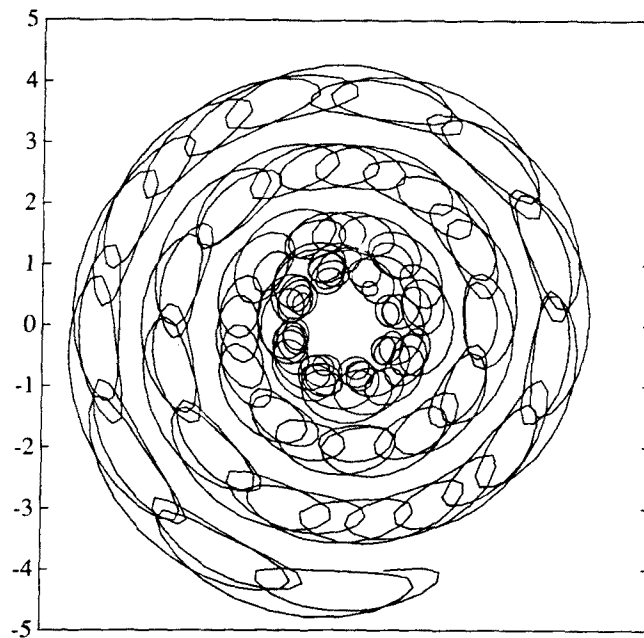
Figure A.6 (x, y) projection of the mode $e_1(t)$. Polar coordinates have been used, with the scaling $r \sim \mathrm{argsinh}(r)$. Time span: 1625 days.
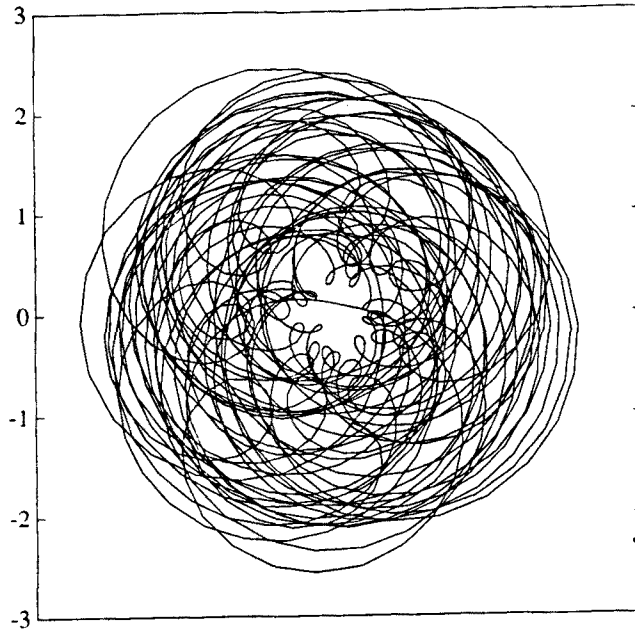
Figure A.7 (x, y) projection of the mode $e_2(t)$. Time span: 1625 days.
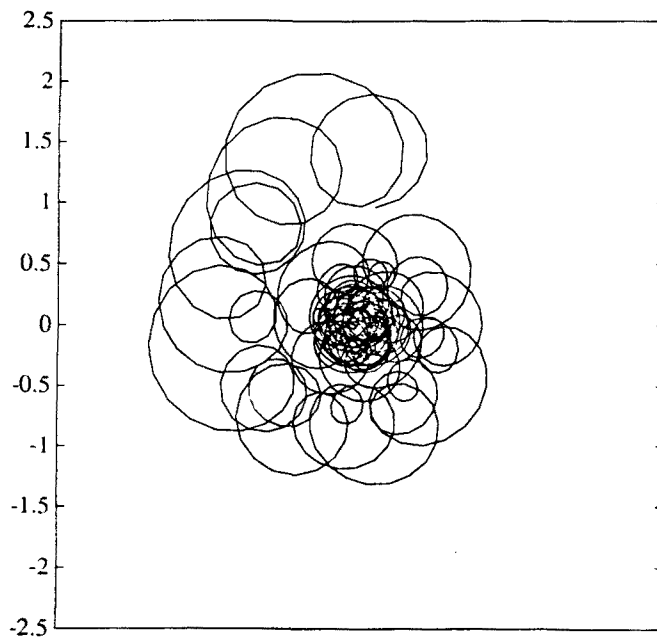
Figure A.8 (x, y) projection of the real part of the mode $e_3(t)$ (Note that $\text{Re}(e_3) = \text{Re}(e_4)$). Time span: 1625 days.
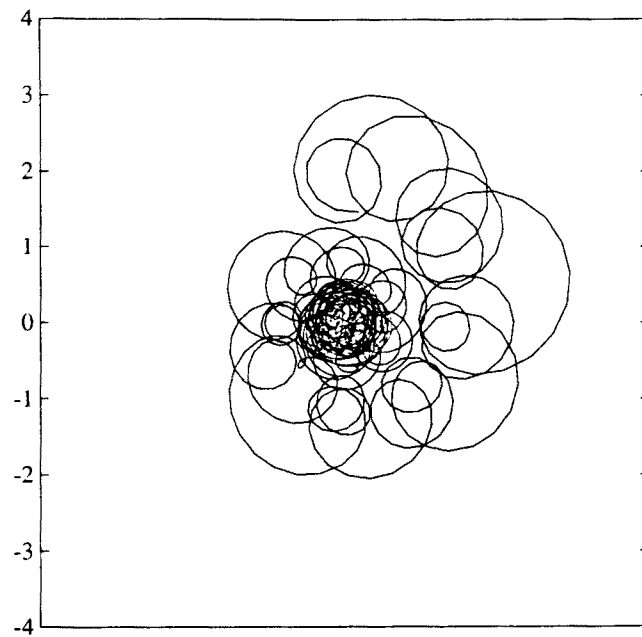
Figure A.9 (x, y) projection of the imaginary part of the mode $e_3(t)$ (Note that $\text{Im}(e_3) = -\text{Im}(e_4)$). Time span: 1625 days.
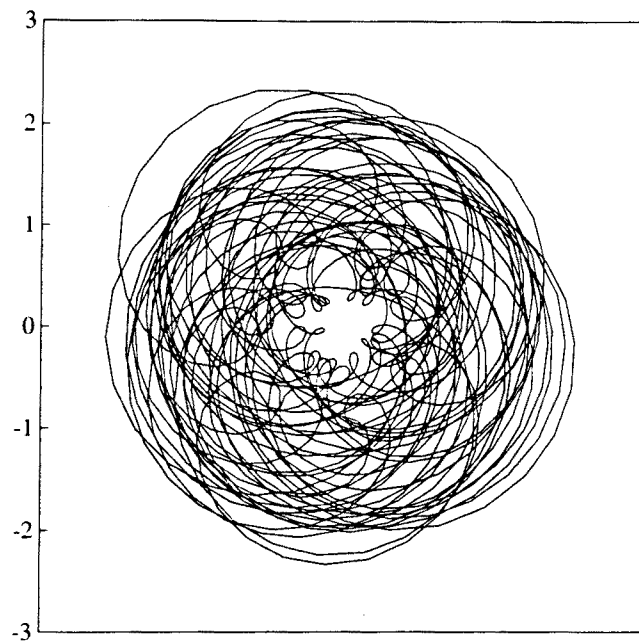
Figure A.10 (x, y) projection of the mode $e_5(t)$. Time span: 1625 days.
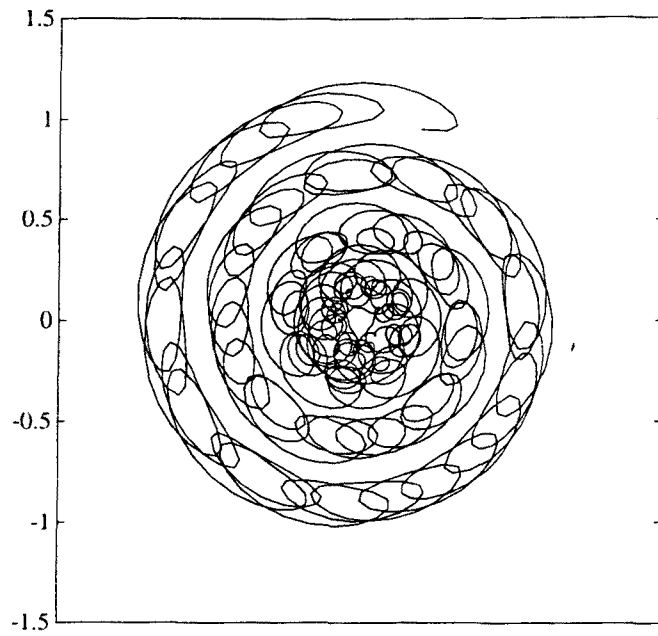
Figure A.11 (x, y) projection of the mode $e_6(t)$. Polar coordinates have been used, with the scaling $r \sim \mathrm{argsinh}(20r)/\mathrm{argsinh}(20)$. Time span: 1625 days.

A.2.2 Results for a Time Interval of 3000 Days

Here we have included, as an example, the results of the parallel shooting for a time interval of 3000 days. The time step used for this parallel shooting is of two days. Figure A.12 shows the final orbit of that numerical refinement, having the same behaviour that the one found in A.2.1.

Stability

As in the preceding section, we have performed the same study that the one of Chapter 3. Figure A.13 gives the distance to the initial conditions in adimensional units as a function of time. After 2772 days approximately the orbit passes near the initial point. As before, we consider the variational matrix after that time interval as if it were a monodromy matrix. Figure A.14 is a plot of the spectral radius of the variational matrix, that shows the same pattern that we have already found in previous sections. To estimate the maximal Lyapunov exponent the values of $\ln \rho(t)/t$ have been computed. A plot of them can be found in Figure A.15.

To obtain information about the stable, central and unstable manifolds we can look at the eigenvalues and eigenvectors of the variational matrix after 2772 days. The numerical values for the eigenvalues are

$$\lambda_1 = -4808.194, \quad \lambda_3 = 0.663928 + 0.747796i, \quad \lambda_5 = -0.048693,$$
$$\lambda_2 = -20.536687, \quad \lambda_4 = 0.663928 - 0.747796i, \quad \lambda_6 = -0.000208.$$

The corresponding eigenvectors are

e_1	e_2	e_3, e_4	e_5	e_6
0.881819	0.909740	$-0.172440 \pm 0.169469i$	0.464372	0.259074
-0.453662	-0.383081	$-0.079694 \pm 0.434039i$	0.861879	0.943558
-0.004352	0.004269	$0.979920 \pm 0.000000i$	-0.070925	-0.069959
0.001261	0.026596	$-0.026213 \pm 0.112874i$	0.169519	0.188344
0.128297	0.157289	$-0.049862 \pm 0.087807i$	0.086250	0.043980
0.010250	0.012531	$-0.022291 \pm 0.873178i$	-0.017819	-0.016630

The study of the behavior in those directions has been done in the usual way (see A.2.1). The results can be found in Figures A.16 to A.21.
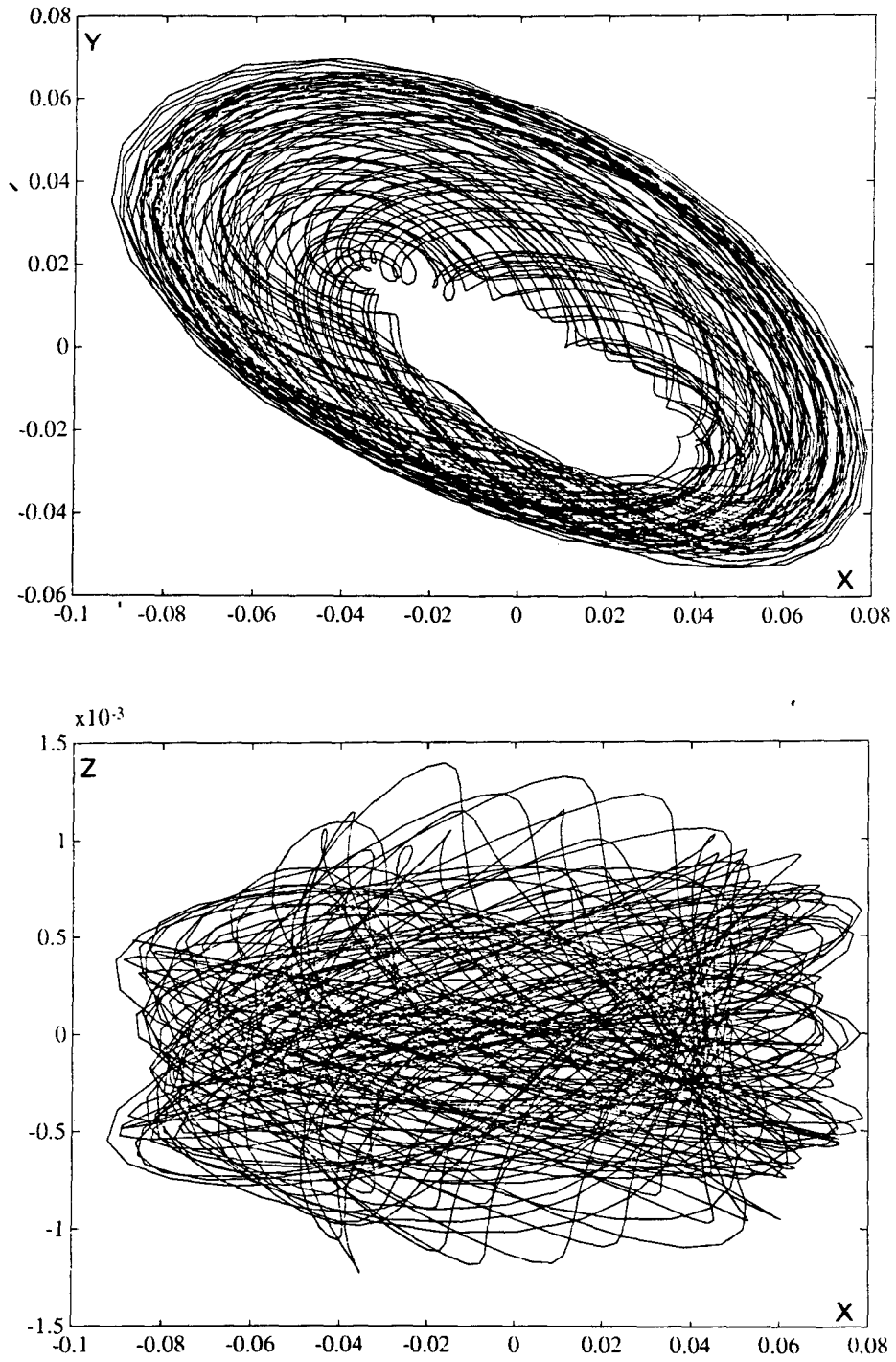
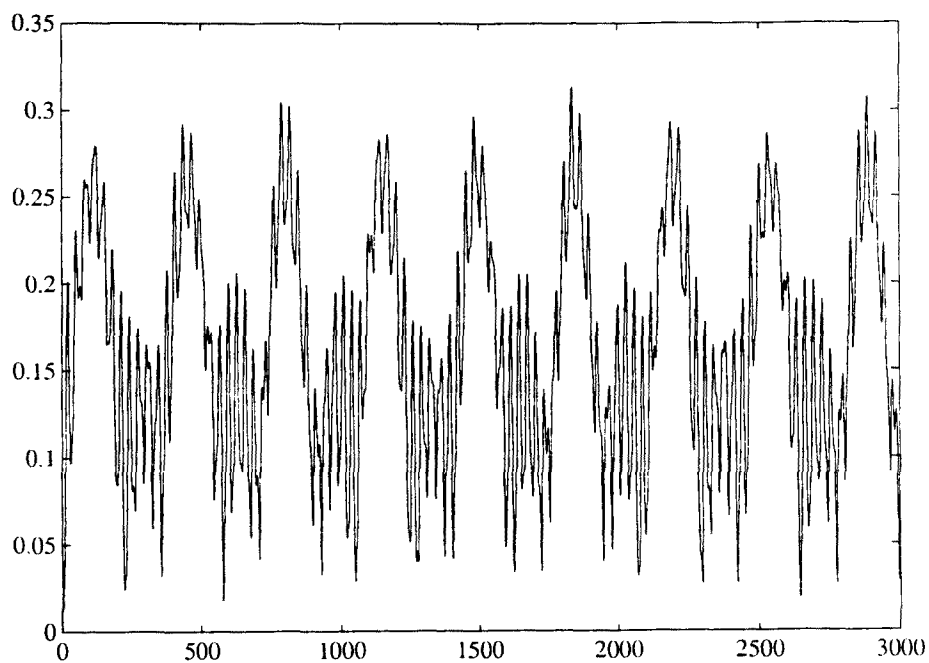**Figure A.12** Numerical refinement of the planar orbit for a time span of 3000 days.

Figure A.13 Distance to the initial conditions in adimensional units as a function of time (in days) from 2000.0 to 2008.2.
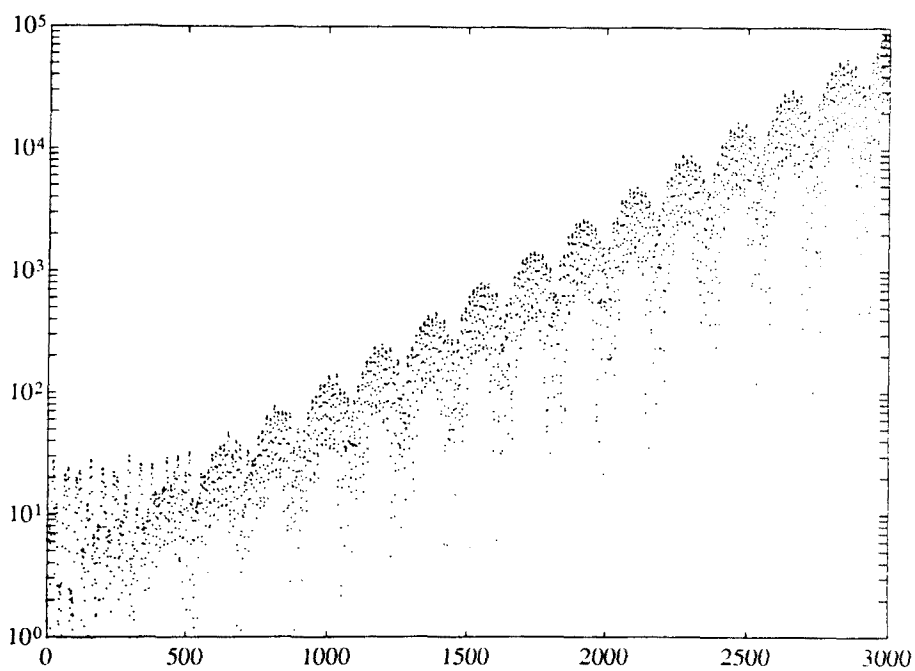

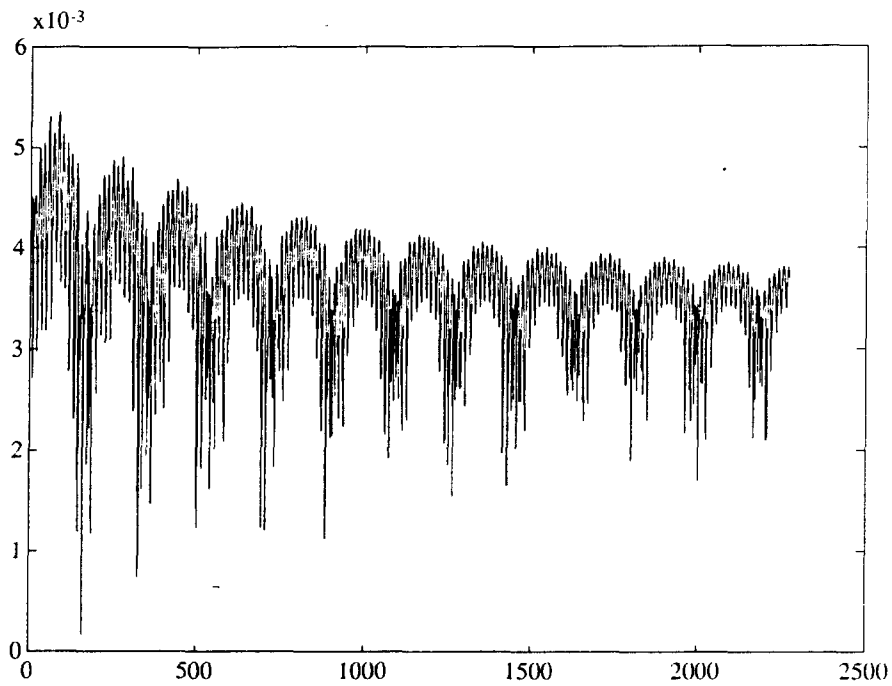**Figure A.14** Spectral radius of the variational matrix as a function of time (in days) from 2000.0 to 2008.2.

Figure A.15 Values of $\ln \rho(t)/t$, where the time t (in days) ranges from 2002.0 to 2008.2, and $\rho(t)$ is the spectral radius of the variational matrix.
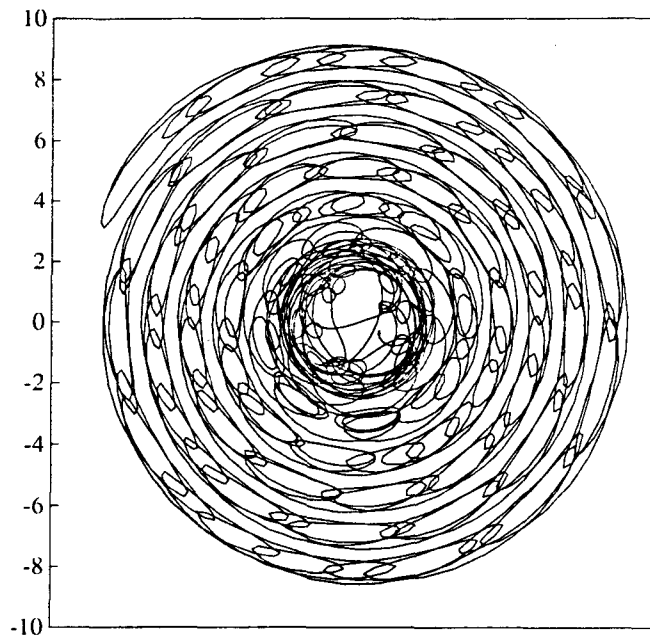

Figure A.16 (x, y) projection of the mode $e_1(t)$. Polar coordinates have been used, with the scaling $r \sim \text{argsinh}(r)$. Time span: 2772 days.
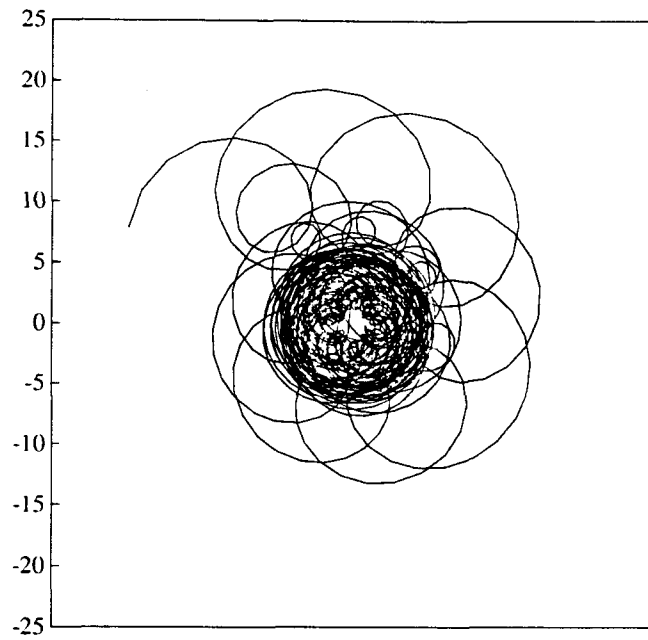
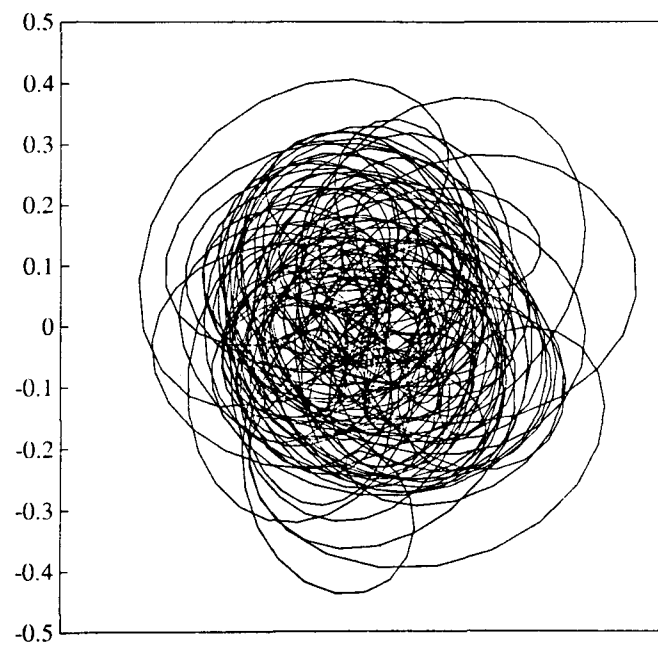**Figure A.17** (x, y) projection of the mode $e_2(t)$. Time span: 2772 days.

Figure A.18 (x, y) projection of the real part of the mode $e_3(t)$ (Note that $\text{Re}(e_3) = \text{Re}(e_4)$). Time span: 2772 days.
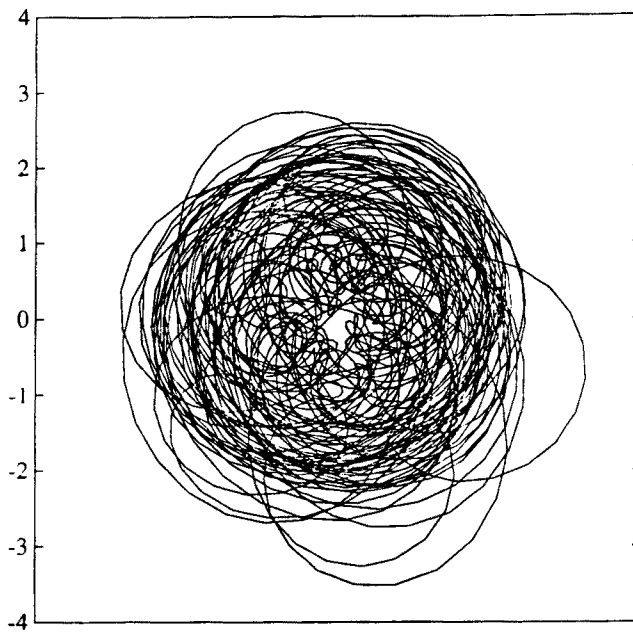
Figure A.19 (x, y) projection of the imaginary part of the mode $e_3(t)$ (Note that $\text{Im}(e_3) = -\text{Im}(e_4)$). Time span: 2772 days.
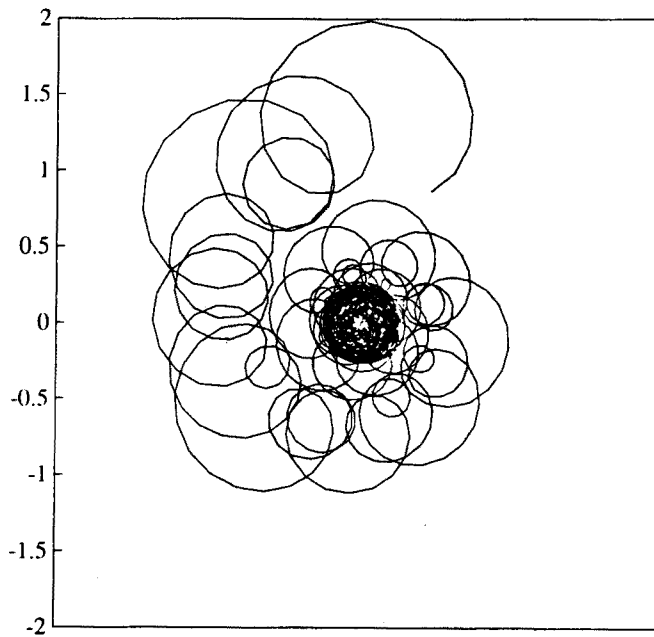

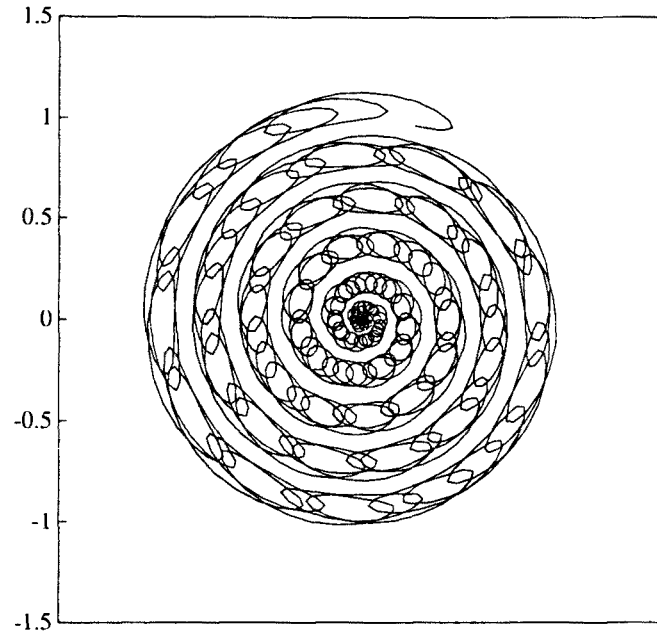**Figure A.20** (x, y) projection of the mode $e_5(t)$. Time span: 2772 days.

Figure A.21 (x, y) projection of the mode $e_6(t)$. Polar coordinates have been used, with the scaling $r \sim \text{argsinh}(100r)/\text{argsinh}(100)$. Time span: 2772 days.